Dynamical Studies in Relativistic Cosmology

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Abstract

We conduct three investigations in Relativistic Cosmology – that is the Einstein Field Equations applied to the largest scales with source field typically taken to be a perfect fluid and fundamental observers comoving with the preferred fluid four-velocity.

We show using a tetrad analysis of the evolution equations for the dynamical variables and all the constraints these satisfy in classical General Relativity, that there are no new consistent perfect fluid cosmologies with the kinematic variables and the electric and/or magnetic parts of the Weyl curvature all rotationally symmetric about a common axis in an open neighbourhood $\mathcal{U}$ of an event. The consistent solutions of this kind are either locally rotationally symmetric, or most generally are subcases of the Szekeres model – an inhomogeneous dust model with no Killing symmetries. This result and its obvious future generalisations provides an input into the equivalence problem in cosmology necessary for a mathematically consistent understanding of probability and a measure set for universes required in quantum cosmology, for instance. We investigate such generalisations and find that similar results hold under some further assumptions dependent on the level of generalisation. In particular, we examine situations where either the electric part or the magnetic part of the free gravitational field are not rotationally symmetric, and also make a brief comment on the most general case where only the shear is rotationally symmetric.

We use a tetrad analysis to show that the well-known result that holds for relativistic shear-free dust cosmologies in Einstein's classical theory – either the expansion vanishes or the flow is irrotational – has an analogue in the Kaluza-Klein universe model, which has its roots presumably in string theory (or $\mathcal{M}$-theory), recently proposed by Randall and Sundrum. The Big Bang singularity of General Relativity can not be avoided in these so-called brane universes *in the situation where we neglect non-local tidal effects on the dynamics* by allowing the vorticity to spin up as the singularity is approached in shear-free cases. Moreover, we show that in the general case of a shearing perfect fluid, the singularity at the start of the universe is approached even more strongly than in classical General Relativity in the case of no tidal interaction.

Finally, we reconsider the issue of proving large scale spatial homogeneity of the universe in classical General Relativity, given isotropic observations about us and the possibility of source evolution both in numbers and luminosities. We use a spherically symmetric dust universe model (compatible with observations) for our investigation and we solve the field equations on the null cone analytically for the first time. Two theorems make precise the freedom available in constructing cosmological models that will fit the observations. They make quite clear that homogeneity cannot be proven without either a fully determinate theory of source evolution, or availability of distance measures that are independent of source evolution. We contrast this goal with the standard approach that assumes spatial homogeneity *a priori*, and determines source evolution functions on the basis of this assumption.

This work was done under the supervision of Professor George F. R. Ellis.

Contents

Conventions

Geometric Variables in N Dimensions

Base Quantities

We will denote a geometric or tensor object F in N dimensions by

$$^{N}F\,.$$

Thus in three dimensions, the Ricci scalar is ^{3}R. In four dimensions we will drop this convention as most of the time we will be dealing with quantities in four dimensional spacetime.

Subspace Quantities

The projection of a (typically) four-dimensional geometric or physical field F with respect to a flow with integral curves $\mathbf{u}$ which has associated projection operator $\mathbf{h}$ will be represented by a pre-superscript $^{\perp}$.

$$^{\perp}F = \mathbf{h}(F)\,.$$

Thus the projection of the Ricci scalar in four dimensions with respect to the flow lines $\mathbf{u}$ is $^{\perp}R$.

Indices Conventions

Index Ranges

Since we will be dealing with tetrad bases and related coordinate bases, we need to distinguish between the two. We follow the convention of [17], [47] and [57]. Lowercase Latin letters will range from $0-3$ and Greek indices range from $1-3$ (in the rest space). That is

$$a, i = 0, 1, 2, 3\,;$$

$$\alpha, \mu = 1, 2, 3\,.$$

Uppercase Latin letters may take on any range N – specified in the context. That is

$$A = 0, 1, 2, 3, \ldots N\,.$$

Letters from the first half of each alphabet (a, b,c ... or α, β, γ ...) are used for a general local tetrad basis and letters from the second half of each alphabet (i, j, k ... or μ, ν, κ ...) are used for the local coordinate basis.

Einstein Summation and Range Convention

Unless stated otherwise we will follow the Einstein summation convention; for ease of notation when dealing with sums of indexed objects,

$$\Sigma_{A=0}^{N} T^{A...B}{}_{C...D} U_{A...E}{}^{F...G} = T^{A...B}{}_{C...D} U_{A...E}{}^{F...G} .$$

When an index is repeated, this requires summing on that index. Thus the summa sign is not required in most cases. Dummy indices are implicitly summed and indices which do not occur twice (as a superscript and subscript) denote the numerical size (number of components or equations, etc.) of the quantity. The specific range of each free index is given according to the index range notation stated in the previous section and will always be finite.

Covariant and Contravariant Vectors

The pair of repeated indices in an indexed expression will almost always occur as a pair where one is on top and the other is at the bottom. This is in contrast to other fields of study, like classical fluid mechanics for example, where this distinction is not made. The reason for making this convention is to make clear the distinction between covariant quantities which are indexed at the bottom (for example a row vector with (three) components v_α) and contravariant quantities (for example the 1-form or column vector with (three) components v^α). Thus a distinction between a vector space of covariant vectors and its dual space of contravariant vectors is made.

So we may form the summed product of a row vector v_A and a column vector w^A – that is, a $1 \times N$ and an $N \times 1$ matrix respectively – by the expression $v_A w^A$. Contrariwise, the product of two column vectors, for example $\Sigma_{\alpha=1}^{3} v^\alpha w^\alpha$, cannot be represented according to the summation convention without introduction of a two-index object, say the identity matrix $\delta_{\alpha\beta} = \text{diag}\ [1, 1, 1]$, which allows us to form this product correctly as $\delta_{\alpha\beta} v^\alpha w^\beta = \Sigma_{\alpha=1}^{3} v^\alpha w^\alpha$. We will see that the two index object which is generally required is a metric function (like $\delta_{\alpha\beta}$ – the Euclidean metric).

Mixed notation

In tensor calculus, the indices on objects are related to objects of the same type – collections of numbers, functions, and so on. We will also use the convention on products which are composed of different types of object. In particular, basis vectors of an N-dimensional vector space may be written $\{\mathbf{e}_A\}_{A=0}^{N-1}$. Any vector in the space $\mathbf{v}$ may then be written as a linear combination of its components with respect to the basis $\mathbf{v} = v^A \mathbf{e}_A$ where the v^A are numbers, but the $\mathbf{e}_A$ are vectors. And componentwise we will write a vector with respect to a basis as $v^B = v^A e_A{}^B$ where the $e_A{}^B$ represent the components of the N basis vectors.

Exception to the Rule

We may wish occasionally to describe a function of N variables evaluated at the coordinate position (x^A). We will denote this by $f = f(x^A)$. The index A in this situation is subject to neither the summation nor the range conventions.

Bracketed Indices

Round brackets denote symmetrised indices, and square brackets denote skew-symmetrised indices. The symmetric part of a rank n tensor is the sum of all permutations divided by the number of permutations (i.e. $n!$). The antisymmetric part of a rank n tensor is the sum of all even permutations minus the sum of all odd permutations over $n!$. We will mostly encounter rank two tensors in this work. For a rank two tensor A_{ij},

$$\begin{aligned} A_{(ij)} &= \frac{1}{2}\left(A_{ij} + A_{ij}\right) \\ A_{[ij]} &= \frac{1}{2}\left(A_{ij} - A_{ij}\right) . \end{aligned}$$

Also frequently encountered is the antisymmetric part of a tensor on three indices. For example

$$A_{[ijk]} = \frac{1}{6}\left(A_{ijk} + A_{kij} + A_{jki} - A_{jik} - A_{kji} + A_{ikj}\right) .$$

What is also used quite often nowadays is a notation for the projected symmetric tracefree (PSTF) part of a tensor – angle brackets on the indices[1]. In a four-dimensional space-time for a rank two tensor,

$$A_{<ij>} = h^k{}_i h^l{}_j (A_{(kl)} - \frac{1}{3} A h_{kl}) .$$

where the trace A is defined as

$$A = A_{mn} h^{mn} .$$

Component-wise this looks like

$$A_{<ij>} = \begin{cases} 0 & \text{if} \quad i = 0 \text{ or } j = 0 \\ A_{(\mu\nu)} & \text{if} \quad i \neq 0 \text{ or } j \neq 0 \\ A^\mu{}_\mu & = 0 . \end{cases}$$

[1]This has its origins in relativistic kinetic theory where PSTF tensors are required for the definition of an irreducible basis for the moments of a distribution function.

Special Uses

The angle bracket notation is often used as well to denote

- the projection of a vector (as opposed to the $\perp$ notation).

$$v_{<a>} = {}^{\perp}v_a = (0, v_1, v_2, v_3)\,;$$

- in four dimensions, the tracefree part of projected symmetric tensor objects $V_{\alpha\beta}$

$$V_{<\alpha\beta>} = V_{(\alpha\beta)} - \frac{1}{3}Vh_{\alpha\beta}\,.$$

Derivatives

Covariant derivatives are denoted by a subscript semi-colon or by the ∇ operator and partial derivatives by a subscript comma or by the ∂ operator.

Constraint Equation Labelling

- We will, by and large, for constraints which hold for the completely general field equations follow the '$(C)^{\alpha\beta}$' labelling as in [30].
- To indicate that an equation only holds on a hypersurface $x^a = c^a$ (a constant), we will use the symbol

$$\stackrel{a}{=}\,.$$

Units

Throughout most of this work we will use so-called cosmological units in which the speed of light in vacuum and Einstein's gravitational constant are unity.

$$c = 8\pi G = \kappa = 1\,.$$

Chapter 1

Introduction

> The Universe is not to be narrowed down to the limits of understanding which has been man's practice up to now; but the understanding must be stretched and enlarged to take in the image of the universe as it is discovered – Sir Francis Bacon.

Cosmology has always concerned itself with a description of the universe in its entirety; that is, on the largest scales. Scientific cosmology attempts to apply the *laws of physics* on those scales as opposed to the more philosophical or metaphysical approaches which centuries ago prevailed. The scientific method with its roots in experiment, arguably first employed by Galileo, is probably responsible for the great successes enjoyed by modern physical theories especially evident today in technological applications. With these blatant successes on a terrestrial scale, science has become bold enough to apply these physical theories on the largest scales and – because of the universal upper speed limit that exists in theory and experiment – back to the furthest time scales. Whilst superficially this might seem to have put the physicist on some shaky ground because of the physical improbability (and sometime impossibility) of performing experiments on these scales, paradoxically, it is precisely because of this bold belief in the universality of physical laws that modern cosmology has been placed firmly in the realm of science. The predictions of the standard cosmological model has been confirmed in many ways, most notably the prediction of the $2.7K$ Cosmic Microwave Background Radiation – stemming from the Hot Big Bang which started the universe – has been detected and studied in some detail. Other successful predictions are the observed element abundances and the Hubble expansion of galaxies.

There are four known fundamental forces of nature – the strong and weak nuclear forces, electromagnetism and gravity. The first two, as is apparent from the adjective 'nuclear' in their names are short range forces. Only electromagnetism or gravity could possibly be dominating the dynamics of large scale structures with which cosmology is concerned. Cosmological points are generally taken to be galaxies. Within galaxies, neutral and ionised cosmic gas (e.g. HI and HII regions) constitutes a large fraction of their masses, and the dynamics of ionised gas is strongly influenced by pressure and magnetic fields. Now, if these galaxies had a net positive or negative charge on them, then electromagnetism would dominate. These electromagnetic effects are, however, not observed – galaxies do not behave as if they repel each other (as like charges would); indeed they form bound systems. It seems unnatural that on larger (cosmological) scales matter would behave fundamentally different to what it does on the smaller scales. Thus we may reasonably assert that gravity is the only force necessary to describe the behaviour of the physical cosmos.

Modern cosmology uses the theory of General Relativity (GR) which incorporates the Einstein Field Equations (EFE) to describe gravity. At a particular scale of physical interest, more detail is provided by other physical theories which apply in that physical domain. There are essentially two approaches in modern cosmology which have become standard in recent times: physical cosmology and relativistic cosmology. By physical cosmology we refer to approaches where the emphasis is on the physical fields that are cosmological in some domain of interest which have little or no interaction with the full nonlinearity of the field equations that govern the generally curved inhomogeneous spacetime itself. These approaches often include the assumption of a standard spatially homogeneous and isotropic model of constant curvature [1] – the Friedmann-Lemaître-Robertson-Walker (FLRW) model – or small deviations from this model[2]. Naturally, this approach is theoretically successful because of the numerous symmetries imposed on the spacetime. On the other hand, it is likely that this approach would tend to overlook the potentially rich effects which the field equations should contain due to their intrinsic complexity. Relativistic cosmology strives to include the full effects deriving from the curvature of space which is intrinsic to Einstein's theory. This form of cosmology is often more geometric than physical and would consider more general models than the standard model. It is this approach which we follow here. Part justification for this is that, although a standard FLRW model (or an almost-FLRW model) is often good enough in descriptions of physical effects or observations that occur, there are some situations where more general models or methods provide valuable insights, or even correct previously accepted beliefs obtained from not being careful with the standard model assumptions[3]. Alternatively, relativistic cosmology can be seen as providing a foundation for determining whether or not an almost-Newtonian (or post-Newtonian) approach will provide a good approximation to relativistic theory or not. Moreover, it is an elegant way of probing the universe.

For a long time it was considered useful to solve the field equations exactly for arbitrary metrics to obtain exact solutions. These exact solutions are often useful, but many give no consideration to physical restrictions and are thus not much more than exercises in solving differential equations and cannot honestly be said to promote the aim of cosmological research. For example, Delgaty and Lake [12] have shown that of 127 static, spherically symmetric solutions of the EFE chosen, – candidates for neutron star models – only nine satisfied their criteria of physical acceptability. Many 'new' exact solutions obtained by writing

[1] Often the zero-curvature (flat) model is chosen.
[2] See for example the excellent text by Weinberg [87].
[3] See for example [63] and the chapter on inhomogeneous observations, Chapter 12.

down a metric and solving the resultant field equations are often re-discoveries of well known solutions in different coordinates. Krasinski [48] reports, in particular, that the spherically symmetric dust (Lemaître-Tolman-Bondi) solution of the EFE, first discovered by Lemaître [52] in 1933, has since been independently rediscovered more than twenty times. Nowadays there are plenty of exact solutions to choose from for any particular application. What is more useful than finding more exact solutions perhaps is investigating what classes of solutions are allowed for realistic matter fields. This could possibly provide insights into the structure of the field equations themselves. Also, one should note that, although it is generally believed that the universe started off smoothly (homogeneous and isotropic) and then developed inhomogeneity and isotropy through some physical process (for example, quantum fluctuations after inflation), it is possible that the universe may have started off chaotic and then got smoothed out by physical processes. A knowledge of more general models and, in particular, the relationship between these and the standard model is useful.

In the field of relativistic cosmology, group symmetry methods have been widely applied in classification of models. It is also fairly common to use a method based on a dynamical tensor description of the field equations initiated by J. Ehlers [14] and others in the sixties (for example [17]). We will review the covariant treatment of this approach following [18] mainly and then the tetrad formalism and ONT approach as in for example [17, 21]. This part of the work may be omitted by readers familiar with this subject, and may be used just as an occasional reference when certain key equations are required.

A relevant field of study is the attempt to provide a classification of all solutions of the Einstein field equations. This is of course more than providing a mere catalogue of solutions as it relates to the equivalence problem in relativity. A general problem in cosmology, and specifically in quantum cosmology, is the absence of a measure on the set of allowed solutions. This makes assertions about probabilities for universes to exist mathematically and physically unsound. Thus an invariant classification of the solution size and types of allowed universes in this field is of great value. We provide some input to this field in the cosmological scenario. This is work done in collaboration with George Ellis, Henk van Elst and Mattias Marklund. We do a tetrad consistency analysis on cosmological models where some dynamical tensors exhibit rotational symmetry making them partially locally rotationally symmetric. The physical background to this work is as follows. In principle, we may invariantly classify all cosmologies by using a non-degenerate shear eigentetrad; and if the shear is completely degenerate – that is it vanishes – then that simplifies the equations greatly and leads to well-known and well-studied solutions. The only other situation that remains to be covered is the one where the shear picks out one preferred direction. Ultimately, this is what we aim towards clarifying[4].

Relativistic cosmology normally incorporates the Einstein Field Equations as a description of gravity. However, it is believed by many that the reason why gravity cannot be unified as yet with the other fundamental forces is because it is an effective low energy limit of a higher order theory. The most popular

[4]The first installment of this project, Chapter 6, has been published in *Classical and Quantum Gravity* [64].

candidates currently for this unified theory are higher-dimensional string theories where a field fitting the description of the graviton pops out naturally. As one can expect, a lot of theoretical research has gone into these theories, which show some mathematical interconnectedness, from the particle or high-energy physics side. Some of these models have been used as toy models of the universe as well. Perhaps what is lacking here is an input from the relativity side. Normally, one requires that the extra dimensions in these theories be compactified in some way; and a background is assumed which is usually uninteresting to a relativist. A recent development has been that this "compactification" can be achieved, in effect, by using curved space. We have done some investigations in this model, considered as a correction to the standard relativistic cosmology. We thought that an interesting question to ask is whether we can use standard results from GR which seem to be intrinsic to the theory to probe corrections to GR from a cosmological perspective. The most profound thing here is of course the existence of the initial Big Bang singularity and also a related issue to that is the very surprising result that for shear-free dust, either the vorticity vanishes or the universe is static. We use this result to probe the recently popular brane models (which are assumed to arise from a higher-dimensional string theory). We find the class of solutions allowed for shear-free dust in a simplified brane cosmology framework. To do this, we derive the tetrad equations up to the differential Bianchi identities and then do a consistency analysis. This is part of work done in collaboration with Bruce Bassett.

The concluding part is somewhat different to the above programmes: a study of source evolution in an inhomogeneous universe[5]. Arguably the most successful working hypothesis in modern cosmology is the Copernican principle and its derivatives, the cosmological principle, etc. Loosely stated, it says something like this: physics should not differ from what it is here in other parts of the universe. Of course, on the face of it, this seems a reasonable assumption, but one should be careful what one means by "physics" here. If we assume too much too often, then we are on dangerous ground since we may lose track of what our assumptions were to start off with. A case in point here is the assumption of a homogeneous and isotropic universe as a model of the real universe. This is indeed a very successful model, passing many independent tests, and can be traced back directly to the Copernican hypothesis. However, the real observed universe is *not* isotropic and homogeneous on all scales and, indeed, in many instances, the inhomogeneity of the universe has to be taken into account when building models of physical processes, for instance. We suggest that maybe the models of galaxy evolution may be suffering somewhat under this type of problem. We show quite clearly that homogeneity cannot be proven without either a fully determinate theory of source evolution, or availability of distance measures that are independent of source evolution. We contrast this goal with the standard approach that assumes spatial homogeneity *a priori*, and determines source evolution functions on the basis of this assumption. We use a simple spherically symmetric dust solution of the Einstein field equations for this. We obtain the field equations on the null

[5]This thesis would have had a central theme – that of consistency checks – if we had not included this section.

cone for the most general solutions here for the first time[6]. This was work done in collaboration with Charles Hellaby and George Ellis.

[6]This work has been published in *Monthly Notices of the Royal Astronomical Society* [65].

Part I

Background

Chapter 2

Covariant Cosmology

2.1 Metric Tensor and its Connection

2.1.1 Metric Structure

The spacetime of general relativity consists of a manifold with a rank two *metric tensor* **g**, which is symmetric, together with a manifold. We require g_{ij} to be nonsingular in the sense that it is invertible. Raising and lowering of indices is done by the metric tensor and the *inverse metric tensor* g^{ij} defined by

$$g^{ij} g_{jk} = \delta^i{}_k \,. \tag{2.1}$$

The metric tensor field provides us with an inner product for two contravariant or covariant vectors at each point P of a manifold $\mathcal{M}$. The inner product of two vectors v^i and w^i is

$$g_{ij} v^i w^j = g^{ij} v_i w_j = v_i w^i = v^i w_i \,. \tag{2.2}$$

Since the metric tensor describes the spacetime field, it gives information about distances, times elapsed, angles and other geometric quantities – precisely what a physicist would want a metric to describe. It is however not positive definite, but instead has signature $[-1, +1, +1, +1]$ in classical GR theories. A non-zero vector v^i in a space with Lorentzian scalar product is called

$$\left\{ \begin{array}{l} \textit{timelike} \\ \textit{null} \text{ or } \textit{lightlike} \\ \textit{spacelike} \end{array} \right\} \quad \text{if} \quad \left\{ \begin{array}{l} g_{ij} v^i v^j < 0 \\ g_{ij} v^i v^j = 0 \\ g_{ij} v^i v^j > 0 \end{array} \right\} . \tag{2.3}$$

The length of a vector $\mathbf{v}$ is given by

$$\sqrt{|g_{ij} v^i v^j|} = \sqrt{|g^{ij} v_i v_j|} = \sqrt{|v_i v^i|} \,. \tag{2.4}$$

The angle ϑ between two non-null vectors v^i and w^j is

$$\cos \vartheta = \frac{g_{ij} v^i w^j}{\sqrt{|g_{kl} v^k w^l|} \sqrt{|g_{mn} v^m w^n|}} \,. \tag{2.5}$$

Any two vectors (null or not) are orthogonal if their inner product is zero.

2.1.2 The Metric Connection

Any non-degenerate symmetric rank two tensor field has associated with it an affine connection $\Gamma^i{}_{jk}$. The affine connection in GR is defined with respect to a local inertial coordinate system $\xi^i(t, \mathbf{x})$ as

$$\Gamma^i{}_{jk} = \frac{\partial x^i}{\partial \xi^r} \partial_j \partial_k \xi^r \,. \tag{2.6}$$

These coefficients allows for a definition of covariant derivative with respect to that field. In general relativity, the metric tensor g_{ij} is generally chosen to have a torsion free connection – the *Christoffel symbols*[1]

$$\Gamma^i{}_{[jk]} = 0 \,. \tag{2.7}$$

This connection is the only one which may be expressed in terms of (partial) derivatives of the metric tensor [87].

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left[\partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{jk} \right] . \tag{2.8}$$

Conversely (2.8) implies clearly that the partial derivatives of the metric tensor may be expressed in terms of the Christoffel symbols:

$$\partial_k g_{ij} = g_{mj} \Gamma^m{}_{ki} + g_{mi} \Gamma^m{}_{kj} \,. \tag{2.9}$$

We could think of the metric tensor as defining a connection which corrects for the variations in the coordinate basis (or that provides for a rule for parallel transport of vectors).

2.1.3 Covariant Derivative

For any vector field v_i, with contravariant components v^i, the *covariant derivative* is

$$\begin{aligned} \nabla_j v_i &= \partial_j v_i - \Gamma^k{}_{ij} v_k \\ \nabla_j v^i &= \partial_j v^i + \Gamma^i{}_{kj} v^k \,. \end{aligned} \tag{2.10}$$

More generally, for a rank n type (r, s) tensor

$$\begin{aligned} &\nabla_k T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = \partial_k T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} + \Gamma^{i_1}{}_{lk} T^{l \ldots i_r}{}_{j_1 \ldots j_s} + \ldots \Gamma^{i_r}{}_{lk} T^{i_1 \ldots l}{}_{j_1 \ldots j_s} \\ &- \Gamma^l{}_{j_1 k} T^{i_1 \ldots i_r}{}_{l \ldots j_s} - \ldots \Gamma^l{}_{j_s k} T^{i_1 \ldots i_r}{}_{j_1 \ldots l} \,. \end{aligned} \tag{2.11}$$

The covariant derivative is defined such that

$$\nabla_k g_{ij} = 0 \,, \tag{2.12}$$

which one can check by substitution of (2.8) into (2.11).

[1] These are often denoted by $\Gamma^i{}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$

2.2 Cosmological Assumptions

We will mean by a *cosmology* a spacetime which satisfies all of the following requirements [14, 17, 21]:

- It is a self-consistent solution of field equations, typically the Einstein field equations ('EFE'), relating any matter source fields which are represented by an energy–momentum–stress tensor T_{ab} to the Ricci curvature tensor R_{ab} and its trace R as

$$R_{ij} - \tfrac{1}{2} R\, g_{ij} = T_{ij} \; . \tag{2.13}$$

- It is filled with matter energy of some sort; which we will represent as a perfect fluid with energy density μ and isotropic pressure p.

- There is a unit timelike vector field — the preferred 4-velocity $\mathbf{u}$ — describing the 4-velocity of the *fundamental observers* in the cosmology. This can always be non-ambiguously defined as the timelike eigendirection of the Ricci tensor R_{ab} if we assume that the first and second attributes in this list hold with

$$(\mu + p) > 0 \; .$$

- Observations have shown that the Universe is expanding. We will take this to mean that the (isotropic) expansion of the 4-velocity field $\mathbf{u}$, described by the rate of expansion scalar Θ (defined further down) is positive:

$$\Theta > 0 \; .$$

2.3 Preferred Four-velocity

In modern cosmology, an added structure on the spacetime of general relativity is that there is a preferred four velocity for the matter content. The matter contained in the universe is considered in most applications to be a fluid with four-velocity u_i describing the velocity of an observer moving with the fluid. This vector can be taken to be a unit vector.

$$u_i u^i = -1 \, . \tag{2.14}$$

This can always be non-ambiguously defined as the eigendirection of the Ricci tensor R_{ij} if we take for granted that $\mu + p \neq 0$ (which we will). We shall use a superscript dot to denote the covariant derivative in the u^i direction since it is normally proper time in cosmology.

$$\left(\dot{T}_{ij\ldots l}\right) \equiv T_{ij\ldots l;m} u^m \equiv \nabla_m T_{ij\ldots l} u^m \, . \tag{2.15}$$

We can define a tensor which projects into the rest space of an observer by

$$h_{ij} = g_{ij} + u_i u_j \, . \tag{2.16}$$

This tensor is orthogonal to u_i,

$$h_{ij}u^i = 0\,; \tag{2.17}$$

it is a projection operator –

$$h^i{}_j h^j{}_k = h^i_k \tag{2.18}$$

and

$$h^i{}_i = 3\,. \tag{2.19}$$

We denote the projection of any tensor in a four-dimensional cosmology by a pre-superscript $\perp$, so

$$^{\perp}T_{ij...l} \equiv h^k{}_i h^m{}_j ... h^n{}_l T_{km...n}\,. \tag{2.20}$$

2.4 Irreducible Splitting of Rank One and Rank Two Tensors

2.4.1 Vector Splitting

When one has a preferred timelike vector field $\mathbf{u}$ with projector h_{ij}, any spacetime vector v^i can be covariantly split relative to the preferred four-velocity as

$$v^i = au^i + b^i, \quad \text{where} \quad a = -u_i v^i,\ b^i = h^i{}_j v^j \Leftrightarrow b^i u_i = 0\,. \tag{2.21}$$

This is just the decomposition of a vector $\mathbf{v}$ into a part orthogonal to, and a part parallel to another vector $\mathbf{u}$, familiar from vector analysis.

2.4.2 Rank Two Tensor Splitting

In a similar fashion, any rank two tensor V_{ij} can be split up relative to a vector $\mathbf{u}$ as follows

$$V_{ij} = Au_iu_j + B_iu_j + u_iC_j + D_{ij}, \tag{2.22}$$

where

$$A = V_{ij}u^iu^j,\ B_iu^i = 0 = C_iu^i \tag{2.23}$$

and

$$D_{ij} = h_i{}^k h_j{}^l V_{kl} \Leftrightarrow D_{ij}u^i = 0 = D_{ij}u^j\,. \tag{2.24}$$

Furthermore we may split V_{ij} up into a symmetric and (traceless) skew-symmetric part[2].

$$V_{ij} = V_{(ij)} + V_{[ij]}\,. \tag{2.25}$$

The symmetric part can have its trace $V = V_{kl}h^{kl}$ removed

$$V_{(ij)} = \frac{1}{3}Vh_{ij} + V_{<ij>}\,. \tag{2.26}$$

[2]Or equivalently we may prefer to split the projected part D_{ij}.

Here we are using the diagonal brackets to denote the projected symmetric, tracefree part of V_{ij} which is defined as:

$$V_{<ij>} = \left(V_{(kl)} - \frac{1}{3}V_{mn}h^{mn}h_{kl}\right)h_i{}^k h_j{}^l\,. \tag{2.27}$$

Thus any rank two tensor we may write in covariant irreducible form as

$$V_{ij} = Au_iu_j + B_iu_j + u_iC_j + \frac{1}{3}V_{kl}h^{kl}h_{ij} + V_{<ij>} + V_{[kl]}h^k{}_ih^l{}_j\,. \tag{2.28}$$

2.5 Fluid Kinematics

When we perform the above splitting procedure on the covariant derivative of the fluid four-velocity itself – which is a rank two tensor – thus writing this in irreducible form, we get the quantities that describe the fluid kinematics.

$$u_{i;j} = -\omega_{ij} + \sigma_{ij} + \frac{1}{3}\theta h_{ij} - \dot{u}_iu_j\,. \tag{2.29}$$

The new tensor quantities defined here are all orthogonal to the fluid flow; i.e. spacelike. The vorticity tensor $\omega_{ij} = -\omega_{ji} = \omega_{[ij]}$ – that is, it is skew-symmetric[3]. The rate of shear tensor σ_{ij} is symmetric, $\sigma_{ij} = \sigma_{ji} = \sigma_{(ij)}$, and tracefree $\sigma^i{}_i = 0$: i.e. $\sigma_{ij} = \sigma_{<ij>}$. The expansion scalar is the divergence of the fluid flow lines $\theta = u^i{}_{;i}$ and $\dot{u}_i$ can be recognised as the acceleration vector of the four-velocity. The completely skew-symmetric tensor η^{ijkl} is specified by

$$\eta_{ijkl} = \eta_{[ijkl]}\,, \eta_{0123} = \sqrt{|det g_{ij}|}\,. \tag{2.30}$$

Using the volume element tensor η^{ijkl}, which is also often called the permutation tensor, we note that the vorticity tensor ω_{ij} is equivalent to a vector defined by $\omega^i \equiv \frac{1}{2}\eta^{ijkl}u_j\omega_{kl}$ which lies in the rest space of u^i and defines the instantaneous axis of rotation of the fluid due to vorticity. We define the tensor magnitudes

$$2\omega^2 = \omega^i{}_j\omega_i{}^j\,, 2\sigma^2 = \sigma^i{}_j\sigma_i{}^j \text{ and } \dot{u}^2 = \dot{u}^i\dot{u}_i\,. \tag{2.31}$$

2.6 The Riemann-Christoffel Curvature Tensor

The *Ricci identities* demonstrate that generally in a curved manifold when one takes second covariant derivatives of a vector quantity v^i, these do not commute, but instead give rise to a *curvature tensor* **R**

$$v^i{}_{;kl} - v^i{}_{;lk} = -R^i{}_{jkl}v^j \tag{2.32}$$

that is in terms of the ∇ operator

$$2\nabla_{[k}\nabla_{l]}v^i = R^i{}_{jkl}v^j\,. \tag{2.33}$$

[3]The components of the vorticity vector as defined here have the opposite sign as compared to [17, 30, 57]. This is so as to have ω^i having the same sign as in Newtonian mechanics.

This tensor, called the Riemann-Christoffel tensor or Riemann tensor, is the only tensor which can be constructed from the metric tensor g_{ij} and its first and second derivatives and is linear in the second derivatives [87]. Of course, one cannot construct a new tensor from g_{ij} and its first derivatives only, because a locally inertial coordinate system can always be chosen in which g_{ij} has the form η_{ij}, and thus the new tensor may be formed out of η_{ij} alone. Invoking the principle of equivalence, we see that there can be no such tensor in GR [87]. The justification for calling this the curvature tensor is the fact that any vector when parallel transported around an arbitrary small closed curve at some point will retain the same alignment if $R^i{}_{jkl} = 0$ at that point. Thus, for example, if two gyroscopes placed in different orbits around the earth are aligned together initially at a point and later non-aligned when their orbits intersect, then the difference in alignment is a measure of the curvature induced by the earths gravitational field. With our definition of covariant derivative (2.11) and from the Ricci identities, we may determine that the components of the Riemann tensor are given by

$$R^i{}_{jkl} = \partial_l \Gamma^i{}_{jk} - \partial_k \Gamma^i{}_{jl} + \Gamma^m{}_{jk}\Gamma^i{}_{lm} - \Gamma^m{}_{jl}\Gamma^i{}_{km} \,. \tag{2.34}$$

If we now substitute (2.8) into the above expression (2.34) and the well known identity arising from the definition of the inverse metric tensor (2.1), viz.

$$g_{ij}\partial_k g^{jl} = -g^{jl}\partial_k g_{ij} \,, \tag{2.35}$$

with equation (2.9); and recalling that partial derivatives commute (??), then after some simplification we may write the components of the Riemann tensor in terms of a sum of second partial derivatives of the metric tensor and products of the metric tensor and Christoffel symbols –

$$R_{ijkl} = \frac{1}{2}\left(\partial_j\partial_l g_{ik} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_i\partial_k g_{jl}\right) + g_{mn}\left(\Gamma^m{}_{ki}\Gamma^n{}_{jl} - \Gamma^m{}_{li}\Gamma^n{}_{jk} \,.\right) \tag{2.36}$$

2.6.1 Algebraic and Differential Identities

The Riemann tensor components are related by the cyclic identities

$$R^i{}_{[jkl]} = 0 \,, \tag{2.37}$$

that is

$$R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} = 0 \,; \tag{2.38}$$

by the symmetry relations

$$R_{ijkl} = R_{klij} \,; \tag{2.39}$$

and by the antisymmetry relations

$$R_{[ij][kl]} = 0 \,, \tag{2.40}$$

– by this notation we mean –

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{jilk}\,, \tag{2.41}$$

where $R_{ijkl} = g_{mi}R^m{}_{jkl}$ was used to lower the index. These algebraic relations can be read off from (2.36). Together they show that there is only one rank two tensor, the Ricci tensor, and essentially only one scalar, the Ricci-scalar, which can be formed from the Riemann tensor in four dimensions. Differentially they are related by the Bianchi Identities

$$\nabla_{[m}R_{ij]kl} = 0\,. \tag{2.42}$$

2.6.2 Number of Independent Components

In a space of N dimensions a rank r tensor in general has N^r components. For example R_{ijkl} has N^4 components. But these components are not independent because of the relations (2.37), (2.40) and (2.41). To count the number of independent components of the Riemann tensor we follow Weinberg [87] by adopting the 'Petrov notation' [67]. In this notation one thinks of the Riemann tensor as a matrix $R_{\{ij\}\{kl\}}$ in the 'indices' ij and kl. The antisymmetry relation (2.41) shows that each 'index' is like an N-dimensional antisymmetric matrix. Thus the number of components of this Riemann 'matrix' is given by the sum of the so-called triangular numbers $\{1, 2, 3, ..., N-1\}$; with the triangle being formed on either side of the diagonal but excluding the diagonal. The number of independent components are thus the area of the triangle $A = \frac{1}{2}N(N-1)$. The symmetry relation (2.40) shows that the Riemann 'matrix' $R_{\{ij\}\{kl\}}$ is like an $A \times A$ symmetric matrix. Thus the total number of components of R_{ijkl} is given again by the sum of the triangular numbers (with the diagonal included this time) – $\frac{1}{2}A(A+1) = \frac{1}{8}N(N-1)(N^2-N+2)$ – subject to the cyclic symmetry constraint (2.37) which is completely antisymmetric because of (2.40) and (2.41). So (2.37) provides $\begin{pmatrix} N \\ 4 \end{pmatrix} = \dfrac{N(N-1)(N-2)(N-3)}{4!}$ constraints. Thus the number of independent components of the Riemann tensor in N dimensions R_N is

$$R_N = \frac{1}{12}N^2(N^2-1)\,. \tag{2.43}$$

In particular, for a four dimensional spacetime the number of independent components of the Riemann tensor is $R_4 = 20$.

2.6.3 Decomposition

The Riemann curvature tensor may be decomposed into the completely tracefree *Weyl conformal curvature tensor* C_{IJKL}, the *Ricci curvature tensor* obtained by contracting on the first and third indices of the Riemann tensor

$$R_{IJ} = R^K{}_{IKJ}\,, \tag{2.44}$$

and the *Ricci curvature scalar* $R = g^{IJ}R_{IJ}$ as

$$R_{IJKL} = C_{IJKL} + g_{I[K}R_{L]J} + g_{J[L}R_{K]I} - \frac{1}{N-1}Rg_{I[K}g_{L]J} \tag{2.45}$$

$$= C_{IJKL} + \frac{1}{N-2}\left(g_{IK}R_{JL} - g_{IL}R_{JK} - g_{JK}R_{IL} + g_{JL}R_{IK}\right) - \frac{1}{(N-1)(N-2)}\left(g_{IK}g_{JL} - g_{IL}g_{JK}\right)R \tag{2.46}$$

where we have used the fact that the Ricci tensor is symmetric because of the symmetry property of the Riemann tensor (2.40). Clearly

$$C^I{}_{JIL} = 0\,.$$

The number of components of the Weyl tensor is equal to $R_N - \frac{1}{2}N(N+1) = \frac{1}{12}N(N+1)(N+2)(N-3)$ which for $N = 4$ amounts to ten. It thus follows that the Weyl tensor and Ricci tensor each has ten independent components. We may naturally think of the Ricci tensor as being the trace of the Riemann tensor. When $R_{IJ} = 0$, the conformal tensor is the same as the Riemann tensor. The Einstein field equations relate the matter content to the Ricci tensor, but not the Weyl tensor – which thus describes the 'free gravitational field'. The Weyl tensor is also called the conformal tensor. The justification for this is that g_{IJ} is proportional to a constant matrix locally [4] iff $C_{IJKL} = 0$ everywhere.

2.6.4 Curvature Invariants

The components of the Riemann tensor gives the curvature of the spacetime, but is dependent on the introduction of a specific coordinate system; i.e. not in an invariant way. To invariantly characterise a spacetime we need to find all curvature invariants which may be formed by the curvature tensor and the metric tensor – the so-called *curvature scalars*. However, whilst the curvature tensor determines the scalars, the curvature scalars do not determine curvature[5]. In a coordinate system x^I these tensors will have $R_N + A$ independent components [6] subject to constraints due to gauge transformations $x^I \to x^{I'}$. The coordinate transformations leave the quantities $\frac{\partial x^{I'}}{\partial x^I}$ arbitrary; so this arbitrariness consists of N^2 equations. Thus the number of scalar invariants is equal to $\frac{1}{12}N(N-1)(N-2)(N-3)$. For $N = 4$ there are 14 scalar invariants, some of which are, however, differentially related to each other. Coordinate axes may be chosen such that R_{ij} and g_{ij} are diagonal with the components of g_{ij} being ± 1 or 0. The 14 curvature invariants then consist of the 10 Weyl tensor components and the 4 (non-degenerate) eigenvalues of the Ricci tensor. For a fluid filled cosmology – the case we will concern ourselves with in this work – with $\mu + p \neq 0$, the timelike eigenvalue of the Ricci tensor defines a unique direction [18] (the fluid four-velocity), but the other 3 eigenvalues of the Ricci tensor may – and in most tractable situations, do – become degenerate. Under these circumstances the above scheme must be adapted.

[4] Recall that a conformal mapping is one which preserves angles, but not lengths.

[5] In particular, the solutions of the field equations describing colliding plane waves have vanishing curvature scalars, but singular $R_{abc}{}^d$ [36].

[6] Remember g_{IJ} is symmetric; therefore the total number of independent components = A.

2.6.5 Electric and Magnetic Parts of the Weyl Tensor

When one has a preferred timelike vector field $\mathbf{u}$ with projector h_{ij}, the Weyl conformal curvature tensor may be decomposed with respect to the group of spatial rotations in a $1+3$ covariant way [14, 18], into its symmetric tracefree 'electric' E_{ij} and 'magnetic' H_{ij} parts. The Weyl conformal curvature tensor may be decomposed with respect to the group of spatial rotations relative to the preferred 4-velocity into its symmetric tracefree 'electric' and 'magnetic' parts, $\mathbf{E}$ and $\mathbf{H}$, respectively, according to [14, 18]

$$E_{ij} := C_{kmln}\, h^k{}_i\, u^m\, h^l{}_j\, u^n = E_{(ij)} \tag{2.47}$$

$$H_{ij} := (-\tfrac{1}{2}\,\eta_{kmop}\, C^{op}{}_{ln})\, h^k{}_i\, u^m\, h^l{}_j\, u^n = H_{(ij)}\,. \tag{2.48}$$

Thus we may write the Weyl tensor as

$$C^{ij}{}_{kl} = 4\, u^{[i}\, u_{[k}\, E^{j]}{}_{l]} + 4\, h^{[i}{}_{[k}\, E^{j]}{}_{l]} + 2\,\varepsilon^{ijm}\, u_{[k}\, H_{l]m} + 2\,\varepsilon_{klm}\, u^{[i}\, H^{j]m}\,, \tag{2.49}$$

where $\varepsilon_{\mu\nu\kappa}$ is the 3-space permutation symbol obtained by projecting η_{ijkl} into the rest 3-space orthogonal to $\mathbf{u}$, $\varepsilon_{ijk} = \eta_{lijk}\, u^l$. The second Bianchi identity differentially relates components of the Riemann tensor:

$$\nabla_{[i} R_{jk]lm} = 0\,. \tag{2.50}$$

As well as entailing the matter conservation equations $\nabla_j T^{ij} = 0 = \nabla_j (R^{ij} - \frac{1}{2}\, R\, g^{ij})$, given the $1+3$ decompositions (2.47) and (2.48), this relation provides evolution and constraint equations for $\mathbf{E}$ and $\mathbf{H}$ [18, 21].

2.7 Energy-Momentum Tensor

On some scale and at various epochs, classical relativity assumes that the matter contained in the universe behaves like a gas. It is common to model the post-decoupling universe as a fluid[7]. This approximation assumes that the gas is sufficiently close to equilibrium to allow for a well defined (smooth) fluid four-velocity and energy density. The stress-energy-momentum tensor – which is symmetric – for a generally viscous, or imperfect, fluid comoving with the preferred timelike congruence (commonly the fluid flow lines) in non-equilibrium state is given in terms of dynamical variables with direct physical interpretation as

$$T_{ij} = \mu u_i u_j + p h_{ij} + 2 q_{<i} u_{j>} + \pi_{ij}\,. \tag{2.51}$$

Its trace $T = T^i{}_i$ is

$$T = -\mu + 3p\,. \tag{2.52}$$

The total energy density, μ, is the contribution of T_{ij} fully contracted in the fluid four-velocity direction

$$\mu = T_{ij} u^i u^j\,, \tag{2.53}$$

[7] Specifically a dust model with a test field of free-streaming photons over it is deemed appropriate.

p is the isotropic pressure in the three-space h_{ij}.

$$p = \frac{1}{3} T_{ij} h^{ij}\,, \tag{2.54}$$

q_i, the energy current density is the contribution contracted in the u^i direction and projected.

$$q_i = q_{<i>} = -T_{kj} h^k{}_i u^j\,. \tag{2.55}$$

The anisotropic pressure π_{ij} is the projected tracefree symmetric part of T_{ij} perpendicular to the four-velocity.

$$\pi_{ij} = \pi_{<ij>} = T_{kl}[h^k{}_{(i} h^l{}_{j)} - \frac{1}{3} h_{ij} h^{kl}]\,; \tag{2.56}$$

$\pi_{ij} = \pi_{ji}$ and $\pi^i{}_i = 0$.

2.7.1 Perfect Fluid

For most cosmological epochs, the anisotropic dissipative terms may be neglected. A perfect fluid has only scalar contributions to its energy-momentum, i.e.

$$q_i = \pi_{ij} = 0\,;$$

so that

$$T_{ij} = \mu u_i u_j + p h_{ij}\,. \tag{2.57}$$

An equation of state relating these quantities generally are required to specify the physics of the perfect fluid which can, and is, often stated in terms of thermodynamic quantities: for example, $p = p(\mu, T)$ where T is the temperature. The fluid flow will be isentropic (i.e. reversible) if there is a barotropic equation of state $p = p(\mu)$; for example an ideal gas has $p = \beta\mu^\gamma$, β and γ constant. Dust is a perfect isentropic fluid for which the isotropic pressure also vanishes ($p = 0$). So for dust, the energy-momentum tensor takes on the very simple form

$$T_{ij} = \mu u_i u_j\,. \tag{2.58}$$

2.8 Einstein Field Equations

The Einstein Field Equations relate the matter fields to the only algebraically independent rank two (and lower) curvature tensors; i.e. the metric tensor g_{ij}, the trace of the Riemann tensor – the Ricci tensor R_{ij} and the trace of the Ricci tensor – R.

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = T_{ij} \tag{2.59}$$

where Λ is a constant called the cosmological constant and is often set to zero (as we will do subsequently). If we take the trace of this equation we find the relationship between the Ricci-scalar and the trace of the energy-momentum tensor:

$$R = 4\Lambda - T \tag{2.60}$$

and so the EFE may equivalently be written as

$$R_{ij} = T_{ij} - \frac{1}{2}T g_{ij} + \Lambda g_{ij}\,. \tag{2.61}$$

If we now make the standard assumption that the source of the gravitational field in our universe is a fluid with a timelike-directed unit flow vector u^i, we may write the Ricci tensor as

$$R_{ij} = \frac{1}{2}(\mu + 3p - 2\Lambda)u_i u_j + 2q_{(i}u_{j)} + \frac{1}{2}(\mu - p + 2\Lambda)h_{ij} + \pi_{ij}\,. \tag{2.62}$$

We can link these relations with the kinematic quantities through the Ricci identities applied to $\mathbf{u}$, but since in the next chapter we give the tetrad form of the full set of equations, this seems rather like overkill and we thus omit it.

Chapter 3

Tetrad Formalism

A set of four contravariant vectors $\{\mathbf{e}_a\}_{a=0}^{4}$ that are linearly independent at each point of spacetime is generally called a tetrad. These vectors, taken as tangent vectors, define the directional derivatives – for any field f, $\mathbf{e}_a(f) = e^a{}_i\partial_i$ in a particular coordinate system. The use of a tetrad formalism is often useful when dealing with covariant quantities. Any covariant quantity can be expressed in terms of its individual tetrad components which then become scalar invariants if the tetrad is uniquely defined. A choice of tetrad allows for simplifying the field equations perhaps leading to clues as to what coordinates to use concomitant to the symmetries in the spacetime. Indeed, in many ways the covariant and tetrad approaches complement each other [26]. In GR, two choices of tetrad basis are useful. The one is especially useful for describing gravitational radiation and massless fields – the Newman Penrose formulation – based on complex null bases and the other is the Orthonormal Tetrad (ONT) used mostly in cosmology. For cosmological purposes where one has a preferred four-velocity at every event, it is often convenient to use this frame which aligns one of its legs with the principal cosmological direction and is an orthonormal basis. Application of a tetrad to the spacetime makes all components of a tensor geometric scalar objects in the frame.

3.1 Tetrad Description

We follow the tetrad methods of [17], but use the notation for the spatial rotation coefficients developed in [23, 57, 30], and utilised to great effect by Wainwright and collaborators (see [84] for a survey).

3.1.1 Generic Tetrad Equations

We choose a basis set of four contravariant vector fields $\{\, \mathbf{e}_a \,\}$ that are linearly independent at each event of the spacetime manifold $\mathcal{M}$ — a tetrad. These are defined as directional derivatives for any field f by

$\mathbf{e}_a(f) := e_a{}^i\,\partial_i f$ in a particular local coordinate system[1] $\{x^i\}$. Any covariant quantity can be expressed in terms of its components in this tetrad basis which then become individual scalar fields. Thus, for example, the tetrad components of the energy–momentum–stress tensor T_{ab} for a perfect fluid comoving with the preferred timelike congruence $\mathbf{u}$ is given by

$$T_{ab} = T_{ij}\, e_a{}^i\, e_b{}^j \ ,$$

and the relations

$$e_a{}^i\, e^a{}_j = \delta^i{}_j \ , \quad e_a{}^i\, e^b{}_i = \delta_a{}^b \ , \tag{3.1}$$

define the inverse coordinate components $e^a{}_i$ of the tetrad basis $\{\mathbf{e}_a\}$. Any change of tetrad basis by a (nonsingular) transformation $\mathbf{\Lambda}$,

$$\mathbf{e}_{a'} = \Lambda^{-1}{}_{a'}{}^{a}\mathbf{e}_a \tag{3.2}$$

relates components of a vector in the new and old bases as

$$v^{a'} = \Lambda^{a'}{}_a v^a \tag{3.3}$$

where $\Lambda^{a'}{}_a$ and $\Lambda^{-1}{}_{a'}{}^a$ are mutually inverse matrices such that

$$v^a\mathbf{e}_a = v^{a'}\mathbf{e}_{a'} \ . \tag{3.4}$$

Tetrad indices are raised and lowered using the tetrad components of the metric tensor

$$g_{ab} = g_{ij}\, e_a{}^i\, e_b{}^j \ , \tag{3.5}$$

with the inverse metric tensor defined as $g_{ac}\, g^{bc} = \delta_a{}^b$. Associated with the four contravariant tetrad vectors, there are their covariant coordinate components which are obtained by lowering the coordinate index with the metric tensor, $e_{ai} = g_{ij}\, e_a{}^j$. From the association of the tetrad vectors with directional derivatives we write for any vector field $\mathbf{v}$,

$$\mathbf{e}_a(v_b) = e_a{}^i\, e_b{}^j\, \nabla_i v_j + e^c{}_j\,(e_a{}^i \nabla_i e_b{}^j)\, v_c \tag{3.6}$$

$$:= \nabla_a v_b + \Gamma^c{}_{ab}\, v_c \ , \tag{3.7}$$

defining the Ricci rotation coefficients $\Gamma^a{}_{bc}$. From the metricity condition for the covariant derivative, $\nabla_a g_{bc} = 0$, we can show that the $\Gamma_{abc} = g_{ad}\,\Gamma^d{}_{bc}$ are antisymmetric in their first and last indices (which was the convention in [17]):[2]

$$\Gamma_{abc} = -\,\Gamma_{cba} \ . \tag{3.8}$$

[1]Remember: when referring to tetrad components in the spacetime, we use Latin indices from the first half of the alphabet: $a,\, b,\, c \in \{0,\, 1,\, 2,\, 3\}$; in the spatial sections we use Greek letters: $\alpha,\, \beta,\, \gamma \in \{1,\, 2,\, 3\}$. For the local coordinate spacetime description we use Latin indices from the second half of the alphabet: $i,\, j,\, k \in \{0,\, 1,\, 2,\, 3\}$.

[2]They are defined in [30, 84, 21] such that $\Gamma_{abc} = -\,\Gamma_{bac}$.

The commutators of the basis vector fields are defined by their actions on a geometric object f

$$[\mathbf{e}_a, \mathbf{e}_b](f) := \mathbf{e}_a(\mathbf{e}_b(f)) - \mathbf{e}_b(\mathbf{e}_a(f)) \ . \tag{3.9}$$

This commutator, being a tangent vector itself, can be expanded in terms of the same basis $\mathbf{e}_a$ and is a vector field, the Lie derivative of $\mathbf{e}_b$ with respect to $\mathbf{e}_a$, $L_{\mathbf{e}_a}\mathbf{e}_b$ — giving the difference between $\mathbf{e}_b$ and the vector field produced by dragging it along by $\mathbf{e}_a$ — which can be described by its tetrad components, the commutation functions γ, according to

$$[\mathbf{e}_a, \mathbf{e}_b](f) := \gamma^c{}_{ab}\, \mathbf{e}_c(f) \ , \quad \gamma^c{}_{ab} = \gamma^c{}_{[ab]} \ . \tag{3.10}$$

From the zero-torsion connection condition, $\nabla_{[a}\nabla_{b]}f = 0$, it follows that the commutation functions γ are expressible as linear combinations of the Ricci rotation coefficients —

$$\gamma^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} \ . \tag{3.11}$$

The Riemann curvature tensor is defined according to the convention adopted in [17] by the Ricci identity,

$$2\,\nabla_{[a}\nabla_{b]}v^c := R^c{}_{dab}\, v^d \ .$$

Applying this to a tetrad vector $\mathbf{e}_a$ we find that[3]

$$R^a{}_{bcd} = \mathbf{e}_c(\Gamma^a{}_{db}) - \mathbf{e}_d(\Gamma^a{}_{cb}) + \Gamma^a{}_{ce}\,\Gamma^e{}_{db} - \Gamma^a{}_{de}\,\Gamma^e{}_{cb} + \Gamma^a{}_{eb}\,\gamma^e{}_{dc} \ . \tag{3.12}$$

The tetrad vector fields $\{\, \mathbf{e}_a \,\}$ must obey the Jacobi identity[4]

$$[\mathbf{e}_a, [\mathbf{e}_b, \mathbf{e}_c]] + [\mathbf{e}_b, [\mathbf{e}_c, \mathbf{e}_a]] + [\mathbf{e}_c, [\mathbf{e}_a, \mathbf{e}_b]] = 0 \ . \tag{3.13}$$

We will see later that the Jacobi identity is equivalent to the Riemann tensor symmetry $R_{a[bcd]} = 0$ (also called the first Bianchi identity).

