UNIVERSITY OF CAPE TOWN

Department of Mathematics
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The Pullback Closure, Perfect Morphisms and Completions

by

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'The time has come,' the Walrus said,
'To talk of many things:
Of shoes — and ships — and sealing-wax —
Of cabbages — and kings —
And why the sea is boiling hot —
And whether pigs have wings.'

Lewis Carroll
"Through The Looking Glass"

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Introduction

Closure operations within objects of various categories have played an important role in the development of Categorical Topology. Notably they have been used to characterise epimorphisms and investigate cowellpoweredness in specific categories, to generalise Hausdorff separation through diagonal theorems, and to extend topological notions such as compactness of objects and perfectness of morphisms to abstract categories.

The categorical theory of factorisation structures for families of morphisms which developed in the 1970's laid the foundation for an axiomatic theory of categorical closure operators. This theory drew together many endeavours involving closure operations, and was coalesced in [Dikranjan, Giuli 1987]. The literature on categorical closure operators continues to extend the theory as well as apply it to problems in Category Theory.

Central to our thesis is a particular closure operator (in the sense of [Dikranjan, Giuli 1987]) which we name the "pullback closure operator". Its construction is not entirely new, but no author has studied this operator in its own right. We investigate some of the operator's properties, present several examples and then apply it in two areas of Categorical Topology.

First we use the pullback closure operator to establish links between two previously disjoint theories of perfect morphisms. One theory, which developed in the 1970's, exploits the orthogonality properties and functor related properties of perfect continuous maps. Another theory, which has developed more recently, generalises the closure and compactness properties of perfect continuous maps. (We should note that this does not include the recent work in [Clementino, Giuli, Tholen 1995] which takes another approach to perfect morphisms via closure operators.) Our investigations centre around finding conditions that are sufficient to ensure that the links between these two theories can be utilised.

Our second use of the pullback closure operator is in pursuing the precategorical ideas expressed in [Birkhoff 1937], and some developments of these ideas in [Brümmer, Giuli, Herrlich 1992] and [Brümmer, Giuli 1992], to build a theory of completion of objects in an abstract category. In this context the pullback closure operator is shown to be appropriate in characterising complete objects, illuminating links with previously studied completion notions and describing epimorphisms in the category in which we are working. (In fact the pullback closure operator can be used to describe epimorphisms in even wider contexts.)

Our methodology is what has been termed colloquially as "doing topology in categories". Topological notions and results are expressed in the language of category theory.

Using these reformulations, new results are pursued at the level of categories, and are then applied in specific topological or algebraic contexts. Within this, our approach has been to make as few global assumptions as possible. The pullback closure operator is strictly a tool, in the sense that when assumptions are made, they concern the underlying categories, functors and classes of morphisms and objects and not the operator itself.

We now provide a survey of the individual Chapters:

In **Chapter 0** we establish the categorical framework for subsequent chapters. A few deviations from conventional usage are highlighted and a brief exposition on categorical closure operators is given.

Chapter 1 introduces the *pullback closure operator* induced by a pointed endofunctor on a category. A number of topological and algebraic examples of endofunctors and their associated pullback closure operators are given. Some are closure operators that have not been considered before, others are well known closure operators that have not previously been described as pullback operators. We investigate the links between the pullback closure operator and other previously studied categorical closure operators.

Our attention is then focussed on properties of the pullback closure operator. Our main concern is to establish conditions on the category and on the endofunctor associated with the pullback closure operator that ensure that the operator itself will exhibit particular properties. Examples demonstrate our results. The chapter is concluded with a short study of the interrelation between the pullback closure operator and factorisation theory relative to pointed endofunctors.

In **Chapter 2** we apply the pullback closure operator to the study of perfect morphisms in a category. Categorical generalisations of the topological notion of a perfect map are many. We highlight five possible generalisations, and choose as central to our investigations a notion of *perfect morphism relative to a pointed endofunctor* that owes its formulation to a well known result of [Henriksen, Isbell 1958]. A basic theory of such perfect morphisms is developed – this is not an entirely new endeavour.

The Chapter then systematically explores how perfect morphisms of the chosen type relate to previous studies of perfect morphisms. The work of [Herrlich 1972, 1974], [Nel 1974] and [Strecker 1972, 1974, 1976] generalised the orthogonality properties of perfect continuous maps, while [Manes 1974], [Herrlich, Salicrup, Strecker 1987] and [Dikranjan, Giuli 1991b] generalised the closure and compactness properties of perfect continuous maps. By establishing sufficient criteria on the underlying category and endofunctor we provide results that show how the pullback closure operator links the orthogonality properties of a perfect morphism with its closure and compactness properties. Theorem 2.7.1 summarises the results of the Chapter before a number of topological and algebraic examples are given.

In **Chapter 3** the definition of a *strong functorial completion* is introduced. This precedes an exposition on the completion of objects in an abstract category, that has as its chief influence the precategorical work of [Birkhoff 1937]. A secondary influence on our exposition is the work of [Brümmer, Giuli, Herrlich 1992] and [Brümmer, Giuli 1992]. In this Chapter the role of the pullback closure operator induced by a strong functorial completion is seen to be central. It describes essential extensions, characterises "complete" objects and illuminates links with previous studies in categorical completion theory. Furthermore in some instances the pullback closure operator can be used to describe epimorphisms in the category in which we are working. The Chapter ends with a short investigation of situations – not necessarily involving strong functorial completions – where density with respect to the pullback closure operator characterises epimorphisms in a category.

At the end of each Chapter there is a short section where unsolved problems are raised.

Chapter 0

Preliminaries

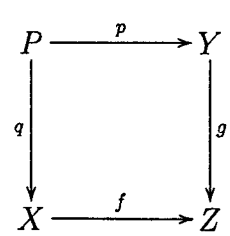
Categories

Our categorical terminology and notation is that of [Adámek, Herrlich, Strecker 1990]. There are a few deviations from their usage and some additional conventions which we highlight now.

Identities – morphisms, functors and natural transformations – are denoted by the numeral "1" with an appropriate subscript. For example $1_X : X \to X$ is the identity morphism on X.

For a product of objects $X \times Y$ in a category $\mathbf{X}$, and $\mathbf{X}$-morphisms $f : Z \to X$ and $g : Z \to Y$, $\langle f, g\rangle : Z \to X \times Y$ is the unique morphism such that $\pi_1\langle f, g\rangle = f$ and $\pi_2\langle f, g\rangle = g$ (where π_1 and π_2 are the projection morphisms). When the product $X \times X$ exists for an object X in $\mathbf{X}$, the unique morphism $\langle 1_X, 1_X\rangle$ is termed the *diagonal of* X and is denoted by Δ_X.

If the diagram below is a pullback square, we will at times adopt the convention of stating that the source $(P, (p, q))$ is "the pullback of f along g" or "the pullback of the sink $((f, g), Z)$".



We will at times talk of a class of objects in a category $\mathbf{X}$ as being a subcategory of $\mathbf{X}$. In such an instance we simply mean the full subcatory of $\mathbf{X}$ with that particular object class.

If $\mathbf{A}$ is a subcategory of $\mathbf{X}$, we say that a morphism $f : X \to Y$ in $\mathbf{X}$ is $\mathbf{A}$-*cancellable* if for any pair of morphisms $u, v : Y \to A$ with codomain $A \in Ob\mathbf{A}$, $uf = vf \Rightarrow u = v$.

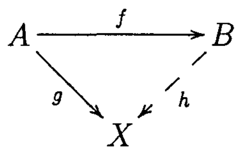
The reader should note that since many categorical constructions such as pullbacks, products, closure operators and reflectors are essentially unique – although not actually unique – they are often treated as being unique in order to make the expression of results less constrained.

The set theoretic foundations for our category theory are as sketched out in the introduction to [Adámek, Herrlich, Strecker 1990]. In particular we assume a hierarchy of sets, classes and conglomerates where each set is a class and each class is in turn a conglomerate. Notationally, we write "$\subseteq$" for set theoretic inclusion, the symbol "$\subset$" being reserved for *strict* inclusion. A similar convention applies to the symbols "$\leq$", "$<$", "$\sqsubseteq$" and "$\sqsubset$".

Orthogonal morphisms

If $\mathcal{X}$ is a class of objects in a category $\mathbf{X}$, then $\mathcal{X}_\perp$ denotes the class of $\mathbf{X}$-morphisms which are orthogonal to every $\mathcal{X}$-object in the following sense.

An $\mathbf{X}$-morphism $f : A \to B$ is in $\mathcal{X}_\perp$ if for any $X \in \mathcal{X}$ and $\mathbf{X}$-morphism $g : A \to X$ there is a *unique* morphism $h : B \to X$ such that $hf = g$. Such an h will be termed an "extension of f to X over g".



This notation is in fairly common use, what is not common however is the notation $\mathcal{X}_{\perp_w}$ which we use to denote those morphisms whose orthogonality to $\mathcal{X}$ is weaker in that the extension h in the diagram above is not necessarily unique. Some authors term these the $\mathcal{X}$-extendable morphisms.

Concrete categories used in examples

The following categories are used in examples during the course of the thesis.

<u>AbGrp</u>	Abelian groups and group homomorphisms.
<u>HCGrp</u>	Compact Hausdorff topological groups and continuous homomorphisms.
<u>Frm</u>	Frames and frame homomorphisms.
<u>Grp</u>	Groups and group homomorphisms.

<u>HAUS</u>	Hausdorff topological spaces and continuous maps.
<u>HCOMP</u>	Compact Hausdorff topological spaces and continuous maps.
<u>IND</u>	Indiscrete topological spaces and continuous maps.
<u>R-MOD</u>	Left R-modules over a ring R with unity and module homomorphisms.
<u>SET</u>	Sets and functions.
<u>SOB</u>	Sober topological spaces and continuous maps.
<u>TFAB</u>	Torsion free Abelian groups and group homomorphisms.
<u>TOP</u>	Topological spaces and continuous maps.
<u>TOP_0</u>	T_0 topological spaces and continuous maps.
<u>TOPGRP</u>	Topological groups and continuous homomorphisms.
<u>TOPGRP_0</u>	Separated topological groups and continuous homomorphisms.
<u>TYCH</u>	Tychonoff topological spaces and continuous maps.
<u>UNIF_0</u>	Separated uniform spaces and uniformly continuous maps.
<u>ZDIM_0</u>	Zerodimensional Hausdorff topological spaces and continuous maps.

Closure operators

The concept of categorical closure operator we work with was first introduced in [Dikranjan, Giuli 1987]. The outline given here is essentially what can be found in that paper. We present it here not only to make the thesis more self contained, but also because our notation differs on occasion.

The setting

We work in a category $\mathbf{X}$. $\mathcal{M}$ is a fixed class of $\mathbf{X}$-morphisms that is the second component of a factorisation structure, $(\mathbf{E},\mathcal{M})$, for sinks in $\mathbf{X}$. We recall here some of the valuable properties possessed by $\mathcal{M}$.

Theorem 1. (cf. [Adámek, Herrlich, Strecker 1990] Theorem 15.14 for the dual result.) *Let $\mathcal{M}$ be a class of $\mathbf{X}$-morphisms. There is a conglomerate $\mathbf{E}$ of sinks in $\mathbf{X}$ such that $(\mathbf{E},\mathcal{M})$ is a factorisation structure for sinks in $\mathbf{X}$ iff $\mathcal{M}$ satisfies the following:*

(a) *$\mathcal{M}$ is closed under composition.*

(b) *Pullbacks of $\mathcal{M}$-morphisms along any $\mathbf{X}$-morphism exist and are again in $\mathcal{M}$.*

(c) *Multiple pullbacks (intersections) of arbitrary families of $\mathcal{M}$-morphisms exist and are again in $\mathcal{M}$.*

We also know that $\mathcal{M}$ has these additional properties:

(d) $Iso\mathbf{X} \subseteq \mathcal{M} \subseteq Mono\mathbf{X}$.

(e) $\mathcal{M}$ is a coessential class, i.e. if $mn \in \mathcal{M}$ and $m \in \mathcal{M}$ then $n \in \mathcal{M}$.

Some authors term conditions (b) and (c) together as the property that $\mathbf{X}$ is $\mathcal{M}$-*complete.*

Any sink factorisation structure on $\mathbf{X}$ can obviously be restricted to a morphism factorisation structure. Throughout $(\mathcal{E}, \mathcal{M})$ will denote the restriction of $(\mathbf{E}, \mathcal{M})$ to morphisms in $\mathbf{X}$. $\mathcal{M}$ represents *subobjects* in the category $\mathbf{X}$, and closure operators act on these subobjects.

For any $X \in Ob\mathbf{X}$, the class $\mathcal{M}_X$ of all $\mathcal{M}$-morphisms with codomain X is endowed with a preorder as follows: $M \xrightarrow{m} X \leq N \xrightarrow{n} X$ iff there is an $\mathcal{M}$-morphism $j : M \to N$ such that $nj = m$. The morphisms m and n are isomorphic if both $m \leq n$ and $n \leq m$. We identify isomorphic subobjects, and in the sequel will simply write $m = n$ in such instances. Note that item (c) of the theorem above ensures that each $\mathcal{M}_X$ is in fact a complete preorder.

An $\mathbf{X}$-morphism $f : X \to Y$ can be used to define two functors, $f(-) : \mathcal{M}_X \to \mathcal{M}_Y$ and its right adjoint $f^{-1}(-) : \mathcal{M}_Y \to \mathcal{M}_X$. For any $m \in \mathcal{M}_X$, $f(m)$ is the $\mathcal{M}$-component of the $(\mathcal{E}, \mathcal{M})$ factorisation of the composition fm, while for $m \in \mathcal{M}_Y$, $f^{-1}(m)$ is the pullback of m along f.

Closures

Definition 1. We present three equivalent ways of defining/describing a closure operator on $\mathbf{X}$ with respect to $\mathcal{M}$.

(1) A closure operator is a family of operators $([-]_X : \mathcal{M}_X \to \mathcal{M}_X)_{X \in Ob\mathbf{X}}$ such that for each $X \in Ob\mathbf{X}$:

  (i) $m \leq [m]_X$ for all $m \in \mathcal{M}_X$.

  (ii) $m \leq n \Rightarrow [m]_X \leq [n]_X$ for any $m, n \in \mathcal{M}_X$.

  (iii) $f([m]_X) \leq [f(m)]_Y$ for any $f : X \to Y$ in $\mathbf{X}$ and $m \in \mathcal{M}_X$.

(2) View $\mathcal{M}$ as a comma category, with morphisms $(f, g) : (M \xrightarrow{m} X) \to (N \xrightarrow{n} Y)$ pairs of $\mathbf{X}$-morphisms ($f : M \to N$ and $g : X \to Y$) such that $gm = nf$. With this in mind, a closure operator is a functor $C : \mathcal{M} \to \mathcal{M}$ that commutes with the codomain functor $U : \mathcal{M} \to \mathbf{X}$ (i.e. $UC = U$) and is endowed with a natural transformation $\gamma : 1_{\mathcal{M}} \to C$ such that $U\gamma = 1_U$.

(3) A closure operator is a family of operators $([-]_X : \mathcal{M}_X \to \mathcal{M}_X)_{X \in Ob\mathbf{X}}$ such that:

  (i) $m \leq [m]_X$ for all $X \in Ob\mathbf{X}$ and $m \in \mathcal{M}_X$.

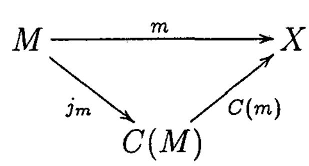
(ii) For any $M \xrightarrow{m} X$ and $N \xrightarrow{n} Y$ in $\mathcal{M}$, and morphisms $f : M \to N$ and $g : X \to Y$ in $\mathbf{X}$ such that $nf = gm$ there is a unique morphism $\overline{f}$ such that $[n]_Y\overline{f} = g[m]_X$.

Note. The adjointness of $f(-)$ and $f^{-1}(-)$ enables us to state (1)(iii) equivalently as follows: $[f^{-1}(m)]_X \leq f^{-1}([m]_Y)$ for any $f : X \to Y$ in $\mathbf{X}$ and $m \in \mathcal{M}_Y$. Some authors refer to this property as continuity with respect to the closure operator, others as the morphism consistency of the closure operator. Details of (1) and (2) can be found in [Dikranjan, Giuli 1987], while (3) is described in [Castellini, Koslowski, Strecker 1994].

Closure operators will be denoted by a single letter, say C. $C(m)$ will denote the operation of that operator on $m \in \mathcal{M}$. We place a preorder on all closure operators on $\mathbf{X}$ with respect to $\mathcal{M}$, saying that $C \sqsubseteq D$ iff $C(m) \leq D(m)$ for every $m \in \mathcal{M}$.

Closed and dense morphisms

Given a closure operator C on $\mathbf{X}$ with respect to $\mathcal{M}$, there are two classes of morphisms that are of particular interest to us. The *C-closed* morphisms are those $m \in \mathcal{M}$ such that $C(m) = m$. The *C-dense* morphisms are those morphisms $f : X \to Y$ for which $C(f(1_X)) = 1_Y$. Consider the closure of $m \in \mathcal{M}$ shown below.



C is said to be *idempotent* if for any $m \in \mathcal{M}$, $C(m)$ is C-closed. We say that C is *weakly hereditary* if for any $m \in \mathcal{M}$, j_m is C-dense. An important fact is that the idempotent, weakly hereditary closure operators are in bijective correspondence with those factorisation structures $(\mathcal{F}, \mathcal{N})$ for morphisms in $\mathbf{X}$ such that $\mathcal{N} \subseteq \mathcal{M}$.

The *idempotent hull* of C – written $\hat{C}$ – is the smallest (with respect to $\sqsubseteq$) idempotent closure operator D such that $C \sqsubseteq D$. Similarly, the *weakly hereditary core* of C – written $\check{C}$ – is the largest weakly hereditary closure operator D such that $D \sqsubseteq C$. The $\mathcal{M}$-completeness of $\mathbf{X}$ ensures that both $\hat{C}$ and $\check{C}$ exist (cf. [Dikranjan, Giuli 1987] Proposition 4.1).

Regular closure

Any subcategory $\mathbf{A}$ of $\mathbf{X}$ induces a closure operator $C_{\mathbf{A}}$ on $\mathbf{X}$ with respect to $\mathcal{M}$, called the *regular closure* induced by $\mathbf{A}$. Assuming that *RegMono*$\mathbf{X} \subseteq \mathcal{M}$, for $M \xrightarrow{m} X \in \mathcal{M}$, $C_{\mathbf{A}}(m)$ is defined as follows:

$$C_{\mathbf{A}}(m) := \bigwedge\{n \in \mathcal{M} \mid m \leq n \text{ and } n \text{ is } \mathbf{A}\text{-regular}\}.$$

$N \xrightarrow{n} X \in \mathcal{M}$ is termed **A**-regular if there is a pair of morphisms $u, v : X \to A$ with codomain in **A** such that n is the equaliser of u and v.

Apart from the basic reference of [Dikranjan, Giuli 1987], further information on the foundations of closure operator theory can be found in [Castellini 1986], [Koslowski 1988], [Dikranjan, Giuli, Tholen 1989] and the survey [Holgate 1992]. Many subsequent publications covering more specialised investigations are cited at the end of this thesis.

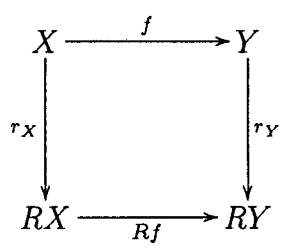
Chapter 1

Pullback Closure

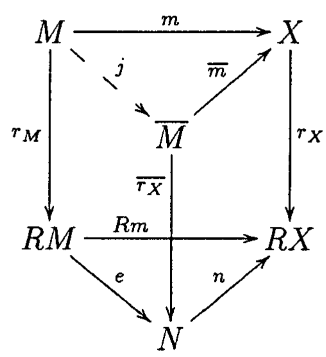
In this chapter we introduce and study the closure operator that is central to our thesis. A variety of examples are given and comparison is made with other closure operators found in the literature. Theoretical investigations are restricted to results that lay the foundation for the following chapters.

1.1 The pullback closure operator

The particular closure operator we are concerned with is induced by a pointed endofunctor (R, r) on $\mathbf{X}$, that is a functor $R : \mathbf{X} \to \mathbf{X}$ and a natural transformation $r : 1_{\mathbf{X}} \to R$. In particular, for any $\mathbf{X}$-morphism $f : X \to Y$, (R, r) induces the following commutative square.



1.1.1 Definition. Let (R, r) be a pointed endofunctor on $\mathbf{X}$. For $M \xrightarrow{m} X \in \mathcal{M}$, construct the diagram below, where $ne = Rm$ is the $(\mathcal{E}, \mathcal{M})$-factorisation of Rm and $\overline{m}$ is the pullback of n along r_X.



Put $\Phi_{(R,r)}(m) := \overline{m}$.

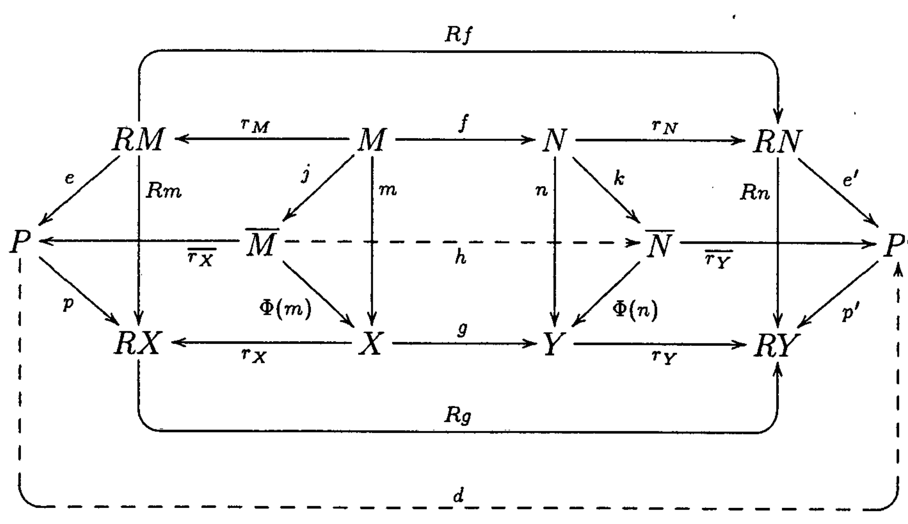
We will make use of the notation in the above diagram in future constructions. Often – especially in diagrams – we will write Φ for $\Phi_{(R,r)}$; no confusion should arise.

1.1.2 Proposition. *If (R, r) is any pointed endofunctor on $\mathbf{X}$ then $\Phi_{(R,r)}$ is a closure operator on $\mathbf{X}$ with respect to $\mathcal{M}$.*

Proof. We demonstrate that $\Phi_{(R,r)}$ fulfills the conditions of the third description of a closure operator given in Chapter 0.

(i) Let $M \overset{m}{\to} X \in \mathcal{M}$. In the diagram of Definition 1.1.1 the pullback induces the morphism $j : M \to \overline{M}$ with $\Phi_{(R,r)}(m)j = m$ whence $m \leq \Phi_{(R,r)}(m)$.

(ii) Let $m : M \to X$ and $n : N \to Y$ be morphisms in $\mathcal{M}$, and let $f : M \to N$ and $g : X \to Y$ be $\mathbf{X}$-morphisms such that $nf = gm$. Construct both $\Phi_{(R,r)}(m)$ and $\Phi_{(R,r)}(n)$.



Since $Rgpe = RgRm = R(gm) = R(nf) = RnRf = p'e'Rf$, the $(\mathcal{E}, \mathcal{M})$ diagonalisation property gives that there is a unique $d : P \to P'$ such that $de = e'Rf$ and $p'd = Rgp$. Thus we have that $p'd\overline{r_X} = Rgp\overline{r_X} = Rgr_X\Phi_{(R,r)}(m) = r_Y g\Phi_{(R,r)}(m)$, so since $p'\overline{r_Y} = r_Y\Phi_{(R,r)}(n)$ is a pullback square there is a unique $h : \overline{M} \to \overline{N}$ such that $\overline{r_Y}h = d\overline{r_X}$ and $\Phi_{(R,r)}(n)h = g\Phi_{(R,r)}(m)$. The second equality verifies that h is exactly the morphism whose existence we needed to establish. □

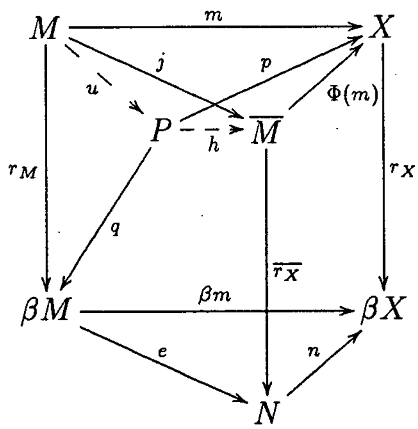
1.2 Examples

In this section we give a number of examples of $\Phi_{(R,r)}$ for different (R, r) in topological and algebraic settings. Some of the closures described are ones that have not been considered before, others are well known closure operations whose description as a pullback operator is new. Most of the examples are of reflectors which we view as endofunctors on the category **X**.

1.2.1 Čech-Stone compactification. Let Φ be the pullback closure on <u>Tych</u> induced by the Čech-Stone compactification, β : <u>Tych</u> $\to$ <u>HComp</u>.

Claim. Φ is the usual topological closure on <u>Tych</u>.

Proof. In <u>Tych</u>, $\mathcal{M}$ is the class of embeddings. The diagram below shows the construction of $\Phi(m)$ for an embedding $m : M \to X$ in <u>Tych</u>, $ne = \beta m$ is the (*Surjection, Embedding*) factorisation of βm. $(P, (p, q))$ is the pullback of βm along r_X. The map $u : M \to P$ is induced by this pullback since $\beta m r_M = r_X m$. It is well known that $pu = m$ is the (*Dense C^*-embedding, Perfect map*) factorisation of m. (Combine [Herrlich 1972] Theorem 1.1 (P2) and [Gillman, Jerison 1960] Theorem 6.4.)



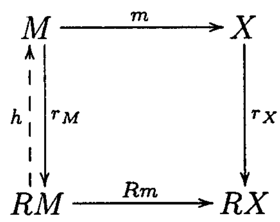
M can be viewed as a subspace of X. N is just the image of βM under βm, and n is the embedding of this image in βX. Our knowledge of pullbacks in $\underline{\text{Top}}$ (cf. [Adámek, Herrlich, Strecker 1990] Example 11.12) gives us the internal descriptions below.

- $P = \{(x, y) \in \beta M \times X \mid \beta m(x) = r_X(y)\}$
- $\overline{M} = \{(x, y) \in N \times X \mid n(x) = r_X(y)\} = \{(x, y) \in \beta m[\beta M] \times X \mid x = r_X(y)\}$
- $u : M \to P$ maps $a \longmapsto (r_M(a), m(a)) = (r_M(a), a)$
- $j : M \to \overline{M}$ maps $a \longmapsto (\beta m(r_M(a)), a)$
- $h : P \to \overline{M}$ maps $(x, y) \longmapsto (\beta m(x), y)$
- p, q, $\Phi(m)$ and $\overline{r_X}$ are restrictions of the appropriate projection maps, P and $\overline{M}$ inherit their topologies through them.

Now h is clearly onto, and u is dense, so hu is dense. But since $\Phi(m)hu = pu = \Phi(m)j$ and because $\Phi(m)$ is an embedding, $hu = j$ and so j is dense. This gives that $\overline{M} \subseteq cl_X M$ (the usual topological closure).

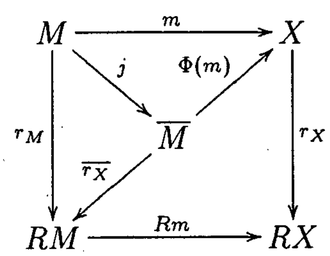
On the other hand, since N is the continuous image of a compact space, N is compact. Then since βX is Hausdorff, n must be a closed embedding. But closed embeddings are closed under pullbacks, so $\Phi(m)$ is closed, which gives that $cl_X M \subseteq cl_X \overline{M} = \overline{M}$, and hence $cl_X M = \overline{M}$. □

1.2.2 $\underline{\text{Top}_0}$ **reflection.** Let Φ be the pullback closure on $\underline{\text{Top}}$ induced by the $\underline{\text{Top}_0}$ reflection. First we note that this reflection preserves $\mathcal{M}$-morphisms, i.e. embeddings. To verify this consider the diagram below, where (R, r) is the $\underline{\text{Top}_0}$ reflection.



Since r_M is initial and surjective we can construct a section h of r_M. Then since h is an extremal monomorphism (i.e. an embedding) and both m and r_X are initial, the fact that $r_X m h = Rm r_M h = Rm$ gives that Rm is initial. Hence since RM is a $\underline{\text{Top}_0}$ space, Rm is an embedding.

This makes calculation of Φ much easier. The diagram below shows the construction of $\Phi(m)$ for a subspace embedding $m : M \to X$ in $\underline{\text{Top}}$.



$\overline{M} = \{(x,y) \in RM \times X \mid Rm(x) = r_X(y)\}$ is the set underlying the pullback closure of M. A more convenient description follows once we see that $\overline{M} \cong \tilde{M} := \{x \in X \mid \exists\, a \in M$ for which $cl_X\{a\} = cl_X\{x\}\}$.

Let $h : \tilde{M} \to \overline{M}$ be the function that sends $x \longmapsto (r_M(a), x)$, where $a \in M$ is such that $cl_X\{a\} = cl_X\{x\}$. (If we have $a, b \in M$ such that $cl_X\{a\} = cl_X\{b\} = cl_X\{x\}$ then $cl_M\{a\} = cl_M\{b\}$ so $r_M(a) = r_M(b)$, ensuring that h is well defined.) Pick $(x,y) \in \overline{M}$, then since r_M is onto we can choose $a \in r_M^{-1}(x)$ and note that $r_X(a) = r_X m(a) = Rmr_M(a) = Rm(x) = r_X(y)$. Because of the construction of the $\underline{\text{TOP}_0}$ reflection this just means that $cl_X\{a\} = cl_X\{y\}$ so $y \in \tilde{M}$ and obviously $h(y) = (x,y)$. This shows that h is onto, it is clearly injective and so $\overline{M} \cong \tilde{M}$.

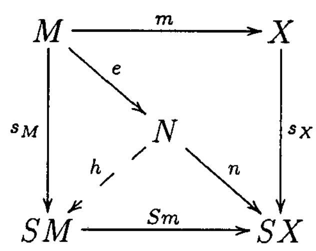
This gives a convenient internal description of Φ. More concisely we could write this as $\overline{M} \cong \tilde{M} = r_X^{-1}[r_X[M]]$.

1.2.3 Sobrification. Throughout this example, $R : \underline{\text{TOP}} \to \underline{\text{TOP}_0}$ will denote the $\underline{\text{TOP}_0}$ reflection as above, while $S : \underline{\text{TOP}_0} \to \underline{\text{SOB}}$ will denote the sobrification of a $\underline{\text{TOP}_0}$ space. More details on the sobrification can be found in [Hoffmann 1976] or [Fedeli 1992]. Again, $\mathcal{M}$ is the class of embeddings.

We make use of the b-closure or front closure introduced in [Baron 1968]. In particular we use the fact that (*b-dense, b-closed embedding*) is a factorisation structure for morphisms in both $\underline{\text{TOP}}$ and $\underline{\text{TOP}_0}$ (cf. [Dikranjan, Giuli 1987] Example 6.3.B(1) and [Holgate 1992] Example 2.3.13(2)).

Claim 1. S maps embeddings to b-closed embeddings.

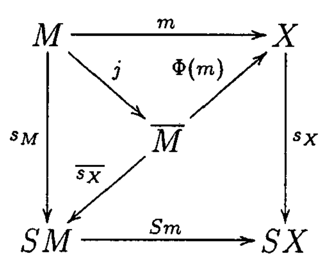
Proof. The diagram below shows the image of an embedding $m : M \to X$ in $\underline{\text{TOP}_0}$ under S.



Let $ne = Sms_M$ be the (*b-dense, b-closed embedding*) factorisation of Sms_M. Since n is an extremal monomorphism and $\underline{\text{SOB}}$ is epireflective in $\underline{\text{TOP}}_0$, N is a sober space. But e is a b-dense *embedding* since $ne = s_X m$ is an embedding, thus there is an isomorphism $h : N \to SM$ with $he = s_M$ and (since e is an epimorphism) $Smh = n$ ([Hoffmann 1976] Proposition 3.1.2). From this it follows that Sm is a b-closed embedding. □

Claim 2. $\Phi_{(S,s)}$ is the b-closure on $\underline{\text{TOP}}_0$.

