

**Option Pricing Models with Jumps in the Context of  
the JSE's Top40 Index**

by

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To my Dad.

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## Abstract

In an attempt to provide a more realistic model of stock price processes, jumps are added to the geometric Brownian motion process on which the Black and Scholes model is based. The addition of jumps to this process has a significant effect on the resulting implied volatility smile. The improvement of adding jumps to the underlying process is investigated in this dissertation in relation to the Top40 index traded on the Johannesburg Stock Exchange. As a comparison, two other models are investigated – the pure-jump Variance Gamma model and the model of Britten-Jones Neuberger which places bounds on option prices.

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# List of Abbreviations

B&S	Black and Scholes (Model)	p.1
MJD	Merton's Jump Diffusion (Model)	p.2
JSE	Johannesburg Stock Exchange	p.2
GBM	Geometric Brownian Motion	p.7
RN	Risk-Neutral	p.8
EMM	Equivalent Martingale Measure	p.9
SDE	Stochastic Differential Equation	p.11
PIDE	Partial-Integro Differential Equation	p.29
VG	Variance-Gamma (Model)	p.30
BJN	Britten-Jones Neuberger (Model)	p.33

# Chapter 1

## Introduction

The Black and Scholes (B&S) model of option prices first published in 1973 had a significant effect on the pricing of derivatives in the financial sector. This was largely due to its simplicity and the fact that the model results in a single price. This single price is a necessary consequence of one of its main assumptions – i.e. that arbitrage does not exist. Another one of its main assumptions is that the size of the changes in the logarithm of the returns of the underlying are determined by sampling from the normal distribution. The result of this is that there is no excess kurtosis in this model. This is a flaw since the log-returns on virtually all securities traded on unconstrained markets do show significant excess kurtosis. One of the reasons for the observed heavy-tailed distribution in log-returns of securities is that crashes, to varying degrees, occur in the market on a systematic basis.

Another important assumption of the B&S model is that the volatility of the underlying is constant (where the volatility is the standard deviation of annualised returns). If the market prices matched the model prices, the volatilities implied by market prices would also be constant. However, to a large extent, the implied volatilities observed from market prices are not constant. If the model can somehow match the excess kurtosis that is observed in the log-returns of the underlying, the observed smile can also be accounted for

to some extent. The reason that the smile is not entirely accounted for is that another factor drives the presence of the smile which cannot be captured by simply modelling the returns in the underlying correctly. This is the fear amongst market traders of large and sudden crashes in the market. Such a large crash occurred in 1987 which had a significant effect on the steepness of the smile. This steepness has remained fairly consistent since then which confirms that the fear of large crashes has not lessened.

Various attempts have been made to find a better model of the changes that occur in the returns of traded securities which will give more market-consistent option prices, without allowing arbitrage opportunities. Although no model has been able to supersede the B&S model, one model that accounts for some of these deviations is the jump diffusion model which was first put forward by Merton in 1976 (subsequently referred to as the MJD model). This model incorporated jumps in the underlying which were random in frequency and amplitude. The timing of the jumps were driven by the Poisson distribution while the size of the jumps were lognormally distributed. By using an equilibrium approach, Merton was able to find quite a simple option pricing formula. The equilibrium approach meant that the jumps in the individual security being considered was not correlated to jumps in the rest of the market. This may have been a fair assumption at the time this paper was published, but it is not particularly reasonable following the large market-wide jumps that have been observed subsequent to 1987. However, the market price of jump risk that many researchers feel is the cause of the steepness of the smile can be captured in the jump parameters used in the MJD model. One of the aims of this dissertation is to investigate how the MJD jump parameters change when they are fitted to the log-returns in the underlying to when they are fitted to the traded options prices. The security that will be used in this analysis is the Top40 index which is actively traded on the Johannesburg Stock Exchange (JSE).

A very important result of the B&S approach is that their assumption that the continuous-time random process which is driven by the normal distribution leads to the fact that all

option prices can be perfectly replicated by a specific portfolio. This portfolio consists of a correct weighting between the risk-free bond and the underlying security. In a complete market setting, unique prices can be obtained which is convenient if a single price is required. Adding jumps to the underlying process, however, moves one away from the complete market setting. This is purely due to the fact that a perfectly replicating portfolio can no longer be set up as one cannot hedge across a random-sized jump that may occur at any time. Unique prices are therefore not possible and bounds will need to be placed on the option prices.

This dissertation will begin with a literature review of the jump diffusion model. This will include a discussion of fundamental underlying option-pricing principles and a short review of models that lead to incomplete markets due to the presence of jumps. The jump diffusion model will then be evaluated by comparing it with the benchmark B&S model, both from the point of view of fitting the model to the log-returns of the underlying and also in terms of fitting the generated smile to that observed for the traded options. All analysis in this dissertation will be done in relation to the Top40 and futures options on this index.

In terms of jump processes, a model that has drawn much interest and has been shown to be successful in terms of fitting market option pricing data is the Variance Gamma model. This model was first introduced by Madan and Seneta [62]. This model will be discussed in Chapter 4 and its ability to fit market data will be investigated and compared with the jump diffusion model.

A particularly interesting model presented by Britten-Jones and Neuberger (1999) investigates the bounds that can be obtained in an incomplete market setting which is due to the presence of jumps. Chapter 5 of this dissertation will investigate and discuss this model and briefly consider its performance in relation to the smile of options on Top40 futures. Conclusions and suggestions for further research shall follow this discussion.

Relevant Matlab coding can be found on the included CD-Rom. The programmes provide implied volatility smiles for the main models discussed in this dissertation (i.e. Merton's jump diffusion model, the variance gamma model and the Britten-Jones and Neuberger model). Parameters can be changed by the user in order to see the effect on the implied volatility smile. Computation speeds for the different models are discussed in the relevant sections.

## Chapter 2

### Literature Review

This literature review will begin with a discussion of the concepts behind the important Risk-Neutral pricing approach and complete markets. This theory forms the backbone to the seminal 1973 B&S pricing approach. Since Merton's 1976 jump diffusion model builds onto this approach, a discussion of these fundamental concepts is required. Merton's jump diffusion model will then be discussed in a separate section within this literature review due to its significant relevance to this dissertation topic. A review of important literature will be divided into two sections: those that are placed in complete markets where unique prices can be obtained and those that are placed in incomplete markets where the price must be placed within bounds.

# 1 Risk-Neutral Pricing and Complete Markets

Options have been traded since the 17th century<sup>1</sup> and there has therefore been a long-standing demand for good mathematical models that, not only match closely the paths of the underlying price process, but do not allow the possibility of arbitrage. An arbitrage opportunity can be defined as a financial opportunity where (1) the cost of entering into the opportunity is zero, (2) there is zero chance of a loss, (3) there is some chance of a profit [49].

In terms of modelling the returns in stock prices, the model that has shown the most success has been Geometric Brownian Motion. In 1828, Robert Brown was the first to observe that pollen seeds suspended in water exhibited an incessant, irregular swimming motion. In 1905, Einstein showed that the random motion of molecules on larger suspended particles corresponded precisely with Brownian Motion. In terms of finance, the generated paths *look* very similar to the paths of prices of securities that are traded on markets around the world. In 1900, 5 years before Einstein's paper, Bachelier used Arithmetic Brownian Motion to model options on French government bonds [68] – i.e.,

$$dS_t = \mu dt + \sigma dz(t) \tag{2.1}$$

Here,  $S_t$  denotes the price of the security,  $\sigma$  its percentage volatility and  $\mu$  is the drift of the underlying security. Both these parameters are assumed to be constant. The change in the underlying is driven by  $dz$  which is distributed  $\sim N(0, dt)$ . In 1923, Norbert Wiener used Fourier series to mathematically construct Brownian motion. Although this model showed some real potential for pricing options, its shortcoming was that the resulting theoretical paths of the stock price were allowed to reach negative values which is not possible in actual markets. Regardless of its potential, this approach went unnoticed by financial economists for more than half a century. In 1965, Samuelson modified Bachelier's model

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<sup>1</sup>Earliest evidence of the use of options were on Greek olive presses around 600 BC. Options on tulip bulbs in Holland in about 1600 which arose out of a surge in speculation of their prices has also been documented.

and introduced *Geometric* Brownian Motion (GBM) to model the returns on stock prices, i.e.,

$$dS_t = \mu S_t dt + \sigma S_t dz(t) \quad (2.2)$$

where the parameters are the same as (2.1). Apart from the fact that GBM paths can never become negative, it also has strong economic reasons for being a successful model of stock prices. The GBM model is consistent with stock splits, where if the number of shares increase by a certain percentage, the stock price changes by an inverse of the same amount *in percentage terms*. This is in contrast to interest rate paths, where changes are in absolute terms and they can become negative. Samuelson's option pricing model required two parameters. These were the expected rate of return of the underlying and the rate at which the option's value at exercise should be discounted back to the pricing date. These two factors depended on the unique risk characteristics of the underlying and the option. Neither factor is observable in the market. The value of these parameters will depend on the risk preference of the individual trader. Consensus on the price could therefore not be obtained. However, in 1973, B&S were able to get around this problem via a different approach. They were able to *prove* that, using GBM to model the underlying prices, the resulting prices are free from arbitrage. In addition to this, the only parameter that requires estimation is the volatility of the underlying - all the other parameters are observable in the market. It is this connection that made their approach so successful. The resulting option pricing formula is also a simple one which is found in closed-form and the prices obtained matched (especially at that time) very closely the prices of options traded on markets. Despite its many shortcomings, no other model has been able to influence the derivative pricing field as much as the B&S model.

In terms of pricing options, suppose that  $f$  is the price of a call option or any other derivative contingent on  $S$  (i.e.  $f = f(S, t)$ ). Applying Ito's Lemma to  $f(S, t)$  yields,

$$df = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \mu S + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \frac{\partial f}{\partial S} \sigma S dz. \quad (2.3)$$

Under these dynamics, the change in the option value is also driven by the *same*  $\sim N(0, dt)$

noise,  $dz$ , used in (2.2). As discussed in the seminal paper by Harrison and Kreps [46], it was shown that this noise can be eliminated by creating a portfolio that is weighted correctly in terms of the amount of underlying and risk-free bonds held. Therefore, because the noise has been eliminated, a riskless portfolio is created. It must be noted that correct weighting must occur as often as changes in the underlying model occur. Since the model is set in a continuous time environment, this must occur over infinitesimally short periods of time. This means that the loss or gain from the underlying will perfectly offset the gain or loss from the option value at the end of the infinitesimally short discrete time step, with the value of the portfolio being known with certainty. The portfolio will therefore have zero market risk (i.e. zero *beta*) and its expected return will therefore equal to the risk-free interest rate. Should this not hold, arbitrage opportunities will be introduced. It was shown in Harrison and Kreps [46] that a portfolio can be constructed which perfectly hedges the trader's obligation at maturity. These authors called this a *replicating portfolio*.

The concept of *complete markets* was introduced by Harrison and Pliska [47]. Option prices in these markets can be perfectly replicated by a riskless portfolio. It is assumed in general that, although there are sometimes arbitrage opportunities (e.g. the trader has calculated an incorrect option price), markets are free from arbitrage. This is a fair assumption due to the fact that arbitrage opportunities that may exist are considered rare and fleeting (they are rapidly closed out by market traders). From a hedging perspective, this needs to be the case or else there will be strategies leading to infinite riskless profits. If the model did allow arbitrage, it would indicate a fundamental flaw with the model [35].

As soon as an option cannot be perfectly replicated, incompleteness is introduced. This can occur in various ways [35]: (1) moving away from continuous time and into discrete time finance, (2) introducing transaction costs, (3) not enough traded assets available, (4) trading restrictions limit the class of portfolios that can be constructed and (5) introducing jumps.

The idea of Risk-Neutral (RN) Valuation was first introduced by Cox, et al. [31]. The

pricing problem using martingale theory and formalising the Risk-Neutral approach was developed by Harrison and Kreps [46] and then further by Harrison and Pliska [47]. In what follows is a brief background to RN-pricing which is fundamental to current option pricing theory. More comprehensive discussions in relation to RN valuation can be found in references such as Bingham and Kiesel [17].

In terms of a suitable financial market model, a general frictionless model is chosen where investors are allowed to trade continuously up to some finite horizon  $T$ . Uncertainty in the financial market is modelled by a probability space,  $(\Omega, \mathcal{F}, \mathbb{P})$  and a filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  which satisfies the usual conditions of right-continuity and completeness.

A self-financing trading strategy  $\phi$  is called an arbitrage opportunity if the wealth process  $V_\phi$  satisfies the following set of conditions:

$$V_\phi(0) = 0, \quad \mathbb{P}(V_\phi(T) \geq 0) = 1, \quad \text{and} \quad \mathbb{P}(V_\phi(T) > 0) > 0$$

Arbitrage opportunities represent the limitless creation of wealth through risk-free profits and thus should not be present in a well-functioning market.

The main tool in investigating arbitrage opportunities is the concept of an Equivalent Martingale Measure (EMM). A probability measure  $\mathbb{Q}$  defined on  $(\Omega, \mathcal{F})$  is an EMM if,

- $\mathbb{Q}$  is equivalent to  $\mathbb{P}$
- the discounted price process  $S$  is a  $\mathbb{Q}$ -local martingale

The Fundamental Theorem of Asset Pricing states that, for a financial market model with bounded prices, there exists an EMM if and only if the no-arbitrage condition holds.

The EMM approach means that it is possible for any other event to change in probability (i.e. events that don't occur with probability 1 or 0). An entirely different path with an associated different probability of occurring can be chosen and this new path will have

the *same underlying properties* as the original path. Conveniently, this preservation of properties under a change of measure applies to Brownian Motion.

The volatility can be estimated from a given path which is what makes the B&S pricing formula so successful. The cost of hedging is determined by the amount of vibration and the amount of vibration is assumed to be always the same. From a hedging perspective, this can be interpreted as saying that all paths are the same – there are no lucky or unlucky paths. One can be totally indifferent to which path occurs which is ideal since there is only one path to observe – i.e. the historical stock price process. Also, because of this indifference to which path is taken, the measure can be changed so that the new path drifts at the rate of the riskless bond. By making this change, the risk preferences of the individual traders will not even enter the pricing equation and therefore the need to determine any of the traders' risk preference in relation to the rest of the market is bypassed. This is very beneficial since measuring trader's risk preferences is not a simple exercise. To find the option pricing formula, the expectation of the payoffs under this measure (which is now been placed in what is referred to as the *Risk-Neutral* world), is then taken and the simple B&S pricing formula is obtained. The arbitrage price process of any attainable claim is given by the risk-neutral valuation formula,

$$\Pi(t) = S_0(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right]$$

Because of the effectiveness of this approach, traders will always attempt to price any option in this way. Even if a closed-form solution cannot be obtained (e.g. in the case of exotic or even simple American options where the payoff is path-dependant), Monte Carlo techniques can be used which involves generating *many* paths in the Risk Neutral world, obtaining the corresponding payoff and using the risk-free rate to discount this to find the required price at time  $t_0$ .

*Implied volatility* is a term which will be used throughout this dissertation. It is the volatility that is obtained by turning the B&S equation around – i.e. instead of knowing

the volatility and solving for the price, the price is known and the volatility is solved for by any standard root-finding technique (e.g. Newton-Bailey).

## 2 Merton's Jump-Diffusion Model

This section will describe in more detail Merton's 1976 jump diffusion model as introduced in Merton [67]. Merton observed that the arrival of information that is normally received by the market can be considered to fall into two categories of information:

- **normal news** which is considered more smooth in its arrival and is due to, for example, (1) temporary imbalances between supply and demand, (2) changes in capitalisation rates, (3) changes in economic outlook and (4) other information that causes marginal changes in the stock's value.
- **abnormal news** as a result of the arrival of important new information about the stock that has more than a marginal effect on price. This typically arrives in more discrete time intervals and its effect is more of a one-off shock to the price. Examples of these are the Exxon Valdez oil-spill disaster or news of a large accounting scandal (e.g. Enron).

In order to account for both of these types of movements in traded securities, Merton [67] developed the jump-diffusion model where the asset-price dynamics consist of a diffusion component as well as jumps of unpredictable sizes that occur at random times.

