THE RELEVANCE OF THE PAULI GROUP IN DYNAMICAL SYSTEMS WITH PSEUDO-FERMIONS

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Abstract. The group of Wolfgang Pauli is well known in mathematical physics, because it describes some relevant symmetries in quantum dynamical systems. It is less known its structure of finite 2-group of order 16, which may be decomposed in the central product of two of its subgroups. From this perspective, the Pauli group has an interesting structure at an algebraic level as well. Here a topological perspective is added to the literature. It is described the Pauli group as an appropriate quotient of the fundamental group of 3-dimensional Riemannian surfaces constructed as two distinct orbit spaces of the 3-dimensional sphere $\mathbb{S}^3$; one orbit space comes from the free action of the quaternion group Q_8 on $\mathbb{S}^3$; another orbit space comes from a similar action of the cyclic group $\mathbb{Z}(4)$ of order 4 on $\mathbb{S}^3$. Applications are illustrated for Pseudo-fermionic operators, introducing a relevant framework of quantum mechanics. This suggests a physical interpretation for the topological decomposition, which has been found at an abstract level.

DECLARATION

I, Yanga Bavuma , hereby declare that the work on which this thesis is based is my original work (except where acknowledgements indicate otherwise) and that neither the whole work nor any part of it has been submitted for another degree in this or any other university. I authorise the University to reproduce for the purpose of research either the whole or any portion of the contents in any manner whatsoever.

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INTRODUCTION

The Pauli matrices were first introduced in an equation for the dynamics of spin particles in an external magnetic field (see [32, 37]), and then generalised to many dynamical systems in several different ways, especially they turn out to have an important role in the quantum information theory and in quantum mechanics (see [27, 28, 35, 40]). Pauli matrices form a nonabelian group of order 16, known as Pauli group, and structural decompositions of this group have been studied at an algebraic level and at a topological level in the present thesis.

If it is true that the notion of central product can be found in [24, 25, 43] and is well known in group theory, it is also true that it wasn't easy to recognise it in the Pauli group of order 16. In fact we conjecture that the notion of central product may have a significant influence on certain dynamical systems involving pseudo-fermions, which we are going to illustrate. Our conjecture is presented in the last section of the present thesis, since it needs a few technicalities before to be formulated. The division of each section has been designed to introduce properly our final conjecture.

We start with Chapter 1, revising some basic properties of group actions and quotient spaces which will be key tools for the construction we use for our first main theorem. Our main idea is to find a fundamental group on Riemannian manifolds, so we spend time in Chapter 2 introducing the fundamental group of a topological space and discussing the various ways in which one can calculate the fundamental group of an arbitrary topological space. We also recall some combinatorial group theory and formal languages: they allow us to handle abstract groups quite naturally. Then the celebrated Theorem of Seifert and Van Kampen is presented as a powerful tool for calculating the fundamental group of a space.

In Chapter 3 we bridge concretely topology and algebra to find our desired Riemannian manifold; roughly speaking this joins two quotient 3-manifolds via an ad hoc construction. Chapter 4 discusses some basic mathematical descriptions of "ordinary" (non relativistic) quantum mechanical systems before focusing on pseudo-fermionic systems. These systems will be where we apply the mathematical framework we had hitherto built up to. We note that this is one description of a quantum mechanical system and the reader can find a more useful description using algebras in [1].

Chapter 5 will be spent solely investigating the Pauli group at the level of its group presentation, leaving the proofs for the main results in Chapter 6. Then we end with Chapter 7, illustrating our conjecture and new ideas for further investigations. In fact some of the main results of the present thesis were published in a recent contribution [5], which puts the basis for a first step towards a unifying approach between the notion of central product and the theory of the pseudo-fermionic operators. We left out most proofs of results which can be found in the bibliography, promoting a more direct approach to the subject of study.

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1. Basic properties of group actions

In every algebraic and topological structure, we usually look for substructures which are stable under prescribed properties, i.e. subgroups and subspaces, and then we look for the stability for structures which are obtained by imposing equivalence relations on the original structure, i.e. quotient groups and quotient spaces. We begin with the formal notion.

Definition 1.1 (See [34], Definition 5.1)**.** Suppose that $f : X \to Y$ is a surjective mapping from a topological space X onto a set Y. The *quotient topology* on Y with respect to f is the family

$$\mathcal{U}_f = \{U : f^{-1}(U) \text{ is open in } X\} \tag{1.1}$$

While the induced topology on a subspace is the weakest topology that would make embedding that subspace in the original space continuous, the quotient topology is the opposite extreme.

Proposition 1.2 (See [34], Exercise 5.3(a))**.** *Suppose that Y is given the quotient topology with respect to the mapping $f : X \to Y$. Then Y has the strongest topology such that f is continuous.*

We can start exploring the connections between algebra and topology by formalising the idea of transformation on a topological space by the notion of a group action on a topological space.

Definition 1.3 (See [34], Definition 5.6)**.** Let X be a set and let G be a group. We say that G *acts* on X and that X is a G-set if there is a function from $G \times X$ to X, denoted by $(g, x) \to g \cdot x$, such that

(i) $1 \cdot x = x$ for all $x \in X$, where 1 is the identity element of G,
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$.

If in addition the action is continuous and the map $\theta_g : x \in X \mapsto g \cdot x \in X$ is an homeomorphism for all $g \in G$, we say that X is a G-*space*.

The previous definition of a G-action is strictly speaking that of a *left* G-action and one can see easily that the above axioms are not always satisfied. For example:

Example 1.4. Let S_3 act on $\mathbb{R}^3$ by permuting co-ordinates: $(\sigma, (x_1, x_2, x_3)) \in S_3 \times \mathbb{R}^3 \mapsto \mu_\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \in \mathbb{R}^3$.

Let $\sigma = (12)$ and $\sigma' = (23)$ and $(x_1, x_2, x_3) = (8, 2, 6)$ then

$$\begin{aligned} \mu_\sigma(\mu_{\sigma'}(x)) &= \mu_{(12)}(\mu_{(23)}(8, 2, 6)) \\ &= \mu_{(12)}(8, 6, 2) \\ &= (6, 8, 2) \end{aligned} \tag{1.2}$$

Whereas $\mu_{(12)} \circ \mu_{(23)} = \mu_{(123)}$, and we have

$$\mu_{(123)}(8, 2, 6) = (2, 6, 8) \tag{1.3}$$

so clearly the axiom of associativity is violated. So this does not define an action on $\mathbb{R}^n$.

A first fact that we can observe is that it is possible to restrict the actions of groups in a natural way. The following result describes the actions, restricted to subgroups.

Lemma 1.5 (See [34], Exercise 5.7(b))**.** *Let H be a subgroup of a group G. For $h \in H$, $g \in G$ define $h \cdot g$ to be hg. This defines an action of H on G.*

Let's now study the orbits of the actions, which are subsets of points that can be transformed into one another.

Definition 1.6 (See [34], Definition 5.6)**.** Let G be a group that acts on the set X (from the left). For $x \in X$, we define the *orbit of x under the action of G* by

$$\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\} \tag{1.4}$$

Of course, $\mathrm{Orb}_G(x) \subseteq X$ and we note that the orbits partition X, because of the relation

$$y \sim x \ \Leftrightarrow \ y \in \mathrm{Orb}_G(x).$$

This means that two elements $x, y \in X$ are related by $\sim$ if and only if $y = g \cdot x$ for some $g \in G$. More precisely we can check,

Reflexive: : $x = 1 \cdot x$ shows that $x \sim x$ for all $x \in X$.
Symmetric:: If $y \sim x$, then there is $g \in G$ such that $g \cdot x = y$. If we multiply by g^{-1} from the left, then $x = g^{-1} \cdot (g \cdot x) = g^{-1} \cdot y$, and so $x \sim y$.
Transitive:: If $y \sim x$ and $x \sim z$, then there are $g_1, g_2 \in G$ such that $g_1 \cdot x = y$ and $g_2 \cdot z = x$. By substituting, we have $y = g_1 \cdot (g_2 \cdot z) = (g_1 g_2) \cdot z$ and so $y \sim z$. Thus $\sim$ is transitive.

We have all that we need in order to describe quotient spaces that arise from actions of groups. The following is an example of the application of this description. This Proposition will be followed with a proof to help with the intuition:

Proposition 1.7 (See [34], Exercise 5.9(a))**.** *Let X be the infinite strip*

$$\left\{(x,y) \in \mathbb{R}^2 : -\frac{1}{2} \leq y \leq \frac{1}{2}\right\}$$

in $\mathbb{R}^2$ with $\mathbb{Z}$ acting on it by $m \cdot (x,y) = (m + x, (-1)^m y)$. Then the quotient space $X/\mathbb{Z}$ is homeomorphic to the Möbius strip.

Proof. Let $M/_\sim$ be the equivalence classes of the unit square with the relation $\sim$, such that: $(x,y)_\sim(x',y')$ if and only if $(x,y) = (x',y')$ or $\{x,x'\} = \{0,1\}$ and $y = 1 - y'$. Then $M/_\sim$ is the Möbius strip. Consider the following two functions:

$$f : [(x,y)]_{X/\mathbb{Z}} \in X/\mathbb{Z} \longmapsto \left[\left(x - \lfloor x \rfloor, (-1)^{\lfloor x \rfloor} y + \frac{1}{2}\right)\right]_\sim \in M/_\sim$$

$$g : [(x,y)]_\sim \in M/_\sim \longmapsto \left[\left(x, y - \frac{1}{2}\right)\right]_{X/\mathbb{Z}} \in X/\mathbb{Z}$$

We note that f is well defined: if $(x,y) \in X$ and $n \in \mathbb{Z}$, then $[(x,y)]_{X/\mathbb{Z}} = [(n + x, (-1)^n y)]_{X/\mathbb{Z}}$ and we get

$$\begin{aligned} f([(n+x,(-1)^n y)]_{X/\mathbb{Z}}) &= [(n+x-(\lfloor n+x\rfloor),(-1)^{\lfloor n+x\rfloor}(-1)^n y+\frac{1}{2})]_\sim \\ &= [(x-\lfloor x\rfloor,(-1)^{2n+\lfloor x\rfloor}y+\frac{1}{2})]_\sim \\ &= [(x-\lfloor x\rfloor,(-1)^2 n(-1)^{\lfloor x\rfloor}y+\frac{1}{2})]_\sim \\ &= [(x-\lfloor x\rfloor,(-1)^{\lfloor x\rfloor}y+\frac{1}{2})]_\sim \end{aligned} \tag{1.5}$$

Also g is well defined: since the equivalence classes that do not contain points from the left and right edges only contain one point, we will consider only the points on the left and right edges. If $[(0,y)]_\sim = [(1,1-y)]_\sim$, then $g([(0,y)]_\sim) = [(0,y-\frac{1}{2})]_{X/\mathbb{Z}}$ and $g([(1,1-y)]_\sim) = [(1,1-y-\frac{1}{2})]_{X/\mathbb{Z}} = [(1,\frac{1}{2}-y)]_{X/\mathbb{Z}}$.

Since $(1,\frac{1}{2}-y) = (1+0,(-1)^1(y-\frac{1}{2})) = 1\cdot(0,y-\frac{1}{2})$, we have $[(1,\frac{1}{2}-y)]_{X/\mathbb{Z}} = [(0,y-\frac{1}{2})]_{X/\mathbb{Z}}$. Now an argument of routine allows us to conclude that f and g are the inverses of the other and so they are bijective. Moreover they are continuous since they involve elementary functions in their definition. □

Theorem 1.8 (See [34], Theorem 5.5)**.** *Let $f : X \to Y$ be a continuous function between the topological spaces X and Y. Suppose that X and Y have equivalence relations $\sim_X$ and $\sim_Y$ respectively such that $x \sim_X x'$ if and only if $f(x) \sim_Y f(x')$, then $X/\sim_X$ is homeomorphic to a subspace of $Y/\sim_Y$.*

Proof. Define a function $F : X/\sim_X \to Y/\sim_Y$ by $F[x] = [f(x)]$, where the square brackets denote equivalence classes. F is well defined since if $[x] = [x']$ then $x \sim_X x'$, thus $f(x) \sim_Y f(x')$ and $[f(x)] = [f(x')]$. We shall prove that F is injective. First assume that $F[x] = F[x']$, which means that $[f(x)] = [f(x')]$, i.e. $f(x) \sim_Y f(x')$. But then $x \sim_X x'$ and $[x] = [x']$. To prove that F is continuous we consider the natural quotient maps $\pi_X : X \to X/\sim_X$ and $\pi_Y : Y \to Y/\sim_Y$ which are continuous. Clearly $F\pi_X = \pi_Y f$ and since f is continuous we deduce that $F\pi_X$ is continuous and hence F is continuous by the universal mapping property of quotients. The fact that $F^{-1}|_{F(X/\sim_X)}$ is continuous follows in a similar way because $F^{-1}|_{F(X/\sim_X)}\pi_Y = \pi_X f^{-1}|_{f(X/\sim_X)}$. Thus $X/\sim_X \cong F(X/\sim_X) \subseteq Y/\sim_Y$. That is the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X/_{\sim X} \arrow[r, "F"] & Y/_{\sim Y} \end{tikzcd} \tag{1.6}$$

□

Now we need to describe an important aspect in algebraic topology: actions of groups on topological spaces induce what it is called a *covering map* and covering maps allow us to calculate the *fundamental group* of a given space. We will see the formalization of these notions in the following chapters, but for the moment we describe an important property.