3.1.2 Orthonormal Tetrad

A choice of tetrad $\{\, \mathbf{e}_a \,\}$ adapted to the geometry or specific dynamics of a given spacetime can simplify the EFE and matter equations and lead perhaps to clues as to what coordinates to use concomitant to any existing symmetries. For cosmological purposes where one has a preferred 4-velocity at every event,

[3]This is obtained by using the Leibniz rule as follows. Any covariant derivative of a tensor which is covariantly expressed as $\nabla_i T_{jk}$ has tetrad components $\nabla_a T_{bc}$ computed as

$$\begin{aligned} \nabla_a T_{bc} &= e_a{}^i\, e_b{}^j\, e_c{}^k\, \nabla_i T_{jk} \\ &= e_a{}^i \nabla_i (T_{jk}\, e_b{}^j\, e_c{}^k) - (e_a{}^i \nabla_i e_b{}^j)\, T_{jk}\, e_c{}^k - (e_a{}^i \nabla_i e_c{}^k)\, T_{jk}\, e_b{}^j \\ &= \mathbf{e}_a(T_{bc}) - \Gamma^d{}_{ab}\, T_{dc} - \Gamma^d{}_{ac}\, T_{bd} \ . \end{aligned}$$

[4]These are listed individually in appendix A.3 and B for an orthonormal tetrad.

it is convenient to use the orthonormal tetrad ('ONT') which aligns its timelike leg $\mathbf{e}_0$ with the principal cosmological direction $\mathbf{u}$ – explicitly, $\mathbf{e}_0 = \mathbf{u}$ – and is an orthonormal basis. In what follows we shall refer to any relations between dynamic scalar fields which have the $\mathbf{e}_0$ frame derivative in them as evolution equations and any relations which do not have this frame derivative in them are termed constraints. This has become standard. We note that when $\mathbf{e}_0 = \mathbf{u}$ has non-vanishing vorticity this dynamical system terminology is doubtful; and that is because, for non-zero vorticity, the local rest tangent spaces do not mesh together to form global surfaces orthogonal to the fluid flow. An orthonormal tetrad is a set of basis vectors $\{\mathbf{e}_a\} = \{\mathbf{e}_0, \mathbf{e}_\alpha\}$ such that

$$\mathbf{e}_0 \cdot \mathbf{e}_0 = -\,\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha = -\,1 \quad \text{(no summation)}$$
$$\mathbf{e}_a \cdot \mathbf{e}_b = 0 \quad (a \neq b)\ ;$$

the $\{\mathbf{e}_\alpha\}$ form a spatial triad. Freedom to choose the spatial frame $\{\mathbf{e}_\alpha\}$ remains. Then quantities become simpler in their tetrad form

$$h_{a0} = h_{0a} = 0 \Leftarrow h_{ab}u^b = 0 \tag{3.14}$$
$$h_{\alpha\beta} = \delta_{\alpha\beta} \Leftarrow u_\alpha u_\beta = 0\,. \tag{3.15}$$

From the orthonormality of the frame we see that the metric g has constant dimensionless physical components — it is of the Minkowski form

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b = e_a{}^i e_{bi} = \eta_{ab} = \text{diag}\,[-1,\,1,\,1,\,1]\,. \tag{3.16}$$

$\Gamma_{00\alpha}$ may be interpreted as the fluid acceleration[5] $\dot{\mathbf{u}}$ and $\Gamma_{\alpha\beta 0}$ as the dot product of the spatial triad with the spatial gradient of the fluid 4-velocity:

$$\Gamma_{\alpha 00} \;=\; \dot{u}_\alpha \tag{3.17}$$
$$\Gamma_{\alpha\beta 0} \;=\; \tfrac{1}{3}\,\Theta\,\delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}\ . \tag{3.18}$$

The rate of expansion scalar Θ is the divergence of the matter fluid flow lines $\Theta := \delta^{\alpha\beta}\nabla_\alpha u_\beta$. The rate of shear tensor $\boldsymbol{\sigma}$ is symmetric, $\sigma_{\alpha\beta} = \sigma_{(\alpha\beta)}$, and tracefree, $\sigma^\alpha{}_\alpha = 0$. The rate of vorticity tensor $\boldsymbol{\omega}$ is skew-symmetric, $\omega_{\alpha\beta} = \omega_{[\alpha\beta]}$. Thus, using the 3-space permutation tensor $\varepsilon_{\alpha\beta\gamma}$, $\omega_{\alpha\beta}$ can without loss of information be written as a vector defined by[6] $\omega_\alpha := \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma}\,\omega^{\beta\gamma}$ which lies in the rest 3-space of $\mathbf{u}$ and defines the instantaneous axis of rotation of the fluid due to vorticity. The skew-symmetry of the Ricci rotation coefficients (3.8) allows us to define the rate of rotation of the spatial triad $\{\mathbf{e}_\alpha\}$ as seen by an observer with 4-velocity $\mathbf{u}$:

$$\Gamma_{\alpha 0\beta} = \mathbf{e}_\alpha \cdot \dot{\mathbf{e}}_\beta = \varepsilon_{\alpha\beta\gamma}\,\Omega^\gamma\,. \tag{3.19}$$

[5]We identify ∇_0 as the covariant time derivative along $\mathbf{u}$ and denote it by an overdot '˙'.

[6]The components of the vorticity vector as defined here have the opposite sign as compared to [17, 30, 57]. Thus it is a polar vector with respect to reflections in a right-handed (as opposed to left-handed) spatial triad $\{\mathbf{e}_\alpha\}$.

The quantity Ω^α is not part of the dynamics of the spacetime, but part of the kinematics of the spatial triad $\{\mathbf{e}_\alpha\}$. Indeed an observer comoving with the fluid 4-velocity $\mathbf{u}$ may always choose the local angular velocity of her reference spatial triad $\{\mathbf{e}_\alpha\}$ with respect to a second one defined by a triad of gyroscopes to be vanishing, $\Omega^\alpha = 0$. This is called Fermi-propagation of the spatial triad along $\mathbf{u}$. It has the effect of eliminating any Coriolis-type effects that may show up when one does experiments in a rotating frame. The purely spatial commutation functions may be decomposed into an object which is skew symmetric $\varepsilon^{\alpha\beta\gamma}\, a_\gamma$ and a symmetric object $n_{\alpha\beta} = n_{(\alpha\beta)}$ as follows:

$$\gamma^\alpha{}_{\beta\gamma} = 2\, a_{[\beta}\, \delta^\alpha{}_{\gamma]} + \varepsilon_{\beta\gamma\delta}\, n^{\delta\alpha}\ . \tag{3.20}$$

With this decomposition, the spatial Ricci rotation coefficients can be expressed as

$$\Gamma_{\alpha\beta\gamma} = 2\, a_{[\alpha}\, \delta_{\gamma]\beta} + \varepsilon_{\beta\delta[\alpha}\, n^\delta{}_{\gamma]} + \tfrac{1}{2}\, \varepsilon_{\alpha\gamma\delta}\, n^\delta{}_\beta\ . \tag{3.21}$$

From (3.10) and (3.11) – using the fact that the spacetime connection is torsion-free: $\nabla_{[a}\nabla_{b]}f = 0$ for any scalar f – the commutators acting as differential operators on a field f may conveniently be expressed as

$$[\,\mathbf{e}_0, \mathbf{e}_\alpha\,](f) = \gamma^c{}_{0\alpha}\, \mathbf{e}_c = \dot{u}_\alpha\ \mathbf{e}_0(f) - [\ \tfrac{1}{3}\, \Theta\, \delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \varepsilon_\alpha{}^\beta{}_\gamma\, (\omega^\gamma + \Omega^\gamma)\ ]\ \mathbf{e}_\beta(f) \tag{3.22}$$

$$[\,\mathbf{e}_\alpha, \mathbf{e}_\beta\,](f) = \gamma^c{}_{\alpha\beta}\, \mathbf{e}_c = 2\, \varepsilon_{\alpha\beta\gamma}\, \omega^\gamma\ \mathbf{e}_0(f) + [\ 2\, a_{[\alpha}\, \delta^\gamma{}_{\beta]} + \varepsilon_{\alpha\beta\delta}\, n^{\delta\gamma}\ ]\ \mathbf{e}_\gamma(f)\ ; \tag{3.23}$$

utilising the interpretations of these variables given by equations (3.17), (3.18), (3.19) and (3.20).

Because the rotation coefficients Γ_{abc} are skew in the first and last indices, $\Gamma_{abc} = -\Gamma_{cba}$, they have 24 independent nonzero components; when expressed in an orthonormal basis, they can all be obtained from the quantities $\dot{u}_\alpha$, ω^α, $\sigma_{\alpha\beta}$, θ, Ω^α, $n_{\alpha\beta}$ and a_α. We recall that $\sigma_{\alpha\beta} = \sigma_{(\alpha\beta)}$ and $\sigma^\alpha{}_\alpha = 0$, $n_{\alpha\beta} = n_{(\alpha\beta)}$.

The dynamics resides in the commutation functions γ. These commutation functions obey identities and field equations: to wit, the Jacobi identity (3.39) provides evolution and constraint equations for the vorticity and the purely spatial commutators $\gamma^\alpha{}_{\beta\gamma}$ — the latter of which we have expressed in decomposed form as a^α and $n_{\alpha\beta}$. The components of the Riemann curvature tensor, given by (3.12), yield evolution equations for the expansion and shear, and constraint equations for the expansion, shear and vorticity which we give in terms of $\mathbf{E}$ and $\mathbf{H}$. The second Bianchi identity gives evolution and constraint equations for $\mathbf{E}$ and $\mathbf{H}$. The purpose is to use these identities and field equations to find the γ. Then we may be able to find the $e_a{}^i$ from (3.9), (3.10) and other remaining equations, and the $e^a{}_i$ from (3.1), which then determine the coordinate components of the metric, g_{ij}, from (3.5) and (3.16). A suitable choice of tetrad (on top of the orthonormality choice) will hopefully help to accomplish these aims.

3.1.3 Einstein Field Equations

Contraction of (3.12) gives the tetrad Ricci tensor components $R_{bc} = R_b{}^a{}_{ca}$ and thus the Einstein Field Equations in tetrad form.

$$\begin{aligned} R_{bc} &= \mathbf{e}_c\Gamma^a{}_{ab} - \mathbf{e}_a\Gamma^a{}_{cb} + \Gamma^a{}_{ce}\Gamma^e{}_{ab} - \Gamma^a{}_{ae}\Gamma^e{}_{cb} + \Gamma^a{}_{eb}\gamma^e{}_{ac} \\ &= T_{bc} - \frac{1}{2}Tg_{bc} + \Lambda g_{bc}\,. \end{aligned} \tag{3.24}$$

3.1.3.1 Energy-Momentum Tensor

In the orthonormal frame, we may write (2.51) in decomposed form as

$$T_{00} = \mu\,. \tag{3.25}$$

Thus the energy density is T_{00}, the component of the energy-momentum tensor in the fluid four velocity direction. Similarly,

$$T_{0\alpha} = -q_\alpha \tag{3.26}$$

$$T_{\alpha\beta} = p\delta_{\alpha\beta} + \pi_{\alpha\beta}\,. \tag{3.27}$$

The trace $T = T^a{}_a$ is given by

$$T^a{}_a = -\mu + 3p\,. \tag{3.28}$$

3.1.3.2 Einstein Field Equations for a Perfect Fluid Source

We have the tetrad components of the Einstein Field equations with vanishing Λ holding. This may be written as

$$R_{ab} - \frac{1}{2}R\eta_{ab} = T_{ab} \quad \Leftrightarrow \quad R_{ab} = T_{ab} - \frac{1}{2}T\eta_{ab}\,. \tag{3.29}$$

A fluid has a Ricci curvature tensor, a Ricci curvature scalar and a tracefree Ricci curvature tensor given by, using $u_a = -\delta^0{}_a$ and $h_{ab} = \eta_{ab} + \delta^0{}_a\delta^0{}_b$

$$R_{ab} = (\mu + p)\,\delta^0{}_a\delta^0{}_b + \frac{1}{2}(\mu - p)\,\eta_{ab} + 2q_{<a}\delta^0{}_{b>} + \pi_{ab} \tag{3.30}$$

$$R = (\mu - 3p) \tag{3.31}$$

$$R_{ab} - \frac{1}{4}R\eta_{ab} = (\mu + p)\left(\delta^0{}_a\delta^0{}_b + \frac{1}{4}\eta_{ab}\right) + 2q_{<a}\delta^0{}_{b>} + \pi_{ab} \tag{3.32}$$

respectively.

Perfect Fluid In particular, a perfect fluid matter source has a Ricci curvature tensor, a Ricci curvature scalar and a tracefree Ricci curvature tensor given by

$$R_{ab} = (\mu + p)\, \delta^0{}_a \delta^0{}_b + \frac{1}{2}(\mu - p)\, \eta_{ab} \tag{3.33}$$

$$R = (\mu - 3p) \tag{3.34}$$

$$R_{ab} - \frac{1}{4} R \eta_{ab} = (\mu + p)\left(\delta^0{}_a \delta^0{}_b + \frac{1}{4}\eta_{ab}\right) \tag{3.35}$$

respectively. This is with respect to the barycentric four-velocity u_a or the timelike eigenvector of the energy-momentum tensor T_{ab}. The four-velocity represents the average motion of rest-mass – the total momentum measured relative to this frame vanishes. The timelike eigenvector of T_{ab} choice is defined by $q_a = 0$; that is we choose the average motion of relativistic energy.

3.1.4 Jacobi Identities

The constancy of the tetrad components of the metric tensor also gives us the inverse of (3.11), which reads

$$\Gamma_{abc} = \frac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca})\,. \tag{3.36}$$

The cyclic identities of the Riemann tensor (2.37) thus become on using equation (3.36)

$$\partial_{[d}(\gamma^a{}_{bc]}) + \gamma^e{}_{[bc}\gamma^a{}_{d]e} = 0\,. \tag{3.37}$$

Since $\gamma^a{}_{bc} = -\gamma^a{}_{cb}$, this can be written as

$$\partial_b(\gamma^a{}_{cd}) + \partial_c \gamma^a{}_{db} + \partial_d \gamma^a{}_{bc} + \gamma^e{}_{bc}\gamma^a{}_{de} + \gamma^e{}_{cd}\gamma^a{}_{be} + \gamma^e{}_{db}\gamma^a{}_{ce} = 0\,. \tag{3.38}$$

If we now identify $\partial_a = \mathbf{e}_a$, then a short calculation shows, from (3.9) and (3.10) that the cyclic Riemann identities are equivalent to the Jacobi identities for the basis vector fields $\{\mathbf{e}_a\}$.

$$[\mathbf{e}_a, [\mathbf{e}_b, \mathbf{e}_c]] + [\mathbf{e}_c, [\mathbf{e}_a, \mathbf{e}_b]] + [\mathbf{e}_b, [\mathbf{e}_c, \mathbf{e}_a]] = 0\,. \tag{3.39}$$

We use equation (3.38) to write out the Jacobi identities in full in an ONT. We denote each equation obtained from (3.38) by $\binom{a}{bcd}$ following [17]. It is clear from (3.37) that there are 16 independent nontrivial equations. There are three which can be considered as vorticity propagation equations;

$$\begin{aligned} &\binom{0}{0\gamma\delta} : \quad \mathbf{e}_0(\omega_{\gamma\delta}) + \mathbf{e}_{[\delta}(\dot{u}_{\gamma]}) + 2\omega_{\epsilon[\gamma}\left[\varepsilon^\epsilon{}_{\delta]\phi}\Omega^\phi - \left(\sigma^\epsilon{}_{\delta]} + \tfrac{1}{3}\theta\delta^\epsilon{}_{\delta]}\right)\right] + \tfrac{1}{2}\left(2a_{[\gamma}\delta^\epsilon{}_{\delta]} + \varepsilon_{\gamma\delta\beta}n^{\beta\epsilon}\right)\dot{u}_\epsilon = 0 \\ &\gamma \neq \delta\,; \end{aligned} \tag{3.40}$$

nine which may be thought of as propagation equations for the purely spatial commutators;

$$\begin{aligned} \binom{\alpha}{0\gamma\delta} : \quad & \mathbf{e}_0\left(2a_{[\gamma}\delta^\alpha{}_{\delta]} - \varepsilon_{\gamma\delta\epsilon}n^{\epsilon\alpha}\right) + 2\mathbf{e}_{[\delta}\left(\varepsilon^\alpha{}_{\gamma]\epsilon}\Omega^\epsilon - \left(-\omega^\alpha{}_{\gamma]} + \sigma^\alpha{}_{\gamma]} + \tfrac{1}{3}\theta\delta^\alpha{}_{\gamma]}\right)\right) \\ \gamma \neq \delta \quad & + \left[-\varepsilon^\alpha{}_{\epsilon\phi}\Omega^\phi - \left(-\omega^\alpha{}_\epsilon + \sigma^\alpha{}_\epsilon + \tfrac{1}{3}\theta\delta^\alpha{}_\epsilon\right)\right]\left[2a_{[\gamma}\delta^\epsilon{}_{\delta]} - \varepsilon_{\gamma\delta\beta}n^{\beta\epsilon}\right] \\ & + 2\dot{u}_{[\delta}\left[-\varepsilon^\alpha{}_{\gamma]\beta}\Omega^\beta - \left(-\omega^\alpha{}_{\gamma]} + \sigma^\alpha{}_{\gamma]} + \tfrac{1}{3}\theta\delta^\alpha{}_{\gamma]}\right)\right] \\ & + 2\left[-\varepsilon^\epsilon{}_{[\gamma|\beta|}\Omega^\beta - \left(-\omega^\epsilon{}_{[\gamma} + \sigma^\epsilon{}_{[\gamma} + \tfrac{1}{3}\theta\delta^\epsilon{}_{[\gamma}\right)\right]\left[2a_{[\delta]}\delta^\alpha{}_{\epsilon]} - \varepsilon_{\delta]\epsilon\phi}n^{\phi\alpha}\right] = 0\,; \end{aligned} \tag{3.41}$$

one equation which we may consider as the vorticity constraint equation –

$$\binom{0}{123}\ :\quad \mathbf{e}_1(\omega_{23})+\mathbf{e}_2(\omega_{31})+\mathbf{e}_3(\omega_{12})-\omega_{23}(2a_1+\dot{u}_1)-\omega_{31}(2a_2+\dot{u}_2)-\omega_{12}(2a_3+\dot{u}_3)=0 \quad (3.42)$$

and three equations which are essentially constraint equations for the purely spatial commutators –

$$\begin{aligned}\binom{\alpha}{123}\ :\quad & \mathbf{e}_1\left(2a_{[2}\delta^\alpha{}_{3]}+n^{1\alpha}\right)+\mathbf{e}_2\left(2a_{[3}\delta^\alpha{}_{1]}+n^{2\alpha}\right)+\mathbf{e}_3\left(2a_{[1}\delta^\alpha{}_{2]}+n^{3\alpha}\right)\\ & -2a_1\left(2a_{[2}\delta^\alpha{}_{3]}+n^{1\alpha}\right)-2a_2\left(2a_{[3}\delta^\alpha{}_{1]}+n^{2\alpha}\right)-2a_3\left(2a_{[1}\delta^\alpha{}_{2]}+n^{3\alpha}\right)\\ & =2\omega_{12}\left[-\varepsilon^\alpha{}_{3\beta}\Omega^\beta-\left(-\omega^\alpha{}_3+\sigma^\alpha{}_3+\tfrac{1}{3}\theta\delta^\alpha{}_3\right)\right]+2\omega_{23}\left[-\varepsilon^\alpha{}_{1\beta}\Omega^\beta-\left(-\omega^\alpha{}_1+\sigma^\alpha{}_1+\tfrac{1}{3}\theta\delta^\alpha{}_1\right)\right]\\ & +2\omega_{31}\left[-\varepsilon^\alpha{}_{2\beta}\Omega^\beta-\left(-\omega^\alpha{}_2+\sigma^\alpha{}_2+\tfrac{1}{3}\theta\delta^\alpha{}_2\right)\right].\end{aligned} \quad (3.43)$$

If we identify

$$\begin{aligned}\omega_{23} &= \omega^1\\ \omega_{31} &= \omega^2\\ \omega_{12} &= \omega^3\end{aligned}$$

since

$$\frac{1}{2}\eta^{abcd}u_b\omega_{cd}=\frac{1}{2}\eta^{a0\gamma\delta}\omega_{\gamma\delta}\equiv-\frac{1}{2}\varepsilon^{\alpha\gamma\delta}\omega_{\gamma\delta}=\omega^\alpha\,, \quad (3.44)$$

then we may rewrite the vorticity propagation and constraint equations, respectively, as follows:

$$\mathbf{e}_0(\omega^\alpha)-\frac{1}{2}\varepsilon^{\alpha\delta\gamma}\left(\mathbf{e}_\gamma(\dot{u}_\delta)-a_\gamma\dot{u}_\delta\right)+\frac{1}{2}n^{\alpha\beta}\dot{u}_\beta=\omega^\gamma\left(-\varepsilon^\alpha{}_{\delta\gamma}\Omega^\delta+\sigma^\alpha{}_\gamma-\frac{2}{3}\theta\delta^\alpha{}_\gamma\right), \quad (3.45)$$

$$(C_2):=\qquad \mathbf{e}_\alpha(\omega^\alpha)-(2a_\alpha+\dot{u}_\alpha)\omega^\alpha=0\,, \quad (3.46)$$

where for the first time we use the constraint labelling which has become standard in for example [30]. The spatial commutator constraints can be written as

$$(C_J)^\alpha:=\qquad \mathbf{e}_\beta(n^{\beta\alpha})-\varepsilon^{\alpha\gamma\beta}\mathbf{e}_\beta(a_\gamma)-2a_\beta n^{\beta\alpha}+2\omega^\beta\left(-\varepsilon^\alpha{}_{\beta\gamma}\Omega^\gamma+\sigma^\alpha{}_\beta+\frac{1}{3}\theta\delta^\alpha{}_\beta\right)=0 \quad (3.47)$$

We see that if the vorticity vanishes (or if the bracketed quantity on the right hand side of the spatial commutator constraint expression vanishes) on some initial hypersurface, then if

$$\begin{aligned}a_\alpha=0, &\quad n^{\alpha\beta}\ \text{is 'tetrad divergence' free;}\\ n^{\alpha\beta}=0, &\quad a_\alpha\ \text{is 'tetrad curl' free.}\end{aligned} \quad (3.48)$$

3.1.5 Riemann (Ricci) Identities

The equations below summarizes the tetrad formulation comprehensively and may be found in [21]. The nine fluid kinematical variables θ, $\sigma_{\alpha\beta}$ and ω_α have evolution equations provided by the Ricci field equations.

The vorticity evolution equations are equivalent to the Jacobi identities $\binom{0}{0\gamma\delta}$.

$$\begin{aligned}
\mathbf{e}_0(\theta) &= \mathbf{e}_\alpha(\dot{u}^\alpha) - \frac{1}{3}\theta^2 + (\dot{u}_\alpha - 2a_\alpha)(\dot{u}^\alpha) - (\sigma^\alpha{}_\beta\sigma^\beta{}_\alpha) + 2(\omega_\alpha\omega^\alpha) - \frac{1}{2}(\mu + 3p) + \Lambda \quad (3.49)\\
\mathbf{e}_0(\sigma^{\alpha\beta}) &= \delta^{\gamma<\alpha}\mathbf{e}_\gamma(\dot{u}^{\beta>}) - \frac{2}{3}\theta\sigma^{\alpha\beta} - \sigma^{<\alpha}{}_\gamma\sigma^{\beta>\gamma} - \omega^{<\alpha}\omega^{\beta>} + (\dot{u}^{<\alpha} + a^{<\alpha})\dot{u}^{\beta>}\\
&\quad -(E^{\alpha\beta} - \frac{1}{2}\pi^{\alpha\beta}) + \varepsilon^{\gamma\delta<\alpha}\left[2\Omega_\gamma\sigma^{\beta>}{}_\delta - n^{\beta>}{}_\gamma\dot{u}_\delta\right] \quad (3.50)\\
\mathbf{e}_0(\omega^\alpha) &= \frac{1}{2}\varepsilon^{\alpha\beta\gamma}\mathbf{e}_\beta(\dot{u}_\gamma) - \frac{2}{3}\theta\omega^\alpha + \sigma^\alpha{}_\beta\Omega^\beta - \frac{1}{2}n^\alpha{}_\beta\dot{u}^\beta - \frac{1}{2}\varepsilon^{\alpha\beta\gamma}\left[a_\beta\dot{u}_\gamma - 2\Omega_\beta\Omega_\gamma\right]. \quad (3.51)
\end{aligned}$$

We obtain the so-called *div*$\boldsymbol{\sigma}$ constraint[7] $(C_1)^\alpha$, also known as the momentum constraint in Hamiltonian treatments of the EFE; the *div*$\boldsymbol{\omega}$ constraint (C_2) which can also be found from the Jacobi Identity $\binom{0}{123}$, and the so-called $\mathbf{H}$-constraint $(C_3)^{\alpha\beta}$

$$\begin{aligned}
(C_1)^\alpha &:= (\mathbf{e}_\beta - 3a_\beta)(\sigma^{\alpha\beta}) - \frac{2}{3}\delta^{\alpha\beta}\mathbf{e}_\beta(\theta) - n^\alpha{}_\beta w^\beta + q^\alpha\\
&\quad +\varepsilon^{\alpha\beta\gamma}\left[(\mathbf{e}_\beta + 2\dot{u}_\beta - a_\beta)(\omega_\gamma) - n_{\beta\delta}\sigma^\delta{}_\gamma\right] = 0 \quad (3.52)\\
(C_2) &:= (\mathbf{e}_\alpha - \dot{u}_\alpha - 2a_\alpha)(\omega^\alpha) = 0 \quad (3.53)\\
(C_3)^{\alpha\beta} &:= H^{\alpha\beta} + (\delta^{\gamma<\alpha}\mathbf{e}_\gamma + 2\dot{u}^{<\alpha} + a^{<\alpha})(\omega^{\beta>}) - \frac{1}{2}n^\gamma{}_\gamma\sigma^{\alpha\beta}\\
&\quad +3n^{<\alpha}{}_\gamma\sigma^{\beta>\gamma} - \varepsilon^{\gamma\delta<\alpha}\left[(\mathbf{e}_\gamma - a_\gamma)(\sigma^{\beta>}{}_\delta) + n^{\beta>}{}_\gamma w_\delta\right] = 0. \quad (3.54)
\end{aligned}$$

More directly, from the EFE, we get what corresponds to the PSTF part of the once-contracted Gauss embedding equation (in the case of vanishing vorticity). In this case the trace of the Gauss embedding equation (C_G) is known as the generalised equation of motion or generalised Friedman equation; it is known as the Hamiltonian or energy constraint in Hamiltonian treatments.

$$\begin{aligned}
(C_G)^{\alpha\beta} &:= \delta^{\gamma<\alpha}\mathbf{e}_\gamma(a^{\beta>}) + 2n^{<\alpha}{}_\gamma n^{\beta>\gamma} - n^\gamma{}_\gamma n^{<\alpha\beta>} - \varepsilon^{\gamma\delta<\alpha}(\mathbf{e}_{|\gamma|} - 2a_{|\gamma|})(n^{\beta>}{}_\delta)\\
&\quad +\frac{1}{3}\theta\sigma^{\alpha\beta} - \sigma^{<\alpha}{}_\gamma\sigma^{\beta>\gamma} - \omega^{<\alpha}\omega^{\beta>} + 2\omega^{<\alpha}\Omega^{\beta>} - (E^{\alpha\beta} + \frac{1}{2}\pi^{\alpha\beta}) = 0 \quad (3.55)\\
(C_G) &:= 2(2\mathbf{e}_\alpha - 3a_\alpha)(a_\alpha) - (n^\alpha{}_\gamma n^\gamma{}_\alpha - n^\alpha{}_\alpha)\\
&\quad +\frac{2}{3}\theta^2 - 2\sigma^2 + 4\omega^2 - 4\omega_\alpha\Omega^\alpha - 2\mu - 2\Lambda = 0. \quad (3.56)
\end{aligned}$$

3.1.6 Bianchi Identities

If we include the gravito-electric and gravito-magnetic fields in the description of the field equations as variables, we need to provide evolution equations for these quantities. These are provided by the Bianchi Identities. In tetrad components these look like

$$R_{ab[cd;e]} = 0. \quad (3.57)$$

Doing this in an ONT has become known as extending the $1+3$ orthonormal frame formalism. The equations may be found in [57, 21]; see also [26, 30].

[7] The equation numbering is as in [30].

3.1.6.1 Once Contracted Bianchi Identities

The Bianchi identities once contracted provide propagation and constraint equations for the components of the decomposed Weyl curvature variables $\mathbf{E}$ and $\mathbf{H}$. If one naively counts these equations, we find that there are 12 equations, but both of these fields are tracefree so that there are actually only 10 equations for the 10 independent components of the Weyl tensor. Explicitly these equations come from contracting the Bianchi Identities on the first index and the index after the semi-colon:

$$R_{bc;d} - R_{bd;c} + R^{a}{}_{bcd;a} = 0\,. \tag{3.58}$$

This equation is obtained from the definition of the Ricci tensor (2.44) and from the anti-symmetric Riemann symmetry (2.41). Explicitly, in terms of the gravito-electric and -magnetic fields,

$$\begin{aligned}
\mathbf{e}_0\left(E^{\alpha\beta}+\frac{1}{2}\pi^{\alpha\beta}\right) &= \varepsilon^{\gamma\delta<\alpha}\mathbf{e}_\gamma\left(H^{\beta>}{}_\delta\right) - \frac{1}{2}\delta^{\gamma<\alpha}\mathbf{e}_\gamma\left(q^{\beta>}\right) - \frac{1}{2}(\mu+p)\sigma^{\alpha\beta} - \theta\left(E^{\alpha\beta}+\frac{1}{6}\pi^{\alpha\beta}\right)\\
&\quad +3\sigma^{<\alpha}{}_\gamma\left(E^{\beta>\gamma}-\frac{1}{6}\pi^{\beta>\gamma}\right)+\frac{1}{2}n^\gamma{}_\gamma H^{\alpha\beta} - 3n^{<\alpha}{}_\gamma H^{\beta>\gamma} - \frac{1}{2}\left(2\dot{u}^{<\alpha}+a^{<\alpha}\right)q^{\beta>}\\
&\quad +\varepsilon^{\gamma\delta<\alpha}\left[(2\dot{u}_\gamma - a_\gamma)H^{\beta>}{}_\delta + (\omega_\gamma+2\Omega_\gamma)(E^{\beta>}{}_\delta + \frac{1}{2}\pi^{\beta>}{}_\delta) + \frac{1}{2}n^{\beta>}{}_\gamma q_\delta\right] \qquad (3.59)\\
\mathbf{e}_0\left(H^{\alpha\beta}\right) &= -\varepsilon^{\gamma\delta<\alpha}\mathbf{e}_\gamma\left(E^{\beta>}{}_\delta - \frac{1}{2}\pi^{\beta>}{}_\delta\right) - \theta H^{\alpha\beta} + 3\sigma^{<\alpha}{}_\gamma H^{\beta>\gamma} + \frac{3}{2}\omega^{<\alpha}q^{\beta>}\\
&\quad -\frac{1}{2}n^\gamma{}_\gamma\left(E^{\alpha\beta}-\frac{1}{2}\pi^{\alpha\beta}\right) + 3n^{<\alpha}{}_\gamma\left(E^{\beta>\gamma}-\frac{1}{2}\pi^{\beta>\gamma}\right)\\
&\quad +\varepsilon^{\gamma\delta<\alpha}\left[a_\gamma\left(E^{\beta>}{}_\delta - \frac{1}{2}\pi^{\beta>}{}_\delta\right) - 2\dot{u}^\gamma E^{\beta>}{}_\delta + \frac{1}{2}\sigma^{\beta>}{}_\gamma q_\delta + (\omega_\gamma + 2\Omega_\gamma)H^{\beta>}{}_\delta\right] \qquad (3.60)
\end{aligned}$$

Also the frame analogue of the so-called $divE$ and $divH$ relations labelled $(C_4)^\alpha$ and $(C_5)^\alpha$, respectively, are obtained from the once-contracted Bianchi identities.

$$\begin{aligned}
(C_4)^\alpha :=&\quad (\mathbf{e}_\beta - 3a_\beta)\left(E^{\alpha\beta}+\frac{1}{2}\pi^{\alpha\beta}\right) - \frac{1}{3}\delta^{\alpha\beta}\mathbf{e}_\beta(\mu) + \frac{1}{3}\theta q^\alpha - \frac{1}{2}\sigma^\alpha{}_\beta q^\beta - 3\omega_\beta H^{\alpha\beta}\\
&\quad -\varepsilon^{\alpha\beta\gamma}\left[\sigma_{\beta\delta}H^\delta{}_\gamma - \frac{3}{2}\omega_\beta q_\gamma + n_{\beta\delta}\left(E^\delta{}_\gamma + \frac{1}{2}\pi^\delta{}_\gamma\right)\right] = 0 \qquad (3.61)\\
(C_5)^\alpha :=&\quad (\mathbf{e}_\beta - 3a_\beta)\left(H^{\alpha\beta}\right) + (\mu+p)\omega^\alpha + 3\omega_\beta\left(E^{\alpha\beta}-\frac{1}{6}\pi^{\alpha\beta}\right) - \frac{1}{2}n^\alpha{}_\beta q^\beta\\
&\quad +\varepsilon^{\alpha\beta\gamma}\left[\frac{1}{2}(\mathbf{e}_\beta - a_\beta)(q_\gamma) + \sigma_{\beta\delta}\left(E^\delta{}_\gamma + \frac{1}{2}\pi^\delta{}_\gamma\right) - n_{\beta\delta}H^\delta{}_\gamma\right] = 0 \qquad (3.62)
\end{aligned}$$

3.1.6.2 Twice Contracted Bianchi Identities

The twice contracted Bianchi Identities provide evolution equations for the four matter fields μ and q^α

$$\mathbf{e}_0(\mu) = -\mathbf{e}_\alpha(q^\alpha) - \theta(\mu+p) - 2(\dot{u}_\alpha - a_\alpha)q^\alpha - \sigma^\alpha{}_\beta\pi^\beta{}_\alpha \tag{3.63}$$

$$\begin{aligned}
\mathbf{e}_0(q^\alpha) &= -\delta^{\alpha\beta}\mathbf{e}_\beta(p) - \mathbf{e}_\beta(\pi^{\alpha\beta}) - \frac{4}{3}\theta q^\alpha - \sigma^\alpha{}_\beta q^\beta - (\mu+p)\dot{u}^\alpha\\
&\quad -(\dot{u}_\beta - 3a_\beta)\pi^{\alpha\beta} - \varepsilon^{\alpha\beta\gamma}\left[(\omega_\beta - \Omega_\beta)q_\gamma - n_{\beta\delta}\pi^\delta{}_\gamma\right]\,. \qquad (3.64)
\end{aligned}$$

3.1.7 Comments

We note that the evolution equation for $\mathbf{E}$ does not occur on its own, but involves the matter field variable π as well. There is in fact no way of resolving this into separate evolution equations for both quantities in the phenomenological fluid approximation, without equations of state, but a common *ad hoc* way of resolving this is to assume that the pressure variable is simply coupled to the shear[8] – the historical reason for calling this the *anisotropic* pressure. If one wants to resolve this problem in a reasonably rigorous fashion, a kinetic theory treatment instead of the fluid description must at least be employed. The alternative, of course, is to look at perfect fluids where the anisotropic pressure along with the relativistic energy flux q^a vanishes.

There are no evolution equations for the acceleration $\dot{u}_\alpha$. This is not too unexpected because we may obtain any desired value for $\dot{u}_\alpha$ by appropriate choice of $\mathbf{u}$. For the often-used perfect fluid however, if needed one may obtain evolution equations for $\dot{u}_\alpha$ from (3.64) and the commutators (3.22) combined with (3.63).

There are also no evolution equations for the spatial frame rotation variables Ω^α. Ω^α is the local angular velocity of the frame $\{\mathbf{e}_\alpha\}$ representing the freedom of fixing the rotating spatial frame. We may always, for instance, choose it to be a non-rotating frame – Fermi-propagation ($\Omega^\alpha = 0$).

[8]This is to ensure that the Second Law of Thermodynamics holds.

Part II

Consistency Studies I: Classical GR

Chapter 4

Introduction

Our long-term aim is to determine all perfect fluid cosmologies which cannot be invariantly defined by the existence of a unique shear eigentetrad. This consideration is of relevance to the equivalence problem of spacetimes [6, 43] and its application to relativistic cosmology. Key symmetries of spacetimes are their continuous isotropies, and cosmologies are either *isotropic* (and then have a Robertson–Walker metric), *locally rotationally symmetric*, or 'LRS' (and then are all known up to the *form* of their metric), or are *anisotropic* (see, e.g., [17, 28, 21] for a discussion of these cases). In the case of LRS cosmologies, there is at each event (relative to the family of fundamental observers) precisely one preferred spatial direction, and all physical properties and observations are invariant under rotation about this direction. It follows, in general, that these spacetimes are invariant under multiply transitive groups of isometries [17, 75, 28]. The question that is interesting from both the physical point of view, and in terms of determining the equivalence of cosmological spacetimes, is *how weak* we can make the assumptions of rotational symmetry and still determine explicitly the family of spacetimes involved; in other words, how few physical and geometrical quantities we can make *rotationally symmetric* — where rotationally symmetric means the quantity concerned is either isotropic, or invariant under arbitrary rotations about a preferred axis.

Central to the equivalence problem formalism are the components of the spacetime Riemann curvature tensor and its covariant derivatives in a standard tetrad. In [17] it was shown that a spacetime will be LRS if all tensors algebraically defined by the spacetime Riemann curvature tensor and their covariant derivatives up to the *third order* are rotationally symmetric (note that the fluid velocity field is algebraically determined by the curvature tensor, through the Einstein field equations). The ultimate aim is to weaken this assumption by considering perfect fluid cosmologies in which *only* the shear tensor (and not its covariant derivatives) has this symmetry, since the fluid shear plays a central role in the dynamics of a generic cosmology. All cosmologies that do *not* satisfy this restriction can be invariantly defined through tensor components relative to the unique shear eigentetrad. In this thesis we consider this general project but can only make a detailed deduction of a *subcase* of the general project; in detail, we firstly consider perfect fluid cosmologies in which all the fluid kinematical variables and the Weyl curvature tensor are rotationally symmetric about the same axis (the Ricci curvature tensor components are automatically so, because of the perfect fluid assumption). We make no similar assumption about their covariant derivatives. We find all perfect fluid cosmologies satisfying this restriction. We then attempt to generalise this to the situation where the shear and either one of the Weyl curvature variables are of this form. We then briefly comment on the situation where only the shear is rotationally symmetric.

4.1 Symmetry Assumptions

A dynamical tensor field is *rotationally symmetric* if there is a degeneracy in that tensor quantity in terms of its eigenvalues, but it is not isotropic (not all eigenvalues are zero). The justification for this terminology is based on the fact that a spacetime is LRS if all tensor quantities, as well as their covariant derivatives, are either isotropic or rotationally symmetric about the *same* axis, with at least one *not being isotropic* [17, 28].

Definition 4.1.1 *A dynamical rank two tensor field* $\mathbf{T}$ *is* rotationally symmetric *(RS) if there is a degeneracy in that tensor quantity in terms of its eigenvalues. Thus it assumes the following form:*

$$(T_{22} - T_{33}) = T_{12} = T_{23} = T_{31} = 0\,. \tag{4.1}$$

We arbitrarily pick out the $\mathbf{e}_1$ *direction as the axis of rotation.*

The question we ask is: 'Under what conditions are all or a combination of the matter fluid acceleration, vorticity and shear and/or the spacetime electric and magnetic Weyl curvatures simultaneously *either* rotationally symmetric *or* isotropic, but the spacetime itself is not?'. Our ultimate aim is to be able to uniquely classify all perfect fluid cosmologies which have *degenerate shear*. Thus we have written out the evolution and constraint equations for the tetrad commutation functions γ as well as for $\mathbf{E}$ and $\mathbf{H}$, which are obtained from the Jacobi, Ricci and second Bianchi identities for rotationally symmetric shear. These are given in full in appendix A.3.

We consider the special case where, in addition to the shear being degenerate, we may or may not have any combination of the electric and magnetic Weyl curvatures degenerate about the same axis, with both the acceleration and vorticity aligned with this axis. But we place *no* restrictive requirements on their derivatives. We call a spacetime which has this degeneracy a *partially locally rotationally symmetric* ('PLRS') spacetime. Note that this definition allows a PLRS spacetime to be LRS and Petrov type D if $\mathbf{E}$ and $\mathbf{H}$ are RS about the same axis.

4.1.1 Rotationally Symmetric Shear Eigentetrads

An ONT $\{\,\mathbf{e}_a\,\} = \{\,\mathbf{u},\,\mathbf{e}_\alpha\,\}$ may be chosen such that the shear tensor $\boldsymbol{\sigma}$ assumes *diagonal form*. The three eigendirections of the shear, if unique, then invariantly define the axes of the spatial triad $\{\,\mathbf{e}_\alpha\,\}$. However, if the shear becomes *degenerate*, this results in the directions of the spatial triad not being fully specified. In particular, if all three shear components are the same, then, since the shear is tracefree, it vanishes, and the spatial triad is consequently free by a general rotation. This is a severe restriction on the dynamics and has been clarified in the dust case where $p = 0$ by Ellis [17] while it has been investigated for irrotational expanding perfect fluids with equation of state $p = p(\mu)$ by Collins and Wainwright [9]. Recently Senovilla et. al. [?] provided a fully covariant proof showing that any shear-free perfect fluid with the acceleration proportional to the vorticity vector (including the simpler case of vanishing acceleration) must be either non-expanding or non-rotating. The other potential situation, which concerns us here and in later work, is when the shear is degenerate in one plane, say the $\mathbf{e}_2\,/\,\mathbf{e}_3$–plane: then $\sigma_{22} = \sigma_{33}$. The tetrad is now free by a rotation in this plane while the tetrad vector $\mathbf{e}_1$ is uniquely defined. Any change of tetrad basis by a (non-singular) transformation $\mathbf{\Lambda} = \mathbf{\Lambda}(x^i)$ relates components of a vector $\mathbf{v}$ in the new and old bases by

$$v^{a\prime} = \Lambda^{a\prime}{}_{a}\, v^{a} \ ,$$

leaving $\mathbf{v}$ invariant:

$$\mathbf{v} = v^{a\prime}\, \mathbf{e}_{a\prime} = v^{a}\, \mathbf{e}_{a} \ .$$

The Lorentz matrix $\Lambda^{a\prime}{}_{a}$ has an inverse $\Lambda^{-1}{}_{a\prime}{}^{a}$. The tetrad freedom now is that of a spatial rotation given by

$$\mathbf{e}_{a\prime} = \Lambda^{-1}{}_{a\prime}{}^{a}\, \mathbf{e}_{a} \ ,$$

where

$$\Lambda^{-1}{}_{a\prime}{}^{a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix} . \tag{4.2}$$

The effect of this freedom is that the possibility of invariantly classifying perfect fluid cosmologies with this feature is now uncertain: the shear itself does *not* define a unique direction in the $\mathbf{e}_2\,/\,\mathbf{e}_3$–plane. However, its derivatives, or other covariantly defined quantities, may do so. Similar considerations arise as regards PLRS degeneracies in other tensor fields such as $\mathbf{E}$ and $\mathbf{H}$.

4.2 PLRS Perfect Fluid Spacetimes

The effect of restricting the geometry of a spacetime is reflected by setting certain geometrically defined tensor components to zero. However, once dynamical restrictions have been imposed on a spacetime (as in the discussion to follow), *new constraints* result as reductions from the set of general identities and field equations. And then these new constraints, which are often differential expressions, need to be checked for consistency with the remaining relations in the set by taking derivatives particularly along the matter fluid flow lines $\mathbf{u}$. Again, these consistency checks could provide further constraints. We continue this programme until we either only obtain identities, or inconsistency has been demonstrated.

A fluid spacetime geometry with a preferred fluid four velocity, is *Locally Rotationally Symmetric (LRS)* if at each point of the spacetime in an open neighbourhood, there exists a non-discrete subgroup of the Lorenz group in the tangent space of the manifold which leaves invariant the curvature tensor and all its covariant derivatives up to the third order. This corresponds to the existence of a triad basis with a degeneracy which leaves, in addition to $\mathbf{e}_0 = \mathbf{u}$, a preferred spatial direction $\mathbf{e}_1$ invariant. Our study is an attempt to list all perfect fluid cosmologies that provide solutions to the EFE which have the tensor fields $\boldsymbol{\sigma}$, $\mathbf{E}$ and/or $\mathbf{H}$ as well as the vector fields $\dot{\mathbf{u}}$ and $\boldsymbol{\omega}$, *but not necessarily their covariant derivatives*, pointing in the same direction. In our investigations we find that all consistent cosmological solutions to the EFE satisfying this criterion, bar one — the Szekeres solution[1] — are in fact LRS spacetimes in the definition of Ellis [17] and Stewart and Ellis [75].

Definition 4.2.1 *We assume that in an open neighbourhood $\mathcal{U}$ of an event of an expanding perfect fluid spacetime $\mathcal{M}$ that*

1. *The shear is rotationally symmetric.*
2. *At least one of the Weyl curvature variables are rotationally symmetric about the same axis as the shear.*
3. *The fluid kinematical variables all point in the same invariant direction defined by the shear.*

That is to say, we have set $\boldsymbol{\sigma}$ rotationally symmetric and $\mathbf{E}$ and/or $\mathbf{H}$ may also be rotationally symmetric about the same axis. In addition, the vectors $\dot{\mathbf{u}}$ and $\boldsymbol{\omega}$ are aligned with the axis of rotational symmetry. In accordance with the definition given above, cosmological spacetimes $\mathcal{M}$ in which there is a tetrad such that these restrictions are satisfied are partially locally rotationally symmetric ('PLRS') *cosmologies. Those PLRS spacetimes that are* not *LRS will be referred to as* strictly PLRS. *The* weak PLRS *case involves setting either the shear $\sigma_{\alpha\beta}$, electric part of the Weyl tensor ($E_{\alpha\beta}$) or magnetic part of the Weyl tensor ($H_{\alpha\beta}$) to be rotationally symmetric. The* strong PLRS *case is as above for the weak form but also has covariant derivatives of $\sigma_{\alpha\beta}$, $E_{\alpha\beta}$ and/or $H_{\alpha\beta}$ vanishing. For a spacetime to be LRS, the tensor quantities concerned and some of their covariant derivatives must be rotationally symmetric to some order. A spin-off of the work in this thesis has to do with how few derivatives are needed.*

Remarks:

(i) Using a $\boldsymbol{\sigma}$–eigentetrad is equivalent to using an $\mathbf{E}$– or $\mathbf{H}$–eigentetrad when they are all RS. In this case, the spacetime is of Petrov type D and thus we overlap in many instances with the vast literature which has built up around this topic. We refer in particular to Wainwright [83] where a classification scheme for solutions with this property based on the Newman-Penrose formalism has been suggested. See also the series [10, 11, 78].

[1] See chapter 5.1.

(ii) If either of the vector quantities $\dot{\mathbf{u}}$ or $\boldsymbol{\omega}$ do *not* point in the invariant direction defined by the degenerate tensor quantities, then they define another invariant direction. The tetrad may then be invariantly defined by aligning the free legs with this new direction. And thus, for the situations we want to start from to be PLRS, we must have that $\dot{u}_2 = \dot{u}_3 = 0$; that is, $\dot{\mathbf{u}} \parallel \mathbf{e}_1$. Similarly, we must have $\omega_2 = \omega_3 = 0$; corresponding to $\boldsymbol{\omega} \parallel \mathbf{e}_1$. If these conditions do not hold, it would negate the possibility of the spacetime being LRS and thus compromise our notion of partial symmetry.

(iii) The present setup arbitrarily adapts to the spatial $\mathbf{e}_1$–axis. However, this is only a matter of convention and by a cyclic permutation of indices $1 \to 2 \to 3 \to 1$ one can easily adapt to any of the other spatial axes as well.

We will first discuss a known PLRS spacetime in the next chapter and then we shall deal with the cases below in turn as individual chapters.

Chapter 6: All tensors RS.

Chapter 7: Shear and gravito-magnetic field RS.

Chapter 8: Shear and gravito-electric field RS.

Finally, we will comment on the case which has only the shear RS and summarise our results in Chapter 9.

Chapter 5

The Known Dynamics of Irrotational PLRS Dust

We strive to find solutions to the dynamically restricted Einstein field equations. In particular, we seek to further clarify dust solutions with some type of rotational symmetry; a seminal paper in this regard being [17]. The dust case is interesting because here we already have a known model which satisfies our definition of partial symmetry: the Szekeres model. The question is how many other such models are there?

For dust, $p = 0$, we can see from the contracted Bianchi Identities (3.64) that

$$\dot{u}_\alpha = 0$$

(for non-zero μ). This allows for considerable simplification of the dynamical equations and related equations in the ONT. Also from the contracted Bianchi Identities (3.63) we infer that the dust is conserved as

$$\mathbf{e}_0(\mu) = -\theta\mu \,. \tag{5.1}$$

Of the 24 rotation coefficients (or commutation functions) there only remain 21 which are nonzero. And these 21 nonzero independent commutation functions are expressed in terms of 21 fluid dynamical and frame kinematical quantities. We assume from here on that the cosmological constant vanishes. We will consider the dynamical subcase where, in addition to restricting ourselves to dust ($p = 0 \Rightarrow \dot{u}_\alpha = 0$), we also specify that the dust must be irrotational ($\omega_\alpha = 0$) and weakly Partially Locally Rotationally Symmetric (PLRS)[1] with all tensors rotationally symmetric . When $\omega = 0$, in addition to the dust condition ${\gamma^0}_{0\alpha} = 0$, we also have that

$${\gamma^0}_{12} = {\gamma^0}_{23} = {\gamma^0}_{31} = 0 \,. \tag{5.2}$$

[1]This is described in Definition 4.2.1 in Chapter 4.

