Coverings and Descent Theory of Finite Spaces

ABSTRACT. This thesis presents the categorical Galois theory of the reflection of the category of finite topological spaces into the category of discrete finite topological spaces. This turns out to be nothing but the equivalence between the category of coverings of a connected finite topological space and the actions of the fundamental group of that space. Since some descent theory is necessary for categorical Galois theory, this thesis also contains an account of some of the descent theory of finite topological spaces. The reader is assumed to know the basics of category theory, but no descent theory or categorical Galois theory, or even internal category theory, but to be somewhat familiar with coverings and fundamental groups, and the notion of "locally" for topological spaces.

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Introduction

In Chapter 6 of [**GalTheo**] Janelidze shows how the classical theory of covering maps of locally connected topological spaces can be described as the categorical Galois theory of the reflection of the category of locally connected topological spaces into the category of sets. The purpose of this thesis is to do much the same thing for an even nicer (for the purposes of categorical Galois theory) subcategory of the category of topological spaces, the category of finite topological spaces **FinTop**. In doing so, the author hoped to gain a better understanding of both some portions of categorical Galois theory and the descent theory used therein.

0.1. Descent theory

Janelidze and Sobral did a similar thing in investigating descent theory for finite topological spaces in their paper [**FinPreI**], from which I take primarily the identification of the category of finite topological spaces with the category of finite preorders, but also the description of étale maps of finite topological spaces as discrete fibrations of finite preorders. However, neither from that paper, nor from Chapter 6 of [**GalTheo**] was I able to satisfy myself that I understood how, or why, descent theory can be thought of as giving a category theory account of the classical notion of "locally in $\mathbb{M}$", which is an idea the author first encountered in Carboni, Janelidze, Kelly and Paré's paper [**LocStab**].[1]

So that is one thing this thesis tries to accomplish, in the simple setting of the category of finite topological spaces. In sections 4.1 and 4.2 of this thesis, I was able to give, at least to my own satisfaction, an account of how the beginnings of descent theory might be discovered from, or at least motivated by, the classical notion of "locally in $\mathbb{M}$" for continuous maps of finite topological spaces, where $\mathbb{M}$ is (almost)

[1] Of course, having written this thesis I now better see how to think about statements in those papers that went over my head, especially in [**FinPreI**].

any subclass of continuous maps. What makes this particularly easy to do is that every point a in a finite topological space A has a smallest open set $\downarrow a$ containing it, unlike in most topological spaces, where arbitrary intersections of open sets aren't open, because they can involve infinitely many sets. Thus for finite topological spaces, a map $f : A \to B$ is locally in $\mathbb{M}$ if and only if for each $b \in B$ its restriction to $f^{-1}(\downarrow b) \to \downarrow b$ is in $\mathbb{M}$. And this restriction is simply the pullback of $f : A \to B$ along the inclusion $\downarrow b \hookrightarrow B$. Then I initially took what I think is the most obvious route: to say that each such pullback is in $\mathbb{M}$ is simply to say that the single pullback of $f : A \to B$ along the obvious projection $\coprod_B \downarrow b \to \downarrow b$ is in $\Sigma\mathbb{M}$.[2]

This was satisfactory to the author for two reasons.

Firstly, one can easily verify that this $\coprod_B \downarrow b \to \downarrow b$ is an effective descent morphism in **FinTop** according to the characterization given in [**FinPreI**], which is equivalent to: $p : E \to B$ in **FinTop** is of effective descent if and only if for each $b \in B$ and $b' \in \downarrow b$ and $b'' \in \downarrow b'$, there's $e' \in \downarrow e$ and $e'' \in \downarrow e'$ in E such that $pe = b$, $pe' = b'$ and $pe'' = b''$.

Secondly, I began to understand what I had not from my study of Chapter 6 of [**GalTheo**]: why it needed to introduce the class $\Sigma\mathbb{M}$, which the paper [**LocStab**] had made no mention of.

There was still a big mystery though: whilst one can say that a map is in $\mathbb{M}^*$ (and understand this to mean "locally in $\mathbb{M}$") if and only if its pullback along some effective descent morphism $p : E \to B$ is in $\Sigma\mathbb{M}$, descent theory does not thereby establish an equivalence between maps in $\mathbb{M}^*/B$ and $(\Sigma\mathbb{M})/E$! But now that I was thinking about the specific effective descent morphism $\coprod_B \downarrow b \to \downarrow b$ it became clear that the conditions one needs to place upon a map in $\Sigma\mathbb{M}$ so that p can "collapse" it into a map in $\mathbb{M}^*$, classically expressed as requiring that the maps in $\mathbb{M}$ that make up the map in $\Sigma\mathbb{M}$ agree on their intersections, can be captured by the condition that there be equivalence relations on both the domain and codomain of the map in $\Sigma\mathbb{M}$ that are compatible with each other (and the topologies) in such a way that all this data can be used to construct the glued together map in $\mathbb{M}^*$ as the unique fill-in the map in $\Sigma\mathbb{M}$ induces between the quotients of the two equivalence

[2] $\Sigma\mathbb{M}$ is all maps isomorphic to a map of the form $\coprod_X f_x : \coprod_X A_x \to \coprod_X B_x$ with each $f_x : A_x \to B_x$ in $\mathbb{M}$.

relations. An effective descent morphism then, is a morphism such that composing the two processes (pulling back and inducing with quotients) in either order always yields isomorphic results.

0.2. Internal category theory

I mentioned that the equivalence relations on the domain and codomain of the map in $\Sigma\mathbb{M}$ need to be compatible with the topologies. This is, of course, best expressed by simply requiring that they be internal equivalence relations in the category of finite topological spaces. So this requires a little internal category theory; more than I think is described explicitly in even the additional chapters (XI and XII) Mac Lane added to the second edition of [**CatWork**], but nothing I didn't already know from my previous studies of Barr exactness, in particular of the Barr exactness of toposes.

But the compatibility of the equivalence relations with *each other* is best expressed by requiring, roughly, that they, with the map in $g : C \to E$ in $\Sigma\mathbb{M}$, form a kind of internal action of $\bar{B}$ (the internal equivalence relation formed by the kernel pair of $p : E \to B$) on C. Such objects are described, briefly, at the end of XII.1 in [**CatWork**], where they are called "left $\bar{B}$-objects" or "internal base-valued functors" or an object C and an "action map" with codomain C. But I found the account in 4.1-4.3 of them in [**LocStab**] of them as discrete opfibrations of precategories to be more illuminating, once I had worked out for myself some of the details there elided. Section 1.1-1.6 of Chapter 1 of this thesis therefore contain these details, in order to show a hypothetical reader who only knows the Chapters I-X of [**CatWork**] enough about internal category theory to describe the category of actions $\mathbb{M}^{\bar{B}}$, each object of which is collapsed by the quotienting (or now, we may say, more generally, coequalizing) process sketched above into a unique map in $\mathbb{M}^{*}$. The final section of Chapter 1 then gives an account of effective descent morphisms that avoids the use of monads, but is rather very direct, so that it can easily be seen to agree with the development of the concept in the concrete instance of Chapter 4.

However I do not refer to $\Sigma\mathbb{M}$ in 1.7, nor do I actually use the effective descent morphism $\coprod_B \downarrow b \to \downarrow b$ in 4.1 or 4.2., despite it having been helpful in my process of coming to understand some descent theory. This is because of the detail that makes the category of finite topological spaces a setting for such a simple categorical Galois theory: the existence, for every connected finite space, of a universal covering, with connected domain.

0.3. Universal coverings

The beginnings of my study of the topic of this thesis actually began when my supervisor, Professor Janelidze, suggested I look for links between the categorical Galois theory of finite topological spaces and Paré's paper [**UCC**]. In that paper Paré constructs for, any connected category B, a functor he calls the universal covering of that category. It is the comma category projection $(b, \eta_B^\pi) \to B$, where the functor $\eta_B^\pi : B \to \pi B$ is the unit of the reflection $\pi : \mathbf{Cat} \to \mathbf{Gpd}$ that gives what Paré calls the "fundamental groupoid" of B, because its morphisms are equivalence classes of paths in B, and $b : 1 \to \pi B$ is a functor that selects any point in πB.

In 3.1 and 3.2 I work out the details of the equivalence **FinTop** $\sim$ **FinPreord** of the category of finite topological spaces with the category of finite preorders. In short, a finite space has a preorder defined by "$x \to y$ exactly when $x \in\downarrow y$", and a finite preorder has a topology with basic open sets $\{x \in A | x \to y\}$. Then it is straightforward to show, as I do in 3.2, that a continuous map of finite spaces is the same as a monotonic ($fx \to fy$ if $x \to y$) map of finite preorders. But, as a preorder can be though of as a category with each hom-set either $\cong 1$ or empty, we can think of **FinPreord** as a full subcategory of **Cat**. Therefore Paré's universal covering category construction can be applied to finite topological spaces (just as long as we show that the category (b, η_B^π) it produces is in fact a preorder whenever B is a preorder). This I do, in detail, in 3.4-3.6, also showing that the universal covering so constructed is indeed a covering in the classical sense (locally a trivial covering, i.e. locally a projection of a number of copies of open sets onto an open set) and also showing in 3.6 that coverings of finite spaces are precisely the maps that, as maps of preorders, are both discrete fibrations and discrete opfibrations. This insight is due to Clementino, Hofmann and Montoli's paper [**CMiCoRA**], where they give a similar result for a broad class of categories that does include **FinPreord** as a particularly simple instance.

Since, as mentioned above, a discrete fibration of finite preorders is the same thing as an étale map of finite spaces, when dealing with coverings it is often convenient to work in **Etale**, the subcategory of **FinTop** with the same objects, but only étale maps as morphisms. In 4.3, after showing that every surjective étale map in **FinTop** is an effective descent morphism, I show that the effective descent morphisms in **Etale** are precisely the surjectives.

0.4. Categorical Galois theory

But the importance of Paré's $(b, \eta_B^\pi) \to B$ to this thesis is that it is a universal covering in the categorical Galois theory sense. Let me explain what that means by describing my Chapter 2, on some of the abstract categorical Galois theory of an admissible reflection $I : \mathbf{C} \to \mathbf{X}$, and my Chapter 5, on the categorical Galois theory of the reflection $I : \mathbf{FinTop} \to \mathbf{FinSet}$ that sends each finite topological space to its set of connected components.

In categorical Galois theory, a reflection $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ is said to be admissible when for each $E \in \mathbf{C}$ the obvious functor $I^E : \mathbf{C}/E \to \mathbf{X}/IE$ that $I : \mathbf{C} \to \mathbf{X}$ induces has a fully faithful right adjoint H^E. In 2.1 I show that a right adjoint for I^E *must* be defined as $H^E(X, u) = (E \times_{(\eta_E, u)} X, \pi_1)$. I then show that H^E is fully faithful exactly when I always takes the other pullback projection $\pi_2 : E \times_{(\eta_E, u)} X \to X$ to an isomorphism. In 5.1 I use this to show that the connected component reflection $I : \mathbf{FinTop} \to \mathbf{FinSet}$ is admissible, because $H^E(X, u) = (\coprod_X ux, \pi_1)$, so that $I\pi_2 : I(\coprod_X ux) \cong IX$. We can think of $u : X \to IE$ as selecting components of E, and then $\pi_1 : \coprod_X ux \to E$ covers (some of) E with a disjoint union of those components. If E is connected, the replete image of H^E is thus precisely the trivial coverings of E, as classically defined, and we have $\mathbb{M}/E \sim \mathbf{FinSet}/IE \sim \mathbf{FinSet}$ (the last equivalence because for connected E, $IE \cong 1$).

In the abstract, in 2.1, *defining* $\mathbb{M}/E$ as the replete image of H^E we also have $\mathbb{M}/E \sim \mathbf{X}/IE$. In 2.3 I give the details of how this equivalence extends to an equivalence $\mathbb{M}^{\bar{B}} \sim \mathbf{X}^{I\bar{B}}$. Since, from descent theory, (A, f) is in $\mathbb{M}^*/B \subset \mathbf{C}/B$ precisely when there's some effective descent morphism $p : E \to B$ such that it is in the image $\mathrm{Spl}(E, p)$ of $p^\# : \mathbb{M}^{\bar{B}} \to \mathbf{C}/B$, we could here give the fundamental theorem of categorical Galois theory in its general form: $\mathrm{Spl}(E, p) \sim \mathbf{X}^{I\bar{B}}$.

Instead, in 2.2 I simplify the abstract categorical Galois theory by assuming the existence of a universal covering for each B with $IB \cong 1$: a single effective descent morphism $p : E \to B$, with $\mathbb{M}^*/B = \mathrm{Spl}(E, p)$ (and moreover, with $IE \cong 1$). This large assumption is justified because it does hold for $I : \mathbf{FinTop} \to \mathbf{FinSet}$: Paré's universal covering construction is a universal covering in this categorical Galois theory sense. Thus in 2.4, assuming the existence of these universal coverings, I prove a simplified form of the fundamental theorem: $\mathbb{M}^*/B \sim \mathbf{X}^{I\bar{B}}$. I think of this as really being $\mathbb{M}^*/B \sim \mathbb{M}^{\bar{B}} \sim \mathbf{X}^{I\bar{B}}$, where $p^\# : \mathbb{M}^*/B \sim \mathbb{M}^{\bar{B}}$, which works because $p : E \to B$ is a universal covering, and does not need $\Sigma\mathbb{M}$, because $IE \cong 1$.[3] $I\bar{B}$ is called the Galois (pre)group(oid) and denoted $\mathrm{Gal}(E, p)$.

In 5.3 I use the theory of Chapter 2, to show, first, that for a connected finite topological space E, $\mathrm{Gal}(E, p) \cong \mathrm{Aut}(E, p)$, the group of automorphisms $g : E \to E$ with $pg = g$, also known as the Chevalley fundamental group of B. Then specifically using the universal covering $(b, \pi B) \to B$ I show that also $\mathrm{Gal}(E, p) \cong \pi_1(B, b)$, the Poincaré group of B, consisting of equivalence classes of looping paths around and between the various "holes" in the space B. This can be considered a key result of this thesis: that in **FinTop** the Chevalley fundamental group, Poincaré fundamental group and Galois group coincide.

Finally, in 5.4 I describe concretely what the fundamental theorem $\mathbb{M}^*/B \sim \mathbf{FinSet}^{\mathrm{Gal}(E,p)}$ means when $\mathbb{M}^*$ is the coverings in **FinTop**: every covering $f : A \to B$ of a connected finite space B can is constructed via the "quotienting" process described earlier, from an ordinary group action of $\mathrm{Gal}(E, p)$ on the set IA of the components of A.

0.5. Some reflections

Writing now at the end of this process, I can see that there's much in this thesis that is more general than it needs to be. I can see it might have been nicer to spend less time on internal category theory, descent

[3]So this is why I do not use $\Sigma\mathbb{M}$ in this thesis.

theory and abstract categorical Galois theory, and thus have more time to spend on the details of the categorical Galois theory of finite spaces. For instance, I believe the literature on finite spaces is relatively slender, as they are often regarded as boring, so I could have spent some time comparing my results, and the categorical Galois theory way of dealing with coverings of finite spaces, to results and methods in that literature. I would have also liked to find deeper connections between Paré's paper [**UCC**] and categorical Galois theory. But I am assured that these kinds of regrets are just part of doing a Masters thesis, and thereby learning enough that when you're done with it, it seems to be done in entirely the wrong way!

CHAPTER 1

From Internal Category Theory to Effective Descent Morphisms

1.1. Internal precategories

Definition 1. An *internal precategory* P in a category $\mathbf{C}$ is a functor $P : \mathbb{P} \to \mathbf{C}$ where $\mathbb{P}^{\mathrm{op}}$ is the following subcategory of the simplicial category Δ:

$$\begin{tikzcd} 1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & 2 \arrow[l, "\sigma_0" description] \arrow[r, shift left=3, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=3, "\delta_2"'] & 3 \end{tikzcd}$$

Using the notation

$$n + 1 \mapsto P_n$$

$$\delta_n \mapsto d_n$$

$$\sigma_n \mapsto s_n$$

for $P : \mathbb{P} \to \mathbf{C}$, an internal precategory P in $\mathbf{C}$ can thus be thought of as consisting of

$$P = (P_0, P_1, P_2, e, d, c, q, m, r)$$

with

(1) an object P_0 in $\mathbf{C}$ called the *object-of-objects*
(2) an object P_1 in $\mathbf{C}$ called the *object-of-arrows*
(3) an object P_3 in $\mathbf{C}$ that I will call the *object-of-composable-pairs*
(4) a morphism $m = d_1 : P_2 \to P_1$ in $\mathbf{C}$ called *composition*
(5) a morphism $e = s_0 : P_0 \to P_1$ in $\mathbf{C}$ called *identity*
(6) a morphism $d = d_0 : P_1 \to P_0$ in $\mathbf{C}$ called *domain*
(7) a morphism $c = d_1 : P_1 \to P_0$ in $\mathbf{C}$ called *codomain*
(8) a morphism $q = d_0 : P_2 \to P_1$ in $\mathbf{C}$ I will call *first-arrow*
(9) a morphism $r = d_2 : P_2 \to P_1$ in $\mathbf{C}$ I will call *second-arrow*

such that the following diagrams in $\mathbf{C}$ commute

$$\begin{tikzcd} P_2 \arrow[r, "r"] \arrow[d, "q"'] & P_1 \arrow[d, "d"] \\ P_1 \arrow[r, "c"'] & P_0 \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "m"] \arrow[d, "q"'] & P_1 \arrow[d, "d"] \\ P_1 \arrow[r, "d"'] & P_0 \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "r"] \arrow[d, "m"'] & P_1 \arrow[d, "c"] \\ P_1 \arrow[r, "c"'] & P_0 \end{tikzcd}$$

$$\begin{tikzcd} P_0 \arrow[r, "e"] \arrow[dr, "1"'] & P_1 \arrow[d, "d"] \\ & P_0 \end{tikzcd}$$

$$\begin{tikzcd} P_0 \arrow[r, "e"] \arrow[dr, "1"'] & P_1 \arrow[d, "c"] \\ & P_0 \end{tikzcd}$$

Example 2. Each simplicial object $S : (\Delta^+)^{\mathrm{op}} \to \mathbf{C}$ defines an internal precategory

$$(S_0, S_1, S_2, s_0 : S_0 \to S_1, d_0, d_1 : S_1 \to S_0, d_0, d_1, d_2 : S_2 \to S_1)$$

in $\mathbf{C}$, though clearly, because $\mathbb{P}$ is a proper subcategory of $(\Delta^+)^{\mathrm{op}}$, not every internal precategory is a simplicial object. In particular, a simplicial set $S : (\Delta^+)^{\mathrm{op}} \to \mathbf{Set}$ defines an internal precategory in $\mathbf{Set}$. Of course, an internal precategory in $\mathbf{Set}$ (i.e. a functor $P : \mathbb{P} \to \mathbf{Set}$) is simply called a precategory.