Proof. Knowing the above result, we form $\Phi_{(S,s)}(m)$ for an embedding $M \xrightarrow{m} X$ by taking the pullback shown below.

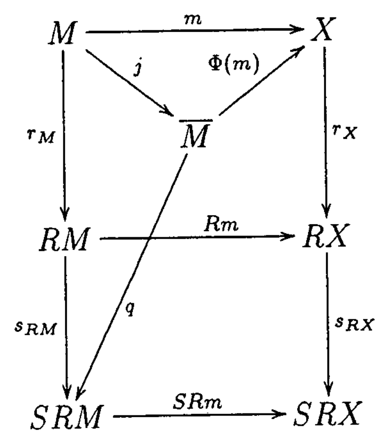


Since b-closed embeddings are closed under pullbacks, $\Phi_{(S,s)}(m)$ is b-closed. But $\overline{s_X}j = s_M$ is b-dense and $\overline{s_X}$ is an embedding, so j is b-dense. This means that $\Phi_{(S,s)}(m)j = m$ is a (*b-dense, b-closed embedding*) factorisation of m, and so by the uniqueness of factorisations $\Phi_{(S,s)}$ must be the b-closure. □

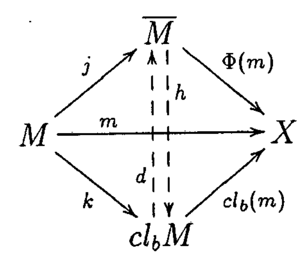
The $\underline{\text{TOP}}_0$ reflection and the sobrification of a $\underline{\text{TOP}}_0$ space compose to give a reflector $SR : \underline{\text{TOP}} \to \underline{\text{SOB}}$. This reflector induces a pullback closure Φ on $\underline{\text{TOP}}$.

Claim 3. Φ induced by the reflector $SR : \underline{\text{TOP}} \to \underline{\text{SOB}}$ is the b-closure on $\underline{\text{TOP}}$.

Proof. The diagram below shows the construction of $\Phi(m)$ for an embedding m in $\underline{\text{TOP}}$. Since both reflectors preserve embeddings, so does their composition.



We have two factorisations of m, one given by the formation of $\Phi(m)$, the other the (*b-dense, b-closed embedding*) factorisation. (cl_b denotes the b-closure.)



The regular closure induced on TOP by the sober spaces is the b-closure. (This is since SOB contains the Sierpinski dyad and is contained in TOP_0, cf. [Salbany 1976] and [Dikranjan, Giuli 1987].) So by the remark made in 1.3.1 below, since $\mathcal{E}$ is the class of surjective continuous maps, we have that $\Phi \sqsubseteq cl_b$. This gives a map $h : \overline{M} \to cl_bM$ such that $cl_b(m)h = \Phi(m)$. On the other hand k is b-dense and as was shown earlier, since SRm is a b-closed embedding $\Phi(m)$ is too. Thus by the (*b-dense, b-closed embedding*) diagonalisation property there is a unique $d : cl_bM \to \overline{M}$ such that $dk = j$ and $\Phi(m)d = cl_b(m)$. From this it follows that $cl_bM \cong \overline{M}$ and Φ is the b-closure as claimed. □

1.2.4 Endofunctors induced by congruence relations. In a number of algebraic categories, families of congruence relations induce pointed endofunctors that in turn give rise to interesting pullback closure operators. We describe these constructions in an arbitrary variety $\mathbf{X}$ and then demonstrate the theory in categories of Frames, Modules and Groups.

By a variety $\mathbf{X}$, we mean a monadic construct as in [Adámek, Herrlich, Strecker 1990] Definition 24.12. (Theoretical details can be found in their Chapter 20.)

Following the progression of [Adámek, Herrlich, Strecker 1990] Proposition 20.34, Theorem 20.32 and Proposition 20.12(8) we see that a variety **X** is a (*RegEpi, MonoSource*)-category, that the regular epimorphisms are just the surjective **X**-morphisms and so (*Surjection, Embedding*) is a factorisation structure for morphisms in **X**. Propositions 20.34 and 20.35 tell us furthermore that **X** is wellpowered and intersections of families of monomorphisms (embeddings) exist. Thus we let $\mathcal{M}$ be the class of all embeddings in **X**, and $(\mathcal{E}, \mathcal{M})$ is the (*Surjection, Embedding*) factorisation structure through which we define subobjects in **X**.

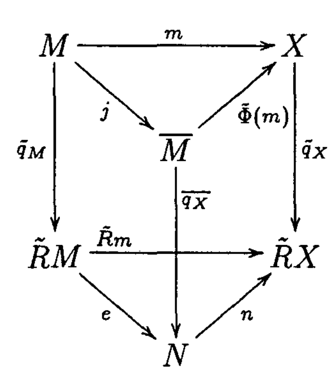
Products and pullbacks in **X** are formed as in SET. The product and pullback objects inherit an initial structure by virtue of the previously mentioned Proposition 20.12(8). While the objects of **X** are not themselves sets, for simplicity in the discussion below we will at times treat an **X**-object X as a set. In such an instance we mean the object in SET underlying X.

A congruence relation $\sim_X$ on an object X in **X** is an equivalence relation on X such that $\sim_X$ is a subobject of $X \times X$. A family $(\sim_X)_{X \in Ob\mathbf{X}}$ will be termed a *natural family of congruence relations* if for each $X \in Ob\mathbf{X}$, $\sim_X$ is a congruence relation on X, and for any **X**-morphism $f : X \to Y$, $x \sim_X y \Rightarrow f(x) \sim_Y f(y)$.

Any natural family $(\sim_X)_{X \in Ob\mathbf{X}}$ of congruence relations on **X** gives rise to a pointed endofunctor $(\tilde{R}, \tilde{q})$ on **X**. For $X \in Ob\mathbf{X}$, $\tilde{q}_X : X \to \tilde{R}X$ is the quotient map from X to $X/\sim_X$. For a morphism $f : X \to Y$ in **X**, $\tilde{R}f : \tilde{R}X \to \tilde{R}Y$ is the map that takes the equivalence class $[x]_{\sim_X} \in X/\sim_X$ to the equivalence class $[f(x)]_{\sim_Y} \in Y/\sim_Y$. The fact that f preserves the congruence relation guarantees that $\tilde{R}f$ is well defined. The construction of $\tilde{R}f$ ensures that at the SET level $\tilde{R}f\tilde{q}_X = \tilde{q}_Y f$ and so since $\tilde{q}_X$ is final we conclude that $\tilde{R}f$ is an **X**-morphism. The fact that $\tilde{R}$ is indeed a functor is quite clear.

The pullback closure operator $\Phi_{(\tilde{R},\tilde{q})}$ – which we will denote simply by $\tilde{\Phi}$ – is easy to describe.

Let $m : M \to X$ be an embedding in **X** and take the $(\mathcal{E}, \mathcal{M})$ factorisation $ne = \tilde{R}m$. Since $\tilde{q}_M \in \mathcal{E}$, $ne\tilde{q}_M = \tilde{R}m\tilde{q}_M = \tilde{q}_X m$ is the $(\mathcal{E}, \mathcal{M})$ factorisation of $\tilde{q}_X m$, and so $\tilde{\Phi}(m) = \tilde{q}_X^{-1}(\tilde{q}_X(m))$.



We will use m and M interchangeably to denote the subobject $m : M \to X$. As long as the codomain X is understood, this is unambiguous since any subobject M has the initial $\mathbf{X}$-structure with respect to its inclusion in X. Taking this approach we can describe $\tilde{\Phi}$ as follows:

$$\tilde{\Phi}(M) = \tilde{q}_X^{-1}[\tilde{q}_X[M]] = \{x \in X \mid \exists n \in M \text{ such that } x \sim_X n\} = \bigcup\{[n]_{\sim_X} \mid n \in M\}.$$

It is clear that $\tilde{\Phi}$ is an idempotent closure operator and also that it is completely additive. (By completely additive we mean that for any family $(M_i)_{i \in I}$ of subobjects of an object X in $\mathbf{X}$, $\tilde{\Phi}(\bigcup_{i \in I} M_i) = \bigcup_{i \in I} \tilde{\Phi}(M_i)$.)

We say that a natural family $(\sim_X)_{X \in Ob\mathbf{X}}$ of congruence relations is *hereditary* if for any embedding $m : M \to X$ it is true that $\sim_M = \sim_X \cap (M \times M)$. The pullback closure operator $\tilde{\Phi}$ induced by such a family is easily seen to be an hereditary closure operator on $\mathbf{X}$ with respect to $\mathcal{M}$. (This means that for any chain of embeddings $M \xrightarrow{m} N$ and $N \xrightarrow{n} X$ the closure $\tilde{\Phi}_N(M)$ of M in N is $\tilde{\Phi}_X(M) \cap N$ – the closure of M in X restricted to N.)

Endofunctors of the type $(\tilde{R}, \tilde{q})$ and their induced closure operators are not restricted to varieties. The example given above for the $\underline{\text{TOP}_0}$ reflection in $\underline{\text{TOP}}$ is another closure operator of this type. In varieties, however, they play a prominent role. We now look at this scenario in three particular varietal categories.

(1) Frames. Congruence relations in the category $\underline{\text{FRM}}$ are often described by means of nuclei. A nucleus k on a frame L is a unary operation with the three properties below.

(i) For any $a \in L$, $a \leq k(a)$.

(ii) For any $a \in L$, $k(a) = k(k(a))$.

(iii) For any $a, b \in L$, $k(a \wedge b) = k(a) \wedge k(b)$.

Any nucleus k induces a congruence relation Θ on the frame L as follows:

$$(a, b) \in \Theta \Leftrightarrow k(a) = k(b).$$

This is in fact one direction of a bijective correspondence between nuclei and congruence relations on the frame L. (More information on these matters can be found in [Johnstone 1982] or [Banaschewski 1988].) Thus any natural family $(\sim_L)_{L \in Ob\underline{\mathbf{FRM}}}$ of congruence relations can be described as a natural family of nuclei $(k_L)_{L \in Ob\underline{\mathbf{FRM}}}$. (For nuclei, naturality means that for any frame homomorphism $f : L \to M$, $k_L(a) = k_L(b) \Rightarrow k_M(f(a)) = k_M(f(b))$.)

For a natural family $(k_L)_{L \in Ob\underline{\mathrm{FRM}}}$ of nuclei, the pointed endofunctor $(\tilde{R}, \tilde{q})$ and the resulting pullback closure operator $\tilde{\Phi}$ have very simple descriptions in terms of the actions of the nuclei.

- For a frame L, $\tilde{R}L = \mathrm{Fix}k_L = \{a \in L \mid k_L(a) = a\}$, while $\tilde{q}_L$ is simply $k_L : L \to \mathrm{Fix}k_L$.
- The image $\tilde{R}f$ of a frame homomorphism $f : L \to M$ maps $a \in \mathrm{Fix}k_L$ to $\tilde{R}f(a) = k_M(f(a))$.
- For an embedding $m : M \to L$, $\tilde{\Phi}(M) = k_L^{-1}[k_L[M]]$.

If for every frame L we let k_L be the identity function on L, then the pullback closure operator induced by the resulting family of nuclei is the discrete closure operator. At the other extreme, the indiscrete closure operator is induced by the family $(k_L)_{L \in Ob\underline{\mathrm{FRM}}}$ of all nuclei such that $k_L(a) = e_L$ for all $a \in L$ (where e_L is the top element of the frame L).

The spectrum functor $\Sigma : \underline{\mathrm{FRM}} \to \underline{\mathrm{TOP}}$ and its right adjoint $\Omega : \underline{\mathrm{TOP}} \to \underline{\mathrm{FRM}}$ induce a more interesting closure operator on $\underline{\mathrm{FRM}}$. The composition $\Sigma\Omega : \underline{\mathrm{TOP}} \to \underline{\mathrm{TOP}}$ is the sobrification reflector in $\underline{\mathrm{TOP}}$ while the functor $\Omega\Sigma : \underline{\mathrm{FRM}} \to \underline{\mathrm{FRM}}$ is the reflector in $\underline{\mathrm{FRM}}$ to the spatial frames. We restrict our attention to this second reflection. More information on these functors and the theory below can be found in [Banaschewski 1988] or [Johnstone 1982].

An element s in a frame L is a *prime element* if for any $a, b \in L$, $a \wedge b \leq s \Rightarrow a \leq s$ or $b \leq s$. For $a \in L$ define $\Sigma_a := \{s \mid s \text{ is a prime element in } L \text{ and } a \not\leq s\}$. This facilitates the following internal descriptions.

- For a frame L, $\Omega\Sigma L = \{\Sigma_a \mid a \in L\}$. Meet and join in $\Omega\Sigma L$ is set-theoretic intersection and union. Note that $\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b}$ and $\bigcup_{i \in I} \Sigma_{a_i} = \Sigma_{\bigvee_{i \in I} a_i}$.
- For a frame homomorphism $h : L \to M$, $\Omega\Sigma h$ maps $\Sigma_a \in \Omega\Sigma L$ to $\Sigma_{h(a)}$.
- For a frame L, the unit of the adjunction $\eta_L : L \to \Omega\Sigma L$ maps $a \in L$ to Σ_a. η_L is always onto, and is the reflection of L to the subcategory of spatial frames.

We can describe this reflection in terms of nuclei. For an element a in a frame L, set

$$k_L(a) := \bigwedge\{s \mid s \text{ is prime and } a \leq s\}.$$

Claim 1. k_L is a nucleus on the frame L.

Proof. It is clear that for any $a \in L$, $a \leq k_L(a)$ and $k_L(k_L(a)) = k_L(a)$. Let $a, b \in L$, $k_L(a \wedge b) = \bigwedge\{s \mid s \text{ is prime and } a \wedge b \leq s\} = \bigwedge\{s \mid s \text{ is prime and } a \leq s \text{ or } b \leq s\} = k_L(a) \wedge k_L(b)$. □

Claim 2. For $a, b \in L$, $k_L(a) = k_L(b) \Leftrightarrow \Sigma_a = \Sigma_b$.

Proof. $\Sigma_a = \Sigma_b \Leftrightarrow \{s \mid s \text{ is prime and } a \nleq s\} = \{t \mid t \text{ is prime and } b \nleq t\} \Leftrightarrow \{s \mid s \text{ is prime and } a \leq s\} = \{t \mid t \text{ is prime and } b \leq t\} \Leftrightarrow k_L(a) = k_L(b)$. □

This result tells us that k_L is just the nucleus induced by the kernel of $\eta_L : L \to \Omega\Sigma L$. From this observation it follows that – since η_L is onto – $\Omega\Sigma L \cong \mathrm{Fix}k_L$, and the reflection $\eta_L : L \to \Omega\Sigma L$ can be equivalently described as $k_L : L \to \mathrm{Fix}k_L$.

So $(k_L)_{L \in Ob\underline{\mathrm{FRM}}}$ is a natural family of nuclei in $\underline{\mathrm{FRM}}$ and the pullback closure operator $\Phi_{(\Omega\Sigma,\eta)}$ is just $\tilde{\Phi}$ induced by this family of nuclei. For a subframe M of L we have that

$$\tilde{\Phi}(M) = k_L^{-1}[k_L[M]] = \{a \in L \mid \exists m \in M \text{ such that } \Sigma_a = \Sigma_m\}.$$

(2) Modules. For any ring R with unity, $\underline{R\text{-MOD}}$ is the category of all left R-modules and module homomorphisms. For any module M in $\underline{R\text{-MOD}}$, we write the group operation as addition "+", and the additive identity as "0".

[Dikranjan, Giuli 1991a] investigates closure operators on $\underline{R\text{-MOD}}$, in particular closure operators induced by preradicals. A preradical r on $\underline{R\text{-MOD}}$ is a subfunctor of the identity functor. Fundamentals of preradicals are dealt with in [Bican, Jambor, Kepka, Němec 1974] as well as in [Dikranjan, Giuli 1991a]. It turns out that preradicals are in one-one correspondence with natural families of congruence relations on $\underline{R\text{-MOD}}$, and the pullback closure induced by such a family is one of those studied in [Dikranjan, Giuli 1991a].

Let $(\sim_M)_{M \in Ob\underline{\text{R-MOD}}}$ be a natural family of congruence relations on $\underline{R\text{-MOD}}$. For a module M, set $\tilde{r}M := [0]_{\sim_M}$, and for a module homomorphism $f : M \to N$ put $\tilde{r}f := f|_{\tilde{r}M}$ (f restricted to $\tilde{r}M$). Using the fact that f preserves the relation $\sim_M$ we see that $a \in \tilde{r}M \Rightarrow a \sim_M 0 \Rightarrow f(a) \sim_N f(0) = 0 \Rightarrow f(a) \in \tilde{r}N$ so $\tilde{r}f$ is well defined. The assignment $\tilde{r} : \underline{R\text{-MOD}} \to \underline{R\text{-MOD}}$ is trivially seen to be functorial, from which it follows that $\tilde{r}$ is a preradical on $\underline{R\text{-MOD}}$.

Conversely, let r be a preradical on $\underline{R\text{-MOD}}$. Define a relation $\sim_M^r$ on a module M as follows:

$$a \sim_M^r b \Leftrightarrow a + rM = b + rM.$$

It is not difficult to see that the collection $(\sim_M^r)_{M \in Ob\underline{\text{R-MOD}}}$ is a natural family of congruence relations on $\underline{R\text{-MOD}}$.

The two assignments we have described above are in fact mutually inverse to each other.

- On the one hand let $\tilde{r}$ be the preradical induced by the natural family of congruence relations $(\sim_M)_{M \in Ob\underline{\text{R-MOD}}}$ on $\underline{R\text{-Mod}}$. In a module M:

$$a \sim_M^{\tilde{r}} b \Leftrightarrow a + \tilde{r}M = b + \tilde{r}M \Leftrightarrow a + [0]_{\sim_M} = b + [0]_{\sim_M} \Leftrightarrow a \sim_M b.$$

- On the other hand let $(\sim_M^r)_{M \in Ob\underline{\text{R-MOD}}}$ be the natural family of congruence relations on $\underline{R\text{-Mod}}$ induced by the preradical r. Let $\tilde{r}$ be the preradical induced by this family of relations. For a module M:

$$\tilde{r}M = [0]_{\sim_M^r} = \{a \in M \mid a + rM = 0 + rM\} = rM.$$

  For a module homomorphism $f : M \to N$, $\tilde{r}f = f|_{\tilde{r}M} = f|_{rM} = rf$.

A preradical r is termed *hereditary* if for any embedding $n : N \to M$ in $\underline{R\text{-Mod}}$ it is true that $rN = rM \cap N$. It is easily confirmed that a natural family of congruence relations $(\sim_M)_{M \in Ob\underline{\text{R-MOD}}}$ is hereditary iff the associated preradical $\tilde{r}$ is an hereditary preradical.

With the above observations in mind, the pullback closure operator induced by a natural family $(\sim_M)_{M \in Ob\underline{\text{R-MOD}}}$ of congruence relations on $\underline{R\text{-Mod}}$ can be viewed as being induced by the associated preradical $\tilde{r}$. For a submodule M of N the pullback closure of M in N can thus be described as follows:

$$\tilde{\Phi}(M) = \{n \in N \mid \exists m \in M \text{ such that } n + \tilde{r}N = m + \tilde{r}N\} = M + \tilde{r}N.$$

This particular closure operator was denoted $C_{\tilde{r}}$ in [Dikranjan, Giuli 1991a] (Definition (1) on page 54), where it is studied quite extensively.

A preradical r is called an *idempotent radical* if for any module M, $rrM = rM$ and $r(M/rM) = 0$. Idempotent radicals correspond to torsion theories on $\underline{R\text{-Mod}}$ (cf. [Dikranjan, Giuli 1991a] Section 1). (Closure operators induced by torsion theories are studied in [Fay 1988] and [Fay, Walls 1989].) The torsion-free reflection in the category of Abelian groups – which is just the category $\underline{\mathbf{Z}\text{-Mod}}$ – is a particular example of a pointed endofunctor $(\tilde{R}, \tilde{q})$ as described above that is induced by an idempotent radical.

The torsion free reflection of an Abelian group G is the quotient $\tilde{q}_G : G \to G/{\sim_G}$, where the congruence relation $\sim_G$ is defined as follows:

$$a \sim_G b \Leftrightarrow \text{there is a nonzero integer } n \text{ such that } na = nb.$$

The induced preradical $t : \underline{\text{AbGrp}} \to \underline{\text{AbGrp}}$ that maps G to $tG := [0]_{\sim_G}$ and $G \xrightarrow{h} H$ to $th := h|_{tG}$ is an idempotent, hereditary radical. The resultant pullback closure operator is an idempotent, hereditary and completely additive closure operator on $\underline{\text{AbGrp}}$.

(3) Groups. The general theory under discussion is easily applied in the category of groups and group homomorphisms. One example that will be of interest in later sections is the Abelian reflection in $\underline{\text{GRP}}$.

For a group G, C_G will denote the *commutator subgroup* of G, that is the subgroup generated in G by the set $\{g \cdot h \cdot g^{-1} \cdot h^{-1} \mid g,\ h \in G\}$. (Here we write the group operation as multiplication.) Define a congruence relation $\sim_G$ on G as follows:

$$g \sim_G h \Leftrightarrow g \cdot C_G = h \cdot C_G.$$

The resultant collection $(\sim_G)_{G \in Ob\underline{\text{GRP}}}$ is a natural family of congruence relations on $\underline{\text{GRP}}$, and the pointed endofunctor $(\tilde{R}, \tilde{q})$ induced by this family is the reflector from $\underline{\text{GRP}}$ to $\underline{\text{ABGRP}}$.

For a subgroup M of G, the pullback closure $\tilde{\Phi}(M) = \{g \in G \mid \exists m \in M$ such that $g \cdot C_G = m \cdot C_G\} = M \cdot C_G$. We will see later that this closure operator is not weakly hereditary.

Further examples of pullback closures induced by endofunctors in concrete categories are given in Chapter 3.

1.3 Relation to other closures

There are other closure investigations that are similar to $\Phi_{(R,r)}$, but the pullback operator itself has not yet been systematically studied. Here we consider similar endeavours and how they relate to the study of $\Phi_{(R,r)}$ for given (R, r).

1.3.1 Regular closure. The regular closure induced by a subcategory is without doubt the most widely studied of all categorical closure operators. In most cases the regular closure $C_{\mathbf{A}}$ can be viewed as being induced by an epireflective subcategory $\mathbf{A}$ of $\mathbf{X}$ (cf. [Giuli, Mantovani, Tholen 1988] Theorem 1.1). In these instances, if $(R, r) : \mathbf{X} \to \mathbf{A}$ is the epireflection it is worth comparing $C_{\mathbf{A}}$ and $\Phi_{(R,r)}$.

It is not difficult to see that if $\mathcal{E}$ is a class of epimorphisms then $\Phi_{(R,r)} \sqsubseteq C_{\mathbf{A}}$. (It is often the case that $\mathcal{E}$ is a class of epimorphisms since to define regular closures one needs to assume that *RegMono*$\mathbf{X} \subseteq \mathcal{M}$ and if $\mathbf{X}$ has equalisers this is equivalent to assuming that $\mathcal{E} \subseteq Epi\mathbf{X}$ – [Holgate 1992] Proposition 2.3.1.) On the other hand there are various occasions when $C_{\mathbf{A}} \sqsubseteq \Phi_{(R,r)}$, as for example with the pullback and regular closures induced by $\underline{\text{HCOMP}}$ in $\underline{\text{TYCH}}$ and $\underline{\text{SOB}}$ in $\underline{\text{TOP}}$.

In general though the pullback closure is not always regular as the examples given demonstrate. It also seems to be more sensitive to the subcategory inducing it than the regular closure is. Whereas a regular closure $C_{\mathbf{A}}$ for any subcategory $\mathbf{A}$ of $\mathbf{X}$ can be viewed as being induced by the strongly epireflective hull of $\mathbf{A}$ in $\mathbf{X}$, the pullback closures induced by $\mathbf{A}$ and its hull may well differ, as for example with $\underline{\text{Sob}}$ and $\underline{\text{Top}_0}$ in $\underline{\text{Top}}$.

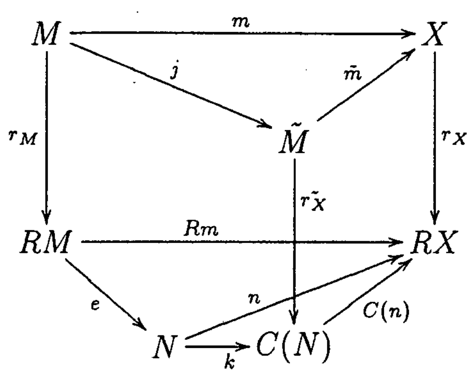
1.3.2 Lifting of a closure operator. In [Dikranjan, Giuli 1984] the observation was made (Lemma 2.1) that for the regular closure $C_{\mathbf{A}}$ induced by an epireflective subcategory $\mathbf{A}$ of $\underline{\text{Top}}$, $C_{\mathbf{A}}(M) = r_X^{-1}[C_{\mathbf{A}}(r_X[M])]$ for any embedding $m : M \to X$ (r_X is the reflection morphism). This result was extended to any topological construct in [Dikranjan, Giuli, Tozzi 1988] (Proposition 2.3). One value of this fact is that certain properties of the regular closure can be determined by studying its behaviour in the reflective subcategory.

These results lead to the obvious generalisation of taking a subobject's image under a reflection, performing a closure operation on that image in the reflective subcategory and then lifting the resultant closure back along the reflection map. In other words, for any reflection $(R, r) : \mathbf{X} \to \mathbf{A}$, closure operator C on $\mathbf{A}$ and subobject $m : M \to X$ of $X \in Ob\mathbf{X}$, form $r_X^{-1}(C(r_X(m)))$. This idea is introduced in [Dikranjan 1992] (Definition 3.5) where mention is also made of the possibility of performing this along any adjunction.

In [Stramaccia 1988] the r-closure is introduced and studied. The r-closure is a particular instance of performing this operation where the reflection is in $\underline{\text{Top}}$, and the closure being "lifted" is the usual topological closure.

The pullback closure obviously overlaps with this idea of lifting a closure operator. Specifically, if the endofunctor (R, r) is an $\mathcal{E}$-reflection, then $\Phi_{(R,r)}$ is just the lifting along (R, r) of the discrete closure operator.

1.3.3 Modification of a closure operator. Another view of the results about the preimage of a regular closure mentioned above is that the procedure being performed is one of modifying the closure along the reflection map. Since $C_{\mathbf{A}}(M) = r_X^{-1}[C_{\mathbf{A}}(r_X[M])]$, the regular closure $C_{\mathbf{A}}$ is in this case seen to be unaffected by the modification. With this view in mind, the obvious generalisation is instead of taking an epireflection on a topological category, to take a pointed endofunctor (R, r) on $\mathbf{X}$ and perform the following construction to modify the closure operator C on $\mathbf{X}$ with respect to $\mathcal{M}$ along (R, r).



Here $M \xrightarrow{m} X \in \mathcal{M}$, $ne = Rm$ is the $(\mathcal{E}, \mathcal{M})$-factorisation of Rm and $\tilde{m}$ is the pullback of $C(n)$ along r_X. The assignment $m \longmapsto \tilde{m}$ will give another closure operator on $\mathbf{X}$ with respect to $\mathcal{M}$. We are not aware that this modification has been studied at all, but believe it is used in the forthcoming book [Dikranjan, Tholen 1995].

$\Phi_{(R,r)}$ is simply the modification of the discrete closure on $\mathbf{X}$ with respect to $\mathcal{M}$ along (R, r). As our thesis endeavours to demonstrate, the resulting operator even in this case is far from trivial and worthy of study in its own right.

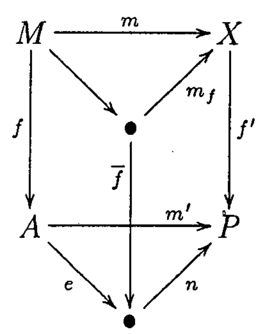
1.3.4 Splitting closure. In [Brümmer, Giuli 1993b] the splitting closure operator induced by a class $\mathcal{P} \subseteq Ob\mathbf{X}$ is introduced. Although not the original definition, it is shown there that for $M \xrightarrow{m} X \in \mathcal{M}$ the splitting closure of m in X can be described as follows:

$$spl(m) = \bigwedge\{f^{-1}(f(m)) \mid f : X \rightarrow P \in \mathcal{P}\}.$$

This coincides with a closure operator introduced in a specific example in [Castellini 1988] (Proposition 2.13) where $\mathcal{P}$ was the torsion free Abelian groups. Later in [Castellini, Hajek 1994] (Proposition 3.3) it was used in the more general form above – but in a construct – as part of a factorisation of the connectedness-disconnectedness Galois connection.

In Proposition 5 of [Brümmer, Giuli 1993b] it is shown that if $\mathbf{P}$ is a reflective subcategory of $\mathbf{X}$ with reflection $(R, r) : \mathbf{X} \rightarrow \mathbf{P}$ and $\mathcal{P} = Ob\mathbf{P}$ then for any $M \xrightarrow{m} X \in \mathcal{M}$, $spl(m) = r_X^{-1}(r_X(m))$. (The same result can be found in [Castellini, Hajek 1994] (Proposition 3.8).) Thus as in 1.3.2 above, if $\mathbf{P}$ is an $\mathcal{E}$-reflective subcategory then the splitting closure induced by $Ob\mathbf{P}$ and the pullback closure induced by the reflection coincide. In general though, the two closures are different. (For example if (R, r) is the Čech-Stone compactification in <u>TYCH</u> then $\Phi_{(R,r)}$ is the usual topological closure while the corresponding splitting closure is discrete.)

1.3.5 Orthogonal closure. Recently in [Sousa 1994] the orthogonal closure $O_{\mathbf{A}}$ induced by a subcategory $\mathbf{A}$ of $\mathbf{X}$ was introduced. Assuming that $\mathbf{X}$ has pushouts, for $M \xrightarrow{m} X \in \mathcal{M}$ construct $O_{\mathbf{A}}(m)$ as follows.



For every morphism $f : M \to A$ with codomain $A \in Ob\mathbf{A}$, form the pushout square $m'f = f'm$. Take the $(\mathcal{E}, \mathcal{M})$ factorisation $ne = m'$ and let m_f be the pullback of n along f'. $O_{\mathbf{A}}(m)$ is then defined to be the intersection of all such m_f.

Given the similarity in construction to the pullback closure, it is not surprising that these two operators are often comparable and at times coincide. For instance it is easy to see that if we have an endofunctor (R, r) such that $RX \in \mathbf{A}$ for every $X \in Ob\mathbf{X}$ then $O_{\mathbf{A}} \sqsubseteq \Phi_{(R,r)}$.

Using results from [Sousa 1994] (in particular Corollary 4.3) and Chapter 3 of this thesis it is not difficult to see that if there is a conglomerate $\mathbf{M}$ of sources in $\mathbf{X}$ such that $\mathbf{X}$ is an $(\mathcal{E}$,$\mathbf{M})$-category, if $\mathbf{A}$ is $\mathcal{M}$-reflective in $\mathbf{X}$ with reflector (R, r) and if $\mathcal{M}$ is stable under pushout then $O_{\mathbf{A}} \cong \Phi_{(R,r)}$. Generally though, the scopes of the two operators are quite different.

1.4 Properties of $\Phi_{(R,r)}$

1.4.1 Many investigations can be made into properties of a closure operator. We will restrict ourselves to weak heredity and idempotence. This is because idempotent, weakly hereditary closure operators are in one to one correspondence with morphism factorisation structures $(\mathcal{F}, \mathcal{N})$ on $\mathbf{X}$ for which $\mathcal{N} \subseteq \mathcal{M}$. Such factorisations derived from $\Phi_{(R,r)}$ will be of interest in the following chapters.

1.4.2 Definition. Let (R, r) be a pointed endofunctor on $\mathbf{X}$.

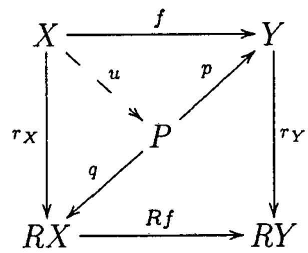
(1) $\text{Fix}(R, r) = \{X \in Ob\mathbf{X} \mid r_X : X \to RX \text{ is an isomorphism}\}$.