5.1 Szekeres Models

It is known that the Szekeres models would fall under our Definition 4.2.1 of PLRS spacetimes. Indeed, part of our investigation concerns determining exactly under what circumstances generalisations of LRS spacetimes are *not* in the Szekeres class of known solutions. With this in mind it seems useful for us to review these classes of known solutions. The deviation from LRS in Szekeres is provided by the non-vanishing of the $\mathbf{e}_2$– and $\mathbf{e}_3$–gradients of the energy density μ. In fact, these models have no Killing vectors [4]. We look at the characterisation of the Szekeres models [79, 78, 33, 2] which we have written in terms of our variables and formalism. This class of solutions was first proposed as solutions of the metric

$$ds^2 = -\,dt^2 + e^{2\alpha}\,dr^2 + e^{2\beta}\,(dy^2 + dz^2)\;,$$

where $\alpha = \alpha(t,r,y,z)$ and $\beta = \beta(t,r,y,z)$ and

$$G_{ij} = \mu\,u_i\,u_j \quad u_i = (-1,0,0,0) \quad \text{and} \quad \omega = 0\,.$$

We choose the natural dual orthonormal basis

$$\mathbf{e}^0 = dt\;,\quad \mathbf{e}^1 = e^{\alpha}dr\;,\quad \mathbf{e}^2 = e^{\beta}dy\;,\quad \mathbf{e}^3 = e^{\beta}dz\;,$$

such that

$$\mathbf{e}_0 = \frac{\partial}{\partial t}\quad \mathbf{e}_1 = e^{-\alpha}\,\frac{\partial}{\partial r}\,,\quad \mathbf{e}_2 = e^{-\beta}\,\frac{\partial}{\partial y}\,,\quad \mathbf{e}_3 = e^{-\beta}\,\frac{\partial}{\partial z}\,.$$

Note that all four basis vector fields $\mathbf{e}_a$ are hypersurface orthogonal (as Szekeres' metric ansatz is diagonal). We apply the commutator equations (A.3) – (A.8) to this basis: Thus from the first three we deduce that $\sigma_{\alpha\beta} = \Omega_\alpha = 0 \quad (\alpha \neq \beta)$ and then

$$\begin{aligned}\Theta &= \alpha_t + 2\,\beta_t\\ \sigma_{11} &= \tfrac{2}{3}\,(\alpha_t - \beta_t)\\ \sigma_{22} = \sigma_{33} &= -\tfrac{1}{3}\,(\alpha_t - \beta_t)\;;\end{aligned}$$

that is to say, this basis is a *Fermi-propagated* degenerate shear eigentetrad. From the last three commutator relations we find the specialisations

$$\begin{aligned}&n_{23} = n_{11} = n_{22} = n_{33} = 0\\ &a_1 = -\,e^{-\alpha}\,\beta_r\\ &a_2 = -\,\tfrac{1}{2}\,e^{-\beta}\,(\alpha_y + \beta_y) \quad \text{and} \quad a_3 = -\,\tfrac{1}{2}\,e^{-\beta}\,(\alpha_z + \beta_z)\\ &n_{31} = \tfrac{1}{2}\,e^{-\beta}\,(\alpha_y - \beta_y) \quad \text{and} \quad n_{12} = -\,\tfrac{1}{2}\,e^{-\beta}\,(\alpha_z - \beta_z)\;.\end{aligned} \tag{5.3}$$

The remaining field equations will determine the nature of $\mathbf{E}$ and $\mathbf{H}$. In fact we can show that *when the shear is rotationally symmetric and the tetrad is Fermi-propagated,*[2] *then the electric Weyl curvature*

[2] As is the case for the natural orthonormal basis for the Szekeres metric.

is immediately rotationally symmetric, a result which may be found in [?]. That is $E_{12} = E_{23} = E_{31} = (E_{22} - E_{33}) = 0$. Furthermore, for this tetrad

$$\text{(A.55), (A.56)} \Rightarrow H_{23} = 0$$
$$\text{(A.52), (A.53) and (A.54)} \Rightarrow H_{11} = H_{22} = H_{33} = 0 \ ,$$

and

$$\text{(A.30)} \Rightarrow a_1 H_{12} = 0$$
$$\text{(A.32)} \Rightarrow a_1 H_{31} = 0 \ .$$

From the above we see that we have a splitting in the type of solutions as was the case in Szekeres' original paper [79]. We may identify his class I and class II by whether a_1 vanishes or not. This shows that Szekeres' class I and II solutions may be invariantly classified since the tetrad is uniquely determined; we see from appendix A.3.2 that a_1 is an invariant under a spatial rotation in the e_2 / e_3–plane.

5.1.1 The Case $a_1 \neq 0$

This corresponds to the Szekeres class I [79], given through $\beta_r \neq 0$, as is evidenced by (5.3). In this case the magnetic Weyl curvature is zero: $\mathbf{H} = 0$. The remaining field equations are listed below. The contracted Bianchi identity (A.9) gives

$$\mu_t + (\alpha_t + 2\beta_t)\,\mu = 0 \ ,$$

the Raychaudhuri equation (from one nontrivial linear combination of $R_{0\alpha0\alpha}$ (no sum))

$$\alpha_{tt} + 2\,\beta_{tt} + \alpha_t^2 + 2\,\beta_t^2 + \tfrac{1}{2}\,\mu = 0 \ .$$

The $\dot{\sigma}$–equation yields (from the other nontrivial linear combination of $R_{0\alpha0\alpha}$ (no sum))

$$\alpha_{tt} - \beta_{tt} + \alpha_t^2 - \beta_t^2 + E_{11} = 0 \ .$$

while the $\dot{\mathbf{E}}$–equation, from (A.24), is

$$(E_{11})_t + 3\,\beta_t\,E_{11} + \tfrac{1}{3}\,\mu\,(\alpha_t - \beta_t) = 0 \ .$$

Moreover, from (A.55) and (A.56)

$$\beta_{tr} - \beta_r\,(\alpha_t - \beta_t) = 0 \ ,$$

from (A.57)

$$\alpha_{ty} + \alpha_y\,(\alpha_t - \beta_t) = 0 \ ,$$

from (A.58)

$$\beta_{ty} = 0 \ ,$$

from (A.59)

$$\alpha_{tz} + \alpha_z\,(\alpha_t - \beta_t) = 0\ ,$$

from (A.60)

$$\beta_{tz} = 0\ ,$$

from (A.62)

$$\alpha_{yz} - \alpha_y\,\beta_z - \alpha_z\,\beta_y + \alpha_y\,\alpha_z = 0\ ,$$

from (A.64)

$$\left(e^{-\alpha}\,\beta_r\right)_z = 0\ ,$$

from (A.66)

$$\left(e^{-\alpha}\,\beta_r\right)_y = 0\ ,$$

from (A.67)

$$e^{-2\beta}\,(\beta_{yy} + \beta_{zz}) + e^{-2\alpha}\,\beta_r^2 - \beta_t^2 + \tfrac{1}{3}\,\mu - E_{11} = 0\ ,$$

from (A.69) − (A.68)

$$\alpha_{yy} - \alpha_{zz} - 2\,\alpha_y\,\beta_y + 2\,\alpha_z\,\beta_z + \alpha_y^2 - \alpha_z^2 = 0\ ,$$

and from (A.69) + (A.68)

$$\begin{aligned} &2\,e^{-2\alpha}\,\beta_{rr} + e^{-2\beta}\,\alpha_{yy} + e^{-2\beta}\,\alpha_{zz} + 2\,e^{-2\alpha}\,\beta_r^2 + e^{-2\beta}\,\alpha_y^2 + e^{-2\beta}\,\alpha_z^2 \\ &\quad - 2\,e^{-2\beta}\,\alpha_r\,\beta_r - 2\,\alpha_t\,\beta_t + \tfrac{2}{3}\,\mu + E_{11} = 0\ . \end{aligned}$$

The second Bianchi identity gives the following constraints; from (A.71)

$$(E_{11})_r - \tfrac{1}{3}\,\mu_r + 3\,\beta_r\,E_{11} = 0\ ,$$

from (A.72) and (A.38)

$$(E_{11})_y = \tfrac{1}{3}\,\mu_y = -\,\alpha_y\,E_{11}\ ,$$

and from (A.73) and (A.40)

$$(E_{11})_z = \tfrac{1}{3}\,\mu_z = -\,\alpha_z\,E_{11}\ .$$

The generalised Friedmann equation (A.70) gives:

$$\begin{aligned} &-2\,e^{-2\alpha}\,(2\,\beta_{rr} + 3\,\beta_r^2 - 2\,\alpha_r\,\beta_r) - 2\,e^{-2\beta}(\alpha_{yy} + \beta_{yy} + \alpha_{zz} + \beta_{zz} + \alpha_y^2 + \alpha_z^2) \\ &\quad + 2\,\beta_t\,(2\,\alpha_t + \beta_t) - 2\,\mu = 0\ . \end{aligned}$$

All other equations are identically satisfied. The solution to these equations is known; see [79] and [33]. The corresponding tetrad relations of the above equations obtained as a subclass under our investigation now follow. First we list the evolution equations:

$$
\begin{aligned}
\mathbf{e}_0(\mu) &= -\Theta\,\mu \quad \text{(from (A.9))}\\
\mathbf{e}_0(\Theta_1) &= -{\Theta_1}^2 - E_{11} - \tfrac{1}{6}\mu \quad \text{(from (A.10))}\\
\mathbf{e}_0(\Theta_2) &= -{\Theta_2}^2 + \tfrac{1}{2}E_{11} - \tfrac{1}{6}(\mu + 3p) \quad \text{(from (A.11))}\\
\mathbf{e}_0(E_{11}) &= -3\,\Theta_2\,E_{11} - \tfrac{1}{2}\mu\sigma_{11} \quad \text{(from (A.24))}\\
\mathbf{e}_0(a_1) &= -\Theta_2\,a_1 \quad \text{(from (A.15) and (A.55))}\\
\mathbf{e}_0(a_2 - n_{31}) &= -\Theta_1\,(a_2 - n_{31}) \quad \text{(from (A.17) and (A.57))}\\
\mathbf{e}_0(a_3 + n_{12}) &= -\Theta_1\,(a_3 + n_{12}) \quad \text{(from (A.18) and (A.59))}\\
\mathbf{e}_0(a_2 + n_{31}) &= -\Theta_2\,(a_2 + n_{31}) \quad \text{(from (A.19) and (A.58))}\\
\mathbf{e}_0(a_3 - n_{12}) &= -\Theta_2\,(a_3 - n_{12}) \quad \text{(from (A.20) and (A.60))}\ .
\end{aligned}
$$

The constraints should also be noted. Of particular interest are the relations between the gradients of E_{11} and μ. Firstly, in the invariant direction defined by the PLRS restrictions,

$$\mathbf{e}_1(E_{11}) - \tfrac{1}{3}\,\mathbf{e}_1(\mu) - 3\,a_1\,E_{11} = 0 \quad \text{(from (A.71))}.$$

Then we note the defining characteristic of the Szekeres models — the gradients of E_{11} or μ do not vanish in the other directions. Instead,

$$
\begin{aligned}
\mathbf{e}_2(E_{11}) = \tfrac{1}{3}\,\mathbf{e}_2(\mu) = (a_2 - n_{31})\,E_{11} &\qquad \text{(from (A.38) and (A.39))}\\
\mathbf{e}_3(E_{11}) = \tfrac{1}{3}\,\mathbf{e}_3(\mu) = (a_3 + n_{12})\,E_{11} &\qquad \text{(from (A.40) and (A.41))}.
\end{aligned}
$$

The other notable constraints follow.

$$
\begin{aligned}
&\mathbf{e}_1(\Theta_2) + \tfrac{3}{2}\,a_1\,\sigma_{11} = 0 \quad \text{(from (A.55))}\\
&\mathbf{e}_2(\Theta_2) = \mathbf{e}_3(\Theta_2) = 0 \quad \text{(from (A.58) and (A.60))}\\
&\mathbf{e}_2(\Theta_1) - \tfrac{3}{2}\,(a_2 - n_{31})\,\sigma_{11} = 0 \quad \text{(from (A.57))}\\
&\mathbf{e}_3(\Theta_1) - \tfrac{3}{2}\,(a_3 + n_{12})\,\sigma_{11} = 0 \quad \text{(from (A.59))}\\
&\mathbf{e}_2(a_3) - \mathbf{e}_3(a_2) - a_2\,n_{12} - a_3\,n_{31} = 0 \quad \text{(from (A.61) and (6.49))}\\
&\mathbf{e}_1(a_3 - n_{12}) + 2\,a_1\,n_{12} = 0 \quad \text{(from (A.63))}\\
&\mathbf{e}_1(a_2 + n_{31}) - 2\,a_1\,n_{31} = 0 \quad \text{(from (A.65))}\\
&\mathbf{e}_3(a_2 - n_{31}) + 2\,(a_3 + n_{12})\,n_{31} = 0 \quad \text{(from (A.62))}\\
&\mathbf{e}_2(a_2 + n_{31}) + \mathbf{e}_3(a_3 - n_{12}) + \Theta_2^2 - a_1^2\\
&-(a_2 + n_{31})^2 - (a_3 - n_{12})^2 + E_{11} - \tfrac{1}{3}\,\mu = 0 \quad \text{(from (A.67))}\\
&\mathbf{e}_3(a_3 + n_{12}) - \mathbf{e}_2(a_2 - n_{31}) - 2\,[\,(a_2 - n_{31})\,n_{31} + (a_3 + n_{12})\,n_{12}\,] = 0 \quad \text{(from (A.68) – (A.69))}\\
&\mathbf{e}_3(a_3 + n_{12}) + \mathbf{e}_2(a_2 - n_{31}) + 2\,\mathbf{e}_1(a_1) + 2\,\Theta_1\,\Theta_2\\
&-2\,a_1^2 - 2\,[\,(a_2 - n_{31})\,a_2 + (a_3 + n_{12})\,a_3\,] - E_{11} - \tfrac{2}{3}\,\mu = 0 \quad \text{(from (A.68) + (A.69))}\ .
\end{aligned}
$$

Finally, the generalised Friedmann equation (A.70) reads:

$$4\,\mathbf{e}_1(a_1) + 4\,\mathbf{e}_2(a_2) + 4\,\mathbf{e}_3(a_3) - 6\,a_1^2 - 6\,a_2^2 - 6\,a_3^2 - 2\,n_{31}^2 - 2\,n_{12}^2 + \tfrac{2}{3}\,\Theta^2 - \frac{3}{2}\,\sigma_{11}^2 - 2\,\mu = 0\ .$$

5.1.2 The Case $a_1 = 0$

Here we are dealing with Szekeres class II [79] as can be seen from (5.3). These are generalisations of the Kantowski–Sachs [42] homogeneous and anisotropic (cosmological) models. They do not enter our discussion at any stage and are thus of little relevance to us here.

Chapter 6

Perfect Fluid Cosmologies with All Tensors LRS

We show that there are no new consistent perfect fluid cosmologies with *the kinematic variables and the electric and magnetic parts of the Weyl curvature all rotationally symmetric about a common axis* in an open neighbourhood $\mathcal{U}$ of an event. The consistent solutions of this kind are either locally rotationally symmetric, or are subcases of the Szekeres model. The prescriptions to the curvature variables makes the solutions we are studying here of Petrov type D. Thus this work may be viewed in conjunction with the rather vast literature that has built up around this. In particular, Wainwright classified the Petrov type D perfect fluids in which the fluid velocity, the vorticity vector and one of the eigenvalues of the shear tensor lie in the two-plane generated by the principal null directions [83].

We shall generally take $\sigma \neq 0$; we will see that with vanishing shear these types of universes specialize to the well known spatially homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker ('FLRW') cosmologies. The momentum conservation equations reduce in $\mathcal{U}$ to (A.42) – (A.44). Moreover, combining the $\mathbf{H}$–constraint equations (A.53) with (A.54) and (A.55) with (A.56), we find that

$$(n_{22} - n_{33})\,(\sigma_{11}^2 + \tfrac{4}{9}\,\omega_1^2) = n_{23}\,(\sigma_{11}^2 + \tfrac{4}{9}\,\omega_1^2) = 0;$$

which means, if we assume for now that $\sigma\omega \neq 0$ that[1]

$$(n_{22} - n_{33}) = n_{23} = 0\ . \tag{6.1}$$

We will find that the cases with $\omega_1 = 0$ are physically uninteresting. The related evolution equations obtained from (A.22) and (A.23) as well as (A.15) and (A.16), respectively, now reduce in $\mathcal{U}$ to constraints

[1]This restriction on $n_{\alpha\beta}$ (a direct result of the present PLRS restrictions) suggests that for these spacetimes no consistent solutions to the EFE exist that contain gravitational radiation; it is the transverse components $(n_{22} - n_{33})$, n_{23}, $(\sigma_{22} - \sigma_{33})$ and σ_{23} which typically form those connection characteristic eigenfields that propagate along null rays (cf. [29]). If we relax the PLRS restrictions so that $\mathbf{E}$ is not rotationally symmetric, we find in Chapter 7 that this still holds. Whilst if we relax the PLRS restrictions so that $\mathbf{H}$ is not rotationally symmetric, the results are as yet inconclusive; see Chapter 8.

on the e_2– and e_3–gradients of the Fermi-rotation variables Ω_2 and Ω_3; namely

$$(e_2 + 2\,n_{31})\,(\Omega_2) - (e_3 - 2\,n_{12})\,(\Omega_3) \;=\; 0 \tag{6.2}$$

$$(e_2 + 2\,n_{31})\,(\Omega_3) + (e_3 - 2\,n_{12})\,(\Omega_2) \;=\; 0\;. \tag{6.3}$$

We proceed to check the consistency of the PLRS subcase of the EFE by mainly computing the time evolution of all new constraints. In the process of doing this we will fix the tetrad freedom and thus invariantly classify the solutions. For each of the cases we consider below we will check the transformation behaviour of the commutation functions and other tensor quantities and use this to fix the freedom conveniently.

So we now turn to the issue of choice of spatial triad $\{\, e_\alpha \,\}$. As e_1 is presently a uniquely defined vector, $\dot{e}_1$ is fixed. This will thus have an invariantly defined direction, say $\mathbf{X}$. For any given tetrad choice, the components of this fixed direction in the e_2– and e_3–directions are $\mathbf{X} \cdot e_2$ and $\mathbf{X} \cdot e_3$, respectively. We have the freedom to set one of these components (or any other quantity which does not behave like a scalar under a rotation) to zero in $\mathcal{U}$ by rotating the spatial triad $\{\, e_\alpha \,\}$ in the $e_2\,/\,e_3$–plane. Alternatively, as can be seen from the transformation property of Ω_1,[2] we may choose to set $\Omega_1 = 0$ in $\mathcal{U}$ which fixes the e_0–gradient of φ everywhere; and then for example we choose either n_{22} or n_{33} in the 3-space by fixing the e_1–gradient of φ on a spatial hypersurface. This can be done provided the choices are consistent with the commutators. And so on.

For ease of notation we define

$$\Theta_\alpha := \tfrac{1}{3}\,\Theta + \sigma_{\alpha\alpha} \quad \text{(no summation)}\;.$$

6.1 General Perfect Fluid

This is the most general case where we will take throughout that $\dot{\mathbf{u}} \neq 0$ and $\omega \neq 0$. For $\omega = 0$ see section 6.2, for $\dot{\mathbf{u}} = 0$ refer to section 6.3. We assume a perfect fluid, leaving for the moment the equation of state unspecified.

6.1.1 Constraint Analysis

We proceed by fully fixing the tetrad freedom by choosing e_3 orthogonal to the projection of the fixed vector $\mathbf{X}$ in the $e_2\,/\,e_3$–plane. Hence, we rotate the spatial triad such that

$$\Omega_2 = e_3 \cdot \dot{e}_1 = -\,e_1 \cdot \dot{e}_3 = 0\;. \tag{6.4}$$

[2] See appendix A.3.2.

Under the given assumptions of PLRS symmetry, and with the above tetrad choice, the $\dot{\mathbf{E}}$–equations (A.29) and (A.30) yield the new constraints

$$\mathrm{e}_2(H_{11}) = 0 \qquad (6.5)$$

$$(a_2 - n_{31})\, H_{11} = 0 \;, \qquad (6.6)$$

while from the $\dot{\mathbf{H}}$–equations (A.38) and (A.39), the new constraints

$$\mathrm{e}_2(\mu) = 3\,(a_2 - n_{31})\, E_{11} \qquad (6.7)$$

$$\mathrm{e}_2(E_{11}) = \tfrac{1}{3}\, \mathrm{e}_2(\mu) \qquad (6.8)$$

arise. The algebraic condition (6.6) suggests that we distinguish between two subcases according to

$$\mathbf{A]}\;\; (a_2 - n_{31}) = 0 \quad \text{or} \quad \mathbf{B]}\;\; H_{11} = 0 \;.$$

A] $(a_2 - n_{31}) = 0$: So now $(a_2 + n_{31}) = 2\,a_2$. From the $\dot{\boldsymbol{\omega}}$–equation (A.49) we see that $\mathrm{e}_2(\dot{u}_1) = 0$ and from (6.7) we have $\mathrm{e}_2(\mu) = 0$. The $\dot{\boldsymbol{\sigma}}$–equation (A.47) then shows that also $\Omega_3\, \sigma_{11} = 0$ must hold, providing a split into further subcases according to

$$\mathbf{A1]}\;\; \Omega_3 = 0 \quad \text{or} \quad \mathbf{A2]}\;\; \sigma_{11} = 0 \;.$$

A1] $\Omega_3 = 0$: This has the implications from (A.48) that $\mathrm{e}_3(\dot{u}_1) = 0$, which implies from (A.50) that $(a_3 + n_{12}) = 0$, as we assumed $\dot{u}_1 \neq 0$. We note that the following e_2– and e_3–gradients of certain quantities must vanish: from (6.5) and (A.32) we get that $\mathrm{e}_2(H_{11}) = \mathrm{e}_3(H_{11}) = 0$ and from (A.38) – (A.41) we find that $\mathrm{e}_2(E_{11}) = \mathrm{e}_3(E_{11}) = 0$ and $\mathrm{e}_2(\mu) = \mathrm{e}_3(\mu) = 0$. Then (A.17) and (A.18) reduce to

$$\mathrm{e}_2(\Theta_1) = \mathrm{e}_3(\Theta_1) = 0 \;. \qquad (6.9)$$

Now we check the propagation property of the vanishing e_2– and e_3–gradients of the energy density. We use the commutator relations (A.4) and (A.5) operating on μ and the energy conservation equation (A.9) to show that $\mathrm{e}_2(\mu) = \mathrm{e}_3(\mu) = 0$ if $(\mu + p)\, \mathrm{e}_2(\Theta) = (\mu + p)\, \mathrm{e}_3(\Theta) = 0$, since the momentum conservation equations (A.43) and (A.44) must hold. Thus, if we want to stick to purely cosmological solutions as we have defined it in the introduction, we must have $\mathrm{e}_2(\Theta) = \mathrm{e}_3(\Theta) = 0$. Now this means that $\mathrm{e}_2(\omega_1) = \mathrm{e}_3(\omega_1) = 0$, which we obtain from (A.58) and (A.60), suitably combined with (6.9). This result is crucial because now we can show that the solutions contained in the present PLRS subclass are *not* cosmological ones. To do so we find the commutator (A.6) most useful. We first apply this commutator relation to ω_1, yielding

$$2\,\Theta_2\, \omega_1 - a_1\, n_{11} = 0 \;, \qquad (6.10)$$

where we have used the vorticity evolution equation (A.14), and the constraint on its gradient in the $\mathbf{e}_1$–direction given by (A.51). We then apply the commutator (A.6) to the energy density μ, using (A.9), and find that

$$2\,\Theta\,\omega_1\,(\mu+p) - n_{11}\,\mathbf{e}_1(\mu) = 0 \; . \tag{6.11}$$

Finally we apply the commutator (A.6) to the electric Weyl curvature component E_{11} and substitute from (A.24), (A.71) and (6.11) to get

$$2\,\Theta_2\,\omega_1\,[\,(\mu+p) - 3\,E_{11}\,] + 3\,a_1\,n_{11}\,E_{11} = 0 \; . \tag{6.12}$$

We now use (6.10) in (6.12) to get $\Theta_2\,\omega_1\,(\mu+p) = 0 \Rightarrow \Theta_2 = 0$, as we assumed $(\mu+p) > 0$ and $\omega_1 \neq 0$. Substituting back into (6.10) now gives $a_1\,n_{11} = 0$. We argue that this necessarily means $n_{11} = 0$. If instead we started from $a_1 = 0$, then (A.55) shows that $n_{11}\,\omega_1 = 0 \Rightarrow n_{11} = 0$; hence, $n_{11} = 0$ in any case. But then (A.61) shows $\Theta_1 = 0$, and thus, since we already have $\Theta_2 = 0$, we find with the Raychaudhuri equation (A.13) that in $\mathcal{U}$

$$\Theta = 0 \; , \tag{6.13}$$

violating the premise that our cosmology be an expanding one. We conclude that there exist no cosmologically viable solutions in the present PLRS subclass.

A2] $\sigma_{11} = 0$: From the $\dot{\boldsymbol{\sigma}}$– and $\dot{\boldsymbol{\omega}}$–equations, respectively (A.48) and (A.50), we must have

$$(a_3 + n_{12})\,\dot{u}_1 - \Omega_3\,\omega_1 = 0 \; , \tag{6.14}$$

and from the second Bianchi identities (A.31) and (A.32),

$$(a_3 + n_{12})\,H_{11} + \Omega_3\,E_{11} = 0 \; . \tag{6.15}$$

Combining these two equations, we get that either $(a_3 + n_{12}) = \Omega_3 = 0$ which would be a subcase of **A1]** and is thus non-cosmological, or

$$\dot{u}_1\,E_{11} + \omega_1\,H_{11} = 0 \; . \tag{6.16}$$

An important algebraic relation is given by the **H**–constraint (A.52); that is

$$2\,(\dot{u}_1 + a_1)\,\omega_1 + H_{11} = 0 \; . \tag{6.17}$$

The tetrad choice employed has the effect of eliminating the gradients generally in the $\mathbf{e}_2$–direction of important scalars. In particular, from (A.59) and (A.60) we get

$$\mathbf{e}_2(\omega_1) = 0 = \mathbf{e}_3(\Theta) \; . \tag{6.18}$$

Now, from (A.49) we get that

$$\mathbf{e}_2(\dot{u}_1) = 0 \; , \tag{6.19}$$

and from (A.75) we get

$$\mathbf{e}_2(H_{11}) = 0 \ . \tag{6.20}$$

From (A.38), suitably combined with (A.39), we get

$$\mathbf{e}_2(E_{11}) = \mathbf{e}_2(\mu) = 0 \ . \tag{6.21}$$

If we now take the $\mathbf{e}_2$–gradient of equation (6.17) and substitute the three equations (6.18), (6.19) and (6.20), we get the useful result

$$\mathbf{e}_2(a_1) = 0 \ . \tag{6.22}$$

A relation for the $\mathbf{e}_3$–gradient of the vorticity is provided by (A.57) and (A.58):

$$\mathbf{e}_3(\omega_1) + (a_3 + n_{12})\,\omega_1 = 0 \ . \tag{6.23}$$

We get relations involving the $\mathbf{e}_3$–gradients of E_{11} and μ from combining (A.40) and (A.41) suitably, thus yielding

$$\mathbf{e}_3(E_{11}) - \tfrac{1}{3}\,\mathbf{e}_3(\mu) - 3\,\Omega_3\,H_{11} = 0 \ , \tag{6.24}$$

$$\mathbf{e}_3(\mu) - 3\,(a_3 + n_{12})\,E_{11} + 3\,\Omega_3\,H_{11} = 0 \ . \tag{6.25}$$

The commutators provide vital information here. We find from the commutator (A.6) acting on ω_1 that

$$-\tfrac{2}{3}\,\Theta\,\omega_1 + a_1\,n_{11} = 2\,a_2\,(a_3 + n_{12}) \ , \tag{6.26}$$

where we have used the evolution equation for the remaining component of the vorticity (A.14), the constraints on the gradients of the vorticity provided by (A.51), (6.18) and (6.23), and then substituted (A.61) with (A.62) appropriately. We find from the commutator (A.6) acting on the magnetic Weyl curvature component H_{11} that

$$3\,H_{11}\,(-\tfrac{2}{3}\,\Theta\,\omega_1 + a_1\,n_{11}) - n_{11}\,(\mu + p)\,\omega_1 - 12\,a_2\,E_{11}\,\Omega_3 = 0 \ ,$$

using a relation for the $\mathbf{e}_3$–gradient of H_{11} provided by (A.32), noting the constraint on the $\mathbf{e}_2$–gradient of H_{11} (6.20), the constraint on the $\mathbf{e}_1$–gradient of H_{11} given by (A.74) and then using the evolution equation for H_{11} provided by (A.33). We also needed the first part of (6.21) and (6.3). Combining the above with (6.26) we get the useful result

$$n_{11}\,(\mu + p)\,\omega_1 = -\,18\,a_2\,E_{11}\,\Omega_3 \ , \tag{6.27}$$

where we have also used (6.15). We now take the commutator (A.8) operating on ω_1 and substitute (6.18), (6.19), (6.22), the vorticity constraint equation (A.51) and (6.23) into the resultant expression and we find that $n_{33}\,(a_3 + n_{12})\,\omega_1 = 0$. Now if $(a_3 + n_{12}) = 0$, then from (6.14) we must have $\Omega_3 = 0$ and thus this would be a class already dealt with in section **A2**] and those were non-cosmological. So we conclude that $n_{33} = 0$. We now take the $\mathbf{e}_3$–gradient of (6.16) and get the key result

$$\dot{u}_1\, \mathbf{e}_3(\mu) = 0 \quad \Rightarrow \quad \mathbf{e}_3(\mu) = 0 \tag{6.28}$$

by using, in addition to (6.16), equations (6.15), (6.23) – (6.25), (A.50) and (A.32). We proceed to check the consistency of (6.24) and (6.25) in the combined form $\mathbf{e}_3(E_{11}) - 3\,(a_3 + n_{12})\,E_{11} = 0$. We propagate this and find

$$\tfrac{3}{2}\, n_{11}\,[\,(a_3 + n_{12})\,H_{11} - \Omega_3\,E_{11}\,] = 0\ , \tag{6.29}$$

where we have used the $\dot{\mathbf{E}}$-equation (A.24),the constraints on the $\mathbf{e}_2$–gradients of E_{11} and μ given respectively by (6.21) and (6.18), the constraints on the $\mathbf{e}_3$–gradient of H_{11} given by (A.32), and the Jacobi Identities (A.18) together with (A.59); the evolution of the $\mathbf{e}_3$–gradient of E_{11} is obtained by applying the commutator (A.5) to E_{11} and using the equations (A.24) with (A.66). Now we may rewrite (6.29) by using (6.15) obtaining $n_{11}\,(a_3 + n_{12})\,H_{11} = 0$.

- We show that this means that $n_{11} = 0$. If not, then $(a_3 + n_{12})\,H_{11} = 0$ which means that $\Omega_3\,E_{11} = 0$ (from (6.15)), which then contradicts (6.27).

We have seen here that $n_{11} = 0$ is required. But now, again from (6.27), we must have $a_2\,\Omega_3\,E_{11} = 0$. So either a_2 or $\Omega_3\,E_{11}$ vanishes. We consider these possibilities below.

- If $\Omega_3\,E_{11} = 0$, then from (6.15) we require $(a_3 + n_{12})\,H_{11} = 0$. If now $H_{11} = 0$, then (A.27) tells us that these solutions are not cosmological since they require $(\mu + p)\,\omega_1 = 0 \Rightarrow (\mu + p) = 0$. And if $(a_3 + n_{12}) = 0$, then from (6.26) we get in $\mathcal{U}$ $\Theta\,\omega_1 = 0 \Rightarrow \Theta = 0$, and we are clearly in the non-cosmological realm again.

- If, on the other hand, $a_2 = 0$, then again from (6.26) we must have in $\mathcal{U}$ $\Theta\,\omega_1 = 0 \Rightarrow \Theta = 0$. So none of the solutions here are of relevance to us.

B] $H_{11} = 0$**:** Immediately we see from (A.27) that $(\mu + p) + 3\,E_{11} = 0$. Also, from (A.33), we get that $n_{11}\,E_{11} = 0$. Now if $E_{11} = 0$, then $(\mu + p) = 0$ which is not allowed. So $n_{11} = 0$. We also get from (A.31) that $\Omega_3\,E_{11} = 0$, and once again we deduce that since $E_{11} = 0 \Rightarrow (\mu + p) = 0$, it follows that we must have $\Omega_3 = 0$. Moreover $\mathbf{e}_3(\dot{u}_1) = 0$ from (A.48); and $\mathbf{e}_2(\dot{u}_1) = 0$ from (A.47). This tells us from (A.49) that $(a_2 - n_{31}) = 0$ and from (A.50) that $(a_3 + n_{12}) = 0$. And now from (A.62) we get that $\Theta_1\,\omega_1 = 0 \Rightarrow \Theta_1 = 0$. We get from (A.38) and (A.39) that $\mathbf{e}_2(E_{11}) = \mathbf{e}_2(\mu) = 0$ and from (A.40) and (A.41) that $\mathbf{e}_3(E_{11}) = \mathbf{e}_3(\mu) = 0$. If we now take the above two relations for the gradients of μ and substitute them into the commutator (A.6) applied to μ using (A.9), we get $\Theta\,\omega_1\,(\mu + p) = 0$ and so in $\mathcal{U}$

$$\Theta\,(\mu + p) = 0\ ; \tag{6.30}$$

in other words, there are no solutions in this PLRS subclass that are of cosmological interest.

6.1.2 Summary

We conclude that there are no rotating and accelerating perfect fluid cosmologies which are PLRS as we have limited the definition of Definition 4.2.1.

6.2 Irrotational Accelerating Perfect Fluid

These models have $\boldsymbol{\omega} = 0$. We assume that $\dot{\mathbf{u}} \neq 0$. An immediate implication here for the commutation functions $n_{\alpha\beta}$, in addition to (6.1), is that $n_{11} = 0$, from (A.14). Now from the $\mathbf{H}$-constraint (A.52) it follows that $H_{11} = 0$, reducing the $\dot{\mathbf{H}}$–equation (A.33) to a trivial statement. A useful point of departure is provided here by the $\dot{\mathbf{E}}$–equations (A.29) and (A.31). That is, $\Omega_2\, E_{11} = \Omega_3\, E_{11} = 0$. A brief argument below will show that this means that in $\mathcal{U}$

$$\Omega_2 = \Omega_3 = 0\,. \tag{6.31}$$

- The argument goes as follows. If $E_{11} = 0$, then from (A.24) we have $(\mu{+}p)\,\sigma_{11} = 0 \Rightarrow \sigma_{11} = 0$. We find that the following gradients vanish: $\mathbf{e}_1(\mu) = 0 = \mathbf{e}_1(\Theta)$ from (A.71) and (A.55), respectively. And now from the commutator relation (A.3) acting on μ we get $\Theta\,(\mu + p) = 0$; that is, this is a non-cosmological subcase. In this last statement, we have employed the assumption that the matter fluid has a barotropic equation of state, $p = p(\mu)$.

So we must have (6.31) holding — which then implies that

$$\mathbf{e}_2(\dot{u}_1) = \mathbf{e}_3(\dot{u}_1) = 0 \tag{6.32}$$

from (A.47) and (A.48). In turn the effect of the above is that $(a_2 - n_{31}) = (a_3 + n_{12}) = 0$ which derives from (A.49) and (A.50). So we may write $(a_2 + n_{31}) = 2\,a_2$ and $(a_3 - n_{12}) = 2\,a_3$.

6.2.1 Tetrad Choice and Constraint Analysis

From the above we see that we are free to employ along $\mathbf{u}$ a Fermi-propagated spatial triad $\{\,\mathbf{e}_\alpha\,\}$ by setting $\Omega_1 = \mathbf{e}_0(\varphi)$ via a spatial rotation by an angle φ in the $\mathbf{e}_2\,/\,\mathbf{e}_3$-plane so that $\Omega_1' = 0$. We note the following consistency conditions:

$$\begin{aligned} n_{11} &= 0 \quad \text{(from (A.21))} \\ (a_2 - n_{31}) &= 0 \quad \text{(from (A.17) and (A.57))} \\ (a_3 + n_{12}) &= 0 \quad \text{(from (A.18) and (A.59))}\,. \end{aligned}$$

We can further use the tetrad freedom on a hypersurface $x^0 = c^0$ to set $n_{33} = 0$. This quantity is conserved as we can see from (A.23); so we are allowed to do this since there is no loss of generality from this choice. We can also set $a_3 = 0$ on a 2-surface $x^0 = c^0$, $x^1 = c^1$. This we can do with impunity since firstly $\mathbf{e}_0(a_3) = -\Theta_2\, a_3$ from (A.20) and (A.60); secondly $\mathbf{e}_1(a_3) = a_1\, a_3$ from (A.63) and (A.64). We conclude $a_3 = 0$ everywhere. We may summarise: the only non-zero commutator functions are $\dot{u}_1$, Θ_1, Θ_2, a_1 and a_2. Of these quantities, only a_2 does not have its $\mathbf{e}_2$– and $\mathbf{e}_3$–gradients vanishing; as can be seen from (6.32), (A.57) and (A.58), (A.59) and (A.60), (A.64) and (A.66). We may now proceed to use the remaining tetrad freedom to set $\mathbf{e}_3(a_2) = 0$ on the line $x^0 = c^0$, $x^1 = c^1$, $x^2 = c^2$. This we can do by observing that the $\mathbf{e}_0$–, $\mathbf{e}_1$– and $\mathbf{e}_2$–derivatives of this quantity are conserved respectively by applying the respective commutators (A.5), (A.7) and (A.6) to a_2. We also need (A.19) and (A.58), (A.65) and (A.67) to see this. Finally we set $\mathbf{e}_2(a_2) = 0$ at an event $x^i = c^i$. This can be done because firstly, the $\mathbf{e}_0$ derivative of $\mathbf{e}_2(a_2)$ is driven by a multiple of $\mathbf{e}_2(a_2)$ (from (A.4) operating on a_2 and (A.19)); secondly, the $\mathbf{e}_1$–derivative of $\mathbf{e}_2(a_2)$ is driven by a multiple of $\mathbf{e}_2(a_2)$ (from (A.8) operating on a_2 and (A.65)); thirdly, the $\mathbf{e}_2$–derivative of $\mathbf{e}_2(a_2)$ is driven by a multiple of $\mathbf{e}_2(a_2)$ (from taking $\mathbf{e}_2$ of (A.67)); lastly, the $\mathbf{e}_3$–derivative of $\mathbf{e}_2(a_2)$ vanishes (from (A.6) operating on a_2). The remaining commutator functions have their gradients in the $\mathbf{e}_2$ / $\mathbf{e}_3$–plane vanishing and the only equations remaining constrain quantities in the one invariant spatial direction (from (A.55), (A.68) and (A.65)). There is also an algebraic relation (A.67) determining E_{11} in terms of μ, say. The curvature variables which remain non-zero in $\mathcal{U}$ are p, μ, and E_{11}. All of these variables also have their gradients vanishing in the $\mathbf{e}_2$ / $\mathbf{e}_3$–plane. This is obvious from (A.43) and (A.44), and (A.38) combined with (A.39), as well as (A.40) combined with (A.41). The remaining constraints on the various gradients in the fixed directions are given by (A.42) and (A.71). The consistent solutions in $\mathcal{U}$ we obtain in this fashion are the LRS class II solutions of Stewart and Ellis [75].

6.2.2 Summary

We conclude that the only irrotational accelerating perfect fluid cosmologies that are PLRS in this chapter are the expanding solutions in LRS class II of Stewart and Ellis [75] (see also [28] and [59]). To see this, we had to assume (only) in the shear-free subcase that the fluid was barotropic. But we have not made this assumption anywhere else; thus this is not central to our argument.

6.3 Rotating Dust

These models have $p = 0$. From part of the twice-contracted Bianchi identities (momentum conservation equations), $\mathbf{e}_\alpha(p) = 0 \Rightarrow \dot{u}^\alpha = 0$. We assume here that $\omega \neq 0$. Since we are dealing with dust we note that our tetrad choice is that of Ellis in [17] [3] which allows us to easily recognise in a standard form the

[3] Described in his Theorem 3.1.

LRS spacetimes which he discussed in that paper when we find them. Because we demand $\Theta > 0$, we refrain from identifying his rotating dust solutions of LRS class II which require $\Theta = 0$. Recalling this tetrad choice, in addition to $\mathbf{u} = \mathbf{e}_0$, $\boldsymbol{\omega} = \omega_1\,\mathbf{e}_1$. Since we are dealing with degenerate shear in the $\mathbf{e}_2\,/\,\mathbf{e}_3$–plane, this amounts to aligning the vorticity vector with the preferred spatial direction singled out by the shear tensor. We immediately note from the Jacobi identities (A.49) and (A.50) that in $\mathcal{U}$

$$\Omega_2 = \Omega_3 = 0\ .$$

6.3.1 Tetrad Choice and Constraint Analysis

We are free to propagate the spatial triad $\{\,\mathbf{e}_\alpha\,\}$ along $\mathbf{u}$ as anti-rotating by choosing $\mathbf{e}_0(\varphi) = (\omega_1 + \Omega_1)$, that is $(\omega_1 + \Omega_1)' = 0$. We can further use the tetrad freedom on a 3-surface $x^0 = c^0$ to set $n'_{33} = 0$ by choosing $\mathbf{e}_1(\varphi)$ such that $\mathbf{e}_1(\varphi) = -\,n_{33}$. This quantity is conserved, as we can see from (A.23); so we are allowed to do this since there is no loss of generality from this choice. But now we may not as yet proceed with a further tetrad specification as in [17], where $(a_3 - n_{12}) = 0$ on a 2-surface $x^0 = c^0$, $x^1 = c^1$, because this would constrain the geometry. In particular, it actually requires $\mathbf{e}_2(\omega_1) = 0$ for $(a_3 - n_{12}) = 0$ to hold – using (A.20) and (A.60). So we proceed by leaving the freedom unfixed for now and see what the implications are from consistency checks. From setting $\boldsymbol{\sigma}$ to be rotationally symmetric we do not get any immediate constraints. But $\boldsymbol{\sigma}$ feeds into $\mathbf{E}$, and from setting $\mathbf{E}$ to be rotationally symmetric we also get new constraints. Specifically we get

$$\begin{aligned}
(a_2 - n_{31})\,H_{11} &= 0 \quad \text{(from the } \dot{\mathbf{E}}\text{–equation (A.29) combined with (A.75))} \\
(a_3 + n_{12})\,H_{11} &= 0 \quad \text{(from the } \dot{\mathbf{E}}\text{–equation (A.31) combined with (A.76))}\ .
\end{aligned}$$

So naturally here we have a split into

$$\textbf{A]}\ \ H_{11} = 0 \quad \text{or} \quad \textbf{B]}\ \ (a_2 - n_{31}) = (a_3 + n_{12}) = 0\ .$$

A] $H_{11} = 0$: For H_{11} to vanish, we must have from the $\dot{\mathbf{H}}$–equation (A.33) that $n_{11}\,E_{11} = 0$.

- A brief argument now shows that we are not interested in $E_{11} = 0$. It goes as follows. If $E_{11} = 0$, then from (A.24) we have $\mu\,\sigma_{11} = 0 \Rightarrow \sigma_{11} = 0$. For consistency we now require from the $\dot{\boldsymbol{\sigma}}$–equations (A.10) and (A.11) that $\omega_1 = 0$; that is, this case is dealt with elsewhere. In fact this leads to Robertson–Walker solutions as we have already noted.

Then we conclude that $E_{11} \neq 0$ must hold here and hence $n_{11} = 0$. This has the immediate implication from the $\mathbf{H}$–constraint equation (A.52) that $a_1 = 0$. Now we get from the second Bianchi identity the following new constraints

$$3\,(a_2 - n_{31})\,E_{11} - \mathbf{e}_2(\mu) \;=\; 0 \quad \text{(from (A.38) and (A.39))} \tag{6.33}$$
$$3\,(a_3 + n_{12})\,E_{11} - \mathbf{e}_3(\mu) \;=\; 0 \quad \text{(from (A.40) and (A.41))}\ . \tag{6.34}$$

We propagate (6.33) twice along $\mathbf{u}$, using the necessary evolution equations. We use (A.17), (A.24), and (A.4) operating on μ with (A.9), to get

$$\omega_1\,(a_3 + n_{12})\,(3\,E_{11} - \mu) + 2\,\mu\,\mathbf{e}_3(\omega_1) = 0\ . \tag{6.35}$$

We now need (A.14), (A.18), (A.24), (A.9), and (A.5) operating on ω_1, to get for consistency of (6.35) that

$$-\,\omega_1\,(a_2 - n_{31})\,(3\,E_{11} - \mu) + 4\,\mu\,\mathbf{e}_2(\omega_1) = 0\ . \tag{6.36}$$

We propagate (6.34) twice along $\mathbf{u}$, using the necessary evolution equations. These are (A.18), (A.24), and (A.5) operating on μ with (A.9), which thus yields

$$\omega_1\,(a_2 - n_{31})\,(3\,E_{11} - \mu) + 2\,\mu\,\mathbf{e}_2(\omega_1) = 0\ . \tag{6.37}$$

We now need (A.14), (A.17), (A.24), (A.9), and (A.4) operating on ω_1 to get for consistency of (6.37) that

$$\omega_1\,(a_3 + n_{12})\,(3\,E_{11} - \mu) - 4\,\mu\,\mathbf{e}_3(\omega_1) = 0\ . \tag{6.38}$$

We form linear combinations of the above four constraints to facilitate our task at this point.

$$\begin{aligned}
\mu\,\mathbf{e}_3(\omega_1) &= 0 \quad \text{(from (6.35) − (6.38))} \quad \Rightarrow \quad \mathbf{e}_3(\omega_1) = 0\\
\mu\,\mathbf{e}_2(\omega_1) &= 0 \quad \text{(from (6.36) + (6.37))} \quad \Rightarrow \quad \mathbf{e}_2(\omega_1) = 0\\
(a_3 + n_{12})\,(3\,E_{11} - \mu) &= 0 \quad \text{(from } 2\times \text{ (6.35) + (6.38))}\\
(a_2 - n_{31})(3\,E_{11} - \mu) &= 0 \quad \text{(from (6.36) } - 2\times \text{ (6.37))}\ .
\end{aligned}$$

- A brief argument now shows that $3\,E_{11} - \mu = 0$ is not applicable. It goes as follows. We note that $\mathbf{e}_2(\omega_1) = \mathbf{e}_3(\omega_1) = 0$. So from the commutator (A.6) acting on ω_1 and incorporating the $\dot{\boldsymbol{\omega}}$–equation (A.14) into this, we get that $\Theta_2 = 0$. The consistency of this requires from (A.11) that $3\,E_{11} - \mu + 3\,\omega_1^2 = 0$, and if $3\,E_{11} - \mu = 0$ it must necessarily follow that $\omega_1 = 0$; and this case is dealt with elsewhere.

So we must conclude that $(a_2 - n_{31}) = (a_3 + n_{12}) = 0$. This is severely restrictive. We get from the commutator (A.6) acting on ω_1 that $\Theta_2 = 0$, using the $\dot{\boldsymbol{\omega}}$–equation (A.14). And now from (A.61) we must also have $\Theta_1 = 0$. These last two results, in particular, imply that in $\mathcal{U}$

$$\Theta = 0\ , \tag{6.39}$$

which means that spacetimes in this PLRS subclass are non-cosmological.

B] $(a_2 - n_{31}) = (a_3 + n_{12}) = 0$: Firstly we note from the relations obtained from (A.57) and (A.59) that

$$\mathbf{e}_2(\Theta_1) = \mathbf{e}_3(\Theta_1) = 0 \; . \tag{6.40}$$

The critical constraints are obtained from the second Bianchi identity: (A.38) combined with (A.39), and (A.40) combined with (A.41) once again. They read, respectively,

$$\begin{aligned} \mathbf{e}_2(E_{11}) = \mathbf{e}_2(\mu) &= 0 \\ \mathbf{e}_3(E_{11}) = \mathbf{e}_3(\mu) &= 0 \; . \end{aligned}$$

We proceed to check the preservation along $\mathbf{u}$ of $\mathbf{e}_2(\mu) = \mathbf{e}_3(\mu) = 0$ by using evolution equations obtained from the commutator relations (A.4) and (A.5) when acting on μ. We also require from this the relations given by (A.9), (A.57), (A.59), (A.58) and (A.60). We get that

$$\begin{aligned} \mathbf{e}_0(\mathbf{e}_2(\mu)) &= (\tfrac{1}{2}\,\sigma_{11} - \tfrac{4}{3}\,\Theta)\,\mathbf{e}_2(\mu) - 2\,\mathbf{e}_3(\omega_1) \quad \Rightarrow \quad \mathbf{e}_3(\omega_1) = 0 \\ \mathbf{e}_0(\mathbf{e}_3(\mu)) &= (\tfrac{1}{2}\,\sigma_{11} - \tfrac{4}{3}\,\Theta)\,\mathbf{e}_3(\mu) + 2\,\mathbf{e}_2(\omega_1) \quad \Rightarrow \quad \mathbf{e}_2(\omega_1) = 0 \; . \end{aligned}$$

Taking these results and putting them back into (A.57), (A.59), (A.58) and (A.60), and recalling (6.40), we have $\mathbf{e}_2(\Theta_1) = \mathbf{e}_3(\Theta_1) = \mathbf{e}_2(\Theta_2) = \mathbf{e}_3(\Theta_2) = 0$. We note now from (A.75) and (A.76) that it is apparent that the gradients of H_{11} are also degenerate in this fashion: $\mathbf{e}_2(H_{11}) = \mathbf{e}_3(H_{11}) = 0$. We check the time consistency of $\mathbf{e}_2(E_{11}) = \mathbf{e}_3(E_{11}) = 0$ by using (A.71) and the commutators (A.4) and (A.5) acting on E_{11}. The results are that

$$\begin{aligned} \mathbf{e}_0(\mathbf{e}_2(E_{11})) &= -\tfrac{3}{2}\,H_{11}\,\mathbf{e}_2(n_{11}) \\ \mathbf{e}_0(\mathbf{e}_3(E_{11})) &= -\tfrac{3}{2}\,H_{11}\,\mathbf{e}_3(n_{11}) \; . \end{aligned}$$

If now $H_{11} = 0$, then we have a case already dealt with in **A**]. So we must therefore conclude that $\mathbf{e}_2(n_{11}) = \mathbf{e}_3(n_{11}) = 0$. Now here again crucial algebraic constraints are obtained from the commutator (A.6). Acting on ω_1 we get

$$\Theta_2\,\omega_1 = a_1\,n_{11} \tag{6.41}$$

from the $\dot{\boldsymbol{\omega}}$–equation (A.14) and the constraint on the remaining non-zero gradient of the vorticity (A.51). Acting on n_{11} we get

$$n_{11}\,(\sigma_{11}\,\omega_1 + a_1\,n_{11}) = 0 \tag{6.42}$$

from the evolution equation for n_{11} (A.21) and the constraint on the remaining non-zero gradient of n_{11} (A.61). We show that we do not get any relevant solutions here. We start with equation (6.42).

- We first show that $n_{11} = 0$ leads to trivial solutions. If $n_{11} = 0$, then from (A.61) we must have $\Theta_1 = 0$ and then from (6.41) this means that $\Theta_2 = 0$ and so this cannot be cosmological because consequently in $\mathcal{U}$

$$\Theta = 0 \; . \tag{6.43}$$

- If, on the other hand, $n_{11} \neq 0$, and instead we have from (6.42) that $\sigma_{11}\,\omega_1 + a_1\,n_{11} = 0$, then we immediately get from (6.41) that $\Theta\,\omega_1 = 0$; that is, in $\mathcal{U}$

$$\Theta = 0 \ . \tag{6.44}$$

So these are also not cosmological solutions because they are not expanding.

6.3.2 Summary

We conclude that there are no rotating dust cosmologies which are PLRS such that all tensors point in the same direction.

6.4 Irrotational Dust

We shall now consider irrotational dust spacetimes under the PLRS restrictions of this chapter: that is to say, spacetimes where $\boldsymbol{\omega} = 0$ and $p = 0 \Rightarrow \dot{\mathbf{u}} = 0$. We can show that the only consistent irrotational dust solutions of the EFE which have $\boldsymbol{\sigma}$ and *both* $\mathbf{E}$ and $\mathbf{H}$ rotationally symmetric in the same plane are known solutions.[4] Thus here we may take $\boldsymbol{\sigma} \neq 0$ throughout, because $\boldsymbol{\sigma} = 0$ leads to the FLRW solutions [17]. Proceeding since, in particular, in an open neighbourhood $\mathcal{U}$ of an event of $\mathcal{M}$ we have $E_{12} = E_{31} = 0$, we find from (A.48) and (A.47) that in $\mathcal{U}$

$$\Omega_2 = \Omega_3 = 0 \ .$$

6.4.1 Tetrad Choice and Constraint Analysis

We can state that for the models we are interested in here, the shear eigentetrad is Fermi-propagated along $\mathbf{u}$. We see from the transformation behaviour of the commutation functions that we may choose $\Omega_1' = 0$ by setting $\mathbf{e}_0(\varphi) = \Omega_1$ by a spatial rotation by an angle φ in the plane of rotational symmetry as before. We may also choose $n_{33}' = 0 \Leftrightarrow \mathbf{e}_1(\varphi) = -\,n_{33}$, since now the evolution equation (A.23) for n_{33} becomes involutive. So $n_{33} = 0$. We will show that the present PLRS subclass recovers the well known Szekeres dust solutions [79], which we have reviewed in Chapter 5.1. From the $\dot{\mathbf{E}}$-equation (A.29) we find $\mathbf{e}_2(H_{11}) = \frac{3}{2}\,(a_2 - n_{31})\,H_{11}$, while (A.30) gives $\mathbf{e}_2(H_{11}) = 0$, leading to the more useful algebraic result

$$(a_2 - n_{31})\,H_{11} = 0 \ . \tag{6.45}$$

Then from (A.31) we find $\mathbf{e}_3(H_{11}) + \frac{3}{2}\,(a_3 + n_{12})\,H_{11} = 0$, while (A.32) gives $\mathbf{e}_3(H_{11}) = 0$, now yielding the algebraic result

$$(a_3 + n_{12})\,H_{11} = 0 \ . \tag{6.46}$$

[4] We may with relative ease generalise this result to the situation where only $\boldsymbol{\sigma}$ and $\mathbf{H}$ are rotationally symmetric, which we do in Chapter 7.