Merton proposed a theoretical model which superimposed discrete jumps over the diffusion process used in the B&S approach. These jumps are random in amplitude and arrival rate. The underlying stochastic differential equation (SDE) for this process can be written as,

$$dS/S = \mu dt + \sigma dz(t) + A(t)dN_t \quad (2.4)$$

Here, all the parameters are the same as in (2.2), apart from  $A(t)$  which is the random amplitude of the jump, and  $dN_t$  which is the increment of a counting process. The counting process  $N_t$  is introduced which represents the number of times a jump has occurred between now ( $t = 0$ ) and time  $t$ .  $N_0$  is therefore set to equal zero. It is assumed that the jump process is Poisson in nature, which provides a suitable model for the arrival of these *excessive* jumps. The smooth variation (or *normal news*) is modelled by the continuous-time diffusion process (i.e. Brownian Motion). If  $\lambda$  is the frequency of the jumps, then the probability of  $k$  jumps having occurred out to time  $t$  is given by,

$$\mathbb{P}[N_t = k] = \exp(-\lambda t)(\lambda t)^k/k! \quad (2.5)$$

For the Poisson processes, the occurrence of a jump is independent of the occurrence of previous jumps, and the probability of two jumps occurring simultaneously is zero. Let it be assumed that, out to time  $t$ ,  $j$  jumps have occurred. Over the next small time interval,  $dt$ , another jump will occur with probability  $\lambda dt$  and no jumps will occur with probability  $1 - \lambda dt$ . The *change* in the counting process will therefore be 1 with probability  $\lambda dt$  and 0 with probability  $1 - \lambda dt$ . Therefore,

$$\mathbb{E}[dN_t] = 1 \cdot \lambda dt + 0 \cdot (1 - \lambda dt) = \lambda dt.$$

An arbitrary function  $f(S)$  of the stock price is now considered. If the dynamics of the underlying are of a mixed jump diffusion nature, the SDE will contain the Ito-like component in (2.3) and a new term arising from the jump events,

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2 + [\Delta_A(f)] dN_t \quad (2.6)$$

The  $\Delta$  symbol in the last term is a differencing operator whose action on the argument is to produce a jump of magnitude  $A(t)$  in the variable  $S$ :

$$\Delta_A(f) = f(S_t + A, t) - f(S_t, t)$$

Equation (2.6) can be rewritten as,

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2 + [f(S_t + A, t) - f(S_t, t)] dN_t \quad (2.7)$$

Some assumptions are now made about  $A(t)$ . After a jump occurs, the stock price moves from  $S_t$  to  $S_t + A(t)$ . Let it be assumed that the quantity  $A(t)$  is of the form,

$$A(t) = S_t(Y - 1)$$

This means that the change in the stock level before and after the jump is given by

$$S_{\text{after}} - S_{\text{before}} = S_{\text{before}}(Y - 1)$$

i.e. the incremental change in the stock price is scaled up or down depending on the value of  $Y$ . This is particularly appropriate in the case of equity and forex markets, where the jumps in the index are meaningful in terms of the relative, rather than their absolute magnitude<sup>2</sup>. If  $Y$  is greater than 1, the jump will be upwards; less than 1 will be downwards. It must also be noted that  $Y$  can never be negative or else the stock can go negative which is never possible. The process in (2.4) therefore becomes,

$$dS/S = \mu dt + \sigma dz(t) + (Y - 1)dN_t \quad (2.8)$$

As discussed earlier, in order to price options under this process, an EMM needs to be found and discounted expectations are then taken under this measure. In order to find the EMM, the expected change in  $S$  is considered over a small time period,

$$\mu S \Delta t + \mathbb{E}(Y - 1)\lambda S \Delta t$$

which follows from  $\mathbb{E}(\Delta N_t) = \lambda \Delta t$  as shown earlier. For the ratio  $Se^{-rt}$  to be a martingale,  $S$  must grow at the constant risk-free rate,  $r$ . The expected change must therefore be  $rS \Delta t$  which means that the drift of the process must be

$$\mu = -\mathbb{E}(Y - 1)\lambda + r \quad (2.9)$$

The price of a European call option at expiry  $T$  is obtained by evaluating

$$e^{-rT} \mathbb{E}(S_T - K)^+$$

---

<sup>2</sup>This follows the same reasoning as Samuelson's introduction of GBM in 1965. If there is a stock split or an index gets revalued, then nothing has fundamentally happened here. However, the same is not true of interest rates where the absolute level does have an intrinsic meaning [33].

where  $S_T$  is evolved as per (2.8), but with drift as per (2.9). Girsanov's Theorem can be used to make this required change of measure.

The properties of  $Y$  can be suitably described as a log-normally distributed variable with expectation  $\theta$  and variance  $\delta^2$ , i.e.,

$$\begin{aligned}\mathbb{E}[\ln(Y)] &= \theta \\ \text{var}[\ln(Y)] &= \delta^2\end{aligned}$$

A benefit of assuming that the variable  $Y$  is lognormally distributed is that, if the log is evolved, the interaction between the Brownian part and the jump part will be kept separate. Using Ito's Lemma on  $f = \ln(S)$  and using the previous definitions  $A(t) = S_t(Y - 1)$  (or  $A(t) + S_t = Y S_t$ ), from (2.7),

$$\begin{aligned}d\ln(S) &= \frac{1}{S}dS - \frac{1}{2} \frac{1}{S^2}(dS)^2 + (\ln(S + A) - \ln(S))dN_t \\ &= \mu dt + \sigma dz - \frac{1}{2}\sigma^2 dt + (\ln(S + A) - \ln(S))dN_t\end{aligned}\quad (2.10)$$

Using standard properties of logarithms, the term  $[\ln(S + A) - \ln(S)]$  can be written as

$$[\ln(S + A) - \ln(S)] = [\ln(SY) - \ln(S)] = \ln(Y)$$

and therefore, (2.10) can be rewritten as,

$$d\ln(S) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz(t) + \ln(Y)dN_t.$$

The solution to this equation is,

$$\begin{aligned}\int_0^T d\ln(S) &= \ln(S_T) - \ln(S_0) \\ &= \int_0^T (\mu - \frac{1}{2}\sigma^2)ds + \sigma dz(s) + \ln(Y)dN_s \\ \Rightarrow \ln S_T &= \ln S_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma Z(t) + \sum_{j=1}^{N_T} \ln Y_j\end{aligned}\quad (2.11)$$

where  $Z(t) \sim N(0, t)$  and  $Y_j$  are independent log-normally distributed variables. This means that the terminal distribution of  $\ln S$  is,

$$\ln S_0 + (\mu - r)T + (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z_0 + \sum_{j=1}^{N_T} \ln(\exp(\theta - \frac{1}{2}\delta^2 + \delta Z_j)) \quad (2.12)$$

where  $Z_j$  are independent draws from  $N(0, 1)$ . All the normal distributions can be grouped together to obtain,

$$\ln S_T = \ln S_0 + (\mu - r)T + (r - \frac{1}{2}\sigma^2)T + \quad (2.13)$$

$$+ (\theta - \frac{1}{2}\delta^2)N_T + \sqrt{\sigma^2 T + N_T \delta^2} Z_0 \quad (2.14)$$

where  $Z_0$  is a standard normal variable. If  $N_T$  is fixed, this is the terminal distribution of a stock in a B&S world with volatility,

$$\hat{\sigma} = \sqrt{\sigma^2 + N_T \delta^2 / T}$$

and initial spot,

$$S_0 e^{(\mu-r)T} e^{(\theta - \frac{1}{2}\delta^2)N_T}$$

Integrating over the possible values of  $Z(t)$ , a B&S price for the option is obtained, but with initial spot and volatility as specified above. Along with the distribution for  $N_T$ , the formula for the price of a call option is given as,

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T} (\lambda' T)^n}{n!} BS(S_0, \hat{\sigma}(n), \hat{r}(n), T, K) \quad (2.15)$$

where

$$\hat{\sigma}(n) = \sqrt{\delta^2 + n\delta^2/T}$$

$$\hat{r}(n) = r - \lambda(\exp(\theta + \frac{1}{2}\delta^2) - 1) + n(\theta + \frac{1}{2}\delta^2)/T$$

$$\lambda' = \lambda \exp(\theta + \frac{1}{2}\delta^2)$$

and  $BS(\cdot)$  is the standard B&S formula. The Matlab code for the MJD model can be found in the included CD-Rom (see file *Merton-prices*). As shown in Ball and Torous [8], the

series in (2.15) can be truncated after  $n = 10$  while still maintaining double precision in the option prices. With a CPU speed of 447MHz, Matlab takes 0.05 seconds to evaluate an option price. Section 3.2 discusses in more detail how the implied parameters from the option prices are obtained and how long it takes for these parameters to be backed out.

### 3 Incompleteness in Markets with Jumps

One of the assumptions of the B&S model is that trading takes place continuously. Although in the real world, trading does not take place continuously, it would be unreasonable to put aside the B&S formula based on the fact that continuous trading is purely theoretical and that no empirical time series has a continuous sample path [67]. In Merton and Samuelson [69], it was shown that the continuous-trading solution approximates the discrete-trading solution asymptotically. When the stock price dynamics are not modelled by a stochastic process with a *continuous sample path*, the B&S solution is not valid – even in the continuous limit. Any ‘jump’ stochastic process defined in continuous time would cause problems for this continuous trading assumption. Including these jumps would allow a probability of a stock price change of extraordinary magnitude, even if the time interval between successive observations was reduced to zero.

Despite the potential that the MJD shows as an improvement over the B&S model, there are technical difficulties with this model. The problems with jump-diffusion processes are (1) the impossibility to build a riskless portfolio, (2) the difficulty with parameter estimation and (3) the difficulty in obtaining closed-form solutions to pricing of options. In order to deal with these difficulties, various authors have taken the following approaches:

- Assume that the jump-risk is non-systematic (i.e. they are not correlated with the market portfolio) and so the underlying grows at the risk-free interest rate (Merton [67]).

- Find the minimum variance of the portfolio for hedging and valuation purposes (Davis [35]).
- Specify a utility function of the investor (single agent optimality of a detailed equilibrium description) (Naik and Lee [71]).
- Use a dynamic programming with an exogenous risk-adjusted discount rate, or with a ‘market estimated’ discount rate as a proxy (El Karoui and Quenez [40]).
- Follow Merton’s 1976 assumption but reflect the market price of risk through the jump parameters (Andersen and Andreasen [3]).

In terms of the last item, the *market price of risk* is a feature of incomplete markets, where a price is attached to the possibility of a jump occurring. In real markets, this will not remain constant as it depends on the risk preferences of the market participants which changes as new information arrives. As mentioned in Section 2.1, obtaining the market price of risk is not required in the B&S world since the risk preference of the traders does not even feature in the option pricing equation. In Merton’s 1976 model, the price of risk associated with the jump part of the process is taken as zero. As discussed previously, this assumption does not hold very strongly and, in the context of the market which contains extraordinary jumps, the market price of risk can be calculated through various techniques. The way jump parameters in the MJD model could reflect the market price of risk is outlined in Andersen and Andreasen [3]. This is done by fitting the volatilities implied by the MJD model to the market volatilities. The results of applying this technique to options on the Top40 futures will be discussed in Section 3.2.

## 4 Volatility Smiles

Before the crash of 1987, the implied volatility across strike-stock price ratios was fairly constant [75]. This meant that there was no excessive *relative* demand for either deep in-, at- and out-the-money options and the resulting prices were quite consistent with the constant-volatility model of B&S. However, when the crash of 1987 occurred, there was suddenly a high demand for deeper out-the-money put options which served as protection against sudden downward jumps. The subsequent demand for these options increased the price of these options *relative* to those at- and deep in-the-money. This relative increase in put options is carried over to deep in-the-money *call* options which can be shown by using the put-call parity [49]. This argument is valid for market as well as model prices.

Merton describes the effect on the option price, should the trader assume that there are no jumps in the market and that the B&S pricing solution is correct. The trader would use historical volatility of the underlying and the jumps would be incorporated in the volatility of the underlying. In general, for deep out-the-money options, there is relatively little probability that the stock price will exceed the exercise price before expiration if the underlying process is continuous (i.e. no discrete jumps). However, including the possibility of a large, definite jump in price significantly increases this probability, and hence makes the option more valuable. Similarly, for deep in-the-money options, there is relatively little probability that the stock would decline below the exercise price prior to expiration if the underlying process is continuous. The ‘insurance’ value of the option would be therefore be virtually zero. However, this changes if the possibilities of a discrete jump are incorporated. These differences will be magnified as one goes to short-maturity options [67].

Stylized facts of option prices can be summarised as follows:

- Leptokurtosis: Asset log-returns are not normally distributed, but exhibit a substantial degree of excess kurtosis [67]. Kurtosis is the relative size of the outliers to

the variance of the process. Skewness in the log-returns also results in a deviation from the assumption of normally distributed log-returns. Skewness and kurtosis are defined below:

$$\beta = \frac{\mathbb{E}((X_t - \mathbb{E}X_t)^3)}{(\mathbb{E}((X_t - \mathbb{E}X_t)^2))^{3/2}}$$

$$\kappa = \frac{\mathbb{E}((X_t - \mathbb{E}X_t)^4)}{(\mathbb{E}((X_t - \mathbb{E}X_t)^2))^2}$$

- Clustering: Financial time series alternate between periods of consistently low and high volatility [42]. This variation of volatility can be linked to the arrival of information and high trading volumes.
- Long Memory: Despite the abovementioned clustering effect, a high degree of persistency of the volatility towards a certain average does exist [5].
- Leverage effects: Black [19] suggested that volatilities and asset returns are negatively correlated. Falling stock prices are assumed by agents to entail more uncertainty, and therefore volatility.

Kurtosis results in fat-tailed distributions which produce the well-known ‘smile’ effect where options which are away from the money trade at higher prices than those which are at the money. The smile is the plot of implied volatilities from a range of options of the same maturity across different strike prices. For a symmetric smile, it is noticed that the options at-the-money seem to trade at the lowest implied volatilities, and the in- and out-the-money options seem to trade at the highest volatilities. Since the options are all written on the same underlying variable, there should be no plausible reason for this, other than the fact that there is a fundamental difference between the market and the model. This observed difference is consistent with fatter tails in the actual distribution of many equity returns than that assumed by the GBM process as specified in (2.2). When plotted against the strike price, the graph of implied volatilities appears U-shaped as a ‘smile’. A one-sided smile is called a ‘smirk’ which is caused by call options which are deeper in-the-money having a higher volatility than options which are deeper out-the-money. This type of smile

has been shown to be quite typical for options on equity and also on foreign exchange between developing and developed markets [75].

A significant assumption that Merton made in his 1976 jump diffusion model was that stock-price jumps were a diversifiable risk and that risk-aversion of the market traders would not affect the frequency of the jumps in the model. This is effectively saying that the jump intensity associated with the pricing measure should be the same as that of the underlying. Although Merton's assumption of undiversifiable risk could be justified when he first developed and published his model in 1976, its justification is no longer possible considering the crashes that have occurred since 1987. These crashes had an effect on the whole market and cannot therefore be considered as undiversifiable risk. The very strong downward sloping smirks present in the equity markets reflects an increased risk aversion to downward jumps amongst market participants.

Very briefly, the nature of smiles can be divided into two categories. Implied volatility is always a function of strike price. However, smiles do change depending on how the initial stock price changes. Smiles can depend on strike and stock price in the following way:

$$\sigma(K) \quad \text{and} \quad \sigma(K/S)$$

where  $K$  is the strike price and  $S$  is the spot price. In the first case, the smile is only a function of strike and does not change if  $S$  changes – these smiles are called *sticky* [56]. If, for a given  $K$  and  $S$  which corresponds to an at-the-money ratio, we are at the bottom of the smile and both  $K$  and  $S$  are increased so that we are still at-the-money, we will move up the sides of the 'U'. This is because the smile does not depend on  $S$  but only on  $K$ . These smiles are more applicable to interest-rate markets than equity markets [33].

The second case is called *floating* which provides a better fit for the behaviour of equity smiles [56]. The risk-aversion of traders against the potential of sudden downward movements has a significant effect on the shape of these smiles. These sudden downward movements are always relative to the *initial spot price* and not some other price closer

to maturity. There is some evidence that that implied volatilities increase when spot decreases which suggests that there is a sticky component to equity smiles [56]. This could be explained by the fact that a lower stock price brings a corporation closer to bankruptcy which will make the stock more risky and lead to higher volatility in the underlying and hence also in the volatility of the option on that underlying.

Equity cases are (by and large) floating [75]. The model under consideration should therefore generate a future smile surface which approximately maintains its shape as a function of the future level of at-the-moneyness. The MJD model implies deterministic future smiles which float perfectly [56]. This actually implies a flaw with the model since future smiles are not deterministic, even though there is a high degree of confidence based on the behaviour of the smile to date that they may remain the same.

Skewness results in an asymmetric distribution in the asset's log-returns, which often makes it difficult for symmetric stochastic processes to fit both out-the-money calls and puts [32]. If one is fit well, then the other will usually not match existing parameter choices, producing a smirk. By using three additional jump parameters, Das [32] showed that options can be priced to be consistent with an arbitrary range of distributional shapes providing for consistency with both skewness and kurtosis.

The following two sections will describe important literature that relates to jumps in the option pricing theory. Literature that relates to jumps can be divided into two parts: those that can be placed in complete markets where a unique price can be obtained, and those where an incomplete market setting is created and bounds are placed on the option prices. The tightness and accuracy of these bounds will depend on the approach of the model and on the underlying assumptions.

## 5 Obtaining Unique Option Prices in Models with Jumps

### 5.1 Theory Relating to Jumps

Although the assumptions behind the B&S model have been questioned by market participants since its first publication in 1973, their approach had a significant effect on the world of derivative pricing. *Many* improvements to the B&S model were attempted following the publication of their seminal paper. However, the focus on improvements to the B&S model was intensified after the crash of 1987 when smiles in implied volatility became much more pronounced. One way of accounting for the smiles was by making the volatility of the underlying stochastic (i.e. random and no longer constant). Generalised volatility models allow the volatility to take on some general form (i.e.  $\sigma = \sigma(t)$  in (2.2)) which can be deterministic in time or even stochastic (which is often chosen to be mean-reverting). This is aimed at providing the desired levels of skewness and kurtosis. The stochastic volatility approach was first developed by Scott [78], Hull and White [50] and Wiggins [83] and later expanded on by Heston [48]. Although these models did show an improvement in terms of matching observed smiles, for reasonable choices of coefficients they tend to produce large smiles for long maturities and shallow smiles for short maturities. Many studies have shown that this is, in fact, not the case (Das and Sundaram [34], Jorion [55], Bates [12] and Ball and Torous [8]). This inability to match the smiles that were being observed in the market led to research into other ways of accounting for the smiles. Adding jumps to the underlying was considered to be a practical and sensible enhancement of the stochastic volatility model.

As early as 1976, Cox and Ross [30] developed an option-pricing model for a jump process *without* a diffusion term. In this model, the only relevant source of uncertainty is the time of the jump. Jump sizes were made to be fixed or predictable in magnitude and there

was no instantaneous uncertainty about the direction of the jump. Since there is only one source of uncertainty (i.e. the jump times), there exists a self-financing strategy consisting of the underlying asset and the riskless bond which replicates the payoff on a given option. The value of the replicating portfolio is known at all times and the price of the option can therefore be determined in closed-form.

