Low-dimensional internal categorial structures in weakly Mal'cev sesquicategories

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Abstract

In this work, the author introduces *pseudocategory* as a generalization for an internal category in dimension 2. First, a pseudocategory is defined [Ch1][1] as a system, consisting of a precategory diagram together with special 2-cells in a 2-category, satisfying some coherence conditions: if the 2-category is of the form Cat(B), of internal categories, internal functors and internal natural transformations in some category B, then a pseudocategory in (internal to) Cat(B) simultaneously generalizes internal bicategory in B and internal double-category in B (it is a pseudo-double-category in B, using the terminology of M. Grandis and R. Paré); later, a pseudocategory is considered in the more general context of a sesquicategory [Ch2], with one of the main results of this thesis being the description of pseudocategories in (internal to) a weakly Mal'cev sesquicategory [Ch9].

The notions of weakly Mal'cev category and weakly Mal'cev sesquicategory are also new concepts that are introduced here. Weakly Mal'cev categories, generalize Mal'cev categories, and seem to be an appropriate setting for the study of internal categories and precategories: an internal category (here, as in a Mal'cev category) is completely determined by its underlying reflexive graph; but (here, unlike in a Mal'cev category) not every internal category is an internal groupoid [Ch3]. A weakly Mal'cev sesquicategory is specially designed to mimic, as an axiomatic abstraction, a sesquicategory of the form Cat(B), with the 2-cell structure given by internal transformations, not necessarily natural, and B a weakly Mal'cev category [Ch8]. In fact, if thinking that a weakly Mal'cev category is a kind of partially enriched category, in the sense that there are many partial ternary operations in each hom-set, then a sesquicategory is weakly Mal'cev when the 2-cells are also enriched in the previous sense.

In [Ch1] and [Ch6] the name *tetracategory* is used for a structure with objects, morphisms, 2-cells, pseudo-cells, and tetra-cells, obtained as an abstraction for the category PsCat of pseudocategories, pseudofunctors, natural transformations, pseudo-natural transformations and modifications. An equivalence of categories PsCat(A)∼PsMor(A) is proved in [Ch6] between PsCat(A) the category of pseudocategories in A, an additive 2-category with kernels [Ch5], and PsMor(A) an *ad hoc* category of "pseudomorphisms" in A. This result may be seen as the 2-dimensional analogue for the well known equivalence Cat(A)∼Mor(A) if A is just an additive category with kernels. In particular, for the case A=Cat(Ab), the general result gives us a description for the tetracategory of internal bicategories in abelian groups, where it is immediate to observe that homotopies between 2-chains indeed correspond to pseudo-natural transformations.

At the end, a final note is added for the case of internal bicategories in Groups, that can be derived from the more general result of pseudocategories in a weakly Mal'cev sesquicategory, taking the sesquicategory of crossed-modules with derivations as 2-cells.

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[1] Troughout this thesis a reference such as [Chn] refers to Chapter number n.

For Susan: *adeo venusto, ut nil supra*

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'I would also have added a word of advice concerning the way to read this work, which is that I would like it first to be read rapidly in its entirety, like a novel, without the reader forcing his attencion too much or stopping at the difficulties which he may encounter in it, simply in order to have a broad view of the matters I have treated in it. And after that, if the reader judges that these matters merit examination, and is curious to know their causes, he can read the book a second time, in order to notice the sequence of my reasonings.'

Descartes, 'Letter from the Author', The Principles of Philosophy

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Introduction

State of art: a perspective written by G. Janelidze

The study of internal categorical structures in "nice" algebraic categories and their abstract counterparts has a long history involving several areas of mathematics from homotopy theory to universal algebra. Its categorical-algebraic side includes, but is not limited to, the following:

1. As mentioned in [J4], referring to a discussion with F. Borceux, internal categories in groups were first described as crossed modules by R. Lavendhomme. Since internal categories in groups are the same as internal groups in categories, this description turned out to be a reformulation of a result on so-called categorical groups known in homotopy theory independently. This and related results on crossed modules and crossed complexes are very central in homotopical algebra, as one can see from many important papers of R. Brown, J.-L. Loday, T. Porter, and their collaborators and followers.
2. Some internal groupoids in varieties of groups with multiple operations that occur in Galois theory of generalized central extensions were described in [J1] in situations where the theory of crossed modules could not be applied yet. Internal crossed modules in semi-abelian categories were invented only many years later in [J4].
3. Generalizing the description of internal categories in Mal'tsev varieties [J2], the description of certain internal categories and all internal groupoids in congruence modular varieties has been obtained in [JP1]. Some of the results of [J2] were also generalized to Mal'tsev categories by A. Carboni, M. C. Pedicchio, and N. Pirovano in [CPP]; that paper also contained important new ideas later used by M. C. Pedicchio in her categorical version of the Smith commutator theory.
4. A new categorical structure called pseudogroupoid was introduced in [JP2] in order to extend Pedicchio's categorical approach to commutator theory beyond the Mal'tsev case and even beyond the congruence modular case.
5. Remarkable further developments and improvements of results in various directions had later been made in a number of papers of M. Gran and his

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> collaborators (see [G] and references there). One of those collaborators is D. Bourn, whose previous and recent independent work made an enormous contribution, not just to the topic of this discussion, but to categorical algebra in general.

In spite of the successful investigations at semi-abelian, Mal'tsev, and even congruence modular level, the abelian case remains important. The study of higher categorical structures in the abelian case was first suggested in [J3]; it was motivated by observing that they usually form presheaf categories - which of course helps to compare them. As far as we know, apart from the classical result "strict n-categories = n-complexes" and (a part of) the present work, there is only one result in this direction, due to S. Crans [C], describing the so-called internal teisi in the category of abelian groups.

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A word from the author

As a Master student, under the supervision of Professor G. Janelidze, my task was to describe internal bicategories in Groups. After some months of investigation it turn out that I was not to be ready yet for such a work with groups. The issue was then shifted to abelian groups and the following result was obtained: an internal bicategory in Ab is completely determined, up to an isomorphism, by a diagram in Ab of the form

$$\begin{tikzcd}
A \arrow[r, "d_1"] & B \arrow[r, "d_0"] \arrow[l, bend right, "\lambda"'] \arrow[l, bend left, "\rho"] & C \arrow[ll, bend left, "\eta"]
\end{tikzcd}$$

satisfying $d_0d_1 = 0$ (Chapter 4).

Even in abelian groups I have found two major difficulties in obtaining this result. The first one was in formalizing the precise definition of internal bicategory, the second one was in checking all the coherence conditions involved, namely the pentagon coherence condition, since the rest of the structure was already determined.

With the experience of writing down the definition for internal bicategory, and checking the calculations many times, I started to realize that the definition was simpler to encode in Cat(C) rather than directly in a category C, because in this way the vertical composition was intrinsic to the objects of Cat(C) and there was no longer the necessity of manipulating it explicitly. However, it would not give the exact definition of internal bicategory, and a pseudo-double-category in the sense of Grandis and Paré was obtained instead. A pseudo-double-category simultaneously generalizes bicategory and double-category, involving objects, morphisms, 2-cells, pseudo-cells and square-cells. By that time it was also clear to me that instead of Cat(C), I could consider an abstract 2-category. This way I arrived to the definition of pseudocategory internal to a 2-category, and the results for bicategory or pseudo-double-category would be obtained if calculating it in a 2-category of the form Cat(C). The problem of a complicated definition was solved: a pseudocategory in a 2-category is just a slight modification of the definition of internal category, where the commutative diagrams representing the unitary and associative axioms are replaced by an additional structure of special 2-cells that fill in the diagrams and add some coherence axioms; but for the concrete case of describing internal bicategories in Ab, the problem of manipulating with 2-cells in Cat(Ab), or even in Mor(Ab), was still somehow difficult and demanding. The solution for this was to axiomatize the 2-category Mor(Ab), to work with its axioms and properties, to prove the results, and only then to interpret them in Mor(Ab). This way I formalized the concept of an additive 2-category and described pseudocategories in an additive 2-category with kernels (Chapter 5).

The study of internal categorical structures in categories was now moved to the study of internal categorical structures in 2-categories, involving objects,

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morphisms and 2-cells, as part of the structure, and identities between morphisms and 2-cells as part of the axioms. To develop a theory of pseudocategories we should also have their morphisms and so on. The next step was to define pseudofunctors, natural transformations, pseudo-natural transformations and modifications (Chapter 1). In this way, a category denoted PsCat(C), of pseudocategories and pseudofunctors in a 2-category C is obtained. With the extra structure of natural and pseudo-natural transformations and also modifications, this category is in fact a kind of tetracategory with objects, morphisms, 2-cells, pseudo-cells and tetra-cells where each hom-set is enriched in PsCat.

With the theory settled it was then desirable to know what is PsCat(Cat(Ab)). It turns out to be a presheaf category where the 2-cells are also involved (Chapter 6).

With the abelian case completed, it was congenital to investigate the non abelian case.

The first problem was to find an appropriate setting to work with. We need to use 2-cells. I first tried the semi-abelian context, but it turned out with many difficulties to handle and it was not clear how to introduce 2-cells. This later problem of introducing 2-cells was very restrictive, and, slowly, I started to realize that a big part of the theory of pseudocategories could be developed not only in a 2-category, but also in a sesquicategory, or a category with a 2-cell structure. The experience in calculating with 2-cells in additive 2-categories also suggest me to use an additive notation for the general vertical composition of 2-cells, and this simple change of notation proved to be very useful in introducing the idea of a category with many different 2-cell structures (Chapter 2).

For each possible different 2-cell structure, given on a particular category C, in order to make it a sesquicategory, we obtain a (possibly) different category PsCat(C). For example: if considering C with the discrete 2-cell structure (only identity 2-cells) then PsCat(C)=Cat(C); while if considering C with the codiscrete 2-cell structure (exactly one 2-cell between each pair of parallel morphisms) then PsCat(C) is equal to PreCat(C), the category of internal precategories in C.

In order to go beyond the abelian world it was clear that we needed to find an appropriate setting; at the same time rich enough to do calculations, strong enough to compute internal categories as well as precategories, and giving the possibility to introduce a 2-cell structure with the very same properties.

The result was a new concept that was called *weakly Mal'cev category* (Chapter 3). It is such that an internal category in there, as it happens in a Mal'cev category, is completely determined by its underlying reflexive graph, however, unlike in a Mal'cev category, not every internal category is an internal groupoid. Moreover, an internal precategory (Chapter 7) is determined by a diagram of the form

$$\begin{tikzcd} C_1 \arrow[loop above, "u"] \arrow[loop below, "v"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \tag{1}$$

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satisfying

$$\begin{aligned} de = 1_{C_0} = ce \\ du = d = dv \\ cu = c = cv \\ ue = ve \end{aligned} \tag{2}$$

plus an admissibility property for the triple (u, e, v).

The main axiom defining a weakly Mal'cev category is a property about morphisms and it is suitable to be used also with respect to the 2-cells: the result is what we call weakly Mal'cev sesquicategory (Chapter 8).

Lastly, considering the category of crossed modules with derivations (an example of a weakly Mal'cev sesquicategory, that it is in fact a 2-category) I was finally able to describe pseudocategories in Cat(Gr), obtaining this way a description of internal bicategories in Groups (Chapter 9).

Organization of this thesis

The thesis is divided into three conceptual parts. Each part contains three chapters. Each chapter is written as an article and can be read independently from the others. Each chapter also contains its own references and the marks of the form [Chn] are used to distinguish between external references and the internal references from one chapter to another.

The first part "Establishing the framework" gives, as suggested, an account of the definitions, notation and assumptions that will be used in this thesis. The tetracategory of pseudocategories in (internal to) a 2-category is presented in Chapter 1 and extended to the more general context of a sesquicategory in Chapter 2, while the concept of weakly Mal'cev category is introduced in Chapter 3.

The second part "In the Abelian World" is devoted to a survey of studies in the abelian case. An internal bicategory in Ab is calculated in Chapter 4, a pseudocategory in an additive 2-category with kernels is computed in Chapter 5, and the whole tetracategory of pseudocategories is described in Chapter 6.

The third part "Beyond the Abelian World" goes to the more general context of a weakly Mal'cev sesquicategory, and describes pseudocategories in there. One of the important examples of a weakly Mal'cev sesquicategory (that is in fact a 2-category) is the category of crossed modules with derivations as 2-cells. Pseudocategories in there give in particular a description for internal bicategories in Groups.

A diagram showing the connection between chapters may be displayed as follows, where an arrow from chapter i to chapter j indicates at least one reference

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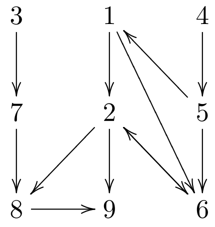
in chapter j to chapter i.



Each chapter may be summarized as follows (using the order 4, 5, 1, 6, 2, 3, 7, 8, 9, suggested by the diagram).

Chapter 4 consists of a preprint with the English version of the author's Master thesis describing "Internal Bicategories in Ab". It is included here only as background material, containing a definition of internal bicategory, illustrating the tools and techniques that were used to work in the 2-category Mor(Ab), of morphisms of abelian groups, though by that time they were not understood as such.

Chapter 5 consist of the author's paper "Weak categories in additive 2-categories with kernels" published in the Fields Institute Communications, Vol.43, 387-410, 2004, and it contains: the definition of pseudocategory as a generalization of internal category in dimension 2; the definition of additive 2-category as a frame work to the study of internal categorical structures in dimension 2, within an additive context; the description of internal pseudocategories in an additive 2-category with kernels, with further particularization to the case Mor(Ab).
It is worth noting that the requirement for the existence of kernels is only used to prove that each split epi is, up to isomorphism, a biproduct projection. Thus, if we restrict the study of internal pseudocategories to the particular class of split epis that are given by product projections, the result then generalizes to arbitrary categories - but we will not discuss this further in here.

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Chapter 1 consists of the author's paper "Pseudo-categories" published in the Journal of Homotopy and Related Structures, Vol.1(1), 47-78, 2006, and it contains the formal definitions of pseudo-functor, natural transformation, pseudo-natural transformation, and modification, in the context of a 2-category (Chapter 2 shows that it is possible to extend this concepts to the more general context of a sesquicategory).

Chapter 6 consists of a paper "The (tetra)category of pseudocategories in an additive 2-category with kernels", submitted to the journal Applied Categorical Structures for the special issue CT2007 and it contains a description of the (tetra)category PsCat(A) of peudocategories, pseudofunctors, natural transformations, pseudo-natural transformations and modifications, as defined in Chapter 1, in an additive 2-category, A, with kernels, as formalized in Chapter 5. The results thus obtained extend the well known categorical equivalence Cat(Ab)∼Mor(Ab) to the 2-dimensional case.

Chapter 2 contains a generalization of the notion of 2-Ab-category to a category with 2-cells "enriched" (in a suitable sense) in any category with a forgetful functor into Sets. A 2-Ab-category is completely determined by a triple

$$(\mathbf{A}, H, D)$$

in which **A** is an Ab-category,

$$H : \mathbf{A}^{op} \times \mathbf{A} \longrightarrow Ab$$

is an Ab-functor and

$$D : H \longrightarrow \hom_{\mathbf{A}}$$

is a natural transformation, such that

$$D(x)\, y = xD(y) \tag{3}$$

for every appropriate x and y (writing gxf for $H(f,g)(x)$). Condition (3) is very strong and it is responsible for the naturality of the 2-cells. If ignoring it, the triple $(\mathbf{A}, H, D)$ no longer gives a 2-category in general, since there is no way to horizontally compose the 2-cells − an appropriate name for it would be sesqui-Ab-category. The interest in removing restriction (3) is to be able to consider examples such as abelian 2-chain complexes with homotopies. This practice of considering a 2-Ab-category as a system $(\mathbf{A}, H, D)$, lead us to consider more general settings, such as **A** a category, with the property that there is a functor

$$map : \mathbf{A}^{op} \times \mathbf{A} \longrightarrow \text{Groups}$$

[we may think of $map(A, B)$ as the set of maps from A to B, not necessarily homomorphisms, and **A** an algebraic variety containing a group operation, giving componentwise a group operation to $map(A, B)$] such that

$$\hom(A, B) \subseteq Umap(A, B), \quad \text{naturally for all } A \text{ and } B,$$

[with U :Groups⟶Sets, the forgetful functor],

$$H : \mathbf{A}^{op} \times \mathbf{A} \longrightarrow \text{Groups}$$

a functor and

$$D : H \longrightarrow map$$

a natural transformation. We then define a 2-cell from A to B as a pair (x, f) such that $x \in H(A, B)$, $f : A \longrightarrow B$ is a morphism in **A** and the element $D(x) + f$ of the group $map(A, B)$ belongs to $\hom(A, B)$; it is the codomain of $(x, f) : f \longrightarrow D(x) + f$. The example of crossed modules is obtained in this way.

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Chapter 3 introduces the notion of weakly Mal'cev category, giving a characterization of internal categories in there. A weakly Mal'cev category has two features: (a) it has pullbacks of split epimorphisms along split epimorphisms, that is, every diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd} , \quad fr = 1_C = gs \tag{4}$$

can be completed into a commutative square of split epimorphisms

$$\begin{tikzcd} P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

such that the up and left square is a pullback diagram; (b) in every such completed square, the pair (e_1, e_2) is jointly epimorphic.
As a consequence, for every object D and triple of morphisms (h, l, k)

$$\begin{tikzcd} A \arrow[dr, "h"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "l"] & B \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

such that $hr = l = ks$ there is at most one morphism

$$\varphi : P \longrightarrow D$$

with the property that $\varphi e_1 = h$ and $\varphi e_2 = k$. In the case of existence of such a morphism, φ, it is written as $\varphi = \begin{bmatrix} h & l & k \end{bmatrix}$ and the triple (h, l, k) is said to be *admissible* with respect to (4).
An internal category in a weakly Mal'cev category is completely determined by a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce$$

with the property that the triple $(1_{C_1}, e, 1_{C_1})$ is admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} , \tag{5}$$

and furthermore, it is an internal groupoid if and only if the triple $(\pi_2, 1_{C_1}, \pi_1)$ is admissible with respect to

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, shift left, "{[1\, e\, 1]}"] & C_1 \arrow[l, shift left, "e_2"] \arrow[r, shift right, "e_1"'] & C_1 \times_{C_0} C_1 \arrow[l, shift right, "{[1\, e\, 1]}"'] \end{tikzcd} .$$

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Chapter 7 investigates all the possible reasonable variations in the intermediate axioms between an internal category and a precategory, in the context of a weakly Mal'cev category.

An internal precategory in a weakly Mal'cev category is determined by a diagram of the form (1), satisfying conditions (2) and the admissibility of the triple (u, ue, v) with respect to (5). In particular if $ue = e = ve$, then the associativity axiom is equivalent to having

$$\begin{aligned} uu &= u \\ uv &= vu \\ v &= vv. \end{aligned}$$

Chapter 8 introduces the notion of weakly Mal'cev sesquicategory as a weakly Mal'cev category, together with a 2-cell structure satisfying the property that the pair (e_1, e_2) is jointly epimorphic also with respect to the 2-cells. The main example of such a structure is obtained by considering in Cat(B), for B a weakly Mal'cev category, the 2-cell structure of internal transformations, not necessarily natural. The pentagon coherence condition

$$\begin{tikzcd}
 & f(g(hk)) \arrow[rr, "f\alpha_{g,h,k}"] \arrow[dl, "\alpha_{f,g,hk}"'] & & f((gh)k) \arrow[dr, "\alpha_{f,gh,k}"] & \\
(fg)(hk) \arrow[drr, "\alpha_{fg,h,k}"'] & & & & (f(gh))k \arrow[dll, "\alpha_{f,g,h}k"] \\
 & & ((fg)h)k & &
\end{tikzcd}$$

is also investigated here.

Cahpter 9 gives the main results in the weakly Mal'cev context: establishing an equivalence between the set of coherence conditions

$$\begin{tikzcd}
1(fg) \arrow[rr, "\alpha_{1,f,g}"] \arrow[dr, "\lambda_{fg}"'] & & (1f)g \arrow[dl, "\lambda_f g"] & f(g1) \arrow[rr, "\alpha_{f,g,1}"] \arrow[dr, "f\rho_g"'] & & (fg)1 \arrow[dl, "\rho_{fg}"] \\
 & fg & & & fg &
\end{tikzcd}$$

and the set of coherence conditions

$$\begin{tikzcd}
f(1g) \arrow[rr, "\alpha_{f,1,g}"] \arrow[dr, "f\lambda_g"'] & & (f1)g \arrow[dl, "\rho_f g"] & & 1(f1) \arrow[r, "\alpha_{1,f,1}"] \arrow[dl, "1\rho_f"'] & (1f)1 \arrow[dr, "\lambda_f 1"] & \\
 & fg & & 1f \arrow[drr, "\lambda"'] & & & f1 \arrow[dll, "\rho"] \\
 & & & & & f &
\end{tikzcd} ;$$

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and proving that either one of the equivalent sets of conditions above, completely determine the associativity isomorphism α and hence, the pentagon coherence condition becomes a property of λ and ρ, the left and right identity isomorphisms.
Assuming $11 = 1$ instead of $11 \cong 1$, the pentagon coherence condition is trivially satisfied and a pseudocategory (where we have to discard the requirement of coherence for all, if not in the presence of a natural 2-cell structure, see remark below) is completely determined by a diagram of the form (1), together with invertible 2-cells

$$\rho : u \longrightarrow 1_{C_1} \ , \ \lambda : v \longrightarrow 1_{C_1}$$

such that

$$\begin{aligned} d\lambda &= 0_d = d\rho \\ c\lambda &= 0_c = c\rho \\ \lambda e &= 0_e = \rho e \end{aligned}$$

$$\begin{aligned} \lambda v &= v\lambda \\ \rho + \lambda u &= \lambda + v\rho \\ \lambda + \rho v &= \rho + u\lambda \\ \rho u &= u\rho \end{aligned} \tag{6}$$

plus the admissibility of the following tripes

$$\begin{aligned} &(u, e, v) \\ &(\alpha_2, 0_e, \alpha_3) \\ &\big(\alpha_1, 0_e, \begin{bmatrix}\alpha_2 & 0_e & \alpha_3\end{bmatrix}\big) \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= -\rho u \\ \alpha_2 &= -u\lambda + \lambda u \\ \alpha_3 &= \lambda v. \end{aligned}$$

Remarks. The 2-cell α is determined by the admissible triple $\big(\alpha_1, 0_e, \begin{bmatrix}\alpha_2 & 0_e & \alpha_3\end{bmatrix}\big)$. The set of conditions (6) ensures the naturality of λ and ρ with respect to each other, that is $\lambda \circ \lambda, \lambda \circ \rho, \rho \circ \rho, \rho \circ \lambda$, and this is sufficient to give the above result, however it does not guarantee the commutativity of all the diagrams involving instances of α, λ, ρ, possible nested with the composition map $m = \begin{bmatrix}u & e & v\end{bmatrix}$ (obtained from the admissibility of the triple (u, e, v)), as in the Mac Lane's Coherence Theorem. To achieve that, it

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is necessary (and sufficient) to ask for the naturality of α, λ, ρ with respect to all such 2-cells that can be obtained by appropriate instances of $\alpha, \lambda, \rho, 0_m$. For example to have coherence for: five (non identity) elements, we should have the naturality of α with respect to $m\,(0_1 \times \alpha)$ and $m\,(\alpha \times 0_1)$; for six (non identity) elements, we need α to be natural with respect to $m\,(0_1 \times m\,(0_1 \times \alpha))$, $m\,(0_1 \times m\,(\alpha \times 0_1))$, $m\,(m\,(0_1 \times \alpha) \times 0_1)$ and $m\,(m\,(\alpha \times 0_1) \times 0_1)$; *et cetera.*

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Part I

Establishing the framework

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Chapter 1

Pseudocategories

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This chapter is a paper with the same title published in the Journal of Homotopy and Related Structures, Vol 1 (1), 47-78, 2006. It is presented here exactly as it is published. A single remark has to do with notation: instead of the term pseudo-modification, just modification is used in the remaining of the thesis; also pseudocategory and pseudofunctor are used instead of pseudo-category and pseudo-functor. The references [5], [6] and [7] correspond respectively to Chapters 4, 5 and Appendix A of this thesis.

Title: Pseudo-Categories

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Key words: bicategory, double category, weak category, pseudo-category, pseudo double category, pseudo-functor, pseudo-natural transformation, pseudo-modification, weakly cartesian closed.

Abstract: We provide a complete description of the category of pseudo-categories (including pseudo-functors, natural and pseudo-natural transformations and pseudo modifications). A pseudo-category is a non strict version of an internal category. It was called a weak category and weak double category in some earlier papers. When internal to Cat it is at the same time a generalization of a bicategory and a double category. The category of pseudo-categories is a kind of "tetracategory" and it turns out to be cartesian closed in a suitable sense.

1.1 Introduction

The notion of pseudo-category[1] considered in this paper is closely related and essentially is a special case of several higher categorical structures studied for example by Grandis and Paré [2], Leinster [3], Street [6],[7], among several others. We have arrived to the present definition of pseudo-category (which some authors would probably call a pseudo double category) while describing internal bicategories in Ab [3]. We even found it easier, for our particular purposes, to work with pseudo-categories than to work with bicategories. Defining a pseudo-category we begin with a 2-category, take the definition of an internal category there, and replace the equalities in the associativity and identity axioms by the existence of suitable isomorphisms which then have to satisfy some coherence conditions. That is, let **C** be a 2-category, a pseudo-category in (internal to) **C** is a system

$$(C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)$$

where C_0, C_1 are objects of **C**,

$$d, c : C_1 \longrightarrow C_0 \ , \ e : C_0 \longrightarrow C_1 \ , \ m : C_1 \times_{C_0} C_1 \longrightarrow C_1$$

are morphisms of **C** , with $C_1 \times_{C_0} C_1$ the object in the pullback diagram

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] \\
C_1 \arrow[r, "d"] & C_0
\end{tikzcd} ;$$

$$\begin{aligned}
\alpha \ &: \ m\,(1_{C_1} \times_{C_0} m) \longrightarrow m\,(m \times_{C_0} 1_{C_1}) \ , \\
\lambda \ &: \ m\,\langle ec, 1_{C_1} \rangle \longrightarrow 1_{C_1} \ , \ \rho : m\,\langle 1_{C_1}, ed \rangle \longrightarrow 1_{C_1},
\end{aligned}$$

are 2-cells of **C** (which are isomorphisms), the following conditions are satisfied

$$de = 1_{c_0} = ce, \tag{1.1}$$

$$dm = d\pi_2 \ , \ cm = c\pi_1, \tag{1.2}$$

$$\begin{aligned} d \circ \lambda &= 1_d = d \circ \rho, \\ c \circ \lambda &= 1_c = c \circ \rho, \end{aligned} \tag{1.3}$$

$$d \circ \alpha = 1_{d\pi_3} \ , \ c \circ \alpha = 1_{c\pi_1}, \tag{1.4}$$

$$\lambda \circ e = \rho \circ e, \tag{1.5}$$

---

[1] In the previous work [4] the word "weak" was used with the same meaning. We claim that "pseudo" is more apropriate because it is the intermediate term between precategory and internal category. Also it agrees with the notion of pseudo-functor, already well established.

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and the following diagrams commute

$$\begin{tikzcd}
& \bullet \arrow[rr, "m\circ(1_{C_1}\times_{C_0}\alpha)"] \arrow[dl, "\alpha\circ(1_{C_1}\times_{C_0}1_{C_1}\times_{C_0}m)"'] & & \bullet \arrow[dr, "\alpha\circ(1_{C_1}\times_{C_0}m\times_{C_0}1_{C_1})"] & \\
\bullet \arrow[drr, "\alpha\circ(m\times_{C_0}1_{C_1}\times_{C_0}1_{C_1})"'] & & & & \bullet \arrow[dll, "m\circ(\alpha\times_{C_0}1_{C_1})"] \\
& & \bullet & &
\end{tikzcd} \tag{1.6}$$

$$\begin{tikzcd}
\bullet \arrow[rr, "\alpha\circ(1_{C_1}\times_{C_0}<ec{,}1_{C_1}>)"] \arrow[dr, "m\circ(1_{C_1}\times_{C_0}\lambda)"'] & & \bullet \arrow[dl, "m\circ(\rho\times_{C_0}1_{C_1})"] \\
& \bullet &
\end{tikzcd} \tag{1.7}$$

Examples:

1. When **C**=Set with the discrete 2-category structure (only identity 2-cells) one obtains the definition of an ordinary category since α, λ, ρ are all identities;

2. When **C**=Set with the codiscrete 2-category structure (exactly one 2-cell for each pair of morphisms) one obtain the definition of a precategory (see Chapter 7) since α, λ, ρ always exist and the coherence conditions are trivially satisfied;
   (This result applies equally to any category)

3. When **C**=Grp considered as a 2-category: every group is a (one object) category and the inclusion functor

   $$\text{Grp} \longrightarrow \text{Cat}$$

   induces a 2-category structure in Grp, where a 2-cell

   $$\tau : f \longrightarrow g \ \ , \ (f, g : A \longrightarrow B \text{ group homomorphisms})$$

   is an element $\tau \in B$, such that for every $x \in A$,

   $$g(x) = \tau f(x) \tau^{-1}.$$

   With this setting, a pseudo-category in Grp is described (see [7]) by a group homomorphism

   $$\partial : X \longrightarrow B,$$

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an arbitrary element

$$\delta \in \ker \partial$$

and an action of B in X (denoted by $b \cdot x$ for $b \in B$ and $x \in X$) satisfying

$$\begin{aligned} \partial\left(b \cdot x\right) &= b\partial\left(x\right)b^{-1} \\ \partial\left(x\right) \cdot x' &= x + x' - x \end{aligned}$$

for every $b \in B,\ x, x' \in X$. Note that the difference to a crossed module (description of an internal category in Grp) is that in a crossed module the element $\delta = 1$.

The pseudo-category so obtained is as follows: objects are the elements of B, arrows are pairs $(x, b) : b \longrightarrow \partial x + b$ and the composition of $(x', \partial x + b) : \partial x + b \longrightarrow \partial x' + \partial x + b$ with $(x, b) : b \longrightarrow \partial x + b$ is the pair $(x' + x - \delta + b \cdot \delta, b) : b \longrightarrow \partial x' + \partial x + b$. The isomorphism between $(0, \partial x + b) \circ (x, b) = (x, b) \circ (0, b)$ and (x, b) is the element $(\delta, 0) \in X \rtimes B$. Associativity is satisfied, since $(x'', \partial x' + \partial x + b) \circ ((x', \partial x + b) \circ (x, b)) = ((x'', \partial x' + \partial x + b) \circ (x', \partial x + b)) \circ (x, b)$.

4. When **C**=Mor(Ab) the 2-category of morphisms of abelian groups, the above definition gives a structure which is completely determined by a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "\partial"] \arrow[d, "k_1"'] & A_0 \arrow[d, "k_0"] \\ B_1 \arrow[r, "\partial'"] & B_0 \end{tikzcd}$$

together with three morphisms

$$\begin{aligned} \lambda, \rho &: A_0 \longrightarrow A_1, \\ \eta &: B_0 \longrightarrow A_1, \end{aligned}$$

satisfying conditions

$$\begin{aligned} k_1\lambda &= 0 = k_1\rho, \\ k_1\eta &= 0, \end{aligned}$$

and it may be viewed as a structure with objects, vertical arrows, horizontal arrows and squares, in the following way (see also Section 5 of Chapter 5 of this thesis for more details)

$$\begin{tikzcd} b \arrow[rr, "{(b,x)}"] \arrow[dd, "\binom{b}{d}"'] & & b + k_0\left(x\right) \arrow[dd, "\binom{b+k_0(x)}{d+k_1(y)}"] \\ & \begin{pmatrix} b & x \\ d & y \end{pmatrix} & \\ b + \partial'\left(d\right) \arrow[rr, "{\left(b+\partial'(d), x+\partial(y)\right)}"'] & & * \end{tikzcd} ,$$

where $*$ stands for $b + \partial'\left(d\right) + k_0\left(x + \partial\left(y\right)\right) = b + k_0\left(x\right) + \partial'\left(d + k_1\left(y\right)\right)$.

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5. When **C**=Top (with homotopy classes as 2-cells) we find the following particular example. Let X be a space and consider the following diagram

$$\begin{tikzcd} X^I \times_X X^I \arrow[r, "m"] & X^I \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X \arrow[l, "e"'] \end{tikzcd}$$

where X^I is equipped with the compact open topology and $X^I \times_X X^I$ with the product topology (I is the unit interval), with

$$X^I \times_X X^I = \{\langle g, f\rangle \mid f(0) = g(1)\}$$

and d, e, c, m defined as follows

$$\begin{array}{rcl} d(f) & = & f(0) \\ c(f) & = & f(1) \\ e_x(t) & = & x \\ m(f,g) & = & \left\{ \begin{array}{c} g(2t), t < \frac{1}{2} \\ f(2t-1), t \geq \frac{1}{2} \end{array} \right. \end{array}$$

with $f, g : I \longrightarrow X$ (continuous maps) and $x \in X$. The homotopies α, λ, ρ are the usual ones.

6. When **C**=Cat the objects C_0 and C_1 are (small) categories, and the morphisms d, c, e, m are functors. We denote the objects of C_0 by the first capital letters in the alphabet (possible with primes) $A, A', B, B', ...$ and the morphisms by first small letters in the alphabet $a : A \longrightarrow A', b : B \longrightarrow B', ...$. We will denote the objects of C_1 by small letters as $f, f', g, g', ...$ and the morphisms by small greek letters as $\varphi : f \longrightarrow f', \gamma : g \longrightarrow g', ...$. We will also consider that the functors d and c are defined as follows

$$\begin{tikzcd} C_1 & & C_0 \\ & & a : A \longrightarrow A' \\ \varphi : f \longrightarrow f' \arrow[urr, "d"] \arrow[drr, "c"'] & & \\ & & b : B \longrightarrow B' \end{tikzcd}$$

hence, the objects of C_1 are arrows $f : A \longrightarrow B, f' : A' \longrightarrow B'$, that we will always represent using inplace notation as $A \xrightarrow{f} B$, $A' \xrightarrow{f'} B'$ to distinguish from the morphisms of C_0, and thus the morphisms of C_1 are of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"] & B' \arrow[from=1-1, to=2-2, phantom, "\Downarrow \varphi"] \end{tikzcd} .$$

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The functor e sends $a : A \longrightarrow A'$ to

$$\begin{tikzcd}
A \arrow[rr, "id_A"] \arrow[d, "a"'] & {} \arrow[d, Rightarrow, "id_a"] & A \arrow[d, "a"] \\
A' \arrow[rr, "id_{A'}"'] & {} & A'
\end{tikzcd},$$

while the functor m sends $\langle \gamma, \varphi \rangle$ to $\gamma \otimes \varphi$ as displayed in the diagram below

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "a"'] & {} \arrow[d, Rightarrow, "\varphi"] & B \arrow[rr, "g"] \arrow[d, "b"] & {} \arrow[d, Rightarrow, "\gamma"] & C \arrow[d, "c"] & & A \arrow[rr, "g\otimes f"] \arrow[d, "a"'] & {} \arrow[d, Rightarrow, "\gamma\otimes\varphi"] & C \arrow[d, "c"] \\
A' \arrow[rr, "f'"'] & {} & B' \arrow[rr, "g'"'] & {} & C' & \longmapsto & A' \arrow[rr, "g'\otimes f'"'] & {} & C'
\end{tikzcd}.$$

Each component of α is of the form

$$\begin{tikzcd}
A \arrow[rr, "h\otimes(g\otimes f)"] \arrow[d, "1_A"'] & {} \arrow[d, Rightarrow, "\alpha_{h,g,f}"] & D \arrow[d, "1_D"] \\
A \arrow[rr, "(h\otimes g)\otimes f"'] & {} & D
\end{tikzcd},$$

while the components of λ and ρ are given by

$$\begin{tikzcd}
A \arrow[rr, "id_B\otimes f"] \arrow[d, "1_A"'] & {} \arrow[d, Rightarrow, "\lambda_f"] & B \arrow[d, "1_B"] & , & A \arrow[rr, "f\otimes id_A"] \arrow[d, "1_A"'] & {} \arrow[d, Rightarrow, "\rho_f"] & B \arrow[d, "1_B"] \\
A \arrow[rr, "f"'] & {} & B & & A \arrow[rr, "f"'] & {} & B
\end{tikzcd}.$$

Thus, a description of pseudo-category in Cat is as follows.

A pseudo-category in Cat is a structure with
- objects: $A, A', A'', B, B', ...$
- morphisms: $a : A \longrightarrow A', a' : A' \longrightarrow A'', b : B \longrightarrow B', ...$
- *pseudo-morphisms*: $A \xrightarrow{f} B$, $A' \xrightarrow{f'} B'$, $B \xrightarrow{g} C$, ...
- and cells:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "a"'] & {} \arrow[d, Rightarrow, "\varphi"] & B \arrow[d, "b"] & , & A' \arrow[rr, "f'"] \arrow[d, "a'"'] & {} \arrow[d, Rightarrow, "\varphi'"] & B' \arrow[d, "b'"] & , & B \arrow[rr, "g"] \arrow[d, "b"'] & {} \arrow[d, Rightarrow, "\gamma"] & C \arrow[d, "c"] & , ... \\
A' \arrow[rr, "f'"'] & {} & B' & & A'' \arrow[rr, "f''"'] & {} & B'' & & B' \arrow[rr, "g'"'] & {} & C' &
\end{tikzcd}$$

where objects and morphisms form a category

$$\begin{aligned}
a''(a'a) &= (a''a')\,a, \\
1_{A'}a &= a1_A;
\end{aligned}$$

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pseudo-morphisms and cells also form a category

$$\begin{aligned} \varphi''(\varphi'\varphi) &= (\varphi''\varphi')\varphi, \\ 1_{f'}\varphi &= \varphi 1_f, \end{aligned}$$

with 1_f being the cell

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\Downarrow 1_f"] & B \arrow[d, "1_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} ;$$

for each pair of *pseudo-composable* cells γ, φ, there is a pseudo-composition $\gamma \otimes \varphi$

$$\begin{tikzcd} A \arrow[r, "g\otimes f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \gamma\otimes\varphi"] & C \arrow[d, "c"] \\ A' \arrow[r, "g'\otimes f'"'] & C' \end{tikzcd} ;$$

satisfying

$$\begin{aligned} (\gamma'\gamma) \otimes (\varphi'\varphi) &= (\gamma' \otimes \varphi')(\gamma \otimes \varphi), \\ 1_{g\otimes f} &= 1_g \otimes 1_f; \end{aligned} \tag{1.8}$$

for each morphism $a : A \longrightarrow A'$, there is a pseudo-identity id_a

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow id_a"] & A \arrow[d, "a"] \\ A' \arrow[r, "id_{A'}"'] & A' \end{tikzcd}$$

satisfying

$$\begin{aligned} id_{1_A} &= 1_{id_A} \\ id_{a'a} &= id_{a'} id_a; \end{aligned}$$

there is a special cell $\alpha_{h,g,f}$ for each triple of composable pseudo-morphisms h, g, f

$$\begin{tikzcd} A \arrow[r, "h\otimes(g\otimes f)"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\Downarrow \alpha_{h,g,f}"] & D \arrow[d, "1_D"] \\ A \arrow[r, "(h\otimes g)\otimes f"'] & D \end{tikzcd} ,$$

natural in each component, i.e., the following diagram of cells

$$\begin{tikzcd} h \otimes (g \otimes f) \arrow[r, "\alpha_{h,g,f}"] \arrow[d, "\eta\otimes(\gamma\otimes\varphi)"'] & (h \otimes g) \otimes f \arrow[d, "(\eta\otimes\gamma)\otimes\varphi"] \\ h' \otimes (g' \otimes f') \arrow[r, "\alpha_{h',g',f'}"] & (h' \otimes g') \otimes f' \end{tikzcd}$$

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commutes for every triple of pseudo-composable cells φ, γ, η

$$\begin{tikzcd}
A \arrow[rr, "f" description] \arrow[d, "a"'] & {} \arrow[d, Rightarrow, "\varphi"] & B \arrow[rr, "g" description] \arrow[d, "b"] & {} \arrow[d, Rightarrow, "\gamma"] & C \arrow[rr, "h" description] \arrow[d, "c"] & {} \arrow[d, Rightarrow, "\eta"] & D \arrow[d, "d"] \\
A' \arrow[rr, "f'" description] & {} & B' \arrow[rr, "g'" description] & {} & C' \arrow[rr, "h'" description] & {} & D'
\end{tikzcd} ;$$

to each pseudo-morphism $f : A \longrightarrow B$ there are two special cells

$$\begin{tikzcd}
A \arrow[rr, "id_B \otimes f" description] \arrow[d, "1_A"'] & {} \arrow[d, Rightarrow, "\lambda_f"] & B \arrow[d, "1_B"] \\
A \arrow[rr, "f" description] & {} & B
\end{tikzcd} , \qquad
\begin{tikzcd}
A \arrow[rr, "f \otimes id_A" description] \arrow[d, "1_A"'] & {} \arrow[d, Rightarrow, "\rho_f"] & B \arrow[d, "1_B"] \\
A \arrow[rr, "f" description] & {} & B
\end{tikzcd} ,$$

natural in f, that is, to every cell φ as above, the following diagrams of cells commute

$$\begin{tikzcd}
id_B \otimes f \arrow[r, "\lambda_f"] \arrow[d, "id_{1_B} \otimes \varphi"'] & f \arrow[d, "\varphi"] \\
id_{B'} \otimes f' \arrow[r, "\lambda_{f'}"] & f'
\end{tikzcd} , \qquad
\begin{tikzcd}
f \otimes id_A \arrow[r, "\rho_f"] \arrow[d, "\varphi \otimes id_{1_A}"'] & f \arrow[d, "\varphi"] \\
f' \otimes id_{B'} \arrow[r, "\lambda_{f'}"] & f'
\end{tikzcd} .$$

And furthermore, the following conditions are satisfied whenever the compositions are defined

$$\begin{tikzcd}
& f \otimes (g \otimes (h \otimes k)) \arrow[rr, "f \otimes \alpha_{g,h,k}"] \arrow[dl, "\alpha_{f,g,h \otimes k}"'] & & f \otimes ((g \otimes h) \otimes k) \arrow[dr, "\alpha_{f,g \otimes h,k}"] & \\
(f \otimes g) \otimes (h \otimes k) \arrow[drr, "\alpha_{f \otimes g,h,k}"'] & & & & (f \otimes (g \otimes h)) \otimes k \arrow[dll, "\alpha_{f,g,h} \otimes k"] \\
& & ((f \otimes g) \otimes h) \otimes k & &
\end{tikzcd}$$

$$\begin{tikzcd}
f \otimes (1 \otimes g) \arrow[rr, "\alpha_{f,1,g}"] \arrow[dr, "f \otimes \lambda_g"'] & & (f \otimes 1) \otimes g \arrow[dl, "\rho_f \otimes g"] \\
& f \otimes g &
\end{tikzcd} .$$

Examples of pseudo-categories internal to Cat include the usual bicategories of Spans, Bimodules, homotopies, ... where in each case it is also allowed to

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consider the natural morphisms between the objects in order to obtain a vertical categorical structure. For example in the case of spans we would have sets as objects, maps as morphisms, spans $A \longleftarrow S \longrightarrow B$ as pseudo-morphisms and the cells being triples (h, k, l) with the following two squares commutative

$$\begin{tikzcd}
A \arrow[d, "h"'] & S \arrow[l] \arrow[r] \arrow[d, "k"'] & B \arrow[d, "l"] \\
A' & S' \arrow[l] \arrow[r] & B
\end{tikzcd}.$$

A pseudo-category in Cat has the following structures: a category (with objects and morphisms); a category (with pseudo-morphisms and cells); a bicategory (considering only the morphisms that are identities); a double category (if all the special cells are identity cells).

Other examples as Cat (with modules as pseudo-morphisms) may be found in [3] or [2].

The present description of pseudo double category (internal pseudo-category in Cat) is the same given by Leinster [3] and differs from the one considered by Grandis and Paré [2] in the sense that they also have

$$id_A = id_A \otimes id_A.$$

In the following sections we will provide a complete description of pseudo-functors, natural and pseudo-natural transformations and pseudo-modifications. We prove that all the compositions are well defined (except for the horizontal composition of pseudo-natural transformations which is only defined up to an isomorphism). In the end we show that the category of pseudo-categories (internal to some ambient 2-category **C**) is Cartesian closed up to isomorphism. We will give all the definitions in terms of the internal structure to some ambient 2-category and also explain what is obtained in the case where the ambient 2-category is Cat. While doing some proofs we will make use of Yoneda embedding and consider the diagrams in Cat rather than in the abstract ambient 2-category.

We will also freely use known definitions and results from [1],[3],[4] and [10].

1.2 Pseudo-functors

Let **C** be a 2-category and suppose

$$\begin{aligned} C &= (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)\,, \\ C' &= \left(C'_0, C'_1, d', c', e', m', \alpha', \lambda', \rho'\right) \end{aligned} \tag{1.9}$$

are two pseudo-categories in **C**.

A pseudo-functor $F : C \longrightarrow C'$ is a system

$$F = (F_0, F_1, \mu, \varepsilon)$$

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where $F_0 : C_0 \longrightarrow C_0', F_1 : C_1 \longrightarrow C_1'$ are morphisms of $\mathbf{C}$,

$$\mu : F_1 m \longrightarrow m' \left(F_1 \times_{F_0} F_1 \right) \ , \ \varepsilon : F_1 e \longrightarrow e' F_0,$$

are 2-cells of $\mathbf{C}$ (that are isomorphisms[2]), the following conditions are satisfied

$$\begin{aligned} d' F_1 &= F_0 d, \\ c' F_1 &= F_0 c, \end{aligned} \tag{1.10}$$

$$\begin{aligned} d' \circ \mu &= 1_{F_0 d \pi_2}, \\ c' \circ \mu &= 1_{F_0 c \pi_1}, \end{aligned} \tag{1.11}$$

$$\begin{aligned} d' \circ \varepsilon &= 1_{F_0}, \\ c' \circ \varepsilon &= 1_{F_0}, \end{aligned} \tag{1.12}$$

and the following diagrams commute

$$\begin{tikzcd} & \bullet \arrow[r, "\mu\left(1 \times_{C_0} m\right)"] \arrow[dl, "F_1 \alpha"'] & \bullet \arrow[dr, "m'\left(1_{F_1} \times \mu\right)"] & \\ \bullet \arrow[dr, "\mu\left(m \times_{C_0} 1\right)"'] & & & \bullet \arrow[dl, "\alpha'\left(F_1 \times_{F_0} F_1 \times_{F_0} F_1\right)"] \\ & \bullet \arrow[r, "m'\left(\mu \times 1_{F_1}\right)"'] & \bullet & \end{tikzcd} \quad , \tag{1.13}$$

$$\begin{tikzcd} \bullet \arrow[r, "F_1 \rho"] \arrow[d, "\mu\langle 1, ed \rangle"'] & \bullet \\ \bullet \arrow[r, "m'\left(1_{F_1} \times \varepsilon\right)"] & \bullet \arrow[u, "\rho' F_1"'] \end{tikzcd} \quad , \tag{1.14}$$

$$\begin{tikzcd} \bullet \arrow[r, "F_1 \lambda"] \arrow[d, "\mu\langle ec, 1 \rangle"'] & \bullet \\ \bullet \arrow[r, "m'\left(\varepsilon \times 1_{F_1}\right)"] & \bullet \arrow[u, "\lambda' F_1"'] \end{tikzcd} \quad .$$

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[2] Some authors (example Grandis and Paré in [2, 5]) consider the notion of pseudo - which corresponds to the present one - but also consider the notions of lax and colax where the 2-cells may not be isomorphisms.

Consider the particular case of **C**=Cat. Let

$$\begin{tikzcd}
A \arrow[r, dash] \arrow[d, "a"'] & f \arrow[r] \arrow[d, Rightarrow, "\varphi"] & B \arrow[d, "b"] \\
A' \arrow[r, dash] & f' \arrow[r] & B'
\end{tikzcd}$$

be a cell in the pseudo-category C. A pseudo-functor $F : C \longrightarrow C'$, consists of four maps (sending objects to objects, morphisms to morphisms, pseudo-morphisms to pseudo-morphisms and cells to cells - that we will denote only by F to keep notation simple)

$$\begin{tikzcd}
FA \arrow[r, dash] \arrow[d, "Fa"'] & Ff \arrow[r] \arrow[d, Rightarrow, "F\varphi"] & FB \arrow[d, "Fb"] \\
FA' \arrow[r, dash] & Ff' \arrow[r] & FB'
\end{tikzcd} ;$$

a special cell $\mu_{f,g}$

$$\begin{tikzcd}
FA \arrow[r, dash] \arrow[d, "1"'] & F(g\otimes f) \arrow[r] \arrow[d, Rightarrow, "\mu_{f,g}"] & FC \arrow[d, "1"] \\
FA \arrow[r, dash] & Fg\otimes Ff \arrow[r] & FC
\end{tikzcd}$$

to each pair of composable pseudo-morphisms f, g; a special cell ε_A

$$\begin{tikzcd}
FA \arrow[r, dash] \arrow[d, "1"'] & F(id_A) \arrow[r] \arrow[d, Rightarrow, "\epsilon_A"] & FA \arrow[d, "1"] \\
FA \arrow[r, dash] & id_{FA} \arrow[r] & FA
\end{tikzcd}$$

to each object A, and satisfying the commutativity of the following diagrams

$$\begin{tikzcd}
& F(f\otimes(g\otimes h)) \arrow[r, "F(\alpha_{f,g,h})"] \arrow[dl, "\mu_{f,g\otimes h}"'] & F((f\otimes g)\otimes h) \arrow[dr, "\mu_{f\otimes g,h}"] & \\
F(f)\otimes F(g\otimes h) \arrow[dr, "F(f)\otimes\mu_{g,h}"'] & & & F(f\otimes g)\otimes F(h) \arrow[dl, "\mu_{f,g}\otimes F(h)"] \\
& F(f)\otimes(F(g)\otimes F(h)) \arrow[r, "\alpha'_{Ff,Fg,Fh}"'] & (F(f)\otimes F(g))\otimes F(h) &
\end{tikzcd},$$

$$\begin{tikzcd}
F(f\otimes id_A) \arrow[r, "F(\rho_f)"] \arrow[d, "\mu_{f,id_A}"'] & F(f) \\
F(f)\otimes F(id_A) \arrow[r, "F(f)\otimes\varepsilon_A"] & F(f)\otimes id_{F(A)} \arrow[u, "\rho'_{Ff}"']
\end{tikzcd},$$

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$$\begin{tikzcd}
F\left(id_B \otimes f\right) \arrow[r, "F(\lambda_f)"] \arrow[d, "\mu_{id_B,f}"'] & F\left(f\right) \\
F\left(id_B\right) \otimes F\left(f\right) \arrow[r, "\varepsilon_B \otimes F(f)"] & id_{F(B)} \otimes F\left(f\right) \arrow[u, "\lambda'_{Ff}"']
\end{tikzcd},$$

whenever the pseudo-compositions are defined.

Return to the general case.

Let $F : C \longrightarrow C'$ and $G : C' \longrightarrow C''$ be pseudo-functors in a 2-category $\mathbf{C}$. Consider C and C' as in (1.9) and let

$$\begin{aligned}
C'' &= \left(C''_0, C''_1, d'', c'', e'', m'', \alpha'', \lambda'', \rho''\right), \\
F &= \left(F_0, F_1, \mu^F, \varepsilon^F\right), \\
G &= \left(G_0, G_1, \mu^G, \varepsilon^G\right).
\end{aligned}$$

The composition of the pseudo-functors F and G is defined by the formula

$$GF = \left(G_0F_0, G_1F_1, \left(\mu^G \circ \left(F_1 \times_{F_0} F_1\right)\right) \cdot \left(G_1 \circ \mu^F\right), \left(\varepsilon^G \circ F_0\right) \cdot \left(G_1 \circ \varepsilon^F\right)\right) \tag{1.15}$$

where $\circ$ represents the horizontal composition in $\mathbf{C}$ and $\cdot$ represents the vertical composition, as displayed in the diagram below

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "m"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[rd, phantom, "\mu^F \Downarrow"] & C_1 \arrow[d, "F_1"'] \arrow[rd, phantom, "\varepsilon^F \Downarrow"] & C_0 \arrow[l, "e"'] \arrow[d, "F_0"] \\
C'_1 \times_{C'_0} C'_1 \arrow[r, "m'"] \arrow[d, "G_1 \times_{G_0} G_1"'] \arrow[rd, phantom, "\mu^G \Downarrow"] & C'_1 \arrow[d, "G_1"'] \arrow[rd, phantom, "\varepsilon^G \Downarrow"] & C'_0 \arrow[l, "e'"'] \arrow[d, "G_0"] \\
C''_1 \times_{C''_0} C''_1 \arrow[r, "m''"] & C''_1 & C''_0 \arrow[l, "e''"']
\end{tikzcd}.$$

Proposition 1 *The above formula to compose pseudo-functors is well defined.*

Proof. Consider the system

$$GF = \left(G_0F_0, G_1F_1, \mu^{GF}, \varepsilon^{GF}\right)$$

with $\mu^{GF}, \varepsilon^{GF}$ as in (1.15). We will show that GF is a pseudo-functor from the pseudo-category C to the pseudo-category C''.

It is clear that $G_0F_0 : C_0 \longrightarrow C''_0, G_1F_1 : C_1 \longrightarrow C''_1$, are morphisms of the ambient 2-category $\mathbf{C}$ and $\mu^{GF} : G_1F_1m \longrightarrow m''\left(G_1F_1 \times_{G_0F_0} G_1F_1\right)$, $\varepsilon^{GF} : G_1F_1e \longrightarrow e''G_0F_0$ are 2-cells of $\mathbf{C}$ and they are isomorphisms.

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Conditions (1.10) are satisfied and

$$\begin{aligned}
d''\mu^{GF} &= d\left(\left(\mu^{G}\circ\left(F_1\times_{F_0}F_1\right)\right)\cdot\left(G_1\circ\mu^{F}\right)\right)\\
&= \left(d\circ\mu^{G}\circ\left(F_1\times_{F_0}F_1\right)\right)\cdot\left(d\circ G_1\circ\mu^{F}\right)\\
&= \left(1_{G_0d'\pi_2'}\circ\left(F_1\times_{F_0}F_1\right)\right)\cdot\left(G_0\circ d'\circ\mu^{F}\right)\\
&= \left(1_{G_0d'\pi_2'\left(F_1\times_{F_0}F_1\right)}\right)\cdot\left(G_0\circ 1_{F_0d\pi_2}\right)\\
&= 1_{G_0d'F_1\pi_2}\cdot 1_{G_0F_0d\pi_2}\\
&= 1_{G_0F_0d\pi_2},
\end{aligned}$$

as well $c''\mu^{GF} = 1_{G_0F_0c\pi_1}$, hence (1.11) holds. Also

$$\begin{aligned}
d''\varepsilon^{GF} &= d''\left(\left(\varepsilon^{G}\circ F_0\right)\cdot\left(G_1\circ\varepsilon^{F}\right)\right)\\
&= \left(d''\circ\varepsilon^{G}\circ F_0\right)\cdot\left(d''G_1\circ\varepsilon^{F}\right)\\
&= \left(1_{G_0}\circ F_0\right)\cdot\left(G_0d'\circ\varepsilon^{F}\right)\\
&= 1_{G_0F_0}\cdot\left(G_0\circ 1_{F_0}\right)\\
&= 1_{G_0F_0}\cdot 1_{G_0F_0} = 1_{G_0F_0},
\end{aligned}$$

and similarly $c''\varepsilon^{GF} = 1_{G_0F_0}$, so conditions (1.12) are satisfied.

Commutativity of diagrams (1.13), (1.14) follows from Yoneda Lemma and

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the commutativity of the following diagrams

$$\begin{tikzcd}
 & H_{f\otimes(g\otimes h)} \arrow[rrr, "H(\alpha_{f,g,h})"] \arrow[d] \arrow[dl, "\mu^H_{f,g\otimes h}"'] & & & H_{(f\otimes g)\otimes h)} \arrow[d] \arrow[dr, "\mu^H_{f\otimes g,h}"] & \\
 & G_{F_f\otimes F_{g\otimes h}} \arrow[dl] \arrow[dr] & & & G_{F_{f\otimes g}\otimes F_h} \arrow[dl] \arrow[dr] & \\
H_f\otimes H_{g\otimes h} \arrow[dr] \arrow[ddr, "H_f\otimes\mu^H_{g,h}"'] & & (1) \arrow[r, "G(\alpha'_{Ff,Fg,Fh})"] \arrow[dl] & (2) \arrow[dr] & & H_{f\otimes g}\otimes H_h \arrow[dl] \arrow[ddl, "\mu^H_{f,g}\otimes H_h"] \\
 & F_f\otimes G_{F_g\otimes F_h} \arrow[d] & & & G_{F_f\otimes F_g}\otimes H_h \arrow[d] & \\
 & H_f\otimes(H_g\otimes H_h) \arrow[rrr, "\alpha''_{Hf,Hg,Hh}"] & & & (H_f\otimes H_g)\otimes H_h &
\end{tikzcd}$$

$$\begin{tikzcd}
H_{id_B\otimes f} \arrow[rrr, "H(\lambda_f)"] \arrow[dr] \arrow[ddd, "\mu^H_{id_B,f}"'] & & & H_f \\
 & G_{F_{id_B}\otimes F_f} \arrow[r] \arrow[ddl] & G_{id_{F_B}\otimes F_f} \arrow[d] & \\
 & & G_{id_{F_B}}\otimes H_f \arrow[dr] & \\
H_{id_B}\otimes H_f \arrow[urr] \arrow[rrr, "\epsilon^H_B\otimes Hf"'] & & & id_{H_B}\otimes H_f \arrow[uuu, "\lambda''_{Hf}"']
\end{tikzcd}$$

$$\begin{tikzcd}
H_{f\otimes id_A} \arrow[rrr, "H(\rho_f)"] \arrow[dr] \arrow[ddd, "\mu^H_{f,id_A}"'] & & & H_f \\
 & G_{F_f\otimes F_{id_A}} \arrow[r] \arrow[ddl] & G_{F_f\otimes id_{F_A}} \arrow[d] & \\
 & & H_f\otimes G_{id_{F_A}} \arrow[dr] & \\
H_f\otimes H_{id_A} \arrow[urr] \arrow[rrr, "Hf\otimes\epsilon^H_A"'] & & & F_f\otimes id_{H_A} \arrow[uuu, "\rho''_{Hf}"']
\end{tikzcd}$$

where $(1) = G_{F_f\otimes(F_g\otimes F_h)}$ and $(2) = G_{(F_f\otimes F_g)\otimes F_h}$. We also use the abbreviations $H = GF$ and F_f or Ff instead of $F(f)$ to save space in the diagram. ■

Composition of pseudo-functors is associative and there is an identity pseudo-functor for every pseudo-category, namely the pseudo-functor

$$1_C = (1_{C_0}, 1_{C_1}, 1_m, 1_e)$$

for the pseudo-category

$$C = (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)\,.$$

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Given a 2-category $\mathbf{C}$, we define the category $\mathrm{PsCat}(\mathbf{C})$ consisting of all pseudo-categories and pseudo-functors internal to $\mathbf{C}$.

1.3 Natural and pseudo-natural transformations

Let $\mathbf{C}$ be a 2-category and suppose

$$\begin{aligned} C &= (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho), \\ C' &= \left(C_0', C_1', d', c', e', m', \alpha', \lambda', \rho'\right) \end{aligned} \tag{1.16}$$

are pseudo-categories in $\mathbf{C}$ and

$$\begin{aligned} F &= \left(F_0, F_1, \mu^F, \varepsilon^F\right), \\ G &= \left(G_0, G_1, \mu^G, \varepsilon^G\right) \end{aligned} \tag{1.17}$$

are pseudo-functors from C to C'.

A **natural transformation** $\theta : F \longrightarrow G$ is a pair $\theta = (\theta_0, \theta_1)$ of 2-cells of $\mathbf{C}$

$$\begin{aligned} \theta_0 &: F_0 \longrightarrow G_0 \\ \theta_1 &: F_1 \longrightarrow G_1 \end{aligned}$$

satisfying

$$\begin{aligned} d' \circ \theta_1 &= \theta_0 \circ d \\ c' \circ \theta_1 &= \theta_0 \circ c \end{aligned}$$

and the commutativity of the following diagrams of 2-cells

$$\begin{tikzcd} \bullet \arrow[r, "\theta_1 \circ m"] \arrow[d, "\mu^F"'] & \bullet \arrow[d, "\mu^G"] \\ \bullet \arrow[r, "m' \circ \left(\theta_1 \times_{\theta_0} \theta_1\right)"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "\theta_1 \circ e"] \arrow[d, "\varepsilon^F"'] & \bullet \arrow[d, "\varepsilon^G"] \\ \bullet \arrow[r, "e' \circ \theta_0"] & \bullet \end{tikzcd} .$$

A **pseudo-natural transformation** $T : F \longrightarrow G$ is a pair

$$T = (t, \tau)$$

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where $t : C_0 \longrightarrow C_1'$ is a morphism of $\mathbf{C}$,

$$\tau : m' \langle G_1, td \rangle \longrightarrow m' \langle tc, F_1 \rangle$$

is a 2-cell (that is an isomorphism); the following conditions are satisfied

$$\begin{aligned} d't &= F_0 \\ c't &= G_0 \end{aligned} \tag{1.18}$$

$$\begin{aligned} d' \circ \tau &= 1_{d'F_1} \\ c' \circ \tau &= 1_{c'G_1} \end{aligned} \tag{1.19}$$

and the following diagrams of 2-cells are commutative[3]

$$\begin{tikzcd}
 & \bullet \arrow[r, "\alpha^{-1}\left(G_1 \times_{G_0} G_1 \times_{G_0} t\right)"] \arrow[dl, "m'\left\langle \mu_G^{-1}, td\pi_2 \right\rangle"'] & \bullet \arrow[dr, "m'\langle G_1\pi_1, \tau\pi_2 \rangle"] & \\
\bullet \arrow[d, "\tau m"'] & & & \bullet \arrow[d, "\alpha\left(G_1 \times_{G_0} t \times_{F_0} F_1\right)"] \\
\bullet \arrow[dr, "m'\langle tc\pi_1, \mu_F \rangle"'] & & & \bullet \arrow[dl, "m'\langle \tau\pi_1, F_1\pi_2 \rangle"] \\
 & \bullet \arrow[r, "\alpha\left(t \times_{F_0} F_1 \times_{F_0} F_1\right)"'] & \bullet &
\end{tikzcd} , \tag{1.20}$$

$$\begin{tikzcd}
 & \bullet \arrow[r, "\tau e"] \arrow[dl, "\lambda' t"'] & \bullet \arrow[dr, "m'\langle 1_t, \varepsilon_F \rangle"] & \\
\bullet \arrow[drr, "m'\langle \varepsilon_G, 1_t \rangle"'] & & & \bullet \arrow[dl, "\rho' t"] \\
 & & \bullet &
\end{tikzcd} . \tag{1.21}$$

In the case $\mathbf{C}$=Cat: let W, W' be two pseudo-categories in Cat, and $F, G : W \longrightarrow W'$ two pseudo-functors. Given a cell

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \varphi"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

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[3] $G_1 \times_{G_0} t \times_{F_0} F_1 : C_1 \times_{C_0} C_0 \times_{C_0} C_1 \longrightarrow C_1 \times_{C_0} C_1 \times_{C_0} C_1$
$t \times_{F_0} F_1 \times_{F_0} F_1 : C_0 \times_{C_0} C_1 \times_{C_0} C_1 \longrightarrow C_1 \times_{C_0} C_1 \times_{C_0} C_1$
$G_1 \times_{G_0} G_1 \times_{G_0} t : C_1 \times_{C_0} C_1 \times_{C_0} C_0 \longrightarrow C_1 \times_{C_0} C_1 \times_{C_0} C_1$

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in W, we will write

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\Downarrow F\varphi"] & FB \arrow[d, "Fb"] & & GA \arrow[r, "Gf"] \arrow[d, "Ga"'] \arrow[dr, phantom, "\Downarrow G\varphi"] & GB \arrow[d, "Gb"] \\
FA' \arrow[r, "Ff'"] & FB' & \text{and} & GA' \arrow[r, "Gf'"] & GB'
\end{tikzcd}$$

for the image of φ under F and G.

The description of natural and pseudo-natural transformations in this particular case is as follows:

- While a natural transformation $\theta : F \longrightarrow G$ is a family of cells

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\Downarrow \theta_f"] & FB \arrow[d, "\theta_B"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd} ,$$

one for each pseudo-morphism f in W, such that for every cell φ in W, the square

$$\begin{tikzcd}
Ff \arrow[r, "\theta_f"] \arrow[d, "F\varphi"'] & Gf \arrow[d, "G\varphi"] \\
Ff' \arrow[r, "\theta_{f'}"] & Gf'
\end{tikzcd}$$

is commutative as displayed in the picture below

$$\begin{tikzcd}
{} \arrow[rr, "Ff"] \arrow[d, "Fa"'] \arrow[ddr, "\theta_A"'] & {} \arrow[d, Rightarrow, "F\varphi"'] \arrow[ddr, Rightarrow, "\theta_f"] & {} \arrow[d, "Fb"] \arrow[ddr, "\theta_B"] & \\
{} \arrow[rr, "Ff'"] \arrow[ddr, "\theta_{A'}"'] & {} \arrow[ddr, Rightarrow, "\theta_{f'}"] & {} \arrow[ddr, "\theta_{B'}"] & \\
 & {} \arrow[rr, "Gf"] \arrow[d, "Ga"'] & {} \arrow[d, Rightarrow, "G\varphi"] & {} \arrow[d, "Gb"] \\
 & {} \arrow[rr, "Gf'"] & {} & {}
\end{tikzcd} ;$$

and furthermore, given two composable pseudo-morphisms g, f and an object A in W, the following squares are commutative

$$\begin{tikzcd}
F(g \otimes f) \arrow[r, "\mu^F_{g,f}"] \arrow[d, "\theta_{g\otimes f}"'] & Fg \otimes Ff \arrow[d, "\theta_g \otimes \theta_f"] \\
G(g \otimes f) \arrow[r, "\mu^G_{g,f}"] & Gg \otimes Gf
\end{tikzcd}$$

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$$\begin{tikzcd}
F\left(id_A\right) \arrow[r, "\varepsilon_A^F"] \arrow[d, "\theta_{id_A}"'] & id_{FA} \arrow[d, "id_{\theta_A}"] \\
G\left(id_A\right) \arrow[r, "\varepsilon_A^G"'] & id_{GA}
\end{tikzcd} .$$

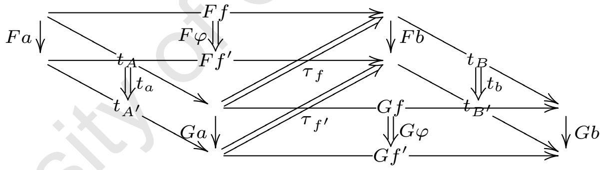
- Rather a pseudo-natural transformation $T : F \longrightarrow G$ consists of two families of cells

$$\begin{tikzcd}
FA \arrow[r, "t_A"{name=U}] \arrow[d, "Fa"'] & GA \arrow[d, "Ga"] \\
FA' \arrow[r, "t_{A'}"'{name=D}] & GA'
\arrow[Rightarrow, from=U, to=D, "t_a"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
FA \arrow[r, "Gf \otimes t_A"{name=U}] \arrow[d, "1"'] & GB \arrow[d, "1"] \\
FA \arrow[r, "t_B \otimes Ff"'{name=D}] & GB
\arrow[Rightarrow, from=U, to=D, "\tau_f"]
\end{tikzcd}$$

with a a morphism and f a pseudo-morphism of W, as displayed in the following picture



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such that (t is a functor)

$$\begin{aligned} t_{a'a} &= t_a t_{a'} \\ t_{1_A} &= 1_{t_A} \end{aligned}$$

(τ is natural)

$$\begin{tikzcd}
Gf \otimes t_A \arrow[r, "\tau_f"] \arrow[d, "G\varphi \otimes t_a"'] & t_B \otimes Ff \arrow[d, "t_b \otimes F\varphi"] \\
Gf' \otimes t_{A'} \arrow[r, "\tau_{f'}"] & t_{B'} \otimes Ff'
\end{tikzcd} ,$$

and for every two composable pseudo-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, the following diagrams of cells in W' are commutative

$$\begin{tikzcd}
& \left(G\left(g\right)\otimes G\left(f\right)\right)\otimes t_A \arrow[r] \arrow[dl] & G\left(g\right)\otimes\left(G\left(f\right)\otimes t_A\right) \arrow[dr] & \\
G\left(g\otimes f\right)\otimes t_A \arrow[d] & & & G\left(g\right)\otimes\left(t_B\otimes F\left(f\right)\right) \arrow[d] \\
t_C\otimes F\left(g\otimes f\right) \arrow[dr] & & & \left(G\left(g\right)\otimes t_B\right)\otimes F\left(f\right) \arrow[dl] \\
& t_C\otimes\left(F\left(g\right)\otimes F\left(f\right)\right) \arrow[r] & \left(t_C\otimes F\left(g\right)\right)\otimes F\left(f\right) &
\end{tikzcd}$$

$$\begin{tikzcd}
& G\left(id_A\right)\otimes t_A \arrow[rr] \arrow[dl] & & t_A\otimes F\left(id_A\right) \arrow[dr] & \\
id_{GA}\otimes t_A \arrow[drr] & & & & t_A\otimes id_{FA} \arrow[dll] \\
& & t_A & &
\end{tikzcd}.$$

Return to the general case.

Let $\mathbf{C}$ be a 2-category and suppose C, C', C'' are pseudo-categories in $\mathbf{C}$ and $F, G, H : C \longrightarrow C'$, $F', G' : C' \longrightarrow C''$ are pseudo-functors. Natural transformations $\theta, \theta', \dot{\theta}$

$$\begin{tikzcd}
C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\theta"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\dot{\theta}"] & C' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\theta'"] & C''
\end{tikzcd}$$

may be composed horizontally with $\theta' \circ \theta = \left(\theta'_0, \theta'_1\right) \circ \left(\theta_0, \theta_1\right) = \left(\theta'_0 \circ \theta_0, \theta'_1 \circ \theta_1\right)$ obtained from the horizontal composition of 2-cells of $\mathbf{C}$, and vertically with $\dot{\theta} \cdot \theta = \left(\dot{\theta}_0, \dot{\theta}_1\right) \cdot \left(\theta_0, \theta_1\right) = \left(\dot{\theta}_0 \cdot \theta_0, \dot{\theta}_1 \cdot \theta_1\right)$ obtained from the vertical composition of 2-cells of $\mathbf{C}$. Clearly both compositions are well defined, are associative, have identities and satisfy the middle interchange law. This fact may be stated as in the following theorem.

Theorem 2 *Let* $\boldsymbol{C}$ *be a 2-category. The category PsCat(*$\boldsymbol{C}$*) (with pseudo-categories, pseudo-functors and natural transformations) is a 2-category.*

Composition of pseudo-natural transformations is much more delicate.

Again let $\mathbf{C}$ be a 2-category and suppose C, C' are pseudo-categories in $\mathbf{C}$, $F, G, H : C \longrightarrow C'$ are pseudo-functors (as above) and consider the pseudo-natural transformations

$$F \xrightarrow{T} G \xrightarrow{S} H$$

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with

$$T=(t,\tau)\,,\quad S=(s,\sigma)\,.$$

Vertical composition of pseudo-natural transformations S and T is defined as

$$S\otimes T=\left(m'\left\langle s,t\right\rangle,\sigma\otimes\tau\right) \tag{1.22}$$

where

$$\sigma\otimes\tau=\alpha\left\langle sc,tc,F_1\right\rangle\cdot m'\left\langle 1_{sc},\tau\right\rangle\cdot\alpha^{-1}\left\langle sc,G_1,td\right\rangle\cdot m'\left\langle\sigma,1_{td}\right\rangle\cdot\alpha\left\langle H_1,sd,td\right\rangle. \tag{1.23}$$

The above formula in the case **C**=Cat is expressed as follows

$$(s\otimes t)_a=s_a\otimes t_a,\qquad \begin{tikzcd} FA \arrow[r, "t_A"] \arrow[d, "Fa"'] & GA \arrow[r, "s_A"] \arrow[d, "Ga"] & HA \arrow[d, "Ha"] \\ FA' \arrow[r, "t_{A'}"'] \arrow[u, phantom, shift left=8, "\Downarrow t_a"] & GA' \arrow[r, "s_{A'}"'] \arrow[u, phantom, shift left=8, "\Downarrow s_a"] & HA' \end{tikzcd}\;;$$

and

$$(\sigma\otimes\tau)_f=\alpha\left(s_B\otimes\tau_f\right)\alpha^{-1}\left(\sigma_f\otimes t_A\right)\alpha,$$

as displayed in the following picture

$$\begin{tikzcd} Hf\otimes(s_A\otimes t_A) \arrow[r, "(\sigma\tau)_f"] \arrow[d, "\alpha"'] & (s_B\otimes t_B)\otimes Ff \\ (Hf\otimes s_A)\otimes t_A \arrow[d, "\sigma_f\otimes t_A"'] & s_B\otimes(t_B\otimes Ff) \arrow[u, "\alpha"'] \\ (s_B\otimes Gf)\otimes t_A \arrow[r, "\alpha^{-1}"'] & s_B\otimes(Gf\otimes t_A) \arrow[u, "s_B\otimes\tau_f"'] \end{tikzcd}\quad.$$

Return to the general case.

Theorem 3 *The vertical composition of pseudo-natural transformations is well defined.*

Proof. Consider C,C' as in (1.16), F,G as in (1.17), $H=(H_0,H_1,\mu_H,\varepsilon_H)$ and S,T as above. Clearly $(st)=m'\left\langle s,t\right\rangle:C_0\longrightarrow C_1'$ is a morphism of **C** and $\sigma\tau:m'\left\langle H_1,(st)\,d\right\rangle\longrightarrow m'\left\langle(st)\,c,F_1\right\rangle$ is a 2-cell of **C** that is an isomorphism (is defined as a composition of isomorphisms).

Conditions (1.18) and (1.19) are satisfied

$$\begin{aligned} d'm'\left\langle s,t\right\rangle &= d'\pi_2'\left\langle s,t\right\rangle \\ &= d't \\ &= F_0, \end{aligned}$$

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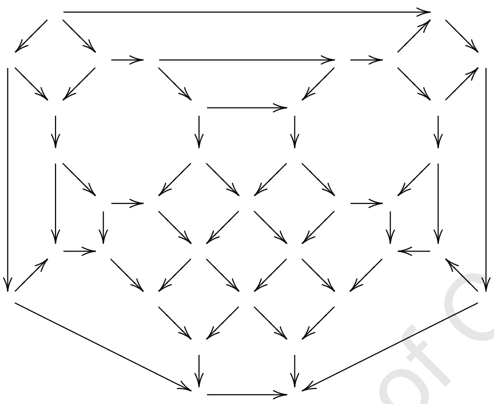
also $c'm'\langle s,t\rangle = c's = H_0$, and

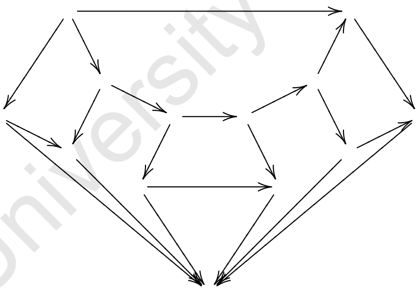
$$\begin{aligned} d' \circ (\sigma \otimes \tau) &= d' \circ \left(\alpha \langle sc, tc, F_1\rangle \cdot m' \langle 1_{sc}, \tau\rangle \cdot \alpha^{-1} \langle sc, G_1, td\rangle \cdot m' \langle \sigma, 1_{td}\rangle \cdot \alpha \langle H_1, sd, td\rangle\right) \\ &= (d' \circ \alpha \langle sc, tc, F_1\rangle) \cdot (d' \circ m' \langle 1_{sc}, \tau\rangle) \cdot \\ &\qquad \left(d' \circ \alpha^{-1} \langle sc, G_1, td\rangle\right) \cdot (d' \circ m' \langle \sigma, 1_{td}\rangle) \cdot (d' \circ \alpha \langle H_1, sd, td\rangle) \\ &= 1_{d'F_1} \cdot 1_{d'F1} \cdot 1_{d'td} \cdot 1_{d'td} \cdot 1_{d'td} \\ &= 1_{d'F_1} \cdot 1_{d'td} = 1_{d'F_1} \cdot 1_{F_0 d} = 1_{d'F_1} \cdot 1_{d'F_1} = 1_{d'F_1} \end{aligned}$$

with similar computations for $c' \circ (\sigma \otimes \tau) = 1_{c'H_1}$.

Commutativity of diagrams (1.20) and (1.21) is obtained using Yoneda Lemma, writing the respective diagrams and adding all the possible arrows to fill them in order to obtain the following *mask* and *diamond*



(mask)



(diamond)

in which squares commute by naturality, hexagons commute by definition of $(\sigma \otimes \tau)$, octagons commute because S, T are pseudo-natural transformations, pentagons in the diamond commute by the same reason and all the other pentagons and triangles commute by coherence. ■

The horizontal composition of pseudo-natural transformations is only defined up to an isomorphism and it will be considered at the end of this paper.

In the next section we define square pseudo-modification (simply called pseudo-modification) and show that given two pseudo-categories C, C', we ob-

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tain a pseudo-category by considering the pseudo-functors as objects, natural transformations as morphisms, pseudo-natural transformations as pseudo-morphisms and pseudo-modifications as cells. So, in particular, we will show that the vertical composition of pseudo-natural transformations is associative and has identities up to isomorphism. We also show that PsCat is Cartesian closed up to isomorphism, that is, instead of an isomorphism of categories PsCat$(A \times B, C) \cong$PsCat$(A$,PsCAT$(B, C))$ we get an equivalence of categories PsCat$(A \times B, C) \sim$PsCat$(A$,PsCAT$(B, C))$.

1.4 Pseudo-modifications

Let **C** be a 2-category. Suppose C, C' are pseudo-categories in **C**, $F, G, H, K : C \longrightarrow C'$ are pseudo-functors, $T = (t, \tau) : F \longrightarrow G$, $T' = (t', \tau') : H \longrightarrow K$ are pseudo-natural transformations and $\theta = (\theta_0, \theta_1) : F \longrightarrow H, \theta' = \left(\theta'_0, \theta'_1\right) : G \longrightarrow K$ are two natural transformations.

A **pseudo-modification** Φ (that will be represented as)

$$\begin{tikzcd}
F \arrow[r, "T"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \Phi"] & G \arrow[d, "\theta'"] \\
H \arrow[r, "T'"] & K
\end{tikzcd}$$

is a 2-cell of **C**

$$\Phi : t \longrightarrow t'$$

satisfying

$$\begin{aligned} d' \circ \Phi &= \theta_0 \\ c' \circ \Phi &= \theta'_0 \end{aligned} \tag{1.24}$$

and the commutativity of the square

$$\begin{tikzcd}
\bullet \arrow[r, "\tau"] \arrow[d, "m'\left\langle \theta'_1, \Phi \circ d \right\rangle"'] & \bullet \arrow[d, "m'\langle \Phi \circ c, \theta_1 \rangle"] \\
\bullet \arrow[r, "\tau'"] & \bullet
\end{tikzcd} . \tag{1.25}$$

Consider the case where **C**=Cat. Suppose W, W' are two pseudo-categories in Cat, $F, G, H, K : W \longrightarrow W'$ are pseudo-functors, $T : F \longrightarrow G, T' : H \longrightarrow K$ are pseudo-natural transformations and $\theta : F \longrightarrow G, \theta' : H \longrightarrow K$ are natural transformations.