Now the commutator (A.6) is helpful at this point. If we operate on H_{11}, we find that $n_{11}\,\mathbf{e}_1(H_{11}) = 0$, which subdivides the class into

$$\mathbf{A]}\ \ n_{11} = 0 \quad \text{or} \quad \mathbf{B]}\ \ \mathbf{e}_1(H_{11}) = 0 \ .$$

A] $n_{11} = 0$: It is easy to see from (A.21) that presently $n_{11} = 0$ is conserved along $\mathbf{u}$. Hence, the $\mathbf{H}$–constraint (A.52) gives $H_{11} = 0$, which the $\dot{\mathbf{H}}$–equation (A.33) preserves; thus $\mathbf{H} = 0$. Now we look at (A.39), which reads

$$\mathbf{e}_2(E_{11}) = \tfrac{1}{3}\,\mathbf{e}_2(\mu) \ , \tag{6.47}$$

and we use (5.4), (A.58), (5.4), and the commutator (A.4) acting on E_{11} and μ, to check the time evolution property of this last relation:

$$\mathbf{e}_0(\mathbf{e}_2(E_{11}) - \tfrac{1}{3}\,\mathbf{e}_2(\mu)) = -\,4\,\Theta_2\,[\,\mathbf{e}_2(E_{11}) - \tfrac{1}{3}\,\mathbf{e}_2(\mu)\,] \ .$$

This is solved by $\mathbf{e}_2(E_{11}) = \frac{1}{3}\,\mathbf{e}_2(\mu)$, and we may now use (A.72) with the above to get $\mathbf{e}_2(E_{11}) = \frac{3}{2}\,(a_2 - n_{31})\,E_{11} - \frac{1}{6}\,\mathbf{e}_2(\mu)$, which, in this manner, shows the consistency of (A.38). Now we look at (A.41), which reads

$$\mathbf{e}_3(E_{11}) = \tfrac{1}{3}\,\mathbf{e}_3(\mu) \ , \tag{6.48}$$

and we use (5.4), (A.60), (5.4), and the commutator (A.5) acting on E_{11} and μ, to check the time evolution property of this last relation:

$$\mathbf{e}_0(\mathbf{e}_3(E_{11}) - \tfrac{1}{3}\,\mathbf{e}_3(\mu)) = -\,4\,\Theta_2\,[\,\mathbf{e}_3(E_{11}) - \tfrac{1}{3}\,\mathbf{e}_3(\mu)\,] \ .$$

This is solved by $\mathbf{e}_3(E_{11}) = \frac{1}{3}\,\mathbf{e}_3(\mu)$, and we may now use (A.73) with the above to get $\mathbf{e}_3(E_{11}) = \frac{3}{2}\,(a_3 + n_{12})\,E_{11} - \frac{1}{6}\,\mathbf{e}_3(\mu)$, which, in this manner, shows the time consistency of (A.40). We have from (A.66) and (A.64) that along $\mathbf{u}$, and, by substitution into the relevant commutators, that everywhere $\mathbf{e}_2(a_1) = \mathbf{e}_3(a_1) = 0$. We may at this point use some of the remaining freedom in the commutators to (on a 2-surface $x^0 = c^0$, $x^1 = c^1$) set $\mathbf{e}_3(a_2 + n_{31}) = \mathbf{e}_2(a_3 - n_{12})$. This can be done because firstly

$$\mathbf{e}_0(\mathbf{e}_3(a_2 + n_{31}) - \mathbf{e}_2(a_3 - n_{12})) = -\,2\,\Theta_2\,[\,\mathbf{e}_3(a_2 + n_{31}) - \mathbf{e}_2(a_3 - n_{12})\,] \ ,$$

using the commutator (A.5) on $(a_2 + n_{31})$, the commutator (A.4) on $(a_3 - n_{12})$, and (5.4) and (5.4). Secondly,

$$\mathbf{e}_1(\mathbf{e}_3(a_2 + n_{31}) - \mathbf{e}_2(a_3 - n_{12})) = 2\,a_1\,[\,\mathbf{e}_3(a_2 + n_{31}) - \mathbf{e}_2(a_3 - n_{12})\,] \ ,$$

using the commutator (A.7) on $(a_2 + n_{31})$, the commutator (A.8) on $(a_3 - n_{12})$, and (A.65), (A.63) and (A.61). Thus

$$\mathbf{e}_3(a_2 + n_{31}) = \mathbf{e}_2(a_3 - n_{12}) \ . \tag{6.49}$$

There is still tetrad freedom remaining, but it is not obvious how one could utilise this freedom. The commutation functions which remain non-zero and the curvature variables they are coupled to in $\mathcal{U}$ are

Θ_1, Θ_2, a_α, n_{31}, n_{12}, μ, and E_{11}. This may be recognised as the Szekeres class I of solutions [79] (and its subcase, Ellis' dust spacetimes of LRS class II [17]), which is discussed in Chapter 5.1. The evolution equations and remaining constraint equations of these quantities are also given there.

B] $\mathbf{e}_1(H_{11}) = 0$: Immediately we see from the constraint on H_{11} by the second Bianchi identity (A.74) that $a_1 H_{11} = 0$ which provides us with the subdivision

$$\mathbf{B1]}\ \ a_1 = 0 \quad \text{or} \quad \mathbf{B2]}\ \ H_{11} = 0\ .$$

B1] $a_1 = 0$: We still have (6.45) and (6.46) holding — $(a_2 - n_{31}) H_{11} = (a_3 + n_{12}) H_{11} = 0$. Now if $H_{11} = 0$, then we are dealing with **B2]**. So instead here we must have $(a_2 - n_{31}) = (a_3 + n_{12}) = 0$. We may use the tetrad freedom to set $(a_3 - n_{12}) = 0$. From (A.20) and (A.60) we have $(a_3 - n_{12}) = 0$ without any further new constraints. And now, from (A.63), we have $(a_3 - n_{12}) = 0$ without any further new constraints. So this choice is allowed. Thus we may say here that

$$a_3 = n_{12} = 0 \quad \text{and} \quad (a_2 + n_{31}) = 2\,a_2\ .$$

Now from (A.38) combined with (A.39), (A.40) combined with (A.41), the commutator (A.6) operating on μ, and then using (A.36), we get that $\mathbf{e}_\alpha(\mu) = \mathbf{e}_\alpha(E_{11}) = 0$. Also, from (A.57) and (A.58) we have $\mathbf{e}_2(\Theta_1) = \mathbf{e}_2(\Theta_2) = 0 \Rightarrow \mathbf{e}_2(\Theta) = \mathbf{e}_2(\sigma_{11}) = 0$, while from (A.59) and (A.60) follows $\mathbf{e}_3(\Theta_1) = \mathbf{e}_3(\Theta_2) = 0 \Rightarrow \mathbf{e}_3(\Theta) = \mathbf{e}_3(\sigma_{11}) = 0$, which then, from the commutator (A.6) acting on Θ and σ_{11}, gives $\mathbf{e}_\alpha(\Theta) = \mathbf{e}_\alpha(\sigma_{11}) = 0$. Moreover, it is clear from (A.29) and (A.31) that $\mathbf{e}_\alpha(H_{11}) = 0$. The constraints (A.61), (A.64) and (A.66) imply $\mathbf{e}_\alpha(n_{11}) = 0$. By applying the commutators (A.3), (A.4) and (A.5) on the variables $f \in \{\mu, E_{11}, H_{11}, \Theta, \sigma_{11}, n_{11}\}$, and utilising their respective evolution equations (A.9), (A.24), (A.33), (A.10), (A.11) and (A.21), we can show that $\mathbf{e}_\alpha(f) = 0$. Now from (A.65) we have $\mathbf{e}_1(a_2) = 0$, which allows us to use the remaining freedom to set $\mathbf{e}_3(a_2) = 0$. We can do this because

$$\mathbf{e}_0(\mathbf{e}_3(a_2)) = -\,2\,\Theta_2\,\mathbf{e}_3(a_2) \quad \text{(from (A.5) on } a_2\text{, (A.19) and (A.60))}\ ,$$

$$\mathbf{e}_1(\mathbf{e}_3(a_2)) = 0 \quad \text{(from (A.7) on } a_2 \text{ and (A.65))}\ ,$$

and

$$\mathbf{e}_2(\mathbf{e}_3(a_2)) = 6\,a_2\,\mathbf{e}_3(a_2) \quad \text{(from (A.6) on } a_2\text{, (A.67) and (A.65))}\ .$$

Finally we set $\mathbf{e}_2(a_2) = 0$ at an event $x^i = c^i$. This can be done because firstly, the $\mathbf{e}_0$–derivative of $\mathbf{e}_2(a_2)$ is given by a multiple of $\mathbf{e}_2(a_2)$ (from (A.4) operating on a_2 and (A.19)); secondly, the $\mathbf{e}_1$–derivative of $\mathbf{e}_2(a_2)$ vanishes (from (A.8) operating on a_2 and (A.65)); thirdly, the $\mathbf{e}_2$–derivative of $\mathbf{e}_2(a_2)$ is given by a multiple of $\mathbf{e}_2(a_2)$ (from taking $\mathbf{e}_2$ of (A.67)); lastly, the $\mathbf{e}_3$–derivative of $\mathbf{e}_2(a_2)$ vanishes (from (A.6) operating on a_2). The remaining non-zero commutation functions and curvature variables they are coupled to in $\mathcal{U}$ are μ, Θ_1, Θ_2, a_2, n_{11}, E_{11} and H_{11}. The remaining constraints (A.52), (A.68) and (A.67)) are all algebraic relations. These constitute the subclass of spatially homogeneous cosmologies within the dust LRS class II of Ellis [17].

B2] $H_{11} = 0$**:** From the $\dot{\mathbf{H}}$–equation (A.33) it follows that $n_{11}\,\sigma_{11} = 0$. Now if $n_{11} = 0$ then we are dealing with case **A]**, and if in $\mathcal{U}$ $\sigma_{11} = 0$, these are the FLRW solutions and are well known.

6.4.2 Summary

We conclude that the only non-trivial irrotational dust cosmologies that are PLRS according to the restrictions imposed in this chapter are known. These are the Szekeres dust spacetimes [79], which are strictly PLRS, and Ellis' dust spacetimes in LRS class II [17]. Moreover, this last study confirms that local rotational symmetry results if, in an open neighbourhood $\mathcal{U}$ of an event of $\mathcal{M}$, all covariantly defined tensors determined from the Riemann tensor algebraically and by their covariant derivatives only up to *second* order are rotationally symmetric, generalizing a corresponding result in [17]. This may be traced back to the fact that, in particular, since $\nabla_1 \mathbf{e}_1$ is covariantly defined, it follows that if the covariant derivatives are also LRS, then

$$\Gamma_{211} = \mathbf{e}_2 \cdot \nabla_1 \mathbf{e}_1 = 0 = \mathbf{e}_3 \cdot \nabla_1 \mathbf{e}_1 = \Gamma_{311}$$

which from (3.21) corresponds to

$$(a_2 - n_{31}) = (a_3 + n_{12}) = 0 \ ,$$

eliminating the Szekeres models and leading to the LRS solutions found above. Note that these conditions also apply when $p \neq 0 \Leftrightarrow \dot{u}_1 \neq 0$. According to Malcolm Mac Callum, Cahen and Defrise [?] have shown this result and also claimed to have improved this in most cases to first derivatives only (but without proof). Also, the results of Goode and Wainwright for the type D case (Cf. [?]) (which is what we are considering here) have been shown by MacCallum to require only the first derivative.

6.5 Conclusion

The only consistent solutions here for the general class described in this chapter — perfect fluid cosmologies with all tensors LRS and pointing in the same direction — are known solutions. All of these solutions are either LRS or they belong to the Szekeres class of cosmological dust spacetimes. Thus this chapter may be viewed as a form of classification scheme for inhomogeneous cosmologies which incorporates the Szekeres dust solutions. Contrats this with the scheme developed by Szafron and Collins, where restrictions placed on submanifolds achieve this result [?, 11, 78]. A main result is that for all *spatially inhomogeneous* PLRS spacetimes in $\mathcal{U}$

$$\mathbf{H} = 0 \ .$$

This generalises similar results obtained for the Szekeres dust solutions in [33], and for perfect fluid spacetimes in LRS class II in [28]. It has also been demonstrated that perfect fluid cosmologies are LRS if tensors and their covariant derivatives up to second order are rotationally symmetric, generalizing a similar result which established the number of derivatives of the Riemann tensor required, as three (obtained in [17]) and confirming previous generalisations.

There are no strictly PLRS cosmologies here which are rotating. Also, the cases which have vanishing shear are fairly trivial: they are either not cosmologies at all in our understanding or they are of the simple FLRW kind. This last result requires the assumption of a barotropic equation of state $p = p(\mu)$ in the irrotational accelerating perfect fluid shear-free case (although it is probably possible to relax this requirement).

The results of this study is germane to the equivalence problem in cosmology. It shows that, in principle, all cosmological solutons of the Einstein Field Equations with perfect fluid matter source (barotropic, for now) may be invariantly defined using one of the dynamical tensors. This is so, since, in the case where the vector quantities point in the direction of symmetry defined by the rotationally symmetric tensors,

- if the tensors are all isotropic[5], the cosmological solutions are known and trivial: they are isotropic with an RW metric.
- if any one of these dynamical tensors is isotropic and the others are rotationally symmetric, then the solutions are known.
- if they are all rotationally symmetric, they are well-known and most generally are the Szekeres solutions in the dust case or are LRS otherwise.

If any two dynamical vectors or tensors, $\mathbf{x}_1$ and $\mathbf{x}_2$ say, point in different directions, then in principle, a tetrad may be invariantly defined by (in addition to choosing $\mathbf{e}_0 = u$) choosing $\mathbf{e}_1 = \mathbf{x}_1$ and $\mathbf{e}_2 = \mathbf{x}_2$.

We have seen that the only strictly PLRS solution is the Szekeres dust model. In later work we relax the requirements of this chapter systematically to see what partial symmetry results.

We note that the main results of this chapter could have been obtained in a more compact fashion by using the null tetrad formalism of Newman and Penrose[6] since the spacetime is of Petrov type D and perfect fluid. However, our intention to generalise these in later work (where, in particular, the Petrov type is changed) is easier accomplished in our formalism.

[5] We are adapting the term *isotropic* here to refer to individual tensors which are rotationally symmetric in more than one spatial plane. This should not be confused with the idea of isotropic spacetimes.

[6] Our thanks to an unknown referee for pointing this out.

Chapter 7

Perfect Fluid Cosmologies with RS Shear and H

We consider the consistent cosmologies which have σ and $\mathbf{H}$ both RS. We find that this forces $\mathbf{E}$ to be RS as well and thus this generalises the results of the previous chapter. We assume that in an open set $\mathcal{U}$

$$\sigma_{22} - \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0\,, \tag{7.1}$$

$$H_{22} - H_{33} = H_{12} = H_{31} = H_{23} = 0\,. \tag{7.2}$$

That is to say, we have set

$$\sigma_{ab} \text{ and } H_{ab} \text{ LRS;} \tag{7.3}$$

and also to be *RS in the same plane* initially. Since this holds on an open set $\mathcal{U}$, it means that, in addition to the above, the relevant derivatives must satisfy

$$\mathbf{e}_a(\sigma_{\beta\gamma}) = \mathbf{e}_a(H_{\beta\gamma}) = 0\,, (\beta \neq \gamma); \tag{7.4}$$

$$\mathbf{e}_a(\sigma_{22} - \sigma_{33}) = \mathbf{e}_a(H_{22} - H_{33}) = 0\,. \tag{7.5}$$

We now proceed by again assuming our cosmologies are perfect fluid. Thus

$$q_a = \pi_{ab} = 0\,. \tag{7.6}$$

In what follows we also assume that

$$\mu + p \neq 0 \quad \& \quad \theta \neq 0; \tag{7.7}$$

two of our requirements for a standard modern cosmology. We shall sometimes refer to the above stated conditions, without further restrictions, as the general class for this chapter.

We proceed to check the consistency of this subcase of the EFE by following the time evolution of all new constraints. In the process of doing this we will fix the tetrad freedom and thus invariantly classify the solutions. The tetrad chosen is aligned with the shear (or equivalently with the gravito-magnetic field). The tetrad freedom is that of a rotation given by (4.2).

7.1 General Perfect Fluid

This is the most general case where we will take throughout that

$$\omega \neq 0 \quad \& \quad \dot{u}_a \neq 0\,. \tag{7.8}$$

For $\omega = 0$ see section 7.2. For $\dot{u}_a = 0$ see section 7.3. Now, as before, if either or both of $\dot{u}_2$ or $\dot{u}_3$ do not vanish, then we cannot possibly get LRS symmetry. And thus we must have for the situations to be PLRS that

$$\dot{u}_2 = \dot{u}_3 = 0\,. \tag{7.9}$$

A similar argument holds for ω and for this we must have

$$\omega_2 = \omega_3 = 0\,. \tag{7.10}$$

We assume a perfect fluid, leaving the equation of state unspecified.

7.1.1 Constraint Analysis and Tetrad Choice

The immediate relations of interest pertain to (A.11) – (A.12) and (A.46) which read respectively

$$E_{22} - E_{33} + 2n_{23}\dot{u}_1 = 0 \tag{7.11}$$

$$E_{23} - \frac{1}{2}(n_{22} - n_{33})\dot{u}_1 = 0 \tag{7.12}$$

Because of (7.9), it follows that from (A.43) and (A.44) that

$$\mathbf{e}_2(p) = \mathbf{e}_3(p) = 0\,. \tag{7.13}$$

Upon application of the commutators (A.8) and (A.7) to p, the implications of the above are that

$$\dot{u}_1\mathbf{e}_2(\mu) - 2\omega\Omega_2(\mu + p) = 0 \tag{7.14}$$

$$\dot{u}_1\mathbf{e}_3(\mu) - 2\omega\Omega_3(\mu + p) = 0\,. \tag{7.15}$$

The most useful relations to continue from here are obtained from (A.54) – (A.53) and (A.55) + (A.56); namely

$$2n_{23}\omega_1 + \frac{3}{2}(n_{22} - n_{33})\sigma_{11} = 0 \tag{7.16}$$

$$(n_{22} - n_{33})\omega_1 - 3n_{23}\sigma_{11} = 0\,. \tag{7.17}$$

It follows that

$$n_{23} = n_{22} - n_{33} = 0 \tag{7.18}$$

and therefore from (7.11) and (7.12) we get the fairly significant information which reads

$$E_{23} = E_{22} - E_{33} = 0\,. \tag{7.19}$$

Since $\mathbf{e}_0(E_{23}) = \mathbf{e}_0(E_{22} - E_{33}) = 0$, we get from (A.25) − (A.26) and (A.27) + (A.28) that

$$\Omega_3 E_{12} + \Omega_2 E_{31} = 0 \tag{7.20}$$

$$\Omega_3 E_{31} - \Omega_2 E_{12} = 0 \tag{7.21}$$

which implies that

$$\Omega_2 = \Omega_3 = 0 \tag{7.22}$$

since if not, then we are dealing with a case already dealt with in Chapter 6. So now (7.14) and (7.15) show that

$$\dot{u}_1 \mathbf{e}_2(\mu) = \dot{u}_1 \mathbf{e}_3(\mu) = 0 \Rightarrow \mathbf{e}_2(\mu) = \mathbf{e}_3(\mu) = 0\,. \tag{7.23}$$

Taking the $\mathbf{e}_0$ frame derivative of the above, we require (A.4) and (A.5) operating on μ and use (A.9) to find

$$(\mu + p)\mathbf{e}_2(\theta) = (\mu + p)\mathbf{e}_3(\theta) = 0 \Rightarrow \mathbf{e}_2(\theta) = \mathbf{e}_3(\theta) = 0\,. \tag{7.24}$$

The tetrad freedom is of course that of rotation in the $\mathbf{e}_2 / \mathbf{e}_3$ plane given by (4.2) which for this case results in the transformation of the commutator functions as given in appendix A.3.2. We fully specify the tetrad by performing a rotation which sets

$$a_3 + n_{12} = 0\,. \tag{7.25}$$

It follows from (A.50) that

$$\mathbf{e}_3(\dot{u}_1) = 0 \tag{7.26}$$

which then implies from (A.48) that

$$E_{31} = 0\,. \tag{7.27}$$

And now, from (A.18) we get

$$\mathbf{e}_3(\sigma_{11}) = 0 \tag{7.28}$$

which means from (A.59) that

$$a_2 - n_{31} = 0 \tag{7.29}$$

(since $\omega \neq 0$). This in turn sets

$$\mathbf{e}_2(\dot{u}_1) = 0 \tag{7.30}$$

from (A.49); which then from (A.47) requires

$$E_{12} = 0\,. \tag{7.31}$$

Therefore all directed quantities are RS in the same plane and this has been dealt with in Chapter 6.

7.1.2 Summary

We find that for a rotating fluid, if we let the shear, gravito-magnetic, vorticity and acceleration fields pick out the same direction, then this sets the electric field to be RS as well. Typically what happens is that because the acceleration is orthogonal to the plane of rotational symmetry, from the twice contracted Bianchi identities – (A.42) and (A.43) – the pressure gets directed in the invariant direction. This then feeds into the gradients of the density, making it point in the same direction (if the tetrad is not rotating outside the plane of rotation) which in turn tells the gradients of the expansion to fall in line, as it were. These LRS-like gradients are enough to force the electro-magnetic field to be RS.

7.2 Irrotational Accelerating Perfect Fluid

These models have $\omega_a = 0$. We assume here that $\dot{u}_a \neq 0$. But once again we must have

$$\dot{u}_2 = \dot{u}_3 = 0 \tag{7.32}$$

or else we cannot have PLRS solutions. This means, of course, from the contracted Bianchi Identities (A.43) and (A.44) that

$$\mathbf{e}_2(p) = \mathbf{e}_3(p) = 0 \tag{7.33}$$

For $\dot{u}_a = \omega_a = 0$, see section 7.4.

7.2.1 Tetrad Choice and Constraint Analysis

From (A.14) we get that

$$n_{11}\dot{u}_1 = 0 \Rightarrow n_{11} = 0 \tag{7.34}$$

From (A.53) – (A.54) and (A.55) – (A.56) we get, respectively, that

$$(n_{22} - n_{33})\sigma_{11} = n_{23}\sigma_{11} = 0 \tag{7.35}$$

so that we have a division here according to

$$\text{A]}\ n_{23} = n_{22} - n_{33} = 0 \quad \text{or} \quad \text{B]}\ \sigma_{11} = 0$$

A] $n_{23} = n_{22} - n_{33} = 0$

Immediately we get from (A.11) – (A.12) that

$$E_{22} - E_{33} = 0 \tag{7.36}$$

and from (A.46) that

$$E_{23} = 0 \,. \tag{7.37}$$

The implications of the last two, in turn, is that

$$\Omega_2 E_{31} + \Omega_3 E_{12} = 0 \quad \text{(from (A.25) − (A.26);)} \tag{7.38}$$

$$\Omega_3 E_{31} + \Omega_2 E_{12} = 0 \quad \text{(from (A.27) − (A.28))} \tag{7.39}$$

which implies that

$$\Omega_2 = \Omega_3 = 0 \tag{7.40}$$

because if not, then we are dealing with situation previously covered in Chapter 6. Crucial here is (A.52) which establishes that

$$H_{11} = 0 \,. \tag{7.41}$$

We then get from (A.75) and (A.76) that

$$E_{31}\sigma_{11} = E_{12}\sigma_{11} = 0 \tag{7.42}$$

which means that

$$\sigma_{11} = 0 \tag{7.43}$$

or else all tensors are RS. We note from (A.55), (A.57) and (A.59) that

$$\mathbf{e}_\alpha(\theta) = 0 \,. \tag{7.44}$$

We now take the $\mathbf{e}_2$ frame derivative of (A.42), using the commutator (A.8) applied to the pressure p, (A.47) and (A.49) to get

$$\dot{u}_1 \mathbf{e}_2(\mu) = 0 \Rightarrow \mathbf{e}_2(\mu) = 0 \tag{7.45}$$

Also taking the $\mathbf{e}_3$ frame derivative of (A.42), using the commutator (A.7) applied to the pressure p, (A.48) and (A.50) we get

$$\dot{u}_1 \mathbf{e}_3(\mu) = 0 \Rightarrow \mathbf{e}_3(\mu) = 0 \,. \tag{7.46}$$

We find it most expedient to check (A.39) and (A.41) which read, respectively,

$$\mathbf{e}_2(E_{11}) - 2(\dot{u}_1 + a_1)E_{12} = 0 \tag{7.47}$$

$$\mathbf{e}_3(E_{11}) - 2(\dot{u}_1 + a_1)E_{31} = 0 \,. \tag{7.48}$$

We check the consistency of the above two relations using evolution equations for the gradients of E_{11} obtained from applying the commutators (A.4) and (A.5) to E_{11}; other evolution equations needed – (A.29), (A.31) and (A.15). We find that

$$E_{12}\left[\mathbf{e}_0(\dot{u}_1)\right] = 0 \tag{7.49}$$

$$E_{31}\left[\mathbf{e}_0(\dot{u}_1)\right] = 0\,. \tag{7.50}$$

And now, since $E_{12} = E_{31} = 0$ leads to well-known solutions covered elsewhere, we require

$$\mathbf{e}_0(\dot{u}_1) = 0\,. \tag{7.51}$$

We now check consistency of (A.47) and (A.48) using the above relation as an evolution equation for the acceleration. We know from (A.47) and (A.48) that respectively

$$\mathbf{e}_2(\dot{u}_1) - E_{12} = 0 \tag{7.52}$$

$$\mathbf{e}_2(\dot{u}_1) - E_{12} = 0\,. \tag{7.53}$$

We apply the commutators (A.4) and (A.5) to $\dot{u}_1$ and require the evolution equations (A.31) and (A.29) respectively; and we find that for consistency

$$\frac{1}{3}\mathbf{e}_2(\dot{u}_1) - E_{12} = 0 \tag{7.54}$$

$$\frac{1}{3}\mathbf{e}_2(\dot{u}_1) - E_{12} = 0\,. \tag{7.55}$$

So, it turns out that

$$E_{12} = E_{31} = 0 \tag{7.56}$$

and this has been dealt with elsewhere.

B] $\sigma_{11} = 0$

From (A.52) we have

$$H_{11} = 0\,. \tag{7.57}$$

We note from (A.55), (A.57) and (A.59) that

$$\mathbf{e}_\alpha(\theta) = 0\,. \tag{7.58}$$

We now take the $\mathbf{e}_2$ frame derivative of (A.42), using the commutator (A.8) applied to the pressure p, (A.47) and (A.49) to get

$$\dot{u}_1\mathbf{e}_2(\mu) = 0 \Rightarrow \mathbf{e}_2(\mu) = 0 \tag{7.59}$$

Also taking the $\mathbf{e}_3$ frame derivative of (A.42), using the commutator (A.7) applied to the pressure p, (A.48) and (A.50) we get

$$\dot{u}_1\mathbf{e}_3(\mu) = 0 \Rightarrow \mathbf{e}_3(\mu) = 0\,. \tag{7.60}$$

It is convenient at this stage to use the tetrad freedom as follows. Set

$$a_2 - n_{31} = 0\,. \tag{7.61}$$

The implications are that

$$\mathbf{e}_2(\dot{u}_1) = 0 \quad \text{(from (A.49).} \tag{7.62}$$

It follows that

$$E_{12} = 0 \quad \text{(from (A.47)} \tag{7.63}$$

and for consistency of this we must have from the Bianchi identity (A.31) –

$$\Omega_3(E_{11} - E_{22}) + \Omega_2 E_{23} - \Omega_1 E_{31} = 0\,. \tag{7.64}$$

From (A.11) – (A.12) –

$$E_{22} - E_{33} + 2n_{23}\dot{u}_1 = 0\,. \tag{7.65}$$

From (A.46) –

$$E_{23} - \frac{1}{2}(n_{22} - n_{33})\dot{u}_1 = 0\,; \tag{7.66}$$

and also then from (A.48) –

$$E_{31} - \mathbf{e}_3(\dot{u}_1) = 0\,. \tag{7.67}$$

We check the consistency of the vanishing density gradients.

$$\mathbf{e}_2(\mu) = \mathbf{e}_3(\mu) = 0 \tag{7.68}$$

if

$$\Omega_3\mathbf{e}_1(\mu) = \Omega_2\mathbf{e}_1(\mu) = 0 \tag{7.69}$$

– where we have used the commutators (A.4) and (A.5) on μ and the density evolution equation (A.9). The implications are that

$$\Omega_2 = \Omega_3 = 0 \tag{7.70}$$

because if alternatively $\mathbf{e}_1(\mu) = 0$, then (assuming $p = p(\mu)$) from (A.42) it follows that $(\mu + p)\dot{u}_1 = 0 \Rightarrow \mu + p = 0$; i.e. this is non-cosmological. We have from (A.31)

$$\Omega_1 E_{31} = 0\,. \tag{7.71}$$

We look at

B1] $\Omega_1 = 0$ or B2] $E_{31} = 0$

B1] $\Omega_1 = 0$

We check the $\mathbf{e}_0$ consistency of (A.41). For this we need to apply the commutator (A.4) to E_{23} and the (A.5) commutator to E_{22}; we also need (A.28), (A.15), (A.19), (A.20), (A.25), (A.26) and (A.29). We get after some algebra,

$$E_{31}\left[\mathbf{e}_0(\dot{u}_1)\right] = 0\,. \tag{7.72}$$

A brief argument will now show that this means that $E_{31} = 0$. If instead the acceleration is static, then from checking the e_0 consistency of (A.48) we get after applying the commutator (A.5) to $\dot{u}_1$ and utilising (A.29) that $\theta E_{31} = 0 \Rightarrow E_{31} = 0$ (because $\theta \neq 0$ is always assumed). Hence we have reached a contradiction.

And thus

$$E_{31} = 0 \tag{7.73}$$

which implies that

$$e_3(\dot{u}_1) = 0 \quad \text{(from (A.48)}. \tag{7.74}$$

It follows from (A.50) that

$$a_3 + n_{12} = 0\,. \tag{7.75}$$

We evolve (7.65) and (7.66). Using (A.15) – (A.16) and (A.28) it follows that

$$n_{23}\left[e_0(\dot{u}_1) + \frac{2}{3}\theta\dot{u}_1\right] = 0\,, \tag{7.76}$$

and using (A.22) + (A.23) and (A.25) – (A.26), it follows that

$$(n_{22} - n_{33})\left[e_0(\dot{u}_1) + \frac{2}{3}\theta\dot{u}_1\right] = 0\,. \tag{7.77}$$

Now if $n_{23} = n_{22} - n_{33} = 0$, then this implies that $E_{23} = E_{22} - E_{33} = 0$ which has already been dealt with elsewhere. So we must have for generality

$$e_0(\dot{u}_1) = -\frac{2}{3}\theta\dot{u}_1\,. \tag{7.78}$$

If we now check the consistency of $\sigma = 0$; that is – take the e_0 derivative of (A.10) – (A.12) using the evolution equation for the acceleration obtained above along with (A.3) on $\dot{u}_1$, (A.15), (A.24) and (A.25) we find that

$$\theta\dot{u}_1{}^2 = 0 \Rightarrow \theta = 0 \tag{7.79}$$

so this is non-cosmological.

B2] $E_{31} = 0$

The immediate implications are that

$$e_3(\dot{u}_1) = 0 \tag{7.80}$$

which arises from (A.48). This in turn implies

$$a_3 + n_{12} = 0\,. \tag{7.81}$$

Now the new constraint (A.33) reads

$$\frac{1}{2}(n_{22} - n_{33})(E_{22} - E_{33}) + 2n_{23}E_{23} = 0\,. \tag{7.82}$$

We evolve this using evolution equations obtained from (A.15) – (A.16), (A.22) + (A.23), (A.25) – (A.26) and (A.27) to get

$$\Omega_1\dot{u}_1(E_{22} - E_{33}) + 2n_{23}\Omega_1(E_{22} - E_{33}) = 0 \tag{7.83}$$

which implies that

$$\dot{u}_1(E_{22} - E_{33}) + 2n_{23}(E_{22} - E_{33}) = 0 \tag{7.84}$$

since if $\Omega_1 = 0$, then we are dealing with case **B1**]. It follows that

$$\dot{u}_1 + 2n_{23} = 0 \tag{7.85}$$

because if on the other hand $E_{22} - E_{33} = 0$, then from (A.25) – (A.26) we get $\Omega_1 E_{23} = 0$ which implies[1] $E_{23} = 0$ and this has been dealt with in Chapter 6. We note that one of the implications of the above are that

$$\mathbf{e}_2(n_{23}) = \mathbf{e}_3(n_{23}) = 0\,. \tag{7.86}$$

Also we find that the evolution of the acceleration is now determined by this relation:

$$\mathbf{e}_0(\dot{u}_1) = -\frac{1}{3}\theta\dot{u}_1 - 2\Omega_1(n_{22} - n_{33}) \tag{7.87}$$

using (A.15) – (A.16). We now evolve (7.65) and (7.66). Using (A.22) + (A.23) and (A.27) it follows that

$$\frac{1}{3}\theta n_{23} + (n_{22} - n_{33})\Omega_1 = 0 \tag{7.88}$$

which implies that now

$$\mathbf{e}_0(\dot{u}_1) = -\frac{2}{3}\theta\dot{u}_1\,. \tag{7.89}$$

If we now check the consistency of $\sigma = 0$; that is – take the $\mathbf{e}_0$ derivative of (A.10) – (A.12) using the evolution equation for the acceleration obtained directly above along with (A.3) on $\dot{u}_1$, (A.15), (A.24) and (A.25) we find that

$$\theta\dot{u}_1{}^2 = 0 \Rightarrow \theta = 0\,; \tag{7.90}$$

so this is non-cosmological.

7.2.2 Summary

There are no new cosmologies here. To see this, we assumed a barotropic equation of state at some stage.

[1]– since $\Omega_1 \neq 0$ once again –

7.3 Rotating Dust

These models have $\dot{u}_a = 0$. From the contracted Bianchi identities (conservation of momentum equations), $\dot{u}_a = 0 \Rightarrow \mathbf{e}_\alpha p = 0$. Thus the pressure has to be constant. This is equivalent to having a non-vanishing cosmological constant. We will assume this is zero here and subsequently in similar cases. We assume here that $\omega_a \neq 0$. For $\dot{u}_a = \omega_a = 0$ see section 7.4. Since we are dealing with dust we may choose again to use the tetrad[2] used in [17].

7.3.1 Tetrad Choice and Constraint Analysis

So we go about it by setting, in addition to $\mathbf{u} = \mathbf{e}_0$,

$$\boldsymbol{\omega} = \omega \mathbf{e}_1 \,, \quad \omega_2 = \omega_3 = 0 \,.$$

Since we are dealing with degenerate shear in the $\mathbf{e}_2$ / $\mathbf{e}_3$-plane, this amounts to aligning the shear with the vorticity. Any other choice would not be PLRS according to Definition 4.2.1. With these specifications, we will see that $\mathbf{E}$ is forced to be RS for mathematical consistency. First of all, we get from (A.11) − (A.12) that

$$E_{22} - E_{33} = 0 \,. \tag{7.91}$$

Secondly, from (A.46) it follows that

$$E_{23} = 0 \,. \tag{7.92}$$

We will see that these new restraints on the dynamics fix the tetrad to be rotating in the $\mathbf{e}_2$ / $\mathbf{e}_3$-plane only (initially). We find from the constraint equations (A.53) − (A.54) that

$$\frac{3}{2}(n_{22} - n_{33})\sigma_{11} + 2n_{23}\omega_1 = ;0 \Rightarrow n_{22} - n_{33} = 0 \tag{7.93}$$

since $\omega \neq 0$. And from the constraint equations (A.55) + (A.56) it follows that

$$-3n_{23}\sigma_{11} + (n_{22} - n_{33})\omega_1 = 0 \Rightarrow n_{23} = 0 \,; \tag{7.94}$$

since $\omega \neq 0$. Now we can show that $\Omega_2 = \Omega_3 = 0$ as claimed above. From (A.25) − (A.26) and (A.27) + (A.28) we must have that respectively

$$\Omega_3 E_{12} + \Omega_2 E_{31} = 0 \tag{7.95}$$

$$\Omega_3 E_{31} - \Omega_2 E_{12} = 0 \,, \tag{7.96}$$

the implications of which are that (since $E_{12} = E_{31} = 0$ has been dealt with elsewhere)

$$\Omega_2 = \Omega_3 = 0 \,. \tag{7.97}$$

[2] of his Theorem 3.1.

But now from (A.48) and (A.47) it follows that

$$E_{12} = E_{31} = 0 \tag{7.98}$$

and thus this case has been considered elsewhere because all tensor quantities are now RS.

7.3.2 Summary

Consistency requires that $\mathbf{E}$ be RS as well. These cosmologies have been described in Chapter 6.

7.4 Irrotational Dust

We shall now consider irrotational dust with $\boldsymbol{\sigma}$ and $\mathbf{H}$ RS.

$$\boldsymbol{\omega} = 0 = \dot{u}_a \Rightarrow p = 0\,. \tag{7.99}$$

With these specifications, $\mathbf{E}$ is forced to be RS. First of all, we get from (A.11) $-$ (A.12) that

$$E_{22} - E_{33} = 0\,. \tag{7.100}$$

Secondly, from (A.46) it follows that

$$E_{23} = 0\,. \tag{7.101}$$

We will see that these new restraints on the dynamics fix the tetrad to be rotating in the $\mathbf{e}_2\,/\,\mathbf{e}_3$-plane only (initially). We find from the constraint equations (A.53) $-$ (A.54) that

$$(n_{22} - n_{33})\sigma_{11} = 0 \Rightarrow n_{22} - n_{33} = 0 \tag{7.102}$$

and from the constraint equations (A.55) + (A.56) it follows that

$$n_{23}\sigma_{11} = 0 \Rightarrow n_{23} = 0\,. \tag{7.103}$$

We have used here the fact that if $\sigma = 0$, then (from (A.48) and (A.47)) $E_{12} = E_{31} = 0$ which means that $\mathbf{E}$ is RS and this has been dealt with in Chapter 6.

Now we can show that $\Omega_2 = \Omega_3 = 0$ as claimed above. From (A.25) $-$ (A.26) and (A.27) + (A.28) we must have that respectively

$$\Omega_3 E_{12} + \Omega_2 E_{31} = 0 \tag{7.104}$$

$$\Omega_3 E_{31} - \Omega_2 E_{12} = 0\,, \tag{7.105}$$

the implications of which are that (since $E_{12} = E_{31} = 0$ has been dealt with elsewhere)

$$\Omega_2 = \Omega_3 = 0\,. \tag{7.106}$$

But now from (A.48) and (A.47) it follows that

$$E_{12} = E_{31} = 0 \tag{7.107}$$

and thus this case has been considered elsewhere because all tensor quantities are now RS.

7.4.1 Summary

Consistency requires that $\mathbf{E}$ be RS as well. These cosmologies have been dealt with in some detail in Chapter 6.

7.5 Summary

We find that there are no fully PLRS cosmologies which arise from relaxing the RS requirement on the gravito-electric field. The evolution of RS shear and the H-constraint essentially results in algebraic relations which feed into the gravito-electric field and, in fact, forces it to be RS.

Thus the results of the previous chapter, where all tensors pointed in the same direction, are true here as well, when we relax that requirement on the gravito-electric field.

Chapter 8

Perfect Fluid Cosmologies with RS Shear and E

We have seen in Chapter 7 that when we relax the requirements of Chapter 6 such that $\mathbf{E}$ is not RS, the results of Chapter 6 remain unchanged. We now consider the situation where $\mathbf{H}$ is not RS. We expect to recover inhomogeneous cosmologies not encountered before. An example of a perfect fluid cosmology with coinciding eigenvalues the shear and nonzero $\mathbf{E}$ and $\mathbf{H}$ is provided by the diagonal Abelian $\mathcal{G}_2$ solutions with radiation equation of state by Senovilla [72].

We attempt to find all the consistent cosmologies which have $\boldsymbol{\sigma}$ and $\mathbf{E}$ both LRS. We assume that in an open set $\mathcal{U}$

$$\sigma_{22} - \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0\,, \tag{8.1}$$

$$E_{22} - E_{33} = E_{12} = E_{31} = E_{23} = 0\,. \tag{8.2}$$

That is to say, we have set

$$\sigma_{ab} \text{ and } E_{ab} \text{ LRS;} \tag{8.3}$$

and also to be *LRS in the same plane* initially. In addition, the relevant derivatives of these quantities must also vanish. We assume our cosmologies are filled with a perfect fluid, as before. Thus

$$q_a = \pi_{ab} = 0\,. \tag{8.4}$$

In what follows we again demand, since we are only interested in cosmological solutions, that

$$\mu + p \neq 0 \;\;\&\;\; \theta \neq 0\,. \tag{8.5}$$

We proceed to check the consistency of this subcase of the EFE by following the time evolution of all new constraints. The tetrad chosen is aligned with the shear (or equivalently with the gravito-electric field and the tetrad freedom is that of a rotation given by (4.2) once again.

8.1 Rotating, Accelerating Perfect Fluid

We will take throughout that

$$\omega \neq 0 \ \ \& \ \ \dot{u}_a \neq 0 . \tag{8.6}$$

For $\omega = 0$ see section 8.2. For $\dot{u}_a = 0$ see section 8.3. We assume also that

$$\dot{u}_2 = \dot{u}_3 = 0 , \tag{8.7}$$

and

$$\omega_2 = \omega_3 = 0 . \tag{8.8}$$

We thus have a rotating accelerating fluid without any assumptions on the equation of state.

8.1.1 Constraint Analysis

The acceleration constrains the dynamics to a large extent. From (A.45) and (A.46) it follows respectively that

$$n_{23} = n_{22} - n_{33} = 0 \tag{8.9}$$

which implies from (A.53) − (A.54) and (A.55) + (A.56) shows us that $\mathbf{H}$ has to have the following form:

$$H_{22} - H_{33} = H_{23} = 0 . \tag{8.10}$$

Thus, from here on, if $H_{12} = H_{31} = 0$, then $\mathbf{H}$ is rotationally symmetric and we would be dealing with cases covered in Chapter 6. But now, for time consistency of the above constraints on $\mathbf{H}$, we find, from their evolution equations (A.34) − (A.35) and (A.36) + (A.37), that

$$\Omega_3 H_{12} + \Omega_2 H_{31} = 0 \tag{8.11}$$

$$\Omega_3 H_{31} - \Omega_2 H_{12} = 0 , \tag{8.12}$$

so either $H_{12} = H_{31} = 0$ (which has been dealt with before), or

$$\Omega_A = 0 ; \tag{8.13}$$

which results in killing two of the gradients of $\dot{u}_1$:

$$\mathbf{e}_A(\dot{u}_1) = 0 \tag{8.14}$$

from (A.47) and (A.48)[1]. This in turn forces

$$a_2 - n_{31} = a_3 + n_{12} = 0 \tag{8.15}$$

[1]We find it convenient from here on to use capital Latin letters to refer to indices in the plane of rotation; $A, B, C = 2, 3$.

from (A.49) and (A.50). For these to be consistent with their time evolution and other equations, from (A.17) and (A.18), means that we must have

$$\mathbf{e}_A(\Theta_1) = 0\,. \tag{8.16}$$

And then, finally, from (A.57) and (A.59)

$$H_{12} = H_{31} = 0 \tag{8.17}$$

is forced. So now $\mathbf{H}$ is rotationally symmetric and this reduces to the models dealt with in Chapter 6.

8.1.2 Summary

We see that the acceleration constrains the dynamics to the extent that setting only the shear and gravito-electric field to be rotationally symmetric is only as large as the set covered by setting all the tensors to be rotationally symmetric and the acceleration and vorticity pointing in the same axis of rotation defined by the shear or gravito-electric field.

8.2 Irrotational Perfect Fluid

These models have $\omega_a = 0$. We assume here that $p \neq 0 \Rightarrow \dot{u}_a \neq 0$. But once again we assume that

$$\dot{u}_2 = \dot{u}_3 = 0\,. \tag{8.18}$$

For $\dot{u}_a = \omega_a = 0$ see section 8.4.

8.2.1 Tetrad Choice and Constraint Analysis

The constraint arising from setting the vorticity to vanish, (A.14), requires

$$n_{11} = 0\,, \tag{8.19}$$

(A.11) – (A.12) demands

$$n_{23} = 0 \tag{8.20}$$

and from (A.46) we need

$$n_{22} - n_{33} = 0\,. \tag{8.21}$$

So now the $\mathbf{H}$-constraint field equations (A.52), (A.53) and (A.54) set

$$H_{11} = H_{22} = H_{33} = 0 \tag{8.22}$$

and (A.55) + (A.56) set

$$H_{23} = 0\,. \tag{8.23}$$

We fix the frame freedom by rotating the tetrad such that

$$H_{12} = 0\,. \tag{8.24}$$

We now find from (A.75) that

$$n_{33}H_{31} = 0 \Rightarrow n_{33} = 0 \tag{8.25}$$

or else $\mathbf{H} = 0$ and we are in a case already dealt with in Chapter 6. Also, from (A.28) it follows that

$$(a_3 - n_{12})H_{31} = 0 \Rightarrow a_3 - n_{12} = 0\,. \tag{8.26}$$

Since $H_{11} = 0$ it is necessary for consistency with its evolution equation, that

$$\Omega_2 H_{31} = 0 \Rightarrow \Omega_2 = 0 \tag{8.27}$$

from (A.33). We can now use (A.48) to show that

$$\mathbf{e}_3(\dot{u}_1) = 0 \tag{8.28}$$

and this, in turn, implies from (A.50) that

$$a_3 + n_{12} = 0\,. \tag{8.29}$$

Furthermore, (A.36) + (A.37) gives

$$\Omega_3 H_{31} = 0 \Rightarrow \Omega_3 = 0 \tag{8.30}$$

so that (A.47) gives

$$\mathbf{e}_2(\dot{u}_1) = 0 \tag{8.31}$$

which, in turn, implies from (A.49) that

$$(a_2 - n_{31})\dot{u}_1 = 0 \Rightarrow a_2 - n_{31} = 0\,. \tag{8.32}$$

Finally, the Jacobi identity (A.17) requires

$$\mathbf{e}_2(\Theta_1) = 0 \tag{8.33}$$

and then (A.57) forces

$$H_{31} = 0\,; \tag{8.34}$$

that is, the gravito-magnetic field vanishes.

8.2.2 Summary

The acceleration constrains the allowed set of cosmologies drastically. The only ones here are known ones dealt with in Chapter 6.

8.3 Rotating Dust

These models have $p = 0$. It follows that $\dot{u}_a = 0$. We assume here that $\omega_a \neq 0$. For $\dot{u}_a = \omega_a = 0$ see section 8.4.

8.3.1 Tetrad Choice and Constraint Analysis

Since we are dealing with dust we prefer to use the tetrad utilised in [17]. So we go about it by setting, in addition to $\mathbf{u} = \mathbf{e}_0$,

$$\boldsymbol{\omega} = \omega \mathbf{e}_1 \,, \quad \omega_2 = \omega_3 = 0 \,.$$

Since we are dealing with degenerate shear in the $\mathbf{e}_2 \,/\, \mathbf{e}_3$ plane this amounts to aligning the shear with the vorticity. We note the following:

$$\Omega_A = 0 \tag{8.35}$$

from (A.49) and (A.50). Thus the constraints from setting the shear to be rotationally symmetric are all satisfied: (A.45), (A.46), (A.47) and (A.48). So all we need to check for time consistency of this set is the evolution of the vanishing components of the gravito-electric field. This problem, however, soon becomes intractable even with the aid of computer algebra. We will consider the weakened situation[2] where

$$\mathbf{e}_A(\omega_1) = 0 \tag{8.36}$$

is assumed. It turns out – as we will see – that this is quite a strong restriction. From $[\mathbf{e}_0, \mathbf{e}_A](\omega_1)$ and the vorticity evolution (A.14), this requires that

$$\mathbf{e}_A(\Theta_2) = 0 \,. \tag{8.37}$$

We now get that the following components of the gravito-magnetic field must vanish:

$$H_{12} = H_{31} = 0 \tag{8.38}$$

which follows from (A.58) and (A.60). We now proceed by fixing $\mathbf{e}_0(\varphi)$ everywhere such that

$$\Omega_1 + \omega_1 = 0 \tag{8.39}$$

and then we fix $\mathbf{e}_1(\varphi)$ such that

$$n_{33} = 0 \tag{8.40}$$

on a 3-surface $x^0 = c^0$. From its evolution equation (A.23) this holds everywhere without introducing new constraints. A fundamental relation is obtained by applying the commutator (A.6) to ω_1.

$$[\mathbf{e}_2 \,, \mathbf{e}_3](\omega_1) = 0 = 2\omega_1(-2\Theta_2\omega_1 + n_{11}a_1) \Rightarrow 2\Theta_2\omega_1 = n_{11}a_1 \tag{8.41}$$

[2] We looked at relaxing the situation to where the density gradients in the plane of rotation vanished, but this also turned out to be problematic.

where we have used the vorticity evolution and conservation equations (A.14) and (A.51). We note that one implication of this is that

$$\mathbf{e}_A(a_1 n_{11}) = 0 \tag{8.42}$$

We apply the other spatial commutators (A.8) and (A.7) to ω_1, using the vorticity conservation equation (A.51) again, giving respectively:

$$\begin{aligned} \mathbf{e}_2(a_1) &= (a_2 - n_{31})a_1 \\ \mathbf{e}_3(a_1) &= (a_3 + n_{12})a_1 \end{aligned} \tag{8.43}$$

We will also apply these commutators to Θ_2. We apply the spatial commutators (A.8) and (A.7) to Θ_2, using the field equations (A.55) – (A.56) and also (8.42) and (8.43), giving respectively:

$$\tfrac{3}{2}a_1\mathbf{e}_2(\sigma_{11}) = (a_2 - n_{31})n_{11}\omega_1 \tag{8.44}$$

$$\tfrac{3}{2}a_1\mathbf{e}_3(\sigma_{11}) = (a_3 + n_{12})n_{11}\omega_1 \,. \tag{8.45}$$

We apply the commutator (A.6) to Θ_2:

$$[\mathbf{e}_2\,,\mathbf{e}_3](\Theta_2) = 0 = 2\omega_1(-\Theta_2^2 + \omega_1^2 + \tfrac{1}{2}E_{11} - \tfrac{1}{6}\mu) - n_{11}(\tfrac{3}{2}a_1\sigma_{11} + \tfrac{1}{2}n_{11}\omega_1) \tag{8.46}$$

from the evolution equation for Θ_2 given by (A.11) and the equation giving the $\mathbf{e}_1$ gradient of Θ_2, that is (A.55) – (A.56). Applying $[\mathbf{e}_0\,,\mathbf{e}_A]$ to Θ_2, using the evolution equation (A.11), we find that

$$\mathbf{e}_A(\tfrac{1}{2}E_{11} - \tfrac{1}{6}\mu) = 0\,. \tag{8.47}$$

It follows from (8.42), (8.43), (8.44) and (8.45), that upon taking $\mathbf{e}_A$ derivatives of (8.46), we get that

$$\omega_1 n_{11}[\mathbf{e}_3(n_{11}) + (a_3 + n_{12})n_{11}] = 0 \tag{8.48}$$

$$\omega_1 n_{11}[\mathbf{e}_2(n_{11}) + (a_2 - n_{31})n_{11}] = 0\,. \tag{8.49}$$

So from (8.42), (8.43), (8.48) and (8.49) we get a division here according to

A] $a_1 = n_{11} = 0$ or **B]** $\mathbf{e}_A(n_{11}) + (a_A + \varepsilon^{1B}{}_A n_{B1})n_{11} = 0 = \mathbf{e}_A(a_1) - (a_A + \varepsilon^{1B}{}_A n_{B1})n_{11}$.