Proposition 1.9 (See [34], Exercise 5.13(b))**.** *Let X be a G-space with G finite group. Then the natural quotient map $\pi : x \in X \mapsto \pi(x) = \mathrm{Orb}_G(x) \in X/G$ is both a closed and open mapping.*

Proof. Let U be a closed set in X. Consider $\pi^{-1}(\pi(U))$. Then

$$\begin{aligned}\pi^{-1}(\pi(U)) &= \{x \in X \ : \ \pi(x) \in \pi(U)\} \\ &= \{x \in X \ : \ x = g \cdot y \text{ for some } y \in U, g \in G\} \\ &= \{x \in X \ : \ x \in g \cdot U, \ g \in G\} = \bigcup_{g \in G} g \cdot U.\end{aligned} \tag{1.7}$$

The action of each $g \in G$ is a homeomorphism, so if U is closed then so is $\pi^{-1}(\pi(U))$ (since G is finite) and hence $\pi(U)$ is closed in X/G. Now one can apply exactly the same steps, concluding that p is also open. ☐

To generalise the above result, we can say that

Proposition 1.10 (See [34], See Theorem 5.12)**.** *If X is a G-space, where G is an infinite group. Then the canonical projection $\pi : X \to X/G$ is always an open mapping, but is not necessarily a closed mapping.*

Proof. In the solution of the exercise above, if U is open, then so is $\pi^{-1}(\pi(U))$, regardless the cardinality of G, since the union of the $g \cdot U$ is a union of opens anyway. In general, if U is closed, then this union is no longer closed if it is not made by finitely many $g \cdot U$, so here the cardinality of G must be finite. ☐

2. Calculating the fundamental group of a space

In order to prove our main result we need to briefly talk about the fundamental group, after which we will give a few methods of which to calculate these fundamental groups. These methods will be used later to reverse construct the manifold used in the proof of our first main theorem. We will begin by defining all the elements needed to properly define the fundamental group.

2.1. **The fundamental group.** We begin by introducing the concept of a "homotopy" which is a formalization of the concept of "deforming with continuity".

Definition 2.1 (See [34], Definitions 13.1, 13.2, 13.5)**.** Two continuous maps $f_0, f_1 : X \to Y$ are said to be *homotopic* if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$. When this happens, we briefly write $f_0 \simeq f_1$

Suppose that $A \subseteq X$. We say that f_0 is *homotopic relative* to A, if f_0 is homotopic to f_1 and A is fixed by F in its first variable, that is, $F(a,t) = f_0(a)$ for all $a \in A$ and $t \in [0,1]$. In this case, we write $f_0 \simeq_A f_1$.

In the special case of $[0,1]$ as domain of f_0, f_1 and X as codomain of f_0, f_1 and $\{0,1\} = A$, we say that $f_0 \simeq_{\{0,1\}} f_1$ (or briefly $f_0 \sim f_1$), if there is a continuous map $F : (t,s) \in [0,1] \times [0,1] \to X$ such that $F(t,0) = f_0(t)$, $F(t,1) = f_1(t)$, $F(0,s) = f_0(0)$, $F(1,s) = f_0(1)$ for all $s,t \in [0,1]$. This means that F is deforming the path f_0 with continuity to the path f_1 (with continuity within X), fixing the extremes $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$ along the deformation.

Now we have a formal meaning for “deforming” a space: a space X is *homotopically equivalent* to a space Y, if there are two continuous functions $f_0 : X \to Y$ and $f_1 : Y \to X$ such that $f_0 \circ f_1 \simeq 1_Y$ and $f_1 \circ f_0 \simeq 1_X$. Now we can even talk about "contracting" a space: if X is homotopically equivalent to a point, we say that X *is contractible* (or that X can be contracted to a point). A cone is a clear example of contractible space.

One way is to check that a space X is contractible if and only if the identity map $1 : X \to X$ is homotopic to a constant map. The proof of this fact is elementary and involves just the way in which one has to write the homotopy.

The next proposition gives us a clear intuition of how we can characterize two continuous maps that are homotopic in $\mathbb{R}^n$.

Proposition 2.2 (See [34], Exercise 13.10 (o))**.** *Let Y be a subspace of $\mathbb{R}^n$ and $f, g : X \to Y$ be two continuous maps. If for each $x \in X$, $f(x)$ and $g(x)$ can be joined by a straight-line segment in Y, then $f \simeq g$. In particular, any two maps $f, g : X \to \mathbb{R}^n$ must be homotopic.*

Because of the gluing lemma [34, Lemma 12.2], we may define the multiplication between paths. This is very important in defining the operation of elements in a fundamental group.

Lemma 2.3 (See [34], Lemma 12.1)**.** *If f and g are two paths in X for which the final point of f coincides with the initial point of g then the function $f * g : [0,1] \to X$ defined by*

$$(f * g)(t) = \begin{cases} f(2t) & \textit{if } 0 \leq t \leq \frac{1}{2}; \\ \\ g(2t-1) & \textit{if } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{2.1}$$

is a path in X.

We do not repeat here the proof of the above lemma, but we stress the fact that the re-parametrization that we need, in order to define a new path, by gluing two paths such that the final point of the first is equal to the starting point of the other, may be done in a different way. Below there is another possible re-parametrization, which can allow us to define the multiplication of two paths.

Proposition 2.4 (See [34], Exercise 13.4 (c))**.** *Let* $0 < s < 1$. *Given twp paths* p *and* q *with* $p(1) = q(0)$ *in* X,

$$h(t) = \begin{cases} p(t/s) & 0 \leq t \leq s; \\ \\ q((t-s)/(1-s)) & s \leq t \leq 1. \end{cases} \tag{2.2}$$

defines a multiplication path $h \sim p * q$.

For a path f let $\overline{f}$ be the path given by

$$\overline{f}(t) = f(1-t) \tag{2.3}$$

for all $t \in [0,1]$. It is easy to see that $f \simeq g$ if and only if $\overline{f} \simeq \overline{g}$. Moreover, if f is a loop, then $f * \overline{f}$ gives the constant path to the base point of the loop, which means that we may look at $\overline{f}$, up to equivalence classes modulo the homotopy relative to $\{0,1\}$, as the inverse elements of f with respect to the operation $*$, which we have just defined. The operation $*$ is in fact compatible with the homotopies of loops. The following proposition gives us an important result for the compositions of functions.

Proposition 2.5 (See [34], Exercise 13.4 (e))**.** *If* f_0 *and* f_1 *are paths from* X *to* Y *homotopic relative to* A *and* $g : Y \to Z$ *is a continuous map, then* $g \circ f_0$ *is homotopic relative to* A *with* $g \circ f_1$.

As we would expect, the composition of functions that are homotopic to one another will preserve the homotopy relation.

Proposition 2.6 (See [34], Exercise 13.4 (f))**.** *Suppose that* $f_0 \simeq f_1$ *from* X *to* Y, *and* $g_0 \simeq g_1$ *from* Y *to* Z. *Then* $g_0 \circ f_0 \simeq g_1 \circ f_1$, *defined from* X *to* Z.

What we have seen until now will allow us to define the *fundamental group of a space* X *based at the point* x_0 as the set

$$\pi(X, x_0) = \{[f] \mid f : [0,1] \to X \text{ is a loop with basis at } x_0 = f(0) = f(1)\} \tag{2.4}$$

of equivalence classes

$$[f] = \{g : [0,1] \to X \mid g \sim f \text{ and } g(0) = g(1) = x_0\} \tag{2.5}$$

of loops which are homotopic to f (relative to $\{0,1\}$), and, using the $*$ product as the binary operation, by having $[f][g] = [f * g]$ where $[f], [g] \in \pi(X, x_0)$, where the neutral element is $[\varepsilon_{x_0}]$, where ε_{x_0} denotes the constant loop at x_0, and inverse $[\overline{f}]$ for each element $[f] \in \pi(X, x_0)$. This is remarkable because now we can extract algebraic properties from topological spaces via this fundamental group.

Now we may explore if we can find a relationship between the fundamental groups of spaces from an underlying topological relationship. Let us see that when $\varphi : X \to Y$ is a map between X and another topological space Y and take $[f], [g] \in \pi(X, x_0)$. The compositions $\varphi \circ f$ and $\varphi \circ g$ turn out to be loops in Y and, if $f \sim g$, then also $\varphi \circ f \sim \varphi \circ g$. The details can be checked easily. This allows us to define what we call the *induced homomorphism*, when we have a continuous map like φ, and it is given by

$$\varphi_* : [f] \in \pi(X, x_0) \mapsto \varphi_*([f]) = [\varphi \circ f] \in \pi(Y, \varphi(x_0)) \tag{2.6}$$

which turns out to be a homomorphism of groups with respect to $*$. And there is an unsurprising relationship between the fundamental groups of homotopically equivalent spaces. '

Proposition 2.7 (See [34], Theorem 15.13)**.** *If $\varphi : X \to Y$ is a homotopy equivalence then $\varphi_* : \pi(X, x) \to \pi(Y, \varphi(x))$ is an isomorphism for any $x \in X$.*

2.2. **The fundamental group of a covering space.** Given two topological spaces $\widetilde{X}$ and X, a continuous surjective map $p : \widetilde{X} \to X$ is called a *covering map* if for all $x \in X$, there exists an open neighborhood U of x such that

$$p^{-1}(U) = \bigcup_{i \in I} U_i, \text{ where } U_i \cap U_j = \emptyset \ \forall i \neq j \tag{2.7}$$

and

$$p|U_i : U_i \to U \text{ is homeomorphism } \forall i \in I.$$

In this situation the space $\widetilde{X}$ is called *a covering space* for X. For any point x,

$$p^{-1}(x) = \{\widetilde{x} \in \widetilde{X} \mid p(\widetilde{x}) = x\} \tag{2.8}$$

is necessarily a discrete space, called *the fiber over x*. The open neighborhoods U of x, that we use in the definition of covering map, are said to be *evenly covered neighborhoods* and they form an open cover of the space X. The homeomorphic copies in $\widetilde{X}$ of an evenly covered neighborhood U are called the *sheets*, or *leaves*, over U. The following proposition gives an example of a covering map with the proof to help build the intuition.

Example 2.8. The following map

$$\varphi : t \in \mathbb{R} \longmapsto \varphi(t) = e^{2\pi i t} \in \mathbb{C}^* \tag{2.9}$$

induces a covering map for the torus group $\mathbb{T}$ (see [34, Exercise 17.9 (c)]). The torus group is the quotient group of the additive group of the reals $\mathbb{R}$ through its subgroup $\mathbb{Z}$. It turns out that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a topological group, which intuitively can be thought of as just a group with a topology, but will be made more rigorous in Definition 3.1. In fact $\mathbb{T}$ has the quotient topology (see Definition 1.1) of the natural topology of $\mathbb{R}$ via the canonical projection $\pi : x \in \mathbb{R} \mapsto x + \mathbb{Z} \in \mathbb{T}$. Moreover $\mathbb{T}$ is a group such that the addition $(x, y) \in \mathbb{T} \times \mathbb{T} \mapsto x + y \in \mathbb{T}$ is a continuous map wrt the product topology in $\mathbb{T} \times \mathbb{T} = \mathbb{T}^2$, and the opposite $x \in \mathbb{T} \mapsto -x \in \mathbb{T}$ is continuous. One can see in addition that $\mathbb{T}$ has a compact Hausdorff topology, since $\mathbb{T}$ is continuous image of $\mathbb{R}$ which is Hausdorff, and because π produces a compact topology on $\mathbb{R}/\mathbb{Z}$. Briefly $\mathbb{T}$ is a compact abelian topological (Hausdorff) group [31, Exercise E1.6]. Note also that (2.9) is a continuous homomorphism between the topological groups $\mathbb{R}$ (wrt addition) and $\mathbb{C}^*$ (wrt multiplication). Moreover its kernel is $\ker \varphi = \mathbb{Z}$, so Theorem 1.8 shows that there is an isomorphism of groups which is also homeomorphism at the level of topological spaces and shows that $\mathbb{T}$ and $\mathbb{S}^1$ are isomorphic as groups, and homeomorphic as topological spaces. A fact we use in our example in 2.18, later.

In order to get more information on the fundamental groups of a given topological spaces, we shall recall that if $f : t \in I = [0,1] \mapsto f(t) \in X$ is a path in X, then we may consider the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "p"] & & X \\ & I \arrow[ul, "\widetilde{f}"] \arrow[ur, "f"'] & \end{tikzcd} \tag{2.10}$$

and define the *lifting* $\widetilde{f} : t \in [0,1] \mapsto \widetilde{f}(t) \in \widetilde{X}$ of f as the continuous map, making (2.10), i.e, $p \circ \widetilde{f} = f$ and with $p(\widetilde{x}_0) = f(0) = x_0$ the starting point of the path f. Now one can explore the relationship between the fundamental group on $\widetilde{X}$ and that on X. Here are two fundamental propositions on the existence and uniqueness of the liftings in a prescribed topological space.

Proposition 2.9 (See [34], Theorem 17.6, Existence and Uniqueness of Liftings)**.** *Let $p : \widetilde{X} \to X$ be a covering.*

(i) *Given a path $f : I \to X$ and $a \in \widetilde{X}$ with $p(a) = f(0)$, there is a unique path $\tilde{f} : I \to \widetilde{X}$ such that $p\tilde{f} = f$ and $\tilde{f}(0) = a$.*
(ii) *Given a continuous map $F : I \times I \to X$ and $a \in \widetilde{X}$ with $p(a) = F(0,0)$, there is a unique continuous map $\widetilde{F} : I \times I \to \widetilde{X}$ such that $p\widetilde{F} = F$ and $\widetilde{F}(0,0) = a$.*

Another classical situation which one encounters in studying liftings is described by the following result.