EXAMPLE 3. If, for a morphism $p : E \to B$ in $\mathbf{C}$ the pullbacks

$$\begin{tikzcd}
E \times_B E \times_B E \arrow[r, "r"] \arrow[d, "q"'] & E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\
E \times_B E \arrow[r, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

exist, then $\bar{B} = (E, E \times_B E, E \times_B E \times_B E, e, d, c, q, m, r)$ is an internal precategory in $\mathbf{C}$, where $e : E \to E \times_B E$ is the unique fill-in in

$$\begin{tikzcd}
E \arrow[dr, dashed, "e"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\
& E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

and $m : E \times_B E \times_B E \to E \times_B E$ is the unique fill-in in

$$\begin{tikzcd}
E \times_B E \times_B E \arrow[dr, dashed, "m"] \arrow[drr, bend left, "cr"] \arrow[ddr, bend right, "dq"'] & & \\
& E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

since all the commuting diagrams for an internal precategory in $\mathbf{C}$ appear in the above diagrams.

DEFINITION 4. A *morphism* $f : P \to P'$ *of internal precategories* in $\mathbf{C}$ is a natural transformation $f : P \dot{\to} P' : \mathbb{P} \to \mathbf{C}$.

Since there are only three objects in $\mathbb{P}$, the natural transformation $f : P \dot{\to} P' = (P'_0, P'_1, P'_2, e', d', c', q', m', r') : \mathbb{P} \to \mathbf{C}$ consists of just three morphisms

- $f_0 : P_0 \to P'_0$
- $f_1 : P_1 \to P'_1$
- $f_2 : P_2 \to P'_2$

in **C** such that the following diagrams commute

$$\begin{tikzcd} P_1 \arrow[r, "d"] \arrow[d, "f_1"'] & P_0 \arrow[d, "f_0"] \\ P_1' \arrow[r, "d'"'] & P_0' \end{tikzcd}$$

$$\begin{tikzcd} P_1 \arrow[r, "c"] \arrow[d, "f_1"'] & P_0 \arrow[d, "f_0"] \\ P_1' \arrow[r, "c'"'] & P_0' \end{tikzcd}$$

$$\begin{tikzcd} P_0 \arrow[r, "e"] \arrow[d, "f_0"'] & P_1 \arrow[d, "f_1"] \\ P_0' \arrow[r, "e'"'] & P_1' \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "m"] \arrow[d, "f_2"'] & P_1 \arrow[d, "f_1"] \\ P_2' \arrow[r, "m'"'] & P_1' \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "q"] \arrow[d, "f_2"'] & P_1 \arrow[d, "f_1"] \\ P_2' \arrow[r, "q'"'] & P_1' \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "r"] \arrow[d, "f_2"'] & P_1 \arrow[d, "f_1"] \\ P_2' \arrow[r, "r'"'] & P_1' \end{tikzcd}$$

DEFINITION 5. The internal precategories in **C** and their morphisms form the functor category $\mathbf{C}^{\mathbb{P}}$, called the *category of internal precategories in* **C**.

THEOREM 6. *Any functor* $I : \mathbf{C} \to \mathbf{X}$ *induces a functor* $I : \mathbf{C}^{\mathbb{P}} \to \mathbf{X}^{\mathbb{P}}$ *that carries an internal precategory*

$$P = (P_0, P_1, P_2, e, d, c, q, m, r)$$

in $\mathbf{C}$ *to the internal precategory*

$$IP = (IP_0, IP_1, IP_2, Ie, Id, Ic, Iq, Im, Ir)$$

in $\mathbf{X}$*, and a morphism*

$$f : P \to P' \text{ with } f_0 : P_0 \to P_0', f_1 : P_1 \to P_1', f_2 : P_2 \to P_2'$$

of internal precategories to a morphism

$$If : IP \to IP' \text{ with } If_0 : IP_0 \to IP_0', If_1 : IP_1 \to IP_1', If_2 : IP_2 \to IP_2'$$

of internal precategories.

Proof. IP is just the composition of $P : \mathbf{Q} \to \mathbf{C}$ with $I : \mathbf{C} \to \mathbf{X}$, and thus a functor in $\mathbf{X}^{\mathbb{P}}$, and If as given above is obviously a natural transformation $IP \to IP'$. □

My understanding is that this theorem is the motivation for the definition of an internal precategory. An internal precategory is everything about an internal category that is preserved by any functor, but no more.[1]

1.2. Internal categories

Here's the extra structure that an (internal) category possesses over an (internal) precategory.

Definition 7. An *internal category* $C = (C_0, C_1, C_2, m, e, d, c, q, r)$ in a category $\mathbf{C}$ is an internal precategory C in $\mathbf{C}$ such that

$$\begin{tikzcd} C_2 \arrow[r, "r"] \arrow[d, "q"'] & C_1 \arrow[d, "d"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

[1]Another candidate for such a concept would be a simplicial object, since every category is a simplicial set, and a functor $I : \mathbf{C} \to \mathbf{X}$ takes a simplicial object $S : (\Delta^+)^{\mathrm{op}} \to \mathbf{C}$ to a simplicial object $IS : (\Delta^+)^{\mathrm{op}} \to \mathbf{X}$ But I suppose there may be *internal* categories that are not simplicial objects, due to not having the iterated pullbacks $C_3, C_4, \ldots$. And perhaps it's easier to work with $\mathbb{P}$ than the whole of $(\Delta^+)^{\mathrm{op}}$.

is a pullback in $\mathbf{C}$, the following pullback in $\mathbf{C}$ exists

$$\begin{tikzcd}
C_3 \arrow[r, "t"] \arrow[d, "s"'] & C_2 \arrow[d, "q"] \\
C_2 \arrow[r, "r"'] & C_1
\end{tikzcd}$$

and the following diagrams in $\mathbf{C}$ commute

$$\begin{tikzcd}
C_1 \arrow[r, "{(1,ec)}"] \arrow[dr, "1"'] & C_2 \arrow[d, "m"] \\
& C_1
\end{tikzcd}$$

$$\begin{tikzcd}
C_1 \arrow[r, "{(ed,1)}"] \arrow[dr, "1"'] & C_2 \arrow[d, "m"] \\
& C_1
\end{tikzcd}$$

$$\begin{tikzcd}
C_3 \arrow[r, "{(qs,mt)}"] \arrow[d, "{(ms,rt)}"'] & C_2 \arrow[d, "m"] \\
C_2 \arrow[r, "m"'] & C_1
\end{tikzcd}$$

and a *morphism* $f : C \to C'$ *of internal categories* in $\mathbf{C}$ is just a morphism $f : C \to C'$ of precategories in $\mathbf{C}$.

The full subcategory of $\mathbf{C}^{\mathbb{P}}$ with each object an internal category is denoted $\mathrm{Cat}(\mathbf{C})$ and called the *category of internal categories in* $\mathbf{C}$.

The diagrams that induce the fill-ins used in this definition are:

$$\begin{tikzcd}
C_1 \arrow[dr, "{(1,ec)}"] \arrow[drr, bend left, "ec"] \arrow[ddr, bend right, "1"'] & & \\
& C_2 \arrow[r, "r"] \arrow[d, "q"'] & C_1 \arrow[d, "d"] \\
& C_1 \arrow[r, "c"'] & C_0
\end{tikzcd}$$

which works because $dec = 1c = c1$,

$$\begin{tikzcd}
C_1 \arrow[dr, dashed, "{(ed,1)}"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "ed"'] & & \\
& C_2 \arrow[r, "r"] \arrow[d, "q"'] & C_1 \arrow[d, "d"] \\
& C_1 \arrow[r, "c"'] & C_0
\end{tikzcd}$$

which works because $ced = 1d = d1$,

$$\begin{tikzcd}
C_3 \arrow[dr, dashed, "{(ms,rt)}"] \arrow[drr, bend left, "rt"] \arrow[ddr, bend right, "ms"'] & & \\
& C_2 \arrow[r, "r"] \arrow[d, "q"'] & C_1 \arrow[d, "d"] \\
& C_1 \arrow[r, "c"'] & C_0
\end{tikzcd}$$

which works because $cms = crs = cqt = drt$,

$$\begin{tikzcd}
C_3 \arrow[dr, dashed, "{(qs,mt)}"] \arrow[drr, bend left, "mt"] \arrow[ddr, bend right, "qs"'] & & \\
& C_2 \arrow[r, "r"] \arrow[d, "q"'] & C_1 \arrow[d, "d"] \\
& C_1 \arrow[r, "c"'] & C_0
\end{tikzcd}$$

which works because $dmt = dqt = drs = cqs$.

Up to an isomorphism, the data for an internal category C can be limited to (C_0, C_1, m, e, d, c) since then C_2, q and r can be formed as a pullback of d, c. And the data for a morphism $f : C \to C'$ of internal categories can be limited to $f_0 : C_0 \to C'_0$ and $f_1 : C_1 \to C'_1$ since then $f_2 : C_2 \to C'_2$ is induced as the unique-fill in to the pullback $q, r : C_2 \to C_1$.

Example 8. Cat(**Set**) $\sim$ **Cat**, which is one of the justifications for the definition of internal categories.

EXAMPLE 9. If, for a morphism $p : E \to B$ in $\mathbf{C}$ the following pullbacks in $\mathbf{C}$ exist

$$\begin{tikzcd}
E \times_B E \times_B E \times_B E \arrow[r, "t"] \arrow[d, "s"'] & E \times_B E \times_B E \arrow[r, "r"] \arrow[d, "q"'] & E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\
E \times_B E \times_B E \arrow[r, "r"'] & E \times_B E \arrow[r, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

then the internal precategory $\bar{B} = (E, E\times_B E, E\times_B E\times_B E, e, d, c, q, m, r)$ constructed in a previous example is an internal category in $\mathbf{C}$.

For $e : E \to E \times_B E$ was defined to be the unique morphism with $de = 1$ and $ce = 1$, and $m : E \times_B E \times_B E \to E \times_B E$ was defined to be the unique morphism with $dm = dq$ and $cm = cr$.

Then

$$dm(1, ec) = dq(1, ec) = d$$

and

$$cm(1, ec) = cr(1, ec) = cec = c$$

so

$$m(1, ec) = 1$$

And

$$dm(ed, 1) = dq(ed, 1) = ded = d$$

and

$$cm(ed, 1) = cr(ed, 1) = c$$

so

$$m(ed, 1) = 1$$

Finally

$$dm(ms, rt) = dq(ms, rt) = dms = dqs$$

$$cm(ms, rt) = cr(ms, rt) = crt = cmt$$

and

$$dm(qs, mt) = dq(qs, mt) = dqs$$

$$cm(qs, mt) = cr(qs, mt) = cmt$$

so

$$m(ms, rt) = m(qs, mt)$$

1.3. Internal groupoids

Definition 10. An *internal groupoid* $G = (G_0, G_1, e, d, c, m, i)$ in $\mathbf{C}$ is an internal category (G_0, G_1, e, d, c, m) in $\mathbf{C}$ with an additional morphism $i : G_1 \to G_1$, called *inverse* such that the following diagrams commute

$$\begin{tikzcd} G_1 \arrow[r, "i"] \arrow[dr, "1"'] & G_1 \arrow[d, "i"] \\ & G_1 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, "1"] \arrow[d, "i"'] & G_1 \arrow[d, "d"] \\ G_1 \arrow[r, "c"'] & G_0 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, "i"] \arrow[d, "1"'] & G_1 \arrow[d, "d"] \\ G_1 \arrow[r, "c"'] & G_0 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, "d"] \arrow[d, "(1,i)"'] & G_0 \arrow[d, "e"] \\ G_2 \arrow[r, "m"'] & G_1 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, "c"] \arrow[d, "(i,1)"'] & G_0 \arrow[d, "e"] \\ G_2 \arrow[r, "m"'] & G_1 \end{tikzcd}$$

and a *morphism* $f : G \to G' = (G'_0, G'_1, e', d', c', m', i')$ *of internal groupoids* is a morphism $f : G \to G'$ of precategories such that the following diagram also commutes

$$\begin{tikzcd} G_1 \arrow[r, "i"] \arrow[d, "f_1"'] & G_1 \arrow[d, "f_1"] \\ G_1' \arrow[r, "i'"'] & G_1' \end{tikzcd}$$

The category of internal groupoids in $\mathbf{C}$ and their morphisms is denoted $\mathrm{Gpd}(\mathbf{C})$

EXAMPLE 11. $\mathbf{Gpd} \sim \mathrm{Gpd}(\mathbf{Set})$, which is one of the justifications for the definition of an internal groupoid.

DEFINITION 12. If $\mathbf{C}$ has a terminal object 1, an *internal group* G in $\mathbf{C}$ is an internal groupoid with $G_0 \cong 1$.

DEFINITION 13. The full subcategory of $\mathrm{Gpd}(\mathbf{C})$ with all the objects that are internal groups in $\mathbf{C}$ is called the *category of internal groups in* $\mathbf{C}$ and denoted $\mathrm{Grp}(\mathbf{C})$.

EXAMPLE 14. Of course $\mathbf{Grp} \sim \mathrm{Grp}(\mathbf{Set})$

1.4. Internal pregroupoids

Now I want to remove some properties from the concept of internal groupoids to get internal pregroupoids, just as the concept of internal categories weakens to the concept of internal precategories by removing all properties that aren't simply commuting diagrams, and I want to do this in such a way that the following two facts be true.

FACT 15. *Any functor* $I : \mathbf{C} \to \mathbf{X}$ *carries an internal pregroupoid* G *in* $\mathbf{C}$ *to an internal pregroupoid* IG *in* $\mathbf{X}$.

FACT 16. *An internal pregroupoid* G *is an internal groupoid if and only if it's an internal category.*

To that end, recall that the definition of an internal groupoid involves constructing $(1, i) : G_1 \to G_2$ and $(i, 1) : G_1 \to G_2$ as in the unique fill-ins in the diagrams

$$\begin{tikzcd}
G_1 \arrow[dr, "{(1,i)}"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "1"'] & & \\
& G_2 \arrow[r, "r"] \arrow[d, "q"'] & G_1 \arrow[d, "d"] \\
& G_1 \arrow[r, "c"'] & G_0
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \arrow[dr, "{(i,1)}"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "i"'] & & \\
& G_2 \arrow[r, "r"] \arrow[d, "q"'] & G_1 \arrow[d, "d"] \\
& G_1 \arrow[r, "c"'] & G_0
\end{tikzcd}$$

But if G is just an internal precategory, we can't do this, because

$$\begin{tikzcd}
G_2 \arrow[r, "r"] \arrow[d, "q"'] & G_1 \arrow[d, "d"] \\
G_1 \arrow[r, "c"'] & G_0
\end{tikzcd}$$

is not a pullback, but merely commutes.

So instead let's replace $(1, i) : G_1 \to G_2$ and $(i, 1) : G_1 \to G_2$ with two new morphisms $k : G_1 \to G_2$ and $l : G_1 \to G_2$ that make commute the same diagrams as $(1, i) : G_1 \to G_2$ and $(i, 1) : G_1 \to G_2$ respectively:

$$\begin{tikzcd}
G_1 \arrow[r, "k"] \arrow[dr, "1"'] & G_2 \arrow[d, "q"] \\
& G_1
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \arrow[d, "k"'] \arrow[dr, "i"] & \\
G_2 \arrow[r, "r"'] & G_1
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \arrow[r, "d"] \arrow[d, "k"'] & G_0 \arrow[d, "e"] \\
G_2 \arrow[r, "m"'] & G_1
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \arrow[r, "l'"] \arrow[dr, "i"'] & G_2 \arrow[d, "q"] \\
& G_1
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \arrow[d, "l"'] \arrow[dr, "1"] & \\
G_2 \arrow[r, "r"'] & G_1
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \arrow[r, "c"] \arrow[d, "l"'] & G_0 \arrow[d, "e"] \\
G_2 \arrow[r, "m"'] & G_1
\end{tikzcd}$$

The triangles come from the pullbacks defining $(1, i)$ and $(i, 1)$, and the squares are from the definition of an internal groupoid.