A] $a_1 = n_{11} = 0$: Since $a_1 n_{11} = 0$, it follows from (8.41) that

$$\Theta_2 = 0\,. \tag{8.50}$$

Also, from (A.52),

$$H_{11} = 0\,. \tag{8.51}$$

But now, from (A.29) and (A.31) we get that

$$(a_3 + n_{12})H_{23} - (a_2 - n_{31})H_{33} = 0 \tag{8.52}$$

$$(a_2 - n_{31})H_{23} + (a_3 + n_{12})H_{33} = 0 \tag{8.53}$$

where we have also used the fact that $\mathbf{H}$ is tracefree. So either $\mathbf{H} = 0$ (which was covered in Chapter 6) or

$$a_2 - n_{31} = 0 = a_3 + n_{12} \tag{8.54}$$

which now means from the Jacobi Identity (A.61), that

$$\omega_1\Theta_1 = 0 \Rightarrow \Theta_1 = 0\,; \tag{8.55}$$

and since $\Theta_2 = 0$, this implies that $\omega\Theta = 0$; that is

$$\Theta = 0 \tag{8.56}$$

and so this is non-cosmological.

B] $\mathbf{e}_A(n_{11}) + (a_A + \varepsilon^{1B}{}_A n_{B1})n_{11} = 0 = \mathbf{e}_A(a_1) - (a_A + \varepsilon^{1B}{}_A n_{B1})n_{11}$**:** Now we use the commutators $[\mathbf{e}_0\,,\mathbf{e}_A]$ on a_1 to check the consistency with evolution of (8.48) and (8.49), using (8.44) and (8.45), (A.15) + (A.16), (A.17), (A.18) and (A.57), (A.59). We get

$$\omega_1 n_{11}[\tfrac{1}{2}(a_2 - n_{31}) + (a_3 + n_{12})] = 0 \tag{8.57}$$

$$\omega_1 n_{11}[\tfrac{1}{2}(a_3 + n_{12}) - (a_2 - n_{31})] = 0 \tag{8.58}$$

so that either

$$\mathbf{B1]}\;\; n_{11} = 0 \quad \text{or} \quad \mathbf{B2]}\;\; a_A + \varepsilon^{1B}{}_A n_{B1} = 0\,.$$

B1] $n_{11} = 0$**:** Immediately we find from (8.41) that here

$$\Theta_2 = 0\,. \tag{8.59}$$

Also from (8.44) and (8.45), since $a_1 = 0$ is case **A]**, it follows that

$$\mathbf{e}_A(\sigma_{11}) = 0; \tag{8.60}$$

therefore, since $\Theta_2 = 0$, we get that

$$\mathbf{e}_A(\Theta) = 0\,. \tag{8.61}$$

But now from the Raychaudhuri equation (A.13), it must be that

$$\mathbf{e}_A(\mu) = 0 \tag{8.62}$$

and then from (A.39) and (A.41),

$$\mathbf{e}_A(E_{11}) = 0 \tag{8.63}$$

and so from (A.38) and (A.40) we find that

$$(a_A + \varepsilon^{1B}{}_A n_{B1})E_{11} = 0\,. \tag{8.64}$$

Now if $a_A + \varepsilon^{1B}{}_A n_{B1} = 0$, then we are dealing with a special case of **B2]** (for which, in fact, $\Theta = 0$), so

$$E_{11} = 0\,. \tag{8.65}$$

But now, from (A.14) and (A.11) it follows that

$$\mathbf{e}_0(\mu) = 0 \Rightarrow \Theta\mu = 0 \tag{8.66}$$

and so this is non-cosmological. We stop any further investigation here.

B2] $a_A + \varepsilon^{1B}{}_A n_{B1} = 0$**:** Let us first note that if $a_1 = 0$, then we have case **A]** again. This is because (A.15) + (A.16) and (A.55) − (A.56) demonstrates that $n_{11} = 0$. So from (8.44) and (8.45), the gradients of the shear vanish:

$$\mathbf{e}_A(\sigma) = 0\,. \tag{8.67}$$

Since $\mathbf{e}_A(\Theta_2) = 0$, we must have

$$\mathbf{e}_A(\Theta) = 0 \tag{8.68}$$

which for consistency with evolution requires

$$\mathbf{e}_A(\mu) = 0 \tag{8.69}$$

from the Raychaudhuri equation (A.13). In addition, we get from (A.39) and (A.41) that the gradients of E_{11} satisfy

$$\mathbf{e}_A(E_{11}) = 0 \tag{8.70}$$

and this leads to

$$\mathbf{e}_A(H_{11}) = 0 \tag{8.71}$$

from (A.35) and (A.31). We propagate the constraint on the gradients of E_{11} using the commutator $[\mathbf{e}_0\,, \mathbf{e}_A]$ on E_{11} and the evolution equation for E_{11}; that is (A.24). We find that

$$\mathbf{e}_A(\tfrac{1}{2}n_{22}(H_{22} - H_{33}) + 2n_{23}H_{23}) = 0 \tag{8.72}$$

which when we use (A.53) − (A.54) and (A.55) + (A.56) to replace the gravito-magnetic field terms, can be written as

$$\sigma[\tfrac{3}{4}\mathbf{e}_A(n_{22}^2) + 3\mathbf{e}_A(n_{23}^2)] = 0 \tag{8.73}$$

and since $\sigma = 0$ was dealt with in [17] and shown to be the Gödel universe, we deduce that

$$\tfrac{3}{4}(n_{22}\mathbf{e}_A(n_{22}) + 3n_{23}\mathbf{e}_A(n_{23}) = 0\,. \tag{8.74}$$

If we take the $\mathbf{e}_2$ frame derivative of (8.74) for $A = 3$ and the $\mathbf{e}_3$ frame derivative of (8.74) for $A = 2$, we get respectively

$$\mathbf{e}_3(n_{22})\mathbf{e}_2(n_{22}) + 4\mathbf{e}_3(n_{23})\mathbf{e}_2(n_{23}) = -2\omega\Theta_1(n_{22}^2 + 4n_{23}^2) + 2a_1n_{11}n_{22}^2 = 2\omega\Theta_1(n_{22}^2 + 4n_{23}^2) - 2a_1n_{11}n_{22}^2 \tag{8.75}$$

by applying the commutator (A.6) to n_{22} and using (A.15) – (A.16), (A.22), (A.61), (A.62) and (A.69) – (A.68). We now replace a_1n_{11} by using (8.41) and we find that

$$(\Theta_1 - 2\Theta_2)n_{22}^2 + 4\Theta_1n_{23}^2 = 0 \tag{8.76}$$

which implies that

$$(\Theta_1 - 2\Theta_2)\mathbf{e}_A(n_{22}^2) + 4\Theta_1\mathbf{e}_A(n_{23}^2) = 0 \tag{8.77}$$

which in turn implies from (8.74) that

$$\Theta_2\mathbf{e}_A(n_{23}^2) = 0\,. \tag{8.78}$$

But $\Theta_2 = 0 \Rightarrow a_1n_{11} = 0$ and we have already seen that this leads to trivial solutions ($\Theta = 0$). We discard this, and instead consider

$$\mathbf{e}_A(n_{23}^2) = 0 \Rightarrow \mathbf{e}_A(n_{22}^2) = 0 \tag{8.79}$$

clearly, from (8.74). Now if $n_{22} = n_{23} = 0$, then $\mathbf{H}$ is rotationally symmetric. So we need only consider

B2a] $\mathbf{e}_A(n_{23}) = \mathbf{e}_A(n_{22}) = 0$ or **B2b]** $n_{23} = 0\,, n_{22} \neq 0$ or **B2c]** $n_{23} \neq 0\,, n_{22} = 0\,.$

B2a] $\mathbf{e}_A(n_{23}) = \mathbf{e}_A(n_{22}) = 0$: We may now use some of the remaining freedom to set $a_3 = 0$ on a 2-surface $x^0 = c^0$, $x^1 = c^1$. From (A.20) and (A.63) we can see that this holds everywhere:

$$a_3 = 0\,. \tag{8.80}$$

From (A.64) and (A.66) we require that

$$a_2n_{22} = a_2n_{23} = 0 \Rightarrow a_2 = 0 \tag{8.81}$$

or else $\mathbf{H}$ is rotationally symmetric again. We now apply the $[\mathbf{e}_2\,,\mathbf{e}_3]$ commutator to n_{22} and n_{23}.

$$[\mathbf{e}_2\,,\mathbf{e}_3](n_{22}) = 2(2\Theta_2 - \Theta_1)\omega n_{22} + 2n_{11}n_{23}(n_{22} - n_{11}) \tag{8.82}$$

$$[\mathbf{e}_2\,,\mathbf{e}_3](n_{23}) = 2(2\Theta_2 - \Theta_1)\omega n_{23} - \tfrac{1}{2}n_{11}n_{22}(n_{22} - n_{11})\,. \tag{8.83}$$

(These two above equations are obtained from utilising (A.15) – (A.16), (A.22), (A.69) – (A.68), (A.62) and (A.61).) So either $n_{23} = n_{22} = 0 \Rightarrow \mathbf{H}$ rotationally symmetric, or

$$n_{11} - n_{22} = 0 = 2\Theta_2 - \Theta_1 \tag{8.84}$$

which from (A.61) and (A.62) gives

$$\omega\Theta_1 = 0 \Rightarrow \Theta_1 = 0 \tag{8.85}$$

and since $\Theta_2 = 0$, it follows that this is non-cosmological:

$$\Theta = 0\,. \tag{8.86}$$

B2b] $n_{23} = 0\,, n_{22} \neq 0$**:** The implication from (8.74) is that

$$\mathbf{e}_A(n_{22}) = 0 \tag{8.87}$$

and then we have from (A.64) and (A.66) that to avoid **H** becoming rotationally symmetric

$$a_A = 0\,. \tag{8.88}$$

Then from (A.69) − (A.68) we need

$$n_{22}(n_{11} - n_{22}) = 0 \Rightarrow n_{11} = n_{22} \tag{8.89}$$

which from (A.61) and (A.62) combined with (8.41) gives

$$\omega\Theta_1 = 0 \Rightarrow \Theta_1 = 0\,; \tag{8.90}$$

but (8.76) implies that

$$n_{22}^2\Theta_2 = 0 \Rightarrow \Theta_2 = 0 \tag{8.91}$$

and since $\Theta_1 = 0$, it follows that this is non-cosmological:

$$\Theta = 0\,. \tag{8.92}$$

B2c] $n_{23} \neq 0\,, n_{22} = 0$**:** The implication from (8.74) is that

$$\mathbf{e}_A(n_{22}) = 0 \tag{8.93}$$

and then we have from (8.76) that to avoid **H** becoming rotationally symmetric

$$\Theta_1 = 0\,. \tag{8.94}$$

From (A.61) and (A.62) we find

$$n_{11}n_{23} = 0 \tag{8.95}$$

and so to avoid $n_{23} = 0 \Rightarrow H_{23} = H_{22} - H_{33} = 0$, we must have

$$n_{11} = 0 \tag{8.96}$$

and these models have been shown previously to be trivial (static).

8.3.2 Summary

We have had to relax the restrictions on the gradients of the vorticity to make some progress here. We assumed $\mathbf{e}_A(\omega) = 0$. This is an assumption on the second derivative of the curvature tensor (since we have defined the fluid four-velocity as the timelike eigenvector of the Ricci tensor) and it turned out to be very constrictive on the allowed cosmologies. Indeed, there are no cosmologies here at all – since we have already shown that the case with $\mathbf{H}$ rotationally symmetric has no cosmological solutions.

8.4 Irrotational Dust

We shall now consider irrotational dust for which

$$\omega = 0 = p \Rightarrow \dot{u}_a = 0\,. \tag{8.97}$$

We may always take

$$\sigma \neq 0 \tag{8.98}$$

throughout this section; if not, then $\mathbf{H}$ is LRS (from (A.53) – (A.54), (A.55) + (A.56), (A.57) + (A.58) and (A.59) + (A.60)). This has been dealt with in Chapter 6. In fact, this is FLRW.

8.4.1 Tetrad Choice and Constraint Analysis

So now, from (A.48) and (A.47) we get

$$\Omega_2\sigma_{11} = \Omega_3\sigma_{11} = 0 \Rightarrow \Omega_2 = \Omega_3 = 0 \tag{8.99}$$

respectively. No other useful constraints arise from setting the shear to be rotationally symmetric. We note the following algebraic relations that hold between the gravito-magnetic field and some of the commutator functions. From (A.53) – (A.54) it follows that

$$H_{22} - H_{33} + \tfrac{3}{2}(n_{22} - n_{33})\sigma_{11} = 0 \tag{8.100}$$

and from (A.55) + (A.56) it follows that

$$H_{23} - \tfrac{3}{2}n_{23}\sigma_{11} = 0\,. \tag{8.101}$$

We rotate the tetrad such that

$$n_{22} - n_{33} = 0 \Rightarrow H_{22} - H_{33} = 0 \tag{8.102}$$

from (8.100). Consistency of this requires from (A.34) – (A.35) that

$$\Omega_1 H_{23} = 0 \tag{8.103}$$

which splits this set into two cases

$$\textbf{A]}\ \ H_{23} = 0 \quad \text{or} \quad \textbf{B]}\ \ \Omega_1 = 0\,.$$

A] $H_{23} = 0$: Immediately from (8.101) we get that

$$n_{23} = 0\,. \tag{8.104}$$

The four crucial constraints from arising from setting **E** rotationally symmetric are

$$\mathrm{e}_3(H_{12}) + \mathrm{e}_2(H_{31}) - (a_3 + 3n_{12})H_{12} - (a_2 - 3n_{31})H_{31} = 0$$

from (A.25) − (A.26), (8.105)

$$\mathrm{e}_3(H_{31}) - \mathrm{e}_2(H_{12}) - (a_3 + 3n_{12})H_{31} + (a_2 - 3n_{31})H_{12} = 0$$

from (A.27) + (A.28), (8.106)

$$\mathrm{e}_1(H_{12}) - 4a_1H_{12} - \tfrac{1}{2}(2n_{33} - 5n_{11})H_{31} + \tfrac{3}{2}(a_2 - n_{31})H_{11} = 0$$

from (A.29) − 2× (A.30), (8.107)

$$\mathrm{e}_1(H_{31}) - 4a_1H_{31} + \tfrac{1}{2}(2n_{33} - 5n_{11})H_{12} + \tfrac{3}{2}(a_3 + n_{12})H_{11} = 0$$

from (A.31) − 2× (A.32). (8.108)

We have propagated these without success, eventually running out of computing power. So we must reconcile ourselves to accepting these as intractable for the level of technology available to us. We instead proceed here to prove a lesser result, which dramatically reduces the labour required on this topic in future studies. We can demonstrate that, in principle, the tetrad may be invariantly defined. The idea is to base this on the gradients of μ in the e_2 and e_3 directions. We have seen that $a_2 - n_{31}$, $a_3 + n_{12}$, and $\mathrm{e}_2(\mu)$ and $\mathrm{e}_3(\mu)$ constrain each other through the Bianchi identities. In, fact, we have seen in Chapter 6, that if all tensors are rotationally symmetric, then the only PLRS spacetime which occurs is the Szekereres model, where

$$\mathrm{e}_2(\mu) = \mathrm{e}_3(\mu) = 0 \Leftrightarrow a_2 - n_{31} = a_3 + n_{12} = 0\,. \tag{8.109}$$

The same statement holds here.

> The implication from right to left is straightforward. If $a_2 - n_{31} = a_3 + n_{12} = 0$, then from (A.17) and (A.18) combined with (A.57) and (A.59), it follows that $H_{12} = H_{31} = 0$ and thus **H** is now rotationally symmetric. This has been dealt with in Chapter 6 where it was shown that this is a fully LRS spacetime with, in particular, $\mathrm{e}_2(\mu) = \mathrm{e}_3(\mu) = 0$.
>
> On the other hand, if we start off with $\mathrm{e}_2(\mu) = \mathrm{e}_3(\mu) = 0$, then for consistency of this, by applying $[\mathrm{e}_0, \mathrm{e}_A]$ to μ, and using the energy conservation equation (A.9), it must be that $\mathrm{e}_A(\Theta) = 0$. Now we propagate this by applying $[\mathrm{e}_0, \mathrm{e}_A]$ to Θ, and using the Raychaudhuri equation (A.13) and it follows that $\mathrm{e}_A(\sigma) = 0$. So now from the four equations (A.57) – (A.60) it follows that $a_2 - n_{31} = a_3 + n_{12} = 0$ (or else $\sigma = 0$ and $H_{12} = H_{31} = 0$).

Thus if the gradients don't vanish, we can use them to fix the tetrad, and if they do vanish, then we are dealing with LRS spacetimes – these are well-known.

B] $\Omega_1 = 0$**:** We will show that here, as well as in **A]**, that

$$\mathbf{e}_2(\mu) = \mathbf{e}_3(\mu) = 0 \Leftrightarrow a_2 - n_{31} = a_3 + n_{12} = 0\,. \tag{8.110}$$

The implication from right to left is not quite as straightforward as before. If $a_2 - n_{31} = a_3 + n_{12} = 0$, then from (A.17) and (A.18) combined with (A.57) and (A.59), it follows that $H_{12} = H_{31} = 0$. And now, for consistency of this, we get from their evolution equations (A.38) – (A.41) that $\mathbf{e}_2(\mu) = \mathbf{e}_3(\mu) = 0$.

In the other direction, the proof is exactly the same as in case **A]** directly above.

We now go on to map out these models which have

$$a_2 - n_{31} = a_3 + n_{12} = \mathbf{e}_2(\mu) = \mathbf{e}_3(\mu) = H_{12} = H_{31} = \mathbf{e}_A(\Theta_1) = \mathbf{e}_A(\Theta_2) = 0\,.$$

We note that if, at any stage, $H_{23} = \frac{3}{2} n_{23}\sigma_{11}$ vanishes, then we are back to case **A]**. Note that from the Bianchi identities (A.38) – (A.41) it follows also that

$$\mathbf{e}_A(E_{11}) = 0 \tag{8.111}$$

and from the Bianchi identities, (A.29) and (A.31), the gradients of H_{11} are constrained as follows:

$$\mathbf{e}_A(H_{11}) = 0\,. \tag{8.112}$$

To summarise so far:

$$\mathbf{e}_A(f) = 0 \quad \text{for } f = \{\mu\,, \Theta_1\,, \Theta_2\,, E_{11}\,, H_{11}\}.$$

The $[\mathbf{e}_2, \mathbf{e}_3]$ commutator, applied to these functions f above, now provides a subdivision. Either

B1] $n_{11} = 0 \Rightarrow H_{11} = 0$ or **B2]** $\mathbf{e}_\alpha(f) = 0 \Rightarrow a_1 = 0$.

B1] $n_{11} = 0$**:** From the H-constraint relation (A.52) we get that

$$H_{11} = 0\,. \tag{8.113}$$

And now it follows from (A.27) that

$$n_{33}H_{23} = 0 \Rightarrow n_{33} = 0\,. \tag{8.114}$$

We now apply the commutator $[\mathbf{e}_0, \mathbf{e}_A]$ on E_{11} using the evolution equation for E_{11}, that is (A.24), and the Bianchi identities (A.30) and (A.32) which gives

$$\mathbf{e}_A(n_{23}) + 4a_A n_{23} = 0\,; \tag{8.115}$$

which is consistent with other equations if

$$n_{23}\Theta_2 a_A = 0 \Rightarrow \Theta_2 a_A = 0 \tag{8.116}$$

from the evolution equation for n_{23} – (A.15) – (A.16), the evolution equations for a_A – (A.19) and (A.20) and using the $[\mathbf{e}_0, \mathbf{e}_A]$ commutator on n_{23}. Now if $\Theta_2 = 0$, then from (A.55) – (A.56), we get $a_1 = 0$. This is dealt with in **B2**]. Indeed, we may assume from here on in this subcase that a_1 does not vanish. So, instead we must have

$$a_A = 0 \tag{8.117}$$

which implies that

$$\mathbf{e}_A(H_{23}) = 0 \tag{8.118}$$

as well (from (A.30) and (A.32)). We recognise this case as the diagonal Abelian G_2 dust cosmologies with LRS shear. From (A.63) and (A.65) and (8.115)) we get the following gradients vanishing:

$$\mathbf{e}_A(a_1) = \mathbf{e}_A(n_{23}) = 0\,. \tag{8.119}$$

So all quantities (commutator functions and source variables) have vanishing gradients in the plane of rotation. These models have been discussed elsewhere – see, for example, [29] and references therein. We perform further consistency checks to find an expression for $\mathbf{e}_1(\Theta_1)$ –

$$\mathbf{e}_1(\Theta_1) + 3a_1\sigma_{11} = 0 \tag{8.120}$$

as a result of applying the propagator $[\mathbf{e}_0\,, \mathbf{e}_1]$ on n_{23} and comparing with the evolution equations (A.15) and (A.16), along with (A.55). Further propagation of this is consistent. We find an expression for the spatial gradient of the density by evolving (A.25) – (A.26) (an expression for the gradient of H_{23}), using the commutator $[\mathbf{e}_0\,, \mathbf{e}_1]$ on H_{23}, the relevant evolution equations, after which we also substitute for H_{23} from (A.55) and (A.56); and we use (8.120) as well. It follows that

$$\mathbf{e}_1(\mu) + 6a_1(2E_{11} - 3\sigma_{11}^2) = 0\,. \tag{8.121}$$

This then specifies the last spatial gradient of the remaining fields. We now take higher derivatives of this last expression and find algebraic constraints that must hold, by eliminating all gradients in the resulting expression using appropriate relations on these. At the first derivative level, we get

$$\sigma_{11}\Theta_2 + 4E_{11} + 2n_{23}^2 = 0\,. \tag{8.122}$$

The best way of proceeding now is to eliminate E_{11} in the above by substituting equation (A.67), thus obtaining

$$\Theta_2(3\sigma + \tfrac{4}{3}\Theta) + 4a_1^2 - 2n_{23}^2 + \tfrac{4}{3}\mu = 0\,. \tag{8.123}$$

We differentiate the above using the necessary evolution equations, replace H_{23} in terms of n_{23} and eliminate n_{23} using (8.123); so

$$\tfrac{5}{54}(3\sigma - 2\Theta)(\Theta^2 + 3\sigma\Theta - 9\sigma^2) + \tfrac{1}{3}(5\Theta + 27\sigma)a_1^2 + \tfrac{5}{18}(3\sigma + 2\Theta)\mu = 0\,. \tag{8.124}$$

We differentiate this, replace H_{23}, and in the resulting expression, eliminate E_{11} using (8.122) and n_{23} using (8.123) and μ using (8.124). Thus

$$-\tfrac{25}{36}\sigma(3\sigma - 2\Theta)^2(6\sigma - \Theta) - \tfrac{1}{30}a_1^2(2106a_1^2 - 5175\sigma^2 - 230\Theta^2 + 1455\sigma\Theta) = 0\,. \tag{8.125}$$

We see now that, in principle, three more derivatives should give us a conclusive result here. Indeed, the situation is aided by the fact that the equation immediately above, when viewed as cubic in Θ has only one real solution. But the algebra gets extensive now, so we turn once again to an algebraic computing package (Maple). The resulting expressions serve no illustrative purpose at all, so we have not included these. We obtain them, however, by differentiating successive constraints and eliminating the same variables that were eliminated to get previous constraints as before, and then we use (8.125) to eliminate Θ. We solve for σ in the resulting expression, obtaining two real solutions

$$\sigma = \pm\sqrt{\frac{26}{25}}a_1\,. \tag{8.126}$$

We then differentiate again, and eliminate all variables in the same fashion as before, substitute for σ from the above equation, and we get

$$a_1 = 0\,. \tag{8.127}$$

Thus we may conclude that these solutions reduce to spatially homogeneous models dealt with in **B2]**.

B2] $\mathbf{e}_\alpha(f) = 0$: The immediate implication is that

$$a_1 = 0 \tag{8.128}$$

which follows, in particular, from (A.55) – (A.56). So now the Jacobi identity (A.61) gives

$$\mathbf{e}_1(n_{11}) = 0 \tag{8.129}$$

which means that from (A.62) that

$$n_{23}(n_{11} - 2n_{33}) = 0 \Rightarrow n_{11} - 2n_{33} = 0\,. \tag{8.130}$$

We note from (A.69) – (A.68) that

$$\mathbf{e}_1(n_{23}) = 0 \tag{8.131}$$

whilst (A.25) – (A.26) demands

$$\mathbf{e}_1(H_{23}) = 0\,. \tag{8.132}$$

Crucial here now is (A.27) which reads

$$n_{23}\sigma_{11}n_{11} = 0 \Rightarrow n_{11} = 0 \tag{8.133}$$

when combined with (8.130) and (A.52) and (A.55); which means that

$$H_{11} = 0 \tag{8.134}$$

from (A.52). So in effect, we deal with a subcase of **B1]** for which $a_1 = 0$. However, we consider this case independently. We note that $n_{11} = 0$ implies from (8.130) that

$$n_{33} = 0\,. \tag{8.135}$$

We now propagate (8.115) using the necessary evolution equations; that is, (A.4) and (A.5) applied to n_{23}, which has evolution given by (A.15) – (A.16), the evolution equations for a_A – (A.19) and (A.20). We find

$$n_{23}\Theta_2 a_A = 0 \Rightarrow \Theta_2 a_A = 0 \tag{8.136}$$

So now we have a further division according to

$$\mathbf{B2a]}\ \ a_A = 0 \quad \text{or} \quad \mathbf{B2b]}\ \ \Theta_2 = 0\,.$$

B2a] $a_A = 0$: It follows from (8.115) that

$$\mathbf{e}_A(n_{23}) = 0 \tag{8.137}$$

and from (A.32), (A.30) we get that

$$\mathbf{e}_A(H_{23}) = 0\,. \tag{8.138}$$

So now every remaining quantity is homogeneous:

$$\mathbf{e}_\alpha(f) = 0 \ \text{ for } \ f = \{\mu\,, E_{11}\,, H_{23}\,, \Theta_1\,, \Theta_2\,, n_{23}\}$$

so this must be a Bianchi universe.

B2b] $\Theta_2 = 0$: Again from (8.115) it follows that

$$\mathbf{e}_A(n_{23}) = 0 \tag{8.139}$$

but now from (A.64) and (A.66) we get

$$n_{23}a_A = 0 \Rightarrow a_A = 0 \tag{8.140}$$

thus effectively making this a subcase of **B2a]** for which

$$n_{23} = 0 \tag{8.141}$$

from (A.67) thus making this a case covered in Chapter 6 where all tensors pointed in the same direction.

8.4.2 Summary

We have attempted to describe this set under the most general conditions. However, we found that this was not possible as yet. We have made a simplifying assumption, which in its weakest form amounts to assuming that the density gradients in the plane of rotation vanish – a condition on the first derivative of the Riemann tensor. The results have been that we retrieve well-known solutions: an Abelian G_2 subcase and a subcase of the Bianchi models. The other possibilities are all LRS and were found in Chapter 6.

8.5 Summary

What we have managed to illustrate in the case of a dust is weaker than the results we obtained for an accelerating perfect fluid. To make headway in the irrotational dust case, we made the additional assumption that the density gradients in the plane of rotation vanishes – a condition on the first derivative of the Riemann tensor. This was found to be a severe constraint however, and only known solutions are covered by this assumption. In the rotating case, we made the even stronger assumption (a condition on the second derivative of the Riemann tensor) that the gradients of the vorticity be rotationally symmetric: $\mathbf{e}_A(\omega) = 0$. This was very restrictive – again only known solutions were covered by this; the most general a subcase of the Abelian G_2 solutions which when checked for consistency belongs to the class of spatially homogeneous models.

It would be very useful if we could generalise this result and show[3] that $\mathbf{e}_A(\omega) = 0 \Rightarrow \mathbf{e}_A(\mu) = 0$ or at least classify all solutions for which the density gradients are rotationally symmetric in the irrotational case. The accelerating perfect fluid models, whether rotating or not, are only as general as those dealt with in Chapter 6 where all tensors were rotationally symmetric.

[3]It seems likely that this could be true.

Chapter 9

Final Comments and Summary

9.1 Rotationally Symmetric Shear Dust Cosmologies

Since the solutions which have more than one tensor quantity LRS have now been delineated, we continue here by looking at a way of generalising those solutions. We shall look at what the implications are for setting the shear only to be rotationally symmetric. We make the standard assumption of a perfect fluid cosmology. We start our investigations for dust. It follows that

$$E_{22} - E_{33} = E_{23} = 0 \tag{9.1}$$

from (A.45) and (A.46). It appears most appropriate to rotate the tetrad such that

$$\Omega_2 = 0\,, \tag{9.2}$$

which has the effect of setting one of the off-diagonal gravito-electric components to zero:

$$E_{31} = 0 \tag{9.3}$$

from (A.48). We also note that

$$\tfrac{3}{2}\Omega_3\sigma - E_{12} = 0 \tag{9.4}$$

from (A.47). So, since shear-free dust is well-known, our problem amounts to finding the spacetimes for which the tetrad can never be Fermi-propagated: if it is, then $\mathbf{E}$ is RS.

We proceed to check the consistency of this subcase of the EFE by following the time evolution of all new constraints. Not unexpectedly, we run into serious problems very quickly. Algebraic computing is insufficient to solve this most general problem. The main problem is that the higher order derivatives keep increasing in order without allowing a convenient factorisation until one runs out of computing power. This is not too surprising, as we encountered the same problem in the previous chapter where the shear and $\mathbf{E}$ only were RS. We can try and generalise the results of that chapter by assuming

$$\mathbf{e}_A(\mu) = 0\,. \tag{9.5}$$

This then gives us a "handle" on the problem by giving a convenient fork in the class of allowed solutions from the $[\mathbf{e}_2, \mathbf{e}_3]$ commutator; viz.

$$n_{11}\mathbf{e}_1(\mu) = 0\,. \tag{9.6}$$

Let us take the case

$$\mathbf{e}_1(\mu) = 0 \tag{9.7}$$

and see where this leads us. We find that this implies in the end that $n_{11} = 0$ and thus (9.6) has really only one solution. We propagate $\mathbf{e}_\alpha(\mu) = 0$ using (A.3) on the energy density and compare with what one gets form taking the spatial gradient of its evolution equation, (A.9). Clearly

$$\mathbf{e}_\alpha(\Theta) = 0\,. \tag{9.8}$$

This must also be consistent with other relations. We use the same commutator and then the Raychaudhuri equation (A.13) and we find that also

$$\mathbf{e}_\alpha(\sigma) = 0\,. \tag{9.9}$$

These relations are very restrictive. We use the relations entailing the $\mathbf{H}$-constraint; i.e. (A.55) – (A.60), and find that all of the following holds:

$$H_{31} = H_{12} = a_2 - n_{13} = a_3 + n_{12} = 0\,. \tag{9.10}$$

Now (A.18) gives

$$\Omega_3(n_{22} - n_{11}) = 0 \Rightarrow n_{22} - n_{11} = 0 \tag{9.11}$$

(or else we are in Chapter 8). Since $E_{22} - E_{33} = 0$, we require for consistency, from (A.25) – (A.26) that

$$2\mathbf{e}_1(H_{23}) - \tfrac{3}{2}n_{33}H_{33} + \tfrac{3}{2}n_{22}H_{22} + \tfrac{1}{2}(n_{11} - n_{33})(H_{33} - H_{11}) = 0 \tag{9.12}$$

which, using the $\mathbf{H}$-constraint relations – (A.52), (A.53), (A.54), (A.55) – gives us the following expression for the $\mathbf{e}_1$ gradient of n_{23}:

$$\mathbf{e}_1(n_{23}) = \tfrac{1}{2}n_{33}(n_{33} + n_{11}) + n_{11}^2\,. \tag{9.13}$$

if we compare this with (A.69) – (A.68), which reads

$$\mathbf{e}_1(n_{23}) = \tfrac{1}{2}(n_{33} + n_{11})n_{33}\,, \tag{9.14}$$

it follows that

$$n_{11}(n_{11} + n_{33}) = 0\,. \tag{9.15}$$

Now, let us say, that $n_{11} \neq 0$, then

$$(n_{11} + n_{33}) = 0 \tag{9.16}$$

and it follows that

$$\mathbf{e}_1(n_{23}) = n_{11}^2\,; \tag{9.17}$$

but now from (A.62) and (A.61), we get that

$$n_{23}n_{11} = 0 \Rightarrow n_{23} = 0\,, \tag{9.18}$$

and now, from (9.17) it follows that

$$n_{11} = 0\,. \tag{9.19}$$

Indeed, this implies that

$$H_{11} = 0 \tag{9.20}$$

from (A.52).

So we may say that this class is characterised by the vanishing of the density gradients in the plane of rotation and

$$n_{11} = H_{11} = 0\,. \tag{9.21}$$

When we now attempt to analyse this set, we unfortunately can find no way to continue in a significant fashion. The problem at the time of writing is intractable. We thus end our studies on this topic here.

9.2 Summary

We have considered the problem of characterising cosmological spacetimes by use of a shear eigentetrad. This is relevant to the equivalence problem placed in a cosmological context. Cosmologies which have vanishing shear are fairly well-understood and those for which the shear is not degenerate, solve the problem in effect, because all quantities are then invariantly classified in terms of this tetrad. The hope was originally to fully classify all accelerating rotating perfect cosmologies which have rotationally symmetric shear. However, we considered situations where a combination of the Weyl curvature variables were also rotationally symmetric. We called these spacetimes Partially Locally Rotationally Symmetric Spacetimes. However, this is a very difficult problem to solve in full generality and we were eventually obliged to consider a somewhat weaker problem which was successfully completed. Our results are summarised in Table 9.1. In effect, the work we have done shows very clearly how restrictive the class of PLRS spacetimes are. The only inhomogeneous model we recovered was the Szekeres model (generally). When the assumption of vanishing density gradients in the plane of rotation was added in some cases, again this was shown to be remarkably restrictive. These results may possibly be generalised further. The immediate investigation concerning us at the time of writing is to relax the assumption on the vorticity gradients in the irrotational dust case where $\boldsymbol{\sigma}$ and $\mathbf{E}$ are RS to only an assumption on the density gradients as in the irrotational case; i.e. we want to show that $\mathbf{e}_A(\mu) = 0 \Rightarrow \mathbf{e}_A(\omega) = 0$. We certainly expect that if we impose weak PLRS symmetry, this should lend itself to further investigation.

σ, $\mathbf{H}$ RS $\Rightarrow$ $\mathbf{E}$ RS		σ, $\mathbf{E}$ RS	
INHOMOGENEOUS		ROTATING	
$p = p(\mu) \neq 0$	LRS II (Stewart and Ellis [75])	$p \neq 0$	$\Rightarrow \mathbf{H}$ RS
$p = 0$	Szekeres; LRS II (Ellis [17])	$p = 0$, $\mathbf{e}_A(\omega) = 0$	$\Rightarrow \mathbf{H}$ RS
HOMOGENEOUS		IRROTATIONAL	
$p \neq 0$	Bianchi LRS (Kantowski and Sachs [42])	$p \neq 0$	$\Rightarrow \mathbf{H}$ RS
$p = 0$	Bianchi LRS II (Ellis [17])	$p = 0$, $\mathbf{e}_A(\mu) = 0$	$\Rightarrow \mathbf{H}$ RS; Bianchi, $H_{23} \neq 0$ ([23])

Table 9.1: PLRS class of solutions satisfying Definition 4.2.1 covered in this work.

An interesting question one can also ask – which is somewhat tangential to our study – is to consider situations where the various dynamical quantities are RS, *but not in the same plane*[1]. Indeed, there are Abelian G_2 solutions ascribed to Feinstein and Senovilla [31, 72] which have this property. On the other hand, it is unclear from the analysis that we have done in this part of this thesis, that with σ and $\mathbf{E}$ RS in the same plane, whether this negates the possibility of models which exhibit gravitational radiation. Thus it is still an open question whether under the general conditions stated in Definition 4.2.1, the quantities essential [29] for gravitational radiation vanish or not.

[1] As remarked by H. van Elst.

Part III

Consistency Studies II: Dust Brane World

Chapter 10

Introduction

The evolution in theories of gravity over the last century has been remarkable. The major developments from Newtonian gravity to General Relativity and then to string theory, have lead to radical paradigm shifts. Nevertheless, if one takes the view that none of these theories is *the* theory of gravity, but are rather steps in a sequence of ever more accurate, predictive and widely-applicable theories, then the focus naturally shifts towards understanding properties of the sequence, rather than each single theory.

In particular, a fascinating issue is how each successive step contains the preceding one in one or more limits and whether results proven for one theory, hold in the more general context of the enveloping theory, i.e. are the results stable to inter-theory deformations. If the one theory is in some sense a linearisation of the other, we would like to consider issues of linearisation instability, in some sense. Of central interest in this regard in cosmology is the existence of the singularity at the start of the universe, the Big Bang.

To illustrate this, consider General Relativity and its Newtonian gravity limit. This limit is extremely subtle since one must relinquish diffeomorphism invariance and recover an absolute, uncurved space and time in the Newtonian limit. Further, there exist theorems which, while valid in Newtonian theory, are not valid in the full theory of General Relativity.

In particular, we wish to address this issue in the context of higher-dimensional brane cosmologies. Visser [82] constructed an exotic Kaluža-Klein model with non-compact extra dimensions in which gravity was confined to a four dimensional sub-space. This model was recently unearthed and popularised in slightly different form by Randall and Sundrum [68], motivated by Horava-Witten cosmology [39, 40, 51] and has subsequently been extensively developed.

The conceptual power behind these models invoking one or more branes is their ability to resolve the hierarchy problem, to explain naturally the weakness of gravity on the brane, and perhaps most importantly, to provide an alternative to compactification of the extra dimensions so crucial to string theory.

We discuss the stability of the famous $\omega\Theta = 0$ result for shear-free dust in General Relativity to modifications of the field equations motivated by higher-dimensional brane models. The most recent generalisation of this in classical GR is the paper by Senovilla et. al. [?] in which they elegantly show, in a completely covariant fashion, that the result is true for all perfect fluids with the vorticity and acceleration vectors aligned. They also demonstrate that these results are not necessarily true in the Newtonian case, although they do hold in most cases. In fact, in Newtonian cosmology, the only solutions where they do not hold are homogeneous.

We find that for situations of direct physical interest, these models have similar problems to GR cosmology (and in some repect may even be worse).

Chapter 11

Shear-free Brane Cosmology

11.1 Introduction

The perturbative string theory framework is the first fundamental theory to make a prediction for any of the constants of nature. The fact that only $d = 4$ dimensions are visible out of a posited $D = 10$ might have killed string theory had it not been for its excellent short-distance behaviour and the natural appearance of the graviton. Instead the extra $D - d = 6$ dimensions were therefore required to be compact, with volumes determined by the tiny string scale ℓ_s, of order the Planck scale L_{pl}. This naturally hides the extra dimensions from the probes of low-energy physics, but arguably removes quantum gravity from the arena of the physics one might conduct in the foreseeable future.

Work on extra dimensions was re-invigorated by the realisation [1] that large compact extra dimensions could provide a natural explanation of the weakness of gravity while solving the hierarchy problem through the large volume in the extra dimensions[1].

Randall and Sundrum (RS) implemented a solution to the hierarchy problem by invoking two branes with the weak scale on the visible brane exponentially suppressed by the distance to the hidden brane. Further, building on much earlier work by Visser [82], they discussed how linearised gravitational waves can be trapped on a 3-brane, providing a linearised *brane-geon* in a bulk formed from copies of 5-d anti-de Sitter (AdS_5). Five dimensional cosmologies are particularly in vogue in the wake of M-theory solutions such as the Horavâ-Witten scenario [39, 40] in which the $E_8 \times E_8$ string is dual to the 11-d supergravity theory on $\mathbf{R}^{10} \times S^1/\mathbf{Z}_2$. Compactification on a Calabi-Yau 3-fold leaves a 5-dimensional spacetime [51] which may contain branes naturally located at orbifold fixed points.

The advantage of the RS implementation over Visser's is partially one of elegance. Hawking *et al* [38] have discussed how the RS solution appears naturally via the AdS/CFT correspondence. In particular they neatly showed using the Euclidean path-integral approach how the RS AdS_5 action with counter-terms

[1] In the non-compact scenarios gravity is confined to a sub-manifold.

is equivalent to 4-d GR on a domain wall coupled to a conformal field theory (CFT) with corrections due to the counter-term. Perhaps the fundamental interest in these solutions is their provision of an alternative to compactification of the extra dimensions so fundamental to string theory.

A major motivation for the work in this chapter, is the potential existence of rotating, shear-free, and expanding cosmologies which are absent in standard GR. In Newtonian theory such cosmologies can exist. As a cosmic singularity is approached backwards in time, the vorticity dominates the energy density of standard matter, the collapse can be halted and the singularity avoided.

If one examines the shear-free generalisation of the Friedmann equation in General Relativity [18], it appears that a similar result is possible. However, whilst the Einstein field equations are integrable in the sense that the evolution commutes with the constraints equations arising from the Bianchi and Ricci identities [58, 32], this is not true when additional constraints are included, such as imposing the flow to be shear-free, $\sigma_{ab} = 0$.

In a classic result, Ellis [17] extended a conjecture of Gödel to find the result that $\omega\Theta = 0$ for a spacetime filled with dust. Thus, either the universe is expanding ($\Theta \neq 0$) and irrotational ($\omega = 0$), or the contrary, an example of which is the static, rotating, Gödel universe [37].

Hence Newtonian theory yields results which are spurious when embedded inside GR. The question we wish to begin to address is whether results of GR, and in particular the $\omega\Theta = 0$ result, holds in modern generalisations of GR inspired by string theory. Our results for brane cosmologies, summarised in Fig. 1, imply that the $\omega\Theta = 0$ result *does* hold in many situations and that the initial singularity cannot be avoided by invoking vorticity.

It is therefore natural to ask whether these brane models suffer from the same restrictions as cosmologies in GR. We will show that indeed the shear-free brane assumption (which contains Friedmann-Lemaître-Robertson-Walker (FLRW) models as sub-cases) is a very powerful constraint which leads to a very special and small class of brane cosmologies.

As we have seen before in this thesis in previous chapters, shear plays a special role in the dynamics of a perfect fluid; most evidently so for dust. The evolution equation for the shear (3.50) shows explicitly how the free gravitational field, in the form of the gravito-electric field, enters the dynamics of the fluid with which we model the matter content of the universe. It is natural to ask what happens if the shear is forced to vanish[2]. The fundamental point is that this not only affects the evolution equations of the theory, but alters the constraint equations arising from the Bianchi and Ricci identities, (3.59) and (3.50) respectively. Thus the $\dot{\sigma}$ shear evolution equation, becomes a new constraint - and the requirement that this constraint commutes with the new evolution equations is the source of additional restrictions.

A theorem of Gödel's [17] states: in classical general relativity, a dust-filled universe with homoge-

[2]Or indeed to ask what happens if it is rotationally symmetric as we have done in Chapter 4.

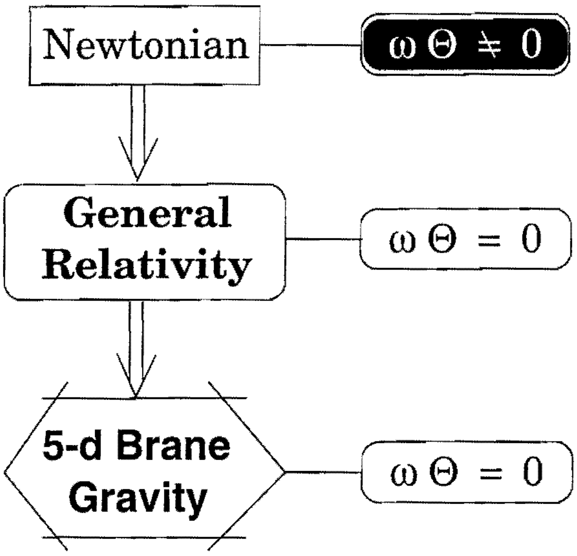

Figure 11.1: A schematic diagram of the structure of shear-free cosmologies. While Newtonian gravity allows non-singular rotating, shear-free, models, these are outlawed in General Relativity due to the additional constraints that must be satisfied in GR. We show that these constraints have the same effect in brane cosmologies with the Einstein-Hilbert action imposed, showing that the result $\omega\Theta = 0$ is stable and the initial singularity cannot be removed with non-zero vorticity, $\omega \neq 0$.

neous space sections and nonzero expansion scalar, Θ, must be irrotational if the shear vanishes.

This result was proved under the more general circumstance where the assumption of homogeneous space sections was not assumed by Ellis in 1967 [17]. It was shown that shear-free dust is very special: either irrotational or non-expanding. This result is interesting for many reasons. It is a key illustration of the difference between Newtonian gravity and Einstein's theory; since in Newtonian theory, shear-free solutions which are expanding and rotating are indeed allowed. The key point here is that whereas in Newtonian theory, one only has to solve for one potential, in Einstein's theory there are ten partial differential equations which have to be satisfied for a solution to exist; these extra nine being identities in Newtonian theory [21].

11.1.1 Field Equations

The geometric framework we follow is that of Shiromizu *et al* [70]. The 5-dimensional bulk[3] is assumed to be an Einstein manifold with negative cosmological constant. Matter is assumed confined to a (3+1) dimensional brane which breaks Poincaré invariance in 5-d. A $\mathbf{Z}_2$ symmetry about the brane is assumed, and hence the brane may naturally lie at an orbifold fixed point.

The field equations for the brane are given by exploiting the Gauss-Codacci equations which relate the Riemann tensors of the bulk and the brane and give derivatives of the brane extrinsic curvature in terms of the bulk Ricci tensor.

[3]We will use capitals to denote quantities in the bulk: A, B, $C = 0, 1, 2, 3, 4$.

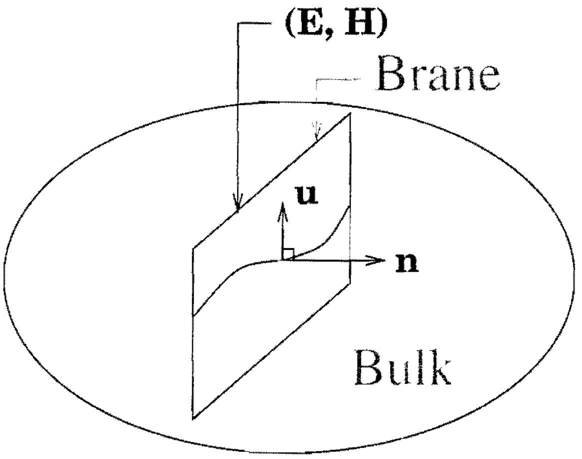

Figure 11.2: Schematic geometry showing the bulk, brane (2 spacelike dimension suppressed) and the spacelike, **n**, and timelike, **u**, vector fields used to define the brane and 3 + 1 spitting. **E** and **H** are the electric and magnetic parts of the brane Weyl tensor.

The brane is assumed everywhere orthogonal to a spacelike vector field n_A (see Fig 2). Exploiting the $\mathbf{Z}_2$ symmetry then yields equations resembling Einstein's equations in the four-dimensional brane subspace [4] [70]:

$$G_{ab} = -\Lambda\, g_{ab} + \kappa\, T_{ab} + \kappa_5^4\, \pi_{ab} - \mathcal{E}_{ab} \tag{11.1}$$

where G_{ab} is the usual Einstein tensor and T_{ab} the standard energy-momentum tensor. π_{ab} is a term quadratic in T_{ab} (and should not be confused with the anisotropic pressure of an imperfect fluid, although it behaves in a similar fashion with respect to the Ricci and Bianchi equations) given by:

$$\pi_{ab} = -\frac{1}{4}T_{ac}T_b{}^c + \frac{1}{12}T_{ab} + \frac{1}{8}g_{ab}T_{cd}T^{cd} - \frac{1}{24}g_{ab}T^2\,, \tag{11.2}$$

where $T \equiv T^\mu_\mu$. Finally the $\mathcal{E}_{ab}$ term is the limiting value near the brane of a projection of the bulk Weyl tensor from the bulk; *viz*

$$\mathcal{E}_{AB} \equiv {}^{(5)}C_{ACBD}n^C n^D\,, \tag{11.3}$$

which is symmetric and tracefree: $\mathcal{E}_{[AB]} = 0$ and $\mathcal{E}_A{}^A = 0$. This represents the free, non-local gravitational field in the bulk. Further $\kappa = \dfrac{8\pi}{M_4^2}$ and $\kappa_5^2 = \dfrac{8\pi}{M_5^3}$, where M_4 and M_5 are the four- and five-dimensional Planck masses respectively.

11.1.2 The Brane

Since the model is essentially a domain wall inserted into a 5-d manifold satisfying Einstein's vacuum equations, it can be shown [70] that the brane energy-momentum tensor satisfies the usual conservation equations – i.e.

$$T^{ab}{}_{;b} = 0\,; \tag{11.4}$$

[4] Our notation differs slightly from [70].

where the semi-colon represents covariant differentiation with respect to the brane metric, g_{ab}. Thus, from the contracted Bianchi Identities,

$$\nabla_b G^{ab} = 0\,, \tag{11.5}$$

it follows that

$$\kappa_5^4\,\nabla_b \pi^{ab} = \nabla_b \mathcal{E}^{ab}\,, \tag{11.6}$$

which imply that a vanishing $\nabla_b \mathcal{E}^{ab}$ is only consistent with homogeneous density (if T_{ab} has a perfect fluid form). This in some sense demonstrates already how non-local effects affect the classical GR dynamics – by not being there, it constrains the class of allowed solutions. It is an indication to us of the essential difference on the brane between the two theories; which is the typical type of thing we wish to explore.

The differential geometric structure of GR is unchanged; thus $\mathbf{G}$ is unchanged and also the Riemann curvature tensor $\mathbf{R}$ may be decomposed into the completely tracefree 4-d Weyl conformal curvature tensor C_{abcd}, the Ricci curvature tensor $R_{ab} \equiv R^c{}_{acb}$ and the Ricci curvature scalar $R \equiv g^{ab}R_{ab}$ in the usual fashion

$$R_{abcd} = C_{abcd} + R_{a[c}\,g_{d]b} - R_{b[c}\,g_{d]a} - \tfrac{1}{3}\,R\,g_{a[c}\,g_{d]b}\,. \tag{11.7}$$

11.2 Cosmology

We now discuss cosmology of the brane using the orthonormal tetrad formalism employed in part I of this thesis. Θ, the expansion, [5] is taken to be non-zero; indeed

$$\Theta > 0 \tag{11.8}$$

is evident from current observations. The four-velocity of the fundamental observers u_a allows us to define the prejection tensor into the rest space of an observer [6]. by

$$h_{ab} = g_{ab} + u_a u_b\,. \tag{11.9}$$

We assume throughout most of this chapter that the universe can be modeled as dust –

$$T_{ab} = \varrho u_a u_b\,. \tag{11.10}$$

This means that the conservation equations (11.4) read

$$\partial_0(\varrho) = -\Theta\,\varrho \tag{11.11}$$

$$\varrho\dot{u}_\alpha = 0 \Rightarrow \dot{u}_\alpha = 0\,. \tag{11.12}$$

[5]In a FLRW geometry $\Theta = 3H$ where H is the *Hubble constant*.

[6]There exists an analogous projection tensor from the bulk onto the brane.

The second order energy momentum tensor π_{ab} now takes the form

$$\pi_{ab} = \tfrac{1}{12}\,\kappa_5^4\,(\varrho^2\,u_a\,u_b + \varrho(\varrho + 2p)h_{ab})\,, \tag{11.13}$$

which means that the conservation equation (11.6) reads

$$\begin{aligned} u_a\,\nabla_b\mathcal{E}^{ab} &= 0 \\ \tfrac{1}{6}\,\kappa_5^4\,\varrho\,\partial_b(\varrho)\,h^{ab} &= \nabla_b\mathcal{E}^{ab}\,. \end{aligned} \tag{11.14}$$

The 4-d Weyl conformal curvature tensor may be decomposed with respect to the group of spatial rotations relative to the preferred 4-velocity into its symmetric tracefree 'electric' and 'magnetic' parts, $\mathbf{E}$ and $\mathbf{H}$, respectively, according to [14, 18]

$$E_{ab} \equiv C_{cedf}\,h^c{}_a\,u^e\,h^d{}_b\,u^f = E_{(ab)} \tag{11.15}$$

$$H_{ab} \equiv (-\tfrac{1}{2}\,\eta_{cegh}\,C^{gh}{}_{df})\,h^c{}_a\,u^e\,h^d{}_b\,u^f = H_{(ab)}\,, \tag{11.16}$$

with the completely skew[7] spacetime permutation tensor η_{abcd} specified by

$$\eta_{abcd} = \eta_{[abcd]} \quad \text{and} \quad \eta_{0123} = 1\,,\ \eta^{0123} = -1\,,$$

ss before. Thus we may write the 4-d Weyl tensor as

$$C^{ab}{}_{cd} = 4\,u^{[a}\,u_{[c}\,E^{b]}{}_{d]} + 4\,h^{[a}{}_{[c}\,E^{b]}{}_{d]} + 2\,\varepsilon^{abe}\,u_{[c}\,H_{d]e} + 2\,\varepsilon_{cde}\,u^{[a}\,H^{b]e}\,, \tag{11.17}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the 3-space permutation symbol obtained by projecting η_{abcd} into the rest 3-space orthogonal to $\mathbf{u}$, $\varepsilon_{abc} \equiv \eta_{defg}\,u^d\,h^e{}_a\,h^f{}_b\,h^g{}_c$;

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{[\alpha\beta\gamma]} \quad \text{and} \quad \varepsilon_{123} = +1\,.$$

The second Bianchi identity differentially relates components of the Riemann tensor:

$$\nabla_{[a}R_{bc]de} = 0\,. \tag{11.18}$$

As well as entailing the matter conservation equations above, given the 1+3 decomposition of the curvature variables, (11.15) and (11.16), this relation provides evolution and constraint equations for $\mathbf{E}$ and $\mathbf{H}$ [18, 21].