Corollary 2.10 (See [34], Theorem 17.7, Monodromy Theorem)**.**
Let $p : \widetilde{X} \to X$ be a covering, and suppose that $f, g : I \to X$ are homotopic paths relative to $\{0,1\}$. If $\tilde{f}(0) = \tilde{g}(0)$, then $\tilde{f}(1) = \tilde{g}(1)$.

Now we can give some useful facts. In a path connected space X with covering space $\widetilde{X}$ and covering map $p : \widetilde{X} \to X$ such that $p(\widetilde{x}_0) = x_0$, the induced homomorphism

$$p_* : [\widetilde{f}] \in \pi(\widetilde{X}, \widetilde{x}_0) \to p_*([\widetilde{f}]) = [p \circ \widetilde{f}] \in \pi(X, x_0) \tag{2.11}$$

turns out to be (well defined and) injective. Therefore, First Isomorphism Theorem for groups [43, Theorem 1.4.3] shows that

$$\frac{\pi(\widetilde{X}, \widetilde{x}_0)}{\ker p_*} \simeq p_*(\pi(\widetilde{X}, \widetilde{x}_0)) \tag{2.12}$$

Certain G-spaces can give us covering spaces. Suppose that X is a G-space.

Definition 2.11. We say that the action of G on X is *properly discontinuous* if for each $x \in X$ there is an open neighbourhood V of x such that $g \cdot V \cap g' \cdot V = \emptyset$ for all $g, g' \in G$ with $g \neq g'$.

A significant amount of covering maps may originate from properly discontinuous actions on path connected spaces, where discrete groups are acting. In particular, we can have as our covering space the G-space X and cover the quotient space X/G. Therefore, most of the times, we can reduce the calculation of the fundamental group of a given topological space may by looking at it as the quotient space of orbits, providing appropriate actions of discrete groups. In fact we can say,

Proposition 2.12 (See [34], Theorem 17.1)**.** *Let X be a G-space. If the action of G on X is properly discontinuous then $p : X \to X/G$ is a covering.*

We say that G acts *freely* on X (or is a *free action*) if $g \cdot x \neq x$ for all $x \in X$, $g \in G$, $g \neq 1$ and in particular (as will be relevant for proving our main theorem) we find,

Proposition 2.13 (See [34], Theorem 17.2)**.** *If G is a finite group acting freely on a Hausdorff space X then the action of G on X is properly discontinuous.*

2.3. The fundamental group of an orbit space.

Next we explore a method of calculation that will be explicitly used for our construction in our main theorem. Throughout this subsection we shall assume X is a path connected space with a properly discontinuous action of G on it. Thus $p : X \to X/G$ is a covering. So now we investigate the relation between G and the fundamental group of the orbit space X/G.

Let $x_0 \in X$ and $y_0 = p(x_0) \in X/G$. Notice that

$$p^{-1}(y_0) = \{g \cdot x_0 : g \in G\}. \tag{2.13}$$

If $[f] \in \pi(X/G, y_0)$ then there is a unique lift $\tilde{f}$ of f that begins at $x_0 \in X$. The element $\tilde{f}(1) \in p^{-1}(y_0)$ and so there is a unique element $g_f \in G$ such that $\tilde{f}(1) = g_f \cdot x_0$. The correspondence $f \to g_f$ therefore defines a function

$$\varphi : \pi(X/G, y_0) \to G. \tag{2.14}$$

Proposition 2.14 (See [34], Theorem 19.1)**.** *The function $\varphi : \pi(X/G, y_0) \to G$ is a homomorphism of groups.*

The kernel of the homomorphism φ is given next.

Lemma 2.15 (See [34], Lemma 19.2)**.** *The kernel of $\varphi : \pi(X/G, y_0) \to G$ is the subgroup $p_*\pi(X, x_0)$.*

So since $p_*\pi(X, x_0)$ is the kernel of $\pi(X/G, y_0)$ we can then use the First Isomorphism Theorem to show the following relationship between this fundamental group and the group G.

Proposition 2.16 (See [34], Theorem 19.3)**.** *The groups $\pi(X/G, y_0)/p_*\pi(X, x_0)$ and G are isomorphic.*

We end this subsection noting a very useful result.

Corollary 2.17 (See [34], Corollary 19.4)**.** *If X is simply connected then $\pi(X/G, y_0) \cong G$.*

2.4. **Theorem of Seifert-Van Kampen.** There is a last relevant method that we will use to calculate the fundamental group of a path connected topological space. This is known as the theorem of Seifert and Van Kampen and it is a fundamental construction in the main result of the present thesis.

Given a set of distinct symbols, $\{a, b, c, ...\}$, that we call *generators*, and a set of (finite) "words" made from these symbols, $\{P(a, b, c, ...), Q(a, b, c, ...), R(a, b, c, ...), ...\}$, that we call *relators*, we can show that any group G is isomorphic to a presentation of the form $\langle a, b, c, ... \mid P(a, b, c, ...), Q(a, b, c, ...), R(a, b, c, ...), ...\rangle$ (see [36], Chapter 1.2). This is true for any group G and it is possible to prove that a group presentation can be seen always as an appropriate quotient group of a free group via a normal subgroup, which originates from the relations.

In combinatorial group theory, if X is a topological space; U and V are open, path connected subspaces of X; $U \cap V$ is nonempty and path-connected; $w \in U \cap V$; then the natural inclusions $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ for the following commutative diagram

$$\begin{tikzcd}
U \cap V \arrow[r, "i_1"] \arrow[d, "i_2"'] & U \arrow[d, "j_1"] \\
V \arrow[r, "j_2"] & X
\end{tikzcd} \tag{2.15}$$

induce another commutative diagram on the corresponding fundamental groups, given by

$$\begin{tikzcd}
\pi(U \cap V, w) \arrow[r, "(i_1)_*"] \arrow[d, "(i_2)_*"'] & \pi(U, w) \arrow[d, "(j_1)_*"] \\
\pi(V, w) \arrow[r, "(j_2)_*"] & \pi(X, w)
\end{tikzcd} \tag{2.16}$$

Another way to look at $\pi(X, w)$ here is as the *free product with amalgamation* (see [45, Chapter 7] for more details on this notion) of $\pi(U, w)$ and $\pi(V, w)$ so that, given group presentations:

$$\pi(U, w) = \langle u_1, \cdots, u_k \mid \alpha_1, \cdots, \alpha_l \rangle; \tag{2.17}$$

$$\pi(V, w) = \langle v_1, \cdots, v_m \mid \beta_1, \cdots, \beta_n \rangle; \tag{2.18}$$

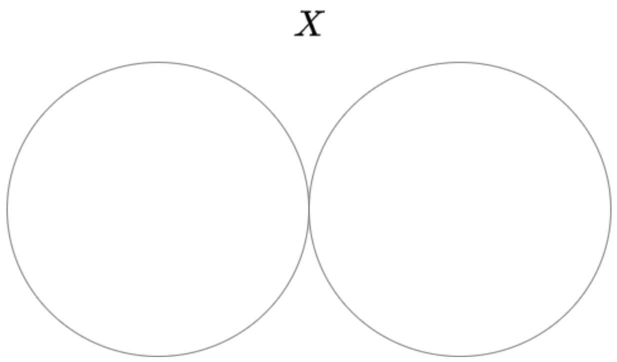
$$\pi(U \cap V, w) = \langle w_1, \cdots, w_p \mid \gamma_1, \cdots, \gamma_q \rangle; \tag{2.19}$$

one can describe $\pi(X, w)$ in terms of generators and relators as

$$\begin{aligned} \pi(X, w) = \langle u_1, \ldots, u_k, v_1, \cdots, v_m \mid \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_n, \\ (i_1)_*(w_1)(j_1)_*(w_1)^{-1}, \ldots, (i_2)_*(w_p)(j_2)_*(w_p)^{-1} \rangle. \end{aligned} \tag{2.20}$$

The way in which one can get the presentation of this last group, beginning from the original data is indeed known as the Seifert-Van Kampen Theorem. Its proof is quite technical and can be found in [34, Chapters 23, 24, 25 and 26]. This means now that if we cannot calculate the fundamental group of a given space directly, if we can calculate the fundamental groups of some subspaces of that space, and there is a way to recover what the fundamental group of the space is.

Example 2.18. Let X be two circles joined at a point. This space is clearly path connected. And we can take our base point to be x_0, the point where the two circles join.



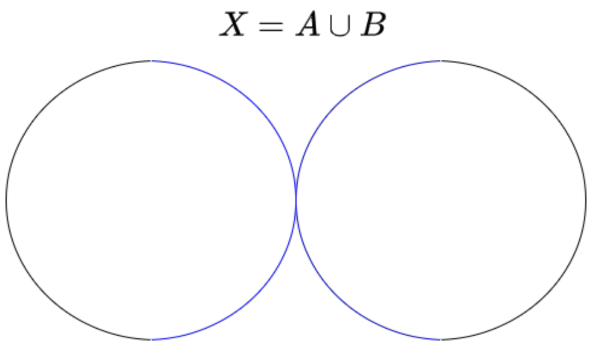
(2.21)

We can decompose our X into two path-connected open subsets A and B in such a way that A and B are homotopic to $\mathbb{S}^1$ (see Example 2.9) as follows

(2.22)

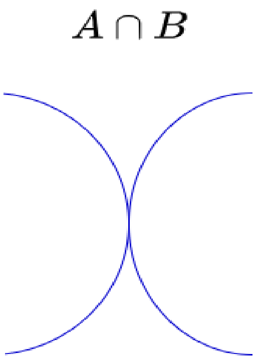
One can see that the $X = A \cup B$ and that by 2.7

$$\pi(A, x_0) = \mathbb{Z} = \langle a \mid \emptyset \rangle \text{ and } \pi(B, x_0) = \mathbb{Z} = \langle b \mid \emptyset \rangle \tag{2.23}$$



(2.24)

And also that their intersection $A \cap B$ is non-empty and path connected,



(2.25)

And this space is contractible so

$$\pi(A \cap B, x_0) = \mathbf{1} = \langle \emptyset \mid \emptyset \rangle. \tag{2.26}$$

So by the Theorem of Seifert-Van Kampen,

$$\pi(X, x_0) = \langle \{a\} \cup \{b\} \mid \emptyset \cup \emptyset \cup \emptyset \rangle = \langle a, b \mid \emptyset \rangle. \tag{2.27}$$

Which is just the free group on two generators, F_2.

On the other hand, we may use Proposition 2.14, 2.16 and Corollary 2.17 in order to get directly the fundamental group of certain topological spaces for which we have enough information on the actions of groups on them. This is exactly the method which we used in the present context, avoiding computational methods which are proper of the geometric group theory.

2.5. Some facts on central products and free products in group theory.

We have seen that the Theorem of Seifert-Van Kampen has a formalization at the level of finitely presented groups (finite generators and relators) and involves also the notion of free product between groups. We are going to mention another notion of product between groups here: this, as the free product, is very abstract but we are going to give a topological interpretation when we will be dealing with the Pauli group. This means that it is relevant to look at combinatorial aspects of products of groups and we refer to [36] and [25, Chapter 2.5] for more details.

Lemma 2.19 (See [36], Theorem 2.1). *Let G have the presentation $\langle a, b, c, ... \mid P, Q, R, ...\rangle$, where $a, b, c, ...$ are generators of G and $P = P(a, b, c, ...), Q = Q(a, b, c, ...), R = R(a, b, c, ...), ...$ are words in G that give us the relators. Let N be the normal subgroup of G generated by the words $S(a, b, c, ...), T(a, b, c, ...), ...$ in G, then the quotient group G/N has the presentation $\langle a, b, c, ... \mid P, Q, R, ..., S, T, ...\rangle$.*

A group with a presentation that has a set of n generators but has an empty set of relators is said to be the *free group on n generators*, F_n, $n \geq 1$. In particular, we can describe the quotients of the free groups and this gives a powerful tool, in order to think at arbitrary groups via appropriate quotients of free groups.

Corollary 2.20 (See [36], Corollary 2.1). *If F is the free group on $a, b, c, ...$ and N is the normal subgroup of F generated by $P(a, b, c, ...)$, $Q(a, b, c, ...)$, $R(a, b, c, ...)$, ..., then $F/N = \langle a, b, c, ... \mid P, Q, R, ...\rangle$.*

This notation and framework will be used extensively in our main theorem and its proof. We now get to present a free product of groups in terms of their presentations.

Definition 2.21 (See [36], Chapter 4)**.** The *free product* of A and B, denoted $A * B$ of the groups

$$A = \langle a_1, a_2, ..., a_n \mid R_1, R_2, ..., R_p \rangle$$

and

$$B = \langle b_1, b_2, ..., b_m \mid S_1, S_2, ..., S_q \rangle$$

is the group

$$A * B = \langle a_1, ..., a_n, b_1, ..., b_m \mid R_1, ..., R_p, S_1, ..., S_q \rangle.$$

A and B are called free factors of $A * B$.

We can formulate a few constructions using this notion of a free product. For instance the Theorem of Seifert-Van Kampen is formulated as a free product (with amalgamation, see [36], Chapter 4.2) and a more familiar object can also be formulated in terms of a free product.