(2) $\Sigma_R = \{f \in Mor\mathbf{X} \mid Rf \text{ is an isomorphism}\}$.

(3) We say that (R,r) is *idempotent* if $RX \in \text{Fix}(R,r)$ for every $X \in Ob\mathbf{X}$.

(4) (R,r) is said to be *well-pointed* if for any $X \in Ob\mathbf{X}$, $r_{RX} = Rr_X$.

(5) If $\mathcal{A}$ is a class of $\mathbf{X}$-morphisms, (R,r) will be called $\mathcal{A}$-*direct* if for any $X \xrightarrow{f} Y \in \mathcal{A}$ the pullback $(P,(p,q))$ in the diagram below of Rf along r_Y exists and the morphism $u : X \to P$, induced since $r_Y f = Rfr_X$, is in Σ_R.



If $\mathcal{A} = Mor\mathbf{X}$ then we will simply call (R,r) *direct*.

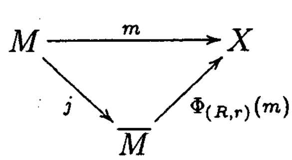
1.4.3 Remark. These notions are not ours. (1) and (2) are in common use, although some authors view $\text{Fix}(R,r)$ as the corresponding full subcategory. Σ_R is the notation of [Cassidy, Hébert, Kelley 1985]. (4) comes from [Kelly 1980]. Directness is a concept that Brümmer and Giuli have recently introduced, $\mathcal{A}$-directnes is a natural generalisation of their notion. It is discussed in more detail in the next chapter.

We can now give some useful criteria for weak heredity and idempotence of $\Phi_{(R,r)}$ for certain (R,r).

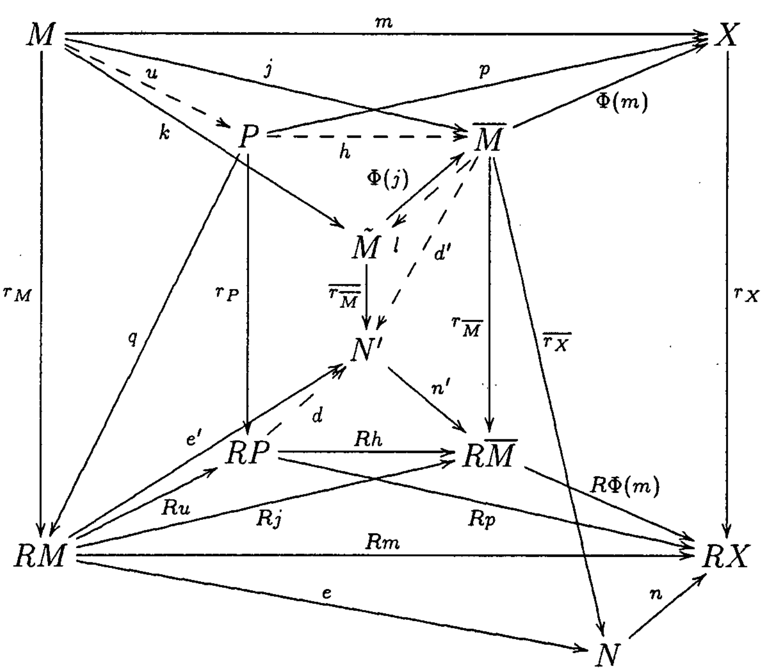
For the rest of the thesis we will assume that the pair (R,r) denotes a pointed endofunctor on $\mathbf{X}$.

1.4.4 Lemma. *If the pullback $(P,(p,q))$ of Rm along r_X exists for any $M \xrightarrow{m} X$ in $\mathcal{M}$ and the induced $u : M \to P$ is such that $Ru \in \mathcal{E}$, then if $\mathcal{E}$ is closed under pullbacks, $\Phi_{(R,r)}$ is weakly hereditary.*

Proof. Take $M \xrightarrow{m} X \in \mathcal{M}$ and form its $\Phi_{(R,r)}$-closure.



To show that $\Phi_{(R,r)}$ is weakly hereditary, we now construct $\Phi_{(R,r)}(j)$ and show that this is an isomorphism. The diagram below details the necessary constructions. By assumption, we can take the pullback $(P,(p,q))$ of Rm along r_X. As always, $ne = Rm$ is the $(\mathcal{E},\mathcal{M})$ factorisation needed to construct $\Phi_{(R,r)}(m)$, while $n'e' = Rj$ is the factorisation used to construct $\Phi_{(R,r)}(j)$.



Since $r_X\Phi_{(R,r)}(m) = n\overline{r_X}$ is a pullback square and $neq = Rmq = r_Xp$, there is a unique $h : P \longrightarrow \overline{M}$ such that $\Phi_{(R,r)}(m)h = p$ and $\overline{r_X}h = eq$. Considering the two pullbacks $(\overline{M},(\Phi_{(R,r)}(m),\overline{r_X}))$ and $(P,(p,q))$, since $r_X\Phi_{(R,r)}(m)h = r_Xp = Rmq = neq$, h is the pullback of e along $\overline{r_X}$ ([Adámek, Herrlich, Strecker 1990] Proposition 11.10(2)) and so since $\mathcal{E}$ is closed under pullbacks, $h \in \mathcal{E}$.

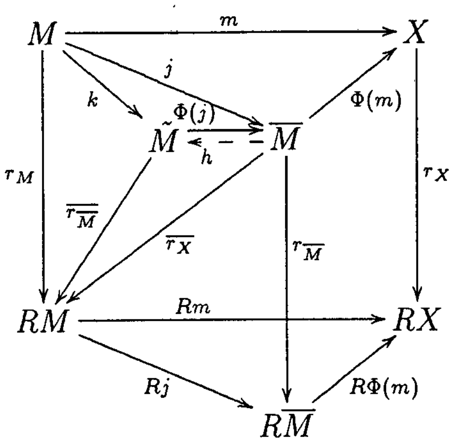
Next note that $\Phi_{(R,r)}(m)hu = pu = m = \Phi_{(R,r)}(m)j$ gives $hu = j$ and thus $RhRu = R(hu) = Rj = n'e'$, so since by assumption $Ru \in \mathcal{E}$, there is a unique $d : RP \longrightarrow N'$ such that $n'd = Rh$ and $dRu = e'$. This then gives that $n'dr_P = Rhr_P = r_{\overline{M}}h$ and then since $h \in \mathcal{E}$, a unique $d' : \overline{M} \longrightarrow N'$ exists with $n'd' = r_{\overline{M}}$ and $d'h = dr_P$.

Now since $n'd' = r_{\overline{M}}$ and $\Phi_{(R,r)}(j)$ is the pullback of n' along $r_{\overline{M}}$, there is a unique $l : \overline{M} \longrightarrow \tilde{M}$ such that $\overline{r_{\overline{M}}}l = d'$ and $\Phi_{(R,r)}(j)l = 1_{\overline{M}}$. This last equality tells us that $\Phi_{(R,r)}(j)$ is a retraction, but it is also a monomorphism, hence an isomorphism and $\Phi_{(R,r)}$ is weakly hereditary. □

1.4.5 Corollary. *If $\mathcal{E}$ is closed under pullbacks and (R,r) is $\mathcal{M}$-direct, then $\Phi_{(R,r)}$ is weakly hereditary.*

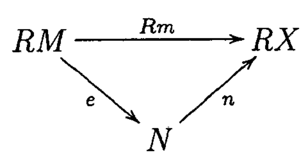
1.4.6 Proposition. *If (R,r) preserves $\mathcal{M}$-morphisms, then $\Phi_{(R,r)}$ is weakly hereditary.*

Proof. Again we form the closure of j in the diagram below. The preservation of $\mathcal{M}$-morphisms by (R,r) makes the constructions simpler.



Now $R\Phi_{(R,r)}(m)r_{\overline{M}} = r_X\Phi_{(R,r)}(m) = Rm\overline{r_X} = R(\Phi_{(R,r)}(m)j)\overline{r_X} = R\Phi_{(R,r)}(m)Rj\overline{r_X}$, and so since $R\Phi_{(R,r)}(m) \in \mathcal{M} \subseteq Mono\mathbf{X}$ we have that $Rj\overline{r_X} = r_{\overline{M}}$. But then since $Rj\overline{r_M} = r_{\overline{M}}\Phi_{(R,r)}(j)$ is a pullback square there is a unique $h : \overline{M} \to \tilde{M}$ such that $\overline{r_M}h = \overline{r_X}$ and $\Phi_{(R,r)}(j)h = 1_{\overline{M}}$. By this second equality $\Phi_{(R,r)}(j)$ is a retraction, and thus an isomorphism. □

1.4.7 Lemma. *If (R,r) is idempotent, and for any $M \xrightarrow{m} X \in \mathcal{M}$ the $(\mathcal{E},\mathcal{M})$ factorisation depicted below of Rm gives $N \in \mathrm{Fix}(R,r)$, then $\Phi_{(R,r)}$ is idempotent.*



Proof. Since (R,r) is idempotent, RX in the above diagram is in $\mathrm{Fix}(R,r)$, as is N. Thus n is $\Phi_{(R,r)}$-closed since in the diagram below both r_N and r_X are isomorphisms, so $Rn \cong n \in \mathcal{M}$ and the square is trivially a pullback.

$$\begin{array}{ccc} N & \xrightarrow{n} & RX \\ {\scriptstyle r_N}\downarrow & & \downarrow{\scriptstyle r_X} \\ RN & \xrightarrow{Rn} & R^2X \end{array}$$

But for any closure operator C, C-closed $\mathcal{M}$-morphisms are closed under pullbacks, so $\Phi_{(R,r)}(m)$ being the pullback of n along r_X is $\Phi_{(R,r)}$-closed and hence $\Phi_{(R,r)}$ is idempotent. □

1.4.8 Corollary. *If (R,r) is a reflection to a full subcategory of* $\mathbf{X}$ *which is closed under $\mathcal{E}$-images or $\mathcal{M}$-subobjects, then $\Phi_{(R,r)}$ is idempotent.*

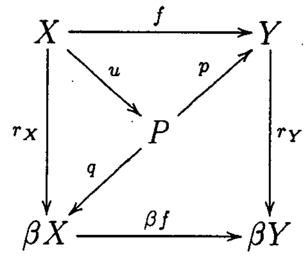
1.4.9 Corollary. *If (R,r) is idempotent and preserves $\mathcal{M}$-morphisms, then $\Phi_{(R,r)}$ is idempotent.*

1.4.10 Remark. The condition of an endofunctor preserving $\mathcal{M}$-morphisms is rather strong. We introduce it here, not just because it provides useful sufficient criteria but because pointed endofunctors with this property are investigated further in Chapter 3.

1.5 Examples

This section relates the criteria for idempotence and weak heredity established in the preceding results to the examples of section 1.2 above.

1.5.1 Čech-Stone compactification. As was mentioned in 1.2.1, in <u>TYCH</u> the pullback $(P,(p,q))$ shown below of βf along r_Y for any continuous map $f : X \to Y$ induces the (Dense C^*-embedding, Perfect map) factorisation pu of f.



It is also well known (cf. [Gillman, Jerison 1960] Theorem 6.7) that since X is densely C^*-embedded in P, $\beta X \cong \beta P$ with isomorphism βu and so the Čech-Stone compactification is a direct endofunctor. Since in this setting $\mathcal{E}$ is the class of surjective continuous maps, which is closed under pullbacks, Corollary 1.4.5 gives that Φ induced by β is weakly hereditary.

On the other hand the image of any HCOMP space in another HCOMP space is an HCOMP space, so by Lemma 1.4.7, Φ must be idempotent. (Of course we know already that Φ is the usual closure which is both hereditary and idempotent.)

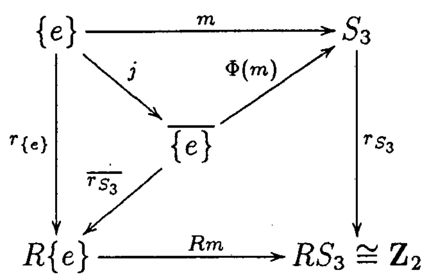
1.5.2 TOP$_0$ reflection and sobrification. As was noted in Examples 1.2.2 and 1.2.3, both the TOP$_0$ reflection and its composition with the sobrification of a TOP$_0$ space are endofunctors that preserve embeddings, so by Proposition 1.4.6 and Corollary 1.4.9 they induce weakly hereditary, idempotent pullback closures. Again, these observations can easily be made from the examples themselves.

1.5.3 $\mathcal{E}$-reflections. If (R, r) is an $\mathcal{E}$-reflection, then the full reflective subcategory Fix(R, r) is closed under $\mathcal{M}$-subobjects, so by Corollary 1.4.8 $\Phi_{(R,r)}$ will be an idempotent closure. This is clearly the case in Example 1.2.2 and the specific examples given in 1.2.4.

1.5.4 Reflectors that are not direct. Not all reflectors are direct, or even $\mathcal{M}$-direct.

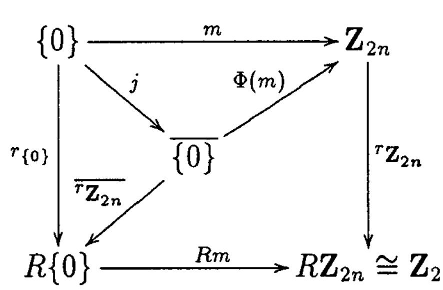
(1) Let (R, r) : GRP $\to$ ABGRP be the Abelian reflection in the category of groups described in Example 1.2.4(3) above. As in that example, for a group G, C_G denotes the commutator subgroup of G.

Consider now the example where S_3 is the group of permutations on 3 elements and A_3 is the subgroup of S_3 consisting of even permutations. One can easily verify that $C_{S_3} \cong A_3$ and hence that $RS_3 \cong S_3/A_3 \cong \mathbf{Z}_2$ (the cyclic group of order 2). Take the $\Phi_{(R,r)}$-closure of the neutral element $\{e\}$ in S_3. (No factorisation of Rm is necessary since it is injective.)



It is clear that the closure $\overline{\{e\}} = A_3$. If we then act (R,r) on the inclusion $j : \{e\} \to A_3$ we have $Rj = j$. (Both $\{e\}$ and A_3 are Abelian.) This shows firstly that (R,r) is not $\mathcal{M}$-direct, and secondly that the closure of $\{e\}$ in A_3 is $\{e\}$, so $\Phi_{(R,r)}$ is not weakly hereditary. We already know from Example 1.2.4(3) that $\Phi_{(R,r)}$ is idempotent, since <u>AbGrp</u> is closed under subgroups this observation also follows from Corollary 1.4.8.

(2) [Cassidy, Hébert, Kelly 1985] Example 4.2 demonstrates that if **A** is the category of Abelian groups of exponent 2, then the reflector (R,r) : <u>AbGrp</u> $\to$ **A** is not direct. For an Abelian group G, the reflection is given by the quotient $r_G : G \to G/2G$. If $\mathbf{Z}_n$ represents the cyclic group of order n, it is easy to see that for any $n \in \mathbf{N}$, $R\mathbf{Z}_{2n} = \mathbf{Z}_{2n}/2\mathbf{Z}_{2n} \cong \mathbf{Z}_2$. So if we take the inclusion $m : \{0\} \to \mathbf{Z}_{2n}$ and form its $\Phi_{(R,r)}$-closure we have the diagram below. (As above Rm is injective.) Clearly $\overline{\{0\}} \cong \mathbf{Z}_n$.



Consider then $m : \{0\} \to \mathbf{Z}_8$. The $\Phi_{(R,r)}$-closure of $\{0\}$ in $\mathbf{Z}_8$ is $\mathbf{Z}_4$ and then the closure of $\{0\}$ in $\mathbf{Z}_4$ is $\mathbf{Z}_2$. This demonstrates that (R,r) is not $\mathcal{M}$-direct and also that $\Phi_{(R,r)}$ is not weakly hereditary. Since **A** is closed under subobjects, Corollary 1.4.8 tells us that $\Phi_{(R,r)}$ is however idempotent.

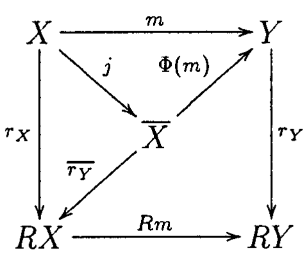
(3) For a topological example, let (R,r) : <u>Top</u> $\to$ <u>ZDim$_0$</u> be the <u>ZDim$_0$</u> reflection in <u>Top</u>. (For a topological space X, $r_X : X \to RX$ is the first component of the (*Epi, ExtrMonoSource*) factorisation of the source of all continuous maps from X to the two point discrete space.) Let X be a singleton topological space, and let Y be the space with underlying set $\mathbf{N} \cup \{\infty_1, \infty_2\}$ (the natural numbers with two points at infinity). The topology on Y is generated by the following open neighbourhood bases:

- For $n \in \mathbf{N}$, simply $\{\{n\}\}$ (i.e. $\mathbf{N}$ is discrete as a subspace of Y).

- For ∞_1, $\mathcal{B}_1 = \{\{m \mid m \geq n\} \cup \{\infty_1\} \mid n \in \mathbf{N}\}$.
- For ∞_2, $\mathcal{B}_2 = \{\{m \mid m \geq n\} \cup \{\infty_2\} \mid n \in \mathbf{N}\}$.

Obviously $RX = X$, and it is not difficult to see that RY is the space with underlying set $\mathbf{N} \cup \{\infty\}$ and topology that again has $\mathbf{N}$ as a discrete subspace, while ∞ has the neighbourhood base $\mathcal{B} = \{\{m \mid m \geq n\} \cup \{\infty\} \mid n \in \mathbf{N}\}$. (This is just the one point compactification of $\mathbf{N}$.)

Now consider the embedding $m : X \to Y$ that maps the single point in X to ∞_1. Form the $\Phi_{(R,r)}$-closure of X in Y.



$\overline{X} \cong \{\infty_1, \infty_2\}$ and inherits the discrete topology as a subspace of Y. This means that $R\overline{X} = \overline{X}$, and so since $RX = X$, Rj is not an isomorphism and (R, r) is not $\mathcal{M}$-direct. Furthermore the $\Phi_{(R,r)}$-closure of X in $\overline{X}$ is simply X, so $\Phi_{(R,r)}$ is not weakly hereditary. Again applying Corollary 1.4.8 we see that $\Phi_{(R,r)}$ is idempotent.

1.6 $\Phi_{(R,r)}$ and factorisation theory

Closure operators are intimately linked to factorisation theory. Pointed endofunctors also have strong links with factorisation structures and in this last section of the chapter, we investigate some of the resulting interconnections.

1.6.1 Definition. Let $\mathcal{A}$ be a class of $\mathbf{X}$-morphisms. $\mathcal{A}^{\downarrow}$ is the class of morphisms g in $\mathbf{X}$ for which any commuting square $vf = gu$ with $f \in \mathcal{A}$ has a unique diagonal. Dually we will speak of the class $\mathcal{A}^{\uparrow}$.

1.6.2 Proposition. (Well known, cf. for example [Strecker 1972] Proposition 1.) *Let $\mathcal{A}$ be a class of* $\mathbf{X}$*-morphisms.*

(a) *Iso*$\mathbf{X} \subseteq \mathcal{A}^{\downarrow}$.

(b) $\mathcal{A}^{\downarrow}$ *is closed under composition.*

(c) $\mathcal{A}^{\downarrow}$ *is closed under pullbacks along any* $\mathbf{X}$*-morphism.*

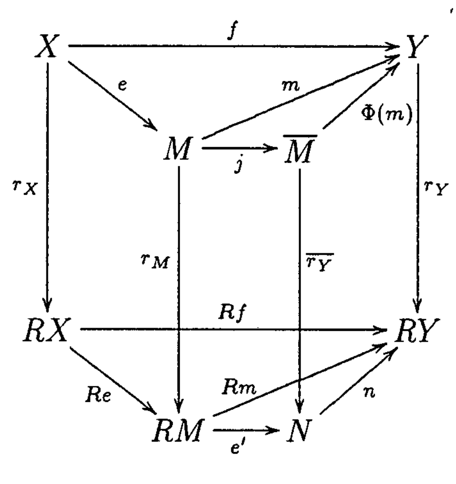
(d) $\mathcal{A}^{\downarrow}$ *is closed under multiple pullbacks.*

(e) $\mathcal{A}^{\downarrow}$ *is closed under products.*

1.6.3 A number of authors have investigated when the pair $(\Sigma_R, \Sigma_R^{\downarrow})$ forms a factorisation structure for morphisms in $\mathbf{X}$. We look at this question in Chapter 2. For now we are concerned with the strong links that exist between the morphisms in $\Sigma_R^{\downarrow} \cap \mathcal{M}$ and the $\Phi_{(R,r)}$-closed morphisms.

1.6.4 Proposition. $\Sigma_R \subseteq \{\Phi_{(R,r)}\textit{-dense}\}$.

Proof. Let $X \xrightarrow{f} Y \in \Sigma_R$, and $me = f$ be its $(\mathcal{E}, \mathcal{M})$ factorisation. We must show that $\Phi_{(R,r)}(m)$ is an isomorphism. The construction is shown below, with $ne' = Rm$ the $(\mathcal{E}, \mathcal{M})$ factorisation that gives $\Phi_{(R,r)}(m)$.



But, $ne'Re = RmRe = R(me) = Rf$ which is an isomorphism. Thus n is a retraction as well as a monomorphism, hence it is an isomorphism and so is $\Phi_{(R,r)}(m)$. □

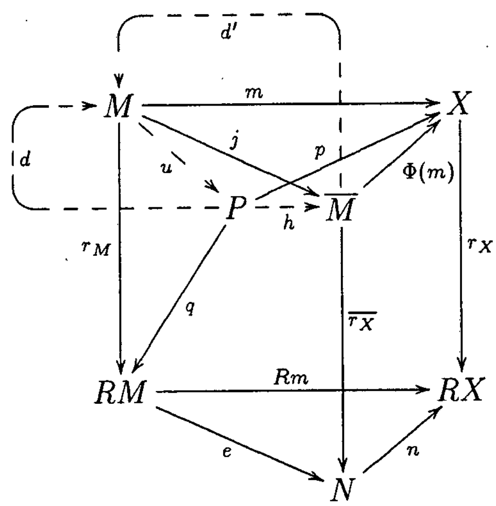
1.6.5 Corollary. $\{\Phi_{(R,r)}\textit{-closed}\} \subseteq \Sigma_R^{\downarrow}$.

Proof. For any closure operator C on $\mathbf{X}$ with respect to $\mathcal{M}$, it is true that $\{C\text{-}\textit{closed}\} \subseteq \{C\text{-}\textit{dense}\}^{\downarrow}$ ([Holgate 1992] Lemma 2.2.9). Then since the operation $(-)^{\downarrow}$ reverses order we see that $\{\Phi_{(R,r)}\textit{-closed}\} \subseteq \{\Phi_{(R,r)}\textit{-dense}\}^{\downarrow} \subseteq \Sigma_R^{\downarrow}$. □

1.6.6 Proposition. *If (R,r) is $\mathcal{M}$-direct and $\mathcal{E}$ is closed under pullbacks then,* $\Sigma_R^{\downarrow} \cap \mathcal{M} = \{\Phi_{(R,r)}\text{-}closed\}$.

Proof. In light of the above corollary, we only need to show that $\Sigma_R^{\downarrow} \cap \mathcal{M} \subseteq \{\Phi_{(R,r)}$-*closed*$\}$.

Take $m \in \Sigma_R^{\downarrow} \cap \mathcal{M}$. Form both $\Phi_{(R,r)}(m)$ and the pullback $(P,(p,q))$ of Rm along r_X. u is the unique morphism such that $pu = m$ and $qu = r_M$ and h is the unique morphism such that $\overline{r_X}h = eq$ and $\Phi_{(R,r)}(m)h = p$.



Since both $(P,(p,q))$ and $(\overline{M},(\Phi_{(R,r)}(m),\overline{r_X}))$ are pullbacks and $r_X\Phi_{(R,r)}(m)h = r_Xp = Rmq = neq$, h is the pullback of e along $\overline{r_X}$ ([Adámek, Herrlich, Strecker 1990] Proposition 11.10(2)). Thus since $\mathcal{E}$ is closed under pullbacks, $h \in \mathcal{E}$. Now (R,r) is $\mathcal{M}$-direct, so $u \in \Sigma_R$, and then because $pu = m \in \Sigma_R^{\downarrow}$ there is a unique $d : P \rightarrow M$ such that $md = p$ and $du = 1_M$. Furthermore $md = p = \Phi_{(R,r)}(m)h$ and $h \in \mathcal{E}$, so by the $(\mathcal{E},\mathcal{M})$ diagonalisation property there is a unique $d' : \overline{M} \rightarrow M$ such that $md' = \Phi_{(R,r)}(m)$ and $d'h = d$.

Now, $md' \in \mathcal{M} \Rightarrow d' \in \mathcal{M}$, but $d'hu = du = 1_M$ so d' is a retraction and a monomorphism, hence an isomorphism and m is $\Phi_{(R,r)}$-closed. □

1.6.7 Considering Proposition 1.6.2 and Theorem 1 of Chapter 0, we see immediately that the properties of $\Sigma_R^{\downarrow}$ and $\mathcal{M}$ are such that the class $\Sigma_R^{\downarrow} \cap \mathcal{M}$ satisfies conditions (a), (b) and (c) of Theorem 1. This means that for any (R,r) on $\mathbf{X}$, $((\Sigma_R^{\downarrow} \cap \mathcal{M})^{\uparrow}, \Sigma_R^{\downarrow} \cap \mathcal{M})$ is a factorisation structure for morphisms in $\mathbf{X}$.

A well known feature of closure operator theory is the bijective correspondence between morphism factorisation structures $(\mathcal{F},\mathcal{N})$ on $\mathbf{X}$, where $\mathcal{N} \subseteq \mathcal{M}$ and idempotent, weakly hereditary closure operators on $\mathbf{X}$ with respect to $\mathcal{M}$. (This is outlined in [Dikranjan, Giuli 1987].) Since $\Sigma_R^{\downarrow} \cap \mathcal{M} \subseteq \mathcal{M}$, the factorisation structure $((\Sigma_R^{\downarrow} \cap \mathcal{M})^{\uparrow}, \Sigma_R^{\downarrow} \cap \mathcal{M})$ corresponds to an idempotent, weakly hereditary closure operator which we will denote by C_R.

1.6.8 Proposition. (a) *C_R is contained in the idempotent hull of $\Phi_{(R,r)}$.*

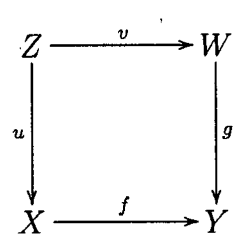
(b) *If $\{\Phi_{(R,r)}$-closed$\} = \Sigma_R^{\downarrow} \cap \mathcal{M}$ then C_R is the idempotent hull of $\Phi_{(R,r)}$.*

Proof. (a) Knowing that the idempotent hull of $\Phi_{(R,r)}$ has exactly the same closed morphisms as $\Phi_{(R,r)}$ (cf. [Dikranjan, Giuli, Tholen 1989] Theorem 2.12), the result follows since $\{\Phi_{(R,r)}$-*closed*$\} \subseteq \Sigma_R^{\downarrow} \cap \mathcal{M} = \{C_R$-*closed*$\}$.

(b) Clear in the light of (a). □

1.7 Notes and problems

1.7.1 The condition that $\mathcal{E}$ be stable under pullback which is used in some results of Section 1.4 is recurrent in this thesis. This property is also utilised in [Clementino, Giuli, Tholen 1995], where it is demonstrated (Proposition 1.5) that $\mathcal{E}$ is closed under pullbacks iff $(\mathcal{E}, \mathcal{M})$ satisfies the *Beck-Chevalley Property.* This property states that for any pullback square in $\mathbf{X}$



and any $m \in \mathcal{M}_X$ it is true that $v(u^{-1}(m)) = g^{-1}(f(m))$. A consequence of this condition is that for any $f : X \to Y$ in $\mathbf{X}$ and any $m \in \mathcal{M}_Y$, $f(f^{-1}(m)) = m$ ([Clementino, Giuli, Tholen 1995] Proposition 1.3).

1.7.2 Proposition 1.6.8 gives a condition under which C_R is the idempotent hull of $\Phi_{(R,r)}$. In particular if (R, r) is $\mathcal{M}$-direct and $\mathcal{E}$ is closed under pullbacks then Proposition 1.6.6 leads us to conclude that C_R is the idempotent hull of $\Phi_{(R,r)}$. We also know from Corollary 1.4.5 that under these conditions $\Phi_{(R,r)}$ is weakly hereditary. These observations lead us to pose the following

Problem A: Is it generally true that C_R is the weakly hereditary core of the idempotent hull of $\Phi_{(R,r)}$?

Chapter 2

Perfect Morphisms

This chapter develops a theory of perfect morphisms relative to a pointed endofunctor (R, r) on the category $\mathbf{X}$. The purpose is partly to further develop the general theory of perfect morphisms in a category, it is also to demonstrate how the pullback closure $\Phi_{(R,r)}$ can be used to reconcile different categorical approaches to perfectness.

2.1 Different notions of perfect morphism

2.1.1 Since the introduction of perfect continuous maps in the middle of this century, topologists have built up a large theory concerning them. In particular a number of characterisations – and indeed definitions – of perfect maps have been given. (They have also been studied under various names such as proper maps and fitting maps.) Thus when categorical topologists in the 1970's set about generalising the notion of a perfect map, a number of different generalisations were possible. A particularly good summary of these can be found in [Herrlich 1974]. Below is an outline of five characterisations that will be used in our investigations.

For this section, we consider perfect continuous maps in TYCH. (R, r) is the pointed endofunctor induced by the Čech-Stone compactification, the reflection in TYCH to HCOMP, which is the paradigmatic example for the rest of this chapter. For a continuous $f : X \to Y$ in TYCH, the following are five different ways of characterising f as a perfect map.

(1) **f is a closed map and for any $y \in Y$, $f^{-1}(y)$ is compact.** This is usually considered to be the definition of a perfect map. Until recently no attempts had been made to generalise this definition. To our knowledge, [Dikranjan, Giuli

1991b] was the first endeavour to make a more general study of morphisms that preserve closure and have compact preimages of points.

(2) **For any space Z the map $f \times 1_Z : X \times Z \to Y \times Z$ is closed.** [Bourbaki 1966] used this as the definition of a perfect map, and their Theorem 1 (Chapter 1 §10.2) showed the equivalence of this definition with the one given in (1). The first attempts to generalise this characterisation were made in [Brown 1973] – for sequential closure – and [Manes 1974] – in categories of "structured sets".

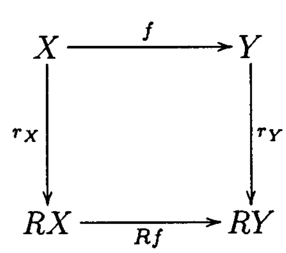
[Herrlich, Salicrup, Strecker 1987] use the closure associated with a factorisation structure for morphisms to take these generalisations further in an hereditary construct. More recently since the formal study of categorical closure operators began, [Dikranjan, Giuli 1991b] investigated the interrelation of this notion with the one in (1) above. They also restrict themselves to certain constructs. Some improvements on their joint results were made in [Dikranjan 1989].

(3) **f is orthogonal to every compact extendable epimorphism.** In categorical notation we could write that $f \in (\underline{\mathrm{HC{\small OMP}}}_{\perp_w} \cap Epi)^{\downarrow} = (\mathrm{Fix}(R,r)_{\perp_w} \cap Epi)^{\downarrow} = \{$*Dense C^*-embeddings*$\}^{\downarrow}$. [Herrlich 1972] made use of this characterisation of perfect maps to find a categorical generalisation. (An appendix to [Nakagawa 1974] introduces independently an equivalent notion.)

A string of papers [Strecker 1972], [Herrlich 1974], [Nel 1974], [Strecker 1974] and finally [Strecker 1976] exploited this line of study. The final paper introduced perfect sources. Collections of such sources occur as the second part of factorisation structures for sources in $\mathbf{X}$, and so are in one-one correspondence with epireflective subcategories of $\mathbf{X}$.

(4) **$f \in \Sigma_{\mathbf{R}}^{\downarrow}$.** This is a result of the fact that in our setting, $\{$*Dense C^*-embeddings*$\} = \Sigma_R$. While this is obviously strongly related to (3) above, we have not found any author who has specifically generalised this fact by considering an arbitrary endofunctor or even reflector (R, r).