Amin [2] developed a discrete time model which superimposed jumps onto the seminal binomial model of Cox, et al. [31]. This provided a model that was a *limiting* jump diffusion process.

Bates [12] built onto Heston's [48] stochastic volatility model by allowing the underlying to jump. It was felt that such a model would be able to generate adequate kurtosis at short maturities (via the jump component) and at moderate maturities (via the stochastic volatility component). By allowing the underlying to undergo discrete jumps, the parameters became much more sensible compared with the stochastic volatility model in order to match the observed levels of kurtosis. However, based on the data analysed, it was not conclusive as to whether the addition of jumps was an improvement over the model without jumps. This approach also leads to quite a complex pricing formula which is not particularly practical for traders to use on a regular basis.

This model was expanded on by Fang [44] by also making the jump intensity rate ( $\lambda$  introduced in (2.5)) mean reverting and stochastic. This is a complex model and there is little confidence that the stochastic jump intensity rate is an improvement (as shown in Sepp [81]). However, this paper did show that a model with a time-independent intensity performs slightly better than one where the intensity is constant.

A simplification of this model was investigated by Maheu and McCurdy [63] in the context of the discrete GARCH model. In their model, an autoregressive conditional jump intensity factor is incorporated which allows for the expected arrival rate of the jumps to vary and also allows for grouping of the jumps. This study found that a good fit to the data could

be obtained and that a rich variety of implied distributions could be generated.

An interesting model was presented by Naik [70] where the *volatility* is subject to random and discontinuous shifts over time. Naik assumed that  $\{\sigma(t)\}$  is a right-continuous Markov chain with left limits and with a rate matrix  $A$ . The volatility process in this model remains in one state for a random amount of time and then shifts discretely to another state. The rate matrix governs the probabilities of transition of the volatility process from the current state to another. By choosing the parameters that make up the rate matrix, different levels of persistence can be modelled in the various volatility states. Correlation between jumps in the volatility and jumps in the underlying can also be incorporated into this model. He considered the case where the volatility risk is assumed to be diversifiable and therefore not priced and he also considered the case where the volatility is undiversifiable. In the latter case, the market is incomplete since there are two sources of noise driving the underlying process (the standard Brownian Motion noise and the Poisson noise which determines when jumps in the volatility occur). For this reason, the market price of risk enters the pricing equation for the latter case.

A similar jump diffusion model was put forward by Duffie, et al. [37] where independent or correlated jumps occur in both the stochastic price and the stochastic volatility. This is known as the double-jump model. Their pricing formulas are known in closed form up to the computation of a one-dimensional inverse Fourier transform. This model was investigated by Eraker, et al. [43] which showed that a remarkable fit can be obtained to various volatility surfaces.

The double exponential jump diffusion model is presented in Kou [59] where the underlying process can be described as,

$$dS/S(t-) = \mu dt + \sigma dz(t) + d\left(\sum_{i=1}^{N_t} (V_i - 1)\right)$$

where  $dz(t)$  is a standard Brownian motion,  $N_t$  is a Poisson process with rate  $\lambda$  and  $\{V_i\}$  is a sequence of independent and identically distributed (i.i.d.) non-negative random variables

such that  $Y = \log(V)$  has an asymmetric double exponential distribution with the density,

$$f_Y = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \quad \eta_1 > 1, \eta_2 > 0,$$

where  $p, q \geq 0$  represents the probabilities of upward and downward jumps, respectively (and hence  $p + q = 1$ ). The requirement that  $\eta_1 > 1$  is imposed to ensure that  $\mathbb{E}(V) < \infty$  and  $\mathbb{E}(S_t) < \infty$ . This essentially means that the average upward jump cannot exceed 100% – a reasonable assumption. The double exponential distribution has two interesting properties. Firstly, it inherently possesses the leptokurtic feature. Secondly, it possesses the memoryless property. This is the property that allows closed-form solutions for various option-pricing problems, including barrier, lookback and perpetual American options to be obtained under this model and not under other models, including Merton's 1976 jump diffusion model.

One of the fundamental assumptions of Merton's 1976 model is that the risk associated with the jump in the underlying is diversifiable – i.e. that any jumps in the underlying asset is uncorrelated with any jumps in the market portfolio. However, this assumption is violated if the security under consideration is the market portfolio itself. The paper by Naik and Lee [71] looks at the case where trading only in one underlying (e.g. a market index) and a riskless bond is possible. They use a general equilibrium framework to derive option prices when the underlying asset is the market portfolio with discontinuous returns. Markets can be made complete by adding another option with a different strike or maturity on the same underlying. However, these authors were interested in what the pricing equation becomes should trading not be possible in other securities. This is essentially accomplished by including the risk preferences of traders in the pricing approach. Naik and Lee are forced to use a general equilibrium argument in their paper since they show that a model with independent diffusion and jump uncertainties cannot be priced simply by a no-arbitrage argument.

This dissertation will be evaluating options on the Top40 futures which are traded on the JSE. Although the shares that make up the Top40 index cannot be considered to be the

whole market, it would be interesting to use Naik and Lee's model to evaluate the market price of risk through their risk aversion factor in the context of options on the Top40 futures. In this dissertation, however, Merton's 1976 model will be fitted to the market prices and the market's risk aversion factor will be observed through the resulting jump parameter values. This is discussed further in Section 3.2.

Simplification in the computation of the option value can be achieved by evaluating it as an integral in Fourier space instead of the conventional  $S$ -space, where  $S$  is the terminal security price is described in Lewis [60]. His result applies to any European-style option under any Lévy process with a known characteristic function. Lévy processes are the class of all stationary, independent increment processes. In general, they are a combination of a linear drift, a Brownian motion and a jump process. Two types of Lévy processes with a jump component can be distinguished. Type A (Poisson subclass), we have  $\int_{-\infty}^{\infty} \mu(x)dx < \infty$ . We can then write  $\mu(x) = \lambda f(x)$ , where  $\lambda = \int_{-\infty}^{\infty} \mu(x)dx$  is the Poisson intensity (the mean jump arrival rate), and  $f(x)dx = dF(x)$ , where  $F(x)$  is a cumulative probability distribution. Examples of this subclass are:

- Merton's 1976 jump diffusion model with lognormally distributed jumps:

$$\mu(x) = \lambda f(x) = \lambda \frac{1}{\sqrt{2\pi\delta^2}} \exp[-(x - Y)^2/2\delta^2]$$

where  $Y$  is the same as in (2.8),

- Kou's 2000 jump diffusion model with exponentially distributed jumps:

$$\mu(x) = \frac{1}{2\eta} \exp[-|y - \kappa|/\eta]$$

Type B is the subclass where no Poisson intensity exists ( $\int_{-\infty}^{\infty} \mu(x)dx = \infty$ ) and no Poisson intensity can be defined. An example of this is the Finite Moment Logstable Process

presented in Carr and Wu [25] where,

$$\mu(x) = \frac{c_{\pm}}{|x|^{1+Y}}, \quad c_{\pm} = \begin{cases} c_+, & x > 0 \\ c_-, & x < 0 \end{cases} \quad 1 < Y < 2 \quad (2.16)$$

If  $Y = 2$ , the process is Brownian motion.

## 5.2 Empirical Results

Ever since the publication of B&S paper, much empirical research has been undertaken in order to determine which of their assumptions has the most significant effect on the pricing inaccuracies. Papers that find errors due to stock log-returns deviating from the normal distribution as well as the assumption of constant volatility not holding too strongly find evidence that jumps exist and should be incorporated into the pricing approach. As early as 1967, Press observed that the characteristics of normally distributed jumps which occur at times that are determined by a Poisson distribution show some empirical agreement with the log-returns in security prices.

Many studies have shown that the stochastic volatility models developed by Hull and White [50] and Heston [48] cannot, on their own, account for the implied-pricing distributions found in the option data. Crashes similar to that of 1987 which was largely responsible for the smile becoming more pronounced have continued, to varying extents. It is this fear of sudden changes that has caused the volatility smile to persist [10]. Bates [12] studied this model in the context of options on the deutsche mark and concluded that the stochastic volatility model cannot explain the observed smile for realistic parameters. He went on to show that, by including jumps in the underlying process, the smile could be better explained. Empirical support for the stochastic volatility model was found by Bakshi, et al. [6], but they also showed that jumps are critical near an option's expiry. It was shown by Duffie, et al. [37] that models including a jump are superior to those with only stochastic volatility. A thorough review of the literature available in relation to implied distributions

is provided in Jackwerth [51] and supported the notion that jumps need to be included. It was also shown in Jones [54] that the single-parameter models are inadequate and that volatility cannot adjust to compensate for the volatility smile on its own.

It was shown in Jackwerth and Rubinstein [52] that moves of more than 4 standard deviations and more occur 100 times more frequently than would be predicted by a lognormal distribution. They also suggest that the existence of the smirk in option's implied volatilities is due to investors placing a higher probability on crashes. This supports the Bates' (2001) hypothesis that investors are willing to overpay to reduce the risk of a crash occurring. It appears that investors include this risk of a large jump in their pricing approach and demand compensation for this undiversifiable scenario.

Other studies have examined the effect of including jumps when time to maturity decreases. It was shown in Bakshi, et al. [6] that jump models generate the best fit to observed data. In addition to this, Chernov and Ghysels [26] showed evidence that supports the Heston (1993) model which includes jumps. Empirical studies into Bates' seminal 1996 model include Fang [44] and Duffie, et al. [37].

More recently, there has been an increased interest into jump models for option pricing. A premium for both jumps and volatility was studied separately by Pan [72]. In this paper, the jump risk premium was determined by jointly analysing short-dated option prices with eight or more days until expiration on the S&P 500 index. Pan showed that the jump is correlated to the volatility by showing the interaction between jumps and the volatility smile.

In the context of the degree of risk-aversion on the S&P 500, Aït-Sahalia, et al. [1] investigate the RN density estimated in complete markets to the R-N density inferred from the time series density of the index. They point out that, if investors are risk-averse, the inferred densities will be different. They reject the hypothesis that options on the S&P 500 are efficiently priced given the S&P 500 index dynamics. However, by adding a jump com-

ponent to the index dynamics, they are able to partly reconcile the differences between the index and option-implied R-N densities.

Maximum likelihood estimation techniques are used in Ball and Torous [8], Jarrow and Rosenfeld [53] and Jorion [55] to find the jump parameters in the log-returns of the underlying. Ball and Torous [8] found that the jump parameters for various stocks are not significant over the purely diffusion model. On the other hand, Jorion [55] finds that jumps in the US\$/DM are quite significant, and (to a lesser extent) also in the CRSP value-weighted index. He finds that this significance increases as the sampling time decreases (i.e. jumps in weekly sampled data are more significant than jumps in monthly sampled data). Investigations into jumps in the returns has not been limited to equity. The presence of jumps in the returns of the Fed Funds interest rate are investigated by Das [33] using the same maximum likelihood techniques as the above-mentioned authors to acquire the jump parameters. Jumps in the returns were found to be very significant for this security. Efficient method of moments techniques were used by Andersen and Andreasen [3] to determine the jump parameters in the S&P 500 index and they find that the stochastic volatility jump diffusion process provides an acceptable characterisation for this index.

A Bayesian approach is used by Eraker, et al. [43] to uncover the return jump as well as an implied jump in the volatility. These authors show that it is the jumps, rather than the diffusion, that drives the conditional distribution of the underlying asset. Bates [14] shows that the jump parameters implied by option prices are larger following a Bear market than following a Bull market. The S&P 500 index is investigated by Carr and Wu [25] and they show that the options on this index are priced as if a downward jump is expected. This study is backed up by that of Andersen and Andreasen [3] who come to the same conclusion.

There has also been some focus on the use of Fourier Transform methods to solve the Partial Integro-Differential Equation (PIDE) for European call options in the context of the MJD

model. The PIDE is obtained by using martingale pricing such that the option value is represented as an integral of a discounted probability density times the payoff function. The Feynman-Kač theorem is then applied to find the appropriate PIDE. Implementation of this technique on the S&P 500 is demonstrated by Carr and Madan [23] and Andersen and Andreasen [3], while Sepp [81] considers options on the DAX.

### 5.3 Pure Jump Models

**Variance Gamma (VG).** This model was introduced by Madan and Seneta [62]. In this model, the log-returns on the prices are distributed normally, conditional on a variance that is distributed as a gamma variate. This leads to higher kurtosis (fatter tails) for daily returns, finite moments of all orders and good empirical fit. Log-returns over longer periods approach that of the normal distribution – a feature observed in the market. The process is pure jump which can be approximated by a compound Poisson process with high jump frequency and low jump magnitudes. This approach leads to option prices which are different at-the-money compared to in- or out-the-money. Due to this model's different approach and the fact that close fits to real data can be obtained, it will be given more consideration in Chapter 4.

**The Hyperbolic Model.** An interesting approach in the context of discrete jump processes is that described in the paper by Eberlein, et al. [39]. They investigate a purely discontinuous process where the Brownian motion is completely replaced by the hyperbolic Lévy motion. Smiles that reflect those observed in the market can be generated by this process and the various parameters needed to drive the underlying process can be estimated through maximum likelihood estimation techniques. Due to the fact that the process is purely discontinuous, the market setting is incomplete<sup>3</sup>. For this reason, a drift parameter enters their pricing formula which reflects the risk-preference of the trader. They go on to

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<sup>3</sup>An exception to purely discontinuous processes leading to incomplete markets is the Poisson-driven process as in the Cox-Ross (1976) approach mentioned at the beginning of this section.

compare the hyperbolic model with the B&S model and show some improvements in the performance of the hyperbolic model in the context of a few options traded in the German market.

## 6 Obtaining Bounds on Option Prices in Incomplete Markets due to Jumps

Merton [66] developed a set of restrictions on option pricing formulas under the assumption that investors prefer more to less. The restrictions imposed by Merton are necessary conditions for a formula to be consistent with a rational pricing theory.

In incomplete markets, the bounds on prices that are obtained by merely requiring absence of arbitrage are extremely wide. These bounds are not very practical in deciding on the price of a particular option [18]. Therefore, there is a requirement to develop a method of obtaining reasonable pricing bounds for derivative assets.

Eberlein and Jacod [38] showed that non-trivial bounds do not exist on European claim prices in a model where prices are driven by a purely discontinuous Lévy process with unbounded jumps. The problem of the range of viable prices for mixed diffusion dynamics is investigated by Bellamy and Jeanblanc [15]. Their upper bound is shown to be the trivial one as per Eberlein and Jacod [38] where a long position is taken in the underlying asset. However, their lower bound is not trivial and is shown to be the corresponding B&S function evaluated at the current spot price. They also point out that literature that incorporates jumps must either provide a more satisfying way of pricing the jump risk or to construct, as does Dritschel and Protter [36], a complete model which allows jumps in the stock price dynamics. Setting up a minimal-cost superhedging portfolio that succeeds with a high probability was investigated by Föllmer and Leukert [45]. Their approach is considered to be a dynamic version of the *Value-at-Risk* concept.

The bounds when a jump is present is also explored by Masson and Perrakis [64]. Bounds on options in the context of stochastic control methods is investigated by Karoui and Quenez [57]. Their main result is that the maximum price is the smallest price that allows the seller to hedge completely by a controlled portfolio of the basic securities (i.e. a riskless bond and the underlying). As in Eberlein and Jacod [38], their minimum price also corresponds to the purchase price.

A method of ruling out, not only those prices which introduce arbitrage, but also prices which can be considered to be ‘deals which are too good’ was introduced by Cochrane and Saá-Requejo [29]. They define a good deal as an asset price process with a high (bounded) Sharpe ratio and set about finding the upper and lower bound for all arbitrage-free price processes. Their setting was that of a diffusion model driven by a multidimensional Wiener process. Within this framework, they derive a pricing PDE which is then studied in detail and, in some cases, solved numerically. A similar approach to obtaining asset price bounds, based on gains-loss-ratios, is presented in Bernardo and Ledoit [16]. The analysis of Cochrane and Saá-Requejo [29] is currently being extended by Bjørk and Slinko [18] by incorporating the possibility of jumps in the random process model that is used to describe the financial market under consideration. They use numerical techniques to obtain bounds in a jump diffusion setting and show where the MJD model fits within these bounds.

A way of incorporating the risk preferences of the traders into the pricing of the option is demonstrated by Carr, et al. [24]. They build on the Cochrane et al. (2000) model by considering opportunities that have very low associated risk compared with the potential returns (i.e. *good deals* in Cochrane et al’s (2000) model). These opportunities are assumed to be acceptable by all investors. They use a family of probability measures on the state space which they call *test measures*. These represent the possible views held by market participants and are used in conjunction with a fixed floor in order to determine if the return of the particular investment opportunity in question outweighs its associated risk. The difficulty with this approach is how to select the test measures which represent

the probabilities associated with the possible states of the world. The underlying way in which these measures are sampled will have an effect on the prices obtained and this approach needs careful consideration and justification.

Britten-Jones and Neuberger (BJN) [21] introduced a way of looking at option pricing which enjoys several useful and desirable features. The BJN approach relies on the concept of the dominating strategy outlined in Pliska [74] which allows the determination of the upper and lower bounds for the price of the contingent claim where Risk-Neutral valuation is not possible or appropriate (i.e. in the context of incomplete markets). Conveniently, they also provide the tools for treating jump-processes. The volatilities implied by this model are quite interesting and the model can incorporate jumps. For this reason, Chapter 5 investigates this model in more detail.