Lemma 2.22 (See [36], Problems for Section 4.1 (13))**.** *Let $A = \langle S_A \mid R_A \rangle$, $B = \langle S_B \mid R_B \rangle$, and let $[A, B] = \{aba^{-1}b^{-1} \mid a \in A, b \in B\}$ be the normal subgroup of commutators of $A * B$, then*

$$A * B/[A, B] = \langle S_A \cup S_B \mid R_A \cup R_B \cup [S_A, S_B] \rangle \simeq A \times B \tag{2.28}$$

Where $A \times B$ is the regular direct product of A and B.

Next we need the notion of a central product, but we'll need a more helpful (for our purposes) way of looking at it: as a quotient of a direct product.

Lemma 2.23 (See [24], Theorem 19.1)**.** *Let G_1, G_2, and H be groups, let $\varepsilon_i : G_i \to H$ be an epimorphism, and write $K_i = \ker(\varepsilon_i)(i = 1, 2)$. Let $D = G_1 \times G_2$ and $G = \{(g_1, g_2) : g_i \in G_i$ and $\varepsilon_1(g_1) = \varepsilon_2(g_2)\}$. Then G is a subgroup of D, and there exist epimorphisms $\delta_i : G \to G_i$ such that*

(a) $\ker(\delta_1) = G \cap G_2 \cong K_2$ *and* $\ker(\delta_2) = G \cap G_1 \cong K_1$,
(b) $\ker(\delta_1)\ker(\delta_2) = K_1 \times K_2$, *and*
(c) $G/(K_1 \times K_2) \cong H$.

Following [24], the subgroup G of $G_1 \times G_2$ constructed in Lemma 2.23 is a *subdirect product of G_1 and G_2*. On the other hand,

Definition 2.24 (See [25], Theorem 5.3)**.** Let H, K, M be groups with $M \subseteq Z(H)$ ($Z(H) = \{h \in H \mid hx = xh \quad \forall x \in H\}$ denotes the center of H) and suppose there is an isomorphism θ of M onto a subgroup of $Z(K)$. Then if we identify M with its image $\theta(M)$, we call a group G of the form $G = HK$ with $M = H \cap K \subseteq Z(G)$ the *central product of H and K* (with respect to M). And denote $G = H \circ K$.

Another way to understand this definition is an arbitrary group G is a central product of its subgroups H and K, if $G = HK$ and $[H, K] = 1$. Note that both H and K are normal in G in Definition 2.24; moreover $H \cap K \leqslant Z(H) \cap Z(K)$.

Lemma 2.25 (See [24], Proposition 19.5)**.** *Let H, K be subgroups of a group G, $D = H \times K$ and $\bar{H} = H \times 1$ and $\bar{K} = 1 \times K$. Then the following statements are equivalent:*

(a) *G is a central product of H and K;*
(b) *There exists an epimorphism $\varepsilon : D \to G$ such that $\varepsilon(\bar{H}) = H$, $\varepsilon(\bar{K}) = K$;*

In particular, if G is a central product of H and K, then G is a subdirect product of H and K.

Lemma 2.25 basically says that a group which can be written as the central product of two groups H and K must be necessarily isomorphic to a quotient of $H \times K$.

We can also use the general notion of a central product to allow us to construct presentations.

Lemma 2.26. *If $G = A \circ B$ with $A = \langle S \mid R_A \rangle$ and $B = \langle T \mid R_B \rangle$ are presentations for A and B, then $G = \langle S \cup T \mid R_A \cup R_B \cup R_C \cup R_\varepsilon \rangle$, where $R_C = \{(a, b) \in S \times T \mid ab = ba\}$ and $R_\varepsilon = \{(a, b) \in S \times T \mid ab = 1\}$.*

Proof. From Lemma 2.25 if we have the epimorphism $\varepsilon : (a, b) \in A \times B \mapsto ab \in G$, then $G = (A \times B)/\ker(\varepsilon)$. A presentation for $A \times B$ is of the form $\langle S \cup T \mid R_A \cup R_B \cup R_C \rangle$, because in the definition of $A \times B$ we require $[A, B] = 1$. Now $\ker(\varepsilon)$ induces the additional relation R_ε and so $(A \times B)/\ker(\varepsilon)$ is presented as claimed, because of Lemma 2.19. □

We will appreciate the application of the general notions of products, which we introduce here, only in the final part of the present thesis, where mathematical models are constructed on the basis of the structural information at the level of groups.

3. GROUP ACTIONS ON SPHERES

Next we introduce some interesting results related to groups on spheres since we will be investigating group actions on spheres in some of our main results later.

3.1. **Topologies on algebraic objects.**

Groups with topologies serve a fundamental purpose in Mathematics and Physics. Especially compact groups, the structure of which can be found in much detail in [31]. One would think that it suffices that we just endow a group with an arbitrary topology, but care must be taken in defining these objects, as the compatibility of the topology with the underlying algebraic structure must be ensured.

Definition 3.1 (See [31], Definitions 1.1)**.**

(i) A *topological group* is a group G together with a topology such that multiplication $(x, y) \in G \times G \mapsto xy \in G$ and inversion $x \in G \mapsto x^{-1} \in G$ are continuous functions.

(ii) A *compact group* is a topological group whose topology is compact and Hausdorff.

(iii) A *locally compact group* is a topological group whose topology is Hausdorff and the identity has a compact neighbourhood.

In continuance with our focus on group actions, here we start by looking at a similar notion that focuses on group actions on a (topological) vector space.

A *topological vector space* is a vector space $\mathbb{V}$ over some field $\mathbb{F}$, which is a topological group with respect to addition and for which scalar multiplication $(r, x) \mapsto r \cdot x : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$ is continuous. A continuous linear self-map of $\mathbb{V}$ is called an *endomorphism* of $\mathbb{V}$. The vector space $\mathrm{Hom}(\mathbb{V}, \mathbb{V}) \subseteq \mathbb{V}^{\mathbb{V}}$ of these endomorphisms is a topological vector space relative to the structure induced from the topological vector space product $\mathbb{V}^{\mathbb{V}}$. One writes $\mathcal{L}_p(\mathbb{V})$ for this topological vector space.

Definition 3.2 (See [31], Definition 2.1)**.** Let G be a topological group and $\mathbb{V}$ a topological vector space. A *representation* of G on $\mathbb{V}$ is a continuous map $\pi : G \to \mathcal{L}_p(\mathbb{V})$ satisfying $\pi(\mathbf{1}) = \mathrm{id}_{\mathbb{V}}$ and $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in G$.

The relationship of a representation of G on $\mathbb{V}$ and a group action of G on $\mathbb{V}$ is easy to see when you define the group action $\phi : (g, v) \in G \times \mathbb{V} \mapsto \phi(g, v) = \pi(g)(v)\mathbb{V}$. Satisfaction of the axioms is easy to check. This relationship allows us to look at groups that act on a (topological) vector space as groups of invertible matrices which then reduces the investigation of group actions on (topological) vector spaces to investigations of a linear algebraic nature instead. Let us quickly just remind the reader of some groups of matrices of special interest. A square matrix A over a field $\mathbb{F}$ in $\mathcal{M}_{n\times n}(\mathbb{F})$ $(\mathcal{M}_{n\times n}(\mathbb{R}))$ is a *unitary (orthogonal) matrix* if and only if $A^{-1} = A^*$ $(A^{-1} = A^T)$. Where A^* is the conjugate transpose of A and A^T is the regular transpose of A.

Topologies that arise from metrics in the context of vector spaces can give us a very useful class of spaces called Banach spaces. We begin with recalling some basic notions of analysis.

Definition 3.3 (See [46], See Definitions 24.1 and 24.2)**.** A sequence (x_n) in a metric space (M, d) is *Cauchy* if and only if for each $\varepsilon > 0$, there is some positive integer

N such that $d(x_n, x_m) < \varepsilon$ whenever $m, n \geq N$. A metric space (M, d) is *complete* if and only if every Cauchy sequence in M converges.

Definition 3.4. Let $\mathbb{F}$ be a field and $\mathbb{A}$ be a vector space over $\mathbb{F}$ with an additional internal operation

$$\bullet : (\mathbf{x}, \mathbf{y}) \in \mathbb{A} \times \mathbb{A} \mapsto \mathbf{x} \bullet \mathbf{y} \in \mathbb{A}$$

such that

$$(\mathbf{x}+\mathbf{y})\bullet\mathbf{z} = \mathbf{x}\bullet\mathbf{z}+\mathbf{y}\bullet\mathbf{z}, \quad \mathbf{x}\bullet(\mathbf{y}+\mathbf{z}) = \mathbf{x}\bullet\mathbf{y}+\mathbf{x}\bullet\mathbf{z}, \quad (a\mathbf{x})\bullet(b\mathbf{y}) = (ab)(\mathbf{x}\bullet\mathbf{y}) \tag{3.1}$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{A}$ and $a, b \in \mathbb{F}$. In this situation $\mathbb{A}$ is an *algebra* on $\mathbb{F}$

Next we define a norm which is similar to a metric, and in fact a norm always induces a metric on a normed space.

Definition 3.5 (See [22], Definition 1.1)**.** If $\mathbb{V}$ is a vector space over a field $\mathbb{F}$, a *norm* is a function $\| \cdot \| : \mathbb{V} \to [0, \infty)$ having the properties:

(i) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{V}$.
(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in \mathbb{V}$.
(iii) $x = 0$ if $\|x\| = 0$

And a *Banach algebra over* $\mathbb{F}$ is an algebra $(\mathbb{B}, \| \cdot \|)$ endowed with a norm which makes the underlying vector space complete, and satisfies $\|xy\| \leq \|x\| \|y\|$.

Example 3.6. On the 2-dimensional Euclidean space $\mathbb{R}^2$, for $x = (x_1, x_2)$ the function $\|x\| := \sqrt{x_1^2 + x_2^2}$ is a norm, the Euclidean norm. This norm induces the usual metric on $\mathbb{R}^2$.

Finding a Riemannian structure, in connection with a group, has a relevant meaning in several models of quantum mechanics. For instance, Chepilko and Romanenko [19] produced a series of contributions, illustrating how certain processes of quantization and some sophisticated variational principles may be easily understood in presence of Riemannian manifolds and groups. Such structures can be found in the context of Lie groups. When introducing the notion of a Lie group it is important first to recall the topological notion of a manifold.

Definition 3.7 (See [34], Definition 11.1)**.** Let n be a non-negative integer. An *n-dimensional manifold* is a Hausdorff space in which each point has an open neighbourhood homeomorphic to the open n-dimensional disc $\mathring{D}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$. Note that $\mathring{D}^n \cong \mathbb{R}^n$, so that we could equally require that each point has a neighbourhood homeomorphic to $\mathbb{R}^n$. For brevity we talk about an *n-manifold*. In particular, compact connected 2-dimensional manifolds are called *surfaces*.

Roughly speaking, a *Lie group* is a group that is also a differentiable manifold. The applications of Lie groups are widely documented and understood, so we need not mention them here, and common examples of Lie groups would be the groups of unitary and orthogonal matrices. And when we speak of the dimension of a Lie group, we refer to the topological dimension of the group as a manifold. From [31] we can characterise Lie groups. To do this we first need the following notion,

Definition 3.8 (See [31], Definition 2.37)**.** Let G be a topological group. We say that G *has no small subgroups* (or is an *NSS-group*), respectively, *no small normal subgroups* if there is a neighbourhood U of the identity such that for every subgroup, respectively, normal subgroup H of G the relation $H \subseteq U$ implies $H = \{\mathbf{1}\}$.

The notion above is purely topological and works perfectly in the present context of study. In fact one can define a compact Lie group, according to the following result.

Proposition 3.9 (See [31], Corollary 2.40). *For a compact group G, the following statements are equivalent:*

1. *G is isomorphic as a topological group to a (compact) group of orthogonal (or unitary) matrices.*
2. *G has a faithful (injective) finite dimensional orthogonal (or unitary) representation.*
3. *G has a faithful (injective) finite dimensional representation.*
4. *G is isomorphic as a topological group to a closed subgroup of the multiplicative group of some Banach algebra $\mathbb{B}$.*
5. *There is a Banach algebra $\mathbb{B}$ and an injective morphism $j : G \to \mathbb{B}^{-1}$ into the multiplicative group of $\mathbb{B}$.*
6. *G has no small subgroups.*
7. *G has no small normal subgroups.*

Therefore we are perfectly consistent to introduce a compact Lie group in the following way:

Definition 3.10. A compact group G is called a *compact Lie group* if it satisfies one, and therefore all, of the equivalent conditions of Proposition 3.9

A topological (Hausdorff) group G is *locally euclidean* if there is an identity neighbourhood which is homeomorphic to $\mathbb{R}^n$.

Proposition 3.11 (See [31], Theorem 9.57). *A locally euclidean compact group is a compact Lie group.*

Corollary 3.12 (See [31], Theorem 9.58). *A compact group is isomorphic (as a topological group) to a group of matrices if and only if it is locally euclidean.*

The n-torus is the abelian group $\mathbb{T}^n$ where $\mathbb{T}$ has been discussed in Example 2.9 .

The next are very important definitions for our description of quantum mechanical systems. An *inner product* in a complex vector space $\mathcal{V}$ is a strictly positive, symmetric, bilinear functional on $\mathcal{V}$.