A pseudo-modification Φ

$$\begin{tikzcd}
F \arrow[r, "T"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \Phi"] & G \arrow[d, "\theta'"] \\
H \arrow[r, "T'"] & K
\end{tikzcd}$$

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is a family of cells

$$\begin{tikzcd}
FA \arrow[r, "t_A"{name=U}] \arrow[d, "\theta_A"'] & GA \arrow[d, "\theta'_A"] \\
HA \arrow[r, "t'_A"'{name=D}] & KA
\arrow[Rightarrow, from=U, to=D, "\Phi_A"]
\end{tikzcd}$$

of W', for each object A in W, where the square

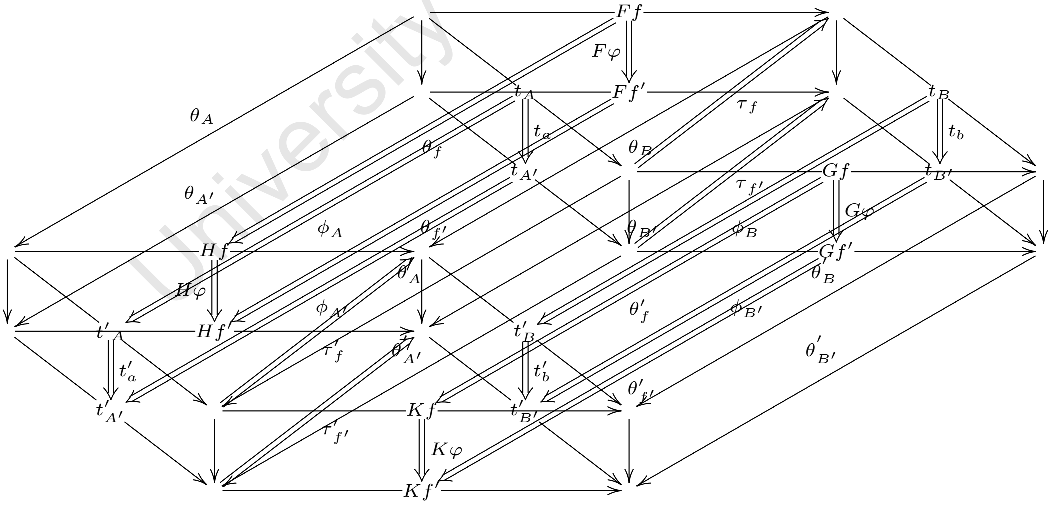
$$\begin{tikzcd}
t_A \arrow[r, "\Phi_A"] \arrow[d, "t_a"'] & t'_A \arrow[d, "t'_a"] \\
t_{A'} \arrow[r, "\Phi_{A'}"] & t'_{A'}
\end{tikzcd} , \tag{1.26}$$

commutes for every morphism $a : A \longrightarrow A'$ in W (naturality of Φ) and the square

$$\begin{tikzcd}
Gf \otimes t_A \arrow[r, "\tau_f"] \arrow[d, "\theta'_f \otimes \Phi_A"'] & t_B \otimes Ff \arrow[d, "\Phi_B \otimes \theta_f"] \\
Kf \otimes t'_A \arrow[r, "\tau'_f"'] & t'_B \otimes Hf
\end{tikzcd} , \tag{1.27}$$

commutes for every pseudo-morphism $f : A \longrightarrow B$ in W.

Both squares (1.26) and (1.27) may be displayed together with full information, for a φ in W, as follows



(1.28)

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Return to the general case.

Let **C** be a 2-category and consider C, C' two pseudo-categories in **C** as in (1.16). Suppose T, T', T'' are pseudo-natural transformations between pseudo-functors from C to C': we define for

$$T \xrightarrow{\Phi} T' \xrightarrow{\Phi'} T''$$

a composition $\Phi'\Phi$ as the composition of 2-cells in **C**, and clearly it is well defined, is associative and has identities. Now for $\theta, \theta', \theta''$ natural transformations between pseudo-functors from C to C', we define for

$$\theta \xrightarrow{\Phi} \theta' \xrightarrow{\Psi} \theta''$$

a pseudo-composition $\Psi \otimes \Phi = m' \langle \Psi, \Phi \rangle$.

Proposition 4 *Let **C** be a 2-category and suppose Ψ, Φ are pseudo-modifications*

$$\begin{tikzcd}
F \arrow[rr, "T"] \arrow[d, "\theta"'] & {} \arrow[d, Rightarrow, "\Phi"] & G \arrow[rr, "S"] \arrow[d, "\theta'"] & {} \arrow[d, Rightarrow, "\Psi"] & H \arrow[d, "\theta''"] \\
F' \arrow[rr, "T'"] & {} & G' \arrow[rr, "S'"] & {} & H'
\end{tikzcd}$$

with F, G, H, F', G', H' pseudo-functors from C to C' (pseudo-categories as in (1.16)*), S, T, S', T' pseudo-natural transformations and $\theta, \theta', \theta''$ natural transformations as considered above.*
The formula

$$\Psi \otimes \Phi = m' \langle \Psi, \Phi \rangle$$

for pseudo-composition of pseudo-modifications is well defined.

Proof. Recall that the composition of pseudo-modifications is given by

$$S \otimes T = (m' \langle s, t \rangle, (\sigma \otimes \tau))$$

with $(\sigma \otimes \tau)$ given as in (1.23), hence

$$m' \langle \Psi, \Phi \rangle : m' \langle s, t \rangle \longrightarrow m' \langle s', t' \rangle$$

is a 2-cell of **C** as required.
Conditions (1.24) are satisfied,

$$\begin{aligned}
d'm' \circ \langle \Psi, \Phi \rangle &= d'\pi_2' \circ \langle \Psi, \Phi \rangle = d' \circ \Phi = \theta_0 \\
c'm \circ \langle \Psi, \Phi \rangle &= c'\pi_1' \langle \Psi, \Phi \rangle = c' \circ \Psi = \theta_0''.
\end{aligned}$$

To prove commutativity of square (1.25) we use Yoneda Lemma and the following diagram, obtained by adapting (1.25) to the present case and filling its

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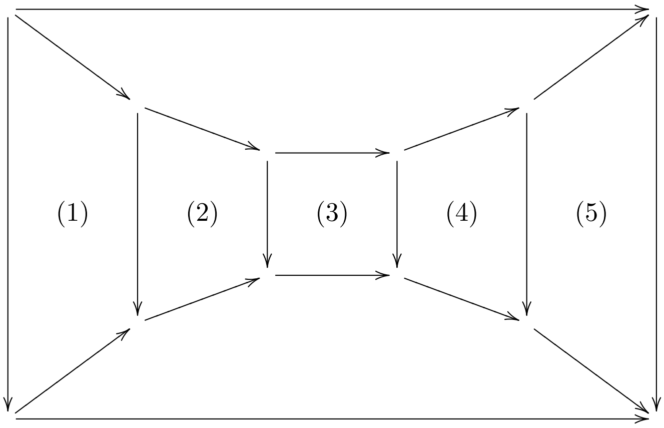
interior



where hexagons commute by definition of $(\sigma \otimes \tau)$ and $(\sigma' \otimes \tau')$, squares $(1), (3), (5)$ commute by naturality of α' while squares $(2), (4)$ commute because Ψ, Φ are pseudo-modifications (satisfy (1.27)) together with the fact that pseudo-composition (in C') satisfies the middle interchange law (1.8). ■

Composition of pseudo-natural transformations is not associative, however there is a special pseudo-modification for each triple of composable pseudo-natural transformations.

Proposition 5 *Let **C** be 2-category and suppose $F, G, H, K : C \longrightarrow C'$ are pseudo-functors in **C** and that $S = (s, \sigma), T = (t, \tau), U = (u, \upsilon)$ are pseudo-natural transformations as follows*

$$F \xrightarrow{S} G \xrightarrow{T} H \xrightarrow{U} K.$$

The 2-cell $\alpha'_{U,T,S} = \alpha' \langle u, t, s \rangle$ is a pseudo-modification

$$\begin{tikzcd} F \arrow[r, "U\otimes(T\otimes S)"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \alpha'_{U,T,S}"] & K \arrow[d, "1"] \\ F \arrow[r, "(U\otimes T)\otimes S"'] & K \end{tikzcd} ,$$

and it is natural in S, T, U, in the sense that the square

$$\begin{tikzcd} U \otimes (T \otimes S) \arrow[r, "\alpha'\langle u,t,s\rangle"] \arrow[d, "\varphi\otimes(\gamma\otimes\delta)"'] & (U \otimes T) \otimes S \arrow[d, "(\varphi\otimes\gamma)\otimes\delta"] \\ U' \otimes (T' \otimes S') \arrow[r, "\alpha'\left\langle u',t',s'\right\rangle"] & (U' \otimes T') \otimes S' \end{tikzcd}$$

commutes for every pseudo-modification $\varphi : U \longrightarrow U', \gamma : T \longrightarrow T', \delta : S \longrightarrow S'$.

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Proof. The 2-cell $\alpha' \langle u,t,s\rangle$ is obtained from

$$\begin{tikzcd} C_0 \arrow[r, "\langle u,t,s\rangle"] & C_1' \times_{C_0'} C_1' \times_{C_0'} C_1' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \alpha'"] & C_1, \end{tikzcd}$$

and

$$\begin{aligned} U \otimes (T \otimes S) &= (m\,(1 \times m)\,\langle u,t,s\rangle\,, (\upsilon \otimes (\tau \otimes \sigma))) \\ (U \otimes T) \otimes S &= (m\,(m \times 1)\,\langle u,t,s\rangle\,, ((\upsilon \otimes \tau) \otimes \sigma))\,, \end{aligned}$$

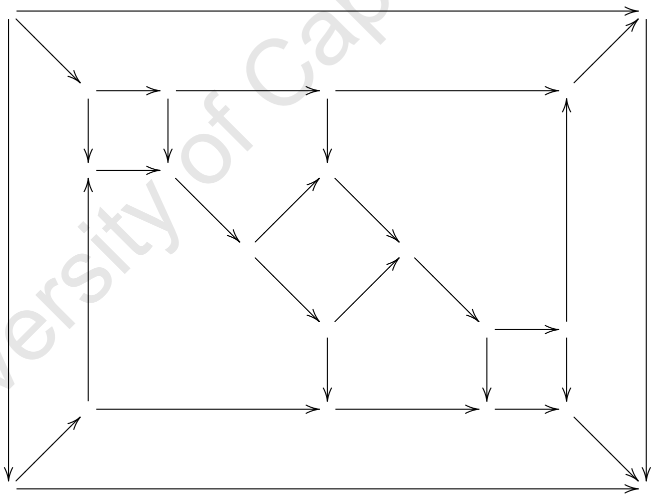
hence

$$\alpha' \langle u,t,s\rangle : m\,(1 \times m)\,\langle u,t,s\rangle \longrightarrow m\,(m \times 1)\,\langle u,t,s\rangle$$

is a 2-cell of **C**.
Conditions (1.24) are satisfied

$$\begin{aligned} d' \circ \alpha' \circ \langle u,t,s\rangle &= 1_{d'\pi_3'}\,\langle u,t,s\rangle = 1_{d's} = 1_{F_0} \\ c' \circ \alpha' \circ \langle u,t,s\rangle &= 1_{c'\pi_1'}\,\langle u,t,s\rangle = 1_{c'u} = 1_{K_0}. \end{aligned}$$

Commutativity of (1.25) follows from Yoneda Lemma and the commutativity of the following diagram



where hexagons commute because S, T, U are pseudo-natural transformations, squares commute by naturality and pentagons by coherence.

To prove naturality we observe that

$$\begin{aligned} ((\varphi \otimes \gamma) \otimes \delta) \cdot (\alpha' \langle u,t,s\rangle) &= (m' \langle m \langle \varphi, \gamma\rangle, \delta\rangle) \cdot (\alpha' \langle u,t,s\rangle) \\ &= (m'\,(m' \times 1)\,\langle \varphi, \gamma, \delta\rangle) \cdot (\alpha' \langle u,t,s\rangle) \\ &= \left(1_{m'(m' \times 1)} \cdot \alpha'\right) \circ \left(\langle \varphi, \gamma, \delta\rangle \cdot 1_{\langle u,t,s\rangle}\right) \\ &= \alpha' \circ \langle \varphi, \gamma, \delta\rangle \end{aligned}$$

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and

$$\begin{aligned}(\alpha' \langle u', t', s' \rangle) \cdot (\varphi \otimes (\gamma \otimes \delta)) &= (\alpha' \langle u', t', s' \rangle) \cdot (m' \langle \varphi, m' \langle \gamma, \delta \rangle \rangle) \\ &= (\alpha' \langle u', t', s' \rangle) \cdot (m' (1 \times m') \langle \varphi, \gamma, \delta \rangle) \\ &= \left(\alpha' \cdot 1_{m'(1\times m')}\right) \circ \left(1_{\langle u',t',s' \rangle} \langle \varphi, \gamma, \delta \rangle\right) \\ &= \alpha' \circ \langle \varphi, \gamma, \delta \rangle .\end{aligned}$$

■

For every pseudo-functor there is a pseudo-identity pseudo-natural transformation and a pseudo-identity pseudo-modification.

Proposition 6 *Consider a pseudo-functor* $F = (F_0, F_1, \mu_F, \varepsilon_F) : C \longrightarrow C'$ *in a 2-category* $\boldsymbol{C}$ *(with* C, C' *pseudo-categories in* $\boldsymbol{C}$ *as in* (1.16)*). The pair* $\left(e'F_0, \left(\lambda'^{-1}\rho'\right) \circ F_1\right)$ *is a pseudo-natural transformation in PsCat(*$\boldsymbol{C}$*)*

$$id_F = \left(e'F_0, \lambda'^{-1}\rho' F_1\right) : F \longrightarrow F,$$

and the 2-cell $1_{e'F_0} : e'F_0 \longrightarrow e'F_0$ *is a pseudo-modification in PsCat(*$\boldsymbol{C}$*)*

$$\begin{tikzcd} F \arrow[r, "id_F"] \arrow[d, "1"'] & F \arrow[d, "1"] \\ F \arrow[r, "id_F"'] & F \arrow[from=1-1, to=2-2, phantom, "\Downarrow 1_{id_F}"] \end{tikzcd} .$$

Proof. Clearly $e'F_0 : C_0 \longrightarrow C'_1$ is a morphism of $\mathbf{C}$, and

$$\lambda'^{-1}\rho' F_1 : m \langle F_1, e'd'F_1 \rangle \longrightarrow m' \langle e'c'F_1, F_1 \rangle$$

is a 2-cell (that is an isomorphism) of $\mathbf{C}$.

Conditions (1.18) and (1.19) are satisfied,

$$\begin{aligned} d'e'F_0 &= F_0 \\ c'e'F_0 &= F_0 \end{aligned}$$

$$\begin{aligned} d' \circ \left(\lambda'^{-1}\rho' F_1\right) &= d' \circ \left(\lambda'^{-1}\rho'\right) \circ F_1 \\ &= \left(d'\lambda'^{-1} F_1\right) \cdot \left(d'\rho' F_1\right) \\ &= \left(1_{d'F_1}\right) \cdot \left(1_{d'F_1}\right) \\ &= \left(1_{d'F_1}\right), \end{aligned}$$

and similarly for $c' \circ \left(\lambda'^{-1}\rho' F_1\right) = 1_{c'F_1}$.

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Commutativity of (1.20) is obtained using Yoneda Lemma and the commutativity of the diagram

$$\begin{tikzcd}
& & F_{g\otimes f}\otimes id_{FA} \arrow[rr] \arrow[dll] \arrow[dr] & & id_{FC}\otimes F_{g\otimes f} \arrow[drr] & & \\
(F_g\otimes F_f)\otimes id_{FA} \arrow[d] \arrow[drrr] & & & F_{g\otimes f} \arrow[ur] \arrow[d] & & & id_{FC}\otimes (F_g\otimes F_f) \arrow[d] \\
F_g\otimes (F_f\otimes id_{FA}) \arrow[drr] & & & F_g\otimes F_f \arrow[lll] \arrow[urrr] \arrow[dr] & & & (id_{FC}\otimes F_g)\otimes F_f \arrow[lll] \\
& & F_g\otimes (id_{FB}\otimes F_f) \arrow[ur] \arrow[rr] & & (F_g\otimes id_{FB})\otimes F_f \arrow[urr] & &
\end{tikzcd} ,$$

while (1.21) follows in a similar way as observed in the diagram

$$\begin{tikzcd}
& F(id_A)\otimes id_{FA} \arrow[rr] \arrow[dl] & & id_{FA}\otimes F(id_A) \arrow[dr] & \\
id_{FA}\otimes id_{FA} \arrow[drr] & & & & id_{FA}\otimes id_{FA}. \arrow[dll] \\
& & id_{FA} & &
\end{tikzcd}$$

This proves that id_F is a pseudo-natural transformation. To prove $1_{id_F} = 1_{e'F_0}$ is a pseudo-modification we note that

$$1_{e'F_0} : e'F_0 \longrightarrow e'F_0$$

is a 2-cell of $\mathbf{C}$,

$$\begin{aligned} d'\circ 1_{e'F_0} &= 1_{d'e'F_0} = 1_{F_0}, \\ c'\circ 1_{e'F_0} &= 1_{c'e'F_0} = 1_{F_0}. \end{aligned}$$

To prove commutativity of square (1.25) we use Yoneda Lemma and the commutativity of the following square

$$\begin{tikzcd}
Ff\otimes id_{FA} \arrow[r, "\lambda'^{-1}_{Ff}\rho'_{Ff}"] \arrow[d, "1_{Ff}\otimes 1_{id_{FA}}"'] & id_{FB}\otimes Ff \arrow[d, "1_{id_{FB}}\otimes 1_{Ff}"] \\
Ff\otimes id_{FA} \arrow[r, "\lambda'^{-1}_{Ff}\rho'_{Ff}"] & id_{FB}\otimes Ff
\end{tikzcd} .$$

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Proposition 7 *Let* $\boldsymbol{C}$ *be a 2-category and suppose* $F, G : C \longrightarrow C'$ *are pseudo-functors in* $\boldsymbol{C}$*.*
For every pseudo-natural transformation

$$T = (t, \tau) : F \longrightarrow G$$

there are two special pseudo-modifications

$$\begin{tikzcd}
F \arrow[rr, "id_G \otimes T"] \arrow[d, "1"'] & {} \arrow[d, Rightarrow, "\lambda_T"] & G \arrow[d, "1"] \\
F \arrow[rr, "T"'] & {} & G
\end{tikzcd}, \quad
\begin{tikzcd}
F \arrow[rr, "T \otimes id_F"] \arrow[d, "1"'] & {} \arrow[d, Rightarrow, "\rho_T"] & G \arrow[d, "1"] \\
F \arrow[rr, "T"'] & {} & G
\end{tikzcd},$$

with $\lambda_T = \lambda' \circ t, \rho_T = \rho' \circ t$ *both natural in* T*.*

Proof. It is clear that $\lambda' \circ t : m' \langle t, e' F_0 \rangle \longrightarrow t$ is a 2-cell of $\mathbf{C}$, and

$$\begin{aligned} d' \circ \lambda' \circ t &= 1_{d't} = 1_{F_0} \\ c' \circ \lambda' \circ t &= 1_{c't} = 1_{G_0}. \end{aligned}$$

The commutativity of square (1.25) is obtained from the commutativity of diagram

$$\begin{tikzcd}
G_f \otimes (id_{G_A} \otimes t_A) \arrow[rrrrrr] \arrow[dr] \arrow[ddd] & & & & & & (id_{G_B} \otimes t_B) \otimes F_f \arrow[ddd] \\
& (G_f \otimes id_{G_A}) \otimes t_A \arrow[dr] \arrow[ddl] & & & & id_{G_B} \otimes (t_B \otimes F_f) \arrow[ur] \arrow[ddr] & \\
& & (id_{G_B} \otimes G_f) \otimes t_A \arrow[rr] \arrow[dll] & & id_{G_B} \otimes (G_f \otimes t_A) \arrow[ur] \arrow[dllll] & & \\
G_f \otimes t_A \arrow[rrrrrr] & & & & & & t_B \otimes F_f
\end{tikzcd}$$

In order to prove naturality of λ_T consider a internal pseudo-modification

$$\begin{tikzcd}
F \arrow[rr, "T"] \arrow[d, "\theta"'] & {} \arrow[d, Rightarrow, "\Phi"] & G \arrow[d, "\theta'"] \\
H \arrow[rr, "T'"'] & {} & K
\end{tikzcd}$$

as defined in (1.24); then, on the one hand we have

$$\begin{aligned} \Phi \cdot \left(\lambda' \circ t\right) &= \left(1_{C_1'} \circ \Phi\right) \cdot \left(\lambda' \circ 1_t\right) \\ &= \left(1_{C_1'} \cdot \lambda'\right) \circ \left(\Phi \cdot 1_t\right) \\ &= \lambda' \circ \Phi \end{aligned}$$

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and on the other hand we have

$$\begin{aligned}
(\lambda' \circ t') \cdot (m' \langle e'\theta'_0, \Phi\rangle) &= (\lambda' \circ t') \cdot (m' \langle e'c', 1_{C'_1}\rangle \circ \Phi) \\
&= \left(\lambda' \cdot 1_{m'\left\langle e'c', 1_{C'_1}\right\rangle}\right) \circ (1_{t'} \cdot \Phi) \\
&= \lambda' \circ \Phi.
\end{aligned}$$

The proof on rho is similar. ■

The three last propositions lead us to the following theorem.

Theorem 8 *Let* $\boldsymbol{C}$ *be a 2-category, and consider* C, C' *two pseudo-categories in* $\boldsymbol{C}$*. The data:*

- *objects: pseudo-functors from* C *to* C'*;*
- *morphisms: natural transformations (between pseudo-functors from* C *to* C'*);*
- *pseudo-morphisms: pseudo-natural transformations (between pseudo-functors from* C *to* C'*);*
- *cells: pseudo-modifications (between such natural and pseudo-natural transformations);*

form a pseudo-category (in Cat).

Proof. Natural transformations and pseudo-functors form a category: theorem 2. pseudo-modifications and pseudo-natural transformations also form a category: the composition is associative and has identities (that inherit the structure of 2-cells of the ambient 2-category).

For every pseudo-natural transformation $T = (t, \tau) : F \longrightarrow G$, the identity pseudo-modification is $1_T = 1_t$

$$\begin{tikzcd}
F \arrow[rr, "T"] \arrow[d, "1"'] & {} \arrow[d, Rightarrow, "1_T"] & G \arrow[d, "1"] \\
F \arrow[rr, "T"'] & {} & G
\end{tikzcd} .$$

For each pair of pseudo-composable pseudo-modifications Φ, Ψ, there is a (well defined - proposition 4) pseudo-composition $\Phi \otimes \Psi = m' \langle \Phi, \Psi \rangle$ satisfying (1.8)

$$(\Phi\Phi') \otimes (\Psi\Psi') = m' \langle \Phi\Phi', \Psi\Psi' \rangle$$

$$\begin{aligned}
(\Phi \otimes \Psi)(\Phi' \otimes \Psi') &= (m' \langle \Phi, \Psi \rangle)(m' \langle \Phi', \Psi' \rangle) \\
&= (1_{m'} 1_{m'}) \circ (\langle \Phi, \Psi \rangle \langle \Phi', \Psi' \rangle) \\
&= m' \langle \Phi\Phi', \Psi\Psi' \rangle ;
\end{aligned}$$

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and $1_{T\otimes S}=1_T\otimes 1_S$,

$$1_{m\langle t,s\rangle}=1_m\circ 1_{\langle t,s\rangle}=1_m\circ\langle 1_t,1_s\rangle=m\langle 1_t,1_s\rangle .$$

For each natural transformation $\theta : F\longrightarrow G$ there is a pseudo-modification

$$\begin{tikzcd}
F \arrow[r, "id_F"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow id_\theta"] & F \arrow[d, "\theta"] \\
G \arrow[r, "id_G"'] & G
\end{tikzcd}$$

with $id_\theta = e'\theta_0$, satisfying

$$id_{1_F}=e'1_{F_0}=1_{e'F_0}=1_{id_F},$$

$$id_{\theta'\theta}=e'\circ\left(\theta_0'\theta_0\right)=\left(e'\circ\theta_0'\right)\left(e'\circ\theta_0\right)=id_{\theta'}id_\theta .$$

By Proposition 5 there is a special pseudo-modification $\alpha_{T,U,S}=\alpha\langle T,U,S\rangle$ for each triple of composable pseudo-natural transformations T,U,S, natural in each component and satisfying the pentagon coherence condition.

By Proposition 7 there are two special pseudo-modifications λ_T,ρ_T to each pseudo-natural transformation $T : F\longrightarrow G$, natural in T and satisfying the triangle coherence condition. ■

1.5 Conclusion and final remarks

The mathematical object PsCat that we have just defined has the following structure:

- objects: $A,B,C,...$
- morphisms: $f : A\longrightarrow B,...$
- 2-cells: $\theta : f\longrightarrow g,...(f,g : A\longrightarrow B)$
- pseudo-cells: $f \xrightarrow{T} g$, ...
- tetra cells: $$\begin{tikzcd}
f \arrow[r, "T"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \Phi"] & g \arrow[d, "\theta'"] \\
f' \arrow[r, "T'"'] & g'
\end{tikzcd} , ...$$

where objects, morphisms and 2-cells form a 2-category and for each pair of objects A,B, the morphisms, 2-cells, pseudo-cells and tetra cells from A to B form a pseudo-category.

Two questions arise at this moment:

- What is happening from PsCat$(B,C)\times$PsCat(A,B) to PsCat(A,C)?

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- What is the relation between PsCat$(A\times B, C)$ and PsCat$(A,$PsCAT$(B,C))$?

The answer to the second question is easy to find out. If starting with a pseudo-functor in PsCat$(A\times B, C)$, say

$$h : A\times B \longrightarrow C,$$

by going to PsCat$\big(A, C^B\big)$ and coming back we will obtain either

$$h\,(c,g)\otimes h\,(f,b)$$

or

$$h\,(f,d)\otimes h\,(a,g)$$

instead of $h\,(f,g)$ as displayed in the diagram below

$$\begin{tikzcd}
(a,b) \arrow[d, "{(f,g)}"] & h(a,b) \arrow[r, "{h(f,b)}"] \arrow[d, "{h(a,g)}"'] \arrow[dr, "{h(f,g)}" description] \arrow[dr, phantom, shift left=3ex, "\overset{\mu}{\Rightarrow}"] \arrow[dr, phantom, shift right=3ex, "\Downarrow \mu"] & h(c,b) \arrow[d, "{h(c,g)}"] \\
(c,d) & h(a,d) \arrow[r, "{h(f,d)}"'] & h(c,d)
\end{tikzcd}$$

And since they are all isomorphic via μ and τ we have that the relation is an equivalence of categories.

A similar phenomena happens when trying to define horizontal composition of pseudo-natural transformations (while trying to answer the first question): there are two equally good ways to define a horizontal composition and they differ by an isomorphism.

Let **C** be a 2-category and C, C', C''pseudo-categories in **C**, consider S, T pseudo-natural transformations as in

$$\begin{tikzcd}
C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow T"] & C' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\downarrow S"] & C''
\end{tikzcd}$$

there are two possibilities to define horizontal composition

$$S\circ_{w1} T = m''\left\langle sG_0, F_1 t'\right\rangle$$

and

$$S\circ_{w2} T = m''\left\langle G_1' t, sF_0\right\rangle$$

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as displayed in the following picture

$$\begin{tikzcd}
C_1 \arrow[d, "F_1"', "G_1"] & C_0 \arrow[l, "e"'] \arrow[d, "F_0"', "G_0"] \arrow[dl, phantom, "\swarrow t"] \\
C'_1 \arrow[d, "F'_1"', "G'_1"] & C'_0 \arrow[l, "e'"'] \arrow[d, "F'_0"', "G'_0"] \arrow[dl, phantom, "\swarrow s"] \\
C''_1 & C''_0 \arrow[l, "e''"']
\end{tikzcd} .$$

Hence we have two isomorphic functors from $\mathrm{PsCat}(B,C)\times\mathrm{PsCat}(A,B)$ to $\mathrm{PsCat}(A,C)$ both defining a horizontal composition.

We note that this behaviour, of composition beeing defined up to isomorphism, also occurs while trying to compose homotopies. So one can expect further relations between the theory of pseudo-categories and homotopy theory to be investigated.

For instance the category Top itself may be viewed as a structure with objects (spaces), morphisms (continuous mappings), 2-cells (homotopy classes of homotopies), pseudo-cells (simple homotopies) and tetra cells (homotopies between homotopies).

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Chapter 2

Sesquicategory: a category with a 2-cell structure

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Title: Sesquicategory: a category with a 2-cell structure

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Key words: sesquicategory, 2-cell structure, cartesian 2-cell structure, natural 2-cell, pseudocategory.

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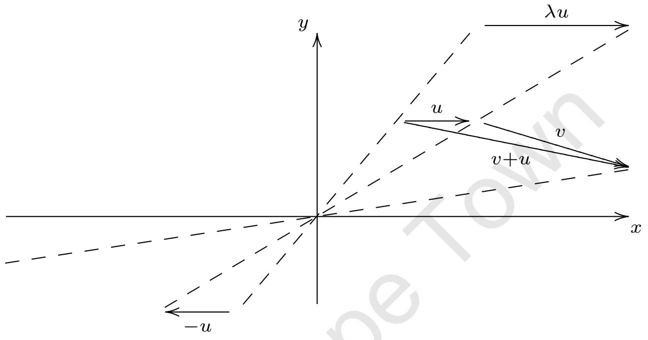
Abstract: For a given (fixed) category, we consider the category of all 2-cell structures (over it) and study some naturality properties. A category with a 2-cell structure is a sesquicategory; we use additive notation for the vertical composition of 2-cells; instead of a law for horizontal composition we consider a relation saying which pairs of 2-cells can be horizontally composed; for a 2-cell structure with every 2-cell invertible, we also consider a notion of commutator, measuring the obstruction for horizontal composition. We compare the concept of naturality in an abstract 2-cell structure with the example of internal natural transformations in a category of the form Cat(**B**), of internal categories in some category **B**, and show that they coincide. We provide a general construction of 2-cell structures over an arbitrary category, under some mild assumptions. In particular, the canonical 2-cell structures over groups and crossed-modules, respectively "conjugations" and "derivations", are instances of these general constructions. We define cartesian 2-cell structure and extend the notion of pseudocategory from the context of a 2-category

(as in [Ch1]) to the more general context of a sesquicategory. Some remarks on coherence are also given.

2.1 Introduction

In this article we use a different notation for the vertical composition of 2-cells: instead of the usual dot '$\cdot$' we use plus '$+$'. To support this we present the following analogy between geometrical vectors in the plane and 2-cells between morphisms in a category.



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Two geometrical vectors in the plane can be added only if the end point of the second (u as in the picture above) is the starting point of the first one (v as in the picture) and in that case the resulting vector (the sum) goes from the starting point of the second to the end point of the first: exactly the same as with 2-cells

$$\begin{tikzcd}
\cdot \arrow[r, bend left=60, "\operatorname{dom} u", ""{name=A, below}] \arrow[r, ""{name=B, above}, ""{name=C, below}] \arrow[r, bend right=60, "\operatorname{cod} v"', ""{name=D, above}] & \cdot \arrow[r, mapsto] & \cdot \arrow[r, bend left=60, "\operatorname{dom} u", ""{name=E, below}] \arrow[r, bend right=60, "\operatorname{cod} v"', ""{name=F, above}] & \cdot \; ;
\arrow[Rightarrow, from=A, to=B, "u"]
\arrow[Rightarrow, from=C, to=D, "v"]
\arrow[Rightarrow, from=E, to=F, "v+u"]
\end{tikzcd}$$

In some sense the analogy still holds for scalar multiplication

$$\begin{tikzcd}
\cdot \arrow[r, "\lambda"] & \cdot \arrow[r, bend left=60, "\operatorname{dom} u", ""{name=A, below}] \arrow[r, bend right=60, "\operatorname{cod} u"', ""{name=B, above}] & \cdot \arrow[r, "\rho"] & \cdot \arrow[r, mapsto] & \cdot \arrow[r, bend left=60, "\rho \operatorname{dom}(u) \lambda", ""{name=C, below}] \arrow[r, bend right=60, "\rho \operatorname{cod}(u) \lambda"', ""{name=D, above}] & \cdot \; ,
\arrow[Rightarrow, from=A, to=B, "u"]
\arrow[Rightarrow, from=C, to=D, "\rho u \lambda"]
\end{tikzcd}$$

and for inverses (in the case they exist)

$$\begin{tikzcd}
\cdot \arrow[r, bend left=60, "\operatorname{dom} u", ""{name=A, below}] \arrow[r, bend right=60, "\operatorname{cod} u"', ""{name=B, above}] & \cdot \arrow[r, mapsto] & \cdot \arrow[r, bend left=60, "\operatorname{cod} u", ""{name=C, below}] \arrow[r, bend right=60, "\operatorname{dom} u"', ""{name=D, above}] & \cdot \; .
\arrow[Rightarrow, from=A, to=B, "u"]
\arrow[Rightarrow, from=C, to=D, "-u"]
\end{tikzcd}$$

Concerning horizontal composition, there is still an analogy with some relevance: it is, in some sense, analogous to the cross product of vectors − in the sense that it raises in dimension (see the introduction of [6] and its references for further discussion on this). Given 2-cells, u and v

$$\begin{tikzcd}
\cdot \arrow[r, bend left, "\mathrm{dom}\, u"] \arrow[r, bend right, "\mathrm{cod}\, u"'] \arrow[r, phantom, "\Downarrow u"] & \cdot \arrow[r, bend left, "\mathrm{dom}\, v"] \arrow[r, bend right, "\mathrm{cod}\, v"'] \arrow[r, phantom, "\Downarrow v"] & \cdot
\end{tikzcd}$$

the horizontal composition $v \circ u$ should be a 3-cell, from the 2-cell

$$\mathrm{cod}\,(v)\, u + v\, \mathrm{dom}\,(u) \tag{2.1}$$

to the 2-cell

$$v\, \mathrm{cod}\,(u) + \mathrm{dom}\,(v)\, u. \tag{2.2}$$

In some cases (2.1) and (2.2) coincide (as it happens in a 2-category) and this is the reason why one may think of a horizontal composition, but it is an illusion; to overcome this we better consider a relation $v \circ u$ saying that the 2-cell v is natural with respect to u, defined as

$$v \circ u \Longleftrightarrow (2.1) = (2.2)\,,$$

in this sense, the horizontal composition is only defined for those pairs (v, u) that are in relation $v \circ u$, with the composite being then given by either (2.1) or (2.2).

This is a geometrical intuition. An algebraic intuition is also provided in Proposition 10.

This article is organized as follows.

For a fixed category, **C**, we define a 2-cell structure (over **C**, as to make it a sesquicategory) and give a characterization of such a structure as a family of sets, together with maps and actions, satisfying some conditions. It generalizes the characterization of 2-Ab-categories as a family of abelian groups, together with group homomorphisms and laws of composition as given in [Ch5] and [Ch6] where the strong condition

$$D\,(x)\, y = xD\,(y)$$

is no longer required. A useful consequence is that the example of chain complexes, say of order 2, can be considered in this more general setting. Of course, this condition is equivalent to the naturality condition, and the results obtained in [Ch5] and [Ch6] heavily rest on this assumption, so one must be careful in removing it. For this we introduce and study the concept of a 2-cell being natural with respect to another 2-cell, and the concept of natural 2-cell, as one being natural with respect to all. Next we compare this notions when **C** is a category of the form Cat(**B**), of internal categories in some category **B**, and conclude

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that if the 2-cell structure is the canonical one (internal transformations, not necessarily natural) then a natural 2-cell corresponds to a natural transformation, and furthermore, it is sufficient to check if a given transformation is natural with respect to a particular 2-cell (from the "category of arrows"), to determine if it is natural.

We give a general process for constructing 2-cell structures in arbitrary categories, and for the purposes of latter discussions we will restrict our study to the 2-cell structures obtained this way. In order to argue that we are not restricting too much, we show that the canonical 2-cell structures over groups and crossed-modules, that are respectively "conjugations" and "derivations", are captured by this construction.

We introduce the notion of cartesian 2-cell structure, in order to consider 2-cells of the form $u \times_w v$ that are used in the coherence conditions involved in a pseudocategory.

At the end we extend the notion of pseudocategory from the context of a 2-category to the more general context of a category with a 2-cell structure (sesquicategory).

All the notions defined in [Ch1]: pseudofunctor, natural and pseudo-natural transformation, modification, may also be extended in this way. However some careful is needed when dealing with coherence issues. For example MacLane's Coherence Theorem, saying that it suffices to consider the coherence for the *pentagon* and *middle triangle* is no longer true in general, since it uses the fact that α, λ, ρ are natural. One way to overcome this difficulty is to impose the naturality for α, λ, ρ in the definition, so that in [Ch1] (introduction, definition of pseudocategory in a 2-category) instead of saying

"...α, λ, ρ are 2-cells (which are isomorphisms)..."

we have to say

"...α, λ, ρ are natural and invertible 2-cells ..."

We will not study deeply all the consequences of this. Instead we will restrict ourselves to the study of 2-cell structures such that all 2-cells are invertible (since the main examples are groups, abelian groups, 1-chain complexes and crossed modules) and hence the question of α, λ, ρ being invertible becomes intrinsic to the 2-cell structure. The issue of naturality is more delicate. To prove the results in [Ch5], [Ch6] and [Ch9], we will only need that λ and ρ to be natural with respect to each other, that is

$$\lambda \circ \lambda, \lambda \circ \rho, \rho \circ \lambda, \rho \circ \rho.$$

If interested in the Coherence Theorem, we can always use the reflexion

$$\text{2-cellstruct}(\mathbf{C}) \xrightarrow{I} \text{nat-2-cellstruct}(\mathbf{C})$$

of the category of 2-cell structures over $\mathbf{C}$ (sesquicategories "with base $\mathbf{C}$"), into the subcategory of natural 2-cell structures over $\mathbf{C}$ (2-categories "with base $\mathbf{C}$"),

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sending each 2-cell structure to its "naturalization"; which if $\mathbf{C} = \mathbf{1}$ becomes the familiar reflexion of monoids into commutative monoids

$$\mathrm{Mon} \xrightarrow{I} \mathrm{CommMon}$$

and if restricting further to invertible 2-cells gives the reflection

$$\mathrm{Grp} \xrightarrow{I} \mathrm{Ab}$$

of groups into abelian groups.

All these considerations will be examined in [Ch9] when describing pseudo-categories in weakly Mal'cev sesquicategories.

2.2 2-cell structures and sesquicategories

Let $\mathbf{C}$ be a fixed category.

Definition 9 (2-cell structure) *A 2-cell structure over* $\mathbf{C}$ *is a system*

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

where

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set$$

is a functor and

$$\begin{tikzcd} H \times_{\mathrm{hom}} H \arrow[r, "+"] & H \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathrm{hom}_{\mathbf{C}} \arrow[l, "0"'] \end{tikzcd}$$

are natural transformations, such that

$$(\mathrm{hom}_{\mathbf{C}}, H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

is a category object in the functor category $Set^{C^{op} \times C}$ *or, in other words, an object in* $Cat\left(Set^{C^{op} \times C}\right)$.

Proposition 10 *Giving a 2-cell structure over a category* $\mathbf{C}$*, is to give, for every pair* (A, B) *of objects in* $\mathbf{C}$*, a set* $H(A, B)$*, together with maps*

$$\begin{tikzcd} H(A,B) \times_{\mathrm{hom}(A,b)} H \arrow[r, "+"] & H \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathrm{hom}(A,B) \arrow[l, "0"'] \end{tikzcd},$$

and actions

$$\begin{array}{ccc} H(B,C) \times \mathrm{hom}(A,B) & \longrightarrow & H(A,C) \\ (x,f) & \longmapsto & xf \\ \mathrm{hom}(B,C) \times H(A,B) & \longrightarrow & H(A,C) \\ (g,y) & \longmapsto & gy \end{array}$$

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satisfying the following conditions

$$\begin{gathered}
\operatorname{dom}(gy) = g\operatorname{dom}(y)\ ,\ \operatorname{dom}(xf) = \operatorname{dom}(x)\,f \\
\operatorname{cod}(gy) = g\operatorname{cod}(y)\ ,\ \operatorname{cod}(xf) = \operatorname{cod}(x)\,f \\
g0_f = 0_{gf} = 0_g f \\
(x+x')\,f = xf + x'f\ ,\ g\,(y+y') = gy + gy'
\end{gathered} \tag{2.3}$$

$$\begin{aligned}
g'(gy) &= (g'g)\,y\ ,\ (xf)\,f' = x\,(ff') \\
g'(xf) &= (g'x)\,f \\
1_C x &= x = x1_B
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\operatorname{dom}(0_f) &= f = \operatorname{cod}(0_f) \\
\operatorname{dom}(x+x') &= x'\ ,\ \operatorname{cod}(x+x') = x \\
0_{\operatorname{cod}x} + x &= x = x + 0_{\operatorname{dom}x} \\
x + (x'+x'') &= (x+x') + x''.
\end{aligned} \tag{2.5}$$

Proof. For every $f : A' \longrightarrow A$, $g : B \longrightarrow B'$ and $x \in H(A,B)$, write

$$H(f,g)(x) = gxf$$

and it is clear that the set of conditions (2.3) asserts the naturality of dom, cod, 0, +; the set of conditions (2.4) asserts the functoriality of H and the set of conditions (2.5) asserts the axioms for a category. ■

Definition 11 (sesquicategory) *A* sesquicategory *is a pair* $(\mathbf{C},\mathbf{H})$ *where* $\mathbf{C}$ *is a category and* $\mathbf{H}$ *a 2-cell structure over it.*

Observation: A sesquicategory as defined, is the same as a sesquicategory in the sense of Ross Street [7], that is, a category $\mathbf{C}$ together with a functor $\mathbf{H}$ into Cat, such that the restriction to Set gives $\hom_{\mathbf{C}}$, as displayed in the following picture

$$\begin{tikzcd}
 & Cat \arrow[d] \\
\mathbf{C}^{op} \times \mathbf{C} \arrow[ur, "\mathbf{H}"] \arrow[r, "\hom"'] & Set
\end{tikzcd}.$$

Proposition 12 *A category* $\mathbf{C}$ *with a 2-cell structure*

$$\mathbf{H} = (H, \operatorname{dom}, \operatorname{cod}, 0, +),$$

is a 2-category if and only if the naturality condition

$$\operatorname{cod}(x)\,y + x\operatorname{dom}(y) = x\operatorname{cod}(y) + \operatorname{dom}(x)\,y \tag{naturality condition}$$

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holds for every $x \in H(B,C)$, $y \in H(A,B)$ *,and every triple of objects* (A,B,C) *in* $\mathbf{C}$*, as displayed in the diagram below*

$$\begin{tikzcd} A \arrow[r, bend left=50, "\mathrm{dom}\, y"{name=U1}] \arrow[r, bend right=50, "\mathrm{cod}\, y"'{name=D1}] & B \arrow[r, bend left=50, "\mathrm{dom}\, x"{name=U2}] \arrow[r, bend right=50, "\mathrm{cod}\, x"'{name=D2}] & C . \arrow[Rightarrow, from=U1, to=D1, "y"] \arrow[Rightarrow, from=U2, to=D2, "x"] \end{tikzcd}$$

Proof. If $\mathbf{C}$ is a 2-category, the naturality condition follows from the horizontal composition of 2-cells and conversely, given a 2-cell structure over $\mathbf{C}$, in order to make it a 2-category one has to define a horizontal composition and it is defined as

$$x \circ y = \mathrm{cod}\,(x)\, y + x\, \mathrm{dom}\,(y)$$

or

$$x \circ y = x\, \mathrm{cod}\,(y) + \mathrm{dom}\,(x)\, y$$

provided the naturality condition is satisfied for every appropriate x, y. The middle interchange law also follows from the naturality condition. ■

It may happen that the naturality condition does not hold for all possible x and y, but only for a few; thus the following definitions.

Let $\mathbf{C}$ be a category and $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ a 2-cell structure over it.

Definition 13 *A 2-cell* $\delta \in H(A,B)$ *is* natural with respect to *a 2-cell* $z \in H(X,A)$*, when*

$$\mathrm{cod}\,(\delta)\, z + \delta\, \mathrm{dom}\,(z) = \delta\, \mathrm{cod}\,(z) + \mathrm{dom}\,(\delta)\, z,$$

in that case one writes $\delta \circ z$*.*

Definition 14 *A 2-cell* $\delta \in H(A,B)$ *is* natural *when it is natural with respect to all possible* $z \in H(X,A)$ *for all* $X \in \mathbf{C}$*, i.e.,* δ *is a natural 2-cell if and only if* $\delta \circ z$ *for all possible* z*.*

2.3 Examples

We shall now see how the above notions of naturality are related, in the case where $\mathbf{C} = Cat\,(\mathbf{B})$ for some category $\mathbf{B}$, with the 2-cell structure given by the internal (natural) transformations.

Example 15 *Consider* $\mathbf{C} = Cat\,(\mathbf{B})$ *the category of internal categories in some category* $\mathbf{B}$*. The objects are*

$$A = (A_0, A_1, d, c, e, m)\,,\ B = (B_0, B_1, d, c, e, m)\,,\ ...$$

and morphisms

$$f = (f_1, f_0) : A \longrightarrow B, ...$$

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Consider the following 2-cell structure over $\mathbf{C}$*:*

$$\begin{aligned}
H(A,B) &= \{(k,t,h) \mid t : A_0 \longrightarrow B_1;\ h,k \in \hom_{\mathbf{C}}(A,B)\,;\ dt = h_0, ct = k_0\} \\
H(f,g)(k,t,h) &= (gkf, g_1 t f_0, ghf) \\
\operatorname{dom}(k,t,h) &= h \\
\operatorname{cod}(k,t,h) &= k \\
0_h &= (h, eh_0, h) \\
(k,t,h) + (h,s,l) &= (k, m\langle t,s\rangle, l)
\end{aligned}$$

where $f : A' \longrightarrow A, g : B \longrightarrow B', h,k,l : A \longrightarrow B$ *are morphisms in* $Cat(\mathbf{B})$ *and* $t,s : A_0 \longrightarrow B_1$ *are morphisms in* $\mathbf{B}$.
Observe that in particular, for every $A = (A_0, A_1, d, c, e, m)$ *there is* $A^{\rightarrow} = (A_1, A_1, 1, 1, 1, 1)$ *and the two morphisms*

$$d^{\rightarrow} = (ed, d) : A^{\rightarrow} \longrightarrow A$$

and

$$c^{\rightarrow} = (ec, c) : A^{\rightarrow} \longrightarrow A.$$

Proposition 16 *In the context of the previous example, a 2-cell* $\mathbf{t} = (k,t,h) \in H(A,B)$ *is an internal natural transformation* $\mathbf{t} : h \longrightarrow k$*, if and only if it is natural with respect to the 2-cell*

$$(c^{\rightarrow}, 1_{A_1}, d^{\rightarrow}) \in H(A^{\rightarrow}, A).$$

Proof. Consider $\mathbf{t} = (k,t,h) \in H(A,B)$ and $\mathbf{z} = (g,z,f) \in H(X,A)$,

$$\begin{tikzcd}
\cdots & X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & X_0 \arrow[l] \arrow[ld, "z"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \\
\cdots & A_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "k_1"] & A_0 \arrow[l] \arrow[ld, "t"'] \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "k_0"] \\
\cdots & B_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & B_0 \arrow[l]
\end{tikzcd}$$

by definition

$$\begin{aligned}
\mathbf{t} \circ \mathbf{z} \quad &\Leftrightarrow \quad (kg, k_1 z, kf) + (kf, tf_0, hf) = (kg, tg_0, hg) + (hg, h_1 z, hf) \\
&\Leftrightarrow \quad (kg, m\langle k_1 z, tf_0\rangle, hf) = (kg, m\langle tg_0, h_1 z\rangle, hf) \\
&\Leftrightarrow \quad m\langle k_1 z, tf_0\rangle = m\langle tg_0, h_1 z\rangle \qquad (2.6)
\end{aligned}$$

and also by definition t is an internal natural transformation when

$$m\langle k_1, td\rangle = m\langle tc, h_1\rangle \qquad (2.7)$$

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which is equivalent to saying that (k, t, h) is natural relative to $\left(c^{\rightarrow}, 1_{A_1}, d^{\rightarrow}\right)$,as displayed below

$$\begin{tikzcd}
\cdots & A_1 \arrow[r, equal] \arrow[d, shift right, "ed"'] \arrow[d, shift left, "ec"] & A_1 \arrow[ld, "1"'] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & . \\
\cdots & A_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & A_0 \arrow[l]
\end{tikzcd}$$

■

Corollary 17 *Every internal natural transformation is a natural 2-cell.*

Proof. Simply observe that

$$(2.7) \Longrightarrow (2.6)$$

since

$$\begin{aligned}
m\langle k_1, td\rangle z &= m\langle tc, h_1\rangle z \\
m\langle k_1 z, tdz\rangle &= m\langle tcz, h_1 z\rangle \\
m\langle k_1 z, tf_0\rangle &= m\langle tg_0, h_1 z\rangle .
\end{aligned}$$

■

The notion of a category with a 2-cell structure, besides giving a simple characterization of a 2-category as

"2-category"="sesquicategory"+"naturality condition";

it also provides a powerful tool to construct examples in arbitrary situations.

Example 18 *Consider* $\boldsymbol{C}$ *a category and*

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Mon$$

a functor into Mon, the category of monoids, together with a natural transformation

$$D : UH \times \hom_{\mathbf{C}} \longrightarrow \hom_{\mathbf{C}}$$

(where $U : Mon \longrightarrow Set$ *denotes the forgetful functor) satisfying*

$$\begin{aligned}
D(0, f) &= f \\
D(x' + x, f) &= D(x', D(x, f))
\end{aligned}$$

for all $f : A \longrightarrow B$ *in* $\boldsymbol{C}$ *and* $x', x \in H(A, B)$*, with* 0 *the zero of the monoid* $H(A, B)$ *considered in additive notation.*
A 2-cell structure in $\boldsymbol{C}$ *is now given as*

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "D(x{,}f)"', ""{name=L}] & B
\arrow[Rightarrow, from=U, to=L, "(x{,}f)"]
\end{tikzcd}$$

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with vertical composition

$$\begin{tikzcd} A \arrow[rr, bend left=60, "f"{name=U}] \arrow[rr, "{D(x,f)}"{name=M, description}] \arrow[rr, bend right=60, "{D(x',D(x,f))}"'{name=L}] && B \arrow[Rightarrow, from=U, to=M, "{(x,f)}"] \arrow[Rightarrow, from=M, to=L, "{(x',D(x,f))}"] \end{tikzcd}$$

$$(x', D(x,f)) + (x,f) = (x'+x, f)$$

(well defined because $D(x'+x,f) = D(x', D(x,f))$*), with identity 2-cells*

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "f"'{name=L}] & B \arrow[Rightarrow, from=U, to=L, "{(0,f)}"] \end{tikzcd}$$

well defined because $D(0,f) = f$*, and the left and right actions of morphisms in 2-cells,*

$$\begin{tikzcd} A' \arrow[r, "h"] & A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "{D(x,f)}"'{name=L}] & B \arrow[r, "g"] \arrow[Rightarrow, from=U, to=L, "{(x,f)}"] & B' \end{tikzcd}$$

$$g(x,f)h = (gxf, gfh) = (H(h,g)(x), gfh).$$

If in addition,

$$D(y,g)x + yf = yD(x,f) + gx \tag{2.8}$$

for all x, y, f, g *pictured as*

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U1}] \arrow[r, bend right=50, "{D(x,f)}"'{name=L1}] & B \arrow[r, bend left=50, "g"{name=U2}] \arrow[r, bend right=50, "{D(y,g)}"'{name=L2}] \arrow[Rightarrow, from=U1, to=L1, "{(x,f)}"] & C \arrow[Rightarrow, from=U2, to=L2, "{(y,g)}"] \end{tikzcd},$$

then, the result is a 2-category.

In some cases, the above example may even be pushed further.

Example 19 *Suppose the functor*

$$\hom_{\mathbf{C}} : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set$$

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may be extended to Mon, that is, there is a functor (denote it by map, and think of the underlying map of a homomorphism)

$$map : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Mon \xrightarrow{U} Set$$

with $\hom \subseteq Umap$*, in the sense that* $\hom(A,B) \subseteq Umap(A,B)$ *naturally for every* $A, B \in \mathbf{C}$*;*
Now, given any functor

$$K : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Mon$$

and any natural transformation

$$D : K \longrightarrow map,$$

define

$$\begin{aligned} H(A,B) &= \{(x,f) \in UK(A,B) \times \hom(A,B) \mid D(x) + f \in \hom(A,B)\} \\ H(h,g)(x,f) &= (gxh, gfh) \end{aligned}$$

and obtain a functor $H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set$*. With obvious* $\mathrm{dom}, \mathrm{cod}, 0, +,$*a 2-cell structure in* $\mathbf{C}$*, is obtained as follows*

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "D(x)+f"'{name=L}] & B \arrow[Rightarrow, from=U, to=L, "{(x,f)}"] \end{tikzcd}$$

where $(x,f) \in H(A,B)$*,*
vertical composition: $(x', D(x) + f) + (x,f) = (x' + x, f)$
identity: $(0,f)$
left and right actions: $g(x,f)h = (gxh, gfh)$*.*
If in addition the property

$$D(y)x + gx + yf = yD(x) + yf + gx \tag{2.9}$$

is satisfied for all $(x,f) \in H_1(A,B)$ *and* $(y,g) \in H_1(A,C)$*, then the resulting structure is a 2-category.*

Remark 20 *In particular, if* $\mathbf{C}$ *is an Ab-category, a 2-Ab-category as defined in [Ch5] and [Ch6] is obtained in this way; in that case the functor* hom *is in fact a functor*

$$\hom : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Ab.$$

Giving a 2-cell structure is then to give a functor (usually required to be an Ab-functor) $H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Ab$*, and a natural transformation* $D : H \longrightarrow \hom$*. This 2-cell structure makes* $\mathbf{C}$ *a 2-category (in fact a 2-Ab-category) if in addition the condition* (2.9) *is satisfied, which in the abelian context simplifies to* $D(y)x = yD(x)$*. Furthermore, as proved in [Ch5], every 2-cell structure (if enriched in Ab) is obtained in this way.*

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The example of Groups

In the case of $\mathbf{C} = Grp$ the category of groups and group homomorphisms, the construction of Example 18 is so general that it includes the canonical 2-cells that are obtained if considering each group as a one object groupoid and each group homomorphism as a functor. In that case, as it is well known, a 2-cell

$$t : f \longrightarrow g$$

from the homomorphism f to the homomorphism g, both from the group A to the group B, is an element $t \in B$ such that

$$tf(x) = g(x)\,t \quad , \quad \text{for all } x \in A.$$

Now, given t and f, the homomorphism g is uniquely determined as

$$g(x) = tf(x)\,t^{-1} = {}^{t}f(x),$$

and hence, this particular 2-cell structure over Grp is an instance of Example 18 with Grp instead of Mon.

To see this just consider H the functor that projects the second argument

$$\begin{aligned} H : Grp^{op} \times Grp &\longrightarrow Grp \\ (A, B) &\longmapsto B \end{aligned}$$

and

$$\begin{aligned} D : B \times \hom(A, B) &\longrightarrow \hom(A, B) \\ (t, f) &\longmapsto {}^{t}f \end{aligned}$$

and it is a straightforward calculation to check that

$$\begin{aligned} D(0, f) &= f \\ D(t + t', f) &= D(t, D(t', f)) \end{aligned}$$

and also, since condition (2.8) is satisfied, the 2-cell structure is natural.

The example of crossed modules

In the case **C**=X-Mod, the category of crossed modules, we have the canonical 2-cell structure given by derivations, and it is an instance of Example 19 with Grp instead of Mon:

The objects in X-Mod are of the form

$$A = \left(X \xrightarrow{d} B, \varphi : B \longrightarrow Aut(X)\right)$$

where $d : X \longrightarrow B$ is a group homomorphism, together with a group action of B in X denoted by $b \cdot x$ satisfying

$$\begin{aligned} d(b \cdot x) &= bd(x)\,b^{-1} \\ d(x) \cdot x' &= x + x' - x; \end{aligned}$$

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a morphism $f : A \longrightarrow A'$ in $X\text{-}Mod$ is of the form

$$f = (f_1, f_0)$$

where $f_1 : X \longrightarrow X'$ and $f_0 : B \longrightarrow B'$ are group homomorphisms such that

$$f_0 d = d' f_1$$

and

$$f_1 (b \cdot x) = f_0 (b) \cdot f_1 (x) .$$

Clearly there are functors

$$map : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Grp$$

sending (A, A') to the group of pairs (f_1, f_0) of maps (not necessarily homomorphisms) $f_1 : UX \longrightarrow UX'$ and $f_0 : UB \longrightarrow UB'$ such that

$$f_0 d = d' f_1,$$

with the group operation defined componentwise

$$(f_1, f_0) + (g_1, g_0) = (f_1 + g_1, f_0 + g_0) .$$

Also there is a functor

$$M : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Grp$$

sending (A, A') to the group $M (A, A') = \{t \mid t : UB \longrightarrow UX' \text{ is a map}\}$, and a natural transformation

$$D : M \longrightarrow map$$

defined by

$$D (A, A') (t) = (td, dt) .$$

Now, define

$$H (A, A') = \{(t, f) \mid t \in M (A, A') \ , \ f = (f_1, f_0) : A \longrightarrow A' \ , \ (td + f_1, dt + f_0) \in \hom (A, A')\} .$$

It is well known that the map $t : B \longrightarrow X'$ is such that

$$t (bb') = t (b) + f_0 (b) \cdot t (b') \quad , \quad \text{for all } b, b' \in B,$$

while $(td + f_1, dt + f_0) \in \hom (A, A')$ asserts that the the pair $(td + f_1, dt + f_0)$ is a morphism of crossed modules

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "td+f_1"'] & B \arrow[d, "dt+f_0"] \\ X' \arrow[r, "d"] & B' \end{tikzcd} \tag{2.10}$$

and it is equivalent to

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- $dt+f_0$ is a homomorphism of groups

$$dt\left(bb'\right)=d\left(t\left(b\right)+f_0\left(b\right)\cdot t\left(b'\right)\right)$$

- $td+f_1$ is a homomorphism of groups

$$t\left(d\left(x\right)d\left(x'\right)\right)=t\left(dx\right)+f_0d\left(x\right)\cdot td\left(x'\right)$$

- the square (2.10) commutes, which is trivial because $\left(f_1,f_0\right)\in\hom\left(A,A'\right)$
- $\left(td+f_1\right)$ preserves the action of $\left(dt+f_0\right)$

$$t\left(bd\left(x\right)b^{-1}\right)=t\left(b\right)+f_0\left(b\right)\cdot t\left(d\left(x\right)\right)+f_0\left(bd\left(x\right)b^{-1}\right)\cdot\left(-t\left(b\right)\right).$$

The commutator

Previous examples apply to arbitrary (even large) categories, provided they admit the functors and the natural transformations as specified. Interesting examples also appear if one tries to particularize the category **C**. For example if **C** has only one object, or if it is a preorder; the first case gives something that particularizes to a (strict) monoidal category (with fixed set of objects) in the presence of the naturality condition; while the second case gives something that particularizes to an enriched category over monoids.
The simplest case, when **C**=**1**, gives Monoids and, Commutative Monoids under the naturality condition; so in particular, if considering only invertible 2-cell structures the result is Groups and Abelian Groups, respectively.

The well known reflection

$$Gr\xrightarrow{I}Ab,$$

accordingly to G. Janelidze, generalizes to a reflexion

$$\text{2-cellstruct}(\mathbf{C})\xrightarrow{I}\text{nat-2-cellstruct}(\mathbf{C})$$

from the category of 2-cell structures over **C**, into the subcategory of natural 2-cell structures over **C**, sending each 2-cell structure to its "naturalization"; and, under the assumption that all the 2-cells are invertible, one may consider for each

$$\begin{tikzcd}
A \arrow[r, bend left=60, ""{name=U1, below}] \arrow[r, bend right=60, ""{name=D1, above}] & B \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, bend right=60, ""{name=D2, above}] & C
\arrow[Rightarrow, from=U1, to=D1, "y"]
\arrow[Rightarrow, from=U2, to=D2, "x"]
\end{tikzcd}$$

the commutator

$$\begin{aligned}
[x,y] &= \left(c_1+d_2-d_1-c_2\right)\left(x,y\right)\\
&= c_1\left(x,y\right)+d_2\left(x,y\right)-d_1\left(x,y\right)-c_2\left(x,y\right)
\end{aligned}$$

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where

$$\begin{aligned} c_1(x,y) &= \operatorname{cod}(x)\,y\ ,\ c_2(x,y) = x\operatorname{cod}(y) \\ d_1(x,y) &= \operatorname{dom}(x)\,y\ ,\ d_2(x,y) = x\operatorname{dom}(y)\,, \end{aligned}$$

and the comparison with $0_{\operatorname{cod}(x)\operatorname{cod}(y)}$ tell us the obstruction that x and y offer to be composed horizontally.

We will not developed this concept further, at the moment we are only observing that in the case of **C** being an Ab-category (see [Ch5],[Ch6] and Remark 20) then the notion of commutator reduces to

$$[x,y] = D(x)\,y - xD(y)\,.$$

In fact the notion of 2-Ab-category (as introduced in [Ch5]) may be pushed further in the direction of a sesquicategory enriched in any category **A** with a "forgetful" functor into Sets.

It is a simple generalization of Example 19 and it is as follows.

For a category **A** with a "forgetful" functor into Sets, $U : \mathbf{A} \longrightarrow$Sets, assume the existence of a functor

$$map : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathbf{A}$$

such that

$$\hom_C(A,B) \subseteq Umap(A,B)$$

(as in Example 19).

If **A** were monoidal and **C** a category enriched in **A** then we would always be in the above conditions, simply by choosing $map = \hom$. It is then reasonably to say that in this more general context, the category **C** is weakly enriched in **A** (for example, in this sense, Groups are weakly enriched in Groups, and every algebraic structure is weakly enriched in itself). In this conditions, we may be interested in considering only 2-cell structures over **C** that are "weakly enriched" in **A** in the same way as **C** is. This concept is obtained if considering only the 2-cell structures that are given by

$$H(A,B) = \{x \in UM(A,B) \mid U\operatorname{dom} x, U\operatorname{cod} x \in \hom(A,B)\}$$

for some $M, \operatorname{dom}, \operatorname{cod}$ being part of an internal category object in $\mathbf{A}^{\mathbf{C}^{op}\times\mathbf{C}}$, of the form

$$\begin{tikzcd} M \times_{map} M \arrow[r, "+"] & M \arrow[r, shift left=2, "\operatorname{dom}"] \arrow[r, shift right=2, "\operatorname{cod}"'] & map, \arrow[l, "0"'] \end{tikzcd}$$

with the obvious restrictions after applying U.

It is interesting now to observe that in the case of $\mathbf{A}$ =Groups the result of this is precisely the construction of Example 19. If $\mathbf{A}$ =Ab and also requiring M to be an Ab-functor, then the result is a 2-Ab-category if also adding the condition

$$D(x)\,y = xD(y)$$

for all appropriate x and y.

Next we formalize the category of 2-cell structures.

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2.4 The category of 2-cell structures

For a fixed category **C**, there is the category 2-cellstruct(**C**) of all possible 2-cell structures over **C**, as well as the subcategory nat-2-cellstruct(**C**) of natural 2-cell structures over C and inv-2-cellstruct(C) of all the invertible 2-cell structures over C. The category 2-cellstruct(**C**) has a initial object (the discrete 2-cell structure) and a terminal object (the codiscrete 2-cell structure). If C is of the form Cat(B) for some category B, it also has the canonical 2-cell structure of internal transformations and the canonical natural 2-cell structure of internal natural transformations.

For the sake of a formal definition: the objects of 2-cellstruct(**C**) are of the form

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

where

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set$$

is a functor and

$$\begin{tikzcd} H \times_{\mathrm{hom}} H \arrow[r, "+"] & H \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathrm{hom}_{\mathbf{C}} \arrow[l, "0"'] \end{tikzcd}$$

are natural transformations, such that

$$(\mathrm{hom}_{\mathbf{C}}, H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

is a category object in the functor category $\mathrm{Set}^{C^{op} \times C}$, in other words is an object in $\mathrm{Cat}\big(Set^{C^{op} \times C}\big)$.
A morphism $\varphi : \mathbf{H} \longrightarrow \mathbf{H}'$ is a natural transformation

$$\varphi : H \longrightarrow H'$$

such that

$$\begin{aligned} \mathrm{dom}' \varphi &= \mathrm{dom} \\ \mathrm{cod}' \varphi &= \mathrm{cod} \\ \varphi 0 &= 0' \\ \varphi + &= +' (\varphi \times \varphi) \, . \end{aligned}$$

We will often write simply H to refer to a 2-cell structure, whenever confusion is unlikely to appear.

The purpose of describing 2-cellstruct(**C**), the category of all 2-cell structures over a given category **C**, is the study of pseudocategories in **C**. The notion of pseudocategory in a category **C** depends of the 2-cell structure considered over **C**. For example, a pseudocategory in **C** with the codiscrete 2-cell structure is a precategory, while if considering the discrete structure it is a internal category. It seems to be interesting to study, for a given category **C**, how the notion of

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pseudocategory changes from a precategory to a internal category by changing the 2-cell structure considered over **C**. This topic is studied in [Ch9] for the case of weakly Mal'cev sesquicategories.

Also, every morphism

$$\varphi : H \longrightarrow H' \tag{2.11}$$

in 2-cellstruct(**C**) induces a functor

$$PsCat\left(\mathbf{C}, H\right) \longrightarrow PsCat\left(\mathbf{C}, H'\right) \tag{2.12}$$

from pseudocategories in **C** relative to the 2-cell structure H to pseudocategories in **C** relative to the 2-cell structure H'.

At this point it would be also interesting to study the notion of equivalent 2-cell structures, saying that (2.11) is an equivalence whenever (2.12) is. We choose to postpone it for a future work.

The notion of a pseudocategory ([Ch1],[Ch5]) rests in the construction of the induced 2-cells between pullback objects, thus the following definition.

2.5 Cartesian 2-cell structure

It will be useful to consider 2-cell structures such that the functor $H\left(D, \_\right) : \mathbf{C} \longrightarrow Set$ preserves pullbacks for every object D in **C**, that is: the functor

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set,$$

giving a 2-cell structure to a category **C**, has the following property

$$H\left(D, A \times_{\{f,g\}} B\right) \overset{\varphi}{\cong} \{(x, y) \in H\left(D, A\right) \times H\left(D, B\right) \mid fx = gy\}$$

for every object D in **C** and pullback diagram

$$\begin{tikzcd}
A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "f"] \\
A \arrow[r, "g"] & C
\end{tikzcd},$$

where φ is required to be a natural isomorphism, that is, for every $h : D \longrightarrow D'$, the following square commutes

$$\begin{tikzcd}
H(D, A \times_C B) \arrow[r, "\cong\varphi"] \arrow[d, "H(h{,}1)"'] & \{(x, y) \mid fx = gy\} \arrow[d] \\
H(D', A \times_C B) \arrow[r, "\cong\varphi"] & \{(x', y') \mid fx' = gy'\}
\end{tikzcd}$$

or in other words, that

$$\langle x, y\rangle h = \langle xh, yh\rangle$$

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as displayed in the diagram below

$$\begin{tikzcd}
D' \arrow[r, "h"] & D \arrow[rrd, Rightarrow, bend left, "y"] \arrow[rd, Rightarrow, dashed, "<x{,}y>"'] \arrow[rdd, Rightarrow, bend right, "x"'] & & \\
& & A \times_C B \arrow[r] \arrow[d] & B \arrow[d, "g"] \\
& & A \arrow[r, "f"'] & C
\end{tikzcd} .$$

In particular, for $D = A' \times_{C'} B'$, and appropriate x, y, z as in

$$\begin{tikzcd}
A' \arrow[d, Rightarrow, "x"'] & A' \times'_C B' \arrow[l, "\pi'_1"'] \arrow[r, "\pi'_2"] \arrow[d, Rightarrow, "x \times_z y"] & B' \arrow[d, Rightarrow, "y"] & , & C' \arrow[d, Rightarrow, "z"] , \\
A & A \times_C B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B & , & C
\end{tikzcd}$$

it follows that $x \times_z y$ is the unique element (2-cell) in $H\left(A' \times_{C'} B', A \times_C B\right)$ satisfying

$$\begin{aligned}
\pi_2 \left(x \times_z y\right) &= y\pi'_2 \\
\pi_1 \left(x \times_z y\right) &= y\pi'_1 .
\end{aligned}$$

Let **C** be a category.

Definition 21 (cartesian 2-cell structure) *A 2-cell structure* $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ *over the category* **C** *is said to be* Cartesian *if the functor* $H(D, \_) : \mathbf{C} \longrightarrow Set$ *preserves pullbacks for every object* D *in* **C**.

2.6 Pseudocategories

The notion of pseudocategory (as introduced in [Ch1]) is only defined internally to a 2-category. Here we extend it to the more general context of a category with a 2-cell structure (or sesquicategory).

First consider three leading examples.

In any category **C**, it is always possible to consider two different 2-cell structures, namely the discrete one, obtained when $H = \hom$ and $\mathrm{dom}, \mathrm{cod}, 0, +$ are all identities, and the codiscrete one, obtained when $H = \hom \times \hom$, dom is second projection, cod is first projection, 0 is diagonal and + is uniquely determined. A pseudocategory, in the first situation becomes an internal category in **C**, while in the second situation becomes a precategory in **C**.

In the case of $\mathbf{C} = Cat$, and choosing the natural transformations to be the 2-cell structure, a pseudocategory becomes a pseudo-double-category (see [Ch1]), which is at the same time a generalization of a double-category and a bicategory.

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At this level of generality, it becomes clear that there is no particular reason why to prefer a specific 2-cell structure in a category instead of another.

For instance, in Top it is usually considered the 2-cell structure obtained from the homotopy classes of homotopies, but other may be consider as well.

Let **C** be a category with a cartesian 2-cell structure $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$.

Definition 22 *A pseudocategory in* **C***, with respect to the 2-cell structure* $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$*, is a system*

$$(C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)$$

where $(C_0, C_1, d, c, e, m, ...)$ *is a thin protocategory (see definition in [Ch7]), and* α, λ, ρ *are natural and invertible 2-cells, in the sense that*

$$\alpha \in H(C_3, C_1) \text{ and } \lambda, \rho \in H(C_1, C_1)$$

with

$$\begin{aligned} \mathrm{dom}(\alpha) &= mm_1 \ , \ \mathrm{cod}(\alpha) = mm_2 \\ \mathrm{dom}(\lambda) &= me_2 \ , \ \mathrm{dom}(\rho) = me_1 \ , \ \mathrm{cod}(\lambda) = 1_{C_1} = \mathrm{cod}(\rho) \end{aligned}$$

satisfying the following conditions

$$\begin{aligned} d\lambda &= 0_d = d\rho \\ c\lambda &= 0_c = c\rho \\ d\alpha &= 0_{d\pi_2 p_2} \ , \ c\alpha = 0_{c\pi_1 p_1} \end{aligned}$$

$$\lambda e = \rho e$$

$$m(\alpha \times 0_1) + \alpha(1 \times m \times 1) + m(0_1 \times \alpha) = \alpha(m \times 1 \times 1) + \alpha(1 \times 1 \times m) \quad (2.13)$$

$$m(\rho \times 0_1) + \alpha i_0 = m(0_1 \times \lambda) . \quad (2.14)$$

Some remarks:

A 2-cell $x \in H(A, B)$ is invertible when there is a (necessarily unique) element

$$-x \in H(A, B)$$

such that $\mathrm{dom}(x) = \mathrm{cod}(-x)$, $\mathrm{cod}(x) = \mathrm{dom}(-x)$ and

$$x + (-x) = 0_{\mathrm{cod}(x)} \ , \ (-x) + x = 0_{\mathrm{dom}(x)};$$

A 2-cell $x \in H(A, B)$ is natural when

$$\mathrm{cod}(x)\, y + x\, \mathrm{dom}(y) = x\, \mathrm{cod}(y) + \mathrm{dom}(x)\, y$$

for every element $y \in H(X, A)$ for every object X in **C**.

The 2-cells α, λ, ρ may also be presented as

$$\begin{tikzcd} C_3 \arrow[r, bend left, "mm_1"{name=U}] \arrow[r, bend right, "mm_2"'{name=D}] & C_1 \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \ , \quad \begin{tikzcd} C_1 \arrow[r, bend left, "me_2"{name=U}] \arrow[r, bend right, "1"'{name=D}] & C_1 \arrow[Rightarrow, from=U, to=D, "\lambda"] \end{tikzcd} \ , \quad \begin{tikzcd} C_1 \arrow[r, bend left, "me_1"{name=U}] \arrow[r, bend right, "1"'{name=D}] & C_1 \arrow[Rightarrow, from=U, to=D, "\rho"] \end{tikzcd} \ .$$

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Equations (2.13) and (2.14) correspond to the internal versions of the famous MacLane's coherence pentagon and triangle, presented diagrammatically as follows

$$\begin{tikzcd}
& \bullet \arrow[rr, "m(0_{C_1}\times_{C_0}\alpha)"] \arrow[dl, "\alpha(1_{C_1}\times_{C_0}1_{C_1}\times_{C_0}m)"'] & & \bullet \arrow[dr, "\alpha(1_{C_1}\times_{C_0}m\times_{C_0}1_{C_1})"] & \\
\bullet \arrow[drr, "\alpha(m\times_{C_0}1_{C_1}\times_{C_0}1_{C_1})"'] & & & & \bullet \arrow[dll, "m(\alpha\times_{C_0}0_{C_1})"] \\
& & \bullet & &
\end{tikzcd} \tag{2.15}$$

$$\begin{tikzcd}
\bullet \arrow[rr, "\alpha i_0"] \arrow[dr, "m(0_{C_1}\times_{C_0}\lambda)"'] & & \bullet \arrow[dl, "m(\rho\times_{C_0}0_{C_1})"] \\
& \bullet &
\end{tikzcd} \tag{2.16}$$

and restated in terms of generalized elements as

$$\begin{tikzcd}
& f(g(hk)) \arrow[rr, "f\alpha_{g,h,k}"] \arrow[dl, "\alpha_{f,g,hk}"'] & & f((gh)k) \arrow[dr, "\alpha_{f,gh,k}"] & \\
(fg)(hk) \arrow[drr, "\alpha_{fg,h,k}"'] & & & & (f(gh))k \arrow[dll, "\alpha_{f,g,h}k"] \\
& & ((fg)h)k & &
\end{tikzcd} \tag{pentagon}$$

$$\begin{tikzcd}
f(1g) \arrow[rr, "\alpha_{f,1,g}"] \arrow[dr, "f\lambda_g"'] & & (f1)g \arrow[dl, "\rho_f g"] \\
& fg &
\end{tikzcd} \tag{midle triangle}$$

where $m\langle f,g\rangle = fg$.

As proved in [1] this two coherence conditions plus the naturality of α, λ, ρ are sufficient to show that every diagram involving instances of α, λ, ρ, possible nested with $m(-\times -)$, commutes; there are other such diagrams that still play an important role. They are the following

$$\begin{tikzcd}
1(fg) \arrow[rr, "\alpha_{1,f,g}"] \arrow[dr, "\lambda_{fg}"'] & & (1f)g \arrow[dl, "\lambda_f g"] \\
& fg &
\end{tikzcd} \tag{2.17}$$

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$$\begin{tikzcd}
 & 1(f1) \arrow[rr, "\alpha_{1,f,1}"] \arrow[dl, "1\rho_f"'] & & (1f)1 \arrow[dr, "\lambda_f 1"] & \\
1f \arrow[drr, "\lambda"'] & & & & f1 \arrow[dll, "\rho"] \\
 & & f & &
\end{tikzcd} \tag{2.18}$$

$$\begin{tikzcd}
f(g1) \arrow[rr, "\alpha_{f,g,1}"] \arrow[dr, "f\rho_g"'] & & (fg)1 \arrow[dl, "\rho_{fg}"] \\
 & fg &
\end{tikzcd} \tag{2.19}$$

and correspond, respectively, (when internalized) to the following equations

$$\begin{aligned}
m\left(\lambda \times 0_{C_1}\right) + \alpha i_2 &= \lambda m, \\
\rho + m e_1 \lambda + \alpha\left(i_2 e_1 = i_1 e_2\right) &= \lambda + m e_2 \rho \\
\rho m + \alpha i_1 &= m\left(0_{C_1} \times \rho\right)
\end{aligned}$$

and since the 2-cells are invertible, the above set of equations may be presented as

$$\begin{aligned}
\alpha i_2 &= -m\left(\lambda \times 0_{C_1}\right) + \lambda m, \\
\alpha\left(i_2 e_1 = i_1 e_2\right) &= -m e_1 \lambda - \rho + \lambda + m e_2 \rho \\
\alpha i_1 &= -\rho m + m\left(0_{C_1} \times \rho\right).
\end{aligned}$$

Note that the definition of pseudocategory as introduced in [Ch1] does not ask for naturality of α, λ, ρ. This is because in there we were assuming that the considered 2-cell structure was a 2-category, and hence every 2-cell was natural. It would be interesting to see what are the exact requirements about the naturality of α, λ, ρ in order to be able to prove MacLane's Coherence Theorem, but we choose not to investigate it here and postpone it for a future work.

We observe that it is not necessary to ask for the naturality of α, λ, ρ in the sense defined above as to be natural with respect to all possible 2-cells. A quick look at the proof of MacLane's Coherence Theorem tells us that it is sufficient to consider naturality (of each one of α, λ, ρ) with respect to α, λ, ρ and instances of $m\left(u \times_{C_1} v\right)$ where u and v are α, λ, ρ or again of the form $m\left(\_ \times_{C_1} \_\right)$.

As mentioned in the introduction of this article, we will not concentrate on this problem since the main examples are 2-cell structures where every 2-cell is natural (as the examples of groups and crossed-modules above) and even if considering some 2-cell structure that it is not natural we may always use the "naturalization reflexion"

$$\text{2-cellstruct}(\mathbf{C}) \xrightarrow{I} \text{nat-2-cellstruct}(\mathbf{C}).$$

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Chapter 3

Weakly Mal'cev Categories

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Title: Weakly Mal'cev Categories

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Key words: Admissible reflexive graph, multiplicative graph, internal category, internal groupoid, weakly Mal'cev category, naturally weakly Mal'cev category, Mal'cev variety of universal algebras.

Abstract: We introduce a notion of *weakly Mal'tsev category*, and show that: (a) every internal reflexive graph in a weakly Mal'tsev category admits at most one multiplicative graph structure in the sense of [15] (see also [16]), and such a structure always makes it an internal category; (b) (unlike the special case of Mal'tsev categories) there are weakly Mal'tsev categories in which not every internal category is an internal groupoid. We also give a simplified characterization of internal groupoids among internal categories in this context.