As $i = rk = ql$ it is clear that if the data for an internal pregroupoid does include k or l it needn't include i explicitly. And then, just using the fact that G is an internal precategory, we do have

$$di = drk = cqk = c$$

and

$$ci = crk = cmk = ced = d$$

But $i = rk = ql$ also suggests that k and l are related in some way; possibly l can be constructed once k is given. Certainly this is the case when the internal pregroupoid G is also an internal category:

$$\begin{tikzcd}
G_1 \arrow[dr, dashed, "l"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "rk"'] & & \\
& G_2 \arrow[r, "r"] \arrow[d, "q"'] & G_1 \arrow[d, "d"] \\
& G_1 \arrow[r, "c"'] & G_0
\end{tikzcd}$$

which works because $crk = cmk = ced = d$, and gives us the two commuting triangles for l:

$$\begin{tikzcd} G_1 \arrow[r, "l"] \arrow[dr, "i"'] & G_2 \arrow[d, "q"] \\ & G_1 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[d, "l"'] \arrow[dr, "1"] & \\ G_2 \arrow[r, "r"'] & G_1 \end{tikzcd}$$

as well as

$$\begin{tikzcd} G_1 \arrow[r, "c"] \arrow[d, "l"'] & G_0 \arrow[d, "e"] \\ G_2 \arrow[r, "m"'] & G_1 \end{tikzcd}$$

and

$$\begin{tikzcd} G_1 \arrow[r, "i"] \arrow[dr, "1"'] & G_1 \arrow[d, "i"] \\ & G_1 \end{tikzcd}$$

Thus with the following definition, Facts 15 and 16 are true:

Definition 17. An *internal pregroupoid* $G = (G_0, G_1, G_2, e, d, c, q, m, r, k)$ is an internal precategory $G = (G_0, G_1, G_2, e, d, c, q, m, r)$ with an additional morphism $k : G_1 \to G_2$ such that the following diagrams commute

$$\begin{tikzcd} G_1 \arrow[r, "k"] \arrow[dr, "1"'] & G_2 \arrow[d, "q"] \\ & G_1 \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, "d"] \arrow[d, "k"'] & G_0 \arrow[d, "e"] \\ G_2 \arrow[r, "m"'] & G_1 \end{tikzcd}$$

and a morphism of internal pregroupoids

$$f : G \to G' = (G'_0, G'_1, G'_2, e', d', c', q', m', r', k')$$

is a morphism of internal precategories

$$f : G \to G'$$

such that also the following diagram commutes

$$\begin{tikzcd} G_1 \arrow[r, "k"] \arrow[d, "f_1"'] & G_2 \arrow[d, "f_2"] \\ G'_1 \arrow[r, "k'"'] & G'_2 \end{tikzcd}$$

Example 18. The internal category

$$\bar{B} = (E, E \times_B E, E \times_B E \times_B E, e, d, c, q, m, r)$$

constructed in a previous example is an internal pregroupoid, with k the unique fill-in in

$$\begin{tikzcd} E \times_B E \arrow[dr, dashed, "k"] \arrow[drr, bend left, "(c,d)"] \arrow[ddr, bend right, "1"'] & & \\ & E \times_B E \times_B E \arrow[r, "r"] \arrow[d, "q"'] & E \times_B E \arrow[d, "d"] \\ & E \times_B E \arrow[r, "c"'] & E \end{tikzcd}$$

where (c, d) is the unique fill-in in

$$\begin{tikzcd} E \times_B E \arrow[dr, dashed, "(c,d)"] \arrow[drr, bend left, "d"] \arrow[ddr, bend right, "c"'] & & \\ & E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

because

$$dmk = dqk = d = ded$$

and

$$cmk = crk = c(c, d) = d = ced$$

so

$$mk = ed$$

and $qk = 1$ is already in the diagram above defining k as $(1, (c, d))$.

Thus, as an internal category that's an internal pregroupoid, $\bar{B}$ is an internal groupoid (with $i = (c, d)$).

1.5. Internal equivalence relations

An equivalence relation is a relation that's also a precategory (and thus reflexive) and a category (and thus transitive) and a pregroupoid (and thus, as it is a category, a groupoid, and so symmetric).

Thus a first step towards defining internal equivalence relations can be:

DEFINITION 19. An *internal reflexive relation R in* $\mathbf{C}$ consists of

(1) an object R_0 in $\mathbf{C}$ which I'll call the *objects-of-objects*
(2) an object R_1 in $\mathbf{C}$ which I'll call the *object-of-arrows*
(3) a morphism $e : R_0 \to R_1$ in $\mathbf{C}$ which I'll call *identity*
(4) a morphism $d : R_1 \to R_0$ in $\mathbf{C}$ which I'll call *domain*
(5) a morphism $c : R_1 \to R_0$ in $\mathbf{C}$ which I'll call *codomain*

such that

(1) $de = 1 = ce$
(2) d and c are *jointly monic* i.e. if $df = dg$ and $cf = cg$ then $f = g$.

EXAMPLE 20. An internal reflexive relation R in $\mathbf{Set}$ is an ordinary reflexive relation R_1 on the set R_0. The maps $d, c : R_1 \to R_0$ defined by $xRy \mapsto x$ and $xRy \mapsto y$ are jointly monic. $e : R_0 \to R_1$ is $x \mapsto xRx$.

EXAMPLE 21. An internal precategory $P = (P_0, P_1, P_2, e, d, c, q, m, r)$ is an internal reflexive relation if and only if d and c are jointly monic.

EXAMPLE 22. The internal category $\bar{B} = (E, E \times_B E, E \times_B E \times_B E, e, d, c, q, m, r)$ is an internal reflexive relation. For if $df = dg$ and $cf = cg$, then $f = g$, because $pd = pc$ is a pullback.

DEFINITION 23. An *internal equivalence relation* is an internal reflexive relation, which is also an internal pregroupoid, and also an internal category (and thus an internal groupoid).

EXAMPLE 24. Of course an internal equivalence relation in **Set** is an ordinary equivalence relation.

EXAMPLE 25. $\bar{B} = (E, E \times_B E, E \times_B E \times_B E, e, d, c, q, m, r)$ is an internal equivalence relation as it's an internal reflexive relation and an internal groupoid.

Note that a functor $I : \mathbf{C} \to \mathbf{X}$ does not, in general, preserve internal reflexive relations, since d, c jointly monic does not imply Id, Ic jointly monic. The important consequence of this for us is that though $I\bar{B}$ is always an internal pregroupoid, and often an internal groupoid, it is not usually an internal equivalence relation, because it is not usually an internal reflexive relation.

1.6. Internal actions

DEFINITION 26. A *discrete fibration* is a morphism $f : P \to P'$ of internal precategories in $\mathbf{C}$ such that the following two diagrams are pullbacks

$$\begin{tikzcd} P_1 \arrow[r, "c"] \arrow[d, "f_1"'] & P_0 \arrow[d, "f_0"] \\ P_1' \arrow[r, "c'"'] & P_0' \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "r"] \arrow[d, "f_2"'] & P_1 \arrow[d, "f_1"] \\ P_2' \arrow[r, "r'"'] & P_1' \end{tikzcd}$$

DEFINITION 27. A *discrete opfibration* is a morphism $f : P \to P'$ of internal precategories in $\mathbf{C}$ such that the following two diagrams are pullbacks

$$\begin{tikzcd} P_1 \arrow[r, "d"] \arrow[d, "f_1"'] & P_0 \arrow[d, "f_0"] \\ P_1' \arrow[r, "d'"'] & P_0' \end{tikzcd}$$

$$\begin{tikzcd}
P_2 \arrow[r, "q"] \arrow[d, "f_2"'] & P_1 \arrow[d, "f_1"] \\
P_2' \arrow[r, "q'"'] & P_1'
\end{tikzcd}$$

EXAMPLE 28. A discrete opfibration $f : P \to G$ in **Set** with G a group is an ordinary group action $c : G \times P_0 \to P_0$. Based on this it seems somewhat reasonable to, in general, call a discrete opfibration $f : P \to P'$ in $\mathbf{C}$ an action of the internal precategory P' on the object P_0 in $\mathbf{C}$.

DEFINITION 29. The *category of actions of a precategory P on objects of* $\mathbf{C}$ is denoted $\mathbf{C}^P$ and is the full subcategory of $\mathbf{C}^{\mathbb{P}}/P$ containing all the objects (P', f) such that $f : P' \to P$ is a discrete opfibration in $\mathbf{C}$.

EXAMPLE 30. Here is an important way to construct an object in $\mathbf{C}^{\bar{B}}$. Given $\bar{B}$ and a morphism $f : A \to B$ in $\mathbf{C}$ construct the pullbacks

$$\begin{tikzcd}
E \times_B E \times_B E \times_B A \arrow[r, "q'"] \arrow[d, "f_2"'] & E \times_B E \times_B A \arrow[r, "d'"] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \times_B E \arrow[r, "q"'] & E \times_B E \arrow[r, "d"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

and then form c', e', m' and r' as the unique fill-ins in the following diagrams

$$\begin{tikzcd}
E \times_B E \times_B A \arrow[dr, dashed, "c'"] \arrow[drr, bend left, "p'd'"] \arrow[ddr, bend right, "cf_1"'] & & \\
& E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
E \times_B A \arrow[dr, dashed, "e'"] \arrow[drr, bend left, "1'"] \arrow[ddr, bend right, "ef_0"'] & & \\
& E \times_B E \times_B A \arrow[r, "d'"] \arrow[d, "f_1"'] & E \times_B A \arrow[d, "f_0"] \\
& E \times_B E \arrow[r, "d"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
E \times_B E \times_B E \times_B A \arrow[dr, dashed, "m'"] \arrow[drr, bend left, "d'q'"] \arrow[ddr, bend right, "mf_2"'] & & \\
& E \times_B E \times_B A \arrow[r, "d'"] \arrow[d, "f_1"'] & E \times_B A \arrow[d, "f_0"] \\
& E \times_B E \arrow[r, "d"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
E \times_B E \times_B E \times_B A \arrow[dr, dashed, "r'"] \arrow[drr, bend left, "c'q'"] \arrow[ddr, bend right, "rf_2"'] & & \\
& E \times_B E \times_B A \arrow[r, "d'"] \arrow[d, "f_1"'] & E \times_B A \arrow[d, "f_0"] \\
& E \times_B E \arrow[r, "d"'] & E
\end{tikzcd}$$

In the above diagrams we already have some of the identities of a precategory

$$d'e' = 1$$

$$d'm' = d'q'$$

$$d'r' = c'q$$

and all of the identities for $f = (f_0, f_1, f_2)$ to be a morphism of precategories

$$f_0 d' = d f_1$$

$$f_0 c' = c f_1$$

$$f_1 e' = e f_0$$

$$f_1 q' = q f_1$$

$$f_1 m' = m f_2$$

$$f_1 r' = r f_2$$

What remains is to show that $c'e' = 1$ and $c'm' = c'r$.

For the first, as $f_0c'e' = cf_1e' = cef_0 = f_0$ and $p'c'e' = p'd'e' = p'$, the diagram

$$\begin{tikzcd}
E \times_B A \arrow[dr, dashed, "c'e' = 1"] \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "f_0"'] & & \\
& E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

indicates that $c'e' = 1$ as the fill-in is unique.

For the second, as

$$f_0c'm' = cf_1m' = cmf_2 = crf_2 = cf_1r' = f_0c'r'$$

and

$$p'c'r' = p'd'r' = p'c'q' = p'd'q' = p'd'm' = p'c'm'$$

the diagram

$$\begin{tikzcd}
E \times_B A \times_B A \arrow[dr, dashed, "c'm' = c'r'"] \arrow[drr, bend left, "p'c'r'"] \arrow[ddr, bend right, "f_0c'r'"'] & & \\
& E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

indicates that $c'm' = c'r$ as the fill-in is unique.

Thus

$$A' = (E \times_B A, E \times_B E \times_B A, E \times_B E \times_B E \times_B A, e', d', c', q', m', r')$$

is a precategory, and $f' = (f_0, f_1, f_2) : A' \to \bar{B}$ is a discrete opfibration, and (A', f') is an object in $\mathbf{C}^{\bar{B}}$.

In fact $p^{\#}(A, f) = (A', f')$ is the object part of a functor $p^{\#} : \mathbf{C}/B \to \mathbf{C}^{\bar{B}}$, but that gets us into descent theory, which is the subject of the next section.

Example 31. Substituting $p : E \to B$ for $f : A \to B$ in the above work reveals that $p^{\#}(E, p)$ is a representation of $\bar{B}$ as an action of $\bar{B}$ on $E \times_B E$.

Example 32. Here is the object $p^{\#}(A, f)$ when $\mathbf{C} = \mathbf{Set}$.

$\bar{B}$ is an ordinary equivalence relation $\sim$ on the set E, so the pullbacks

$$\begin{tikzcd}
E \times_B E \times_B E \times_B A \arrow[r, "q'"] \arrow[d, "f_2"'] & E \times_B E \times_B A \arrow[r, "d'"] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \times_B E \arrow[r, "q"'] & E \times_B E \arrow[r, "d"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

are, in terms of the elements of the sets

$$\begin{tikzcd}
(x \sim y \sim z, a) \arrow[r, "q'"] \arrow[d, mapsto, "f_2"'] & (x \sim y, a) \arrow[r, "d'"] \arrow[d, mapsto, "f_1"'] & (x, a) \arrow[r, mapsto, "p'"] \arrow[d, mapsto, "f_0"'] & a \arrow[d, mapsto, "f"] \\
x \sim y \sim z \arrow[r, mapsto, "q"'] & x \sim y \arrow[r, mapsto, "d"'] & x \arrow[r, mapsto, "p"'] & px = fa
\end{tikzcd}$$

the commuting diagram

$$\begin{tikzcd}
E \times_B E \times_B E \times_B A \arrow[r, "r'"] \arrow[d, "f_2"'] & E \times_B E \times_B A \arrow[r, "c'"] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \times_B E \arrow[r, "r"'] & E \times_B E \arrow[r, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

is

$$\begin{tikzcd}
(x \sim y \sim z, a'') \arrow[r, "r'"] \arrow[d, mapsto, "f_2"'] & (y \sim z, a') \arrow[r, "c'"] \arrow[d, mapsto, "f_1"'] & (z, a) \arrow[r, mapsto, "p'"] \arrow[d, mapsto, "f_0"'] & a \arrow[d, mapsto, "f"] \\
x \sim y \sim z \arrow[r, mapsto, "r"'] & y \sim z \arrow[r, mapsto, "c"'] & z \arrow[r, mapsto, "p"'] & pz = fa
\end{tikzcd}$$

the commuting square

$$\begin{tikzcd}
E \times_B E \times_B E \times_B A \arrow[r, "m'"] \arrow[d, "f_2"'] & E \times_B E \times_B A \arrow[d, "f_1"'] \\
E \times_B E \times_B E \arrow[r, "m"'] & E \times_B E
\end{tikzcd}$$

is

$$\begin{tikzcd}
(x \sim y \sim z, a') \arrow[r, "m'"] \arrow[d, mapsto, "f_2"'] & (x \sim z, a) \arrow[d, mapsto, "f_1"'] \\
x \sim z \arrow[r, mapsto, "m"'] & x \sim z
\end{tikzcd}$$

and the commuting square

$$\begin{tikzcd} E \times_B E \times_B A \arrow[d, "f_1"'] & E \times_B A \arrow[l, "e'"'] \arrow[d, "f_0"] \\ E \times_B E & E \arrow[l, "e"] \end{tikzcd}$$

is

$$\begin{tikzcd} (x \sim x, a) \arrow[d, mapsto, "f_1"'] & (x, a) \arrow[l, mapsto, "e'"'] \arrow[d, mapsto, "f_0"] \\ x \sim x & x \arrow[l, mapsto, "e"] \end{tikzcd}$$

Thus $c' : E \times_B E \times_B A \to E \times_B A$ takes an $a \in A$, and $x \in E$ such that $fa = px$, a $y \in E$ such that $x \sim y$ and "pushes" a along $x \sim y$ to give an $a' \in A$ such that $fa' = py$. And it does this in such a way that it pushing a along $x \sim y$ and then $y \sim z$ is the same as pushing a along $x \sim z$, and pushing a along $x \sim x$ just gives a.

Thus the discrete opfibration $f' : A' \to \bar{B}$ can be thought of as a functor $F : \bar{B} \to \mathbf{Set}$, where the equivalence relation $\bar{B}$ is regarded as a category with a set of objects E and set of arrows $E \times_B E$, $Fx = f^{-1}(x)$, and $F(x \sim y) : Fx \to Fy$ takes $a \in Fx$ to $c(x \sim y, a) \in Fy$.

In a very similar, and hardly more complicated way, a discrete opfibration $F : C' \to C$ in $\mathbf{Set}$ with C a category can be thought of as a functor $F : C \to \mathbf{Set}$. Thus we can think of a discrete opfibration $F : C' \to C$ in $\mathbf{C}$, with C an internal category as a functor $F : C \to \mathbf{C}$. Thus such a discrete opfibration is sometimes called an internal base-valued functor – it's like a picture, internal to $\mathbf{C}$, of a functor from the internal category C to the "base" category $\mathbf{C}$. This also explains the notation $\mathbf{C}^P$ for the category of discrete opfibrations over P.

1.7. Effective descent morphisms

DEFINITION 33. The assignment on objects $p^{\#}(A, f) = (A', f')$ explained in Example 30 can be turned into a functor $p^{\#} : \mathbf{C}/B \to \mathbf{C}^{\bar{B}}$. When this functor is part of an adjoint equivalence $\mathbf{C}/B \sim \mathbf{C}^{\bar{B}}$, then $p : E \to B$ is called an *effective descent morphism.*

Instead of describing $p^{\#}$ on morphisms, let me just construct a functor $p_{\#} : \mathbf{C}^{\bar{B}} \to \mathbf{C}/B$ that will (when p is an effective descent morphism) be in adjoint equivalence with $p^{\#}$ (and thus imply that $p^{\#}$ is then a functor).