Definition 3.13 (See [22], Definition 1.1). If $\mathbb{V}$ is a vector space over $\mathbb{F}$, an *inner product* on $\mathbb{V}$ is a function $\langle\cdot,\cdot\rangle : \mathbb{V}\times\mathbb{V} \to \mathbb{F}$ such that for all $a, b \in \mathbb{F}$ and $x, y, z \in \mathbb{V}$ the following are satisfied:

(a) $\langle ax + by, z\rangle = a^*\langle x, z\rangle + b^*\langle y, z\rangle$,
(b) $\langle x, ay + bz\rangle = a\langle x, y\rangle + b\langle x, z\rangle$,
(c) $\langle x, x\rangle \geq 0$
(d) $\langle x, y\rangle = \langle y, x\rangle^*$,
(e) If $\langle x, x\rangle = 0$, then $x = 0$.

Here "$*$" denotes the complex conjugate.

An *inner product space* is a complex vector space $\mathbb{V}$ and an inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{V}$. We continue this subsection with the definition of a Hilbert space, which is a class of spaces that are fundamental in the description of quantum systems as shall be seen later. Although we cite [22] for these general definitions, more extensive theory of Hilbert spaces can be found in [29].

Definition 3.14 (See [22], Definition 1.6)**.** A *Hilbert space*, $\mathcal{H}$, is an inner product space which, as a metric space, is complete.

The following result shows that not all the spheres admit a nice algebraic structure, which is compatible with the topological structure.

Corollary 3.15 (See [31], Theorem 9.59)**.**

(i) *If the dimension of a locally euclidean compact connected group is less than 3, then it is a one-torus or a two-torus.*
(ii) *The only group on a compact surface is the two torus.*
(iii) *The only compact connected groups which contain an open set homeomorphic to euclidean three-space are: the three-torus, the group* SO(3) *of rotations of euclidean three-space, and the group* SU(2) *of isometries of two dimensional complex Hilbert space.*
(iv) *(The Sphere Group Theorem) The only groups on a sphere are: the two element group $\mathbb{S}^0 = \{1, -1\}$, the circle group $\mathbb{S}^1$, and the group $\mathbb{S}^3$ of quaternions of norm one ($\cong$ SU(2)).*
(v) *Let G be an n-dimensional compact connected group and assume that G has a homeomorphic copy K contained in $\mathbb{R}^{n-1}$. Then G is a Lie group and $\mathbb{R}^{n+1} \setminus K$ has two components U_b and U_u where matters can be arranged so that $U_b \cup K$ is compact while U_u is unbounded.*

In particular the proposition:

Proposition 3.16. *The only groups on a sphere are: the two element group $\mathbb{S}^0 = \{1, -1\}$, the circle group $\mathbb{S}^1$ ($\simeq$ SO(2)), and the group $\mathbb{S}^3$ of quaternions of norm one ($\simeq$ SU(2)).*

It is important to note here that the groups $\mathbb{S}^1$ and $\mathbb{S}^3$ are compact connected Lie groups, and we know from Lie theory that a structure of differential manifold can be introduced on compact connected Lie groups via the [32, Chapters 5 and 6]. Therefore the relevance of $\mathbb{S}^3$ appears again among all the possible spheres $\mathbb{S}^n$, which possesses a group structure, which will be useful for our construction later. In fact groups acting on spheres deserve attention in literature. There are two conditions that a finite group G acting freely on $\mathbb{S}^n$ must satisfy, according to [30, Page 75].

Remark 3.17. The following conditions are well known

(a) Every abelian subgroup of G is cyclic. This is equivalent to saying that G contains no subgroup $\mathbb{Z}(p) \times \mathbb{Z}(p)$ with p prime.
(b) G contains at most one element of order 2.

So we have in $\mathbb{S}^3$ not only a topological structure, but indeed a group structure as well. Riemannian manifolds and groups can be used in quantum mechanics to better explain certain processes of quantization and some sophisticated variational principles, which can be found in a series of papers, [19], produced by Chepilko and Romanenko. $\mathbb{S}^3$ indeed is a Riemannian surface and we will show that the space that we will construct inherits in a natural way a Riemannian structure of its own. In [41] we see that models that have Riemannian manifolds and a group of symmetries strongly simplify the structure of the *Hamiltonian*, which is the operator for the total energy (potential and kinetic) of a quantum mechanical dynamical system.

the usual description of a (non relativistic) quantum mechanical system is usually given by a fixed Hilbert space.

3.2. **Actions of finite groups on the three dimensional sphere and computations of fundamental groups.** To construct our space for one of the main results in the present PhD thesis, we first need to look at two group actions on the sphere $\mathbb{S}^3$. One action is the group action of Q_8, the quaternions of order 8, with presentation

$$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, ba = a^{-1}b \rangle \tag{3.2}$$

on $\mathbb{S}^3$ and the other the group action of $\mathbb{Z}(4)$, with presentation

$$\mathbb{Z}(4) = \langle a \mid a^4 = 1 \rangle \tag{3.3}$$

the cyclic group of order 4, on $\mathbb{S}^3$.

The quaternion algebra $\mathbb{H}$ (on $\mathbb{R}$) is a different way to endow an algebraic (and topological) structure on the usual euclidean space $\mathbb{R}^4$. Specifically it consists of all the linear combinations $a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $a, b, c, d \in \mathbb{R}$ and $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms a standard basis for the vector space $\mathbb{H}$, hence

$$\mathrm{span}(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) = \mathrm{span}((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)) = \mathbb{H}. \tag{3.4}$$

In addition to the pointwise sum of elements of $\mathbb{H}$ and to the usual scalar multiplication on $\mathbb{R}$, we have the (internal) multiplication in $\mathbb{H}$, given by

$$\begin{aligned} \mathbf{x} \bullet \mathbf{y} &= (a_1\mathbf{1} + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \bullet (a_2\mathbf{1} + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)\mathbf{1} + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} \\ &\quad + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathbf{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k}. \end{aligned} \tag{3.5}$$

From this rule, one can check that

$$\mathbf{i}\bullet\mathbf{i} = \mathbf{i}^2 = -\mathbf{1},\ \ \mathbf{j}\bullet\mathbf{j} = \mathbf{j}^2 = -\mathbf{1},\ \ \mathbf{k}\bullet\mathbf{k} = \mathbf{k}^2 = -\mathbf{1},\ \ \mathbf{i}\bullet\mathbf{j} = \mathbf{k},\ \ \mathbf{j}\bullet\mathbf{k} = \mathbf{i},\ \ \mathbf{k}\bullet\mathbf{i} = \mathbf{j}. \tag{3.6}$$

and that every nonzero element of $\mathbb{H}$ has an inverse (w.r.t. $\bullet$) of the form

$$(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}), \tag{3.7}$$

and so every nonzero element of $\mathbb{H}$ has a multiplicative inverse. In this context the usual sphere $\mathbb{S}^3 = \{(x, y, z, t) \mid x^2 + z^2 + y^2 + t^2 = 1\} \subseteq \mathbb{R}^4$ can be regarded as the set of elements of $\mathbb{H}$ with norm equal to one. On $\mathbb{H}$ we may introduce the norm $\| \ \|$ which is defined as the usual Euclidean norm in $\mathbb{R}^4$, but on $\mathbb{H}$ now the norm $\| \ \|$ becomes multiplicative, i.e.

$$\|\mathbf{x} \bullet \mathbf{y}\| = \|\mathbf{x}\| \ \ \|\mathbf{y}\| \ \text{ for all } \ \mathbf{x}, \mathbf{y} \in \mathbb{H}, \tag{3.8}$$

so the multiplication of vectors whose norm is equal to one will result in a vector whose norm equals one. From [30], we know that such a norm allows us to give a topological structure on $\mathbb{H}$ and $\mathbb{S}^3$ may be regarded as topological subspace of $\mathbb{H}$, but also as a group with respect to the algebraic structure of $\mathbb{H}$, because the operation

$$(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^3 \times \mathbb{S}^3 \longmapsto \mathbf{x} \bullet \mathbf{y} \in \mathbb{S}^3$$

is well defined and endow $\mathbb{S}^3$ of a structure of group.

The following fact is known, but we offer a direct argument:

Lemma 3.18. *The groups Q_8 and $\mathbb{Z}(4)$ act freely on $\mathbb{S}^3$.*

Proof. The map $(\mathbf{x},\mathbf{y}) \in \mathbb{S}^3 \times \mathbb{S}^3 \longmapsto \mathbf{x}\bullet\mathbf{y} \in \mathbb{S}^3$ makes $\mathbb{S}^3$ a group, and an appropriate restriction of this map to a subgroup G of $\mathbb{S}^3$ allows us to define an action on $\mathbb{S}^3$ of the form

$$(g, x) \in G \times \mathbb{S}^3 \longmapsto g \cdot x = g \bullet x \in \mathbb{S}^3. \tag{3.9}$$

In particular we look at (3.6) and note that this happens when G is chosen as

$$Q_8 = \langle \mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{i} \bullet \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \bullet \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \bullet \mathbf{i} = \mathbf{j} \,\rangle, \tag{3.10}$$

producing the action

$$(q, x) \in Q_8 \times \mathbb{S}^3 \longmapsto q \cdot x = q \bullet x \in \mathbb{S}^3. \tag{3.11}$$

The rest follows from the fact that $\mathbb{H} \setminus \{0\}$ contains inverses for all its elements and so the equation $g \bullet x = x$ is only true for $g = 1$ or $x = 0$ where $g, x \in \mathbb{H}$. This shows that Q_8 acts freely on S^3. In order to show the second part of the result, we consider

$$h : (z_0, z_1) \in \mathbb{S}^3 \longmapsto (e^{\frac{2\pi i}{4}} z_0, e^{\frac{2\pi i 3}{4}} z_1) \in \mathbb{S}^3$$

and observe that $\mathbb{S}^3$, regarded as a group, contains cyclic subgroups of order four, so we may define the action of $\mathbb{Z}(4)$ on $\mathbb{S}^3$ by

$$(n, (z_0, z_1)) \in \mathbb{Z}(4) \times \mathbb{S}^3 \longmapsto n \cdot (z_0, z_1) = h^n(z_0, z_1) \in \mathbb{S}^3. \tag{3.12}$$

We need to show that the only element that fixes points in this action is the identity element. This can be checked easily, since

$$\begin{aligned} h^n(z_0, z_1) = (z_0, z_1) \quad &\Longleftrightarrow \quad (e^{\frac{2\pi i n}{4}} z_0, e^{\frac{2\pi i 3n}{4}} z_1) = (z_0, z_1) \\ &\Longleftrightarrow \quad e^{\frac{2\pi i n}{4}} z_0 = z_0, \quad e^{\frac{2\pi i 3n}{4}} z_1 = z_1. \end{aligned} \tag{3.13}$$

Since z_0 or z_1 are different from zero, we have for all $k \geq 0$

$$e^{\frac{2\pi i n}{4}} z_0 = z_0 \quad \Longleftrightarrow \quad \frac{2\pi i n}{4} = 0 + 2\pi i k \quad \Longleftrightarrow \quad n = 4k;$$

$$e^{\frac{2\pi i 3n}{4}} z_1 = z_1 \quad \Longleftrightarrow \quad \frac{2\pi i 3n}{4} = 0 + 2\pi i k \quad \Longleftrightarrow \quad 2\pi i 3n = 8\pi i k.$$

Clearly $n \equiv 0 \mod 4$ and therefore the only fixed points are under the identity action of $\mathbb{Z}(4)$. This means that $\mathbb{Z}(4)$ acts freely on $\mathbb{S}^3$. ☐

The following corollary comes naturally as an application of Proposition 2.13 and Corollary 2.17.

Corollary 3.19. *There are properly discontinuous actions of Q_8 and $\mathbb{Z}(4)$ on $\mathbb{S}^3$. Moreover, the orbit spaces $\mathbb{S}^3/Q_8$ and $\mathbb{S}^3/\mathbb{Z}(4)$ are compact and path connected.*

Therefore we may conclude that

Lemma 3.20. *The fundamental group $\pi(\mathbb{S}^3/Q_8)$ of the space of orbits $\mathbb{S}^3/Q_8$ via the action (3.11) is isomorphic to Q_8. Moreover $\pi(\mathbb{S}^3/\mathbb{Z}(4)) \cong \mathbb{Z}(4)$ via the action (3.12).*

Proof. We may apply Corollary 2.17, that is, we have an orbit space X/G produced by a properly discontinuous action of a group G on a simply connected space X, then the fundamental group of the orbit space X/G is isomorphic to the underlying group G. ☐

Because of Remark 3.17, it is not surprising that $\mathbb{Z}(4)$ acts freely on $\mathbb{S}^3$ since it is a finite cyclic group of order four. On the other hand, it is useful to note that:

Remark 3.21. Surprisingly, the dihedral group D_8 of order 8 does not act freely on $\mathbb{S}^3$ since it fails on condition (b) of Remark 3.17: There are in fact involutions in D_8. Structurally D_8 is very similar to Q_8, e.g. $[Q_8, Q_8] \cong [D_8, D_8] \cong \mathbb{Z}(2)$, $Z(Q_8) \cong Z(D_8) \cong \mathbb{Z}(2)$ and $Q_8/Z(Q_8) \cong D_8/Z(D_8) \cong \mathbb{Z}(2) \times \mathbb{Z}(2)$ but the presence of more involutions makes the difference between Q_8 and D_8.

Actions of groups on spheres is a well studied area of topology. Many contributions have been made by Bruno P. Zimmermann and others [42, 48, 50, 49, 51], developing theories for properties that finite (or even infinite) groups must have in order to act on low dimensional spheres, i.e. $\mathbb{S}^2$, $\mathbb{S}^3$ and $\mathbb{S}^4$.