## Chapter 3

# Assessment of the Jump Diffusion Model in the Context of the Top40

This chapter investigates how well option prices on the JSE's Top40 futures are described by the MJD model as described in Section 2.2. The performance of this model is compared to the performance of the industry's benchmark B&S model. As per (2.4), the dynamics of the MJD model can be described by the equation,

$$dS/S = \mu dt + \sigma dz(t) + A(t)dN(t) \quad (3.1)$$

The assumption that the price of the underlying can be modelled with the GBM (2.2) is crucial to the pricing approach of B&S. A significant feature of the GBM process is that the log-returns in the underlying are normally distributed. Since the publication of B&S's seminal paper in 1973, there have been countless investigations into the validity of this assumption. The overwhelming conclusion has been that the log-returns are not perfectly normally distributed. The extent to which this assumption does not hold obviously depends largely on what security is being considered – whether it is a stock, an interest rate or an exchange rate and there are significant variations within these fields.

Figure 3.1 is an illustration of how well the log-returns in the Top40 index which is traded on the JSE fit the standard normal distribution. A straight line would indicate that the log-returns are perfectly normal. As can be seen, this is clearly not the case for the log-returns in the Top40 index. Log-returns in this index are therefore not perfectly normally distributed which indicates that some other model may show an improvement over the B&S model. This chapter investigates whether Merton's 1976 jump-diffusion model could prove to be a more suitable model.

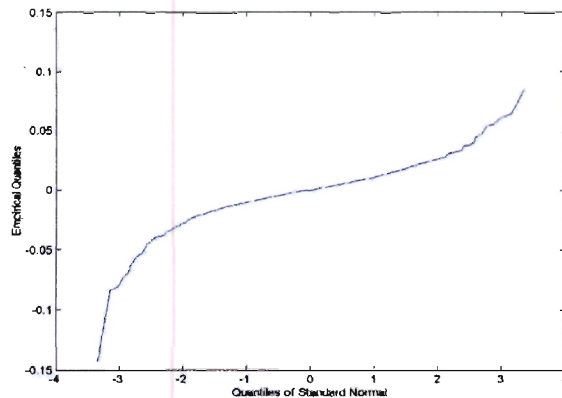


Figure 3.1: Fitted Densities for the Top40 Index (Q-Q plots).

In support of Figure 3.1, Table 3.1 shows the descriptive statistics for the daily Top40 index prices for the data set *June 1995–September 2004* (a total of 2404 prices). If the B&S assumption of lognormally distributed returns was correct, the higher moments (skewness and excess kurtosis) of the log-returns should be close to the standard normal distribution (i.e. zero in both cases). As can be seen in Table 3.1, this is not the case for the Top40 index.

The results in Table 3.1 are quite consistent with what is expected from returns on equity. A difference between the standard deviation and skewness from monthly to daily data is to be expected as these parameters are a function of the frequency at which the data is sampled. The monthly figures are consistently slightly higher than the daily figures in all

Statistic	dTop40 <sub>m</sub>	dTop40 <sub>d</sub>
Minimum	-0.278	-0.133
Maximum	0.211	0.088
Std. Deviation	0.234	0.211
Skewness	-0.358	-0.502
Excess Kurtosis	<b>2.54</b>	<b>8.57</b>

Table 3.1: **Descriptive Statistics for Top40 Index Data.**  $dTop40$  refers to relative changes in the underlying. The standard deviations for the returns in the  $dTop40$  columns are annualised. The subscripts “m” and “d” refers to the sampling frequency – monthly and daily, respectively.

of these parameters. The annualised drift for the underlying was found to be 16.2%. The most striking result of Table 3.1 is both the *level* of the excess kurtosis and the *amount* it increases when the returns are sampled daily instead of monthly. An increase in kurtosis implies that larger outliers occur more frequently than would be suggested by the standard normal distribution. As discussed in Section 2.4, this increase in kurtosis between daily data and monthly data is an important factor in determining a suitable model for the pricing of derivatives.

These results indicate that a model that could incorporate an increase in the level of kurtosis in the underlying might show an improvement over the benchmark B&S model. A potential alternative model that accounts for an increase in both the overall levels of kurtosis and when maturity decreases is the MJD model. This is because, for any sized interval in the MJD environment, there is a chance that a large jump could take place. In relation to the normal variance evaluated over smaller intervals, therefore, the effect on the conditional kurtosis will be significant. As the sampling interval increases, the magnitude of the jump in relation to the diffusion shock decreases, and the kurtosis will revert to normal values.

Figure 3.2 shows the difference between the B&S and the Merton prices for various strike/stock price ratios and times to maturity. It can clearly be seen here that, in re-

lation to Merton's model, the B&S prices are too high at-the-money, but too low in- and out-the-money. In relation to Merton's 1976 model, the risk of bigger stock price jumps in B&S's model is therefore underestimated (both up and down in this case). This difference is significant for shorter maturities which rapidly decreases as the time to maturity increases – a common property of the MJD model. This effect will be observed when the Merton model is used to fit market volatilities later in Section 3.2.

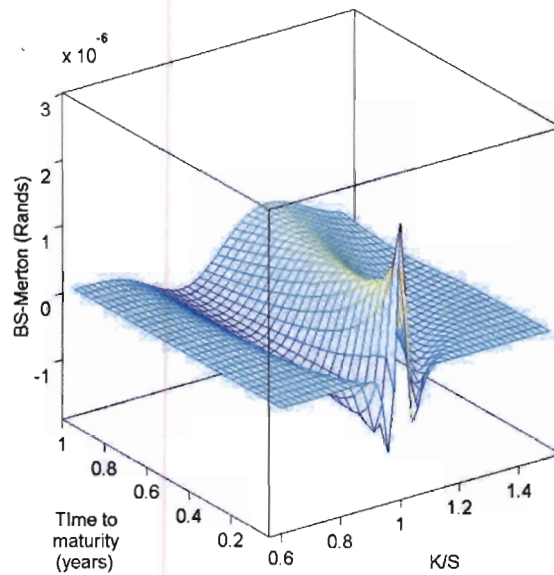


Figure 3.2: **Differences Between B&S Prices and Merton Prices.** The parameters are  $\mu = 5\%$ ,  $\sigma = 15\%$ ,  $\theta = -5\%$ ,  $\delta = 5\%$  and  $\lambda = 0.2$ .

Figure 3.3 shows the implied smile for options on the Top40 futures for various maturities. This data was obtained from the SAFEX website [77]. It can clearly be seen that the steepness of the skew in the smile decreases as the maturity increases.

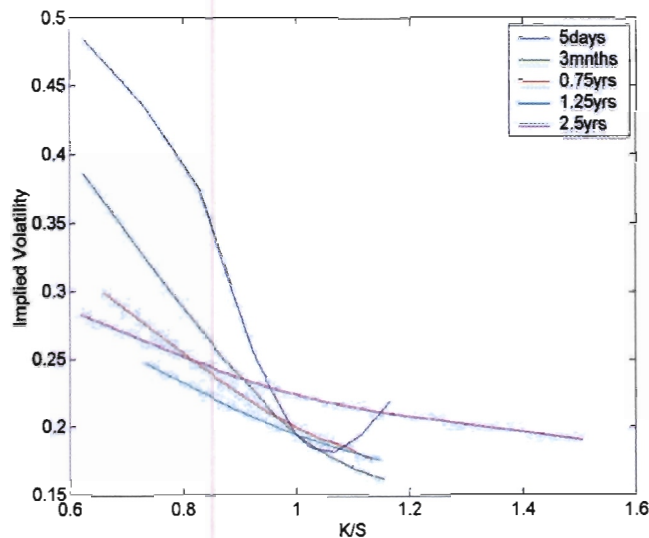


Figure 3.3: Implied Volatility Smile in Options on the Top40 Futures

## 1 Fitting Parameters to the Returns of the Underlying

The following section is aimed at establishing whether there is statistical evidence that the MJD process provides a better fit to the returns in the Top40 futures prices than the GBM process.

Assuming that discrete jumps are present in the underlying, the parameters that feature in the MJD model need to be determined, along with whether they are statistically significant. Also, whether the MJD process shows an improvement over the GBM process also needs to be determined. This section will use a statistical approach to estimate the jump parameters from historical data of the underlying. Since this approach purely looks at the underlying prices, it does not take into account the feeling of market traders as to how big and frequent future jumps may be. This can be achieved through an implied approach where

the parameters are estimated by considering the market options data – i.e. in the same way that implied volatility is obtained, it is possible to get implied MJD parameters. This approach will be discussed in Section 3.2.

Because the definition of jumps in the context of the MJD model corresponds to rare and unusual news, there is often a lack of data to estimate the above-mentioned jump parameters. For this reason, different statistical approaches yield different results. There are several parameter-optimising methods discussed in the literature, from the classical school using Kalman Filters to the Bayesian school using Markov Chain Monte Carlo. Methods of Moments estimation techniques are also used frequently in this type of optimisation problem. However, as demonstrated by Das [33], it is not possible to isolate the individual jump parameters. This is due to the fact that the second jump moment (i.e.  $\mathbb{E}(Y^2)$ ) always enters as a sum with the volatility parameter in the second, third and fourth moments and is therefore not separately identified. This approach is therefore not particularly meaningful and will not be considered in this dissertation. Instead, a simple discrete-time approach using maximum likelihood estimation techniques will be used to parameterise the Top40 index data in the context of the MJD model. The MJD model is estimated by using a Bernoulli approximation which was first introduced by Ball and Torous [8]. The assumption being made here is that in each time interval either only one jump occurs or no jumps occur. Daily data will be sampled which justifies this assumption. This approach makes the estimation procedure highly tractable, stable and convergent [8].

As shown in Underhill and Bradfield [82], p. 122, in the limit, the sum of Bernoulli variables tends towards the Poisson distribution. Figure 3.4 shows an example of a generated path where jumps have been augmented into the diffusion process. This follows the MJD path as described previously (see (2.11)). These paths have been generated by using a Bernoulli random variable to determine whether a random jump which is lognormally distributed with mean ( $\theta$ ) and variance ( $\delta^2$ ) occurs on any given day. Since daily data is being sampled, the probability of a jump occurring on any day is  $\lambda$  (see (2.5)). For example, if  $\lambda = 0.1$ ,

a jump will occur on average once every 10 days. Define  $x_t$  as the logarithm of the price

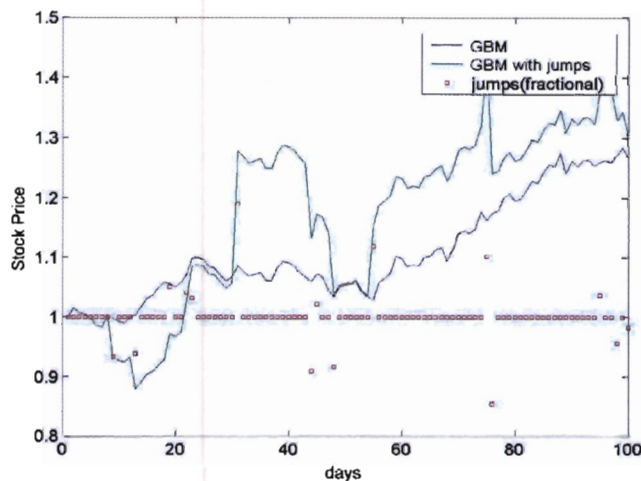


Figure 3.4: Adding Jumps to the Geometric Brownian Motion Process

relatives  $\ln(S_t/S_{t-1})$ , where  $S$  is the Top40 futures price. For the case where the price follows a GBM process,  $dS/S = \alpha dt + \sigma dz$ , so that  $x_t \sim N(\mu, \sigma^2)$ , where  $\mu = \alpha - \sigma^2/2$  (both defined per unit time). In discrete time, this can be written as,

$$\ln(S_t/S_{t-1}) = \mu + \sigma z$$

where  $z$  is a standard normal variable.

If the underlying prices follow a diffusion process with a constant drift of  $\mathbb{E}(\Delta S/S) = \alpha$  and a constant variance  $\text{var}(\Delta S/S) = \sigma^2$ , the logarithm of the price relatives,  $x_t = \ln(S_t/S_{t-1})$  is normally distributed with mean  $\mu = \alpha - \sigma^2/2$  and variance  $\sigma^2$ . The parameter values for this process need to be determined. With  $T$  independent observations, the logarithm of the likelihood function  $L(\gamma; x)$ , viewed as a function of the parameter vector  $\gamma = (\mu, \sigma^2)$  is given by [55],

$$l_{GBM} = -\frac{T}{2} \ln(2\pi) + \sum_{t=1}^T \ln \left[ \frac{1}{\sqrt{\sigma^2}} \exp \left( -\frac{(x_t - \mu)^2}{2\sigma^2} \right) \right] \quad (3.2)$$

noting again that  $\mu = \alpha - \sigma^2/2$  and  $x_t = \ln(S_t/S_{t-1})$  [55].

As in (2.2), the MJD process can be modelled as  $dS/S = \mu dt + \sigma dz + A(t)dN(t)$ , where the Poisson process is characterised by a mean number of jumps occurring per unit time  $\lambda$  as well as a jump size, which is assumed independently lognormally distributed  $\ln Y \sim N(\theta, \delta^2)$ . Therefore,  $x_t$  can be written as,

$$\ln(S_t/S_{t-1}) = \mu + \sigma z + \sum_{i=1}^{n_t} Y_i$$

where  $n_t$  is the actual number of jumps that occur during the interval and  $\mu$  and  $\sigma$  are as per the diffusion case.

For the MJD case, the log-likelihood function is [55],

$$l_{MJD} = -T\lambda - \frac{T}{2}\ln(2\pi) + \sum_{t=1}^T \ln \left[ \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \frac{1}{\sqrt{\sigma^2 + \delta^2 j}} \exp\left(\frac{-(x_t - \mu - \theta j)^2}{2(\sigma^2 + \delta^2 j)}\right) \right] \quad (3.3)$$

In order to numerically optimise the above function, the infinite sum has to be truncated after some value of  $N$ . Double precision in the results will be obtained if truncation occurs at  $N = 10$  [8].

If the whole parameter set is defined as  $\Gamma$ , then numerical second derivatives of the likelihood function evaluated at the optimal parameter set  $\Gamma^* = \{\mu^*, \sigma^*, \theta^*, \delta^*, \lambda^*\}$  provides the Hessian matrix:  $X = \frac{\partial^2 L}{\partial \Gamma \partial \Gamma}$ .

The log of the likelihood functions, given in (3.2) and (3.3) are maximised over the appropriate parameter set. The standard errors are then computed from the Hessian matrix as  $\sqrt{\text{diag}(-X^{-1})}$ . These standard errors are then used in determining the t-statistics associated with each parameter by obtaining  $\Gamma^*/(\text{s.e.})'$ . Matlab's optimisation function *fminunc* conveniently outputs the Hessian matrix when  $L$  is maximised.

Table 3.2 shows the results for both the GBM and MJD processes with their associated t-statistics in parenthesis. The data set considered here was for the daily Top40 Index

prices from June 1995 to September 2004, amounting to a total of 2404 observations. The unconditional annualised drift in the MJD process under the discretised process is

Parameter	Pure Diffusion	Jump Diffusion
$\mu$	0.1053 (1.69)	0.1579 (3.62)
$\sigma$	0.2125 (49.55)	0.1460 (104.51)
$\theta$	–	-0.001 (0.49)
$\delta$	–	0.0209 (15.91)
$\lambda$	–	0.1961 (6.90)
Log-Likelihood	7155	7378

Table 3.2: Results of Maximum Likelihood Estimation for Parameter Estimation for Top 40 Index Prices for June 1995 - September 2004

$\mu + \lambda\theta/dt = 14.5\%$ . This is close to the annualised drift of the underlying futures over the 9.5 years of the sampled data (16.2%). In addition to this, the total volatility of the MJD process is  $\sigma + \lambda\delta$  which is 15.0%. Comparing this with the annualised standard deviation of the sampled data (12.7%) is again evidence that the maximum likelihood technique is giving sensible results.

Analysing the data in Table 3.2, all the parameters were found to be significant to within the 95% confidence interval (i.e. all the t-statistics are greater than the threshold t-statistic of 1.65), apart from the drift of the jumps – i.e. there is not a high level of confidence in this value. As shown in Bates [12], this is not too critical since it is the jump rate frequency ( $\lambda$ ) that has a more significant effect on the prices of options. This is well above the statistically significant threshold of 1.65 and confidence is therefore high that this value is statistically meaningful.

The drift of the diffusion part of the MJD process increases when the jump parameters are included. This is offset by the slightly lower diffusion volatility and the negative mean of the jumps (-0.1%). These occur once every 5 days with an annualised volatility of 2.1%. Although the size of the jumps are not large in the sampled data for the Top40 index (compared with, for example, the ZAR-USD exchange rate, where  $\delta$  was found to be 5.1%), the effect of including the jumps has a significant effect on the log-likelihood value as can be seen in Table 3.2. The relatively small jumps are consistent with the Top40 index, since this index is made up of stocks from different sectors. Jumps that occur in one stock can often be reduced (to some extent) by the diversification that is inherent in this index.

The Likelihood Ratio Statistic can be introduced in order to compare the significance of the improvement of the MJD over the GBM process. According to Jorion [55], this statistic can be defined as,

$$\Lambda = -2(\ln L(\gamma, x) - \ln L(\gamma^*, x)) \quad (3.4)$$

where  $\gamma$  and  $\gamma^*$  are the maximum likelihood estimates for the GBM and the MJD parameters, respectively, and  $x$  is the data set. Under the null hypothesis that the returns in the futures prices are consistent with a lognormal diffusion process without a Bernoulli jump structure (i.e. parameter set  $\gamma$ ),  $\Lambda$  is asymptotically  $\chi^2$ -distributed with degrees of freedom equal to the difference in the number of parameters between the two models (i.e. 3 in this case). With  $\Gamma$  being defined as the whole parameter space, it is clear that  $\gamma \in \Gamma$  and  $\gamma^* \in \Gamma$ . It is for this reason that this asymptotic result holds. The value of  $\Lambda$  in (3.4) is 446 which indicates that the MJD process is a significant improvement over the GBM process (at the 99% significance level, the  $\chi^2$ -stat is 12.84).