(5) **f is the pullback of its image under (R, r).** More precisely, the diagram below is a pullback square.



The fact that this characterises perfect maps was first proved in [Henriksen, Isbell 1958] (Lemma 1.5). In their non-categorical language the result essentially stated that $Rf[RX \setminus r_X[X]] \subseteq (RY \setminus r_Y[Y])$. (This gives a pullback since each r_X is an embedding.) A number of authors, [Blaszczyk, Mioduszewski 1971], [Franklin 1971], [Tsai 1973] and [Hager 1975] took this approach to generalising perfect maps in relatively restricted settings.

[Strecker 1976] calls this notion of perfectness R-strongly perfect and extends it to sources. He gives a few results that relate this notion to the one in (3) that was so widely studied. It seems that no-one took these ideas any further apart from the recent work in [Brümmer, Giuli 1993a]. This last notion of a perfect map is the central one that we will use in the investigations of this chapter.

2.1.2 Definition. Let (R, r) be any pointed endofunctor on our category $\mathbf{X}$. A morphism $f : X \to Y$ in $\mathbf{X}$ will be called *weakly (R, r)-perfect* if $f \in \Sigma_R^{\downarrow}$. We will call f *(R, r)-perfect* if the commutative square below is a pullback.

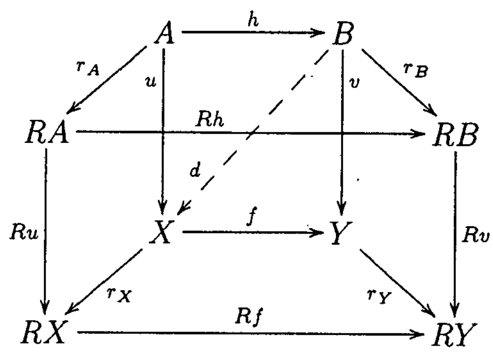
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle r_X}\downarrow & & \downarrow{\scriptstyle r_Y} \\ RX & \xrightarrow[Rf]{} & RY \end{array}$$

2.2 Basic results

There are numerous results in topology regarding properties of perfect maps and their relation to compact spaces. A number of these are easily generalised in our setting, which we do here for both weakly (R, r)-perfect and (R, r)-perfect morphisms, taking the class $\text{Fix}(R, r)$ as the analogue of the compact Hausdorff spaces. First we make an important observation.

2.2.1 Proposition. *If $f : X \to Y$ in $\mathbf{X}$ is (R, r)-perfect, then f is weakly (R, r)-perfect.*

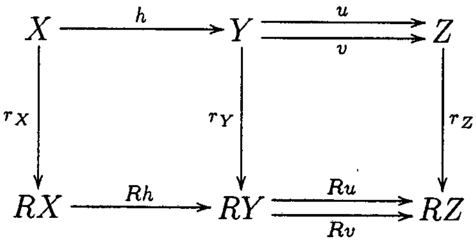
Proof. Let $f : X \to Y$ be (R, r)-perfect and $A \xrightarrow{h} B \in \Sigma_R$ with morphisms u and v such that $vh = fu$.



$RfRu(Rh)^{-1}r_B = Rvr_B = r_Yv$ so since f is (R,r)-perfect there is a unique $d : B \to X$ such that $fd = v$ and $r_Xd = Ru(Rh)^{-1}r_B$. Thus we have $fdh = vh = fu$ and $r_Xdh = Ru(Rh)^{-1}r_Bh = Rur_A = r_Xu$, so since $(X,(f,r_X))$ is a monosource $dh = u$ and d is a diagonal for the square $fu = vh$. It is a unique diagonal since any other diagonal d^* would give $R(d^*h) = Ru \Rightarrow Rd^* = Ru(Rh)^{-1} \Rightarrow Rd^*r_B = Ru(Rh)^{-1}r_B \Rightarrow r_Xd^* = Ru(Rh)^{-1}r_B$ and so by the uniqueness condition on d, $d^* = d$, giving that f is weakly (R,r)-perfect. □

2.2.2 Lemma. *Any morphism $h : X \to Y$ in Σ_R is* Fix(R,r)*-cancellable.*

Proof. Take $X \xrightarrow{h} Y \in \Sigma_R$ and morphisms $u, v : Y \to Z$ such that $uh = vh$ and Z is in Fix(R,r). We must show that $u = v$. Consider the diagram below.



Since $uh = vh$, $RuRh = R(uh) = R(vh) = RvRh$, but Rh is an isomorphism, so $Ru = Rv$. Thus $Rur_Y = Rvr_Y$ and so $r_Zu = r_Zv$, but r_Z is an isomorphism since $Z \in$ Fix(R,r) so $u = v$. □

2.2.3 Proposition. *The class of weakly (R,r)-perfect morphisms contains all* **X** *isomorphisms and is closed under composition, pullbacks, multiple pullbacks and products in* **X**.

Proof. Proposition 1.6.2. □

2.2.4 Remark. It is easy to see that the class of (R, r)-perfect morphisms contains all isomorphisms and is closed under composition. We need to assume various properties for (R, r) before the other results follow. The next section will deal with this.

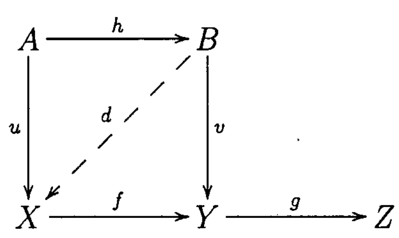
2.2.5 Proposition. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathbf{X}$ such that their composition gf is (weakly) (R, r)-perfect.*

(a) *If g is a monomorphism then f is (weakly) (R, r)-perfect.*

(b) *If g is (weakly) (R, r)-perfect then f is (weakly) (R, r)-perfect.*

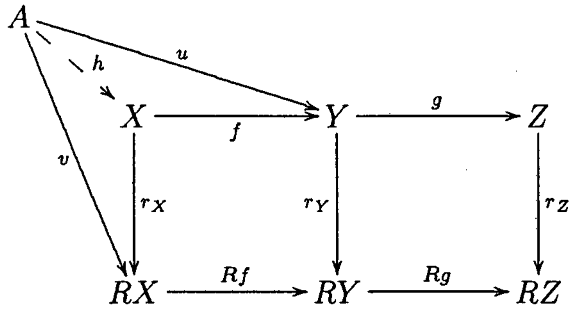
(c) *If f is a retraction then g is weakly (R, r)-perfect.*

Proof. (a) Let gf be weakly (R, r)-perfect . Assume g is a monomorphism and that we have $A \xrightarrow{h} B \in \Sigma_R$ with morphisms u and v such that $fu = vh$.



Since $gfu = gvh$ and gf is weakly (R, r)-perfect, there is a unique morphism $d : B \to X$ such that $gfd = gv$ and $dh = u$. But since g is a monomorphism d is a unique diagonal for the square $fu = vh$ and f is weakly (R, r)-perfect.

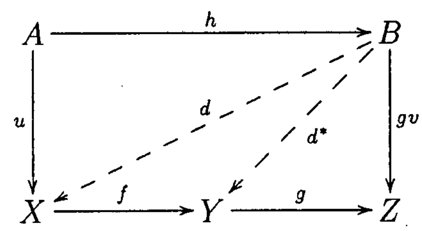
Now let gf be (R, r)-perfect, we must show that the left hand square in the diagram below is a pullback.



Say we have a source $(A, (u, v))$ such that $r_Y u = Rfv$ then $r_Z gu = Rgr_Y u = RgRfv = R(gf)v$, so since gf is (R, r)-perfect, there is a unique $h : A \to X$ such that $gfh = gu$

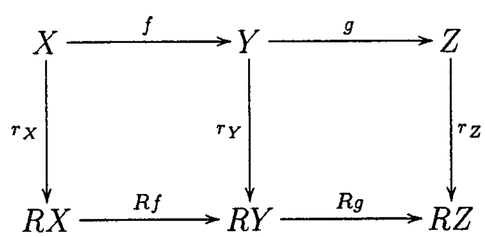
and $r_X h = v$. Since g is a monomorphism, h is also the unique morphism such that $r_X h = v$ and $fh = u$, so f is (R, r)-perfect.

(b) Let gf be weakly (R, r)-perfect and $A \xrightarrow{h} B \in \Sigma_R$ with morphisms u and v such that $fu = vh$. Consider the diagram below.

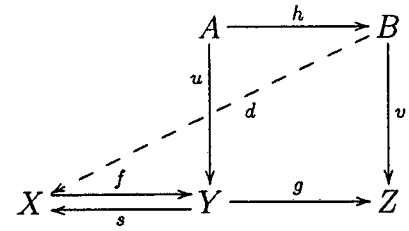


Since gf is weakly (R, r)-perfect, there is a unique morphism $d : B \to X$ such that $dh = u$ and $gfd = gv$. Also since g is weakly (R, r)-perfect, there is a unique morphism $d^* : B \to Y$ such that $d^*h = fu$ and $gd^* = gv$. But then since $fdh = fu$, $gfd = gv$, $vh = fu$ and $gv = gv$ the uniqueness condition on d^* gives that $fd = v = d^*$. Thus d is a unique diagonal for the square $fu = vh$ and f is weakly (R, r)-perfect.

On the other hand if both gf and g are (R, r)-perfect then in the diagram below both the outer square and the right hand square are pullbacks, so the left hand square is a pullback and f is (R, r)-perfect.



(c) Say gf is weakly (R, r)-perfect and we have $A \xrightarrow{h} B \in \Sigma_R$ and morphisms u and v such that $gu = vh$.



f has a right inverse s, so $gfsu = gu = vh$ and thus there is a unique $d : B \to X$ such that $dh = su$ and $gfd = v$. Put $d^* := fd$ then $d^*h = fdh = fsu = u$ and $gd^* = gfd = v$

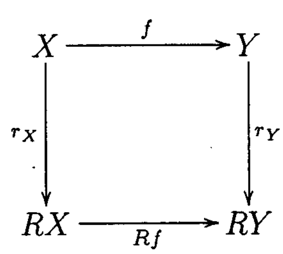
making d^* a diagonal for the square $gu = vh$. It is a unique diagonal since any other d' such that $d'h = u$ and $gd' = v$ would give $sd'h = su$ and $gfsd' = v$ and so by the uniqueness condition on d, $sd' = d$ and then $d' = fsd' = fd = d^*$. □

2.2.6 Proposition. *Let $f : X \to Y$ be a morphism in* $\mathbf{X}$ *with codomain Y in* $\mathrm{Fix}(R, r)$.

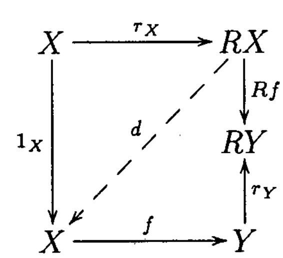
(a) *f is (R, r)-perfect iff X is in* $\mathrm{Fix}(R, r)$.

(b) *If (R, r) is idempotent and well-pointed then f is weakly (R, r)-perfect iff X is in* $\mathrm{Fix}(R, r)$.

Proof. (a) Clear since if r_Y is an isomorphism, then the square below is a pullback iff r_X is an isomorphism.



(b) The reverse implication is immediate since if X is in $\mathrm{Fix}(R, r)$ then by (a) f is (R, r)-perfect, so by Proposition 2.2.1 f is weakly (R, r)-perfect. On the other hand assume that (R, r) is idempotent and well-pointed and that f is weakly (R, r)-perfect.



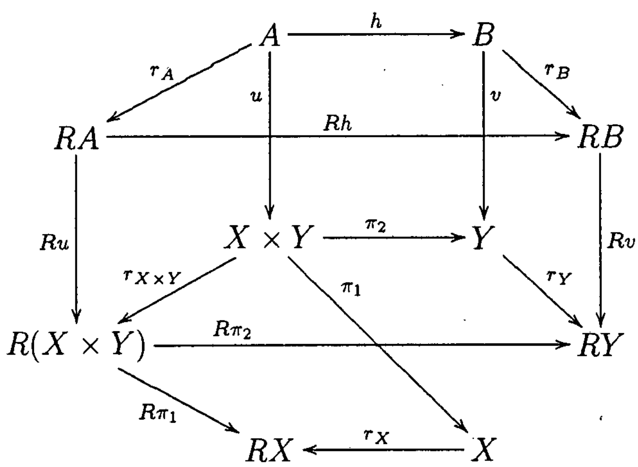
$f1_X = r_Y^{-1}Rfr_X$ so since $r_X \in \Sigma_R$ and (R, r) is well-pointed ($Rr_X = r_{RX}$ which is an isomorphism) there is a unique $d : RX \to X$ such that $dr_X = 1_X$ and $fd = r_Y^{-1}Rf$. But $r_X d r_X = r_X$ and so since $r_X \in \Sigma_R$ and $RX \in \mathrm{Fix}(R, r)$, Lemma 2.2.2 gives that $r_X d = 1_{RX}$, thus r_X is an isomorphism and X is in $\mathrm{Fix}(R, r)$. □

2.2.7 Corollary. *Let* $\mathbf{X}$ *have a terminal object* T *such that* $T \cong RT$. *(If* (R,r) *is idempotent and well-pointed) an* $\mathbf{X}$*-object* X *is in* $\mathrm{Fix}(R,r)$ *iff the unique morphism* $X \xrightarrow{t_X} T$ *is (weakly)* (R,r)*-perfect.*

2.2.8 Remark. In most instances it is the case that $T \cong RT$ (for example if (R,r) is pointwise epimorphic). It is worth noting that this condition is not needed to prove that $X \in \mathrm{Fix}(R,r) \Rightarrow X \xrightarrow{t_X} T$ is (R,r)-perfect. (An alternative proof can be given.) Also the assumption of idempotence and well-pointedness is only needed for the one direction in the weakly (R,r)-perfect case.

2.2.9 Proposition. *Let* $\mathbf{X}$ *have products of pairs. If* $X \in \mathrm{Fix}(R,r)$ *then for any* $Y \in Ob\mathbf{X}$*, the projection* $\pi_2 : X \times Y \to Y$ *is weakly* (R,r)*-perfect.*

Proof. Let $X \in \mathrm{Fix}(R,r)$ and $Y \in Ob\mathbf{X}$. Say we have $A \xrightarrow{h} B \in \Sigma_R$ and morphisms u and v such that $\pi_2 u = vh$.



Put $d := \langle r_X^{-1} R\pi_1 Ru(Rh)^{-1} r_B, v\rangle$ (the unique morphism $d : B \to X \times Y$ such that $\pi_1 d = r_X^{-1} R\pi_1 Ru(Rh)^{-1} r_B$ and $\pi_2 d = v$). Now $\pi_1 dh = r_X^{-1} R\pi_1 Ru(Rh)^{-1} r_B h = r_X^{-1} R\pi_1 Rur_A = r_X^{-1} R\pi_1 r_{X\times Y} u = r_X^{-1} r_X \pi_1 u = \pi_1 u$ and $\pi_2 dh = vh = \pi_2 u$, so since $(X \times Y, (\pi_1, \pi_2))$ is a monosource $dh = u$ and d is a diagonal for the square $\pi_2 u = vh$.

Say we have a morphism d^* such that $d^* h = u$ and $\pi_2 d^* = v$ then since $\pi_1 d^* h = \pi_1 dh$ and $h \in \Sigma_R$ and $X \in \mathrm{Fix}(R,r)$ Lemma 2.2.2 gives that $\pi_1 d^* = \pi_1 d$. We also have that $\pi_2 d^* = v = \pi_2 d$, so $d^* = d$ and the diagonal is unique proving that π_2 is weakly (R,r)-perfect as claimed. □

2.2.10 Remark. The extent of the work done on perfectness is such that these results in various guises have appeared in many publications considering different definitions of perfectness. This section has been an attempt to show that similar results hold true in our context.

[Blaszczyk, Mioduszewski 1971], [Franklin 1971], [Tsai 1973] and [Hager 1975] in particular all have some results similar to ours in their contexts. Of these, [Franklin 1971] has the closest similarity to our work because of the categorical methods employed there. Good summaries of the basic results in topology can be found in [Bourbaki 1966] §10 and [Engelking 1989] §3.7.

2.3 $\mathcal{A}$-direct endofunctors

2.3.1 We return now to the notion of an $\mathcal{A}$-direct endofunctor. In Chapter 1 $\mathcal{A}$-directness (in particular $\mathcal{M}$-directness) was useful for the effect it had on the pullback closure. Here we see how important it is in the interrelation of (R, r)-perfect and weakly (R, r)-perfect morphisms.

As was mentioned in Chapter 1, the definition of $\mathcal{A}$-directness is a natural generalisation of the notion of a direct reflection that Brümmer and Giuli have recently introduced. In [Cassidy, Hébert, Kelly 1985] Section 4, the notion of a *simple* reflection is considered. The difference between a simple reflection and a direct reflection is that the definition of a simple reflection is made in a category that has pullbacks, whereas in the case of directness pullbacks are built into the definition. This makes directness a somewhat more versatile concept.

In both [Brümmer, Giuli 1993a] and [Cassidy, Hébert, Kelly 1985] a number of results regarding directness and simplicity are given. We now reproduce some of these in our context, and give an overview of the scope of the notion of $\mathcal{A}$-directness in a category $\mathbf{X}$.

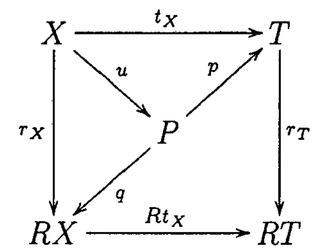
2.3.2 Proposition. *For any (R, r) on $\mathbf{X}$ the following are true.*

(a) *(R, r) is Σ_R-direct.*

(b) *If (R, r) is $\mathcal{A}$-direct and $\mathcal{B} \subseteq \mathcal{A}$ then (R, r) is $\mathcal{B}$-direct.*

(c) *Let (R, r) be pointwise epimorphic. If $\mathbf{X}$ has a terminal object T such that for every $X \in Ob\mathbf{X}$, the unique morphism $X \xrightarrow{t_X} T \in \mathcal{A}$, then if (R, r) is $\mathcal{A}$-direct, (R, r) is idempotent.*

Proof. (a) and (b) are trivial. To verify (c), let $X \in Ob\mathbf{X}$ and perform the constructions in the following diagram:

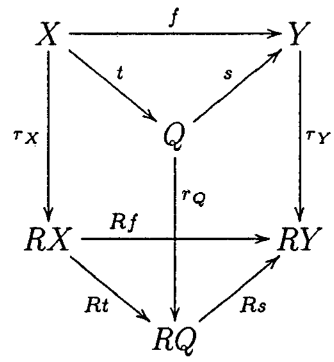


Since $t_X \in \mathcal{A}$ and (R,r) is $\mathcal{A}$-direct, we form the pullback $(P,(p,q))$ of the sink $((Rt_X, r_T), RT)$. Because r_T is an epimorphism and T is a terminal object, r_T is an isomorphism, so q is an isomorphism too. But Ru is an isomorphism so $RqRu = R(qu) = Rr_X = r_{RX}$ is an isomorphism and (R,r) is idempotent as claimed. □

The following two propositions are a dissection of the result in [Brümmer, Giuli 1993a] that a reflection (R,r) is direct in $\mathbf{X}$ iff $\mathbf{X}$ has (Σ_R, (R,r)-perfect) factorisations.

2.3.3 Proposition. *Let $\mathcal{A}$ be a class of $\mathbf{X}$-morphisms and (R,r) a pointed endofunctor on $\mathbf{X}$. If every $f \in \mathcal{A}$ is (Σ_R, (R,r)-perfect) factorisable, then (R,r) is $\mathcal{A}$-direct.*

Proof. Take any $f \in \mathcal{A}$ and form its (Σ_R, (R,r)-perfect) factorisation, $st = f$ and the image of this under R.



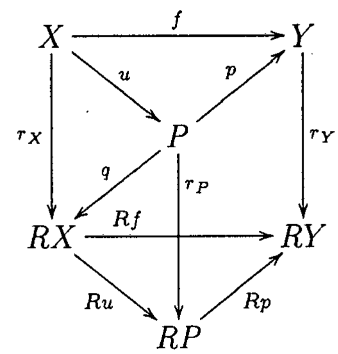
Since s is (R,r)-perfect, the square $r_Y s = Rsr_Q$ is a pullback. But Rt is an isomorphism so the source $(Q,(s,(Rt)^{-1}r_Q))$ is the pullback of Rf along r_Y. Thus since t is in Σ_R and this is for any $f \in \mathcal{A}$, (R,r) is $\mathcal{A}$-direct. □

2.3.4 Proposition. *If (R,r) is a pointed endofunctor on $\mathbf{X}$ which is $\mathcal{A}$-direct for some class $\mathcal{A}$ of $\mathbf{X}$-morphisms, then every $f \in \mathcal{A}$ is (Σ_R, (R,r)-perfect) factorisable if* *either one of the following conditions holds:*

(i) *(R, r) is idempotent; or*

(ii) $\Sigma_R \subseteq Epi\mathbf{X}$.

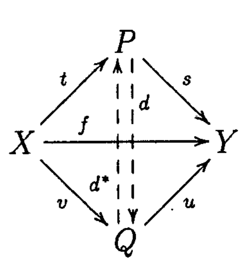
Proof. Let $f \in \mathcal{A}$. Since (R, r) is $\mathcal{A}$-direct we can form the pullback $(P, (p, q))$ of Rf along r_Y below and know that $u \in \Sigma_R$.



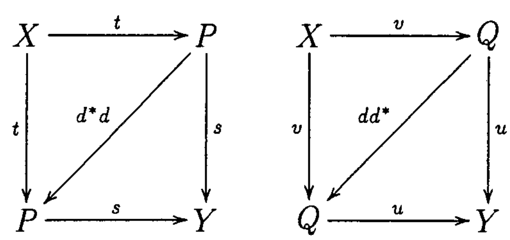
Now $Ruqu = Rur_X = r_P u$ so if (i) holds then by Lemma 2.2.2 $Ruq = r_P$. If (ii) holds then also $Ruq = r_P$. But Ru is an isomorphism which means that the square $r_Y p = Rpr_P$ is a pullback, and so $pu = f$ is a (Σ_R, (R, r)-perfect) factorisation of f. □

2.3.5 Theorem. *If (R, r) is idempotent or $\Sigma_R \subseteq Epi\mathbf{X}$, then for a class $\mathcal{A}$ of $\mathbf{X}$-morphisms, (R, r) is $\mathcal{A}$-direct iff every $\mathcal{A}$-morphism has a unique (up to unique isomorphism) (Σ_R, (R, r)-perfect) factorisation. In particular (R, r) is direct iff (Σ_R, (R, r)-perfect) is a factorisation structure for morphisms in $\mathbf{X}$.*

Proof. The reverse implication is immediate from Proposition 2.3.3. To verify the forward implication, consider two (Σ_R, (R, r)-perfect) factorisations $st = uv = f$ of $f \in \mathcal{A}$. (We know by Proposition 2.3.4 that there is at least one such factorisation.)



By Proposition 2.2.1 both s and u are in $\Sigma_R^{\downarrow}$ so there are unique morphisms $d : P \to Q$ and $d^* : Q \to P$ such that $dt = v$, $ud = s$, $d^*v = t$ and $sd^* = u$. This means that both diagrams below commute and so by the uniqueness of diagonals $d^*d = 1_P$ and $dd^* = 1_Q$, making the factorisations uniquely isomorphic.



The observation for direct (R, r) follows from [Adámek, Herrlich, Strecker 1990] Proposition 14.7 since $Iso\mathbf{X} \subseteq \Sigma_R \cap \{(R, r)\text{-}perfect\}$ and both Σ_R and $\{(R, r)\text{-}perfect\}$ are closed under composition. □

2.3.6 Remarks. (1) Let (R, r) be idempotent or $\Sigma_R \subseteq Epi\mathbf{X}$, then if (R, r) is direct we have that $\{(R, r)\text{-}perfect\} = \Sigma_R^{\downarrow} = \{\,Weakly\ (R, r)\text{-}perfect\}$. Just how intimately this condition is interlinked with directness is underlined in the next result.

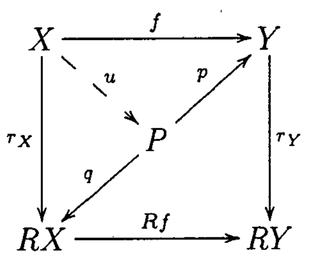
(2) If (R, r) is pointwise monomorphic, then $\Sigma_R \subseteq Epi\mathbf{X}$. (Mimic the proof of Lemma 2.2.2.) On the other hand if (R, r) is pointwise epimorphic and $\mathbf{X}$ has a terminal object T for which every $X \xrightarrow{t_X} T \in \mathcal{A}$, then if (R, r) is $\mathcal{A}$-direct it is idempotent by Proposition 2.3.2(c). Thus the the conclusion of Theorem 2.3.5 holds for either pointwise monomorphic (R, r), or pointwise epimorphic (R, r) with the above terminal object condition. (Note that the idempotence of (R, r) is only used in the forward implication of the theorem.)

(3) A valuable consequence of this result is that by giving sufficient criteria on (R, r) for the coincidence of (R, r)-perfect and weakly (R, r)-perfect morphisms, it enables us to combine the results for these two morphism classes obtained in Section 2.2. For example we now know that for direct and idempotent (R, r) the (R, r)-perfect morphisms are closed under arbitrary products.

(4) In their more specialised setting, [Cassidy, Hébert, Kelly 1985] Theorem 4.1 contains the forward implication of this theorem. That same Theorem 4.1 contains the following result in the case (R, r) is a reflection.

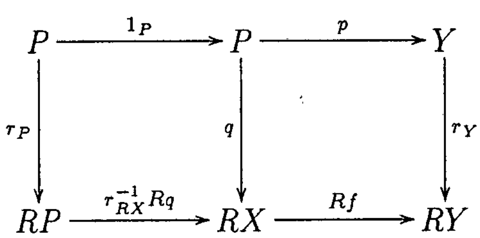
2.3.7 Theorem. *If* $\mathbf{X}$ *has pullbacks and* (R, r) *is idempotent and well-pointed, then* (R, r) *is direct iff* $\{(R, r)\text{-}perfect\} = \{\,Weakly\ (R, r)\text{-}perfect\}$.

Proof. The forward implication follows from Theorem 2.3.5, as was remarked above. To prove the reverse implication, take a morphism $f : X \to Y$ in $\mathbf{X}$ and form the usual pullback shown below.

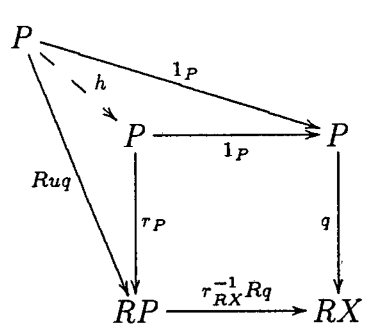


Since (R, r) is idempotent and well-pointed, Proposition 2.2.6 tells us that Rf is weakly (R, r)-perfect. (Both RX and RY are in $\mathrm{Fix}(R, r)$.) But weakly (R, r)-perfect morphisms are closed under pullbacks (Proposition 2.2.3) so p is weakly (R, r)-perfect, and thus by assumption p is (R, r)-perfect.

Now $Rfr_{RX}^{-1}Rq = r_{RY}^{-1}R^2fRq = r_{RY}^{-1}R(Rfq) = r_{RY}^{-1}R(r_Yp) = r_{RY}^{-1}r_{RY}Rp = Rp$, so consider the diagram below.



The outer square is a pullback since p is (R, r)-perfect and the right hand square is a pullback by formation. Thus since $r_{RX}^{-1}Rqr_P = q$ (i.e. all commutes) the left hand square is a pullback. But $RqRu = R(qu) = R(r_X) = r_{RX}$ so $q = r_{RX}^{-1}RqRuq$ and the diagram below commutes.



We have just shown that the square $q1_P = r_{RX}^{-1}Rqr_P$ is a pullback so there is a unique $h : P \to P$ such that $r_Ph = Ruq$ and $1_Ph = 1_P$. So obviously $Ruq = r_P$ and thus $R(Ruq) = R(r_P)$ which implies that $r_{RP}Rur_{RX}^{-1}Rq = R^2uRq = r_{RP}$ giving that $Rur_{RX}^{-1}Rq = 1_{RP}$. Thus Ru has a left and right inverse and u is in Σ_R making (R, r) direct as claimed. □

2.3.8 Remark. As was mentioned above, for the case (R, r) is a reflection this theorem can be found in [Cassidy, Hébert, Kelly 1985]. They give a number of results concerning directness (or simpleness as they term it), ending with a characterisation of localisations in a finitely complete category. The emphasis of this thesis is somewhat different to theirs, so without going deeply into functor theory we summarise parts of their Theorems 4.3, 4.5 and 4.7 in the following result.

2.3.9 Theorem. [Cassidy, Hébert, Kelly 1985] *If* $\mathbf{X}$ *is finitely complete and* (R, r) *is a reflection, then for the following statements we have that (a)* $\Leftrightarrow$ *(b) and each statement implies its successor.*

(a) (R, r) *is a localisation (i.e. preserves finite limits).*

(b) Σ_R *is closed under pullbacks.*

(c) *For any* $X \in Ob\mathbf{X}$*, the pullback of* r_X *along any* $\mathbf{X}$*-morphism is in* Σ_R.

(d) Σ_R *is closed under pullbacks along* $\Sigma_R^{\downarrow}$*-morphisms.*

(e) (R, r) *is direct.*

Further sufficient criteria for the directness of a reflector (R, r) are established in [Brümmer, Giuli 1993a]. We refrain from reproducing them here as they still await publication and have little bearing on the exposition we are giving.

2.4 Early categorical investigations

As was mentioned at the beginning of this chapter, early categorical investigations into perfectness generalised the fact that in <u>TYCH</u> the perfect maps are exactly those in the class $(\underline{\text{HCOMP}}_{\perp_w} \cap Epi)^{\downarrow}$.

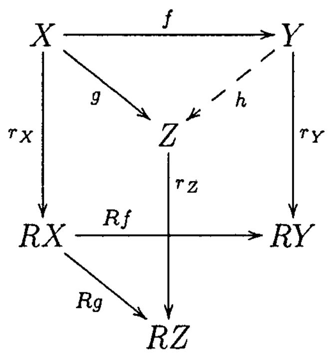
For a class $\mathcal{X}$ of $\mathbf{X}$-objects, a morphism in $\mathbf{X}$ was called *$\mathcal{X}$-perfect* iff it was in the class $(\mathcal{X}_{\perp_w} \cap Epi\mathbf{X})^{\downarrow}$. This notion was introduced in [Herrlich 1972] and in its final investigations was extended to sources in [Strecker 1976]. Theorem 4 of [Strecker 1976] touches on some of the links between this notion of $\mathcal{X}$-perfectness and our present notion of (R, r)-perfectness. In this section we explore these matters further. For any class $\mathcal{X}$

of **X**-objects we will use the term $\mathcal{X}$-perfect as above, this should cause no confusion with the term (R, r)-perfect already being used.

The links we want to explore are between $\text{Fix}(R, r)$-perfect morphisms and (weakly) (R, r)-perfect morphisms. Crucial to this is understanding how Σ_R relates to the class $\text{Fix}(R, r)_{\perp_w} \cap Epi\mathbf{X}$.

2.4.1 Proposition. $\Sigma_R \subseteq \text{Fix}(R, r)_\perp \subseteq \{\text{Fix}(R, r)\text{-}\textit{cancellable}\}$.

Proof. Let $f : X \to Y$ be in Σ_R and let $g : X \to Z$ have codomain Z in $\text{Fix}(R, r)$. Operate (R, r) on these morphisms to get the diagram below.