4. Pseudo-Hermitian physical systems

In quantum mechanics, physical systems are modeled on some fixed Hilbert space $\mathcal{H}$ over the field $\mathbb{C}$ of complex numbers. In this model the probability distribution of the outcomes for each possible outcome is called a *quantum state* and is represented by a vector $\psi \in \mathcal{H}$. Any quantum state that cannot be written as a mixture of other quantum states is called a *pure quantum state*, while those that can are called *mixed quantum states*. Any physical quantity that can be observed in this system is represented by a linear self-adjoint operator on $\mathcal{H}$ with norm $\langle\cdot,\cdot\rangle$. A linear operator $\mathbf{A} : \mathcal{H} \to \mathcal{H}$ is *bounded* if there is a constant k such that $\|\mathbf{A}\psi\| \leq k\|\psi\|$ (where $\|\cdot\|$ is the norm induced by the inner product of $\mathcal{H}$. For any bounded operator $\mathbf{A}$ of $\mathcal{H}$, there is a unique bounded operator $\mathbf{A}^*$, called the *adjoint* of $\mathbf{A}$, such that

$$\langle\psi, \mathbf{A}\varphi\rangle = \langle\mathbf{A}^*\psi, \varphi\rangle \tag{4.1}$$

for all $\psi, \varphi \in \mathcal{H}$. If $\mathbf{A}^* = \mathbf{A}$ then $\mathbf{A}$ is said to be *self-adjoint*. Also we know that $\mathbf{A}^*$ exists from the Riesz Theorem (see [28], Appendix A.4.3). A linear operator, $\mathbf{A}$, defined on all of $\mathcal{H}$ that has the property that $\langle\psi, \mathbf{A}\varphi\rangle = \langle\mathbf{A}\psi, \varphi\rangle$ for all $\psi, \varphi \in \mathcal{H}$, then $\mathbf{A}$ is automatically bounded (See [28], Corollary 9.9). This poses a problem because formulas for these operators quickly show us that some of these operators can not be bounded. To get around that technical difficulty we must adjust the class of operators we will use in our description.

Definition 4.1. An *unbounded operator* $\mathbf{A}$ on $\mathcal{H}$ is a linear map from a dense subspace $\mathrm{Dom}(\mathbf{A}) \subseteq \mathcal{H}$ into $\mathcal{H}$.

We call these "unbounded operator", but really they are "not necessarily bounded" as it is possible that $\mathrm{Hom}(\mathbf{A}) = \mathcal{H}$. So when we try to define the adjoint of an unbounded operator we run into the problem that: for a given $\psi \in \mathcal{H}$, the linear functional $\langle\psi, \mathbf{A}\cdot\rangle$ may not be bounded, in which case we cannot use the Riesz Theorem to define $\mathbf{A}^*x$. So the adjoint of $\mathbf{A}$ might not be defined on all of $\mathcal{H}$ but on some subspace of $\mathcal{H}$, like $\mathbf{A}$ itself.

Definition 4.2 (See [28], Definition 3.2)**.** For an unbounded operator $\mathbf{A}$ on $\mathcal{H}$, the *adjoint* $\mathbf{A}^*$ of $\mathbf{A}$ is defined as follows. A vector $\psi \in \mathcal{H}$ belongs to the domain $\mathrm{Dom}(\mathbf{A}^*)$ of $\mathbf{A}^*$ if the linear functional

$$\langle\psi, \mathbf{A}\cdot\rangle : \mathcal{H} \to \mathbb{C}$$

defined on $\mathrm{Dom}(\mathbf{A})$, is bounded. For $\psi \in \mathrm{Dom}(\mathbf{A}^*)$, let $\mathbf{A}^*\psi$ be the unique vector β such that

$$\langle\beta, \psi\rangle = \langle\psi, \mathbf{A}\varphi\rangle \tag{4.2}$$

for all $\varphi \in \mathrm{Dom}(\mathbf{A})$.

If $\langle\psi, \mathbf{A}\cdot\rangle$ is bounded, then since $\mathrm{Dom}(\mathbf{A})$ is dense, the Bounded Linear Transformation Theorem (see [28], Theorem A.36) tells us that $\langle\psi, \mathbf{A}\cdot\rangle$ has a unique bounded extension to all of $\mathcal{H}$. Then by the Riesz Theorem we will have unique β. So now we have what we need to define self-adjointness in this general case.

Definition 4.3 (See [28], Definition 3.3)**.** An unbounded operator $\mathbf{A}$ on $\mathcal{H}$ is *self-adjoint* if

$$\langle\psi, \mathbf{A}\varphi\rangle = \langle\mathbf{A}\psi, \varphi\rangle, \tag{4.3}$$

$\mathrm{Dom}(\mathbf{A}^*) = \mathrm{Dom}(\mathbf{A})$ and $\mathbf{A}^*\psi = \mathbf{A}\psi$ for all $\psi, \varphi \in \mathrm{Dom}(\mathbf{A})$.

These operators have many properties that are useful for physical descriptions. Not least of which that their eigenvalues are guaranteed to be real. The physical meaning of $\langle\psi, \mathbf{A}\psi\rangle$ becomes the expectation value for the measurements of the observable $\mathbf{A}$ in the quantum state ψ, and the eigenvalue λ represents one of the possible values for this measurement.

Definition 4.4. Given a Hilbert space, $\mathcal{H}$, and $\mathbf{A}$ an operator on $\mathcal{H}$, is said to be a *positive operator* if $\langle\psi, \mathbf{A}\psi\rangle \geq 0$ for every $\psi \in \mathcal{H}$.

So far though we have implicitly considered a wave function ψ at a fixed time. Wave functions can and do evolve in time. In classical mechanics this time-evolution of the system is governed by the Hamiltonian (or energy) function H, through the equations of Hamilton. So the assumption in quantum mechanics is that there exists an operator, $\mathbf{H}$, on the quantum Hilbert space, $\mathcal{H}$, that has an analogous role, called the *Hamiltonian operator* for the system. After a few hypotheses and derivations we are guided by the following proposition in quantum mechanics:

Proposition 4.5 (See [28], Axiom 5)**.** *The time-evolution of the wave function ψ in a quantum system is given by the Schrödinger equation,*

$$\frac{d\psi}{dt} = \frac{1}{i\hbar}\boldsymbol{H}\psi. \tag{4.4}$$

Here $\boldsymbol{H}$ is the operator corresponding to the classical (time independent) Hamiltonian H, and $\hbar$ is the reduced Plank constant.

So Equation (4.4) above is an ordinary differential equation and one solution leads to a proposition we will present with the caveat that its proof requires delicate consideration as we are working with operators that may or may not be bounded, and the implications thereof. In the case where ψ is also dependent on the position, then we consider the partial differential equation $\frac{\partial\psi}{dt} = \frac{1}{i\hbar}\mathbf{H}\psi$ instead. However for our purpose it will suffice to consider it as presented in Equation (4.4), and to present the one solution in this form:

Proposition 4.6 (See [28], Claim 3.17)**.** *Suppose $\boldsymbol{H}$ is a self-adjoint operator on $\mathcal{H}$. If a reasonable meaning can be given to the expression $e^{-it\boldsymbol{H}/\hbar}$, then the Schrödinger equation can be solved by setting*

$$\psi(t) = e^{-it\boldsymbol{H}/\hbar}\psi_0. \tag{4.5}$$

where ψ_0 is an initial condition quantum state.

That ends all the notions that we need for a description of ordinary (relativistic) quantum mechanics. Stressing the importance played by the role of self-adjoint operators.

Remark 4.7. Heisenberg however had a different view of the dynamics of a quantum system. He thought of the operators (quantum observable) as evolving in time instead of the quantum states (vectors in the Hilbert space). And in his interpretation each self-adjoint operator $\mathbf{A}$ evolves in time according to the operator-valued differential equation

$$\frac{d\mathbf{A}(t)}{dt} = \frac{1}{i\hbar}\left[\mathbf{A}(t), \mathbf{H}\right], \tag{4.6}$$

where $\mathbf{H}$ is the Hamiltonian operator of the system, and where $[\mathbf{A},\mathbf{B}] = \mathbf{AB}-\mathbf{BA}$ is the commutator (See [28], Definition 3.20). Note that since $\mathbf{H}$ commutes with itself, the operator $\mathbf{H}$ remains constant in time, even in this interpretation. However this turns out to have the same physical content as Schrödinger's interpretation.

However recently an increasing number of physicists and mathematicians started to be interested to a *generalized version* of quantum mechanics, that is, to the situation in which the hamiltonian of the model under consideration is not necessarily self-adjoint. Examples of a non-self-adjoint Hamiltonian operators with real and discrete eigenvalues, $\mathbf{H}_1 = \mathbf{p}^2 + i\mathbf{x}^3$ and $\mathbf{H}_2 = \mathbf{p}^2 - \mathbf{x}^4$ were found and studied. This lead to a lot of research in this area, in particular by Carl M. Bender (see [13, 14, 15, 16]), Ali Mostafazadeh (see [38, 39]) and even more general in works such as Jean A. Dieudonné (see [23]). Consequently many more non-self-adjoint Hamiltonians have been discovered.

4.1. **Some elementary theory of pseudo-fermionic operators.**

In particle physics, elementary particles and composite particles are broken up into two (overlapping) classes: bosons and fermions. Roughly speaking bosons are force carrier particles, while fermions are particles that are associated with matter. For our purposes we will be looking at the case of pseudo-fermions, which are a generalization of the case of fermions.

Typically on a two dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ (of fermion states), we can define *lowering* and *raising* operators, which will lower or raise the eigenvalues (respectively) associated with an eigenstate by acting on the state. These come as a pair, say $\mathbf{c}$ and $\mathbf{c}^*$ respectively. The lowering operator, $\mathbf{c}$ lowers the eigenvalue of a given state by acting on it, and the raising operator $\mathbf{c}^*$ raises the eigenvalue of a given state by acting on it, and it is the adjoint of the lowering operator.

Remark 4.8. The *fermion* operators must satisfy a canonical anticommutation relations (CAR) $\{\mathbf{c},\mathbf{c}^*\} := \mathbf{cc}^* + \mathbf{c}^*\mathbf{c} = \mathbb{1}$, $\{\mathbf{c},\mathbf{c}\} = \{\mathbf{c}^*,\mathbf{c}^*\} = 0$.

However in the case of *pseudo-fermions* we replace the CAR with the following rules, [10]:

$$\{\mathbf{a},\mathbf{b}\} = \mathbb{1}, \quad \{\mathbf{a},\mathbf{a}\} = 0, \quad \{\mathbf{b},\mathbf{b}\} = 0, \tag{4.7}$$

where we can have that $\mathbf{b} \neq \mathbf{a}^*$.

Example 4.9 (See [8], Chapter 3.5).

$$\mathbf{a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \beta & -\beta^2 \\ 1 & -\beta \end{pmatrix} \tag{4.8}$$

with $\beta \neq 0$.

So often in quantum mechanics operators that are used are unbounded, and so cannot be defined on all of $\mathcal{H}$. Similar problems, however, don't exist for fermions, who only have bounded operators which simplifies the maths a bit.

We observe that there exists a non zero vector $\varphi_0 \in \mathcal{H}$ such that $\mathbf{a}\varphi_0 = 0$, as well as a non zero vector $\Psi_0 \in \mathcal{H}$ such that $\mathbf{b}^*\Psi_0 = 0$. This is because $\mathbf{a}$ and $\mathbf{b}^*$ have non-trivial kernels.

Definition 4.10. A set $\mathcal{E} = \{e_n \in \mathcal{H}, n \geq 0\}$ is a (Schauder) basis for $\mathcal{H}$ if any vector $f \in \mathcal{H}$ can be written, uniquely, as an (in general) infinite linear combination of the $e_n's$: $f = \sum_{n=0}^{\infty} c_n(f)e_n$. Here $c_n(f)$ are complex numbers depending on the vector f we want to expand.

A particular useful type of bases are the so-called orthonormal ones. $\mathcal{E}$ is an orthonormal basis if $\langle e_n, e_m\rangle = \delta_{n,m}$, and if $\mathcal{E}$ is a basis for $\mathcal{H}$. In this particular case, we find that $c_n(f) = \langle e_n, f\rangle$. Notice that, in our notation, the scalar product is linear in its second variable.

Definition 4.11. A slightly extended version of the orthonormal bases is the one given by the so-called Riesz bases: $\mathcal{F} = \{f_n \in \mathcal{H}, n \geq 0\}$ is a Riesz basis if a bounded operator T on $\mathcal{H}$ exists, with bounded inverse, and an orthonormal basis $\mathcal{E} = \{e_n \in \mathcal{H}, n \geq 0\}$ such that $f_n = Te_n$, for all $n \geq 0$.

In this case, it is clear that $\langle f_n, f_m\rangle \neq \delta_{n,m}$, in general and detailed analysis on operators defined by Riesz bases can be found in [9, 21].