3.1 Introduction

A weakly Mal'cev category (WMC) is defined by the following two axioms:

1. Existence of pullbacks of split epis along split epis.
2. Every induced canonical pair of morphisms into a pullback, is jointly epimorphic.

Every Mal'cev category is weakly Mal'cev: it has finite limits (see [9] Definition 2.2.3, p.142) and the induced canonical pair of morphisms into the pullback (see [9] Lemma 2.3.1, p.151) is strongly epimorphic.

Examples of weakly Mal'cev categories that are not Mal'cev, are due to G. Janelidze, and are the following:

Commutative monoids with cancelation.

A category with objects (A, p, e) where A is a set, p a ternary operation, e a unary operation and where the following axioms are satisfyed

$$\begin{aligned} p(x,y,y) &= e(x) \quad , p(x,x,y) = e(y) \\ e(x) &= e(y) \implies x = y. \end{aligned}$$

Note that p becomes a Mal'cev operation when e is the identity.

The setting of a weakly Mal'cev category seems to be the most appropriate to study internal categories (see final note for further discussion).

The main purpose of this paper is to introduce the concept of weakly Mal'cev category and describe some of its properties, establishing a convenient notation for *ad hoc* calculations.

In order to stress the significance of the proposed notion, we compare some of its properties with analogous and well known properties in the context of Mal'cev categories. They are the following (see the references, in particular [9],[13],[10],[12],[5],[6],[7] and [14]).

In the context of a Mal'cev category:

1. every internal category is a groupoid;
2. every multiplicative graph is an internal category;
3. every reflexive graph admits at most one multiplication.

In the context of a weakly Mal'cev category:

1. every reflexive graph admits at most one multiplication (here denoted by admissible, in that case);
2. every multiplicative graph (or admissible reflexive graph) is already an internal category;
3. not every admissible reflexive graph (or multiplicative graph, or internal category) is an internal groupoid, nevertheless there is an intrinsic description of the admissible reflexive graphs with the property of being a groupoid.

In commutative monoids with cancelation, an example of a internal category that is not a internal groupoid is the less or equal relation in the natural numbers considered as a preorder.

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In a weakly Mal'cev category, given a diagram of the form

$$\begin{tikzcd}
 & B \arrow[d, shift right, "g"'] \\
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

with $fr = 1_C = gs$, we may form the pullback (of a split epi along a split epi)

$$\begin{tikzcd}
A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\
A \arrow[r, shift left, "f"] \arrow[u, shift right, "e_1"'] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

with projections π_1 and π_2, and where e_1, e_2 are the canonical induced morphisms, that is, they are such that

$$\begin{aligned}
\pi_1 e_1 &= 1_A \quad , \quad \pi_2 e_1 = sf \\
\pi_1 e_2 &= rg \quad , \quad \pi_2 e_2 = 1_B .
\end{aligned}$$

The pair (e_1, e_2) is jointly epimorphic by definition. Then, for every triple of morphisms (h, l, k)

$$\begin{tikzcd}
A \arrow[dr, "h"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "l"] & B \arrow[dl, "k"] \\
 & D &
\end{tikzcd}$$

such that $hr = l = ks$, there is at most one morphism

$$\alpha : A \times_C B \longrightarrow D$$

such that

$$\begin{aligned}
\alpha e_1 &= h \\
\alpha e_2 &= k
\end{aligned}$$

which is denoted by

$$\alpha = \begin{bmatrix} h & l & k \end{bmatrix}$$

when it exists. It is also convenient to specify the morphisms f and g; so that in general we will say: the triple of morphisms (h, l, k), as above, has the property (or not) that the morphism

$$\begin{bmatrix} h & l & k \end{bmatrix}$$

exists, with respect to

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"']
\end{tikzcd} .$$

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In the case $\begin{bmatrix} h & l & k \end{bmatrix}$ exists we will say that the triple (h, l, k) is admissible.

With this notation, the notion of admissible reflexive graph (that is an internal category) is the following:

In a weakly Mal'cev category, a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce$$

is said to be admissible when the triple

$$(1_{C_1}, e, 1_{C_1})$$

is admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} ;$$

It is then a multiplicative graph with multiplication

$$C_1 \times_{C_0} C_1 \xrightarrow{[1\ e\ 1]} C_1$$

which automatically satisfies the axioms of an internal category, and furthermore, it has the property of being a groupoid if and only if the triple

$$(\pi_2, 1_{C_1}, \pi_1)$$

is admissible with respect to

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, shift left, "{[1\,e\,1]}"] & C_1 \arrow[l, shift left, "e_2"] \arrow[r, shift right, "e_1"'] & C_1 \times_{C_0} C_1 \arrow[l, shift right, "{[1\,e\,1]}"'] \end{tikzcd} .$$

In the presence of a Mal'cev operation, a general morphism

$$\begin{bmatrix} h & l & k \end{bmatrix} : A \times_C B \longrightarrow D,$$

in case of existence, is given by

$$\begin{bmatrix} h & l & k \end{bmatrix} (a, c, b) = h(a) - l(c) + k(b),$$

so that in particular the multiplication $\begin{bmatrix} 1 & e & 1 \end{bmatrix}$, in case of existence, is given by

$$\begin{bmatrix} 1 & e & 1 \end{bmatrix} \left(\cdot \xleftarrow{f} x \xleftarrow{g} \cdot \right) = f - 1_x + g,$$

while inverses (assuming $\begin{bmatrix} \pi_2 & 1_{C_1} & \pi_1 \end{bmatrix}$ exists) are given by

$$\begin{aligned} f^{-1} &= \begin{bmatrix} \pi_2 & 1_{C_1} & \pi_1 \end{bmatrix} \begin{bmatrix} e_1 \\ 1_{C_1} \\ e_2 \end{bmatrix} (f) \\ &= (\pi_2 e_1 - 1 + \pi_1 e_2)(f) \\ &= (ed - 1 + ec)(f) \\ &= ed(f) - f + ec(f) \\ &= 1_x - f + 1_y \end{aligned}$$

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for arrows $x \xrightarrow{f} y$ in C_1.

This paper is organized as follows: First we introduce the notion and deduce some properties of weakly Mal'cev categories; next we prove the equivalence between internal categories and admissible reflexive graphs; later we show the connection with Mal'cev categories; at the end we describe internal groupoids in weakly Mal'cev categories.

3.2 The notion of a weakly Mal'cev category

Let $\mathbf{C}$ be a given category.

Definition 23 (split span) *A* split span *is a diagram in* $\mathbf{C}$ *of the form*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}$$

such that

$$fr = 1_C = gs.$$

Definition 24 (split square) *A* split square *is a diagram in* $\mathbf{C}$ *of the form*

$$\begin{tikzcd} P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

such that

$$\begin{aligned} fr &= 1_C = gs \\ gp_2 &= fp_1 \\ e_2s &= e_1r \\ p_2e_2 &= 1_B \\ p_2e_1 &= sf \\ p_1e_1 &= 1_A \\ p_1e_2 &= rg, \end{aligned}$$

in other words, it is a double split epi, in the sense that it is a split epi in the category of split epis in $\mathbf{C}$*.*

The term *split pullback* will be used to refer to a split square as above, such that

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

is a pullback diagram.

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Definition 25 (weakly Mal'cev category) *A category* **C** *is weakly Mal'cev when:*

1. It has pullbacks of split epis along split epis;

2. For every split square

$$\begin{tikzcd} P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd} ,$$

if (P, p_1, p_2) is a pullback, then the pair (e_1, e_2) is jointly epimorphic, that is, given $u, v : P \longrightarrow D$,

$$\begin{cases} ue_2 = ve_2 \\ ue_1 = ve_1 \end{cases} \Longrightarrow u = v.$$

Proposition 26 *In a weakly Mal'cev category, given a split span*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd} , \tag{3.1}$$

for every object D, and triple of morphisms (h, l, k)

$$\begin{tikzcd} A \arrow[dr, "h"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "l"] & B \arrow[dl, "k"] \\ & D & \end{tikzcd} ,$$

such that $hr = l = ks$, there exists at most one morphism, denoted by $\begin{bmatrix} h & l & k \end{bmatrix}$ when it exists, from the pullback

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

to the object D,

$$\begin{bmatrix} h & l & k \end{bmatrix} : A \times_C B \longrightarrow D,$$

with the property that

$$\begin{aligned} \begin{bmatrix} h & l & k \end{bmatrix} e_1 &= h \\ \begin{bmatrix} h & l & k \end{bmatrix} e_2 &= k, \end{aligned}$$

where $e_1 = \langle 1, sf \rangle = \begin{bmatrix} 1 \\ f \\ sf \end{bmatrix} : A \longrightarrow A \times_C B$, and $e_2 = \langle rg, 1 \rangle = \begin{bmatrix} rg \\ g \\ 1 \end{bmatrix} : B \longrightarrow A \times_C B$ are the induced morphisms into the pullback.

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Proof. The pullback $A \times_C B$, being a pullback of a split epi (g, s) along a split epi (f, r), exists in a weakly Mal'cev category, and e_1, e_2, the induced morphisms into the pullback, make the diagram

$$\begin{tikzcd}
A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\
A \arrow[r, shift left, "f"] \arrow[u, shift right, "e_1"'] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

a split square; to prove that $\begin{bmatrix} h & l & k \end{bmatrix}$ if it exists is unique, suppose the existence of

$$p, q : A \times_C B \longrightarrow D$$

satisfying

$$\begin{aligned} pe_1 &= h,\ pe_2 = k \\ qe_1 &= h,\ qe_2 = k, \end{aligned}$$

by definition of weakly Mal'cev, the pair (e_1, e_2) is jointly epimorphic and hence $p = q$. ■

Note that the morphism l, being determined by either h or k, is explicitly used to avoid always having to choose between hr and ks. Also, if h and k do not satisfy $hr = ks$ then there is no morphism $p : A \times_C B \longrightarrow D$ satisfying $pe_1 = h$ and $pe_2 = k$ because if it existed it would imply that $hr = ks$ since $e_1 r = e_2 s$.

Relative to a split span

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"']
\end{tikzcd}, \tag{3.2}$$

the notation

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_0 \\ \beta_2 \end{bmatrix}$$

for a morphism *into* the pullback

$$\begin{tikzcd}
D \arrow[dr, "\beta"] \arrow[drr, bend left, "\beta_2"] \arrow[ddr, bend right, "\beta_1"'] & & \\
& A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}.$$

induced by $\beta_1, \beta_0, \beta_2$ with

$$f\beta_1 = \beta_0 = g\beta_2;$$

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and

$$\alpha = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}$$

for a morphism *from* the pullback

$$\begin{tikzcd}
& C \arrow[dl, "r"'] \arrow[dr, "s"] & \\
A \arrow[r, "e_1"] \arrow[dr, "\alpha_1"'] & A \times_C B \arrow[d, dashed, "? \mid \alpha"] & B \arrow[l, "e_2"'] \arrow[dl, "\alpha_2"] \\
& D' &
\end{tikzcd}$$

induced (when it exists) by $\alpha_1, \alpha_0, \alpha_2$ with

$$\alpha_1 r = \alpha_0 = \alpha_2 s,$$

seems to be appropriate due to the following facts:

- every split span

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"']
\end{tikzcd}$$

  determines a split pullback

$$\begin{tikzcd}
A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\
A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[u, shift right, "s"'] \arrow[l, shift left, "r"]
\end{tikzcd}$$

  where

$$\begin{aligned}
\pi_1 &= \begin{bmatrix}1 & r & rg\end{bmatrix}\\
\pi_0 &= \begin{bmatrix}f & 1 & g\end{bmatrix}\\
\pi_2 &= \begin{bmatrix}sf & s & 1\end{bmatrix}\\
e_1 &= \begin{bmatrix}1\\ f\\ sf\end{bmatrix},\ e_0 = \begin{bmatrix}r\\ 1\\ s\end{bmatrix},\ e_2 = \begin{bmatrix}rg\\ g\\ 1\end{bmatrix}\\
1_{A\times_C B} &= \begin{bmatrix}\pi_1\\ \pi_0\\ \pi_2\end{bmatrix} = \begin{bmatrix}e_1 & e_0 & e_2\end{bmatrix} = \begin{bmatrix}1 & r & rg\\ f & 1 & g\\ sf & s & 1\end{bmatrix};
\end{aligned}$$

- for every $u : D' \longrightarrow D''$, the composite $u\alpha : A \times_C B \longrightarrow D''$ is given by the formula

$$u\begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix} = \begin{bmatrix}u\alpha_1 & u\alpha_0 & u\alpha_2\end{bmatrix},$$

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whenever both sides are defined, in the sense that from $\begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}$ we can deduce the existence of $\begin{bmatrix}u\alpha_1 & u\alpha_0 & u\alpha_2\end{bmatrix}$, but given $\begin{bmatrix}u\alpha_1 & u\alpha_0 & u\alpha_2\end{bmatrix}$ we can only write $u\begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}$ provided that the existence of $\begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}$ is already ensured.

- for every $v : \overline{D} \longrightarrow D$ the composite $\beta v : \overline{D} \longrightarrow A \times_C B$ is given by the formula
$$\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} v = \begin{bmatrix}\beta_1 v\\ \beta_0 v\\ \beta_2 v\end{bmatrix};$$
- it is sometimes useful to write the composite $\alpha\beta : D \longrightarrow D'$ as a formal formula
$$\begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = \alpha_1\beta_1 - \alpha_0\beta_0 + \alpha_2\beta_2,$$
it is not the case that it defines a Mal'cev operation, but for the following special cases one has

$$\alpha_1 = \alpha e_1 = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}1\\ f\\ sf\end{bmatrix} = \alpha_1 - \alpha_0 f + \alpha_2 sf = \alpha_1 - \alpha_0 f + \alpha_0 f$$

$$\alpha_0 = \alpha e_0 = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}r\\ 1\\ s\end{bmatrix} = \alpha_1 r - \alpha_0 + \alpha_2 s = \alpha_0 - \alpha_0 + \alpha_0$$

$$\alpha_2 = \alpha e_2 = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}rg\\ g\\ 1\end{bmatrix} = \alpha_1 rg - \alpha_0 g + \alpha_2 = \alpha_0 g - \alpha_0 g + \alpha_2$$

$$\beta_1 = \pi_1\beta = \begin{bmatrix}1 & r & rg\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = \beta_1 - r\beta_0 + rg\beta_2 = \beta_1 - r\beta_0 + r\beta_0$$

$$\beta_0 = \pi_0\beta = \begin{bmatrix}f & 1 & g\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = f\beta_1 - \beta_0 + g\beta_2 = \beta_0 - \beta_0 + \beta_0$$

$$\beta_2 = \pi_2\beta = \begin{bmatrix}sf & s & 1\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = sf\beta_1 - s\beta_0 + \beta_2 = s\beta_0 - s\beta_0 + \beta_2,$$

and in general, for a triple of morphisms

$$x, y, z : D \longrightarrow D'$$

such that there exists

$$\beta : D \longrightarrow A \times_C B \quad \text{and} \quad \alpha : A \times_C B \longrightarrow D'$$

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with

$$\alpha_1\beta_1 = x\ ,\ \ \alpha_0\beta_0 = y\ ,\ \ \alpha_2\beta_2 = z,$$

and writing

$$\alpha\beta = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = \alpha_1\beta_1 - \alpha_0\beta_0 + \alpha_2\beta_2 = x - y + z$$

it is clear that there is a partially defined (relative to the split span (3.2)) ternary operation in hom (D, D'),

$$(x, y, z) \longmapsto x - y + z,$$

but in general there is no reason for this to satisfy the Mal'cev axioms $x - z + z = x$ and $x - x + z = z$; however, it does so if the category $\mathbf{C}$ is a Mal'cev variety of universal algebras, since in that case

$$\begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}(a, c, b) = p\left(\alpha_1(a), \alpha_0(c), \alpha_2(b)\right),$$

with p the Mal'cev operation on D';

- also, given another split span

$$\begin{tikzcd} A' \arrow[r, shift left, "f'"] & C' \arrow[l, shift left, "r'"] \arrow[r, shift right, "s'"'] & B' \arrow[l, shift right, "g'"'] \end{tikzcd}$$

and a morphism $\gamma = \begin{bmatrix}\gamma_1 & \gamma_0 & \gamma_2\end{bmatrix} : A' \times_{C'} B' \longrightarrow D$, the composite $\beta\gamma$,

$$\begin{tikzcd} A' \arrow[dr, "\gamma_1"'] & C' \arrow[l, "r'"'] \arrow[r, "s'"] \arrow[d, "\gamma_0"] & B' \arrow[dl, "\gamma_2"] \\ & D \arrow[dl, "\beta_1"'] \arrow[d, "\beta_0"] \arrow[dr, "\beta_2"] & \\ A \arrow[r, "f"] & C & B \arrow[l, "g"'] \end{tikzcd} \tag{3.3}$$

is given by

$$\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix}\begin{bmatrix}\gamma_1 & \gamma_0 & \gamma_2\end{bmatrix} = \begin{bmatrix}\beta_1\gamma_1 & \beta_1\gamma_0 & \beta_1\gamma_2\\ \beta_0\gamma_1 & \beta_0\gamma_0 & \beta_0\gamma_2\\ \beta_2\gamma_1 & \beta_2\gamma_0 & \beta_2\gamma_2\end{bmatrix} : A' \times_{C'} B' \longrightarrow A \times_C B$$

with components relative to the specified split spans, as displayed in (3.3).

Moreover, the following proposition describes the form of the morphisms between pullbacks, relative to the specified split spans.

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Proposition 27 *In a WMC (Weakly Mal'cev Category), given two split spans*

$$\begin{tikzcd}
A' \arrow[r, shift left, "f'"] & C' \arrow[l, shift left, "r'"] \arrow[r, shift right, "s'"'] & B' \arrow[l, shift right, "g'"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"']
\end{tikzcd} ,$$

a morphism $\varphi : A' \times_{C'} B' \longrightarrow A \times_C B$ *is of the form*

$$\varphi = \begin{bmatrix} \pi_1\varphi e_1' & \pi_1\varphi e_0' & \pi_{1\varphi}e_2' \\ \pi_0\varphi e_1' & \pi_0\varphi e_0' & \pi_0\varphi e_2' \\ \pi_2\varphi e_1' & \pi_2\varphi e_0' & \pi_2\varphi e_2' \end{bmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{10} & \varphi_{12} \\ \varphi_{01} & \varphi_{00} & \varphi_{02} \\ \varphi_{21} & \varphi_{20} & \varphi_{22} \end{bmatrix}$$

and it determines the following commutative diagram

$$\begin{tikzcd}
A' \arrow[d, "\varphi_{11}"'] \arrow[dr, "\varphi_{01}"] \arrow[drr, "\varphi_{21}" near end] & C' \arrow[l, "r'"'] \arrow[r, "s'"] \arrow[d, "\varphi_{00}"] \arrow[dl, "\varphi_{10}"'] \arrow[dr, "\varphi_{20}"] & B' \arrow[d, "\varphi_{22}"] \arrow[dl, "\varphi_{02}"'] \arrow[dll, "\varphi_{12}"' near end] \\
A \arrow[r, "f"'] & C & B \arrow[l, "g"]
\end{tikzcd} ; \tag{3.4}$$

Conversely, given a commutative diagram as above, it determines a morphism $\varphi : A' \times_{C'} B' \longrightarrow A \times_C B$ *of the form*

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{10} & \varphi_{12} \\ \varphi_{01} & \varphi_{00} & \varphi_{02} \\ \varphi_{21} & \varphi_{20} & \varphi_{22} \end{bmatrix}$$

if and only if the morphisms $\begin{bmatrix} \varphi_{11} & \varphi_{10} & \varphi_{12} \end{bmatrix}$, $\begin{bmatrix} \varphi_{01} & \varphi_{00} & \varphi_{02} \end{bmatrix}$ *and* $\begin{bmatrix} \varphi_{21} & \varphi_{20} & \varphi_{22} \end{bmatrix}$ *exists.*

In particular, given a commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "h"'] & C' \arrow[d, "l"] & B' \arrow[l, "g'"'] \arrow[d, "k"] \\
A \arrow[r, "f"'] & C & B \arrow[l, "g"]
\end{tikzcd}$$

the induced morphism $h \times_l k : A' \times_{C'} B' \longrightarrow A \times_C B$ *is given by*

$$h \times_l k = \begin{bmatrix} h\pi_1' \\ l\pi_0' \\ k\pi_2' \end{bmatrix} = \begin{bmatrix} h & hr' & hr'g' \\ fh & l & gk \\ ks'f' & ks' & k \end{bmatrix} .$$

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Proof. A given morphism $\varphi : A' \times_{C'} B' \longrightarrow A \times_C B$ is always determined as a morphism into the pullback

$$\begin{tikzcd}
A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_0"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

by the components

$$\varphi = \begin{bmatrix} \pi_1\varphi \\ \pi_0\varphi \\ \pi_2\varphi \end{bmatrix}.$$

Since each one of the components is a morphism from the pullback $A' \times_{C'} B'$, they are determined by the canonical morphisms e'_1, e'_0, e'_2

$$\begin{tikzcd}
A' \times'_C B' & B' \arrow[l, "e'_2"'] \\
A' \arrow[u, "e'_1"] & C' \arrow[l, "r'"] \arrow[u, "s'"'] \arrow[ul, "e'_0"']
\end{tikzcd}$$

and

$$\varphi = \begin{bmatrix} \pi_1\varphi \\ \pi_0\varphi \\ \pi_2\varphi \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \pi_1\varphi e'_1 & \pi_1\varphi e'_0 & \pi_1\varphi e'_2 \end{bmatrix} \\ \begin{bmatrix} \pi_0\varphi e'_1 & \pi_0\varphi e'_0 & \pi_0\varphi e'_2 \end{bmatrix} \\ \begin{bmatrix} \pi_2\varphi e'_1 & \pi_2\varphi e'_0 & \pi_2\varphi e'_2 \end{bmatrix} \end{bmatrix}.$$

In the same way one obtains

$$\varphi = \begin{bmatrix} \varphi e'_1 & \varphi e'_0 & \varphi e'_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \pi_1\varphi e'_1 \\ \pi_0\varphi e'_1 \\ \pi_2\varphi e'_1 \end{bmatrix} & \begin{bmatrix} \pi_1\varphi e'_0 \\ \pi_0\varphi e'_0 \\ \pi_2\varphi e'_0 \end{bmatrix} & \begin{bmatrix} \pi_1\varphi e'_2 \\ \pi_0\varphi e'_2 \\ \pi_2\varphi e'_2 \end{bmatrix} \end{bmatrix}$$

so that one can simply write

$$\varphi = \begin{bmatrix} \pi_1\varphi e'_1 & \pi_1\varphi e'_0 & \pi_{1\varphi} e'_2 \\ \pi_0\varphi e'_1 & \pi_0\varphi e'_0 & \pi_0\varphi e'_2 \\ \pi_2\varphi e'_1 & \pi_2\varphi e'_0 & \pi_2\varphi e'_2 \end{bmatrix}.$$

To prove that

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{10} & \varphi_{12} \\ \varphi_{01} & \varphi_{00} & \varphi_{02} \\ \varphi_{21} & \varphi_{20} & \varphi_{22} \end{bmatrix}$$

with the components φ_{ij} as in the commutative diagram (3.4), exists if and only if

$$\begin{bmatrix} \varphi_{11} & \varphi_{10} & \varphi_{12} \end{bmatrix}, \begin{bmatrix} \varphi_{01} & \varphi_{00} & \varphi_{02} \end{bmatrix}, \begin{bmatrix} \varphi_{21} & \varphi_{20} & \varphi_{22} \end{bmatrix}$$

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exists, observe that given φ, they are respectively $\pi_1\varphi, \pi_0\varphi, \pi_2\varphi$, conversely, given such morphisms, there is φ and it is given by

$$\varphi = \begin{bmatrix} \begin{bmatrix}\varphi_{11} & \varphi_{10} & \varphi_{12}\end{bmatrix} \\ \begin{bmatrix}\varphi_{01} & \varphi_{00} & \varphi_{02}\end{bmatrix} \\ \begin{bmatrix}\varphi_{21} & \varphi_{20} & \varphi_{22}\end{bmatrix} \end{bmatrix}.$$

Finally, given $h \times_l k : A' \times_{C'} B' \longrightarrow A \times_C B$, by the previous argument, it is of the form

$$h \times_l k = \begin{bmatrix} \pi_1 (h \times_l k) \\ \pi_0 (h \times_l k) \\ \pi_2 (h \times_l k) \end{bmatrix}$$

and by the properties of $h \times_l k$ one has

$$\begin{bmatrix} \pi_1 (h \times_l k) \\ \pi_0 (h \times_l k) \\ \pi_2 (h \times_l k) \end{bmatrix} = \begin{bmatrix} h\pi_1' \\ l\pi_0' \\ k\pi_2' \end{bmatrix} = \begin{bmatrix} h\begin{bmatrix}1 & r' & r'g'\end{bmatrix} \\ l\begin{bmatrix}f' & 1 & g'\end{bmatrix} \\ k\begin{bmatrix}s'f' & s' & 1\end{bmatrix} \end{bmatrix}$$

and since $fh = lf'$ and $lg' = gk$ one has

$$h \times_l k = \begin{bmatrix} h & hr' & hr'g' \\ fh & l & gk \\ ks'f' & ks' & k \end{bmatrix}.$$

■

3.3 Internal categories in weakly Mal'cev categories

The abbreviation WMC stands for Weakly Mal´cev Category. A triple of morphisms (h, l, k) as in Proposition 26 is said to be *admissible* with respect to the split span (3.1) if the morphim $\begin{bmatrix}h & l & k\end{bmatrix}$ exists. By abuse of notation we will also say that a reflexive graph is admissible when the triple $(1, e, 1)$ is admissible.

Definition 28 (Admissible Reflexive Graph) *In a WMC, a reflexive graph*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

$$de = 1_{C_0} = ce$$

is said to be admissible *if the triple* $(1, e, 1)$ *is admissible, that is, the morphism*

$$\begin{bmatrix}1 & e & 1\end{bmatrix}$$

exists relative to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd}.$$

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Theorem 29 *In a WMC, every admissible reflexive graph is an internal category. More specifically, given the admissible reflexive graph*

$$\begin{tikzcd}
C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

it is possible to construct the internal category

$$\begin{tikzcd}
C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[l, shift right=2, "e_2" description] \arrow[l, shift left=2, "e_1" description] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

where

$$\begin{aligned}
1_{C_2} &= \begin{bmatrix} 1_{C_1} & e & ec \\ d & 1_{C_0} & c \\ ed & e & 1_{C_1} \end{bmatrix} \\
\pi_1 &= \begin{bmatrix} 1 & e & ec \end{bmatrix} \\
\pi_2 &= \begin{bmatrix} ed & e & 1 \end{bmatrix} \\
e_1 &= \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} \quad , \quad e_2 = \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} \\
m &= \begin{bmatrix} 1 & e & 1 \end{bmatrix}.
\end{aligned}$$

Furthermore, every internal category is of this form.

Proof. The pullback C_2 always exists in a WMC, since d and c are split epis. In a WMC, every split pullback is of the form presented above, and $m = \begin{bmatrix} 1 & e & 1 \end{bmatrix}$ is well defined because the reflexive graph is admissible by hypothesis. In order to be an internal category we have to prove that the following conditions hold

$$\begin{aligned}
dm &= d\pi_2 \\
cm &= c\pi_1 \\
me_1 &= 1 \\
me_2 &= 1 \\
m\,(1 \times_{C_0} m) &= m\,(m \times_{C_0} 1)
\end{aligned}$$

and we observe:

$$\begin{aligned}
dm &= d\begin{bmatrix} 1 & e & 1 \end{bmatrix} \\
&= \begin{bmatrix} d & 1 & d \end{bmatrix} \\
&= \begin{bmatrix} ded & de & d \end{bmatrix} \\
&= d\begin{bmatrix} ed & e & 1 \end{bmatrix} \\
&= d\pi_2;
\end{aligned}$$

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$$\begin{aligned} cm &= c\begin{bmatrix}1 & e & 1\end{bmatrix} \\ &= \begin{bmatrix}c & 1 & c\end{bmatrix} \\ &= \begin{bmatrix}c & ce & cec\end{bmatrix} \\ &= c\begin{bmatrix}1 & e & ec\end{bmatrix} \\ &= c\pi_1; \end{aligned}$$

$$\begin{aligned} me_1 &= \begin{bmatrix}1 & e & 1\end{bmatrix} e_1 = 1 \\ me_1 &= \begin{bmatrix}1 & e & 1\end{bmatrix} e_2 = 1; \end{aligned}$$

In order to compare $m\,(1 \times m)$ and $m\,(m \times 1)$, from the split span

$$\begin{tikzcd} C_2 \arrow[r, shift left, "\pi_2"] & C_1 \arrow[l, shift left, "e_2"] \arrow[r, shift right, "e_1"'] & C_2 \arrow[l, shift right, "\pi_1"'] \end{tikzcd}$$

construct the split pullback

$$\begin{tikzcd} C_3 \arrow[r, shift left, "p'_2"] \arrow[d, shift right, "p'_1"'] & C_2 \arrow[l, shift left, "e'_2"] \arrow[d, shift right, "\pi_1"'] \\ C_2 \arrow[u, shift right, "e'_1"'] \arrow[r, shift left, "\pi_2"] & C_1 \arrow[u, shift right, "e_1"'] \arrow[l, shift left, "e_2"] \end{tikzcd}$$

and observe that from Proposition 27 and the commutativity of the following diagram

$$\begin{tikzcd} C_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] & C_2 \arrow[l, "\pi_1"'] \arrow[d, "m"] \\ C_1 \arrow[r, "d"'] & C_0 & C_1 \arrow[l, "c"'] \end{tikzcd}$$

we have

$$\begin{aligned} (1 \times_{C_0} m) &= \pi_1 \times_c m = \begin{bmatrix} \pi_1 p'_1 \\ c\pi_2 p'_1 \\ m p'_2 \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 & ec & \pi_1 e_2 \pi_1 \\ d\pi_1 & c & cm \\ me_1\pi_2 & me_1 & m \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 & ec & ecm \\ \pi_0 & c & cm \\ \pi_2 & 1 & m \end{bmatrix} \\ &= \begin{bmatrix} 1_{C_2} & e_2 & e_2 m \end{bmatrix} \end{aligned}$$

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and similarly form the commutativity of the diagram

$$\begin{tikzcd}
C_2 \arrow[r, "\pi_2"] \arrow[d, "m"'] & C_1 \arrow[d, "d"] & C_2 \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] \\
C_1 \arrow[r, "d"'] & C_0 & C_1 \arrow[l, "c"']
\end{tikzcd}$$

we have

$$(m \times_{C_0} 1) = m \times_d \pi_2 = \begin{bmatrix} e_1 m & e_1 & 1_{C_2} \end{bmatrix}$$

so that

$$m \begin{bmatrix} 1_{C_2} & e_2 & e_2 m \end{bmatrix} = \begin{bmatrix} m & me_2 & me_2 m \end{bmatrix} = \begin{bmatrix} m & 1 & m \end{bmatrix}$$

and

$$m \begin{bmatrix} e_1 m & e_1 & 1_{C_2} \end{bmatrix} = \begin{bmatrix} me_1 m & me_1 & m \end{bmatrix} = \begin{bmatrix} m & 1 & m \end{bmatrix}.$$

To see that every internal category is obtained in this way simply observe that the morphism m is determined by me_1 and me_2. ■

Proposition 30 *In a WMC, a morphism of admissible reflexive graphs is also a morphism of internal categories. More specifically, given a morphism of admissible reflexive graphs*

$$\begin{tikzcd}
C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[l, "e" description] \arrow[d, "f_0"'] \\
C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e" description]
\end{tikzcd}$$

then

$$\begin{tikzcd}
C_2 \arrow[r, shift left=3, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=3, "\pi_1"'] \arrow[d, "f_2"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[l, "e" description] \arrow[d, "f_0"'] \\
C_2' \arrow[r, shift left=3, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=3, "\pi_1"'] & C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e" description]
\end{tikzcd}$$

is a morphism of internal categories, where

$$f_2 = \begin{bmatrix} f_1 & f_1 e & f_1 ec \\ f_0 d & f_0 & f_0 c \\ f_1 ed & f_1 e & f_1 \end{bmatrix}.$$

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Proof. The morphism $f_2 : C_2 \longrightarrow C_2'$, being a morphism between pullbacks it is of the form

$$\begin{aligned} f_2 &= \begin{bmatrix} \pi_1 f_2 e_1' & \pi_1 f_2 e_0' & \pi_1 f_2 e_2' \\ \pi_0 f_2 e_1' & \pi_0 f_2 e_0' & \pi_0 f_2 e_2' \\ \pi_2 f_2 e_1' & \pi_2 f_2 e_0' & \pi_2 f_2 e_2' \end{bmatrix} \\ &= \begin{bmatrix} f_1 & f_1 e & f_1 ec \\ f_0 d & f_0 & f_0 c \\ f_1 ed & f_1 e & f_1 \end{bmatrix} \\ &= \begin{bmatrix} e_1 f_1 & e_1 e f_0 & e_2 f_1 \end{bmatrix} \end{aligned}$$

and hence we have $f_1 m = m f_2$:

$$\begin{aligned} f_1 m &= f_1 \begin{bmatrix} 1 & e & 1 \end{bmatrix} = \begin{bmatrix} f_1 & f_1 e & f_1 \end{bmatrix} \\ m f_2 &= m \begin{bmatrix} e_1 f_1 & e_1 e f_0 & e_2 f_1 \end{bmatrix} \\ &= \begin{bmatrix} m e_1 f_1 & m e_1 e f_0 & m e_2 f_1 \end{bmatrix} \\ &= \begin{bmatrix} f_1 & e f_0 & f_1 \end{bmatrix}. \end{aligned}$$

■

Theorem 31 *In a WMC, $\mathbf{C}$, the following is an equivalences of categories:*

$$Cat(\mathbf{C}) \sim AdmRGrph(\mathbf{C}).$$

Proof. The equivalence is established by the previous results. ■

Corollary 32 *In a WMC, a internal natural transformation $t : f \longrightarrow g$, corresponds to a morphism $t : C_0 \longrightarrow C_1'$, as displayed in the following picture*

$$\begin{tikzcd}
C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & C_0 \arrow[l, "e" description] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \arrow[dl, "t"] \\
C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e" description]
\end{tikzcd}$$

such that

$$dt = f_0 \quad , \quad ct = g_0$$

and in addition

$$\begin{bmatrix} 1 & e & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ ectd \\ td \end{bmatrix} = \begin{bmatrix} 1 & e & 1 \end{bmatrix} \begin{bmatrix} tc \\ edtc \\ f_1 \end{bmatrix}.$$

Proof. It is simply the interpretation of the condition

$$m \langle tc, f_1 \rangle = m \langle g_1 td \rangle$$

for a internal natural transformation $t : f \longrightarrow g$ in the weakly Mal'cev context.
■

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Theorem 33 *If* $\mathbf{B}$ *is a WMC then Cat*$(\mathbf{B})$ *is a WMC .*

Proof. It has pullbacks of split epis along split epis because they are computed componentwise and because the pullback of admissible reflexive graphs is again admissible. The proof rest on the simply observation that given the split span

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}$$

in Cat($\mathbf{B}$) where

$$\begin{aligned} A &= \left(A_2 \xrightarrow{[1\ e\ 1]} A_1 \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A_0 \right) \\ B &= \left(B_2 \xrightarrow{[1\ e\ 1]} B_1 \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} B_0 \right) \\ C &= \left(C_2 \xrightarrow{[1\ e\ 1]} C_1 \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} C_0 \right) \end{aligned}$$

we have

$$(A \times_C B)_2 \cong A_2 \times_{C_2} B_2$$

where

$$(A \times_C B)_2 = (A_1 \times_{C_1} B_1) \times_{\left(A_0 \times_{C_0} B_0\right)} (A_1 \times_{C_1} B_1)$$

and

$$A_2 \times_{C_2} B_2 = (A_1 \times_{A_0} A_1) \times_{\left(C_1 \times_{C_0} C_1\right)} (B_1 \times_{B_0} B_1) \,.$$

To show that the pair (e_1, e_2) is jointly epimorphic simply observe that the morphisms are given componentwise and since $\mathbf{B}$ is weakly Mal'cev by hypothesis then the morphisms form A_2, C_2 and B_2 are completely determined by its component from A_1, C_1, B_1. It remains to show that given

$$\left(\begin{bmatrix} h_1 & l_1 & k_1 \end{bmatrix}, \begin{bmatrix} h_0 & l_0 & k_0 \end{bmatrix}\right)$$

it satisfies

$$\begin{bmatrix} h_0 & l_0 & k_0 \end{bmatrix} d \times_d d = d \begin{bmatrix} h_1 & l_1 & k_1 \end{bmatrix}, \text{ etc.}$$

which is trivial because

$$h_0 d = d h_1, \text{ etc.}$$

■

3.4 The connection with Mal'cev categories

The following is a result from [9] p.151, adapted to correspond to the present notation.

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Lemma 34 *Let* $\mathbf{C}$ *be a Mal´cev category. Consider the following diagram*

$$\begin{tikzcd}
A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\
A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

where:
$fr = 1_C = gs$,
The up and left square is a pullback,
$e_1 = \langle 1, ed \rangle \ , \ e_2 = \langle ec, 1 \rangle \,$,
then the pair (e_1, e_2) *is jointly strongly epimorphic in* $\mathbf{C}$.

From here one concludes that every Mal'cev category is a weakly Mal'cev category.

Several notions related with Mal'cev categories may be extended to weakly Mal'cev categories.

Definition 35 (Naturally Weakly Mal'cev Category) *A category* $\mathbf{C}$ *is said to be naturally weakly Mal'cev when:*
It has pullbacks of split epis along split epis;
For every split square

$$\begin{tikzcd}
P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\
A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

if the up and left square is a pullback then the down and right square is a pushout.

As a consequence, the morphism

$$\begin{bmatrix} h & l & k \end{bmatrix}$$

always exists relative to a split span

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"']
\end{tikzcd}$$

and hence, in a naturally weakly Mal'cev category $\mathbf{B}$ we have the equivalence of categories

$$Cat\,(\mathbf{B}) \sim RGrph\,(\mathbf{B})\,.$$

Proposition 36 *In a Mal'cev variety of universal algebras, with Mal'cev operation*

$$\begin{aligned} p \quad &: \quad X \times X \times X \longrightarrow X \\ p\,(x, x, z) \quad &= \quad z \ , \ p\,(x, z, z) = x, \end{aligned}$$

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the morphism

$$\begin{bmatrix} h & l & k \end{bmatrix}$$

exists relative to the split span

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift left, "s"'] & B \arrow[l, shift left, "g"'] \end{tikzcd}$$

if and only if

$$\begin{aligned} & p\left(\theta\left(h\left(a_1\right),...,h\left(a_n\right)\right),\theta\left(l\left(c_1\right),...,l\left(c_n\right)\right),\theta\left(k\left(b_1\right),...,k\left(b_n\right)\right)\right) = \\ & \qquad = \theta\left(p\left(h\left(a_1\right),l\left(c_1\right),k\left(b_1\right)\right),....,p\left(h\left(a_n\right),l\left(c_n\right),k\left(b_n\right)\right)\right) \end{aligned}$$

for all n-ary operation θ*, for all* $n \in \mathbb{N}_0$*, and for all*

$$\begin{aligned} a_1,...,a_n &\in A \\ b_1,...,b_n &\in B \\ c_1,...,c_n &\in C \end{aligned}$$

with

$$f\left(a_i\right) = c_i = g\left(b_i\right).$$

Proof. First observe that if $\begin{bmatrix} h & l & k \end{bmatrix} : A \times_C B \longrightarrow D$ exists, it is given by

$$\begin{bmatrix} h & l & k \end{bmatrix}(a,c,b) = p\left(h\left(a\right),l\left(c\right),k\left(b\right)\right).$$

In fact we have

$$\begin{aligned} (a,c,b) &= \left(p\left(a,r\left(c\right),rg\left(b\right)\right),p\left(f\left(a\right),c,g\left(b\right)\right),p\left(sf\left(a\right),s\left(c\right),b\right)\right) \\ &= p\left(\left(a,f\left(a\right),sf\left(a\right)\right),\left(r\left(c\right),c,s\left(c\right)\right),\left(rg\left(b\right),g\left(b\right),b\right)\right) \\ &= p\left(e_1\left(a\right),e_1 r\left(c\right),e_2\left(b\right)\right) \end{aligned}$$

so that

$$\begin{aligned} \begin{bmatrix} h & l & k \end{bmatrix}(a,c,b) &= \begin{bmatrix} h & l & k \end{bmatrix} p\left(e_1\left(a\right),e_1 r\left(c\right),e_2\left(b\right)\right) \\ &= p\left(h\left(a\right),l\left(c\right),k\left(b\right)\right). \end{aligned}$$

In order to be an homomorphism of universal algebras on also has

$$\begin{aligned} & \begin{bmatrix} h & l & k \end{bmatrix}\left(\theta\left(a_1,...,a_n\right),\theta\left(c_1,...,c_n\right),\theta\left(b_1,...,b_n\right)\right) = \\ & \qquad = \theta\left(\begin{bmatrix} h & l & k \end{bmatrix}\left(a_1,c_1,b_1\right),...,\begin{bmatrix} h & l & k \end{bmatrix}\left(a_n,c_n,b_n\right)\right) \end{aligned}$$

and by definition of $\begin{bmatrix} h & l & k \end{bmatrix}$ it becomes

$$\begin{aligned} & p\left(h\theta\left(a_1,...,a_n\right),l\theta\left(c_1,...,c_n\right),k\theta\left(b_1,...,b_n\right)\right) = \\ & \qquad = \theta\left(p\left(ha_1,lc_1,kb_1\right),...,p\left(ha_n,lc_n,kb_n\right)\right) \end{aligned}$$

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and since h, l, k are homomorphisms of universal algebra one obtains the result. ■

As a simple observation one concludes that an internal category in a Mal'cev variety of universal algebras is a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce,$$

such that

$$p\left(\theta\left(a_1, ..., a_n\right), \theta\left(ed\left(a_1\right), ..., ed\left(a_n\right)\right), \theta\left(b_1, ..., b_n\right)\right) = \\ = \theta\left(p\left(a_1, ed\left(a_1\right), b_1\right), ..., p\left(a_n, ed\left(a_n\right), b_n\right)\right)$$

for all n-ary operation θ, for all $n \in \mathbb{N}_0$, and for all

$$\begin{array}{rcl} a_1, ..., a_n & \in & C_1 \\ b_1, ..., b_n & \in & C_1 \end{array}$$

with

$$d\left(a_i\right) = c\left(b_i\right).$$

Proposition 37 *In the category of Groups, the morphism*

$$\begin{bmatrix} h & l & k \end{bmatrix} : A \times_C B \longrightarrow D$$

exists relative to the split span

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}$$

if and only if

$$h\left(1 - rf\right)(a) + k\left(-sg + 1\right)(b) = k\left(-sg + 1\right)(b) + h\left(1 - rf\right)(a)$$

that is $h\left(1 - rf\right)$ *and* $k\left(-sg + 1\right)$ *commute.*
Furthermore if considering the split epi $(f, r) : A \longrightarrow C$ *as a semidirect product projection*

$$\begin{tikzcd} X \rtimes C \arrow[r, shift left] & C \arrow[l, shift left] \end{tikzcd}$$

then h *is of the form* $[h \;\; l]$ *and*

$$\begin{bmatrix} [h \;\; l] & l & k \end{bmatrix} = [h \;\; k]$$

exists if and only if

$$ksg\left(b\right) + h\left(x\right) - ksg\left(b\right) = k\left(b\right) + h\left(x\right) - k\left(b\right).$$

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Proof. In the category of Groups we have

$$p\left(x,y,z\right)=x-y+z$$

and to be a homomorphism is sufficient to check for the binary operation

$$\theta\left(x,y\right)=x+y$$

so that from the previous result one has that $\begin{bmatrix}h & l & k\end{bmatrix}$ exists if and only if

$$\begin{aligned}&\left(h\left(a_1\right)+h\left(a_2\right)\right)-\left(l\left(c_1\right)+l\left(c_2\right)\right)+\left(k\left(b_1\right)+k\left(b_2\right)\right)=\\&\qquad=\left(h\left(a_1\right)-l\left(c_1\right)+k\left(b_1\right)\right)+\left(h\left(a_2\right)-l\left(c_2\right)+k\left(b_2\right)\right)\end{aligned}$$

or equivalently

$$\begin{aligned}&h\left(a_2\right)-l\left(c_2\right)-l\left(c_1\right)+k\left(b_1\right)=\\&\qquad\qquad=-l\left(c_1\right)+k\left(b_1\right)+h\left(a_2\right)-l\left(c_2\right)\end{aligned}$$

or even

$$\begin{aligned}&h\left(a_2\right)-hrf\left(a_2\right)-ksg\left(b_1\right)+k\left(b_1\right)=\\&\qquad\qquad=-ksg\left(b_1\right)+k\left(b_1\right)+h\left(a_2\right)-hrf\left(a_2\right)\end{aligned}$$

and finally, one obtains the result

$$h\left(1-rf\right)\left(a_2\right)+k\left(-sg+1\right)\left(b_1\right)=k\left(-sg+1\right)\left(b_1\right)+h\left(1-rf\right)\left(a_2\right)$$

that is $h\left(1-rf\right)$ and $k\left(-sg+1\right)$ commute.
Since in Groups split epis are the same as semidirect product projections the result may even be simplified to the case where the split span is of the form

$$\begin{tikzcd}
A\cong X\rtimes C \arrow[r, shift left, "{[0\,1]}"] \arrow[dr, swap, "{[h\,l]}"] & C \arrow[l, shift left, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \arrow[r, shift right, swap, "s"] \arrow[d, "l"] & B \arrow[l, shift right, swap, "g"] \arrow[dl, swap, "k"] \\
& D &
\end{tikzcd}$$

and then $a=\left(x,c\right)$, and

$$a+a'=\left(x+c\cdot x',cc'\right)$$

so that the above formula becomes

$$\begin{bmatrix}h & l\end{bmatrix}\left(a-rf\left(a\right)\right)+k\left(-sg\left(b\right)+b\right)=k\left(-sg\left(b\right)+b\right)+\begin{bmatrix}h & l\end{bmatrix}\left(a-rf\left(a\right)\right)$$

which simplifies to

$$\begin{bmatrix}h & l\end{bmatrix}\left(x,0\right)+k\left(-sg\left(b\right)+b\right)=k\left(-sg\left(b\right)+b\right)+\begin{bmatrix}h & l\end{bmatrix}\left(x,0\right)$$

and then to

$$h\left(x\right)+k\left(-sg\left(b\right)+b\right)=k\left(-sg\left(b\right)+b\right)+h\left(x\right)$$

or equivalently

$$ksg\left(b\right)+h\left(x\right)-ksg\left(b\right)=k\left(b\right)+h\left(x\right)-k\left(b\right).$$

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3.5 Internal groupoids in weakly Mal'cev categories

See [9] p. 420 and [10].

Definition 38 *Let* **C** *be a category with pullbacks of split epis along split epis. A internal groupoid is a internal category*

$$\begin{tikzcd} C_2 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

together with a morphism $t : C_1 \longrightarrow C_1$*, satisfying the following conditions*

$$\begin{aligned} dt &= c \\ ct &= d \\ m\begin{bmatrix} t \\ c \\ 1 \end{bmatrix} &= ed \quad , \quad m\begin{bmatrix} 1 \\ d \\ t \end{bmatrix} = ec. \end{aligned}$$

Proposition 39 *([9] p.149) In a category* **C** *with pullbacks, an internal category admits at most one structure of internal groupoid.*

In a WMC, given an admissible reflexive graph (C_1, C_0, d, e, c) one may try to find the morphism

$$t : C_1 \longrightarrow C_1$$

provided it exists.

In the case of a Mal'cev variety it would be of the form

$$t\left(x\right) = ed\left(x\right) - x + ec\left(x\right)$$

which suggest us, in a WMC, to look for something of the form

$$t = \begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix} \begin{bmatrix} e_1 \\ 1 \\ e_2 \end{bmatrix} = \pi_2 e_1 - 1 + \pi_1 e_2 = ed - 1 + ec$$

and a suitable configuration where it make sense.

The challenge is to find an appropriate split span that agrees with the following diagram

$$\begin{tikzcd} & C_1 \arrow[dl, "e_1"'] \arrow[d, "1"] \arrow[dr, "e_2"] & \\ C_2 \arrow[dr, "\pi_2"'] & C_1 \arrow[d, "1"] & C_2 \arrow[dl, "\pi_1"] \\ & C_1 & \end{tikzcd} .$$

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The answer is

$$\begin{tikzcd} C_2 \arrow[r, shift left, "m"] & C_1 \arrow[l, shift left, "e_2"] \arrow[r, shift right, "e_1"'] & C_2 \arrow[l, shift right, "m"'] \end{tikzcd} \tag{3.5}$$

with the respective pullback denoted by

$$\begin{tikzcd} C_m \arrow[r, shift left, "p'_2"] \arrow[d, shift right, "p'_1"'] & C_2 \arrow[l, shift left, "e'_2"] \arrow[d, shift right, "m"'] \\ C_2 \arrow[u, shift right, "e'_1"'] \arrow[r, shift left, "m"] & C_1 \arrow[l, shift left, "e_2"] \arrow[u, shift right, "e_1"'] \end{tikzcd} .$$

Observe that in Sets the object C_m is exactly the set of commutative squares. The only question is whether or not the morphism

$$\begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix}$$

exists with respect to (3.5). The answer is given by the following proposition.

Proposition 40 *In a WMC, for an internal category*

$$\begin{tikzcd} C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, shift left=0, "m"] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[l, shift right=2, "e_2"'] \arrow[l, shift left=2, "e_1"] \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & C_0 \arrow[l, "e"'] \end{tikzcd}$$

the following are equivalent:
1. It is a groupoid.
2. There is a morphism $t : C_1 \longrightarrow C_1$ such that

$$\begin{aligned} ct &= d \\ m \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} &= ec. \end{aligned}$$

3. The morphism

$$\begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix}$$

exists with respect to the split span

$$\begin{tikzcd} C_2 \arrow[r, shift left, "m"] & C_1 \arrow[l, shift left, "e_2"] \arrow[r, shift right, "e_1"'] & C_2 \arrow[l, shift right, "m"'] \end{tikzcd} .$$

Proof. We will prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).
(1) $\implies$ (2) is trivial by definition of groupoid.
(2) $\implies$ (3) First observe that conditions

$$ct = d \ , \ m \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} = ec \ , \ me_2 = 1 \text{ and } de = 1 = ce$$

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imply $te = e$:

$$\begin{aligned} me_2 &= 1 \Leftrightarrow m \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} = 1 \implies m \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} te = te \Leftrightarrow m \begin{bmatrix} ecte \\ cte \\ te \end{bmatrix} = te \Leftrightarrow \\ &\Leftrightarrow \quad m \begin{bmatrix} e \\ de \\ te \end{bmatrix} = te \Leftrightarrow m \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} e = te \Leftrightarrow ece = te \Leftrightarrow e = te; \end{aligned}$$

Now, the commutativity of the following diagram

$$\begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\pi_2"'] & C_1 \arrow[d, "d"] & C_2 \arrow[l, "m"'] \arrow[d, "t\pi_2"] \\ C_1 \arrow[r, "d"'] & C_0 & C_1 \arrow[l, "c"'] \end{tikzcd}$$

induces the morphism

$$\pi_2 \times_d t\pi_2 : C_m \longrightarrow C_2$$

and we have

$$m\,(\pi_2 \times_d t\pi_2) = \begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix},$$

since by Proposition 27

$$\begin{aligned} (\pi_2 \times_d t\pi_2) &= \begin{bmatrix} \pi_2 & \pi_2 e_2 & \pi_2 e_2 m \\ d\pi_2 & d & ct\pi_2 \\ t\pi_2 e_1 m & t\pi_2 e_1 & t\pi_2 \end{bmatrix} \\ &= \begin{bmatrix} \pi_2 & 1 & m \\ d\pi_2 & d & d\pi_2 \\ tedm & ted & t\pi_2 \end{bmatrix} = \begin{bmatrix} \pi_2 & 1 & m \\ d\pi_2 & d & d\pi_2 \\ ed\pi_2 & ed & t\pi_2 \end{bmatrix} \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} 1 & e & 1 \\ d & 1 & d \\ ted & te & t \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} 1 & e & 1 \\ d & 1 & d \\ ed & e & t \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} e_1 & e_1 e & \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{aligned}$$

and composing with m gives the result

$$\begin{aligned} m\,(\pi_2 \times_d t\pi_2) &= m \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} e_1 & e_1 e & \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \pi_2 & 1 & \begin{bmatrix} 1 & e & ec \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix}. \end{aligned}$$

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(3) $\Longrightarrow$ (1) Given $\begin{bmatrix}\pi_2 & 1 & \pi_1\end{bmatrix}$ define

$$t = \begin{bmatrix}\pi_2 & 1 & \pi_1\end{bmatrix}\begin{bmatrix}e_1\\ 1\\ e_2\end{bmatrix}$$

and we have to prove

$$dt = c\ ,\ ct = d\ ,\ m\begin{bmatrix}t\\ c\\ 1\end{bmatrix} = ed\ ,\ m\begin{bmatrix}1\\ d\\ t\end{bmatrix} = ec.$$

First observe that $m = \begin{bmatrix}1 & e & 1\end{bmatrix}$ and

$$1_{C_m} = \begin{bmatrix}1_{C_2} & e_2 & e_2m\\ m & 1_{C_1} & m\\ e_1m & e_1 & 1_{C_2}\end{bmatrix} = \left[\begin{array}{ccc|c|ccc} 1_{C_1} & e & ec & ec & ec & e & ec\\ d & 1_{C_0} & c & e & c & 1 & c\\ ed & e & 1_{C_1} & 1 & 1 & e & 1\\ \hline 1 & e & 1 & 1_{C_0} & 1 & e & 1\\ \hline 1 & e & 1 & 1 & 1_{C_1} & e & ec\\ d & 1 & d & d & d & 1_{C_0} & c\\ ed & e & ed & ed & ed & e & 1_{C_1}\end{array}\right].$$

Denote by l_i the ith line in the (7×7) identity of C_m and observe the following

$$l_i\begin{bmatrix}e_1\\ 1\\ e_2\end{bmatrix} = l_i\left[\begin{array}{c}1_{C_1}\\ d\\ ed\\ \hline 1_{C_1}\\ \hline ec\\ c\\ 1_{C_1}\end{array}\right] = \begin{cases}1_{C_1} & \text{if} \quad \text{i=1}\\ d & \text{if} \quad \text{i=2}\\ ed & \text{if} \quad \text{i=3}\\ 1_{C_1} & \text{if} \quad \text{i=4}\\ ec & \text{if} \quad \text{i=5}\\ c & \text{if} \quad \text{i=6}\\ 1_{C_1} & \text{if} \quad \text{i=7}\end{cases}.$$

Also let

$$\begin{aligned} l &= \begin{bmatrix}\pi_2 & 1 & \pi_1\end{bmatrix} = \left[\begin{array}{ccc|c|ccc} ed & e & 1 & 1 & 1 & e & ec\end{array}\right]\\ \varepsilon &= \begin{bmatrix}e_1\\ 1\\ e_2\end{bmatrix}\end{aligned}$$

so that

$$t = l\varepsilon.$$

To show that $dt = c$ we have

$$dt = dl\varepsilon = \left[\begin{array}{ccc|c|ccc} d & 1 & d & d & d & 1_{C_0} & c\end{array}\right]\varepsilon = l_6\varepsilon = c;$$

To show that $ct = d$ we have

$$ct = cl\varepsilon = \left[\begin{array}{ccc|c|ccc} d & 1_{C_0} & c & e & c & 1 & c\end{array}\right]\varepsilon = l_2\varepsilon = d;$$

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To prove that $m\begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} = ed$ first observe that

$$\begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} = \begin{bmatrix} l\varepsilon \\ dl\varepsilon \\ l_7\varepsilon \end{bmatrix}$$

(note that we could also try to choose $1_{C_1} = l_1\varepsilon$ or $1_{C_1} = l_4\varepsilon$ but then

$$\begin{bmatrix} l \\ dl \\ l_1 \end{bmatrix} \text{ and } \begin{bmatrix} l \\ dl \\ l_4 \end{bmatrix}$$

would not be defined), and then

$$\begin{aligned} \begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} &= \begin{bmatrix} l \\ dl \\ l_7 \end{bmatrix}\varepsilon = \left[\begin{array}{ccc|c|ccc} ed & e & 1 & 1 & 1_{C_1} & e & ec \\ d & 1 & d & d & d & 1_{C_0} & c \\ ed & e & ed & ed & ed & e & 1_{C_1} \end{array}\right]\varepsilon \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & 1_{C_2} \end{bmatrix}\varepsilon \end{aligned}$$

so that

$$\begin{aligned} m\begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} &= m\begin{bmatrix} e_1\pi_2 & e_1 & 1_{C_2} \end{bmatrix}\varepsilon \\ &= \begin{bmatrix} \pi_2 & 1 & m \end{bmatrix}\varepsilon \\ &= \left[\begin{array}{ccc|c|ccc} ed & e & 1_{C_1} & 1 & 1 & e & 1 \end{array}\right]\varepsilon \\ &= l_3\varepsilon = ed. \end{aligned}$$

A similar calculation shows $m\begin{bmatrix} 1 \\ d \\ t \end{bmatrix} = ec$, in fact

$$\begin{aligned} \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} &= \begin{bmatrix} 1 \\ ct \\ t \end{bmatrix} = \begin{bmatrix} l_1\varepsilon \\ cl\varepsilon \\ l\varepsilon \end{bmatrix} = \begin{bmatrix} l_1 \\ cl \\ l \end{bmatrix}\varepsilon \\ &= \left[\begin{array}{ccc|c|ccc} 1_{C_1} & e & ec & ec & ec & e & ec \\ d & 1_{C_0} & c & e & c & 1 & c \\ ed & e & 1_{C_1} & 1 & 1 & e & ec \end{array}\right]\varepsilon \\ &= \begin{bmatrix} 1_{C_2} & e_2 & e_2\pi_1 \end{bmatrix}\varepsilon \end{aligned}$$

and hence

$$\begin{aligned} m\begin{bmatrix} 1 \\ d \\ t \end{bmatrix} &= m\begin{bmatrix} 1_{C_2} & e_2 & e_2\pi_1 \end{bmatrix}\varepsilon \\ &= \begin{bmatrix} m & 1 & \pi_1 \end{bmatrix}\varepsilon \\ &= \left[\begin{array}{ccc|c|ccc} 1 & e & 1 & 1 & 1_{C_1} & e & ec \end{array}\right]\varepsilon \\ &= l_5\varepsilon = ec. \end{aligned}$$

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■

3.6 Conclusion

We conclude by saying once again that the notion of weakly Mal'cev category is introduced with the unique purpose to have a setting (easy to handle) where a multiplicative graph, that is a diagram of the form

$$\begin{tikzcd}
C_2 \arrow[r, shift left=6, "\pi_2"] \arrow[r, shift left=3, leftarrow, "e_2" description] \arrow[r, "m" description] \arrow[r, shift right=3, leftarrow, "e_1" description] \arrow[r, shift right=6, "\pi_1"'] & C_1 \arrow[r, shift left=3, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=3, "c"'] & C_0
\end{tikzcd}$$

where

$$\begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\
C_1 \arrow[r, shift left, "d"] \arrow[u, shift right, "e_1"'] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"']
\end{tikzcd}$$

is a split pullback and $me_1 = 1_{C_1} = me_2$, is already an internal category, that is, the conditions

$$\begin{aligned} dm &= d\pi_2 \\ cm &= c\pi_1 \\ m(1 \times m) &= m(m \times 1) \end{aligned}$$

are automatically satisfied.

However, the original motivation was to have a setting where a reflexive graph would admit at most one multiplication, so that the two axioms in the definition of a WMC are thus explained:

− the existence of pullbacks of split epis along split epis is used to construct the pullback C_2 of the split epi (c, e) along the split epi (d, e);

− the requirement that the pair (e_1, e_2) is jointly epimorphic is used to uniquelly determine the morphism m, provided it exists, from the two components me_1 and me_2.

It was then an happy surprise to observe that preservation of domain and codomain as well as associativity would automatically follow.

There is still many comparisons to be made in order to decide if this is in fact a good notion for a category and if it does not coincide with something already known. For example, it would be interesting to find out what other conditions are needed in orther to have that every internal category is an internal groupoid: our guess would be that the pair (e_1, e_2) should be strongly epimorphic. We leave this and other question for a future work.

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Part II

In the Abelian World

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Chapter 4

Internal Bicategories in Ab

Preprint CM03/I-24, Aveiro Universitiy, 2003.
Title: Internal Bicategories in Ab
Author: N. Martins-Ferreira
Reflexive 2-graph, horizontal composition, associativity isomorphism, coherence conditions, internal bicategory.

Abstract: We describe internal bicategories in the category of abelian groups and in particular present them as presheaf categories.

4.1 Introduction

This paper is a slightly modified version of the author's Master Thesis at Aveiro University, under the supervision of Professor G. Janelidze. It is organized as follows. The section Preliminaries presents an explicit definition of internal bicategory and introduces some well known concepts as the equivalence between the categories Mor(Ab) and Cat(Ab). A description of internal reflexive 2-graphs in Ab, as 2-complexes, is presented in section Reflexive 2-Graphs. The following three sections give a description of internal reflexive 2-graphs with horizontal composition and an associativity isomorphism satisfying the coherence condition. It turns out that it is a very complex structure, even in Ab. However, the inclusion of the left and right isomorphisms with the triangle coherence condition, completely determines the horizontal composition and the associativity isomorphism. The conclusion is that an internal bicategory in Ab is completely determined by a 2-complex

$$Z \xrightarrow{\partial_1} Y \xrightarrow{\partial_0} X \qquad , \qquad \partial_0\partial_1 = 0,$$

and three morphisms

$$\begin{aligned} \lambda_1, \rho_1 &: Y \longrightarrow Z, \\ q &: X \longrightarrow Z. \end{aligned}$$

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4.2 Preliminaries

A bicategory [1,2,3] consists of 0-cells $A, B, C, \ldots$, 1-cells $f, g, h, \ldots$ and 2-cells $\sigma, \tau, \ldots$ arranged as in the following picture

$$\begin{tikzcd}[column sep=large]
A \arrow[r, bend left=60, "g", ""{name=G, below}] \arrow[r, "g'" description, ""{name=G1, above}, ""{name=G2, below}] \arrow[r, bend right=60, "g''"', ""{name=G3, above}] & B \arrow[r, bend left=60, "f", ""{name=F, below}] \arrow[r, "f'" description, ""{name=F1, above}, ""{name=F2, below}] \arrow[r, bend right=60, "f''"', ""{name=F3, above}] & C
\arrow[from=G, to=G1, "\tau"'] \arrow[from=G2, to=G3, "\tau'"'] \arrow[from=F, to=F1, "\sigma"'] \arrow[from=F2, to=F3, "\sigma'"']
\end{tikzcd}.$$

It forms a reflexive 2-graph and the 2-cells compose vertically along 1-cells and horizontally along 0-cells.

Vertical composition is associative

$$\tau\left(\tau'\tau''\right) = \left(\tau\tau'\right)\tau'',$$

and has right and left identities

$$\tau 1_f = \tau = 1_g \tau.$$

Horizontal composition

$$\begin{tikzcd}
A \arrow[r, bend left=40, "g", ""{name=G, below}] \arrow[r, bend right=40, "g'"', ""{name=G1, above}] & B \arrow[r, bend left=40, "f", ""{name=F, below}] \arrow[r, bend right=40, "f'"', ""{name=F1, above}] & C \arrow[r, mapsto, "*"] & A \arrow[r, bend left=40, "f*g", ""{name=H, below}] \arrow[r, bend right=40, "f'*g'"', ""{name=H1, above}] & C
\arrow[from=G, to=G1, "\tau"] \arrow[from=F, to=F1, "\sigma"] \arrow[from=H, to=H1, "\sigma*\tau"]
\end{tikzcd} \tag{4.1}$$

is associative up to a natural isomorphism

$$\alpha_{f,g,h} : f*(g*h) \longrightarrow (f*g)*h$$

and 1-cells $1_A,\ 1_B$, act as right and left identities up to natural isomorphisms

$$\begin{aligned} \lambda_f &: 1_B * f \longrightarrow f, \\ \rho_f &: f * 1_A \longrightarrow f. \end{aligned}$$

The isomorphisms α, λ, ρ are required to satisfy the coherence conditions

$$\begin{tikzcd}
f*(g*(h*k)) \arrow[r, "\alpha"] \arrow[d, "1_f*\alpha"'] & (f*g)*(h*k) \arrow[r, "\alpha"] & ((f*g)*h)*k \\
f*((g*h)*k) \arrow[rr, "\alpha"] & & (f*(g*h))*k \arrow[u, "\alpha*1_k"']
\end{tikzcd},$$

$$\begin{tikzcd}
f*(1_B*g) \arrow[r, "\alpha"] \arrow[dr, "1_f*\lambda"] & (f*1_B)*g \arrow[d, "\rho*1_g"] \\
 & f*g
\end{tikzcd},$$

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and

$$\lambda_{1_f} = \rho_{1_f}.$$

Definition 41 *An internal bicategory in $\mathbb{A}$ consists of the following data:*
A reflexive 2-graph

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "c_1"'] & C_1 \arrow[l, "e_1"'] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "c_0"'] & C_0 \arrow[l, "e_0"'] \end{tikzcd} \ , \ d_0 d_1 = d_0 c_1 \ , \ c_0 c_1 = c_0 d_1 \ , \ d_i e_i = 1_{C_i} = c_i e_i \ , \ i = 0, 1; \tag{4.2}$$

A morphism $m : C_2 \times_{C_1} C_2 \longrightarrow C_2$ such that the diagram

$$C = \left(\begin{tikzcd} C_2 \times_{C_1} C_2 \arrow[r, "m"] & C_2 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "c_1"'] & C_1 \arrow[l, "e_1"'] \end{tikzcd} \right)$$

is an internal category in $\mathbb{A}$;
An internal functor $\mu = (\mu_2, \mu_1) : C \times_{C_0} C \longrightarrow C$, from the internal category[1]

$$C \times_{C_0} C = \left(\begin{tikzcd} C_4 \arrow[r, "m \times m"] & C_2 \times_{C_0} C_2 \arrow[r, shift left=3, "d_1 \times d_1"] \arrow[r, shift right=3, "c_1 \times c_1"'] & C_1 \times_{C_0} C_1 \arrow[l, "e_1 \times e_1"'] \end{tikzcd} \right)$$

to the internal category C, with the morphism $\mu_1 : C_1 \times_{C_0} C_1 \longrightarrow C_1$ satisfying the commutativity of the diagram

$$\begin{tikzcd} C_1 \arrow[d, "c_0"'] & C_1 \times_{C_0} C_1 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\mu_1"] & C_1 \arrow[d, "d_0"] \\ C_0 & C_1 \arrow[l, "c_0"'] \arrow[r, "d_0"] & C_0 \end{tikzcd} ;$$

Three natural isomorphisms

$$\begin{aligned} \alpha &: \mu (1 \times \mu) \longrightarrow \mu (\mu \times 1), \\ \lambda &: \mu \langle ec, 1 \rangle \longrightarrow 1, \\ \rho &: \mu \langle 1, ed \rangle \longrightarrow 1, \end{aligned}$$

where $\langle ec, 1 \rangle = (\langle e_1 e_0 c_0 c_1, 1_{C_2} \rangle, \langle e_0 c_0, 1_{C_1} \rangle) : C \longrightarrow C \times_{C_0} C$ and $\langle 1, ed \rangle = (\langle 1_{C_2}, e_1 e_0 d_0 d_1 \rangle, \langle 1_{C_1}, e_0 d_0 \rangle) : C \longrightarrow C \times_{C_0} C$;
The following conditions are satisfied

$$(\alpha \circ (\mu \times 1 \times 1)) \cdot (\alpha \circ (1 \times 1 \times \mu)) = (\mu \circ (\alpha \times 1)) \cdot (\alpha \circ (1 \times \mu \times 1)) \cdot (\mu \circ (1 \times \alpha)), \tag{4.3}$$

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[1] C_4 stands for $(C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2) \cong (C_2 \times_{C_0} C_2) \times_{(C_1 \times_{C_0} C_1)} (C_2 \times_{C_1} C_2)$

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$$(\mu \circ (\rho \times 1)) \cdot (\alpha \circ (1 \times \langle ec, 1 \rangle)) = \mu \circ (1 \times \lambda)\,, \tag{4.4}$$

$$\lambda \circ e = \rho \circ e. \tag{4.5}$$

The notion of internal bicategory in Ab will be described following the above definition. First we describe internal reflexive 2-graphs in Ab and observe that the vertical composition is determined by it. Next we describe the horizontal composition. Afterwords we describe the associative isomorphism and its coherence condition. After that we describe the left and right identity isomorphisms and finally we present the description of internal bicategories in Ab.

Some well known facts about internal categories in Ab are presented in order to establish notation.

A split epi diagram in Ab,

$$\begin{tikzcd} C_2 \arrow[r, shift left, "d"] & C_1 \arrow[l, shift left, "e"] \end{tikzcd}\ ,\ de = 1_{C_1}$$

is isomorphic to

$$\begin{tikzcd} K \oplus C_1 \arrow[r, shift left, "(0\ 1)"] & C_1 \arrow[l, shift left, "\binom{0}{1}"] \end{tikzcd},$$

where $K = \ker d$.

This fact is used to show that in Ab, a reflexive graph

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_1 \arrow[l, "e"'] \end{tikzcd}\quad ,\quad de = 1_{C_1} = ce$$

is given up to an isomorphism by

$$\begin{tikzcd} K \oplus C_1 \arrow[r, shift left=2, "(0\ 1)"] \arrow[r, shift right=2, "(\partial\ 1)"'] & C_1 \arrow[l, "\binom{0}{1}"'] \end{tikzcd}$$

and hence, it is completely determined by a morphism

$$K \xrightarrow{\partial} C_1.$$

In Ab, a pullback diagram with a split epi as one of the components

$$\begin{tikzcd} C_2 \times_{C_1} C_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "c"] \\ C_2 \arrow[r, shift left, "d"] & C_1 \arrow[l, shift left, "e"] \end{tikzcd},$$

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is isomorphic to

$$\begin{tikzcd}
K \oplus C_2 \arrow[r, "(0\ 1)"] \arrow[d, "1\oplus c"'] & C_2 \arrow[d, "c"] \\
K \oplus C_1 \arrow[r, "(0\ 1)"] & C_1
\end{tikzcd}.$$

Considering C_2 as $K \oplus C_1$, an arrow $\langle f, g\rangle : D \longrightarrow C_2 \times_{C_1} C_2$, with $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : D \longrightarrow K \oplus C_1$, is of the form

$$D \xrightarrow{\binom{f_1}{g}} K \oplus C_2.$$

A morphism $m : C_2 \times_{C_1} C_2 \longrightarrow C_2$ satisfying $m\,(1 \times_{C_1} m) = m\,(m \times_{C_1} 1)$, $m\,\langle 1, ed\rangle = 1_{C_2} = m\,\langle ec, 1\rangle$ and $dm = d\pi_1, cm = c\pi_2$ is completely determined and it is of the form

$$m = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is used in the proof of the well known equivalence of categories

$$\mathrm{Cat(Ab)} \sim \mathrm{Mor(Ab)}$$

establishing that an internal category in Ab is completely determined by a morphism

$$K \xrightarrow{\partial} C_1$$

and it is given as

$$\begin{tikzcd}
K \oplus K \oplus C_1 \arrow[r, "m"] & K \oplus C_1 \arrow[r, shift left=2, "(0\ 1)"] \arrow[r, shift right=2, "(\partial\ 1)"'] & C_1 \arrow[l, "\binom{0}{1}"']
\end{tikzcd}.$$

An internal functor in Ab is given by two morphisms g_1, g_2 making commutative the diagram

$$\begin{tikzcd}
K \arrow[r, "\partial"] \arrow[d, "g_2"'] & C_1 \arrow[d, "g_1"] \\
K' \arrow[r, "\partial'"] & C_1'
\end{tikzcd}.$$

A natural transformation $\tau : f \longrightarrow g$ is given by an arrow $\tau : C_1 \longrightarrow K'$ such that

$$\partial' \tau = g_1 - f_1 \ , \ \tau \partial = g_2 - f_2.$$

Horizontal composition is given by

$$\sigma \circ \tau = \sigma g_1 + f_2' \tau = g_2' \tau + \sigma f_1,$$

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with $\sigma : f' \longrightarrow g'$, and the vertical composition is given by

$$\tau' \cdot \tau = \tau' + \tau.$$

4.3 Reflexive 2-Graphs

An internal reflexive 2-graph is given by a diagram

$$\begin{tikzcd}
C_2 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "c_1"'] & C_1 \arrow[l, "e_1"' description] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "c_0"'] & C_0 \arrow[l, "e_0"' description]
\end{tikzcd}$$

where the arrows $d_0, e_0, c_0, d_1, e_1, c_1$ satisfy the identities

$$\begin{aligned} d_0e_0 &= 1_{C_0} = c_0e_0, \\ d_1e_1 &= 1_{C_1} = c_1e_1, \\ d_0d_1 = d_0c_1 &, \ c_0d_1 = c_0c_1. \end{aligned} \tag{4.6}$$

It is well known that an internal reflexive 2-graph in Ab is completely determined by a 2-complex

$$Z \xrightarrow{\partial_1} Y \xrightarrow{\partial_0} X \quad , \qquad \partial_0\partial_1 = 0,$$

where $Z = \ker d_1, Y = \ker d_0, X = C_0$.

Proposition 42 *An internal reflexive 2-graph in Ab is given (up to an isomorphism) by*

$$\begin{tikzcd}
Z \oplus Y \oplus X \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "\partial_1^*"'] & Y \oplus X \arrow[l, "i_1"' description] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "\partial_0^*"'] & X \arrow[l, "i_0"' description]
\end{tikzcd} \tag{4.7}$$

where $p_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, i_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \partial_1^* = \begin{pmatrix} \partial_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, p_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}, i_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \partial_0^* = \begin{pmatrix} \partial_0 & 1 \end{pmatrix}$ *and with* $\partial_1\partial_0 = 0$.

Proof. A reflexive 2-graph is formed by two reflexive graphs

$$\begin{tikzcd}
C_2 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "c_1"'] & C_1 \arrow[l, "e_1"' description] & \text{and} & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "c_0"'] & C_0. \arrow[l, "e_0"' description]
\end{tikzcd}$$

They are, up to an isomorphism, of the form

$$\begin{tikzcd}
Z \oplus (Y \oplus X) \arrow[r, shift left=2, "(0\ 1)"] \arrow[r, shift right=2, "(\partial\ 1)"'] & (Y \oplus X) \arrow[l, "\binom{0}{1}"' description] & \text{and} & Y \oplus X \arrow[r, shift left=2, "(0\ 1)"] \arrow[r, shift right=2, "(\partial_0\ 1)"'] & X, \arrow[l, "\binom{0}{1}"' description]
\end{tikzcd}$$

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where $Z = \ker d_1, Y = \ker d_0, X = C_0$.

Using conditions (4.6) we have

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_1 & 1 & 0 \\ \partial_2 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \partial_0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \partial_0 & 1 \end{pmatrix} \begin{pmatrix} \partial_1 & 1 & 0 \\ \partial_2 & 0 & 1 \end{pmatrix},$$

which means that $\partial_2 = 0$ and $\partial_0 \partial_1 = 0$. ■

4.4 Horizontal Composition

Considering the reflexive 2-graph obtained in Proposition 42, the internal category

$$\begin{tikzcd} C = \Bigg(C_2 \times_{C_1} C_2 \arrow[r, "m"] & C_2 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "c_1"'] & C_1 \Bigg) \arrow[l, "e_1"' description] \end{tikzcd}$$

is given by the morphism

$$\begin{pmatrix} \partial_1 \\ 0 \end{pmatrix} : Z \longrightarrow Y \oplus X. \tag{4.8}$$

Similarly, the internal category

$$\begin{tikzcd} C \times_{C_0} C = \Bigg(C_4 \arrow[r, "m \times m"] & C_2 \times_{C_0} C_2 \arrow[r, shift left=2, "d_1 \times d_1"] \arrow[r, shift right=2, "c_1 \times c_1"'] & C_1 \times_{C_0} C_1 \Bigg) \arrow[l, "e_1 \times e_1"' description] \end{tikzcd}$$

is given by the morphism

$$\partial = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_1 \\ 0 & 0 \end{pmatrix} : Z \oplus Z \longrightarrow Y \oplus Y \oplus X. \tag{4.9}$$

The horizontal composition is given by a functor $\mu = (\mu_2, \mu_1)$, which means that the diagram

$$\begin{tikzcd} Z \oplus Z \arrow[r, "\partial"] \arrow[d, "\mu_2"'] & Y \oplus Y \oplus X \arrow[d, "\mu_1"] \\ Z \arrow[r, "\binom{\partial_1}{0}"] & Y \oplus X \end{tikzcd}$$

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is commutative. Furthermore, the morphism μ_1 is such that the diagram

$$\begin{tikzcd}
Y \oplus X \arrow[d, "(\partial_0\ 1)"'] & Y \oplus Y \oplus X \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \partial_0 & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\mu_1"'] & Y \oplus X \arrow[d, "(0\ 1)"] \\
X & Y \oplus X \arrow[l, "(\partial_0\ 1)"'] \arrow[r, "(0\ 1)"] & X
\end{tikzcd}$$

commutes. This means that $\mu_1 = \begin{pmatrix} f_1 & g_1 & h \\ 0 & 0 & 1 \end{pmatrix}$, while $\mu_2 = (f_2\ g_2)$ with $f_2, g_2 : Z \longrightarrow Z$, $\quad f_1, g_1 : Y \longrightarrow Y$, $h : X \longrightarrow Y$ satisfying

$$\begin{aligned}
\partial_1 f_2 &= f_1 \partial_1 \ \ , \ \partial_1 g_2 = g_1 \partial_1, \\
\partial_0 f_1 &= \partial_0 = \partial_0 g_1 \ \ , \ \partial_0 h = 0.
\end{aligned}$$

We have just proved the following.

Proposition 43 *In Ab, an internal reflexive 2-graph with composition (not necessarily associative or with identities) is completely determined by a diagram*

$$\begin{tikzcd}
Z \arrow[loop above, "f_2"] \arrow[loop below, "g_2"] \arrow[r, "\partial_1"] & Y \arrow[loop above, "f_1"] \arrow[loop below, "g_1"] \arrow[r, shift left, "\partial_0"] & X \arrow[l, shift left, "h"]
\end{tikzcd} \tag{4.10}$$

with

$$\begin{aligned}
\partial_0 \partial_1 &= 0, \\
\partial_1 f_2 &= f_1 \partial_1 \ \ , \ \partial_1 g_2 = g_1 \partial_1, \\
\partial_0 f_1 &= \partial_0 = \partial_0 g_1 \ \ , \ \partial_0 h = 0.
\end{aligned} \tag{4.11}$$

4.5 Associativity Isomorphism

In this section we describe the isomorphism α without considering yet the coherence condition.