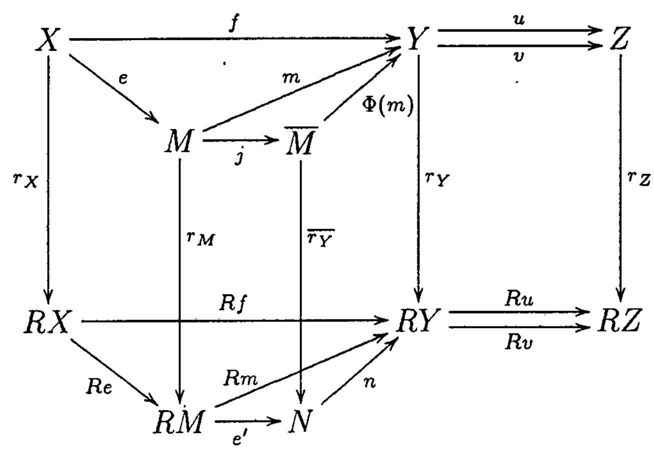


Put $h := r_Z^{-1} Rg (Rf)^{-1} r_Y$ then $hf = r_Z^{-1} Rg (Rf)^{-1} r_Y f = r_Z^{-1} Rg r_X = g$. Lemma 2.2.2 tells us then that since $Z \in \text{Fix}(R, r)$, h is a unique extension so $f \in \text{Fix}(R, r)_\perp$.

If we have a morphism $f : X \to Y$ in $\text{Fix}(R, r)_\perp$ and morphisms $u, v : Y \to Z$ with codomain in $\text{Fix}(R, r)$ such that $uf = vf$, then since $uf = vf$ is a morphism from X to Z it follows immediately that $u = v$ is the unique extension of f to Z over $uf = vf$. Thus f is $\text{Fix}(R, r)$-cancellable. □

2.4.2 Proposition. *If* $\mathcal{E} \subseteq Epi\mathbf{X}$ *then* $\{\Phi_{(R,r)}\text{-}\textit{dense}\} \subseteq \{\text{Fix}(R, r)\text{-}\textit{cancellable}\}$.

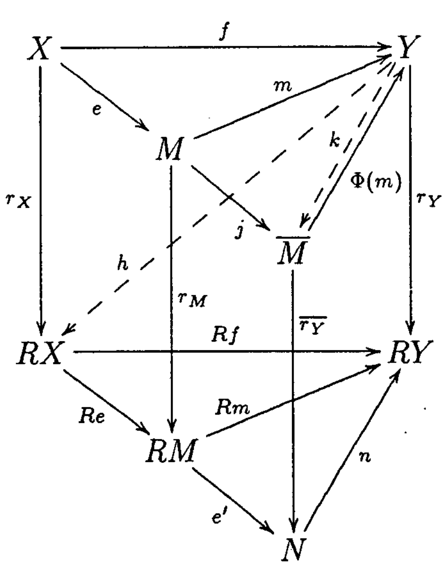
Proof. Let $f : X \to Y$ be $\Phi_{(R,r)}$-dense, and consider morphisms $u, v : Y \to Z$ with codomain in $\text{Fix}(R, r)$ such that $uf = vf$. The diagram below shows the construction of $\Phi_{(R,r)}(m)$ where $me = f$ is the $(\mathcal{E}, \mathcal{M})$ factorisation of f and ne' is the $(\mathcal{E}, \mathcal{M})$ factorisation of Rm.



Since e is an epimorphism, $um = vm$ and thus $RuRm = RvRm$. Again since e' is an epimorphism we have that $Run = Rvn$. Hence $Run\overline{r_Y} = Rvn\overline{r_Y} \Rightarrow Rur_Y\Phi_{(R,r)}(m) = Rvr_Y\Phi_{(R,r)}(m) \Rightarrow r_Zu\Phi_{(R,r)}(m) = r_Zv\Phi_{(R,r)}(m)$. But since Z is in $\mathrm{Fix}(R,r)$ and f is $\Phi_{(R,r)}$-dense, both r_Z and $\Phi_{(R,r)}(m)$ are isomorphisms, so $u = v$ and f is $\mathrm{Fix}(R,r)$-cancellable. □

2.4.3 Proposition. *If* (R,r) *is idempotent then* $\mathrm{Fix}(R,r)_\perp \subseteq \{\Phi_{(R,r)}\text{-dense}\}$.

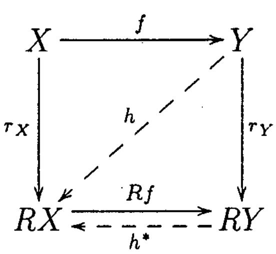
Proof. Let $me = f$ be the $(\mathcal{E},\mathcal{M})$ factorisation of a morphism $f : X \to Y$ in $\mathrm{Fix}(R,r)_\perp$. The diagram below shows the construction of $\Phi_{(R,r)}(m)$ which we need to show is an isomorphism.



Since RX is in $\mathrm{Fix}(R,r)$ there is a (unique) $h : Y \to RX$ such that $hf = r_X$ which gives that $ne'Rehf = RmRehf = Rfhf = Rfr_X = r_Yf$. But since $f \in \mathrm{Fix}(R,r)_{\perp}$ it is $\mathrm{Fix}(R,r)$-cancellable (Proposition 2.4.1) so we conclude that $ne'Reh = r_Y$. This means that there is a unique $k : Y \to \overline{M}$ such that $\overline{r_Y}k = e'Reh$ and $\Phi_{(R,r)}(m)k = 1_Y$. Hence $\Phi_{(R,r)}(m)$ is an isomorphism and f is indeed $\Phi_{(R,r)}$-dense as claimed. □

2.4.4 Proposition. *If (R,r) is idempotent and well-pointed then* $\mathrm{Fix}(R,r)_{\perp} \subseteq \Sigma_R$.

Proof. Construct the diagram below for $f : X \to Y$ in $\mathrm{Fix}(R,r)_{\perp}$. Since (R,r) is idempotent, $RX \in \mathrm{Fix}(R,r)$, so there is a unique $h : Y \to RX$ such that $hf = r_X$.



By Proposition 2.4.1 f is $\mathrm{Fix}(R,r)$-cancellable, so $Rfhf = Rfr_X = r_Yf \Rightarrow Rfh = r_Y$. But since (R,r) is both idempotent and well-pointed, $r_Y \in \Sigma_R \subseteq \mathrm{Fix}(R,r)_{\perp}$ so there is a unique $h^* : RY \to RX$ such that $h^*r_Y = h$. Then because r_Y is $\mathrm{Fix}(R,r)$-cancellable and $Rfh^*r_Y = Rfh = r_Y$ we see that $Rfh^* = 1_{RY}$. Similarly $h^*Rfr_X = h^*r_Yf = hf = r_X$ implies that $h^*Rf = 1_{RX}$ and so Rf is an isomorphism and $f \in \Sigma_R$ as claimed. □

These results combine to give us the following valuable result which generalises Proposition 3.3 of [Brümmer, Giuli 1992] which is given for the case that (R,r) is a reflection.

2.4.5 Proposition. *If (R,r) is idempotent and well-pointed then:*

$$\mathrm{Fix}(R,r)_{\perp_w} \cap \{\mathrm{Fix}(R,r)\text{-}\mathit{cancellable}\} = \mathrm{Fix}(R,r)_{\perp} = \Sigma_R.$$

If in addition $\mathcal{E} \subseteq Epi\mathbf{X}$, these classes are also equal to $\mathrm{Fix}(R,r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}\mathit{dense}\}$.

Proof. Propositions 2.4.1 and 2.4.4 combine to give that $\Sigma_R = \mathrm{Fix}(R,r)_{\perp}$ and with the knowledge of Proposition 2.4.1 it is clear that $\mathrm{Fix}(R,r)_{\perp_w} \cap \{\mathrm{Fix}(R,r)\text{-}\mathit{cancellable}\} = \mathrm{Fix}(R,r)_{\perp}$. Furthermore Proposition 2.4.3 gives that $\mathrm{Fix}(R,r)_{\perp} \subseteq \mathrm{Fix}(R,r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}\mathit{dense}\}$ and if $\mathcal{E} \subseteq Epi\mathbf{X}$ then Proposition 2.4.2 completes the argument by showing that $\mathrm{Fix}(R,r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}\mathit{dense}\} \subseteq \mathrm{Fix}(R,r)_{\perp_w} \cap \{\mathrm{Fix}(R,r)\text{-}\mathit{cancellable}\}$. □

2.4.6 Corollary. *If (R,r) is idempotent and well-pointed, then $\{(R,r)\text{-perfect}\} \subseteq \{\text{Weakly } (R,r)\text{-perfect}\} \subseteq \{\mathrm{Fix}(R,r)\text{-perfect}\}$. If in addition $\Sigma_R \subseteq Epi\mathbf{X}$, then $\{\text{Weakly } (R,r)\text{-perfect}\} = \{\mathrm{Fix}(R,r)\text{-perfect}\}$.*

Proof. The first inclusion is already known to us (Proposition 2.2.1). By the above proposition, $\Sigma_R \cap Epi\mathbf{X} = \mathrm{Fix}(R,r)_{\perp} \cap Epi\mathbf{X} = \mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X}$, from which it follows that $\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X} \subseteq \Sigma_R$ and so $\Sigma_R^{\downarrow} \subseteq (\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X})^{\downarrow}$. This establishes the second inclusion.

If $\Sigma_R \subseteq Epi\mathbf{X}$ then obviously $\Sigma_R = \Sigma_R \cap Epi\mathbf{X} = \mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X}$ and so {*Weakly* (R,r)*-perfect*} = {Fix(R,r)*-perfect*}. □

2.4.7 Remarks. (1) If in addition to the conditions of the above corollary, (R,r) is direct then we know from Remark 2.3.6(1) that (R,r)-perfect morphisms also coincide with Fix(R,r)-perfect morphisms. This is revealed in Theorem 4 of [Strecker 1976] in the case of (R,r) being a reflection.

If furthermore $\mathcal{E} \subseteq Epi\mathbf{X}$ – which is usually the case – then Σ_R can be described as the $\Phi_{(R,r)}$-dense Fix(R,r)-extendable morphisms and ($\Phi_{(R,r)}$*-dense* Fix(R,r)*-extendable*, (R,r)*-perfect*) is a factorisation structure for morphisms in $\mathbf{X}$.

(2) It is not generally the case that $\Sigma_R \subseteq Epi\mathbf{X}$. For example if (R,r) is the $\underline{\mathrm{TOP}_0}$ reflection in $\underline{\mathrm{TOP}}$ then any embedding of a point into any indiscrete space with more than 1 point is in Σ_R while obviously it is not an epimorphism in $\underline{\mathrm{TOP}}$. As was noted in Remark 2.3.6(2) if (R,r) is pointwise monomorphic then we do have that $\Sigma_R \subseteq Epi\mathbf{X}$.

(3) It is also worth noting that in general {$\Phi_{(R,r)}$*-dense*} $\neq$ {Fix(R,r)*-cancellable*}, this is something typical of the regular closure (cf. [Dikranjan, Giuli 1987] Remark(2) p.137). As an example, let (R,r) be the $\underline{\mathrm{TOP}_0}$ reflection again. Take the space $\mathbf{N} \cup \{\infty\}$ which has the topology generated by basic opens of the form $U_n = \{m \in \mathbf{N} \mid m \geq n\} \cup \{\infty\}$ for $n \in \mathbf{N}$. The topological embedding of $\mathbf{N}$ into $\mathbf{N} \cup \{\infty\}$ is b-dense but not $\Phi_{(R,r)}$-dense and it is well known that in $\underline{\mathrm{TOP}}$ the b-dense maps are $\underline{\mathrm{TOP}_0}$-cancellable.

2.4.8 It is of course theoretically possible to have $\Sigma_R^{\downarrow} = (\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X})^{\downarrow}$ without necessarily having that $\Sigma_R = \mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X}$. In most of the examples we consider, this cannot happen.

It is not difficult to see that the class $\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X}$ contains all isomorphisms and is closed under composition, pushouts and cointersections (cf. [Strecker 1972] Proposition 1 (viii)). Thus if $\mathbf{X}$ has pushouts and cointersections, $\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X}$ is the first component of a factorisation structure for sources in $\mathbf{X}$ ([Adámek, Herrlich, Strecker 1990] Theorem 15.14).

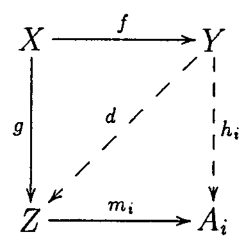
In such cases, the source factorisation structure induces the factorisation structure $((\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X}), (\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X})^{\downarrow})$ for morphisms in $\mathbf{X}$. Thus if we had that $\Sigma_R^{\downarrow} = (\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X})^{\downarrow}$ it would mean that $\Sigma_R \subseteq \Sigma_R^{\downarrow\uparrow} = \mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X}$.

If furthermore the conditions of Proposition 2.4.4 hold, then $\text{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X} \subseteq \Sigma_R$, and equality would follow.

2.4.9 So far we have only considered possible links between these notions from one perspective, namely given an endofunctor (R,r) how $\text{Fix}(R,r)$-perfect and (weakly) (R,r)-perfect morphisms relate. What if we have an arbitrary class $\mathcal{X}$ of $\mathbf{X}$-objects and consider the $\mathcal{X}$-perfect morphisms?

If in our category $\mathbf{X}$ both pushouts of $(\mathcal{X}_{\perp_w} \cap Epi\mathbf{X})$-morphisms along any $\mathbf{X}$-morphism and cointersections of arbitrary families of $(\mathcal{X}_{\perp_w} \cap Epi\mathbf{X})$-morphisms exist then there is a conglomerate $\mathbf{M}$ of sources in $\mathbf{X}$ such that $((\mathcal{X}_{\perp_w} \cap Epi\mathbf{X}), \mathbf{M})$ is a factorisation structure for sources in $\mathbf{X}$ (cf. [Strecker 1972] Proposition 1(vii) and [Adámek, Herrlich, Strecker 1990] Theorem 15.14). This means that $\mathcal{X}$ has a $(\mathcal{X}_{\perp_w} \cap Epi\mathbf{X})$-reflective hull in $\mathbf{X}$. Denote the objects of this hull by $E(\mathcal{X})$ and let $(R_{\mathcal{X}}, r)$ be the pointed endofunctor induced by the reflector.

Since $\mathcal{X} \subseteq E(\mathcal{X})$ obviously $E(\mathcal{X})_{\perp_w} \subseteq \mathcal{X}_{\perp_w}$. On the other hand consider $X \xrightarrow{f} Y \in \mathcal{X}_{\perp_w} \cap Epi\mathbf{X}$ and $g : X \to Z$ with codomain Z in $E(\mathcal{X})$. The formation of $E(\mathcal{X})$ is such that there is a source $(m_i : Z \to A_i)_{i \in I} \in \mathbf{M}$ with each $A_i \in \mathcal{X}$, so we have the diagram below.



Because $f \in \mathcal{X}_{\perp_w}$, for each $i \in I$ there is a morphism $h_i : Y \to A_i$ such that $h_i f = m_i g$. Then by the $((\mathcal{X}_{\perp_w} \cap Epi\mathbf{X}), \mathbf{M})$ diagonalisation property there is a morphism $d : Y \to Z$ such that in particular $df = g$, giving that $f \in E(\mathcal{X})_{\perp_w}$.

So we can conclude that $E(\mathcal{X})_{\perp_w} \cap Epi\mathbf{X} = \mathcal{X}_{\perp_w} \cap Epi\mathbf{X}$ which means (rewriting $E(\mathcal{X})$ as $\text{Fix}(R_{\mathcal{X}}, r)$) that the $\mathcal{X}$-perfect morphisms, $(\mathcal{X}_{\perp_w} \cap Epi\mathbf{X})^{\downarrow} = (\text{Fix}(R_{\mathcal{X}}, r) \cap Epi\mathbf{X})^{\downarrow}$, which are just the $\text{Fix}(R_{\mathcal{X}}, r)$-perfect morphisms.

Since $(R_{\mathcal{X}}, r)$ is a reflection it fulfills the conditions of Corollary 2.4.6 above, so we can conclude that $\{(R_{\mathcal{X}}, r)\textit{-perfect}\} \subseteq \{\textit{Weakly } (R_{\mathcal{X}}, r)\textit{-perfect}\} \subseteq \{\mathcal{X}\textit{-perfect}\}$. Moreover if $\Sigma_{R_{\mathcal{X}}}$ is a class of epimorphisms then $\{\textit{Weakly } (R_{\mathcal{X}}, r)\textit{-perfect}\} = \{\mathcal{X}\textit{-perfect}\}$ and these in turn equal the $(R_{\mathcal{X}}, r)$-perfect morphisms if $(R_{\mathcal{X}}, r)$ is a direct reflection.

2.5 Closure preservation

Our investigations thus far have related (R, r)-perfect morphisms to categorical generalisations of perfectness that exploit the orthogonality properties of perfect continuous maps. Closure operator theory provides a useful tool to use in generalising the closure and compactness properties of perfect maps. We now turn our attention to investigations of this nature, specifically linking (R, r)-perfect morphisms with the $\Phi_{(R,r)}$-closure.

This section considers the Bourbaki definition of a perfect map $f : X \to Y$, namely that for any space Z the map $f \times 1_Z : X \times Z \to Y \times Z$ is closed. For this reason we assume throughout this section that our category $\mathbf{X}$ has products of pairs.

2.5.1 Definition. Let C be a closure operator on $\mathbf{X}$ with respect to $\mathcal{M}$. An $\mathbf{X}$-morphism $f : X \to Y$ is called *C-preserving* if for any $M \xrightarrow{m} X \in \mathcal{M}$, $f(C(m)) \cong C(f(m))$.

This definition is not our own. Initially the term "C-closed preserving" was used in [Castellini 1990], where the context of idempotent closures allowed an equivalent definition of C-preservation to be given, namely that $f : X \to Y$ is C-preserving if for any C-closed $M \xrightarrow{m} X \in \mathcal{M}$, $f(m)$ is C-closed. [Dikranjan, Giuli 1989] introduced the more general form of the definition we use. The form used in [Castellini 1990] is of course more reminiscent of the definition of a closed map in Topology, and is akin to the ideas used in [Manes 1974] and [Herrlich, Salicrup, Strecker 1987].

2.5.2 Proposition. ([Dikranjan, Giuli 1991b] Proposition 3.2.) *Let C be a closure operator on $\mathbf{X}$ with respect to $\mathcal{M}$, and let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathbf{X}$.*

(a) *If both f and g are C-preserving, then gf is C-preserving.*

(b) *If gf is C-preserving and g is a monomorphism, then f is C-preserving.*

Proof. (a) Clear since for $M \xrightarrow{m} X \in \mathcal{M}$, $gf(C(m)) = g(f(C(m))) = g(C(f(m))) = C(g(f(m))) = C(gf(m))$.

(b) Any $\mathbf{X}$-morphism is continuous with respect to C. This combined with the fact that gf is C-preserving gives that $g(C(f(m))) \leq C(g(f(m))) = C(gf(m)) = gf(C(m))$. But g is a monomorphism so we conclude that $C(f(m)) \leq f(C(m))$. The C-continuity of f then gives that $f(C(m)) \leq C(f(m))$ and so equality follows. □

2.5.3 Remark. A number of authors have stated the above result. Although [Dikranjan, Giuli 1991b] prove it in the context of certain constructs, their proof is essentially what we have given.

2.5.4 Proposition. *For $f : X \to Y$ in $\mathbf{X}$, f is (R,r)-perfect $\Rightarrow$ $f \times 1_Z$ is (R,r)-perfect for every $Z \in Ob\mathbf{X}$ if any one of the following holds.*

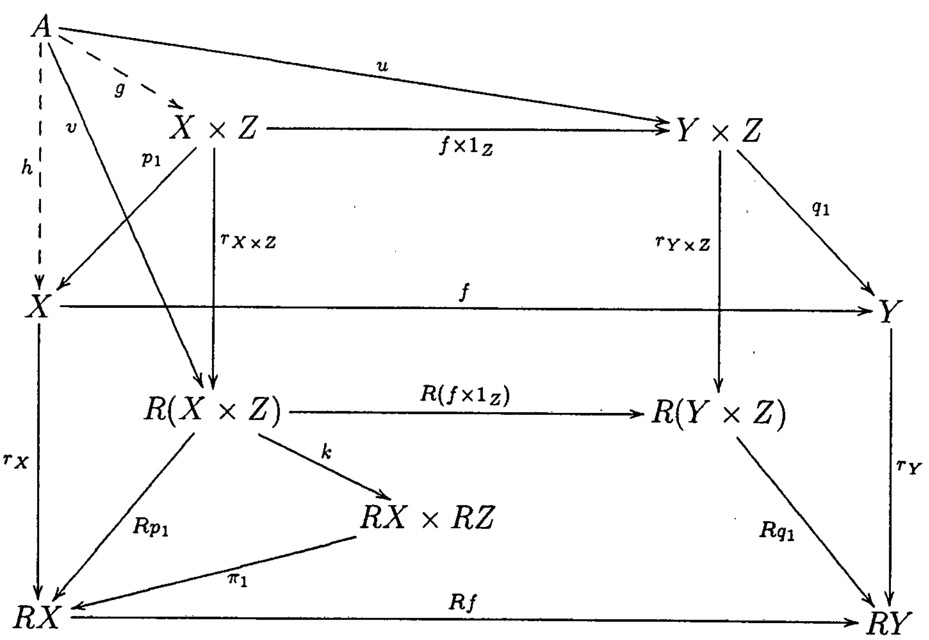
(i) *(R,r) is direct and idempotent.*

(ii) *(R,r) is direct and $\Sigma_R \subseteq Epi\mathbf{X}$.*

(iii) *For any $A, B \in Ob\mathbf{X}$ the canonical morphism $k : R(A \times B) \to RA \times RB$ is a monomorphism.*

Proof. Let $f : X \to Y$ be (R,r)-perfect and pick $Z \in Ob\mathbf{X}$. By Proposition 2.2.1 f is weakly (R,r)-perfect, thus by Proposition 2.2.3 $f \times 1_Z$ is weakly (R,r)-perfect. If either (i) or (ii) holds then Remark 2.3.6(1) makes it clear that $f \times 1_Z$ will then be (R,r)-perfect.

Assume that (iii) holds. To show that $f \times 1_Z$ is (R,r)-perfect consider the diagram below where $u : A \to Y \times Z$ and $v : A \to R(X \times Z)$ are such that $r_{Y \times Z} u = R(f \times 1_Z)v$. The morphisms p_1, q_1 and π_1 are projections.



Since f is (R,r)-perfect and $r_Y q_1 u = Rq_1 r_{Y \times Z} u = Rq_1 R(f \times 1_Z)v = R(q_1(f \times 1_Z))v = R(fp_1)v = RfRp_1 v$, there is a unique $h : A \to X$ such that $fh = q_1 u$ and $r_X h = Rp_1 v$.

Put $g := \langle h, q_2 u\rangle : A \to X \times Z$ and note the following.

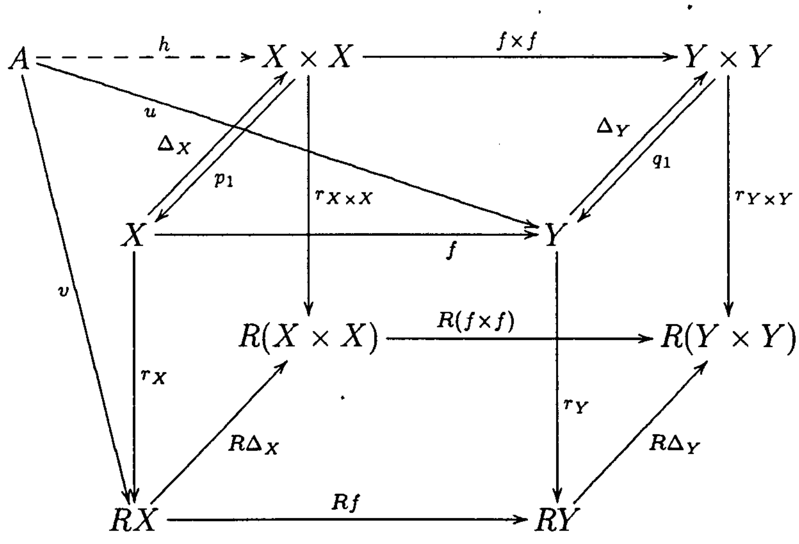
Firstly $q_1(f \times 1_Z)g = fp_1 g = fh = q_1 u$ and $q_2(f \times 1_Z)g = 1_Z p_2 g = q_2 u$, so since projections form a monosource, $(f \times 1_Z)g = u$. Also $\pi_1 k r_{X\times Z} g = Rp_1 r_{X\times Z} g = r_X p_1 g = r_X h = Rp_1 v = \pi_1 k v$ and $\pi_2 k r_{X\times Z} g = Rp_2 r_{X\times Z} g = r_Z p_2 g = r_Z q_2 u = Rq_2 r_{Y\times Z} u = Rq_2 R(f \times 1_Z)v = R1_Z Rp_2 v = Rp_2 v = \pi_2 k v$ so since the projections π_i form a monosource and k is a monomorphism we conclude that $r_{X\times Z} g = v$.

To verify that we have a pullback, we must show that g is unique in this role. Say $g^* : A \to X \times Z$ is such that $(f \times 1_Z)g^* = u$ and $r_{X\times Z} g^* = v$. This gives $fp_1 g^* = q_1(f \times 1_Z)g^* = q_1 u$ and $r_X p_1 g^* = Rp_1 r_{X\times Z} g^* = Rp_1 v$ so by the uniqueness condition on h, $p_1 g^* = h$. On the other hand $p_2 g^* = 1_Z p_2 g^* = q_2(f \times 1_Z)g^* = q_2 u$, so in fact $g^* = \langle h, q_2 u\rangle = g$. □

The implication in the above proposition can be reversed. If our category $\mathbf{X}$ has finite products – not just products of pairs – and thus a terminal object T, we simply put $Z = T$ and clearly $f \times 1_Z$ is (R, r)-perfect $\Rightarrow f$ is (R, r)-perfect. It is possible, however, to prove this without the existence of a terminal object.

2.5.5 Lemma. *Let $f : X \to Y$ in $\mathbf{X}$. If $f \times f$ is (R, r)-perfect then f is (R, r)-perfect.*

Proof. Consider the diagram below with $u : A \to Y$ and $v : A \to RX$ such that $r_Y u = Rfv$. The morphisms $\Delta_X : X \to X \times X$ and $\Delta_Y : Y \to Y \times Y$ are diagonals, note that $\Delta_Y f = (f \times f)\Delta_X$. The morphisms $p_1 : X \times X \to X$ and $q_1 : Y \times Y \to Y$ are again projections.



Now, $r_{Y\times Y}\Delta_Y u = R\Delta_Y r_Y u = R\Delta_Y Rfv = R(\Delta_Y f)v = R((f \times f)\Delta_X)v = R(f \times f)R\Delta_X v$. So since $f \times f$ is (R, r)-perfect, there is a unique $h : A \to X \times X$ such that

$(f \times f)h = \Delta_Y u$ and $r_{X \times X} h = R\Delta_X v$.

Put $g := p_1 h$, then $fg = fp_1 h = q_1(f \times f)h = q_1 \Delta_Y u = 1_Y u = u$ and $r_X g = r_X p_1 h = Rp_1 r_{X \times X} h = Rp_1 R\Delta_X v = R(p_1 \Delta_X)v = R1_X v = v$.

If g^* is such that $fg^* = u$ and $r_X g^* = v$ then $(f \times f)\Delta_X g^* = \Delta_Y f g^* = \Delta_Y u$ and $r_{X \times X}\Delta_X g^* = R\Delta_X r_X g^* = R\Delta_X v$ so by the uniqueness of h, $\Delta_X g^* = h$. Hence $g^* = 1_X g^* = p_1 \Delta_X g^* = p_1 h = g$, and f is (R, r)-perfect. □

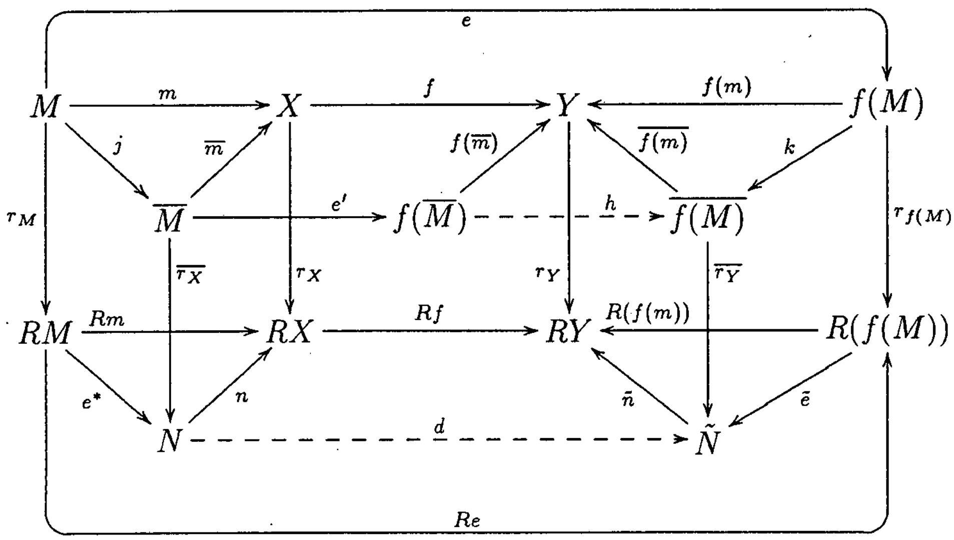
2.5.6 Proposition. *Let $f : X \to Y$ in $\mathbf{X}$. If $f \times 1_Z$ is (R, r)-perfect for every $Z \in Ob\mathbf{X}$, then f is (R, r)-perfect.*

Proof. It is clear that $f \times f \cong (f \times 1_Y)h(f \times 1_X)$ where $h : Y \times X \to X \times Y$ is the canonical coordinate-switching isomorphism between the two products. But we are told that $f \times 1_Y$ and $f \times 1_X$ are (R, r)-perfect, obviously h is too. Hence their composition is (R, r)-perfect, and since (R, r)-perfect morphisms are closed under composition with isomorphisms, $f \times f$ is (R, r)-perfect. The Lemma above then gives that f is (R, r)-perfect. □

So, there are conditions on (R, r) under which a morphism $f : X \to Y$ is (R, r)-perfect iff $f \times 1_Z : X \times Z \to Y \times Z$ is (R, r)-perfect for every $Z \in Ob\mathbf{X}$. While this on its own is of interest, it is also a valuable step towards the final result of this section.

2.5.7 Proposition. *Let $f : X \to Y$ be (R, r)-perfect. If $\mathcal{E}$ is stable under pullback in $\mathbf{X}$, and for every $e \in \mathcal{E}$, $Re \in \mathcal{E}$ then f is $\Phi_{(R,r)}$-preserving.*

Proof. Let $f : X \to Y$ be (R, r)-perfect, and $M \xrightarrow{m} X \in \mathcal{M}$. We must show that $f(\Phi_{(R,r)}(m)) \cong \Phi_{(R,r)}(f(m))$. The constructions are shown in the diagram below ($f(\overline{m}) = f(\Phi_{(R,r)}(m))$ and $\overline{f(m)} = \Phi_{(R,r)}(f(m))$). By virtue of the fact that $\Phi_{(R,r)}$ is a closure operator, $f(\Phi_{(R,r)}(m)) \leq \Phi_{(R,r)}(f(m))$ so there is a morphism $h : f(\overline{M}) \to \overline{f(M)}$ such that $\overline{f(m)}h = f(\overline{m})$. We must show that this h is an isomorphism.



$Rfne^* = RfRm = R(fm) = R(f(m)e) = R(f(m))Re = \tilde{n}\tilde{e}Re$ so there is an $(\mathcal{E}, \mathcal{M})$-diagonal $d : N \to \tilde{N}$ such that $\tilde{n}d = Rfn$ and $de^* = \tilde{e}Re$. Now, $\tilde{n}d\overline{r_X} = Rfn\overline{r_X} = Rfr_X\overline{m} = r_Y f\overline{m} = r_Y f(\overline{m})e' = r_Y \overline{f(m)}he' = \tilde{n}\overline{r_Y}he'$ thus since $\tilde{n}$ is a monomorphism, $d\overline{r_X} = \overline{r_Y}he'$.