11.2.1 Shear-free Dust

We now consider the dynamically interesting subcase of vanishing shear. The resulting tetrad relations may be found in appendix B. Again, consistency checks provide further constraints. We continue this programme until we either only obtain identities, or inconsistency has been demonstrated.

[7] Round brackets denote symmetrized indices and square brackets denote skew-symmetrized indices.

11.2.1.1 Recovering the Relativistic Result: $\omega\Theta = 0$

We would like to investigate the effect of $\mathcal{E}_{ab}$ on GR cosmology eventually; and as a first step towards this objective, we look at the case where it has obviously no effect. If the divergence of $\mathcal{E}_{ab}$ vanishes, then from (11.14) it follows that the dust on the brane is homogeneous:

$$\mathbf{e}_\alpha(\varrho) = 0\,. \tag{11.19}$$

We now assume, for simplicity, that we can neglect the tidal interaction; that is

$$\mathcal{E}_{ab} = 0\,.$$

This is a first step towards proving the more general case where we include possible non-local effects bulk on the brane dynamics. Since now the only difference to the standard theory is the presence of the high energy correction to the density, which appears with the same sign everywhere as the density, we expect the usual Gödel proposal to hold with only this minor modification. This is indeed what happens. However, we will go through the analysis carefully as it may be useful in later work for comparison.

From (B.52) – (B.53) we get

$$E_{22} = E_{33} \tag{11.20}$$

and from (B.54) it follows that

$$E_{23} = 0 \tag{11.21}$$

and then from (B.55) and (B.56),

$$E_{12} = E_{31} = 0\,. \tag{11.22}$$

And the final constraint resulting from killing the shear reads

$$3\,E_{11} + 2\,\omega_1^2 = 0\,, \tag{11.23}$$

from (B.51) – (B.52) This tells us that if the vorticity should now vanish, then $\mathbf{E}$ also does so – illustrating how these two fields constrain each other in the absence of shear. From density evolution and $[\mathbf{e}_0, \mathbf{e}_\alpha]$ operating on the density ϱ, it follows that

$$\mathbf{e}_\alpha(\Theta) = 0 \tag{11.24}$$

and this commutator now applied to Θ gives

$$\mathbf{e}_\alpha(\omega) = 0 \tag{11.25}$$

from the Raychaudhuri equation (B.12). As we will see, the vanishing of the vorticity gradients is a key result for this study. We find now, from (B.59) that

$$a_1 = 0 \tag{11.26}$$

and from (B.60) that

$$H_{11} = 0 \tag{11.27}$$

which results in setting

$$n_{11} = 0 \tag{11.28}$$

from (B.65) – (B.67). Then, (B.66) and (B.68) imply

$$H_{12} = H_{31} = 0 \tag{11.29}$$

which in turn implies from (B.65) and (B.67) that

$$a_2 - n_{31} = a_3 + n_{12} = 0\,. \tag{11.30}$$

The effect of all of this is to constrain the dynamics as follows:

$$\omega\,\Theta = 0 \tag{11.31}$$

reads the Jacobi identity (B.69). This is the standard GR result and it provides us with two neat options for proceeding from here.

Static rotating dust If $\Theta = 0$, then the Raychaudhuri equation (B.12) demands that

$$-\tfrac{1}{2}\,\varrho\,(\kappa + \tfrac{1}{3}\,\kappa_5^4\,\varrho) + 2\,\omega_1^2 + \Lambda = 0\,. \tag{11.32}$$

We choose $\mathbf{e}_1(\varphi)$ such that

$$n_{33} = 0\,. \tag{11.33}$$

Note that

$$\mathbf{e}_0(\gamma^a{}_{bc}) = \mathbf{e}_0(\varrho, E_{11}) = 0\,.$$

In fact, from the equations labelled $[123]\,\alpha\beta$, $\alpha \neq \beta$, we have

$$\mathbf{e}_a(\varrho, E_{11}) = 0$$

From (B.62) – (B.61) we may derive that

$$H_{22} = n_{23}\omega_1 \tag{11.34}$$

and from (B.63) + (B.64) we may write

$$-2\,H_{23} = n_{22}\omega_1\,. \tag{11.35}$$

We combine the above two equations with (B.82) and find

$$[\varrho(\kappa + \tfrac{1}{6}\kappa_5^4\varrho) + 3\,E_{11} + \tfrac{1}{2}n_{22}^2 + 2n_{23}^2]\omega_1 = 0 \tag{11.36}$$

and compare this with (B.77) + (B.76):

$$\varrho(2\kappa + \tfrac{1}{6}\kappa_5^4\varrho) + E_{11} + \tfrac{1}{2}n_{22}^2 + 2n_{23}^2 + \tfrac{2}{3}\Lambda = 0 \tag{11.37}$$

which gives us the result

$$\kappa\varrho - 2\,E_{11} + \tfrac{2}{3}\Lambda = 0\,. \tag{11.38}$$

It also follows from taking gradients of (11.36) and (11.37) that

$$n_{22}\mathbf{e}_\alpha(n_{22}) + 4n_{23}\mathbf{e}_\alpha(n_{23}) = 0\,. \tag{11.39}$$

For $\alpha = 2, 3$, we get from (B.72) and (B.74) a division according to[8]

$$\textbf{A]}\;\; n_{22} = n_{23} = 0 \quad \text{or} \quad \textbf{B]}\;\; \mathbf{e}_A(n_{23}) = 2\varepsilon^{1B}{}_A a_B n_{22} \;\text{ and }\; \mathbf{e}_A(n_{22}) = -8\varepsilon^{1B}{}_A a_B n_{23}\,,\;.$$

A] $n_{22} = n_{23} = 0$: So now

$$n_{22} = n_{23} = H_{22} = H_{33} = H_{23} = 0\,. \tag{11.40}$$

We set

$$a_3 = 0 \tag{11.41}$$

on a 2-surface $x^0 = c^0$, $x^1 = c^1$ since

$$\mathbf{e}_0(a_3) = 0 \quad \text{from (B.19)}$$
$$\mathbf{e}_1(a_3) = 0 \quad \text{from (B.71)}\,.$$

We set

$$\mathbf{e}_3(a_2) = 0 \tag{11.42}$$

on a line $x^0 = c^0$, $x^1 = c^1$, $x^2 = c^2$ since

$$\mathbf{e}_0(\mathbf{e}_3(a_2)) = 0 \quad \text{from } [\mathbf{e}_0, \mathbf{e}_3](a_2) \text{ and (B.18)}$$
$$\mathbf{e}_1(\mathbf{e}_3(a_2)) = 0 \quad \text{from } [\mathbf{e}_3, \mathbf{e}_1](a_2) \text{ and (B.73)}$$
$$\mathbf{e}_2(\mathbf{e}_3(a_2)) = 10a_2\mathbf{e}_3(a_2) \quad \text{from } [\mathbf{e}_2, \mathbf{e}_3](a_2) \text{ and (B.75)}\,.$$

This is a Gödel-type universe. The commutators are

$$[\mathbf{e}_0, \mathbf{e}_\alpha] = 0 \tag{11.43}$$
$$[\mathbf{e}_2, \mathbf{e}_3] = 2\omega_1\mathbf{e}_0 + 2a_2\mathbf{e}_3 \tag{11.44}$$
$$[\mathbf{e}_1, \mathbf{e}_A] = 0\,. \tag{11.45}$$

The field equations now become

$$\tfrac{1}{2}\varrho(\kappa + \tfrac{1}{6}\kappa_5^4\varrho) = \omega_1^2 = -\Lambda\,. \tag{11.46}$$

[8] The indices $A\,, B = 2\,,3$.

B] $\mathrm{e}_A(n_{23}) = 2\varepsilon^{1B}{}_A a_B n_{22}$ **and** $\mathrm{e}_A(n_{22}) = -8\varepsilon^{1B}{}_A a_B n_{23}$: We set

$$a_3 = 0 \tag{11.47}$$

on a 2-surface $x^0 = c^0$, $x^1 = c^1$ since

$$\mathrm{e}_0(a_3) = 0 \quad \text{from (B.19)}$$
$$\mathrm{e}_1(a_3) = 3n_{23}a_3 \quad \text{from (B.71)}.$$

Now we apply the commutator $[\mathrm{e}_2, \mathrm{e}_3]$ firstly to n_{22} to find

$$n_{22}[\mathrm{e}_2(a_2) - 2a_2^2] = 0 \tag{11.48}$$

and then we apply the commutator $[\mathrm{e}_2, \mathrm{e}_3]$ on n_{23} and find

$$n_{23}[\mathrm{e}_2(a_2) - 2a_2^2] = 0\,. \tag{11.49}$$

Thus to avoid case **A]**, it follows that

$$\mathrm{e}_2(a_2) = 2a_2^2 \tag{11.50}$$

We set

$$\mathrm{e}_3(a_2) = 0 \tag{11.51}$$

on a line $x^0 = c^0$, $x^1 = c^1$, $x^2 = c^2$ since

$$\mathrm{e}_0(\mathrm{e}_3(a_2)) = 0 \quad \text{from } [\mathrm{e}_0, \mathrm{e}_3](a_2) \text{ and (B.18)}$$
$$\mathrm{e}_1(\mathrm{e}_3(a_2)) = 2n_{23}\mathrm{e}_3(a_2) \quad \text{from } [\mathrm{e}_3, \mathrm{e}_1](a_2) \text{ and (B.73)}$$
$$\mathrm{e}_2(\mathrm{e}_3(a_2)) = 6a_2\mathrm{e}_3(a_2) \quad \text{from } [\mathrm{e}_2, \mathrm{e}_3](a_2) \text{ and (B.75)}.$$

Furthermore, we set

$$\mathrm{e}_2(a_2) = 0 \tag{11.52}$$

at a point $x^0 = c^0$, $x^1 = c^1$, $x^2 = c^2$, $x^3 = c^3$ since

$$\mathrm{e}_0(\mathrm{e}_2(a_2)) = 0 \quad \text{from } [\mathrm{e}_0, \mathrm{e}_3](a_2) \text{ and (B.18)}$$
$$\mathrm{e}_1(\mathrm{e}_2(a_2)) = 2n_{23}\mathrm{e}_2(a_2) \quad \text{from } [\mathrm{e}_3, \mathrm{e}_1](a_2) \text{ and (B.73)}$$
$$\mathrm{e}_2(\mathrm{e}_2(a_2)) = 4a_2\mathrm{e}_2(a_2) \quad \text{clearly}$$
$$\mathrm{e}_3(\mathrm{e}_2(a_2)) = 0 \quad \text{from } [\mathrm{e}_2, \mathrm{e}_3](a_2) \text{ and (B.75)}.$$

If we now combine (B.77) + (B.76) with (B.75), we get

$$\tfrac{1}{4}E_{11} + \tfrac{3}{2}\omega_1^2 - \tfrac{1}{3}\varrho(\kappa + \tfrac{1}{12}\kappa_5^4\varrho) - \tfrac{1}{3}\Lambda = 0\,. \tag{11.53}$$

We now take (11.23), (11.32), (11.38) and (11.53) together to tell us that

$$\Lambda = -6E_{11} = 4\omega_1^2 = -\kappa\varrho = \tfrac{1}{12}\kappa_5^4\varrho^2 \tag{11.54}$$

In particular

$$\varrho(\kappa + \tfrac{1}{12}\kappa_5^4\varrho) = 0 \Rightarrow \varrho = 0 \tag{11.55}$$

which is not allowed by our assumptions.

Irrotational dust Shear-free irrotational dust is just the usual dust Robertson-Walker models with a higher order correction to the energy density. The dynamics will be exactly the same with just the appropriate corrections $\kappa\varrho \to \varrho(\kappa + \frac{1}{3}\kappa_5^4\varrho)$ in the Raychaudhuri equation and $\kappa\varrho \to \varrho(\kappa + \frac{1}{12}\kappa_5^4\varrho)$ in the Friedmann equation added.

11.2.2 Rotating Perfect Fluid

As a final comment, we try and get a feel for the dynamics of an accelerating fluid in the presence of vorticity.

If we now consider the full problem, with both $\mathcal{E}_{ab} \neq 0$, $\omega_{ab} \neq 0$, then the chance remains that the result $\omega\,\Theta = 0$ may *not* hold due to contamination from the bulk. While this would be a very interesting result, it will not change the main issue of *physical* importance regarding avoidance of the initial, Big Bang, singularity. Consider the Raychaudhuri equation (syn. momentum constraint) in the absence of shear[9]:

$$\mathrm{e}_0(\Theta) = -\tfrac{1}{3}\Theta^2 + 2\omega^2 - \tfrac{1}{2}\varrho(\kappa + \tfrac{1}{3}\kappa_5^4\varrho) + 2\mathcal{E}_{00} + \Lambda\,. \tag{11.56}$$

For a barotropic equation of state with $p = (\gamma - 1)\varrho$, conservation of energy will give $\varrho \propto a^{-3\gamma}$, where a is the average scale factor defined as $3H = 3\dot{a}/a = \Theta$. Hence, unless $\gamma \leq 1$ and we have large negative pressure, the ϱ^2 term will increase more rapidly than the vorticity as the singularity is approached since $\omega^2 \propto a^{-4\gamma}$ from the vorticity conservation equation (A.14). Of course, including pressure introduces new terms into the Raychaudhuri equation, but the vorticity and density terms are still the most dominant at early times relative to the other standard dynamical quantities. The one term that *can* make a difference is the contribution to the expansion evolution from the bulk tidal term; that is $\mathcal{E}_{00}$. This term may act in either direction in gravitational collapse and may grow more rapidly than any of the others possibly. But it may be again that negative pressure is required to allow the vorticity to spin up sufficiently rapidly to overcome the ϱ^2 term.

[9]Indeed the argument that follows holds if shear is present as well.

11.3 Conclusion

We have performed a non-perturbative tetrad analysis of the constraints arising from the Bianchi and Ricci identities showing that rotation cannot remove the initial singularity from shear-free dust brane cosmologies where the tidal interaction has been neglected. The general relativistic result $\omega\,\Theta = 0$ is shown to hold in the brane when the bulk has no tidal interaction with the brane, such as occurs with an AdS bulk. While these results *may* not hold for more general bulk geometries, the brane vorticity cannot spin up rapidly enough to avoid the initial singularity due to the quadratic high-energy corrections to the stress tensor.

Thus these brane cosmologies suffer from the same troubles[10] as General Relativity, and one must consider higher-order corrections to the *Einstein tensor* if one wishes to avoid the singularity. We may hypothesise that for situations involving more than one brane and no tidal interaction, our results would still hold, given that these models all essentially work by modifying only the energy momentum tensor.

There is much room here for further study in this framework we have developed. The UV problems of GR suggest that non-local modifications to the Einstein-Hilbert action will be necessary in order to unify it with quantum mechanics. The most immediately obvious and meaningful generalisation then would be to include non-local effects in our analysis, which would come from not assuming that the tidal term from the bulk vanishes; i.e. $\mathcal{E}_{ab} \neq 0$. However, we have encountered some technical problems in this regard as the equations which close up the "dynamical system" are not easily and transparently derived. This is as a result of our decision to take the brane universe theories at face value; we merely analyse the effect these universe models would have on our brane only and do not comment at all about the gravitational behaviour in the bulk (in full). Even so, we may proceed in a somewhat *ad hoc*, but satisfactory, way of including these non-local tidal effects by assuming that the spatial components of $\mathcal{E}_{ab}$ vanish (since the evolution equations for these quantities are what is lacking) and do the tetrad analysis as before. Indeed, this is work in progress at the time of writing.

[10] Indeed, the problems are more severe here, because in GR, the singularity may at least *in principle* be stopped by a strong enough vorticity.

Part IV

Inhomogeneous Observations

Chapter 12

Inhomogeneity or Source Evolution?

12.1 Introduction

The observational cosmology programme can be traced back to the seminal paper by Kristian and Sachs [50]. Their realisation was that fundamentally all observations take place on the null cone, and thus ideally we can reconstruct the observed universe by performing real observations and fitting the data to the theory on the null cone; thus, in principle, from there being able to determine the space-time geometry by extrapolation. This approach was adopted and taken much further by Ellis and collaborators [24]. Recently, with bold astronomical initiatives harnessing numerous technological advances, this programme has been partially realised and become a new and rapidly expanding field. Indeed many experiments are now being performed on what may be considered cosmological, as opposed to astronomical, scales. The standard approach for astronomers so far appears to have been that a homogeneous and isotropic universe model is sufficient to describe the observed data. However, the real observed universe is inhomogeneous and the observational cosmology approach appears ideally suited to pose and attempt to answer questions on this issue.

We reconsider the issue of proving large scale spatial homogeneity of the universe, given isotropic observations about us and the possibility of source evolution both in numbers and luminosities. Two theorems make precise the freedom available in constructing cosmological models that will fit the observations. They make quite clear that homogeneity cannot be proven without either a fully determinate theory of source evolution, or availability of distance measures that are independent of source evolution. We contrast this goal with the standard approach that assumes spatial homogeneity *a priori*, and determines source evolution functions on the basis of this assumption. Ever since the earliest cosmological models, the Einstein and de Sitter models, we have been trying to fit observations to the Friedmann-Lemaître-Robertson-Walker (FLRW) spatially homogeneous and isotropic family of models. The successes of reproducing a Hubble redshift-distance law, calculating the correct cosmic helium & deuterium abundances, and the prediction of a cosmic microwave background radiation (CMBR), have convinced us of its validity as a bulk description of the universe. However, *proving* that the geometry of the universe is FLRW on the largest scales is not easy. In fact, the history of observational cosmology shows that each time improved instruments permit deeper surveys, the new data soon reveals inhomogeneities on the new scale.

The best evidence for homogeneity comes from limits on anisotropy of both galaxy counts and the CMBR, obtained in each case by comparison of observations in different directions. However this is strictly speaking only evidence for isotropy about the earth; homogeneity follows only if we introduce a Copernican principle, either for galaxies or for the Cosmic Background Radiation [19, 15]. Without this assumption, the models indicated are isotropic about us, but allow a spatial variation of the geometry and matter content that is spherically symmetric about our position.[1] The Copernican principle is not really in dispute on a sub-horizon scale,[2] but could be incorrect on a super-horizon scale if we accept theories such as chaotic inflation ([53]; see also [20]). Therefore one would like to actually prove homogeneity for the observable region of the universe, rather than assuming it on principle, which is essentially what happens in the usual approach. Similar issues have been discussed by Goodman [34].

There are several problems with demonstrating homogeneity from observed data. The deeper observations are not only fainter and redshifted, they are also affected by proper motion, reddening and absorption due to interstellar matter. These contribute to selection effects which are tricky to compensate for. But the main problem at large distances is the evolution of sources, since deeper observations are received from earlier cosmic epochs. Evolution can take place in source colours, luminosities, and sizes; at high redshift it can affect the type of source as well as their numbers. However in this paper, for simplicity, we shall only consider bolometric observations of one type of source.

How does the number and brightness of the observable sources relate to the local density at different times? Recent evidence for a sharp fall off of the space density of quasars above a redshift of $z = 5$ [86, 71, 44, 73], *could* be taken as evidence of inhomogeneity, though most attribute it to source evolution. The large population of faint blue objects found by sensitive optical surveys is thought to be young starforming galaxies at high redshifts, and therefore constitute evidence for evolution [41, 61, 62] — *if* we assume the universe has a FLRW geometry. Without evolution, these observations are inconsistent with that geometry. One difficulty is that redshifts of faint objects are scarce and difficult to obtain. A redshift of $z \approx 2$ was deduced [60] by combining number counts versus magnitudes in 3 colours with galaxy evolution models and cosmological models, and comparing with the few measured redshifts available. Again this required the assumption of homogeneity.[3] Indeed, deducing the effects of source evolution by comparing observations with predictions in a FLRW model is a standard technique. Similarly studies of a luminosity-size relation also assume a FLRW model — recent examples are [74, 81]. However this cannot

[1] In a universe that is isotropic but inhomogeneous, there are anthropic reasons why we might be near the centre, as argued in [22].

[2] Although one can construe the current uncertainties in the values of cosmological parameters such as H_0 and Ω as being evidence for different values on different scales — i.e. inhomogeneity, we are not claiming this here.

[3] It turns out many of the sample are not at high redshift, but there is still a "high redshift tail" to the sample [66].)

lead to certainty [19]. The new discovery of a radio galaxy which is apparently an immature giant elliptical galaxy at $z = 4.41$ [69], provides a more striking example of probable source evolution. The problem is to show that this is not rather evidence of spatial inhomogeneity, manifested in a change of the evolutionary history or the nature of objects observed at larger spatial distances from us. Also, claims of a periodicity on top of the Hubble law in the redshift-distance relation [5] indicate significant deviations from standard FLRW observational relations, which could be due to spatial inhomogeneities (e.g. [16]) or to a temporal variation in the cosmic expansion rate.

If suitably smoothed observations are isotropic, the principal observations of discrete sources one can hope to make are the number counts $n(z)$ of sources as a function of "distance", conveniently taken as given by cosmological redshift z, and the magnitudes and angular diameters of sources, also as a function of redshift. If the assumed linear size and absolute luminosities of the sources are correct, the latter two give the luminosity and area distances $R(z)$, which should be equal. This is rather fortunate since in practice it is often difficult to separate the two measures — one has to define an edge of a galaxy image in order to measure its luminosity, and conversely, one often defines the edge relative to the central brightness; and both measures are significant in determining selection effects [25]. In any case, at the largest distances angular diameters cannot be measured, and it is the luminosity distance that is used.

We consider two types of source evolution; absolute luminosity $L(z)$, and mass per source $m(z)$, i.e. total density over source number density, which represents evolution in source number via sources turning on, galaxy mergers etc. Since source evolution is likely to be determined as a function of age τ, these functions could usefully be expressed as $L(\tau(z))$ and $m(\tau(z))$; however it is analytically easier to solve the observational equations if they are considered as functions of the observable z. This also helps to emphasise that if large-scale inhomogeneity were in fact to occur, the age of the universe would vary with spatial position and so becomes difficult to handle.[4]

Earlier work (section 15.3 of [24], section 7 of [76] and see also [56] where a serious error in [76] was corrected, in particular) showed that if the observational relations are isotropic and of the FLRW form, then the universe is indeed homogeneous, provided we can assume the matter stress tensor is that of pressure-free matter.[5] However that analysis did not fully consider the effects of source evolution.

In this work we show that any given isotropic set of observations $n(z)$ & $R(z)$, together with any given evolution functions $L(z)$ and $m(z)$, can be fitted by a spherically symmetric dust cosmology — a Lemaître-Tolman-Bondi (LTB) model — in which observations are spherically symmetric about us because we are located near the central world-line[6]. Thus we show that any spherically symmetric observations we may eventually make can be accommodated by appropriate inhomogeneities in a LTB model — irrespective

[4]Since inhomogeneity has been introduced, it is even possible the evolution functions are also position dependent.
[5]The result should also obtain for barotropic perfect fluids, but not necessarily for imperfect fluids.
[6]Despite claims to the contrary in the literature, this is a perfectly possible situation [19].

comoving coordinates is the Lemaître-Tolman-Bondi (LTB) [52, 80, 3] metric

$$ds^2 = -dt^2 + \frac{(R'(t,r))^2}{1+2E(r)}dr^2 + R^2(t,r)\,d\Omega^2, \tag{12.2}$$

where $R'(t,r) = \partial R(t,r)/\partial r$, and $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$. The function $R = R(t,r)$ is the areal radius, since the proper area of a sphere of coordinate radius r on a time slice of constant t is $4\pi R^2$.

The expansion Θ is

$$\Theta = \frac{2\dot{R}}{R} + \frac{\dot{R}'}{R'} \tag{12.3}$$

and the shear σ through which the inhomogeneity enters is given by

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R}\right). \tag{12.4}$$

Solving the Einstein field equations gives a generalised 'Friedmann' equation for $R(t,r)$,

$$\dot{R}(t,r) = \pm\sqrt{\frac{2M(r)}{R(t,r)} + 2E(r)}, \tag{12.5}$$

and an expression for the density

$$4\pi\rho(t,r) = \frac{M'(r)}{R^2(t,r)R'(t,r)} \tag{12.6}$$

Eq (12.5) can be solved in terms of a parameter $\eta = \eta(t,r)$:

$$R(t,r) = \frac{M(r)}{\mathcal{E}(r)}\phi_0(t,r), \qquad \xi(t,r) = \frac{(\mathcal{E}(r))^{3/2}\,(t - t_B(r))}{M(r)} \tag{12.7}$$

where[8]

$$\mathcal{E}(r) = \begin{cases} 2E(r), \\ 1, \\ -2E(r), \end{cases} \quad \phi_0 = \begin{cases} \cosh\eta - 1, \\ (1/2)\eta^2, \\ 1-\cos\eta, \end{cases} \quad \xi = \begin{cases} \sinh\eta - \eta, \\ (1/6)\eta^3, \\ \eta - \sin\eta, \end{cases} \quad \text{when} \begin{cases} E > 0 \\ E = 0 \\ E < 0 \end{cases} \tag{12.8}$$

for hyperbolic, parabolic and elliptic solutions respectively.

The LTB model is characterised by 3 arbitrary functions of coordinate radius r. $E = E(r) \geq -1/2$ has a geometric role, determining the local 'embedding angle' of spatial slices, and also a dynamic role, determining the local energy per unit mass of the dust particles, and hence the type of evolution of R. $M = M(r)$ is the effective gravitational mass with comoving radius r. $t_B = t_B(r)$ is the local time at which $R = 0$, i.e. the local time of the big bang — we have a non-simultaneous bang surface. Specification of these three arbitrary functions — $M(r)$, $E(r)$ and $t_B(r)$ — fully determines the model, and whilst all have some type of physical or geometric interpretation, they admit a freedom to choose the radial coordinate, leaving two physically meaningful choices, e.g. $r = r(M)$, $E = E(M)$, $t_B = t_B(M)$.

[8]Strictly speaking, the hyperbolic, parabolic and elliptic solutions obtain when $RE/M > 0$, $= 0$, & < 0 respectively, since $E = 0$ at a spherical origin in both hyperbolic and elliptic models.

12.2.1 The Observer's Null Cone

We now take R and ρ as given on the observer's past null cone, and we wish to express the 3 arbitrary LTB functions in terms of them, so characterising the LTB model that fits the observations.

We generalise the gauge choice used in [63] to the case where the spatial sections are in general non-flat, i.e. all values of E. Human observations of the sky are essentially a single event on cosmological scales, so we only need to be able to locate a single null cone; we don't need a general solution. On radial null geodesics, $ds^2 = 0 = d\theta^2 = d\phi^2$; so from (12.2) if the past null cone of the observation event $(t = t_0,\ r = 0)$ is given by $t = \hat{t}(r)$, then $\hat{t}$ satisfies

$$d\hat{t} = -\frac{R'(\hat{t}(r), r)}{\sqrt{1+2E}}\, dr = -\frac{\widehat{R'}}{\sqrt{1+2E}}\, dr\,. \tag{12.9}$$

We will denote a quantity evaluated on the observer's null cone, $t = \hat{t}(r)$, by a $\hat{}$; for example $R(\hat{t}(r), r) \equiv \hat{R}$. Now if we choose r so that, on the past light cone of (t_0, r),

$$\frac{R'(\hat{t}(r), r)}{\sqrt{1+2E}} = \frac{\widehat{R'}}{\sqrt{1+2E}} = 1\,, \tag{12.10}$$

then the incoming radial null geodesics are given by

$$\hat{t}(r) = t_0 - r\,. \tag{12.11}$$

With our coordinate choice (12.10), the density (12.6) and the Friedmann equation (12.5) become

$$4\pi\hat{\rho}\hat{R}^2 = \frac{M'}{\sqrt{1+2E}} \tag{12.12}$$

$$\hat{\dot{R}} = \pm\sqrt{\frac{2M}{\hat{R}} + 2E}\,. \tag{12.13}$$

The gauge equation is found from the total derivative of R on the null cone

$$\frac{d\hat{R}}{dr} = \widehat{R'} + \hat{\dot{R}}\,\frac{d\hat{t}}{dr} \tag{12.14}$$

and with (12.11) and (12.13) substituted, it follows that

$$\frac{d\hat{R}}{dr} - \sqrt{1+2E} = -\hat{\dot{R}} = -\pm\sqrt{\frac{2M}{\hat{R}} + 2E}\,. \tag{12.15}$$

When we solve this for $2E(r)$ by squaring both sides and rearranging, we get

$$1 + 2E = \left\{\frac{1}{2}\left[\left(\frac{d\hat{R}}{dr}\right)^2 + 1\right] - \frac{M}{\hat{R}}\right\}^2 \Big/ \left(\frac{d\hat{R}}{dr}\right)^2\,. \tag{12.16}$$

This expression will tell us under what circumstances (or for which regions) the spatial sections are hyperbolic $1 + 2E > 1$, parabolic $1 + 2E = 1$ or elliptic $1 + 2E < 1$, based on data obtained from the null cone.

We now use the expression for the density on the null cone to find a linear, first order differential equation for $M(r)$. Eliminating $1+2E$ between (12.16) and (12.12), we get

$$\frac{dM}{dr} + \left(4\pi\hat{\rho}\hat{R} \,/\, \frac{d\hat{R}}{dr}\right) M = \left(2\pi\hat{\rho}\hat{R}^2 \,/\, \frac{d\hat{R}}{dr}\right) \left[\left(\frac{d\hat{R}}{dr}\right)^2 + 1\right] . \tag{12.17}$$

By evaluating (12.7) and (12.8) on the null cone we find

$$\hat{R} = \frac{M}{\mathcal{E}}\,\hat{\phi}_0 \,, \qquad \hat{\xi} = \frac{\mathcal{E}^{3/2}\,\tau}{M} \tag{12.18}$$

where

$$\mathcal{E}(r) = \begin{cases} 2E(r), \\ 1, \\ -2E(r), \end{cases} \quad \hat{\phi}_0 = \begin{cases} \cosh\hat{\eta} - 1, \\ (1/2)\hat{\eta}^2, \\ 1 - \cos\hat{\eta}, \end{cases} \quad \hat{\xi} = \begin{cases} \sinh\hat{\eta} - \hat{\eta}, \\ (1/6)\hat{\eta}^3, \\ \hat{\eta} - \sin\hat{\eta}, \end{cases} \quad \text{when} \begin{cases} E > 0 \\ E = 0 \\ E < 0 \end{cases} \tag{12.19}$$

and

$$\tau(r) \equiv \hat{t}(r) - t_B(r) = t_0 - r - t_B(r) \tag{12.20}$$

can be interpreted as proper time from the bang surface to the past null cone along the particle world lines. Thus, with M given by (12.17) and then E by (12.16), we can solve for $\hat{\eta}$ from

$$\hat{\phi}_0 = \frac{\mathcal{E}\hat{R}}{M} \tag{12.21}$$

and (12.19), $\tau(r)$ from

$$\tau = \frac{M}{\mathcal{E}^{3/2}}\,\hat{\xi} \tag{12.22}$$

with (12.19) again, and hence $t_B(r)$ from (12.20).

12.2.2 Origin Conditions

At the origin of spherical coordinates, $r = 0$, where $R(t,0) = 0$ and $\dot{R}(t,0) = 0$ for all t, we assume that the density is non-zero, that the type of time evolution (hyperbolic, parabolic or elliptic) is not different from its immediate neighbourhood, and that all functions are smooth — i.e. functions of r have zero first derivative there. Thus eq (12.7) tells us that RE/M and $E^{3/2}/M$ must be finite at $r = 0$, (12.5) shows us that $E \to 0$ and hence $M \to 0$ and $E \sim M^{2/3}$ at $r = 0$. Eqs (12.14) and (12.15) become

$$\left.\frac{d\hat{R}}{dr}\right|_{r=0} = \widehat{R'}|_{r=0} = \left.\frac{d\hat{R}}{dr}\right|_{r=0} = \sqrt{1+2E} = 1 \,, \tag{12.23}$$

and thus $\hat{R} \sim r$ to lowest order near $r = 0$. From (12.12) we find

$$M' \approx 4\pi\hat{\rho}_0 r^2 \,, \qquad M \sim \frac{4}{3}\pi\hat{\rho}_0 r^3 \tag{12.24}$$

and so

$$E \sim \left(\frac{4}{3}\pi\hat{\rho}_0\right)^{2/3} r^2 \tag{12.25}$$

We verify these origin conditions satisfy (12.17) to order r^2 and (12.16) trivially to order r^0.

12.2.3 Redshift-distance Formula

We use the fact that in the geometric optics limit, for two light rays emitted on the worldline at r_{em} with time interval $\delta t_{em} = t^+(r_{em}) - t^-(r_{em})$ and observed on the central worldline with time interval $\delta t_{ob} = t^+(0) - t^-(0)$

$$1 + z = \frac{\delta t_{ob}}{\delta t_{em}}. \tag{12.26}$$

The incoming radial null geodesics are given by

$$dt = -R'(t,r) \,/\, \sqrt{1+2E}\; dr\,,$$

so for two successive light rays, $-$ & $+$, passing through two nearby comoving worldlines r_A & $r_B = r_A + dr$ at times t_A^-, t_B^-, t_A^+ & t_B^+

$$d(\delta t) = \delta t_B - \delta t_A = dt^+ - dt^- = \left[-R'(t^+,r) + R'(t^-,r)\right] \,/\, \sqrt{1+2E}\; dr$$

Consequently

$$d \ln \delta t = -\frac{\partial}{\partial t}\left[R'(t,r)\right] \,/\, \sqrt{1+2E}\; dr$$

which means that, integrating along the light ray and applying this to the log of (12.26), the redshift is given by

$$\ln(1+z) = \int_0^{r_{em}} \dot{R}'(t,r) \,/\, \sqrt{1+2E}\, dr \tag{12.27}$$

for the central observer at $r = 0$, receiving signals from an emitter at $r = r_{em}$.

We need to find the redshift z explicitly in terms of observables. We differentiate (12.5) with respect to r:

$$\frac{\dot{R}'}{\sqrt{1+2E}} = \frac{1}{\dot{R}}\left(\frac{M'}{R\sqrt{1+2E}} - \frac{MR'}{R^2\sqrt{1+2E}} + \frac{E'}{\sqrt{1+2E}}\right) \tag{12.28}$$

so when evaluated on the observer's past null cone, we get

$$\frac{\widehat{\dot{R}'}}{\sqrt{1+2E}} = \frac{1}{\hat{\dot{R}}}\left[\frac{M'}{\hat{R}\sqrt{1+2E}} - \frac{M}{\hat{R}^2} + (\sqrt{1+2E}\,)'\right] \tag{12.29}$$

Now, from (12.16), the derivative of $\sqrt{1+2E}$ is given by

$$(\sqrt{1+2E}\,)' = \frac{d^2\hat{R}}{dr^2} - M' \,/\, \left(\hat{R}\frac{d\hat{R}}{dr}\right) + \frac{M}{\hat{R}^2} - \sqrt{1+2E}\,\frac{d^2\hat{R}}{dr^2} \,/\, \frac{d\hat{R}}{dr} \tag{12.30}$$

so, after eliminating M' by substituting from equation (12.12), it follows that

$$\begin{aligned} \frac{\widehat{\dot{R}'}}{\sqrt{1+2E}} &= \frac{1}{\hat{\dot{R}}}\left(4\pi\hat{\rho}\hat{R} - 4\pi\hat{\rho}\hat{R}\sqrt{1+2E} \,/\, \frac{d\hat{R}}{dr} + \frac{d^2\hat{R}}{dr^2} - \frac{d^2\hat{R}}{dr^2}\sqrt{1+2E} \,/\, \frac{d\hat{R}}{dr}\right) \\ &= -\left(4\pi\hat{\rho}\hat{R} + \frac{d^2\hat{R}}{dr^2}\right) \,/\, \left(\frac{d\hat{R}}{dr}\right) \end{aligned} \tag{12.31}$$

where we have used equation (12.15) to provide the second equality. From (12.27) it now follows that

$$\frac{d}{dr}\left[\ln(1+z)\right] = -\left[\frac{d^2\hat{R}}{dr^2} + 4\pi\hat{\rho}\hat{R}\right] \Big/ \left(\frac{d\hat{R}}{dr}\right) , \qquad z(0) = 0 . \tag{12.32}$$

which theoretically gives the redshift in terms of coordinate radius r, directly from $\hat{R}(r)$ and $\hat{\rho}(r)$, viz

$$\ln(1+z) = -\int_0^r \left[\frac{d^2\hat{R}}{dr^2} + 4\pi\hat{\rho}\hat{R}\right] \Big/ \left(\frac{d\hat{R}}{dr}\right) dr . \tag{12.33}$$

However, we will be given observations in terms of z, rather than the unobservable coordinate r. This will be addressed in the next section.

12.3 Observables and Source Evolution

For simplicity we shall confine ourselves to one type of cosmic source and only consider bolometric luminosities. We do not include a dark matter component explicitly. We shall assume that the luminosity of each source can evolve with time, and that the number density of sources can also evolve. The former we write as an absolute bolometric luminosity L, and the latter we shall represent as an evolving mass per source, m, which gives the total local density when multiplied by the source number density. As mentioned, we assume isotropy about the earth (once our proper motion has been accounted for), and also that the post decoupling universe is well described by zero pressure matter — "dust". The particles of this dust are galaxies (or perhaps clusters of galaxies). This means we can use the simplest inhomogeneous cosmology — the LTB metric, which is spherically symmetric and inhomogeneous in the radial direction only, and is written in comoving coordinates.

The two source evolution functions are most naturally expressed as functions of local proper time since the big bang, $L(\tau)$ and $m(\tau)$. However, in a LTB model the time of the bang may vary from point to point, so that the age of objects at redshift z is uncertain both because the bang time is uncertain and because the location of the null cone is uncertain. The proper time from bang to null cone will be a function of redshift, $\tau(z)$, and the projections of the evolution functions on the null cone we will write as $\hat{L}$ and $\hat{m}$. Of course, $\tau(z)$ is unknown until we have solved for the LTB model that fits the data. However, for the sake of simplicity, we will take $\hat{L}$ and $\hat{m}$ to be given as functions of z, to illustrate how the 3 quantities, cosmic evolution, cosmic spatial variation, and source evolution are mixed together in the luminosity and number count observations, ℓ and n. A treatment dealing with evolution functions based on τ would involve solving a much more complicated set of differential equations in parallel.

12.3.1 Relating Observables to the LTB Model

The area distance or equivalently the diameter distance is the true linear extent of the source over the measured angular size. This is by definition the same as the areal radius in the LTB model R, which multiplies the angular displacements to give proper distances tangentially. The projection onto the observer's null cone gives the observable quantity $\hat{R}$. The luminosity distance is theoretically the same as the diameter distance [18], and is measurable provided we know the true absolute luminosity of the source at the time of emission $\hat{L}$. If the observed apparent luminosity is $\ell(z)$ then

$$\hat{R}(z) = \sqrt{\frac{\hat{L}}{\ell}}. \tag{12.34}$$

Let the observed number density of sources in redshift space be $n(z)$ per steradian per unit redshift interval, so that the number observed in a given redshift interval and solid angle is

$$n d\Omega dz \tag{12.35}$$

and over the whole sky this is

$$4\pi n dz\,. \tag{12.36}$$

Thus the total rest mass between z and $z + dz$ is

$$4\pi \hat{m} n dz \tag{12.37}$$

where $\hat{m}(z) = m(\tau(z))$ is the mass per source — i.e. the true density over the source number density. This primarily represents the evolution in the number density of sources. Given a local proper density $\rho = \rho(t, r)$, and its value on the null cone $\hat{\rho}$, the total rest mass between r and $r + dr$ is

$$\hat{\rho}\widehat{d^3V} = \hat{\rho}\frac{4\pi \hat{R}^2 \widehat{R'}}{\sqrt{1+2E}}dr \tag{12.38}$$

where $\widehat{d^3V}$ is the proper volume on a constant time slice, evaluated on the null cone. Hence by (12.37), (12.38) and (12.10)

$$\hat{R}^2\hat{\rho} = \hat{m}n\frac{dz}{dr}. \tag{12.39}$$

Thus we may substitute for $\hat{R}$ and $\hat{\rho}$ from (12.34) and (12.39).

We transform (12.32) to be in terms of redshift z instead of coordinate r by writing it as

$$\frac{d\hat{R}}{dr}\frac{dz}{dr} + \frac{d^2\hat{R}}{dr^2}(1+z) + 4\pi\hat{\rho}\hat{R}(1+z) = 0$$

and applying

$$\frac{d\hat{R}}{dz}\frac{dz}{dr} = \frac{d\hat{R}}{dr}, \qquad \frac{d^2\hat{R}}{dr^2} = \frac{d\hat{R}}{dz}\frac{d^2z}{dr^2} + \frac{d^2\hat{R}}{dz^2}\left(\frac{dz}{dr}\right)^2$$

to get

$$\frac{d\hat{R}}{dz}\frac{d^2z}{dr^2}(1+z) + \left[\frac{d^2\hat{R}}{dz^2}(1+z) + \frac{d\hat{R}}{dz}\right]\left(\frac{dz}{dr}\right)^2 = -4\pi\hat{\rho}\hat{R}(1+z) \tag{12.40}$$

Integrating with respect to r and using (12.39) gives

$$\int_0^z \frac{d}{dr}\left[\frac{d\bar{z}}{dr}\frac{d\hat{R}}{d\bar{z}}(1+\bar{z})\right] dr = -\int_0^z 4\pi\hat{\rho}(\bar{z})\hat{R}(\bar{z})(1+\bar{z})\frac{dr}{d\bar{z}}d\bar{z} \tag{12.41}$$

$$\frac{dz}{dr}\frac{d\hat{R}}{dz}(1+z) - 1 = -4\pi\int_0^z \frac{\hat{m}(\bar{z})n(\bar{z})}{\hat{R}(\bar{z})}(1+\bar{z})\,d\bar{z} \tag{12.42}$$

and we used the origin conditions $[(dz/dr)(d\hat{R}/dz)]_0 = [d\hat{R}/dr)]_0 = 1$, and $z(0) = 0$. It follows that

$$\frac{dz}{dr} = \left[\frac{d\hat{R}}{dz}(1+z)\right]^{-1}\left\{1 - 4\pi\int_0^z \frac{\hat{m}(\bar{z})n(\bar{z})}{\hat{R}(\bar{z})}(1+\bar{z})\,d\bar{z}\right\} \tag{12.43}$$

Note that this equation differs from the analogous one in Stoeger et al [76][9] — their equation (32) — by a factor of $(1+z)$, and perhaps aptly illustrates the difference in the coordinate systems. To get the full model we have to solve the null Raychaudhuri equation (12.40) to get $r(z)$ (and thus $z(r)$). Equation (12.43) is a first integral of (12.40). This has to be integrated one more time to obtain $r(z)$. We must also specify boundary conditions at the origin $r = 0$, which we have already used in getting to (12.43):

$$\frac{dz}{dr}(0) = \frac{dz}{d\hat{R}}(0)\frac{d\hat{R}}{dr}(0) = 1\Big/\frac{d\hat{R}}{dz}(0)$$

and also

$$z(0) = 0 \quad \Leftrightarrow \quad r(z=0) = 0\,.$$

so that, integrating dr/dz gives

$$r(z) = \int_0^z \left[\frac{d\hat{R}}{d\tilde{z}}(1+\tilde{z})\right]\left\{1 - 4\pi\int_0^{\tilde{z}} \frac{\hat{m}(\bar{z})n(\bar{z})}{\hat{R}(\bar{z})}(1+\bar{z})\,d\bar{z}\right\}^{-1} d\tilde{z}\,. \tag{12.44}$$

12.4 The Theorems

Theorem (A): Subject to the conditions of appendix D, for any given isotropic observations $\ell(z)$ & $n(z)$ with any given source evolution functions $\hat{L}(z)$ & $\hat{m}(z)$, a set of LTB functions can be found to make the LTB observational relations fit the observations.

Proof: — Algorithm (A): To obtain the LTB mass, energy and bangtime functions (M, E and t_B respectively) from observational data and source evolution we would proceed as follows.

- Take the discrete observed data for $\ell(z,\theta,\phi)$ and $n(z,\theta,\phi)$, average it over all angles to obtain $\ell(z)$ and $n(z)$, and fit it to some smooth analytic functions, such as polynomials. We may wish to first correct the data for known distortions and selection effects due to proper motions, absorption, shot noise, image distortion, etc;

[9]There they use M_0 which equals $8\pi\hat{m}n/\hat{R}^2$ in the current notation.

- Choose evolution functions $\hat{L}(z)$ and $\hat{m}(z)$ based on whatever theoretical arguments may be mustered;
- Determine $\hat{R}(z)$ from $\hat{L}(z)$ and $\ell(z)$ using (12.34);
- Solve (12.44) for $r(z)$ and hence $z(r)$, then convert functions of z to functions of r — see appendix D for existence conditions;
- Solve (12.17) and (12.39) for $M(r)$ — existence conditions are given in appendix D;
- Determine $E(r)$ from (12.16);
- Solve for $\hat{\eta}$ from (12.21) and (12.19);
- Solve for $\tau(r)$ from (12.22) and (12.19) — $L(\tau)$ and $m(\tau)$ could now be found;
- Determine $t_B(r)$ from (12.20).

In practice, these equations would be solved numerically, and in parallel rather than sequentially, nevertheless the above would determine the numerical procedure within each integration step. □

By determining the 3 arbitrary functions, we have specified the LTB model that fits the given observations and evolution functions. This result simply asserts we can construct a (generally inhomogeneous) spherically symmetric exact solution of the field equations that will fit any given source observations combined with any chosen source evolution functions.

We assert, without proof, that if the given observations and source evolution functions are reasonable, then the LTB arbitrary functions will generate a reasonable LTB model. Our definition of 'reasonable' is intentionally rather vague. By reasonable observations we obviously include the actual data, suitably processed to account for selection effects. We also include 'realistic' hypothetical alternatives, but not functions that are wildly different from reality. Reasonable evolution functions are hard to define since the actual ones are not well known, especially at larger z values. By a reasonable LTB model, we mostly mean that the density and expansion rate will be within realistic ranges. A less crucial criterion is that there will be no shell crossings too close to the past null cone. Evolving the model a long time away from the null cone, either forwards or backwards, may introduce shell crossings because the data is imprecise. In general we don't expect shell crossings on the large scale — i.e. two or more different large scale flows of galaxies in the same region — nevertheless it is conceivable and in that case the LTB description is inapplicable.[10]

Corollary (B): A LTB model can be found to fit the observations with zero evolution — $\hat{m}$ = constant, $\hat{L}$ = constant.

[10]Data can be extended through a shell crossing [8], but not within the LTB formalism.

Proof: This is an obvious consequence of (A). □

Given realistic data, these models will be inhomogeneous. Indeed this is the reason that non-zero evolution functions have been introduced (otherwise, observations are incompatible with a FLRW universe).

Theorem (C): Subject to the conditions of appendix D, for any given isotropic observations $\ell(z)$ & $n(z)$, and any given LTB model, source evolution functions $\hat{L}(z)$ & $\hat{m}(z)$ can be found that make the LTB observational relations fit these observations.

Proof: — Algorithm (C): We adapt the above algorithmic procedure to prove this.

- As before, average the data over all angles, and fit it to smooth functions $\ell(z)$ and $n(z)$;
- Specify two of the three functions $M(r)$, $E(r)$ and $t_B(r)$, the third being determined by the coordinate condition (12.10). It seems expedient to choose $M(r)$ and $E(r)$.
- Determine $\hat{R}(r)$ from the first order differential equation in $\hat{R}$ and its r derivative — equation (12.15).[11] The functions should be chosen to satisfy the origin conditions of section 12.2.2 — see existence conditions in appendix D;
- Calculate $\hat{\rho}(r)$ from (12.12);
- Solve for $t_B(r)$ as well as $\tau(r)$ from (12.21), (12.22) and (12.20)) with (12.19) defining $\hat{\eta}(r)$;
- Integrate (12.33) to get $z(r)$ — appendix D gives the existence conditions;
- Use the given $\ell(z)$ and $n(z)$, to find $\hat{L}(z)$ from (12.34) and $\hat{m}(z)$ from (12.39). From these and $\tau(z)$ solve for $L(\tau)$ and $m(\tau)$, if needed. □

Again we assert that if the given observations and LTB model are 'reasonable', then the derived evolution functions will be 'reasonable'. The idea is that we can vary the LTB model to which we fit the observations to some extent, but still keep the required source evolution functions within a 'realistic' range.

Corollary (D): Source evolution functions can be found that make the dust FLRW observational relations fit any observations.

[11] Though we don't strictly know the sign of $\widehat{\dot{R}} = \sqrt{2M/\hat{R} + 2E}$, it is fairly safe to assume it is positive on our past null cone on the large scales we are considering.

Proof: An obvious consequence of (C). □

Loosely put these theorems say

(i) You can always fit isotropic observations with an LTB model, whatever the source evolution;

(ii) If you fiddle the source evolution hard enough, you can fit the observations to any LTB or dust FLRW model.

Although theorem (C) is an extreme case, and is likely to generate highly unphysical evolution functions if the LTB model is chosen arbitrarily, it is just a generalisation of (D) which is regularly used in an attempt to determine evolution functions from cosmological observations. Theorem (C) highlights the dangers of this approach.

A complication arises if the redshift is not monotonically increasing with distance. We have seen from the well behaved numerical example in [63] for a parabolic case, that $\hat{R}(z)$ and $\hat{\rho}(z)$ may not be single valued, and that the $\hat{R} - z$ and $\hat{\rho}$–z plots can loop. However, in compiling the real observational data, we merely add all the galaxies we see at a particular redshift, to get a number count. Similarly, we merely take an average over the luminosities observed at a particular redshift, ascribing the variation to natural scatter in intrinsic properties and observational error, rather than to a multiply valued function. Thus we make $\hat{R}(z)$ and $\hat{\rho}(z)$ single valued by construction. So the data functions we are trying to fit may not lead to such a good model. In other words, assuming we succeed in constructing a well behaved LTB model from the data, it may not be the LTB model that best represents the real universe. It seems unlikely — though not entirely impossible — that there will be a reliable way of de-convolving the superposed parts of these observational data curves, or even of discerning whether loops are present. It is hard to predict how likely this scenario is.

12.5 Conclusion

We have shown that a LTB model (a Lemaître-Tolman-Bondi spherically symmetric dust cosmology) can be found to fit any given set of observations of source counts $n(z)$ and luminosity/area distance $\hat{R}(z)$, averaged over all angles, and any evolution functions for source luminosity $\hat{L}(z)$ and mass per source $\hat{m}(z)$. In other words, even if we accept isotropy, then demonstrating homogeneity — rather than assuming it must hold because of the Copernican principle — requires more than these observations. Conversely, our result can be used to determine the degree of inhomogeneity from the observations and given source evolution functions.

If the demonstration of homogeneity depends on knowing the source evolution, and validation of source evolution theories depends on knowing the cosmological model is homogeneous, then neither is proved. Thus we need methods of validating source evolution models that don't depend on assumptions of homogeneity to establish the age at any given z. Similarly deep cosmological distance measures that don't depend on luminosity and are not influenced by source evolution would help pin down the cosmological model better. There are various promising developments, in particular:

(a) distance measurement by Supernovae;

(b) determinations of cosmological parameters via gravitational lensing measurements;

(c) accurate measurements of the Sunyaev-Zel'dovich effect;[12]

(d) observation of CMBR Doppler peaks by the MAP and COBRAS/SAMBA[13] satellites. This will only determine parameters in the neighbourhood of $z = 1000$, but is independent of source evolution all the same.