So take an object (P, f) in $\mathbf{C}^{\bar{B}}$, i.e. pullbacks as in

$$\begin{tikzcd}
E \times_B E \times_B E \times_E C \arrow[r, "q'"] \arrow[d, "f_2"'] & E \times_B E \times_E C \arrow[r, "d'"] \arrow[d, "f_1"'] & C \arrow[d, "f_0"] \\
E \times_B E \times_B E \arrow[r, "q"'] & E \times_B E \arrow[r, "d"'] & E
\end{tikzcd}$$

and commuting diagrams

$$\begin{tikzcd}
E \times_B E \times_B E \times_E C \arrow[r, "r'"] \arrow[d, "f_2"'] & E \times_B E \times_E C \arrow[r, "c'"] \arrow[d, "f_1"'] & C \arrow[d, "f_0"] \\
E \times_B E \times_B E \arrow[r, "r"'] & E \times_B E \arrow[r, "c"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
E \times_B E \times_B E \times_E C \arrow[r, "m'"] \arrow[d, "f_2"'] & E \times_B E \times_E C \arrow[d, "f_1"'] & C \arrow[l, "e'"'] \arrow[d, "f_0"] \\
E \times_B E \times_B E \arrow[r, "m"'] & E \times_B E & E \arrow[l, "e"]
\end{tikzcd}$$

such that also

$$d'e' = 1 = c'e'$$
$$c'q' = d'r'$$
$$d'q' = d'm'$$
$$c'r' = c'm'$$

Then suppose that d', c' has a coequalizer $p' : C \to D$, as in

$$\begin{tikzcd}
E \times_B E \times_E C \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & C \arrow[r, "p'"] \arrow[d, "f_0"'] & D \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

Since $pf_0d' = pdf_1 = pcf_1 = pf_0c'$, there's a unique $g : D \to B$ such that $pf_0 = gp'$, as in

$$
\begin{tikzcd}
E \times_B E \times_E C \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & C \arrow[r, "p'"] \arrow[d, "f_0"'] & D \arrow[d, dashed, "g"] \\
E \times_B E \arrow[r, shift left, "c"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}
$$

So $p_\#(P, f) = (D, g)$ defines $p_\# : \mathbf{C}^{\bar{B}} \to \mathbf{C}/B$ on objects.

A morphism $h : (P, f) \to (P', f')$ in $\mathbf{C}^{\bar{B}}$ is a morphism

$$h : P \to P' = (C', E\times_B E\times_E C', E\times_B E\times_B E\times_E C', e'', d'', c'', q'', m'', r'')$$

in $\mathbf{C}^{\mathbb{P}}$ with $f'h = f$, i.e. morphisms

$$h_0 : C' \to C$$

$$h_1 : E \times_B E \times_E C' \to E \times_B E \times_E C$$

$$h_2 : E \times_B E \times_B E \times_E C' \to E \times_B E \times_B E \times_E C$$

in $\mathbf{C}$ with, amongst other identities, $h_0d'' = d'h_1$ and $h_0c'' = c'h_1$.

Thus in the diagram

$$
\begin{tikzcd}
E \times_B E \times_E C' \arrow[r, shift left, "d''"] \arrow[r, shift right, "c''"'] \arrow[d, "h_1"'] & C' \arrow[r, "p''"] \arrow[d, "h_0"'] & D' \arrow[d, dashed, "p_\# h"'] \arrow[dd, bend left=60, "g'"] \\
E \times_B E \times_E C \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & C \arrow[r, "p'"] \arrow[d, "f_0"'] & D \arrow[d, "g"] \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}
$$

we have $p'h_0d'' = p'd'h_1 = p'c'h_1 = p'h_0c''$, and so $p_\#h : D' \to D$ can be the unique fill-in with $p'h_0 = p_\#h \circ p''$.

The uniqueness off such fill-ins takes care of the functorality of $p_\# : \mathbf{C}^{\bar{B}} \to \mathbf{C}/B$. This nice fact is why I chose to describe this functor on arrows, rather than $p^\#$.

Under what circumstances is $(p^\#, p_\#) : \mathbf{C}/B \to \mathbf{C}^{\bar{B}}$ an equivalence?

Firstly, in the construction of $p^{\#}(A, f)$ in the diagram

$$\begin{tikzcd}
E \times_B E \times_B A \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

$p' : E \times_B A \to A$ needs to already be a coequalizer of

$$d', c' : E \times_B E \times_B A \to E \times_B A$$

so that $p_{\#}p^{\#}(A, f) \cong (A, f)$

Secondly, in the construction of $p_{\#}(P, f)$ in the diagram

$$\begin{tikzcd}
E \times_B E \times_E C \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & C \arrow[r, "p'"] \arrow[d, "f_0"'] & D \arrow[d, dashed, "g"] \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

the right hand square needs to already be a pullback, so that

$$p^{\#}p_{\#}(P, f) \cong (P, f)$$

.

So one way to see if a morphism $p : E \to B$ is an effective descent morphism is to check these two conditions; there are other, perhaps better ways, but this will do nicely for our purposes. Here we use it to prove that effective descent morphisms are pullback stable (after the following Lemma):

LEMMA 34. *In the construction of $p^{\#}(A, f)$ in the diagram*

$$\begin{tikzcd}
E \times_B E \times_B A \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

d', c' is a kernel pair of p'.

PROOF. Add to the above diagram a kernel pair u,v of p', g the unique morphism such that $ug = d'$, $vg = c'$, h the unique morphism such that $dh = f_0u$, $ch = f_0v$, and then k the unique morphism such that $f_1k = h$, $d'k = u$ to obtain the following diagram:

$$\begin{tikzcd}
(E \times_B A) \times_A (E \times_B A) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[dd, dashed, bend right, "h"'] \arrow[d, dashed, shift left, "k"] & E \times_B A \arrow[r, "p'"] \arrow[d, equal] & A \arrow[d, equal] \\
E \times_B E \times_B A \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] \arrow[u, dashed, shift left, "g"] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

Now $d'kg = ug = d'$ and $f_1kg = hg$. But $dhg = f_0ug = f_0d' = df_1$ and $chg = f_0vg = f_0c' = cf_1$, and thus $hg = f_1$. So we have $d'kg = d'1$ and $f_1kg = f_11$, and so $kg = 1$.

Similarly $ugk = d'k = u$ and $vgk = c'k$. But $f_0c'k = cf_1k = ch = f_0v$ and $p'c'k = p'd'k = p'u = p'v$, and thus $c'k = v$. So we have $ugk = u1$ and $vgk = v1$, and so $gk = 1$.

Thus d', c' is a kernel pair of p'. $\square$

THEOREM 35. *In a category with pullbacks, the pullback of an effective descent morphism is again an effective descent morphism.*

PROOF. For an effective descent morphism $p : E \to B$ and a morphism $f : A \to B$, construct $p^{\#}(A, f)$ as usual, and then, for any $f' : A' \to A$ construct $p'^{\#}(A', f')$ (using the fact that by the above Lemma 34, d', c' are a kernel pair of p') yielding the diagram

$$\begin{tikzcd}
E \times_B E \times_B A' \arrow[r, shift left, "d''"] \arrow[r, shift right, "c''"'] \arrow[d, "f_1'"'] & E \times_B A' \arrow[r, "p''"] \arrow[d, "f_0'"'] & A' \arrow[d, "f'"] \\
E \times_B E \times_B A \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

As the two squares on the right are, by construction, pullbacks, so is the whole rectangle on the right. Similarly the left rectangle with d and d'' is a pullback, and of course the left rectangle with c and c'' commutes. As p is an effective descent morphism, this implies that p'' is a coequalizer of d'', c''.

Now, after constructing $p^{\#}(A, f)$, take any (P, f') in $\mathbf{C}^{\bar{A}}$ (here the above Lemma 34 ensures that $\overline{A}$ is an internal equivalence relation) and construct $p_{\#}(P, f)$, yielding the diagram

$$\begin{tikzcd}
E \times_B E \times_E C \arrow[r, shift left, "d''"] \arrow[r, shift right, "c''"'] \arrow[d, "f_1'"'] & C \arrow[r, "p''"] \arrow[d, "f_0'"'] & D \arrow[d, dashed, "g'"] \\
E \times_B E \times_B A \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

As p is an effective descent morphism, the right hand rectangle and the lower right hand square are both pullbacks. Thus the top right hand square is a pullback.

These two facts show that p', the pullback of p, is an effective descent morphism. $\square$

Now for, as I understand it, the actual purpose of effective descent morphisms.

Let $\mathbb{M}$ be a pullback closed class of morphisms in $\mathbf{C}$. For an effective descent morphism $p : E \to B$, let $\mathbb{M}^{\bar{B}}$ denote the full subcategory of $\mathbf{C}^{\bar{B}}$ of discrete fibrations $f : A \to \bar{B}$ with $f_0 : A_0 \to E$ in $\mathbb{M}$ (and thus, since $\mathbb{M}$ is pullback closed, f_1 and f_2 in $\mathbb{M}$).

Definition 36. A morphism $f : A \to B$ in $\mathbf{C}$ is said to be in $\mathbb{M}^*$ when there's an effective descent morphism $p : E \to B$ such that $p^{\#}(A, f)$ is in $\mathbb{M}^{\bar{B}}$. The full subcategory of $\mathbf{C}/B$ of objects (A, f) with $f : A \to B$ in $\mathbb{M}^*$ is denoted $\mathbb{M}^*/B$.

As I understand it, the purpose of this is to give a category theory approach to "locally". Often an $f : A \to B$ in $\mathbb{M}^*$ is said to be locally in $\mathbb{M}$, and often this more or less agrees with the classical notion, as it does in the case of the categorical Galois theory of the reflection $I : \mathbf{FinTop} \to \mathbf{FinSet}$ of the category of finite topological spaces into the category of finite sets.

But also note that we certainly do not, in general have an effective descent morphism $p : E \to B$ such that $p^{\#} : \mathbb{M}^*/B \sim \mathbb{M}^{\bar{B}}$ – there is not generally one effective descent morphism that works for all $f : A \to B$ in $\mathbb{M}^*$, despite the fact that for every effective descent morphism $p : E \to B$, by definition $p^{\#} : \mathbf{C}/B \sim \mathbf{C}^{\bar{B}}$. However, when effective descent morphisms are used in categorical Galois theory there often is an effective descent morphism such $p : E \to B$ such that $p^{\#} : \mathbb{M}^*/B \sim \mathbb{M}^{\bar{B}}$, and there will be in the case of the reflection $I : \mathbf{FinTop} \to \mathbf{FinSet}$ that this thesis studies. Thus it is not necessary to go further into the topic of descent theory, except for the following theorem:

Theorem 37. *If* $\mathbb{M}$ *is pullback closed, then so is* $\mathbb{M}^*$*.*

Proof. Given $f : A \to B$ and $p : E \to B$ such that the pullback of f along p is in $\mathbb{M}$, and any $g : B' \to B$, construct the commuting cube

$$\begin{tikzcd}
 & E \times_B A \arrow[rr] \arrow[dd] & & A \arrow[dd, "f"] \\
E' \times_{B'} A \arrow[ur, dashed] \arrow[rr, crossing over] \arrow[dd, "\pi_1'"'] & & A' \arrow[ur] & \\
 & E \arrow[rr, "p"'] & & B \\
E' \arrow[ur] \arrow[rr, "p'"'] & & B' \arrow[ur, "g"'] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in which the back square, bottom square, right square, and front square are pullbacks. Then the left square created by the unique fill-in indicated is also a pullback, as the left square and back square together form the same commuting rectangle as the front square and right square. Thus as $\mathbb{M}$ is pullback stable, π_1' is in $\mathbb{M}$. By Theorem 35, p' is an effective descent morphism. Thus the pullback of f along g is in $\mathbb{M}^*$. $\square$

CHAPTER 2

Categorical Galois Theory

2.1. Admissibility

A reflection $(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$ induces, for each E in $\mathbf{C}$ an obvious functor $I^E : \mathbf{C}/E \to \mathbf{X}/IE$ defined by

$$h : (A, f) \to (C, g) \qquad \mapsto \qquad Ih : (IA, If) \to (IC, Ig)$$

since $Ig \circ Ih = If$ if $g \circ h = f$.

Under what circumstances does $I^E : \mathbf{C}/E \to \mathbf{X}/IE$ itself have a right adjoint $H^E : \mathbf{X}/IE \to \mathbf{C}/E$? And when is H^E, like H, fully faithful?

One might naturally start by looking at the equivalent of I^E for H; I mean the functor defined by

$$w : (X, u) \to (Y, v) \qquad \mapsto \qquad Hw : (HX, Hu) \to (HY, Hv)$$

But this is a functor

$$\mathbf{X}/IE \to \mathbf{C}/HIE$$

How might we go from an object in $\mathbf{C}/HIE$ onward to an object in $\mathbf{C}/E$? The only relation between HIE and E that we always have, in this situation, is the unit $\eta_E : E \to HIE$ of the original reflection.

So we can try pulling back $Hu : HX \to HIE$ along $\eta_E : E \to HIE$ to turn (HX, Hu) in $\mathbf{C}/HIE$ into an object $(E \times_{(\eta_E, Hu)} HX, \pi_1)$ in $\mathbf{C}/E$.

In fact this is the only possible right adjoint for I^E, as the following theorem shows.

Theorem 38. *For any categories $\mathbf{C}$ and $\mathbf{X}$, a reflection*

$$(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$$

induces for an object E in $\mathbf{C}$ an adjunction

$$(I^E, H^E, \eta^E, \varepsilon^E, \varphi^E) : \mathbf{C}/E \to \mathbf{X}/IE$$

in which I^E *is given by*

$$h : (A, f) \to (C, g) \qquad \mapsto \qquad Ih : (IA, If) \to (IC, Ig)$$

exactly when $\mathbf{C}$ *has pullbacks of all* $Hu : HX \to HIE$ *along the unit* $\eta_E : E \to HIE$ *of the reflection.*

Proof. The right adjoint H^E exists when for each $(X, u) \in \mathbf{X}/IE$ there's an object $H^E (X, u) = (C, g)$ and a universal arrow $\varepsilon^E_{(X,u)} : (IC, Ig) \to (X, u)$

Referring to the familiar diagram for universals:

$$\begin{tikzcd}
(C,g) \\
(A,f) \arrow[u, dashed, "v'"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(X,u) & (IC,Ig) \arrow[l, "\varepsilon^E_{(X,u)}"'] \\
 & (IA,If) \arrow[ul, "v"] \arrow[u, "Iv'"']
\end{tikzcd}$$

we see this is the case when for each $u : X \to IE$ there's $g : C \to E$ and $w = \varepsilon^E_{(X,u)} : IC \to X$ with $uw = Ig$ such that for each $f : A \to E$ and $v : IA \to X$ with $uv = If$ there's a unique $v' : A \to C$ such that $gv' = f$ and $w \circ Iv' = v$.

Utilizing the original adjunction's natural bijection $\varphi : \hom (IA, X) \cong \hom (A, HX)$ this translates to:

For each $u : X \to IE$ there's $g : C \to E$ and $w : C \to X$ with $u \circ w = Ig \circ \eta_C = \eta_E \circ g$ such that for each $f : A \to E$ and $v : A \to HX$ with $Hu \circ v = \eta_E \circ f$ there's a unique $v' : A \to C$ such that $g \circ v' = g$ and $w \circ v' = v$, as illustrated in the following diagram:

$$\begin{tikzcd}
A \arrow[dr, dashed, "v'"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "f"'] & & \\
 & C \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "u"] \\
 & E \arrow[r, "\eta_E"'] & IE
\end{tikzcd}$$

In other words, $g : C \to E$ is the pullback of $u : X \to IE$ along $\eta_E : E \to IE$. $\square$

Corollary 39. *Referring to the above theorem and proof,* H^E *is fully faithful if and only if each* Iw *is an isomorphism.*

PROOF. In the pullback in the proof of the above theorem, $w = \varphi(\varepsilon^E_{(X,u)})$, so $\varepsilon^E_{(X,u)} = \varphi^{-1}(w) = \varepsilon_X \circ Iw$. $\square$

DEFINITION 40. The reflection

$$(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$$

is called *admissible* when for each E in $\mathbf{C}$ the induced adjunction

$$(I^E, H^E, \eta^E, \varepsilon^E, \varphi^E) : \mathbf{C}/E \to \mathbf{X}/IE$$

exists and has H^E is fully faithful.

Now the actual class of morphisms $\mathbb{M}$ that we'll primarily be interested in:

DEFINITION 41. The (replete) image of a fully faithful H^E is denoted $\mathbb{M}/E$, and called the *trivial coverings* of E.

So by definition

COROLLARY 42. *An admissible reflection*

$$(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$$

induces for each E in $\mathbf{C}$ an equivalence

$$(I^E, H^E, \eta^E, \varepsilon^E, \varphi^E) : \mathbb{M}/E \sim \mathbf{X}/IE$$

DEFINITION 43. $\mathbb{M}$ is the class of all morphisms that are trivial coverings of their codomain.