But f is (R, r)-perfect and both $(\overline{M}, (\overline{m}, \overline{r_X}))$ and $(\overline{f(M)}, (\overline{f(m)}, \overline{r_Y}))$ are pullbacks, so $d\overline{r_X} = \overline{r_Y}he'$ is a pullback square. Also since $Re \in \mathcal{E}$, $de^* = \tilde{e}Re \in \mathcal{E} \Rightarrow d \in \mathcal{E}$, so since $\mathcal{E}$ is pullback stable $he' \in \mathcal{E}$. This gives $h \in \mathcal{E}$, but $h \in \mathcal{M}$ so h is an isomorphism and the result follows. □

2.5.8 Corollary. *Let $\mathcal{E}$ be stable under pullback in $\mathbf{X}$, and let $Re \in \mathcal{E}$ for every $e \in \mathcal{E}$. If $f : X \to Y$ is (R, r)-perfect then $f \times 1_Z : X \times Z \to Y \times Z$ is $\Phi_{(R,r)}$-preserving for every $Z \in Ob\mathbf{X}$ as long as any one of the following holds.*

(i) *(R, r) is direct and idempotent.*

(ii) *(R, r) is direct and $\Sigma_R \subseteq Epi\mathbf{X}$.*

(iii) *For any $A, B \in Ob\mathbf{X}$ the canonical morphism $k : R(A \times B) \to RA \times RB$ is a monomorphism.*

Proof. Just combine Propositions 2.5.4 and 2.5.7. □

2.6 Compact preimages

The last topological notion we want to look at is the usual definition of a perfect continuous map, namely that $f : X \to Y$ is perfect if it is closed and for every $y \in Y$, $f^{-1}(y)$ is compact. For this we need to have a concept of a point in an **X**-object X, and we simply use a morphism $T \xrightarrow{m} X \in \mathcal{M}$ from a terminal object into X for this end.

Thus in this section we investigate when a morphism $f : X \to Y$ is closure preserving and for any $T \xrightarrow{m} Y \in \mathcal{M}$, $f^{-1}(T)$ is "compact". Again this necessitates the assumption that **X** has products of pairs. T will denote a terminal object in **X**. Technically T may not exist, or there may be no $T \xrightarrow{m} Y \in \mathcal{M}$. Such cases are merely uninteresting, they do not affect the validity of our results as the relevant properties are vacuously fulfilled.

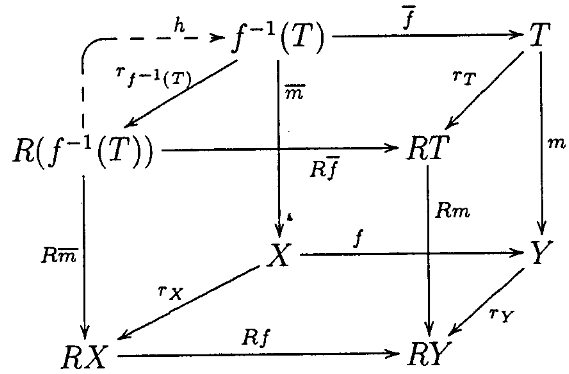
2.6.1 Definition. Let C be a closure operator on **X** with respect to $\mathcal{M}$. An **X**-object X is called *C-compact* if for every $Z \in Ob\mathbf{X}$ the projection $\pi_2 : X \times Z \to Z$ is C-preserving.

This definition exploits the well known result of [Mrówka 1959] for compactness in topological spaces. [Manes 1974] was the first to use it in categorical generality. [Herrlich, Salicrup, Strecker 1987], [Castellini 1990] and [Dikranjan, Giuli 1989] were the key authors in developing it to its present common usage.

As we have seen previously, in our context $\text{Fix}(R, r)$ objects behave with respect to (R, r)-perfect morphisms as we would expect "compact" objects to. With this in mind we explore results for both $\text{Fix}(R, r)$ objects and $\Phi_{(R,r)}$-compact objects.

2.6.2 Proposition. *Let (R, r) be pointwise epimorphic. If $f : X \to Y$ is (R, r)-perfect then for any $T \xrightarrow{m} Y \in \mathcal{M}$, $f^{-1}(T) \in \text{Fix}(R, r)$.*

Proof. Let $T \xrightarrow{m} Y \in \mathcal{M}$ and consider the following diagram. $((f^{-1}(T), (\overline{f}, \overline{m}))$ is the pullback of m along f.) We want to show that $r_{f^{-1}(T)}$ is an isomorphism.



Since (R, r) is pointwise epimorphic, $T \cong RT$. Now, $r_Y m (r_T)^{-1} R\overline{f} = RmR\overline{f} = RfR\overline{m}$, so since both $r_Y f = Rf r_X$ and $f\overline{m} = m\overline{f}$ are pullback squares, there is a unique $h : R(f^{-1}(T)) \to f^{-1}(T)$ such that $\overline{f}h = (r_T)^{-1}R\overline{f}$ and $r_X \overline{m} h = R\overline{m}$.

Since f is (R, r)-perfect, $\overline{m}$ is the unique morphism for which $f\overline{m} = m\overline{f}$ and $r_X\overline{m} = R\overline{m}r_{f^{-1}(T)}$. But $f\overline{m}hr_{f^{-1}(T)} = m\overline{f}hr_{f^{-1}(T)} = m(r_T)^{-1}R\overline{f}r_{f^{-1}(T)} = m\overline{f}$ and $r_X\overline{m}hr_{f^{-1}(T)} = R\overline{m}r_{f^{-1}(T)}$ so $\overline{m} = \overline{m}hr_{f^{-1}(T)}$. Thus since $\overline{m}$ is a monomorphism, this gives that $hr_{f^{-1}(T)} = 1_{f^{-1}(T)}$. So $r_{f^{-1}(T)}$ is an epimorphism and a section, hence an isomorphism. □

2.6.3 Corollary. *Let (R, r) be pointwise epimorphic, $Re \in \mathcal{E}$ for every $e \in \mathcal{E}$ and $\mathcal{E}$ be stable under pullback. If $f : X \to Y$ is (R, r)-perfect then f is $\Phi_{(R,r)}$-preserving and $f^{-1}(T) \in \mathrm{Fix}(R, r)$ for any $T \xrightarrow{m} Y \in \mathcal{M}$.*

Proof. Combine the above proposition with Proposition 2.5.7. □

This corollary can then be extended to $\Phi_{(R,r)}$-compactness once we know the following.

2.6.4 Lemma. *If {Weakly (R, r)-perfect} $\subseteq$ {$\Phi_{(R,r)}$-preserving}, then* $\mathrm{Fix}(R, r) \subseteq$ *{$\Phi_{(R,r)}$-compact}.*

Proof. Let $X \in \mathrm{Fix}(R, r)$. By Proposition 2.2.9 the projection $\pi_2 : X \times Z \to Z$ is weakly (R, r)-perfect for every $Z \in Ob\mathbf{X}$. Thus by assumption it is $\Phi_{(R,r)}$-preserving and so X is $\Phi_{(R,r)}$-compact. □

We of course have conditions under which {Weakly (R, r)-perfect} $\subseteq$ {(R, r)-perfect} $\subseteq$ {$\Phi_{(R,r)}$-preserving}, and so can deduce the following result.

2.6.5 Corollary. *Let (R, r) be direct, idempotent and pointwise epimorphic, and let*

$Re \in \mathcal{E}$ for all $e \in \mathcal{E}$ and $\mathcal{E}$ be stable under pullback. If $f : X \to Y$ is (R,r)-perfect then f is $\Phi_{(R,r)}$-preserving and $f^{-1}(T)$ is $\Phi_{(R,r)}$-compact for any $T \overset{m}{\to} Y \in \mathcal{M}$.

Proof. Since (R,r) is direct and idempotent {Weakly (R,r)-perfect} $\subseteq$ {(R,r)-perfect}. Proposition 2.5.7 then ensures that {(R,r)-perfect} $\subseteq$ {$\Phi_{(R,r)}$-preserving} and the result follows by combining Corollary 2.6.3 and Lemma 2.6.4. □

We can get to this result via another route, and slightly different conditions on (R,r) are required. The requirement that (R,r) is pointwise epimorphic is dropped and we assume that the codomain of our perfect morphism is in some sense "separated" with respect to $\Phi_{(R,r)}$. We also assume the existence of a terminal object T.

2.6.6 Proposition. *Let $f : X \to Y$ in $\mathbf{X}$. Let (R,r) be direct and idempotent, let $Re \in \mathcal{E}$ for all $e \in \mathcal{E}$ and $\mathcal{E}$ be stable under pullback, and assume that every $T \overset{m}{\to} Y \in \mathcal{M}$ is $\Phi_{(R,r)}$-closed. f is (R,r)-perfect $\Rightarrow$ $f \times 1_Z$ is $\Phi_{(R,r)}$-preserving for every $Z \in Ob\mathbf{X}$ $\Rightarrow$ f is $\Phi_{(R,r)}$-preserving and for every $T \overset{m}{\to} Y \in \mathcal{M}$, $f^{-1}(T)$ is $\Phi_{(R,r)}$-compact.*

Proof. Let $f : X \to Y$ in $\mathbf{X}$. We already know that under the given conditions f is (R,r)-perfect $\Rightarrow$ $f \times 1_Z$ is $\Phi_{(R,r)}$-preserving for every $Z \in Ob\mathbf{X}$ (Corollary 2.5.8).

Assume that $f \times 1_Z$ is $\Phi_{(R,r)}$-preserving for every $Z \in Ob\mathbf{X}$. Putting $Z = T$ it is clear that f is $\Phi_{(R,r)}$-preserving. To see that $f^{-1}(T)$ is $\Phi_{(R,r)}$-compact, consider the diagram below for $T \overset{m}{\to} Y \in \mathcal{M}$ and $Z \in Ob\mathbf{X}$. (Again, $(f^{-1}(T), (\overline{f}, \overline{m}))$ is the pullback of m along f.)

$$\begin{array}{ccc} f^{-1}(T) \times Z & \xrightarrow{\overline{f} \times 1_Z} & T \times Z \\ {\scriptstyle \overline{m} \times 1_Z} \downarrow & & \downarrow {\scriptstyle m \times 1_Z} \\ X \times Z & \xrightarrow{f \times 1_Z} & Y \times Z \end{array}$$

This commutes since $f\overline{m} = m\overline{f}$. By assumption, m is $\Phi_{(R,r)}$-closed so by Corollary 1.6.5, $m \in \Sigma_R^{\downarrow} \cap \mathcal{M}$. Both $\Sigma_R^{\downarrow}$ and $\mathcal{M}$ are closed under pullbacks and products so $\overline{m} \in \Sigma_R^{\downarrow} \cap \mathcal{M}$ and thus $\overline{m} \times 1_Z \in \Sigma_R^{\downarrow} \cap \mathcal{M}$.

So $\overline{m} \times 1_Z$ is weakly (R,r)-perfect, but under the given assumptions this means that it is an (R,r)-perfect morphism. Our standing assumptions then ensure that it is $\Phi_{(R,r)}$-preserving. Proposition 2.5.2 (a) tells us that $\Phi_{(R,r)}$-preserving morphisms are closed under composition, so $(f \times 1_Z)(\overline{m} \times 1_Z)$ is $\Phi_{(R,r)}$-preserving and hence $(m \times 1_Z)(\overline{f} \times 1_Z)$ is $\Phi_{(R,r)}$-preserving. But $m \times 1_Z \in \mathcal{M}$, so by Proposition 2.5.2 (b) $\overline{f} \times 1_Z$ is $\Phi_{(R,r)}$-preserving.

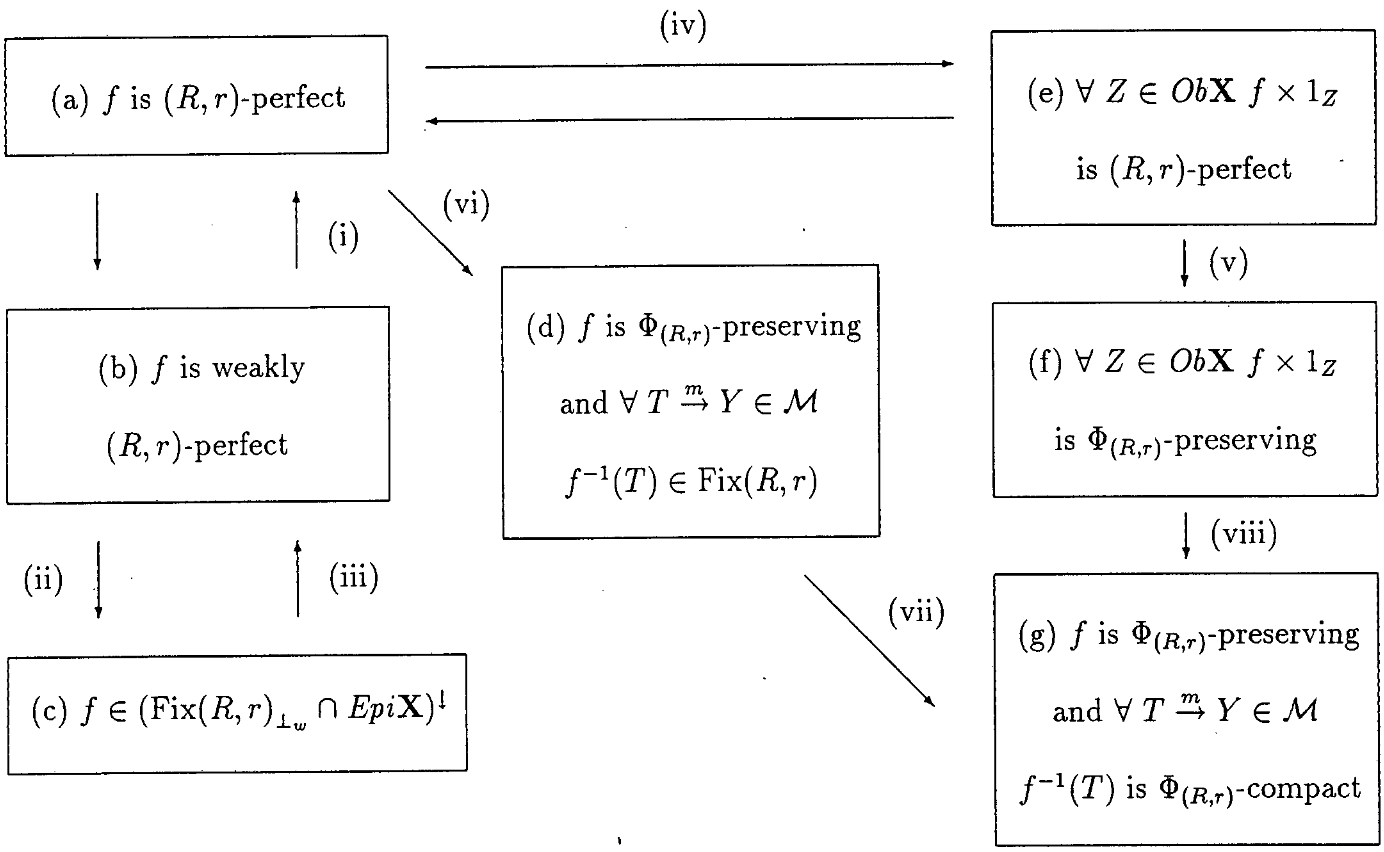
Now we observe that $\overline{f} \times 1_Z \cong \pi_2 : f^{-1}(T) \times Z \to Z$, and so $f^{-1}(T)$ is $\Phi_{(R,r)}$-compact. □

2.6.7 Remark. Noting that under the conditions of this result $\Phi_{(R,r)}$ is weakly hereditary, we can draw the same conclusion from [Dikranjan, Giuli 1991b] Theorem 4.6. Our proof is considerably different, and of course their result is restricted to transportable constructs.

2.7 Summary of results

The interrelation of the various notions of perfect morphism that we have been investigating can best be presented in the following theorem that summarises the results of this chapter.

2.7.1 Theorem. *Let $f : X \to Y$ in $\mathbf{X}$. The properties of f in the boxes below imply others along the arrows drawn. The numerals alongside certain arrows represent the conditions that are sufficient for the associated implication to hold.*



(i) (R,r) is direct and either $\Sigma_R \subseteq Epi\mathbf{X}$ or (R,r) is idempotent.

(ii) (R,r) is idempotent and well-pointed.

(iii) (ii) and $\Sigma_R \subseteq Epi\mathbf{X}$.

(iv) (i) or for any $A, B \in Ob\mathbf{X}$ the canonical morphism $k : R(A \times B) \to RA \times RB$ is a monomorphism.

(v) $\mathcal{E}$ is stable under pullback and $Re \in \mathcal{E}$ for every $e \in \mathcal{E}$.

(vi) (v) and (R,r) is pointwise epimorphic.

(vii) (i) and (v)

(viii) (v), (R,r) is direct and idempotent, $\mathbf{X}$ has a terminal object T and each $T \overset{m}{\to} Y \in \mathcal{M}$ is $\Phi_{(R,r)}$-closed.

For (e), (f) and (g) to be accessed we need to assume that **X** has products of pairs.

Proof. This is simply a summary of the following results.

$(a) \Rightarrow (b)$: Proposition 2.2.1.

$(b) \Rightarrow (c)$: Corollary 2.4.6.

$(c) \Rightarrow (b)$: Corollary 2.4.6.

$(b) \Rightarrow (a)$: Remark 2.3.6(1).

$(a) \Rightarrow (d)$: Corollary 2.6.3.

$(a) \Rightarrow (e)$: Proposition 2.5.4.

$(e) \Rightarrow (a)$: Proposition 2.5.6.

$(e) \Rightarrow (f)$: Proposition 2.5.7.

$(d) \Rightarrow (g)$: Under conditions (i) and (v), Remark 2.3.6(1) and Proposition 2.5.7 give that { *Weakly* (R, r)*-perfect*} $\subseteq$ {$\Phi_{(R,r)}$*-preserving*}, and we then apply Lemma 2.6.4.

$(f) \Rightarrow (g)$: Proposition 2.6.6. □

2.7.2 Remark. These many conditions may seem a little cluttered, but for certain (R, r) there are a number of conditions that are fulfilled simultaneously in which case the following clear deductions can be made.

(1) If (R, r) is a direct reflection, $\mathcal{E}$ is stable under pullback and for every $e \in \mathcal{E}$, $R(e) \in \mathcal{E}$, then all implications except $(c) \Rightarrow (b), (a) \Rightarrow (d)$ and $(f) \Rightarrow (g)$ follow immediately from the theory.

(2) If in addition to (1) above, (R, r) is an epireflection then only $(c) \Rightarrow (b)$ and $(f) \Rightarrow (g)$ cannot be concluded from the theorem.

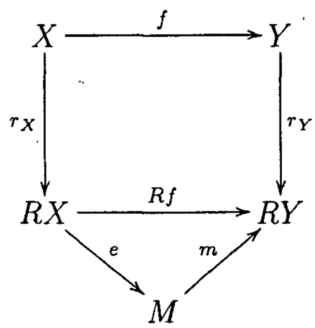
(3) If moreover (R, r) is a bireflection, only $(f) \Rightarrow (g)$ does not automatically hold. It is notable that condition (viii) is unnecessarily strong, in the examples below we show that the implication $(f) \Rightarrow (g)$ can in fact hold without (viii) being fulfilled.

2.8 Examples

We conclude with a number of examples that demonstrate the theory of Chapter 2. In each instance we have specifically investigated whether or not conditions (i) to (viii) of Theorem 2.7.1 are satisfied. Only (ii) and (iv) are satisfied by all examples. For the failure of each of the other conditions – with the exception of (viii) – we have been able to show in an example that the associated implication is not true. While this does not establish the necessity of the conditions given, it does give credence to the emphasis we have placed on them.

2.8.1 Čech-Stone compactification. Let (R, r) denote the Čech-Stone compactification in the category $\mathbf{X}$ of Tychonoff spaces and continuous maps. It was noted in Example 1.5.1 that (R, r) is a direct reflection. Since (R, r) is pointwise monomorphic it follows that $\Sigma_R \subseteq Epi\mathbf{X}$ and that (R, r) is pointwise epimorphic too. Being a reflection, (R, r) is both idempotent and well-pointed.

$(\mathcal{E}, \mathcal{M})$ is the *(Surjection, Embedding)* factorisation structure for morphisms in $\mathbf{X}$. It is well known that the surjective continuous maps are stable under pullback. If $f : X \to Y$ is a surjective continuous map, consider the following diagram depicting the image of f under (R, r).



Let $me = Rf$ be the $(\mathcal{E}, \mathcal{M})$ factorisation of Rf. Knowing that e is surjective, RX is compact and M is Hausdorff, we conclude that M is a compact space. Thus since RY is a Hausdorff space, the embedding $m : M \to RY$ is closed. But $Rfr_X = r_Y f$ is dense, so Rf is a dense map and m must thus be an isomorphism. This tells us that $Rf \in \mathcal{E}$.

Lastly, note that since every space X in $\mathbf{X}$ is Hausdorff, any embedding of a terminal object (i.e. a singleton space) into X is $\Phi_{(R,r)}$-closed because, as we saw in Example 1.2.1, $\Phi_{(R,r)}$ is the usual topological closure.

What we have demonstrated is that in this setting all conditions (i) to (viii) in Theorem 2.7.1 are satisfied, hence all the implications are true. It is of course well known that in this particular example the different notions of perfectness in the theorem in fact coincide.

Knowing what we do about (R,r) and $\mathcal{E}$, we see from Proposition 2.4.5 that $\Sigma_R = \mathrm{Fix}(R,r)_\perp = \mathrm{Fix}(R,r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}dense\}$ which is the class of dense HCOMP-extendable morphisms. From Theorem 2.3.5 it then follows that (*Dense* HCOMP-*extendable,* (R,r)-*perfect*) is a factorisation structure for morphisms in $\mathbf{X}$. (This fact was used earlier to verify that (R,r) is direct, so we cannot claim it as a consequence of our results. It does none-the-less demonstrate how the theory holds together.)

2.8.2 TOP$_0$ **reflection.** In this example, let (R,r) be the TOP$_0$ reflector in $\mathbf{X} =$ TOP. $\mathcal{E}$ is again the class of surjective continuous maps. In [Brümmer, Giuli 1993a] it is shown that (R,r) is direct. Knowing this, that (R,r) is an $\mathcal{E}$-reflection and that surjective continuous maps are stable under pullback, it is immediately clear that conditions (i), (ii), (iv), (v), (vi) and (vii) of Theorem 2.7.1 are true.

Let $j : \{\bullet\} \to I_2$ be an embedding of a singleton space into a two point indiscrete space. The following observations are immediate:

- $j \in \Sigma_R$;
- j is not $\Phi_{(R,r)}$-closed;
- j is not (R,r)-perfect; and
- $j \in (\mathrm{Fix}(R,r)_{\perp_w} \cap Epi\mathbf{X})^{\downarrow}$.

Thus conditions (iii) and (viii) in Theorem 2.7.1 do not hold, nor does the implication $(c) \Rightarrow (b)$.

The properties of (R,r) and $\mathcal{E}$ are such that both Theorem 2.3.5 and Proposition 2.4.5 apply in this situation. Thus we conclude that ($\Phi_{(R,r)}$-*dense* TOP$_0$-*extendable,* (R,r)-*perfect*) is a factorisation structure for morphisms in TOP.

2.8.3 Uniform completion. Let (R,r) be the completion reflector in UNIF$_0$. In this setting, $\mathcal{E}$ is the class of surjective uniformly continuous maps, and $\mathcal{M}$ is the class of uniform embeddings.

$\mathcal{E}$ is stable under pullback in UNIF$_0$ and (R,r) preserves $\mathcal{E}$-morphisms for the same reasons that the Čech-Stone compactification functor preserves surjective continuous maps. (Since for a UNIF$_0$ space X, RX has compact Hausdorff topology.) That (R,r) is direct is shown in [Brümmer, Giuli 1993a]. These observations in conjunction with the fact that (R, r) is a bireflection ensure that all conditions (i) to (vii) of Theorem 2.7.1 are satisfied.

We verify in Example 3.5.1 below that $\Phi_{(R,r)}$ is the underlying topological closure. Knowing this it is clear that condition (viii) also holds, since every $\underline{\text{UNIF}_0}$ space is Hausdorff. Thus all the conclusions of Theorem 2.7.1 can be made in this setting.

We can also conclude from Theorem 2.3.5 and Proposition 2.4.5 that (*Dense complete-extendable,* (R, r)*-perfect*) is a factorisation structure for morphisms in $\underline{\text{UNIF}_0}$. (Dense means with respect to the underlying topology.)

In [Hager 1975] (Section 2) it is observed that a uniformly continuous map $f : X \to Y$ is (R, r)-perfect iff for any Cauchy filter $\mathcal{U}$ in X, $\mathcal{U}$ converges in X if $f(\mathcal{U})$ converges in Y. It is also noted there that the uniformly continuous maps that are perfect in $\underline{\text{TYCH}}$ are the (S, s)-perfect maps, where (S, s) is the Samuel compactification in $\underline{\text{UNIF}}$.

2.8.4 Sobrification. Let (S, s) be the sobrification reflector in $\underline{\text{TOP}_0}$. As with the other examples thus far $(\mathcal{E}, \mathcal{M})$ is the (*Surjection, Embedding*) factorisation structure for morphisms restricted to $\underline{\text{TOP}_0}$. Also in this setting $\mathcal{E}$ is stable under pullback, and again in [Brümmer, Giuli 1993a] it is shown that (S, s) is direct.

Since in addition to the above (S, s) is a bireflection, conditions (i) to (iv) of Theorem 2.7.1 hold. (S, s) does not, however, preserve surjective continuous maps as the following example shows.

Let X be the natural numbers $\mathbf{N}$ endowed with the discrete topology. Let Y be the natural numbers endowed with the co-finite topology. Both are $\underline{\text{TOP}_0}$ spaces. X is clearly a sober space. Y however is not, since $\mathbf{N}$ is a closed irreducible subset of Y yet it cannot be expressed as the closure of a single point.

SY has underlying set $\mathbf{N} \cup \{\bullet\}$. U is an open set in SY iff $\{\bullet\} \subseteq U$ and $U \cap \mathbf{N}$ is open in Y.

The identity function on $\mathbf{N}$, $1_{\mathbf{N}} : X \to Y$ is a surjective $\underline{\text{TOP}_0}$ morphism, yet clearly $S1_{\mathbf{N}} : SX \to SY$ is not surjective. Thus (S, s) does not satisfy conditions (v) to (viii) of Theorem 2.7.1.

Consider now the spaces X and Y, both with underlying set $\mathbf{N} \cup \{\infty\}$ ($n \leq \infty \; \forall \, n \in \mathbf{N}$). Let X have the discrete topology and let Y have the upper topology, namely the topology with open sets of the form $U_n := \{m \in \mathbf{N} \mid n \leq m\} \cup \{\infty\}$ for $n \in \mathbf{N}$. Both X and Y are sober spaces.

Let $1_{\mathbf{N} \cup \{\infty\}}$ be the identity function on $\mathbf{N} \cup \{\infty\}$. Then $1_{\mathbf{N} \cup \{\infty\}} : X \to Y$ is a $\underline{\text{SOB}}$-morphism and is thus (S, s)-perfect (cf. Proposition 2.2.6). Observe now that $\mathbf{N}$ is a

b-closed subset of X yet it is not b-closed in Y, thus since the b-closure is idempotent this means that $1_{\mathbf{N}\cup\{\infty\}} : X \to Y$ is not b-closure preserving. We saw in Example 1.2.3 that $\Phi_{(S,s)}$ is the b-closure, thus we have an example of an (S,s)-perfect map that is not $\Phi_{(S,s)}$-preserving. From this we conclude that in this example neither of the implications $(a) \Rightarrow (d)$ and $(e) \Rightarrow (f)$ of Theorem 2.7.1 holds.

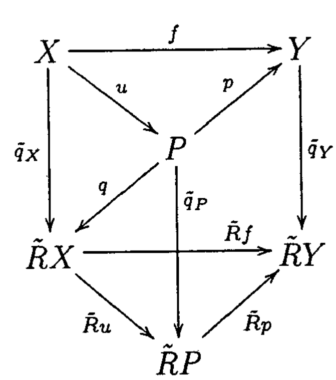
It has been shown (cf. [Fedeli 1992] Corollary 2 and [Dikranjan, Giuli 1989] Example 3.2) that in $\underline{\text{Top}_0}$ the b-compact spaces are properly contained in the sober spaces. Let X be a sober space that is not b-compact. The map $f : X \to \{\bullet\}$ of X onto a singleton space then gives a simple example to show that the implication $(d) \Rightarrow (g)$ of Theorem 2.7.1 does not hold either.

The properties of (S,s) are such that we can conclude from Theorem 2.3.5 and Proposition 2.4.5 that (*b-dense* $\underline{\text{Sob}}$*-extendable,* (S,s)*-perfect*) is a factorisation structure for morphisms in $\underline{\text{Top}_0}$.

2.8.5 Endofunctors induced by congruence relations in varieties. In the notation of Example 1.2.4, let $(\tilde{R},\tilde{q})$ denote the pointed endofunctor induced by a natural family $(\sim_X)_{X\in Ob\mathbf{X}}$ of congruence relations in a variety $\mathbf{X}$.

We saw in Example 1.5.4 that $(\tilde{R},\tilde{q})$ need not be direct. The first claim below helps characterise exactly when in fact $(\tilde{R},\tilde{q})$ will be direct. First we establish some notation.

The following diagram shows the image of an $\mathbf{X}$-morphism $f : X \to Y$ under $(\tilde{R},\tilde{q})$. The source $(P,(p,q))$ is the pullback of the sink $((\tilde{R}f,\tilde{q}_Y),\tilde{R}Y)$.



Our knowledge of pullbacks in $\mathbf{X}$ allows us to give the following internal descriptions.

- $P = \{([x]_{\sim_X}, y) \in \tilde{R}X \times Y \mid \tilde{R}f([x]_{\sim_X}) = \tilde{q}_Y(y)\} = \{([x]_{\sim_X}, y) \in \tilde{R}X \times Y \mid f(x) \sim_Y y\}$.
- The morphism $u : X \to P$ maps x to $([x]_{\sim_X}, f(x))$.

- The morphism $\tilde{R}u : \tilde{R}X \to \tilde{R}P$ maps $[x]_{\sim_X}$ to $[([x]_{\sim_X}, f(x))]_{\sim_P}$.

Claim 1. The pointed endofunctor $(\tilde{R}, \tilde{q})$ induced by a natural family $(\sim_X)_{X \in Ob\mathbf{X}}$ of congruence relations is direct iff the following conditions hold:

(a) $(\tilde{R}, \tilde{q})$ is idempotent; and

(b) For points $([x]_{\sim_X}, y)$ and $([z]_{\sim_X}, w)$ in the pullback object P of the diagram above, $([x]_{\sim_X}, y) \sim_P ([z]_{\sim_X}, w) \Leftrightarrow x \sim_X z$.

Proof. $\Rightarrow$: Assume that $(\tilde{R}, \tilde{q})$ is direct. Being a variety, $\mathbf{X}$ has a terminal object T. $(\tilde{R}, \tilde{q})$ is pointwise epimorphic, so Proposition 2.3.2(c) tells us that $(\tilde{R}, \tilde{q})$ is idempotent. (Note that this means that for any $X \in Ob\mathbf{X}$, $\sim_{\tilde{R}X}$ is discrete.)

To show (b), let $([x]_{\sim_X}, y) \sim_P ([z]_{\sim_X}, w)$ in P. Then, since we have a natural family, $[x]_{\sim_X} = q(([x]_{\sim_X}, y)) \sim_{\tilde{R}X} q(([z]_{\sim_X}, w)) = [z]_{\sim_X}$, but $\sim_{\tilde{R}X}$ is discrete so $x \sim_X z$. On the other hand let $([x]_{\sim_X}, y)$ and $([z]_{\sim_X}, w)$ be points in P and assume that $x \sim_X z$. Since $(\tilde{R}, \tilde{q})$ is direct, $\tilde{R}u$ is an isomorphism so there are points $a, b \in X$ such that $\tilde{R}u([a]_{\sim_X}) = [([x]_{\sim_X}, y)]_{\sim_P}$ and $\tilde{R}u([b]_{\sim_X}) = [([z]_{\sim_X}, w)]_{\sim_P}$. This means that $([a]_{\sim_X}, f(a)) \sim_P ([x]_{\sim_X}, y)$ and $([b]_{\sim_X}, f(b)) \sim_P ([z]_{\sim_X}, w)$, so since $(\sim_X)_{X \in Ob\mathbf{X}}$ is a natural family and $\sim_{\tilde{R}X}$ is discrete we conclude (by taking images under q) that $[a]_{\sim_X} = [x]_{\sim_X}$ and $[b]_{\sim_X} = [z]_{\sim_X}$. But $x \sim_X z$ so it follows that $[a]_{\sim_X} = [b]_{\sim_X}$ and thus that $[([x]_{\sim_X}, y)]_{\sim_P} = \tilde{R}u([a]_{\sim_X}) = \tilde{R}u([b]_{\sim_X}) = [([z]_{\sim_X}, w)]_{\sim_P}$, in other words that $([x]_{\sim_X}, y) \sim_P ([z]_{\sim_X}, w)$.