(e) the increasing number of source evolution studies that look for tell-tale signs of early stages of galaxy evolution, such as intense star formation, etc.

Once again, the FLRW assumption is usually if not always made in analyses of these effects. A re-analysis that permits inhomogeneity would be very worthwhile, as these techniques may well provide information complementary to the principal cosmological measures, that would help separate out the effects of cosmic evolution, spatial inhomogeneity, and source evolution. Some of these issues are discussed in [34].

In fact, it is already difficult to constrain the values of H_0, q_0 and Λ within a homogeneous dust model because of the uncertainty in source evolution, as pointed out in [49]. In this case the value of Λ affects the time evolution of the scale factor, and so the deviation of the angular diameter-redshift relation from expectation for a $\Lambda = 0$ FLRW model could be due to non-zero Λ or to source evolution. Similarly the possible presence of non-baryonic dark matter — or for that matter, the possibility that gravity obeys field equations other than Einstein's — could significantly affect the cosmic time evolution, and introduce further uncertainty.

The introduction of multi-colour observations does not resolve the problem in any simple way. If we have observations in various colour bands — say U B & V — then we must replace the source luminosity evolution function by a set of evolution functions for the luminosity in each colour. Thus, if we find deviations of the observations from FLRW expectations, we still have a freedom to attribute this either to inhomogeneity or to source evolution. It's true that young galaxies with lots of star formation are very blue. But, having introduced colour observations, and permitted evolution in colour, we must also admit the possibility of spatial inhomogeneities in the intrinsic colours of sources. We come back to the same problem — are the differences between observations in different colours due to source evolution or spatial

[12] Indeed a very interesting option is to use the bounds on the inhomogeneity obtained from the Sunyaev-Zel'dovich effect to constrain the LTB model chosen, bearing in mind that this method still suffers from excessive error due to absorption effects.

[13] i.e. Planck Survey

of what source evolution may occur[7]. Conversely we show that, given any spherically symmetric geometry and any set of observations, we can find evolution functions that will make the model compatible with the observations.

The purpose is to demonstrate explicity — developing the ideas in [19] — that the relationship between the large scale isotropy of observations and large scale cosmic homogeneity is weaker than is commonly assumed. Indeed, apart from any other problems, we can't have a good demonstration of homogeneity without observational tests of our source evolution theories that are independent of cosmological model, or distance measures that are not influenced by source evolution. This emphasises the conclusion that if the demonstration of homogeneity depends on knowing the source evolution, and validation of source evolution theories depends on knowing the cosmological model is homogeneous, then neither is proved. Indeed if we do not make the FLRW assumption, our results can be used to determine the degree of inhomogeneity from the observations and any given source evolution functions. If we do make the FLRW assumption, they can be used to determine the source evolution functions required to make the observations compatible with that model. *The latter is the way theory is usually run.* The point of our work is to emphasise that there are other options, and so such source evolution results should be viewed with caution.

12.2 The LTB Model and its Null Cone

We here outline the metric and our notation and null cone solution; for more details in this notation see [63]. The model falls under the class II LRS solutions in Ellis [17], a non-rotating twist-free family of solutions.

The universe is spherically symmetric, but in general radially inhomogeneous. This means that it is isotropic about one worldline – the centre of symmetry. The field equations are

$$\begin{aligned}
2\frac{\dot{C}\dot{R}}{CR} + \frac{1+\dot{R}^2}{R^2} - \frac{1}{C^2}\left(2\frac{R''}{R} + \frac{R'^2}{R^2} - 2\frac{C'R'}{CR}\right) &= \kappa {T^t}_t = 8\pi G\rho \\
2\frac{\ddot{R}}{R} + \frac{1+\dot{R}^2}{R^2} - \frac{R'^2}{C^2R^2} &= \kappa {T^r}_r = 0 \\
\frac{\ddot{C}}{C} + \frac{\ddot{R}}{R} + \frac{\dot{C}\dot{R}}{CR} - \frac{1}{C^2}\left(\frac{R''}{R} - \frac{C'R'}{CR}\right) &= \kappa {T^\theta}_\theta = \kappa {T^\phi}_\phi = 0 \\
-2\left(\frac{\dot{R}'}{R} - \frac{\dot{C}R'}{CR}\right) &= \kappa {T^t}_r = 0\,. \qquad (12.1)
\end{aligned}$$

The general spherically symmetric metric for an irrotational dust matter source in synchronous

[7] This is within the spirit of the programme of observational cosmology set out by Kristian and Sachs in their fundamental work [50] and developed further by Ellis, Stoeger, et al [24].

inhomogeneity? The only difference here is that cosmic evolution is fairly easily factored out, as the redshift is measured.

We are not here asserting that the observable universe *is* inhomogeneous, nor are we suggesting that source evolution studies that assume homogenity are not worthwhile.

The purpose of this work is to emphasise that we don't have unquestionable evidence for spatial homogeneity; and that we can't have a good demonstration of homogeneity — or even homogeneity on average — without a reliable theory of source evolution, supported by measurements that are independent of cosmological model, and/or cosmic distance measures that don't depend on knowing the luminosity evolution of sources. Our best basis for assuming spatial homogeneity is the Stoeger-Maartens-Ellis theorem or "almost EGS theorem" [77], which says that, if the universe is expanding and the CMBR (cosmic microwave background radiation) is almost isotropic for all observers since decoupling, then the universe is almost homogeneous, and more specifically, the scale of CMBR anisotropy puts a limit on the degree of cosmic inhomogeneity. But this result depends on a weak form of the Copernican principle[14]; and however convincing that principle is in general terms, we shouldn't overstate it. This line of thought says that the earth is just another planet around the sun, but it doesn't say all planets are the same size or composition. It says that our galaxy and our supercluster are one among many, but allows several types of galaxy and considerable variety in galaxy clustering. Thus the principle does not insist on uniformity on any scale, or even that the observable portion of the universe has a density particularly close to the "global average" — assuming we can define such a thing. And above all, while it may be true in the real universe, it is also possible that this is not so.

Recently, high-redshift Type Ia supernovae has caused excitement because of their potential as standard candles. Given that the universe is homogeneous this data can be used to determine the cosmological parameters. Célérier [7] showed, using an LTB model as an example, that the present data can be taken as implying a positive cosmological constant in a homogeneous universe or, on the other hand, as evidence of inhomogeneity; indeed, these data in no way constrain Λ in an inhomogeneous universe.

We are entitled to deduce homogeneity on the basis of untested philosophical principles, such as a Copernican principle; but we must be quite clear what we are doing when we make such a deduction, and how it relates to possible observational tests. This work helps throw light on the latter issue.

[14] On the other hand, this form of the weak Copernican principle is partially testable and falsifiable via the Sunyaev-Zel'dovich effect [55].

Part V

Chapter 13

Final Word

To what purpose should I trouble myself in searching out the secrets of the stars, having death or slavery continually before my eyes? – Anaximenes to Pythagoras

We would like to just briefly summarise the work done. The results of the three individual pieces of work have already been discussed in some detail in Chapters 9, 11 and 12 where options for further study was also mentioned. Perhaps what we can do here is just repeat in a more concise manner what was said there in each chapter.

We have made some progress in trying to characterise classical GR spacetimes where there are degeneracies in certain tensor quantities, which is not assumed for their covariant derivatives. We showed that (under some additional assumptions in the more general case of the shear and gravito-electric field RS only) that this class is small. Indeed the only inhomogeneous PLRS spacetime is the Szekeres spacetime. In the process we have also strengthened the result ascribed to Ellis that three derivatives of the Riemann tensor are needed to characterise local rotational symmetry; only two are required, in fact.

Our next study concerned a popular and topical issue; that of brane cosmologies. We demonstrated that these models which are essentially different to classical GR, in the sense that effects can propagate through the bulk and interact non-locally with the brane, suffer from similar problems to GR in the profound aspects of the theory. Certain classical results from GR, which do not hold in Newtonian gravity, carry over to the new scenario unchanged. As mentioned in the conclusion to that work, there is much scope for further study here.

Our final piece of work had a cautionary flavour to it. It is a demonstration of how the non-linearity of classical GR complicates issues. We showed that source evolution theories are incomplete without considering the effects that inhomogeneity should certainly have on individual models. We used a spherically symmetric dust solution of the EFE to model the post-decoupling universe, which is admissible if one suspends belief in a Copernican hypothesis. Thus it is perhaps also a comment on this fundamental assumption in cosmology. We suggest that the time is perhaps right for us to try and find ways of proving or disproving this hypothesis.

All in all, we have tried to improve our understanding of the universe by using theories, incorporating models, which are complex enough to give us a feel for what things should be like in the real universe, yet simple enough for us to make some headway in analytical studies.

Part VI

Appendices

Appendix A

Explicit Orthonormal Frame Expressions

A.1 Ricci rotation coefficients

In an ONF, the Ricci rotation coefficients are antisymmetric in the first and last indices (with our convention for defining the Riemann tensor). That is

$$\Gamma_{abc} = -\Gamma_{cba}$$

They are related to the commutation functions $\gamma^a{}_{bc}$ by

$$\Gamma_{abc} = \frac{1}{2}\left(\gamma_{abc} + \gamma_{cab} - \gamma_{bca}\right)$$

The nontrivial components are listed below.

$$\begin{array}{ll}
\Gamma_{100} = \dot{u}_1 & \Gamma_{102} = \Omega^3 \\
\Gamma_{200} = \dot{u}_2 & \Gamma_{103} = -\Omega^2 \\
\Gamma_{300} = \dot{u}_3 & \Gamma_{203} = \Omega^1 \\
\Gamma_{110} = \sigma_{11} + \frac{1}{3}\theta & \Gamma_{123} = \frac{1}{2}(n^1{}_1 + n^3{}_3 - n^2{}_2) \\
\Gamma_{220} = \sigma_{22} + \frac{1}{3}\theta & \Gamma_{132} = \frac{1}{2}(-n^1{}_1 - n^2{}_2 + n^3{}_3) \\
\Gamma_{330} = \sigma_{33} + \frac{1}{3}\theta & \Gamma_{213} = \frac{1}{2}(-n^2{}_2 - n^3{}_3 + n^1{}_1) \\
\Gamma_{120} = -\omega_3 + \sigma_{12} & \Gamma_{112} = n^1{}_3 - a_2 \\
\Gamma_{130} = \omega_2 + \sigma_{13} & \Gamma_{113} = -n^1{}_2 - a_3 \\
\Gamma_{210} = \omega_3 + \sigma_{12} & \Gamma_{122} = n^2{}_3 + a_1 \\
\Gamma_{230} = -\omega_1 + \sigma_{23} & \Gamma_{133} = -n^2{}_3 + a_1 \\
\Gamma_{310} = -\omega_2 + \sigma_{13} & \Gamma_{223} = n^1{}_2 - a_3 \\
\Gamma_{320} = \omega_1 + \sigma_{23} & \Gamma_{233} = n^1{}_3 + a_2
\end{array} \tag{A.1}$$

A.2 Commutation functions

The commutation functions are antisymmetric in their lower indices

$$\gamma^a{}_{bc} = -\gamma^a{}_{cb}\,.$$

They can be obtained from the Ricci rotation coefficients Γ_{abc} by

$$\gamma^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb}$$

The nontrivial components (in an ONF) are listed below.

$$\begin{array}{ll}
\gamma^0{}_{01} = \dot{u}_1 & \gamma^1{}_{02} = \Omega^3 + \omega_3 - \sigma_{12} \\
\gamma^0{}_{02} = \dot{u}_2 & \gamma^1{}_{03} = -\Omega^2 - \omega_2 - \sigma_{13} \\
\gamma^0{}_{03} = \dot{u}_3 & \gamma^2{}_{03} = \Omega^1 + \omega_1 - \sigma_{23} \\
\gamma^0{}_{12} = 2\omega_3 & \gamma^3{}_{01} = \Omega^2 + \omega_2 - \sigma_{13} \\
\gamma^0{}_{13} = -2\omega_2 & \gamma^3{}_{02} = -\Omega^1 - \omega_1 - \sigma_{23} \\
\gamma^0{}_{23} = 2\omega_1 & \gamma^2{}_{01} = -\Omega^3 - \omega_3 - \sigma_{12} \\
\gamma^1{}_{01} = -(\sigma_{11} + \frac{1}{3}\theta) & \gamma^3{}_{23} = n^1{}_3 + a_2 \\
\gamma^2{}_{02} = -(\sigma_{22} + \frac{1}{3}\theta) & \gamma^3{}_{13} = -n^2{}_3 + a_1 \\
\gamma^3{}_{03} = -(\sigma_{33} + \frac{1}{3}\theta) & \gamma^2{}_{23} = n^1{}_2 - a_3 \\
\gamma^3{}_{12} = n^3{}_3 & \gamma^2{}_{12} = n^2{}_3 + a_1 \\
\gamma^2{}_{13} = -n^2{}_2 & \gamma^1{}_{13} = -n^1{}_2 - a_3 \\
\gamma^1{}_{23} = n^1{}_1 & \gamma^1{}_{12} = n^1{}_3 - a_2
\end{array} \quad (A.2)$$

A.3 Evolution and constraint equations for the reduced set

The equations here are essentially obtained by writing out the independent components of the Jacobi identity (3.39), Riemann curvature tensor (3.12) and second Bianchi identity (2.50)[1] subject to the symmetry requirements listed below, germane to the whole programme discussed in the Introduction. The equations are written in terms of $\mathbf{E}$ and $\mathbf{H}$ as well as μ and p and thus we also require (2.47), (2.48), (2.45) and (2.49), incorporating the EFE (2.13). We denote each equation obtained from the Jacobi relations (3.39) by $\binom{a}{bcd}$, following [17]. The reason for this labelling is because the Jacobi identity is equivalent to the Riemann tensor symmetry $R_{a[bcd]} = 0$. The equations from the second Bianchi identity are labelled $\boxed{[abc]\,de}$ according to the indices in $\nabla_{[a}R_{bc]de} = 0$. There are some equations which are linear combinations of others in the set. They can be traced by utilising the fact that $\boldsymbol{\sigma}$, $\mathbf{E}$ and $\mathbf{H}$ are all tracefree. The cyclic Riemann symmetry $R_{a[bcd]} = 0$ is equivalent to (A.51). The generalised Friedmann equation is given by (A.67) + (A.68) + (A.69). The Raychaudhuri equation (A.13) is equivalent to (A.10) + (A.11) + (A.12).

Restrictions imposed: All equations that are to follow are written out for an expanding spacetime geometry with a perfect fluid matter source that are subject to the specific restrictions[2] that

- A matter-comoving ONT $\{\mathbf{e}_a\} = \{\mathbf{u}, \mathbf{e}_\alpha\}$ which diagonalises $\boldsymbol{\sigma}$ is chosen,

[1] We have to take into account the following symmetries of the Riemann curvature tensor:

$$R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc}\ .$$

[2] The 1+3 orthonormal frame equations for perfect fluid spacetime geometries in matter-comoving description *without* the specialisation stated are available online at [27].

- $\boldsymbol{\sigma}$ is rotationally symmetric in the $\mathbf{e}_2 / \mathbf{e}_3$–plane; meaning, $\sigma_{22} = \sigma_{33}$ $\left(= -\frac{1}{2}\sigma_{11}\right)$,
- $\omega_2 = \omega_3 = 0$, that is, $\boldsymbol{\omega} \parallel \mathbf{e}_1$,
- $\dot{u}_2 = \dot{u}_3 = 0$; equivalently, $\dot{\mathbf{u}} \parallel \mathbf{e}_1$.

For brevity, we have used the variables Θ_1 and Θ_2 in some sets of equations as well as σ_{11} and Θ in other areas where the degeneracy in the eigenvalues of $\boldsymbol{\sigma}$ allows for some simplification — namely in the second Bianchi identities. To get from the former to the latter set of variables is accomplished by the relations

$$\sigma_{11} = \tfrac{2}{3}(\Theta_1 - \Theta_2) \quad \text{and} \quad \Theta = \Theta_1 + 2\,\Theta_2 \ .$$

A.3.1 Commutators

From (3.22) and (3.23) it follows explicitly that

$$
\begin{aligned}
[\,\mathbf{e}_0, \mathbf{e}_1\,](f) &= \dot{u}_1\,\mathbf{e}_0(f) - \Theta_1\,\mathbf{e}_1(f) - \Omega_3\,\mathbf{e}_2(f) + \Omega_2\,\mathbf{e}_3(f) && \text{(A.3)}\\
[\,\mathbf{e}_0, \mathbf{e}_2\,](f) &= \Omega_3\,\mathbf{e}_1(f) - \Theta_2\,\mathbf{e}_2(f) - (\omega_1 + \Omega_1)\,\mathbf{e}_3(f) && \text{(A.4)}\\
[\,\mathbf{e}_0, \mathbf{e}_3\,](f) &= -\,\Omega_2\,\mathbf{e}_1(f) + (\omega_1 + \Omega_1)\,\mathbf{e}_2(f) - \Theta_2\,\mathbf{e}_3(f) && \text{(A.5)}\\
[\,\mathbf{e}_2, \mathbf{e}_3\,](f) &= 2\,\omega_1\,\mathbf{e}_0(f) + n_{11}\,\mathbf{e}_1(f) - (a_3 - n_{12})\,\mathbf{e}_2(f) + (a_2 + n_{31})\,\mathbf{e}_3(f) && \text{(A.6)}\\
[\,\mathbf{e}_3, \mathbf{e}_1\,](f) &= (a_3 + n_{12})\,\mathbf{e}_1(f) + n_{22}\,\mathbf{e}_2(f) - (a_1 - n_{23})\,\mathbf{e}_3(f) && \text{(A.7)}\\
[\,\mathbf{e}_1, \mathbf{e}_2\,](f) &= -\,(a_2 - n_{31})\,\mathbf{e}_1(f) + (a_1 + n_{23})\,\mathbf{e}_2(f) + n_{33}\,\mathbf{e}_3(f)\ . && \text{(A.8)}
\end{aligned}
$$

A.3.2 Transformation properties of commutation functions

In the present work we choose a shear eigentetrad which is degenerate and thus not fully specified. We take (4.2) and substitute into the commutation relations (A.3) – (A.8) to obtain the transformation behaviour of the remaining non-zero commutation functions under the rotation (4.2). We get that

$$
\begin{aligned}
\dot{u}'_1 &\to \dot{u}_1\\
\Theta'_1 &\to \Theta_1\\
\Theta'_2 &\to \Theta_2\\
\omega'_1 &\to \omega_1\\
\Omega'_1 &\to \Omega_1 - \mathbf{e}_0(\varphi)\\
a'_1 &\to a_1\\
n'_{11} &\to n_{11}\\
\Omega'_2 &\to \cos\varphi\,\Omega_2 + \sin\varphi\,\Omega_3\\
\Omega'_3 &\to -\,\sin\varphi\,\Omega_2 + \cos\varphi\,\Omega_3
\end{aligned}
$$

$$
\begin{aligned}
(a_2 - n_{31})' &\rightarrow \cos\varphi\,(a_2 - n_{31}) + \sin\varphi\,(a_3 + n_{12}) \\
(a_3 + n_{12})' &\rightarrow -\sin\varphi\,(a_2 - n_{31}) + \cos\varphi\,(a_3 + n_{12}) \\
(a_2 + n_{31})' &\rightarrow \cos\varphi\,(a_2 + n_{31} - \mathbf{e}_3(\varphi)) + \sin\varphi\,(a_3 - n_{12} + \mathbf{e}_2(\varphi)) \\
(a_3 - n_{12})' &\rightarrow -\sin\varphi\,(a_2 + n_{31} - \mathbf{e}_3(\varphi)) + \cos\varphi\,(a_3 - n_{12} + \mathbf{e}_2(\varphi)) \\
n'_{22} &\rightarrow \cos^2\varphi\, n_{22} + \sin^2\varphi\, n_{33} + 2\,\cos\varphi\,\sin\varphi\, n_{23} + \mathbf{e}_1(\varphi) \\
n'_{33} &\rightarrow \sin^2\varphi\, n_{22} + \cos^2\varphi\, n_{33} - 2\,\cos\varphi\,\sin\varphi\, n_{23} + \mathbf{e}_1(\varphi) \\
n'_{23} &\rightarrow -\cos\varphi\,\sin\varphi\,(n_{22} - n_{33}) + (\cos^2\varphi - \sin^2\varphi)\, n_{23}\ .
\end{aligned}
$$

A.3.3 Evolution equations

A.3.3.1 Energy density evolution

Conservation of energy requires

$$\mathbf{e}_0(\mu) = -\,\Theta\,(\mu + p)\ . \tag{A.9}$$

A.3.3.2 Shear and expansion evolution in terms of E

$$R_{0101} = -\,(\mathbf{e}_0 + \Theta_1)\,(\Theta_1) + (\mathbf{e}_1 + \dot u_1)\,(\dot u_1) = E_{11} + \tfrac{1}{6}\,(\mu + 3p) \tag{A.10}$$

$$R_{0202} = -\,(\mathbf{e}_0 + \Theta_2)\,(\Theta_2) + \omega_1^2 - (a_1 + n_{23})\,\dot u_1 = E_{22} + \tfrac{1}{6}\,(\mu + 3p) \tag{A.11}$$

$$R_{0303} = -\,(\mathbf{e}_0 + \Theta_2)\,(\Theta_2) + \omega_1^2 - (a_1 - n_{23})\,\dot u_1 = E_{33} + \tfrac{1}{6}\,(\mu + 3p)\ . \tag{A.12}$$

A.3.3.3 Expansion evolution

The expansion evolution is obtained by writing out $R_0{}^{\alpha}{}_{0\alpha}$:

$$\mathbf{e}_0(\Theta) = -\,\tfrac{1}{3}\,\Theta^2 + (\mathbf{e}_1 + \dot u_1 - 2\,a_1)\,(\dot u_1) - \tfrac{3}{2}\,\sigma_{11}^2 + 2\,\omega_1^2 - \tfrac{1}{2}\,(\mu + 3p)\ . \tag{A.13}$$

A.3.3.4 Vorticity and spatial commutation function evolution

$$\binom{0}{023} \quad \mathbf{e}_0(\omega_1) = -\,2\,\Theta_2\,\omega_1 - \tfrac{1}{2}\,n_{11}\,\dot u_1 \tag{A.14}$$

$$\binom{2}{012} \quad \mathbf{e}_0(a_1 + n_{23}) = -\,\Theta_1\,(a_1 + n_{23}) - (\mathbf{e}_1 + \dot u_1)\,(\Theta_2) + (\mathbf{e}_2 - a_2 + n_{31})\,(\Omega_3) + (\omega_1 + \Omega_1)\,(n_{22} - n_{33}) + \Omega_2\,(a_3 - n_{12}) \tag{A.15}$$

$$\binom{3}{013} \quad \mathbf{e}_0(a_1 - n_{23}) = -\,\Theta_1\,(a_1 - n_{23}) - (\mathbf{e}_1 + \dot u_1)\,(\Theta_2) - (\mathbf{e}_3 - a_3 - n_{12})\,(\Omega_2) - (\omega_1 + \Omega_1)\,(n_{22} - n_{33}) - \Omega_3\,(a_2 + n_{31}) \tag{A.16}$$

$$\binom{1}{012} \quad \mathbf{e}_0(a_2 - n_{31}) = -\,\Theta_2\,(a_2 - n_{31}) - \mathbf{e}_2(\Theta_1) - (\mathbf{e}_1 + \dot u_1 - a_1 - n_{23})(\Omega_3)$$

$$
\begin{aligned}
& & & - (\omega_1 + \Omega_1)\,(a_3 + n_{12}) - \Omega_2\,(n_{33} - n_{11}) && \text{(A.17)}\\
\binom{1}{013} \quad & \mathbf{e}_0(a_3 + n_{12}) &=\;& - \Theta_2\,(a_3 + n_{12}) - \mathbf{e}_3(\Theta_1) + (\mathbf{e}_1 + \dot{u}_1 - a_1 + n_{23})\,(\Omega_2) && \\
& & & + (\omega_1 + \Omega_1)\,(a_2 - n_{31}) + \Omega_3\,(n_{11} - n_{22}) && \text{(A.18)}\\
\binom{3}{023} \quad & \mathbf{e}_0(a_2 + n_{31}) &=\;& - (\mathbf{e}_2 + a_2 + n_{31})\,(\Theta_2) + (\mathbf{e}_3 - a_3 + n_{12})\,(\omega_1 + \Omega_1) && \\
& & & + \Omega_3\,(a_1 - n_{23}) + \Omega_2\,(n_{33} - n_{11}) && \text{(A.19)}\\
\binom{2}{023} \quad & \mathbf{e}_0(a_3 - n_{12}) &=\;& - (\mathbf{e}_3 + a_3 - n_{12})\,(\Theta_2) - (\mathbf{e}_2 - a_2 - n_{31})\,(\omega_1 + \Omega_1) && \\
& & & - \Omega_2\,(a_1 + n_{23}) - \Omega_3\,(n_{11} - n_{22}) && \text{(A.20)}\\
\binom{1}{023} \quad & \mathbf{e}_0(n_{11}) &=\;& - (\tfrac{1}{3}\,\Theta - 2\,\sigma_{11})\,n_{11} - (\mathbf{e}_2 - 2\,n_{31})\,(\Omega_2) - (\mathbf{e}_3 + 2\,n_{12})\,(\Omega_3) && \text{(A.21)}\\
\binom{2}{013} \quad & \mathbf{e}_0(n_{22}) &=\;& - \Theta_1\,n_{22} - (\mathbf{e}_1 + \dot{u}_1 + 2\,n_{23})\,(\omega_1 + \Omega_1) - (\mathbf{e}_3 - 2\,n_{12})\,(\Omega_3) && \text{(A.22)}\\
\binom{3}{012} \quad & \mathbf{e}_0(n_{33}) &=\;& - \Theta_1\,n_{33} - (\mathbf{e}_1 + \dot{u}_1 - 2\,n_{23})\,(\omega_1 + \Omega_1) - (\mathbf{e}_2 + 2\,n_{31})\,(\Omega_2)\;. && \text{(A.23)}
\end{aligned}
$$

A.3.3.5 Evolution equations for E and H

$$
\begin{aligned}
\boxed{[023]\,23} \quad & (\mathbf{e}_0 + 3\Theta_2)(E_{11}) = (\mathbf{e}_2 - a_2 - n_{31})(H_{31}) - (\mathbf{e}_3 - a_3 + n_{12})(H_{12}) \\
& \quad -\tfrac{1}{2}(\mu + p)\sigma_{11} + 2\Omega_2 E_{31} - 2\Omega_3 E_{12} \\
& \quad -\tfrac{3}{2} n_{11} H_{11} + \tfrac{1}{2}(n_{22} - n_{33})(H_{22} - H_{33}) + 2 n_{23} H_{23} && \text{(A.24)}\\
\boxed{[031]\,31} \quad & (\mathbf{e}_0 + \Theta)(E_{22}) = -(\mathbf{e}_1 + 2\dot{u}_1 - a_1 + n_{23})(H_{23}) + (\mathbf{e}_3 - a_3 - n_{12})(H_{12}) \\
& \quad +\tfrac{1}{4}(\mu + p)\sigma_{11} + \tfrac{3}{2}\sigma_{11} E_{33} + 2\Omega_3 E_{12} - (\omega_1 + 2\Omega_1) E_{23} \\
& \quad -\tfrac{3}{2} n_{22} H_{22} + \tfrac{1}{2}(n_{33} - n_{11})(H_{33} - H_{11}) + 2 n_{31} H_{31} && \text{(A.25)}\\
\boxed{[012]\,12} \quad & (\mathbf{e}_0 + \Theta)(E_{33}) = (\mathbf{e}_1 + 2\dot{u}_1 - a_1 - n_{23})(H_{23}) - (\mathbf{e}_2 - a_2 + n_{31})(H_{31}) \\
& \quad +\tfrac{1}{4}(\mu + p)\sigma_{11} + \tfrac{3}{2}\sigma_{11} E_{22} - 2\Omega_2 E_{31} + (\omega_1 + 2\Omega_1) E_{23} \\
& \quad -\tfrac{3}{2} n_{33} H_{33} + \tfrac{1}{2}(n_{11} - n_{22})(H_{11} - H_{22}) + 2 n_{12} H_{12} && \text{(A.26)}\\
\boxed{[031]\,12} \quad & (\mathbf{e}_0 + \Theta + \tfrac{3}{2}\sigma_{11})(E_{23}) = -(\mathbf{e}_1 + \dot{u}_1 - a_1 + n_{23})(H_{33}) + (\mathbf{e}_3 - 2a_3 - 2n_{12})(H_{31}) \\
& \quad +\tfrac{1}{2}(\mu + p)\omega_1 - \omega_1(E_{33} - E_{11}) - \Omega_1(E_{33} - E_{22}) - \Omega_2 E_{12} + \Omega_3 E_{31} \\
& \quad -\tfrac{1}{2}(3n_{22} + n_{33} - n_{11}) H_{23} - (a_2 + n_{31}) H_{12} - (a_1 - n_{23}) H_{11} + \dot{u}_1 H_{22} && \text{(A.27)}\\
\boxed{[012]\,31} \quad & (\mathbf{e}_0 + \Theta + \tfrac{3}{2}\sigma_{11})(E_{23}) = (\mathbf{e}_1 + \dot{u}_1 - a_1 - n_{23})(H_{22}) - (\mathbf{e}_2 - 2a_2 + 2n_{31})(H_{12}) \\
& \quad -\tfrac{1}{2}(\mu + p)\omega_1 + \omega_1(E_{22} - E_{11}) - \Omega_1(E_{33} - E_{22}) + \Omega_3 E_{31} - \Omega_2 E_{12} \\
& \quad -\tfrac{1}{2}(3n_{33} - n_{11} + n_{22}) H_{23} + (a_3 - n_{12}) H_{31} + (a_1 + n_{23}) H_{11} - \dot{u}_1 H_{33} && \text{(A.28)}\\
\boxed{[012]\,23} \quad & (\mathbf{e}_0 + \Theta)(E_{31}) = (\mathbf{e}_1 + \dot{u}_1 - 2a_1 - 2n_{23})(H_{12}) - (\mathbf{e}_2 - a_2 + n_{31})(H_{11}) \\
& \quad -\Omega_2(E_{11} - E_{33}) - \Omega_3 E_{23} + (2\omega_1 + \Omega_1) E_{12} \\
& \quad -\tfrac{1}{2}(3n_{33} + n_{11} - n_{22}) H_{31} - (a_3 + n_{12}) H_{23} - (a_2 - n_{31}) H_{22} && \text{(A.29)}\\
\boxed{[023]\,12} \quad & (\mathbf{e}_0 + 3\Theta_2)(E_{31}) = (\mathbf{e}_2 - a_2 - n_{31})(H_{33}) - (\mathbf{e}_3 - 2a_3 + 2n_{12})(H_{23})
\end{aligned}
$$

$$-\Omega_2(E_{11}-E_{33})+(\Omega_1-\omega_1)E_{12}-\Omega_3E_{23}$$
$$-\tfrac{1}{2}(3n_{11}-n_{22}+n_{33})H_{31}+(\dot{u}_1+a_1-n_{23})H_{12}+(a_2+n_{31})H_{22} \quad (A.30)$$

[031] 23
$$(\mathbf{e}_0+\Theta)(E_{12})=-(\mathbf{e}_1+\dot{u}_1-2a_1+2n_{23})(H_{31})+(\mathbf{e}_3-a_3-n_{12})(H_{11})$$
$$-\Omega_3(E_{22}-E_{11})+\Omega_2E_{23}-(2\omega_1+\Omega_1)E_{31}$$
$$-\tfrac{1}{2}(3n_{22}-n_{33}+n_{11})H_{12}+(a_2-n_{31})H_{23}+(a_3+n_{12})H_{33} \quad (A.31)$$

[023] 31
$$(\mathbf{e}_0+3\Theta_2)(E_{12})=(\mathbf{e}_2-2a_2-2n_{31})(H_{23})-(\mathbf{e}_3-a_3+n_{12})(H_{22})$$
$$-\Omega_3(E_{22}-E_{11})+(\omega_1-\Omega_1)E_{31}+\Omega_2E_{23}$$
$$-\tfrac{1}{2}(3n_{11}+n_{22}-n_{33})H_{12}-(\dot{u}_1+a_1+n_{23})H_{31}-(a_3-n_{12})H_{33} \quad (A.32)$$

[023] 01
$$(\mathbf{e}_0+3\Theta_2)(H_{11})=-(\mathbf{e}_2-a_2-n_{31})(E_{31})+(\mathbf{e}_3-a_3+n_{12})(E_{12})$$
$$+\tfrac{3}{2}n_{11}E_{11}-\tfrac{1}{2}(n_{22}-n_{33})(E_{22}-E_{33})-2n_{23}E_{23}+2\Omega_2H_{31}-2\Omega_3H_{12} \quad (A.33)$$

[031] 02
$$(\mathbf{e}_0+\Theta)(H_{22})=(\mathbf{e}_1+2\dot{u}_1-a_1+n_{23})(E_{23})-(\mathbf{e}_3-a_3-n_{12})(E_{12})$$
$$+\tfrac{3}{2}\sigma_{11}H_{33}+\tfrac{3}{2}n_{22}E_{22}-\tfrac{1}{2}(n_{33}-n_{11})(E_{33}-E_{11})$$
$$-2n_{31}E_{31}-(2\Omega_1+\omega_1)H_{23}+2\Omega_3H_{12} \quad (A.34)$$

[012] 03
$$(\mathbf{e}_0+\Theta)(H_{33})=-(\mathbf{e}_1+2\dot{u}_1-a_1-n_{23})(E_{23})+(\mathbf{e}_2-a_2+n_{31})(E_{31})$$
$$+\tfrac{3}{2}\sigma_{11}H_{22}+\tfrac{3}{2}n_{33}E_{33}-\tfrac{1}{2}(n_{11}-n_{22})(E_{11}-E_{22})$$
$$-2n_{12}E_{12}+(2\Omega_1+\omega_1)H_{23}-2\Omega_2H_{31} \quad (A.35)$$

[012] 02
$$(\mathbf{e}_0+\Theta+\tfrac{3}{2}\sigma_{11})(H_{23})=-(\mathbf{e}_1+\dot{u}_1-a_1-n_{23})(E_{22})+(\mathbf{e}_2-2a_2+2n_{31})(E_{12})$$
$$+\omega_1(H_{22}-H_{11})-\Omega_1(H_{33}-H_{22})+\Omega_3H_{31}-\Omega_2H_{12}-\tfrac{1}{6}\mathbf{e}_1(\mu)$$
$$+\tfrac{1}{2}(3n_{33}-n_{11}+n_{22})E_{23}-(a_3-n_{12})E_{31}-(a_1+n_{23})E_{11}+\dot{u}_1E_{33} \quad (A.36)$$

[031] 03
$$(\mathbf{e}_0+\Theta+\tfrac{3}{2}\sigma_{11})(H_{23})=(\mathbf{e}_1+\dot{u}_1-a_1+n_{23})(E_{33})-(\mathbf{e}_3-2a_3-2n_{12})(E_{31})$$
$$-\omega_1(H_{33}-H_{11})-\Omega_1(H_{33}-H_{22})-\Omega_2H_{12}+\Omega_3H_{31}+\tfrac{1}{6}\mathbf{e}_1(\mu)$$
$$+\tfrac{1}{2}(3n_{22}+n_{33}-n_{11})E_{23}+(a_2+n_{31})E_{12}+(a_1-n_{23})E_{11}-\dot{u}_1E_{22} \quad (A.37)$$

[012] 01
$$(\mathbf{e}_0+\Theta)(H_{31})=-(\mathbf{e}_1+\dot{u}_1-2a_1-2n_{23})(E_{12})+(\mathbf{e}_2-a_2+n_{31})(E_{11})$$
$$-\Omega_2(H_{11}-H_{33})-\Omega_3H_{23}+(2\omega_1+\Omega_1)H_{12}+\tfrac{1}{6}\mathbf{e}_2(\mu)$$
$$+\tfrac{1}{2}(3n_{33}+n_{11}-n_{22})E_{31}+(a_3+n_{12})E_{23}+(a_2-n_{31})E_{22} \quad (A.38)$$

[023] 03
$$(\mathbf{e}_0+3\Theta_2)(H_{31})=-(\mathbf{e}_2-a_2-n_{31})(E_{33})+(\mathbf{e}_3-2a_3+2n_{12})(E_{23})$$
$$-\Omega_2(H_{11}-H_{33})-\Omega_3H_{23}-(\omega_1-\Omega_1)H_{12}-\tfrac{1}{6}\mathbf{e}_2(\mu)$$
$$+\tfrac{1}{2}(3n_{11}-n_{22}+n_{33})E_{31}-(\dot{u}_1+a_1-n_{23})E_{12}-(a_2+n_{31})E_{22} \quad (A.39)$$

[031] 01
$$(\mathbf{e}_0+\Theta)(H_{12})=(\mathbf{e}_1+\dot{u}_1-2a_1+2n_{23})(E_{31})-(\mathbf{e}_3-a_3-n_{12})(E_{11})$$
$$-\Omega_3(H_{22}-H_{11})+\Omega_2H_{23}-(2\omega_1+\Omega_1)H_{31}-\tfrac{1}{6}\mathbf{e}_3(\mu)$$
$$+\tfrac{1}{2}(3n_{22}-n_{33}+n_{11})E_{12}-(a_2-n_{31})E_{23}-(a_3+n_{12})E_{33} \quad (A.40)$$

[023] 02
$$(\mathbf{e}_0+3\Theta_2)(H_{12}) = -(\mathbf{e}_2-2a_2-2n_{31})(E_{23})+(\mathbf{e}_3-a_3+n_{12})(E_{22})$$
$$-\Omega_3(H_{22}-H_{11})+\Omega_2H_{23}+(\omega_1-\Omega_1)H_{31}+\tfrac{1}{6}\mathbf{e}_3(\mu)$$
$$+\tfrac{1}{2}(3n_{11}+n_{22}-n_{33})E_{12}+(\dot{u}_1+a_1+n_{23})E_{31}+(a_3-n_{12})E_{33}\ . \quad \text{(A.41)}$$

A.3.4 Constraint equations

A.3.4.1 Momentum conservation constraints

We contract the second Bianchi identity (2.50) twice and find that for PLRS spacetimes according to Definition 4.2.1

$$\mathbf{e}_1(p)+(\mu+p)\,\dot{u}_1 = 0 \quad \text{(A.42)}$$
$$\mathbf{e}_2(p) = 0 \quad \text{(A.43)}$$
$$\mathbf{e}_3(p) = 0\ . \quad \text{(A.44)}$$

A.3.4.2 Vorticity, spatial commutation function and shear constraints

$$R_{0202}-R_{0303} = (E_{22}-E_{33}) = -2\,n_{23}\,\dot{u}_1 \quad \text{(A.45)}$$
$$R_{0203}-\binom{0}{023} = E_{23} = \tfrac{1}{2}\,(n_{22}-n_{33})\,\dot{u}_1 \quad \text{(A.46)}$$
$$R_{0102} = E_{12} = \mathbf{e}_2(\dot{u}_1)+\Omega_2\,\omega_1+\tfrac{3}{2}\,\Omega_3\,\sigma_{11} \quad \text{(A.47)}$$
$$R_{0103} = E_{31} = \mathbf{e}_3(\dot{u}_1)+\Omega_3\,\omega_1-\tfrac{3}{2}\,\Omega_2\,\sigma_{11} \quad \text{(A.48)}$$
$$\binom{0}{012} \quad 0 = \tfrac{1}{2}\,\mathbf{e}_2(\dot{u}_1)-\tfrac{1}{2}\,(a_2-n_{31})\,\dot{u}_1+\Omega_2\,\omega_1 \quad \text{(A.49)}$$
$$\binom{0}{013} \quad 0 = \tfrac{1}{2}\,\mathbf{e}_3(\dot{u}_1)-\tfrac{1}{2}\,(a_3+n_{12})\,\dot{u}_1+\Omega_3\,\omega_1 \quad \text{(A.50)}$$
$$\binom{0}{123} \quad 0 = (\mathbf{e}_1-\dot{u}_1-2\,a_1)\,(\omega_1) \quad \text{(A.51)}$$
$$R_{0123} = \tfrac{3}{2}\,n_{11}\,\sigma_{11}+2\,(\dot{u}_1+a_1)\,\omega_1 = -H_{11} \quad \text{(A.52)}$$
$$R_{0231} = -(\mathbf{e}_1-a_1+n_{23})\,\omega_1-\tfrac{3}{4}\,n_{11}\,\sigma_{11}-\tfrac{3}{4}\,(n_{22}-n_{33})\,\sigma_{11} = -H_{22} \quad \text{(A.53)}$$
$$R_{0312} = -(\mathbf{e}_1-a_1-n_{23})\,\omega_1-\tfrac{3}{4}\,n_{11}\,\sigma_{11}+\tfrac{3}{4}\,(n_{22}-n_{33})\,\sigma_{11} = -H_{33} \quad \text{(A.54)}$$
$$R_{0331} = \mathbf{e}_1(\Theta_2)+\tfrac{3}{2}\,(a_1-n_{23})\,\sigma_{11}+\tfrac{1}{2}\,(n_{11}+n_{22}-n_{33})\,\omega_1 = -H_{23} \quad \text{(A.55)}$$
$$R_{0212} = -\mathbf{e}_1(\Theta_2)-\tfrac{3}{2}\,(a_1+n_{23})\,\sigma_{11}+\tfrac{1}{2}\,(n_{22}-n_{33}-n_{11})\,\omega_1 = -H_{23} \quad \text{(A.56)}$$
$$R_{0112} = \mathbf{e}_2(\Theta_1)-\tfrac{3}{2}\,(a_2-n_{31})\,\sigma_{11}+(a_3+n_{12})\,\omega_1 = -H_{31} \quad \text{(A.57)}$$
$$R_{0323} = -\mathbf{e}_2(\Theta_2)+\mathbf{e}_3(\omega_1) = -H_{31} \quad \text{(A.58)}$$
$$R_{0131} = -\mathbf{e}_3(\Theta_1)+\tfrac{3}{2}\,(a_3+n_{12})\,\sigma_{11}+(a_2-n_{31})\,\omega_1 = -H_{12} \quad \text{(A.59)}$$
$$R_{0223} = \mathbf{e}_3(\Theta_2)+\mathbf{e}_2(\omega_1) = -H_{12} \quad \text{(A.60)}$$
$$\binom{1}{123} \quad \mathbf{e}_1(n_{11})+\mathbf{e}_2(a_3+n_{12})-\mathbf{e}_3(a_2-n_{31})-2\,a_1\,n_{11}-2\,a_2\,n_{12}-2\,a_3\,n_{31}$$

$$= -2\,\Theta_1\,\omega_1 \tag{A.61}$$

$$\begin{aligned} R_{3112} &= -\,\mathbf{e}_3(a_2 - n_{31}) + \tfrac{1}{2}\,\mathbf{e}_1(n_{11} + n_{22} - n_{33}) - n_{11}\,(a_1 - n_{23}) \\ &\quad - a_1\,(n_{22} - n_{33}) - n_{23}\,(n_{22} + n_{33}) - 2\,n_{31}\,(a_3 + n_{12}) + \Theta_1\,\omega_1 \\ &= -\,E_{23} \end{aligned} \tag{A.62}$$

$$\begin{aligned} \binom{2}{123} \quad & \mathbf{e}_2(n_{22}) - \mathbf{e}_1(a_3 - n_{12}) + \mathbf{e}_3(a_1 + n_{23}) - 2\,a_1\,n_{12} - 2\,a_2\,n_{22} - 2\,a_3\,n_{23} \\ & = -\,2\,\Omega_3\,\omega_1 \end{aligned} \tag{A.63}$$

$$\begin{aligned} R_{2312} &= -\,\mathbf{e}_3(a_1 + n_{23}) - \tfrac{1}{2}\,\mathbf{e}_2(n_{11} + n_{22} - n_{33}) + n_{11}\,(a_2 - n_{31}) \\ &\quad + (a_2 + n_{31})\,(n_{22} - n_{33}) + 2\,n_{23}\,(a_3 - n_{12}) - 2\,\Omega_3\,\omega_1 \\ &= -\,E_{31} \end{aligned} \tag{A.64}$$

$$\begin{aligned} \binom{3}{123} \quad & \mathbf{e}_3(n_{33}) - \mathbf{e}_2(a_1 - n_{23}) + \mathbf{e}_1(a_2 + n_{31}) - 2\,a_1\,n_{31} - 2\,a_2\,n_{23} - 2\,a_3\,n_{33} \\ & = 2\,\Omega_2\,\omega_1 \end{aligned} \tag{A.65}$$

$$\begin{aligned} R_{2331} &= -\,\mathbf{e}_2(a_1 - n_{23}) - \tfrac{1}{2}\,\mathbf{e}_3(n_{22} - n_{33} - n_{11}) - n_{11}\,(a_3 + n_{12}) \\ &\quad + (a_3 - n_{12})\,(n_{22} - n_{33}) - 2\,n_{23}\,(a_2 + n_{31}) - 2\,\Omega_2\,\omega_1 \\ &= -\,E_{12} \end{aligned} \tag{A.66}$$

$$\begin{aligned} R_{2323} &= \mathbf{e}_2(a_2 + n_{31}) + \mathbf{e}_3(a_3 - n_{12}) + \Theta_2^2 + \omega_1^2 \\ &\quad - (a_1 + n_{23})\,(a_1 - n_{23}) - (a_2 + n_{31})^2 - (a_3 - n_{12})^2 \\ &\quad - \tfrac{3}{4}\,n_{11}^2 + \tfrac{1}{2}\,n_{11}\,(n_{22} + n_{33}) + \tfrac{1}{4}\,(n_{22} - n_{33})^2 - 2\,\Omega_1\,\omega_1 \\ &= -\,E_{11} + \tfrac{1}{3}\,\mu \end{aligned} \tag{A.67}$$

$$\begin{aligned} R_{3131} &= \mathbf{e}_3(a_3 + n_{12}) + \mathbf{e}_1(a_1 - n_{23}) + \Theta_2\,\Theta_1 \\ &\quad - (a_2 + n_{31})\,(a_2 - n_{31}) - (a_3 + n_{12})^2 - (a_1 - n_{23})^2 \\ &\quad + \tfrac{1}{4}\,n_{11}^2 + \tfrac{1}{4}\,(n_{22} - n_{33})\,(2\,n_{11} - 3\,n_{22} - n_{33}) \\ &= -\,E_{22} + \tfrac{1}{3}\mu \end{aligned} \tag{A.68}$$

$$\begin{aligned} R_{1212} &= \mathbf{e}_1(a_1 + n_{23}) + \mathbf{e}_2(a_2 - n_{31}) + \Theta_2\,\Theta_1 \\ &\quad - (a_3 + n_{12})\,(a_3 - n_{12}) - (a_1 + n_{23})^2 - (a_2 - n_{31})^2 \\ &\quad + \tfrac{1}{4}\,n_{11}^2 - \tfrac{1}{4}\,(n_{22} - n_{33})\,(2\,n_{11} - 3\,n_{33} - n_{22}) \\ &= -\,E_{33} + \tfrac{1}{3}\,\mu\,. \end{aligned} \tag{A.69}$$

A.3.4.3 Generalised Friedmann equation

This is obtained by writing out $R^{\alpha\beta}{}_{\alpha\beta}$ (which corresponds to summing (A.67), (A.68) and (A.69)).

$$\begin{aligned} & 4\,\mathbf{e}_1(a_1) + 4\,\mathbf{e}_2(a_2) + 4\,\mathbf{e}_3(a_3) - 6\,a_1^2 - 6\,a_2^2 - 6\,a_3^2 \\ & \quad - 2\,n_{23}^2 - 2\,n_{31}^2 - 2\,n_{12}^2 - \tfrac{1}{2}\,n_{11}^2 - \tfrac{1}{2}\,(n_{22} - n_{33})^2 + n_{11}\,(n_{22} + n_{33}) \end{aligned}$$

$$+\tfrac{2}{3}\,\Theta^2 - \tfrac{3}{2}\,\sigma_{11}^2 + 2\,\omega_1^2 - 4\,\Omega_1\,\omega_1 = 2\,\mu\ . \tag{A.70}$$

A.3.4.4 Constraints on E and H

[123] 23

$$(\mathbf{e}_1 - 3a_1)(E_{11}) + (\mathbf{e}_2 - 3a_2 + n_{31})(E_{12}) + (\mathbf{e}_3 - 3a_3 - n_{12})(E_{31}) = \tfrac{1}{3}\mathbf{e}_1(\mu) + (n_{22} - n_{33})E_{23} - (E_{22} - E_{33})n_{23} + 3\omega_1 H_{11} \tag{A.71}$$

[123] 31

$$(\mathbf{e}_2 - 3a_2)(E_{22}) + (\mathbf{e}_3 - 3a_3 + n_{12})(E_{23}) + (\mathbf{e}_1 - 3a_1 - n_{23})(E_{12}) = \tfrac{1}{3}\mathbf{e}_2(\mu) + (n_{33} - n_{11})E_{31} - (E_{33} - E_{11})n_{31} + 3\omega_1 H_{12} - \tfrac{3}{2}\sigma_{11}H_{31} \tag{A.72}$$

[123] 12

$$(\mathbf{e}_3 - 3a_3)(E_{33}) + (\mathbf{e}_1 - 3a_1 + n_{23})(E_{31}) + (\mathbf{e}_2 - 3a_2 - n_{31})(E_{23}) = \tfrac{1}{3}\mathbf{e}_3(\mu) + (n_{11} - n_{22})E_{12} - (E_{11} - E_{22})n_{12} + 3\omega_1 H_{31} + \tfrac{3}{2}\sigma_{11}H_{12} \tag{A.73}$$

[123] 01

$$(\mathbf{e}_1 - 3a_1)(H_{11}) + (\mathbf{e}_2 - 3a_2 + n_{31})(H_{12}) + (\mathbf{e}_3 - 3a_3 - n_{12})(H_{31}) = -(\mu + p)\omega_1 + (n_{22} - n_{33})H_{23} - (H_{22} - H_{33})n_{23} - 3\omega_1 E_{11} \tag{A.74}$$

[123] 02

$$(\mathbf{e}_2 - 3a_2)(H_{22}) + (\mathbf{e}_3 - 3a_3 + n_{12})(H_{23}) + (\mathbf{e}_1 - 3a_1 - n_{23})(H_{12}) = (n_{33} - n_{11})H_{31} - (H_{33} - H_{11})n_{31} - 3\omega_1 E_{12} + \tfrac{3}{2}\sigma_{11}E_{31} \tag{A.75}$$

[123] 03

$$(\mathbf{e}_3 - 3a_3)(H_{33}) + (\mathbf{e}_1 - 3a_1 + n_{23})(H_{31}) + (\mathbf{e}_2 - 3a_2 - n_{31})(H_{23}) = (n_{11} - n_{22})H_{12} - (H_{11} - H_{22})n_{12} - 3\omega_1 E_{31} - \tfrac{3}{2}\sigma_{11}E_{12}\ . \tag{A.76}$$

Appendix B

Evolution and Constraint Equations: Brane Dust

The equations here are, once again, essentially obtained by writing out the independent components of the Jacobi identity (3.39), Riemann curvature tensor (3.12) and second Bianchi identity (11.18)[1]. The equations are written in terms of $\mathbf{E}$ and $\mathbf{H}$ as well as ϱ; and thus we also require (11.15), (11.16), (11.7) and (11.17), incorporating the field equations (11.1).

Restrictions imposed: All equations that are to follow are written out for a shear-free spacetime geometry with a dust matter source that are subject to the specific restriction that

- A matter-comoving ONT $\{\, \mathbf{e}_a \,\} = \{\, \mathbf{u},\, \mathbf{e}_\alpha \,\}$ spatially aligned with $\boldsymbol{\omega}$ is chosen – $\omega_2 = \omega_3 = 0$, that is, $\boldsymbol{\omega} \parallel \mathbf{e}_1$.

These equations have been adapted directly from those which hold for a classical perfect fluid cosmology with rotationally symmetric shear; these may be found in [64].