These results show that the patterns of higher-order moments (i.e. skewness and kurtosis) cannot be generated by purely diffusion models alone. Jump models can be considered as a refinement of the diffusion process and can be used in addition to other diffusion models (e.g. stochastic volatility).

As discussed in Das [33], the behaviour of the kurtosis in the returns can be used to demonstrate the difference between the diffusion based class of models and the jump diffusion model. Considering first any diffusion process: as  $T$  tends to zero, the conditional skewness and kurtosis also goes to zero. In the case of the stochastic volatility model, when the time interval is very small, the volatility of the volatility does not have much of an opportunity to have any significant effect. This makes the higher moments negligible. However, as  $T$  increases, these moments start having more and more of an effect. For larger values of  $T$ , this diffusion model starts tending towards the standard Gaussian model and the skewness and kurtosis return to normal values. The graph for this process will therefore be humped-shaped, with the early and later values being normal, but the intermediate values will be higher.

In the case of the MJD model, at small values of  $T$ , the probability of a large jump taking place (relative to the normal variance) is significant. The conditional skewness and kurtosis at shorter maturities will be higher than that for diffusion processes. As  $T$  increases, the effect of this jump on the stock price in relation to the effect of the normal variance reduces which will mean that the skewness and kurtosis will revert back to normal values as  $T$  increases. Therefore, for the MJD process, the graph of skewness and kurtosis should decline monotonically with  $T$ .

In order to test this approach, the level of kurtosis present in different time interval windows for the sampled Top40 index data was determined. The specified interval window is moved through the data set and an average of the kurtosis for each window is taken – this provides one of the points on the graph. The procedure is then repeated for a larger interval window and the next point on the graph is obtained. The Matlab code for this is shown in the included CD-Rom (see file *Variation-of-Kurtosis*). Figure 3.5 shows how the kurtosis in the returns decreases consistently as the time interval considered is increased from 1 to 255 days. As discussed, this backs up the claim that jumps are definitely present in the returns of the underlying. This outcome does not mean that diffusion processes are not valid, but

rather than jumps play a significant part in the observed path of the index. This provides evidence that a model which incorporates stochastic volatility *and* discrete jumps in the underlying may prove to be even more suitable. Evidence of this improvement is shown by Bates [12] in the context of options on Deutsche mark futures.

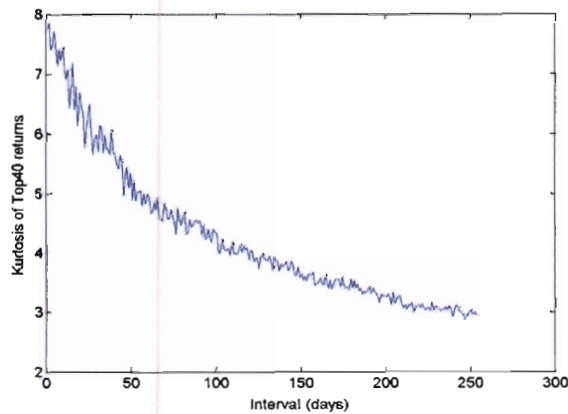


Figure 3.5: Variation of Kurtosis in the Top40 Index by Time Interval

## 2 Obtaining Jump Parameters Implied by Market Prices

In this section, the ability of the jump diffusion model to fit the smile generated by market prices will be investigated. How the resulting best-fit parameters for the options compare to the parameters for the underlying data obtained previously in Section 3.1 will also be considered.

The data used for this analysis was downloaded from the Safex website: <http://safex.co.za/>. The appropriate formula for calculating the implied volatility is the 'SAFEX Black' formula which is essentially the B&S formula, but with the dividend yield and the interest rate equal

to zero [9]. The implied volatilities were confirmed to equal those given by the prices before parameterisation was carried out.

The volatility of this data is implied and not realised [9]. At the end of each day, SAFEX use the last implied traded volatility for an at or near the money option for margining purposes. All options for the Top40 futures contracts for that expiry are then margined at a predefined smile, which depends on the at-the-money volatility at that time. Because the resulting skew is more of a theoretical skew and not one that is determined by market forces, the data used in the subsequent analysis was not altered in any way.

If the MJD model is under consideration where the parameters are constant, then everything is defined relative to the current value of spot and the current time. If  $Y$  in (2.8) is chosen to be less than 1, then downward-sloping smiles are obtained. The sharpness of the smile will increase as the time to maturity decreases. As time to maturity increases, the smile becomes more horizontal. This is due to the diffusion component of the model becoming more prevalent in relation to the effect of the jumps. Figure 3.6 shows an example of how steep and skew an implied volatility smile can become by using some fairly mild parameters in the MJD model. This also demonstrates how quickly the smile flattens out as time to maturity is increased.

Many of the studies that ignore short-term options cite Rubenstein [76], who excluded all options with less than 21 days. However, overall trading activity in short-dated options is significant<sup>1</sup>.

To determine the jump parameters in Merton's 1976 model, a least-squares best fit approach is undertaken as outlined in Andersen and Andreasen [3]. Here, all the parameters (except for the drift of the diffusion part) are optimised such that the Euclidean distance between the volatility backed out using Merton's 1976 model prices as the input to the Black76 formula for futures (from now on referred to as *MertonVol*) and the market's

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<sup>1</sup>For the options sampled on the Top40 futures, 5-day options accounted for 15% of total option volume.

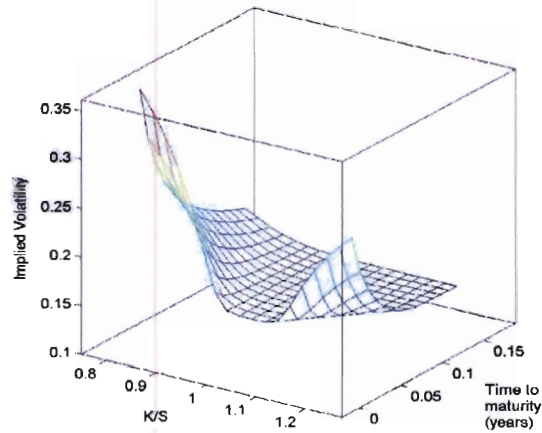


Figure 3.6: *Smile generated by Merton's Model.* The parameters are  $\mu = 5\%$ ,  $\sigma = 15\%$ ,  $\theta = -5\%$ ,  $\delta = 5\%$  and  $\lambda = 0.2$ .

implied volatility (*MarketVol*) is as small as possible. This is done according to,

$$\frac{1}{N} \min \left\{ \sum_1^N (\hat{\sigma} - \sigma)^2 \right\} \quad (3.5)$$

where  $\hat{\sigma}$  are the MertonVols,  $\sigma$  are the MarketVols and  $N$  is the number of options considered in the optimisation. Matlab's *fminsearch* function was used to find the abovementioned function. This optimization is an unconstrained nonlinear technique which uses a simplex search method. For a data set of 228 points, minimising this function is numerically intensive. In order to find a solution as quickly as possible, the shortest maturities data set (5 days) was initially considered. This was then used as a starting point for the whole day's data set (including all the maturities). Interestingly, the parameter set did not change much from this initial starting point when the longer maturities were included. Backing out the implied parameters for the MJD model for the data set of 229 data option prices took just over 46 minutes.

The focus of the results are for the September 2004 Top40 futures options data. However, the implied parameters for a month either side of this data set were determined to see how much the implied parameters change over time. This gives an indication of how the market

price of risk can change over time. These results are presented in Table 3.3. The percentage changes in parenthesis refer to the deviation from the September implied parameters.

Parameter	Aug.	Sept.	Oct.
$\sigma$	0.237 (3%)	<b>0.230</b> -	0.241 (5%)
$\theta$	-0.399 (20%)	<b>-0.320</b> -	-0.285 (-12%)
$\delta$	0.065 (8%)	<b>0.060</b> -	0.068 (12%)
$\lambda$	0.138 (5%)	<b>0.131</b> -	0.087 (-51%)

Table 3.3: Implied Parameters for the MJD Model from Top40 Options on Futures Prices (September 2004)

It can clearly be seen that there are significant changes between the implied parameters over the three data sets. The most significant in terms of the size of the change and in terms of its relevance to the market price of risk is the jump frequency ( $\lambda$ ) which drops rapidly from September to October. The above result shows that the MJD model can capture quite effectively the changing market price of risk.

Figure 3.7 shows the difference between the B&S prices and the Merton prices with these parameters. The difference between this graph and that of Figure 3.2 is significant. For these implied parameters, the prices from the Merton model are always less than the Black model. The best-fit parameters for September options can now be compared with those previously found for the underlying futures. As discussed in Section 3.1, the parameters for the underlying were found to be  $\sigma = 0.146$ ;  $\theta = -0.001$ ;  $\delta = 0.021$ ;  $\lambda = 0.196$ . The significantly lower downward mean drift for the jumps in the options (-0.320 c.f. -0.001) in conjunction with the higher volatility of the jumps in the options (0.060 c.f. 0.021) is consistent with the hypothesis that traders in these options are fearful of large downward jumps. This feature is confirmed by Andersen and Andreasen [3] in their study of options on the S&P 500 where the shape of this smile is very similar to that of options on the

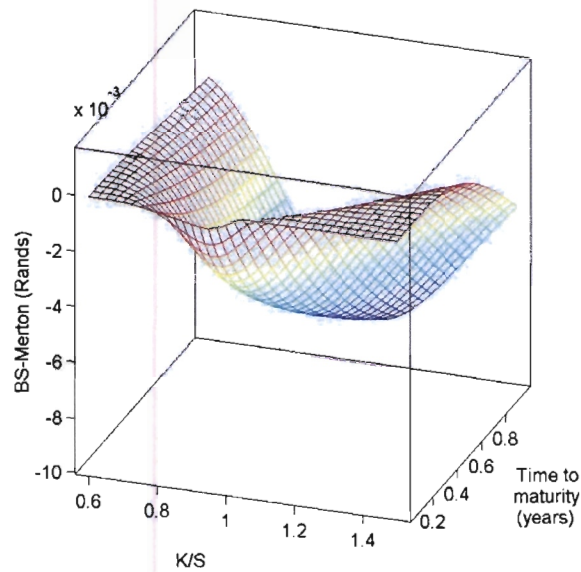


Figure 3.7: Differences Between B&S Prices and Merton Prices for Fitted Parameters (September 2004 Options on Top40 Futures)

Top40 futures. The frequency of the jumps are slightly longer for options (once every 7.6 days c.f. once every 5 days for the underlying parameters).

The fit between the shortest maturity (1 week) and the next maturity (3 months) is shown in Figure 3.8. It is clear that the jump diffusion model was able to fit the shorter maturity options much better than the longer maturity options. As mentioned previously, this is inherent in the jump diffusion model – i.e. that the smile obtained from the model prices reduces rapidly as the maturity increases. The standard deviation between the MertonVols and the MarketVols was found to be 14%. In an attempt to ascertain how well the MertonVols fitted the MarketVols, the distance between a constant volatility parameter (ConstantVol) and the MarketVol was minimised (as per (3.5)). ConstantVol was found to be 0.238 – not too different from the volatility obtained by optimising the smile of the jump diffusion model (c.f. 0.230). The averaged sum of the distances between ConstantVol and

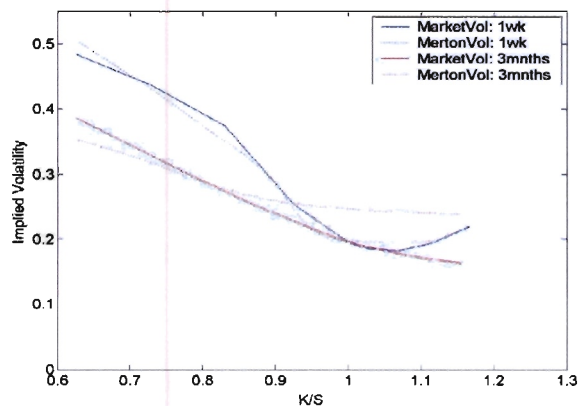


Figure 3.8: Fit of Parameterised Merton Volatilities to Market Volatilities (September 2004 Options on Top40 Futures)

MarketVol was found to be 0.00353, while between MertonVol and MarketVol was found to be 0.00168 – indicating that the Merton model provides a significantly better fit than the benchmark model of B&S.

As discussed in Section 2.3, there are an infinite number of measures to choose from to price options. The jump intensity can be considered to be equivalent to a risk-aversion factor to jumps. As illustrated in Bates [10], the standard deviation of the jump size is (in some cases) not important, whereas the jump arrival frequency has a more relevant impact on options valuation. A risk-averse investor will use a greater arrival intensity in the model than an investor preferring to take on risk. For this reason, the jump arrival frequency,  $\lambda$ , was determined for each data set. These are shown in Table 3.4. As can clearly be seen,  $\lambda$  drops consistently as the time to maturity increases which is consistent with fears of downward jumps being more pronounced for shorter maturities<sup>2</sup>.

<sup>2</sup>Although the significance of these results may be questionable due to the small sample size ( $\sim 30$  data points per maturity), the downward trend in  $\lambda$  is clearly present.

Maturity (years)	$\lambda$
0.02	0.700
0.25	0.515
0.5	0.322
0.75	0.178
1	0.042
1.5	0.090

Table 3.4: Variation of Implied  $\lambda$  for Different Option Maturities

## Chapter 4

# Assessment of the Variance Gamma Model in the Context of the Top40

Actual stock prices move in small jumps and stock price paths are not in reality continuous. Even the jump diffusion model differs from real stock price paths due to the fact that the jump diffusion model's path process is continuous, apart from at the discrete and relatively infrequent jump times. The Variance Gamma (VG) process that will be discussed in this chapter is not set in continuous time. However, it is a pure jump process with an infinite number of jumps in any size time interval. The need for a pure-jump process to model stock prices has been observed by Bakshi, et al. [6], where it is pointed out that pure diffusion based models struggle to explain the smile effects – especially in short-dated option prices. If one wants to explore models that are not diffusion processes, but are still martingales, then a random time-changed Brownian motion needs to be considered [56]. The VG model is an example of this approach.

Time in the VG model is made to be random which represents the random nature of the arrival of new information. There is much evidence that trading volume can be considered to be a proxy for the arrival of information (see, for example, Karpoff [58]). Statistical

evidence exists that stock price returns are more normal when rescaled to use trading volume for the time parameter instead of normal time.

The process used to model the arrival of information needs to have certain properties. Market information accumulates and is not generally forgotten. The random process should therefore be a monotone increasing function. Independence is still important in the random time process – i.e. the rate of information arrival in one period should not depend on the amount of information at the start of that period.

Let  $Y(t; \sigma, \theta)$  be a Brownian motion with a drift  $\theta$  and variance rate  $\sigma^2$ . If  $W(t)$  is a standard Brownian motion, the process  $Y(t; \sigma, \theta)$  can be written in terms of  $W(t)$  as,

$$Y(t; \sigma, \theta) = \theta t + \sigma W(t)$$

The VG process is obtained on evaluating the process  $Y$  at an independent random time given by a gamma process. For this, we define the process  $G(t; \nu)$  with independent increments, identically distributed over non-overlapping intervals of length  $h$ . The increments,  $G(t+h; \nu) - G(t; \nu) = g$ , has a gamma density,

$$p(g, h) = \frac{g^{h/\nu-1} \exp(-g/\nu)}{\nu^{h/\nu} \Gamma(h/\nu)}$$

where  $\Gamma(\cdot)$  is the gamma function. The mean of the gamma density is  $h$  and the variance is  $\nu h$ . Therefore, the average random time change in  $h$  units of calendar time is  $h$  and its variance is proportional to the length of the interval.

The variance gamma process  $X(t; \sigma, \nu, \theta)$  is defined as,

$$\begin{aligned} X(t; \sigma, \nu, \theta) &= Y(G(t; \nu); \sigma, \theta) \\ &= \theta G(t; \nu) + \sigma W(G(t; \nu)) \end{aligned} \tag{4.1}$$

This is equivalent to a Brownian motion with drift  $\theta$  and variance rate  $\sigma^2$  evaluated at the gamma time  $G(t; \nu)$ . The parameters  $\theta$  and  $\nu$  control the skewness and kurtosis features of this process.

The characteristic function of  $X_t$ ,  $\phi_{X(t)}(u) \stackrel{D}{=} \mathbb{E}[\exp(iuX(t))]$ , is given by,

$$\phi_X(u) = \left( \frac{1}{1 - i\theta\nu u + (\sigma^2\nu/2)u^2} \right)^{t/\nu} \quad (4.2)$$

for  $-\infty < u < \infty$ .

If  $\theta$  is zero, the characteristic function is real-valued and the process is therefore symmetric and there is no skewness.

The PDF of  $X_t$ , defined by (4.1) is available in closed-form and is derived by Madan et al. [61],

$$f_X(x) = \frac{2\exp(\theta(x-c)/\sigma^2)}{\sigma\sqrt{2\pi\nu^{1/\nu}}\Gamma(1/\nu)} \left( \frac{|x-c|}{\sqrt{2\sigma^2/\nu + \theta^2}} \right)^{1/\nu-1/2} K_{1/\nu-1/2} \left( \frac{|x-c|\sqrt{2\sigma^2/\nu + \theta^2}}{\sigma^2} \right) \quad (4.3)$$

for  $-\infty < x < \infty$ , where  $\nu > 0$  and the other parameters are the same as those described previously and  $K(\cdot)$  is a modified Bessel function of the second kind. The differential equation,

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} - (z^2 + \nu^2)y = 0$$

where  $\nu$  is a real constant, is called the modified Bessel's equation, and its solutions are known as modified Bessel functions.