THEOREM 44. *$f : A \to B$ is in $\mathbb{M}$ if and only if the following square is a pullback*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HIA \arrow[d, "HIf"] \\ B \arrow[r, "\eta_B"'] & HIB \end{tikzcd}$$

PROOF. By definition $f : A \to B$ is in $\mathbb{M}$ exactly (A, f) is in the image of H^B, i.e. when there's some $u : X \to IB$ in $\mathbf{X}$ such that

$$\begin{tikzcd} A \arrow[r, "\pi_2"] \arrow[d, "f"'] & HX \arrow[d, "Hu"] \\ B \arrow[r, "\eta_B"'] & HIB \end{tikzcd}$$

is a pullback. Then considering

$$\begin{tikzcd} A \arrow[dr, dashed, "1"] \arrow[drr, bend left, "\varepsilon_{HX} \circ HI\pi_2 \circ \eta_A"] \arrow[ddr, bend right, "f"'] & & \\ & A \arrow[r, "\pi_2"] \arrow[d, "f"'] & HX \arrow[d, "Hu"] \\ & B \arrow[r, "\eta_B"'] & HIB \end{tikzcd}$$

indicates that $\pi_2 \cong \eta_A$ and $u \cong If$. $\square$

THEOREM 45. $\mathbb{M}$ *is pullback closed*

PROOF. Suppose that both squares in

$$\begin{tikzcd} D \arrow[r] \arrow[d, "h"'] & C \arrow[r, "\eta_C"] \arrow[d, "g"'] & HIC \arrow[d, "HIg"] \\ B \arrow[r, "f"'] & E \arrow[r, "\eta_E"'] & HIE \end{tikzcd}$$

are pullbacks. Then of course the outer rectangle is a pullback. But the outer rectangle is the same as the outer rectangle in

$$\begin{tikzcd} D \arrow[r, "\eta_D"] \arrow[d, "h"'] & HID \arrow[r] \arrow[d, "HIh"'] & HIC \arrow[d, "HIg"] \\ B \arrow[r, "\eta_B"'] & HIB \arrow[r, "HIf"'] & HIE \end{tikzcd}$$

As the right-hand square here is a pullback, so is the left hand square, so $h : D \to B$ is in $\mathbb{M}$. $\square$

2.2. Universal coverings

Recall that $f : A \to B$ is said to be in $\mathbb{M}^*$ if there's an effective descent morphism $p : E \to B$ such that in the pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$\pi_1 : E \times_B A \to E$ is in $\mathbb{M}$.

DEFINITION 46. A morphism in $\mathbb{M}^*$ is called a *covering*.

DEFINITION 47. If E has no non-trivial coverings (i.e. if whenever $f : A \to E$ is in $\mathbb{M}^*$ it's actually in $\mathbb{M}$) then an effective descent morphism $p : E \to B$ that is itself in $\mathbb{M}^*$ is called a *universal covering.*

Before seeing why it's called a *universal* covering, we need:

THEOREM 48. $\mathbb{M}^*$ *is pullback closed*

PROOF. $\mathbb{M}$ is pullback closed; so apply Theorem 37. $\square$

Thus if $f : A \to B$ is in $\mathbb{M}^*$ and $p : E \to B$ is a universal covering, in the pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$\pi_1 : E \times_B A \to E$ is in $\mathbb{M}^*$. But since E has no non-trivial coverings, $\pi_1 : E \times_B A \to E$ is thus in $\mathbb{M}$. Thus

THEOREM 49. *When $p : E \to B$ is a universal covering, there is an equivalence* $p^{\#} : \mathbb{M}^*/B \sim \mathbb{M}^{\bar{B}}$

Note also that as a universal covering is itself a covering, in the pullback

$$\begin{tikzcd} E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

both $c : E \times_B E \to E$ and $d : E \times_B E \to E$ are in $\mathbb{M}$.

2.3. More induced equivalences

Now that we have, if $p : E \to B$ is a universal covering, the equivalence

$$(p^{\#}, p_{\#}) : \mathbb{M}^*/B \sim \mathbb{M}^{\bar{B}}$$

we want to use the equivalence

$$(I^E, H^E, \eta^E, \varepsilon^E) : \mathbb{M}/E \sim \mathbf{X}/IE$$

to make an equivalence

$$(I^{\bar{B}}, H^{\bar{B}}, \eta^{\bar{B}}, \varepsilon^{\bar{B}}) : \mathbb{M}^{\bar{B}} \sim \mathbf{X}^{I\bar{B}}$$

so that we can compose them into an equivalence

$$(I^{\bar{B}}p^{\#}, p_{\#}H^{\bar{B}}) : \mathbb{M}^*/B \sim \mathbf{X}^{I\bar{B}}$$

which is a simplified version of the fundamental theorem of categorical Galois theory that's more than enough for the application to the reflection $I : \mathbf{FinTop} \to \mathbf{FinSet}$.

To that end, we start with

THEOREM 50. *The reflection*

$$(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$$

extends to a reflection

$$(I^{\mathbb{P}}, H^{\mathbb{P}}, \eta^{\mathbb{P}}, \varepsilon^{\mathbb{P}}) : \mathbf{C}^{\mathbb{P}} \to \mathbf{X}^{\mathbb{P}}$$

PROOF. Just let

(1) $I^{\mathbb{P}} : \mathbf{C}^{\mathbb{P}} \to \mathbf{X}^{\mathbb{P}}$ send $P : \mathbb{P} \to \mathbf{C}$ to $IP : \mathbb{P} \to \mathbf{X}$.
(2) $H^{\mathbb{P}} : \mathbf{X}^{\mathbb{P}} \to \mathbf{C}^{\mathbb{P}}$ send $P : \mathbb{P} \to \mathbf{X}$ to $HP : \mathbb{P} \to \mathbf{C}$.
(3) $\eta^{\mathbb{P}}_P : P \to H^{\mathbb{P}}I^{\mathbb{P}}P$ be the natural transformation $P \to HIP$ with components
    (a) $(\eta^{\mathbb{P}}_P)_0 = \eta_{P_0} : P_0 \to HIP_0$
    (b) $(\eta^{\mathbb{P}}_P)_1 = \eta_{P_1} : P_1 \to HIP_1$
    (c) $(\eta^{\mathbb{P}}_P)_2 = \eta_{P_2} : P_2 \to HIP_2$
(4) $\varepsilon^{\mathbb{P}}_P : I^{\mathbb{P}}H^{\mathbb{P}}P \to P$ be the natural transformation $IHP \to P$ with components
    (a) $(\varepsilon^{\mathbb{P}}_P)_0 : IHP_0 \cong P_0$
    (b) $(\varepsilon^{\mathbb{P}}_P)_1 : IHP_1 \cong P_1$
    (c) $(\varepsilon^{\mathbb{P}}_P)_2 : IHP_2 \cong P_2$

□

The following is basically a restatement of Theorem 38:

COROLLARY 51. *The reflection*

$$(I^{\mathbb{P}}, H^{\mathbb{P}}, \eta^{\mathbb{P}}, \varepsilon^{\mathbb{P}}) : \mathbf{C}^{\mathbb{P}} \to \mathbf{X}^{\mathbb{P}}$$

induces for $Q : \mathbb{P} \to \mathbf{C}$ in $\mathbf{C}^{\mathbb{P}}$ an adjunction

$$(I^{Q}, H^{Q}, \eta^{Q}, \varepsilon^{Q}) : \mathbf{C}^{\mathbb{P}}/Q \to \mathbf{X}^{\mathbb{P}}/IQ$$

exactly when $\mathbf{C}^{\mathbb{P}}$ *has pullbacks of all* $u : P \to IQ$ *in* $\mathbf{X}^{\mathbb{P}}$ *along the unit* $\eta^{\mathbb{P}}_P : Q \to H^{\mathbb{P}} I^{\mathbb{P}} Q$ *of the reflection, i.e. as pullbacks in the functor category* $\mathbf{C}^{\mathbb{P}}$ *are constructed pointwise, exactly when* $\mathbf{C}$ *has*

(1) *pullbacks in* $\mathbf{C}$ *of all* $u : X \to IQ_0$ *in* $\mathbf{X}$ *along* $\eta_{P_0} : P_0 \to HIP_0$
(2) *pullbacks in* $\mathbf{C}$ *of all* $u : X \to IQ_1$ *in* $\mathbf{X}$ *along* $\eta_{P_1} : P_1 \to HIP_1$
(3) *pullbacks in* $\mathbf{C}$ *of all* $u : X \to IQ_2$ *in* $\mathbf{X}$ *along* $\eta_{P_2} : P_2 \to HIP_2$

Since an admissible reflection $(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$ does have all those pullbacks, and the functors H^{P_0}, H^{P_1}, H^{P_2} are fully faithful we immediately have:

COROLLARY 52. *For an admissible reflection*

$$(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$$

$$H^Q : \mathbf{X}^{\mathbb{P}}/IQ \to \mathbf{C}^{\mathbb{P}}/Q$$

is fully faithful, or to put it another way an admissible reflection

$$(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$$

extends to a reflection

$$(I^{\mathbb{P}}, H^{\mathbb{P}}, \eta^{\mathbb{P}}, \varepsilon^{\mathbb{P}}) : \mathbf{C}^{\mathbb{P}} \to \mathbf{X}^{\mathbb{P}}$$

which is also admissible.

Any $f : P \to Q$ in the image of H^Q has

(1) $f_0 : P_0 \to Q_0$ in the image of H^{Q_0}, i.e. in $\mathbb{M}/Q_0$
(2) $f_1 : P_1 \to Q_1$ in the image of H^{Q_1}, i.e. in $\mathbb{M}/Q_1$
(3) $f_2 : P_2 \to Q_2$ in the image of H^{Q_2}, i.e. in $\mathbb{M}/Q_2$

Thus for an admissible reflection $(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$ we denote the (replete) image of H^Q by $\mathbb{M}/Q$, and so by definition

COROLLARY 53. *An admissible reflection*

$$(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$$

induces for each $Q : \mathbb{P} \to \mathbf{C}$ *an equivalence*

$$(I^Q, H^Q, \eta^Q, \varepsilon^Q) : \mathbb{M}/Q \sim \mathbf{X}/IQ$$

And clearly

THEOREM 54. *The equivalence* $(I^Q, H^Q, \eta^Q, \varepsilon^Q) : \mathbb{M}/Q \sim \mathbf{X}/IQ$ *restricts to an equivalence* $(I^Q, H^Q, \eta^Q, \varepsilon^Q) : \mathbb{M}^Q \sim \mathbf{X}^{IQ}$

COROLLARY 55. *An admissible reflection* $(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$ *induces for each* $\bar{B}$ *in* $\mathbf{C}^{\mathbb{P}}$ *an equivalence* $(I^{\bar{B}}, H^{\bar{B}}, \eta^{\bar{B}}, \varepsilon^{\bar{B}}) : \mathbb{M}^{\bar{B}} \sim \mathbf{X}^{I\bar{B}}$

2.4. The fundamental theorem of categorical Galois theory

Combining Theorem 49 and Corollary 55 gives this simplified version of the fundamental theorem of categorical Galois theory:

THEOREM 56. *When we have an admissible reflection* $(I, H, \eta, \varepsilon, \varphi) : \mathbf{C} \to \mathbf{X}$ *and a universal covering* $p : E \to B$ *we have an equivalence*

$$(I^{\bar{B}} p^{\#}, p_{\#} H^{\bar{B}}) : \mathbb{M}^{*}/B \sim \mathbf{X}^{I\bar{B}}$$

Since any functor $I : \mathbf{C} \to \mathbf{X}$ carries internal pregroupoids in $\mathbf{C}$ to internal pregroupoids in $\mathbf{X}$, and $\bar{B} : \mathbb{P} \to \mathbf{C}$, as an internal equivalence relation, is certainly an internal pregroupoid, $I\bar{B} : \mathbb{P} \to \mathbf{X}$ is also an internal pregroupoid. Thus, at least for the purposes of this thesis (there is a more general definition, i.e. more general fundamental theorem, with $\mathrm{Spl}(E, p)$ in place of $\mathbb{M}^{*}/B$, but that is not needed in this thesis) we can say:

DEFINITION 57. When the equivalence $\mathbb{M}^{*}/B \sim \mathbf{X}^{I\bar{B}}$ exists $I\bar{B}$ is called the *Galois pregroupoid*, and denoted $\mathrm{Gal}(E, p)$

THEOREM 58. *When the Galois pregroupoid* $\mathrm{Gal}(E, p)$ *is an internal category in* $\mathbf{X}$*, then it is an internal groupoid in* $\mathbf{X}$*, called the Galois groupoid.*

THEOREM 59. $\mathrm{Gal}(E, p)$ *is an internal category (and thus groupoid) in* $\mathbf{X}$ *if and only if* $I : \mathbf{C} \to \mathbf{X}$ *takes the pullbacks*

$$\begin{tikzcd}
E \times_B E \times_B E \arrow[r, "r"] \arrow[d, "q"'] & E \times_B E \arrow[d, "d"] \\
E \times_B E \arrow[r, "c"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
E \times_B E \times_B E \arrow[r, "t"] \arrow[d, "s"'] & E \times_B E \times_B E \arrow[d, "q"] \\
E \times_B E \times_B E \arrow[r, "r"'] & E \times_B E
\end{tikzcd}$$

in **C** *to pullbacks*

$$\begin{tikzcd}
I(E \times_B E \times_B E) \arrow[r, "Ir"] \arrow[d, "Iq"'] & I(E \times_B E \arrow[d, "Id"] \\
I(E \times_B E) \arrow[r, "Ic"'] & IE
\end{tikzcd}$$

$$\begin{tikzcd}
I(E \times_B E \times_B E) \arrow[r, "It"] \arrow[d, "Is"'] & I(E \times_B E \times_B E) \arrow[d, "Iq"] \\
I(E \times_B E \times_B E) \arrow[r, "Ir"'] & I(E \times_B E)
\end{tikzcd}$$

PROOF. The only other conditions $\mathrm{Gal}(E, p) = I\bar{B}$ must satisfy in order to be an internal category in **X** are that the diagrams

$$\begin{tikzcd}
I(E \times_B E) \arrow[r, "(1, IeIc)"] \arrow[dr, "1"'] & I(E \times_B E \times_B E) \arrow[d, "Im"] \\
& I(E \times_B E)
\end{tikzcd}$$

$$\begin{tikzcd}
I(E \times_B E) \arrow[r, "(IeId, 1)"] \arrow[dr, "1"'] & I(E \times_B E \times_B E) \arrow[d, "Im"] \\
& I(E \times_B E)
\end{tikzcd}$$

$$\begin{tikzcd}
I(E \times_B E \times_B E \times_B E) \arrow[r, "(IqIs, ImIt)"] \arrow[d, "(ImIs, IrIt)"'] & I(E \times_B E \times_B E) \arrow[d, "Im"] \\
I(E \times_B E \times_B E) \arrow[r, "Im"'] & I(E \times_B E)
\end{tikzcd}$$

must commute. But the uniqueness of the fill-ins means that each is actually the I-image of the corresponding fill-in in **C**. Thus the diagrams are the I-images of the corresponding diagrams in **C**, and so as $I : \mathbf{C} \to \mathbf{X}$ is a functor, they still commute $\square$

THEOREM 60. *If* **X** *has a terminal object, and* $IE \cong 1$, *then a Galois groupoid* $\mathrm{Gal}(E, p)$ *is an internal group in* **X**, *called the Galois group.*

CHAPTER 3

Some Properties of Finite Spaces

This chapter contains mostly non-category theory details of finite topological spaces that will be helpful to know when applying categorical Galois theory (and before that, descent theory) to them.

3.1. Étale maps

Each point a in a finite topological space A has a smallest open set $\downarrow a$ containing it. $\downarrow a$ is the intersection of all the open sets that contain a. It is open because as the space is finite there are only finitely many open sets, so it is a finite intersection of open sets. Every open set in $U \subset A$ is $U = \bigcup \{\downarrow a \subset A | a \in U\}$. So $\{\downarrow a \subset A | a \in A\}$ is a base for the topology on A. Thus

THEOREM 61. *A map $f : A \to B$ of finite topological spaces is continuous if and only if for each $b \in B$, the set $f^{-1}(\downarrow b)$ is open in A.*

Note, of course, that for any map $f : A \to B$ of finite topological spaces $f^{-1}(\downarrow b) = \bigcup \{\{a\} \subset A | a \in f^{-1}(\downarrow b)\} = \bigcup \{\{a\} \subset A | fa \in \downarrow b\}$, and when f is continuous this becomes $f^{-1}(\downarrow b) = \bigcup \{\downarrow a \subset A | fa \in \downarrow b\}$.
Recall the following definition from topology:

DEFINITION 62. A map $f : A \to B$ of topological spaces is *étale* if and only if for each $a \in A$ there is an open set $a \in U$ in A such that fU is open in B and the restriction of f to $U \to fU$ is an isomorphism.

The following is obvious:

THEOREM 63. *An étale map of topological spaces is continuous.*

In the case of finite spaces, the definition of étale can become

THEOREM 64. *A map $f : A \to B$ of finite topological spaces is étale if and only if for each $a \in A$, $f(\downarrow a)$ is open in B and the restriction of f to $\downarrow a \to f(\downarrow a)$ is an isomorphism.*

Or even simpler:

COROLLARY 65. *A map $f : A \to B$ of finite topological spaces is étale if and only if for each $a \in A$ the restriction of f to $\downarrow a \to f(\downarrow a)$ is a bijection.*

3.2. Finite spaces as finite preorders

Define a relation $\to$ on a finite space A by $a \to a'$ if and only if $a \in \downarrow a'$. This relation is a preorder. It is reflexive, because $a \in \downarrow a$, and transitive, because if $a \in \downarrow a'$ and $a' \in \downarrow a''$ then $\downarrow a' \subset \downarrow a''$ so that $a \in \downarrow a''$. On the other hand, any finite preordered set A defines a finite topological space A with basic open sets *defined* as $\downarrow a = \{a' \in A | a' \to a\}$. It's clear that composing these two processes, either way, yields an isomorphism, so that we can say that every finite topological space is isomorphic to a finite preordered set. Thus we will freely talk of a finite topological space A as a finite preordered set A, and vice versa.

Even better, the morphisms of finite spaces correspond exactly to the morphisms of the associated preordered sets:

THEOREM 66. *A map $f : A \to B$ of finite topological spaces is continuous if and only if for each $a \to a'$ in A, $fa \to fa'$ in B.*

PROOF. Suppose $f : A \to B$ is a morphism of the preorders associated with the topologies on A and B. Consider for each $b \in B$ the set $f^{-1}(\downarrow b)$. I want to show that $f^{-1}(\downarrow b)$ is open, i.e. the union of basic open sets. So taking any $a \in f^{-1}(\downarrow b)$, I need to show $\downarrow a \subset f^{-1}(\downarrow b)$. So consider any $a' \in \downarrow a$. In terms of the preorder this means $a' \to a$ in A, so that $fa' \to fa$ in B. In terms of the preorder $a \in f^{-1}(\downarrow b)$ means $fa \to b$. Thus we have $fa' \to fa \to b$, and thus by transitivity $fa' \to b$, which in terms of the topology, is $a' \in f^{-1}(\downarrow b)$.