Given a reflexive 2-graph with an horizontal composition, that is, given a diagram as (4.10) satisfying conditions (4.11), in order to analyze the isomorphism

$$\alpha : \mu\,(1 \times \mu) \longrightarrow \mu\,(\mu \times 1)\,,$$

we have to describe the functors $\mu\,(\mu \times 1)$ and $\mu\,(1 \times \mu)$.

Similarly to (4.9), the category of composable triples is described by the morphism

$$\overline{\partial} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 0 & \partial_1 \\ 0 & 0 & 0 \end{pmatrix} : Z \oplus Z \oplus Z \longrightarrow Y \oplus Y \oplus Y \oplus X. \tag{4.12}$$

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The functor $(1 \times \mu)$ is obtained from the functors

$$\begin{tikzcd}
Z \arrow[r, "\binom{\partial_1}{0}"] \arrow[d, "1"'] & Y \oplus X \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\
Z \arrow[r, "\binom{\partial_1}{0}"] & Y \oplus X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Z \oplus Z \arrow[r, "\partial"] \arrow[d, "(f_2\ g_2)"'] & Y \oplus Y \oplus X \arrow[d, "\mu_1"] \\
Z \arrow[r, "\binom{\partial_1}{0}"] & Y \oplus X
\end{tikzcd} .$$

Since the category of horizontal composable triples is determined by the morphism (4.12) we have that $(1 \times \mu)$ is given by

$$\begin{tikzcd}
Z \oplus Z \oplus Z \arrow[r, "\overline{\partial}"] \arrow[d, "(1\times\mu)_2"'] & Y \oplus Y \oplus Y \oplus X \arrow[d, "(1\times\mu)_1"] \\
Z \oplus Z \arrow[r, "\partial"] & Y \oplus Y \oplus X
\end{tikzcd}$$

with $(1 \times \mu)_2 = 1 \oplus \mu_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f_2 & g_2 \end{pmatrix}$ and with $(1 \times \mu)_1$ obtained from

$$\begin{tikzcd}
Y \oplus X \arrow[d, "1"] & Y \oplus Y \oplus Y \oplus X \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_0 & \partial_0 & 1 \end{smallmatrix}\right)"'] \arrow[r, "(0\ 1_{Y^2 \oplus X})"] \arrow[d, "1 \times_{C_0} \mu_1"'] & Y \oplus Y \oplus X \arrow[d, "\mu_1"] \\
Y \oplus X & Y \oplus Y \oplus X \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \partial_0 & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] & X
\end{tikzcd} ,$$

which gives

$$1 \times_{C_0} \mu_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f_1 & g_1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Similarly for $\mu \times 1$ we have

$$\mu_2 \times 1 = \begin{pmatrix} f_2 & g_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

whereas

$$\mu_1 \times 1 = \begin{pmatrix} f_1 & g_1 & h\partial_0 & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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is given as in the diagram

$$\begin{tikzcd}
Y \oplus Y \oplus X \arrow[d, "\mu_1"] & Y \oplus Y \oplus Y \oplus X \arrow[l, "\left(\begin{smallmatrix} 1_{Y^2} & 0 & 0 \\ 0 & \partial_0 & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\mu_1 \times_{C_0} 1"'] & Y \oplus X \arrow[d, "1"] \\
Y \oplus X & Y \oplus Y \oplus X \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \partial_0 & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] & X
\end{tikzcd} .$$

Computing the composition of internal functors in Ab we obtain

$$\mu\left(1 \times \mu\right) = \left(\begin{pmatrix} f_2 & g_2 f_2 & g_2^2 \end{pmatrix}, \begin{pmatrix} f_1 & g_1 f_1 & g_1^2 & h + g_1 h \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$$

and

$$\mu\left(\mu \times 1\right) = \left(\begin{pmatrix} f_2^2 & f_2 g_2 & g_2 \end{pmatrix}, \begin{pmatrix} f_1^2 & f_1 g_1 & f_1 h \partial_0 + g_1 & h + f_1 h \\ 0 & 0 & 0 & 1 \end{pmatrix}\right).$$

The isomorphism α is given by an arrow

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \end{pmatrix} : Y^3 \oplus X \longrightarrow Z$$

with

$$\begin{pmatrix} \partial_1 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \end{pmatrix} =$$
$$= \begin{pmatrix} f_1^2 & f_1 g_1 & f_1 h \partial_0 + g_1 & h + f_1 h \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} f_1 & g_1 f_1 & g_1^2 & h + g_1 h \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \end{pmatrix} \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 0 & \partial_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} f_2^2 & f_2 g_2 & g_2 \end{pmatrix} - \begin{pmatrix} f_2 & g_2 f_2 & g_2^2 \end{pmatrix},$$

that is

$$\begin{array}{llll} \partial_1 \alpha_1 = f_1^2 - f_1, & \partial_1 \alpha_2 = f_1 g_1 - g_1 f_1, & \partial_1 \alpha_3 = f_1 h \partial_0 + g_1 - g_1^2, & \partial_1 \alpha_0 = f_1 h - g_1 h \\ \alpha_1 \partial_1 = f_2^2 - f_2, & \alpha_2 \partial_1 = f_2 g_2 - g_2 f_2, & \alpha_3 \partial_1 = g_2 - g_2^2. & \end{array}$$

This gives the following.

Proposition 44 *In Ab, an internal 2-reflexive graph with horizontal composition and an isomorphism for associativity is described by a diagram as* (4.10), *together with morphisms* $\alpha_1, \alpha_2, \alpha_3 : Y \longrightarrow Z$, $\alpha_0 : X \longrightarrow Z$ *satisfying conditions* (4.11) *and*

$$\begin{array}{llll} \partial_1 \alpha_1 = f_1^2 - f_1, & \partial_1 \alpha_2 = f_1 g_1 - g_1 f_1, & \partial_1 \alpha_3 = f_1 h \partial_0 + g_1 - g_1^2, & \partial_1 \alpha_0 = f_1 h - g_1 h \\ \alpha_1 \partial_1 = f_2^2 - f_2, & \alpha_2 \partial_1 = f_2 g_2 - g_2 f_2, & \alpha_3 \partial_1 = g_2 - g_2^2. & \end{array}$$

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4.6 Coherence Condition

Given an internal reflexive 2-graph with a horizontal composition and an associativity isomorphism as in the previous section, we now analyze the coherence condition (4.3).

Let us begin with $(\alpha \circ (\mu \times 1 \times 1))$ where the internal functor

$$(\mu \times 1 \times 1) = \left(\mu_2 \times 1 \times 1, \begin{pmatrix} f_1 & g_1 & h\partial_0 & h\partial_0 & h \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right),$$

represents the identity natural transformation of $\mu \times 1 \times 1$, that is $1_{\mu \times 1 \times 1} = 0 : Y^4 \oplus X \longrightarrow Z^3$. Using the horizontal composition of natural transformations we have that $\alpha \circ (\mu \times 1 \times 1)$ is given by

$$\begin{pmatrix} \alpha_1 f_1 & \alpha_1 g_1 & \alpha_1 h\partial_0 + \alpha_2 & \alpha_1 h\partial_0 + \alpha_3 & \alpha_1 h + \alpha_0 \end{pmatrix} : Y^4 \oplus X \longrightarrow Z.$$

In order to evaluate $(\alpha \circ (1 \times 1 \times \mu))$ we have that

$$(1 \times 1 \times \mu) = \left(1 \times 1 \times \mu_2, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f_1 & g_1 & h \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

and so $(\alpha \circ (1 \times 1 \times \mu))$ consists of

$$\alpha \circ (1 \times 1 \times \mu) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 f_1 & \alpha_3 g_1 & \alpha_3 h + \alpha_0 \end{pmatrix} : Y^4 \oplus X \longrightarrow Z$$

The internal functor $(1 \times \mu \times 1)$ is given by

$$\left(1 \times \mu_2 \times 1, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & f_1 & g_1 & h\partial_0 & h \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right),$$

hence $(\alpha \circ (1 \times \mu \times 1))$ consists of

$$\alpha \circ (1 \times \mu \times 1) = \begin{pmatrix} \alpha_1 & \alpha_2 f_1 & \alpha_2 g_1 & \alpha_2 h\partial_0 + \alpha_3 & \alpha_2 h + \alpha_0 \end{pmatrix} : Y^4 \oplus X \longrightarrow Z.$$

In the natural transformation $\mu (\alpha \times 1)$, 1 represents the identity natural transformation of 1_C and it is given by the two morphisms

$$0 : Y \longrightarrow Z, 0 : X \longrightarrow Z.$$

This means that $(\alpha \times 1)$ is given by

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0\partial_0 & \alpha_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : Y^4 \oplus X \longrightarrow Z^2,$$

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while $\mu(\alpha \times 1)$ consists of

$$\begin{pmatrix} f_2\alpha_1 & f_2\alpha_2 & f_2\alpha_3 & f_2\alpha_0\partial_0 & f_2\alpha_0 \end{pmatrix} : Y^4 \oplus X \longrightarrow Z,$$

considering μ as $1_\mu = 0 : Y^2 \oplus X \longrightarrow Z$.

Similarly $(1 \times \alpha)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \end{pmatrix} : Y^4 \oplus X \longrightarrow Z^2$$

and then $\mu(1 \times \alpha)$ consists of

$$\begin{pmatrix} 0 & g_2\alpha_1 & g_2\alpha_2 & g_2\alpha_3 & g_2\alpha_0 \end{pmatrix} : Y^4 \oplus X \longrightarrow Z.$$

Condition (4.3) can be written as

$$\begin{aligned} &\begin{pmatrix} \alpha_1 f_1 & \alpha_1 g_1 & \alpha_1 h\partial_0 + \alpha_2 & \alpha_1 h\partial_0 + \alpha_3 & (\alpha_1 h + \alpha_0) \end{pmatrix} + \\ &+ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 f_1 & \alpha_3 g_1 & (\alpha_3 h + \alpha_0) \end{pmatrix} = \begin{pmatrix} f_2\alpha_1 & f_2\alpha_2 & f_2\alpha_3 & f_2\alpha_0\partial_0 & f_2\alpha_0 \end{pmatrix} + \\ &+ \begin{pmatrix} \alpha_1 & \alpha_2 f_1 & \alpha_2 g_1 & \alpha_2 h\partial_0 + \alpha_3 & (\alpha_2 h + \alpha_0) \end{pmatrix} + \begin{pmatrix} 0 & g_2\alpha_1 & g_2\alpha_2 & g_2\alpha_3 & g_2\alpha_0 \end{pmatrix} \end{aligned}$$

that is

$$\begin{aligned} &\alpha_1 f_1 = f_2\alpha_1, \quad \alpha_1 g_1 + \alpha_2 = f_2\alpha_2 + \alpha_2 f_1 + g_2\alpha_1, \\ &\alpha_1 h\partial_0 + \alpha_2 + \alpha_3 f_1 = f_2\alpha_3 + \alpha_2 g_1 + g_2\alpha_2, \\ &\alpha_1 h\partial_0 + \alpha_3 g_1 = f_2\alpha_0\partial_0 + \alpha_2 h\partial_0 + g_2\alpha_3, \\ &\alpha_1 h + \alpha_0 + \alpha_3 h = f_2\alpha_0 + \alpha_2 h + g_2\alpha_0. \end{aligned} \tag{4.13}$$

This gives the following.

Proposition 45 *In Ab, an internal reflexive 2-graph with a horizontal composition and an associative isomorphism satisfying the pentagon coherence condition, is determined by a diagram of the form*

$$\begin{tikzcd} Z \arrow[r, "\partial_1"] \arrow[loop above, "f_2"] \arrow[loop below, "g_2"] & Y \arrow[r, shift left, "\partial_0"] \arrow[loop above, "f_1"] \arrow[loop below, "g_1"] & X \arrow[l, shift left, "h"] \end{tikzcd}$$

together with morphisms $\alpha_1, \alpha_2, \alpha_3 : Y \longrightarrow Z$, $\alpha_0 : X \longrightarrow Z$ satisfying the following conditions

$$\partial_0\partial_1 = 0, \tag{I}$$

$$\begin{aligned} \partial_1 f_2 &= f_1\partial_1 \;\; , \;\; \partial_1 g_2 = g_1\partial_1, \\ \partial_0 f_1 &= \partial_0 = \partial_0 g_1 \;\; , \;\; \partial_0 h = 0, \end{aligned} \tag{II}$$

$$\begin{aligned} &\partial_1\alpha_1 = f_1^2 - f_1, \quad \partial_1\alpha_2 = f_1 g_1 - g_1 f_1, \\ &\partial_1\alpha_3 = f_1 h\partial_0 + g_1 - g_1^2, \quad \partial_1\alpha_0 = f_1 h - g_1 h, \\ &\alpha_1\partial_1 = f_2^2 - f_2, \quad \alpha_2\partial_1 = f_2 g_2 - g_2 f_2, \quad \alpha_3\partial_1 = g_2 - g_2^2, \end{aligned} \tag{III}$$

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$$\begin{array}{rcl}\alpha_1 f_1 & = & f_2\alpha_1, \\ \alpha_1 g_1 + \alpha_2 & = & f_2\alpha_2 + \alpha_2 f_1 + g_2\alpha_1, \\ \alpha_1 h\partial_0 + \alpha_2 + \alpha_3 f_1 & = & f_2\alpha_3 + \alpha_2 g_1 + g_2\alpha_2, \\ \alpha_1 h\partial_0 + \alpha_3 g_1 & = & f_2\alpha_0\partial_0 + \alpha_2 h\partial_0 + g_2\alpha_3, \\ \alpha_1 h + \alpha_0 + \alpha_3 h & = & f_2\alpha_0 + \alpha_2 h + g_2\alpha_0. \end{array} \tag{IV}$$

Condition (I) arrived from the reflexive 2-graph; conditions (II) arrived from the horizontal composition; conditions (III) arrived from the associativity isomorphism and conditions (IV) arrived from coherence condition.

4.7 Identity Isomorphisms

In this section we consider the structure obtained in the previous one and introduce the identity isomorphisms

$$\begin{array}{rcl}\lambda & : & \mu\left\langle ec, 1\right\rangle \longrightarrow 1, \\ \rho & : & \mu\left\langle 1, ed\right\rangle \longrightarrow 1,\end{array}$$

and require them to satisfy the condition (4.4) and (4.5).

The internal functor $\langle ec, 1\rangle$ is obtained from $1 = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$ and $ec = \left(0, \begin{pmatrix} 0 & 0 \\ \partial_0 & 1 \end{pmatrix}\right)$ as in the following diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "0"'] & Y \oplus X \arrow[d, "(\partial_0\ 1)"] \\ X \arrow[r] \arrow[d, "0"'] & X \arrow[d, "\binom{0}{1}"] \\ Z \arrow[r] & Y \oplus X \end{tikzcd}.$$

Its components are $\langle ec, 1\rangle_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : Z \longrightarrow Z \oplus Z$ and $\langle ec, 1\rangle_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} :$ $Y \oplus X \longrightarrow Y \oplus Y \oplus X$ which is obtained from the diagram

$$\begin{tikzcd} Y \oplus X \arrow[r, equal] \arrow[d, "\begin{pmatrix} 0 & 0 \\ \partial_0 & 1 \end{pmatrix}"'] & Y \oplus X \arrow[r, equal] \arrow[d, "\langle ec,1\rangle_1"] & Y \oplus X \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] \\ Y \oplus X & Y \oplus Y \oplus X \arrow[l, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial_0 & 1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & Y \oplus X \end{tikzcd}.$$

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Similarly $\langle 1, ed\rangle_1$ is obtained from

$$\begin{tikzcd}
Y \oplus X \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & Y \oplus X \arrow[r, equal] \arrow[d, "\langle 1, ed\rangle_1"] & Y \oplus X \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\
Y \oplus X & Y \oplus Y \oplus X \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \partial_0 & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] & Y \oplus X
\end{tikzcd}$$

and therefore

$$\langle 1, ed\rangle = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Finally we have

$$\begin{aligned}
\mu \langle ec, 1\rangle &= \left(g_2, \begin{pmatrix} g_1 & h \\ 0 & 1 \end{pmatrix} \right), \\
\mu \langle 1, ed\rangle &= \left(f_2, \begin{pmatrix} f_1 & h \\ 0 & 1 \end{pmatrix} \right).
\end{aligned}$$

This means that the natural transformation λ is given by

$$(\lambda_1\ \lambda_0) : Y \oplus X \longrightarrow Z$$

with

$$\begin{aligned}
\begin{pmatrix} \partial_1 \\ 0 \end{pmatrix} (\lambda_1, \lambda_0) &= \begin{pmatrix} 1 - g_1 & -h \\ 0 & 0 \end{pmatrix}, \\
(\lambda_1, \lambda_0) \begin{pmatrix} \partial_1 \\ 0 \end{pmatrix} &= 1 - g_2.
\end{aligned} \tag{4.14}$$

The natural transformation ρ is given by

$$(\rho_1\ \rho_0) : Y \oplus X \longrightarrow Z$$

with

$$\begin{aligned}
\begin{pmatrix} \partial_1 \\ 0 \end{pmatrix} (\rho_1\ \rho_0) &= \begin{pmatrix} 1 - f_1 & -h \\ 0 & 0 \end{pmatrix}, \\
(\rho_1\ \rho_0) \begin{pmatrix} \partial_1 \\ 0 \end{pmatrix} &= 1 - f_2.
\end{aligned} \tag{4.15}$$

The inclusion of λ and ρ yields four new morphisms $\lambda_1, \rho_1 : Y \longrightarrow Z$, $\lambda_0, \rho_0 : X \longrightarrow Z$ satisfying

$$\begin{aligned}
\partial_1 \lambda_1 &= 1 - g_1\ ,\ \partial_1 \lambda_0 = -h\ ,\ \lambda_1 \partial_1 = 1 - g_2 \\
\partial_1 \rho_1 &= 1 - f_1\ ,\ \partial_1 \rho_0 = -h\ ,\ \rho_1 \partial_1 = 1 - f_2.
\end{aligned}$$

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This four new morphisms also simplify the existent structure since the arrows f_1, f_2, g_1, g_2, h are completely determined by $\partial_1, \rho_1, \lambda_1, \rho_0, \lambda_0$ as follows

$$\begin{aligned} f_1 &= 1 - \partial_1 \rho_1, \\ f_2 &= 1 - \rho_1 \partial_1, \\ g_1 &= 1 - \partial_1 \lambda_1, \\ g_2 &= 1 - \lambda_1 \partial_1, \\ h &= -\partial_1 \rho_0 = -\partial_1 \lambda_0. \end{aligned} \tag{4.16}$$

In order to establish condition (4.4) it is necessary to evaluate $\mu(\rho \times 1)$. From the diagram

$$\begin{tikzcd} Y \oplus X \arrow[d, "(\rho_1\ \rho_0)"'] & Y \oplus Y \oplus X \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \partial_0 & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\rho \times 1"'] & Y \oplus X \arrow[d, "(0\ 0)"] \\ Z & Z \oplus Z \arrow[l] \arrow[r] & Z \end{tikzcd},$$

we conclude that $(\rho \times 1)$ is given by

$$\begin{pmatrix} \rho_1 & \rho_0 \partial_0 & \rho_0 \\ 0 & 0 & 0 \end{pmatrix} : Y^2 \oplus X \longrightarrow Z^2.$$

This means that $\mu(\rho \times 1)$ consist of

$$\begin{pmatrix} f_2 \rho_1 & f_2 \rho_0 \partial_0 & f_2 \rho_0 \end{pmatrix} : Y^2 \oplus X \longrightarrow Z.$$

Using the fact that

$$\langle ec, 1 \rangle = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

the internal functor $(1 \times \langle ec, 1 \rangle)$ is given by

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right),$$

and then $\alpha \circ (1 \times \langle ec, 1 \rangle)$ consist of

$$\alpha \circ (1 \times \langle ec, 1 \rangle) = \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_0 \end{pmatrix} : Y^2 \oplus X \longrightarrow Z.$$

To evaluate $\mu(1 \times \lambda)$, we have that $(1 \times \lambda)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_0 \end{pmatrix} : Y^2 \oplus X \longrightarrow Z^2$$

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and therefore $\mu\,(1\times\lambda)$ consists of

$$\left(\begin{array}{ccc} 0 & g_2\lambda_1 & g_2\lambda_0 \end{array}\right) : Y^2 \oplus X \longrightarrow Z.$$

The condition (4.4) may be written as

$$\left(\begin{array}{ccc} f_2\rho_1 & f_2\rho_0\partial_0 & f_2\rho_0 \end{array}\right) + \left(\begin{array}{ccc} \alpha_1 & \alpha_3 & \alpha_0 \end{array}\right) = \left(\begin{array}{ccc} 0 & g_2\lambda_1 & g_2\lambda_0 \end{array}\right) \qquad (4.17)$$

and the morphisms $\alpha_1, \alpha_3, \alpha_0$ are completely determined by $\partial_1, \rho_1, \lambda_1, \rho_0, \lambda_0$, in the following way

$$\begin{array}{rcl} \alpha_1 & = & -f_2\rho_1 = \rho_1\partial_1\rho_1 - \rho_1, \\ \alpha_3 & = & g_2\lambda_1 - f_2\rho_0\partial_0 = \lambda_1 - \lambda_1\partial_1\lambda_1 - \rho_0\partial_0 + \rho_1\partial_1\rho_0\partial_0, \\ \alpha_0 & = & g_2\lambda_0 - f_2\rho_0 = \lambda_0 - \lambda_1\partial_1\lambda_0 - \rho_0 + \rho_1\partial_1\rho_0. \end{array} \qquad (4.18)$$

We proceed with the analysis of conditions (IV) in Proposition 45.
First condition is trivial

$$\begin{array}{rcl} \alpha_1 f_1 & = & f_2\alpha_1 \Leftrightarrow (\rho_1\partial_1\rho_1 - \rho_1)(1-\partial_1\rho_1) = (1-\rho_1\partial_1)(\rho_1\partial_1\rho_1 - \rho_1) \\ & \Leftrightarrow & \rho_1\partial_1\rho_1 - \rho_1 - \rho_1\partial_1\rho_1\partial_1\rho_1 + \rho_1\partial_1\rho_1 = \rho_1\partial_1\rho_1 - \rho_1 - \rho_1\partial_1\rho_1\partial_1\rho_1 + \rho_1\partial_1\rho_1 \\ & \Leftrightarrow & 0 = 0. \end{array}$$

Second condition is

$$\alpha_1 g_1 + \alpha_2 = f_2\alpha_2 + \alpha_2 f_1 + g_2\alpha_1,$$

after substituting the respective definitions of α's f's and g's we get

$$(\rho_1\partial_1\rho_1 - \rho_1)(1-\partial_1\lambda_1) + \alpha_2 = \alpha_2 - \rho_1\partial_1\alpha_2 + \alpha_2 - \alpha_2\partial_1\rho_1 + (1-\lambda_1\partial_1)(\rho_1\partial_1\rho_1 - \rho_1),$$

that is,

$$\overbrace{\rho_1\partial_1\rho_1 - \rho_1}^{1} - \rho_1\partial_1\rho_1\partial_1\lambda_1 + \rho_1\partial_1\lambda_1 + \alpha_2 =$$

$$= \alpha_2 - \rho_1\partial_1\alpha_2 + \alpha_2 - \alpha_2\partial_1\rho_1 + \overbrace{\rho_1\partial_1\rho_1 - \rho_1}^{1} - \lambda_1\partial_1\rho_1\partial_1\rho_1 + \lambda_1\partial_1\rho_1,$$

simplifying, it becomes

$$\begin{array}{c} -\rho_1\partial_1\rho_1\partial_1\lambda_1 + \rho_1\partial_1\lambda_1 = \\ = -\rho_1\partial_1\alpha_2 + \alpha_2 - \alpha_2\partial_1\rho_1 - \lambda_1\partial_1\rho_1\partial_1\rho_1 + \lambda_1\partial_1\rho_1. \end{array}$$

From condition (III) we know that $\partial_1\alpha_2 = f_1g_{1-}g_1f_1$ and $\alpha_2\partial_1 = f_2g_2 - g_2f_2$, which gives

$$\begin{array}{c} -\rho_1\partial_1\rho_1\partial_1\lambda_1 + \rho_1\partial_1\lambda_1 = \\ = -\rho_1(f_1g_{1-}g_1f_1) + \alpha_2 - (f_2g_2 - g_2f_2)\rho_1 - \lambda_1\partial_1\rho_1\partial_1\rho_1 + \lambda_1\partial_1\rho_1 \end{array}$$

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further we obtain

$$\begin{gathered}-\rho_1\partial_1\rho_1\partial_1\lambda_1+\rho_1\partial_1\lambda_1= \\ =-\rho_1\left(\partial_1\rho_1\partial_1\lambda_1-\partial_1\lambda_1\partial_1\rho_1\right)+\alpha_2-\left(\rho_1\partial_1\lambda_1\partial_1-\lambda_1\partial_1\rho_1\partial_1\right)\rho_1-\lambda_1\partial_1\rho_1\partial_1\rho_1+\lambda_1\partial_1\rho_1,\end{gathered}$$

which becomes

$$\begin{gathered}\overbrace{-\rho_1\partial_1\rho_1\partial_1\lambda_1}^{1}+\rho_1\partial_1\lambda_1= \\ =\overbrace{-\rho_1\partial_1\rho_1\partial_1\lambda_1}^{1}+\overbrace{\rho_1\partial_1\lambda_1\partial_1\rho_1}^{2}+\alpha_2-\overbrace{\rho_1\partial_1\lambda_1\partial_1\rho_1}^{2}+\lambda_1\partial_1\rho_1\partial_1\rho_1-\lambda_1\partial_1\rho_1\partial_1\rho_1+\lambda_1\partial_1\rho_1\end{gathered}$$

and finally it simplifies to

$$\alpha_2=\rho_1\partial_1\lambda_1-\lambda_1\partial_1\rho_1. \tag{4.19}$$

The associativity isomorphism α is completely determined.

The third condition in (IV) is

$$\alpha_1 h\partial_0+\alpha_2+\alpha_3 f_1=f_2\alpha_3+\alpha_2 g_1+g_2\alpha_2,$$

then it becomes

$$\begin{gathered}\left(\rho_1\partial_1\rho_1-\rho_1\right)h\partial_0+\rho_1\partial_1\lambda_1-\lambda_1\partial_1\rho_1+\left(\lambda_1-\lambda_1\partial_1\lambda_1-\rho_0\partial_0+\rho_1\partial_1\rho_0\partial_0\right)\left(1-\partial_1\rho_1\right)= \\ =\left(1-\rho_1\partial_1\right)\left(\lambda_1-\lambda_1\partial_1\lambda_1-\rho_0\partial_0+\rho_1\partial_1\rho_0\partial_0\right)+ \\ +\left(\rho_1\partial_1\lambda_1-\lambda_1\partial_1\rho_1\right)\left(1-\partial_1\lambda_1\right)+\left(1-\lambda_1\partial_1\right)\left(\rho_1\partial_1\lambda_1-\lambda_1\partial_1\rho_1\right),\end{gathered}$$

going further we have

$$\begin{gathered}\rho_1\partial_1\rho_1 h\partial_0-\rho_1 h\partial_0+\overbrace{\rho_1\partial_1\lambda_1-\lambda_1\partial_1\rho_1}^{1}+\overbrace{\lambda_1-\lambda_1\partial_1\lambda_1-\rho_0\partial_0+\rho_1\partial_1\rho_0\partial_0}^{2}- \\ -\overbrace{\lambda_1\partial_1\rho_1}^{3}+\overbrace{\lambda_1\partial_1\lambda_1\partial_1\rho_1}^{4}+\rho_0\partial_0\partial_1\rho_1-\rho_1\partial_1\rho_0\partial_0\partial_1\rho_1= \\ =\overbrace{\lambda_1-\lambda_1\partial_1\lambda_1-\rho_0\partial_0+\rho_1\partial_1\rho_0\partial_0}^{2}+ \\ -\overbrace{\rho_1\partial_1\lambda_1}^{5}+\overbrace{\rho_1\partial_1\lambda_1\partial_1\lambda_1}^{6}+\rho_1\partial_1\rho_0\partial_0-\rho_1\partial_1\rho_1\partial_1\rho_0\partial_0 \\ +\overbrace{\rho_1\partial_1\lambda_1-\lambda_1\partial_1\rho_1}^{1}-\overbrace{\rho_1\partial_1\lambda_1\partial_1\lambda_1}^{6}+\overbrace{\lambda_1\partial_1\rho_1\partial_1\lambda_1}^{7} \\ +\overbrace{\rho_1\partial_1\lambda_1}^{5}-\overbrace{\lambda_1\partial_1\rho_1}^{3}-\overbrace{\lambda_1\partial_1\rho_1\partial_1\lambda_1}^{7}+\overbrace{\lambda_1\partial_1\lambda_1\partial_1\rho_1}^{4}\end{gathered}$$

which may be simplified to

$$\rho_1\partial_1\rho_1 h\partial_0-\rho_1 h\partial_0=\rho_1\partial_1\rho_0\partial_0-\rho_1\partial_1\rho_1\partial_1\rho_0\partial_0$$

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and it is a trivial equation since $\partial_0\partial_1 = 0$ and $-h = \partial_1\rho_0$.

The fourth condition in (IV) is

$$\alpha_1 h\partial_0 + \alpha_3 g_1 = f_2\alpha_0\partial_0 + \alpha_2 h\partial_0 + g_2\alpha_3.$$

First it becomes

$$\begin{gathered}(\rho_1\partial_1\rho_1 - \rho_1)\, h\partial_0 + (\lambda_1 - \lambda_1\partial_1\lambda_1 - \rho_0\partial_0 + \rho_1\partial_1\rho_0\partial_0)\,(1 - \partial_1\lambda_1) = \\ = (1 - \rho_1\partial_1)\,(\lambda_0 - \lambda_1\partial_1\lambda_0 - \rho_0 + \rho_1\partial_1\rho_0)\,\partial_0 + (\rho_1\partial_1\lambda_1 - \lambda_1\partial_1\rho_1)\, h\partial_0 + \\ + (1 - \lambda_1\partial_1)\,(\lambda_1 - \lambda_1\partial_1\lambda_1 - \rho_0\partial_0 + \rho_1\partial_1\rho_0\partial_0)\end{gathered}$$

and then

$$\begin{gathered}\rho_1\partial_1\rho_1 h\partial_0 - \rho_1 h\partial_0 + \overbrace{\lambda_1 - \lambda_1\partial_1\lambda_1 - \rho_0\partial_0 + \rho_1\partial_1\rho_0\partial_0}^{1} - \\ - \overbrace{\lambda_1\partial_1\lambda_1 + \lambda_1\partial_1\lambda_1\partial_1\lambda_1}^{2} + \rho_0\partial_0\partial_1\lambda_1 - \rho_1\partial_1\rho_0\partial_0\partial_1\lambda_1 = \\ = \lambda_0\partial_0 - \lambda_1\partial_1\lambda_0\partial_0 - \rho_0\partial_0 + \rho_1\partial_1\rho_0\partial_0 - \\ - \rho_1\partial_1\lambda_0\partial_0 + \rho_1\partial_1\lambda_1\partial_1\lambda_0\partial_0 + \rho_1\partial_1\rho_0\partial_0 - \rho_1\partial_1\rho_1\partial_1\rho_0\partial_0 \\ + \rho_1\partial_1\lambda_1 h\partial_0 - \lambda_1\partial_1\rho_1 h\partial_0 + \overbrace{\lambda_1 - \lambda_1\partial_1\lambda_1 - \rho_0\partial_0 + \rho_1\partial_1\rho_0\partial_0}^{1} - \\ - \overbrace{\lambda_1\partial_1\lambda_1 + \lambda_1\partial_1\lambda_1\partial_1\lambda_1}^{2} + \lambda_1\partial_1\rho_0\partial_0 - \lambda_1\partial_1\rho_1\partial_1\rho_0\partial_0.\end{gathered}$$

Having in mind that $\partial_0\partial_1 = 0$ and $\partial_1\rho_0 = -h = \partial_1\lambda_0$ it can be simplified to

$$\begin{gathered}\overbrace{\rho_1\partial_1\rho_1 h\partial_0 - \rho_1 h\partial_0}^{1} = \lambda_0\partial_0 + \overbrace{\lambda_1 h\partial_0}^{2} - \rho_0\partial_0 - \rho_1 h\partial_0 + \rho_1 h\partial_0 - \overbrace{\rho_1\partial_1\lambda_1 h\partial_0}^{3} - \\ - \overbrace{\rho_1 h\partial_0 + \rho_1\partial_1\rho_1 h\partial_0}^{1} + \overbrace{\rho_1\partial_1\lambda_1 h\partial_0}^{3} - \lambda_1\partial_1\rho_1 h\partial_0 - \overbrace{\lambda_1 h\partial_0}^{2} + \lambda_1\partial_1\rho_1 h\partial_0,\end{gathered}$$

which results in

$$\lambda_0\partial_0 = \rho_0\partial_0.$$

The fifth condition in (IV) is

$$\alpha_1 h + \alpha_0 + \alpha_3 h = f_2\alpha_0 + \alpha_2 h + g_2\alpha_0,$$

then it becomes

$$\begin{gathered}(\rho_1\partial_1\rho_1 - \rho_1)\, h + \alpha_0 + (\lambda_1 - \lambda_1\partial_1\lambda_1 - \rho_0\partial_0 + \rho_1\partial_1\rho_0\partial_0)\, h = \\ = (1 - \rho_1\partial_1)\,\alpha_0 + (\rho_1\partial_1\lambda_1 - \lambda_1\partial_1\rho_1)\, h + (1 - \lambda_1\partial_1)\,\alpha_0,\end{gathered}$$

that is,

$$\begin{gathered}\rho_1\partial_1\rho_1 h - \rho_1 h + \alpha_0 + \lambda_1 h - \lambda_1\partial_1\lambda_1 h - \rho_0\partial_0 h + \rho_1\partial_1\rho_0\partial_0 h = \\ = \alpha_0 - \rho_1\partial_1\alpha_0 + \rho_1\partial_1\lambda_1 h - \lambda_1\partial_1\rho_1 h + \alpha_0 - \lambda_1\partial_1\alpha_0.\end{gathered}$$

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Since we have $\partial_0 h = 0$ and $\alpha_0 = \lambda_0 - \lambda_1\partial_1\lambda_0 - \rho_0 + \rho_1\partial_1\rho_0$ it gives

$$\begin{gathered}\rho_1\partial_1\rho_1 h - \rho_1 h + \lambda_1 h - \lambda_1\partial_1\lambda_1 h = \\ = (\lambda_0 - \lambda_1\partial_1\lambda_0 - \rho_0 + \rho_1\partial_1\rho_0) - \\ -\rho_1\partial_1(\lambda_0 - \lambda_1\partial_1\lambda_0 - \rho_0 + \rho_1\partial_1\rho_0) + \rho_1\partial_1\lambda_1 h - \\ -\lambda_1\partial_1\rho_1 h - \lambda_1\partial_1(\lambda_0 - \lambda_1\partial_1\lambda_0 - \rho_0 + \rho_1\partial_1\rho_0),\end{gathered}$$

which simplifies to

$$\begin{gathered}\rho_1\partial_1\rho_1 h - \rho_1 h + \lambda_1 h - \lambda_1\partial_1\lambda_1 h = \\ = \lambda_0 - \lambda_1\partial_1\lambda_0 - \rho_0 + \rho_1\partial_1\rho_0 - \\ -\rho_1\partial_1\lambda_0 + \rho_1\partial_1\lambda_1\partial_1\lambda_0 + \rho_1\partial_1\rho_0 - \rho_1\partial_1\rho_1\partial_1\rho_0 + \rho_1\partial_1\lambda_1 h - \\ -\lambda_1\partial_1\rho_1 h - \lambda_1\partial_1\lambda_0 + \lambda_1\partial_1\lambda_1\partial_1\lambda_0 + \lambda_1\partial_1\rho_0 - \lambda_1\partial_1\rho_1\partial_1\rho_0.\end{gathered}$$

Having in mind that $\partial_1\rho_0 = -h = \partial_1\lambda_0$ we end up with

$$\begin{gathered}\overbrace{\rho_1\partial_1\rho_1 h}^{1} - \rho_1 h + \lambda_1 h - \overbrace{\lambda_1\partial_1\lambda_1 h}^{2} = \lambda_0 + \lambda_1 h - \rho_0 - \rho_1 h + \rho_1 h - \overbrace{\rho_1\partial_1\lambda_1 h}^{3} - \rho_1 h + \\ + \overbrace{\rho_1\partial_1\rho_1 h}^{1} + \overbrace{\rho_1\partial_1\lambda_1 h}^{3} - \lambda_1\partial_1\rho_1 h + \lambda_1 h - \overbrace{\lambda_1\partial_1\lambda_1 h}^{2} - \lambda_1 h + \lambda_1\partial_1\rho_1 h\end{gathered}$$

and finally we have

$$\lambda_0 = \rho_0.$$

With this last condition, $\lambda_0\partial_0 = \rho_0\partial_0$ is also trivial and condition (4.5) is satisfied.

It is now a straightforward calculation to check that,with the definitions given in (4.16), (4.18) and (4.19), all equations in (II) and (III) of Proposition 45 are satisfied.

We have seen that the inclusion of the morphisms $\lambda_1, \rho_1, \lambda_0, \rho_0$ with respect to the identity isomorphisms, satisfying condition (4.4) is sufficient to determined all the other morphisms that came from the horizontal composition and the associativity isomorphism.

In the process of describing the associativity isomorphism α we saw that condition (4.4) is sufficient to determine α_1, α_3 and α_0, remaining unknown only α_2. The morphism α_2 was determined by condition (4.3) . This means that we are allowed to replace condition (4.3) by a weaker one that still determines α_2. It is the case of condition

$$\begin{tikzcd}
1_B * (f * 1_A) \arrow[rr, "\alpha"] \arrow[d, "1_B*\rho"'] & & (1_B * f) * 1_A \arrow[d, "\lambda*1_A"] \\
1_B * f \arrow[r, "\lambda"] & f & f * 1_A \arrow[l, "\rho"']
\end{tikzcd}$$

that is written internally as

$$\begin{tikzcd}
\begin{array}{c}\mu(1\times\mu)\langle ec,(1,ed)\rangle \\ = \mu\langle ec,\mu\langle 1,ed\rangle\rangle\end{array} \arrow[rr, "\alpha\circ\langle ec,1,ed\rangle"] \arrow[d, "\mu\circ\langle 1,\rho\rangle"'] & & \begin{array}{c}\mu(\mu\times 1)\langle (ec,1),ed\rangle \\ = \mu\langle\mu\langle ec,1\rangle,ed\rangle\end{array} \arrow[d, "\mu\circ\langle\lambda,1\rangle"] \\
\mu\langle ec,1\rangle \arrow[r, "\lambda"] & 1 & \mu\langle 1,ed\rangle \arrow[l, "\rho"']
\end{tikzcd},$$

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or simply as

$$\rho \cdot (\mu \circ \langle \lambda, 1\rangle) \cdot (\alpha \circ \langle ec, 1, ed\rangle) = \lambda \cdot (\mu \circ \langle 1, \rho\rangle)\,. \tag{4.20}$$

Considering the above condition in the place of (4.3) and analyze it in Ab, we would obtain the following.

The morphism $\langle ec, 1, ed\rangle$ is given by

$$\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

whereas $\langle \lambda, 1\rangle$ and $\langle 1, \rho\rangle$ are

$$\begin{aligned} \langle \lambda, 1\rangle &= \left(\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda_0 \\ 0 \end{pmatrix} \right), \\ \langle 1, \rho\rangle &= \left(\begin{pmatrix} 0 \\ \rho_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \rho_0 \end{pmatrix} \right). \end{aligned}$$

Using the horizontal composition of internal natural transformations in Ab we obtain

$$\begin{aligned} \mu \circ \langle \lambda, 1\rangle &= \left(\mu_2 \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \mu_2 \begin{pmatrix} \lambda_0 \\ 0 \end{pmatrix} \right) = (f_2\lambda_1, f_2\lambda_0)\,, \\ \mu \circ \langle 1, \rho\rangle &= \left(\mu_2 \begin{pmatrix} 0 \\ \rho_1 \end{pmatrix}, \mu_2 \begin{pmatrix} 0 \\ \rho_0 \end{pmatrix} \right) = (g_2\rho_1, g_2\rho_0)\,, \\ \alpha \circ \langle ec, 1, ed\rangle &= \left((\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_0 \right) = (\alpha_2, \alpha_0)\,. \end{aligned}$$

Equation (4.20) can be written as

$$(\rho_1, \rho_2) + (f_2\lambda_1, f_2\lambda_0) + (\alpha_2, \alpha_0) = (\lambda_1, \lambda_0) + (g_2\rho_1, g_2\rho_0)$$

that is

$$\begin{aligned} \alpha_2 &= \lambda_1 - f_2\lambda_1 + g_2\rho_1 - \rho_1, \\ \alpha_0 &= \lambda_0 - f_2\lambda_0 + g_2\rho_0 - \rho_0. \end{aligned}$$

Using the expressions for f_2 and g_2 as in (4.16) we obtain the same result for α_2 as in (4.3), that is

$$\begin{aligned} \alpha_2 &= \rho_1\partial_1\lambda_1 - \lambda_1\partial_1\rho_1, \\ \alpha_0 &= (\rho_1\partial_1 - \lambda_1\partial_1)\, q. \end{aligned}$$

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4.8 Conclusion

The previous sections suggest the following description of internal bicategories in the category of abelian groups.

An internal bicategory in Ab is completely determined by a 2-complex

$$Z \xrightarrow{\partial_1} Y \xrightarrow{\partial_0} X \qquad , \qquad \partial_0\partial_1 = 0,$$

and three morphisms

$$\begin{aligned} \lambda_1, \rho_1 &\;:\; Y \longrightarrow Z \\ (\lambda_0 = \rho_0) &\;=\; q : X \longrightarrow Z. \end{aligned}$$

It is given up to an isomorphism by the diagram

$$\begin{tikzcd} Z \oplus Z \oplus Y \oplus X \arrow[r, "m"] & Z \oplus Y \oplus X \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "c_1"] & Y \oplus X \arrow[l, "e_1"] \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "c_0"] & X, \arrow[l, "e_0"] \end{tikzcd}$$

where

$$d_0 = \begin{pmatrix} 0 & 1 \end{pmatrix} \;,\; e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;,\; c_0 = \begin{pmatrix} \partial_0 & 1 \end{pmatrix},$$

$$d_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;,\; e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \;,\; c_1 = \begin{pmatrix} \partial_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$m = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

the horizontal composition is given by

$$\begin{aligned} \mu_1 &= \begin{pmatrix} f_1 & g_1 & h \\ 0 & 0 & 1 \end{pmatrix} : Y^2 \oplus X \longrightarrow Y \oplus X, \\ \mu_2 &= \begin{pmatrix} f_2 & g_2 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix} : Z^2 \oplus Y^2 \oplus X \longrightarrow Z \oplus Y \oplus X, \end{aligned}$$

whereas the natural isomorphisms are given by

$$\begin{aligned} \alpha &= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ f_1 & g_1 f_1 & g_1^2 & h + g_1 h \\ 0 & 0 & 0 & 1 \end{pmatrix} : Y^3 \oplus X \longrightarrow Z \oplus Y \oplus X, \\ \lambda &= \begin{pmatrix} \lambda_1 & q \\ g_1 & h \\ 0 & 1 \end{pmatrix} : Y \oplus X \longrightarrow Z \oplus Y \oplus X, \\ \rho &= \begin{pmatrix} \rho_1 & q \\ f_1 & h \\ 0 & 1 \end{pmatrix} : Y \oplus X \longrightarrow Z \oplus Y \oplus X, \end{aligned}$$

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with

$$\begin{aligned}
f_1 &= 1-\partial_1\rho_1, \qquad f_2 = 1-\rho_1\partial_1\\
g_1 &= 1-\partial_1\lambda_1, \qquad g_2 = 1-\lambda_1\partial_1\\
h &= -\partial_1\lambda_0 = -\partial_1\rho_0\\
\alpha_1 &= \rho_1\partial_1\rho_1-\rho_1\\
\alpha_2 &= \rho_1\partial_1\lambda_1-\lambda_1\partial_1\rho_1\\
\alpha_3 &= \lambda_1-\lambda_1\partial_1\lambda_1-q\partial_0+\rho_1\partial_1 q\partial_0\\
\alpha_0 &= \rho_1\partial_1\rho_0-\lambda_1\partial_1\lambda_0.
\end{aligned}$$

This result also generalizes in a straightforward way to internal bicategories in additive categories with kernels.

The reader interested in internal structures in Ab may also see Crans work [2] and a more general notion of bicategory that will appear in a paper of the Fields Workshop on Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories.

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Chapter 5

Pseudocategories in additive 2-categories with kernels

Througout this chapter, weak category is to be understood as pseudocategory.

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Title:Weak Categories In Additive 2-Categories With Kernels

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Weak category, additive 2-category, 2-V-category, internal bicategory, weak double category.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Abstract: We introduce a notion of weak category, define additive 2-categories and describe weak categories in them. We make this description more explicit in the case of the additive 2-category of morphisms of abelian groups. In particular we present internal bicategories in the category of abelian groups as presheaf categories.

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5.1 Introduction

Consider the notion of monoidal category as an internal structure in (Cat, $\times$, 1). More generally consider it in an abstract 2-category, not necessarily Cat. We denote this new notion by *weak monoid.* Table 1 describes the notion of monoid and weak monoid (for the cases where it is applicable) in some concrete examples of ambient categories.

Ambient Category	Monoids	Weak Monoids
Set	ordinary monoids	N/A
$\mathcal{O}$-graphs(Set)	ordinary categories (objects are the elements of $\mathcal{O}$)	N/A
Cat	strict monoidal categories	monoidal categories
$\mathcal{O}$-graphs(Cat)	double categories (the vertical structure is $\mathcal{O}$)	*Weak Categories*
$\mathcal{O}$-graphs(Cat) ($\mathcal{O}$ discrete)	2-categories (objects are the elements of $\mathcal{O}$)	bicategories (objects are the elements of $\mathcal{O}$)

Table 1

The term *weak category* appears in this way as a generalization of double category and bicategory.

In this work, after giving explicit description of weak monoids and weak categories, we analyze internal weak categories in additive 2-categories with kernels. We also introduce the notion of additive 2-category by defining 2-Ab-category and more generally 2-V-category where V is a monoidal category.

An example of additive 2-category with kernels is Mor(Ab) and we show that a weak category in Mor(Ab) is completely determined by four abelian groups A_1, A_0, B_1, B_0, together with four group homomorphisms $\partial, \partial', k_1, k_0$, such that the following square is commutative

$$\begin{tikzcd}
A_1 \arrow[r, "\partial"] \arrow[d, "k_1"'] & A_0 \arrow[d, "k_0"] \\
B_1 \arrow[r, "\partial'"] & B_0
\end{tikzcd}$$

and three more group homomorphisms

$$\lambda, \rho : A_0 \longrightarrow A_1 \ , \ \eta : B_0 \longrightarrow A_1,$$

such that

$$\begin{aligned}
k_1\lambda &= 0 = k_1\rho, \\
k_1\eta &= 0.
\end{aligned}$$

This result generalizes at the same time the description of internal double categories and of internal bicategories in Ab. Let the morphisms λ, ρ, η become zero morphisms; then we obtain the known description of internal double categories in Ab (for a similar description of (strict) n-categories see e.g. [6],[2],[7] and references there). Taking instead B_1 to be the trivial group, and obtain the description of internal bicategories in Ab [3].

5.2 Weak Categories

This section begins with the formal definition of weak monoid and shows how the notion of weak category can be regarded as a weak monoid. Nevertheless,

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to consider the category of all weak categories (see [4]), an explicit definition of weak category is required. Last part of the section gives an explicit definition of weak category.

Weak Monoids

An ordinary monoid in a monoidal category $(M, \Box, \mathbf{1})$ is a diagram

$$C\Box C \xrightarrow{m} C \xleftarrow{e} \mathbf{1}$$

(see [1]) in M such that the following diagrams are commutative

$$\begin{tikzcd} C\Box C\Box C \arrow[r, "1\Box m"] \arrow[d, "m\Box 1"'] & C\Box C \arrow[d, "m"] \\ C\Box C \arrow[r, "m"] & C \end{tikzcd} , \tag{5.1}$$

$$\begin{tikzcd} C \arrow[r, "e\Box 1"] \arrow[dr] & C\Box C \arrow[d, "m"] & C \arrow[l, "1\Box e"'] \arrow[dl] \\ & C & \end{tikzcd} . \tag{5.2}$$

In the monoidal category $(\mathrm{Cat}, \times, 1)$, the monoids are precisely the strict monoidal categories, whereas in the monoidal category ($\mathcal{O}$-Graphs, $\times_{\mathcal{O}}, \mathcal{O} = \mathcal{O}$) the monoids are all categories with the fixed set $\mathcal{O}$ of objects.

Consider a monoidal category $(M, \Box, \mathbf{1})$, in which M is a 2-category and $\Box$ is a 2-bifunctor. Replacing the commutativity of the diagrams (5.1) and (5.2) by the existence of suitable 2-cells satisfying the usual coherence conditions for monoidal categories (see [1]), we obtain the notion of *weak monoid.*

Definition 46 *A weak monoid in a monoidal category* $(M, \Box, \mathbf{1})$ *(where* M *is a 2-category and* $\Box$ *is a 2-bifunctor) is a diagram of the form*

$$C\Box C \xrightarrow{m} C \xleftarrow{e} \mathbf{1}$$

together with 2-cells

$$\begin{aligned} \alpha &: m(1\Box m) \longrightarrow m(m\Box 1), \\ \lambda &: m(1\Box e) \longrightarrow 1, \\ \rho &: m(e\Box 1) \longrightarrow 1, \end{aligned}$$

that are isomorphisms satisfying the identity

$$\lambda \circ e = \rho \circ e,$$

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and the commutativity of the following diagrams[1]

$$\begin{tikzcd}
 & \cdot \arrow[rr, "m\circ(1\square\alpha)"] \arrow[dl, "\alpha\circ(1\square1\square m)"'] & & \cdot \arrow[dr, "\alpha\circ(1\square m\square1)"] & \\
\cdot \arrow[drr, "\alpha\circ(m\square1\square1)"'] & & & & \cdot \arrow[dll, "m\circ(\alpha\square1)"] \\
 & & \cdot & &
\end{tikzcd} , \tag{5.3}$$

$$\begin{tikzcd}
\cdot \arrow[r, "\alpha\circ(1\square e\square1)"] \arrow[dr, "m\circ(1\square\lambda)"'] & \cdot \arrow[d, "m\circ(\rho\square1)"] \\
 & \cdot
\end{tikzcd} . \tag{5.4}$$

In the monoidal category (Cat,$\times$, 1) a weak monoid is precisely a monoidal category (not necessarily strict). A weak category (see Table 1) is obtained as a particular case of a weak monoid by considering a weak monoid in the monoidal category of ($\mathcal{O}$-Graphs(Cat), $\times_{\mathcal{O}}, \mathcal{O} = \mathcal{O}$) where $\mathcal{O}$-Graphs(Cat) is the category of internal $\mathcal{O}$-graphs in Cat. In this case we have the notion of weak monoid written as

$$\begin{tikzcd}
C\times_{\mathcal{O}} C \arrow[r, "m"] \arrow[d, shift right, "c\pi_1"'] \arrow[d, shift left, "d\pi_2"] & C \arrow[d, shift right, "c"'] \arrow[d, shift left, "d"] & \mathcal{O} \arrow[l, "e"'] \arrow[d, equal] \\
\mathcal{O} \arrow[r, equal] & \mathcal{O} \arrow[r, equal] & \mathcal{O}
\end{tikzcd}$$

with

$$\begin{aligned} de &= 1_{\mathcal{O}} = ce, \\ dm &= d\pi_2 \ , \ cm = c\pi_1, \end{aligned}$$

and the commutativity of the diagrams for associativity and identities replaced by natural isomorphisms α, λ and ρ satisfying the usual coherence conditions. If $\mathbb{X}$ is a 2-category, then a weak monoid in ($\mathcal{O}$-Graphs($\mathbb{X}$), $\times_{\mathcal{O}}, \mathcal{O} = \mathcal{O}$) is a weak category in $\mathbb{X}$.

Weak Categories

For simplicity we introduce the notion of weak category in several steps. First we define the notion of *precategory*, which is just an internal reflexive graph with

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[1] The reader not familiar with internal constructions may think as if the object C had elements a, b, c, d, and write ab for $m(a, b)$ in order to obtain

$$\begin{tikzcd}
 & a(b(cd)) \arrow[rr, "a\alpha"] \arrow[dl, "\alpha_{a,b,cd}"'] & & a((bc)d) \arrow[dr, "\alpha_{a,bc,d}"] & \\
(ab)(cd) \arrow[drr, "\alpha_{ab,c,d}"'] & & & & (a(bc))d \arrow[dll, "\alpha d"] \\
 & & ((ab)c)d & &
\end{tikzcd} \quad , \quad
\begin{tikzcd}
a(eb) \arrow[r, "\alpha"] \arrow[dr, "a\lambda"'] & (ae)b \arrow[d, "\rho b"] \\
 & ab
\end{tikzcd} .$$

composition. Next we define precategory with associativity (up to isomorphism) and call it *associative precategory*. Afterwards we define *associative precategory with identity*, an associative precategory with (up to isomorphism) left and right identities.

With respect to coherence conditions we specify the usual *pentagon* and *triangle* (which generalize 5.3 and 5.4) but also consider an intermediate coherence condition (that we call *mixed* coherence condition). The mixed coherence condition is important since (in an additive 2-category with kernels) an associative precategory with identity, satisfying the triangle and the mixed coherence conditions, completely determines the structure of weak category.

Finally we will define the notion of weak category by saying that it is an associative precategory with identity satisfying the pentagon and the triangle coherence conditions.

Definition 47 *An internal* precategory *in a category $\mathbb{C}$ is a diagram in $\mathbb{C}$ of the form*

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

with

$$de = 1_{c_1} = ce, \tag{5.5}$$

$$dm = d\pi_2 \text{ , } cm = c\pi_1 \tag{5.6}$$

and where $C_1 \times_{C_0} C_1$ is defined via the pullback diagram

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd}.$$

Definition 48 *An internal* associative precategory*, in a 2-category $\mathbb{C}$, is a system*

$$(C_0, C_1, m, d, e, c, \alpha),$$

where (C_0, C_1, m, d, e, c) is a precategory, (internal to $\mathbb{C}$) and

$$\alpha : m(1 \times_{C_0} m) \longrightarrow m(m \times_{C_0} 1)$$

is an isomorphism with

$$d \circ \alpha = 1_{d\pi_3}, c \circ \alpha = 1_{c\pi_1}. \tag{5.7}$$

Definition 49 *An internal* associative precategory with identity*, in a 2-category $\mathbb{C}$, is a system*

$$(C_0, C_1, m, d, e, c, \alpha, \lambda, \rho),$$

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where $(C_0, C_1, m, d, e, c, \alpha)$ *is an associative precategory, and*

$$\lambda : m\langle ec, 1\rangle \longrightarrow 1_{C_1}\ ,\ \ \rho : m\langle 1, ed\rangle \longrightarrow 1_{C_1}$$

are isomorphisms with

$$\begin{aligned} d\circ\lambda &= 1_d = d\circ\rho, \\ c\circ\lambda &= 1_c = c\circ\rho, \\ \lambda\circ e &= \rho\circ e. \end{aligned} \tag{5.8}$$

Definition 50 *An internal* associative precategory with coherent identity*, in a 2-category* $\mathbb{C}$*, is a system*

$$(C_0, C_1, m, d, e, c, \alpha, \lambda, \rho)\,,$$

forming an associative precategory with identity and satisfying the triangle *and the* mixed *coherence conditions*

$$(m\circ(\rho\times 1))\cdot(\alpha\circ(1\times\langle ec, 1\rangle)) = m\circ(1\times\lambda)\,, \tag{5.9}$$

$$\rho\cdot(m\circ\langle\lambda, 1_{ed}\rangle)\cdot(\alpha\circ\langle ec, 1, ed\rangle) = \lambda\cdot(m\circ\langle 1_{ec}, \rho\rangle)\,. \tag{5.10}$$

Definition 51 *An internal* weak category *in the 2-category* $\mathbb{C}$ *is a system*

$$(C_0, C_1, m, d, e, c, \alpha, \lambda, \rho)\,,$$

forming an associative precategory with identity and satisfying the triangle *and the* pentagon *coherence conditions*

$$(m\circ(\rho\times 1))\cdot(\alpha\circ(1\times\langle ec, 1\rangle)) = m\circ(1\times\lambda)\,,$$

$$(\alpha\circ(m\times 1\times 1))\cdot(\alpha\circ(1\times 1\times m)) = (m\circ(\alpha\times 1))\cdot(\alpha\circ(1\times m\times 1))\cdot(m\circ(1\times\alpha))\,. \tag{5.11}$$

If the 2-cells α, λ, ρ were identities, then this would become nothing but the definition of internal category in $\mathbb{C}$. On the other hand, if we let the object C_0 be terminal, then the notion of internal monoidal category is obtained.

In the case where $\mathbb{C}$ is Cat, if the 2-cells are identities we get the definition of a double category; if the category C_0 is discrete (has only objects and the identity morphism for each object) then the definition of bicategory is obtained. More generally, if $\mathbb{C}$ is the category of internal categories in some category $\mathbb{X}$, i.e., $\mathbb{C}$ =Cat($\mathbb{X}$) then we obtain the definition of double category in $\mathbb{X}$ on the one hand, and the definition of internal bicategory in $\mathbb{X}$ on the other hand.

In what follows, after defining additive 2-categories, we will give a complete description of the above structures inside (=internal to) them.

5.3 Additive 2-categories

In order to define additive 2-category we need the notion of 2-Ab-category. To do so, we give the general notion of a 2-V-category, a 2-category enriched in a monoidal category V.

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2-V-Categories

Let $\mathbb{V} = (\mathbb{V}, \Box, 1)$ be a monoidal category and O a fixed set of objects. A $\mathbb{V}$-category over the set of objects O is given by a system

$$(H, \mu, \varepsilon)$$

where H is a family of objects[2] of $\mathbb{V}$,

$$H = (H\,(A,B) \in \mathbb{V})_{A,B \in O}\,,$$

μ is a family of morphisms of $\mathbb{V}$

$$\mu = \left(\mu_{A,B,C} : H\,(A,B)\,\Box H\,(B,C) \longrightarrow H\,(A,C)\right)_{A,B,C \in O}$$

and ε is another family of morphisms of $\mathbb{V}$

$$\varepsilon = (\varepsilon_A : 1 \longrightarrow H\,(A,A))_{A \in O}\,,$$

such that for every $A, B, C, D \in O$, the following diagrams commute

$$\begin{tikzcd}
H\,(A,B)\,\Box H\,(B,C)\,\Box H\,(C,D) \arrow[r, "1_{H(A,B)}\Box\mu_{B,C,D}"] \arrow[d, "\mu_{A,B,C}\Box 1_{H(C,D)}"'] & H\,(A,B)\,\Box H\,(B,D) \arrow[d, "\mu_{A,B,D}"] \\
H\,(A,C)\,\Box H\,(C,D) \arrow[r, "\mu_{A,C,D}"] & H\,(A,D)
\end{tikzcd} \quad , \qquad (5.12)$$

$$\begin{tikzcd}
H(A,B) \arrow[r, "\epsilon_A\Box 1"] \arrow[dr] & H(A,A)\Box H(A,B) \arrow[d, "\mu_{A,A,B}"] \,, & H(A,B)\Box H(B,B) \arrow[d, "\mu_{A,B,B}"'] & H(A,B) \arrow[l, "1\Box\epsilon_B"'] \arrow[dl] \,. \\
 & H(A,B) & H(A,B) &
\end{tikzcd}$$

$$(5.13)$$

A morphism φ between two $\mathbb{V}$-categories over the set of objects O

$$(H, \mu, \varepsilon) \xrightarrow{\varphi} (H', \mu', \varepsilon')$$

is a family of morphisms of $\mathbb{V}$

$$\varphi = \left(\varphi_{A,B} : H\,(A,B) \longrightarrow H'\,(A,B)\right)_{A,B \in O}$$

such that for every $A, B, C \in O$ the following diagrams are commutative

$$\begin{tikzcd}
H(A,A) \arrow[r, "\varphi_{A,A}"] & H'(A,A)\,, \\
1 \arrow[u, "\epsilon_A"] \arrow[ur, "\epsilon'_A"'] &
\end{tikzcd} \qquad (5.14)$$

---

[2] The object $H(A,B) \in \mathbb{V}$ represents $\hom(A,B)$ of the $\mathbb{V}$-category that is being defined.

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$$\begin{tikzcd}
H(A,B)\,\square H(B,C) \arrow[r, "\varphi_{A,B}\square\varphi_{B,C}"] \arrow[d, "\mu_{A,B,C}"'] & H'(A,B)\,\square H'(B,C) \arrow[d, "\mu'_{A,B,C}"] \\
H(A,C) \arrow[r, "\varphi_{A,C}"] & H'(A,C)
\end{tikzcd} . \tag{5.15}$$

Defining composition in the usual way, the category of all $\mathbb{V}$-categories over the set of objects O, denoted by $(\mathbb{V}, O)$-Cat, can be formed.

Definition 52 *A* 2-$\mathbb{V}$-category *over the set of objects O is an internal category in the category* $(\mathbb{V}, O)$*-Cat.*

A 2-*Ab*-category is obtained by considering the monoidal category $\mathbb{V} = (Ab, \otimes, Z)$.

2-Ab-Categories

Following the previous definition, a 2-Ab-Category over the set of objects O, is an internal category in the category (Ab, O)-Cat, that is, a diagram of the form

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

satisfying the usual axioms for a category. In order to analyze the definition, it is convenient to think of the object C_0 as an ordinary Ab-category (not given in terms of hom objects) and to think of C_1 as given by a system $C_1 = (H, \mu, \varepsilon)$ (see previous section). Since m, d, e, c are morphisms between Ab-categories, for each two objects A, B of C_0 (note that the objects of C_0 are by definition the elements of O), we have the following diagram in the category of abelian groups

$$\begin{tikzcd}
H(A,B) \times_{\hom(A,B)} H(A,B) \arrow[r, "m"] & H(A,B) \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & \hom(A,B) \arrow[l, "e" description]
\end{tikzcd}.$$

Using the well-known equivalence Cat(Ab)$\sim$Mor(Ab), the diagram can be presented as

$$\begin{tikzcd}
\ker d_{A,B} \oplus \ker d_{A,B} \oplus \hom(A,B) \arrow[r, "m"] & \ker d_{A,B} \oplus \hom(A,B) \arrow[r, shift left=2, "(0\ 1)"] \arrow[r, shift right=2, "(D\ 1)"'] & \hom(A,B) \arrow[l, "\binom{0}{1}" description]
\end{tikzcd}$$

with $m = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The group homomorphism $D : \ker d_{A,B} \longrightarrow \hom(A,B)$ sends each 2-cell with zero domain to its codomain.

Applying the commutativity of (5.15) to the cases $\varphi = d, e, c$ we conclude that the horizontal composition is completely determined by the composition

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of the 2-cells in $\ker d$ and by the horizontal composition of each element in $\ker d$ with left and right identity 2-cells. In fact a 2-cell $\tau^* : f \longrightarrow g$ with $f, g : A \longrightarrow B$ may be decomposed into the sum

$$\tau^* = \tau + 1_f \ ,$$

where $\tau \in \ker d_{A,B}$ and $D(\tau) = g - f \quad (\tau : 0 \longrightarrow g - f)$. The horizontal composition (in C_1) of $\sigma^* : f' \longrightarrow g' \,(: B \longrightarrow C)$ and $\tau^* : f \longrightarrow g \,(: A \longrightarrow B)$ is given by

$$\mu(\tau^*, \sigma^*) = \sigma^* \circ \tau^* = (\sigma + 1_{f'}) \circ (\tau + 1_f)$$

and, since horizontal composition is bilinear, we obtain the following formula

$$\sigma^* \circ \tau^* = (\sigma \circ \tau + \sigma \circ 1_f + 1_{f'} \circ \tau) + 1_{f'f}.$$

Also, by condition (5.15) applied to $\varphi = c$, the homomorphism D must satisfy the following conditions

$$\begin{array}{rcl} D(\tau \circ \sigma) & = & D(\tau)\, D(\sigma)\,, \\ D(\tau \circ 1_f) & = & D(\tau)\, f, \\ D(1_g \circ \tau) & = & g D(\tau)\,. \end{array}$$

Moreover, requiring the commutativity of (5.15) for $\varphi = m$ is the same as to require the four middle interchange law. Consider a diagram of the form

$$\begin{tikzcd}
A \arrow[r, bend left=70, "g", ""{name=U1, below}] \arrow[r, "g'" description, ""{name=M1, above}, ""{name=N1, below}] \arrow[r, bend right=70, "g''"', ""{name=D1, above}] & B \arrow[r, bend left=70, "f", ""{name=U2, below}] \arrow[r, "f'" description, ""{name=M2, above}, ""{name=N2, below}] \arrow[r, bend right=70, "f''"', ""{name=D2, above}] & C
\arrow[from=U1, to=M1, "\tau^*"'] \arrow[from=N1, to=D1, "\tau^{*\prime}"'] \arrow[from=U2, to=M2, "\sigma^*"'] \arrow[from=N2, to=D2, "\sigma^{*\prime}"']
\end{tikzcd}$$

in C_1, the four middle interchange law states that

$$(\sigma^{*\prime} \cdot \sigma^*) \circ (\tau^{*\prime} \cdot \tau^*) = (\sigma^{*\prime} \circ \tau^{*\prime}) \cdot (\sigma^* \circ \tau^*)\,.$$

Using the formulas obtained above, we have

$$\begin{aligned} ((\sigma' + \sigma + 1_f) \circ (\tau' + \tau + 1_g)) = {} & ((\sigma' \circ \tau' + \sigma' \circ 1_{g'} + 1_{f'} \circ \tau') + 1_{f'g'}) \cdot \\ & \cdot ((\sigma \circ \tau + \sigma \circ 1_g + 1_f \circ \tau) + 1_{fg})\,, \end{aligned}$$

which extends to

$$\begin{aligned} & \sigma' \circ \tau' + \sigma \circ \tau' + 1_f \circ \tau' + \sigma' \circ \tau + \sigma \circ \tau + 1_f \circ \tau + \sigma' \circ 1_g + \sigma \circ 1_g + 1_{fg} = \\ & \qquad ((\sigma' \circ \tau' + \sigma' \circ 1_{g'} + 1_{f'} \circ \tau' + \sigma \circ \tau + \sigma \circ 1_g + 1_f \circ \tau) + 1_{fg})\,, \end{aligned}$$

and then becomes

$$\sigma \circ \tau' + 1_f \circ \tau' + \sigma' \circ \tau + \sigma' \circ 1_g = \sigma' \circ 1_{g'} + 1_{f'} \circ \tau'.$$

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By substituting

$$g' = D(\tau) + g \ , \ f' = D(\sigma) + f \ ,$$

in the formula above we obtain

$$\sigma \circ \tau' + \sigma' \circ \tau = \sigma' \circ 1_{D(\tau)} + 1_{D(\sigma)} \circ \tau'$$

which is equivalent to

$$\sigma \circ \tau = \sigma \circ 1_{D(\tau)} = 1_{D(\sigma)} \circ \tau.$$

Finally, by the commutativity of (5.13) we have

$$\begin{aligned} \tau \circ 1_A &= \tau \\ 1_C \circ \sigma &= \sigma. \end{aligned}$$

We may summarize the above calculations in the following proposition.

Proposition 53 *Giving a 2-Ab-category is the same as to give the following data:*

- *An Ab-category* $\mathbb{A}$*;*
- *An abelian group* $K(A,B)$*, for each pair of objects* A, B *of* $\mathbb{A}$*;*
- *A group homomorphism* $D_{A,B} : K(A,B) \longrightarrow \hom_{\mathbb{A}}(A,B)$*, for each pair of objects* A, B *of* $\mathbb{A}$*;*
- *Associative and bilinear laws of composition*

  $$g\tau \ , \ \sigma\tau \ , \ \sigma f \in K(A,C)$$

  for each $\tau \in K(A,B), \sigma \in K(B,C), f \in \hom_{\mathbb{A}}(A,B), g \in \hom_{\mathbb{A}}(B,C)$ *with* A, B, C *objects of* $\mathbb{A}$*, satisfying the following conditions*

  $$\begin{aligned} \tau 1_A &= \tau, \\ 1_B \tau &= \tau, \end{aligned} \tag{5.16}$$

  $$\begin{aligned} D(\sigma\tau) &= D(\sigma) D(\tau), \\ D(\sigma f) &= D(\sigma) f, \\ D(g\tau) &= g D(\tau), \end{aligned} \tag{5.17}$$

  $$\sigma\tau = \sigma D(\tau) = D(\sigma)\tau. \tag{5.18}$$

The data given in the above proposition determines a 2-category structure in the Ab-category $\mathbb{A}$. Given two morphisms $f, g : A \longrightarrow B$ of $\mathbb{A}$, a 2-cell from f to g is a pair (τ, f) with τ in $K(A,B)$ and $D(\tau) = g - f$. Note that $K(A,B)$ plays the role of $\ker d_{A,B}$.

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The vertical composition is given by the formula

$$(\sigma, g) \cdot (\tau, f) = (\sigma + \tau, f)$$

whereas the horizontal composition is given by

$$(\tau', f') \circ (\tau, f) = (\tau'\tau + \tau' f + f'\tau\ ,\ f'f)\,.$$

We always write the three different compositions $g\tau, \sigma\tau, \sigma f$ as justaposition, because it is clear from the context. We also use small letters like f, g, h, k to denote the morphisms of $\mathbb{A}$ and small greek letters, like $\alpha, \lambda, \rho, \eta$ to denote the elements of K. Sometimes the same greek letter is used to denote the element of K and the 2-cell itself, e.g. $\alpha = (\alpha, m\,(1 \times m))$.

Definition 54 *An additive 2-category is a 2–Ab-category with:*

- *a zero object;*
- *all binary biproducts;*

In the next section simple properties of additive 2-categories are presented.

Properties of Additive 2-categories

Let $\mathbb{A}$ be an additive 2-category (with K and D as above). As is well known, in an additive category (see [1]), a morphism between iterated biproducts is described as a matrix of its components and composition is just the product of matrices. In an additive 2-category the same is true for the 2-cells since we are able to compose them with the projections and the injections of the biproducts. This means that if we have

$$\tau \in K\,(A_1 \oplus A_2, B_1 \oplus B_2)\,,$$

then we can write

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$$

with

$$\tau_{ij} \in K\,(A_j, B_i)\,.$$

Let us recall:

Proposition 55 *A split epi $X \xrightarrow{u} Y$ (with splitting $Y \xrightarrow{v} X$) in an additive category with kernels is isomorphic to*

$$\begin{tikzcd} \ker u \oplus Y \arrow[r, shift left, "p"] & Y \arrow[l, shift left, "i"] \end{tikzcd}$$

where $p = (0\ ,\ 1)$ *and* $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

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Proposition 56 *Let $X \times_Y Z$ be the object of a pullback diagram in an additive category with kernels, where u is a split epi, with splitting v, as in the following diagram*

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "w"] \\
X \arrow[r, "u"] & Y
\end{tikzcd}.$$

Then $X \times_Y Z \cong \ker u \oplus Z$ and the pullback diagram becomes

$$\begin{tikzcd}
\ker u \oplus Z \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & w \end{smallmatrix}\right)"'] & Z \arrow[d, "w"] \\
\ker u \oplus Y \arrow[r, "(0\ 1)"] & Y
\end{tikzcd}.$$

In the following section we will describe the notion of weak category in an additive 2-category with kernels.

5.4 Weak Categories in Additive 2-categories with Kernels

Let $\mathbb{A}$ be an additive 2-category with kernels. We will identify $\mathbb{A}$ with the data $(\mathbb{A}, K, D)$ of Proposition 53. When it is clear from the context, we will refer to a 2-cell

$$(\tau, f) : f \longrightarrow f + D(\tau)$$

simply by τ.

Precategories

(See Definition 47).

Proposition 57 *An internal precategory in $\mathbb{A}$ is completely determined by four morphisms of $\mathbb{A}$,*

$$\begin{aligned} k &: A \longrightarrow B, \\ f, g &: A \longrightarrow A, \\ h &: B \longrightarrow A, \end{aligned}$$

with

$$kf = k = kg\ ,\ kh = 0, \tag{5.19}$$

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and is given (up to an isomorphism) by

$$\begin{tikzcd} A \oplus A \oplus B \arrow[r, "m"] & A \oplus B \arrow[r, shift left=3, "(0\ 1)"] \arrow[r, shift right=3, "(k\ 1)"'] & B, \arrow[l, "\binom{0}{1}"'] \end{tikzcd} \tag{5.20}$$

where $m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix}$.

Proof. Since the morphism $d : C_1 \longrightarrow C_0$ in Definition 47 is a split epi, using Proposition 55 we conclude that the object C_1 is of the form $A \oplus B$ (considering A as the kernel of d and $C_0 = B$). This means that the underlying reflexive graph of our precategory is of the form

$$\begin{tikzcd} A \oplus B \arrow[r, shift left=3, "(0\ 1)"] \arrow[r, shift right=3, "(k\ 1)"'] & B. \arrow[l, "\binom{0}{1}"'] \end{tikzcd}$$

The object $C_1 \times_{C_0} C_1$ is (by Proposition 56) isomorphic to

$$A \oplus A \oplus B$$

and the projections π_1 and π_2 are given by the diagram

$$\begin{tikzcd} A \oplus B & A \oplus A \oplus B \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & k & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] & A \oplus B. \end{tikzcd}$$

The composition $m : A \oplus A \oplus B \longrightarrow A \oplus B$ is a morphism satisfying $dm = d\pi_2$, which means that

$$m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix}$$

with $f, g : A \longrightarrow A$ and $h : B \longrightarrow A$ arbitrary morphisms of $\mathbb{A}$. Nevertheless, the condition $cm = c\pi_1$ yields

$$kf = k = kg\ ,\ kh = 0.$$

■

Associative Precategories

In order to analyze the 2-cell $\alpha : m\,(1 \times_{C_0} m) \longrightarrow m\,(m \times_{C_0} 1)$ (see Definition 48), the morphisms $m\,(1 \times_{C_0} m)$ and $m\,(m \times_{C_0} 1)$ have to be described. Having in mind (by Proposition 57) that $C_1 \times_{C_0} C_1$ is of the form $A \oplus A \oplus B$ and, using Proposition 56, we conclude that $C_1 \times_{C_0} C_1 \times_{C_0} C_1$ is of the form

$$A \oplus A \oplus A \oplus B.$$

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The projections for $C_1 \times_{C_0} (C_1 \times_{C_0} C_1)$ are given as in the diagram

$$\begin{tikzcd} A \oplus B & A \oplus (A \oplus A \oplus B) \arrow[l, "\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & k & k & 1 \end{smallmatrix}\right)"'] \arrow[r, "(0\ 1)"] & (A \oplus A \oplus B) \end{tikzcd}$$

and the projections for $(C_1 \times_{C_0} C_1) \times_{C_0} C_1$ are given by

$$\begin{tikzcd} A \oplus A \oplus B & A \oplus A \oplus A \oplus B \arrow[l, "p"'] \arrow[r, "\left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)"] & A \oplus B \end{tikzcd}$$

with

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{pmatrix}.$$

The reader may appreciate checking that

$$1 \times_{C_0} m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f & g & h \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$m \times_{C_0} 1 = \begin{pmatrix} f & g & hk & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using matrix multiplication we have that

$$m\,(1 \times_{C_0} m) = \begin{pmatrix} f & gf & g^2 & gh + h \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$m\,(m \times_{C_0} 1) = \begin{pmatrix} f^2 & fg & fhk + g & fh + h \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The isomorphism $(\alpha, m\,(1 \times m)) : m\,(1 \times m) \longrightarrow m\,(m \times 1)$ has α in $K\,(A \oplus A \oplus A \oplus B, A \oplus B)$ and

$$D\,(\alpha) = m\,(m \times 1) - m\,(1 \times m)\,. \tag{5.21}$$

Since α must satisfy

$$d \circ \alpha = 1_{d\pi_3},$$

which may be written as[3]

$$(d\alpha, d\pi_3) = (0, d\pi_3)\,,$$

we conclude that $d\alpha = 0$. Having in mind that $d = (0\ 1)$ and α is a 2×4 matrix, we have

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

---

[3] Note that for any morphism $\varphi : A \longrightarrow A'$, the 2-cell 1_φ is of the form $(0, \varphi)$ and so, $1_{d\pi_3}$ is of the form $(0, d\pi_3)$. The composite $d \circ \alpha$ is of the form $(0, d) \circ (\alpha, m\,(1 \times m)) = (d\alpha, dm\,(1 \times m)) = (d\alpha, d\pi_3)\,.$

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with $\alpha_1, \alpha_2, \alpha_3 \in K(A, A)$ and $\alpha_0 \in K(B, A)$. Similarly, from $c \circ \alpha = 1_{c\pi_1}$, we conclude that $c\alpha = 0$. Furthermore, since $c = (k\ 1)$ we have

$$k\alpha_i = 0\ ,\ i = 0, 1, 2, 3.$$

In order to satisfy condition (5.21), we must also have

$$\begin{aligned} D(\alpha_1) &= f^2 - f, \\ D(\alpha_2) &= fg - gf, \\ D(\alpha_3) &= fhk + g - g^2, \\ D(\alpha_4) &= fh - gh. \end{aligned}$$

We are now ready to establish the following:

Proposition 58 *An internal associative precategory in $\mathbb{A}$ is completely determined by morphisms*

$$\begin{aligned} k &: A \longrightarrow B, \\ f, g &: A \longrightarrow A, \\ h &: B \longrightarrow A, \end{aligned}$$

with

$$kf = k = kg\ ,\ kh = 0,$$

and objects $\alpha_1, \alpha_2, \alpha_3 \in K(A, A)$, $\alpha_0 \in K(B, A)$ with

$$k\alpha_i = 0\ ,\ i = 0, 1, 2, 3$$

$$\begin{aligned} D(\alpha_1) &= f^2 - f, \\ D(\alpha_2) &= fg - gf, \\ D(\alpha_3) &= fhk + g - g^2, \\ D(\alpha_4) &= fh - gh. \end{aligned}$$

Associative Precategories with Identity

In order to analyze the 2-cells for the left and right identities (see Definition 49), we have to describe the morphisms $m\langle ec, 1\rangle$ and $m\langle 1, ed\rangle$ from C_1 to C_1. Proposition 56 yields

$$\langle ec, 1\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\ ,\ \langle 1, ed\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$m\langle ec, 1\rangle = \begin{pmatrix} g & h \\ 0 & 1 \end{pmatrix}\ ,\ m\langle 1, ed\rangle = \begin{pmatrix} f & h \\ 0 & 1 \end{pmatrix}.$$

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Since $(\lambda, m\langle ec,1\rangle) : m\langle ec,1\rangle \longrightarrow 1$ is a 2-cell from $A\oplus B$ to $A\oplus B$, we conclude that λ is in $K(A\oplus B, A\oplus B)$ and

$$D(\lambda)=\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}g&h\\0&1\end{pmatrix}, \tag{5.22}$$

while ρ is in $K(A\oplus B, A\oplus B)$ and

$$D(\rho)=\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}f&h\\0&1\end{pmatrix}. \tag{5.23}$$

In order to satisfy conditions (5.8) λ and ρ must be of the form

$$\begin{aligned}\lambda &= \begin{pmatrix}\lambda_1&\lambda_0\\0&0\end{pmatrix},\\ \rho &= \begin{pmatrix}\rho_1&\rho_0\\0&0\end{pmatrix},\end{aligned}$$

with $\lambda_1,\rho_1\in K(A,A)$ and $\lambda_0,\rho_0\in K(B,A)$ such that

$$\begin{aligned}k\lambda_1 &= 0=k\rho_1,\\ k\lambda_0 &= 0=k\rho_0.\end{aligned} \tag{5.24}$$

From the condition

$$\lambda\circ e=\rho\circ e,$$

we conclude that $\lambda_0=\rho_0$. To simplify notation a new letter, η, is introduced to denote λ_0 and ρ_0. In this way, instead of having $\lambda_0,\rho_0\in K(B,A)$ and one condition $\lambda_0=\rho_0$ we simply have $\eta\in K(B,A)$. Since λ_0 and ρ_0 are not used anymore, we will write λ and ρ instead of λ_1 and ρ_1 respectively.