$I_\nu(z)$  and  $I_{-\nu}(z)$  form a fundamental set of solutions of the modified Bessel's equation for the noninteger  $\nu$ .  $K_\nu(z)$  is a second solution, independent of  $I_\nu(z)$ .

$I_\nu(z)$  and  $K_\nu(z)$  are defined by

$$I_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{\left( \frac{z^4}{4} \right)^k}{k! \Gamma(\nu + k + 1)}$$

$$K_\nu(z) = \left( \frac{\pi}{2} \right) \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

where  $\Gamma(a)$  is the gamma function.

Equation (4.3) is useful for maximum likelihood estimation of parameters from time series, as demonstrated by Madan et al. [61].

The Brownian motion in the traditional form of the geometric Brownian motion model is replaced by the VG process,

$$S_t = S_0 \exp[ct + X(t; \sigma, \nu, \theta)] \quad (4.4)$$

For the discounted stock price process to be a martingale, the stock process must be of the form,

$$S_t = S_0 \exp[ct + X(t; \sigma, \nu, \theta) + \omega t] \quad (4.5)$$

where

$$\omega = \frac{1}{\nu} \ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right)$$

and  $c, \theta$  and  $\sigma (> 0)$  are real constants.

Figure 4.1 shows some generated paths for realistic parameters. The Matlab code for generating these paths can be found on the included CD-Rom (see file *VG-Generated-Paths*).

## 1 Fitting the VG Model to Underlying Data

Conditional on the gamma time change ( $G$ ), the VG variate ( $X(t)$ ) is normally distributed with mean  $\theta G$  and variance  $\sigma\sqrt{G}$ . For the underlying stock, we can therefore write,

$$X(t) = ct + \theta G + \sigma\sqrt{G}z$$

where  $z$  is a standard normal variate, independent of the gamma random variable  $G$  that has mean  $t$  and variance  $\nu t$ .

Through computing the expectations, Madan et al. [61] show that the first four central moments of the return distribution are,

$$\mathbb{E}X_t = c + \theta \quad (4.6)$$

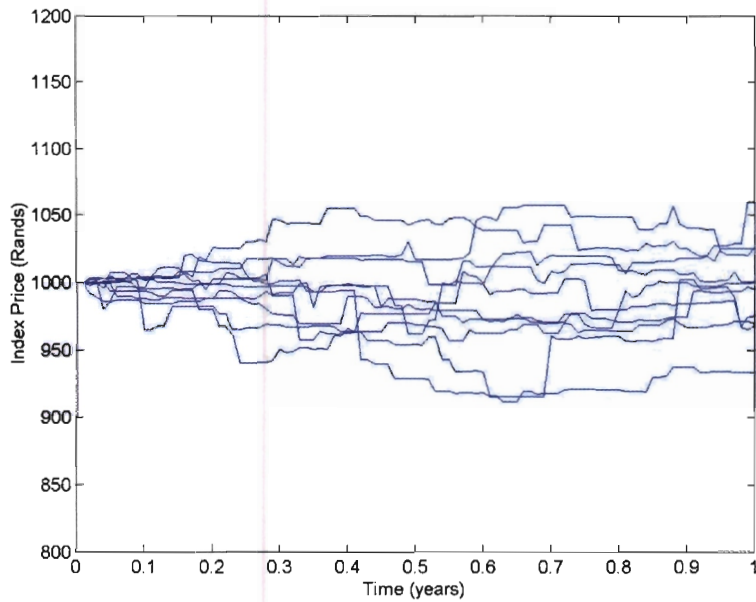


Figure 4.1: *Variance Gamma Paths*. The parameters are  $\nu = 0.1$ ,  $\theta = 0$  and  $\sigma = 5\%$ .

$$\text{var}(X_t) = \sigma^2 + \theta^2\nu \quad (4.7)$$

$$\mathbb{E}((X_t - \mathbb{E}X_t)^3) = 2\theta^3\nu^2 + 3\sigma^2\theta\nu \quad (4.8)$$

$$\mathbb{E}((X_t - \mathbb{E}X_t)^4) = 3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3 + 3\sigma^4 + 6\sigma^2\theta^2\nu + 3\theta^4\nu^2 \quad (4.9)$$

The coefficients of skewness and kurtosis are defined by,

$$\beta = \frac{\mathbb{E}((X_t - \mathbb{E}X_t)^3)}{(\mathbb{E}((X_t - \mathbb{E}X_t)^2))^{3/2}} \quad (4.10)$$

and

$$\kappa = \frac{\mathbb{E}((X_t - \mathbb{E}X_t)^4)}{(\mathbb{E}((X_t - \mathbb{E}X_t)^2))^2}. \quad (4.11)$$

The statistical estimation procedure that will be employed is simply the Method of Moments as detailed in Seneta [80], which relies on the stationarity of the increments. If the case of  $\theta = 0$  is considered, the parameters  $\mathbb{E}X_t = c$  and  $\text{var}X_t = \sigma^2$  in the model of (4.4)

may be estimated from a long observed sequence  $\{X_t\}$ ,  $t = 1, \dots, n$  by,

$$\begin{aligned}\hat{c} &= \bar{X} = \frac{\sum_{t=1}^n X_t}{n} \\ \hat{\sigma}^2 &= \frac{\sum_{t=1}^n (X_t - \bar{X})^2}{n} = \widehat{\text{var}}(X_t).\end{aligned}\quad (4.12)$$

The skewness and kurtosis coefficients, (4.10) and (4.11) can be consistently estimated for the general model of (4.4) by [80],

$$\hat{\beta} = \frac{\sum_{t=1}^n (X_t - \bar{X})^3}{(\sum_{t=1}^n (X_t - \bar{X})^2)^{3/2}} \quad (4.13)$$

and

$$\hat{\kappa} = \frac{\sum_{t=1}^n (X_t - \bar{X})^4}{(\sum_{t=1}^n (X_t - \bar{X})^2)^2} \quad (4.14)$$

where

$$\bar{X} = \frac{\sum_{t=1}^n X_t}{n} = \hat{\mu} \quad (4.15)$$

The moment estimators,  $\bar{X}$ ,  $\widehat{\text{var}}(X_t)$ ,  $\hat{\beta}$  and  $\hat{\kappa}$  in equations (4.12)–(4.15) can be used to estimate  $\mu$ ,  $\text{var}(X_t)$ ,  $\beta$  and  $\kappa$ . These can then be used to solve for  $c$ ,  $\theta$ ,  $\sigma^2$  and  $\nu$  by solving (4.6)–(4.9).

From (4.7)–(4.11), it can be noted that,

$$\beta = \frac{2\theta^3\nu^2 + 3\sigma^2\theta\nu}{(\theta^2\nu + \sigma^2)^{3/2}} \quad (4.16)$$

$$\kappa = \frac{3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3 + 3\sigma^4 + 6\sigma^2\theta^2\nu + 3\theta^4\nu^2}{(\theta^2\nu + \sigma^2)^2}. \quad (4.17)$$

Normally, an iterative procedure is required to solve these equations, but by assuming that  $\theta$  is small,  $\theta^2$ ,  $\theta^3$  and  $\theta^4$  can be ignored. Under the assumption of a small  $\theta$ , we therefore have,

$$\mu = c + \theta \quad (4.18)$$

$$\text{var}(X_t) = \sigma^2 \quad (4.19)$$

$$\beta = \frac{3\theta\nu}{\sigma} \quad (4.20)$$

$$\kappa = 3(1 + \nu) \quad (4.21)$$

Therefore, from  $\widehat{\text{var}}X_t$ ,  $\hat{\kappa}$ ,  $\hat{\beta}$  and  $\hat{\mu}$ , approximations to  $\hat{\sigma}^2$ ,  $\hat{\nu}$ ,  $\hat{\theta}$  and  $\hat{c}$  can be obtained. If  $\hat{\theta}$  is small, then the full equations (4.6)–(4.9) will be approximately true. Under this scenario, there is no need for the independence of increments assumption here.

For the same Top40 Index data set described previously in Chapter 3, we have,

$$\begin{aligned}\hat{\mu} &= 3.614 \times 10^{-4} \\ \widehat{\text{var}}X_t &= 1.4043 \times 10^{-4} \\ \hat{\beta} &= -0.00368 \\ \hat{\kappa} &= 6.1058\end{aligned}$$

Then, from (4.18)–(4.21), the parameters for underlying data can be obtained. These parameters are summarised in Table 4.1.

Parameter	Value
$\hat{\sigma}^2$	$1.4043 \times 10^{-4}$
$\hat{\nu}$	1.0352
$\hat{\theta}$	$-1.4041 \times 10^{-5}$
$\hat{c}$	$3.7546 \times 10^{-4}$

Table 4.1: Parameters Obtained by Fitting VG Model to Top40 Index Prices.

## 2 Option Pricing Using FFT Techniques in the VG Framework

In relation to a European call of maturity  $T$  which is written on the terminal spot price  $S_T$  of some underlying asset, the characteristic function of  $s_T = \ln(S_T)$  is defined by,  $\phi_T(u) = \mathbb{E}[\exp(ius_T)]$ . The characteristic function is derived in Madan, et al. [61] and is given by (4.2). Dynamics of log prices which are given by an infinitely divisible process

of independent increments results in this characteristic function being known analytically. Characteristic functions have been used in a pure diffusion context with stochastic volatility (Heston [48]) and jumps coupled with stochastic volatility (Bates, [12]). In addition to these models, the characteristic function has been used in the variance gamma model (Madan, et al. [61]). This approach is generally much faster than finite difference solutions, partial differential equations or integro-differential equations. Assuming that the characteristic function is known analytically, many authors (e.g. Scott [79]) have numerically determined the risk-neutral probability of finishing in-the-money as,

$$\mathbb{P}(S_T > K) = \Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathbb{R} \left[ \frac{e^{-iu \ln(K)} \phi_T(u)}{iu} \right] du \quad (4.22)$$

Similarly, the delta of the option, denoted as  $\Pi_1$ , is numerically obtained as,

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathbb{R} \left[ \frac{e^{-iu \ln(K)} \phi_T(u-i)}{iu \phi_T(-i)} \right] du \quad (4.23)$$

Assuming no dividends and a constant interest rate,  $r$ , Bakshi and Madan [7] show that, very generally, one may write a call option price in the form,

$$C = S\Pi_1 - Ke^{-rT}\Pi_2 \quad (4.24)$$

*FFT* techniques cannot be used to evaluate the integral, since the integrand is singular at the required evaluation point,  $u = 0$ . In order to be able to use *FFT*, an alternative approach is detailed in Carr and Madan [23] which is described below.

Let  $k$  denote the log of the strike price  $K$  and let  $C_T(k)$  be the desired value of a  $T$  maturity call option with strike  $\exp(k)$ . Let the risk-neutral density of the log price  $s_T$  be  $q_T(s)$ . The characteristic function of this density is defined by,

$$\phi_T(u) = \int_{-\infty}^{\infty} e^{ius} q_T(s) ds \quad (4.25)$$

The initial call value  $C_T(k)$  is related to the risk-neutral density  $q_T(s)$  by,

$$C_T(k) = \int_k^\infty (e^s - e^k) q_T(s) ds \quad (4.26)$$

$C_T(k)$  tends to  $S_0$  as  $k$  tends to  $-\infty$ , and hence the call pricing function is not square integrable. To obtain a square integrable function, the modified call price  $c_T(k)$  is defined by,

$$c_T(k) = e^{\alpha k} C_T(k) \quad (4.27)$$

for  $\alpha > 0$ . For a range of positive values for  $\alpha$ , it is expected that  $c_T(k)$  is square integrable in  $k$  over the entire real line. The Fourier transform of  $c_T(k)$  is defined as follows,

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(s) dk \quad (4.28)$$

An analytical expression for  $\psi_T(v)$  in terms of  $\phi_T$  is firstly obtained and then call prices are obtained numerically using the inverse transform,

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) dv = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \psi_T(v) dv \quad (4.29)$$

Bakshi and Madan [7] show generally that,

$$\psi_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \quad (4.30)$$

Call values are determined by substituting (4.30) into (4.29) and carrying out the required integration. The integration (4.29) is a direct Fourier transform and *FFT* techniques can therefore be applied. If  $\alpha = 0$ , then the denominator vanishes when  $\nu = 0$  which results in a singularity in the integrand. In order to get around this, the use of the factor  $\exp(\alpha k)$  or something similar is required. As explained in Carr and Madan [23], for the modified call value to be integrable in the positive log strike direction, and for it to be square integrable as well,  $\psi(0)$  is required to be finite. From (4.30),  $\psi_T(0)$  is finite provided  $\phi_T(-\alpha(+1)i)$  is finite. From the definition of the characteristic function, this requires that  $\mathbb{E}[S_T^{\alpha+1}] < \infty$ . It is found in Carr and Madan [23] that one fourth of the upper bound serves as a good choice for  $\alpha$ .

The *FFT* is an efficient algorithm for computing the sum,

$$w(k) = \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j) \quad \text{for } k = 1, \dots, N \quad (4.31)$$

where  $N$  is typically a power of 2. The algorithm reduces the number of multiplications in the required  $N$  summations from an order of  $N^2$  to that of  $N\ln_2(N)$  which is a significant reduction. Madan et al. [61] show that (4.29) can be considered as an application of (4.31).

Using the Trapezoidal rule for the integral on the right side of (4.29) and setting  $v_j = \eta(j - 1)$ , the approximation for  $C(k)$  is,

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^N e^{-iv_j k} \psi_T(v_j) \eta \quad (4.32)$$

The effective upper limit for the integration is now,

$$a = N\eta \quad (4.33)$$

The *FFT* returns  $N$  values of  $k$  and a regular spacing of  $\lambda$  is employed, so that the values for  $k$  are,

$$k_u = -b + \lambda(u - 1), \quad \text{for } u = 1, \dots, N \quad (4.34)$$

This gives log strike levels ranging from  $-b$  to  $b$ , where

$$b = \frac{N\lambda}{2} \quad (4.35)$$

Substituting (4.34) into (4.32) gives,

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^N e^{-iv_j(-b+\lambda(u-1))} \psi_T(v_j) \eta, \quad \text{for } u = 1, \dots, N \quad (4.36)$$

Noting that  $v_j = (j - 1)\eta$ , this can be written as,

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^N e^{-i\lambda\eta(j-1)(u-1)} e^{ibv_j} \psi_T(v_j) \eta. \quad (4.37)$$

To apply the Fast Fourier Transform, it can be noted from (4.31) that,

$$\lambda\eta = \frac{2\pi}{N} \quad (4.38)$$

This means that, if a small  $\eta$  is chosen in order to obtain a fine grid for the integration, then the resulting call prices will have strike spacings that are relatively large. This is

undesirable, so it would be ideal to maintain accurate integration with larger values of  $\eta$ . To ensure this is possible, Simpson's Rule weightings are incorporated into the summation. With these weightings and the restriction of (4.38), the call price may be written as,

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j} \psi_T(v_j) \frac{\eta}{3} (3 + (-1)^j - \delta_{j-1}). \quad (4.39)$$

where  $\delta_n$  is the Kronecher delta function that is unity for  $n = 0$  and zero otherwise. The summation of (4.39) is an exact application of the *FFT*. Appropriate choices need to be made for  $\eta$  and  $\alpha$ .

Option prices in the VG setting were obtained using Matlab's *FFT* functions (see file on included CD-Rom *FFT-Prices*). As per Carr and Madan [23], a spacing of  $\eta = 0.25$  provides the optimal speed without affecting the accuracy of the results. The number of nodes in the quadrature was chosen to be  $N = 4096$ , leading to a log strike spacing of  $8\pi/4096 = 0.613\%$ . Figure 4.2 shows the type of smile that can be obtained with typical parameters. Actual strike prices can then be interpolated from the VG strike price matrix to obtain the desired VG prices. The time for Matlab to evaluate a single price is 0.06 seconds, only marginally slower than the MJD model.

The same approach was then taken as in Chapter 3, where the model is fitted to the actual smile using a least-squares approach (see (3.5)). The average fit of 0.000117 was obtained which is significantly better than the fit of 0.00168 obtained with the MJD model. The resulting optimised parameters are as follows,

$$\theta = -0.2631$$

$$\sigma = 0.1683$$

$$\nu = 0.3112$$

In relation to the parameters obtained through methods of moments techniques as summarised in Table 4.1, there was a significant decrease in  $\theta$  and  $\nu$ . This implies a tendency

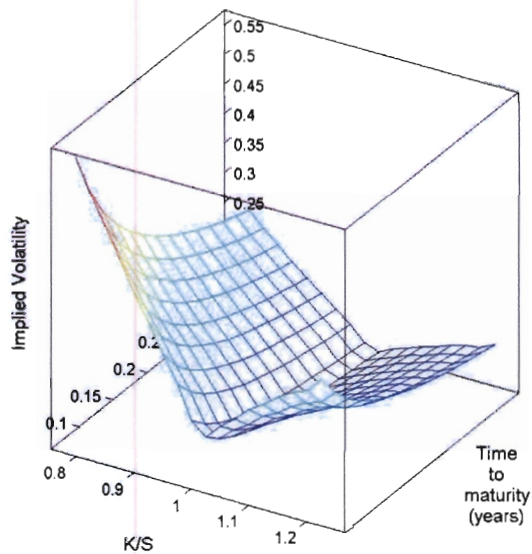


Figure 4.2: *Smile Generated by Using VG Model to Price Options.* The parameters are  $r = 5\%$ ,  $\sigma = 20\%$ ,  $\theta = -20\%$  and  $\nu = 0.2$ .

toward risk aversion amongst market traders. The volatility,  $\sigma$ , actually reduced slightly compared with the annualised volatility for the underlying which was found to be 0.1892.