We can now deduce results that are key to this framework. First we define the following vectors

$$\varphi_1 := \mathbf{b}\varphi_0, \quad \Psi_1 := \mathbf{a}^*\Psi_0, \tag{4.9}$$

as well as the following non-self-adjoint operators

$$\mathbf{N} := \mathbf{ba}, \quad \mathbf{N}^* = \mathbf{a}^*\mathbf{b}^*. \tag{4.10}$$

Note that for $n \geq 2$ the vectors $\mathbf{b}^n\varphi_0$ and $\mathbf{a}^{*n}\Psi_0$ are automatically equal to zero. Now we introduce the self-adjoint operators $\mathbf{S}_\varphi$ and $\mathbf{S}_\Psi$ via their action on a generic $f \in \mathcal{H}$:

$$\mathbf{S}_\varphi f = \sum_{n=0}^{1}\langle\varphi_n, f\rangle\varphi_n, \quad \mathbf{S}_\Psi f = \sum_{n=0}^{1}\langle\Psi_n, f\rangle\Psi_n. \tag{4.11}$$

Then the following results can be easily proved:

$$\mathbf{a}\varphi_1 = \varphi_0, \quad \mathbf{b}^*\Psi_1 = \Psi_0 \tag{4.12}$$

$$\mathbf{N}\varphi_n = n\varphi_n, \quad \mathbf{N}^*\Psi_n = n\Psi_n, \text{ for } n = 0, 1. \tag{4.13}$$

If the normalizations of φ_0 and Ψ_0 are chosen so that $\langle\varphi_0, \Psi_0\rangle = 1$, then

$$\langle\varphi_k, \Psi_n\rangle = \delta_{k,n} \text{ for } k, n = 0, 1. \tag{4.14}$$

$\mathbf{S}_\varphi$ and $\mathbf{S}_\Psi$ are bounded, strictly positive, self-adjoint, and invertible. They satisfy

$$\|\mathbf{S}_\varphi\| \leq \|\varphi_0\|^2 + \|\varphi_1\|^2, \quad \|\mathbf{S}_\Psi\| \leq \|\Psi_0\|^2 + \|\Psi_1\|^2, \tag{4.15}$$

$$\mathbf{S}_\varphi\Psi_n = \varphi_n, \quad \mathbf{S}_\Psi\varphi_n = \Psi_n, \tag{4.16}$$

for $n = 0, 1$, as well as $\mathbf{S}_\varphi = \mathbf{S}_\Psi^{-1}$ and the following intertwining relations

$$\mathbf{S}_\Psi\mathbf{N} = \mathbf{N}^*\mathbf{S}_\Psi, \quad \mathbf{S}_\varphi\mathbf{N}^* = \mathbf{N}\mathbf{S}_\varphi. \tag{4.17}$$

Notice that, being biorthogonal, the vectors of both $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are linearly independent. Hence φ_0 and φ_1 are two linearly independent vectors in a two dimensional Hilbert space, so that $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$. the same argument obviously can be used for $\mathcal{F}_\Psi$. More than this: both these sets are also Riesz bases.

All these formulas show that (i) $\mathbf{N}$ and $\mathbf{N}^\dagger$ behave as fermionic number operators, having eigenvalues 0 and 1; (ii) their related eigenvectors are respectively the vectors in $\mathcal{F}_\varphi = \{\varphi_0, \varphi_1\}$ and $\mathcal{F}_\Psi = \{\Psi_0,\ \Psi_1\}$; (iii) $\mathbf{a}$ and $\mathbf{b}^\dagger$ are lowering operators for P_φ and P_Ψ respectively; (iv) $\mathbf{b}$ and $\mathbf{a}^\dagger$ are raising operators for $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ respectively; (v) the two sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal; (vi) the "very well-behaved" operators $\mathbf{S}_\varphi$ and $\mathbf{S}_\Psi$ map $\mathcal{F}_\varphi$ in $\mathcal{F}_\Psi$ and vice versa; (vii) $\mathbf{S}_\varphi$ and $\mathbf{S}_\Psi$ intertwine between operators which are not self-adjoint.

The following connection between Pseudo-fermions and *ordinary* fermions exists:

Proposition 4.12 (See [8], Theorem 3.5.1)**.** *Let* $\boldsymbol{c}$ *and* $\boldsymbol{T} = \boldsymbol{T}^\dagger$ *be two operators on* $\mathcal{H}$ *such that* $\{\boldsymbol{c}, \boldsymbol{c}^\dagger\} = \mathbb{1}$, $\boldsymbol{c}^2 = 0$, *and* $\boldsymbol{T} > 0$. *Then, defining*

$$\boldsymbol{T}\boldsymbol{c}\boldsymbol{T}^{-1}, \quad \boldsymbol{b} = \boldsymbol{T}\boldsymbol{c}^\dagger\boldsymbol{T}^{-1}, \tag{4.18}$$

these operators satisfy (4.7)*. Viceversa given two operators* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *acting on* $\mathcal{H}$*, satisfying* (4.7)*, it is possible to define two operators,* $\boldsymbol{c}$ *and* $\boldsymbol{T}$*, such that* $\{\boldsymbol{c}, \boldsymbol{c}^\dagger\} = \mathbb{1}$, $\boldsymbol{c}^2 = 0$, $\boldsymbol{T} = \boldsymbol{T}^\dagger$ *is strictly positive, and* (4.18) *holds.*

Remark 4.13. Examples of pseudo-fermions can be found in [8]. In particular, in Chapter 3.5.1, an effective non-self-adjoint hamiltonian describing a two-level atom interacting with an electromagnetic field, from [9], was interpreted in terms of pseudo-fermionic operators, and the structure previously described can easily be recovered in our context of investigation.

Our results are focused on the case of these pseudo-fermions and for theory of pseudo-bosons we refer the reader to the recent works in [4, 2, 3, 11, 8, 12].

5. Some facts on Pauli groups

The Pauli group P is a finite group of order 16 and it is an interesting 2-group, which has relevant properties for dynamical systems and theoretical physics. It is a group that is generated by three matrices, the Pauli matrices, named after the physicist Wolfgang Pauli. These matrices were used in the Pauli equation, which was an equation formulated by Wolfgang Pauli, to give the dynamics of *spin* $\frac{1}{2}$ (see [28], Definition 17.5) particles in an external magnetic field. The Pauli matrices are:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.1}$$

and the generate a finite group P of order 16. One can check that the Pauli matrices satisfy the following rules:

$$X^2 = Y^2 = Z^2 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

$$(YZ)^4 = (ZX)^4 = (XY)^4 = \mathbb{1}, \tag{5.3}$$

$$(XYZ)^4 = [XYZ, X] = [XYZ, Y] = [XYZ, Z] = \mathbb{1}. \tag{5.4}$$

This Pauli group, P, and generalised versions of it are very well known and important in many aspects of quantum theory (see [32, 37]). They have a rich lattice of subgroups and these have applications in quantum information theory (see [40]), with notable examples in quantum error correcting codes and for finding mutually unbiased bases (see [26, 27, 33, 35, 44]).

The Pauli group also happens to be the central product of the cyclic group of order 4, $\mathbb{Z}(4)$, and either the Dihedral group of an octagon, D_8, or the quaternion group Q_8. That is, $P = \mathbb{Z}(4) \circ D_8 = \mathbb{Z}(4) \circ Q_8$. What we would like to do is give a topological decomposition of P as a central product of $\mathbb{Z}(4)$ and Q_8, for the reasons stated in Remark 3.21. And since we will be using the Theorem of Seifert-Van Kampen to get this decomposition, we would like to have a view of these groups at the level of presentations. But first we need a different, but equivalent presentation of P. We will show that the Pauli group P may be presented using Q_8 and $\mathbb{Z}(4)$ only.

Lemma 5.1. *The group P can be presented by*

$$P = \langle u, xy, y \mid u^4 = x^2 = 1, u^2 = y^2, uy = yu, yx = xy, x^{-1}ux = u^{-1} \rangle. \tag{5.5}$$

And the central product factors can be presented as follows

$$Q_8 = \langle u, xy \mid u^4 = 1, u^2 = (xy)^2, (xy)^{-1}u(xy) = u^{-1} \rangle \tag{5.6}$$

from Equation (3.2), and

$$\mathbb{Z}(4) = \langle y \mid y^4 = 1 \rangle. \tag{5.7}$$

from Equation (3.3).

Proof. Consider the Pauli group P and define

$$u = XY, \quad x = Y \quad \text{and} \quad y = XYZ. \tag{5.8}$$

Then we need to show that the relations defining the Pauli matrices can be generated by the relations in the thesis.
First of all we will derive the equations $X^2 = Y^2 = Z^2 = 1$. We begin to note that

$$x^2 = 1 \implies Y^2 = 1, \tag{5.9}$$

so one equation is obtained and (5.9) allows us to conclude that

$$\begin{gathered} x^{-1}ux = u^{-1} \implies Y^{-1}XYY = (XY)^{-1} \implies Y^{-1}XYY = Y^{-1}X^{-1} \implies \\ XYY = X^{-1} \implies XXYY = 1 \implies X^2Y^2 = 1 \implies X^2 1 = 1 \implies X^2 = 1 \end{gathered} \tag{5.10}$$

so a second equation is obtained. Now we need to show that

$$uy = yu \implies XYXYZ = XYZXY \implies XYZ = ZXY, \tag{5.11}$$

in order to derive the following equation

$$\begin{aligned} u^2 = y^2 &\implies XYXY = XYZXYZ \implies XY = ZXYZ \\ &\implies XY = XYZZ \implies 1 = Z^2 \end{aligned} \tag{5.12}$$

So we have shown until now that $X^2 = Y^2 = Z^2 = 1$. Now we go ahead to show the equations $(YZ)^4 = (ZX)^4 = (XY)^4 = 1$. We begin with

$$u^4 = 1 \implies (XY)^4 = 1, \tag{5.13}$$

then with help of (5.11) we derive

$$\begin{aligned} yx = xy &\implies XYZY = YXYZ \implies ZXYY = YXYZ \\ &\implies ZXYY = YZXY \implies ZXY = YZX. \end{aligned} \tag{5.14}$$

Now (5.11) allows us to have two more equations, namely

$$ZXY = YZX \implies ZXYX = YZXX \implies ZXYX = YZ, \tag{5.15}$$

$$ZXY = YZX \iff ZX = YZXY \implies ZX = YXYZ. \tag{5.16}$$

Therefore we find

$$\begin{aligned} (YZ)^4 &= YZYZYZYZ = ZXYXZXYXZXYXZXYX \\ &= (ZXY)XZXYXZXYXZXYX = (YZX)XZXYXZXYXZXYX \\ &= YZXXZXYXZXYXZXYX = YXYXYZZXYX \\ &= YZZXYXZXYXZXYX = Y(ZZ)XYXZXYXZXYX \\ &= YXYXZXYXZXYX = YXYX(ZXY)XZXYX \\ &= YXYX(YZX)XZXYX = YXYXYZXXZXYX \\ &= YXYXYZ(XX)ZXYX = YZ(XX)ZXYXZXYXZXYX \\ &= YXYXY(ZZ)XYX = YXYXYXYX = Y(XYXYXY)X \\ &= Y(XY)^3X = Y(XY)^{-1}X = YY^{-1}X^{-1}X = 1 \end{aligned}$$

and in analogy $(ZX)^4 = 1$. This allows us to conclude that P is equivalent to the presentation in Equation 5.5. ☐

So using the ideas of [24], we can consider the map

$$\varepsilon : (a, b) \in Q_8 \times \mathbb{Z}(4) \mapsto ab \in P \tag{5.17}$$

and we note that $xy = yx$ in P, hence for all $\alpha, \beta, \gamma, \delta \in \{0, 1, 2, 3\}$ we have

$$\begin{aligned} \varepsilon(u^\alpha, y^\beta)\varepsilon((xy)^\gamma, y^\delta) &= (u^\alpha y^\beta)((xy)^\gamma y^\delta) = u^\alpha y^\beta x^\gamma y^{\gamma+\delta} = u^\alpha x^\gamma y^{\beta+\gamma+\delta} \\ &= \varepsilon(u^\alpha (xy)^\gamma, y^{\beta+\delta}) = \varepsilon((u^\alpha, y^\beta)((xy)^\gamma, y^\delta)), \end{aligned} \tag{5.18}$$

we can therefore conclude that ε is homomorphism of groups, because we checked that the map preserves operations on generic generators of $Q_8 \times \mathbb{Z}(4)$. ε is clearly surjective by construction, so $P = Q_8 \circ \mathbb{Z}(4)$ by Lemma 2.25.

6. The main results and their proofs

So our main result is to show that you can find the Pauli group, P, as a certain quotient of the fundamental group of a Riemannian 3-manifold. This gives us a way of geometrically identifying P, and this construction is one of our main contributions. The methods of this construction are of a general nature, but in this paper we focus particularly on P. Our interest lying in the fact that we can avoid abstract notions such as the free product (with amalgamation) and central product, and directly analysing P using low dimensional topology and few combinatorial results instead of computational software. We will refer to the usual *connected sum*, #, between manifolds (see [30, 34]).

Our first main result is the following.

Theorem 6.1. *There exist two compact path connected orbit spaces* $U = \mathbb{S}^3/Q_8$ *and* $V = \mathbb{S}^3/\mathbb{Z}(4)$ *such that the following conditions hold:*

(i) $U \cup V$ *is a compact path connected space with* $U \cap V \neq \emptyset$, $\pi(U \cap V)$ *cyclic of order* 2 *and* $P \simeq \pi(U \cup V)/N$ *for some normal subgroup* N *of* $\pi(U \cup V)$*;*

(ii) $U\#V$ *is a Riemannian manifold of* $\dim(U\#V) = 3$ *and* $P \simeq \pi(U\#V)/L$ *for some normal subgroup* L *of* $\pi(U\#V)$*.*

Both in case (i) and (ii), P *is central product of* $\pi(U)$ *and* $\pi(V)$*.*

We begin to prove Theorem 6.1.