With this new notation, conditions (5.22) and (5.23) become

$$\begin{pmatrix}D(\lambda)&D(\eta)\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}g&h\\0&1\end{pmatrix}$$

and

$$\begin{pmatrix}D(\rho)&D(\eta)\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}f&h\\0&1\end{pmatrix}.$$

This means that λ and ρ completely determine the morphisms f,g,h and we have

$$\begin{aligned}g &= 1-D(\lambda),\\ f &= 1-D(\rho),\\ h &= -D(\eta).\end{aligned}$$

Next, we show that the conditions (5.19) are satisfied with f,g,h given as above. Since

$$kf=k-kD(\rho),$$

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and $kD(\rho) = D(k\rho)$ (by condition (6.2)) and $k\rho = 0$, we have

$$kf = k.$$

The same argument shows that $k = kg$ and $kh = 0$, since $k\lambda = 0$ and $k\eta = 0$.

This suggests the following description of associative precategories with identity in an additive 2-category with kernels:

Proposition 59 *An associative precategory with identity in an additive 2-category with kernels is completely determined by a morphism*

$$A \xrightarrow{k} B$$

together with

$$\begin{array}{rcl} \alpha_1, \alpha_2, \alpha_3, \lambda, \rho & \in & K(A, A), \\ \alpha_0, \eta & \in & K(B, A), \end{array}$$

subject to the following conditions

$$k\alpha_i = 0\ ,\ i = 0, 1, 2, 3$$

$$k\lambda = 0 = k\rho,\ k\eta = 0,$$

$$\begin{array}{rcl} D(\alpha_1) & = & f^2 - f, \\ D(\alpha_2) & = & fg - gf, \\ D(\alpha_3) & = & fhk + g - g^2, \\ D(\alpha_0) & = & fh - gh, \end{array} \tag{5.25}$$

where f, g, h are defined as follows:

$$\begin{array}{rcl} g & = & 1 - D(\lambda), \\ f & = & 1 - D(\rho), \\ h & = & -D(\eta). \end{array} \tag{5.26}$$

Associative precategories with coherent identity

An associative precategory with coherent identity is an associative precategory with identity (see previous section) where the triangle and mixed coherent conditions are satisfied (see Definition 50). We proceed using the description of associative precategory with identity given as in the previous section to describe the triangle and the mixed coherence conditions in additive 2-categories with kernels.

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Triangle Coherence Condition

In order to analyze the triangle coherence condition

$$(m \circ (\rho \times 1)) \cdot (\alpha \circ (1 \times \langle ec, 1 \rangle)) = m \circ (1 \times \lambda)$$

the 2-cells $\rho \times_{C_0} 1$ and $1 \times_{C_0} \lambda$, have to be described. Since they are elements in $K(A \oplus A \oplus B, A \oplus A \oplus B)$, the reader is invited to show that

$$\rho \times_{C_0} 1 = \begin{pmatrix} \rho & \eta k & \eta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 1 \times_{C_0} \lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & \eta \\ 0 & 0 & 0 \end{pmatrix}.$$

We have already seen that

$$\langle ec, 1 \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ so } (1 \times \langle ec, 1 \rangle) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The definition of horizontal composition in an additive 2-category yields

$$\begin{aligned} m \circ (\rho \times 1) &= \begin{pmatrix} f\rho & f\eta k & f\eta \\ 0 & 0 & 0 \end{pmatrix}, \\ m \circ (1 \times \lambda) &= \begin{pmatrix} 0 & g\lambda & g\eta \\ 0 & 0 & 0 \end{pmatrix}, \\ \alpha \circ (1 \times \langle ec, 1 \rangle) &= \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Finally, the coherence condition may be written as

$$\begin{pmatrix} f\rho & f\eta k & f\eta \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & g\lambda & g\eta \\ 0 & 0 & 0 \end{pmatrix},$$

or, equivalently, as

$$\begin{aligned} \alpha_1 &= -f\rho, \\ \alpha_3 &= g\lambda - f\eta k, \\ \alpha_0 &= g\eta - f\eta. \end{aligned}$$

The components $\alpha_1, \alpha_3, \alpha_0$ are completely determined. In the next section we show that the component α_2 is also determined by the mixed coherence condition.

Mixed Coherence Condition

Consider the mixed coherence condition

$$\rho \cdot (m \circ \langle \lambda, 1_{ed} \rangle) \cdot (\alpha \circ \langle ec, 1, ed \rangle) = \lambda \cdot (m \circ \langle 1_{ec}, \rho \rangle).$$

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We have already seen that

$$\langle 1, ed\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}. \text{ Thus } \langle ec, 1, ed\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, we have

$$\langle \lambda, 1_{ed}\rangle = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\langle 1_{ec}, \rho\rangle = \begin{pmatrix} 0 & 0 \\ \rho & \eta \\ 0 & 0 \end{pmatrix}.$$

Using the definition of horizontal composition we obtain

$$\begin{aligned} m \circ \langle \lambda, 1_{ed}\rangle &= \begin{pmatrix} f\lambda & f\eta \\ 0 & 0 \end{pmatrix}, \\ \alpha \circ \langle ec, 1, ed\rangle &= \begin{pmatrix} \alpha_2 & \alpha_0 \\ 0 & 0 \end{pmatrix}, \\ m \circ \langle 1_{ec}, \rho\rangle &= \begin{pmatrix} g\rho & g\eta \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and the mixed coherence condition may be written as

$$\begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f\lambda & f\eta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \alpha_0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} g\rho & g\eta \\ 0 & 0 \end{pmatrix},$$

or, equivalently, as

$$\begin{aligned} \alpha_2 &= \lambda + g\rho - \rho - f\lambda, \\ \alpha_0 &= \eta + g\eta - \eta - f\eta. \end{aligned}$$

Therefor the 2-cell α is completely determined and it is a straightforward calculation checking that $k\alpha_i = 0,\ i = 0, 1, 2, 3$, and that conditions (6.21) are satisfied.

Hence, we have:

Proposition 60 *An associative precategory with coherent identity in an additive 2-category with kernels is completely determined by a morphism*

$$A \xrightarrow{k} B,$$

together with

$$\begin{aligned} \lambda, \rho &\in K(A, A), \\ \eta &\in K(B, A), \end{aligned}$$

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subject to the conditions

$$\begin{aligned} k\lambda &= 0, \\ k\rho &= 0, \\ k\eta &= 0. \end{aligned}$$

It is given (up to an isomorphism) by

$$\begin{tikzcd} A \oplus A \oplus B \arrow[r, "m"] & A \oplus B \arrow[r, shift left=2, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[r, shift right=2, "\begin{pmatrix} k & 1 \end{pmatrix}"'] & B, \arrow[l, "\binom{0}{1}" description] \end{tikzcd}$$

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix}, \ \rho = \begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix},$$

where

$$m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned} g &= 1 - D(\lambda), \\ f &= 1 - D(\rho), \\ h &= -D(\eta), \end{aligned}$$

$$\begin{aligned} \alpha_1 &= \rho^2 - \rho, \\ \alpha_2 &= \rho\lambda - \lambda\rho, \\ \alpha_3 &= \lambda - \lambda^2 - f\eta k, \\ \alpha_0 &= \rho\eta - \lambda\eta. \end{aligned}$$

Weak categories

In this section we show that the pentagon coherence condition does not add new restrictions on the data involved in Proposition 60, i.e. the description of associative precategory with coherent identity is in fact the description of a weak category in an additive 2-category with kernels.

In order to analyze the pentagon coherence condition

$$(\alpha \circ (m \times 1 \times 1)) \cdot (\alpha \circ (1 \times 1 \times m)) = (m \circ (\alpha \times 1)) \cdot (\alpha \circ (1 \times m \times 1)) \cdot (m \circ (1 \times \alpha)),$$

we need some preliminary calculations. Namely, all the arrows in the expression have to be described.

To describe the arrow $m \times_{C_0} 1 \times_{C_0} 1$, we have to analyze its domain $(C_1 \times_{C_0} C_1) \times_{C_0} C_1 \times_{C_0} C_1$ and codomain $C_1 \times_{C_0} C_1 \times_{C_0} C_1$. Using the results obtained in the

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previous sections, we have that the domain, together with its three projections, is given by

$$\begin{tikzcd}
A \oplus A \oplus B & A \oplus A \oplus A \oplus A \oplus B \arrow[l, "\pi_1"'] \arrow[r, "\pi_3"] \arrow[d, "\pi_2"] & A \oplus B \\
 & A \oplus B &
\end{tikzcd},$$

where

$$\begin{aligned}
\pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & k & k & 1 \end{pmatrix}, \\
\pi_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k & 1 \end{pmatrix}, \\
\pi_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

The codomain, together with its projections, is given by

$$\begin{tikzcd}
A \oplus B & A \oplus A \oplus A \oplus B \arrow[l, "\pi_1"'] \arrow[r, "\pi_3"] \arrow[d, "\pi_2"] & A \oplus B \\
 & A \oplus B &
\end{tikzcd},$$

where

$$\begin{aligned}
\pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & k & 1 \end{pmatrix}, \\
\pi_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{pmatrix}, \\
\pi_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Having in mind that

$$\begin{aligned}
m &= \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix} : A \oplus A \oplus B \longrightarrow A \oplus B, \\
1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A \oplus B \longrightarrow A \oplus B,
\end{aligned}$$

we obtain

$$(m \times 1 \times 1) = \begin{pmatrix} f & g & hk & hk & h \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By similar calculations we also have

$$(1 \times m \times 1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & f & g & hk & h \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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and

$$(1\times 1\times m)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & f & g & h\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

We remark that $\alpha\times 1$ is in fact an abbreviation of $\alpha\times_{C_0} 1_{1_{C_1}}$, where $1_{1_{C_1}}$ is the identity 2-cell of the arrow 1_{C_1}. So, it is the pair $(0,1_{C_1})$ with 0 in $K\left(A\oplus B, A\oplus B\right).$

The domain of $\alpha\times 1$ is given (together with its two projections) as in the diagram

$$A\oplus A\oplus A\oplus B\xleftarrow{\pi_1} A\oplus A\oplus A\oplus A\oplus B\xrightarrow{\pi_2} A\oplus B,$$

where

$$\begin{aligned}\pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & k & 1\end{pmatrix},\\ \pi_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},\end{aligned}$$

and the codomain is given as in

$$A\oplus B\xleftarrow{\pi_1} A\oplus A\oplus B\xrightarrow{\pi_2} A\oplus B,$$

where

$$\begin{aligned}\pi_1 &= \begin{pmatrix} 1 & 0 & 0\\ 0 & k & 1\end{pmatrix},\\ \pi_2 &= \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

Hence, we obtain

$$\alpha\times 1=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 k & \alpha_0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

Similarly, we have get

$$1\times\alpha=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

Now that we have all ingredients of our calculation, we begin the main part.

On the one hand, we have to describe

$$\alpha\left(m\times 1\times 1\right)+\alpha\left(1\times 1\times m\right)$$

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and the result is

$$\begin{pmatrix} \alpha_1 f+\alpha_1 & \alpha_1 g+\alpha_2 & \alpha_1 hk+\alpha_2+\alpha_3 f & \alpha_1 hk+\alpha_3+\alpha_3 g & \alpha_1 h+\alpha_0+\alpha_3 h+\alpha_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.27}$$

On the other hand, we need

$$(m(\alpha\times 1))+(\alpha(1\times m\times 1))+(m(1\times\alpha))$$

and the result is

$$\begin{pmatrix} f\alpha_1+\alpha_1 & f\alpha_2+\alpha_2 f+g\alpha_1 & f\alpha_3+\alpha_2 g+g\alpha_2 & f\alpha_0 k+\alpha_2 hk+\alpha_3+g\alpha_3 & f\alpha_0+\alpha_2 h+\alpha_0+g\alpha_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.28}$$

To check whether (5.27) and (5.28) are equal is the same as checking whether the following identities hold

$$\begin{aligned} \alpha_1 f &= f\alpha_1 && (5.29)\\ \alpha_1 g+\alpha_2 &= f\alpha_2+\alpha_2 f+g\alpha_1 && (5.30)\\ \alpha_1 hk+\alpha_2+\alpha_3 f &= f\alpha_3+\alpha_2 g+g\alpha_2 && (5.31)\\ \alpha_1 hk+\alpha_3 g &= f\alpha_0 k+\alpha_2 hk+g\alpha_3 && (5.32)\\ \alpha_1 h+\alpha_3 h+\alpha_0 &= f\alpha_0+\alpha_2 h+g\alpha_0. && (5.33) \end{aligned}$$

We will use

$$\begin{aligned} f &= 1-D(\rho)\ ,\ g=1-D(\lambda)\ ,\ h=-D(\eta)\,,\\ \alpha_1 &= \rho^2-\rho\ ,\ \alpha_2=\rho\lambda-\lambda\rho,\\ \alpha_3 &= \lambda-\lambda^2-\eta k+\rho\eta k,\\ \alpha_0 &= \rho\eta-\lambda\eta. \end{aligned}$$

(see Proposition 60).

The condition (5.29) holds since we have $\rho f=f\rho$.

The condition (5.30) is equivalent to

$$\alpha_1-\alpha_1\lambda+\alpha_2=\alpha_2-\rho\alpha_2+\alpha_2-\alpha_2\rho+\alpha_1-\lambda\alpha_1,$$

which simplifies to

$$-\left(\rho^2-\rho\right)\lambda=-\rho\alpha_2+\alpha_2-\alpha_2\rho-\lambda\left(\rho^2-\rho\right)$$

and then becomes

$$-\rho^2\lambda+\rho\lambda=-\rho^2\lambda+\rho\lambda\rho+\rho\lambda-\lambda\rho-\rho\lambda\rho+\lambda\rho^2-\lambda\rho^2+\lambda\rho$$

which is a trivial condition.

Moreover, the condition (5.31) is equivalent to

$$\alpha_1 hk+\alpha_2+\alpha_3-\alpha_3\rho=\alpha_3-\rho\alpha_3+\alpha_2-\alpha_2\lambda+\alpha_2-\lambda\alpha_2,$$

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which extends to

$$\left(\rho^2-\rho\right)(-\eta)\,k-\alpha_3\rho=-\rho\alpha_3-(\rho\lambda-\lambda\rho)\,\lambda+\rho\lambda-\lambda\rho-\lambda\,(\rho\lambda-\lambda\rho)\,,$$

and also to

$$\begin{aligned}&\left(\rho^2-\rho\right)(-\eta)\,k-\left(\lambda-\lambda^2-\eta k+\rho\eta k\right)\rho=\\&\qquad=-\rho\left(\lambda-\lambda^2-\eta k+\rho\eta k\right)-(\rho\lambda-\lambda\rho)\,\lambda+\rho\lambda-\lambda\rho-\lambda\,(\rho\lambda-\lambda\rho)\,.\end{aligned}$$

Since $k\rho=0$, this condition is also trivial.

The condition (5.32) is equivalent to

$$\alpha_1hk+\alpha_3-\alpha_3\lambda=f\alpha_0k+\alpha_2hk+\alpha_3-\lambda\alpha_3,$$

and then to

$$\begin{aligned}&\left(\rho^2-\rho\right)(-\eta)\,k-\left(\lambda-\lambda^2-\eta k+\rho\eta k\right)\lambda=\\&=(\rho\eta-\lambda\eta)\,k-\rho\,(\rho\eta-\lambda\eta)\,k+(\rho\lambda-\lambda\rho)\,(-\eta)\,k-\lambda\left(\lambda-\lambda^2-\eta k+\rho\eta k\right).\end{aligned}$$

Since $k\lambda=0$ it is trivial again.

The condition (5.33) is equivalent to

$$\alpha_1h+\alpha_3h=-\rho\alpha_0+\alpha_2h+\alpha_0-\lambda\alpha_0,$$

or

$$\begin{aligned}&-\rho^2\eta+\rho\eta-\lambda\eta+\lambda^2\eta+\eta k\eta-\rho\eta k\eta=\\&\qquad=-\rho\,(\rho\eta-\lambda\eta)-\rho\lambda\eta+\lambda\rho\eta+\rho\eta-\lambda\eta-\lambda\,(\rho\eta-\lambda\eta)\,.\end{aligned}$$

Since $k\eta=0$, the condition is trivial.

Finally, we obtain:

Proposition 61 *An associative precategory with coherent identity in an additive 2-category with kernels is a weak category.*

5.5 Examples

In this section we consider internal weak categories in Ab and Mor(Ab) that are examples of additive 2-categories with kernels.

Abelian Groups

According to Proposition 53, taking $\mathbb{A}$=Ab and $D=id:\hom_{\mathrm{Ab}}(A,B)\longrightarrow\hom_{\mathrm{Ab}}(A,B)$, the category Ab of abelian groups is an example of an additive 2-category.

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The data describing a weak category in Ab consists of four morphisms of abelian groups

$$A \xrightarrow{k} B\ ,\ \lambda, \rho : A \longrightarrow A\ ,\ \eta : B \longrightarrow A$$

subject to the conditions

$$\begin{aligned} k\lambda &= 0 = k\rho, \\ k\eta &= 0. \end{aligned}$$

This information can be used to construct the corresponding weak category with the objects being the elements of B, the morphisms pairs $(a, b) \in A \oplus B$

$$b \xrightarrow{(a,b)} k(a) + b,$$

and the composition

$$b \xrightarrow{(a,b)} k(a) + b \xrightarrow{\left(a', k(a)+b\right)} k(a' + a) + b$$

given by

$$(a', k(a) + b)(a, b) = (a' - \rho(a') + a - \lambda(a) - \eta(b), b).$$

For every three composable morphisms

$$b \xrightarrow{(a,b)} k(a) + b = b' \xrightarrow{\left(a',b'\right)} k(a') + b' = b'' \xrightarrow{\left(a'',b''\right)} k(a'') + b'',$$

the 2-cell

$$\alpha : (a'', b'')((a', b')(a, b)) \longrightarrow ((a'', b'')(a', b'))(a, b),$$

is given by

$$\alpha(a'', a', a, b) = \left(\left(\rho^2 - \rho\right)(a'') + (\rho\lambda - \lambda\rho)(a') + \left(\lambda - \lambda^2 - \eta k + \rho\eta k\right)(a) + (\rho\eta - \lambda\eta)(b)\right).$$

The 2-cells λ and ρ for one morphism (a, b)

$$\begin{aligned} \lambda &: (0, b')(a, b) \longrightarrow (a, b) \\ \rho &: (a, b)(0, b) \longrightarrow (a, b) \end{aligned}$$

are given by

$$\begin{aligned} \lambda(a, b) &= (\lambda a + \eta b, 0), \\ \rho(a, b) &= (\rho a + \eta b, 0). \end{aligned}$$

Note that if $B = 0$, we obtain what we called a weak monoid. If the morphisms λ, ρ, η are the zero morphisms, then we obtain an internal category in Ab (which is well known to be just a group homomorphism).

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Morphisms of Abelian Groups

The category Mor(Ab) of morphisms of abelian groups is the category where the objects are morphisms of Ab, say

$$A = \left(A_1 \xrightarrow{\partial} A_0 \right).$$

The arrows are pairs of morphisms of Ab $(f_1, f_0) : A \longrightarrow B$ such that the following square is commutative

$$\begin{tikzcd} A_1 \arrow[r, "\partial"] \arrow[d, "f_1"'] & A_0 \arrow[d, "f_0"] \\ B_1 \arrow[r, "\partial"] & B_0 \end{tikzcd}.$$

For each two arrows $f = (f_1, f_0)$ and $g = (g_1, g_0)$ from A to B, a 2-cell from f to g is a pair $(\tau, f) : f \longrightarrow g$ where $\tau : A_0 \longrightarrow B_1$ is a homomorphism of abelian groups with

$$\begin{aligned} \tau\partial &= g_1 - f_1 \\ \partial\tau &= g_0 - f_0. \end{aligned}$$

In order to be able to see that this category is an example of an additive 2-category we note that, with respect to objects and arrows, it is in fact an additive category. Now, for each pair of objects, A and B, we define the abelian group $K(A, B)$ as

$$K(A, B) = \hom_{Ab}(A_0, B_1)$$

and the homomorphism D as

$$D(\tau) = (\tau\partial, \partial\tau).$$

It can be shown that D satisfies all the conditions in (6.2) and (6.3) if we define the following laws of composition

$$\begin{aligned} g\tau &= g_1\tau, \\ \sigma\tau &= \sigma\partial\tau, \\ \sigma f &= \sigma f_0, \end{aligned}$$

for every $\tau \in K(A, B), \sigma \in K(B, C), f \in \hom(A, B), g \in \hom(B, C).$

A weak category C in the additive 2-category of Mor(Ab) is determined by a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "\partial"] \arrow[d, "k_1"'] & A_0 \arrow[d, "k_0"] \\ B_1 \arrow[r, "\partial'"] & B_0 \end{tikzcd}$$

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together with three morphisms

$$\begin{aligned} \lambda, \rho &: A_0 \longrightarrow A_1, \\ \eta &: B_0 \longrightarrow A_1, \end{aligned}$$

satisfying the conditions

$$\begin{aligned} k_1\lambda &= 0 = k_1\rho, \\ k_1\eta &= 0. \end{aligned}$$

The objects of C are pairs (b, d) with $b \in B_0, d \in B_1$. The morphisms are of the form

$$\begin{pmatrix} b & x \\ d & y \end{pmatrix}$$

with $b \in B_0, d \in B_1, x \in A_0, y \in A_1$.

A weak category in Mor(Ab) may also be viewed as a structure with objects, vertical arrows, horizontal arrows and squares, in the following way

$$\begin{tikzcd} b \arrow[r, "{(b,x)}"] \arrow[d, "{\binom{b}{d}}"'] \arrow[dr, phantom, "{\begin{pmatrix} b & x \\ d & y \end{pmatrix}}"] & b + k_0(x) \arrow[d, "{\binom{b+k_0(x)}{d+k_1(y)}}"] \\ b + \partial'(d) \arrow[r, "{\left(b+\partial'(d), x+\partial(y)\right)}"'] & * \end{tikzcd},$$

where $*$ stands for $b + \partial'(d) + k_0(x + \partial(y)) = b + k_0(x) + \partial'(d + k_1(y))$.

The horizontal composition between squares is given by

$$\begin{pmatrix} b + k_0(x) & x' \\ d + k_1(y) & y' \end{pmatrix} \circ \begin{pmatrix} b & x \\ d & y \end{pmatrix} = \begin{pmatrix} b & f_0(x') + g_0(x) + h_0(b) \\ d & f_1(y') + g_1(y) + h_1(d) \end{pmatrix},$$

where

$$\begin{aligned} f_0 &= 1 - \partial\rho, \\ g_0 &= 1 - \partial\lambda, \\ h_0 &= -\partial\eta, \end{aligned}$$

$$\begin{aligned} f_1 &= 1 - \rho\partial, \\ g_1 &= 1 - \lambda\partial, \\ h_1 &= -\eta\partial. \end{aligned}$$

For each three horizontal arrows

$$b \xrightarrow{(b,x)} b' \xrightarrow{(b',x')} b'' \xrightarrow{(b'',x'')} b'' + k_0(x'')$$

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with $b' = b + k_0(x)$ and $b'' = b' + k_0(x')$ the isomorphism for associativity is given by

$$\begin{tikzcd}[ampersand replacement=\&]
b \arrow[rr, "(b'',x'')\circ((b',x')\circ(b,x))"] \arrow[dd, "\binom{b}{0}"'] \& \& b'' + k_0(x'') \arrow[dd, "\binom{b''+k_0(x'')}{0}"] \\
\& \begin{pmatrix} b \& f_0(x'') + g_0(z) + h_0(b) \\ 0 \& \alpha_1(x'') + \alpha_2(x') + \alpha_3(x) + \alpha_0(b) \end{pmatrix} \& \\
b \arrow[rr, "((b'',x'')\circ(b',x'))\circ(b,x)"'] \& \& b'' + k_0(x'')
\end{tikzcd}$$

where $z = f_0(x') + g_0(x) + h_0(b)$ and

$$\begin{aligned}
\alpha_1 &= -f_1\rho, \\
\alpha_2 &= \lambda + g_1\rho - \rho - f_1\lambda, \\
\alpha_3 &= g_1\lambda - f_1\eta k_0, \\
\alpha_0 &= g_1\eta - f_1\eta.
\end{aligned}$$

The left and right isomorphisms are given, respectively, by

$$\begin{tikzcd}[ampersand replacement=\&]
b \arrow[rr, "(b+k_0(x),0)\circ(b,x)"] \arrow[dd, "\binom{b}{0}"'] \& \& b + k_0(x) \arrow[dd, "\binom{b+k_0(x)}{0}"] \\
\& \begin{pmatrix} b \& g_0(x) + h_0(b) \\ 0 \& \lambda(x) + \eta(b) \end{pmatrix} \& \\
b \arrow[rr, "(b,x)"'] \& \& b + k_0(x)
\end{tikzcd}$$

and

$$\begin{tikzcd}[ampersand replacement=\&]
b \arrow[rr, "(b,x)\circ(b,0)"] \arrow[dd, "\binom{b}{0}"'] \& \& b + k_0(x) \arrow[dd, "\binom{b+k_0(x)}{0}"] \\
\& \begin{pmatrix} b \& f_0(x) + h_0(b) \\ 0 \& \rho(x) + \eta(b) \end{pmatrix} \& \\
b \arrow[rr, "(b,x)"'] \& \& b + k_0(x)
\end{tikzcd} .$$

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Chapter 6

The (tetra)category of pseudocategories in an additive 2-category with kernels

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Abstract: We describe the (tetra) category of pseudo-categories, pseudo-functors, natural transformations, pseudo-natural transformations, and modifications, as introduced in [5], internal to an additive 2-category with kernels, as formalized in [4]. In the context of a 2-Ab-category, we introduce the notion of a pseudo-morphism and prove the equivalence of categories: PsCat(A)∼PsMor(A) between pseudo-categories and pseudo-morphisms in an additive 2-category, A, with kernels – extending thus the well known equivalence Cat(Ab)∼Mor(Ab) between internal categories and morphisms of abelian groups. The leading example of an additive 2-category

with kernels is Cat(Ab). In the case A=Cat(Ab) we obtain a description of the (tetra) category of internal pseudo-double categories in Ab, and particularize it to a description of the (tetra) category of internal bicategories in abelian groups. As expected, pseudo-natural transformations coincide with homotopies of 2-chain complexes (as in [6]).

6.1 Introduction

In [4] we introduced the notion of pseudo-category as an internal category in a 2-category with the unitary and associativity axioms replaced by the existence of suitable 2-cells, that are considered as additional structure, required to be isomorphisms and satisfy some coherence conditions; we also formalized the notion of 2-Ab-category as a 2-category where both morphisms and 2-cells are enriched in Ab, and described pseudo-categories in additive 2-categories with kernels.

In [5] we introduced the notion of pseudo-functor, natural and pseudo-natural transformation and also modification, in order to form the *tetracategory* of pseudo-categories, PsCat.

A tetracategory, as suggested by the structure of PsCat, may have the following definition.

A tetracategory consist of the following data:

- objects: $A, B, C, ...$
- morphisms: $f : A \longrightarrow B, ...$
- 2-cells: $\theta : f \longrightarrow g, ...(f, g : A \longrightarrow B)$
- pseudo-cells: $f \overset{T}{\longrightarrow} g$, ...
- tetra cells: $$\begin{tikzcd} f \arrow[rr, "T" description] \arrow[d, "\theta"'] & {} \arrow[d, Rightarrow, "\Phi"] & g \arrow[d, "\theta'"] \\ f' \arrow[rr, "T'" description] & {} & g' \end{tikzcd} \quad , \cdots$$

  where: objects, morphisms and 2-cells, form a 2-category; for each pair of objects A, B, the morphisms, 2-cells, pseudo-cells and tetra cells from A to B form a pseudo-category; and there are also horizontal compositions involved, but we will not discuss it here.

Here, we describe PsCat($\mathbb{A}$), the tetracategory of pseudo-categories in an additive 2-category, $\mathbb{A}$, with kernels. We define the tetracategory PsMor($\mathbb{A}$) of pseudo-morphisms in $\mathbb{A}$ and show the equivalence of categories PsCat($\mathbb{A}$)$\sim$PsMor($\mathbb{A}$) which canonically extends to the additional structure of 2-cells, pseudo-cells and tetra-cells.

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One of the most important examples of an additive 2-category is Cat(Ab) the category of internal categories in Ab, the category of abelian groups, which is well known to be equivalent to the category of morphisms of abelian groups Cat(Ab)∼Mor(Ab); the result obtained here is therefore a 2-dimensional generalization of it.

The work is organized as follows: section 2 is mainly a brief collection of results, techniques and notations used in [4] that will also be adopted in here; section 3 has the details of the equivalence PsCat($\mathbb{A}$)∼PsMor($\mathbb{A}$) by defining the objects and morphisms in PsMor($\mathbb{A}$) that correspond to pseudo-categories and pseudo-functors in PsCat($\mathbb{A}$); section 4 extends the equivalence of categories to an equivalence of 2-categories by defining 2-cells in PsMor($\mathbb{A}$) that correspond to natural transformations in PsCat($\mathbb{A}$); section 5 gives a description of pseudo-natural transformations and modifications internal to $\mathbb{A}$ with the respective formulas to compose them; although proofs are only presented in Appendix B, where in Appendix A we recall definitions from [5]; section 6 contains the description of BiCat(Ab) which is obtained from PsMor($\mathbb{A}$) by taking $\mathbb{A}$ =Cat(Ab)∼Mor(Ab).

6.2 Preliminaries

In order to establish notation an permit further reference to some specific conditions that hold true in the structure of an additive 2-category we repeat here (from [4]) that an additive 2-category is completely determined by the following data:

- an additive category $\mathbb{A}$;
- an abelian group $H\left(A,B\right)$ and a group homomorphism $D_{A,B}:H(A,B)\longrightarrow \hom(A,B)$, for every two objects $A,B\in\mathbb{A}$;
- for each $A,B,C\in\mathbb{A}$, and $\tau\in H\left(A,B\right),\sigma\in H\left(B,C\right),f\in\hom\left(A,B\right),g\in\hom\left(B,C\right)$ there are defined associative and bilinear laws of composition

$$g\tau,\sigma\tau,\sigma f\in H\left(A,C\right)$$

satisfying

$$\begin{aligned}\tau 1_A &= \tau \\ 1_C\sigma &= \sigma\end{aligned}\tag{6.1}$$

$$\begin{aligned}D\left(\tau\sigma\right) &= D\left(\tau\right)D\left(\sigma\right)\\ D\left(\tau f\right) &= D\left(\tau\right)f\\ D\left(g\tau\right) &= gD\left(\tau\right)\end{aligned}\tag{6.2}$$

$$\sigma\tau=\sigma D\left(\tau\right)=D\left(\sigma\right)\tau.\tag{6.3}$$

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To construct a 2-category structure on $\mathbb{A}$ from the above, consider a 2-cell from $f : A \longrightarrow B$ to $g : A \longrightarrow B$ as an ordered pair (τ, f) with $\tau \in H(A, B)$ and $D(\tau) = g - f$, with vertical composition given by

$$(\sigma, g) \cdot (\tau, f) = (\sigma + \tau, f)$$

and horizontal composition is given by

$$(\tau', f') \circ (\tau, f) = (\tau'\tau + \tau' f + f'\tau \ , \ f'f) \, .$$

Note that we are using justaposition for the three different compositions

$$g\tau, \sigma\tau, \sigma f \in H(A, C)$$

and dot $\cdot$ and circle $\circ$ respectively for the vertical and horizontal compositions in the 2-category (the composition of morphisms gf is also denoted by justaposition). We believe there will be no confusion because as a rule we use small letters as $f, g, h, ..., u, v, w, ...$ to denote the morphisms of $\mathbb{A}$ and small greek letters such as $\alpha, \lambda, \rho, \eta, \sigma, \tau, ...$ to denote the elements of the abelian groups H. Also we will often refer to (τ, f) simply as τ and sometimes even write $\tau = (\tau, f)$. For the sake of simplicity we often denote the identity 2-cell $1_f : f \longrightarrow f$ simply by f.

Before continuing, we recall (also from [4]) that an element

$$\tau \in H(A_1 \oplus A_2, B_1 \oplus B_2)$$

is a matrix

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$$

with

$$\tau_{ij} \in H(A_j, B_i) \, ,$$

which is the same that happens to morphisms in an additive category.

Examples of additive 2-categories are as follows:

- Ab - the category of abelian groups is an additive 2-category if we define

  $$H(A, B) = \hom_{Ab}(A, B)$$

  and

  $$D_{A,B} = id : \hom_{Ab}(A, B) \longrightarrow \hom_{Ab}(A, B) \, .$$

- Ab^2 = Mor(Ab)- the category of morphisms of abelian groups is an additive 2-category if we define

  $$H(A, B) = \hom_{Ab}(A_0, B_1) \, ,$$

  note that

  $$\begin{aligned} A &= \left(A_1 \xrightarrow{\partial} A_0 \right) \\ B &= \left(B_1 \xrightarrow{\partial'} B_0 \right) \end{aligned}$$

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are group homomorphisms of abelian groups, and

$$D_{A,B} : \hom_{Ab}(A_0, B_1) \longrightarrow \hom_{Ab^2}(A, B)$$

sends each $\tau : A_0 \longrightarrow B_1$ to

$$(\tau\partial : A_1 \longrightarrow B_1, \partial'\tau : A_0 \longrightarrow B_0) .$$

- Ab^J- a more general example is obtained by letting J be an index category and $\mathbb{A}$ to be the additive functor category Ab^J: where we may define several 2-category structures by letting

  $$H(A, B) = \hom_{Ab}(\operatorname{col\,im} A, \lim B)$$

  and

  $$D_{A,B} : \hom_{Ab}(\operatorname{col\,im} A, \lim B) \longrightarrow \hom_{\mathbb{A}}(A, B)$$

  sending each

  $$\tau : \operatorname{col\,im} A \longrightarrow \lim B$$

  to

  $$(\pi_j \tau \iota_j : A_j \longrightarrow B_j)_{j \in J} ;$$

  or even by letting the elements of $H(A, B)$ to be families $(\hom_{\mathbb{A}}(A_x, B_y))_\alpha$ indexed by all the arrows $\alpha : x \longrightarrow y$ in J, such that for every pair of composable ones

  $$x \xrightarrow{\alpha} y \xrightarrow{\beta} z$$

  we have that $\tau_\beta A_\beta = B_\alpha \tau_\alpha$ as displayed below

  $$\begin{tikzcd}
  A_x \arrow[r, "A_\alpha"] \arrow[d] & A_y \arrow[r, "A_\beta"] \arrow[d] \arrow[dl, "\tau_\alpha"'] & A_z ; \arrow[d] \arrow[dl, "\tau_\beta"'] \\
  B_x \arrow[r, "B_\alpha"] & B_y \arrow[r, "B_\beta"] & B_z
  \end{tikzcd}$$

  The homomorphisms $D_{A,B}$ send each family $(\tau_\alpha)_\alpha$ to $(\tau_{1_x}) \in \hom_{\mathbb{A}}(A, B)$

In what follows we will always assume that $\mathbb{A}$ is an additive 2-category with kernels. That is, we assume given an additive category with kernels (also denoted by $\mathbb{A}$) as well as abelian groups H, and group homomorphisms $D : H \longrightarrow \hom$ and composition laws as specified above.

6.3 Pseudocategories, pseudofunctors and natural transformations

This section contains the categorical equivalence PsCat($\mathbb{A}$)$\sim$PsMor($\mathbb{A}$) between the category of internal pseudo-categories in $\mathbb{A}$ and the category of pseudo morphisms in $\mathbb{A}$, in a sense to be defined. We recall here the definitions of internal pseudo-category, pseudo-functor and natural transformation and prove

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PsCat(𝔸)∼PsMor(𝔸) by giving explicit functors. It turns out that given an appropriate 2-cell structure to PsMor(𝔸) and extending the functors to 2-functors we still obtain an equivalence of 2-categories.

The section is divided in two parts: first part contains the main result that is the equivalence PsCat(𝔸)∼PsMor(𝔸); second part extends it to natural transformations in PsCat(𝔸) and 2-cells in PsMor(𝔸).

Pseudo-categories, pseudo-functors

Recall from [5] that a internal pseudo-category in 𝔸 is a system

$$(C_1, C_0, m, d, e, c, \alpha, \lambda, \rho)$$

where C_1, C_0 are objects, m, d, e, c are morphisms of 𝔸 displayed as follows

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \overset{\xrightarrow{d}}{\underset{\xrightarrow[c]{}}{\xleftarrow{e}}} C_0, \tag{6.4}$$

α, λ, ρ are 2-cells of 𝔸 (that are isomorphisms)

$$\alpha : m\,(1 \times_{C_0} m) \longrightarrow m\,(m \times_{C_0} 1) \tag{6.5}$$

$$\lambda : m\,\langle ec, 1_{C_1} \rangle \longrightarrow 1_{C_1} \tag{6.6}$$

$$\rho : m\,\langle 1_{C_1}, ed \rangle \longrightarrow 1_{C_1}, \tag{6.7}$$

the following conditions are satisfied

$$de = 1_{c_0} = ce \tag{6.8}$$

$$dm = d\pi_2 \ , \ cm = c\pi_1 \tag{6.9}$$

$$d \circ \lambda = 1_d = d \circ \rho \ , \ c \circ \lambda = 1_c = c \circ \rho \tag{6.10}$$

$$d \circ \alpha = 1_{d\pi_3}, c \circ \alpha = 1_{c\pi_1} \tag{6.11}$$

$$\lambda \circ e = \rho \circ e. \tag{6.12}$$

and the following diagrams (of 2-cells) commute

$$\begin{tikzcd}
 & \bullet \arrow[rr, "m\circ(1_{C_1}\times_{C_0}\alpha)"] \arrow[dl, "\alpha\circ(1_{C_1}\times_{C_0}1_{C_1}\times_{C_0}m)"'] & & \bullet \arrow[dr, "\alpha\circ(1_{C_1}\times_{C_0}m\times_{C_0}1_{C_1})"] & \\
\bullet \arrow[drr, "\alpha\circ(m\times_{C_0}1_{C_1}\times_{C_0}1_{C_1})"'] & & & & \bullet \arrow[dll, "m\circ(\alpha\times_{C_0}1_{C_1})"] \\
 & & \bullet & &
\end{tikzcd} \tag{6.13}$$

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$$\begin{tikzcd}
\bullet \arrow[rr, "\alpha\circ(1_{C_1}\times_{C_0}<ec,1_{C_1}>)"] \arrow[dr, "m\circ(1_{C_1}\times_{C_0}\lambda)"'] & & \bullet \arrow[dl, "m\circ(\rho\times_{C_0}1_{C_1})"] \\
& \bullet &
\end{tikzcd} . \tag{6.14}$$

Suppose given two internal pseudo-categories in $\mathbb{A}$,

$C = (C_1, C_0, m, d, e, c, \alpha, \lambda, \rho)$ and $C' = \left(C_1', C_0', m', d', e', c', \alpha', \lambda', \rho'\right)$.

A pseudo-functor $F : C \longrightarrow C'$ is a system

$$F = (F_1, F_0, \mu, \varepsilon)$$

where $F_0 : C_0 \longrightarrow C_0'$, $F_1 : C_1 \longrightarrow C_1'$ are morphisms of $\mathbb{A}$,

$$\mu : F_1 m \longrightarrow m'\left(F_1 \times_{F_0} F_1\right)\ ,\ \varepsilon : F_1 e \longrightarrow e' F_0,$$

are 2-cells of $\mathbb{A}$ (that are isomorphisms), the following conditions are satisfied

$$d'F_1 = F_0 d\ ,\ c'F_1 = F_0 c, \tag{6.15}$$

$$d' \circ \mu = 1_{F_0 d\pi_2}\ ,\ c' \circ \mu = 1_{F_0 c\pi_1}, \tag{6.16}$$

$$d' \circ \varepsilon = 1_{F_0} = c' \circ \varepsilon \tag{6.17}$$

and the following diagrams (of 2-cells) commute

$$\begin{tikzcd}
& \bullet \arrow[r, "\mu\left(1\times_{C_0}m\right)"] \arrow[dl, "F_1\alpha"'] & \bullet \arrow[dr, "m'\left(1_{F_1}\times\mu\right)"] & \\
\bullet \arrow[dr, "\mu\left(m\times_{C_0}1\right)"'] & & & \bullet \arrow[dl, "\alpha'\left(F_1\times_{F_0}F_1\times_{F_0}F_1\right)"] \\
& \bullet \arrow[r, "m'\left(\mu\times 1_{F_1}\right)"'] & \bullet &
\end{tikzcd} , \tag{6.18}$$

$$\begin{tikzcd}
\bullet \arrow[r, "F_1\rho"] \arrow[d, "\mu\langle 1,ed\rangle"'] & \bullet \\
\bullet \arrow[r, "m'\left(1_{F_1}\times\varepsilon\right)"] & \bullet \arrow[u, "\rho' F_1"']
\end{tikzcd} , \tag{6.19}$$

$$\begin{tikzcd}
\bullet \arrow[r, "F_1\lambda"] \arrow[d, "\mu\langle ec,1\rangle"'] & \bullet \\
\bullet \arrow[r, "m'\left(\varepsilon\times 1_{F_1}\right)"] & \bullet \arrow[u, "\lambda' F_1"']
\end{tikzcd} .$$

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The composition of the pseudo-functors F and G is defined by the formula

$$GF = \left(G_0F_0, G_1F_1, \left(\mu^G \circ (F_1 \times_{F_0} F_1)\right) \cdot \left(G_1 \circ \mu^F\right), \left(\varepsilon^G \circ F_0\right) \cdot \left(G_1 \circ \varepsilon^F\right)\right) \tag{6.20}$$

where $\circ$ represents the horizontal composition in $\mathbb{A}$ and $\cdot$ represents the vertical composition, as displayed in the diagram below

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "m"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\mu^F \Downarrow"] & C_1 \arrow[d, "F_1"'] \arrow[dr, phantom, "\varepsilon^F \Downarrow"] & C_0 \arrow[l, "e"'] \arrow[d, "F_0"] \\
C_1' \times_{C_0'} C_1' \arrow[r, "m'"] \arrow[d, "G_1 \times_{G_0} G_1"'] \arrow[dr, phantom, "\mu^G \Downarrow"] & C_1' \arrow[d, "G_1"'] \arrow[dr, phantom, "\varepsilon^G \Downarrow"] & C_0' \arrow[l, "e'"'] \arrow[d, "G_0"] \\
C_1'' \times_{C_0''} C_1'' \arrow[r, "m''"] & C_1'' & C_0'' \arrow[l, "e''"']
\end{tikzcd} .$$

Composition of pseudo-functors is associative and there is an identity pseudo-functor for every pseudo-category, namely the pseudo-functor

$$1_C = (1_{C_0}, 1_{C_1}, 1_m, 1_e)$$

for the pseudo-category

$$C = (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho) .$$

Denote by $\mathrm{PsCat}(\mathbb{A})$ the category of pseudo-categories and pseudo-functors internal to $\mathbb{A}$. Since $\mathbb{A}$ is an additive 2-category with kernels, conditions (6.8) and (6.9) imply that

$$\begin{aligned}
C_1 &\cong \ker c \oplus C_0 \ , \ C_1 \times_{C_0} C_1 \cong \ker c \oplus \ker c \oplus C_0 \\
d &\cong \begin{pmatrix} 0 & 1 \end{pmatrix} \ , \ c \cong \begin{pmatrix} k & 1 \end{pmatrix} \\
e &\cong \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \ m \cong \begin{pmatrix} u & v & w \\ 0 & 0 & 1 \end{pmatrix} , \\
\pi_1 &\cong \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \ \pi_2 \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 1 \end{pmatrix} , \\
1 \times_{C_0} m &\cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & v & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \ m \times_{C_0} 1 \cong \begin{pmatrix} u & v & wk & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \\
\langle ec, 1_{C_1} \rangle &\cong \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \langle 1_{C_1}, ed \rangle \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} ,
\end{aligned}$$

with $k : \ker c \longrightarrow C_0$ the composite of c with the universal arrow from the kernel, $u, v : \ker c \longrightarrow \ker c, w : \ker c \longrightarrow C_0$ such that

$$ku = k = kv \ , \ kw = 0.$$

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Conditions (6.10), (6.11), (6.12) imply

$$\alpha \cong \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \ \lambda \cong \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix} , \ \rho \cong \begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix}$$

where $\alpha_1, \alpha_2, \alpha_3, \lambda, \rho \in H(\ker c, \ker c)$, $\alpha_4, \eta \in H(C_0, \ker c)$ are such that

$$\begin{aligned} k\alpha_i &= 0 \ , \ i = 0,1,2,3. \\ k\lambda &= k\rho = 0 \ , \ k\eta = 0 \end{aligned}$$

$$\begin{aligned} u &= 1 - D(\rho) \\ v &= 1 - D(\lambda) \\ w &= -D(\eta) \end{aligned}$$

$$\begin{aligned} D(\alpha_1) &= u^2 - u \\ D(\alpha_2) &= uv - vu \\ D(\alpha_3) &= uwk + v - v^2 \\ D(\alpha_4) &= uw - vw. \end{aligned} \tag{6.21}$$

Conditions (6.15), (6.16), and (6.17) imply that a morphism in PsCat($\mathbb{A}$)

$$F = (F_1, F_0, \mu, \varepsilon) : C \longrightarrow C'$$

is of the form

$$F_1 \cong \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \ , \ F_0 = g \ , \ \mu \cong \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \varepsilon \cong \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$$

where $f : \ker c \longrightarrow \ker c'$, $g : C_0 \longrightarrow C_0'$, $\mu_1, \mu_2 \in H(\ker c, \ker c')$ and $\mu_3, \varepsilon \in H(C_0, \ker c)$ are such that

$$k'f = gk$$

$$\begin{aligned} k'\mu_i &= 0 \ , \ i = 1,2 \\ k'\mu_3 &= 0 \ , \ k'\varepsilon = 0 \end{aligned} \tag{6.22}$$

$$\begin{aligned} D(\varepsilon) &= -h \\ D(\mu_1) &= fD(\rho) - D(\rho') f && (6.23) \\ D(\mu_2) &= fD(\lambda) + D(\rho'\varepsilon) k - D(\varepsilon) k - D(\lambda') f && (6.24) \\ D(\mu_3) &= fD(\eta) + D(\rho'\varepsilon) + D(\lambda'\varepsilon) - D(\varepsilon) - D(\eta') g. \end{aligned}$$

Next we define the category PsMor($\mathbb{A}$) so that it is equivalent to the category PsCat($\mathbb{A}$). This result may also be found in [3]. We choose to repeat it here in order to make it self contained and also better readable The description of pseudo-functors is however a new result.

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The definition of PsMor($\mathbb{A}$) is as follows:
-objects:

$$K = (k : A \longrightarrow B, \lambda, \rho, \eta)$$

with $A, B \in \mathbb{A}$, $\lambda, \rho \in H(A, A), \eta \in H(B, A)$, such that

$$k\lambda = k\rho = 0 \ , \ k\eta = 0. \tag{6.25}$$

-morphisms:

$$F : K \longrightarrow K'$$

with $F = (f : A \longrightarrow A', g : B \longrightarrow B', \varepsilon \in H(B, A'))$ such that

$$gk = k'f \ , \ k'\varepsilon = 0. \tag{6.26}$$

The composition is defined by the formula

$$(f', g', \varepsilon')(f, g, \varepsilon) = (f'f, g'g, f'\varepsilon + \varepsilon' g),$$

and the identity of $(A, B, k; \lambda, \rho, \eta)$ is the morphism $(1_A, 1_B, 0)$.

The main result of this section is the following.

Theorem 62 *Let $\mathbb{A}$ be an additive 2-category with kernels. The category PsCat($\mathbb{A}$) is categorically equivalent to the category PsMor($\mathbb{A}$).*

In order to prove PsCat($\mathbb{A}$) $\sim$ PsMor($\mathbb{A}$) we define the two functors

$$\begin{aligned} U &: \text{PsCat}(\mathbb{A}) \longrightarrow \text{PsMor}(\mathbb{A}) \\ V &: \text{PsMor}(\mathbb{A}) \longrightarrow \text{PsCat}(\mathbb{A}) \end{aligned}$$

in the following way

$$\begin{aligned} U(C_1, C_0, m, d, e, c, \alpha, \lambda, \rho) &= (\ker c, C_0, k, \lambda, \rho, \eta) \\ U(F_1, F_0, \mu, \varepsilon) &= (f, g, \varepsilon) \end{aligned}$$

where $k, \lambda, \rho, \eta, f, g, \varepsilon$ are as above, and it is a straightforward calculation to check that U is well defined;
On objects

$$V(A, B, k, \lambda, \rho, \eta) = (A \oplus B, B, m, d, e, c, \alpha, \lambda, \rho)$$

where

$$\begin{aligned} m &= \begin{pmatrix} 1 - D\rho & 1 - D\lambda & -D\eta \\ 0 & 0 & 1 \end{pmatrix} \ , \ e = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ d &= \begin{pmatrix} 0 & 1 \end{pmatrix} \ , \ c = \begin{pmatrix} k & 1 \end{pmatrix} \\ \alpha &= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ \lambda = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix} \ , \ \rho = \begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix} \end{aligned}$$

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and

$$\begin{aligned}\alpha_1 &= \rho\rho - \rho\\ \alpha_2 &= \rho\lambda - \lambda\rho\\ \alpha_3 &= \lambda - \lambda\lambda - \eta k + \rho\eta k\\ \alpha_4 &= \rho\eta - \lambda\eta.\end{aligned}$$

On morphisms

$$V\left(f,g,\varepsilon\right) = \left(\begin{pmatrix} f & -D\varepsilon\\ 0 & g\end{pmatrix}, g, \begin{pmatrix} \mu_1 & \mu_2 & \mu_3\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}\varepsilon\\ 0\end{pmatrix}\right)$$

where

$$\begin{aligned}\mu_1 &= f\rho - \rho' f\\ \mu_2 &= f\lambda - \lambda' f - \varepsilon k + \rho'\varepsilon k\\ \mu_3 &= f\eta + \rho'\varepsilon + \lambda'\varepsilon - \eta' g - \varepsilon.\end{aligned}$$

In order to show that V is well defined one has to prove that conditions (6.14), (6.13), (6.18), (6.19) are satisfied.

The triangle condition

$$(m\circ(\rho\times 1))\cdot(\alpha\circ(1\times\langle ec,1\rangle)) = m\circ(1\times\lambda)$$

may be written as

$$m\begin{pmatrix}\rho & \eta k & \eta\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \alpha\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = m\begin{pmatrix}0 & 0 & 0\\ 0 & \lambda & \eta\\ 0 & 0 & 0\end{pmatrix}$$

and it is equivalent to the following system of equations

$$\begin{cases} -D(\rho)\rho + \rho\rho = 0\\ -D(\rho)\eta k - \lambda\lambda + \rho\eta k + D(\lambda)\lambda = 0\\ -D(\rho)\eta + \rho\eta - \lambda\eta + D(\lambda)\eta = 0\end{cases}$$

which is satisfied because $\mathbb{A}$ is an additive 2-category and we have (6.3).

The pentagon condition

$$(\alpha\circ(m\times 1\times 1))\cdot(\alpha\circ(1\times 1\times m)) = (m\circ(\alpha\times 1))\cdot(\alpha\circ(1\times m\times 1))\cdot(m\circ(1\times\alpha)),$$

may be written as

$$\alpha\begin{pmatrix} u \quad v & wk \quad wk \quad w\\ zeros\,(3,2) & eye(3)\end{pmatrix} + \alpha\begin{pmatrix} eye(2) & zeros(2,3)\\ zeros\,(2) & m\end{pmatrix} =$$

$$= m\begin{pmatrix} \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 k \quad \alpha_4\\ zeros(2,5)\end{pmatrix} + \alpha\begin{pmatrix} 1 \quad 0 \quad 0 & 0 \quad 0\\ 0 \quad u \quad v & wk \quad w\\ zeros(2,3) & eye(2)\end{pmatrix}$$

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where zeros(n,m) represents a $n \times m$ matrix of zeros and eye(n) represents the identity matrix of order n; the above equality of matrices is equivalent to the following set of equations

$$-\rho\rho D(\rho) + \rho D(\rho) + D(\rho)\rho\rho - D(\rho)\rho = 0$$

$$\begin{aligned} &- \rho\rho D(\lambda) + \rho D(\lambda) + D(\rho)\rho\lambda - D(\rho)\lambda\rho - \rho\lambda + \rho\lambda D(\rho) + \lambda\rho \\ &\qquad - \lambda\rho D(\rho) + D(\lambda)\rho\rho - D(\lambda)\rho = 0 \end{aligned}$$

$$\begin{aligned} &- \rho\rho D(\eta)k + \rho D(\eta)k - \lambda D(\rho) + \lambda\lambda D(\rho) + \eta k D(\rho) - \rho\eta k D(\rho) + \\ &\qquad D(\rho)\lambda - D(\rho)\lambda\lambda - D(\rho)\eta k + D(\rho)\rho\eta k + \rho\lambda D(\lambda) - \\ &\qquad \lambda\rho D(\lambda) - \rho\lambda + \lambda\rho + D(\lambda)\rho\lambda - D(\lambda)\lambda\rho = 0 \end{aligned}$$

$$\begin{aligned} &- \rho\rho D(\eta)k + \rho D(\eta)k - \lambda D(\lambda) + \lambda\lambda D(\lambda) + \eta k D(\lambda) - \rho\eta k D(\lambda) \\ &\qquad + \lambda\eta k + D(\rho)\rho\eta k - D(\rho)\lambda\eta k + \rho\lambda D(\eta)k \\ &\qquad - \lambda\rho D(\eta)k - \rho\eta k + D(\lambda)\lambda - D(\lambda)\lambda\lambda - D(\lambda)\eta k + D(\lambda)\rho\eta k = 0 \end{aligned}$$

$$\begin{aligned} &- \rho\rho D(\eta) + \rho D(\eta) - \lambda D(\eta) + \lambda\lambda D(\eta) + \eta k D(\eta) - \\ &\qquad \rho\eta k D(\eta) + D(\rho)\rho\eta - D(\rho)\lambda\eta + \rho\lambda D(\eta) - \\ &\qquad \lambda\rho D(\eta) - \rho\eta + \lambda\eta + D(\lambda)\rho\eta - D(\lambda)\lambda\eta = 0 \end{aligned}$$

which becomes trivial by (6.3) and (6.25).

Condition

$$(\rho' \circ F_1) \cdot (m' \circ (1_{F_1} \times \varepsilon)) \cdot (\mu \circ \langle 1, ed \rangle) = F_1 \circ \rho$$

in PsCat($\mathbb{A}$) may be written as

$$\rho' F_1 + m' \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \\ 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = F_1 \rho$$

and it is equivalent to $-\rho' D(\epsilon) - D(\lambda')\epsilon + \rho'\epsilon + \lambda'\epsilon = 0$ which is trivial.

Condition

$$\left(\lambda' \circ F_1\right) \cdot (m' \circ (\varepsilon \times 1_{F_1})) \cdot (\mu \circ \langle ec, 1 \rangle) = F_1 \circ \lambda$$

may be written as

$$\lambda' F_1 + m' \begin{pmatrix} \varepsilon k & \varepsilon \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = F_1 \lambda$$

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and it is equivalent to

$$\begin{cases} -D(\rho')\epsilon k + \rho'\epsilon k = 0 \\ -\lambda' D(\epsilon) - D(\rho')\epsilon + \rho'\epsilon + \lambda'\epsilon = 0 \end{cases}$$

which is a trivial system.

The hexagon condition

$$\begin{aligned} &(m' \circ (\mu \times 1_{F_1})) \cdot (\mu \circ (m \times_{C_0} 1)) \cdot (F_1 \circ \alpha) = \\ &\qquad = (\alpha' \circ (F_1 \times_{F_0} F_1 \times_{F_0} F_1)) \cdot (m' \circ (1_{F_1} \times \mu)) \cdot (\mu \circ (1 \times_{C_0} m)) \end{aligned}$$

may be written as

$$m' \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 k & \mu_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} u & v & wk & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + F_1 \alpha =$$

$$= \alpha' \begin{pmatrix} f & hk & hk & h \\ 0 & f & hk & h \\ 0 & 0 & f & h \\ 0 & 0 & 0 & g \end{pmatrix} + m' \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & \mu_2 & \mu_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & v & w \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and it is equivalent to the following set of equations

$$-D(\rho')f\rho + D(\rho')\rho' f - f\rho D(\rho) + \rho' f D(\rho) + f\rho\rho - \rho'\rho' f = 0$$

$$\begin{aligned} &-D(\rho')f\lambda + D(\rho')\lambda' f + D(\rho')\epsilon k - D(\rho')\rho'\epsilon k - f\rho D(\lambda) + \rho' f D(\lambda) + f\rho\lambda \\ &-f\lambda\rho + \rho'\rho' D(\epsilon)k - \rho' D(\epsilon)k - \rho'\lambda' f + \lambda'\rho' f + D(\lambda')f\rho \\ &-D(\lambda')\rho' f + f\lambda D(\rho) - \lambda' f D(\rho) - \epsilon k D(\rho) + \rho'\epsilon k D(\rho) = 0 \end{aligned}$$

$$\begin{aligned} &\lambda'\epsilon k - \eta' g k - D(\rho')f\eta k - D(\rho')\rho'\epsilon k - D(\rho')\lambda'\epsilon k + D(\rho')\eta' g k + D(\rho')\epsilon k \\ &-f\rho D(\eta)k + \rho' f D(\eta)k - f\lambda\lambda + f\rho\eta k + \rho'\rho' D(\epsilon)k - \rho' D(\epsilon)k + \rho'\lambda' D(\epsilon)k \\ &-\lambda'\rho' D(\epsilon)k + \lambda'\lambda' f + \eta' k' f - \rho'\eta' k' f + D(\lambda')f\lambda - D(\lambda')\lambda' f - D(\lambda')\epsilon k \\ &+D(\lambda')\rho'\epsilon k + f\lambda D(\lambda) - \lambda' f D(\lambda) - \epsilon k D(\lambda) + \rho'\epsilon k D(\lambda) = 0 \end{aligned}$$

$$\begin{aligned} &-D(\rho')f\eta - D(\rho')\rho'\epsilon - D(\rho')\lambda'\epsilon + D(\rho')\eta' g + D(\rho')\epsilon - f\rho D(\eta) + \rho' f D(\eta) \\ &+f\rho\eta - f\lambda\eta + \rho'\rho' D(\epsilon) - \rho' D(\epsilon) + \rho'\lambda' D(\epsilon) - \lambda'\rho' D(\epsilon) + \lambda' D(\epsilon) - \lambda'\lambda' D(\epsilon) \\ &-\eta' k' D(\epsilon) + \rho'\eta' k' D(\epsilon) - \rho'\eta' g + \lambda'\eta' g + D(\lambda')f\eta + D(\lambda')\rho'\epsilon + D(\lambda')\lambda'\epsilon \\ &-D(\lambda')\eta' g - D(\lambda')\epsilon + f\lambda D(\eta) - \lambda' f D(\eta) - \epsilon k D(\eta) + \rho'\epsilon k D(\eta) = 0 \end{aligned}$$

which is trivial because (6.3) , (6.25) and (6.26).

It follows immediately that

$$UV = 1 \ , \ VU \cong 1$$

with a natural isomorphism.

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Natural transformations

Again, we repeat here the definition of natural transformation. Suppose

$$\begin{aligned} C &= (C_1, C_0, d, c, e, m, \alpha, \lambda, \rho), \\ C' &= \left(C'_1, C'_0, d', c', e', m', \alpha', \lambda', \rho'\right) \end{aligned} \tag{6.27}$$

are pseudo-categories in $\mathbb{A}$ and

$$\begin{aligned} F &= \left(F_1, F_0, \mu^F, \varepsilon^F\right), \\ G &= \left(G_1, G_0, \mu^G, \varepsilon^G\right). \end{aligned} \tag{6.28}$$

are pseudo-functors from C to C'.

A *natural transformation* $\theta : F \longrightarrow G$ is a pair $\theta = (\theta_1, \theta_0)$ of 2-cells of $\mathbb{A}$,

$$\begin{aligned} \theta_0 &: F_0 \longrightarrow G_0 \\ \theta_1 &: F_1 \longrightarrow G_1 \end{aligned}$$

satisfying

$$\begin{aligned} d' \circ \theta_1 &= \theta_0 \circ d \\ c' \circ \theta_1 &= \theta_0 \circ c \end{aligned}$$

and the commutativity of the following diagrams of 2-cells

$$\begin{tikzcd} \bullet \arrow[r, "\theta_1 \circ m"] \arrow[d, "\mu^F"'] & \bullet \arrow[d, "\mu^G"] \\ \bullet \arrow[r, "m' \circ \left(\theta_1 \times_{\theta_0} \theta_1\right)"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, "\theta_1 \circ e"] \arrow[d, "\varepsilon^F"'] & \bullet \arrow[d, "\varepsilon^G"] \\ \bullet \arrow[r, "e' \circ \theta_0"] & \bullet \end{tikzcd} .$$

The vertical and horizontal compositions are defined componentwise with the respective composition of 2-cells in $\mathbb{A}$. Hence PsCat($\mathbb{A}$) with internal pseudo-categories, pseudo-functors and natural transformations is a 2-category.

We may also define a 2-cell structure in PsMor($\mathbb{A}$) so that PsCat($\mathbb{A}$) is still equivalent to PsMor($\mathbb{A}$).

Let $K = (A, B, k, \lambda, \rho, \eta)$, $K' = \left(A', B', k', \lambda', \rho', \eta'\right)$ be two objects of PsMor($\mathbb{A}$), and let $f = (f_1, f_0, \varepsilon_f) : K \longrightarrow K'$, $g = (g_1, g_0, \varepsilon_g) : K \longrightarrow K'$ be two morphisms in PsMor($\mathbb{A}$), a 2-cell $\theta : f \longrightarrow g$, is a pair

$$\theta = (\theta_1, \theta_0),$$

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where $\theta_1 \in H(A, A'), \theta_0 \in H(B, B')$ are such that

$$\begin{aligned} D(\theta_1) &= g_1 - f_1 \\ D(\theta_0) &= g_0 - f_0 \\ k'\theta_1 &= \theta_0 k. \end{aligned} \tag{6.29}$$

Vertical composition between $\theta : f \longrightarrow g$ and $\vartheta : g \longrightarrow h$ is given by

$$\vartheta \cdot \theta = (\vartheta_1 + \theta_1, \vartheta_0 + \theta_0)$$

while horizontal composition between $\theta : f \longrightarrow g$ and $\theta' : f' \longrightarrow g'$ is given by

$$\theta' \circ \theta = \left(\theta'_1\theta_1 + \theta'_1 f_1 + f'_1\theta_1, \theta'_0\theta_0 + \theta'_0 f_0 + f'_0\theta_0\right).$$

The equivalence is obtained by letting

$$V(\theta_1, \theta_0) = \left(\begin{pmatrix} \theta_1 & \varepsilon_f - \varepsilon_g \\ 0 & \theta_0 \end{pmatrix}, \theta_0\right).$$

To see that V is well defined and PsCat($\mathbb{A}$) still equivalent to PsMor($\mathbb{A}$) we have the following.

In PsCat($\mathbb{A}$) condition $d' \circ \theta_1 = \theta_0 \circ d$ implies that

$$\theta_1 = \begin{pmatrix} \theta_1 & \bar{\theta} \\ 0 & \theta_0 \end{pmatrix},$$

where $\theta_1 \in H(A, A'), \bar{\theta} \in H(B, A'), \theta_0 \in H(B, B')$ are such that

$$\begin{aligned} D(\theta_1) &= G_1 - F_1 \\ D(\bar{\theta}) &= D(\varepsilon_F) - D(\varepsilon_G) \\ D(\theta_0) &= G_0 - F_0, \end{aligned} \tag{6.30}$$

while condition $c' \circ \theta_1 = \theta_0 \circ c$ determines that

$$\begin{aligned} k'\theta_1 &= \theta_0 k \\ k'\bar{\theta} &= 0. \end{aligned}$$

Condition $(e' \circ \theta_0) \cdot \varepsilon_F = \varepsilon_G \cdot (\theta_1 \circ e)$ in PsCat($\mathbb{A}$) may be written as

$$\begin{pmatrix} 0 \\ \theta_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_F \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_G \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{\theta} \\ \theta_0 \end{pmatrix},$$

which is equivalent to

$$\bar{\theta} = \varepsilon_F - \varepsilon_G.$$

Then $k'\bar{\theta} = 0$ and $D(\bar{\theta}) = D(\varepsilon_F) - D(\varepsilon_G)$. Condition

$$(m' \circ (\theta_1 \times_{\theta_0} \theta_1)) \cdot \mu_F = \mu_G \cdot (\theta_1 \circ m), \tag{6.31}$$

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in PsCat($\mathbb{A}$) is translated to[1]

$$m'\begin{pmatrix} \theta_1 & (\varepsilon_f-\varepsilon_g)\,k & \varepsilon_f-\varepsilon_g \\ 0 & \theta_1 & \varepsilon_f-\varepsilon_g \\ 0 & 0 & \theta_0 \end{pmatrix}+\mu_f=\mu_g+\begin{pmatrix} \theta_1 & \varepsilon_f-\varepsilon_g \\ 0 & \theta_0 \end{pmatrix}m$$

which is equivalent to the following equations

$$-D(\rho')\theta_1+f_1\rho-\rho' f_1-g_1\rho+\rho' g_1+\theta_1 D(\rho)=0$$

$$-D(\rho')\varepsilon_f k+D(\rho')\varepsilon_g k-D(\lambda')\theta_1+f_1\lambda-\lambda' f_1+\rho'\varepsilon_f k-g_1\lambda+\lambda' g_1-\rho'\varepsilon_g k+\theta_1 D(\lambda)=0$$

$$\begin{aligned}-D(\rho')\varepsilon_f+D(\rho')\varepsilon_g-D(\lambda')\varepsilon_f+D(\lambda')\varepsilon_g-D(\eta')\theta_0+f_1\eta+\rho'\varepsilon_f+{}\\ \lambda'\varepsilon_f-\eta' f_0-g_1\eta-\rho'\varepsilon_g-\lambda'\varepsilon_g+\eta' g_0+\theta_1 D(\eta)=0\end{aligned}$$

that are all trivial because of (6.3) , (6.25), (6.26) and (6.29).

The vertical composition in PsCat($\mathbb{A}$), between $\theta : F \longrightarrow G$ and $\vartheta : G \longrightarrow H$ is given by

$$\vartheta\cdot\theta=\left(\begin{pmatrix} \vartheta_1+\theta_1 & \varepsilon_f-\varepsilon_h \\ 0 & \vartheta_0+\theta_0 \end{pmatrix},\vartheta_0+\theta_0\right)$$

while horizontal composition between $\theta : F \longrightarrow G$ and $\theta' : F' \longrightarrow G'$ is given by

$$\theta'\circ\theta=\left(\begin{pmatrix} \theta_1'\theta_1+\theta_1' f_1+f_1'\theta_1 & * \\ 0 & \theta_0'\theta_0+\theta_0' f_0+f_0'\theta_0 \end{pmatrix},\theta_0'\theta_0+\theta_0' f_0+f_0'\theta_0\right)$$

where $*$ is for

$$-\theta_1'\varepsilon_g+\theta_1'\varepsilon_f-\varepsilon_{g'}\theta_0+\varepsilon_{f'}\theta_0-\theta_1'\varepsilon_f-\varepsilon_{g'}f_0+\varepsilon_{f'}f_0-f_1'\varepsilon_g+f_1'\varepsilon_f-\varepsilon_{f'}\theta_0,$$

which is equal to

$$f_1'\varepsilon_f+\varepsilon_{f'}f_0-g_1'\varepsilon_g-\varepsilon_{g'}g_0.$$

and they are well defined.

---

[1] To describe

$$(\theta_1\times_{\theta_0}\theta_1)\in H\left(A\oplus A\oplus B,A'\oplus A'\oplus B'\right),$$

one notes that the projections of domain and codomain are given as in the following diagram

$$\begin{tikzcd} A\oplus A\oplus B \arrow[r, "\left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}1&0&0\\0&k&1\end{smallmatrix}\right)"'] & A\oplus B \arrow[d, "(k\ 1)"] \\ A\oplus B \arrow[r, "(0\ 1)"'] & B \end{tikzcd}$$

and conditions

$$\begin{aligned}\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}(\theta_1\times_{\theta_0}\theta_1) &= \theta_1\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}\\ \begin{pmatrix}1&0&0\\0&k'&1\end{pmatrix}(\theta_1\times_{\theta_0}\theta_1) &= \theta_1\begin{pmatrix}1&0&0\\0&k&1\end{pmatrix}\end{aligned}$$

will give the result.

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6.4 Pseudo-natural transformations and modifications

Following the results of previous section we answer the following question: what should be the structure of pseudo-cells and tetra-cells that would correspond, via the equivalence stated above, to pseudo-natural transformations and modifications in $\mathrm{PsCat}(\mathbb{A})$.

First we give the detailed definition of $\mathrm{PsMor}(\mathbb{A})$, with objects, morphisms and 2-cells as above, and also add (what we call) pseudo-cells and tetra-cells that will be in correspondence with the pseudo-natural transformations and modifications in $\mathrm{PsCat}(\mathbb{A})$. Next we give the explicit equivalence between both structures. It is easy to see that it is indeed an equivalence, however, it involves a big amount of calculations to see that it is well defined. In order to not overload this presentation with all the calculations involved, we choose to add them only at the end as Appendix B. We also recall in Appendix A (from [5]) the definition of pseudo-natural transformation and modification internal to an arbitrary 2-category.

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The Definition

The complete definition of $\mathrm{PsMor}(\mathbb{A})$ is as follows (and we repeat here the results of previous section):

- objects:
$$K=(A,B,k,\lambda,\rho,\eta)$$
where $A,B\in\mathbb{A}$, $k:A\longrightarrow B$ is a morphism of $\mathbb{A}$, and $\lambda,\rho\in H(A,A),\eta\in H(B,A)$ are such that
$$k\lambda=k\rho=0\ ,\ k\eta=0. \tag{6.32}$$
- morphisms:
a morphism $f:K\longrightarrow K'$ from $K=(A,B,k,\lambda,\rho,\eta)$ to $K'=\left(A',B',k',\lambda',\rho',\eta'\right)$ is of the form
$$f=(f_1,f_0,\varepsilon_f)$$
where $f_1:A\longrightarrow A'$, $f_0:B\longrightarrow B'$ are morphisms of $\mathbb{A}$, and $\varepsilon_f\in H(B,A')$ are such that
$$f_0k=k'f_1\ ,\ k'\varepsilon_f=0. \tag{6.33}$$
- 2-cells:
a 2-cell $\theta:f\longrightarrow g$ from $f=(f_1,f_0,\varepsilon_f)$ to $g=(g_1,g_0,\varepsilon_g)$ both from K to K' is of the form
$$\theta=(\theta_1,\theta_0),$$
where $\theta_1\in H(A,A'),\theta_0\in H(B,B')$ are such that
$$\begin{aligned}\theta_0k&=k'\theta_1\\ D(\theta_1)&=g_1-f_1\\ D(\theta_0)&=g_0-f_0.\end{aligned} \tag{6.34}$$

- pseudo-cells (in correspondence with the pseudo-natural transformations of PsCat($\mathbb{A}$)):
  a pseudo-cell $T : f \longrightarrow g$ from $f = (f_1, f_0, \varepsilon_f)$ to $g = (g_1, g_0, \varepsilon_g)$ is of the form
  $$T = (t, \tau)$$
  where $t : B \longrightarrow A'$ is a morphism of $\mathbb{A}$, and $\tau \in H(A, A')$ are such that
  $$\begin{aligned} k't &= g_0 - f_0 \\ k'\tau &= 0 \\ D(\tau) &= tk - g_1 - D(\rho')\,tk + D(\rho')\,g_1 + f_1 - D(\lambda')\,f_1. \end{aligned} \tag{6.35}$$

- tetra-cells (in correspondence with the modifications of PsCat($\mathbb{A}$)):
  a tetra-cell $\Phi \in H(B, A')$, displayed as follows
  $$\begin{tikzcd} f \arrow[rr, "T"{name=U}] \arrow[d, "\theta"'] & & g \arrow[d, "\theta'"] \\ h \arrow[rr, "T'"'{name=D}] & & l \arrow[Rightarrow, from=U, to=D, "\Phi"] \end{tikzcd}$$
  from the pseudo-cell $T = (t, \tau) : f \longrightarrow g$ to $T' = (t', \tau') : h \longrightarrow l$ and from the 2-cell $\theta = (\theta_1, \theta_0) : f \longrightarrow h$ to $\theta' = (\theta'_1, \theta'_0) : g \longrightarrow l$ with $f = (f_1, f_0, \varepsilon_f)\,, g = (g_1, g_0, \varepsilon_g)\,, h = (h_1, h_0, \varepsilon_h)\,, l = (l_1, l_0, \varepsilon_l)$ morphisms from $K = (A, B, k, \lambda, \rho, \eta)$ to $K' = (A', B', k', \lambda', \rho', \eta')$, is such that
  $$\begin{aligned} D(\Phi) &= t' - t \\ k'\Phi &= \theta'_0 - \theta_0 \\ \Phi k &= \tau' - \tau + \theta'_1 - \theta_1 - \rho' l_1 + \rho' g_1 - \lambda' h_1 - \lambda' f_1 + \rho' t' k - \rho' tk. \end{aligned}$$

Composition of morphisms is given by
$$(g_1, g_0, \varepsilon_g)(f_1, f_0, \varepsilon_f) = (g_1 f_1, g_0 f_0, g_1 \varepsilon_f + \varepsilon_g f_0)\,;$$
the identity morphism of $K = (A, B, k, \lambda, \rho, \eta)$ is
$$1_K = (1_A, 1_B, 0)\,.$$

Consider $K = (A, B, k, \lambda, \rho, \eta)$, $K' = (A', B', k', \lambda', \rho', \eta')$ objects in PsMor($\mathbb{A}$), and $f = (f_1, f_0, \varepsilon_f) : K \longrightarrow K'$, $g = (g_1, g_0, \varepsilon_g) : K \longrightarrow K'$ morphisms in PsMor($\mathbb{A}$).

Vertical composition between $\theta : f \longrightarrow g$ and $\vartheta : g \longrightarrow h$ is given by
$$\vartheta \cdot \theta = (\vartheta_1 + \theta_1, \vartheta_0 + \theta_0)$$
while horizontal composition between $\theta : f \longrightarrow g$ and $\theta' : f' \longrightarrow g'$ is given by
$$\theta' \circ \theta = (\theta'_1 \theta_1 + \theta'_1 f_1 + f'_1 \theta_1, \theta'_0 \theta_0 + \theta'_0 f_0 + f'_0 \theta_0)\,;$$

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identity 2-cell for $f = (f_1, f_0, \varepsilon_f) : K \longrightarrow K'$ is $1_f = (0 \in H(A, A'), 0 \in H(B, B'))$.

Tensor $S \otimes T$ between pseudo-cells $S = (s, \sigma) : g \longrightarrow h$ and $T = (t, \tau) : f \longrightarrow g$ where f, g, h are morphisms from K to K' as above, is the pseudo-cell

$$S \otimes T = (s \otimes t, \sigma \otimes \tau)$$

where

$$\begin{aligned} s \otimes t &= s + t - D(\rho')\, s - D(\lambda')\, t - D(\eta')\, f_0, \\ \sigma \otimes \tau &= \rho'\lambda' tk - \eta' k' f_1 + \rho'\eta' k' f_1 + \tau + \sigma - \rho' sk + \rho'\rho' sk - \rho' g_1 - \lambda' tk + \lambda' g_1. \end{aligned}$$

Identity pseudo-cell for a morphism $f = (f_1, f_0, \varepsilon_f) : K \longrightarrow K'$ is the pseudo-cell

$$id_f = (0, \rho' f_1 - \lambda' f_1).$$

Composition and tensor of tetra-cells:

$$\Phi \cdot \Gamma = \Phi + \Gamma$$

$$\Phi \otimes \Phi' = \Phi + \Phi' - \rho' t - \lambda' s' - \eta' h_0 + \eta' f_0 \tag{6.36}$$

for Φ, Γ, Φ' as follows

$$\begin{tikzcd} f \arrow[r, "T" description] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \Phi"] & g \arrow[d, "\theta'"] \\ h \arrow[r, "T'" description] & l \end{tikzcd}$$

$$\begin{tikzcd} f \arrow[r, "T'" description] \arrow[d, "\theta'"'] \arrow[dr, phantom, "\Downarrow \Gamma"] & g \arrow[d, "\theta''"] \\ h \arrow[r, "T''" description] & l \end{tikzcd}$$

$$\begin{tikzcd} f \arrow[r, "S" description] \arrow[d, "\theta'"'] \arrow[dr, phantom, "\Downarrow \Phi'"] & g \arrow[d, "\theta''"] \\ h \arrow[r, "S'" description] & l \end{tikzcd}$$

Identity tetra-cell (with respect to composition) for the pseudo-cell $T = (t, \tau)$ is

$$1_T = 0 \in H(B, A'),$$

while the identity tetra-cell (with respect to tensor) for the 2-cell $\theta = (\theta_1, \theta_0)$ is

$$id_\theta = 0 \in H(B, A').$$

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The Equivalence

The functor U :PsCat($\mathbb{A}$) $\longrightarrow$PsMor($\mathbb{A}$) is defined, with the obvious restrictions, (for a pseudo-category C, a pseudo-functor F, a natural-transformation $\boldsymbol{\theta}$, a pseudo-natural transformation T and a modification $\mathbf{\Phi}$, in PsCat($\mathbb{A}$), as above and in Appendix A) as follows,

$$\begin{aligned} U(C) &= (\ker c, C_0, k, \lambda, \rho, \eta) \\ U(F) &= (f_1, f_0, \varepsilon) \\ U(\boldsymbol{\theta}) &= (\theta_1, \theta_0) \\ U(T) &= (t, \tau) \\ U(\mathbf{\Phi}) &= \Phi. \end{aligned}$$

Considering $K = (A, B, k, \lambda, \rho, \eta)$ and $K' = \left(A', B', k', \lambda', \rho', \eta'\right)$ objects of PsMor($\mathbb{A}$), $f = (f_1, f_0, \varepsilon_f), g = (g_1, g_0, \varepsilon_g), h = (h_1, h_0, \varepsilon_h)$ and $l = (l_1, l_0, \varepsilon_l)$ morphisms from K to K', $\theta = (\theta_1, \theta_0) : f \longrightarrow h$ to $\theta' = \left(\theta_1', \theta_0'\right) : g \longrightarrow l$ 2-cells, $T = (t, \tau) : f \longrightarrow g$ and $T' = (t', \tau') : h \longrightarrow l$ pseudo-cells and

$$\begin{tikzcd}
f \arrow[r, "T"{name=U}] \arrow[d, "\theta"'] & g \arrow[d, "\theta'"] \\
h \arrow[r, "T'"'{name=D}] & l
\arrow[Rightarrow, from=U, to=D, "\Phi"]
\end{tikzcd}$$

a tetra-cell in PsMor($\mathbb{A}$), we define the functor

$$V : \text{PsMor}(\mathbb{A}) \longrightarrow \text{PsCat}(\mathbb{A})$$

in the following way:

$$\begin{aligned} V(K) &= (A \oplus B, B, m, d, e, c, \alpha, \lambda, \rho) \\ V(f) &= \left(\begin{pmatrix} f_1 & -D\varepsilon \\ 0 & f_0 \end{pmatrix}, f_0, \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_f \\ 0 \end{pmatrix} \right) \\ V(\theta) &= \left(\begin{pmatrix} \theta_1 & \varepsilon_f - \varepsilon_g \\ 0 & \theta_0 \end{pmatrix}, \theta_0 \right) \\ V(T) &= \left(\begin{pmatrix} t \\ f_0 \end{pmatrix}, \begin{pmatrix} \tau & \overline{\tau} \\ 0 & 0 \end{pmatrix} \right) \\ V(\Phi) &= \begin{pmatrix} \Phi \\ \theta_0 \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} m &= \begin{pmatrix} 1 - D\rho & 1 - D\lambda & -D\eta \\ 0 & 0 & 1 \end{pmatrix} \ , \ e = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ d &= \begin{pmatrix} 0 & 1 \end{pmatrix} \ , \ c = \begin{pmatrix} k & 1 \end{pmatrix} \\ \alpha &= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ \lambda = \begin{pmatrix} \lambda & \eta \\ 0 & 0 \end{pmatrix} \ , \ \rho = \begin{pmatrix} \rho & \eta \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{6.37}$$

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with

$$\begin{aligned} \alpha_1 &= \rho\rho - \rho \\ \alpha_2 &= \rho\lambda - \lambda\rho \\ \alpha_3 &= \lambda - \lambda\lambda - \eta k + \rho\eta k \\ \alpha_4 &= \rho\eta - \lambda\eta. \end{aligned} \tag{6.38}$$

and

$$\begin{aligned} \mu_1 &= f_1\rho - \rho' f_1 \\ \mu_2 &= f_1\lambda - \lambda' f_1 - \varepsilon_f k + \rho'\varepsilon_f k \\ \mu_3 &= f_1\eta + \rho'\varepsilon_f + \lambda'\varepsilon_f - \eta' f_0 - \varepsilon_f \end{aligned} \tag{6.39}$$

$$\overline{\tau} = -\rho' t - \varepsilon_f + \lambda'\varepsilon_f + \lambda' t + \varepsilon_g - \rho'\varepsilon_g.$$

See Appendix B for the details showing the equivalence is well defined for pseudo-natural transformations and modifications.