In relation to the time it took to back out the implied parameters, the VG model took just over 22 minutes which is just under half the time it took for the MJD model – an indication of the efficiency of the FFT approach.

## Chapter 5

# Bounds on Option Prices in Incomplete Markets: The BJN Model

This chapter will begin with an outline of the BJN model's framework as discussed in Britten-Jones and Neuberger's (1996) paper. This will lead on to the implementation of the model which includes a discussion on issues relating to the programming and on how well the model can fit the smiles observed in the Top40 futures discussed in Chapter 3. It should be noted that this is not an exhaustive discussion of the model, but merely an overview which gives some insight into the workings of a model that provides bounds on option prices in incomplete markets.

# 1 General Framework for Option Pricing in Markets with Jumps

Let  $S(0)$  be the price today of an asset  $S$ . It is assumed that there exists a forward market in the asset  $S$ . This underlying assumption is required in terms of the hedging argument that will follow and will be used in the dynamic programming approach that will be discussed later. The drawback of this requirement is that the class of options that can be considered is restricted to European claims only. This is not seen to be an issue in the context of this dissertation as only European options are being considered.

The forward price for delivery of the stock at time  $t$  will be denoted by  $S_t$ . A European contingent claim,  $V$ , is a security whose pay-off at time  $t$  only depends on the realisation of  $S$  at  $t$ , i.e.  $V = V(S_t)$ .

Let  $\pi$  be a price sequence, i.e. an ordered collection of prices,  $S_i$ , with  $0 \leq i \leq N$  where  $N$  is finite. A general price sequence can be described by

$$\pi = \{S_0, S_1, S_2, \dots, S_N\}$$

The price sequence is said to be permissible if and only if,

**Condition 1.**  $S_0 = S(0)$

**Condition 2.**  $|\ln S_{j+1} - \ln S_j| \leq d, \quad 0 \leq j \leq N$

**Condition 3.**  $\sum_{j=0}^{N-1} (\ln S_{j+1} - \ln S_j)^2 = v$

Condition 1 simply requires that, for a price sequence to be permissible, its first value should be equal to today's price. Condition 2 specifies that the jump in the log of the stock price between any two steps should never exceed an arbitrarily large but finite quantity,  $d$ . If the underlying process is a pure diffusion,  $d$  will simply scale with the square root of the

time step. However, if the underlying process contains discrete jumps (e.g. from a Poisson process), then Condition 2 is a real restriction and the user must ensure that the quantity  $d$  does not become excessively large.

Regarding Condition 3, if the underlying process were a diffusion with hedging continuous and frictionless, it would be the same as requiring that the time integral of the square of the instantaneous volatility,  $\sigma(t)$ , should equal a known quantity,  $(\sigma_{\text{ave}})^2 t$ :

$$\int_0^t \sigma^2(u) du = (\sigma_{\text{ave}})^2 t$$

As long as the instantaneous volatility is a purely deterministic function of time, the quantity that matters is just  $\sigma_{\text{ave}}^2 t$ . Condition 3 is therefore no more restrictive than requiring deterministic volatility. It assumes that the total magnitude of the moves over the finite re-hedging periods including the jump events is known. Condition 3 is therefore stronger than the above equation.

From the previous 3 conditions, the permissible price sequence can be defined as one where the prices depend on the asset,  $S$ , the maximum jump size,  $d$ , and the total variance,  $v$ . This permissible price sequence is denoted  $P(S, d, v)$ . The conditions for a new term called the *residual volatilities*,  $v_i$  for  $0 \leq i \leq N$  for each step  $i$  is described as follows:

**Condition 1.**  $v_0 = v$

**Condition 2.**  $v_i = v_{i-1} - (\ln S_{i+1} - \ln S_i)^2$

From these two conditions, it follows that,

**Condition 2'.**  $v_i = v_0 - \sum_{k=0}^{i-1} (\ln S_{k+1} - \ln S_k)^2$

BJN introduce the idea that the quantity  $v_i$  conveys information about how much of the total variance is still available to use at time step  $i$ .

A trading strategy,  $H$ , is now introduced which specifies the number of forward contracts held at time step  $i$ , which depends (unspecified at this point) on the stock price and on the residual volatility at time step  $i$  (i.e.  $H = H(S_i, v_i)$ ).

As long as only European options are considered, the following strategy can be considered,

1. selling the contingent claim at time 0 for an amount of cash equal to  $V_0$
2. paying out to the buyer of  $V$  the pay-off (if any) of the contingent claim at time  $T$
3. accumulating the profits and losses arising from the strategy  $H$  (which will be called the hedging profits)

Zero interest rates are assumed to simplify the analysis. The profits from the overall strategy are given by

$$V_0 - V(S_T) + \sum_{j=0}^{N-1} H(S_j, v_j)(S_{j+1} - S_j)$$

Amongst all the admissible strategies  $H$ , an arbitrage strategy is one whereby, for all permissible paths,

$$V_0 - V(S_T) + \sum_{j=0}^{N-1} H(S_j, v_j)(S_{j+1} - S_j) \geq 0$$

where the inequality is strict for at least one path. If this condition were satisfied, then the seller of the option would receive at least as much and sometimes more money up front ( $V_0$ ) than what he has to pay at the end ( $V(S_T)$ ), plus the cost of the strategy (i.e.  $\sum_{j=0, N-1} H(S_j, v_j)(S_{j+1} - S_j)$ ). Any selected trading strategy will be a deterministic function of all possible values of the residual volatility and stock price.

In order to price options sensibly, arbitrage strategies are not allowed. It therefore follows that the price for  $V$  that is obtained today ( $V_0$ ) cannot be larger than the maximum cost incurred in paying the final pay-off of the claim, and accumulating the hedging profits or losses for the chosen strategy. The maximum (supremum) must therefore be over all the

permissible paths:

$$V_0 \leq \sup_{\pi \in P(S,v,d)} \left\{ V(S_T) + \sum_{j=0}^{N-1} H(S_j, v_j)(S_{j+1} - S_j) \right\} \quad (5.1)$$

The above equation must hold for all trading strategies. In general, therefore, the price that is received from the sale of the option today cannot exceed even the lowest (infinum) possible realisation of the quantity  $V(S_T) + \sum_{j=0, N-1} H(S_j, v_j)(S_{j+1} - S_j)$  over all the possible trading strategies. Therefore,

$$V_0 \leq \inf_H \left\{ \sup_{\pi \in P(S,v,d)} \left\{ V(S_T) + \sum_{j=0}^{N-1} H(S_j, v_j)(S_{j+1} - S_j) \right\} \right\} \quad (5.2)$$

Therefore, Equations 5.1 and 5.2 express a minimum upper bound for the price today of the contingent claim in 2 steps: first, for a given strategy, the price that is certain to cover the total payout for every possible admissible path is determined; then the strategies  $H$  are varied in order to reduce the total payout (i.e. the payout arising from the final payment and the re-hedging costs) as much as possible, knowing that the price  $V_0$  will have to be lower also than this value. Finally, this minimum upper bound is defined as,

$$V_0 = \inf_H \left\{ \sup_{\pi \in P(S,v,d)} \left\{ V(S_T) + \sum_{j=0}^{N-1} H(S_j, v_j)(S_{j+1} - S_j) \right\} \right\} \quad (5.3)$$

In order to find the maximum lower bound, the sign in the argument just needs to be reversed, i.e.,

$$V_0 = \sup_H \left\{ \inf_{\pi \in P(S,v,d)} \left\{ V(S_T) + \sum_{j=0}^{N-1} H(S_j, v_j)(S_{j+1} - S_j) \right\} \right\}$$

This approach is conceptually not practical and to overcome this, BJN cast this problem in a different light. Since the current price of the asset,  $S$ , and the total variance determine both the permissible price sequences and the trading strategies,  $H$ , the function  $V(S, v)$  just defined provides the minimum upper bound as a function of  $S$  and  $v$ . More precisely,  $V$  is a function with a domain given by all the possible values of  $S$  and  $v$ . At a future point in time, i.e. as the price evolution unravels, the function itself will not change, only

its domain will. As a consequence, the minimum upper bound in state  $i$  can be written as  $V(S_i, v_i)$ . When the problem is looked at in this way, no special meaning is attached to the initial and final times. In particular, one can focus on two consecutive steps and write,

$$V(S_i, v_i) = \inf_h \left\{ \sup \{ V(S_{i+1}, v_{i+1}) + h(S_i, v_i)(S_{i+1} - S_i) \} \right\} \quad (5.4)$$

In moving from Equation 5.3 to 5.4, the supremum is no longer taken over paths, but over realisation of  $S$  and  $v$  at the next step. For a given  $S_i, v_i$ , it is required to keep track of a finite number of possible terminal destinations  $S_{i+1}, v_{i+1}$ , instead of a multitude of connecting paths. Also, the sum over time steps has disappeared, since a single time step is being dealt with (at a time). Finally, the term  $V(S_{i+1})$  no longer indicates the terminal payout, but simply the value of the minimum upper bound itself at the next time step.

A variable  $z_i$  is now defined as follows,

$$S_{i+1} = S_i \exp(z_i)$$

From Condition 2 on p.66,

$$v_{i+1} = v_i - z_i^2$$

Therefore, Equation 5.4 can be re-written as

$$V(S_i, v_i) = \inf_h \left\{ \sup_{z_i} \{ V(S_i \exp(z_i), v_i - z_i^2) + h(S_i(\exp(z_i) - 1)) \} \right\} \quad (5.5)$$

as long as the quantity  $z_i^2$  does not exceed either  $v_i$  (otherwise the residual volatility  $v_{i+1}$  could become negative), or  $d^2$  (in order to comply with the maximum jump size condition). Since two consecutive ‘time’ slices are now being considered and the supremum simply has to be searched over all admissible values of  $z_i$ , the setting begins to look more like a traditional dynamic programming problem (i.e. a usual tree- or lattice-based backward-induction approach). There still remains, however, the infimum over the trading strategies that makes the problem unique. This is dealt with in the following section.

## 2 Finding the Optimal Hedge

Assume that the optimal hedge ( $h^*$ ) has been found – i.e. the hedge that minimises the quantity  $\sup_{z_i} \{V(S_i \exp(z_i), v_i - z_i^2) + h(S_i(\exp(z_i) - 1))\}$ . Since  $V(S_i, v_i)$  has been defined as an upper bound, the following will hold,

$$V(S_i, v_i) = V(S_i \exp(z_i), v_i - z_i^2) + h^* S_i (\exp(z_i) - 1) \quad (5.6)$$

The amount by which the upper bound  $V(S_i, v_i)$  is greater than the right-hand side is a function  $f$  of the permissible value for  $z_i$ . Therefore,

$$f(z) \equiv V(S_i, v_i) - V(S_i \exp(z_i), v_i - z_i^2) + h^* S_i (\exp(z_i) - 1) \quad (5.7)$$

The following outlines some of the properties of  $f$

**Property 1:**  $f(0) = 0$ . From the definition.

**Property 2:**  $f(z) \geq 0$ . Follows from Equations 5.6 and 5.7.

**Property 3:**  $\frac{\partial f}{\partial z} \Big|_{z=0} = 0$ . This follows from Properties 1 and 2. If the function  $f$  is equal to zero at the origin and greater than or equal to zero for positive or negative values of  $z$ , as long as arbitrarily small in magnitude (Property 2), then the derivative w.r.t.  $z$  at the origin must be equal to 0.

The derivative  $\frac{\partial f}{\partial z}$  can be easily calculated:

$$\frac{df}{dz} = -\frac{\partial V}{\partial S_i} S_i \exp(z_i) + 2z \frac{\partial V}{\partial z} + h^* S_i \exp(z_i)$$

Evaluating this quantity at  $z = 0$  and equating it to zero due to Property 3,

$$\begin{aligned} \frac{df}{dz} \Big|_{z=0} &= -\frac{\partial V}{\partial S_i} S_i + h^* S_i = S_i \left( -\frac{\partial V}{\partial S_i} + h^* \right) = 0 \\ &\Rightarrow h^* = \frac{\partial V}{\partial S_i} \end{aligned} \quad (5.8)$$

Equation 5.8 therefore determines the optimal strategy. The optimal strategy that has just been determined can now be substituted into Equation 5.5,

$$V(S_i, v_i) = \inf_h \left\{ \sup_{z_i} \left\{ V(S_i \exp(z_i), v_i - z_i^2) + \frac{\partial V}{\partial S_i}(S_i(\exp(z_i) - 1)) \right\} \right\}$$

subject to the usual constraint on  $z$  and with the boundary (initial) condition,

$$V(S_i, 0) = V(S_i)$$

### 3 Implementation of BJN Model

Binomial trees are normally constructed by specifying a time step  $\Delta t$  and a volatility. In the BJN model, instead of mapping time on the horizontal axis, the residual volatility ( $v = j(\sigma\sqrt{\Delta t})^2$ ) is used, where  $j$  is a positive integer. BJN use  $\sigma\sqrt{\Delta t}$  as their unit of ‘time’ in the numerical discussions. They justify this on the basis that time and volatility always appear together in this form or as its square in the B&S pricing formula. They explain that real time is made to go faster or slower, by using the numerical device  $v$  according to whether the volatility is higher or lower. This means that the volatility-adjusted time flows at the same rate. When the last node in the tree has been reached, all the available volatility has been used up and therefore the value of the option,  $V$ , is simply equal to the appropriate pay-off condition, depending on the corresponding value of the underlying. One of the crucial assumptions about the BJN model is that the exact amount of total volatility is known throughout the constructed tree (all of which will be used up by option expiry). *How* this volatility will actually occur during the option’s life is not assumed to be known. The BJN assumption is milder than assuming to know the instantaneous volatility function  $\sigma(t)$  at every time step. For a purely diffusive process, it is the same as knowing either  $\sigma_{ave}(T)^2 T$  or  $\int_0^T \sigma(u)^2 du$ , since these two quantities are equivalent. However, this equality is lost when the case of jump-diffusions is considered.

BJN’s grid can be constructed in the following way:

- choose an arbitrary step size  $\delta = \sigma\sqrt{\Delta t}$
- place  $S_i = S_0\exp(i\delta)$  with  $i$  a positive or negative index on the  $y$  axis
- place  $v_j = j\delta^2$  with  $j$  a non-negative index on the  $x$  axis
- for the case of no-jumps, force the stock price at a generic  $(i, j)$  point to be linked only to two possible states  $(i + 1, j - 1)$  and  $(i - 1, j - 1)$

Consider the second-to-last residual volatility slice where only an amount  $\delta^2$  of the total volatility is available. The value of  $V(S_i, 1)$  is normally obtained as a linear combination of the values  $V(S_{i+1}, 0)$  and  $V(S_{i-1}, 0)$ . The weights  $p$  and  $(1 - p)$  can easily be obtained by imposing that

$$pS(i + 1, j - 1) + (1 - p)S(i - 1, j - 1) = S(i, j)$$

noting that forward prices are being considered. Therefore,

$$p = \frac{1 - \exp(-\delta)}{\exp(\delta) - \exp(-\delta)} \quad (5.9)$$

as per Cox, et al. [31].

In order to be able to generalise to the case when jumps are present, a slightly different procedure is used to bring back the option value (this will be needed when using the convex hull described later in this section). A straight line is constructed through the two points  $V(S_{i+1}, 0)$  and  $V(S_{i-1}, 0)$  using the following two relationships,

$$V(i + 1, j - 1) = a + bS(i + 1, j - 1)$$

$$V(i - 1, j - 1) = a + bS(i - 1, j - 1)$$

The values of  $a$  and  $b$  can be found from this 2x2 linear system of equations. It follows that,

$$V(i, j) = a + bS(i, j)$$

This is effectively the same as what is carried out through the standard binomial model.

Moving now to the case of the jump-diffusion process, the construction of the tree remains the same but the destination nodes from a given parent node are different. It is required that, from node  $(i, j)$ , the reachable points should be  $(i + n, j - n^2)$ , with  $n$  a positive or negative but never zero integer. It is also required that the move is not so large such that the residual volatility from a give node,  $v_i$  is exceeded; nor must the maximum allowable (log) jump be greater than  $d$ . These two conditions together require that

$$n^2 \leq (j, d^2)$$

For  $d = 2$ , at  $j = 4$ , the reachable cells will correspond to  $j = 4, 3, 0$ ;  $i = -2, -1, 0, 1, 2$ . This is where the volatility-time tree approach becomes quite beneficial. If the tree had been built with conventional time on the horizontal axis and if jumps are allowed, several possible states would be reachable at the next time slice – thereby making recombination impossible<sup>1</sup>. The last 2 steps are identical to the standard binomial tree. In the second-to-last step, the remaining volatility allows us to only move 1 step toward expiry and the last step is obviously already at expiry (i.e.  $V(S_i, 0) = V(S_i)$ ). The abovementioned description is supported by Tables 5.1 and 5.2 where the following values have been selected:  $\sigma = 0.2$ ,  $T = 0.4$  and  $dt = 0.1$ . The bold-faced figures in Table 5.2 are the option prices that are achievable as per the requirements mentioned above.

Although the resulting underlying mechanism does seem somewhat questionable in terms of the jumps reducing as the final steps are approached (which does not happen in real life), this requirement is merely a restriction that is placed on the permissible states to be consistent with the theory (i.e. that  $n^2 \leq (j, d^2)$ ). The effect of this restriction will be discussed in the following section.