$\Leftarrow$: Assuming that (a) and (b) hold, we need to show that $\tilde{R}u$ is an isomorphism. Let $x, y \in X$, $\tilde{R}u([x]_{\sim_X}) = \tilde{R}u([y]_{\sim_X}) \Rightarrow ([x]_{\sim_X}, f(x)) \sim_P ([y]_{\sim_X}, f(y)) \Rightarrow [x]_{\sim_X} \sim_{\tilde{R}X} [y]_{\sim_X} \Rightarrow [x]_{\sim_X} = [y]_{\sim_X}$, so $\tilde{R}u$ is one-one. On the other hand, pick $[([x]_{\sim_X}, y)]_{\sim_P} \in \tilde{R}P$, we know that $([x]_{\sim_X}, y) \sim_P ([x]_{\sim_X}, f(x))$ so $\tilde{R}u([x]_{\sim_X}) = [([x]_{\sim_X}, y)]_{\sim_P}$ and $\tilde{R}u$ is onto and hence an isomorphism. □

A natural family $(\sim_X)_{X \in Ob\mathbf{X}}$ of congruence relations in $\mathbf{X}$ will be termed *idempotent* if for any $X \in Ob\mathbf{X}$ the relation $\sim_{X/\sim_X}$ is discrete (i.e. if $(\tilde{R}, \tilde{q})$ is idempotent). We will term the family *finitely productive* if for any $X, Y \in Ob\mathbf{X}$ it is true that $(x, y) \sim_{X \times Y} (z, w) \Leftrightarrow x \sim_X z$ and $y \sim_Y w$.

Claim 2. If the natural family $(\sim_X)_{X \in Ob\mathbf{X}}$ is idempotent, hereditary (cf. 1.2.4) and finitely productive then $(\tilde{R}, \tilde{q})$ is direct.

Proof. Clearly such a family and the induced $(\tilde{R}, \tilde{q})$ satisfy conditions (a) and (b) of Claim 1 above. □

For each $X \in Ob\mathbf{X}$, $\tilde{q}_X \in \mathcal{E} \subseteq Epi\mathbf{X}$, and $\mathcal{E}$ (the class of surjective $\mathbf{X}$-morphisms) is stable under pullback. Thus if a natural family $(\sim_X)_{X \in Ob\mathbf{X}}$ of congruence relations and the induced $(\tilde{R}, \tilde{q})$ satisfy (a) and (b) of Claim 1, then conditions (i), (ii) and (iv) to (vii) of Theorem 2.7.1 hold and all implications except $(c) \Rightarrow (b)$ and $(f) \Rightarrow (g)$ are known to be true.

Before we look at some specific examples observe that the group structure in $\underline{R\text{-MOD}}$ and $\underline{\text{GRP}}$ simplifies the description of $\tilde{\Phi}$-preservation and $(\tilde{R}, \tilde{q})$-perfectness for morphisms in these categories. We give the proofs in $\underline{\text{GRP}}$, but these carry over to $\underline{R\text{-MOD}}$ too. The neutral element in a group G is denoted by e_G and the group operation is written multiplicatively.

For the next four results, $(\sim_X)_{X \in Ob\mathbf{X}}$ is assumed to be a natural family of congruence relations on $\mathbf{X} = \underline{\text{GRP}}$, with induced endofunctor $(\tilde{R}, \tilde{q})$.

Claim 3. For a group homomorphism $f : G \to H$ the following are equivalent:

(a) f is $\tilde{\Phi}$ preserving.

(b) $f[[e_G]_{\sim_G}] = [e_H]_{\sim_H}$.

(c) For every $g \in G$, $f[[g]_{\sim_G}] = [f(g)]_{\sim_H}$.

Proof. $(a) \Rightarrow (b)$: Let f be $\tilde{\Phi}$-preserving. Since $[e_G]_{\sim_G}$ is $\tilde{\Phi}$-closed, $f[[e_G]_{\sim_G}]$ is $\tilde{\Phi}$-closed in H, thus from the fact that $e_H \in f[[e_G]_{\sim_G}]$ it follows that $[e_H]_{\sim_H} \subseteq f[[e_G]_{\sim_G}]$. On the other hand if $g \in [e_G]_{\sim_G}$ then $f(g) \sim_H f(e_G) = e_H$ and so $f(g) \in [e_H]_{\sim_H}$ from which (b) follows.

$(b) \Rightarrow (c)$: Since for any $g, h \in G$, $g \sim_G h \Rightarrow f(g) \sim_H f(h)$, it is always true that for $g \in G$, $f[[g]_{\sim_G}] \subseteq [f(g)]_{\sim_H}$. Pick $h \in [f(g)]_{\sim_H}$, $h \sim_H f(g) \Rightarrow h \cdot (f(g))^{-1} \sim_H e_H \Rightarrow$ there is an $x \in G$ such that $x \sim_G e_G$ and $f(x) = h \cdot (f(g))^{-1}$. Thus $h = f(x) \cdot f(g) = f(x \cdot g)$ and $x \cdot g \sim_G e_G \cdot g = g$, giving that $h \in f[[g]_{\sim_G}]$. (Note that we used here the fact that for any group X, $\sim_X$ is a subgroup of $X \times X$.)

$(c) \Rightarrow (a)$: (Since $\tilde{\Phi}$ is idempotent, it is sufficient to show that f preserves $\tilde{\Phi}$-closed subgroups.) Let M be a $\tilde{\Phi}$-closed subgroup of G. Pick $g \in \tilde{\Phi}(f[M])$, there is an $m \in M$ such that $g \sim_H f(m)$ hence according to (c), $g \in f[[m]_{\sim_G}]$. But since M is $\tilde{\Phi}$-closed, $[m]_{\sim_G} \subseteq M$ and so $g \in f[M]$ telling us that $f[M]$ is $\tilde{\Phi}$-closed. □

Claim 4. A group $G \in \text{Fix}(\tilde{R}, \tilde{q}) \Leftrightarrow [e_G]_{\sim_G} = \{e_G\}$.

Proof. $G \in \text{Fix}(\tilde{R}, \tilde{q}) \Leftrightarrow G \cong G/{\sim_G} \Leftrightarrow {\sim_G}$ is discrete $\Leftrightarrow [e_G]_{\sim_G} = \{e_G\}$. □

Claim 5. A homomorphism $f : G \to H$ is $(\tilde{R}, \tilde{q})$-perfect iff f is $\tilde{\Phi}$-preserving and $f^{-1}(e_H) \cap [e_G]_{\sim_G} = \{e_G\}$.

Proof. Using the notation of the diagram at the beginning of this example – writing

X as G and Y as H – $(\tilde{R},\tilde{q})$-perfectness of $f: G \to H$ is equivalent to the fact that the map $u: G \to P$ is an isomorphism.

$\Rightarrow$: $(\tilde{R},\tilde{q})$ preserves $\mathcal{E}$-morphisms, so $\tilde{\Phi}$-preservation follows from Proposition 2.5.7. Pick $g \in f^{-1}(e_H) \cap [e_G]_{\sim_G}$, then $[g]_{\sim_G} = [e_G]_{\sim_G}$ and $f(g) = e_H = f(e_G)$. Thus $u(g) = u(e_G)$, so since u is an isomorphism we conclude that $g = e_G$.

$\Leftarrow$: We want to show that $u: G \to P$ is an isomorphism. Say we have $g, h \in G$ such that $u(g) = u(h)$, then $[g]_{\sim_G} = [h]_{\sim_G}$ and $f(g) = f(h)$. Thus $g \cdot h^{-1} \sim_G e_G$ and $f(g \cdot h^{-1}) = e_H$, so we must have that $g \cdot h^{-1} = e_G$ which means that $g = h$, so u is one-one. On the other hand, pick $([g]_{\sim_G}, h) \in P$, then $h \sim_H f(g)$ so $h \cdot (f(g))^{-1} \sim_H e_H$. Thus according to Claim 3 there is a $z \in [e_G]_{\sim_G}$ such that $f(z) = h \cdot (f(g))^{-1}$, giving $f(z \cdot g) = h$. From this we conclude that $u(z \cdot g) = ([z \cdot g]_{\sim_G}, f(z \cdot g)) = ([g]_{\sim_G}, h)$ and hence u is onto. □

Claim 6. If $(\sim_G)_{G \in Ob\mathbf{X}}$ is an hereditary family, then a homomorphism $f: G \to H$ is $(\tilde{R},\tilde{q})$-perfect iff f is $\tilde{\Phi}$-preserving and $f^{-1}(e_H) \in \text{Fix}(\tilde{R},\tilde{q})$.

Proof. This follows from Claims 4 and 5 since $f^{-1}(e_H) \in \text{Fix}(\tilde{R},\tilde{q}) \Leftrightarrow [e_G]_{\sim_{f^{-1}(e_H)}} = \{e_G\}$, but since $(\sim_G)_{G \in Ob\mathbf{X}}$ is an hereditary family, $[e_G]_{\sim_{f^{-1}(e_H)}} = f^{-1}(e_H) \cap [e_G]_{\sim_G}$. □

Now we look at two specific examples in the categories of <u>GRP</u> and <u>ABGRP</u>.

(1) Let $(\tilde{R},\tilde{q})$ be the reflector from <u>GRP</u> to <u>ABGRP</u> described in Example 1.2.4(3). We saw in Example 1.5.4(1) that $(\tilde{R},\tilde{q})$ is not direct, thus conditions (i), (vii) and (viii) of Theorem 2.7.1 do not hold. Condition (iii) of that theorem does not hold either. Clearly conditions (ii), (v) and (vi) hold, and according to [Hušek, de Vries 1987] Theorem 1, $(\tilde{R},\tilde{q})$ preserves products, so condition (iv) holds too.

We can conclude from Theorem 2.3.7 that the implication $(b) \Rightarrow (a)$ in Theorem 2.7.1 does not hold. In the notation of Example 1.5.4 (1), consider an embedding $m: \mathbf{Z}_2 \to S_3$ (where 0 is mapped to the identity permutation, and 1 is mapped to any one of the three transpositions). Then since the domain of m is an abelian group, it is clear that $m \in (\text{Fix}(\tilde{R},\tilde{q})_{\perp_w} \cap Epi)^{\downarrow}$. Considering the commutative square $1_{S_3} m = m 1_{\mathbf{Z}_2}$ we see, however, that m is not in $\Sigma_{\tilde{R}}^{\downarrow}$ so the implication $(c) \Rightarrow (b)$ of Theorem 2.7.1 also does not hold in this example.

The results above enable us to characterise the $(\tilde{R},\tilde{q})$-perfect homomorphisms as those $f: G \to H$ for which $f[C_G] = C_H$ and $f^{-1}(e_H) \cap C_G = \{e_G\}$ (C_G being the commutator subgroup of G). Note that the family of congruence relations in this example is not hereditary, so we cannot apply the result of Claim 6. In fact the reflection map $\tilde{q}_{S_3}: S_3 \to \mathbf{Z}_2$ is $\tilde{\Phi}$-preserving and $f^{-1}(0) = A_3 \in \text{Fix}(\tilde{R},\tilde{q})$ yet it is not $(\tilde{R},\tilde{q})$-perfect.

(2) Let $(\tilde{R},\tilde{q})$ be the reflector of Example 1.2.4(2) from <u>ABGRP</u> to <u>TFAB</u>. The natural family of congruence relations that induces this reflection is idempotent, hereditary and productive, thus Claim 2 above tells us that $(\tilde{R}, \tilde{q})$ is direct. Hence all conditions in Theorem 2.7.1 except (iii) and (viii) hold.

The result of Claim 6 above tells us that a homomorphism $f : G \to H$ is $\tilde{\Phi}$-perfect iff $f[tG] = tH$ and $f^{-1}(e_H)$ is torsion free.

Since the congruence relations involved are productive, it is not difficult to see that every Abelian group is $\tilde{\Phi}$-compact. Thus for a homomorphism $f : G \to H$, f is $\tilde{\Phi}$-preserving and $f^{-1}(e_H)$ is $\tilde{\Phi}$-compact iff f is $\tilde{\Phi}$-preserving. Note that this tells us that the implication $(f) \Rightarrow (g)$ is true even though condition (viii) does not hold.

The inclusion map $i : \{0\} \to \mathbf{Z}_2 \in (\mathrm{Fix}(\tilde{R}, \tilde{q})_{\perp_w} \cap Epi)^{\downarrow}$ yet it is not in $\Sigma_{\tilde{R}}^{\downarrow}$ (consider the square $1_{\mathbf{Z}_2} i = i 1_{\{0\}}$). So yet again the implication $(c) \Rightarrow (b)$ does not hold.

Theorem 2.3.5 and Propsition 2.4.5 allow us to conclude that ($\tilde{\Phi}$-*dense* <u>TFAB</u>-*extendable,* $(\tilde{R}, \tilde{q})$-*perfect*) is a factorisation structure for morphisms in <u>ABGRP</u>.

2.9 Problems

2.9.1 As has been mentioned in the text, the notion of $\mathcal{A}$-directness for a pointed endofunctor (R, r) is a natural generalisation of the concept of a direct reflection introduced in [Brümmer, Giuli 1993a]. Our main motivation for this generalisation is the fact that $\mathcal{M}$-directness is sufficient for the results of Section 1.4. The only examples we have are of direct reflections, which leads us to pose

Problem B: Is there an example of an endofunctor (R, r) on a category $\mathbf{X}$, and a class $\mathcal{A}$ of $\mathbf{X}$-morphisms such that $\Sigma_R \subset \mathcal{A} \subset Mor\mathbf{X}$ and (R, r) is $\mathcal{A}$-direct yet not direct?

2.9.2 Theorem 2.7.1 leaves many open questions about the possibilities of reversing the implications that have only been shown to hold in one direction. In [Dikranjan, Giuli 1991b] there are results that give sufficient conditions on the closure operator $\Phi_{(R,r)}$ for the implication (g) $\Rightarrow$ (f) to hold. These results are in a somewhat different spirit and setting to ours. Our main interest is in the following

Problem C: Under what conditions will the implication (g) $\Rightarrow$ (a) of Theorem 2.7.1 hold?

2.9.3 We have not demonstrated the necessity of any of the conditions in Theorem 2.7.1. Only for condition (viii) have we been able to give an example that refutes the necessity of the condition. This leaves open a number of problems.

Chapter 3

Strong Functorial Completions

The ideas for this chapter were born from a categorical interpretation of [Birkhoff 1937]. In that paper the author equates the notion of completion with a closure notion. Much the same approach is taken here in a categorical setting.

The notion of an endofunctor that is a "strong functorial completion" is introduced. The pullback closure induced by such an endofunctor then plays an important role in describing $\mathcal{M}$-essential morphisms and "complete" objects.

3.1 Motivation

3.1.1 Definition. An endofunctor (R, r) on $\mathbf{X}$ will be called a *strong functorial completion* if it has the following four properties.

(1) For each $X \in Ob\mathbf{X}$, $r_X \in \mathcal{M}$.

(2) (R, r) is idempotent.

(3) For each $m \in \mathcal{M}$, $Rm \in \mathcal{M}$.

(4) (R, r) is well-pointed.

In one sense this is a rather naïve definition. The basic justification comes from typical examples in set based categories where one visualises a completion procedure as being one that involves remedying a particular "deficiency" of an object by adding certain points. Such a completion procedure should have the following properties.

(1) The completion should be constructed in such a way that each object appears as a subobject of its completion.

(2) There should be an identifiable class of "complete" objects – those which are isomorphic to their completion –and the completion of any object should indeed be "complete".

(3) If A is contained in B then the completion of A should be contained in the completion of B.

The well-pointedness of (R, r) is built into the definition mainly because it ensures that several of the results hold together more tidily. It is of course a property that one could desire for a "well behaved" endofunctor, but its inclusion in the definition is more pragmatic.

The first two points of the definition - each $r_X \in \mathcal{M}$ and idempotence - are very natural and easily justified. Without doubt the strongest – and in a sense most limiting – property of a strong functorial completion is its preservation of $\mathcal{M}$-morphisms. This property is recurrent in both examples of and the theory of completions, and is a characteristic whose value we endeavour to demonstrate below. There are completion notions that do not satisfy this criterion, none-the-less the examples we give show that completions which do, fill an important niche in the broader theory.

The other limitation which is implicit in the definition is that completions should be functorial. The MacNeille completion of a partially ordered set for instance is not functorial despite being an exemplary model for completions. One could drop the functoriality and build a slightly wider theory. The consequence would be that the pullback operator would not be a closure operator. It would perform a closure in individual subobject lattices but morphisms would not relate this closure as they do for closure operators.

A strong functorial completion is very close to being a reflection without, it seems, necessarily having to be one. We have no example of a non-reflective strong functorial completion, but present the theory in this slight generality because it helps to clarify – especially in proofs – how the different properties of the endofunctor contribute to the various results. In fact the careful reader will notice that not all the results below rely on the full strength of the definition. There are occasions (eg. Proposition 3.2.5) when a simpler proof could be given if all conditions of the definition were utilised.

3.1.2 The understanding of a completion process in a set based category as being one that adds points to an object to compensate for some "deficiency", makes completion similar to closure. The essential difference is that completion does not happen within a subobject framework. This is one idea captured by the pullback closure operator induced by a strong functorial completion.

Proposition 1.4.6 and Corollary 1.4.9 tell us that for a strong functorial completion (R, r), the induced $\Phi_{(R,r)}$ is both idempotent and weakly hereditary. These are properties that make working with $\Phi_{(R,r)}$ easier, in particular ($\Phi_{(R,r)}$*-dense*, $\Phi_{(R,r)}$*-closed*) is a factorisation structure for morphisms in $\mathbf{X}$.

For a strong functorial completion (R, r), $\Phi_{(R,r)}$ plays a valuable role relative to essential extensions and in characterising the "complete" objects with respect to (R, r). Our investigations are broken into these two areas.

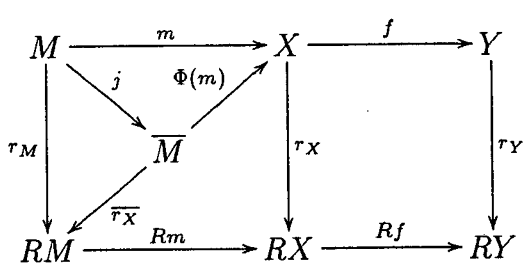
3.2 $\mathcal{M}$-essential morphisms

3.2.1 Definition. (1) $\mathcal{M}^*$ denotes the class all $\mathcal{M}$*-essential* $\mathcal{M}$*-morphisms.* That is those $M \xrightarrow{m} X \in \mathcal{M}$ with the property that for any $\mathbf{X}$-morphism $f : X \to Y$, $fm \in \mathcal{M} \Rightarrow f \in \mathcal{M}$.
(2) For a class $\mathcal{A}$ of $\mathbf{X}$-morphisms, $\mathrm{Inj}(\mathcal{A})$ denotes the class of all $\mathcal{A}$*-injective objects.* These are those objects Z in $\mathbf{X}$ such that for any $f : X \to Y$ in $\mathcal{A}$ and $\mathbf{X}$-morphism $g : X \to Z$, there is an extension $g^* : Y \to Z$ such that $g^* f = g$.

From now until the end of Section 3.3 (R, r) *is always a strong functorial completion.* The only exceptions to this are in Propositions 3.2.6 and 3.2.7 and Corollary 3.2.8.

3.2.2 Proposition. $\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M} \subseteq \mathcal{M}^*$.

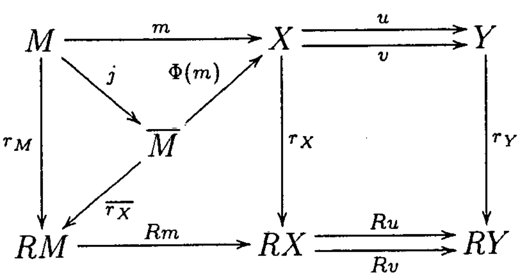
Proof. Let $f : X \to Y$ be a morphism in $\mathbf{X}$, and let $m : M \to X$ be a $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphism such that $fm \in \mathcal{M}$. Consider the diagram below where we form the $\Phi_{(R,r)}$-closure of m.



Since $fm \in \mathcal{M}$, $R(fm) \in \mathcal{M}$. Also, $r_X \in \mathcal{M}$ and $\mathcal{M}$ is closed under pullbacks, so $\overline{r_X} \in \mathcal{M}$, and thus $Rfr_X\Phi_{(R,r)}(m) = RfRm\overline{r_X} \in \mathcal{M}$. But m is $\Phi_{(R,r)}$-dense, so $\Phi_{(R,r)}(m)$ is an isomorphism giving that $r_Y f = Rfr_X \in \mathcal{M}$. Since $r_Y \in \mathcal{M}$ it then follows that $f \in \mathcal{M}$. □

3.2.3 Proposition. $\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M} \subseteq Epi\mathbf{X}$.

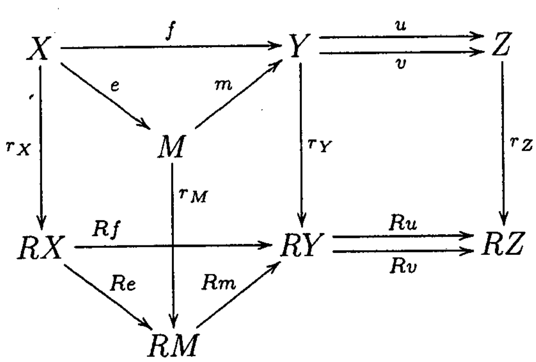
Proof. Let $m : M \to X$ be a $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphism, and let $u, v : X \to Y$ be such that $um = vm$.



$um = vm \Rightarrow RuRm = RvRm \Rightarrow RuRm\overline{r_X} = RvRm\overline{r_X} \Rightarrow Rur_X\Phi_{(R,r)}(m) = Rvr_X\Phi_{(R,r)}(m) \Rightarrow r_Y u\Phi_{(R,r)}(m) = r_Y v\Phi_{(R,r)}(m)$. But $\Phi_{(R,r)}(m)$ is an isomorphism and r_Y is a monomorphism, so it follows that $u = v$ and hence that m is an epimorphism. □

3.2.4 Proposition. *If* $\mathcal{E} \subseteq Epi\mathbf{X}$ *then* $\{\Phi_{(R,r)}\text{-}dense\} \subseteq Epi\mathbf{X}$.

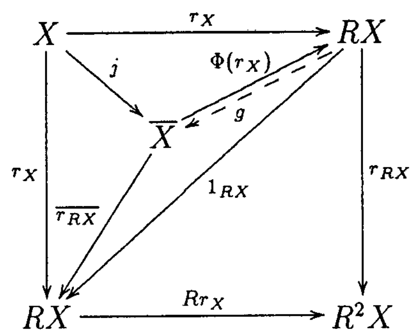
Proof. Given a $\Phi_{(R,r)}$-dense morphism $f : X \to Y$, and morphisms $u, v : Y \to Z$ such that $uf = vf$, construct the following diagram ($me = f$ is the $(\mathcal{E}, \mathcal{M})$ factorisation of f).



Noting that since e is an epimorphism, $uf = vf \Rightarrow um = vm$, the proof follows as for Proposition 3.2.3. □

3.2.5 Proposition. *For every* $X \in Ob\mathbf{X}$, $r_X \in \{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M}$.

Proof. Let $X \in Ob\mathbf{X}$. By assumption, $r_X \in \mathcal{M}$, so we construct its $\Phi_{(R,r)}$-closure.



Since (R,r) is well-pointed, $Rr_X 1_{RX} = r_{RX}$. But $Rr_X \overline{r_{RX}} = r_{RX}\Phi_{(R,r)}(r_X)$ is a pullback square, so there is a unique $g : RX \to \overline{X}$ such that $\overline{r_{RX}}g = 1_{RX}$ and $\Phi_{(R,r)}(r_X)g = 1_{RX}$. This means that $\Phi_{(R,r)}(r_X)$ is a monomorphism and a retraction, hence an isomorphism and r_X is $\Phi_{(R,r)}$-dense. □

These results tell us that (R,r) is pointwise epimorphic – a result that is not surprising – and also that for each $X \in Ob\mathbf{X}$, $r_X : X \to RX$ is an $\mathcal{M}$-essential extension of X (i.e. an $\mathcal{M}$-essential morphism with domain X). Interestingly, this latter fact could be used in the definition of a strong functorial completion in place of the preservation of $\mathcal{M}$-morphisms, as the following proposition shows.

3.2.6 Proposition. *Let (R,r) be an idempotent, well-pointed endofunctor such that $r_X \in \mathcal{M}$ for every $X \in Ob\mathbf{X}$. For such (R,r), $Rm \in \mathcal{M}$* for every *$m \in \mathcal{M} \Leftrightarrow r_X \in \mathcal{M}^*$* for every *$X \in Ob\mathbf{X}$*

Proof. $\Rightarrow$: Follows from Propositions 3.2.5 and 3.2.2.

$\Leftarrow$: For $M \overset{m}{\to} X \in \mathcal{M}$, $Rmr_M = r_X m \in \mathcal{M}$. Thus since $r_M \in \mathcal{M}^*$ it follows that $Rm \in \mathcal{M}$. □

The well-pointedness of (R,r) can similarly be ensured by conditions on individual r_X morphisms.

3.2.7 Proposition. *If (R,r) is an idempotent endofunctor such that $r_X \in \mathcal{M}$ for every $X \in Ob\mathbf{X}$ and $Rm \in \mathcal{M}$ for every $m \in \mathcal{M}$, then the following are equivalent.*

(a) *(R,r) is well-pointed.*

(b) *For every $X \in Ob\mathbf{X}$, r_X is $\Phi_{(R,r)}$-dense.*

(c) *For every $X \in Ob\mathbf{X}$, r_X is epimorphic.*

Proof. (a) $\Rightarrow$ (b): Proposition 3.2.5.

(b) $\Rightarrow$ (c): Follows from Proposition 3.2.3, since the proof of that result does not use the well-pointedness of (R, r).

(c) $\Rightarrow$ (a): For any $X \in Ob\mathbf{X}$, $Rr_X r_X = r_{RX} r_X$, so if r_X is an epimorphism, $Rr_X = r_{RX}$. $\square$

These two propositions have the following interesting corollary.

3.2.8 Corollary. *An endofunctor (R, r) is a strong functorial completion iff it is idempotent and for every $X \in Ob\mathbf{X}$, $r_X \in \mathcal{M}^* \cap Epi\mathbf{X}$.*

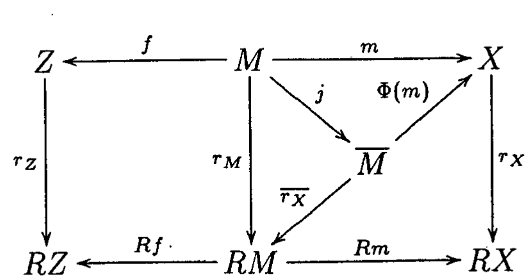
Proof. The forward implication is clear. The reverse implication follows since idempotence and the fact that each $r_X \in \mathcal{M}$ are given, well-pointedness is immediate since each r_X is epimorphic and preservation of $\mathcal{M}$-morphisms then follows from Proposition 3.2.6. $\square$

3.2.9 Remark. The above three results are interesting in that they highlight the power and scope of some of the properties used in the definition of a strong functorial completion. There may also be occasions when they will be useful for determining whether or not a given functor is such a completion. However, the corollary would not, in our opinion, be an instructive form in which to actually define a strong functorial completion.

It has become clear that the $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphisms play a pivotal role in the theory of strong functorial completions. Knowledge of the following result illuminates this observation.

3.2.10 Proposition. $\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M} = \Sigma_R$.

Proof. Let $M \xrightarrow{m} X \in \{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M}$ and let $f : M \to Z$ have codomain Z in $\text{Fix}(R, r)$. Operate (R, r) on both m and f and form $\Phi_{(R,r)}(m)$.



Both r_Z and $\Phi_{(R,r)}(m)$ are isomorphisms, which allows us to consider the composition $(r_Z)^{-1}Rf\overline{r_X}(\Phi_{(R,r)}(m))^{-1}m = (r_Z)^{-1}Rf\overline{r_X}j = (r_Z)^{-1}Rfr_M = f$. Thus since by Proposition 3.2.3 m is an epimorphism, we conclude that $m \in \mathrm{Fix}(R,r)_{\perp}$. Proposition 2.4.4 then tells us that $m \in \Sigma_R$.

On the other hand if $X \xrightarrow{f} Y \in \Sigma_R$ then we know by Proposition 1.6.4 that f is $\Phi_{(R,r)}$-dense, while the fact that $r_Y f = (Rf)^{-1}r_X \in \mathcal{M}$ gives that $f \in \mathcal{M}$. □

At times the $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphisms further coincide with the class $\mathcal{M}^*$. Proposition 3.3.3 provides a characterisation of when this occurs.

3.2.11 Corollary. *If $f : X \to Y \in \{\Phi_{(R,r)}\text{-dense}\} \cap \mathcal{M}$ and $Y \in \mathrm{Fix}(R,r)$ then $f \cong r_X$.*

Proof. Immediate since $Rfr_X = r_Y f$ and both Rf and r_Y are isomorphisms. □

3.2.12 Remark. This corollary tells us that the completion of an object is the unique $\Phi_{(R,r)}$-dense extension of that object into a "complete" object. This is a characteristic of completions that other authors have investigated and is taken further in Section 3.4.

3.3 Complete objects

$\mathrm{Fix}(R,r)$ is the class of $\mathbf{X}$-objects that are "complete" with respect to the strong functorial completion (R,r). The pullback closure can be used rather elegantly to characterise these "complete" objects.

3.3.1 Theorem. *For an object X in $\mathbf{X}$, the following are equivalent.*

(a) $X \in \mathrm{Fix}(R,r)$.

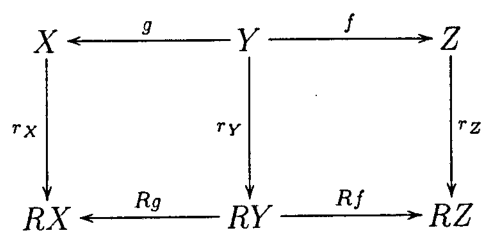
(b) $X \in \mathrm{Inj}(\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M})$.

(c) *Any $\Phi_{(R,r)}$-closed $\mathcal{M}$-morphism $m : M \to X$ has domain $M \in \mathrm{Fix}(R,r)$.*

(d) *Any $\mathcal{M}$-morphism $m : X \to Y$ with domain X is $\Phi_{(R,r)}$-closed (i.e. X is an absolutely $\Phi_{(R,r)}$-closed object).*

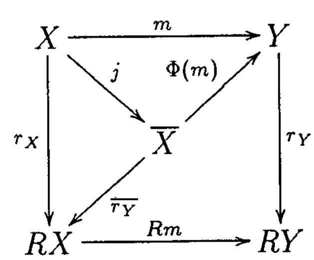
(e) *Any $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphism $m : X \to Y$ with domain X is an isomorphism.*

Proof. (a) $\Rightarrow$ (b): Let $X \in \text{Fix}(R,r)$, let $f : Y \to Z$ be a $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphism and assume there is a morphism $g : Y \to X$. Operate (R,r) on f and g.



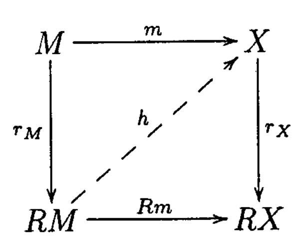
By Proposition 3.2.10, Rf is an isomorphism, and since r_X is an isomorphism too, we can consider the composition $(r_X)^{-1}Rg(Rf)^{-1}r_Zf = (r_X)^{-1}Rgr_Y = g$ which gives that $X \in \text{Inj}(\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M})$.

(a) $\Rightarrow$ (d): Let $X \in \text{Fix}(R,r)$ and let $X \xrightarrow{m} Y \in \mathcal{M}$. Form the $\Phi_{(R,r)}$-closure of m.



Since $\Phi_{(R,r)}$ is weakly hereditary, j is $\Phi_{(R,r)}$-dense, and thus by Proposition 3.2.3 it is an epimorphism. But $\overline{r_Y}j = r_X$ is an isomorphism, so j is a section and hence an isomorphism and m is thus $\Phi_{(R,r)}$-closed.