6.5 The category of bicategories

In this section we present a description for the category of bicategories internal to abelian groups, together with homomorphisms and homotopies between them.

First we observe that a bicategory is a particular case of a pseudo-double category (in the sense of a pseudo-category internal to Cat) where the vertical morphisms are all identity morphisms. Hence, taking the additive 2-category $\mathbb{A}$ to be Cat(Ab), the category of internal categories in abelian groups, we obtain PsCat(Cat(Ab)): pseudo-double categories internal to abelian groups. To obtain bicategories just consider the pseudo-double categories with only identities as vertical morphisms. We illustrate this passage in the strict case where an internal double category in Ab is (equivalent to) a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "d"] \arrow[d, "k_1"'] & A_0 \arrow[d, "k_0"] \\ B_1 \arrow[r, "d"] & B_0 \end{tikzcd} ,$$

and an internal 2-category is obtained by taking B_1 to be the trivial group, giving then a 2-chain complex

$$A_1 \longrightarrow A_0 \longrightarrow B_0$$

where the requirement that composition has to be the zero morphism comes from the commutativity of the square above.

It is now straightforward to interpret the general result and obtain BiCat(Ab) the category of internal bicategories in Ab.

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Pseudo-double-categories

In order to make this passage as clear as possible we will do it in two steps. First we calculate PsMor(Mor(Ab)) and obtain PsDCat(Ab) the category of pseudo-double categories in Ab, next, following the procedure above, we restrict the vertical morphisms to identities and obtain BiCat(Ab).

Using $\mathbb{A}$=Mor(Ab) as described in the example of the first section we have PsMor(Mor(Ab))$\sim$PsCat(Cat(Ab)) as the following structure:

- A pseudo-double-category, $K = (A, B, k, \lambda, \rho, \eta)$, *is* a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "d"] \arrow[d, "k_1"'] & A_0 \arrow[d, "k_0"] \\ B_1 \arrow[r, "d"] & B_0 \end{tikzcd}$$

  in Ab, together with morphisms $\lambda, \rho : A_0 \longrightarrow A_1$, $\eta : B_0 \longrightarrow A_1$ such that

$$k_1\lambda = k_1\rho = 0\ ,\ k_1\eta = 0. \tag{6.40}$$

- A pseudo-double-functor $f : K \longrightarrow K'$ from $K = (A, B, k, \lambda, \rho, \eta)$ to $K' = \left(A', B', k', \lambda', \rho', \eta'\right)$ is of the form

$$f = (f_1, f_0, \varepsilon_f)$$

  where $f_1 : A \longrightarrow A', f_0 : B \longrightarrow B'$ are morphisms in Mor(Ab), $\varepsilon_f : B_0 \longrightarrow A'_1$ is a morphism in Ab, and they are such that

$$f_0k = k'f_1 \text{ and } k'_1\varepsilon_f = 0. \tag{6.41}$$

- A natural transformation $\theta : f \longrightarrow g$ from $f = (f_1, f_0, \varepsilon_f)$ to $g = (g_1, g_0, \varepsilon_g)$ both from K to K' is of the form

$$\theta = (\theta_1, \theta_0)\,,$$

  where $\theta_1 : A_0 \longrightarrow A'_1, \theta_0 : B_0 \longrightarrow B'_1$ are morphisms in Ab, satisfying

$$\begin{aligned} \theta_0 k_0 &= k'_1\theta_1 \\ d'\theta_1 &= g_{10} - f_{10}\ ,\ \theta_1 d = g_{11} - f_{11}, \\ d'\theta_0 &= g_{00} - f_{00}\ ,\ \theta_0 d = g_{01} - f_{01}. \end{aligned} \tag{6.42}$$

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as displayed in the following picture

$$\begin{tikzcd}
A_1 \arrow[rr, "d"] \arrow[dr, "k_1"] \arrow[dd, "f_{11}{,}g_{11}"'] & & A_0 \arrow[ddll, "\theta_1"] \arrow[dr, "k_0"] \arrow[dd, "f_{10}{,}g_{10}"'] & \\
& B_1 \arrow[rr, "d"] \arrow[dd, "f_{01}{,}g_{01}"'] & & B_0 \arrow[ddll, "\theta_0"] \arrow[dd, "f_{00}{,}g_{00}"] \\
A'_1 \arrow[rr, "d'"] \arrow[dr, "k'_1"'] & & A'_0 \arrow[dr, "k'_0"] & \\
& B'_1 \arrow[rr, "d'"] & & B'_0
\end{tikzcd}$$

- A pseudo-natural transformation $T : f \longrightarrow g$ from $f = (f_1, f_0, \varepsilon_f)$ to $g = (g_1, g_0, \varepsilon_g)$ is of the form

$$T = (t, \tau)$$

where $t : B \longrightarrow A'$ is a morphism in Mor(Ab), $\tau : A_0 \longrightarrow A'_1$ is a morphism in Ab, and they are such that

$$\begin{aligned}
k'_0 t_0 &= g_{00} - f_{00} \ , \ k'_1 t_1 = g_{01} - f_{01} \\
k'_1 \tau &= 0 \\
d'\tau &= t_0 k_0 - g_{10} - d'\rho' t_0 k_0 + d'\rho' g_{10} + f_{10} - d'\lambda' f_{10} \\
\tau d &= t_1 k_1 - g_{11} - \rho' d' t_1 k_1 + \rho' d' g_{11} + f_{11} - \lambda' d' f_{11}.
\end{aligned}$$

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- A modification $\Phi : B_0 \longrightarrow A_1'$, displayed as follows

$$\begin{tikzcd}
A_1 \arrow[rr, "d"] \arrow[dr, "k_1"] \arrow[dd, "{f_{11},g_{11},h_{11},l_{11}}"'] & & A_0 \arrow[dr, "k_0"] \arrow[dd, "{f_{10},g_{10},h_{10},l_{10}}"] \arrow[ddll, "{\theta_1,\theta_1',\tau}"'] & \\
& B_1 \arrow[rr, "d"] \arrow[dl, "{t_1,t_1'}"] \arrow[dd, "{f_{01},g_{01},h_{01},l_{01}}"] & & B_0 \arrow[dlll, "\Phi"] \arrow[dl, "{t_0,t_0'}"'] \arrow[ddll, "{\theta_0,\theta_0'}"] \arrow[dd, "{f_{00},g_{00},h_{00},l_{00}}"] \\
A_1' \arrow[rr, "d'"] \arrow[dr, "k_1'"'] & & A_0' \arrow[dr, "k_0'"] & \\
& B_1' \arrow[rr, "d'"] & & B_0'
\end{tikzcd}$$

from the pseudo-natural transformation $T = (t, \tau) : f \longrightarrow g$ to $T' = (t', \tau') : h \longrightarrow l$ and from the natural transformation $\theta = (\theta_1, \theta_0) : f \longrightarrow h$ to $\theta' = (\theta_1', \theta_0') : g \longrightarrow l$ with $f = (f_1, f_0, \varepsilon_f), g = (g_1, g_0, \varepsilon_g), h = (h_1, h_0, \varepsilon_h), l = (l_1, l_0, \varepsilon_l)$ pseudo-doube-functors from $K = (A, B, k, \lambda, \rho, \eta)$ to $K' = (A', B', k', \lambda', \rho', \eta')$, is such that

$$\begin{aligned}
d'\Phi &= t_0' - t_0,\ \Phi d = t_1' - t_1 \\
k_1'\Phi &= \theta_0' - \theta_0 \\
\Phi k_0 &= \tau' - \tau + \theta_1' - \theta_1 - \rho' l_{10} + \rho' g_{10} - \lambda' h_{10} - \lambda' f_{10} + \rho' t_0' k_0 - \rho' t_0 k_0.
\end{aligned}$$

Tensor $S \otimes T$ between pseudo-natural transformations $S = (s, \sigma) : g \longrightarrow h$ and $T = (t, \tau) : f \longrightarrow g$ where f, g, h are pseudo-double-functors from K to K' as above, is

$$S \otimes T = (s \otimes t, \sigma \otimes \tau) \tag{6.43}$$

where

$$\begin{aligned}
(s \otimes t)_0 &= s_0 + t_0 - d'\rho' s_0 - d'\lambda' t_0 - d'\eta' f_{00}, \\
(s \otimes t)_1 &= s_1 + t_1 - \rho' d' s_1 - \lambda' d' t_1 - \eta' d' f_{01}, \\
\sigma \otimes \tau &= \rho' d' \lambda' t_0 k_0 - \eta' k_0' f_{10} + \rho' d' \eta' k_0' f_{10} + \tau + \sigma - \rho' s_0 k_0 + \rho' d' \rho' s_0 k_0 - \rho' g_{10} - \lambda' t_0 k_0 + \lambda' g
\end{aligned}$$

Identity pseudo-natural transformation for $f = (f_1, f_0, \varepsilon_f) : K \longrightarrow K'$ is

$$id_f = (0, \rho' f_{10} - \lambda' f_{10}). \tag{6.44}$$

Composition and tensor of tetra-cells is

$$\Phi \cdot \Gamma = \Phi + \Gamma$$

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$$\Phi \otimes \Phi' = \Phi + \Phi' - \rho' t_0 - \lambda' s_0' - \eta' h_{00} + \eta' f_{00} \tag{6.45}$$

derived from (6.36). Identity tetra-cell (with respect to composition) for the pseudo-cell $T = (t, \tau)$ is

$$1_T = 0 : B_0 \longrightarrow A_1',$$

while the identity tetra-cell (with respect to tensor) for the 2-cell $\theta = (\theta_1, \theta_0)$ is

$$id_\theta = 0 : B_0 \longrightarrow A_1'.$$

Bicategories

From above, as we have already seen, to obtain a internal bicategory in Ab, just take the abelian group B_1 to be the trivial group. Thus, an internal bicategory in Ab is determined by a sequence

$$A_1 \xrightarrow{d} A_0 \xrightarrow{k} B_0$$

satisfying $kd = 0$, three more morphisms $\lambda, \rho : A_0 \longrightarrow A_1, \eta : B_0 \longrightarrow A_1$ and it is constructed as follows (see [4] and [3] for more details). Objects are the elements of B_0, morphisms are pairs $(a, b) \in A_0 \oplus B_0$ with domain b and codomain $k(a) + b$, 2-cells are triples $(x, a, b) \in A_1 \oplus A_0 \oplus B_0$ from the morphism (a, b) to $(d(x) + a, b)$.

Vertical composition is given by the formula

$$(x', d(x) + a, b) \cdot (x, a, b) = (x' + x, a, b)$$

while to evaluate horizontal composition we need to have m as in (6.37) which in this case becomes

$$m_1 = \begin{pmatrix} 1 - \rho d & 1 - \lambda d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ m_0 = \begin{pmatrix} 1 - d\rho & 1 - d\lambda & -d\eta \\ 0 & 0 & 1 \end{pmatrix}$$

an hence the formula giving horizontal composition of 2-cells $(x', a', k(a) + b) \circ (x, a, b)$ is the following

$$(x' - \rho d(x) + x - \lambda d(x), a' - d\rho(a') + a - d\lambda(a) - d\eta(b), b).$$

For every triple of composable morphisms

$$(a'', k(a' + a) + b), (a', k(a) + b), (a, b)$$

there is a 2-cell $\alpha(a'', a', a, b)$ form

$$(\overline{a}, b) = (a'', k(a' + a) + b) \circ ((a', k(a) + b) \circ (a, b))$$

to

$$\left(\overline{\overline{a}}, b\right) = ((a'', k(a' + a) + b) \circ (a', k(a) + b)) \circ (a, b)$$

and it is given by (6.38) as

$$\alpha(a'', a', a, b) = (\alpha_1(a'') + \alpha_2(a') + \alpha_3(a) + \alpha_4, \overline{a}, b)$$

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where

$$\begin{aligned} \alpha_1 &= \rho d\rho - \rho \\ \alpha_2 &= \rho d\lambda - \lambda d\rho \\ \alpha_3 &= \lambda - \lambda d\lambda - \eta k + \rho d\eta k \\ \alpha_4 &= \rho d\eta - \lambda d\eta. \end{aligned} \tag{6.46}$$

Given a morphism (a,b) we also have 2-cells $\lambda(a,b) = (\lambda(a) + \eta(b), a, b)$ and $\rho(a,b) = (\rho(a) + \eta(b), a, b)$ between (a,b) and its composite with left and right identity respectively.

A homomorphism $f : K \longrightarrow K'$ from *the* bicategory

$$K = (A = (A_1 \longrightarrow A_0), B = B_0, k = k_0 : A_0 \longrightarrow B_0, \lambda, \rho, \eta)$$

to $K' = \left(A', B', k', \lambda', \rho', \eta'\right)$ consists of four maps

$$f_{11} : A_1 \longrightarrow A_1',\ f_{10} : A_0 \longrightarrow A_0',\ f_0 : B_0 \longrightarrow B_0',\ \varepsilon_f : B_0 \longrightarrow A_1'$$

such that

$$f_0 k = k' f_{10}\ ,\ f_{10} d = d' f_{11}. \tag{6.47}$$

(While comparing to (6.41) we observe that $k_1' \varepsilon_f = 0$ because $B_1 = 0$ and hence $k_1' : A_1' \longrightarrow 0$.)
A 2-cell (x,a,b) of K is transformed by f into a 2-cell of K' as follows

$$(f_{11}(x) + \varepsilon_f(b), f_{10}(a), f_0(b)),$$

where we observe that $f(0,0,b) = (\varepsilon_f(b), 0, f_0(b))$ gives the 2-cell that compares $f(id_b) = (0, f_0(b))$ with $id_{fb} = (0, f_0(b))$ while the 2-cell comparing $f((a', k(a) + b) \circ (a,b))$ and $f(a', k(a) + b) \circ f(a,b)$ is given by

$$(\mu_1(a') + \mu_2(a) + \mu_3(b), f((a', k(a) + b) \circ (a,b)))$$

(remark the abuse of notation on writing $f(b)$ instead of $f_0(b)$ and $f(a,b)$ instead of $(f_{10}(a), f_0(b))$) where μ_i's are obtained from (6.39) which translates into present situation as

$$\begin{aligned} \mu_1 &= f_{11}\rho - \rho' f_{10} \\ \mu_2 &= f_{11}\lambda - \lambda' f_{10} - \varepsilon_f k + \rho' d' \varepsilon_f k \\ \mu_3 &= f_{11}\eta + \rho' d' \varepsilon_f + \lambda' d' \varepsilon_f - \eta' f_0 - \varepsilon_f. \end{aligned}$$

Natural transformations are only defined between homomorphisms $f, g : K \longrightarrow K'$ satisfying $f_0 = g_0$, and correspond to the usual natural transformations between strict internal categories in Ab; only the vertical structure of 2-cells is involved.

Concerning pseudo-natural transformations, as expected, they correspond to homotopies between 2-complexes, see [6] for the general approach involving n-complexes.

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If $f, g : K \longrightarrow K'$ are two homomorphisms between bicategories, a pseudo-natural transformation T is determined by two group homomorphisms $t : B_0 \longrightarrow A_0'$, $\tau : A_0 \longrightarrow A_1'$ satisfying

$$\begin{aligned} k't &= g_0 - f_0 \\ d'\tau &= t_0k_0 - g_{10} - d'\rho'tk + d'\rho'g_{10} + f_{10} - d'\lambda'f_{10} \\ \tau d &= -g_{11} + \rho'd'g_{11} + f_{11} - \lambda'd'f_{11}. \end{aligned}$$

The 2-cells (x, a, b) in K, are mapped, by f and g, into K' as

$$\left(f_{11}(x) + \varepsilon_f(b), f_{10}(a), f_0(b)\right)$$

and

$$\left(g_{11}(x) + \varepsilon_g(b), g_{10}(a), g_0(b)\right);$$

to relate them, we have the following families of morphisms

$$T_b = \left(t(b), f_0(b)\right) : f_0(b) \longrightarrow g_0(b)$$

and 2-cells

$$T_{(a,b)} = \left(\tau(a) + \overline{\tau}(b), T_{ka+b} \circ (a, b)\right)$$

where

$$\overline{\tau} = -\rho't - \varepsilon_f + \lambda'd'\varepsilon_f + \lambda't + \varepsilon_g - \rho'd'\varepsilon_g.$$

Finally, *a* modification $\Phi : T \longrightarrow T'$, from *the* pseudo-natural transformation $T = (t, \tau) : f \longrightarrow g$ to $T' = (t', \tau') : h \longrightarrow l$ and from *the* natural transformation $\theta = (\theta_1, \theta_0 = 0) : f \longrightarrow h$ to $\theta' = \left(\theta_1', \theta_0' = 0\right) : g \longrightarrow l$ with $f = (f_1, f_0, \varepsilon_f), g = (g_1, g_0, \varepsilon_g), h = (h_1, h_0, \varepsilon_h), l = (l_1, l_0, \varepsilon_l)$ homomorphisms from $K = (A, B, k, \lambda, \rho, \eta)$ to $K' = \left(A', B', k', \lambda', \rho', \eta'\right)$, as displayed below

$$\begin{tikzcd}
A_1 \arrow[r, "d"] \arrow[d, "{f_{11},g_{11},h_{11},l_{11}}"'] & A_0 \arrow[r, "k"] \arrow[d, "{f_{10},g_{10},h_{10},l_{10}}"] \arrow[dl, "{\theta_1,\theta_1',\tau,\tau'}"] & B_0 \arrow[d, "{f_0,g_0,h_0,l_0}"] \arrow[dl, "{t_0,t_0'}"] \\
A_1' \arrow[r, "d'"] & A_0' \arrow[r, "k'"] & B_0'
\end{tikzcd}$$

is a group homomorphism $\Phi : B_0 \longrightarrow A_1'$ satisfying

$$\begin{aligned} d'\Phi &= t' - t \\ \Phi k &= \tau' - \tau + \theta_1' - \theta_1 - \rho'l_{10} + \rho'g_{10} - \lambda'h_{10} - \lambda'f_{10} + \rho't'k - \rho'tk \end{aligned}$$

where the pairs of homomorphisms f, h and g, l must agree on objects in order to the natural transformations θ and θ' to be defined, in other words

$$\begin{aligned} f_0 &= h_0 \\ g_0 &= l_0. \end{aligned}$$

Formulas for tensor composition and identities are easily deduced from (6.43), (6.44) and (6.45) by taking $B_1 = 0$ and its implications.

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Example

For a topological abelian group X consider

$$\begin{aligned} B_0 &= X \\ A_0 &= \{a : \mathbb{N} \longrightarrow X \mid a \text{ is a convergent sequence}\} \\ A_1 &= \{x : \mathbb{N} \longrightarrow X \mid x \text{ is a bounded sequence}\} \end{aligned}$$

and define

$$\begin{aligned} k(a) &= \lim a \\ d(x_n) &= \frac{x_n}{n} \\ \eta(b) &= 0 \\ \lambda(a_n) &= \begin{cases} a_n \text{ if } n \text{ is odd} \\ 0 \text{ otherwise} \end{cases} \\ \rho(a_n) &= \begin{cases} a_n \text{ if } n \text{ is even} \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

to obtain the bicategory where objects are the points in X, a morphism from b to b' is a sequence (a_n) converging to $b' - b$ while composition of two sequences $(a_n) : b \longrightarrow b'$ and $(a'_n) : b' \longrightarrow b''$ is given by

$$a' \circ a : b \longrightarrow b''$$

where

$$a' \circ a = \begin{cases} a'_n - \frac{a'}{n} + a_n \text{ if } n \text{ is even} \\ a'_n + a_n - \frac{a_n}{n} \text{ if } n \text{ is odd} \end{cases} .$$

6.6 Appendix A - Pseudo-natural transformations and modifications

Recall from [5]. In what follows the words objects, morphism and 2-cell refer to a given 2-category where we are defining the concepts of pseudo-natural transformation and modification internal to.

pseudo-natural transformation

A **pseudo-natural transformation** $T : F \longrightarrow G$ from the pseudo-functor $F = \left(F_0, F_1, \mu^F, \varepsilon^F\right) : C \longrightarrow C'$ to the pseudo-functor $G = \left(G_0, G_1, \mu^G, \varepsilon^G\right) : C \longrightarrow C'$is a pair

$$T = (t, \tau)$$

where $t : C_0 \longrightarrow C'_1$ is a morphism,

$$\tau : m' \langle G_1, td \rangle \longrightarrow m' \langle tc, F_1 \rangle$$

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is a 2-cell (that is an isomorphism); the following conditions are satisfied

$$\begin{aligned} d't &= F_0 \\ c't &= G_0 \end{aligned} \tag{6.48}$$

$$\begin{aligned} d' \circ \tau &= 1_{d'F_1} \\ c' \circ \tau &= 1_{c'G_1} \end{aligned} \tag{6.49}$$

and the following diagrams of 2-cells are commutative[2]

$$\begin{tikzcd}
 & \bullet \arrow[r, "\alpha^{-1}\left(G_1\times_{G_0}G_1\times_{G_0}t\right)"] \arrow[dl, "m'\left\langle\mu_G^{-1},1_{td\pi_2}\right\rangle"'] & \bullet \arrow[dr, "m'\langle G_1\pi_1,\tau\pi_2\rangle"] & \\
\bullet \arrow[d, "\tau m"'] & & & \bullet \arrow[d, "\alpha\left(G_1\times_{G_0}t\times_{F_0}F_1\right)"] \\
\bullet \arrow[dr, "m'\left\langle 1_{tc\pi_1},\mu_F\right\rangle"'] & & & \bullet \arrow[dl, "m'\langle\tau\pi_1,F_1\pi_2\rangle"] \\
 & \bullet \arrow[r, "\alpha\left(t\times_{F_0}F_1\times_{F_0}F_1\right)"'] & \bullet &
\end{tikzcd} \tag{6.50}$$

,

$$\begin{tikzcd}
 & \bullet \arrow[rr, "\tau e"] \arrow[dl, "\lambda' t"'] & & \bullet \arrow[dr, "m'\langle 1_t,\varepsilon_F\rangle"] & \\
\bullet \arrow[drr, "m'\langle\varepsilon_G,1_t\rangle"'] & & & & \bullet \arrow[dll, "\rho' t"] \\
 & & \bullet & &
\end{tikzcd} \tag{6.51}$$

.

Tensor of pseudo-natural transformations

The **tensor** $S\otimes T : F \longrightarrow H$ of the pseudo-natural transformation $S=(s,\sigma) : G \longrightarrow H$ with $T=(t,\tau) : F \longrightarrow G$ is the pseudo-natural transformation

$$S\otimes T = \left(m'\langle s,t\rangle, \sigma\otimes\tau\right) \tag{6.52}$$

where

$$\sigma\otimes\tau = \alpha\langle sc,tc,F_1\rangle\cdot m'\langle 1_{sc},\tau\rangle\cdot\alpha^{-1}\langle sc,G_1,td\rangle\cdot m'\langle\sigma,1_{td}\rangle\cdot\alpha\langle H_1,sd,td\rangle . \tag{6.53}$$

---

[2] $G_1\times_{G_0}t\times_{F_0}F_1 : C_1\times_{C_0}C_0\times_{C_0}C_1 \longrightarrow C_1\times_{C_0}C_1\times_{C_0}C_1$
$t\times_{F_0}F_1\times_{F_0}F_1 : C_0\times_{C_0}C_1\times_{C_0}C_1 \longrightarrow C_1\times_{C_0}C_1\times_{C_0}C_1$
$G_1\times_{G_0}G_1\times_{G_0}t : C_1\times_{C_0}C_1\times_{C_0}C_0 \longrightarrow C_1\times_{C_0}C_1\times_{C_0}C_1$

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Identity pseudo-natural transformation

The identity pseudo-natural transformation for the pseudo-functor $F = \left(F_0, F_1, \mu^F, \varepsilon^F\right) : C \longrightarrow C'$ from $C = (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)$ to $C' = \left(C'_0, C'_1, d', c', e', m', \alpha', \lambda', \rho'\right)$ is the pseudo-natural transformation

$$id_F = \left(e'F_0, \lambda'^{-1}\rho' F_1\right) : F \longrightarrow F.$$

Modifications

Suppose C, C' are pseudo-categories, $F, G, H, K : C \longrightarrow C'$ are pseudo-functors, $T = (t, \tau) : F \longrightarrow G$, $T' = (t', \tau') : H \longrightarrow K$ are pseudo-natural transformations and $\theta = (\theta_0, \theta_1) : F \longrightarrow H, \theta' = \left(\theta'_0, \theta'_1\right) : G \longrightarrow K$ are natural transformations.

A **modification** $\Phi : (\theta, T) \longrightarrow \left(\theta', T'\right)$ represented as

$$\begin{tikzcd}
F \arrow[r, "T"] \arrow[d, "\theta"'] & G \arrow[d, "\theta'"] \\
H \arrow[r, "T'"] & K
\arrow[from=1-1, to=2-1, phantom, shift left=8ex, "\Downarrow \Phi"]
\end{tikzcd}$$

is a 2-cell

$$\Phi : t \longrightarrow t'$$

satisfying

$$\begin{aligned} d' \circ \Phi &= \theta_0 \\ c' \circ \Phi &= \theta'_0 \end{aligned} \qquad (6.54)$$

and the commutativity of the square

$$\begin{tikzcd}
\bullet \arrow[r, "\tau"] \arrow[d, "m'\left\langle\theta'_1, \Phi\circ d\right\rangle"'] & \bullet \arrow[d, "m'\langle\Phi\circ c, \theta_1\rangle"] \\
\bullet \arrow[r, "\tau'"] & \bullet
\end{tikzcd} . \qquad (6.55)$$

(Vertical) Composition of Modifications

The (vertical) composition $\Phi' \cdot \Phi : T \longrightarrow T''$ of the modification $\Phi' : T' \longrightarrow T''$ with the modification $\Phi : T \longrightarrow T'$ is the modification

$$\Phi' \cdot \Phi$$

simply the composition of 2-cells.

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Tensor of Modifications

The tensor $\Psi \otimes \Phi : \theta \longrightarrow \theta''$ of the modification $\Psi : \theta' \longrightarrow \theta''$ with the modification $\Phi : \theta \longrightarrow \theta'$ is the modification

$$\Psi \otimes \Phi = m' \langle \Psi, \Phi \rangle .$$

Identity Modification (with respect to composition)

The identity modification (with respect to composition) for the pseudo-natural transformation $T = (t, \tau) : F \longrightarrow G$, is $1_T = 1_t$

$$\begin{tikzcd} F \arrow[r, "T"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow 1_T"] & G \arrow[d, "1"] & . \\ F \arrow[r, "T"] & G & \end{tikzcd}$$

Identity Modification (with respect to tensor)

The identity modification (with respect to tensor) for the natural-transformation $\theta = (\theta_0, \theta_1) : F \longrightarrow G$, is $id_\theta = e'\theta_0$

$$\begin{tikzcd} F \arrow[r, "id_F"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow id_\theta"] & F \arrow[d, "\theta"] \\ G \arrow[r, "id_G"] & G \end{tikzcd}$$

In particular for a pseudo-functor $F = (F_0, F_1, \mu_F, \varepsilon_F) : C \longrightarrow C'$ we have

$$id_{1_F} = e' 1_{F_0} = 1_{e' F_0} = 1_{id_F} .$$

6.7 Appendix B - Calculations

To show that *The Equivalence* (6.4) is well defined for pseudo-cells in PsMor($\mathbb{A}$) and pseudo-natural transformations in PsCat($\mathbb{A}$), we will show that given a pseudo-cell $T : f \longrightarrow g$ as in *The Definition* (6.4), i.e.

$$T = (t, \tau)$$

with $t : B \longrightarrow A'$ a morphism of $\mathbb{A}$, and $\tau \in H(A, A')$ such that

$$\begin{aligned} k't &= g_0 - f_0 \\ k'\tau &= 0 \\ D(\tau) &= tk - g_1 - D(\rho')\, tk + D(\rho')\, g_1 + f_1 - D(\lambda')\, f_1, \end{aligned}$$

defines a pseudo-natural transformation in PsCat($\mathbb{A}$) given by

$$V(T) = \left(\begin{pmatrix} t \\ f_0 \end{pmatrix}, \begin{pmatrix} \tau & \overline{\tau} \\ 0 & 0 \end{pmatrix} \right)$$

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with

$$\overline{\tau} = -\rho' t - \varepsilon_f + \lambda' \varepsilon_f + \lambda' t + \varepsilon_g - \rho' \varepsilon_g,$$

since the converse is straightforward.

Conditions (6.48) and (6.49) are easily seen to be satisfied. Coherence condition (6.51) in PsCat($\mathbb{A}$) translates to

$$\begin{pmatrix} \rho' t + \eta' f_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon_f - D(\lambda')\varepsilon_f \\ 0 \end{pmatrix} + \begin{pmatrix} \overline{\tau} \\ 0 \end{pmatrix} + \begin{pmatrix} -\lambda' t - \eta' f_0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\varepsilon_g + D(\rho')\varepsilon_g \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and it is equivalent to

$$\begin{pmatrix} \rho' t + \varepsilon_f - D(\lambda')\varepsilon_f + \overline{\tau} - \lambda' t - \varepsilon_g + D(\rho')\varepsilon_g \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

witch is consistent with the definition of $\overline{\tau}$.

Condition (6.50) is equivalent to the fact that the following three expressions are trivial

$$\begin{aligned} &-g_1\rho + \rho' g_1 + D(\rho')g_1\rho - D(\rho')\rho' g_1 - \tau D(\rho) + f_1\rho - \rho' f_1 - D(\lambda')f_1\rho \\ &+D(\lambda')\rho' f_1 + \rho'\rho' tk - \rho' tk + \rho'\lambda' f_1 - \lambda'\rho' f_1 + D(\rho')\tau \end{aligned}$$

$$\begin{aligned} &-g_1\lambda + \lambda' g_1 - \rho'\varepsilon_g k + D(\rho')g_1\lambda - D(\rho')\lambda' g_1 + D(\rho')\varepsilon_g k + D(\rho')\rho'\varepsilon_g k - \tau D(\lambda) + f_1\lambda \\ &+\rho'\varepsilon_f k - D(\lambda')f_1\lambda + D(\lambda')\lambda' f_1 - D(\lambda')\rho'\varepsilon_f k + \rho'\rho' tk - \rho'\lambda' D(\varepsilon_f)k + \lambda'\rho' D(\varepsilon_f)k \\ &+\rho'\lambda' g_1 - \lambda'\rho' g_1 + D(\lambda')\tau - \rho'\lambda' tk + \lambda'\rho' tk - \lambda' f_1 - \lambda' tk - D(\rho')\rho' tk \\ &-D(\rho')\varepsilon_f k + D(\rho')D(\lambda')\varepsilon_f k + D(\rho')\lambda' tk - D(\rho')D(\rho')\varepsilon_g k \end{aligned}$$

$$\begin{aligned} &-g_1\eta - \rho'\varepsilon_g - \lambda'\varepsilon_g + \eta' g_0 + D(\rho')g_1\eta + D(\rho')\rho'\varepsilon_g + D(\rho')\lambda'\varepsilon_g - D(\rho')\eta' g_0 + D(\rho')\varepsilon_g \\ &-\tau D(\eta) - D(\lambda')\varepsilon_f + f_1\eta + \rho'\varepsilon_f + \lambda'\varepsilon_f - \eta' f_0 - D(\lambda')f_1\eta - D(\lambda')\rho'\varepsilon_f - D(\lambda')\lambda'\varepsilon_f + D(\lambda')\eta' f_0 \\ &+\rho'\rho' t - \rho'\lambda' D(\varepsilon_f) + \lambda'\rho' D(\varepsilon_f) + \rho'\eta' f_0 - \lambda'\eta' f_0 - \rho'\lambda' D(\varepsilon_g) + \lambda'\rho' D(\varepsilon_g) - \lambda'\lambda' t - \eta' k' t \\ &+\rho'\eta' k' t - D(\lambda')\rho' t + D(\lambda')D(\lambda')\varepsilon_f + D(\lambda')\lambda' t + D(\lambda')\varepsilon_g - D(\lambda')D(\rho')\varepsilon_g - \rho'\lambda' t + \lambda'\rho' t \\ &-D(\rho')\rho' t - D(\rho')\varepsilon_f + D(\rho')D(\lambda')\varepsilon_f + D(\rho')\lambda' t - D(\rho')D(\rho')\varepsilon_g \end{aligned}$$

To see that the first expression is trivial observe the following steps:
-replace $-\tau D\left(\rho\right)$ by $-D\left(\tau\right)\rho$

$$\rho' g_1 - D(\rho')\rho' g_1 - \tau D(\rho) - \rho' f_1 + D(\lambda')\rho' f_1 + \rho'\rho' tk - \rho' tk + \rho'\lambda' f_1 - \lambda'\rho' f_1 + D(\rho')\tau + \tau\rho - tk\rho + D(\rho')tk\rho$$

-simplify the operator D

$$\rho' g_1 - \rho'\rho' g_1 - \rho' f_1 + \rho'\rho' tk - \rho' tk + \rho'\lambda' f_1 + \rho'\tau - tk\rho + \rho' tk\rho$$

-replace $\rho'\tau$ by $\rho' D\left(\tau\right)$

$$-\rho'\rho' g_1 + \rho'\rho' tk + \rho'\lambda' f_1 - tk\rho + \rho' tk\rho - \rho' D(\rho')tk + \rho' D(\rho')g_1 - \rho' D(\lambda')f_1$$

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-simplify the operator D and get the trivial result

$$-tk\rho + \rho' tk\rho.$$

To show the second expression

$$-g_1\lambda+\lambda' g_1+\rho' g_1\lambda-\tau\lambda+f_1\lambda-\lambda' f_1\lambda+\lambda'\lambda' f_1-\lambda'\rho' g_1+\lambda'\tau+\lambda'\rho' tk-\lambda' f_1-\lambda' tk$$

is trivial we observe the following procedure:
-substitute $\tau\lambda$ by $D(\tau)\lambda$

$$\begin{aligned}&\lambda' g_1 - \rho'\varepsilon_g k - D(\rho')\lambda' g_1 + D(\rho')\varepsilon_g k + D(\rho')\rho'\varepsilon_g k - \tau D(\lambda) + \rho'\varepsilon_f k + D(\lambda')\lambda' f_1\\ &-D(\lambda')\rho'\varepsilon_f k + \rho'\rho' tk - \rho'\lambda' D(\varepsilon_f)k + \lambda'\rho' D(\varepsilon_f)k + \rho'\lambda' g_1 - \lambda'\rho' g_1 + D(\lambda')\tau\\ &-\rho'\lambda' tk + \lambda'\rho' tk - \lambda' f_1 - \lambda' tk - D(\rho')\rho' tk - D(\rho')\varepsilon_f k + D(\rho')D(\lambda')\varepsilon_f k\\ &+D(\rho')\lambda' tk - D(\rho')D(\rho')\varepsilon_g k + \tau\lambda - tk\lambda + D(\rho')tk\lambda\end{aligned}$$

-simplify D operator

$$\lambda' g_1 + \lambda'\lambda' f_1 - \lambda'\rho' g_1 + \lambda'\tau + \lambda'\rho' tk - \lambda' f_1 - \lambda' tk - tk\lambda + \rho' tk\lambda$$

-substitute $\lambda'\tau$ by $\lambda' D(\tau)$

$$\lambda'\lambda' f_1 - \lambda'\rho' g_1 + \lambda'\rho' tk - tk\lambda + \rho' tk\lambda - \lambda' D(\rho')tk + \lambda' D(\rho')g_1 - \lambda' D(\lambda')f_1$$

-and simplify D operator to get the trivial result

$$-tk\lambda + \rho' tk\lambda.$$

To show the third expression

$$-g_1\eta + \eta' g_0 + \rho' g_1\eta - \rho'\eta' g_0 - \tau\eta + f_1\eta - \eta' f_0 - \lambda' f_1\eta + \rho'\eta' f_0 - \eta' k' t + \rho'\eta' k' t$$

is trivial, we observe the following procedure:
-substitute $\tau\eta$ by $D(\tau)\eta$

$$\begin{aligned}&-\rho'\varepsilon_g - \lambda'\varepsilon_g + \eta' g_0 + D(\rho')\rho'\varepsilon_g + D(\rho')\lambda'\varepsilon_g - D(\rho')\eta' g_0 + D(\rho')\varepsilon_g\\ &-\tau D(\eta) - D(\lambda')\varepsilon_f + \rho'\varepsilon_f + \lambda'\varepsilon_f - \eta' f_0 - D(\lambda')\rho'\varepsilon_f - D(\lambda')\lambda'\varepsilon_f\\ &+D(\lambda')\eta' f_0 + \rho'\rho' t - \rho'\lambda' D(\varepsilon_f) + \lambda'\rho' D(\varepsilon_f) + \rho'\eta' f_0 - \lambda'\eta' f_0\\ &-\rho'\lambda' D(\varepsilon_g) + \lambda'\rho' D(\varepsilon_g) - \lambda'\lambda' t - \eta' k' t + \rho'\eta' k' t - D(\lambda')\rho' t\\ &+D(\lambda')D(\lambda')\varepsilon_f + D(\lambda')\lambda' t + D(\lambda')\varepsilon_g - D(\lambda')D(\rho')\varepsilon_g - \rho'\lambda' t\\ &+\lambda'\rho' t - D(\rho')\rho' t - D(\rho')\varepsilon_f + D(\rho')D(\lambda')\varepsilon_f + D(\rho')\lambda' t\\ &-D(\rho')D(\rho')\varepsilon_g + \tau\eta - tk\eta + D(\rho')tk\eta\end{aligned}$$

-simplify D operator

$$\eta' g_0 - \rho'\eta' g_0 - \eta' f_0 + \rho'\eta' f_0 - \eta' k' t + \rho'\eta' k' t - tk\eta + \rho' tk\eta$$

and observe that it is a trivial expression due to (6.32) and (6.35).

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Part III

Beyond the Abelian World

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Chapter 7

Internal precategories in weakly Mal'cev categories

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Internal category, precategory, protocategory, sort, homogeneous-sort, adjustable-sort, unitary-sort, bounded-sord, associative, half-associative, admissible precategory, admissible sort.

Abstract: We describe precategories in weakly Mal'cev categories (see [Ch3]). We also analyze all the possible reasonable variations in the axioms that vary from the notion of internal precategory to the notion of internal category. For that purpose we introduce the intermediate notions of homogeneous, unitary, adjustable and bounded precategory and sort. In order to include the axioms for associativity we also consider protocategories, and study associative and half-associative protocategories. In the context of a weakly Mal'cev category we consider admissible precategories and show that the category of admissible precategories is reflective in the category of thin precategories (with premorphisms).

7.1 Introduction

In [Ch3] we introduce the notion of weakly Mal'cev category and prove that a internal category is given by a special case of a reflexive graph. In [Ch9] we describe a pseudocategory in a weakly Mal'cev category with a 2-cell structure (sesquicategory). Depending on the 2-cell structure that is considered, a pseudocategory corresponds to a internal category (discrete 2-cell structure) or to a precategory (codiscrete 2-cell structure). Anticipating this passage, we study in here the notion of an internal precategory in a weakly Mal'cev category, and we also consider all the (reasonable) variations in the axioms between internal category and internal precategory, such as the ones displayed below; the words homogeneous, adjustable, bounded and half-associative are to be considered as labels, introduced to facilitate the reference to the corresponding conditions. We could also have chosen letters such as A, B, C, D or symbols such as (*1), (*2), etc. to refer to, we simply find this way more readable and hope the chosen names to be intuitive, preventing the reader to always going back and confirm which axiom are we referring to. We will use

homogeneous for	$11 = 1$
unitary for	$1x = x = x1$
adjustable for	$(1x)1 = 1(x1)$
left bounded for	$1(1x) = 1x$
right bounded for	$(x1)1 = x1$
bounded for	$1(1x) = 1x$ and $(x1)1 = x1$
associative for	$(xy)\,z = x\,(yz)$
half-associative for	$(xy)\,1 = x\,(y1)$ and $(1x)\,y = 1\,(xy)$.

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The study of internal precategories, in particular, is also relevant for the Categorical Galois Theory of Janelidze [8].

The notion of precategory in the sense of this article is almost the same as in the sense of G. Janelidze except that we also consider the morphisms e_1, e_2 as part of the structure (see below); we will also say thin precategory for the case when C_2 is a pullback and then it coincides with the notion of precategory used by R. Brown.

The notions of protocategory, precategory and reflexive graph are obtained from the simplicial objects of order 3, 2 and 1 as follows.

Let **C** be a category.

A simplicial object of order 3 in **C** is a diagram of the form

$$\begin{tikzcd}
C_3 \arrow[r, "p_2", shift left=9] \arrow[r, <-, "i_2" description, shift left=6] \arrow[r, "m_2" description, shift left=3] \arrow[r, <-, "i_0" description] \arrow[r, "m_1" description, shift right=3] \arrow[r, <-, "i_1" description, shift right=6] \arrow[r, "p_1" description, shift right=9] & C_2 \arrow[r, "\pi_2", shift left=6] \arrow[r, <-, "e_2" description, shift left=3] \arrow[r, "m" description] \arrow[r, <-, "e_1" description, shift right=3] \arrow[r, "\pi_1"', shift right=6] & C_1 \arrow[r, "d", shift left=3] \arrow[r, <-, "e" description] \arrow[r, "c"', shift right=3] & C_0
\end{tikzcd}$$

satisfying the following axioms:

i	$de = 1$	v	$\pi_1 e_1 = 1$	ix	$d\pi_1 = c\pi_2$
ii	$ce = 1$	vi	$me_1 = 1$	x	$\pi_2 e_1 = ed$
iii	$\pi_2 e_2 = 1$	vii	$d\pi_2 = dm$	xi	$\pi_1 e_2 = ec$
iv	$me_2 = 1$	viii	$c\pi_1 = cm$	xii	$e_2 e = e_1 e$

xiii	$p_2 i_2 = 1$	xx	$m_1 i_0 = 1$	xxvii	$\pi_2 p_1 = \pi_1 p_2$
xiv	$p_2 i_0 = e_2 \pi_2$	xxi	$m_1 i_1 = 1$	xxviii	$m m_2 = m m_1$
xv	$p_2 i_1 = e_1 \pi_2$	xxii	$p_1 i_2 = e_2 \pi_1$	xxix	$m p_1 = \pi_1 m_2$
xvi	$m_2 i_2 = 1$	xxiii	$p_1 i_0 = e_1 \pi_1$	xxx	$\pi_1 m_1 = \pi_1 p_1$
xvii	$m_2 i_0 = 1$	xxiv	$p_1 i_1 = 1$	xxxi	$i_2 e_2 = i_0 e_2$
xviii	$m_2 i_1 = e_1 m$	xxv	$\pi_2 p_2 = \pi_2 m_2$	xxxii	$i_1 e_2 = i_2 e_1$
xix	$m_1 i_2 = e_2 m$	xxvi	$\pi_2 m_1 = m p_2$	xxxiii	$i_0 e_1 = i_1 e_1$

where the axioms i and ii are of order 1, the axioms iii-xii are of order 2 and the axioms xiii-xxxiii are of order 3.

In the case **C**=Set and assuming ix and xxvii to represent pullbacks and not just commutative squares, we have

$$\begin{gathered}
\pi_1 (x,y) = x,\ \pi_2 (x,y) = y,\ e_1 (x) = (x,1), e_2 (x) = (1,x), \\
m (x,y) = xy,\ p_2 (x,y,z) = (y,z),\ m_2 (x,y,z) = (xy,z), \\
m_1 (x,y,z) = (x,yz),\ p_1 (x,y,z) = (x,y),\ i_2 (x,y) = (1,x,y), \\
i_0 (x,y) = (x,1,y),\ i_1 (x,y) = (x,y,1).
\end{gathered}$$

If truncated at level 1 we obtain a reflexive graph; If truncating at level 2 and forgetting the two axioms iv and vi we obtain a precategory; if in a precategory we ask for the commutative square ix to be a pullback then we obtain a thin precategory.

Considering the whole structure and removing the axioms iv and vi, xvi to xxi, and xxviii we obtain the notion of a protocategory, which becomes a thin protocategory if the commutative squares ix and xxvii are in fact pullbacks.

We also introduce the notion *sort*, extending the notion of reflexive graph in such a way that thin-precategories in a weakly Mal'cev category correspond to (a particular case of) sorts, in the same way as internal categories correspond to (a particular case of) reflexive graphs [Ch3].

In the context of a weakly Mal'cev category we introduce admissible sort and admissible precategory and prove the following configuration

$$\begin{tikzcd}
ProtoCat \arrow[r, "U"] & PreCat \arrow[r, "U"] & Sort \\
ThinProtoCat \arrow[u, "\iota"] \arrow[r, equal] & ThinPreCat \arrow[u, "\iota"] \arrow[r, "\sim"] \arrow[d, "\iota"] & AdmSort \arrow[u, "\iota"] \arrow[d, "\iota"] \\
AdmPreCat_{pre} \arrow[r, shift left, "I"] & ThinPreCat_{pre} \arrow[l, shift left, "H"] \arrow[r, "\sim"] & AdmSort_{pre}
\end{tikzcd}$$

where the pair (I, H) is a reflection, the subscript *pre* indicates that also premorphisms are considered, U is the forgetful functor, ι is the inclusion functor, and $\sim$ represents an equivalence of categories.

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The main difference to distinguish between *admissible* or *thin* among all the precategories is the study of the induced morphism into the pullback

$$\begin{tikzcd}
C_2 \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "\pi_1"'] & & \\
& C_1 \times_{C_0} C_1 \arrow[r, shift left, "\pi_2'"] \arrow[d, shift right, "\pi_1'"'] & C_1 \arrow[l, shift left, "e_2'"] \arrow[d, shift right, "c"'] \\
& C_1 \arrow[r, shift left, "d"] \arrow[u, shift right, "e_1'"'] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"']
\end{tikzcd} .$$

If it is an isomorphism, we have a thin precategory, if there is a morphism

$$s : C_1 \times_{C_0} C_1 \longrightarrow C_2$$

such that $\theta s = 1$ and

$$\begin{aligned} se_2' &= e_2 \\ se_1' &= e_1, \end{aligned}$$

then we have an admissible precategory (it is possible to transport the structure of $m : C_2 \longrightarrow C_1$ to $ms : C_1 \times_{C_0} C_1 \longrightarrow C_1$).

We will assume that all the definitions, notations and results of [Ch3] are also present here.

7.2 Precategories and sorts

Let **C** be a given category and recall from [Ch3].

Definition 63 (split span) *A* split span *is a diagram in* **C** *of the form*

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"']
\end{tikzcd}$$

such that

$$fr = 1_C = gs.$$

Definition 64 (split square) *A* split square *is a diagram in* **C** *of the form*

$$\begin{tikzcd}
P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\
A \arrow[r, shift left, "f"] \arrow[u, shift right, "e_1"'] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

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such that

$$\begin{aligned} fr &= 1_C = gs \\ gp_2 &= fp_1 \\ e_2 s &= e_1 r \\ p_2 e_2 &= 1_B \\ p_2 e_1 &= sf \\ p_1 e_1 &= 1_A \\ p_1 e_2 &= rg, \end{aligned}$$

in other words, it is a double split epi, in the sense that it is a split epi in the category of split epis in **C**.

The term *split pullback* will be used to refer to a split square as above, such that

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

is a pullback diagram.

Sorts

The name *sort*, in the following definition, is somehow arbitrary: there is no serious interpretation for it; simply it facilitates the exposition of the results to be presented if naming such structure. Also, some properties of the stated structure turn out to play important role in the results to be presented. In that light, the words *homogeneous, unitary, adjustable* and *bounded* were somehow arbitrarily chosen and the only purpose they serve is to facilitate the exposition of the results.

Let **C** be a given category.

Definition 65 (*sort*) *A* sort *is a diagram in* **C** *of the form*

$$\begin{tikzcd} C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

such that

$$\begin{aligned} de &= 1_{C_0} = ce \\ du &= d = dv \\ cu &= c = cv \\ ue &= ve \ \ (= l)\,. \end{aligned}$$

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A sort is said to be:
Homogeneous, when $ue = ve = e$ (or simply $l = e$).
Unitary, when $u = 1_{C_1} = v$ (it then becomes a reflexive graph).
Adjustable, when $uv = vu$.
Left bounded, when $vv = v$.
Right bounded, when $uu = u$.
Bounded, when it is both left and right bounded.

Assuming that it is clear from the context, we will write $(C_1, C_0, d, c, e, u, v)$ and $(C_1', C_0', d, c, e, u, v)$ to talk about two distinct sorts.

A morphism of sorts is of course a pair

$$f = (f_1, f_0) : (C_1, C_0, ...) \longrightarrow (C_1', C_0', ...)$$

where $f_1 : C_1 \longrightarrow C_1'$ and $f_0 : C_0 \longrightarrow C_0'$ are morphisms in $\mathbf{C}$ such that the obvious squares in the following diagram

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\
C_1' \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & C_0'
\end{tikzcd}$$

commute, and furthermore

$$f_1 e = e f_0 \tag{7.1}$$

$$f_1 u = u f_1 \text{ and } f_1 v = v f_1. \tag{7.2}$$

However, it will also be important to consider morphisms between sorts that do not necessarily satisfy (7.1) and (7.2); in that case they will be denoted as *premorphisms* of sorts.

In order to establish some notation let:
Sort($\mathbf{C}$) represent the category of sorts and sort morphisms;
Sort$_{pre}$($\mathbf{C}$) represent the category of sorts and sort premorphisms;
Homogeneous-Sort($\mathbf{C}$)
Unitary-Sort($\mathbf{C}$)
Adjustable-Sort($\mathbf{C}$)
Bounded-Sort($\mathbf{C}$)
represent the respective subcategories of homogeneous, unitary, adjustable and bounded sorts.

It is also possible to intersect two or more such subcategories, so that for example bounded-adjustable-Sort($\mathbf{C}$) represents the subcategory of bounded and adjustable sorts; also, for each case, the category with premorphisms instead of morphisms is also considered, thus Homogeneous-Sort$_{pre}$ ($\mathbf{C}$), etc.

Some immediate results are as follows:

$$\text{Unitary-Sort}\,(\mathbf{C}) = \text{RGraph}\,(\mathbf{C})$$

$$\textit{unitary} \Longrightarrow \begin{cases} \textit{homogeneous} \\ \textit{adjustable} \\ \textit{bound} \end{cases}$$

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In order to obtain some intuition, the reader is invited do consider a sort

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=2, "c"'] & C_0
\end{tikzcd}$$

in the category Set of sets and maps, where an element $x \in C_1$ is pictured as

$$x : a \longrightarrow b,$$

and the maps u and v are defined as left and right multiplication of an element with the respective "identity" arrow of its domain and codomain:

$$\begin{aligned} u(x) &= x * e(b) = x * ed(x) \\ v(x) &= e(a) * x = ec(x) * x; \end{aligned}$$

with this view and writing 1_b for $e(b)$,

$$\begin{array}{rcc} \text{homogeneous} & \Longleftrightarrow & 1_b * 1_b = 1_b \\ \text{unitary} & \Longleftrightarrow & x * 1_a = x = 1_b * x \\ \text{adjustable} & \Longleftrightarrow & 1_b * (x * 1_a) = (1_b * x) * 1_a \\ \text{left bounded} & \Longleftrightarrow & 1_b * (1_b * x) = 1_b * x \\ \text{right bounded} & \Longleftrightarrow & (x * 1_a) * 1_a = x * 1_a . \end{array}$$

Precategories

Let **C** be a given category.

Definition 66 (precategory) *A* precategory *is a diagram in* **C** *of the form*

$$\begin{tikzcd}
C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, leftarrow, shift left=2, "e_2" description] \arrow[r, "m" description] \arrow[r, leftarrow, shift right=2, "e_1" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=2, "c"'] & C_0
\end{tikzcd}$$

such that

$$\begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\
C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"']
\end{tikzcd} \tag{7.3}$$

is a split square, so that in particular

$$de = 1_{C_0} = ce \tag{7.4}$$

and furthermore, the following two conditions are satisfied

$$dm = d\pi_2 \tag{7.5}$$

$$cm = c\pi_1 . \tag{7.6}$$

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A morphism of precategories is of course a triple

$$f=(f_2,f_1,f_0):(C_2,C_1,C_0,d,c,...)\longrightarrow(C_2',C_1',C_0',d,c,...)$$

where $f_2:C_2\longrightarrow C_2'$, $f_1:C_1\longrightarrow C_1'$ and $f_0:C_0\longrightarrow C_0'$ are morphisms in $\mathbf{C}$, such that the obvious squares in the following diagram

$$\begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\pi_1"'] \arrow[d, "f_2"'] & C_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\
C_2' \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\pi_1"'] & C_1' \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & C_0'
\end{tikzcd}$$

commute, and furthermore

$$f_1e=ef_0, \tag{7.7}$$

$$f_1m=mf_2 \tag{7.8}$$

and

$$f_2e_i=e_if_1 \quad , \ i=1,2. \tag{7.9}$$

Here, it is also useful to consider a *premorphism* between two precategories when conditions (7.7) , (7.8) and (7.9) are not necessarily satisfied.

Proposition 67 *Every precategory*

$$\begin{tikzcd}
C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, shift right=4, "\pi_1"'] \arrow[r, "m" description] & C_1 \arrow[l, shift left=2, "e_2" description] \arrow[l, shift right=2, "e_1" description] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

is in particular a sort

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

where $u=me_1$ and $v=me_2$.

Proof. Given a precategory as above, and defining $u=me_1$, $v=me_2$ one has:

$$\begin{aligned}
dv &= dme_2=d\pi_2e_2=d=ded=d\pi_2e_1=dme_1=du\\
cv &= cme_1=c\pi_1e_2=cec=c=c\pi_1e_1=cme_1=cu\\
ue &= me_1e=me_2e=ve.
\end{aligned}$$

■

Proposition 68 *Every premorphism of precategories*

$$(f_2,f_1,f_0):(C_2,C_1,C_0,d,c,...)\longrightarrow(C_2',C_1',C_0',d,c,...)$$

is in particular a premorphism of sorts

$$(f_1,f_0):(C_1,C_0,d,c,...)\longrightarrow(C_1',C_0',d,c,...)\,,$$

and furthermore, if it is a morphism of precategories then it is also a morphism of sorts.

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Proof. Every premorphism of precategories is trivially a premorphism of sorts. To prove that the same holds for morphisms simply observe that

$$\left\{\begin{array}{c} f_1 m = m f_2 \\ f_2 e_i = e_i f_1 \ \ , \ i = 1,2 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} u f_1 = m e_1 f_1 = m f_2 e_1 = f_1 m e_1 = f_1 u \\ v f_1 = m e_2 f_1 = m f_2 e_2 = f_1 m e_2 = f_1 v \end{array}\right. .$$

■

The situation

$$\begin{tikzcd}
\operatorname{Precat}(\mathbf{C}) \arrow[r, phantom, "\subseteq"] \arrow[d] & \operatorname{Precat}_{pre}(\mathbf{C}) \arrow[d] \\
\operatorname{Sort}(\mathbf{C}) \arrow[r, phantom, "\subseteq"] & \operatorname{Sort}_{pre}(\mathbf{C})
\end{tikzcd}$$

will be studied in more detail.

A precategory, being in particular a sort, is said to be:
Homogeneous, when $me_1e = me_2e = e$.
Unitary, when $me_1 = 1_{C_1} = me_2$.
Adjustable, when $me_1me_2 = me_2me_1$.
Left bounded, when $me_2me_2 = me_2$.
Right bounded, when $me_1me_1 = me_1$.
Bounded, when it is both left and right bounded.

Definition 69 (Thin precategory) *A precategory*

$$\begin{tikzcd}
C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[l, shift right=2, "e_2"'] \arrow[l, shift left=2, "e_1"] \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

is thin*, when the split square*

$$\begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\
C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"']
\end{tikzcd}$$

is a split pullback.

A thin precategory is determined, up to isomorphism, by a system (C_0, C_1, d, c, e, m) satisfying conditions $(7.4), (7.5), (7.6)$ with $C_2, \pi_1, \pi_2, e_1, e_2$ obtained by the construction of the pullback and induced morphisms. For simplicity, a thin precategory will be referred to as (C_0, C_1, d, c, e, m), assumed with the hidden structure: $C_2, \pi_1, \pi_2, e_1, e_2$, always with the same notation.

For the reader not familiar with the stated definition of a precategory, a word of guidance: it is not a requirement, for a comfortable reading, to have some intuition for a precategory in general, however, intuition for a *thin* precategory is very easy to grasp in the case of sets; just write $x : a \longrightarrow b$ for an element $x \in C_1$ and for every

$$a \xrightarrow{y} b \xrightarrow{x} c$$

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write

$$\begin{aligned} m(x,y) &= x * y \\ \pi_2(x,y) &= y \\ \pi_1(x,y) &= x \\ e_2(y) &= (1_b, y) \\ e_1(x) &= (x, 1_b)\,. \end{aligned}$$

7.3 Precategories and sorts in weakly Mal'cev categories

The abbreviation WMC stands for Weakly Mal'cev Category (see [Ch3]).

We will use the name *admissible sort* as an abuse of notation, meaning that it is a sort with (u,l,v) an admissible triple.

Definition 70 (Admissible Sort) *In a WMC, a sort*

$$\begin{tikzcd} C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

$$\begin{aligned} de &= 1_{C_0} = ce \\ du &= d = dv \\ cu &= c = cv \\ ue &= ve \ (= l)\,, \end{aligned}$$

is said to be admissible *if the triple*

$$(u,l,v)$$

is admissible (see [Ch3]) relative to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} .$$

Recall from [Ch3] that the triple (u,l,v) being admissible means that there is a, necessarily unique, morphism written as $\begin{bmatrix} u & l & v \end{bmatrix} : C_1 \times_{C_0} C_1 \longrightarrow C_1$ satisfying $\begin{bmatrix} u & l & v \end{bmatrix} e_1 = u$ and $\begin{bmatrix} u & l & v \end{bmatrix} e_2 = v$.

Theorem 71 *In a WMC, every admissible sort is also a thin precategory. More specifically, given the admissible sort*

$$\begin{tikzcd} C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

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it is possible to construct the thin precategory

$$\begin{tikzcd}
C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, shift left=2, leftarrow, "e_2" description] \arrow[r, "m" description] \arrow[r, shift right=2, leftarrow, "e_1" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=2, "c"'] & C_0
\end{tikzcd}$$

where

$$\begin{aligned}
1_{C_2} &= \begin{bmatrix} 1_{C_1} & e & ec \\ d & 1_{C_0} & c \\ ed & e & 1_{C_1} \end{bmatrix} \\
\pi_1 &= \begin{bmatrix} 1 & e & ec \end{bmatrix} \\
\pi_2 &= \begin{bmatrix} ed & e & 1 \end{bmatrix} \\
e_1 &= \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} \quad , \quad e_2 = \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} \\
m &= \begin{bmatrix} u & l & v \end{bmatrix} .
\end{aligned}$$

Proof. The pullback C_2 always exists in a WMC, since d and c are split epis. In an WMC, every split pullback is of the form presented above, and $m = \begin{bmatrix} u & l & v \end{bmatrix}$ is well defined because the sort is admissible by hypothesis.
It remains to prove $dm = d\pi_2$ and $cm = c\pi_1$:

$$\begin{aligned}
dm &= d\begin{bmatrix} u & l & v \end{bmatrix} \\
&= \begin{bmatrix} du & dl & dv \end{bmatrix} \\
&= \begin{bmatrix} d & 1 & d \end{bmatrix} \\
&= \begin{bmatrix} ded & de & d \end{bmatrix} \\
&= d\begin{bmatrix} ed & e & 1 \end{bmatrix} \\
&= d\pi_2
\end{aligned}$$

$$\begin{aligned}
cm &= c\begin{bmatrix} u & l & v \end{bmatrix} \\
&= \begin{bmatrix} cu & cl & cv \end{bmatrix} \\
&= \begin{bmatrix} c & 1 & c \end{bmatrix} \\
&= \begin{bmatrix} c & ce & cec \end{bmatrix} \\
&= c\begin{bmatrix} 1 & e & ec \end{bmatrix} \\
&= c\pi_1 .
\end{aligned}$$

■

Proposition 72 *In a WMC, a premorphism of admissible sorts is also a premorphism of thin precategories, and furthermore, if it is a morphism of admissible sorts then it is also a morphism of thin precategories.*

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More specifically, given a premorphism of admissible sorts

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\
C_1' \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & C_0'
\end{tikzcd}$$

then

$$\begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\pi_1"'] \arrow[d, "f_2"'] & C_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\
C_2' \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\pi_1"'] & C_1' \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & C_0'
\end{tikzcd}$$

is a premorphism of thin precategories, with

$$f_2 = \begin{bmatrix} f_1 & f_1e & f_1ec \\ f_0d & f_0 & f_0c \\ f_1ed & f_1e & f_1 \end{bmatrix} ;$$

and furthermore, if

$$f_1e = ef_0 \ , \ f_1u = uf_1 \ , \ f_1v = vf_1$$

then (f_2, f_1, f_0) *is a morphism of precategories, that is*

$$\begin{aligned} f_1 \begin{bmatrix} u & l & v \end{bmatrix} &= \begin{bmatrix} u & l & v \end{bmatrix} f_2 \\ f_2e_i &= e_if_1 \ , \ i = 1, 2. \end{aligned}$$

Proof. Previous proposition says that given the admissible sorts

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

and

$$\begin{tikzcd}
C_1' \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e" description]
\end{tikzcd}$$

they are also thin precategories of the form

$$\begin{tikzcd}
C_2 \arrow[r, "{[u\,l\,v]}" description] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

and

$$\begin{tikzcd}
C_2' \arrow[r, "{[u\,l\,v]}" description] & C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e" description]
\end{tikzcd}$$

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with the canonical data π_1,π_2, e_1, e_2 given as above.
The morphism $f_2 : C_2 \longrightarrow C_2'$, being a morphism into a pullback and satisfying

$$\begin{aligned} \pi_1 f_2 &= f_1 \pi_1 \\ \pi_2 f_2 &= f_1 \pi_2 \end{aligned}$$

is completely determined by (see [Ch3], Proposition 27)

$$\begin{aligned} f_2 &= \begin{bmatrix} \pi_1 f_2 \\ d\pi_1 f_2 \\ \pi_2 f_2 \end{bmatrix} = \begin{bmatrix} f_1 \pi_1 \\ f_0 d \pi_1 \\ f_1 \pi_2 \end{bmatrix} \\ &= \begin{bmatrix} f_1 & f_1 e & f_1 ec \\ f_0 d & f_0 & f_0 c \\ f_1 ed & f_1 e & f_1 \end{bmatrix}. \end{aligned}$$

Now, assuming

$$f_1 e = e f_0 \ , \ f_1 u = u f_1 \ , \ f_1 v = v f_1$$

one has to prove

$$\begin{aligned} f_1 \begin{bmatrix} u & l & v \end{bmatrix} &= \begin{bmatrix} u & l & v \end{bmatrix} f_2 \\ f_2 e_i &= e_i f_1 \ \ , \ i = 1, 2. \end{aligned}$$

In fact, using $f_1 e = e f_0$, on the one hand

$$\begin{aligned} f_2 &= \begin{bmatrix} f_1 & f_1 e & f_1 ec \\ f_0 d & f_0 & f_0 c \\ f_1 ed & f_1 e & f_1 \end{bmatrix} = \begin{bmatrix} f_1 & e f_0 & ec f_1 \\ d f_1 & f_0 & c f_1 \\ ed f_1 & e f_0 & f_1 \end{bmatrix} = \\ & \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} f_1 & \begin{bmatrix} e \\ 1 \\ e \end{bmatrix} f_0 & \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} f_1 \end{bmatrix} \\ &= \begin{bmatrix} e_1 f_1 & e_1 e f_0 & e_2 f_1 \end{bmatrix}, \end{aligned}$$

while using the fact that (e_1, e_2) is jointly epimorphic, on the other hand

$$f_2 = \begin{bmatrix} f_2 e_1 & f_2 e_1 e & f_2 e_2 \end{bmatrix}$$

so that one concludes

$$\begin{aligned} f_2 e_1 &= e_1 f_1 \\ f_2 e_2 &= e_2 f_1. \end{aligned}$$

To prove $f_1 \begin{bmatrix} u & l & v \end{bmatrix} = \begin{bmatrix} u & l & v \end{bmatrix} f_2$ observe that

$$\begin{aligned} f_1 \begin{bmatrix} u & l & v \end{bmatrix} &= \begin{bmatrix} f_1 u & f_1 l & f_1 v \end{bmatrix} \\ &= \begin{bmatrix} u f_1 & l f_0 & v f_1 \end{bmatrix} \end{aligned}$$

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and from the above calculations

$$\begin{aligned}\begin{bmatrix} u & l & v \end{bmatrix} f_2 &= \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} e_1 f_1 & e_1 e f_0 & e_2 f_1 \end{bmatrix} \\ &= \begin{bmatrix} u f_1 & l f_0 & v f_1 \end{bmatrix}.\end{aligned}$$

■

Definition 73 (Admissible Precategory) *A precategory (in a category with pullbacks of split epis along split epis)*

$$\begin{tikzcd} C_2 \arrow[r, shift left=6, "\pi_2"] \arrow[r, leftarrow, shift left=3, "e_2" description] \arrow[r, "m" description] \arrow[r, leftarrow, shift right=3, "e_1" description] \arrow[r, shift right=6, "\pi_1"'] & C_1 \arrow[r, shift left=3, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=3, "c"'] & C_0 \end{tikzcd}$$

is said to be admissible *if the induced morphism* $\theta : C_2 \longrightarrow C_1 \times_{C_0} C_1$ *from* C_2 *into the split pullback*

$$\begin{tikzcd} C_2 \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "\pi_1"'] & & \\ & C_1 \times_{C_0} C_1 \arrow[r, shift left, "\pi_2'"] \arrow[d, shift right, "\pi_1'"'] & C_1 \arrow[l, shift left, "e_2'"] \arrow[d, shift right, "c"'] \\ & C_1 \arrow[r, shift left, "d"] \arrow[u, shift right, "e_1'"'] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"'] \end{tikzcd}$$

is a split epi, with splitting $s : C_1 \times_{C_0} C_1 \longrightarrow C_2$ *satisfying (besides* $\theta s = 1$*)*

$$\begin{aligned} s e_1' &= e_1 \\ s e_2' &= e_2. \end{aligned}$$

Once again, the terminology *admissible precategory* is used as an abuse of notation for a precategory where the triple $(e_1, e_1 e, e_2)$ is admissible, as shown in the following proposition.

Proposition 74 *In a WMC, a precategory*

$$\begin{tikzcd} C_2 \arrow[r, shift left=6, "\pi_2"] \arrow[r, leftarrow, shift left=3, "e_2" description] \arrow[r, "m" description] \arrow[r, leftarrow, shift right=3, "e_1" description] \arrow[r, shift right=6, "\pi_1"'] & C_1 \arrow[r, shift left=3, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=3, "c"'] & C_0 \end{tikzcd}$$

is admissible if and only if the triple $(e_1, e_1 e, e_2)$ *is admissible with respect to the split span*

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} ,$$

that is, the morphism

$$\begin{bmatrix} e_1 & e_1 e & e_2 \end{bmatrix} : C_1 \times_{C_0} C_1 \longrightarrow C_2$$

exists.

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Proof. Since the splitting $s : C_1 \times_{C_0} C_1 \longrightarrow C_2$ is required to satisfy

$$\begin{aligned} se_1' &= e_1 \\ se_2' &= e_2, \end{aligned}$$

in a WMC, if it exists, it is uniquely determined and given by

$$s = \begin{bmatrix} e_1 & e_1 e & e_2 \end{bmatrix}.$$

It remains to prove $\theta s = 1$ and in fact

$$\theta s = \begin{bmatrix} \pi_1 \\ d\pi_1 \\ \pi_2 \end{bmatrix} \begin{bmatrix} e_1 & e_1 e & e_2 \end{bmatrix} = \begin{bmatrix} 1_{C_1} & e & ec \\ d & 1_{C_0} & c \\ ed & e & 1_{C_1} \end{bmatrix} = 1_{C_1 \times_{C_0} C_1}.$$

■

Theorem 75 *In a WMC,* $\mathbf{C}$*, the following are equivalences of categories:*

$$\begin{aligned} ThinPreCat(\mathbf{C}) &\sim AdmSort(\mathbf{C}) \\ ThinPreCat_{pre}(\mathbf{C}) &\sim AdmSort_{pre}(\mathbf{C}) \end{aligned}$$

and furthermore, thin precategories (with premorphisms) are reflective in admissible precategories (with premorphisms).

Proof. The equivalences of categories are established by Propositions 67, 68, 71 and 72 simply observing that in a WMC, a thin precategory is always an admissible sort.
To prove that thin precategories are reflective in admissible precategories, is equivalent to proving that the functor H (which is well defined since every thin precategory is admissible)

$$AdmSort_{pre}(\mathbf{C}) \xrightarrow{H} AdmPreCat_{pre}(\mathbf{C})$$

has a left adjoint.
Consider the functor

$$AdmPreCat_{pre}(\mathbf{C}) \xrightarrow{I} AdmSort_{pre}(\mathbf{C})$$

defined as follows

$$\begin{aligned} H(C_2, C_1, C_0, d, c, e, m, ...) &= (C_1, C_0, d, c, e, me_1, me_2) \\ H(f_2, f_1, f_0) &= (f_1, f_0). \end{aligned}$$

To see it is well defined it is sufficient to check that $(C_1, C_0, d, c, e, me_1, me_2)$ is an admissible sort, since Proposition 67 ensures it is a sort. In fact

$$\begin{bmatrix} me_1 & me_1 e & me_2 \end{bmatrix}$$

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exists because $(C_2, C_1, C_0, d, c, e, m, ...)$ is an admissible precategory, by previous proposition

$$\begin{bmatrix} e_1 & e_1e & e_2 \end{bmatrix}$$

exists, and so

$$m\begin{bmatrix} e_1 & e_1e & e_2 \end{bmatrix} = \begin{bmatrix} me_1 & me_1e & me_2 \end{bmatrix}$$

also exists.
It remains to prove that the premorphism

$$\begin{tikzcd}
C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, leftarrow, shift left=2, "e_2" description] \arrow[r, "m" description] \arrow[r, leftarrow, shift right=2, "e_1" description] \arrow[r, shift right=4, "\pi_1"'] \arrow[d, "\theta"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=2, "c"'] \arrow[d, equal] & C_0 \arrow[d, equal] \\
C_1 \times_{C_0} C_1 \arrow[r, shift left=4, "\pi'_2"] \arrow[r, leftarrow, shift left=2, "e'_2" description] \arrow[r, "{m[e_1\ e_0\ e_2]}" description] \arrow[r, leftarrow, shift right=2, "e'_1" description] \arrow[r, shift right=4, "\pi'_1"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=2, "c"'] & C_0
\end{tikzcd}$$

has the desired universal property, that is, given

$$(f_2, f_1, f_0) : (C_2, C_1, C_0, d, c, e, m, ...) \longrightarrow H\left(C'_1, C'_0, d, c, e, u, v\right)$$

there is a unique morphism $\overline{f_2} : C_1 \times_{C_0} C_1 \longrightarrow C'_1 \times_{C'_0} C'_1$ such that

$$\overline{f_2}\theta = f_2.$$

In fact

$$\begin{aligned}
\theta &= \begin{bmatrix} \pi_1 \\ d\pi_1 \\ \pi_2 \end{bmatrix} \\
\overline{f_2} &= \begin{bmatrix} f_1 & f_1e & f_1ec \\ f_0d & f_0 & f_0c \\ f_1ed & f_1e & f_1 \end{bmatrix} = \begin{bmatrix} f_1\pi'_1 \\ df_1\pi'_1 \\ f_1\pi'_2 \end{bmatrix} \\
f_2 &= \begin{bmatrix} f_1\pi_1 \\ df_1\pi_1 \\ f_1\pi_2 \end{bmatrix}
\end{aligned}$$

and so

$$\begin{bmatrix} f_1\pi'_1 \\ df_1\pi'_1 \\ f_1\pi'_2 \end{bmatrix} \begin{bmatrix} \pi_1 \\ d\pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} f_1\pi_1 \\ df_1\pi_1 \\ f_1\pi_2 \end{bmatrix}.$$

■

Observation: the universal premorphism $(\theta, 1_{C_1}, 1_{C_0})$ fails to be a morphism of precategories because, in general

$$m\begin{bmatrix} e_1 & e_1e & e_2 \end{bmatrix}\theta = \begin{bmatrix} me_1 & me_1e & me_2 \end{bmatrix} \begin{bmatrix} \pi_1 \\ d\pi_1 \\ \pi_2 \end{bmatrix}$$

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is not equal to m, however it does satisfy

$$\begin{aligned} 1_{C_1}e &= e1_{C_0} \\ \theta e_1 &= e_1' \\ \theta e_2 &= e_2', \end{aligned}$$

since

$$\theta e_1 = \begin{bmatrix} \pi_1 \\ d\pi_1 \\ \pi_2 \end{bmatrix} e_1 = \begin{bmatrix} \pi_1 e_1 \\ d\pi_1 e_1 \\ \pi_2 e_1 \end{bmatrix} = \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} = e_1',$$

and similarly to e_2'.

7.4 Protocategories and associativity

The word protocategory is used here to denote a structure that is an extension of a precategory (as explained in the introduction), where it is possible to talk about associativity, even if not in the presence of pullbacks; the morphisms i_1, i_2 and i_0 are inserted because they will have some important role in further sections.

Let $\mathbf{C}$ be a given category.

Definition 76 (*protocategory*) *A* protocategory *is a diagram in* $\mathbf{C}$ *of the form*

$$\begin{tikzcd}
C_3 \arrow[r, shift left=9, "p_2"] \arrow[r, leftarrow, shift left=6, "i_2" description] \arrow[r, shift left=3, "m_2" description] \arrow[r, leftarrow, "i_0" description] \arrow[r, shift right=3, "m_1" description] \arrow[r, leftarrow, shift right=6, "i_1" description] \arrow[r, shift right=9, "p_1" description] & C_2 \arrow[r, shift left=6, "\pi_2"] \arrow[r, leftarrow, shift left=3, "e_2" description] \arrow[r, "m" description] \arrow[r, leftarrow, shift right=3, "e_1" description] \arrow[r, shift right=6, "\pi_1"'] & C_1 \arrow[r, shift left=3, "d"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=3, "c"'] & C_0
\end{tikzcd}$$

such that

$$\begin{tikzcd}
C_3 \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & C_2 \arrow[l, shift left, "i_2"] \arrow[d, shift right, "\pi_1"'] \\
C_2 \arrow[r, shift left, "\pi_2"] \arrow[u, shift right, "i_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[u, shift right, "e_1"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\
C_1 \arrow[r, shift left, "d"] \arrow[u, shift right, "e_1"'] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"']
\end{tikzcd} \tag{7.10}$$

are split squares, so that in particular

$$de = 1_{C_0} = ce \tag{7.11}$$

and furthermore, the following conditions are satisfied

$$dm = d\pi_2 \tag{7.12}$$

$$cm = c\pi_1, \tag{7.13}$$

$$\begin{aligned} \pi_1 m_2 &= mp_1 \\ \pi_2 m_2 &= \pi_2 p_2 \end{aligned}$$

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$$\begin{aligned}\pi_1 m_1 &= \pi_1 p_1\\ \pi_2 m_1 &= m p_2.\end{aligned}$$

(The morphism i_0 *will be used only in further sections and it satisfies*

$$\begin{aligned}p_1 i_0 &= e_1 \pi_1\\ p_2 i_0 &= e_2 \pi_2\end{aligned}$$

and also $i_0 e_1 = i_1 e_1$ *and* $i_0 e_2 = i_2 e_2$*.)*

Every protocategory is in particular a precategory and it is said to be:
Thin, when both split squares (7.10) are split pullbacks.
Homogeneous, when $me_1e = me_2e = e$.
Unitary, when $me_1 = 1_{C_1} = me_2$.
Adjustable, when $me_1me_2 = me_2me_1$.
Left bounded, when $me_2me_2 = me_2$.
Right bounded, when $me_1me_1 = me_1$.
Bounded, when it is both left and right bounded.

And also:
Associative, when $mm_1 = mm_2$.
Half-associative, when $mm_1i_1 = mm_2i_1$ and $mm_1i_2 = mm_2i_2$.

Considerations over the morphisms between protocategories are the same stated for precategories.