The BJN approach diverges from the binomial method when the process moves beyond the second-to-last step. At these nodes, there are more than two possible destinations

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<sup>1</sup>This is the approach used by Amin [2] in his seminal discrete-time model of the jump-diffusion process where a finite number of points are reachable at the next time slice.

i/j	4	3	2	1	0
4					128.79
3				120.89	
2			113.48		113.48
1		106.53		106.53	
0	100.00		100.00		100.00
-1		93.87		93.87	
-2			88.12		88.12
-3				82.72	
-4					77.65

Table 5.1: BJN stock prices

i/j	4	3	2	1	0
4					28.79
3				20.89	
2			13.48		<b>13.48</b>
1		<b>8.16</b>		6.53	
0	<b>4.74</b>		3.16		0.00
-1		<b>1.53</b>		0.00	
-2			0.00		<b>0.00</b>
-3				0.00	
-4					0.00

Table 5.2: BJN option prices

and it is not possible to calculate the option value as a linear combination of its values at the destination nodes. It is here that the optimal hedging strategy determined by BJN becomes useful. Looking at the expression,

$$V(S_i, v_i) = \sup_{z_i} \{V(S_i \exp(z_i), v_i - z_i^2) - h^* S_i (\exp(z_i) - 1)\} \quad (5.10)$$

$$= \sup_{z_i} \{V(S_i \exp(z_i), v_i - z_i^2) - \frac{\partial V}{\partial S_i} S_i (\exp(z_i) - 1)\} \quad (5.11)$$

The process moves from the final expiry to today, where  $V(S_i, v_i)$  is the new value for the option that needs to be determined and  $V(S_i \exp(z_i), v_i - z_i^2)$  are option values that have already been calculated. A hedging amount of forward contracts  $h^*$  (with  $h^* = \partial V / \partial S_i$ ) is

also being held and therefore the change in the value of the portfolio will be linear in the stock price. The value of the stock price is known to be  $S_i$  in the state where the option value is to be determined. Considering the contents of the  $\sup\{\cdot\}$  operator, if the correct  $z_i$  was chosen, then the new unknown value of  $V(S_i, v_i)$  is linked to the later (known) value,  $V(S_i \exp(z_i), v_i - z_i^2)$  by the linear relationship:

$$V(S_i, v_i) = V(S_i \exp(z_i), v_i - z_i^2) - h^* S_i (\exp(z_i) - 1) \quad (5.12)$$

$$= V(S_i \exp(z_i), v_i - z_i^2) - \frac{\partial V}{\partial S_i} S_i (\exp(z_i) - 1) \quad (5.13)$$

The value of the slope  $h^*$  (the amount of forward contracts held) is not known. However, since  $V(S_i, v_i)$  must be an upper bound, for any possible value of the arrival  $S_i \exp(z_i)$ , its value must be worth at least as much as any of the known reachable values  $V(S_i \exp(z_i), v_i - z_i^2)$ . It must also be a lower upper bound, and putting these two constraints together, the straight line that must be found is the one that has the lowest possible value at  $S_i$ , such that all the  $V(S_i \exp(z_i), v_i - z_i^2)$  points lie on or below this line. This line defines what BJN call the *convex hull*. Figure 5.1 shows how the convex hull function is used to determine the optimal hedging ratio ( $\Delta V / \Delta S$ ) and the new value  $V(S_i, v_i)$ . Each node at each step is recalculated depending on how many points are included in the convex hull function. The constructed convex hull is carried out at each variance step, where the set of known values,  $V(S_i \exp(z_i), v_i - z_i^2)$  are determined by the criterion that  $n^2 \leq (j, d^2)$  as discussed previously. Using the same procedure, one can move all the way back to the root where the required minimum upper bound for the option price will be found. Matlab's *convhull* function provides the convex hull required for the analysis as described above. The Matlab programme for generating prices for this model can be found on the included CD-Rom (see file *BJNcode*). If the argument in Equation 5.5 is reversed so that the sup-inf is found instead of the inf-sup, the maximum lower bound will be obtained. This is illustrated in Figure 5.1, label (2). The time for Matlab to evaluate a single BJN price is 7.03 seconds, considerably slower than the MJD and the VG models.

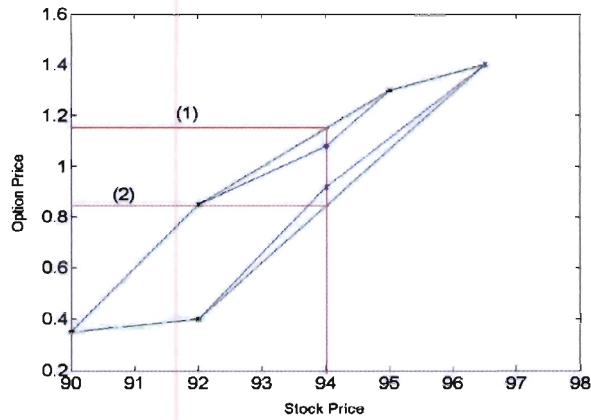


Figure 5.1: *Using the Convex Hull Function to Find the Optimal Hedge Ratio and the New Option Value.* Minimum upper bound (1) and maximum lower bound (2) are shown for the stock value being considered.

## 4 Results from the BJN Model

It was found that the BJN model converges to the binomial model when the number of steps,  $N$ , exceeds about 120 as can be seen in Figure 5.2.

It was found that the prices for options that include jumps are always at least as large as the B&S pricing formula and the value of the option increases with an increase in the jump size. The effect of this price increase reduces as the jump size is increased by the same proportion. This is illustrated in Figure 5.2.

The BJN model can generate some interesting implied volatility surfaces. Reassuringly, symmetric jumps are generated with  $d$  equal to a constant value. However, market smiles are not generally symmetrical as demonstrated in the case of options on the Top40 futures. Fortunately, asymmetrical maximum jump sizes can be incorporated into the BJN model which results in asymmetric smiles. As described in Rebonato [75], the condition,

$$n^2 \leq (j, d^2)$$

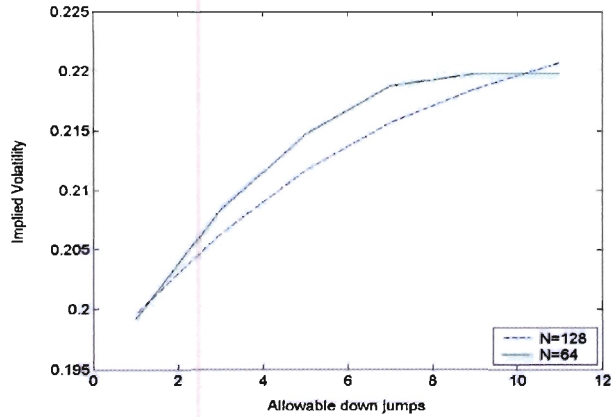


Figure 5.2: Effect on Implied Volatility with Varying Downward Jumps for Different Values of  $N$ .

can be separated into two conditions,

$$n_{\text{up}}^2 \leq (j, d_{\text{up}}^2)$$

$$n_{\text{down}}^2 \leq (j, d_{\text{down}}^2)$$

which in turn implies that,

$$|\ln S_{j+1} - \ln S_j| \leq d_{\text{up}} \quad \text{if } S_{j+1} > S_j$$

$$|\ln S_{j+1} - \ln S_j| \leq d_{\text{down}} \quad \text{if } S_{j+1} < S_j$$

Figure 5.3 shows the implied volatility for the BJN model with different jump up- and down-sizes. Figure 5.4 shows how the smile varies with strike price and time to maturity for  $d_{\text{up}} = 1$  and  $d_{\text{down}} = 5$ . These are all for  $N = 128$ .

The discussion so far has referred to the jump parameter,  $d$  as an integer. The correct way to look at this parameter is actually as a percentage allowable change, which will depend on the total number of nodes and the overall time scale used in the algorithm (i.e.  $dt = N/T$ ). Care must be taken in choosing this parameter to ensure that the required permissible percentage jump size is correct. Also, for larger jumps, the value of  $N$  becomes

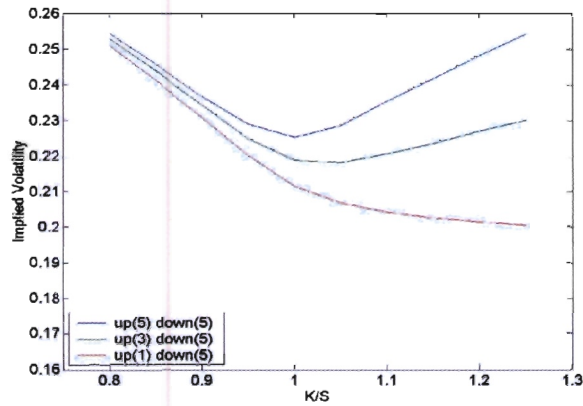


Figure 5.3: *BJN implied volatility with different allowable step-sized up- and down-jumps as inputs.  $N = 128$ , constant volatility ( $\sigma$ ) = 0.2.*

important. If this is not chosen large enough, the generated tree does not incorporate these larger jumps sufficiently. For example, if the jumps size has been chosen as 7 steps, this 7-step jump will only come into play after 64 nodes. If  $N$  has been chosen to be 64, the tree will not have incorporated this larger jump at all (although it would have incorporated up to 6 jumps from node 36).

In order to determine if the BJN model can actually fit market volatilities (e.g. the smile in the options on the Top40 futures), the jump parameters were changed manually to try and provide an acceptable fit. Manual changes to the jump parameters were necessary due to the fact that a least-squares type of fitting process would require excessive programming resource (the amount of programming time to obtain just a single solution was considerable). It was found that an optimisation technique was not really required since it soon became clear that the BJN model could not get close to the steep smile that is implied by the Top40 options data. This was despite including very skew jump parameters (e.g.  $d_{\text{down}} = 3\%$  (or  $\sim 8$  steps),  $d_{\text{up}} = 0.4\%$  (or  $\sim 1$  step)). A further increase in these parameters did not make much difference to the steepness of the generated smile.

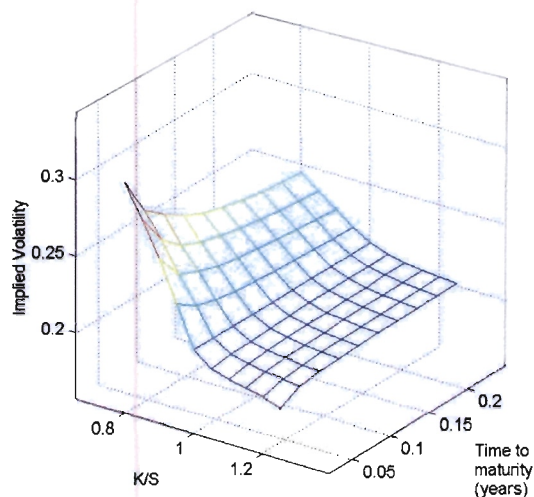


Figure 5.4: *Variation of BJK implied volatility with time to maturity.* Up- and down-jumps are given as 1 and 5 allowable steps, respectively.  $N = 128$ , constant volatility ( $\sigma$ ) = 0.2.

In order to compare the BJK model with the market volatilities, the up and down parameters were chosen to be 1 and 7 steps, respectively. A greater difference between these parameters would become unrealistic. The diffusion volatility was then changed so that the BJK smile was approximately equal to the market smile for the lowest strike price and the shortest maturity. Comparisons were then made using this as the fixed point. In order to quantify the fit of the BJK smile to the market smile, Euclidean sum-of-squares distance as described in Section 3.2 was determined for whole implied volatility data set. For the chosen parameters, the Euclidean distance between the BJK implied volatilities and the market implied volatilities was found to be 0.00189. This is only slightly worse than Merton's model (c.f. 0.00168) found earlier. The fit between the BJK model and the market volatility is illustrated in Figure 5.5 for the 1 week and 3 month maturities. Running the implied volatility fitting procedure for the BJK model took extremely long. For instance, one iteration took just over 10 minutes. For the MJD and the VG models,

over 50 iterations were required, depending of course on the initial parameter set that was chosen. For this reason, backing out market-implied parameters is not feasible for the BJN model.

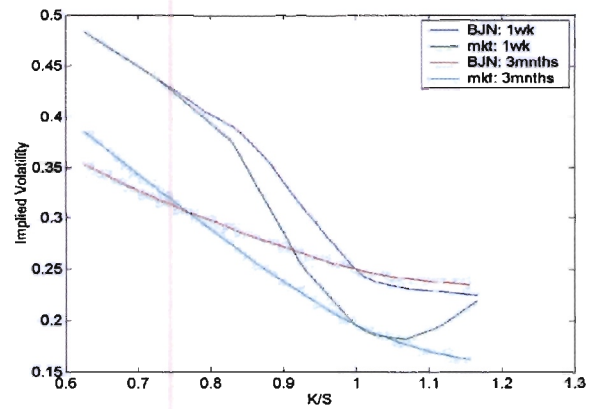


Figure 5.5: Fit of Upper-Bound BJNVol to MarketVol for maturities of 1 week and 3 months.

## Chapter 6

# Conclusion and Recommendations

The introduction of jumps into the returns of the underlying is a sensible enhancement of the pure diffusion model on which the B&S pricing formula is based. Excessive shocks are far more prevalent than suggested by the pure diffusion models and incorporating jumps into the underlying has a direct effect on the volatility smile that is observed in the market. Fitting this smile has been the focus of much literature since the first large shock in 1987 when the smile became more pronounced. Although there are benefits to adding jumps to the underlying, the effect is to make the model more complex - i.e. market incompleteness is introduced, estimating the extra parameters correctly becomes more difficult and the pricing formula becomes more complex. This dissertation was divided into an evaluation of Merton's 1976 seminal MJD model in relation to the benchmark B&S model; an evaluation of the variance gamma model and a model which obtains bounds on prices in incomplete markets due to jumps - i.e. the BJN model. These models were evaluated in relation to futures options on the Top40 index which is an actively traded JSE security.

The descriptive statistics for the Top40 index showed that the excess kurtosis is significantly different to what would be expected if the changes in the returns were driven by a standard normal distribution. The level of the kurtosis increased significantly also when the sampling

frequency was decreased from monthly to daily sampling. This is consistent with the behaviour of the MJD model. In support of this, it was found that the behaviour of the kurtosis in the returns when the sampling interval is increased is consistent with the MJD model. The paths that can be generated by including random-sized discrete jumps can be made to look like those found on the stock market, providing the jump parameters are chosen to be realistic.

The difference between the MJD model and B&S model was demonstrated for some typical parameters. This showed a significant difference between the two models, especially for very short maturities. These differences disappear quite quickly which is an inherent characteristic of the MJD model. The B&S model was shown to be too high at-the-money and too low out-the-money in relation to the MJD model. This is confirmation that the MJD model is capable of generating smiles that may be consistent with the smiles found in the market. Both steep and skew smiles can be generated by pricing options using Merton's model.

One of the objectives of this dissertation was to compare the model's implied stock price process parameters with the model's implied option pricing parameters. This comparison could indicate the risk aversion of the traders at that time. For the fit of the stock price process to the underlying prices, maximum likelihood techniques were used for the MJD model, while methods of moments techniques were used for the VG model. In terms of obtaining the implied option pricing parameters, the Euclidean distance between the volatility implied by the model to the implied volatility of the Top40 futures prices was minimised.

In relation to the parameters obtained for the underlying process, the drift of the implied jumps decreased significantly. Literature has indicated that fear amongst traders of a sudden downward jump is rolled into the price of the options. The large drop in the drift of the implied jump parameter does confirm this fear. There was a significant amount of change in the implied parameters when consecutive months of data was analysed which

confirms that the market price of risk does not remain the same over time. The analysis also confirmed that the MJD model fitted the market implied volatility smile better than the constant volatility of the B&S model. By backing out the implied volatility from the MJD model instead of the B&S model, the smile was reduced which is again confirmation that the Merton model performs better than the B&S model.

The pure point process of the variance gamma model was able to fit the Top40 smile considerably better than the MJD model. The *FFT* techniques applied to the variance gamma model as discussed are relatively fast compared to other numerical techniques. As per the MJD model, the implied variance gamma parameters obtained from the market option prices compared with the underlying parameters confirmed that there is significant fear amongst traders of crashes in the stock market.

In terms of the BJN model, this dissertation showed the workings of this incomplete-market model and how this model can incorporate discrete jumps in the underlying process. It was also shown what this model is capable of in terms of implied volatility smiles. The amount of skew that can be generated by the correct choice of parameters can be quite pronounced. Although proper parameterisation could not be carried out in terms of the Top40 smile, the BJN model did fit this smile quite well for rough estimates of the jump parameters, albeit not as well as Merton's 1976 model.

The Euclidean distance between the different models to the implied volatility smile can be summarised as follows:

- MJD to B&S: 0.00168
- VG to B&S: 0.000117
- BJN to B&S: 0.00189

In relation to the time taken to output option prices when compared with the MJD model

for  $N = 10$ , it was found that the VG model takes slightly longer than the MJD model for the option price to be attained (0.06 vs. 0.05 seconds), whereas the BJN model took 7.03 seconds to perform the same task. In relation to backing out the implied parameters from the market smile, it was found that the VG model is more efficient than the MJD model (22 minutes vs. 46 minutes for a typical parameterisation run). The BJN model takes a very long time to parameterise which is a serious shortcoming of this model. Parameterisation to the market implied volatilities could not be performed in this investigation.

Recommendations for further research:

- It would be interesting to find the implied parameters for a number of options that are traded on the JSE as a comparison with the parameters found on the Top40. In addition to this, it would be interesting to see how the performance of the Merton model varies over an extended time period.
- Stochastic volatility could be introduced in order to determine whether this shows further improvement to the market's implied volatilities.
- The effect of adding discrete jumps into the VG model would be an interesting investigation.
- Any of the more recent jump diffusion models mentioned in the Literature Review could be examined in the same way as this dissertation in order to evaluate their performance.
- A means of parameterising the BJN efficiently could be investigated in more detail.
- Testing the jump model outlined in Bjørk and Slinko (2004) in the context of the South African market could also be very interesting.
- A more

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