(b) $\Rightarrow$ (c): Let $X \in \text{Inj}(\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M})$ and let $m : M \to X$ be a $\Phi_{(R,r)}$-closed $\mathcal{M}$-morphism. Operate (R,r) on $m : M \to X$.



By Proposition 3.2.5, $r_M \in \{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M}$, so there is a morphism $h : RM \to X$ such that $hr_M = m$. Since $r_M \in \mathcal{M}^*$ (Proposition 3.2.2) $h \in \mathcal{M}$. But $RM \in \text{Fix}(R,r)$, so knowing (a) $\Rightarrow$ (d) above we conclude that h is $\Phi_{(R,r)}$-closed. Thus $hr_M = m$ is a ($\Phi_{(R,r)}$-*dense*, $\Phi_{(R,r)}$-*closed*) factorisation of m, but m is $\Phi_{(R,r)}$-closed so it follows that r_M is an isomorphism so $M \in \text{Fix}(R,r)$.

(c) $\Rightarrow$ (a): Trivial since $1_X : X \to X$ is a $\Phi_{(R,r)}$-closed $\mathcal{M}$-morphism.

(d) $\Rightarrow$ (e): Clear.

(e) $\Rightarrow$ (a): Trivial since $r_X : X \to RX$ is a $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphism. □

3.3.2 Remarks. (1) Item (b) in conjunction with Proposition 3.2.2 says that for X in $\mathbf{X}$, $r_X : X \to RX$ is an $\mathcal{M}$-injective hull, that is an $\mathcal{M}$-essential extension into an $\mathcal{M}^*$-injective object.

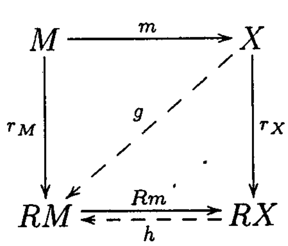
(2) When we can describe $\Phi_{(R,r)}$ the above theorem gives very useful ways of characterising the "complete" objects. In particular, item (e) provides an elegant way of possibly defining "completeness", and in fact a definition in this spirit is used in frame theory (cf. for example [Banaschewski, Pultr 1990]).

The following proposition can be proved partly as a corollary to the above theorem. It characterises when the $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphisms coincide with the class $\mathcal{M}^*$ – exactly when the "complete" objects coincide with the $\mathcal{M}^*$-injective objects. This is pleasing not just because of its bearing on the previous section, but also because essentiality is really an abstract categorical formulation of density.

3.3.3 Proposition. $\mathcal{M}^* = \{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M} \Leftrightarrow \mathrm{Fix}(R,r) = \mathrm{Inj}(\mathcal{M}^*)$.

Proof. $\Rightarrow$: This follows immediately from the equivalence of (a) and (b) in Theorem 3.3.1.

$\Leftarrow$: Knowing Proposition 3.2.2, we need only show that any $m \in \mathcal{M}^*$ is $\Phi_{(R,r)}$-dense. So consider $M \xrightarrow{m} X \in \mathcal{M}^*$ operated on by (R,r).



By assumption $RM \in \mathrm{Inj}(\mathcal{M}^*)$, so there is a morphism $g : X \to RM$ such that $gm = r_M$. Also since $r_X \in \mathcal{M}^*$ (Propositions 3.2.5. and 3.2.2) there is a morphism $h : RX \to RM$ such that $hr_X = g$. Note that since $hr_Xm = gm = r_M \in \mathcal{M}$ and both r_X and m are in $\mathcal{M}^*$, $h \in \mathcal{M}$.

Now $hRmr_M = hr_Xm = gm = r_M$ and r_M is an epimorphism (Propositions 3.2.5 and

3.2.3) so we conclude that $hRm = 1_{RM}$. Thus h is a retraction, hence an isomorphism, and Rm is too. This means that $m \in \Sigma_R$ so by Proposition 3.2.10, m is $\Phi_{(R,r)}$-dense. □

3.4 Firm reflections

Recently [Brümmer, Giuli, Herrlich 1992] and [Brümmer, Giuli 1992] made categorical studies of completions via what they termed *firm* reflections. This concept is best motivated by an example. A separated uniform space X has a unique completion in the sense that any dense uniform embedding of X into a complete uniform space is isomorphic to the completion of X. The notion of firmness exploits the fact that many completions are unique "dense" extensions in this sense.

Corollary 3.2.11 tells us that strong functorial completions exhibit this same uniqueness property – a completion $r_X : X \to RX$ is isomorphic to any $\Phi_{(R,r)}$-dense $\mathcal{M}$-morphism with domain X and "complete" codomain. It is not surprising then that our theory has strong ties with that of firm reflections.

For this section, (R, r) is assumed to be a reflection. We will maintain the notation in use for pointed endofunctors, even though of course some of this is superfluous when (R, r) is a reflection.

The following is the definition of (sub)firmness used in [Brümmer, Giuli 1992] (Definition 1.1). The definition used in [Brümmer, Giuli, Herrlich 1992] was somewhat more specialised.

3.4.1 Definition. Let $\mathcal{U}$ be a class of $\mathbf{X}$-morphisms that is closed under composition and composition with isomorphisms. (R, r) is called a *subfirm $\mathcal{U}$-reflection* if for each $X \in Ob\mathbf{X}$, $r_X \in \mathcal{U}$ and $\mathcal{U} \subseteq \Sigma_R$. A subfirm $\mathcal{U}$-reflection is called a *firm $\mathcal{U}$-reflection* if we in fact have that $\mathcal{U} = \Sigma_R$.

From Propositions 3.2.5 and 3.2.10 we conclude that any reflective strong functorial completion (R, r) is a firm $(\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M})$-reflection. There is however much more to the connections between strong functorial completions and firm reflections.

3.4.2 Theorem. *If $\mathbf{X}$ is an (Epi, ExtrMonoSource) category, and $\mathcal{E} \subseteq Epi\mathbf{X}$ then the following are equivalent.*

(a) *(R, r) is a firm $(Epi\mathbf{X} \cap \mathcal{M})$-reflection.*

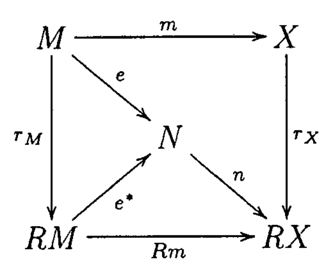
(b) *(R, r) is a subfirm $(Epi\mathbf{X} \cap \mathcal{M})$-reflection.*

(c) *(R, r) is a strong functorial completion and $\{\Phi_{(R,r)}$-dense$\} \cap \mathcal{M} = Epi\mathbf{X} \cap \mathcal{M}$.*

(d) *(R, r) is a strong functorial completion and $\{\Phi_{(R,r)}$-dense$\} = Epi\mathbf{X}$.*

Proof. (a) $\Rightarrow$ (b): Clear.

(b) $\Rightarrow$ (c): To show that (R, r) is a strong functorial completion, we only need to show that it preserves $\mathcal{M}$-morphisms. Take $M \xrightarrow{m} X \in \mathcal{M}$ and consider the following diagram.



Let $ne = r_X m$ be the (*Epi*, *ExtrMono*) factorisation of $r_X m$. Since $\mathbf{X}$ is an (*Epi, ExtrMonoSource*) category and (R, r) is an epireflection, $N \in \text{Fix}(R, r)$, and thus there is a morphism $e^* : RM \to N$ such that $e^* r_M = e$.

Since $\mathcal{E} \subseteq Epi\mathbf{X}$, [Adámek, Herrlich, Strecker 1990] Proposition 14.10 tells us that $ExtrMono\mathbf{X} \subseteq \mathcal{M}$ and so in particular $n \in \mathcal{M}$. But $ne = r_X m \in \mathcal{M}$ so $e \in \mathcal{M}$ and hence $e \in Epi\mathbf{X} \cap \mathcal{M}$. Because (R, r) is a subfirm $(Epi\mathbf{X} \cap \mathcal{M})$-reflection we conclude that $e \in \Sigma_R$. We then see that $(r_N)^{-1} Rer_M = e$ and so $(r_N)^{-1} Re = e^*$ and thus e^* is an isomorphism.

Next we note that $ne^* r_M = ne = r_X m = Rmr_M$ and so since r_M is an epimorphism $ne^* = Rm$. But e^* is an isomorphism so $Rm \in ExtrMono\mathbf{X} \subseteq \mathcal{M}$. Hence (R, r) is a strong functorial completion as claimed.

Proposition 3.2.3 now tells us that $(\{\Phi_{(R,r)}$-*dense*$\} \cap \mathcal{M}) \subseteq Epi\mathbf{X} \cap \mathcal{M}$. However, (R, r) is subfirmly $(Epi\mathbf{X} \cap \mathcal{M})$-reflective so $Epi\mathbf{X} \cap \mathcal{M} \subseteq \Sigma_R$ and we know by Proposition 3.2.10 that $\{\Phi_{(R,r)}$-*dense*$\} \cap \mathcal{M} = \Sigma_R$ so the result follows.

(c) $\Rightarrow$ (d): All we need to show is that $\{\Phi_{(R,r)}$-*dense*$\} = Epi\mathbf{X}$. Since $\mathcal{E} \subseteq Epi\mathbf{X}$, Proposition 3.2.4 tells us that $\{\Phi_{(R,r)}$-*dense*$\} \subseteq Epi\mathbf{X}$. On the other hand let $f : X \to Y \in Epi\mathbf{X}$, and let $me = f$ be its $(\mathcal{E}, \mathcal{M})$ factorisation. Since f is an epimorphism, $m \in Epi\mathbf{X} \cap \mathcal{M}$, and thus by assumption m is $\Phi_{(R,r)}$-dense. This just means that f is $\Phi_{(R,r)}$-dense and so (d) holds.

(d) $\Rightarrow$ (a): Obviously (d) $\Rightarrow$ (c), and knowing (c) it follows from Propositions 3.2.5 and 3.2.10 that (R, r) is a firm $(\{\Phi_{(R,r)}$-*dense*$\} \cap \mathcal{M})$-reflection. Since $\{\Phi_{(R,r)}$-*dense*$\} \cap \mathcal{M} = Epi\mathbf{X} \cap \mathcal{M}$, (a) follows. □

3.4.3 Remarks. (1) The fact that (R,r) preserves $\mathcal{M}$-morphisms in the implication (b) $\Rightarrow$ (c) is revealed in [Brümmer, Giuli, Herrlich 1992] Proposition 2.3. It was really this proposition that lead to the above theorem, and we have used the core ingredients of their proof in ours.

(2) Most of the constructs in which we work satisfy the conditions of the theorem and so many known examples of firm reflections provide examples of strong functorial completions. Moreover $\Phi_{(R,r)}$ can be used to describe the epimorphisms in these constructs, and since $\Phi_{(R,r)}$ is idempotent and weakly hereditary it follows that the extremal monomorphisms are exactly the $\Phi_{(R,r)}$-closed $\mathcal{M}$-morphisms.

3.5 Examples

We conclude this part of the chapter with some examples of strong functorial completions. In the process we describe the pullback closure $\Phi_{(R,r)}$ for some reflectors not yet considered.

3.5.1 Uniform completion. Let (R,r) be the completion reflection in the category $\underline{\text{UNIF}_0}$ of separated uniform spaces and uniformly continuous maps. Subobjects are represented by the class $\mathcal{M}$ of uniform embeddings and the factorisation structure $(\mathcal{E},\mathcal{M})$ factorises a uniformly continuous map through its image, $\mathcal{E}$ being the class of surjective uniformly continuous maps.

The epimorphisms in $\underline{\text{UNIF}_0}$ are the maps which are dense in the underlying topology ([Preuß 1972] Lemma p.402). It is well known (cf. [Engelking 1989] Theorem 8.3.12) that (R,r) is a subfirm $(\{Dense\}\cap\mathcal{M})$-reflection . From this knowledge and Theorem 3.4.2 we can conclude that the uniform completion functor is a firm $(Epi\cap\mathcal{M})$-reflection and thus a strong functorial completion. Moreover, a uniformly continuous map $f: X\rightarrow Y$ is $\Phi_{(R,r)}$-dense iff f is an epimorphism iff f is dense with respect to the topology underlying Y.

Now $\Phi_{(R,r)}$ is an idempotent weakly hereditary closure operator on $\underline{\text{UNIF}_0}$, as is the underlying topological closure viewed as a closure operator K on $\underline{\text{UNIF}_0}$. Hence the fact that $\{\Phi_{(R,r)}\text{-}dense\} = \{K\text{-}dense\}$ implies that $\Phi_{(R,r)} = K$, in other words the pullback closure induced by the uniform completion is the underlying topological closure.

The characterisations of complete uniform spaces given by Theorem 3.3.1 are well known (cf. for example [Engelking 1989] Theorem 8.3.6).

3.5.2 Sobrification. Let (S,s) denote the sobrification reflector in $\underline{\text{TOP}_0}$. It was demonstrated in 1.2.3 that (S, s) preserves embeddings, thus since for each $\underline{\text{TOP}_0}$ space X the sobrification $s_X : X \to SX$ is an embedding it follows that (S, s) is a strong functorial completion.

We could also have drawn the conclusion that (S, s) is a strong functorial completion using Theorem 3.4.2, since [Hoffmann 1976] Proposition 3.1.2 tells us that (S, s) is a firm $(Epi \cap \mathcal{M})$-reflection in $\underline{\text{TOP}_0}$. The fact that the epimorphisms in $\underline{\text{TOP}_0}$ are the $\dot{b}$-dense ($\Phi_{(S,s)}$-dense) morphisms was first observed in [Baron 1968].

The conclusions of Theorem 3.3.1 that characterise sober spaces in $\underline{\text{TOP}_0}$ are noted in Example 1.8(3) of [Brümmer, Giuli, Herrlich 1992].

3.5.3 Cancellative Abelian monoids. The reflection of the Abelian groups in the category $\mathbf{X}$ of cancellative Abelian monoids with homomorphisms that preserve the neutral element is a strong functorial completion. In this setting, subobjects are defined via the (*Surjection, Injection*) factorisation structure for homomorphisms.

Let M be a cancellative Abelian monoid – we write the binary operation as multiplication and neutral element as "e". Define the following relation ρ_M on $M \times M$:

$$(a, b)\rho_M(s, t) \Leftrightarrow a \cdot t = b \cdot s.$$

ρ_M is a congruence relation on $M \times M$ and the quotient $(M \times M)/\rho_M$ is an Abelian group. The injective homomorphism $r_M : M \to (M \times M)/\rho_M$ that maps $a \in M$ to the equivalence class $[(a, e)]_{\rho_M}$ is the reflection of M to $\underline{\text{ABGRP}}$. Let (R, r) denote the associated reflector.

For a homomorphism $f : N \to M$ in $\mathbf{X}$, $Rf : (N \times N)/\rho_N \to (M \times M)/\rho_M$ maps $[(a, b)]_{\rho_N}$ to $[(f(a), f(b))]_{\rho_M}$. Assume that f is an injective homomorphism, then $Rf([(a, b)]_{\rho_N}) = Rf([(s, t)]_{\rho_N}) \Rightarrow (f(a), f(b))\rho_M(f(s), f(t)) \Rightarrow f(a) \cdot f(t) = f(b) \cdot f(s) \Rightarrow f(a \cdot t) = f(b \cdot s) \Rightarrow a \cdot t = b \cdot s \Rightarrow [(a, b)]_{\rho_N} = [(s, t)]_{\rho_N}$ so Rf is injective too. It now follows that (R, r) is a strong functorial completion.

The pullback closure of a submonoid N of M can be described as follows:

$$\Phi_{(R,r)}(N) = \{s \in M \mid \exists a, b \in N \text{ such that } a = b \cdot s\}.$$

Applying Theorem 3.3.1 this closure operator gives a number of descriptions of Abelian groups in the category of cancellative Abelian monoids.

Since the epimorphisms in $\underline{\text{ABGRP}}$ are the surjective homomorphisms, Corollary 3.6.3 below tells us that $Epi\mathbf{X} = \{\Phi\text{-}dense\}$.

From these observations, Theorem 3.4.2 leads us to conclude that (R, r) is a firm $(Epi\mathbf{X} \cap \mathcal{M})$-reflection. This observation is also made in Remark 1.17 of [Brümmer, Giuli 1992]. (The fact that (R,r) is a subfirm $(Epi\mathbf{X} \cap \mathcal{M})$-reflection was noted in [Brümmer, Giuli, Herrlich 1992] Example 1.8(15).)

3.5.4 Uniform completion in Hausdorff topological groups. Not every strong functorial completion fits into the mould of Theorem 3.4.2. Consider the category $\underline{\text{TopGrp}_0}$. Here $\mathcal{M}$ is the class of all group homomorphisms which are topological embeddings, and $(\mathcal{E}, \mathcal{M})$ is the (*Surjection, Embedding*) factorisation structure. (We know that $\mathcal{M}$ has the necessary completeness properties since $\underline{\text{TopGrp}}$ is topological over $\underline{\text{Grp}}$ and thus the completeness properties of the injective group homomorphisms are uniquely lifted to the embeddings in $\underline{\text{TopGrp}}$. Since $\underline{\text{TopGrp}_0}$ is closed under $\mathcal{M}$-subobjects, $\mathcal{M}$-completeness carries over to the subcategory.)

A Hausdorff topological group G is a Tychonoff space (cf. [Preuß 1972] Korollar 9.5.11) and can be endowed with a number of uniformities, all of which are compatible with the topology on G (cf. [Engelking 1989] Example 8.1.17). One of these uniformities is the central uniformity on G. It was shown in [Brümmer, Giuli 1992] Example 1.16 that the groups which are complete with respect to this central uniformity form a firm $(\{Dense\} \cap \mathcal{M})$-reflective subcategory of $\underline{\text{TopGrp}_0}$ (dense in the usual topological sense). Let (R,r) denote this firm reflection.

At the level of uniform spaces, the action of (R,r) is simply the uniform completion. Thus since the uniform completion is a strong functorial completion, (R,r) is an $\mathcal{M}$-reflection and preserves $\mathcal{M}$-morphisms and is itself a strong functorial completion. (More information on these matters can be found in [Nummela 1980].)

Being a strong functorial completion, (R,r) is a firm $(\{\Phi_{(R,r)}\text{-}dense\} \cap \mathcal{M})$-reflection. But it is also a firm $(\{Dense\} \cap \mathcal{M})$-reflection and so it follows that $\{\Phi_{(R,r)}\text{-}dense\} \cap M = \{Dense\} \cap \mathcal{M}$, and hence that $\{\Phi_{(R,r)}\text{-}dense\} = \{Dense\}$. Since both $\Phi_{(R,r)}$ and the topological closure are idempotent weakly hereditary closures we conclude that $\Phi_{(R,r)}$ is the topological closure.

It has been shown in [Uspenskij 1994] that the epimorphisms in $\underline{\text{TopGrp}_0}$ are not the dense homomorphisms, thus our strong functorial completion (R,r) is not a firm $(Epi \cap \mathcal{M})$-reflection.

3.6 Epimorphisms and $\Phi_{(R,r)}$-density

This final section takes a brief look at when epimorphisms in a category $\mathbf{X}$ can be described by means of density with respect to the pullback closure $\Phi_{(R,r)}$ for some (R,r) on $\mathbf{X}$. We saw in Theorem 3.4.2 that for some strong functorial completions (R,r) on $\mathbf{X}$, $\Phi_{(R,r)}$ can be used to describe the epimorphisms in $\mathbf{X}$. Also, in some examples that are not strong functorial completions it is apparent that density with respect to the pullback closure characterises the epimorphisms of the category in which we are working.

Categorical closure operators have been used extensively to describe epimorphisms and investigate cowellpoweredness in different categories. The references [Giuli 1980], [Schröder 1983], [Dikranjan, Giuli 1984], [Castellini 1986], [Dikranjan, Giuli, Tholen 1989], [Dikranjan 1992] and [Dikranjan, Tholen 1995] provide a good overview of these endeavours. In most of these investigations, the regular closure operator induced by a subcategory **A** of **X** has been used to describe the epimorphisms in **A**. The situation with $\Phi_{(R,r)}$ is different in that it describes the epimorphisms in **X**, not some subcategory of **X**.

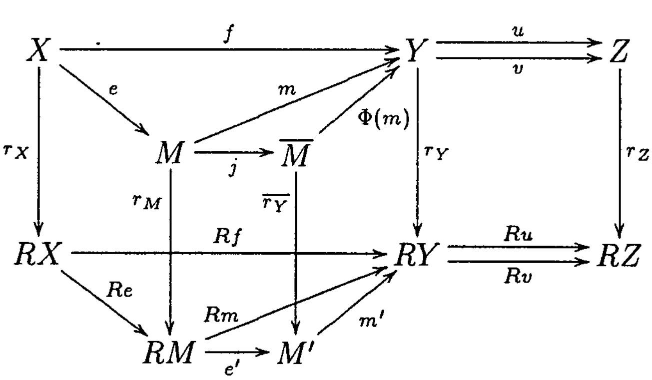
For the purpose of this section, (R, r) is again merely a pointed endofunctor, not a strong functorial completion. For convenience, we view $\text{Fix}(R, r)$ as a full subcategory of **X**.

We assume throughout that $\mathcal{E} \subseteq Epi\mathbf{X}$.

The following proposition extends Proposition 3.2.4 to any pointwise monomorphic endofunctor.

3.6.1 Proposition. *If* (R, r) *is pointwise monomorphic, then* $\{\Phi_{(R,r)}\text{-dense}\} \subseteq Epi\mathbf{X}$.

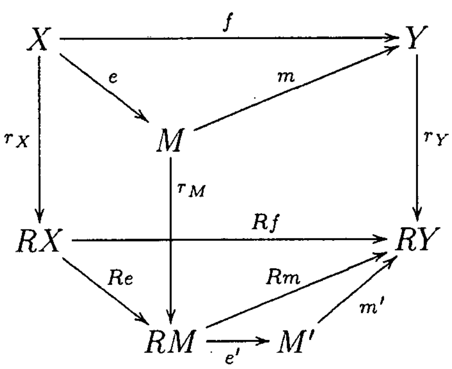
Proof. Let $f : X \to Y$ be a $\Phi_{(R,r)}$-dense **X**-morphism, and let $u, v : Y \to Z$ be such that $uf = vf$. Take the $(\mathcal{E}, \mathcal{M})$ factorisations $me = f$ and $m'e' = Rm$.



Since both e and e' are epimorphisms we see that $uf = vf \Rightarrow um = vm \Rightarrow RuRm = RvRm \Rightarrow Rum' = Rvm' \Rightarrow Rur_Y\Phi_{(R,r)}(m) = Rvr_Y\Phi_{(R,r)}(m) \Rightarrow r_Zu\Phi_{(R,r)}(m) = r_Zv\Phi_{(R,r)}(m)$. But r_Z is a monomorphism and $\Phi_{(R,r)}(m)$ is an isomorphism so $u = v$ and f is an epimorphism as claimed. $\square$

3.6.2 Proposition. *If Epi*$\mathbf{X} \cap$ *Mor*$\mathrm{Fix}(R,r) \subseteq \mathcal{E}$ *and* (R,r) *is idempotent and pointwise epimorphic then Epi*$\mathbf{X} \subseteq \{\Phi_{(R,r)}$*-dense*$\}$.

Proof. Take the $(\mathcal{E},\mathcal{M})$ factorisation $me = f$ of an epimorphism $f : X \to Y$ in $\mathbf{X}$. We must show that m is $\Phi_{(R,r)}$-dense. Consider the diagram below.



The morphisms f, e, r_X, r_M and r_Y are all epimorphisms. Since $Rfr_X = r_Y f$ and $Rer_X = r_M e$ it follows that both Rf and Re are epimorphisms, and so because RX, RY and RM are all in $\mathrm{Fix}(R,r)$ we conclude that both Rf and Re are in $\mathcal{E}$.

Take the $(\mathcal{E},\mathcal{M})$ factorisation $m'e' = Rm$. Since $m'e'Re = RmRe = Rf \in \mathcal{E}$, m' is an isomorphism and thus f is $\Phi_{(R,r)}$-dense. □

3.6.3 Corollary. *If* (R,r) *is a monoreflection and Epi*$\mathbf{X} \cap$ *Mor*$\mathrm{Fix}(R,r) \subseteq \mathcal{E}$ *then Epi*$\mathbf{X} = \{\Phi_{(R,r)}$*-dense*$\}$.

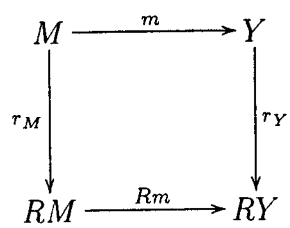
3.6.4 Remark. Note that if the epimorphisms of $\mathrm{Fix}(R,r)$ are contained in $\mathcal{E}$ then *Epi*$\mathbf{X} \cap$ *Mor*$\mathrm{Fix}(R,r) \subseteq \mathcal{E}$. This is the case in the examples we consider below.

The condition that the $\mathbf{X}$-epimorphisms restricted to $\mathrm{Fix}(R,r)$ are $\mathcal{E}$-morphisms is in some cases a necessary condition for the $\mathbf{X}$-epimorphisms to coincide with the $\Phi_{(R,r)}$-dense morphisms as the following proposition shows. By saying that $\mathrm{Fix}(R,r)$ *is closed under* $(\mathcal{E},\mathcal{M})$ *factorisations* we mean that if $f : X \to Y$ is a $\mathrm{Fix}(R,r)$-morphism, then when we take the $(\mathcal{E},\mathcal{M})$ factorisation $me = f$, the domain M of m is in $\mathrm{Fix}(R,r)$.

3.6.5 Proposition. *If* (R,r) *is a monoreflection and* $\mathrm{Fix}(R,r)$ *is closed under* $(\mathcal{E},\mathcal{M})$ *factorisations, then Epi*$\mathbf{X} \cap$ *Mor*$\mathrm{Fix}(R,r) \subseteq \mathcal{E} \Leftrightarrow$ *Epi*$\mathbf{X} = \{\Phi_{(R,r)}$*-dense*$\}$.

Proof. The forward implication follows from Corollary 3.6.3. To verify the reverse implication, let $X \xrightarrow{f} Y \in$ *Epi*$\mathbf{X} \cap$ *Mor*$\mathrm{Fix}(R,r)$ and take the $(\mathcal{E},\mathcal{M})$ factorisation

$me = f$. Since $Epi\mathbf{X} = \{\Phi_{(R,r)}\text{-}dense\}$, $m : M \to Y$ is $\Phi_{(R,r)}$-dense. Consider the diagram below.



By assumption M is in Fix(R,r) so both r_M and r_Y are isomorphisms and the square $Rmr_M = r_Y m$ is a pullback, so since $Rm = r_Y m r_M^{-1} \in \mathcal{M}$, m is $\Phi_{(R,r)}$-closed. This means that m must be an isomorphism and hence that $f \in \mathcal{E}$. □

3.6.6 Examples. This theory applies to a number of concrete examples, some of which we have considered in a different light before.

(1) Epimorphisms in TOP. Let (R,r) be the bireflection in TOP to the full subcategory IND of indiscrete topological spaces. In this setting $(\mathcal{E},\mathcal{M})$ is the (*Surjection, Embedding*) factorisation structure for morphisms in TOP. Clearly $\Phi_{(R,r)}$ is the indiscrete closure operator.

IND is concretely isomorphic to SET so the epimorphisms are surjective, and Corollary 3.6.3 tells us that the epimorphisms in TOP are the $\Phi_{(R,r)}$-dense – in other words surjective – continuous maps. This provides a demonstration of how the epimorphisms in TOP are lifted from those in SET.

(2) Pullback closure in TYCH and HAUS. In Example 1.2.1 we calculated the pullback closure induced by the Čech-Stone compactification (R,r) on TYCH by internal means. The results of this section provide an alternative route to the same result.

Knowing that the epimorphisms in HCOMP are surjective, and that (R,r) is a monoreflection we can conclude from Corollary 3.6.3 that the epimorphisms in TYCH are exactly the $\Phi_{(R,r)}$-dense morphisms. Let K denote the usual topological closure, we know that the epimorphisms in TYCH are K-dense and hence that $\{K\text{-}dense\} = \{\Phi_{(R,r)}\text{-}dense\}$. K is idempotent and weakly hereditary, as is $\Phi_{(R,r)}$ (cf. Example 1.5.1) so it follows that $K = \Phi_{(R,r)}$.

HCOMP is also monoreflective in HAUS and the epimorphisms in HAUS are the dense continuous maps. Thus an argument similar to the one above leads us to conclude that in HAUS, density with respect to the pullback closure operator induced by the HCOMP-reflection is equivalent to density with respect to the topology. We do not know, however, whether or not the pullback closure operator in this setting is weakly hereditary, so all we can conclude is that its weakly hereditary core is the usual topological closure.

(3) Sobrification and b-closure. We have already observed in Example 3.5.2 that the pullback closure operator $\Phi_{(S,s)}$ induced by the sobrification reflector $(S,s) : \underline{\text{TOP}_0} \to \underline{\text{SOB}}$ describes the epimorphisms in $\underline{\text{TOP}_0}$. This conclusion could not, however, be drawn from the results of this section as the epimorphisms in $\underline{\text{SOB}}$ are not necessarily surjective. (In Example 2.8.4 we provided an example of a non-surjective b-dense $\underline{\text{SOB}}$-morphism – the inclusion of $\mathbf{N}$ with discrete topology in $\mathbf{N} \cup \{\infty\}$ with the upper topology.)

(4) Epimorphisms in subcategories of $\underline{\text{TOPGRP}_0}$. The long-standing conjecture that the epimorphisms in $\underline{\text{TOPGRP}_0}$ are the maps which have dense range was recently shown in [Uspenskij 1994] to be false. (For the history of this problem see [Nummela 1978] and [Uspenskij 1994].) The situation in $\underline{\text{TOPGRP}_0}$ is similar to that in $\underline{\text{TYCH}}$ – where a pullback operator describes the epimorphisms – notably we have the following facts.

- The full subcategory $\underline{\text{HCGRP}}$ is epireflective in $\underline{\text{TOPGRP}_0}$, with epireflector the Bohr compactification. This reflector preserves arbitrary products (cf. [Holm 1964] and [Hušek, de Vries 1987]).
- The epimorphisms in $\underline{\text{HCGRP}}$ are the surjective continuous homomorphisms (cf. [Poguntke 1970]).

Let (R,r) denote the Bohr compactification in $\underline{\text{TOPGRP}_0}$. $\mathcal{E}$ is the class of surjective continuous homomorphisms. With the above information, we can conclude from Proposition 3.6.2 that any epimorphism in $\underline{\text{TOPGRP}_0}$ is $\Phi_{(R,r)}$-dense. Let K denote the topological closure. Since any dense continuous homomorphism is an epimorphism, it follows that $K \sqsubseteq \Phi_{(R,r)}$. Uspenskij's example shows that in fact $K \sqsubset \Phi_{(R,r)}$.

Thus we have in $\Phi_{(R,r)}$ a closure different from the usual closure, and every epimorphism is $\Phi_{(R,r)}$-dense. Unfortunately, however, the converse is not true. Let G be the group of 2×2 matrices with determinant 1, and topology inherited from $\mathbf{R}^4$ (cf. [Preuß 1972] p.369). G is a Hausdorff topological group with trivial Bohr compactification, thus (R,r) is not monomorphic (so Corollary 3.6.3 does not apply), and the embedding of the trivial group in G is $\Phi_{(R,r)}$-dense without being epimorphic ([Hušek 1995]).

When restricted to some subcategories of $\underline{\text{TOPGRP}_0}$, (R,r) is pointwise monomorphic. This is the case for example in the subcategory of locally compact Abelian groups ([Holm 1964]), so it follows that there the epimorphisms are exactly the $\Phi_{(R,r)}$-dense continuous homomorphisms ((R,r) being restricted to the subcategory). It is known that in this subcategory the epimorphisms are K-dense ([Uspenskij 1994]), so here $\Phi_{(R,r)}$-density coincides with K-density.

3.7 Problems

3.7.1 We mentioned in 3.1.1 that we know of no strong functorial completion that is not a reflection. An example of a non-reflective strong functorial completion would certainly enrich the theory.

3.7.2 Theorem 3.3.1 demonstrates that the pullback closure induced by a strong functorial completion (R, r) holds enough information to characterise the $\mathrm{Fix}(R, r)$ objects. This leaves the following problem open.

Problem D: What correspondence can be set up between strong functorial completions and the pullback closure operators they induce? Is it possible to characterise those closure operators that correspond to strong functorial completions in this way?

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