On the other hand, suppose $f : A \to B$ is a continuous map of finite spaces. $a \to a'$ means $a \in \downarrow a'$ in A. We want to show that $fa \to fa'$ in B, i.e. that $fa \in \downarrow fa'$. Since f is continuous, $f^{-1}(\downarrow fa')$ is open in A, i.e. a union of basic open sets in A. Now $a' \in f^{-1}(\downarrow fa')$, so $\downarrow a' \in f^{-1}(\downarrow fa')$, i.e. if $a \to a'$, then $fa \in \downarrow fa'$. □

Thus

COROLLARY 67. *The category of finite topological spaces is equivalent to the category of finite preordered sets, i.e.* **FinTop** $\sim$ **FinPreord**

and so we can freely move back and forth between considering a morphism $f : A \to B$ as continuous map of finite topological spaces or a monotonic ($a \to a'$ implies $fa \to fa'$) map of finite preordered sets.

Now suppose a map $f : A \to B$ of finite topological spaces is étale, i.e. each $f(\downarrow a) = \downarrow fa$. Thus if $b \to fa$, there's a unique $x \in A$ with $fx = b$ and $x \to a$. We have just proved

THEOREM 68. *A morphism $f : A \to B$ of finite topological spaces is étale if and only if for each $a \in A$ and $b \to fa$ in B there's unique $x \to a$ with $fx = b$.*

Now recall that a preorder A can be thought of as a category, i.e. as an internal category in **Set**. For a morphism $f : A \to B$ of preorders consider the squares

$$\begin{tikzcd} A_1 \arrow[r, "c"] \arrow[d, "f_1"'] & A_0 \arrow[d, "f_0"] \\ B_1 \arrow[r, "c"'] & B_0 \end{tikzcd}$$

Take any $(b, b') \in B_1\ B_1$ and $a' \in A_0$ such that $f_0a' = fa = c(b, b') = b'$. That is, any $a' \in A$ and $b \to fa'$ in B. The condition that $f : A \to B$ be étale is that there be a unique $a \to a'$ in A such that $fa = b$, i.e. that there be a unique $(a, a') \in A_1$ such that $f_1(a, a') = (b, b')$ and $c(a, a') = a'$. This is just the condition that the above square be a pullback in **Set**. Thus we have proved

THEOREM 69. *A map $f : A \to B$ of finite topological spaces is étale if and only if as a morphism of preorders it is a discrete fibration.*

That immediately makes us wonder what a discrete opfibration is, in terms of the topologies. The condition that

$$\begin{tikzcd} A_1 \arrow[r, "d"] \arrow[d, "f_1"'] & A_0 \arrow[d, "f_0"] \\ B_1 \arrow[r, "d"'] & B_0 \end{tikzcd}$$

be a pullback in **Set** is that for each $(fa, b') \in B_1$ there's a unique $(a, a') \in A_1$ such that $(fa, b') = (fa, fa')$, i.e. that for each $a \in A$

and $fa \to b'$ in B there's a unique $a \to a'$ in A with $fa' = b'$. If we let $\uparrow a = \{a' \in A | a \to a'\}$ and $\uparrow fa = \{b' \in B | fa \to b'\}$ then this says that f restricts to a bijection $f(\uparrow a) \cong \uparrow fa$. (Note that in terms of the topology, $\uparrow a = \{a' \in A | a \to a'\} = \{a' \in A | a \in \downarrow a'\}$ is clearly the closure of $\{a\}$.) Thus we define

Definition 70. A map $f : A \to B$ of finite topological spaces is *"op-étale"* if and only if for each $a \in A$ the restriction of f to $\uparrow a \to f(\uparrow a)$ is a bijection.

And have

Theorem 71. *A map $f : A \to B$ of finite topological spaces is "op-étale" if and only if as a morphism of preorders it is a discrete opfibration.*

And because each closed set $K \subset A$ is the union of the closures of each $a \in K$, we can also say

Theorem 72. *A map $f : A \to B$ of finite topological spaces is "op-étale" if and only if for each $a \in A$ there's a closed set $a \in K$ such that fK is closed in B and the restriction of f to $K \to fK$ is an isomorphism.*

But because the union of infinitely many closed sets is not, in general, closed, we cannot generalize this to topological spaces in general in a reverse of the description of of étale maps.

3.3. $\mathbb{M}^*$, the class of morphisms locally in $\mathbb{M}$

Recall the following definition from topology, where $\mathbb{M}$ is a pullback-stable class of morphisms in **Top**.

Definition 73. A map $f : A \to B$ of topological spaces is *locally* in $\mathbb{M}$ if and only if for each $b \in B$ there's an open set $b \in U \subset B$ such that the restriction of f to $f^{-1}U \to U$ is in $\mathbb{M}$. The class of morphisms locally in $\mathbb{M}$ is denoted $\mathbb{M}^*$.

In the case of finite spaces, this becomes

Theorem 74. *A map $f : A \to B$ of finite topological spaces is in $\mathbb{M}^*$ (i.e locally in $\mathbb{M}$) if and only if for each $b \in B$ the restriction of f to $f^{-1}(\downarrow b) \to \downarrow b$ is in $\mathbb{M}$.*

just so long as $\mathbb{M}^*$ is also pullback-stable (which it will be in our application, due to Theorem 37).

3.4. From "connected" to paths

Recall the following definition from topology:

DEFINITION 75. A topological space $A \neq \emptyset$ is *connected* if whenever $A = U \cup V$ with U, V open sets such that $U \cap V = \emptyset$, then either $U = A$ and $V = \emptyset$ or $U = \emptyset$ and $V = A$.

Now suppose a finite topological space A is connected in this sense. Since the $\downarrow a$ are a base for the topology on A this becomes:

THEOREM 76. *A finite topological space A is connected if and only if for all $a, a' \in A$ there's sequence $a_1, a_2, \ldots, a_n$ of points in A such that $a = a_1$, $a' = a_n$ and for each i either $a_i \in\downarrow a_{i+1}$ or $a_{i+1} \in\downarrow a_i$*

This is perhaps better expressed in terms of the preorder:

COROLLARY 77. *A finite topological space A is connected if and only if for all $a, a' \in A$ there's sequence $a_1, a_2, \ldots, a_n$ of points in A such that $a = a_1$, $a' = a_n$ and for each i either $a_i \to a_{i+1}$ or $a_i \leftarrow a_{i+1}$*

If we make the following definition

DEFINITION 78. A *path* $\alpha : a \to a'$ in a finite topological space is a sequence

$$a \leftrightarrow_0 a_1 \leftrightarrow_1 a_2 \cdots a_n \leftrightarrow_n a'$$

with each $\leftrightarrow_i$ either $\leftarrow$ or $\to$.

Then we can say

COROLLARY 79. *A finite topological space A is connected if and only if for all $a, a' \in A$ there's a path $\alpha : a \to a'$.*

In other words, for finite spaces, connected and path-connected are the same thing.

The following is obvious

THEOREM 80. *A continuous map $f : A \to B$ of finite topological spaces carries each path $\alpha : a \to a'$ in A to the path $f\alpha : fa \to fa'$ in B.*

In detail, if α is the path

$$a \leftrightarrow_0 a_1 \leftrightarrow_1 a_2 \cdots a_n \leftrightarrow_n a'$$

then $f\alpha$ is the path

$$fa \leftrightarrow_0 fa_1 \leftrightarrow_1 fa_2 \cdots fa_n \leftrightarrow_n fa'$$

3.5. Fundamental groupoids

DEFINITION 81. Paths $\alpha : a \to a'$ and $\alpha' : a \to a'$ are said to be equivalent, or in the same *path class* if they are related by the equivalence relation generated by

$$\begin{array}{ccc}
a_{i-1} \to a_i \to a_{i+1} & \sim & a_{i-1} \to a_{i+1} \\
a_i \to a_i & \sim & a_i \\
a_i \to a_{i+1} \leftarrow a_i & \sim & a_i \\
a_i \leftarrow a_{i+1} \to a_i & \sim & a_i
\end{array}$$

EXAMPLE 82. The basic example of two paths that are not equivalent occurs around the "hole"

$$\begin{tikzcd}
& 2 & \\
1 \arrow[ur] \arrow[dr] & & 3 \arrow[ul] \arrow[dl] \\
& 4 &
\end{tikzcd}$$

Here the path $1 \to 2 \leftarrow 3$ is not related to the path $1 \to 4 \leftarrow 3$.

EXAMPLE 83. On the other hand

$$\begin{tikzcd}
& 2 \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & 3 \\
& 4 \arrow[ur] &
\end{tikzcd}$$

is not a "hole" since

$$1 \to 2 \to 3 \quad \sim \quad 1 \to 3 \quad \sim \quad 1 \to 4 \to 3$$

Since the operations defining equivalence of paths don't change the start or end of a path, each path in an equivalence class of paths has the same starting point, and the same ending point. Thus we can make the definition

Definition 84. *The path class* $[\alpha] : a \to a'$ *is the set of all paths equivalent to the path* $\alpha : a \to a'$.

Path classes $[\alpha] : a \to a'$ and $[\alpha'] : a' \to a''$ can be composed to give $[\alpha][\alpha'] = [\alpha\alpha'] : a \to a''$ where $\alpha\alpha' : a \to a''$ is just the path α followed by the path α'.

Every path class $[\alpha] : a \to a'$ has an inverse path class $[\alpha]^{-1} : a' \to a$, since if we denote by $\alpha^{-1} : a' \to a$ the reverse of the path $\alpha : a \to a'$ then $[\alpha]^{-1} = [\alpha^{-1}]$.

Thus

Definition 85. The *fundamental groupoid* πA of a finite topological space A has the same objects as A, but each hom-set $\hom_{\pi A}(a, a')$ is the set of all path classes $a \to a'$, and the composition of $[\alpha] : a \to a'$ with $[\beta] : a' \to a''$ is $[\alpha][\beta] = [\alpha\beta] : a \to a''$, where, of course $\alpha\beta$ is the concatenation of the paths α and β.

Example 86. The fundamental groupoid of the basic "hole"

$$\begin{tikzcd}
 & 2 & \\
1 \arrow[ur] \arrow[dr] & & 3 \arrow[ul] \arrow[dl] \\
 & 4 &
\end{tikzcd}$$

has, for example, $f : \mathbb{Z} \cong \hom(1,1)$, with

$$f(0) = [1 \to 1]$$

$$f(1) = [1 \to 2 \leftarrow 3 \to 4 \leftarrow 1]$$
$$f(-1) = [1 \to 4 \leftarrow 3 \to 2 \leftarrow 1]$$

$$\begin{aligned} f(2) &= [1 \to 2 \leftarrow 3 \to 4 \leftarrow 1][1 \to 2 \leftarrow 3 \to 4 \leftarrow 1] \\ &= [1 \to 2 \leftarrow 3 \to 4 \leftarrow 1 \to 2 \leftarrow 3 \to 4 \leftarrow 1] \end{aligned}$$

$$f(n) = [1 \to 2 \leftarrow 3 \to 4 \leftarrow 1]^n$$

$$
\begin{aligned}
f(m+n) &= [1 \to 2 \leftarrow 3 \to 4 \leftarrow 1]^{m+n} \\
&= [1 \to 2 \leftarrow 3 \to 4 \leftarrow 1]^{n+m} \\
&= f(n+m)
\end{aligned}
$$

Of course the same goes for each hom-set, so that $\mathbb{Z}$ is called the fundamental group of this space.

Or to be precise:

DEFINITION 87. The *Poincaré fundamental group* $\pi_1(B, b)$ of a connected finite topological space B is the group formed by any of the hom-sets of its fundamental groupoid πB.

There is a very nice description of the fundamental groupoids of finite spaces. **FinTop** $\sim$ **FinPreord**, and **FinPreord** can be considered a subcategory of **Cat**. The inclusion **Gpd** $\to$ **Cat** has a left adjoint π : **Cat** $\to$ **Gpd**, that gives the "fundamental groupoid" of a category (see Paré's paper [**UCC**]). When A is not only a category, but a finite preorder, this gives the same fundamental groupoid πA as described above.

For consider, for a finite preorder A, the diagram

$$
\begin{tikzcd}
\textbf{Gpd} & & \textbf{Cat} & \\
\pi A \arrow[d, dashed, "f'"] & & A \arrow[r, "\eta^\pi_A"] \arrow[dr, "f"'] & \pi A \arrow[d, "f'"] \\
G & & & G
\end{tikzcd}
$$

with $\eta^\pi_A(a) = a$ and $\eta^\pi_A(a \to a') = [a \to a']$.

$f' : \pi A \to G$ must be $f'(a) = f(a)$ and $f'([a \to a']) = [fa \to fa']$. Thus η^π_A is a universal.

3.6. Coverings

Recall the following definitions from topology

DEFINITION 88. A *trivial covering* of a connected topological space B is any map isomorphic to the projection $\coprod_X B \to B$ defined by $(b, x) \mapsto b$, for some set X.

DEFINITION 89. A map $f : A \to B$ of topological spaces is called a *covering* of B if it is locally a trivial covering.

In the case of finite spaces, this becomes

THEOREM 90. *A map $f : A \to B$ of finite topological spaces is a covering if and only if for each $b \in B$ the restriction of f to $f^{-1}(\downarrow b) \to \downarrow b$ is a trivial covering, i.e. isomorphic to a projection $\coprod_X \downarrow b \to \downarrow b$ defined by $(b', x) \mapsto b'$.*

It is then quite easy to prove

THEOREM 91. *In* **FinTop** *every covering $f : A \to B$ is étale.*

PROOF. Take any $a \in A$ and $b' \to fa$ in B. As $f : A \to B$ is a covering, its restriction to $f^{-1}(\downarrow fa) \to \downarrow fa$ is isomorphic to a projection $\coprod_X \downarrow fa \to \downarrow fa$. a can only be in one of the components of $f^{-1}(\downarrow fa)$, so it corresponds to $(fa, x) \in \coprod_X \downarrow fa$ for a unique $x \in X$. Then, (b', x) is the only element of $\coprod_X \downarrow fa$ with $(b', x) \to (fa, x)$ that is sent to $b' \in B$ by the projection. So the $a' \in A$ that corresponds to (b', x) is the unique $a' \to a$ with $fa' = b'$. $\square$

And equally easy to prove

THEOREM 92. *In* **FinTop** *every covering $f : A \to B$ is "op-étale".*

PROOF. Take any $a \in A$ and $fa \to b'$ in B. As $f : A \to B$ is a covering, its restriction to $f^{-1}(\downarrow b') \to \downarrow b'$ is isomorphic to a projection $\coprod_X \downarrow b' \to \downarrow b'$. a can only be in one of the components of $f^{-1}(\downarrow b')$, so it corresponds to (fa, x) for a unique $x \in X$. Then (b', x) is the only element of $\coprod_X \downarrow b'$ with $(fa, x) \to (b', x)$ that is sent to $b' \in B$ by the projection. So the $a' \in A$ that corresponds to (b', x) is the unique $a' \in A$ with $a \to a'$ and $fa' = b'$. $\square$

This suggests that we try proving

THEOREM 93. *In* **FinTop** *if $f : A \to B$ is étale and "op-étale", then it is a covering.*

PROOF. Supposing the fibres of $f : A \to B$ are discrete, so that for each $f^{-1}(\downarrow b) \to \downarrow b$, the fibre $f^{-1}b$ is a discrete space. Take any $a \in f^{-1}b$. As f is étale $f(\downarrow a) = \downarrow fa = \downarrow b$. Thus $f^{-1}(\downarrow b) \to \downarrow b$ is isomorphic to the projection $\coprod_{f^{-1}b} \downarrow b \to \downarrow b$ defined by $(b', a) \mapsto b'$. $\square$

So to complete the proof we just need

LEMMA 94. *In* **FinTop** *if $f : A \to B$ is étale or "op-étale", then its fibres are discrete.*

PROOF. Suppose there's $a \to a'$ with $fa = fa'$. We need to show that if f is étale or "op-étale" then $a = a'$. Suppose first that f is étale. We have $a' \in A$ and $fa \to fa'$ in B and so there must be a unique $x \in A$ with $x \to a'$ in A, $fx = fa$ in B. But both a and a' are such x. For $a \to a'$, $fa = fa$, but also $a' \to a'$, $fa' = fa$. As this x is unique, $a = x = a'$. On the other hand, if f is op-étale we have $a \in A$ and $fa \to fa'$ in B and so there must be a unique $y \in A$ with $a \to y$ and $fy = fa'$. And here again, both a and a' are such a y, so $a = y = a'$. □

We have thus characterized the coverings in **FinTop**:

COROLLARY 95. *In* **FinTop** *the coverings are the maps that are both étale and "op-étale".*

THEOREM 96. *If $f : A \to B$ is a covering, $[\beta] : b \to b'$ a path in B, and $a \in f^{-1}(b)$ then there's a unique $[\alpha] : a \to a'$ in A with $f[\alpha] = [\beta]$.*

PROOF. The proof follows from the fact that $f^{-1}(b)$ is discrete, and as $f : A \to B$ is both étale and "op-étale", so for each $b_i \leftrightarrow_i b_{i+1}$ we have unique $a_i \leftrightarrow_i a_{i+1}$ the path class $[\beta]$ "lifts" for each $a \in f^{-1}(b)$ to a unique path class $[\alpha]$ with $f[\alpha] = [\beta]$. □

Using again **FinTop** $\sim$ **FinPreord** $\rightarrowtail$ **Cat** (and Paré's paper [**UCC**]) a particularly important covering of any connected finite topological space B can be constructed as a projection from a certain comma category, which can then be shown to be a preorder. We do this by considering the unit $\eta_B^\pi : B \to \pi B$, which is of course a morphism of preorders, as a functor, and then make the following comma category construction in **Cat**:

$$
\begin{tikzcd}
 & & (b, \eta_B^\pi) \arrow[dll] \arrow[d] \arrow[drr, "p"] & & \\
1 \arrow[r, "b"'] & \pi B & (\pi B)^{\mathbf{2}} \arrow[l] \arrow[r] & \pi B & B \arrow[l, "\eta_B^\pi"]
\end{tikzcd}
$$

An object in (b, η_B^π) is a morphism $[\alpha] : b \to b'$ in πB, and an arrow $[\alpha] \to [\alpha'] : b \to b''$ is a morphism $[b' \to b'']$ in πB such that $[\alpha][b' \to b''] = [\alpha']$. But as B is a preorder, there is no more than one arrow $b' \to b''$ in B, and so (b, η_B^π) is a preorder.