B.1 Tetrad choice and symmetry

An ONT $\{\, \mathbf{e}_a \,\} = \{\, \mathbf{u},\, \mathbf{e}_\alpha \,\}$ may be chosen such that one spacelike leg is aligned with the vorticity. It is then degenerate in one plane, say the $\mathbf{e}_2 \,/\, \mathbf{e}_3$–plane: then $\omega_2 = \omega_3 = 0$. The tetrad is now free by a rotation in this plane while the tetrad vector $\mathbf{e}_1$ is uniquely defined provided $\omega_1 \neq 0$. Any change of tetrad basis by a transformation $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(x^i)$ relates components of a vector $\mathbf{v}$ in the new and old bases by

$$v^{a\prime} \rightarrow \Lambda^{a\prime}{}_a \, v^a \ ,$$

[1] We have to take into account the symmetries of the Riemann curvature tensor:

$$R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc} \ .$$

leaving $\mathbf{v}$ invariant:

$$\mathbf{v} = v^{a\prime}\,\mathbf{e}_{a\prime} = v^{a}\,\mathbf{e}_{a}\ .$$

The Lorentz matrix $\Lambda^{a\prime}{}_{a}$ has an inverse $\Lambda^{-1}{}_{a\prime}{}^{a}$. The tetrad freedom now is that of a spatial rotation given by

$$\mathbf{e}_{a\prime} \to \Lambda^{-1}{}_{a\prime}{}^{a}\,\mathbf{e}_{a}\ ,$$

where

$$\Lambda^{-1}{}_{a\prime}{}^{a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix}\ . \tag{B.1}$$

B.1.1 Transformation properties of commutation functions

We take (B.1) and substitute into the commutation relations (3.22) and (3.23) to obtain the transformation behaviour of the remaining non-zero commutation functions under the rotation (B.1). We get that

$$\begin{aligned}
\Theta' &\to \Theta \\
\omega_1' &\to \omega_1 \\
\Omega_1' &\to \Omega_1 - \mathbf{e}_0(\varphi) \\
a_1' &\to a_1 \\
n_{11}' &\to n_{11} \\
\Omega_2' &\to \cos\varphi\,\Omega_2 + \sin\varphi\,\Omega_3 \\
\Omega_3' &\to -\sin\varphi\,\Omega_2 + \cos\varphi\,\Omega_3 \\
(a_2 - n_{31})' &\to \cos\varphi\,(a_2 - n_{31}) + \sin\varphi\,(a_3 + n_{12}) \\
(a_3 + n_{12})' &\to -\sin\varphi\,(a_2 - n_{31}) + \cos\varphi\,(a_3 + n_{12}) \\
(a_2 + n_{31})' &\to \cos\varphi\,(a_2 + n_{31} - \mathbf{e}_3(\varphi)) + \sin\varphi\,(a_3 - n_{12} + \mathbf{e}_2(\varphi)) \\
(a_3 - n_{12})' &\to -\sin\varphi\,(a_2 + n_{31} - \mathbf{e}_3(\varphi)) + \cos\varphi\,(a_3 - n_{12} + \mathbf{e}_2(\varphi)) \\
n_{22}' &\to \cos^2\varphi\,n_{22} + \sin^2\varphi\,n_{33} + 2\,\cos\varphi\,\sin\varphi\,n_{23} + \mathbf{e}_1(\varphi) \\
n_{33}' &\to \sin^2\varphi\,n_{22} + \cos^2\varphi\,n_{33} - 2\,\cos\varphi\,\sin\varphi\,n_{23} + \mathbf{e}_1(\varphi) \\
n_{23}' &\to -\cos\varphi\,\sin\varphi\,(n_{22} - n_{33}) + (\cos^2\varphi - \sin^2\varphi)\,n_{23}\ .
\end{aligned}$$

We may now specify the tetrad further. If the vorticity does not vanish, then from the Jacobi Identities (B.57) and (B.58) it follows that

$$\Omega_2 = \Omega_3 = 0\,. \tag{B.2}$$

We may then choose $\mathbf{e}_0(\varphi)$ such that

$$\Omega_1 + \omega_1 = 0\,. \tag{B.3}$$

If, on the other hand, the vorticity vanishes, then the tetrad becomes free by a general rotation and we may choose

$$\Omega_\alpha = 0\,. \tag{B.4}$$

B.2 Commutators

From (3.22) and (3.23) it follows explicitly that

$$\begin{aligned}
[\,\mathbf{e}_0,\,\mathbf{e}_1\,](f) &= -\frac{1}{3}\Theta\,\mathbf{e}_1(f) && \text{(B.5)}\\
[\,\mathbf{e}_0,\,\mathbf{e}_2\,](f) &= -\frac{1}{3}\Theta\,\mathbf{e}_2(f) && \text{(B.6)}\\
[\,\mathbf{e}_0,\,\mathbf{e}_3\,](f) &= -\frac{1}{3}\Theta\,\mathbf{e}_3(f) && \text{(B.7)}\\
[\,\mathbf{e}_2,\,\mathbf{e}_3\,](f) &= 2\,\omega_1\,\mathbf{e}_0(f) + n_{11}\,\mathbf{e}_1(f) - (a_3 - n_{12})\,\mathbf{e}_2(f) + (a_2 + n_{31})\,\mathbf{e}_3(f) && \text{(B.8)}\\
[\,\mathbf{e}_3,\,\mathbf{e}_1\,](f) &= (a_3 + n_{12})\,\mathbf{e}_1(f) + n_{22}\,\mathbf{e}_2(f) - (a_1 - n_{23})\,\mathbf{e}_3(f) && \text{(B.9)}\\
[\,\mathbf{e}_1,\,\mathbf{e}_2\,](f) &= -\,(a_2 - n_{31})\,\mathbf{e}_1(f) + (a_1 + n_{23})\,\mathbf{e}_2(f) + n_{33}\,\mathbf{e}_3(f)\ . && \text{(B.10)}
\end{aligned}$$

B.3 Evolution equations

B.3.1 Energy density evolution

Conservation of energy requires

$$\mathbf{e}_0(\varrho) = -\,\Theta\,\varrho\ . \tag{B.11}$$

B.3.2 Expansion evolution

The expansion evolution is obtained by writing out ${R_0}^\alpha{}_{0\alpha}$:

$$\mathbf{e}_0(\Theta) = -\,\tfrac{1}{3}\,\Theta^2 + 2\,\omega_1^2 - \tfrac{1}{2}\,\varrho(\kappa + \tfrac{1}{3}\kappa_5^4\,\varrho) + 2\mathcal{E}_{00} + \Lambda\ . \tag{B.12}$$

B.3.3 Vorticity and spatial commutation function evolution

$$\begin{aligned}
&\tbinom{0}{023} & \mathbf{e}_0(\omega_1) &= -\,\tfrac{2}{3}\,\Theta\,\omega_1 && \text{(B.13)}\\
&\tbinom{2}{012} & \mathbf{e}_0(a_1 + n_{23}) &= -\,(\mathbf{e}_1 + a_1 + n_{23})\,(\tfrac{1}{3}\Theta) && \text{(B.14)}\\
&\tbinom{3}{013} & \mathbf{e}_0(a_1 - n_{23}) &= -\,(\mathbf{e}_1 + a_1 - n_{23})\,(\tfrac{1}{3}\Theta) && \text{(B.15)}\\
&\tbinom{1}{012} & \mathbf{e}_0(a_2 - n_{31}) &= -\,(\mathbf{e}_2 + a_2 - n_{31})\,(\tfrac{1}{3}\Theta) && \text{(B.16)}\\
&\tbinom{1}{013} & \mathbf{e}_0(a_3 + n_{12}) &= -\,(\mathbf{e}_3 + a_3 + n_{12})(\tfrac{1}{3}\Theta) && \text{(B.17)}
\end{aligned}$$

$$\binom{3}{023} \quad \mathbf{e}_0(a_2+n_{31}) = -(\mathbf{e}_2+a_2+n_{31})\left(\tfrac{1}{3}\Theta\right) \tag{B.18}$$
$$\binom{2}{023} \quad \mathbf{e}_0(a_3-n_{12}) = -(\mathbf{e}_3+a_3-n_{12})\left(\tfrac{1}{3}\Theta\right) \tag{B.19}$$
$$\binom{1}{023} \quad \mathbf{e}_0(n_{11}) = -\tfrac{1}{3}\Theta\, n_{11} \tag{B.20}$$
$$\binom{2}{013} \quad \mathbf{e}_0(n_{22}) = -\tfrac{1}{3}\Theta\, n_{22} \tag{B.21}$$
$$\binom{3}{012} \quad \mathbf{e}_0(n_{33}) = -\tfrac{1}{3}\Theta\, n_{33}\ . \tag{B.22}$$

B.3.4 Evolution equations for E and H

[023] 23

$$\mathbf{e}_0(E_{11}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{11})) = (\mathbf{e}_2-a_2-n_{31})(H_{31})-(\mathbf{e}_3-a_3+n_{12})(H_{12})$$
$$-\Theta\left(E_{11}+\tfrac{1}{6}(\mathcal{E}_{00}-\mathcal{E}_{11})\right)-\tfrac{3}{2}n_{11}H_{11}$$
$$+\tfrac{1}{2}(n_{22}-n_{33})(H_{22}-H_{33})+2n_{23}H_{23} \tag{B.23}$$

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$$\mathbf{e}_0(E_{22}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{22})) = -(\mathbf{e}_1-a_1+n_{23})(H_{23})+(\mathbf{e}_3-a_3-n_{12})(H_{12})$$
$$-\Theta\left(E_{22}+\tfrac{1}{6}(\mathcal{E}_{00}-\mathcal{E}_{22})\right)+\omega_1(E_{23}-\tfrac{1}{2}\mathcal{E}_{23}) \tag{B.24}$$
$$-\tfrac{3}{2}n_{22}H_{22}+\tfrac{1}{2}(n_{33}-n_{11})(H_{33}-H_{11})+2n_{31}H_{31} \tag{B.25}$$

[012] 12

$$\mathbf{e}_0(E_{33}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{33})) = (\mathbf{e}_1-a_1-n_{23})(H_{23})-(\mathbf{e}_2-a_2+n_{31})(H_{31})$$
$$-\Theta\left(E_{33}+\tfrac{1}{6}(\mathcal{E}_{00}-\mathcal{E}_{33})\right)-\omega_1(E_{23}-\tfrac{1}{2}\mathcal{E}_{23}) \tag{B.26}$$
$$-\tfrac{3}{2}n_{33}H_{33}+\tfrac{1}{2}(n_{11}-n_{22})(H_{11}-H_{22})+2n_{12}H_{12} \tag{B.27}$$

[031] 12

$$\mathbf{e}_0(E_{23}-\tfrac{1}{2}\mathcal{E}_{23}) = -(\mathbf{e}_1-a_1+n_{23})(H_{33})+(\mathbf{e}_3-2a_3-2n_{12})(H_{31})$$
$$-\Theta(E_{23}-\tfrac{1}{6}\mathcal{E}_{23})+[E_{11}-E_{22}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{33})+\tfrac{1}{2}\varrho(\kappa+\tfrac{1}{6}\kappa_5^4\,\varrho)]\omega_1$$
$$-\tfrac{1}{2}(3n_{22}+n_{33}-n_{11})H_{23}-(a_2+n_{31})H_{12}-(a_1-n_{23})H_{11} \tag{B.28}$$

[012] 31

$$\mathbf{e}_0(E_{23}-\tfrac{1}{2}\mathcal{E}_{23}) = (\mathbf{e}_1-a_1-n_{23})(H_{22})-(\mathbf{e}_2-2a_2+2n_{31})(H_{12})$$
$$-\Theta(E_{23}-\tfrac{1}{6}\mathcal{E}_{23})-[E_{11}-E_{33}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{22})+\tfrac{1}{2}\varrho(\kappa+\tfrac{1}{6}\kappa_5^4\,\varrho)]\omega_1$$
$$-\tfrac{1}{2}(3n_{33}-n_{11}+n_{22})H_{23}+(a_3-n_{12})H_{31}+(a_1+n_{23})H_{11} \tag{B.29}$$

[012] 23

$$\mathbf{e}_0(E_{31}-\tfrac{1}{2}\mathcal{E}_{31}) = (\mathbf{e}_1-2a_1-2n_{23})(H_{12})-(\mathbf{e}_2-a_2+n_{31})(H_{11})$$
$$-\Theta(E_{31}-\tfrac{1}{6}\mathcal{E}_{31})+\omega_1(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})$$
$$-\tfrac{1}{2}(3n_{33}+n_{11}-n_{22})H_{31}-(a_3+n_{12})H_{23}-(a_2-n_{31})H_{22} \tag{B.30}$$

[023] 12

$$\mathbf{e}_0(E_{31}-\tfrac{1}{2}\mathcal{E}_{31}) = (\mathbf{e}_2-a_2-n_{31})(H_{33})-(\mathbf{e}_3-2a_3+2n_{12})(H_{23})$$
$$-\Theta(E_{31}-\tfrac{1}{6}\mathcal{E}_{31})-2\omega_1E_{12}$$
$$-\tfrac{1}{2}(3n_{11}-n_{22}+n_{33})H_{31}+(a_1-n_{23})H_{12}+(a_2+n_{31})H_{22} \tag{B.31}$$

[031] 23

$$\mathbf{e}_0(E_{12}-\tfrac{1}{2}\mathcal{E}_{12}) = -(\mathbf{e}_1-2a_1+2n_{23})(H_{31})+(\mathbf{e}_3-a_3-n_{12})(H_{11})$$
$$-\Theta(E_{12}-\tfrac{1}{6}\mathcal{E}_{12})-\omega_1(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})$$

$$-\tfrac{1}{2}(3n_{22}-n_{33}+n_{11})H_{12}+(a_2-n_{31})H_{23}+(a_3+n_{12})H_{33} \quad \text{(B.32)}$$

[023] 31
$$\mathbf{e}_0(E_{12}-\tfrac{1}{2}\mathcal{E}_{12})=(\mathbf{e}_2-2a_2-2n_{31})(H_{23})-(\mathbf{e}_3-a_3+n_{12})(H_{22})$$
$$-\Theta(E_{12}-\tfrac{1}{6}\mathcal{E}_{12})+2\omega_1 E_{31}$$
$$-\tfrac{1}{2}(3n_{11}+n_{22}-n_{33})H_{12}-(a_1+n_{23})H_{31}-(a_3-n_{12})H_{33} \quad \text{(B.33)}$$

[023] 01
$$(\mathbf{e}_0+\Theta)(H_{11})=-(\mathbf{e}_2-a_2-n_{31})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})+(\mathbf{e}_3-a_3+n_{12})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})$$
$$+\tfrac{3}{2}n_{11}[E_{11}+\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{11})]-\tfrac{1}{2}(n_{22}-n_{33})[E_{22}-E_{33}+\tfrac{1}{2}(\mathcal{E}_{22}-\mathcal{E}_{33})] \quad \text{(B.34)}$$
$$-2n_{13}(E_{23}+\tfrac{1}{2}\mathcal{E}_{23}) \quad \text{(B.35)}$$

[031] 02
$$(\mathbf{e}_0+\Theta)(H_{22})=(\mathbf{e}_1-a_1+n_{23})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23})-(\mathbf{e}_3-a_3-n_{12})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})$$
$$+\tfrac{3}{2}n_{22}[E_{22}+\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{22})]-\tfrac{1}{2}(n_{33}-n_{11})[E_{33}-E_{11}+\tfrac{1}{2}(\mathcal{E}_{33}-\mathcal{E}_{11})]$$
$$-2n_{31}(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})+\omega_1 H_{23} \quad \text{(B.36)}$$

[012] 03
$$(\mathbf{e}_0+\Theta)(H_{33})=-(\mathbf{e}_1-a_1-n_{23})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23})+(\mathbf{e}_2-a_2+n_{31})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})$$
$$+\tfrac{3}{2}n_{33}[E_{33}+\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{33})]-\tfrac{1}{2}(n_{11}-n_{22})[E_{11}-E_{22}+\tfrac{1}{2}(\mathcal{E}_{11}-\mathcal{E}_{22})]$$
$$-2n_{12}(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})-\omega_1 H_{23} \quad \text{(B.37)}$$

[012] 02
$$(\mathbf{e}_0+\Theta)(H_{23})=-\mathbf{e}_1(E_{22}-\tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{22}))+(\mathbf{e}_2-2a_2+2n_{31})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})$$
$$-\omega_1(H_{11}-H_{33})-\tfrac{1}{3}\mathbf{e}_1(\varrho)(\frac{\kappa}{2}+\tfrac{1}{3}\kappa_5^4\,\varrho)-(a_1+n_{23})[E_{22}-E_{11}+\tfrac{1}{2}(\mathcal{E}_{22}-\mathcal{E}_{11})]$$
$$+\tfrac{1}{2}(3n_{33}-n_{11}+n_{22})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23})-(a_3-n_{12})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31}) \quad \text{(B.38)}$$

[031] 03
$$(\mathbf{e}_0+\Theta)(H_{23})=\mathbf{e}_1(E_{33}-\tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{33}))-(\mathbf{e}_3-2a_3-2n_{12})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})$$
$$+\omega_1(H_{11}-H_{22})+\tfrac{1}{3}\mathbf{e}_1(\varrho)(\frac{\kappa}{2}+\tfrac{1}{3}\kappa_5^4\,\varrho)+(a_1-n_{23})[E_{33}-E_{11}+\tfrac{1}{2}(\mathcal{E}_{33}-\mathcal{E}_{11})]$$
$$+\tfrac{1}{2}(3n_{22}+n_{33}-n_{11})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23})+(a_2+n_{31})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12}) \quad \text{(B.39)}$$

[012] 01
$$(\mathbf{e}_0+\Theta)(H_{31})=\mathbf{e}_2(E_{11}-\tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{11}))-(\mathbf{e}_1-2a_1-2n_{23})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})$$
$$+\omega_1 H_{12}+\tfrac{1}{3}\mathbf{e}_2(\varrho)(\frac{\kappa}{2}+\tfrac{1}{3}\kappa_5^4\,\varrho)+(a_2-n_{31})[E_{22}-E_{11}+\tfrac{1}{2}(\mathcal{E}_{22}-\mathcal{E}_{11})]$$
$$+\tfrac{1}{2}(3n_{33}+n_{11}-n_{22})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})+(a_3+n_{12})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23}) \quad \text{(B.40)}$$

[023] 03
$$(\mathbf{e}_0+\Theta)(H_{31})=-\mathbf{e}_2(E_{33}-\tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{33}))+(\mathbf{e}_3-2a_3+2n_{12})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23})$$
$$-2\omega_1 H_{12}-\tfrac{1}{3}\mathbf{e}_2(\varrho)(\frac{\kappa}{2}+\tfrac{1}{3}\kappa_5^4\,\varrho)-(a_2+n_{31})[E_{22}-E_{33}+\tfrac{1}{2}(\mathcal{E}_{22}-\mathcal{E}_{33})]$$
$$+\tfrac{1}{2}(3n_{11}-n_{22}+n_{33})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})-(a_1-n_{23})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12}) \quad \text{(B.41)}$$

[031] 01
$$(\mathbf{e}_0+\Theta)(H_{12})=-\mathbf{e}_3(E_{11}-\tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{11}))+(\mathbf{e}_1-2a_1+2n_{23})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})$$
$$-\omega_1 H_{31}-\tfrac{1}{3}\mathbf{e}_3(\varrho)(\frac{\kappa}{2}+\tfrac{1}{3}\kappa_5^4\,\varrho)-(a_3+n_{12})[E_{33}-E_{11}+\tfrac{1}{2}(\mathcal{E}_{33}-\mathcal{E}_{11})]$$
$$+\tfrac{1}{2}(3n_{22}-n_{33}+n_{11})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})-(a_2-n_{31})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23}) \quad \text{(B.42)}$$

[023] 02
$$(\mathbf{e}_0+\Theta)(H_{12})=\mathbf{e}_3(E_{22}-\tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{22}))-(\mathbf{e}_2-2a_2-2n_{31})(E_{23}+\tfrac{1}{2}\mathcal{E}_{23})$$
$$+2\omega_1 H_{31}+\tfrac{1}{3}\mathbf{e}_3(\varrho)(\frac{\kappa}{2}+\tfrac{1}{3}\kappa_5^4\,\varrho)+(a_3-n_{12})[E_{33}-E_{22}+\tfrac{1}{2}(\mathcal{E}_{33}-\mathcal{E}_{22})]$$
$$+\tfrac{1}{2}(3n_{11}+n_{22}-n_{33})(E_{12}+\tfrac{1}{2}\mathcal{E}_{12})+(a_1+n_{23})(E_{31}+\tfrac{1}{2}\mathcal{E}_{31})\,. \quad \text{(B.43)}$$

B.3.5 Evolution of the bulk tidal components

We expand the equations given by (11.14) which may be considered as evolution equations for $\mathcal{E}_{0a}$. These may be derived from the covariant equations given by Maartens [54].

$$\mathbf{e}_0(\mathcal{E}_{00}) = -\tfrac{4}{3}\Theta\mathcal{E}_{00} + (\mathbf{e}_\alpha - 2a_\alpha)(\mathcal{E}_0{}^\alpha) \tag{B.44}$$

$$\begin{aligned}
\mathbf{e}_0(\mathcal{E}_{01}) &= -\tfrac{4}{3}\Theta\mathcal{E}_{01} + (\mathbf{e}_\alpha - 3a_\alpha)(\mathcal{E}_1{}^\alpha) - a_1\mathcal{E}_{00} \\
&\quad -(n_{22}-n_{33})\mathcal{E}_{23} + n_{23}(\mathcal{E}_{22}-\mathcal{E}_{33}) + n_{31}\mathcal{E}_{12} - n_{12}\mathcal{E}_{31} - \tfrac{\kappa_5^4}{6}\varrho\,\mathbf{e}_1(\varrho) \qquad \text{(B.45)} \\
\mathbf{e}_0(\mathcal{E}_{02}) &= -\tfrac{4}{3}\Theta\mathcal{E}_{02} + (\mathbf{e}_\alpha - 3a_\alpha)(\mathcal{E}_2{}^\alpha) - a_2\mathcal{E}_{00} + 2\omega_1\mathcal{E}_{03} \\
&\quad -(n_{33}-n_{11})\mathcal{E}_{31} + n_{31}(\mathcal{E}_{33}-\mathcal{E}_{11}) - n_{23}\mathcal{E}_{12} + n_{12}\mathcal{E}_{23} - \tfrac{\kappa_5^4}{6}\varrho\,\mathbf{e}_2(\varrho) \qquad \text{(B.46)} \\
\mathbf{e}_0(\mathcal{E}_{03}) &= -\tfrac{4}{3}\Theta\mathcal{E}_{03} + (\mathbf{e}_\alpha - 3a_\alpha)(\mathcal{E}_3{}^\alpha) - a_3\mathcal{E}_{00} - 2\omega_1\mathcal{E}_{02} \\
&\quad -(n_{11}-n_{22})\mathcal{E}_{12} + n_{12}(\mathcal{E}_{11}-\mathcal{E}_{22}) + n_{23}\mathcal{E}_{12} - n_{31}\mathcal{E}_{23} - \tfrac{\kappa_5^4}{6}\varrho\,\mathbf{e}_3(\varrho) \qquad \text{(B.47)}
\end{aligned}$$

B.4 Constraint equations

B.4.1 Momentum conservation constraints

We contract the second Bianchi identity (11.18) twice and find that for dust spacetimes,

$$\begin{aligned}
\varrho\,\dot{u}_1 &= 0 \qquad \text{(B.48)} \\
\varrho\,\dot{u}_2 &= 0 \qquad \text{(B.49)} \\
\varrho\,\dot{u}_3 &= 0\,. \qquad \text{(B.50)}
\end{aligned}$$

B.4.2 Vorticity, spatial commutation function and shear constraints

$$\begin{aligned}
R_{0101} &= -\left(\mathbf{e}_0 + \tfrac{1}{3}\Theta\right)\left(\tfrac{1}{3}\Theta\right) = E_{11} + \tfrac{1}{6}\,\varrho(\kappa + \tfrac{1}{3}\kappa_5^4\,\varrho) - \tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{11}) - \tfrac{1}{3}\Lambda \qquad \text{(B.51)} \\
R_{0202} &= -\left(\mathbf{e}_0 + \tfrac{1}{3}\Theta\right)\left(\tfrac{1}{3}\Theta\right) + \omega_1^2 = E_{22} + \tfrac{1}{6}\,\varrho(\kappa + \tfrac{1}{3}\kappa_5^4\,\varrho) - \tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{22}) - \tfrac{1}{3}\Lambda \qquad \text{(B.52)} \\
R_{0303} &= -\left(\mathbf{e}_0 + \tfrac{1}{3}\Theta\right)\left(\tfrac{1}{3}\Theta\right) + \omega_1^2 = E_{33} + \tfrac{1}{6}\,\varrho(\kappa + \tfrac{1}{3}\kappa_5^4\,\varrho) - \tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{33}) - \tfrac{1}{3}\Lambda \qquad \text{(B.53)}
\end{aligned}$$

The above three equations should be combined with the Raychaudhuri equation to eliminate $\mathbf{e}_0$ for their constraint nature to become apparent.

$$\begin{aligned}
R_{0203} - \binom{0}{023} &= E_{23} + \tfrac{1}{2}\mathcal{E}_{23} = 0 \qquad \text{(B.54)} \\
R_{0102} &= E_{12} + \tfrac{1}{2}\mathcal{E}_{12} = 0 \qquad \text{(B.55)} \\
R_{0103} &= E_{31} + \tfrac{1}{2}\mathcal{E}_{31} = 0 \qquad \text{(B.56)}
\end{aligned}$$

$$\binom{0}{012} \qquad 0 = \Omega_2\, \omega_1 \tag{B.57}$$

$$\binom{0}{013} \qquad 0 = \Omega_3\, \omega_1 \tag{B.58}$$

$$\binom{0}{123} \qquad 0 = (\mathbf{e}_1 - 2\, a_1)\,(\omega_1) \tag{B.59}$$

$$R_{0123} = 2\, a_1\, \omega_1 = -\, H_{11} \tag{B.60}$$

$$R_{0231} = -\,(\mathbf{e}_1 - a_1 + n_{23})\,(\omega_1) = -\, H_{22} \tag{B.61}$$

$$R_{0312} = -\,(\mathbf{e}_1 - a_1 - n_{23})\,(\omega_1) = -\, H_{33} \tag{B.62}$$

$$R_{0331} = \mathbf{e}_1(\tfrac{1}{3}\Theta) + \tfrac{1}{2}\,(n_{11} + n_{22} - n_{33})\,\omega_1 = -\, H_{23} \tag{B.63}$$

$$R_{0212} = -\,\mathbf{e}_1(\tfrac{1}{3}\Theta) + \tfrac{1}{2}\,(n_{22} - n_{33} - n_{11})\,\omega_1 = -\, H_{23} \tag{B.64}$$

$$R_{0112} = \mathbf{e}_2(\tfrac{1}{3}\Theta) + (a_3 + n_{12})\,\omega_1 = -\, H_{31} \tag{B.65}$$

$$R_{0323} = -\,\mathbf{e}_2(\tfrac{1}{3}\Theta) + \mathbf{e}_3(\omega_1) = -\, H_{31} \tag{B.66}$$

$$R_{0131} = -\,\mathbf{e}_3(\tfrac{1}{3}\Theta) + (a_2 - n_{31})\,\omega_1 = -\, H_{12} \tag{B.67}$$

$$R_{0223} = \mathbf{e}_3(\tfrac{1}{3}\Theta) + \mathbf{e}_2(\omega_1) = -\, H_{12} \tag{B.68}$$

$$\binom{1}{123} \qquad \mathbf{e}_1(n_{11}) + \mathbf{e}_2(a_3 + n_{12}) - \mathbf{e}_3(a_2 - n_{31}) - 2\, a_1\, n_{11} - 2\, a_2\, n_{12} - 2\, a_3\, n_{31} = -\,\tfrac{2}{3}\,\Theta\,\omega_1 \tag{B.69}$$

$$\begin{aligned} R_{3112} &= -\,\mathbf{e}_3(a_2 - n_{31}) + \tfrac{1}{2}\,\mathbf{e}_1(n_{11} + n_{22} - n_{33}) - n_{11}\,(a_1 - n_{23}) \\ &\quad -\, a_1\,(n_{22} - n_{33}) - n_{23}\,(n_{22} + n_{33}) - 2\, n_{31}\,(a_3 + n_{12}) + \tfrac{1}{3}\Theta\,\omega_1 \\ &= -\, E_{23} + \tfrac{1}{2}\mathcal{E}_{23} \end{aligned} \tag{B.70}$$

$$\binom{2}{123} \qquad \mathbf{e}_2(n_{22}) - \mathbf{e}_1(a_3 - n_{12}) + \mathbf{e}_3(a_1 + n_{23}) - 2\, a_1\, n_{12} - 2\, a_2\, n_{22} - 2\, a_3\, n_{23} = 0 \tag{B.71}$$

$$\begin{aligned} R_{2312} &= -\,\mathbf{e}_3(a_1 + n_{23}) - \tfrac{1}{2}\,\mathbf{e}_2(n_{11} + n_{22} - n_{33}) + n_{11}\,(a_2 - n_{31}) \\ &\quad +\, (a_2 + n_{31})\,(n_{22} - n_{33}) + 2\, n_{23}\,(a_3 - n_{12}) \\ &= -\, E_{31} + \tfrac{1}{2}\mathcal{E}_{31} \end{aligned} \tag{B.72}$$

$$\binom{3}{123} \qquad \mathbf{e}_3(n_{33}) - \mathbf{e}_2(a_1 - n_{23}) + \mathbf{e}_1(a_2 + n_{31}) - 2\, a_1\, n_{31} - 2\, a_2\, n_{23} - 2\, a_3\, n_{33} = 0 \tag{B.73}$$

$$\begin{aligned} R_{2331} &= -\,\mathbf{e}_2(a_1 - n_{23}) - \tfrac{1}{2}\,\mathbf{e}_3(n_{22} - n_{33} - n_{11}) - n_{11}\,(a_3 + n_{12}) \\ &\quad +\, (a_3 - n_{12})\,(n_{22} - n_{33}) - 2\, n_{23}\,(a_2 + n_{31}) \\ &= -\, E_{12} + \tfrac{1}{2}\mathcal{E}_{12} \end{aligned} \tag{B.74}$$

$$\begin{aligned} R_{2323} &= \mathbf{e}_2(a_2 + n_{31}) + \mathbf{e}_3(a_3 - n_{12}) + \tfrac{1}{9}\Theta^2 + 3\omega_1^2 \\ &\quad -\, (a_1 + n_{23})\,(a_1 - n_{23}) - (a_2 + n_{31})^2 - (a_3 - n_{12})^2 \\ &\quad -\, \tfrac{3}{4}\, n_{11}^2 + \tfrac{1}{2}\, n_{11}\,(n_{22} + n_{33}) + \tfrac{1}{4}\,(n_{22} - n_{33})^2 \\ &= -\, E_{11} + \tfrac{1}{3}\,\varrho(\kappa + \tfrac{1}{12}\kappa_5^4\,\varrho) + \tfrac{1}{2}(\mathcal{E}_{00} + \mathcal{E}_{11}) + \tfrac{1}{3}\Lambda \end{aligned} \tag{B.75}$$

$$\begin{aligned}
R_{3131} &= \mathbf{e}_3(a_3+n_{12})+\mathbf{e}_1(a_1-n_{23})+\tfrac{1}{9}\Theta^2 \\
&\quad -(a_2+n_{31})(a_2-n_{31})-(a_3+n_{12})^2-(a_1-n_{23})^2 \\
&\quad +\tfrac{1}{4}n_{11}^2+\tfrac{1}{4}(n_{22}-n_{33})(2n_{11}-3n_{22}-n_{33}) \\
&= -E_{22}+\tfrac{1}{3}\varrho(\kappa+\tfrac{1}{12}\kappa_5^4\varrho)+\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{22})+\tfrac{1}{3}\Lambda \qquad (B.76)\\
R_{1212} &= \mathbf{e}_1(a_1+n_{23})+\mathbf{e}_2(a_2-n_{31})+\tfrac{1}{9}\Theta^2 \\
&\quad -(a_3+n_{12})(a_3-n_{12})-(a_1+n_{23})^2-(a_2-n_{31})^2 \\
&\quad +\tfrac{1}{4}n_{11}^2-\tfrac{1}{4}(n_{22}-n_{33})(2n_{11}-3n_{33}-n_{22}) \\
&= -E_{33}+\tfrac{1}{3}\varrho(\kappa+\tfrac{1}{12}\kappa_5^4\varrho)+\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{33})+\tfrac{1}{3}\Lambda\,. \qquad (B.77)
\end{aligned}$$

B.4.3 Generalised Friedmann equation

This is obtained by writing out $R^{\alpha\beta}{}_{\alpha\beta}$ (which corresponds to summing (B.75), (B.76) and (B.77)).

$$\begin{aligned}
&4\,\mathbf{e}_1(a_1)+4\,\mathbf{e}_2(a_2)+4\,\mathbf{e}_3(a_3)-6\,a_1^2-6\,a_2^2-6\,a_3^2 \\
&\quad -2\,n_{23}^2-2\,n_{31}^2-2\,n_{12}^2-\tfrac{1}{2}\,n_{11}^2-\tfrac{1}{2}(n_{22}-n_{33})^2+n_{11}(n_{22}+n_{33}) \\
&\quad +\tfrac{2}{3}\Theta^2+6\,\omega_1^2=2\,\varrho(\kappa+\tfrac{1}{12}\kappa_5^4\varrho)+2\mathcal{E}_{00}+2\Lambda\quad . \qquad (B.78)
\end{aligned}$$

B.4.4 Constraints on E and H

[123] 23

$$\begin{aligned}
&\mathbf{e}_1(E_{11}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{11}))+(\mathbf{e}_2-3a_2+n_{31})(E_{12}-\tfrac{1}{2}\mathcal{E}_{12})+(\mathbf{e}_3-3a_3-n_{12})(E_{31}-\tfrac{1}{2}\mathcal{E}_{31})= \\
&(n_{22}-n_{33})(E_{23}-\tfrac{1}{2}\mathcal{E}_{23})-[E_{22}-E_{33}-\tfrac{1}{2}(\mathcal{E}_{22}-\mathcal{E}_{33})]n_{23} \\
&+3a_1(E_{11}-\tfrac{1}{6}\mathcal{E}_{00}-\tfrac{1}{2}\mathcal{E}_{11})+\tfrac{1}{3}\mathbf{e}_1(\varrho)(\kappa+\tfrac{1}{6}\kappa_5^4\varrho)+3\omega_1 H_{11} \qquad (B.79)
\end{aligned}$$

[123] 31

$$\begin{aligned}
&\mathbf{e}_2(E_{22}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{22}))+(\mathbf{e}_3-3a_3+n_{12})(E_{23}-\tfrac{1}{2}\mathcal{E}_{23})+(\mathbf{e}_1-3a_1-n_{23})(E_{12}-\tfrac{1}{2}\mathcal{E}_{12})= \\
&(n_{33}-n_{11})(E_{31}-\tfrac{1}{2}\mathcal{E}_{31})-[E_{33}-E_{11}-\tfrac{1}{2}(\mathcal{E}_{33}-\mathcal{E}_{11})]n_{31} \\
&+3a_2(E_{22}-\tfrac{1}{6}\mathcal{E}_{00}-\tfrac{1}{2}\mathcal{E}_{22})+\tfrac{1}{3}\mathbf{e}_2(\varrho)(\kappa+\tfrac{1}{6}\kappa_5^4\varrho)+3\omega_1 H_{12} \qquad (B.80)
\end{aligned}$$

[123] 12

$$\begin{aligned}
&\mathbf{e}_3(E_{33}-\tfrac{1}{2}(\mathcal{E}_{00}+\mathcal{E}_{33}))+(\mathbf{e}_1-3a_1+n_{23})(E_{31}-\tfrac{1}{2}\mathcal{E}_{31})+(\mathbf{e}_2-3a_2-n_{31})(E_{23}-\tfrac{1}{2}\mathcal{E}_{23})= \\
&(n_{11}-n_{22})(E_{12}-\tfrac{1}{2}\mathcal{E}_{12})-[E_{11}-E_{22}-\tfrac{1}{2}(\mathcal{E}_{11}-\mathcal{E}_{22})]n_{12} \\
&+3a_3(E_{33}-\tfrac{1}{6}\mathcal{E}_{00}-\tfrac{1}{2}\mathcal{E}_{33})+\tfrac{1}{3}\mathbf{e}_3(\varrho)(\kappa+\tfrac{1}{6}\kappa_5^4\varrho)+3\omega_1 H_{31} \qquad (B.81)
\end{aligned}$$

[123] 01

$$\begin{aligned}
&(\mathbf{e}_1-3a_1)(H_{11})+(\mathbf{e}_2-3a_2+n_{31})(H_{12})+(\mathbf{e}_3-3a_3-n_{12})(H_{31})= \\
&-[3E_{11}-\tfrac{1}{2}(\mathcal{E}_{00}-\mathcal{E}_{11})+\varrho(\kappa+\tfrac{1}{6}\kappa_5^4\varrho)]\,\omega_1+(n_{22}-n_{33})H_{23}-(H_{22}-H_{33})n_{23} \qquad (B.82)
\end{aligned}$$

[123] 02

$$\begin{aligned}
&(\mathbf{e}_2-3a_2)(H_{22})+(\mathbf{e}_3-3a_3+n_{12})(H_{23})+(\mathbf{e}_1-3a_1-n_{23})(H_{12})= \\
&-3\,(E_{12}-\tfrac{1}{2}\mathcal{E}_{12})\,\omega_1+(n_{33}-n_{11})H_{31}-(H_{33}-H_{11})n_{31} \qquad (B.83)
\end{aligned}$$

[123] 03

$$\begin{aligned}
&(\mathbf{e}_3-3a_3)(H_{33})+(\mathbf{e}_1-3a_1+n_{23})(H_{31})+(\mathbf{e}_2-3a_2-n_{31})(H_{23})= \\
&-3\,(E_{31}-\tfrac{1}{2}\mathcal{E}_{31})\,\omega_1+(n_{11}-n_{22})H_{12}-(H_{11}-H_{22})n_{12}\,. \qquad (B.84)
\end{aligned}$$

Appendix C

Characterising Homogeneity from Isotropic Observations

We demonstrate the procedure in section 12.4 in the case where the input observations, after correcting for evolution, are in the RW form. It turns out that the LTB arbitrary functions assume their RW form. This amounts to a proof that a radially inhomogeneous dust universe is RW iff the area distance and number count relations as a function of redshift take the RW form. It is a special case of Theorem (B) where we assume that there is no evolution. These RW relations are

$$\hat{R}(z) = \frac{q_0\, z + (1 - q_0)\ (1 - \sqrt{2\, q_0\, z + 1})}{H_0\, {q_0}^2\, (1 + z)^2} \tag{C.1}$$

and

$$4\pi\hat{m}n = \frac{3}{H_0 q_0^3}\frac{\left[q_0\, z + (1 - q_0)\,(1 - \sqrt{2\, q_0\, z + 1})\right]^2 (2q_0 z + 1)^{-1/2}}{(1+z)^3} \tag{C.2}$$

respectively. Then we can integrate the null Raychaudhuri equation (12.40) once obtaining

$$\frac{dz}{dr} = H_0(1+z)^2\sqrt{2q_0 z + 1}\,. \tag{C.3}$$

This may be integrated once again (illustrating with the case $q_0 < \frac{1}{2}$) to obtain

$$\begin{aligned} r \ &= \ \frac{1}{H_0(1-2q_0)}\left[1 - \frac{\sqrt{2q_0 z + 1}}{1+z} + \frac{q_0}{\sqrt{1-2q_0}}\ln\left(\frac{\sqrt{2q_0 z+1} + \sqrt{1-2q_0}}{\sqrt{2q_0 z + 1} - \sqrt{1-2q_0}}\right)\right. \\ &\left. + \frac{q_0}{\sqrt{1-2q_0}}\ln\left(\frac{1-\sqrt{1-2q_0}}{1+\sqrt{1-2q_0}}\right)\right]\,. \end{aligned} \tag{C.4}$$

We continue by solving the first order linear differential equation for $M(z)$ (the effective gravitational mass) (12.17). This equation may be written as

$$\frac{d}{dz}\left[\frac{M(z)}{(1+z)d\hat{R}\,/\,dr}\right] = \frac{2\pi\hat{m}n}{(1+z)} + \frac{2\pi\hat{m}n}{(1+z)(d\hat{R}\,/\,dr)^2}\,. \tag{C.5}$$

We substitute the RW area distance and number count functions into this and find that

$$M(z) = H_0^2 q_0 \hat{R}^3 (1+z)^3 \tag{C.6}$$

and from (12.16) it follows that

$$2E(z) = (1 - 2q_0)H_0^2\hat{R}^2(1+z)^2\,. \tag{C.7}$$

These two relations show that $M \propto (2E)^{3/2}$. We next show that this universe has a simultaneous bangtime. We need (C.4) in addition to (12.22) and (12.19). Restricting ourselves to the case $q_0 < \frac{1}{2}$, it follows that

$$\tau = \frac{1}{H_0(1-2q_0)}\left[\frac{\sqrt{2q_0 z+1}}{1+z} - \frac{q_0}{\sqrt{1-2q_0}}\ln\left(\frac{\sqrt{2q_0 z+1}+\sqrt{1-2q_0}}{\sqrt{2q_0 z+1}-\sqrt{1-2q_0}}\right)\right] \tag{C.8}$$

and thus

$$t_B(r) = t_0 - \frac{1}{H_0(1-2q_0)}\left[1 + \frac{q_0}{\sqrt{1-2q_0}}\ln\left(\frac{1-\sqrt{1-2q_0}}{1+\sqrt{1-2q_0}}\right)\right] \tag{C.9}$$

which means that the bang surface is simultaneous[1]. This, together with $M \propto (2E)^{3/2}$ is all we need to show.

[1]We could set $\tau(0) = t_0$ in which case $t_B(r) = 0$, but this is not necessary.

Appendix D

Conditions for Existence of Solutions

The formulae we check consist of a series of quadratures. Thus one need only prove that the integrand is continuous in order to prove integrability and then deal with any points of possible discontinuity in turn.

D.1 Existence of solutions $r(z)$ and $z(r)$ to equation (12.44)

We know $\hat{R}(z) \to 0$ as $z \to 0$, but from (12.39) we expect $n(z) \sim \hat{R}^2(z)$, assuming $\hat{\rho}$, $\hat{m}(z)$ and dz/dr all $\sim$ constant as $z \to 0$, and of course $\hat{m}$, n, $\hat{R}$ and $(1+z)$ must all be ≥ 0. However the existence condition is less stringent.

Assume that near $z = 0$, $\hat{m}n(1+z)/\hat{R} = S(z)z^{\sigma}$, where σ is a constant and $S(0)$ is finite and non-zero. Then, to leading order near $z = 0$,

$$I_1(z) = \int_0^z \frac{\hat{m}n}{\hat{R}}(1+\overline{z})\,d\overline{z} = \int_0^z S(\overline{z})\overline{z}^{\sigma} d\overline{z}$$

$$I_1 = \left[S(0)\frac{\overline{z}^{\sigma+1}}{(\sigma+1)} \right]_0^z$$

which exists provided $\sigma > -1$. Since all the terms in the integrand are positive, I_1 is monotonically increasing.

We expect $\hat{R}(z)$ to increase from 0 to a maximum, at z_m say, and then decrease asymptotically towards 0. We assume no looping, i.e. $\hat{R}(z)$ is single valued and $d\hat{r}/dz$ doesn't diverge, and that there is only one maximum. Thus $d\hat{R}/dz$ goes from positive to negative values, and approaches 0 asymptotically. It is evident from (12.42) that $4\pi I_1(z) = 1$ where $d\hat{R}/dz = 0$ in LTB models. Therefore we write $d\hat{R}/dz(1+z) = P(z)(z - z_m)^{\alpha}$ and $\{1 - 4\pi I_1\} = Q(z)(z - z_m)^{\beta}$, where α & β are constants, and $P_m = P(z_m)$ & $Q_m = Q(z_m)$ are finite and non-zero. Then to leading order near $z = z_m$, (12.44) is

$$r - r_m = \int_{z_m}^z \frac{P(\overline{z})}{Q(\overline{z})}(\overline{z} - z_m)^{\alpha-\beta} d\overline{z}$$

$$r = r_m + \left[\frac{P_m}{Q_m} \frac{(z - z_m)^{\alpha-\beta+1}}{(\alpha - \beta + 1)} \right]_{z_m}^{z}$$

and so $r(z)$ exists for $\alpha - \beta + 1 > 0$

Our conditions for the existence of $r(z)$ are:

(i) $\hat{m}$, n, $\hat{R}$ and $(1+z)$ are ≥ 0,

(ii) near $z = 0$, $\hat{m}n/\hat{R} \sim z^{\sigma}$ with $\sigma > -1$,

(iii) $d\hat{R}/dz$ is finite everywhere,

(iv) near $z = z_m$, $d\hat{R}/dz(1+z) \sim (z - z_m)^{\alpha}$ and $\{1 - 4\pi I_1\} \sim (z - z_m)^{\beta}$, with $\alpha - \beta + 1 > 0$.

The condition for the existence of $z(r)$ is

(v) $z(r)$ is monotonic.

Conditions (i) & (ii) are manifestly reasonable. Conditions (iii) & (iv) are more problematic. As was shown previously [63], large enough inhomogeneities can create maxima and minima in $z(r)$ and so make $r(z)$ multi-valued, especially near $d\hat{R}/dz = 0$, in which case neither (iii) nor (v) would be satified. However, a multi-valued $r(z)$ manifested itself in a $\hat{R}(z)$ graph that looped. In practice, we don't expect to get a looping $\hat{R}(z)$ from the observational data. The values of ℓ and n at each z are averages over all measured values, and so are single valued by construction. Also $z(r)$ was always single valued in the numerical examples considered in [63], so, if $r(z)$ exists, then inverting it should not be a problem. Unfortunately (iv) is unlikely to be satisfied *exactly* for real data — the maximum in $\hat{R}$ will not be at exactly the same value as the locus of $4\pi I_1 = 1$, so one would have to tweak the fitted function to obtain a numerical solution. In other words, the function $r(z)$ is sensitive to observational error here. This however is not too serious, since there will be a measure of freedom in the smooth functions $\ell(z)$ and $n(z)$ that are fitted to the discrete data. In fact this problem exists even if the universe were genuinely homogeneous — even if we knew the source evolution functions exactly, the best-fit curves obtained from imprecise observational data would need adjustment to obtain a solution.

D.2 Existence of solutions $M(r)$ to equation (12.17)

Eq (12.17) has the form of an inhomogeneous linear first order ODE,

$$\frac{dM}{dr} + a(r)M = b(r)$$

which has the formal solution

$$M = \mu^{-1} \left[M_m \mu_m + \int_{r_m}^{r} b\mu dr \right] , \qquad \mu = e^{\int a dr}$$

where $M_m = M(r_m)$ and $\mu_m = \mu(r_m)$.

However we know that $d\hat{R}/dr$ goes through 0 at the maximum of $\hat{R}(r)$ — at r_m say, so both $a(r)$ and $b(r)$ are divergent there. It is evident that a & b are finite everywhere else, so we just have to show

$M(r)$ exists in the neighbourhood of this divergence. Suppose that, near $r = r_m$, $d\hat{R}/dr$ is of the form $d\hat{R}/dr \sim (r - r_m)^\nu$, where $\nu > 0$ and is constant, so $a(r) = F(r)(r - r_m)^{-\nu}$ and $b(r) = G(r)(r - r_m)^{-\nu}$, where $F(r)$ and $G(r)$ are finite, positive and non-zero. We expect $\nu = 1$. Then to leading order near r_m, for $\nu = 1$,

$$\mu = e^{F_m \ln(r - r_m)} = (r - r_m)^{F_m}$$

where $F_m = F(r_m)$, so

$$M = (r - r_m)^{-F_m} \left[0 + \int_{r_m}^{r} G(r)(r - r_m)^{F_m - 1} dr \right]$$

$$M = (r - r_m)^{-F_m} \left[\frac{G_m (r - r_m)^{F_m}}{F_m} - 0 \right]$$

and thus

$$M = \frac{G_m}{F_m}$$

to leading order. Comparison with (12.17) shows $M_m = \hat{R}_m/2$, which is consistent with the fact that $d\hat{R}/dr = 0$ lies on the apparent horizon $R = 2M$. Thus $M(r)$ exists in the neighbourhood of r_m.

For completeness we consider $\nu \neq 1$, working to leading order.

$$\mu = e^{F_m (r - r_m)^{1-\nu}/(1-\nu)}$$

$$M = e^{-F_m (r - r_m)^{1-\nu}/(1-\nu)} \left[\int_{r_m}^{r} G(r)(r - r_m)^{-\nu} e^{F_m (r - r_m)^{1-\nu}/(1-\nu)} dr \right]$$

$$M = e^{-F_m (r - r_m)^{1-\nu}/(1-\nu)} \left[\frac{G_m e^{F_m (r - r_m)^{1-\nu}/(1-\nu)}}{F_m} \right]_{r_m}^{r}$$

For $0 < \nu < 1$, we again get

$$M = \frac{G_m}{F_m}$$

to leading order, which is the expected value. But for $\nu > 1$ we get a divergence at $r = r_m$.

Thus our conditions for existence of $M(r)$ are that

(i) $\hat{m}$, n, $\hat{R}$ and dz/dr are ≥ 0, which ensure $\hat{\rho} \geq 0$, and

(ii) $\hat{R}(r) = \hat{R}(z(r))$ has a power-law maximum of the form $\hat{R} \sim (r - r_m)^\alpha$ with $1 < \alpha \leq 2$, with a quadratic maximum being the most reasonable.

D.3 Existence of solutions $\hat{R}(r)$ to equation (12.15)

The equation is

$$\frac{d\hat{R}}{dr} = \sqrt{1 + 2E} - \sqrt{\frac{2M}{\hat{R}} + 2E}$$

assuming that we take the positive root on the right — i.e. that large scale recollape has not begun anywhere on our past light cone. Near $r = 0$, E, M & $\hat{R}$ all go to 0, but our origin conditions require $M \sim r^3$, $E \sim r^2$ & $\hat{R} \sim r$, so the solution exists here. Where $\hat{R} = 2M$ the r.h.s. is zero, so $\hat{R}$ has a maximum. We already have $2E \geq -1$ for a well behaved metric, and $2M/\hat{R}$ & $2E$ are separately positive for parabolic and hyperbolic models, while for elliptic models we see from (12.18) & (12.19) that $(-2E)\hat{R}/M = (1 - \cos\eta) \leq 2$, so $\sqrt{2M/\hat{R} + 2E} = \sqrt{M/\hat{R}}\sqrt{2 + 2E\hat{R}/M}$ is always real.

Our only conditions for $\hat{R}(r)$ to exist are:
(i) the origin conditions of section 12.2.2 are satisfied.

D.4 Existence of solutions $z(r)$ to equation (12.33)

The origin conditions ensure that, near $r = 0$, $d^2\hat{R}/dr^2 \sim 0$, $d\hat{R}/dr \sim 1$, and $\hat{\rho} \sim$ constant, so that the integral exists in this neighbourhood.

Where the null cone crosses the apparent horizon, $\hat{R} = 2M$, we have $d\hat{R}/dr = 0$. However, we find from (12.15) & (12.12) that the integrand of (12.33) is

$$\left[\frac{d^2\hat{R}}{dr^2} + 4\pi\hat{\rho}\hat{R}\right] / \left(\frac{d\hat{R}}{dr}\right)$$

$$= \left[\frac{E'}{\sqrt{1+2E}} - \left(\frac{M'}{\hat{R}} - \frac{M}{\hat{R}^2}\left\{\sqrt{1+2E} - \sqrt{\frac{2M}{\hat{R}} + 2E}\right\} + E'\right) / \sqrt{\frac{2M}{\hat{R}} + 2E} + \frac{M'}{\hat{R}\sqrt{1+2E}}\right] / \left(\sqrt{1+2E} - \sqrt{\frac{2M}{\hat{R}} + 2E}\right)$$

$$= \left[\left(\frac{M\sqrt{1+2E}}{\hat{R}^2} - E' - \frac{M'}{\hat{R}}\right)\left\{\sqrt{1+2E} - \sqrt{\frac{2M}{\hat{R}} + 2E}\right\} / \left(\sqrt{1+2E}\sqrt{\frac{2M}{\hat{R}} + 2E}\right)\right] / \left(\sqrt{1+2E} - \sqrt{\frac{2M}{\hat{R}} + 2E}\right)$$

$$= \left[\frac{M\sqrt{1+2E}}{\hat{R}^2} - E' - \frac{M'}{\hat{R}}\right] / \left(\sqrt{1+2E}\sqrt{\frac{2M}{\hat{R}} + 2E}\right)$$

which is well behaved at $\hat{R} = 2M$.

Our conditions for existence of $\ln(1+z)$ are merely
(i) the origin conditions, $E \sim r^2$, $M \sim r^3$ near $r = 0$.

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