In a category with all pullbacks of split epis along split epis,

$$\mathrm{ThinPrecat}\,(\mathbf{C}) = \mathrm{ThinProtoCat}\,(\mathbf{C})$$

and then a thin-protocategory is given by a system

$$(C_0, C_1, d, c, e, m) \tag{7.14}$$

such that

$$de = 1_{C_0} = ce \tag{7.15}$$

$$\begin{aligned}dm &= d\pi_2 &(7.16)\\ cm &= c\pi_1, &(7.17)\end{aligned}$$

with the remaining structure: C_2, C_3, π_1, π_2, e_1, e_2, p_1, p_2, i_1, i_2, m_1, m_2, i_0, being completely determined as follows

$$\begin{aligned}C_2 &= C_1 \times_{C_0} C_1 \text{ , with projections } \pi_1 \text{ and } \pi_2\\ C_3 &= C_2 \times_{C_1} C_2 \text{ , with projections } p_1 \text{ and } p_2\\ m_1 &= 1 \times m\\ m_2 &= m \times 1\\ e_1 &= \langle 1, ed\rangle\\ e_2 &= \langle ec, 1\rangle\\ i_1 &= \langle 1, e_1\pi_2\rangle\\ i_2 &= \langle e_2\pi_1, 1\rangle\\ i_0 &= \langle e_1\pi_1, e_2\pi_2\rangle\,,\end{aligned}$$

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so that a thin protocategory will be referred to simply as a system (7.14) satisfying (7.15), (7.16), (7.17) with the remaining above structure assumed to be present always in this same notation, and even if the category **C** does not have all pullbacks.

In order to obtain some intuition upon the morphisms involved in a protocategory, in addition to the intuition given for a precategory in sets, one adds: for

$$\bullet \xrightarrow{z} \bullet \xrightarrow{y} \bullet \xrightarrow{x} \bullet$$

$$\begin{aligned}
m_1(x,y,z) &= (x, y*z)\\
m_2(x,y,z) &= (x*y, z)\\
p_1(x,y,z) &= (x,y)\\
p_2(x,y,z) &= (y,z)\\
i_1(x,y) &= (x,y,1)\\
i_2(x,y) &= (1,x,y)\\
i_0(x,y) &= (x,1,y).
\end{aligned}$$

Protocategories in weakly Mal'cev categories

Theorem 77 *In a WMC, every admissible sort is also a thin protocategory. More specifically, given the admissible sort*

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

it is possible to construct the thin protocategory

$$\begin{tikzcd}[column sep=large]
C_3 \arrow[r, shift left=6, "p_2"] \arrow[r, shift left=2, "m_2" description] \arrow[r, shift right=2, "m_1" description] \arrow[r, shift right=6, "p_1" description] & C_2 \arrow[l, shift left=4, "i_2" description] \arrow[l, "i_0" description] \arrow[l, shift right=4, "i_1" description] \arrow[r, shift left=4, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[l, shift right=2, "e_2" description] \arrow[l, shift left=2, "e_1" description] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

where the split pullback

$$\begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\
C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"']
\end{tikzcd}$$

is given by (with $e_0 = e_1 e = e_2 e$, $\pi_0 = d\pi_1 = c\pi_2$*)*

$$\begin{aligned}
1_{C_2} &= \begin{bmatrix} 1_{C_1} & e & ec \\ d & 1_{C_0} & c \\ ed & e & 1_{C_1} \end{bmatrix} \quad , \quad e_1 = \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} \quad , \quad e_0 = \begin{bmatrix} e \\ 1 \\ e \end{bmatrix} \quad , \quad e_2 = \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix}\\
\pi_1 &= \begin{bmatrix} 1 & e & ec \end{bmatrix}\\
\pi_0 &= \begin{bmatrix} d & 1 & c \end{bmatrix}\\
\pi_2 &= \begin{bmatrix} ed & e & 1 \end{bmatrix},
\end{aligned}$$

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with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd}$$

while the split pullback

$$\begin{tikzcd} C_3 \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & C_2 \arrow[l, shift left, "i_2"] \arrow[d, shift right, "\pi_1"'] \\ C_2 \arrow[u, shift right, "i_1"'] \arrow[r, shift left, "\pi_2"] & C_1 \arrow[l, shift left, "e_2"] \arrow[u, shift right, "e_1"'] \end{tikzcd}$$

is considered as the composite pullback

$$\begin{tikzcd} C_3 \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & C_2 \arrow[l, shift left, "i_2"] \arrow[d, shift right, "\pi_1"'] \\ C_2 \arrow[u, shift right, "i_1"'] \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[u, shift right, "e_1"'] \arrow[d, shift right, "c"'] \\ C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"'] \end{tikzcd}$$

and it is given by

$$1_{C_3} = \begin{bmatrix} 1_{C_1} & e & ec\pi_1 \\ d & 1_{C_0} & c\pi_1 \\ e_1ed & e_1e & 1_{C_2} \end{bmatrix} = \left[\begin{matrix} \underbrace{\begin{matrix} 1_{C_1} \\ d \\ \begin{bmatrix} ed \\ d \\ ed \end{bmatrix} \end{matrix}}_{i_1e_1} & \underbrace{\begin{matrix} e \\ 1_{C_0} \\ \begin{bmatrix} e \\ 1_{C_0} \\ e \end{bmatrix} \end{matrix}}_{i_1e_1e} & \begin{matrix} \begin{bmatrix} ec & e & ec \end{bmatrix} \\ \begin{bmatrix} c & 1_{C_0} & c \end{bmatrix} \\ \begin{bmatrix} 1_{C_1} & e & ec \\ d & 1_{C_0} & c \\ ed & e & 1_{C_1} \end{bmatrix} \\ \underbrace{\hphantom{ed}}_{i_2e_1} \qquad\qquad \underbrace{\hphantom{1_{C_1}}}_{i_2e_2} \end{matrix} \end{matrix} \right]$$

$$\begin{aligned} \pi_1p_1 &= \begin{bmatrix} 1_{C_1} & e & ec\pi_1 \end{bmatrix} \\ p_1 &= \begin{bmatrix} e_1 & e_1e & e_2\pi_1 \end{bmatrix} \\ p_2 &= \begin{bmatrix} e_1ed & e_1e & 1_{C_2} \end{bmatrix} \end{aligned} \qquad i_1e_1 = \begin{bmatrix} 1_{C_1} \\ d \\ e_1ed \end{bmatrix} \quad , \quad i_1 = \begin{bmatrix} \pi_1 \\ d\pi_1 \\ e_1\pi_2 \end{bmatrix} \quad , \quad i_2 = \begin{bmatrix} ec\pi_1 \\ c\pi_1 \\ 1_{C_2} \end{bmatrix}$$

$$i_0 = \begin{bmatrix} \pi_1 \\ d\pi_1 \\ e_2\pi_2 \end{bmatrix}$$

with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} ,$$

furthermore,

$$m = \begin{bmatrix} u & l & v \end{bmatrix}$$

with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd}$$

and the morphisms m_1 *and* m_2 *are determined as*

$$m_1 = \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & e_2 l & \begin{bmatrix} e_2 u & e_2 l & e_2 v \end{bmatrix} \end{bmatrix}$$

$$m_2 = \begin{bmatrix} e_1 u & e_1 l & \begin{bmatrix} e_1 v & e_1 l & \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} .$$

Every thin protocategory is obtained in this way.

Proof. The morphisms π_1, π_2, e_1, e_2 are as such because they form a split pullback in a WMC, and m is well defined because the sort is given to be admissible.

Concerning C_3, the experience tells us that it is preferable to consider it as the object in the composite pullback

$$\begin{tikzcd} C_3 \arrow[r, "p_2"] \arrow[d, "p_1"'] & C_2 \arrow[d, "\pi_1"] \\ C_2 \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "d"] & C_0 \end{tikzcd}$$

so that $C_3 = C_1 \times_{C_0} C_2$ rather than $C_2 \times_{C_1} C_2$, and it is considered with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} ,$$

the morphisms $p_1, p_2 : C_3 \longrightarrow C_2$ are then determined by

$$\begin{aligned} p_1 &= \begin{bmatrix} p_1 (i_1 e_1) & p_1 (i_1 e_1 e) & p_1 i_2 \end{bmatrix} = \begin{bmatrix} e_1 & e_1 e & e_2 \pi_1 \end{bmatrix} \\ p_2 &= \begin{bmatrix} p_2 (i_1 e_1) & p_2 (i_1 e_1 e) & p_2 i_2 \end{bmatrix} = \begin{bmatrix} e_1 e d & e_1 e & 1 \end{bmatrix} \end{aligned}$$

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and the morphisms $m_1, m_2 : C_3 \longrightarrow C_2$ are determined by

$$\begin{aligned}
m_1 &= \begin{bmatrix} \pi_1 m_1 \\ d\pi_1 m_1 \\ \pi_2 m_1 \end{bmatrix} = \begin{bmatrix} \pi_1 p_1 \\ d\pi_1 p_1 \\ m p_2 \end{bmatrix} \\
&= \begin{bmatrix} \pi_1 e_1 & \pi_1 e_1 e & \pi_1 e_2 \pi_1 \\ d\pi_1 e_1 & d\pi_1 e_1 e & d\pi_1 e_2 \pi_1 \\ m e_1 e d & m e_1 e & m \end{bmatrix} \\
&= \begin{bmatrix} 1 & e & ec\pi_1 \\ d & 1 & c\pi_1 \\ ued & ue & m \end{bmatrix}, \text{ and since } cm = c\pi_1 \text{ and } cl = l \\
&= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} l & \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} m \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & e_2 l & e_2 m \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
m_2 &= \begin{bmatrix} \pi_1 m_2 \\ d\pi_1 m_2 \\ \pi_2 m_2 \end{bmatrix} = \begin{bmatrix} m p_1 \\ d\pi_2 p_1 \\ \pi_2 p_2 \end{bmatrix} \\
&= \begin{bmatrix} m e_1 & m e_1 e & m e_2 \pi_1 \\ d\pi_2 e_1 & d\pi_2 e_1 e & d\pi_2 e_2 \pi_1 \\ \pi_2 e_1 e d & \pi_2 e_1 e & \pi_2 \end{bmatrix} \\
&= \begin{bmatrix} u & l & v\pi_1 \\ d & 1 & d\pi_1 \\ ed & e & \pi_2 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} u & \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} l & \begin{bmatrix} v & l & lc \\ d & 1 & c \\ ed & e & 1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} e_1 u & e_1 l & \begin{bmatrix} e_1 v & e_1 l & \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix}.
\end{aligned}$$

As a consequence of the very definition of WMC, we have that every thin protocategory is obtained in this way. ■

Proposition 78 *In a WMC, the following are equivalent:*
1. C is an associative-thin-protocategory.
2. C is a half-associative-thin-protocategory.

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3. C is an adjustable-admissible-sort such that

$$\begin{aligned} \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} &= uu \qquad (7.18) \\ \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} &= vv. \qquad (7.19) \end{aligned}$$

Proof. (1) $\Longleftrightarrow$ (2) By definition of WMC,

$$mm_1e_i = mm_2e_i \ , \ i = 1, 2 \quad \Longrightarrow mm_1 = mm_2,$$

the converse is trivial.

(1) $\Longleftrightarrow$ (3) By the previous proposition, every admissible sort determines a thin protocategory, conversely, by Proposition 67 and the fact that a thin protocategory is the same as a thin precategory (in a WMC), one concludes that a thin protocategory is also an admissible sort (admissibility is due to the fact that $m = \begin{bmatrix} u & l & v \end{bmatrix}$ exists in the structure of a thin protocategory).
It remains to show that $mm_1 = mm_2$ is equivalent to conditions (7.18), (7.19) plus the fact that the sort is adjustable, which translates as

$$uv = vu.$$

From the previous proposition one has

$$\begin{aligned} mm_1 &= \begin{bmatrix} m \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & vl & vm \end{bmatrix} \\ & \quad \begin{bmatrix} \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & vl & \begin{bmatrix} vu & vl & vv \end{bmatrix} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} mm_2 &= \begin{bmatrix} me_1u & me_1l & \begin{bmatrix} me_1v & me_1l & m \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} uu & ul & \begin{bmatrix} uv & ul & \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix}. \end{aligned}$$

And $mm_1 = mm_2$ if and only if the following three conditions hold

$$\begin{aligned} \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} &= uu \\ uv &= vu \\ \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} &= vv. \end{aligned}$$

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■

Proposition 79 *In a WMC, if a thin protocategory is homogeneous then*

$$\text{adjustable} + \text{bounded} \Longleftrightarrow \text{associative}.$$

Proof. Homogeneous means that

$$l = e \quad (\text{or } ue = e = ve)\,,$$

and then

$$\begin{aligned} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} &= \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} = e_2 \\ \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} &= \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} = e_1, \end{aligned}$$

so that $mm_1 = mm_2$ (see previous result) if an only if

$$\begin{aligned} u &= uu && \text{(rigth bounded)} \\ uv &= vu && \text{(adjustable)} \\ v &= vv && \text{(left bounded)} \end{aligned}$$

■

The following corollary is to be compared with Proposition 57 and 58 of [Ch5]. Observe that the word precategory is used there with the same meaning as thin precategory here.

Corollary 80 *A thin precategory in Ab, the category of abelian groups, is determined by a diagram*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, shift left, "k"] & B \arrow[l, shift left, "h"] \end{tikzcd}$$

such that

$$kf = k = kg \quad , \quad kh = 0$$

and furthermore, it is an associative precategory if the following conditions also hold

$$\begin{aligned} ff &= f \\ fg &= gf \\ gg &= fhk + g \\ fh &= gh. \end{aligned}$$

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Proof. The category Ab is weakly Mal'cev, so by Theorem 75 it is sufficient to describe a sort $(C_0, C_1, d, c, e, u, v)$ with $\begin{bmatrix} u & l & v \end{bmatrix}$, but in Ab a morphism of the form $\begin{bmatrix} x & y & z \end{bmatrix}$ always exist, and since split epis in Ab are simply projections, one has

$$\begin{aligned} C_0 &= B \\ C_1 &= A \oplus B \\ d &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\ e &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ c &= \begin{bmatrix} k & 1 \end{bmatrix} \end{aligned}$$

and since $u, v : A \oplus B \longrightarrow A \oplus B$ are such that

$$du = d = dv,$$

then, they are of the form

$$u = \begin{bmatrix} f & h_1 \\ 0 & 1 \end{bmatrix} \quad , \quad v = \begin{bmatrix} g & h_2 \\ 0 & 1 \end{bmatrix},$$

but condition

$$ue = ve$$

implies $h_1 = h_2 = h$, so that $u\,, v$ and $l = ve = ue$ are given by

$$u = \begin{bmatrix} f & h \\ 0 & 1 \end{bmatrix} \quad , \quad v = \begin{bmatrix} g & h \\ 0 & 1 \end{bmatrix} \quad , \quad l = \begin{bmatrix} h \\ 1 \end{bmatrix}.$$

Conditions

$$cu = c = cv$$

imply

$$kf = k = kg \quad , \quad kh = 0$$

and the result is established for a precategory.

In order to interpret the conditions for associativity, and since Ab is a Mal'cev variety one has

$$\begin{aligned} \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} &= uu \iff u - ld + vld = uu \\ uv &= vu \\ \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} &= vv \iff ulc - lc + v = vv \end{aligned}$$

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that is

$$\begin{aligned}
\begin{bmatrix} f & gh+h \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} ff & fh+h \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} fg & fh+h \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} gf & gh+h \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} fhk+g & fh+h \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} gg & gh+h \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

or equivalently

$$\begin{aligned}
ff &= f \\
fh &= gh \\
gf &= fg \\
gg &= fhk+g.
\end{aligned}$$

■

Internal categories in WMC

A internal category is a unitary and associative thin protocategory, but since in general

$$\textit{unitary} \Longrightarrow \textit{homogeneous} + \textit{adjustable} + \textit{bound}$$

then in a WMC

$$\textit{unitary} \Longrightarrow \textit{associative},$$

so that in a WMC, an internal category is just a unitary thin protocategory, or equivalently a unitary admissible sort, or even an admissible reflexive graph, in the sense that it is a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1 = ce$$

such that

$$\begin{bmatrix} 1 & e & 1 \end{bmatrix}$$

exists with respects to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} .$$

Proposition 81 *In a WMC,* $\mathbf{C}$*, the following is an equivalence of categories*

$$AdmRGraph\left(\mathbf{C}\right) \sim Cat\left(\mathbf{C}\right).$$

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Proof. Simply observe that

$$\begin{array}{rcl} AdmRGraph\left(\mathbf{C}\right) & = & Unitary\text{-}AdmSort\left(\mathbf{C}\right) \\ Unitary\text{-}AdmSort\left(\mathbf{C}\right) & \sim & Unitary\text{-}ThinPreCat\left(\mathbf{C}\right) \\ Unitary\text{-}ThinPreCat\left(\mathbf{C}\right) & = & Unitary\text{-}ThinProtoCat\left(\mathbf{C}\right) \\ Unitary\text{-}ThinProtoCat\left(\mathbf{C}\right) & = & Cat\left(\mathbf{C}\right) \end{array}$$

which follows from Theorem 71 restricting to the unitary subcategory. ∎

7.5 Conclusion

The results presented here, may also be considered in the case of a naturally weakly Mal'cev category, $\mathbf{C}$, were every triple is admissible [Ch3] in order to obtain the following equivalences of categories (see [18] and references there)

$$\begin{array}{rcl} \mathrm{RGraphs}\left(\mathbf{C}\right) & \sim & \mathrm{Cat}\left(\mathbf{C}\right) \sim \mathrm{Groupoids}\left(\mathbf{C}\right) \\ \mathrm{Sorts}\left(\mathbf{C}\right) & \sim & Thin\text{-}\mathrm{PreCat}\left(\mathbf{C}\right), \end{array}$$

and the reflexion

$$\mathrm{PreCat}\left(\mathbf{C}\right) \xrightarrow{I} Thin\text{-}\mathrm{PreCat}\left(\mathbf{C}\right).$$

Recall that a naturally weakly Mal'cev category is a category satisfying the following two axioms: (a) it has pullbacks of split epis; (b) every pullback of split epis is also a pushout.

It is also possible to developed a theory for weakly Mal'cev categories in the same light as [9], [11], [13], [14] for unital, aritmetical, subtractive and Mal'cev categories: simply by everywhere replace the word *strongly epimorphic* by *jointly epimorphic* and always requiring the existence of the less limits possible. For instance we would obtain a *weakly unital category* as a pointed category, with products and such that for each product diagram

$$\begin{tikzcd} A \arrow[r, shift right, "i_1"'] & A \times B \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "i_2"] \end{tikzcd}$$

the induced pair (i_1, i_2) is jointly epimorphic. It is now easily seen that if a category $\mathbf{C}$ has pullbacks of split epis, then, each category $\mathrm{Pt}_C\left(\mathbf{C}\right)$, of points over $C \in \mathbf{C}$, has products, and furthermore, $\mathbf{C}$ is weakly Mal'cev if and only if $\mathrm{Pt}_C\left(\mathbf{C}\right)$ is weakly unital for every $C \in \mathbf{C}$.

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Chapter 8

Weakly Mal'cev sesquicategories

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Abstract: We define weakly Mal'cev sesquicategory as a weakly Mal'cev category (as introduced in [Ch3]) together with a 2-cell structure (as introduced in [Ch2]), required to satisfy the "weakly Mal'cev condition". The result is a sesquicategory where both morphisms and 2-cells have a "weakly Mal'cev enriched structure". The main example of a weakly Mal'cev sesquicategory is Cat(B), the sesquicategory of internal categories, internal functors and internal transformations (not necessarily natural) in a weakly Mal'cev category B. We introduce the notion of alpha-protocategory, generalizing the notion of protocategory [Ch7] to the point were it is possible to consider associativity up to an isomorphism 2-cell. We also compute the Pentagon Coherence Condition in the context of a weakly Mal'cev sesquicategory.

8.1 Introduction

This article has two purposes: (a) to introduce the notion of weakly Mal'cev sesquicategory with a cartesian 2-cell structure [Ch2], as an abstraction for (the

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sesquicategory) Cat(B) of internal categories in a weakly Mal'cev category B [Ch3], obtaining thus an axiomatic setting with the advantage of being easier to manipulate and simpler to calculate; (b) to investigate the associativity axiom in the context of an alpha-protocategory, to be defined, by calculating the pentagonal coherence condition

$$\begin{tikzcd}
& f(g(hk)) \arrow[rr, "f\alpha_{g,h,k}"] \arrow[dl, "\alpha_{f,g,hk}"'] & & f((gh)k) \arrow[dr, "\alpha_{f,gh,k}"] & \\
(fg)(hk) \arrow[drr, "\alpha_{fg,h,k}"'] & & & & (f(gh))k \arrow[dll, "\alpha_{f,g,h}k"] \\
& & ((fg)h)k & &
\end{tikzcd} \qquad \text{(pentagon)}$$

and finding equivalent but simpler conditions when internal to a weakly Mal'cev sesquicategory. This results will then be used in the description of pseudocategories in a weakly Mal'cev sesquicategory [Ch9].

We will freely use the definitions, notations and results from [Ch2], [Ch3] and [Ch7]. This article is organized as follows.

We introduce the notion of weakly Mal'cev sesquicategory; show that Cat(B), with B a weakly Mal'cev category, is an example of such a structure; introduce the notions of α-protocategory and α-sort, and calculate the pentagonal coherence condition. In the context of a weakly Mal'cev sesquicategory, with a cartesian 2-cell structure, the pentagonal coherence condition (see [1], [3], [4])

$$m\,(\alpha \times 0_1) + \alpha\,(1 \times m \times 1) + m\,(0_1 \times \alpha) = \alpha\,(m \times 1 \times 1) + \alpha\,(1 \times 1 \times m)$$

is equivalent to the following set of equations

$$\begin{aligned}
u\alpha_1 + \alpha \begin{bmatrix} 1 \\ d \\ e_1 ld \end{bmatrix} + m \begin{bmatrix} 0_1 \\ 0_d \\ \alpha_0 d \end{bmatrix} &= \alpha_1 u + \alpha \begin{bmatrix} 1 \\ d \\ e_2 ld \end{bmatrix} \\
u\alpha_2 + \alpha_2 u + v\alpha_1 &= \alpha_1 v + \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} \\
u\alpha_3 + \alpha_2 v + v\alpha_2 &= \alpha \begin{bmatrix} lc \\ c \\ e_1 \end{bmatrix} + \alpha_3 u \\
m \begin{bmatrix} \alpha_0 c \\ 0_c \\ 0_1 \end{bmatrix} + \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} + v\alpha_3 &= \alpha \begin{bmatrix} lc \\ c \\ e_2 \end{bmatrix} + \alpha_3 v.
\end{aligned}$$

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with $u = me_1, l = me_1e, v = me_2, \alpha_1 = \alpha i_1 e_1, \alpha_2 = \alpha i_2 e_1, \alpha_3 = \alpha i_2 e_2, \alpha_0 = \alpha i_2 e_2 e$ and e_1, e_2, i_1, i_2 defined as in a protocategory [Ch7].

8.2 Weakly Mal'cev sesquicategories

Let $\mathbf{C}$ be a category and assume all the notations and definitions from [Ch2] and [Ch3].

Definition 82 *A 2-cell structure* $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ *over* $\mathbf{C}$ *is said to be weakly Mal'cev if the following condition is satisfied:*
For every split square

$$\begin{tikzcd}
P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\
A \arrow[r, shift left, "f"] \arrow[u, shift right, "e_1"'] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"']
\end{tikzcd} ,$$

if (P, p_1, p_2) *is a pullback, then the pair* (e_1, e_2) *is jointly epimorphic with respect to the 2-cells, that is, for every* $x, y \in H(P, D')$,

$$\left\{ \begin{array}{l} xe_2 = ye_2 \\ xe_1 = ye_1 \end{array} \right. \Longrightarrow x = y.$$

Definition 83 (weakly Mal'cev sesquicategory) *A pair* $(\mathbf{C}, \mathbf{H})$ *is said to be a weakly Mal'cev sesquicategory when:*
1. $\mathbf{C}$ *is a weakly Mal'cev category;*
2. $\mathbf{H}$ *is a weakly Mal'cev 2-cell structure over* $\mathbf{C}$.

Proposition 84 *In a weakly Mal'cev structured category* $(\mathbf{C}, H, \mathrm{dom}, \mathrm{cod}, 0, +)$*, given a split span*

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"']
\end{tikzcd} ,$$

for every object D *in* $\mathbf{C}$*, and triple of 2-cells* (x, y, z),

$$\begin{tikzcd}
A \arrow[dr, Rightarrow, "x"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, Rightarrow, "y"] & B \arrow[dl, Rightarrow, "z"] \\
& D &
\end{tikzcd} \tag{8.1}$$

satisfying

$$xr = y = zs,$$

there is at most one 2-cell in $H(A \times_C B, D)$*, denoted* $\begin{bmatrix} x & y & z \end{bmatrix}$ *when it exists, with the property*

$$\begin{aligned} \begin{bmatrix} x & y & z \end{bmatrix} e_1 &= x \\ \begin{bmatrix} x & y & z \end{bmatrix} e_2 &= z \end{aligned}$$

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where $e_1 = \begin{bmatrix} 1 \\ f \\ sf \end{bmatrix}, e_2 = \begin{bmatrix} rg \\ g \\ 1 \end{bmatrix}$ *are the induced morphisms into the pullback. Furthermore:*

$$\begin{aligned} \operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix}\right) &= \begin{bmatrix} \operatorname{dom} x & \operatorname{dom} y & \operatorname{dom} z \end{bmatrix} \\ \operatorname{cod}\left(\begin{bmatrix} x & y & z \end{bmatrix}\right) &= \begin{bmatrix} \operatorname{cod} x & \operatorname{cod} y & \operatorname{cod} z \end{bmatrix}; \end{aligned}$$

for every triple of morphisms (h, l, k)

$$\begin{tikzcd} A \arrow[dr, "h"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "l"] & B \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

satisfying $hr = l = ks$ *and such that* $\begin{bmatrix} h & l & k \end{bmatrix}$ *exists,*

$$0_{[h\ l\ k]} = \begin{bmatrix} 0_h & 0_l & 0_k \end{bmatrix};$$

for every appropriate (in the sense that $x' + x, y' + y, z' + z$ *are defined) triple* (x', y', z') *of 2-cells*

$$\begin{tikzcd} A \arrow[dr, Rightarrow, "x'"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, Rightarrow, "y'"] & B \arrow[dl, Rightarrow, "z'"] \\ & D & \end{tikzcd}$$

satisfying $x'r = y' = z's$ *and such that* $\begin{bmatrix} x' & y' & z' \end{bmatrix}$ *exists,*

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} + \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} x' + x & y' + y & z' + z \end{bmatrix}.$$

Proof. Since the category $\mathbf{C}$ is weakly Mal'cev, given the split span

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}$$

one constructs the split square

$$\begin{tikzcd} A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

where

$$\begin{aligned} \pi_1 &= \begin{bmatrix} 1 & r & rg \end{bmatrix} \\ \pi_2 &= \begin{bmatrix} sf & s & 1 \end{bmatrix} \\ e_1 &= \begin{bmatrix} 1 \\ f \\ sf \end{bmatrix}, \quad e_2 = \begin{bmatrix} rg \\ g \\ 1 \end{bmatrix}. \end{aligned}$$

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To prove that $\begin{bmatrix} x & y & z \end{bmatrix}$, if it exists, is unique, assume the existence of

$$\alpha, \alpha' \in H\left(A \times_C B, D\right)$$

with the property that

$$\begin{aligned} \alpha e_1 &= x = \alpha' e_1 \\ \alpha e_2 &= z = a' e_2 \end{aligned}$$

and hence $\alpha = \alpha'$.
To prove that, assuming $\begin{bmatrix} x & y & z \end{bmatrix}$ exists,

$$\operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix}\right) = \begin{bmatrix} \operatorname{dom} x & \operatorname{dom} y & \operatorname{dom} z \end{bmatrix}$$

it is sufficient to prove that

$$\begin{aligned} \operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix}\right) e_1 &= \operatorname{dom} x \\ \operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix}\right) e_2 &= \operatorname{dom} z \end{aligned}$$

and in fact

$$\begin{aligned} \operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix}\right) e_1 &= \operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix} e_1\right) = \operatorname{dom} x \\ \operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix}\right) e_2 &= \operatorname{dom}\left(\begin{bmatrix} x & y & z \end{bmatrix} e_2\right) = \operatorname{dom} z. \end{aligned}$$

The proof for cod is similar.
To prove that if $\begin{bmatrix} h & l & k \end{bmatrix}$ exists, then

$$0_{[h\ l\ k]} = \begin{bmatrix} 0_h & 0_l & 0_k \end{bmatrix}$$

it is sufficient to compute

$$0_{[h\ l\ k]} e_1 = 0_{[h\ l\ k] e_1} = 0_h$$

and

$$0_{[h\ l\ k]} e_2 = 0_{[h\ l\ k] e_2} = 0_k.$$

In order to prove the last assertion, suppose there exists $\begin{bmatrix} x & y & z \end{bmatrix}$ and $\begin{bmatrix} x' & y' & z' \end{bmatrix}$ with

$$\begin{aligned} \operatorname{dom} x' &= \operatorname{cod} x \\ \operatorname{dom} z' &= \operatorname{cod} z \end{aligned}$$

(and consequently $\operatorname{dom} y' = \operatorname{cod} y$) so that

$$\begin{aligned} \operatorname{dom}\begin{bmatrix} x' & y' & z' \end{bmatrix} &= \begin{bmatrix} \operatorname{dom} x' & \operatorname{dom} y' & \operatorname{dom} z' \end{bmatrix} \\ &= \begin{bmatrix} \operatorname{cod} x & \operatorname{cod} y & \operatorname{cod} z \end{bmatrix} \\ &= \operatorname{cod}\begin{bmatrix} x & y & z \end{bmatrix} \end{aligned}$$

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and $\begin{bmatrix} x' & y' & z' \end{bmatrix} + \begin{bmatrix} x & y & z \end{bmatrix}$ is defined; composing it with e_1, one obtains

$$\begin{aligned} \left(\begin{bmatrix} x' & y' & z' \end{bmatrix} + \begin{bmatrix} x & y & z \end{bmatrix} \right) e_1 &= \begin{bmatrix} x' & y' & z' \end{bmatrix} e_1 + \begin{bmatrix} x & y & z \end{bmatrix} e_1 \\ &= x' + x, \end{aligned}$$

similarly with e_2

$$\begin{aligned} \left(\begin{bmatrix} x' & y' & z' \end{bmatrix} + \begin{bmatrix} x & y & z \end{bmatrix} \right) e_2 &= \begin{bmatrix} x' & y' & z' \end{bmatrix} e_2 + \begin{bmatrix} x & y & z \end{bmatrix} e_2 \\ &= z' + z, \end{aligned}$$

and the result follows as

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} + \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} x' + x & y' + y & z' + z \end{bmatrix}.$$

■

Definition 85 (Admissible triple of 2-cells) *A triple* (x, y, z) *as in 8.1, previous proposition, is said to be* admissible *if the 2-cell* $\begin{bmatrix} x & y & z \end{bmatrix}$ *exists in* $H(A \times_C B, D)$.

Definition 86 (cartesian weakly Mal'cev structured category) *A weakly Mal'cev sesquicategory is cartesian if it's 2-cell structure is cartesian (see [Ch2], Definition 21).*

Proposition 87 *A cartesian weakly Mal'cev sesquicategory is a pair* $(\mathbf{C}, \mathbf{H})$ *such that:*
1. $\mathbf{C}$ *is a weakly Mal'cev category;*
2. $\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$ *is a 2-cell structure over* $\mathbf{C}$;
3. For every split square

$$\begin{tikzcd} P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[r, shift left, "f"] \arrow[u, shift right, "e_1"'] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd},$$

if (P, p_1, p_2) *is a pullback, then the square*

$$\begin{tikzcd} H(D,P) \arrow[r, "H(D{,}p_2)"] \arrow[d, "H(D{,}p_1)"'] & H(D,B) \arrow[d, "H(D{,}g)"] \\ H(D,A) \arrow[r, "H(D{,}f)"] & H(D,C) \end{tikzcd} \tag{8.2}$$

is a pullback for every object D *in* $\mathbf{C}$, *and the induced morphism* $\langle H(e_1, D'), H(e_2, D') \rangle$

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into the following pullback

$$\begin{tikzcd}
H(P,D') \arrow[dr, "mono"] \arrow[drr, bend left=10, "{H(e_2,D')}"] \arrow[ddr, bend right=20, "{H(e_1,D')}"'] & & \\
& H(A,D') \times_{H(C,D')} H(B,D') \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H(B,D') \arrow[d, "{H(s,D')}"] \\
& H(A,D') \arrow[r, "{H(r,D')}"] & H(C,D')
\end{tikzcd} .$$

is a monomorphism.

Proof. To say that the induced morphism $\langle H(e_1,D'), H(e_2,D')\rangle$ is a monomorphism is to say that for every $x, y \in H(P,D')$,

$$\left\{ \begin{array}{l} xe_2 = ye_2 \\ xe_1 = ye_1 \end{array} \right. \Longrightarrow x = y,$$

and to say that the square (8.2) is a pullback for every object D in $\mathbf{C}$ is the same as saying that the functor $H(D, \_) : \mathbf{C} \longrightarrow Set$ preserves pullbacks for every object D in $\mathbf{C}$. ■

The example Cat(B)

As proved in [Ch3] if **B** is a weakly Mal'cev category, then so is Cat(**B**). Next we sow that Cat(**B**), with the 2-cell structure of internal transformations (not necessarily natural, just as given in [Ch2] p. 45) is in fact a weakly Mal'cev sesquicategory with a cartesian 2-cell structure.

Theorem 88 *If* **B** *is a weakly Mal'cev category, then:*

1. *The category Cat(**B**) of internal categories and internal functors in **B** is weakly Mal'cev.*
2. *The 2-cell structure given by the internal transformations in **B**,*

$$H(A,B) = \{(k,t,h) \mid t : A_0 \longrightarrow B_1, h, k : A \longrightarrow B, dt = h_0, ct = k_0\}$$

(see [Ch2] p. 45 for further details) is cartesian and weakly Mal'cev.

Proof. The proof of part one is given in [Ch3].
The 2-cell structure $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ as defined in [Ch2] p. 45 is in fact weakly Mal'cev because given a diagram of the form

$$\begin{tikzcd}
A & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, Rightarrow, "\mathbf{t}"] & B \arrow[dl, Rightarrow, "\mathbf{v}"] \\
& D \arrow[from=ul, Rightarrow, "\mathbf{u}"'] &
\end{tikzcd} \quad , \quad \mathbf{u}r = \mathbf{t} = \mathbf{v}s$$

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in Cat($\mathbf{B}$), with

$$A = \left(A_2 \overset{[1\ e\ 1]}{\longrightarrow} A_1 \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} A_0 \right)$$

$$B = \left(B_2 \overset{[1\ e\ 1]}{\longrightarrow} B_1 \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} B_0 \right)$$

$$C = \left(C_2 \overset{[1\ e\ 1]}{\longrightarrow} C_1 \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} C_0 \right)$$

$$A \underset{r_i}{\overset{f_i}{\rightleftarrows}} C \underset{s_i}{\overset{g_i}{\leftrightarrows}} B \qquad , \quad i = 0, 1, 2$$

$$\mathbf{u} = \left(u^c, u, u^d\right), \ \mathbf{v} = \left(v^c, v, v^d\right), \ \mathbf{t} = \left(t^c, t, t^d\right)$$

where

$$\begin{aligned} u^c &= (u_i^c : A_i \longrightarrow D_i)_{i=0,1} \\ u^d &= \left(u_i^d : A_i \longrightarrow D_i\right)_{i=0,1} \\ u &: A_0 \longrightarrow D_1 \end{aligned}$$

$$\begin{aligned} v^c &= (v_i^c : B_i \longrightarrow D_i)_{i=0,1} \\ v^d &= \left(v_i^d : B_i \longrightarrow D_i\right)_{i=0,1} \\ v &: B_0 \longrightarrow D_1 \end{aligned}$$

$$\begin{aligned} t^c &= (t_i^c : C_i \longrightarrow D_i)_{i=0,1} \\ t^d &= \left(t_i^d : C_i \longrightarrow D_i\right)_{i=0,1} \\ t &: C_0 \longrightarrow D_1 \end{aligned}$$

are such that

$$\begin{aligned} du &= u_0^d, \ cu = u_0^c \\ dv &= v_0^d, \ cv = v_0^c \\ dt &= t_0^d, \ ct = t_0^c, \end{aligned}$$

there is at most one 2-cell, represented as

$$\begin{bmatrix} \mathbf{u} & \mathbf{t} & \mathbf{v} \end{bmatrix}$$

from the pullback $A \times_C B$ to the object D in Cat($\mathbf{B}$); in the case of existence, $\begin{bmatrix} \mathbf{u} & \mathbf{t} & \mathbf{v} \end{bmatrix}$ is given by

$$\left(\begin{bmatrix} u^c & t^c & v^c \end{bmatrix}, \begin{bmatrix} u & t & v \end{bmatrix}, \begin{bmatrix} u^d & t^d & v^d \end{bmatrix} \right).$$

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It remains to show that the 2-cell structure is cartesian. To prove it we have to show that the square

$$\begin{tikzcd}
H(D, A \times_C B) \arrow[r, "H(D,\pi_2)"] \arrow[d, "H(D,\pi_1)"'] & H(D, B) \arrow[d, "H(D,g)"] \\
H(D, A) \arrow[r, "H(D,f)"] & H(D, C)
\end{tikzcd}$$

is a pullback diagram in the category of sets for every object D in Cat($\mathbf{B}$). In fact, for every two objects D, D' in $\mathbf{C}$ =Cat($\mathbf{B}$), we have, by definition of H,

$$H\left(D, D'\right) \subseteq \hom_{\mathbf{C}}\left(D, D'\right) \times \hom_{\mathbf{B}}\left(D_0, D_1'\right) \times \hom_{\mathbf{C}}\left(D, D'\right)$$

where an element $(k, t, h) \in H\left(D, D'\right)$ must satisfy

$$dt = h_0,\ ct = k_0;$$

clearly

$$\begin{tikzcd}
\hom(D_0, A_1 \times_{C_1} B_1) \arrow[r] \arrow[d] & \hom(D_0, B_1) \arrow[d] \\
\hom(D_0, A_1) \arrow[r] & \hom(D_0, C_1)
\end{tikzcd}$$

is a pullback and for every fixed $h, k : D \longrightarrow A$ and $h', k' : D \longrightarrow B$ we have that if $t' : D_0 \longrightarrow B_1$ and $t : D_0 \longrightarrow A_1$ are such that

$$dt' = h_0' \quad , \quad ct' = k_0' \text{ and } dt = h_0 \quad , \quad ct = k_0$$

then $\langle t, t' \rangle : D_0 \longrightarrow A_1 \times_{C_1} B_1$ is such that

$$\begin{aligned}
d \times_d d \langle t, t' \rangle &= \langle dt, dt' \rangle = \langle h_0, h_0' \rangle \\
c \times_c c \langle t, t' \rangle &= \langle ct, ct' \rangle = \langle k_0, k_0' \rangle .
\end{aligned}$$

■

8.3 Pentagonal coherence condition

In this section we introduce the concepts of α-protocategory and α-sort and prove that in the context of a weakly Mal'cev sesquicategory there is an equivalence of categories

$$\text{Thin-}\alpha\text{-PreCat}\left(\mathbf{C}\right) \sim \text{Adm-}\alpha\text{-Sort}\left(\mathbf{C}\right)$$

between thin α-protocategories (or thin α-precategories, since in a category with pullbacks of split epis Thin-ProtoCat=Thin-PreCat) and admissible α-sorts.

An α-protocategory is a protocategory (see [Ch7], Definition 76), together with a 2-cell, α, connecting the two morphisms mm_1 and mm_2, from the associativity axiom ([Ch7], p. 208).

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α-protocategory and α-sort

Let $\mathbf{C} = (\mathbf{C}, H, \operatorname{dom}, \operatorname{cod}, 0, +)$ be a weakly Mal'cev sesquicategory with a cartesian 2-cell structure.

Definition 89 (α-protocategory) *A α-protocategory is a pair*

$$(C, \alpha)$$

where C is a protocategory

$$\begin{tikzcd}[column sep=large]
C_3 \arrow[r, "p_2", yshift=1.8ex] \arrow[r, "m_2" description, yshift=0.6ex] \arrow[r, "m_1" description, yshift=-0.6ex] \arrow[r, "p_1"', yshift=-1.8ex] & C_2 \arrow[l, "i_2" description, yshift=1.2ex] \arrow[l, "i_0" description] \arrow[l, "i_1" description, yshift=-1.2ex] \arrow[r, "\pi_2", yshift=1.2ex] \arrow[r, "m" description] \arrow[r, "\pi_1"', yshift=-1.2ex] & C_1 \arrow[l, "e_2" description, yshift=0.6ex] \arrow[l, "e_1" description, yshift=-0.6ex] \arrow[r, "d", yshift=0.8ex] \arrow[r, "c"', yshift=-0.8ex] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

and

$$\alpha \in H(C_3, C_1)$$

is a 2-cell with

$$\operatorname{dom} \alpha = mm_1 \quad , \quad \operatorname{cod} \alpha = mm_2 \quad (or\ mm_1 \overset{\alpha}{\Longrightarrow} mm_2)$$

and satisfying

$$d\alpha = 0_{d\pi_2 p_2} \ , \ c\alpha = 0_{c\pi_1 p_1}.$$

A α-protocategory is said to be *pentagonal* if the MacLane pentagon coherence condition

$$m(\alpha \times 0_1) + \alpha(1 \times m \times 1) + m(0_1 \times \alpha) = \alpha(m \times 1 \times 1) + \alpha(1 \times 1 \times m) \quad (8.3)$$

is satisfied.

Definition 90 (α-sort) *An α-sort is a system*

$$(S, \alpha_1, \alpha_2, \alpha_3)$$

where $S = (C_0, C_1, d, c, e, u, v)$ is a sort,

$$\begin{tikzcd}
C_1 \arrow[r, "u", yshift=0.5ex] \arrow[r, "v"', yshift=-0.5ex] & C_1 \arrow[r, "d", yshift=0.8ex] \arrow[r, "c"', yshift=-0.8ex] & C_0 \arrow[l, "e" description]
\end{tikzcd}$$

and

$$\alpha_i \in H(C_1, C_1), \ i = 1, 2, 3$$

are 2-cells, such that

$$\alpha_1 e = \alpha_2 e = \alpha_3 e \quad (= \alpha_0)$$

$$d\alpha_i = 0_d \ , \ c\alpha_i = 0_c \ , \ i = 1, 2, 3,$$

$$\begin{aligned}
\operatorname{dom}(\alpha_2) &= vu \\
\operatorname{dom}(\alpha_3) &= vv \\
\operatorname{cod}(\alpha_1) &= uu \\
\operatorname{cod}(\alpha_2) &= uv.
\end{aligned}$$

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Proposition 91 *Every α-protocategory is in particular an α-sort.*

Proof. Given an α-protocategory

$$(C_0, C_1, C_2, C_3, d, c, e, m, ..., \alpha)$$

define

$$\begin{aligned} u &= me_1 \quad , \quad v = me_2 \quad , \quad (l = me_1 e), \\ \alpha_1 &= \alpha i_1 e_1, \alpha_2 = \alpha i_2 e_1, \alpha_3 = \alpha i_2 e_2, \quad (\alpha_0 = \alpha i_1 e_1 e) \end{aligned}$$

and obtain an α-sort

$$(C_0, C_1, d, c, e, u, v, \alpha_1, \alpha_2, \alpha_3)\,;$$

In fact $(C_0, C_1, d, c, e, u, v)$ is a sort by Theorem 67, and $\alpha_1, \alpha_2, \alpha_3$ as defined, are 2-cells in $H\,(C_1, C_1)$.
The condition

$$\alpha_1 e = \alpha_2 e = \alpha_3 e \quad (= \alpha_0)$$

follows from the fact that $i_1 e_1 = i_2 e_1$ and $e_1 e = e_2 e$, and so

$$i_1 e_1 e = i_2 e_1 e = i_2 e_2 e.$$

The conditions $d\alpha_i = 0_d$ and $c\alpha_i = 0_c$ follows since

$$d\alpha_1 = d\alpha i_1 e_1 = 0_{d\pi_2 p_2} i_1 e_1 = 0_{d\pi_2 p_2 i_1 e_1} = 0_d$$

$$c\alpha_1 = c\alpha i_1 e_1 = 0_{c\pi_1 p_1} i_1 e_1 = 0_{c\pi_1 p_1 i_1 e_1} = 0_c,$$

with similar calculations on α_2, α_3.
Also, by the previous calculation of m_1, one has

$$\begin{aligned} \mathrm{dom}\,(\alpha_2) &= \mathrm{dom}\,(\alpha)\, i_2 e_1 \\ &= m m_1 i_2 e_1 \\ &= m e_2 u \\ &= vu \end{aligned}$$

and

$$\begin{aligned} \mathrm{dom}\,(\alpha_3) &= \mathrm{dom}\,(\alpha)\, i_2 e_2 \\ &= m m_1 i_2 e_2 \\ &= m e_2 v \\ &= vv \end{aligned}$$

and with respect to cod one has, using the previous calculation of m_2 that

$$\begin{aligned} \mathrm{cod}\,(\alpha_1) &= \mathrm{cod}\,(\alpha)\, i_1 e_1 \\ &= m m_2 i_1 e_1 \\ &= m e_1 u \\ &= uu \end{aligned}$$

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$$\begin{aligned}
\operatorname{cod}(\alpha_2) &= \operatorname{cod}(\alpha)\, i_2 e_1 \\
&= m m_2 i_2 e_1 \\
&= m e_1 v \\
&= uv.
\end{aligned}$$

■

In the following we define admissible α-sort, as such that it corresponds to a thin-α-protocategory, in the same way as an admissible sort corresponds to a thin-protocategory [Ch7].

Definition 92 (admissible α-sort) *In a WMSC (weakly Mal'cev structured category), the α-sort*

$$(S, \alpha_1, \alpha_2, \alpha_3)$$

is said to be admissible when the morphism

$$\begin{bmatrix} u & l & v \end{bmatrix} : C_2 \longrightarrow C_1$$

and the 2-cell

$$\begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix} \in H(C_2, C_1)$$

exists with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} ,$$

the 2-cell

$$\begin{bmatrix} \alpha_1 & \alpha_0 & \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix} \end{bmatrix} \in H(C_3, C_1)$$

exists with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} ,$$

and the additional conditions are satisfied

$$\begin{aligned}
\operatorname{dom}(\alpha_1) &= \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \\
\operatorname{cod}(\alpha_3) &= \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix}.
\end{aligned}$$

The main Theorem

Theorem 93 *In a weakly Mal'cev structured category* $\boldsymbol{C}$ *with 2-cell structure* $(H, \operatorname{dom}, \operatorname{cod}, 0, +)$*, every admissible α-sort is a thin α-protocategory with* $\alpha = \begin{bmatrix} \alpha_1 & \alpha_0 & \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix} \end{bmatrix}$.

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Furthermore, the thin α-protocategory is pentagonal *(i.e. satisfies* (8.3)*) if and only if the following conditions are satisfied:*

$$u\alpha_1 + \alpha \begin{bmatrix} 1 \\ d \\ e_1 ld \end{bmatrix} + m \begin{bmatrix} 0_1 \\ 0_d \\ \alpha_0 d \end{bmatrix} = \alpha_1 u + \alpha \begin{bmatrix} 1 \\ d \\ e_2 ld \end{bmatrix} \tag{8.4}$$

$$u\alpha_2 + \alpha_2 u + v\alpha_1 = \alpha_1 v + \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} \tag{8.5}$$

$$u\alpha_3 + \alpha_2 v + v\alpha_2 = \alpha \begin{bmatrix} lc \\ c \\ e_1 \end{bmatrix} + \alpha_3 u \tag{8.6}$$

$$m \begin{bmatrix} \alpha_0 c \\ 0_c \\ 0_1 \end{bmatrix} + \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} + v\alpha_3 = \alpha \begin{bmatrix} lc \\ c \\ e_2 \end{bmatrix} + \alpha_3 v. \tag{8.7}$$

Proof. The proof is divided into two distinct parts: First it is shown how to construct a thin α-protocategory out of an admissible α-sort. Then, the pentagonal condition is evaluated, by first computing the five 2-cells involved, in the fashion of five lemmas.
Given an admissible α-sort as above, define

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_0 & \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix} \end{bmatrix},$$

and use Theorem 77 to construct the thin protocategory, since an admissible α-sort is in particular an admissible sort. The 2-cell α, as defined, is a 2-cell in $H(C_3, C_1)$, with

$$\begin{aligned} \operatorname{dom}(\alpha) &= \begin{bmatrix} \operatorname{dom}\alpha_1 & \operatorname{dom}\alpha_0 & \begin{bmatrix} \operatorname{dom}\alpha_2 & \operatorname{dom}\alpha_0 & \operatorname{dom}\alpha_3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} m \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & vl & \begin{bmatrix} vu & vl & vv \end{bmatrix} \end{bmatrix} \\ &= mm_1 \end{aligned}$$

observe that $dl = 1$ and

$$\operatorname{dom}(\alpha_0) = \operatorname{dom}(\alpha_1 e) = m \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} e = m \begin{bmatrix} e \\ 1 \\ l \end{bmatrix} = m \begin{bmatrix} ed \\ d \\ 1 \end{bmatrix} l = me_2 l = vl;$$

with respect to cod one also has

$$\begin{aligned} \operatorname{cod}(\alpha) &= \begin{bmatrix} \operatorname{cod}\alpha_1 & \operatorname{cod}\alpha_0 & \begin{bmatrix} \operatorname{cod}\alpha_2 & \operatorname{cod}\alpha_0 & \operatorname{cod}\alpha_3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} uu & ul & \begin{bmatrix} uv & ul & m \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix} \\ &= mm_2. \end{aligned}$$

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The condition $d\alpha = 0_{d\pi_2 p_2}$ is satisfied since

$$\begin{aligned} d\alpha &= \begin{bmatrix} d\alpha_1 & d\alpha_0 & \begin{bmatrix} d\alpha_2 & d\alpha_0 & d\alpha_3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0_d & 0_1 & \begin{bmatrix} 0_d & 0_1 & 0_d \end{bmatrix} \end{bmatrix} \\ &= 0_{[d,1,[d,1,d]]} = 0_{d\pi_2 p_2} \end{aligned}$$

where

$$\pi_2 p_2 = \begin{bmatrix} \pi_2 e_1 ed & \pi_2 e_1 e & \pi_2 \end{bmatrix} = \begin{bmatrix} ed & e & \begin{bmatrix} ed & e & 1 \end{bmatrix} \end{bmatrix},$$

and similarly the condition $c\alpha = 0_{c\pi_1 p_1}$is satisfied since

$$\begin{aligned} c\alpha &= \begin{bmatrix} c\alpha_1 & c\alpha_0 & \begin{bmatrix} c\alpha_2 & c\alpha_0 & c\alpha_3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0_c & 0_1 & \begin{bmatrix} 0_c & 0_1 & 0_c \end{bmatrix} \end{bmatrix} \\ &= 0_{[c,1,[c,1,c]]} = 0_{c\pi_1 p_1}, \end{aligned}$$

where

$$\pi_1 p_1 = \begin{bmatrix} \pi_1 e_1 & \pi_1 e_1 e & \pi_1 e_2 \pi_1 \end{bmatrix} = \begin{bmatrix} 1 & e & ec\pi_1 \end{bmatrix} = \begin{bmatrix} 1 & e & \begin{bmatrix} ec & e & ec \end{bmatrix} \end{bmatrix}.$$

Finally, $m = \begin{bmatrix} u & l & v \end{bmatrix}, \alpha i_2 = \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_1 & \alpha_0 & \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix} \end{bmatrix}$ exists, and

$$\begin{aligned} \operatorname{dom}(\alpha_1) &= \operatorname{dom}(\alpha i_1 e_1) \\ &= \operatorname{dom}(\alpha) i_1 e_1 \\ &= m m_1 i_1 e_1 \end{aligned}$$

$$\operatorname{dom}(\alpha_1) = m \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} = \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix}$$

$$\begin{aligned} \operatorname{cod}(\alpha_3) &= \operatorname{cod}(\alpha) i_2 e_2 \\ &= m m_2 i_2 e_2 \\ &= m \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} u & l & v \end{bmatrix} \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix}. \end{aligned}$$

■

This completes the first part of the proof.

In order to proceed, one has to calculate the five 2-cells involved in the pentagon coherence condition. To compute such morphisms and 2-cells it is useful to have in mind that $C_3 = C_1 \times_{C_0} C_2$ and $C_4 = C_1 \times_{C_0} C_3$ (the reason why they are chosen to be as such is just for a convenience in calculations).

In order to help calculations, it is convenient to have all the required information collected in one place so that it can be quickly and easily accessed every time it is needed. In the following five lemmas, used to construct the morphisms

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and 2-cells involved in the pentagon condition, we will not give explicit reference in each step, we believe that the reader will immediately recognize the piece of information needed at each time and will find no difficulty in locate it in the collection that follows.

The composite pullback diagram is presented in the first place and it provides the quickest way to look for relations of the form say $p_2 i_1 = e_1 \pi_2$ or say $\pi_2 p_1 = \pi_1 p_2$. Next, all the morphisms involved in the diagram are given in terms of its components with respect to the split spans

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"']
\end{tikzcd} ,$$

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"']
\end{tikzcd} ,$$

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "i_1 e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1 p_1"']
\end{tikzcd}$$

and also the identity morphisms $1_{C_2}, 1_{C_3}, 1_{C_4}$.

The collection of information to be used in the next five Lemmas is the following, determining all the structure involved in the definition of a thin-protocategory:

$$\begin{tikzcd}
C_4 \arrow[r, shift left, "p_2'"] \arrow[d, shift right, "p_1'"'] & C_3 \arrow[l, shift left, "e_2'"] \arrow[d, shift right, "p_1"'] \\
C_3 \arrow[u, shift right, "e_1'"'] \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & C_2 \arrow[u, shift right, "i_1"'] \arrow[l, shift left, "i_2"] \arrow[d, shift right, "\pi_1"'] \\
C_2 \arrow[u, shift right, "i_1"'] \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[u, shift right, "e_1"'] \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\
C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[u, shift right, "e"'] \arrow[l, shift left, "e"]
\end{tikzcd} \tag{8.8}$$

$$1_{C_2} = \begin{bmatrix} 1_{C_1} & e & ec \\ d & 1_{C_0} & c \\ ed & e & 1_{C_1} \end{bmatrix} \quad , \quad e_1 = \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} \quad , \quad e_0 = \begin{bmatrix} e \\ 1 \\ e \end{bmatrix} \quad , \quad e_2 = \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix}$$

$$\pi_1 = \begin{bmatrix} 1 & e & ec \end{bmatrix}$$

$$\pi_0 = \begin{bmatrix} d & 1 & c \end{bmatrix}$$

$$\pi_2 = \begin{bmatrix} ed & e & 1 \end{bmatrix}$$

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$$1_{C_3} = \begin{bmatrix} 1_{C_1} & e & ec\pi_1 \\ d & 1_{C_0} & c\pi_1 \\ e_1ed & e_1e & 1_{C_2} \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} 1_{C_1} \\ d \\ \begin{bmatrix} ed \\ d \\ ed \end{bmatrix} \end{matrix}}_{i_1e_1} & \underbrace{\begin{matrix} e \\ 1_{C_0} \\ \begin{bmatrix} e \\ 1_{C_0} \\ e \end{bmatrix} \end{matrix}}_{i_1e_1e} & \begin{matrix} [\; ec & e & ec \;] \\ [\; c & 1_{C_0} & c \;] \\ \left[\begin{matrix} \underbrace{\begin{matrix} 1_{C_1} \\ d \\ ed \end{matrix}}_{i_2e_1} & \begin{matrix} e \\ 1_{C_0} \\ e \end{matrix} & \underbrace{\begin{matrix} ec \\ c \\ 1_{C_1} \end{matrix}}_{i_2e_2} \end{matrix} \right] \end{matrix} \end{bmatrix}$$

$$\begin{aligned} \pi_1p_1 &= \begin{bmatrix} 1_{C_1} & e & ec\pi_1 \end{bmatrix} \\ p_1 &= \begin{bmatrix} e_1 & e_1e & e_2\pi_1 \end{bmatrix} \\ p_2 &= \begin{bmatrix} e_1ed & e_1e & 1_{C_2} \end{bmatrix} \end{aligned} \qquad i_1e_1 = \begin{bmatrix} 1_{C_1} \\ d \\ e_1ed \end{bmatrix} \;,\; i_1 = \begin{bmatrix} \pi_1 \\ d\pi_1 \\ e_1\pi_2 \end{bmatrix} \;,\; i_2 = \begin{bmatrix} ec\pi_1 \\ c\pi_1 \\ 1_{C_2} \end{bmatrix}$$

$$\begin{aligned} 1_{C_4} &= \begin{bmatrix} 1_{C_1} & e & ec\pi_1p_1 \\ d & 1_{C_0} & c\pi_1p_1 \\ i_1e_1ed & i_1e_1e & 1_{C_3} \end{bmatrix} \\ \pi_1p_1p_1' &= \begin{bmatrix} 1_{C_1} & e & ec\pi_1p_1 \end{bmatrix} \\ p_1' &= \begin{bmatrix} i_1e_1 & i_1e_1e & i_2p_1 \end{bmatrix} \\ p_2' &= \begin{bmatrix} i_1e_1ed & i_1e_1e & 1_{C_3} \end{bmatrix} \\ e_1'i_1e_1 &= \begin{bmatrix} 1_{C_1} \\ d \\ i_1e_1ed \end{bmatrix} \;,\; e_1' = \begin{bmatrix} 1_{C_1} & e & ec\pi_1 \\ d & 1_{C_0} & c\pi_1 \\ i_1e_1ed & i_1e_1e & i_1 \end{bmatrix} \\ e_2' &= \begin{bmatrix} ec\pi_1p_1 \\ c\pi_1p_1 \\ 1_{C_3} \end{bmatrix} = \begin{bmatrix} \begin{matrix} ec & e & ec\pi_1 \\ c & 1 & c\pi_1 \end{matrix} \\ \begin{bmatrix} \\ 1_{C_3} \\ \\ \end{bmatrix} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} m_1 &= (1 \times m) \\ &= \begin{bmatrix} m_1i_1e_1 & m_1i_1e_1e & m_1i_2 \end{bmatrix} \\ &= \begin{bmatrix} m_1i_1e_1 & m_1i_1e_1e & \begin{bmatrix} m_1i_2e_1 & m_1i_2e_1e & m_1i_2e_2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & e_2l & e_2m \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & e_2l & \begin{bmatrix} e_2u & e_2l & e_2v \end{bmatrix} \end{bmatrix} \end{aligned}$$

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$$\begin{aligned} m_2 &= (m \times 1) \\ &= \begin{bmatrix} m_2 i_1 e_1 & m_2 i_1 e_1 e & m_2 i_2 \end{bmatrix} \\ &= \begin{bmatrix} m_2 i_1 e_1 & m_2 i_1 e_1 e & \begin{bmatrix} m_2 i_2 e_1 & m_2 i_2 e_1 e & m_2 i_2 e_2 \end{bmatrix} \end{bmatrix} \\ & \begin{bmatrix} e_1 u & e_1 l & \begin{bmatrix} e_1 v & e_1 l & \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{aligned}$$

With the above considerations, the morphisms and 2-cells

$$\begin{aligned} &1 \times 1 \times m \\ &1 \times m \times 1 \\ &m \times 1 \times 1 \\ &\alpha \times 0_1 \\ &0_1 \times \alpha \end{aligned}$$

are calculated as follows:

Lemma 94 *In a WMC, the morphism* $1 \times 1 \times m : C_4 \longrightarrow C_3$ *is given by*

$$\begin{bmatrix} \begin{bmatrix} 1 \\ d \\ e_2 ld \end{bmatrix} & i_2 e_2 l & \begin{bmatrix} \begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} & i_2 e_2 l & i_2 e_2 m \end{bmatrix} \end{bmatrix}.$$

Proof. It is clear that (see diagram (8.8))

$$\begin{aligned} p_1 (1 \times 1 \times m) &= p_1 p_1' \\ p_2 (1 \times 1 \times m) &= (1 \times m) p_2' = m_1 p_2' \end{aligned}$$

to obtain some intuition observe that for generalized elements and writing $m \langle x, y \rangle = xy$ one has

$$\begin{aligned} p_1 (1 \times 1 \times m)(x, y, z, w) &= p_1 (x, y, zw) = (x, y) \\ p_1 p_1' (x, y, z, w) &= p_1 (x, y, z) = (x, y) \end{aligned}$$

and

$$\begin{aligned} p_2 (1 \times 1 \times m)(x, y, z, w) &= p_2 (x, y, zw) = (y, zw) \\ m_1 p_2' (x, y, z, w) &= m_1 (y, z, w) = (y, zw); \end{aligned}$$

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Hence, the morphism $(1 \times 1 \times m) : C_4 \longrightarrow C_3$ being a morphism into the pullback C_3 it is determined by its projections $(\pi_1 p_1, p_2)$ and it is given by

$$\begin{aligned}
(1 \times 1 \times m) &= \begin{bmatrix} \pi_1 p_1 (1 \times 1 \times m) \\ d\pi_1 p_1 (1 \times 1 \times m) \\ p_2 (1 \times 1 \times m) \end{bmatrix} = \begin{bmatrix} \pi_1 p_1 p_1' \\ d\pi_1 p_1 p_1' \\ m_1 p_2' \end{bmatrix} \\
&= \begin{bmatrix} 1 & e & ec\pi_1 p_1 \\ d & 1 & c\pi_1 p_1 \\ m_1 i_1 e_1 ed & m_1 i_1 e_1 e & m_1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & e & \begin{bmatrix} ec & e & ec\pi_1 \end{bmatrix} \\ & & \begin{bmatrix} c & 1 & c\pi_1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} ed & \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} e & \left[\begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \quad e_2 l \quad e_2 m \right] \end{bmatrix}
\end{aligned}$$

since $dl = 1$ then

$$\begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} ed = \begin{bmatrix} ed \\ d \\ ld \end{bmatrix} = \begin{bmatrix} e \\ 1 \\ l \end{bmatrix} d = \begin{bmatrix} ed \\ d \\ 1 \end{bmatrix} ld = e_2 ld$$

and also using $c\pi_1 = cm$, one obtains

$$(1 \times 1 \times m) = \left[\begin{matrix} 1 & e \\ d & 1 \\ \begin{bmatrix} ed \\ d \\ 1 \end{bmatrix} ld & \begin{bmatrix} ed \\ d \\ 1 \end{bmatrix} l \end{matrix} \quad \left[\begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} \quad \begin{bmatrix} ec \\ c \\ \\ e_2 \end{bmatrix} l \quad \begin{bmatrix} ec \\ c \\ \\ e_2 \end{bmatrix} m \right] \right]$$

since $cl = 1$ it becomes

$$\begin{aligned}
(1 \times 1 \times m) &= \left[\begin{bmatrix} 1 \\ d \\ \\ e_2 ld \end{bmatrix} \quad \begin{bmatrix} ec \\ c \\ \\ e_2 \end{bmatrix} l \quad \left[\begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} \quad i_2 e_2 l \quad i_2 e_2 m \right] \right] \\
&= \left[\begin{bmatrix} 1 \\ d \\ e_2 ld \end{bmatrix} \quad i_2 e_2 l \quad \left[\begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} \quad i_2 e_2 l \quad i_2 e_2 m \right] \right].
\end{aligned}$$

■

Lemma 95 *In a WMC, the morphism* $(1 \times m \times 1) : C_4 \longrightarrow C_3$ *is given by*

$$\left[\begin{bmatrix} 1 \\ d \\ e_1 ld \end{bmatrix} \quad i_2 e_1 l \quad \left[i_2 e_1 u \quad i_2 e_1 l \quad \left[i_2 e_1 v \quad i_2 e_1 l \quad \begin{bmatrix} ec \\ c \\ \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \right] \right] \right].$$

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Proof. It is clear that

$$\begin{aligned} p_1\left(1\times m\times 1\right) &= m_1 p_1' \\ p_2\left(1\times m\times 1\right) &= m_2 p_2' \end{aligned}$$

and so one has

$$\begin{aligned} (1\times m\times 1) &= \begin{bmatrix} \pi_1 p_1\left(1\times m\times 1\right) \\ d\pi_1 p_1\left(1\times m\times 1\right) \\ p_2\left(1\times m\times 1\right) \end{bmatrix} = \begin{bmatrix} \pi_1 m_1 p_1' \\ d\pi_1 m_1 p_1' \\ m_2 p_2' \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 m_1 i_1 e_1 & \pi_1 m_1 i_1 e_1 e & \pi_1 m_1 i_2 p_1 \\ d\pi_1 m_1 i_1 e_1 & d\pi_1 m_1 i_1 e_1 e & d\pi_1 m_1 i_2 p_1 \\ m_2 i_1 e_1 e d & m_2 i_1 e_1 e & m_2 \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} & \pi_1 e_2 l & \pi_1 e_2 m p_1 \\ d & cl & cmp_1 \\ e_1 u e d & e_1 u e & m_2 \end{bmatrix} \end{aligned}$$

by definition of π_1, if composed with $\begin{bmatrix} 1 \\ d \\ ld \end{bmatrix}$, it gives the first component

$$\pi_1 \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} = 1$$

and then

$$\begin{aligned} (1\times m\times 1) &= \begin{bmatrix} 1 & ecl & ecmp_1 \\ d & cl & cmp_1 \\ e_1 l d & e_1 l & m_2 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ e_1 l d \end{bmatrix} & \begin{bmatrix} ec \\ c \\ e_1 \end{bmatrix} l & \begin{matrix} ecmp_1 \\ cmp_1 \\ m_2 \end{matrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ e_1 l d \end{bmatrix} & i_2 e_1 l \begin{bmatrix} ecme_1 & ecme_1 e & ecme_2\pi_1 \\ cme_1 & cme_1 e & cme_2\pi_1 \\ e_1 u & e_1 l & \begin{bmatrix} e_1 v & e_1 l & \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ e_1 l d \end{bmatrix} & i_2 e_1 l \begin{bmatrix} \begin{bmatrix} ec \\ c \\ e_1 \end{bmatrix} u & \begin{bmatrix} ec \\ c \\ e_1 \end{bmatrix} l & \begin{bmatrix} ecv & ecve & ecvec \\ cv & cve & cvec \\ e_1 v & e_1 l & \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{aligned}$$

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and since $cl = 1, l = ve, cv = c$ and

$$\begin{bmatrix} ec \\ c \\ e_1 \end{bmatrix} = i_2 e_1$$

one obtains the result

$$(1 \times m \times 1) = \begin{bmatrix} \begin{bmatrix} 1 \\ d \\ e_1 l d \end{bmatrix} & i_2 e_1 l & \begin{bmatrix} i_2 e_1 u & i_2 e_1 l & \begin{bmatrix} i_2 e_1 v & i_2 e_1 l & \begin{bmatrix} ec \\ c \\ \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$

■

Lemma 96 *In a WMC, the morphism* $(m \times 1 \times 1) : C_4 \longrightarrow C_3$ *is given by*

$$\begin{bmatrix} i_1 e_1 u & i_1 e_1 l & \begin{bmatrix} i_1 e_1 v & i_1 e_1 l & \begin{bmatrix} \begin{bmatrix} lc \\ c \\ e_1 \end{bmatrix} & i_1 e_1 l & \begin{bmatrix} lc \\ c \\ e_2 \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$

Proof. It is clear that

$$\begin{aligned} p_1 (m \times 1 \times 1) &= m_2 p_1' \\ p_2 (m \times 1 \times 1) &= p_2 p_2' \end{aligned}$$

and so, one has

$$\begin{aligned} (m \times 1 \times 1) &= \begin{bmatrix} \pi_1 p_1 (m \times 1 \times 1) \\ d\pi_1 p_1 (m \times 1 \times 1) \\ p_2 (m \times 1 \times 1) \end{bmatrix} = \begin{bmatrix} \pi_1 m_2 p_1' \\ d\pi_1 m_2 p_1' \\ p_2 p_2' \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 m_2 i_1 e_1 & \pi_1 m_2 i_1 e_1 e & \pi_1 m_2 i_2 p_1 \\ d... & d... & d... \\ p_2 i_1 e_1 e d & p_2 i_1 e_1 e & p_2 \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 e_1 u & \pi_1 e_1 u e & \begin{bmatrix} \pi_1 m_2 i_2 e_1 & \pi_1 m_2 i_2 e_1 e & \pi_1 m_2 i_2 e_2 \pi_1 \end{bmatrix} \\ du & dl & \begin{bmatrix} d... & d... & d... \end{bmatrix} \\ e_1 e d & e_1 e & \begin{bmatrix} e_1 e d & e_1 e & 1_{C_2} \end{bmatrix} \end{bmatrix} \end{aligned}$$

and since $\pi_1 e_1 = 1$, $l = ue$, $m_2 i_2 e_1 = e_1 v$, $m_2 i_2 e_2 = \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix}$ and hence

$$\pi_1 m_2 i_2 e_2 = \pi_1 \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} = lc,$$

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one obtains

$$\begin{aligned}
(m \times 1 \times 1) &= \left[\begin{bmatrix} 1 \\ d \\ e_1 ed \end{bmatrix} u \quad \begin{bmatrix} 1 \\ d \\ e_1 ed \end{bmatrix} l \quad \begin{bmatrix} v & l & lc\pi_1 \\ dv & dl & c\pi_1 \\ e_1 ed & e_1 e & 1_{C_2} \end{bmatrix} \right] \\
&= \left[i_1 e_1 u \quad i_1 e_1 l \quad \left[i_1 e_1 v \quad i_1 e_1 l \quad \begin{bmatrix} lc & l & lc \\ c & 1 & c \\ \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} & \begin{bmatrix} e \\ 1 \\ e \end{bmatrix} & \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} \end{bmatrix} \right] \right] \\
&= \left[i_1 e_1 u \quad i_1 e_1 l \quad \left[i_1 e_1 v \quad i_1 e_1 l \quad \left[\begin{bmatrix} lc \\ c \\ \\ e_1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ d \\ \begin{bmatrix} ed \\ d \\ ed \end{bmatrix} \end{bmatrix} l \quad \begin{bmatrix} lc \\ c \\ \\ e_2 \end{bmatrix} \right] \right] \right] \\
&= \left[i_1 e_1 u \quad i_1 e_1 l \quad \left[i_1 e_1 v \quad i_1 e_1 l \quad \left[\begin{bmatrix} lc \\ c \\ e_1 \end{bmatrix} \quad i_1 e_1 l \quad \begin{bmatrix} lc \\ c \\ e_2 \end{bmatrix} \right] \right] \right].
\end{aligned}$$

■

Lemma 97 *In a WMSC, the 2-cell* $(0_1 \times \alpha) \in H(C_4, C_2)$ *is given by*

$$\left[\begin{bmatrix} 0_1 \\ 0_d \\ \alpha_0 d \end{bmatrix} \quad e_2 \alpha_0 \quad e_2 \alpha \right],$$

where $\alpha_0 = \alpha_1 e$.

Proof. First observe that $(0_1 \times \alpha)$ is well defined as an induced 2-cell given by the diagram

$$\begin{tikzcd}
C_1 \arrow[d, Rightarrow, "0_1"'] & C_4 \arrow[l, "\pi_1 p_1 p_1'"'] \arrow[r, "p_2'"] \arrow[d, Rightarrow, "(0_1 \times \alpha)"] & C_3 \arrow[d, Rightarrow, "\alpha"] \\
C_1 & C_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C_1
\end{tikzcd}$$

due to the assumption that $H(D, \_) : \mathbf{C} \longrightarrow Set$ preserves pullbacks, which in particular means that

$$H(D, C_2) \cong H(D, C_1) \times_{H(D, C_0)} H(D, C_1)$$

and $(0 \times \alpha)$ is the unique 2-cell in $H(C_4, C_2)$ such that

$$\begin{aligned}
\pi_1 (0 \times \alpha) &= 0_{\pi_1 p_1 p_1'} \\
\pi_2 (0 \times \alpha) &= \alpha p_2'
\end{aligned}$$

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hence

$$\begin{aligned}(0\times\alpha) &= \begin{bmatrix}\pi_1(0\times\alpha)\\ d\pi_1(0\times\alpha)\\ \pi_2(0\times\alpha)\end{bmatrix} = \begin{bmatrix}0_{\pi_1 p_1 p_1'}\\ 0_{d\pi_1 p_1 p_1'}\\ \alpha p_2'\end{bmatrix}\\ &= \begin{bmatrix}0_1 & 0_e & 0_{ec\pi_1 p_1}\\ 0_d & 0_1 & 0_{c\pi_1 p_1}\\ \alpha i_1 e_1 ed & \alpha i_1 e_1 e & \alpha\end{bmatrix}\\ &= \begin{bmatrix}0_1 & 0_e & 0_{ec\pi_1 p_1}\\ 0_d & 0_1 & 0_{c\pi_1 p_1}\\ \alpha_0 d & \alpha_0 & \alpha\end{bmatrix}\end{aligned}$$

and since $c\alpha = 0_{c\pi_1 p_1}$ one has

$$\begin{aligned}(0\times\alpha) &= \begin{bmatrix}0_1 & 0_e & ec\alpha\\ 0_d & 0_1 & c\alpha\\ \alpha_0 d & \alpha_0 & \alpha\end{bmatrix}\\ &= \begin{bmatrix}\begin{bmatrix}0_1\\ 0_d\\ \alpha_0 d\end{bmatrix} & \begin{bmatrix}0_e\\ 0_1\\ \alpha_0\end{bmatrix} & \begin{bmatrix}ec\\ c\\ 1\end{bmatrix}\alpha\end{bmatrix}\end{aligned}$$

and since $c\alpha_0 = 0_1$, one also has

$$(0\times\alpha) = \begin{bmatrix}\begin{bmatrix}0_1\\ 0_d\\ \alpha_0 d\end{bmatrix} & \begin{bmatrix}ec\\ c\\ 1\end{bmatrix}\alpha_0 & e_2\alpha\end{bmatrix}$$

and finally on concludes

$$(0\times\alpha) = \begin{bmatrix}\begin{bmatrix}0_1\\ 0_d\\ \alpha_0 d\end{bmatrix} & e_2\alpha_0 & e_2\alpha\end{bmatrix}.$$

■

Lemma 98 *In a WMSC, the 2-cell* $(\alpha\times 0)\in H(C_4, C_2)$ *is given by*

$$\begin{bmatrix}e_1\alpha & e_1\alpha_0 & \begin{bmatrix}e_1\alpha_2 & e_1\alpha_0 & \begin{bmatrix}e_1\alpha_3 & e_1\alpha_0 & \begin{bmatrix}\alpha_0 c\\ 0_c\\ 0_1\end{bmatrix}\end{bmatrix}\end{bmatrix}\end{bmatrix}.$$

Proof. First observe that $(\alpha\times 0)$

$$\begin{tikzcd}C_3 \arrow[d, Rightarrow, "\alpha"'] & C_4 \arrow[l, "p_1'"'] \arrow[r, "\pi_2 p_2 p_2'"] \arrow[d, Rightarrow, "(\alpha\times 0_1)"] & C_1 \arrow[d, Rightarrow, "0_1"] \\ C_1 & C_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C_1\end{tikzcd}$$

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is such that

$$\begin{aligned}\pi_1(\alpha\times 0) &= \alpha p_1' \\ \pi_2(\alpha\times 0) &= 0_{\pi_2 p_2 p_2'}\end{aligned}$$

so that it is of the form

$$\begin{aligned}(\alpha\times 0) &= \begin{bmatrix}\pi_1(\alpha\times 0)\\ d\pi_1(\alpha\times 0)\\ \pi_2(\alpha\times 0)\end{bmatrix} = \begin{bmatrix}\alpha p_1'\\ d\alpha p_1'\\ 0_{\pi_2 p_2 p_2'}\end{bmatrix} \\ &= \begin{bmatrix}\alpha i_1 e_1 & \alpha i_1 e_1 e & \alpha i_2 p_1\\ d\alpha_1 & d\alpha_0 & d\alpha i_2 p_1\\ 0_{\pi_2 p_2 i_1 e_1 ed} & 0_{\pi_2 p_2 i_1 e_1 e} & 0_{\pi_2 p_2}\end{bmatrix}\end{aligned}$$

and since $\pi_2(p_2 i_1)\, e_1 ed = \pi_2(e_1\pi_2)\, e_1 ed = ededed = ed$ and $d\alpha_1 = 0_d$, then

$$\begin{aligned}(\alpha\times 0) &= \begin{bmatrix}\alpha_1 & \alpha_0 & [& \alpha i_2 e_1 & \alpha i_2 e_1 e & \alpha i_2 e_2 \pi_1 &]\\ 0_d & 0_1 & [& d\alpha_2 & d\alpha_0 & d\alpha_3\pi_1 &]\\ 0_{ed} & 0_e & [& 0_{\pi_2 e_1 ed} & 0_{\pi_2 e_1 e} & 0_{\pi_2} &]\end{bmatrix} \\ &= \left[\begin{matrix}\alpha_1 & \alpha_0\\ 0_d & 0_1\\ 0_{ed} & 0_e\end{matrix}\quad \begin{bmatrix}\alpha_2 & \alpha_0 & \alpha_3\pi_1\\ d\alpha_2 & d\alpha_0 & 0_{d\pi_1}\\ 0_{ed} & 0_e & 0_{\pi_2}\end{bmatrix}\right] \\ &= \left[\begin{bmatrix}1\\ d\\ ed\end{bmatrix}\alpha_1 \quad \begin{bmatrix}1\\ d\\ ed\end{bmatrix}\alpha_0 \quad \left[\begin{bmatrix}1\\ d\\ ed\end{bmatrix}\alpha_2 \quad \begin{bmatrix}1\\ d\\ ed\end{bmatrix}\alpha_0 \quad \begin{bmatrix}\alpha_3 & \alpha_3 e & \alpha_3 ec\\ 0_d & 0_1 & 0_c\\ 0_{ed} & 0_e & 0_1\end{bmatrix}\right]\right] \\ &= \left[e_1\alpha \quad e_1\alpha_0 \quad \left[e_1\alpha_2 \quad e_1\alpha_0 \quad \left[e_1\alpha_3 \quad e_1\alpha_0 \quad \begin{bmatrix}\alpha_0 c\\ 0_c\\ 0_1\end{bmatrix}\right]\right]\right].\end{aligned}$$

■

From Lemmas 94, 95, 96, 97 and 98, one concludes respectively that

$$\alpha(1\times 1\times m) = \left[\alpha \begin{bmatrix}1\\ d\\ \\ e_2 ld\end{bmatrix} \quad \alpha_3 l \quad \left[\alpha \begin{bmatrix}ec\\ c\\ \begin{bmatrix}1\\ d\\ ld\end{bmatrix}\end{bmatrix} \quad \alpha_3 l \quad \alpha_3\begin{bmatrix}u & l & v\end{bmatrix}\right]\right]$$

$$\alpha(1\times m\times 1) = \left[\alpha \begin{bmatrix}1\\ d\\ \\ e_1 ld\end{bmatrix} \quad \alpha_2 l \quad \left[\alpha_2 u \quad \alpha_2 l \quad \left[\alpha_2 v \quad \alpha_2 l \quad \alpha \begin{bmatrix}ec\\ c\\ \begin{bmatrix}lc\\ c\\ 1\end{bmatrix}\end{bmatrix}\right]\right]\right]$$

$$\alpha(m\times 1\times 1) = \left[\alpha_1 u \quad \alpha_1 l \quad \left[\alpha_1 v \quad \alpha_1 l \quad \left[\alpha \begin{bmatrix}lc\\ c\\ e_1\end{bmatrix} \quad \alpha_1 l \quad \alpha \begin{bmatrix}lc\\ c\\ e_2\end{bmatrix}\right]\right]\right]$$

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$$\begin{aligned} m\,(\alpha\times 0_1) &= \begin{bmatrix} m\begin{bmatrix}0_1\\ 0_d\\ \alpha_0 d\end{bmatrix} & v\alpha_0 & v\alpha\end{bmatrix} \\ &= \begin{bmatrix} m\begin{bmatrix}0_1\\ 0_d\\ \alpha_0 d\end{bmatrix} & v\alpha_0 & \begin{bmatrix} v\alpha_1 & v\alpha_0 & \begin{bmatrix} v\alpha_2 & v\alpha_0 & v\alpha_3\end{bmatrix}\end{bmatrix}\end{bmatrix}\end{aligned}$$

$$m\,(0_1\times\alpha) = \begin{bmatrix} u\alpha & u\alpha_0 & \begin{bmatrix} u\alpha_2 & u\alpha_0 & \begin{bmatrix} u\alpha_3 & u\alpha_0 & m\begin{bmatrix}\alpha_0 c\\ 0_c\\ 0_1\end{bmatrix}\end{bmatrix}\end{bmatrix}\end{bmatrix}$$

and the pentagon coherence condition

$$m\,(\alpha\times 0_1)+\alpha\,(1\times m\times 1)+m\,(0_1\times\alpha)=\alpha\,(m\times 1\times 1)+\alpha\,(1\times 1\times m)$$

is equivalent to the following set of equations

$$\begin{aligned} u\alpha_1+\alpha\begin{bmatrix}1\\ d\\ e_1 ld\end{bmatrix}+m\begin{bmatrix}0_1\\ 0_d\\ \alpha_0 d\end{bmatrix} &= \alpha_1 u+\alpha\begin{bmatrix}1\\ d\\ e_2 ld\end{bmatrix} \\ u\alpha_2+\alpha_2 u+v\alpha_1 &= \alpha_1 v+\alpha\begin{bmatrix}ec\\ c\\ \begin{bmatrix}1\\ d\\ ld\end{bmatrix}\end{bmatrix} \\ u\alpha_3+\alpha_2 v+v\alpha_2 &= \alpha\begin{bmatrix}lc\\ c\\ e_1\end{bmatrix}+\alpha_3 u \\ m\begin{bmatrix}\alpha_0 c\\ 0_c\\ 0_1\end{bmatrix}+\alpha\begin{bmatrix}ec\\ c\\ \begin{bmatrix}lc\\ c\\ 1\end{bmatrix}\end{bmatrix}+v\alpha_3 &= \alpha\begin{bmatrix}lc\\ c\\ e_2\end{bmatrix}+\alpha_3 v. \end{aligned}$$

This completes the second part of the proof of Theorem 93.