The following is obvious:

THEOREM 97. *$p : (b, \eta_B^\pi) \to B$ is surjective.*

And the following hardly less so:

THEOREM 98. *$p : (b, \eta_B^\pi) \to B$ is étale.*

PROOF. Take any $[\alpha] \in E$ and any $b \to p[\alpha]$ in B. Then $[\alpha][p[\alpha] \to b]^{-1}$ is the unique path class sent by p to b and with it $\to [\alpha]$ in E. $\square$

THEOREM 99. *$p : (b, \eta_B^\pi) \to B$ is "op-étale".*

PROOF. Take any $[\alpha] \in E$ and any $p[\alpha] \to b$ in B. Then $[\alpha][p[\alpha] \to b]$ is the unique path class with sent by p to b and with $p[\alpha] \to$ it in E. $\square$

COROLLARY 100. *$p : (b, \eta_B^\pi) \to B$ is a covering.*

Of course, as B is connected, the construction $p : (b, \eta_B^\pi) \to B$ is independent of the choice of $b \in B$:

THEOREM 101. $(b, \eta_B^\pi) \cong (b', \eta_B^\pi)$

Finally

THEOREM 102. *For $p : (b, \eta_B^\pi) \to B$*

$$p^{-1}(b) = \hom_{\pi B}(b, b) = \pi_1(B, b)$$

the Poincaré fundamental group of B.

CHAPTER 4

Descent of Finite Spaces and Étale maps

4.1. How to get "locally" by pulling back along certain morphisms

Recall that a continuous map $f : A \to B$ of finite topological spaces is locally in $\mathbb{M}$ if and only if for each $b \in B$ the restriction of f to $f^{-1} \downarrow b \to \downarrow b$ is in $\mathbb{M}$. To begin to bring in some category theory, note that this is equivalent to the following:

THEOREM 103. *A continuous map $f : A \to B$ of finite topological spaces is locally in $\mathbb{M}$ if and only if its pullback along each inclusion $\downarrow b \hookrightarrow B$ is in $\mathbb{M}$, i.e. for each $b \in B$, in the pullback*

$$\begin{tikzcd} f^{-1}\downarrow b \arrow[r, hook] \arrow[d, "f_b"'] & A \arrow[d, "f"] \\ \downarrow b \arrow[r, hook] & B \end{tikzcd}$$

$f_b : f^{-1} \downarrow b \to \downarrow b$ is in $\mathbb{M}$.

Recall the construction, for a connected finite space B, of a universal covering $p : E \to B$, and consider the pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

The inclusion $\downarrow b \hookrightarrow B$ factors (though not necessarily uniquely) through $p : E \to B$. For as $p : E \to B$ is surjective, there's some $[\alpha] \in E$ with $p[\alpha] = b$ (i.e., because B is connected, there's at least one path $\alpha : b_0 \to b$ and so at least one path class $[\alpha] : b_0 \to b$). Then, as $p : E \to B$ is étale, $p(\downarrow [\alpha]) = \downarrow p[\alpha] = \downarrow b$. Now consider the following diagram, which uses one of the factorizations of $\downarrow b \hookrightarrow B$

through $p : E \to B$ to induce a factorization of $f^{-1} \downarrow b \hookrightarrow A$ through $\pi_2 : E \times_B A \to A$:

$$\begin{tikzcd}
f^{-1} \downarrow b \arrow[r, dashed] \arrow[d, "f_b"'] \arrow[rr, hook, bend left] & E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\
\downarrow b \arrow[r] \arrow[rr, hook, bend right] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

As the outside rectangle is a pullback, and the right hand square is a pullback, so is the left hand square. Thus, if we add the hypothesis that the class of morphisms $\mathbb{M}$ be pullback closed, we have

Theorem 104. *A morphism $f : A \to B$ is locally in pullback-closed $\mathbb{M}$ if its pullback along a universal covering $p : E \to B$ is in $\mathbb{M}$.*

Since the only properties of $p : E \to B$ used in establishing this theorem were that $p : E \to B$ is surjective and étale, we immediately have

Theorem 105. *A morphism $f : A \to B$ is locally in pullback-closed $\mathbb{M}$ if its pullback along some surjective étale $p : E \to B$ is in $\mathbb{M}$.*

Thus we might conjecture

Conjecture 106. *A morphism $f : A \to B$ is locally in pullback-closed $\mathbb{M}$ if and only if its pullback along some surjective étale $p : E \to B$ is in $\mathbb{M}$.*

How might a proof of the “only if” part of this conjecture proceed? If we knew that $\mathbb{M}^*$ (the class of morphisms locally in $\mathbb{M}$) was pullback closed, then for any surjective étale $p : E \to B$, in the pullback

$$\begin{tikzcd}
E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

if $f : A \to B$ is in $\mathbb{M}^*$ then so is $\pi_1 : E \times_B A \to E$. If we additionally knew that E was such that any $g : C \to E$ in $\mathbb{M}^*$ was in fact in $\mathbb{M}$ we'd be able to conclude that $\pi_1 : E \times_B A \to E$ is in $\mathbb{M}$. Unfortunately I do not believe this is possible without additional hypotheses on $\mathbb{M}$ (e.g. $\mathbb{M}$ should probably contain only étale maps), or the introduction of $\Sigma\mathbb{M}$ (see Proposition 6.5.2 in [**GalTheo**]). But as it is possible when $\mathbb{M}$ is the class of trivial coverings of finite spaces, I will not consider these complications.

4.2. Why actions are necessary

Suppose $\mathbb{M}$ is such that the above conjecture is true. It is nonetheless not the case that every $g : C \to E$ in $\mathbb{M}$ corresponds to a unique $f : A \to B$ such that there's a surjective étale $p : E \to B$ such that there's some $p' : C \to A$ such that the following diagram is a pullback

$$\begin{tikzcd} C \arrow[r, dashed, "p'"] \arrow[d, "g"'] & A \arrow[d, dashed, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

In short, not every $g : C \to E$ in $\mathbb{M}$ can be collapsed by some $p : E \to B$ into a unique $f : A \to B$ locally in $\mathbb{M}^*$.

But the above diagram suggests that p' be a coequalizer of some $d', c' : D \to C$ such that $pgd' = pgc'$, because then the $f : A \to B$ would be uniquely induced. Of course we'd need some further conditions to ensure that the resulting square is a pullback. What might these conditions be?

Consider the kernel pair of $p : E \to B$

$$\begin{tikzcd} E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

and then consider the diagrams

$$\begin{tikzcd} D \arrow[r, "c'"] \arrow[d, "d'"'] & C \arrow[r, "g"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ C \arrow[r, "p'"'] & A \arrow[r, "f"'] & B \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r, dashed, "g_1"] \arrow[d, "d'"'] \arrow[rr, bend left, "gc'"] & E \times_B E \arrow[r, "c"] \arrow[d, "d"] & E \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[rr, bend right, "fp'"'] & E \arrow[r, "p"'] & B \end{tikzcd}$$

where $g_1 : D \to E \times_B E$ is uniquely induced by $pgc' = fp'c' = fp'd' = pgd'$.

In the first diagram, the left-hand square is a pullback, so if the right-hand square (which is the square we want to be a pullback) is a pullback, the outside rectangle is a pullback. In the second diagram, the right-hand square is a pullback and the outside rectangle is the same as the outside rectangle of the first diagram. Thus if the outside rectangles are pullbacks, so is the left-hand square of the second diagram. So what we have arrived at is that if $d', c' : D \to C$ with $pgd' = pgc'$ are such that their coequalizer $p' : C \to A$ induces a pullback square

$$\begin{tikzcd} C \arrow[r, "p'"] \arrow[d, "g"'] & A \arrow[d, dashed, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

then the square

$$\begin{tikzcd} D \arrow[r, "d'"] \arrow[d, "g_1"'] & C \arrow[d, "g"] \\ E \times_B E \arrow[r, "d"'] & E \end{tikzcd}$$

must be a pullback, and the square

$$\begin{tikzcd} D \arrow[r, "c'"] \arrow[d, "g_1"'] & C \arrow[d, "g"] \\ E \times_B E \arrow[r, "c"'] & E \end{tikzcd}$$

must commute (for that was how g_1 was constructed).

Are these properties sufficient, and, if so, how can they be interpreted?

Recalling that a coequalizer of continuous maps of finite topological spaces $d', c' : C_1 \to C$ is the projection $[-] : C \to C/\sim$ onto the quotient by the least equivalence relation containing $(d, c)\,C$, with the quotient topology, the square

$$\begin{tikzcd} C \arrow[r, "p'"] \arrow[d, "g"'] & A \arrow[d, dashed, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is isomorphic to the square

$$\begin{tikzcd} C \arrow[r, "{[-]}"] \arrow[d, "g"'] & C/\sim \arrow[d, "{[g]}"] \\ E \arrow[r, "{[-]}"'] & E/\sim \end{tikzcd}$$

which is indeed a pullback in **FinTop**. For take any $e \in E$ and $[x] \in C/\sim$ such that$[g][x] = [e]$, i.e. $[gx] = [e]$. As p is étale, $\downarrow e \cong \downarrow [e]$. As $[g]$ is étale, $\downarrow [x] \cong \downarrow [gx]$. Thus $\downarrow e \cong \downarrow [x]$, and so x is the unique point in C with $gx = e$ and $x \in [x]$.

Thus it seems that $g : C \to E$ in $\mathbb{M}$ needs to be paired with an equivalence relation (but not just any equivalence relation!) on C in order to produce via surjective étale $p : E \to B$ a unique $f : A \to B$ in $\mathbb{M}^*$.

This also makes it clearer how a surjective étale $p : E \to B$ mediates a passage between $\mathbb{M}$ and $\mathbb{M}^*$: a suitable $g : C \to E$ in $\mathbb{M}$, with a suitable equivalence relation $\sim$ on C, can be collapsed into $[g] : C/\sim \to E/\sim$; conversely, a suitable $f : A \to B$ in $\mathbb{M}^*$ can be taken apart into a $g : C \to E$ in $\mathbb{M}$ and an equivalence relation on C such that $[g] \cong f$.

Of course these equivalence relations are compatible with the topologies, in the sense that $\downarrow [x] = [\downarrow x]$.

In terms of the descent theory discussed in 1.7 the situation is:

To have

$$\begin{tikzcd} D \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "g_1"'] & C \arrow[d, "g"] \\ E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \end{tikzcd}$$

with $g : C \to E$ in $\mathbb{M}$, and

$$\begin{tikzcd} D \arrow[r, "d'"] \arrow[d, "g_1"'] & C \arrow[d, "g"] \\ E \times_B E \arrow[r, "d"'] & E \end{tikzcd}$$

a pullback, and

$$\begin{tikzcd} D \arrow[r, "c'"] \arrow[d, "g_1"'] & C \arrow[d, "g"] \\ E \times_B E \arrow[r, "c"'] & E \end{tikzcd}$$

commuting is simply to have an object in $\mathbb{M}^{\bar{B}}$; i.e. $c' : D \to C$ is an action of $\bar{B}$ on C. Then the functor $p_{\#} : \mathbb{M}^{\bar{B}} \to \mathbb{M}^*/B$ uses the action to collapse $g : C \to E$ in $\mathbb{M}$ into $f : A \to B$ in $\mathbb{M}^*$, whereas $p^{\#} : \mathbb{M}^*/B \to \mathbb{M}^{\bar{B}}$ takes apart a morphism $f : A \to B$ in $\mathbb{M}^*$ into a morphism $f_0 : E \times_B A \to E$ in $\mathbb{M}$ and an action of $\bar{B}$ on $E \times_B A$.

4.3. Effective descent morphisms

THEOREM 107. *In* **FinTop** *a surjective étale* $p : E \to B$ *is an effective descent morphism.*

PROOF. Recall that for $p : E \to B$ to be an effective descent morphism, in the diagram

$$\begin{tikzcd} E \times_B E \times_B A \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & E \times_B A \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\ E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B \end{tikzcd}$$

$p' : E \times_B A \to A$ needs to already be a coequalizer of $d', c' : E \times_B E \times_B A \to E \times_B A$, and in the diagram

$$\begin{tikzcd} E \times_B E \times_E C \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & C \arrow[r, "p'"] \arrow[d, "f_0"'] & D \arrow[d, dashed, "g"] \\ E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B \end{tikzcd}$$

the right hand square needs to already be a pullback.

The latter has already been proved to be true of any surjective étale $p : E \to B$. For the former, note that as $p : E \to B$ is surjective it is regular-stable, and so the coequalizer of its kernel pair d, c, and $p' : E \times_B A \to A$ is the coequalizer of its kernel pair. So we just need to show that

$$\begin{tikzcd} E \times_B E \times_B A \arrow[r, "c'"] \arrow[d, "d'"'] & E \times_B A \arrow[d, "p'"] \\ E \times_B A \arrow[r, "p'"'] & A \end{tikzcd}$$

is a pullback in **FinTop**.

So take any $(e, a), (e', a)$ such that $pe = pe' = fa$. Then (e, e', a) is the unique point in $E \times_B E \times_B A$ with $d'(e, e', a) = (e, a)$ and $c'(e, e', a) = (e', a)$. □

Not every effective descent morphism in **FinTop** is surjective and étale, but the universal covering $p : E \to B$ has already been shown to be so, and this effective descent morphism will be enough for our purposes.

However, with the definition

DEFINITION 108. **Etale** is the subcategory of **FinTop** with the same objects, but only étale morphisms.

we do have

THEOREM 109. *The effective descent morphisms in* **Etale** *are the surjectives.*

PROOF. Observe that the morphisms in the proof of the above theorem are étale. □

Note that the surjectives in **Etale** are the epimorphisms, and that they are all *regular*, i.e. they are all coequalizers:

THEOREM 110. *In* **Etale** *all epimorphisms are regular.*

PROOF. For $p : E \to B$ an étale surjection, form the kernel pair

$$\begin{tikzcd} E \times_B E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

in **FinTop**.

$(e, e') \in E \times_B E$ if and only if $pe = pe'$, and $(e, e') \to (e'', e''')$ in $E \times_B E$ if and only if $e \to e''$ and $e' \to e'''$ in E.

Then

$$\begin{aligned} \downarrow (e, e') &= \{(e'', e''') \in E \times_B E | (e'', e''') \to (e, e')\} \\ &= \{(e'', e''') \in E \times_B E | e'' \to e, e''' \to e'\} \\ &= \{(e'', e''') \in E \times_B E | e'' \in\downarrow e, e''' \in\downarrow e'\} \end{aligned}$$

and

$$\begin{aligned} d(\downarrow (e, e')) &= d\{(e'', e''') \in E \times_B E | e'' \in\downarrow e, e''' \in\downarrow e'\} \\ &= \{e'' \in E | e'' \in\downarrow e, \exists e', e''' \in E, pe' = pe, pe''' = pe'', e''' \in\downarrow e'\} \end{aligned}$$

But of course $pe = pe$ and $pe'' = pe''$, and $e'' \in\downarrow e$, so

$$\begin{aligned} d(\downarrow (e, e')) &= d\{(e'', e''') \in E \times_B E | e'' \in\downarrow e, e''' \in\downarrow e'\} \\ &= \{e'' \in E | e'' \in\downarrow e\} \\ &= \downarrow e \\ &= \downarrow d(e, e') \end{aligned}$$

Of course same thing goes for $c(e, e') = e'$.

Now to show that $p : E \to B$ is a coequalizer of d, c, consider the diagram

$$\begin{tikzcd} E \times_B E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"] \arrow[dr, "f"'] & B \arrow[d, dashed, "f'"] \\ & & C \end{tikzcd}$$

with $fd = fc$, i.e $fd(e, e') = fc(e, e')$, i.e. if $pe = pe'$ then $fe = fe'$.

$p : E \to B$ is surjective, so for each $b \in B$ there's $e \in E$ with $pe = b$, and $f\{e \in E | pe = b\} = fe$.

So $f' : B \to C$ can (and must) be $f'b = fe$. $\square$

But **Etale** is not exact, as it is not even finitely complete. For though **FinTop** is finitely complete, its limit projections are not always étale (e.g. in any pullback of a map that isn't a surjection),

Thus we could not simply have used Proposition 3.1.(b) of [**FinPreI**] (in an exact category the effective descent morphisms are the regular-stable epimorphisms) to characterize the effective descent morphisms in **Etale**. Though from the above, and from Theorem 35, which tells us that effective descent morphisms in **Etale** are pullback stable, we see that in **Etale** the classes of surjectives, epimorphisms, regular epimorphisms, regular-stable epimorphisms, and effective descent morphisms all coincide.

CHAPTER 5

Coverings of Finite Spaces

5.1. Trivial coverings

THEOREM 111. *The functor*

$$H : \mathbf{FinSet} \to \mathbf{FinTop}$$

that sends a set X to the discrete space X has a left adjoint reflection

$$I : \mathbf{FinTop} \to \mathbf{FinSet}$$

that sends a finite space to its set of (path) connected components IA.

PROOF. Clearly, for each X in **FinSet** we have $IHX \cong X$, so these can be made into a counit $\varepsilon : IH \cong 1$. □

The unit for the reflection $(I, H, \eta, \varepsilon) : \mathbf{FinTop} \to \mathbf{FinSet}$ is the projection $\eta_A : A \to HIA$ that sends each point in A to the component it belongs to.

THEOREM 112. *The reflection $(I, H, \eta, \varepsilon) : \mathbf{FinTop} \to \mathbf{FinSet}$ is admissible.*

PROOF. $H^E : \mathbf{FinSet}/IE \to \mathbf{FinTop}/E$ sends (X, u) to $(\coprod_X ux, \pi_1)$ as in the pullback

$$\begin{tikzcd}
\coprod_X ux \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & HX \arrow[d, "Hu"] \\
E \arrow[r, "\eta_E"'] & HIE
\end{tikzcd}$$

where $\pi_1(e, x) = e$ and $\pi_2(e, x) = x$.

Clearly $I\pi_2 : I(\coprod_X ux) \cong X$, and as this is a counit $\varepsilon^E_{(X,u)}$ (see Chapter 2), H^E is fully faithful. □

Of course we will denote the replete image of H^E by $\mathbb{M}/E$

When E is connected, so that $IE \cong 1$ these are the same as the trivial coverings as classically defined, since the pullback above is then simply

$$\begin{tikzcd} \coprod_X E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & HX \arrow[d] \\ E \arrow[r] & 1 \end{tikzcd}$$

5.2. Coverings

Now we switch to the subcategory **Etale** of **FinTop**, with the same objects, but only étale maps. We couldn't do this in 5.1, because the units $\eta_E : E \to HIE$ are not étale. We need to do this now because we know that the effective descent morphisms in **Etale** are just the surjectives, but we haven't characterized the effective descent morphisms in **FinTop** (though that has been done, by Janelidze and Sobral in their paper [**FinPreI**]). We are justified in doing this because we already know from 3.6 that the coverings are étale. So the category **Etale** has all the morphisms we will need going forward.

THEOREM 113. *In* **Etale**, *a morphism $f : A \to E$ in $\mathbb{M}$ is op-étale*

PROOF. If $f : A \to E$ is in the replete image of H^E, then there is a set X and a map $u : X \to IE$ such that it is isomorphic to $\pi_1 : \coprod_X ux \to E$, with $\pi_1(e, x) = e$. Thus A consists of X components A_x, each of which is isomorphic to a component E_{ux} of E, and the restriction of $f : A \to B$ to A_x is $A_x \cong E_{ux}$. Since $\uparrow a$ is connected, it lies entirely withing one component A_x of A. Thus $f(\uparrow a) \cong\uparrow fa$. □

THEOREM 114. *In* **Etale** *if $p : E \to B$ is surjective, and in the pullback*

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

$\pi_1 : E \times_B A \to E$ is op-étale, then so is $f : A \to B$.

PROOF. Take any $a \in A$ and $fa \to b'$ in B. As $p : E \to B$ is surjective there's $e' \in E$ with $pe' = b'$. As $p : E \to B$ is étale there's then unique $e \to e'$ in E with $pe = fa$. Thus $(e, a) \in E \times_B A$. As $\pi_1 : E \times_B A$ is op-étale and $e \to e'$ in E there's unique $a' \in A$ such that

$(e, a) \to (e', a')$ in $E \times_B A$, i.e. unique $a' \in A$ such that $fa' = pe' = b'$ and $a \to a'$ in A. $\square$

Thus we have

THEOREM 115. *Every morphism $f : A \to B$ in $\mathbb{M}^*$ is op-étale.*

5.3. Fundamental groups and the Galois group

If $p : E \to B$ is a universal covering with connected E then there's a set G such that the following diagram is a pullback

$$\begin{tikzcd} \coprod_G E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

with $d, c : \coprod_G E \to E$ trivial coverings, so that for each $g \in G$, $d(-, g) : E \to E$ and $c(-, g) : E \to E$ are isomorphisms with $pd(-, g) = pc(-, g)$. In fact, amongst the isomorphic pullbacks, choose the one with $d(-, g) = 1 : E \to E$, and then we may as well write $c(e, g) = g(e)$, thinking of each g itself as an isomorphism $g : E \to E$, i.e. G itself a set of automorphisms of E. The commuting of the square then becomes $pg = p$, so that we may further say that G is a set of automorphisms $g : E \to E$ with $pg = p$. In fact, as a set $G \cong \mathrm{Aut}(E, p)$, the group of automorphisms $g : E \to E$ with $pg = p$, known as the Chevalley fundamental group of B.

THEOREM 116. *$p^{\#}(E, p) \in \mathbb{M}^{\bar{B}}$ is isomorphic to the canonical action of the group $\mathrm{Aut}(E, p)$ on its own set of elements.*

PROOF. Recall that to construct $p^{\#}(E, p)$ we first form the pullback

$$\begin{tikzcd} \coprod_{G\times G} E \arrow[r, "v"] \arrow[d, "u"'] & \coprod_G E \arrow[d, "d"] \\ \coprod_G E \arrow[r, "d"'] & E \end{tikzcd}$$

$$\begin{tikzcd} (x, g, h) \arrow[r, mapsto, "v"] \arrow[d, mapsto, "u"'] & (x, g) \arrow[d, mapsto, "d"] \\ (x, h^{-1}) \arrow[r, mapsto, "d"'] & x \end{tikzcd}$$

(The choice of this particular diagram from amongst the isomorphic pullbacks is not essential but makes things neater later; we could have just have well used

$$\begin{tikzcd}
(x,g,h) \arrow[r, mapsto, "v"] \arrow[d, mapsto, "u"'] & (x,h) \arrow[d, mapsto, "d"] \\
(x,g) \arrow[r, mapsto, "d"'] & x
\end{tikzcd}$$

or another version of the pullback, and all that would happen is we'd end up defining $\mathrm{Aut}(E,p)$ terms of a binary operation other than composition.)

Then, since $pcu = pdu = pdv = pcv$, construct $w : \coprod_{G\times G} E \to \coprod_G E$ as the unique fill-in in

$$\begin{tikzcd}
\coprod_{G\times G} E \arrow[dr, dashed, "w"] \arrow[drr, bend left, "cv"] \arrow[ddr, bend right, "cu"'] & & \\
& \coprod_G E \arrow[r, "c"] \arrow[d, "d"'] & E \arrow[d, "p"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
(x,g,h) \arrow[dr, dashed, mapsto, "w"] \arrow[drr, bend left, mapsto, "cv"] \arrow[ddr, bend right, mapsto, "cu"'] & & \\
& (h^{-1}x, gh) \arrow[r, "c"] \arrow[d, mapsto, "d"'] & gx \arrow[d, mapsto, "p"] \\
& h^{-1}x \arrow[r, mapsto, "p"'] & b
\end{tikzcd}$$

Thus the following square commutes

$$\begin{tikzcd}
\coprod_{G\times G} E \arrow[r, "w"] \arrow[d, "u"'] & \coprod_G E \arrow[d, "d"] \\
\coprod_G E \arrow[r, "c"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
(x,g,h) \arrow[r, mapsto, "w"] \arrow[d, mapsto, "u"'] & (h^{-1}x, gh) \arrow[d, mapsto, "d"] \\
(x,h^{-1}) \arrow[r, mapsto, "c"'] & h^{-1}x
\end{tikzcd}$$

Now $Ip^{\#}(E,p)$ is

$$\begin{tikzcd} G \times G \arrow[r, "Iv"] \arrow[d, "Iu"'] & G \arrow[d] \\ G \arrow[r] & E \end{tikzcd}$$

$$\begin{tikzcd} (g,h) \arrow[r, mapsto, "Iv"] \arrow[d, mapsto, "Iu"'] & g \arrow[d, mapsto] \\ h^{-1} \arrow[r, mapsto] & \cdot \end{tikzcd}$$

$$\begin{tikzcd} G \times G \arrow[r, "Iw"] \arrow[d, "Iu"'] & G \arrow[d] \\ G \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} (g,h) \arrow[r, mapsto, "Iw"] \arrow[d, mapsto, "Id"'] & gh \arrow[d, mapsto] \\ h^{-1} \arrow[r, mapsto] & \cdot \end{tikzcd}$$

that is, $Iw : G \times G \to G$ is just the canonical action of the group G on its set of elements. □

Now recall the covering $p : (b, \eta_B^\pi) \to B$ constructed in Section 3.6. Recall that $p^{-1}(b) = \hom_{\pi B}(b,b) = \pi_1(B,b)$, the Poincaré fundamental group of B. Each element of $p^{-1}(b)$ is a looping path class $[\lambda] : b \to b$, and $p^{-1}(b) \subset (b, \eta_B^\pi)$ is discrete. In **FinTop** the following square

$$\begin{tikzcd} \coprod_{p^{-1}(b)}(b \downarrow \pi B) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (b \downarrow \pi B) \arrow[d, "p"] \\ (b \downarrow \pi B) \arrow[r, "p"'] & B \end{tikzcd}$$

with $\pi_1([\alpha],[\lambda]) = [\alpha]$ and $\pi_2([\alpha],[\lambda]) = [\lambda][\alpha]$ is a pullback.

For take any $[\alpha] : b \to b'$ and $[\beta] : b \to b'$.

Then $[\beta][\alpha]^{-1} : b \to b$ is the unique element of $p^{-1}(b)$ such that $\pi_2([\alpha],[\beta][\alpha]^{-1}) = [\beta][\alpha]^{-1}[\alpha] = [\beta]$.

Furthermore, in $\coprod_{p^{-1}(b)}(b \downarrow \pi B)$, $([\alpha],[\lambda]) \to ([\alpha'],[\lambda'])$ if and only if $[\lambda] = [\lambda']$ and $[\alpha] \to [\alpha']$ in $(b \downarrow \pi B)$, i.e. in B $p\alpha \to p\alpha'$ and $[\alpha'] = [\alpha][p\alpha \to p\alpha']$.

But then $[\lambda][\alpha'] = [\lambda][\alpha][p\alpha \to p\alpha']$, so $\pi_2([\alpha],[\lambda]) \to \pi_2([\alpha'],[\lambda])$ in $(b \downarrow \pi B)$.

Conversely, if $[\alpha] \to [\alpha']$ and $[\lambda][\alpha] \to [\lambda'][\alpha']$ in $(b \downarrow \pi B)$, then $[\alpha][p\alpha \to p\alpha'] = [\alpha']$ and $[\lambda'][\alpha'] = [\lambda][\alpha][p\alpha \to p\alpha']$, so $[\lambda'] = [\lambda][\alpha][p\alpha \to p\alpha'][\alpha']^{-1} = [\lambda]$, and so $([\alpha],[\lambda]) \to ([\alpha'],[\lambda'])$ in (b, η_B^π).

Thus we almost have a long promised theorem:

THEOREM 117. *Every connected finite topological space has a universal covering.*

because we've nearly proved

THEOREM 118. $p : (b \downarrow \pi B) \to B$ *is a universal covering.*

PROOF. We already know that it is of effective descent, since when we constructed it in we showed it was surjective (Theorem 97) and étale (98). Then, for any covering $f : A \to B$, the following diagram is a pullback:

$$\begin{tikzcd} \coprod_{f^{-1}(b)}(b \downarrow \pi B) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ (b \downarrow \pi B) \arrow[r, "p"'] & B \end{tikzcd}$$

where $\pi_1([\beta], a) = [\beta]$ but $\pi_2([\beta], a)$ is such that $[\alpha] : a \to \pi_2([\beta], a)$ is the unique path class in A with $f[\alpha] = [\beta]$. $\square$

As universal coverings are isomorphic, repeating the proof of Theorem 116, with $E = (b, \eta_B^\pi)$ and $G = p^{-1}(b)$ we find that $\pi_1(B, b) \cong Ip^{\#}(E, p) \cong \mathrm{Aut}(E, p)$, thus

THEOREM 119. *For any connected finite topological space B the following coincide*

(1) *the Poincaré fundamental group* $\pi_1(B, b)$
(2) *the Chevalley fundamental group* $\mathrm{Aut}(E, p)$
(3) *the Galois group* $\mathrm{Gal}(E, p)$

5.4. Fundamental theorem of Categorical Galois Theory

Now I want to give some concrete description of the fundamental theorem of Categorical Galois Theory for the reflection of the category of finite topological spaces into the category of finite discrete topological spaces. (There is no need to prove the fundamental theorem again; since we have an admissible reflection and universal coverings, we know it's true here from the arguments in Chapter 2.)

In our context the fundamental theorem says there's an equivalence

$$(I^{\bar{B}}p^{\#}, p_{\#}H^{\bar{B}}) : \mathbb{M}^*/B \sim \mathbf{FinSet}^{\mathrm{Gal}(E,p)}$$

In this equivalence, $f : A \to B$ étale and op-étale is sent to an action $Ip^{\#}(A, f)$ of $\mathrm{Gal}(E, p)$ on some finite set.

$p^{\#}(A, f)$ must involve, for some finite set X, the pullbacks

$$\begin{tikzcd}
\coprod_{G\times X} E \arrow[r, "d'"] \arrow[d, "f_1"'] & \coprod_X E \arrow[r, "\pi_2"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
\coprod_G E \arrow[r, "d"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

with G the underlying set of $\mathrm{Gal}(E, p) \cong \mathrm{Aut}(E, p) \cong \pi_1(B, b)$, $f_0(e, x) = e$, $f_1(e, g, x) = (e, g)$, $d(e, g) = e$, and the commuting square

$$\begin{tikzcd}
\coprod_{G\times X} E \arrow[r, "c'"] \arrow[d, "f_1"'] & \coprod_X E \arrow[d, "f_0"'] \\
\coprod_G E \arrow[r, "c"'] & E
\end{tikzcd}$$

And the I-image of this in **FinSet** is (isomorphic to) pullbacks

$$\begin{tikzcd}
G \times X \arrow[r, "Id'"] \arrow[d, "If_1"'] & X \arrow[r, "I\pi_2"] \arrow[d, "If_0"'] & IA \arrow[d] \\
X \arrow[r] & 1 \arrow[r] & 1
\end{tikzcd}$$

so that $X \cong IA$, and a commuting square

$$\begin{tikzcd}
G \times IA \arrow[r, "Ic'"] \arrow[d, "If_1"'] & IA \arrow[d, "If_0"] \\
IA \arrow[r] & 1
\end{tikzcd}$$

in which $Ic' : G \times IA \to IA$, is an ordinary action of the group G on the set IA.

But the pullback

$$\begin{tikzcd} \coprod_{IA} E \arrow[r, "\pi_2"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

in **FinTop** can also be written

$$\begin{tikzcd} \coprod_{f^{-1}(b)} E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ (b, \eta_B^\pi) \arrow[r, "p"'] & B \end{tikzcd}$$

For, as $f : A \to B$ is a covering, the fibre $f^{-1}(b)$ is discrete, and so we can consider it a set and take the coproduct over it. Then, for each $a' \in A$ and path class $[\beta] : b \to fa'$ in B, let $[\alpha] : a \to a'$ be the unique path class with $f[\alpha] = [\beta]$. Then $([\beta], a) \in \coprod_{f^{-1}(b)} E$ is unique, if we define π_2 so that it sends $([\beta], a)$ to a' – the end of the unique path class $[\alpha] : a \to a'$ with $f[\alpha] = [\beta]$.

So $f^{-1}(b) \cong IA$, and since the point $b \in B$ is arbitrarily chosen, we have

THEOREM 120. *The fibres of a covering are discrete and isomorphic to each other.*

Also, the action $Ic' : G \times f^{-1}(b) \to f^{-1}(b)$ sends $([\lambda] : b \to b, a)$ to a', where $[\alpha] : a \to a'$ is the unique path class with $f[\alpha] = [\lambda]$.

In the other direction, $p_\# H^{\bar{B}}$, gives us a method to construct all the coverings of a connected B:

Let $h : G \times X \to X$ be any action of $G = \mathrm{Gal}(E, p)$ on any finite set X. Then form the following in **FinTop**:

$$\begin{tikzcd} \coprod_{G \times X} E \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & \coprod_X E \arrow[d, "f_0"'] \\ \coprod_G E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \end{tikzcd}$$

with $d'(e, g, x) = (e, x)$ and $c'(e, g, x) = (e, h(g, x))$. Form a coequalizer $p' : \coprod_X E \to A$ of d', c', and then the unique fill in $f : A \to B$, as in

$$\begin{tikzcd}
\coprod_{G\times X} E \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "f_1"'] & \coprod_X E \arrow[r, "p'"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\
\coprod_G E \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

This $f : A \to B$ is a covering of B, and every covering of B can be constructed in this way, from an action of $\mathrm{Gal}(E, p)$ on a set.

Bibliography

[CatWork] Mac Lane, S. (1971). *Categories for the Working Mathematician* (Graduate Texts in Mathematics; 5). Springer-Verlag.

[UCC] R. Paré. (1993). Universal covering categories, *Rendiconti dell'Istituto di Matematica dell'Università di Trieste* 25, 1-2, 391-411.

[LocStab] Carboni, A., Janelidze, G., Kelly, G.M. et al. (1997). On Localization and Stabilization for Factorization Systems. *Applied Categorical Structures* 5, 1–58.

[GalTheo] Borceux, F., & Janelidze, G. (2001). *Galois Theories* (Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press.

[FinPreI] Janelidze, George & Sobral, Manuela. (2002). Finite preorders and Topological descent I. *Journal of Pure and Applied Algebra* 175. 187-205.

[CMiCoRA] Clementino, M.M., Hofmann, D. & Montoli, A. (2014). Covering Morphisms in Categories of Relational Algebras. *Applied Categorical Structures* 22, 767–788.