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8.4 Conclusion

It is now interesting to observe that in the case of $l = e$ and $\alpha_0 = 0_e$ some simplifications are observed, namely

$$\begin{aligned}
\alpha \begin{bmatrix} 1 \\ d \\ e_1 l d \end{bmatrix} &= \alpha \begin{bmatrix} 1 \\ d \\ e_1 e d \end{bmatrix} = \alpha i_1 e_1 = \alpha_1 \\
\alpha \begin{bmatrix} 1 \\ d \\ e_2 l d \end{bmatrix} &= \alpha \begin{bmatrix} 1 \\ d \\ e_2 e d \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ d \\ e_1 e d \end{bmatrix} = \alpha_1 \\
m \begin{bmatrix} 0_1 \\ 0_d \\ \alpha_0 d \end{bmatrix} &= m \begin{bmatrix} 0_1 \\ 0_d \\ 0_{ed} \end{bmatrix} = 0_{m e_1} = 0_u \\
\alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} &= \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} \end{bmatrix} = \alpha i_2 e_1 = \alpha_2 \\
\alpha \begin{bmatrix} lc \\ c \\ e_1 \end{bmatrix} &= \alpha \begin{bmatrix} ec \\ c \\ e_1 \end{bmatrix} = \alpha i_2 e_1 = \alpha_2 \\
m \begin{bmatrix} \alpha_0 c \\ 0_c \\ 0_1 \end{bmatrix} &= m \begin{bmatrix} 0_{ec} \\ 0_c \\ 0_1 \end{bmatrix} = 0_v \\
\alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} &= \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} \end{bmatrix} = \alpha i_2 e_2 = \alpha_3 \\
\alpha \begin{bmatrix} lc \\ c \\ e_2 \end{bmatrix} &= \alpha \begin{bmatrix} ec \\ c \\ e_2 \end{bmatrix} = \alpha i_2 e_2 = \alpha_3
\end{aligned}$$

and the set of conditions, equivalent to the pentagon condition (8.3), simplifies to

$$\begin{aligned}
u\alpha_1 + \alpha_1 + 0_u &= \alpha_1 u + \alpha_1 \\
u\alpha_2 + \alpha_2 u + v\alpha_1 &= \alpha_1 v + \alpha_2 \\
u\alpha_3 + \alpha_2 v + v\alpha_2 &= \alpha_2 + \alpha_3 u \\
0_v + \alpha_3 + v\alpha_3 &= \alpha_3 + \alpha_3 v
\end{aligned}$$

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and then, using the fact that 0's are identity 2-cells it simplifies further to

$$\begin{aligned} u\alpha_1 + \alpha_1 &= \alpha_1 u + \alpha_1 \\ u\alpha_2 + \alpha_2 u + v\alpha_1 &= \alpha_1 v + \alpha_2 \\ u\alpha_3 + \alpha_2 v + v\alpha_2 &= \alpha_2 + \alpha_3 u \\ \alpha_3 + v\alpha_3 &= \alpha_3 + \alpha_3 v. \end{aligned}$$

This simplification occurs if one imposes in the definition of precategory that

$$ue = e = ve$$

and in the definition of α-(thin)protocategory that

$$\alpha_i e = 0_e \ , \ i = 1, 2, 3$$

and in the case of a precategory it corresponds to the fact that the result of composing an identity morphism with itself is the identity morphism. This notion seems to be still very important and for instance Grandis and Paré in [2] and [5] use this notion instead of the more general one.

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Chapter 9

Pseudocategories in weakly Mal'cev sesquicategories

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Pseudocategory, coherence conditions, (λ, ρ)-sort, precategory, internal category, (α, λ, ρ)-protocategory, homogeneous pseudocategory.

Abstract: The two coherence conditions involving the left and right triangles, originally considered by MacLane in the definition of a monoidal category and then proved to be a consequence of the pentagon and the middle triangle, turn out to play an important role in the context of a weakly Mal'cev sesquicategory: they completely determine the associativity isomorphism, alpha, and hence the pentagon condition becomes a property of the left and right unit isomorphisms, lambda and rho. We show that in a weakly Mal'cev sesquicategory, if a pseudocategory is homogeneous then the pentagon coherence condition is trivially satisfied. We give a full description for a pseudocategory in a weakly Mal'cev sesquicategory and then interpret the results in the case of crossed modules, obtaining thus, in particular, a description for internal bicategories in groups.

9.1 Introduction

This paper is a sequel of [Ch8] and it presents the study of pseudocategories in a weakly Mal'cev sesquicategory with a cartesian 2-cell structure. A pseudocategory is a complicated structure and it cannot be easily attacked in just one breath. We choose to decompose it into several parts, each one considered simple enough to be handled properly. In [Ch8] we introduce the notions of α-protocategory and α-sort, and study the pentagon coherence condition; here we introduce the notions of (λ, ρ)-sort and (α, λ, ρ)-protocategory, and study the following coherence conditions

$$\begin{tikzcd}
 & f(g(hk)) \arrow[rr, "f\alpha_{g,h,k}"] \arrow[dl, "\alpha_{f,g,hk}"'] & & f((gh)k) \arrow[dr, "\alpha_{f,gh,k}"] & \\
(fg)(hk) \arrow[drr, "\alpha_{fg,h,k}"'] & & & & (f(gh))k \arrow[dll, "\alpha_{f,g,h}k"] \\
 & & ((fg)h)k & &
\end{tikzcd}$$

(pentagon)

$$\begin{tikzcd}
f(1g) \arrow[rr, "\alpha_{f,1,g}"] \arrow[dr, "f\lambda_g"'] & & (f1)g \arrow[dl, "\rho_f g"] \\
 & fg &
\end{tikzcd}$$

(midle triangle)

$$\begin{tikzcd}
1(fg) \arrow[rr, "\alpha_{1,f,g}"] \arrow[dr, "\lambda_{fg}"'] & & (1f)g \arrow[dl, "\lambda_f g"] \\
 & fg &
\end{tikzcd}$$

(left triangle)

$$\begin{tikzcd}
 & 1(f1) \arrow[r, "\alpha_{1,f,1}"] \arrow[dl, "1\rho_f"'] & (1f)1 \arrow[dr, "\lambda_f 1"] & \\
1f \arrow[drr, "\lambda"'] & & & f1 \arrow[dl, "\rho"] \\
 & & f &
\end{tikzcd}$$

(small pentagon)

$$\begin{tikzcd}
f(g1) \arrow[rr, "\alpha_{f,g,1}"] \arrow[dr, "f\rho_g"'] & & (fg)1\ . \arrow[dl, "\rho_{fg}"] \\
 & fg &
\end{tikzcd}$$

(right triangle)

A pseudocategory is a (α, λ, ρ)-(thin)protocategory, satisfying the pentagon and the middle triangle conditions, and where in addition the 2-cell α is required to be invertible and natural, and the 2-cells λ, ρ are also required to be natural (and not just natural with respect to each other, as we will see).

We will prove that in the context of a weakly Mal'cev sesquicategory, with a cartesian 2-cell structure, the 2-cell α is uniquely determined (provided it exists) by each one of the following equivalent conditions.

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1 middle triangle + small pentagon

2 left triangle + right triangle

So that for an (α, λ, ρ)-(thin)protocategory with 1. or 2., the pentagonal condition becomes a property of λ and ρ.

Finally we characterize pseudocategories as admissible (λ, ρ)-sorts satisfying four conditions that are equivalent to the pentagon condition (result obtained in [Ch8]). In the case of homogeneous pseudocategories we prove that all the four conditions are trivial.

9.2 The associativity isomorphism is determined

Let $\mathbf{C} = (\mathbf{C}, H, \mathrm{dom}, \mathrm{cod}, 0, +)$ be a category with a cartesian 2-cell structure. The notion of an (α, λ, ρ)-protocategory is introduced as an extension of α-protocategory [Ch8] where it is also possible to consider the coherence conditions involving the left and right identity isomorphisms. The concept of a (λ, ρ)-sort is designed to correspond to an (α, λ, ρ)-(thin)protocategory, in the same way as an α-sort corresponds to an α-(thin)protocategory, in the case it is admissible (Corollary 104).

Definition 99 ((λ, ρ)-sort) *A (λ, ρ)-sort is a triple*

$$(S, \lambda, \rho)$$

where $S = (C_0, C_1, d, c, e, u, v)$ is a sort ([Ch7], Definition 65),

$$\begin{tikzcd} C_1 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

and

$$\lambda, \rho \in H(C_1, C_1)$$

are invertible 2-cells, such that

$$\mathrm{dom}(\lambda) = v \ , \ \mathrm{dom}(\rho) = u \ , \ \mathrm{cod}(\lambda) = 1_{C_1} = \mathrm{cod}(\rho)$$

and satisfy the following conditions

$$\begin{aligned} d\lambda &= 0_d = d\rho \\ c\lambda &= 0_c = c\rho \\ \lambda e &= \rho e \quad (= \eta)\,, \end{aligned}$$

and in addition we have (see [Ch2])

$$\begin{aligned} \lambda \circ \lambda &\iff \lambda v = v\lambda \\ \lambda \circ \rho &\iff \rho + \lambda u = \lambda + v\rho \\ \rho \circ \lambda &\iff \lambda + \rho v = \rho + u\lambda \\ \rho \circ \rho. &\iff \rho u = u\rho \end{aligned}$$

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Definition 100 **((α,λ,ρ)-protocategory)** *An (α,λ,ρ)-protocategory is a system*

$$(C,\alpha,\lambda,\rho)$$

where (C,α) is an α-protocategory and $(S(C),\lambda,\rho)$ is a (λ,ρ)-sort with $S:ProtoCat\longrightarrow Precat\longrightarrow Sort$ the functor associating to each protocategory $C=(C_0,C_1,d,c,e,m,...)$ a sort $S(C)=(C_0,C_1,d,c,e,u=me_1,v=me_2)$.

For a given (α,λ,ρ)-protocategory we may consider the following equations

$$\begin{aligned} m(\alpha\times 0_1)+\alpha(1\times m\times 1)+m(0_1\times\alpha) &= \alpha(m\times 1\times 1)+\alpha(1\times 1\times m)\\ m(\rho\times 0_1)+\alpha i_0 &= m(0_1\times\lambda) \end{aligned}$$

$$\begin{aligned} m(\lambda\times 0_{C_1})+\alpha i_2 &= \lambda m\\ \rho+me_1\lambda+\alpha i_2 e_1 &= \lambda+me_2\rho\\ \rho m+\alpha i_1 &= m(0_{C_1}\times\rho) \end{aligned}$$

which correspond respectively to the pentagon, middle triangle, left triangle, small pentagon and right triangle coherence conditions (see also the respective diagrams in the introduction). Since λ and ρ are assumed to be isomorphisms we also have

$$\alpha i_0=-m(\rho\times 0_1)+m(0_1\times\lambda)$$

$$\begin{aligned} \alpha i_2 &= -m(\lambda\times 0_{C_1})+\lambda m\\ \alpha i_2 e_1 &= -me_1\lambda-\rho+\lambda+me_2\rho\\ \alpha i_1 &= -\rho m+m(0_{C_1}\times\rho). \end{aligned}$$

Theorem 101 *In a weakly Mal'cev sesquicategory, with a cartesian 2-cell structure, and in the context of a (α,λ,ρ)-(thin)protocategory, the following set of equations are equivalent and uniquely determine α:*

$$\text{(A)}\quad\begin{cases}\alpha i_0=-m(\rho\times 0_1)+m(0_1\times\lambda)\\ \alpha i_2 e_1=-me_1\lambda-\rho+\lambda+me_2\rho\end{cases}$$
$$\text{(B)}\quad\begin{cases}\alpha i_1=-\rho m+m(0_{C_1}\times\rho)\\ \alpha i_2=-m(\lambda\times 0_{C_1})+\lambda m\end{cases}.$$

Proof. In a weakly Mal'cev category, the pair (i_1,i_2) (see [Ch3] and [Ch7]) is jointly epic, hence α is determined by αi_1 and αi_2.

(A) $\Longrightarrow$ (B).

Given (A) we shall prove that

$$\alpha i_1=-\rho m+m(0_{C_1}\times\rho);$$

since we are in a weakly Mal'cev context it is sufficient to show that

$$\alpha i_1 e_1=(-\rho m+m(0_{C_1}\times\rho))e_1$$

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and

$$\alpha i_1 e_2 = \left(-\rho m + m\left(0_{C_1} \times \rho\right)\right) e_2.$$

Since $i_0 e_1 = i_1 e_1$ (by definition of protocategory) we have on the one hand

$$\begin{aligned} \alpha i_1 e_1 &= \alpha i_0 e_1 = \left(-m\left(\rho \times 0_1\right) + m\left(0_1 \times \lambda\right)\right) e_1 \\ &= -m\left(\rho \times 0_1\right) e_1 + m\left(0_1 \times \lambda\right) e_1 \\ &= -m\left\langle \rho, 0_{ed} \right\rangle + m\left\langle 0_1, \lambda ed \right\rangle \end{aligned}$$

and since $0_{ed} = ed\rho$, we have

$$\begin{aligned} \alpha i_1 e_1 &= -m e_1 \rho + m\left\langle 0_1, \lambda ed \right\rangle \\ &= -u\rho + m\left\langle 0_1, \lambda ed \right\rangle, \end{aligned}$$

on the other hand

$$\begin{aligned} \left(-\rho m + m\left(0_{C_1} \times \rho\right)\right) e_1 &= -\rho m e_1 + m\left(0_{C_1} \times \rho\right) e_1 \\ &= -\rho u + m\left\langle 0_1, \rho ed \right\rangle \end{aligned}$$

and they coincide because

$$\rho u = u\rho \quad \text{and} \quad \lambda e = \rho e.$$

This shows $\alpha i_1 e_1 = \left(-\rho m + m\left(0_{C_1} \times \rho\right)\right) e_1$. Next we show $\alpha i_1 e_2 = \left(-\rho m + m\left(0_{C_1} \times \rho\right)\right) e_2$. By definition of split square, $i_2 e_1 = i_1 e_2$, so on the one hand we have

$$\begin{aligned} \alpha i_1 e_2 &= \alpha i_2 e_1 = -m e_1 \lambda - \rho + \lambda + m e_2 \rho \\ &= -u\lambda - \rho + \lambda + v\rho, \end{aligned}$$

while on the other hand

$$\begin{aligned} \left(-\rho m + m\left(0_{C_1} \times \rho\right)\right) e_2 &= -\rho m e_2 + m\left(0_{C_1} \times \rho\right) e_2 \\ &= -\rho v + m\left\langle 0_{ec}, \rho \right\rangle \\ &= -\rho v + m\left\langle ec\rho, \rho \right\rangle \\ &= -\rho v + m\left\langle ec, 1 \right\rangle \rho \\ &= -\rho v + m e_2 \rho = -\rho v + v\rho, \end{aligned}$$

and they coincide because

$$\begin{aligned} \left(-u\lambda - \rho + \lambda + v\rho = -\rho v + v\rho\right) &\iff \left(-u\lambda - \rho + \lambda = -\rho v\right) \\ &\iff \left(\lambda + \rho v = \rho + u\lambda\right) \iff \rho \circ \lambda. \end{aligned}$$

A similar argument holds for αi_2.

(B) $\Longrightarrow$ (A).

Given (B), we have

$$\begin{aligned} \alpha i_2 e_1 &= -m\left(\lambda \times 0_{C_1}\right) e_1 + \lambda m e_1 \\ &= -m\left\langle \lambda, 0_{ed} \right\rangle + \lambda u \\ &= -m\left\langle \lambda, ed\lambda \right\rangle + \lambda u \\ &= -m e_1 \lambda + \lambda u = -u\lambda + \lambda u \end{aligned}$$

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which is equal to

$$-me_1\lambda - \rho + \lambda + me_2\rho = -u\lambda - \rho + \lambda + v\rho$$

since

$$\begin{aligned}(-u\lambda + \lambda u = -u\lambda - \rho + \lambda + v\rho) &\iff (\lambda u = -\rho + \lambda + v\rho)\\ &\iff (\rho + \lambda u = \lambda + v\rho)\\ &\iff \lambda \circ \rho.\end{aligned}$$

To prove $\alpha i_0 = -m(\rho \times 0_1) + m(0_1 \times \lambda)$ it suffices to show that

$$\alpha i_0 e_1 = (-m(\rho \times 0_1) + m(0_1 \times \lambda))\, e_1$$

and

$$\alpha i_0 e_2 = (-m(\rho \times 0_1) + m(0_1 \times \lambda))\, e_2;$$

we show the case e_1, with e_2 being similar. Since $i_0 e_1 = i_1 e_1$ (by definition) we have on the one hand

$$\alpha i_0 e_1 = \alpha i_1 e_1$$

that from (B) gives

$$\begin{aligned}(-\rho m + m(0_{C_1} \times \rho))\, e_1 &= -\rho m e_1 + m(0_{C_1} \times \rho)\, e_1\\ &= -\rho u + m\langle 0_1, \rho e d\rangle,\end{aligned}$$

while on the other hand

$$\begin{aligned}(-m(\rho \times 0_1) + m(0_1 \times \lambda))\, e_1 &= -m(\rho \times 0_1)\, e_1 + m(0_1 \times \lambda)\, e_1\\ &= -me_1\rho + m\langle 0_1, \lambda e d\rangle\\ &= -u\rho + m\langle 0_1, \lambda e d\rangle\end{aligned}$$

and they coincide because

$$u\rho = \rho u \quad \text{and} \quad \lambda e = \rho e.$$

■

Corollary 102 *If the 2-cell α, in the definition of (α, λ, ρ)-thin-protocategory in a weakly Mal'cev sesquicategory, satisfies either one of the equivalent sets of conditions (A) or (B), then it is given by*

$$\alpha = \begin{bmatrix}\alpha_1 & \alpha_0 & \begin{bmatrix}\alpha_2 & \alpha_0 & \alpha_3\end{bmatrix}\end{bmatrix}$$

where

$$\begin{aligned}\alpha_1 &= -\rho u + m\langle 0_1, \eta d\rangle\\ \alpha_0 &= -u\eta + v\eta\\ \alpha_2 &= -u\lambda - \rho + \lambda + v\rho\\ \alpha_3 &= -m\langle \eta c, 0_1\rangle + v\lambda.\end{aligned}$$

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Corollary 103 *A* (α, λ, ρ)*-thin-protocategory in a weakly Mal'cev sesquicategory satisfies the pentagonal coherence condition if the following four conditions hold [Ch8]:*

$$\begin{aligned}
u\alpha_1 + \alpha \begin{bmatrix} 1 \\ d \\ e_1 ld \end{bmatrix} + m \begin{bmatrix} 0_1 \\ 0_d \\ \alpha_0 d \end{bmatrix} &= \alpha_1 u + \alpha \begin{bmatrix} 1 \\ d \\ e_2 ld \end{bmatrix} \\
u\alpha_2 + \alpha_2 u + v\alpha_1 &= \alpha_1 v + \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ld \end{bmatrix} \end{bmatrix} \\
u\alpha_3 + \alpha_2 v + v\alpha_2 &= \alpha \begin{bmatrix} lc \\ c \\ e_1 \end{bmatrix} + \alpha_3 u \\
m \begin{bmatrix} \alpha_0 c \\ 0_c \\ 0_1 \end{bmatrix} + \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} lc \\ c \\ 1 \end{bmatrix} \end{bmatrix} + v\alpha_3 &= \alpha \begin{bmatrix} lc \\ c \\ e_2 \end{bmatrix} + \alpha_3 v.
\end{aligned}$$

Corollary 104 *In a weakly Mal'cev sesquicategory with a cartesian 2-cell structure, a* (λ, ρ)*-sort*

$$(C_0, C_1, d, c, e, u, v, \lambda, \rho)$$

determines an (α, λ, ρ)*-thin-protocategory if the triple of morphisms*

$$(u, l, v)$$

and the triple of 2-cells

$$\alpha_{23} = (\alpha_2, \alpha_0, \alpha_3)$$

are both admissible with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} ,$$

and the triple of 2-cells

$$(\alpha_1, \alpha_0, \alpha_{23})$$

is admissible with respect to the split span

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} ,$$

with α_i*'s as in Corollary 102.*

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Theorem 105 *Let* $(C_0, C_1, d, c, e, u, v, \lambda, \rho)$ *be a* (λ, ρ)*-sort satisfying the admissibility requirements of Corollary 104. If*

$$\lambda e = 0_e = \rho e \quad (or\ \eta = 0_e)$$

then

$$\begin{array}{rcl} \alpha_1 & = & -\rho u \quad (= -u\rho) \\ \alpha_2 & = & -u\lambda + \lambda u \quad (= -\rho v + v\rho) \\ \alpha_3 & = & \lambda v \quad (= v\lambda) \\ \alpha_0 & = & 0_e \end{array}$$

and the coherence conditions of Corollary 103 are all trivial.

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Proof. From Corollary 102, and considering $\eta = 0_e$,

- for α_1 we have:

$$\begin{array}{rcl} \alpha_1 & = & -\rho u + m\langle 0_1, 0_{ed}\rangle = -\rho u + m0_{\langle 0_1, ed\rangle} \\ & = & -\rho u + 0_{me_1} = -\rho u + 0_u = (-\rho + 0_1)\, u \\ & = & \left(-\rho + 0_{\mathrm{dom}(-\rho)}\right) u = -\rho u, \end{array}$$

  and from the definition of (λ, ρ)-sort we have $\rho \circ \rho. \iff \rho u = u\rho$;

- for α_2 we have

$$\begin{array}{rcl} \alpha_2 & = & -u\lambda - \rho + \lambda + v\rho \\ & = & -u\lambda + \lambda u \end{array}$$

  because

$$\begin{array}{rcl} \lambda \circ \rho & \iff & \rho + \lambda u = \lambda + v\rho \\ & \iff & \lambda u = -\rho + \lambda + v\rho, \end{array}$$

  and similarly we have

$$\begin{array}{rcl} \alpha_2 & = & -u\lambda - \rho + \lambda + v\rho \\ & = & -\rho v + v\rho \end{array}$$

  because

$$\begin{array}{rcl} \rho \circ \lambda & \iff & \lambda + \rho v = \rho + u\lambda \\ & \iff & \rho v = -\lambda + \rho + u\lambda \\ & \iff & -\rho v = -u\lambda - \rho + \lambda; \end{array}$$

- for α_3 we have

$$\begin{aligned}\alpha_3 &= -m\left\langle 0_e c, 0_1\right\rangle + v\lambda = -0_{m\langle ec,1\rangle} + v\lambda\\ &= -0_{me_2} + v\lambda = -0_v + v\lambda = v\left(0_1 + \lambda\right)\\ &= v\left(0_{\operatorname{cod}\lambda} + \lambda\right) = v\lambda\end{aligned}$$

  and since $\lambda \circ \lambda$ we also have $\alpha_3 = \lambda v$;

- for α_0 we have

$$\alpha_0 = -u\eta + v\eta = -u0_e + v0_e = -0_{ue} + 0_{ve}$$

  but since, by hypoteses, $\lambda e = 0_e = \rho e$ then $\operatorname{dom}(\lambda e) = e = \operatorname{dom}(\rho e)$, and hence

$$ve = e = ue \quad , \quad (\text{or } l = e)$$

  so that

$$\alpha_0 = -0_e + 0_e = 0_e.$$

We now prove that the four conditions of Corollary 103 are all trivial. The first condition becomes (note that $l = e$)

$$u\alpha_1 + \alpha\begin{bmatrix}1\\ d\\ e_1ed\end{bmatrix} + m\begin{bmatrix}0_1\\ 0_d\\ 0_{ed}\end{bmatrix} = \alpha_1 u + \alpha\begin{bmatrix}1\\ d\\ e_2ed\end{bmatrix}$$

by observing that ([Ch8], p. 231 and followings)

$$i_1e_1 = \begin{bmatrix}1\\ d\\ e_1ed\end{bmatrix} = \begin{bmatrix}1\\ d\\ e_2ed\end{bmatrix}$$

it is clear that the above condition simplifies to

$$u\alpha_1 + \alpha i_1e_1 + m0_{e_1} = \alpha_1 u + \alpha i_1 e_1$$

and using the fact that $\alpha_1 = \alpha i_1e_1,\ \alpha_2 = \alpha i_2e_1,\ \alpha_3 = \alpha i_2e_2,\ \alpha_0 = \alpha i_1e_1e$, simply because ([Ch8], p. 231 and followings)

$$1_{C_3} = \begin{bmatrix}i_1e_1 & i_1e_1e & \begin{bmatrix}i_2e_1 & i_2e_1e & i_2e_2\end{bmatrix}\end{bmatrix}$$

we have

$$u\alpha_1 + \alpha_1 + 0_u = \alpha_1 u + \alpha_1$$

substituting for α_1 gives

$$u\left(-\rho u\right) - \rho u + 0_u = -\rho uu - \rho u$$

now $-\rho u + 0_u = -\rho u$ because $u = \operatorname{dom}(-\rho u)$, and the result is

$$-u\rho u - \rho u = -\rho uu - \rho u$$

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which is trivial because $\rho \circ \rho$.

For the second condition we have

$$u\alpha_2 + \alpha_2 u + v\alpha_1 = \alpha_1 v + \alpha \begin{bmatrix} ec \\ c \\ \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} \end{bmatrix}$$

and from [Ch8], p. 231 and followings, we have

$$u\alpha_2 + \alpha_2 u + v\alpha_1 = \alpha_1 v + \alpha i_2 e_1$$

that is

$$u\alpha_2 + \alpha_2 u + v\alpha_1 = \alpha_1 v + \alpha_2,$$

now, a carefully substitution for α_1 and α_2 gives

$$u(-\rho v + v\rho) + (-\rho v + v\rho) u + v(-\rho u) = -\rho u v + (-u\lambda + \lambda u)$$

which simplifies to

$$uv\rho - \rho v u = -u\lambda + \lambda u,$$

adding $u\lambda$ in front and $\rho v u$ at the end

$$u\lambda + uv\rho = \lambda u + \rho v u$$

collecting u gives

$$u(\lambda + v\rho) = (\lambda + \rho v) u$$

and since $\lambda \circ \rho$ and $\rho \circ \lambda$ we have

$$u(\rho + \lambda u) = (\rho + u\lambda) u$$

which is trivial because $\rho \circ \rho$.

For the third condition we observe that

$$\begin{bmatrix} ec \\ c \\ e_1 \end{bmatrix} = i_2 e_1 \ ,$$

and since $\alpha i_2 e_1 = \alpha_2$ we have

$$u\alpha_3 + \alpha_2 v + v\alpha_2 = \alpha_2 + \alpha_3 u,$$

a carefully substitution of α_2 and α_3 gives

$$u(\lambda v) + (-u\lambda + \lambda u) v + v(-u\lambda + \lambda u) = (-\rho v + v\rho) + (\lambda v) u$$

which simplifies to

$$\lambda u v - v u \lambda = -\rho v + v\rho$$

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removing negative signs and obtaining

$$\rho v + \lambda u v = v \rho + v u \lambda$$

collecting v gives

$$(\rho + \lambda u)\, v = v\, (\rho + u\lambda)$$

and using the fact that $\lambda \circ \rho$ and $\rho \circ \lambda$ it becomes

$$(\lambda + v\rho)\, v = v\, (\lambda + \rho v)$$

which is trivial because $\lambda \circ \lambda$.

Finally we have condition four, which becomes

$$m 0_{e_2} + \alpha_3 + v\alpha_3 = \alpha_3 + \alpha_3 v$$

since $\alpha_3 = \alpha i_2 e_2$ and

$$i_2 e_2 = \begin{bmatrix} ec \\ c \\ e_2 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix}.$$

Now, observe that

$$m 0_{e_2} = 0_{m e_2} = 0_v = 0_{\operatorname{cod} \alpha_3}$$

and hence we have that the fourth condition is simply

$$v\alpha_3 = \alpha_3 v$$

which is trivial because $\lambda v = v\lambda = \alpha_3$. ■

9.3 Pseudocategories in a weakly Mal'cev sesquicategory

We will say that a (λ, ρ)-sort is admissible when it satisfies the admissibility requirements of Corollary 104, if furthermore, the set of conditions in Corollary 103 is also satisfied, with

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_0 & \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix} \end{bmatrix} \tag{9.1}$$

then we will say that the (λ, ρ)-sort is admissible and pentagonal.

Theorem 106 *A pseudocategory [Ch2] in a weakly Mal'cev sesquicategory, with a cartesian 2-cell structure, is completely determined by a pentagonal and admissible* (λ, ρ)*-sort, where in addition the 2-cell* (9.1) *is required to be natural and invertible and the 2-cells* λ, ρ *are required to be natural (and not just natural with respect to each other).*

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Proof. From [1] it is well known that the pentagon condition (see also the introduction of this article) plus the middle triangle condition, imply the left and right triangle conditions and hence, by previous results, a pseudocategory, being in particular an (α, λ, ρ)-(thin)protocategory is completely determined by an admissible (λ, ρ)-sort. ■

In the case where the 2-cell structure considered over the category $\mathbf{C}$ is such that all the 2-cells are invertible and natural (i.e., it is a 2-groupoid) then a pseudocategory is completely determined by an admissible and pentagonal (λ, ρ)-sort, where in this case, the axioms $\lambda \circ \lambda, \lambda \circ \rho, \rho \circ \lambda, \rho \circ \rho$ in the definition of (λ, ρ)-sort are automatically true.

This result can be stated as follows.

Theorem 107 *Let* $\mathbf{C} = (\mathbf{C}, H, \mathrm{dom}, \mathrm{cod}, 0, +)$ *be a weakly Mal'cev category with a weakly Mal'cev, cartesian, invertible and natural 2-cell structure. A pseudocategory in (internal to)* $\mathbf{C}$ *is completely determined by a reflexive graph in* $\mathbf{C}$*,*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e"'] \end{tikzcd} \quad , \quad de = 1_{C_0} = ce$$

together with 2-cells

$$\lambda, \rho \in H(C_1, C_1)$$

such that the following conditions are satisfied (where we write v *for* $\mathrm{dom}(\lambda)$ *and* u *for* $\mathrm{dom}(\rho)$*),*

$$\mathrm{cod}(\lambda) = 1_{C_1} = \mathrm{cod}(\rho)$$

$$\begin{aligned} d\lambda &= 0_d = d\rho \\ c\lambda &= 0_c = c\rho \\ \lambda e &= \rho e \end{aligned}$$

the triples

$$(u, ue, v)$$

and

$$(\alpha_2, \alpha_0, \alpha_3)$$

are admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} ,$$

the triple

$$\left(\alpha_1, \alpha_0, \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix}\right)$$

is admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} ,$$

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with $\alpha_1, \alpha_2, \alpha_3, \alpha_0$ *as in Corollary 102, and the conditions of Corollary 103 are satisfied with*

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_0 & \begin{bmatrix} \alpha_2 & \alpha_0 & \alpha_3 \end{bmatrix} \end{bmatrix}.$$

If we call homogeneous pseudocategory to a pseudocategory (as in [Ch2]) such that

$$\lambda e = 0_e = \rho e$$

then the description of homogeneous pseudocategory in a weakly Mal'cev sesquicategory, with a cartesian, invertible and natural 2-cell structure, is highly simplified, as shown in Theorem 105. This result may also be stated as follows.

Theorem 108 *In a weakly Mal'cev category with a weakly Mal'cev, cartesian, invertible and natural 2-cell structure, a homogeneous pseudocategory is completely determined by a reflexive graph*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1_{C_0} = ce$$

together with 2-cells

$$\lambda, \rho \in H\left(C_1, C_1\right)$$

satisfying the following conditions (where we write v *for* $\operatorname{dom}(\lambda)$ *and* u *for* $\operatorname{dom}(\rho)$*),*

$$\operatorname{cod}(\lambda) = 1_{C_1} = \operatorname{cod}(\rho)$$

$$\begin{aligned} d\lambda &= 0_d = d\rho \\ c\lambda &= 0_c = c\rho \\ \lambda e &= 0_e = \rho e \end{aligned}$$

the triples

$$(u, e, v)$$

and

$$(-u\lambda + \lambda u, 0_e, \lambda v)$$

are admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} ,$$

and the triple

$$\left(-\rho u, 0_e, \begin{bmatrix} -u\lambda + \lambda u & 0_e & \lambda v \end{bmatrix}\right)$$

is admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} .$$

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9.4 Conclusion

We conclude this thesis by analyzing the previous results, describing a homogeneous pseudocategory in a weakly Mal'cev sesquicategory with a cartesian, invertible and natural 2-cell structure, in the cases of groups and crossed modules.

For the case of groups we will use the construction for a 2-cell structure given as in [Ch2], Example 18 and comparing the general result with the particular case described in [ChA].

For the case of crossed modules we will use the construction for a 2-cell structure given as in [Ch2], Example 19, and interpreting further in the concrete example of crossed modules, obtaining this way a description for internal (homogeneous) bicategories in Groups, where homogeneous simply means that the isomorphism

$$id_A \otimes id_A \cong id_A$$

is in fact an identity

$$id_A \otimes id_A = id_A.$$

The example of groups

Let $\mathbf{C}$ be a weakly Mal'cev category and let K

$$K : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Grp \xrightarrow{U} Set$$

be a functor and

$$D : UK \times \hom_{\mathbf{C}} \longrightarrow \hom_{\mathbf{C}}$$

a natural transformation satisfying

$$\begin{aligned} D(0,f) &= f \\ D(x+x',f) &= D(x,D(x',f)). \end{aligned}$$

Consider a 2-cell structure over $\mathbf{C}$ defined as in [Ch2], Example 18, and suppose that the resulting 2-cell structure is also weakly Mal'cev, cartesian, and natural. Is is also clearly invertible, the inverse of a 2-cell

$$(x,f) : f \longrightarrow D(x,f)$$

is given by

$$(-x, D(x,f)) : D(x,f) \longrightarrow f.$$

The naturally condition may be found in (2.9), [Ch2]; cartesian, in this context means that the functor $K(D,\_) : \mathbf{C} \longrightarrow Grp$ preserves pullbacks for every objects $D \in \mathbf{C}$; for weakly Mal'cev we refer to [Ch8], Proposition 87. We are not developing this concepts further because the example of our study in this subsection, the category of groups, clearly satisfies the above requirements.

The result may be stated as follows.

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Proposition 109 *Let* $(\mathbf{C}, K, D)$ *be a system, as above, defining a weakly Mal'cev, cartesian, natural and invertible 2-cell structure over the weakly Mal'cev category* $\mathbf{C}$*. A homogeneous pseudocategory in* $\mathbf{C}$ *is completely determined by an admissible sort, that is a diagram in* $\mathbf{C}$ *of the form*

$$\begin{tikzcd} C_1 \arrow[loop above, "u"] \arrow[loop below, "v"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

satisfying

$$\begin{aligned} de &= 1_{C_0} = ce \\ du &= d = dv \\ cu &= c = cv \\ ue &= e = ve \end{aligned}$$

and the triple

$$(u, e, v)$$

being admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd}, \tag{9.2}$$

together with two elements in the group $K(C_1, C_1)$

$$\lambda, \rho \in K(C_1, C_1)$$

satisfying

$$D(\lambda, v) = 1_{C_1} = D(\rho, u)$$

$$\begin{aligned} d\lambda &= d\rho = c\lambda = c\rho = 0 \in K(C_1, C_0) \\ \lambda e &= \rho e = 0 \in K(C_0, C_1) \end{aligned}$$

with the triples

$$(-uv + vu, e, vv)$$

and

$$(-u\lambda + \lambda u, 0, \lambda v)$$

being admissible with respect to (9.2)*, and the triples*

$$\left(uu, e, \begin{bmatrix} -uv + vu & e & vv \end{bmatrix}\right)$$

and

$$\left(-\rho u, 0, \begin{bmatrix} -u\lambda + \lambda u & 0 & \lambda v \end{bmatrix}\right)$$

are admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd}.$$

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Proof. The proof follows from Theorem 108 with $\boldsymbol{\lambda} = (\lambda, v)$ and $\boldsymbol{\rho} = (\rho, u)$. ■

Some observations for the case $\mathbf{C}=$ *Grp*, K the second projection and $D(x, f) = {}^x f$, as defined in [Ch2]:

- A homogeneous pseudocategory is the same as an internal category, simply because the condition

$$\lambda e = \rho e = 0$$

in Proposition 109 is equivalent to $\lambda = 0$ and $\rho = 0$;

- A pseudocategory is described as in Appendix A ([ChA]).

The example of crossed modules

Let $\mathbf{C}$ be a weakly Mal'cev category and suppose there is an extension of the hom functor into the category of groups, that is a functor

$$map : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Grp$$

such that $\hom_{\mathbf{C}}(A, B) \subseteq map(A, B)$, naturally for all $A, B \in \mathbf{C}$ (see [Ch2], Example 19).

Suppose also that we have given a functor

$$K : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Grp,$$

and a natural transformation

$$D : K \longrightarrow map,$$

defining a 2-cell structure over $\mathbf{C}$, where a 2-cell is a pair

$$(x, f) : f \longrightarrow (D(x) + f) : A \longrightarrow B$$

such that $x \in K(A, B)$, $f : A \longrightarrow B$ and $D(x) + f \in \hom(A, B)$, as constructed in [Ch2], Example 19.

Assume that the resulting 2-cell structure is weakly Mal'cev, cartesian and natural, since it is obviously invertible. We will not develop further this concepts because the example of crossed modules, being (up to equivalence) of the form Cat(Grp) is an example of a weakly Mal'cev 2-category with a cartesian 2-cell structure (by Proposition 88, in [Ch8], and the fact that Grp is a weakly Mal'cev category).

The description of homogeneous pseudocategories in such a structure may be stated as follows.

Proposition 110 *Let $(\mathbf{C}, K, D)$ be a system, as above, defining a weakly Mal'cev, cartesian, natural and invertible 2-cell structure over the weakly Mal'cev category $\mathbf{C}$, as in the construction of Example 19, [Ch2]. A homogeneous pseudocategory in $\mathbf{C}$ is completely determined by a reflexive graph*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd} \quad , \quad de = 1_{C_0} = ce$$

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together with two elements in the group $K(C_1, C_1)$

$$\lambda, \rho \in K(C_1, C_1)$$

such that

$$\begin{aligned} d\lambda &= d\rho = c\lambda = c\rho = 0 \in K(C_1, C_0) \\ \lambda e &= \rho e = 0 \in K(C_0, C_1) \end{aligned} \tag{9.3}$$

with the triples

$$(-uv + vu, e, vv)$$

and

$$(-u\lambda + \lambda u, 0, \lambda v)$$

being admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} ,$$

and the triples

$$\left(uu, e, \begin{bmatrix} -uv + vu & e & vv \end{bmatrix}\right)$$

and

$$\left(-\rho u, 0, \begin{bmatrix} -u\lambda + \lambda u & 0 & \lambda v \end{bmatrix}\right)$$

admissible with respect to

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e_1 e"'] & C_2 \arrow[l, shift right, "c\pi_1"'] \end{tikzcd} ,$$

with $u = -D(\rho) + 1$ *and* $v = -D(\lambda) + 1$.

Proof. The proof follows from Theorem 108 with $\boldsymbol{\lambda} = (\lambda, v)$ and $\boldsymbol{\rho} = (\rho, u)$, where the condition

$$\operatorname{cod}(\boldsymbol{\lambda}) = 1_{C_1} = \operatorname{cod}(\boldsymbol{\rho})$$

implies

$$u = -D(\rho) + 1 \quad \text{and} \quad v = -D(\lambda) + 1.$$

■

Some observations for the case of crossed modules with derivations as described in [Ch2], where we identify a crossed module

$$(X \longrightarrow B, \varphi : B \longrightarrow Aut(X))$$

with a reflexive graph

$$\begin{tikzcd} X \rtimes B \arrow[r, shift left=2, "{[0\ \ 1]}"] \arrow[r, shift right=2, "{[d\ \ 1]}"'] & B \arrow[l, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}" description] \end{tikzcd}$$

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such that $\left[\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \quad 1\right] : X \rtimes (X \rtimes B) \longrightarrow X \rtimes B$ is a homomorphism, or in other words, with an admissible reflexive graph in Grp.

As it is well known, a double split epi in Groups is given, up to isomorphism, as follows

$$\begin{tikzcd}
(Y \rtimes E) \rtimes (X \rtimes B) \arrow[r, shift left, "{[0\,1]}"] \arrow[d, phantom, "\cong"] & X \rtimes B \arrow[l, shift left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] \arrow[dd, shift right, "{[0\,1]}"'] \\
(Y \rtimes X) \rtimes (E \rtimes B) \arrow[d, shift right, "{[0\,1]}"'] & \\
E \rtimes B \arrow[u, shift right, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \arrow[r, shift left, "{[0\,1]}"] & B \arrow[l, shift left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] \arrow[uu, shift right, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"']
\end{tikzcd} .$$

Hence, a split epi in crossed modules, or admissible reflexive graph in Grp, is of the form

$$\begin{tikzcd}
(Y \rtimes E) \rtimes (X \rtimes B) \arrow[r, shift left=2, "{[0\,1]}"] \arrow[r, shift right=2, "{[d\,1]}"'] \arrow[d, phantom, "\cong"] & X \rtimes B \arrow[l] \arrow[dd, shift right, "{[0\,1]}"'] \\
(Y \rtimes X) \rtimes (E \rtimes B) \arrow[d, shift right, "{[0\,1]}"'] & \\
E \rtimes B \arrow[u, shift right, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \arrow[r, shift left=2, "{[0\,1]}"] \arrow[r, shift right=2, "{[d\,1]}"'] & B \arrow[l] \arrow[uu, shift right, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"']
\end{tikzcd} .$$

The description of a homogeneous pseudocategory in crossed modules would give the description of a (homogeneous) pseudo-double-category in Grp, but we are most interested in obtaining a description for a (homogeneous) bicategory in Grp, and in that case we may take $E = 0$, the trivial group (see also [Ch6]).

From the above split epi, with $E = 0$, to obtain a reflexive graph one adds

$$\begin{tikzcd}
Y \rtimes (X \rtimes B) \arrow[r, shift left=2, "{[0\,1]}"] \arrow[r, shift right=2, "{\left[\left[\begin{smallmatrix}d\\0\end{smallmatrix}\right] 1\right]}"'] \arrow[d, phantom, "\cong"] & X \rtimes B \arrow[l] \arrow[dd, shift right=3, "{[0\,1]}"'] \arrow[dd, shift left=3, "{[d\,1]}"] \\
(Y \rtimes X) \rtimes B \arrow[d, shift right=2] \arrow[d, shift left=2] & \\
B \arrow[u] \arrow[r, equal] & B \arrow[uu, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}" description]
\end{tikzcd} \tag{9.4}$$

with different d's of course, displayed as

$$\begin{tikzcd}
Y \arrow[r, "{\left[\begin{smallmatrix}d\\0\end{smallmatrix}\right]}"] \arrow[d] & X \rtimes B \arrow[d, "{[d\,1]}"] \\
0 \arrow[r] & B
\end{tikzcd} .$$

The elements $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$, in order to satisfy conditions (9.3)

$$\begin{aligned}
[0 \quad 1]\boldsymbol{\lambda} &= [0 \quad 1]\boldsymbol{\rho} = [d \quad 1]\boldsymbol{\lambda} = [d \quad 1]\boldsymbol{\rho} = 0 \\
\boldsymbol{\lambda}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] &= \boldsymbol{\rho}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = 0,
\end{aligned}$$

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are given by derivations

$$\begin{aligned}\boldsymbol{\lambda} &= \begin{bmatrix}\lambda & 0\end{bmatrix} : X \rtimes B \longrightarrow Y\\ \boldsymbol{\rho} &= \begin{bmatrix}\rho & 0\end{bmatrix} : X \rtimes B \longrightarrow Y,\end{aligned}$$

with $d\lambda = 0 = d\rho$, and where the morphisms u and v are defined as

$$\begin{aligned}u &= \left(-\begin{bmatrix}\rho & 0\end{bmatrix}\begin{bmatrix}d\\0\end{bmatrix} + 1, -\begin{bmatrix}d\\0\end{bmatrix}\begin{bmatrix}\rho & 0\end{bmatrix} + 1\right)\\ &= \left(-\rho d + 1, \begin{bmatrix}-d\rho + 1 & 0\\ 0 & 1\end{bmatrix}\right)\end{aligned}$$

$$v = \left(-\lambda d + 1, \begin{bmatrix}-d\lambda + 1 & 0\\ 0 & 1\end{bmatrix}\right).$$

We now conclude this thesis by saying that a internal (homogeneous, in the sense that $1_A \otimes 1_A = 1_A$ instead of $1_A \otimes 1_A \cong 1_A$) bicategory in the category of groups is determined by a sequence in groups

$$Y \xrightarrow{d} X \xrightarrow{d} B$$

such that $dd = 0$, together with B-actions on X and Y and an X-action in Y, in such a way that the morphisms of diagram (9.4) are well defined, the appropriate squares commute, the reflexive graphs are admissible, and there is also a pair of maps

$$\lambda, \rho : X \longrightarrow Y$$

such that

$$\begin{aligned}\lambda(x + b \cdot x') &= \lambda(x) + b \cdot (x \cdot \lambda(x')),\\ \rho(x + b \cdot x') &= \rho(x) + b \cdot (x \cdot \rho(x')), \quad x \in X, b \in B,\end{aligned}$$

$$d\lambda = 0 = d\rho,$$

satisfying the admissibility conditions of Proposition 110, with u and v as above.

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Appendix A

Weak categories in Grp

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This notes, originally written in November, 2002, contain the calculations required for describing the notion of pseudocategory (at the time called weak category, so that we didn't change it, exactly as defined in Chapter 5) in the 2-category of groups, with 2-cells being conjugations.

The reason why we choose to include it here, as an Appendix, is because it does not posses the strenght of a Chapter: the results however new, are in a very particular case, and all the required techniques are known; but it is referred in Chapther 1 as unpublished and it is used in Chapter 9 for comparison with the general results, so we belive it is convenient to include it here. Nevertheless, we didn't try to adjust it to the notation or style from the rest of the thesis, it is included as it was originally written. The references are with respect to Chapter 5.

A.1 Introduction

Every group is a particular case of a category; in the sense that a group is a category with only one object, the morphisms are the elements of the group, and the composition is given by the multiplication of the group. Following that idea we consider a 2-category structure over Groups and describe there the notion of weak category.

A.2 The category Groups viewed as a 2-category

In the category Groups we define a 2-category structure as follows:

objects $G, H, ...$ are groups;

morphisms $f : G \longrightarrow H$ are group homomorphisms;

a 2-cell $\tau : f \longrightarrow g : G \longrightarrow H$ is an element of H such that $g(x) = \tau g(x) \tau^{-1}$, for all $x \in G$.

The vertical composition of 2-cells $\sigma \cdot \tau = \sigma\tau$ (the multiplication in H) for $\tau : f \longrightarrow g$ and $\sigma : g \longrightarrow h$, while the horizontal composition of, say

$$\begin{tikzcd} G \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\downarrow \tau"] & H \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\downarrow \sigma"] & K \end{tikzcd}$$

is $\sigma \circ \tau = \sigma f'(\tau) = g'(\tau)\,\sigma$.

Before continuing and investigate the notion of weak category in Groups, it is convenient to state some considerations with respect to semidirect products in groups.

Semidirect Products

Let X and B be two groups and an action of B in X, that is, a homomorphism

$$\begin{array}{rcl} \varphi : B & \longrightarrow & Aut(X) \\ b & \longmapsto & b_X : X \longrightarrow X \\ & & \quad x \longmapsto b \cdot x \end{array}$$

with the following conditions

$$\begin{array}{rcl} 1 \cdot x & = & x \\ bb' \cdot x & = & b \cdot (b' \cdot x) \\ b \cdot (x + x') & = & b \cdot x + b \cdot x' \end{array}$$

where we are considering X with an additive structure and B with a multiplicative structure. The semidirect product of X and B, denoted by $X \rtimes B$, is the set of elements $X \times B$ with the operation defined by

$$(x, b) + (x', b') = (x + b \cdot x', bb').$$

An internal split epi in Groups is isomorphic to a certain semidirect product, as we will see in the next section. So, in the future we will largely use morphisms between semidirect products and their composition. For that matter, the following propositions will be of great assistance.

Proposition 111 *A group homomorphism*

$$f : X \rtimes B \longrightarrow Y \rtimes D$$

is described by four maps, as displayed in the following matrix

$$\begin{array}{cc|c} X & B & \\ \hline f_{11} & f_{12} & Y \\ f_{21} & f_{22} & D \end{array} \tag{A.1}$$

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(with domains above and codomains on the right), where

$$\begin{array}{rcl} f_{22}\left(bb'\right) & = & f_{22}\left(b\right) f_{22}\left(b'\right) \\ f_{21}\left(x+x'\right) & = & f_{21}\left(x\right) f_{21}\left(x'\right) \\ f_{21}\left(b\cdot x'\right) & = & f_{22}\left(b\right) f_{21}\left(x'\right) f_{22}\left(b\right)^{-1} \\ f_{12}\left(bb'\right) & = & f_{12}\left(b\right) + f_{22}\left(b\right)\cdot f_{12}\left(b'\right) \\ f_{11}\left(x+x'\right) & = & f_{11}\left(x\right) + f_{21}\left(x\right)\cdot f_{11}\left(x'\right) \\ f_{11}\left(b\cdot x'\right) & = & f_{12}\left(b\right) + f_{22}\left(b\right)\cdot f_{11}\left(x'\right) - f_{21}\left(b\cdot x'\right)\cdot f_{12}\left(b\right) \end{array}$$

and

$$f\left(x,b\right) = \left(f_{11}\left(x\right) + f_{21}\left(x\right)\cdot f_{12}\left(b\right)\ ,\ f_{21}\left(x\right) f_{22}\left(b\right)\right).$$

Proof. The elements of a semidirect product are pairs, so the homomorphism f is of the form

$$f\left(x,b\right) = \left(f_1\left(x,b\right), f_2\left(x,b\right)\right)$$

where f_1 is a map from $X\times B$ to Y and f_2 is a map from $X\times B$ to D. By definition of multiplication in $X\rtimes B$

$$\begin{array}{rcl} (x,b) & = & (x,1)+(0,b) \\ & = & (0,b)+\left(b^{-1}\cdot x,1\right) \end{array}$$

and since f is a homomorphism on the one hand we have

$$\begin{array}{rcl} f\left(x,b\right) & = & f\left(x,1\right)+f\left(0,b\right) \\ & = & \left(f_1\left(x,1\right), f_2\left(x,1\right)\right) + \left(f_1\left(0,b\right), f_2\left(0,b\right)\right) \end{array}$$

while on the other hand we have

$$\begin{array}{rcl} f\left(x,b\right) & = & f\left(0,b\right)+f\left(b^{-1}\cdot x,1\right) \\ & = & \left(f_1\left(0,b\right), f_2\left(0,b\right)\right) + \left(f_1\left(b^{-1}\cdot x,1\right), f_2\left(b^{-1}\cdot x,1\right)\right). \end{array}$$

If defining

$$\begin{array}{rcl} f_{11}\left(x\right) & = & f_1\left(x,1\right) \\ f_{12}\left(b\right) & = & f_1\left(0,b\right) \\ f_{21}\left(x\right) & = & f_2\left(x,1\right) \\ f_{22}\left(b\right) & = & f_2\left(0,b\right) \end{array}$$

as in $(A.1)$ then the homomorphism f is given eigther by

$$f\left(x,b\right) = \left(f_{11}\left(x\right) + f_{21}\left(x\right)\cdot f_{12}\left(b\right)\ ,\ f_{21}\left(x\right) f_{22}\left(b\right)\right)$$

or

$$f\left(x,b\right) = \left(f_{12}\left(b\right) + f_{22}\left(b\right)\cdot f_{11}\left(b^{-1}\cdot x\right), f_{22}\left(b\right) f_{21}\left(b^{-1}\cdot x\right)\right).$$

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It seems strange to obtain two diferent ways of defining f, but as soon as we impose the conditions for f to be a homomorphism, they will agree. Conditions that we should impose to $f_{ij}, i, j = 1, 2$: since f is a homomorphism, in particular it satisfyes the following condition

$$f\left(0, bb'\right) = f\left(0, b\right) + f\left(0, b'\right)$$

that is

$$\begin{aligned}\left(f_{12}\left(bb'\right), f_{22}\left(bb'\right)\right) &= \left(f_{12}\left(b\right), f_{22}\left(b\right)\right) + \left(f_{12}\left(b'\right), f_{22}\left(b'\right)\right) \\ &= \left(f_{12}\left(b\right) + f_{22}\left(b\right) \cdot f_{12}\left(b'\right), f_{22}\left(b\right) f_{22}\left(b'\right)\right).\end{aligned}$$

Considering the condition

$$f\left(x + x', 1\right) = f\left(x, 1\right) + f\left(x', 1\right)$$

that is

$$\begin{aligned}\left(f_{11}\left(x + x'\right), f_{21}\left(x + x'\right)\right) &= \left(f_{11}\left(x\right), f_{21}\left(x\right)\right) + \left(f_{11}\left(x'\right), f_{21}\left(x'\right)\right) \\ &= \left(f_{11}\left(x\right) + f_{21}\left(x\right) \cdot f_{11}\left(x'\right), f_{21}\left(x\right) f_{21}\left(x'\right)\right)\end{aligned}$$

and finally considering the condition

$$f\left(b \cdot x, b\right) = f\left(0, b\right) + f\left(x, 1\right)$$

that is

$$\begin{aligned}\left(f_{11}\left(b \cdot x\right) + f_{21}\left(b \cdot x\right) \cdot f_{12}\left(b\right)\ ,\ f_{21}\left(b \cdot x\right) f_{22}\left(b\right)\right) &= \left(f_{12}\left(b\right), f_{22}\left(b\right)\right) + \left(f_{11}\left(x\right), f_{21}\left(x\right)\right) \\ &= \left(f_{12}\left(b\right) + f_{22}\left(b\right) \cdot f_{11}\left(x\right), f_{22}\left(b\right) f_{21}\left(x\right)\right)\end{aligned}$$

we conclude that

$$\begin{aligned}f_{21}\left(b \cdot x\right) &= f_{22}\left(b\right) f_{21}\left(x\right) f_{22}\left(b\right)^{-1} \\ f_{11}\left(b \cdot x\right) &= f_{12}\left(b\right) + f_{22}\left(b\right) \cdot f_{11}\left(x\right) - f_{21}\left(b \cdot x\right) \cdot f_{12}\left(b\right)\end{aligned}$$

and we have obtained the stated conditions. To conclude the poof it remains to check that with this conditions f is in fact a homomorphism: it is a straightforward calculation and we omit it. ■

Proposition 112 *The composition of two homomorphisms*

$$X \rtimes B \xrightarrow{f} Y \rtimes D \xrightarrow{g} Z \rtimes E$$

is given by the following formula

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} g_{11}f_{11} + \left(g_{21}f_{11}\right) \cdot g_{12}f_{21} & g_{11}f_{12} + \left(g_{21}f_{12}\right) \cdot g_{12}f_{22} \\ g_{21}f_{11} + g_{22}f_{21} & g_{21}f_{12} + g_{22}f_{22} \end{pmatrix}$$

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Proof. Writting the homomorphisms f and g in the form

$$f\left(x,b\right)=\left(f_{11}\left(x\right)+f_{21}\left(x\right)\cdot f_{12}\left(b\right)\ ,\ f_{21}\left(x\right)f_{22}\left(b\right)\right)$$

$$g\left(y,d\right)=\left(g_{11}\left(y\right)+g_{21}\left(y\right)\cdot g_{12}\left(d\right)\ ,\ g_{21}\left(y\right)g_{22}\left(d\right)\right)$$

and computing

$$gf\left(x,b\right)=g\left(f_{11}\left(x\right)+f_{21}\left(x\right)\cdot f_{12}\left(b\right),f_{21}\left(x\right)f_{22}\left(b\right)\right)$$

we obtain

$$\left(\begin{array}{c} g_{11}\left(f_{11}\left(x\right)+f_{21}\left(x\right)\cdot f_{12}\left(b\right)\right)+g_{21}\left(f_{11}\left(x\right)+f_{21}\left(x\right)\cdot f_{12}\left(b\right)\right)\cdot g_{12}\left(f_{21}\left(x\right)f_{22}\left(b\right)\right)\ , \\ g_{21}\left(f_{11}\left(x\right)+f_{21}\left(x\right)\cdot f_{12}\left(b\right)\right)g_{22}\left(f_{21}\left(x\right)f_{22}\left(b\right)\right); \end{array}\right)$$

if using the properties of the maps that constitute g, we obtain the desired result

$$\left(\begin{array}{cc} g_{11}f_{11}+\left(g_{21}f_{11}\right)\cdot g_{12}f_{21} & g_{11}f_{12}+\left(g_{21}f_{12}\right)\cdot g_{12}f_{22} \\ g_{21}f_{11}+g_{22}f_{21} & g_{21}f_{12}+g_{22}f_{22} \end{array}\right).$$

Corollary 113 *In the previous proposition, if* $g_{21}=0$ *then composition reduces to the usual product of matrices.*

Corollary 114 *In the previous proposition, if* $g_{21}=0$ *and* $f_{21}=0$ *then the composition* gf *becomes*

$$\left(\begin{array}{cc} g_{11}f_{11} & g_{11}f_{12}+g_{12}f_{22} \\ 0 & g_{22}f_{22} \end{array}\right)$$

and so

$$gf\left(x,b\right)=\left(g_{11}f_{11}\left(x\right)+g_{11}f_{12}\left(b\right)+g_{12}f_{22}\left(b\right),g_{22}f_{22}\left(b\right)\right).$$

■

Next we state some useful results about split epis and pullbacks of split epis in the category Groups.

Proposition 115 *Let* $\alpha:A\longrightarrow B$ *be a split epi in Groups, say splited by* $\beta:B\longrightarrow A$*, that is* $\alpha\beta=1_B$*. The object* A *is isomorphic to* $X\rtimes B$ *where* $X=\ker\alpha$ *and the action of* B *in* A *is given by*

$$b\cdot x=\beta\left(b\right)+x-\beta\left(b\right)$$

Proof. The isomorphism is given by

$$\begin{bmatrix} k & \beta \end{bmatrix}:X\rtimes B\longrightarrow A$$

and

$$\begin{bmatrix} l \\ \alpha \end{bmatrix}:A\longrightarrow X\rtimes B$$

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where $k : X \longrightarrow A$ is the kernel of α and $l : A \longrightarrow X$ is the unique map (not a homomorphism) satisfying

$$kl = 1 - \beta\alpha.$$

To see that it is an isomorphism we observe that

$$\begin{bmatrix} l \\ \alpha \end{bmatrix} \begin{bmatrix} k & \beta \end{bmatrix} = \begin{bmatrix} lk & l\beta \\ \alpha k & \alpha\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $lk = 1$ and $l\beta = 0$ because k is monoic and $klk = k$ and $kl\beta = 0$; on the other hand we have

$$\begin{bmatrix} k & \beta \end{bmatrix} \begin{bmatrix} l \\ \alpha \end{bmatrix} = kl + \beta\alpha = 1.$$

It remains to check that $\begin{bmatrix} l \\ \alpha \end{bmatrix}$ is a homomorphism, or equivalently to check if the map $l : A \longrightarrow X$ satisfies the following equation

$$l\left(a + a'\right) = l\left(a\right) + \alpha\left(a\right) \cdot l\left(a'\right).$$

It is in fact the case since after composing with k on both sides we have

$$a + a' - \beta\alpha\left(a + a'\right) = a - \beta\alpha\left(a\right) + \beta\alpha\left(a\right) + a' - \beta\alpha\left(a'\right) - \beta\alpha\left(a\right).$$

■

Proposition 116 *Let $A \times_B C$ be a pullback diagram in Groups, where α is a split epi, say splited by β, as in the following diagram*

$$\begin{tikzcd}
A \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "\delta"] \\
A \arrow[r, "\alpha"] & B
\end{tikzcd}$$

Then $A \times_B C \cong X \rtimes C$ and the pullback diagram becomes

$$\begin{tikzcd}
X \rtimes C \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right)"'] & C \arrow[d, "\delta"] \\
X \rtimes B \arrow[r, "(0\ 1)"] & B
\end{tikzcd}$$

with the action from C in X defined by $c \cdot x = \delta\left(c\right) \cdot x$.

Proof. Since $A \xrightarrow{\alpha} B$ is splited by $\beta : B \longrightarrow A$, then A is isomorphic to a semidirect product, say $X \rtimes B$ with the action from B in X denoted by $b \cdot x$.

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The pullback diagram becomes

$$\begin{tikzcd}
(X \ltimes B) \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "\delta"] \\
X \ltimes B \arrow[r] & B
\end{tikzcd} .$$

The arrow π_2 is splited by $\langle (0\ \delta), 1_C \rangle$ since $\pi_2 \circ \langle (0\ \delta), 1_C \rangle = 1_C$. So the object $(X \rtimes B) \times_B C \cong X \rtimes C$ because X is the kernel of π_2 and the action of C in X is described by the formula

$$((0, \delta(c)), c) + ((x, 1), 1) - (0, \delta(c), c) = ((\delta(c) \cdot x, 0), 0)$$

so

$$c \cdot x = \delta(c) \cdot x$$

and the pullback diagram becomes

$$\begin{tikzcd}
X \rtimes C \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right)"'] & C \arrow[d, "\delta"] \\
X \rtimes B \arrow[r, "(0\ 1)"] & B
\end{tikzcd}$$

this completes the proof. ■

A.3 Weak categories in Groups

We are now in the position of describing the notion of weak category in the 2-category of Groups.

Reflexive Graph

In order to define the structure of weak category in Groups we need two groups C_2 and C_1,three homomorphisms m, d, e, c, and three 2-cells α, λ, ρ.

First we introduce only the groups C_2, C_1 and the homomorphisms m, d, e, c. The analysis of the 2-cells α, λ, ρ and their conditions are postponed for the next section.

Let us call B the group C_1. The homomorphisms d, e state that C_2 is a split epi, so by proposition 115 , C_2 is isomorphic to the semidirect product group

$$X \rtimes B$$

where X is the kernel of $d : C_2 \longrightarrow B$ and the action from B in X, denoted by $b \cdot x$, is given by the formula

$$b \cdot x = e(b) + x - e(b)$$

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so that the product in $X \rtimes B$ is given by $(x,b) \rtimes (x',b') = (x + b \cdot x', bb')$.

This means that up to an isomorphism we may consider the object C_2 as the group $X \rtimes B$, with X any group with an action of B in X , denoted by $b \cdot x$.

The homomorphism $c : C_2 \longrightarrow B$ is then given by a homomorphism from $X \rtimes B \longrightarrow B$ with $d = (0\ 1)\,, e = \binom{0}{1}$ which means that $c = (\partial\ 1)$ (because $ce = 1_B$) where by proposition 111, $\partial : X \longrightarrow B$ is a homomorphism such that

$$\partial\,(b \cdot x) = b\partial\,(x)\,b^{-1}.$$

The resulting reflexive graph is

$$\begin{tikzcd}
X \rtimes B \arrow[r, shift left=2, "(0\ 1)"] \arrow[r, shift right=2, "(\partial\ 1)"'] & B. \arrow[l, "\binom{0}{1}" description]
\end{tikzcd}$$

Composition

To define the group homomorphism $m : C_2 \times_B C_2 \longrightarrow C_2$ we use the fact that C_2 is (up to an isomorphism) of the form $X \rtimes B$ and then, by proposition 116 we have

$$C_2 \times_B C_2 \cong X \rtimes (X \rtimes B)$$

with projections as in the following diagram

$$\begin{tikzcd}
X \rtimes (X \rtimes B) \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix}1&0&0\\0&\partial&1\end{smallmatrix}\right)"'] & X \rtimes B \arrow[d, "(\partial\ 1)"] \\
X \rtimes B \arrow[r, "(0\ 1)"] & B
\end{tikzcd}$$

and the action of $X \rtimes B$ in X given by the formula

$$(x,b) \cdot \bar{x} = \partial\,(x)\,b \cdot \bar{x}.$$

This means that the morphism $m : X \rtimes (X \rtimes B) \longrightarrow X \rtimes B$ is given by a 2×3 matrix

$$m = \begin{pmatrix} f & g & h \\ i & j & k \end{pmatrix}$$

where $\begin{pmatrix} g & h \\ j & k \end{pmatrix}$ is a morphism from $X \rtimes B$ to $X \rtimes B$ and $\begin{pmatrix} f \\ i \end{pmatrix}$ is a morphism from X to $X \rtimes B$. In order to satisfy $dm = d\pi_2$ and $cm = c\pi_1$

$$\begin{tikzcd}
X \rtimes B \arrow[d, "(\partial\ 1)"'] & X \rtimes (X \rtimes B) \arrow[l, "\left(\begin{smallmatrix}1&0&0\\0&\partial&1\end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)"] \arrow[d, "m"] & X \rtimes B \arrow[d, "(0\ 1)"] \\
B & X \rtimes B \arrow[l, "(\partial\ 1)"'] \arrow[r, "(0\ 1)"] & B
\end{tikzcd}$$

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we have that

$$m = \begin{pmatrix} f & g & h \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\partial f = \partial = \partial g \ , \ \partial h = 0.$$

To ensure that m is in fact a group homomorphism we also have the following conditions

$$\begin{gathered} f, g \text{ are homomorphisms} \\ h\left(bb'\right) = h\left(b\right) + b \cdot h\left(b'\right) \\ g\left(b \cdot x\right) = h\left(b\right) + b \cdot g\left(x\right) - h\left(b\right) \\ f\left(b \cdot x\right) = h\left(b\right) + b \cdot f\left(x\right) - h\left(b\right) \\ f\left(\partial\left(x\right) b \cdot \bar{x}\right) = g\left(x\right) + f\left(\bar{x}\right) - g\left(x\right). \end{gathered}$$

Conclusion 117 *A weak category in the 2-category of Groups without identity and associative 2-cells is described by a diagram in Groups of the form*

$$X \xrightarrow{f,g} X \xrightarrow{\partial} B \xrightarrow{h} X$$

with an action of B in X (denoted by $b \cdot x$) satisfying the following conditions

$$\begin{array}{rcl} & & f, g, \partial \textit{ are homomorphisms} \\ h\left(bb'\right) & = & h\left(b\right) + b \cdot h\left(b'\right) \\ \partial f & = & \partial = \partial g \ , \ \partial h = 0 \\ \partial\left(b \cdot x\right) & = & b\partial\left(x\right) b^{-1} \\ g\left(b \cdot x\right) & = & h\left(b\right) + b \cdot g\left(x\right) - h\left(b\right) \\ f\left(b \cdot x\right) & = & h\left(b\right) + b \cdot f\left(x\right) - h\left(b\right) \\ f\left(\partial\left(x\right) \cdot \bar{x}\right) & = & g\left(x\right) + f\left(\bar{x}\right) - g\left(x\right). \end{array}$$

Where the objects are the elements of B, the arrows are pairs $(x, b) : b \longrightarrow \partial x + b$ and the composition of $(x', \partial x + b) : \partial x + b \longrightarrow \partial x' + \partial x + b$ with $(x, b) : b \longrightarrow \partial x + b$ is $(fx' + gx + hb, b) : b \longrightarrow \partial x' + \partial x + b$.

Identity Isomorphisms

Next we will see what does it mean the inclusion of the identity isomorphisms.

For the isomorphism λ we have the following sequence for the composition $m \langle ec, 1 \rangle$

$$(x, b) \longmapsto (0, x, b) \longmapsto \left(g\left(x\right) + h\left(b\right), b\right)$$

which means that $\lambda : \begin{pmatrix} g & h \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is determined by an element in the group $X \rtimes B$ say (λ_1, λ_0) such that for every $(x, b) \in X \rtimes B$,

$$(x, b) = (\lambda_1, \lambda_0) + \left(g\left(x\right) + h\left(b\right), b\right) - (\lambda_1, \lambda_0).$$

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With respect to the isomorphism ρ the composition $m\left\langle 1,ed\right\rangle$ is obtain by the following sequence

$$(x,b)\longmapsto(x,0,b)\longmapsto\left(f\left(x\right)+h\left(b\right),b\right)$$

which means that $\rho:\left(\begin{array}{cc} f & h\\ 0 & 1\end{array}\right)\longrightarrow\left(\begin{array}{cc} 1 & 0\\ 0 & 1\end{array}\right)$ is determined by an element in the group $X\rtimes B$ say (ρ_1,ρ_0) such that for every $(x,b)\in X\rtimes B$,

$$(x,b)=(\rho_1,\rho_0)+\left(f\left(x\right)+h\left(b\right),b\right)-(\rho_1,\rho_0)\,.$$

However, we have to satisfy the condition

$$d\circ\lambda=1_d$$

which means that $\lambda_0=0$, and we also have to satisfy the condition $c\circ\lambda=1_c$ which translates as $\partial\left(\lambda_1\right)=0$. The same is true for ρ so we have

$$\rho_0=0\ ,\ \partial\left(\rho_1\right)=0.$$

With the above conditions we may conclude that f,g,h are completely determined in terms of λ and ρ, in fact we have

$$g\left(x\right)+h\left(b\right)=-\lambda_1+x+b\cdot\lambda_1$$

$$f\left(x\right)+h\left(b\right)=-\rho_1+x+b\cdot\rho_1$$

and in particular (for $(x,1)$ and $(0,b)$)

$$g\left(x\right)=-\lambda_1+x+\lambda_1$$

$$f\left(x\right)=-\rho_1+x+\rho_1$$

$$\begin{aligned} h\left(b\right) &= -\lambda_1+b\cdot\lambda_1\\ &= -\rho_1+b\cdot\rho_1. \end{aligned}$$

If we investigate conditions for f,g,h in the previous section then the conclusion is that all are satisfied except

$$f\left(\partial\left(x\right)\cdot\bar{x}\right)=g\left(x\right)+f\left(\bar{x}\right)-g\left(x\right)$$

which turns out to be

$$\partial\left(x\right)\cdot\bar{x}=\rho_1-\lambda_1+x+\lambda_1-\rho_1+\bar{x}+\rho_1-\lambda_1-x+\lambda_1-\rho_1$$

so, in particular for $x=\lambda_1$ it means that

$$\bar{x}=\rho_1-\lambda_1+\lambda_1+\lambda_1-\rho_1+\bar{x}+\rho_1-\lambda_1-\lambda_1+\lambda_1-\rho_1$$

that is

$$\bar{x}=\rho_1+\lambda_1-\rho_1+\bar{x}+\rho_1-\lambda_1-\rho_1,$$

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considering $\bar{x} = \rho_1 + y - \rho_1$ we have the following result

$$y = \lambda_1 + y - \lambda_1$$

for every $y \in X$.

This result also means that

$$\partial(x) \cdot \bar{x} = \rho_1 + x - \rho_1 + \bar{x} + \rho_1 - x - \rho_1$$

but considering now the particular case of $x = \rho_1$ it becomes

$$\bar{x} = \rho_1 + \bar{x} - \rho_1.$$

The final conclusion may be stated as follows.

Conclusion 118 *A weak category in the 2-category of Groups with left and right identity 2-cells (without coherence conditions) is described by a group homomorphism*

$$X \xrightarrow{\partial} B$$

with an action of B in X (denoted by $b \cdot x$) and two elements in X, λ and ρ satisfying the following conditions

$$\begin{aligned} h(b) &= -\lambda + b \cdot \lambda = -\rho + b \cdot \rho \\ \partial\lambda &= 0 = \partial\rho \\ x &= \lambda + x - \lambda \\ x &= \rho + x - \rho \\ \partial(b \cdot x) &= b\partial(x)\, b^{-1} \\ \partial(x) \cdot \bar{x} &= x + \bar{x} - x. \end{aligned}$$

Where the objects are the elements of B, the arrows are pairs $(x, b) : b \longrightarrow \partial x + b$ and the composition of $(x', \partial x + b) : \partial x + b \longrightarrow \partial x' + \partial x + b$ with $(x, b) : b \longrightarrow \partial x + b$ is $(x' + x + h(b), b) : b \longrightarrow \partial x' + \partial x + b$. The isomorphism between $(0, \partial x + b) \circ (x, b)$ and (x, b) is the element $(\rho, 0) \in X \rtimes B$ while the isomorphism between $(x, b) \circ (0, b)$ and (x, b) is the element $(\lambda, 0) \in X \rtimes B$.

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Associativity Isomorphism

In order to define the 2-cell α for associativity we have to describe the homomorphisms $m(1 \times_{C_1} m)$ and $m(m \times_{C_1} 1)$. For $m(1 \times_{C_1} m)$ we have the following sequence

$$(x'', x', x, b) \mapsto (x'', x' + x + hb, b) \mapsto (x'' + x' + x + hb + hb, b)$$

while for $m(m \times_{C_1} 1)$ we have

$$(x'', x', x, b) \mapsto (x'' + x' + h(\partial(x)\, b), x, b) \mapsto (x'' + x' + h(\partial(x)\, b) + x + hb, b).$$

So, by definition of 2-cell $\alpha = (\alpha_1, \alpha_0) \in X \rtimes B$ is such that for every $(x, b) \in X \rtimes B$ we have

$$(x'' + x' + h(\partial(x)\,b) + x + hb, b) = \alpha + (x'' + x' + x + hb + hb, b) - \alpha. \tag{A.2}$$

However, α must satisfy $d \circ \alpha = 1_{d\pi_3}$ and $c \circ \alpha = 1_{c\pi_1}$, and hence

$$\alpha_0 = 0\ ,\ \partial(\alpha_1) = 0.$$

With respect to the coherence conditions we have the following

$$\alpha \circ (1 \times_{C_!} \langle ec, 1\rangle) = \alpha = (\alpha_1, 0)$$

$$m \circ (\rho \times_{C_1} 1) = m(\rho_1, 0, 0) = (f(\rho_1), 0) = (\rho_1, 0)$$
$$m \circ (1 \times_{C_1} \lambda) = m(0, \lambda_1, 0) = (g(\lambda_1), 0) = (\lambda_1, 0),$$

in general $\alpha \circ 1_f = \alpha$ and

$$m \circ (\alpha \times_{C_1} 1) = m(\alpha_1, 0, 0) = (f(\alpha_1), 0) = (-\rho_1 + \alpha_1 + \rho_1, 0)$$

$$m \circ (1 \times_{C_1} \alpha) = m(0, \alpha_1, 0) = (g(\alpha_1), 0) = (-\lambda_1 + \alpha_1 + \lambda_1, 0).$$

This means that the coherence pentagon states that

$$2\alpha_1 = -\rho_1 + \alpha_1 + \rho_1 + \alpha_1 - \lambda_1 + \alpha_1 + \lambda_1 \tag{A.3}$$

or equivalently (since λ_1 and ρ_1 are in the center of X)

$$2\alpha_1 = \alpha_1 + \alpha_1 + \alpha_1 \Leftrightarrow \alpha_1 = 0$$

and the coherence for identity states that

$$\rho_1 + \alpha_1 = \lambda_1.$$

We may describe α_1 in terms of λ_1 and ρ_1 as

$$\alpha_1 = \lambda_1 - \rho_1 \Leftrightarrow \lambda_1 = \rho_1.$$

The final conclusion is thus stated as follows.

Conclusion 119 *A weak category in the 2-category of Groups is described by a group homomorphism*

$$X \xrightarrow{\partial} B,$$

an action of B in X (denoted by $b \cdot x$) and a distinguished element in X, δ satisfying the following conditions

$$\begin{aligned} \partial\delta &= 0 \\ x &= \delta + x - \delta \\ \partial(b \cdot x) &= b\partial(x)\,b^{-1} \\ \partial(x) \cdot \bar{x} &= x + \bar{x} - x. \end{aligned}$$

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Where the objects are the elements of B*, the arrows are pairs* $(x,b):b\longrightarrow\partial x+b$ *and the composition of* $(x',\partial x+b):\partial x+b\longrightarrow\partial x'+\partial x+b$ *with* $(x,b):b\longrightarrow\partial x+b$ *is* $(x'+x-\lambda+b\cdot\lambda,b):b\longrightarrow\partial x'+\partial x+b$*. The isomorphism between* $(0,\partial x+b)\circ(x,b)=(x,b)\circ(0,b)$ *and* (x,b) *is the element* $(\delta,0)\in X\rtimes B$*. Associativity is satisfied, since* $(x'',\partial x'+\partial x+b)\circ((x',\partial x+b)\circ(x,b))=((x'',\partial x'+\partial x+b)\circ(x',\partial x+b))\circ(x,b)$*.*

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