Proof. Case (i). First of all we note that the assumptions of the theorem of Seifert and Van Kampen are met because of Lemmas 3.18 and 3.20 and Corollary 3.19. We can also see that U, V, $U \cap V$, X are path connected spaces, therefore $\pi(X, x_0)$ is not dependent on the choice of $x_0 \in U \cap V$. Now we construct $\pi(X)$ directly:

$$\pi(U) = Q_8 = \langle u, xy \mid u^4 = 1, u^2 = (xy)^2, (xy)^{-1}u(xy) = u^{-1}\rangle = \langle S_1 \mid R_1\rangle, \tag{6.1}$$

$$\pi(V) = \mathbb{Z}(4) = \langle y \mid y^4 = 1\rangle = \langle S_2 \mid R_2\rangle, \tag{6.2}$$

$$\pi(U \cap V) = \mathbb{Z}(2) = \langle u^2 \mid u^4 = 1\rangle \subseteq \pi(U), \tag{6.3}$$

$$\pi(U \cap V) = \mathbb{Z}(2) = \langle y^2 \mid y^4 = 1\rangle \subseteq \pi(V) \tag{6.4}$$

it is easy to check that

$$(i_1)_* : u^2 \in \pi(U \cap V) \subseteq \pi(U) \mapsto (i_1)_*(u^2) = u^2 \in \pi(U), \tag{6.5}$$

$$(i_2)_* : y^2 \in \pi(U \cap V) \subseteq \pi(V) \mapsto (i_2)_*(y^2) = y^2 \in \pi(V) \tag{6.6}$$

are homomorphisms in accordance with the theorem of Seifert-Van Kampen. Moreover

$$S = \{u^2 \mid u \in \pi(U)\} = \{y^2 \mid y \in \pi(V)\} \tag{6.7}$$

is the set of generators of $\pi(U \cap V)$, but $\mathbb{Z}(2)$ has only one generator so we may deduce that

$$R_S = \{u^2y^2 = 1 \mid u \in \pi(U), y \in \pi(V)\} = \{u^2 = y^2 \mid u \in \pi(U), y \in \pi(V)\} \tag{6.8}$$

are the relations on $\pi(X)$ induced from the generators of $\pi(U \cap V)$ from the theorem of Seifert and van Kampen. We can then conclude that

$$\begin{aligned}\pi(X) &= \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R_S\rangle \\ &= \langle u, xy, y \mid u^4 = y^4 = 1, u^2 = (xy)^2, (xy)^{-1}u(xy) = u^{-1}, u^2 = y^2\rangle.\end{aligned} \tag{6.9}$$

Let N be the normal subgroup generated by $[S_1, S_2]$. By Lemma 2.19 we get the following presentation for the quotient group:

$$\pi(X)/N = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R_3 \cup R_S \rangle$$
$$= \langle u, xy, y \mid u^4 = y^4 = 1, u^2 = (xy)^2, (xy)^{-1}u(xy) = u^{-1}, u^2 = y^2, uy = yu, xyy = yxy \rangle \tag{6.10}$$

where

$$R_3 = \{s_1 s_2 = s_2 s_1 \mid s_1 \in S_1, s_2 \in S_2\}, \tag{6.11}$$

and we claim that (6.10) is equivalent to the following presentation of Lemma 5.1

$$P = \langle u, xy, y \mid u^4 = x^2 = 1, u^2 = y^2, uy = yu, yx = xy, x^{-1}ux = u^{-1} \rangle. \tag{6.12}$$

Since (6.10) and (6.12) have the same generators, the relations in (6.12) will be deduced from the relations in (6.10) and viceversa. Firstly we consider (6.10) and note that

$$\begin{aligned} xyy = yxy \iff xy^2 = yxy \iff xy^2y^3 = yxyy^3 \iff xy^5 = yxy^4 \\ \iff xyy^4 = yxy^4 \iff xy = yx \end{aligned} \tag{6.13}$$

and so x and y commute. Similarly one can see that u commutes with y. Secondly we have

$$\begin{aligned} u^2 = (xy)^2 \iff u^2 = x^2y^2 \iff u^2 = x^2u^2 \\ \iff u^2u^2 = x^2u^2u^2 \iff u^4 = x^2u^4 \iff 1 = x^2. \end{aligned} \tag{6.14}$$

Thirdly we note that

$$\begin{aligned} (xy)^{-1}u(xy) = u^{-1} \iff x^{-1}y^{-1}uxy = u^{-1} \iff x^{-1}y^{-1}uyx = u^{-1} \\ \iff x^{-1}y^{-1}yux = u^{-1} \iff x^{-1}ux = u^{-1} \end{aligned} \tag{6.15}$$

All the other relations in (6.10) are clearly present in Lemma 5.1 so our claim follows and we can now conclude that $P \cong \pi(X)/N$. Now we apply Lemma 2.26 and realize that $P = \pi(U) \circ \pi(V)$.

Case (ii). From [34, Exercise 11.2 (d)] both U and V are 3-manifolds and so [34, Exercise 11.5 (b)] implies that $U \# V$ is a 3-manifold. So now we need to show that $U \# V$ can be endowed with a Riemannian structure. In short the Riemannian metric on $U \# V$ is inherited from the usual metric on the sphere $\mathbb{S}^3$. We start by observing from [18] that the action of Q_8 on the Riemannian sphere $\mathbb{S}^3$ (with the round metric $d_{\mathbb{S}^3}$ that is derived from the Riemannian metric

$$ds^2 = \frac{4\|dx\|^2}{(1 + \|x\|)^2}, \tag{6.16}$$

where $\|dx\|^2$ is the usual Riemannian metric on $\mathbb{R}^2$) produces the Riemannian space of orbits $U = \mathbb{S}^3/Q_8$, with the canonical quotient map $p_U : \mathbb{S}^3 \to U$ with the induced distance function on $u_1, u_2 \in U$,

$$d_U(u_1, u_2) = \inf_{x \in p_U^{-1}(u_1), y \in p_U^{-1}(u_2)} d_{S^3}(x, y) \tag{6.17}$$

and analogously for V with d_V, because the actions are free and properly discontinuous (see Lemma 3.18), and Q_8 and $\mathbb{Z}(4)$ are groups of isometries on $\mathbb{S}^3$ (see [17, Remark 41]).

A pair (M,Γ) where M is a Riemannian manifold and Γ is a (proper) discontinuous group of isometries acting effectively (see [17, Definition 22]) on M is called "good Riemannian orbifold". The underlying space of the "orbifold" is M/Γ. In the case of a good Riemannian orbifold (M,Γ) it follows that for $x,y\in M/\Gamma$,

$$d(x,y)=d_M(\pi^{-1}(x),\pi^{-1}(y)):=\inf_{\tilde{x}\in\pi^{-1}(x),\tilde{y}\in\pi^{-1}(y)} d_M(\tilde{x},\tilde{y}) \tag{6.18}$$

and this is what we have where in our case $M=\mathbb{S}^3$ and Γ as indicated in Lemmas 3.18 and 3.20. This means that we can define a metric on the disjoint union of the subspaces U' and V' (which are U and V without the open balls of a connected sum construction) by:

$$d'(x,y)=\begin{cases} d_{U'}(x,y), & if x,y\in U' \\ d_{V'}(x,y), & if x,y\in V' \\ \infty, & \text{otherwise.}\end{cases} \tag{6.19}$$

Using this metric we find the quotient semi-metric on $U\cup V$ in the sense of [18, Definition 3.1.12], with respect to the relation $\#$

$$d_{\#}(x,y)=\inf\{\sum_{i=1}^{k} d'(p_i,q_i)\mid p_1=x,q_k=y,k\in\mathbb{N}\}, \tag{6.20}$$

This induces a metric space that since $U\cup V$ is compact, this makes $d_{\#}(x,y)$ a Riemannian metric on $U\#V$, (see [18, Exercise 3.1.14]). And then from [34, Exercise 26.6(c)] we have $\pi(U\#V)\simeq\pi(U)*\pi(V)$ and so

$$\pi(U\#V)=Q_8*\mathbb{Z}(4)=\langle u,xy,y\mid u^4=1,u^2=(xy)^2,(xy)^{-1}u(xy)=u^{-1},y^4=1\rangle. \tag{6.21}$$

We can add the relations $uy=yu, xyy=yxy, u^2=y^2$ because of Lemma 2.19, by making a quotient with an appropriate normal subgroup L in $\pi(U\#V)$. Through this we can get the presentation (6.10), which we have shown to be equivalent to the presentation in Lemma 5.1. Therefore $P=\pi(U)\circ\pi(V)$ and Case (ii) follows completely.

□

We need to distinguish between the two cases, (i) and (ii), because $U\cup V$ is a more general case, but with it we cannot guarantee that it will possess the properties of interest, namely, that it will still be a manifold, or that it will have a Riemannian structure.

Our next result concerns dynamical systems involving pseudo-fermionic operators that have P as underlying symmetries. The reader can refer to [8, 10, 7] for more details on the theory of pseudo-fermionic operators.

Theorem 6.2. *There are two dynamical systems S and T involving pseudo-fermions with groups of symmetries respectively $P_\mu\simeq P$ and Q_8 but with the same hamiltonian $H_S=H_T$. In particular, there exist dynamical systems admitting larger groups of symmetries, whose size does not affect the dynamical aspects of the system.*

It is surprising how purely algebraic results can have such a deep connection with physical results, particularly in quantum mechanics. For example, the case of *pseudo-bosons* has been studied very well in recent contributions [2, 3, 11], and in particular [8] makes more concrete the bridge between these realms. Here, however, we focus on the case of pseudo-fermions, which were described before in Section 4.1.

We will show that the Pauli group (as a form of pseudo-fermionic operators) can arise from defining the generators using fermionic operators, not necessarily in a unique manner. And show how the Hamiltonion that arises from this construction, using P, can be reduced to a Hamiltonian involving only Q_8.

The idea is that we consider two operators $\mathbf{a}$ and $\mathbf{b}$ on the Hilbert space $\mathcal{H} = \mathbb{C}^2$ satisfying the following rules:

$$\{\mathbf{a}, \mathbf{b}\} = \mathbf{ab} + \mathbf{ba} = \mathbb{1}, \qquad \mathbf{a}^2 = \mathbf{b}^2 = 0, \tag{6.22}$$

One can check that these operators satisfy the rules of Equation (4.7) and are the basic ingredients now to define the following operators on $\mathcal{H}$:

$$\mu_1 := \mathbf{b} + \mathbf{a}, \qquad \mu_2 := i(\mathbf{b} - \mathbf{a}), \qquad \mu_3 := [\mathbf{a}, \mathbf{b}] = \mathbf{ab} - \mathbf{ba}. \tag{6.23}$$

Here the square brackets are called the *commutator* between $\mathbf{a}$ and $\mathbf{b}$. The main result of this section is that the set $\Gamma_\mu = \{\mu_1, \mu_2, \mu_3\}$ is a concrete realization of the generators of the Pauli group, and we denote the group that they generate P_μ. The proof of this claim is based on several identities which can easily be deduced out of (6.22). To see that these operators generate the Pauli group, we can show that

$$\mu_j^2 = \mathbb{1}, \quad j = 1, 2, 3, \qquad \text{and} \qquad \mu_1\mu_2 = i\mu_3, \quad \mu_2\mu_3 = i\mu_1, \quad \mu_3\mu_1 = i\mu_1. \tag{6.24}$$

We will only prove the first statement and will leave the others to the reader. First

$$\mu_1^2 = (\mathbf{b} + \mathbf{a})^2 = \mathbf{b}^2 + \mathbf{ba} + \mathbf{ab} + \mathbf{a}^2 = \{\mathbf{a}, \mathbf{b}\} = \mathbb{1}, \tag{6.25}$$

since $\mathbf{a}^2 = \mathbf{b}^2 = 0$ and $\{\mathbf{a}, \mathbf{b}\} = \mathbb{1}$. Similarly we can check that $\mu_2^2 = \mathbb{1}$. And lastly the proof that $\mu_3^2 = \mathbb{1}$.

$$\begin{aligned} \mu_2^2 &= (\mathbf{ab} - \mathbf{ba})^2 \\ &= \mathbf{abab} + \mathbf{baba} - \mathbf{abba} - \mathbf{baab} \\ &= (\mathbb{1} - \mathbf{ba})\mathbf{ab} + (\mathbb{1} - \mathbf{ab})\mathbf{ba} \\ &= \mathbf{ab} + \mathbf{ba} \\ &= \mathbb{1}, \end{aligned} \tag{6.26}$$

The proof of the other equalities in (6.22) are similar. However, these equalities are relevant to prove that, indeed, P_μ is a Pauli group. One just needs to see that $(\mu_1\mu_2)^4 = (i\mu_3)^4 = i^4(\mu_3^2)^2 = \mathbb{1}^2 = \mathbb{1}$ and $(\mu_2\mu_3)^4 = (\mu_3\mu_1)^4 = \mathbb{1}$ to prove our claim.

However P_μ is not the only Pauli group which can be constructed out of pseudo-fermionic operators. The operators $\Gamma_\rho = \{\rho_j = \mu_j^\dagger,\ j = 1, 2, 3\}$ also generate a Pauli group, say P_ρ, and one can check these these equalities hold:

$$\rho_j^2 = \mathbb{1}, \quad j = 1, 2, 3, \quad \text{and} \quad \rho_1\rho_2 = i\rho_3, \quad \rho_2\rho_3 = i\rho_1, \quad \text{and} \quad \rho_3\rho_1 = i\rho_1. \tag{6.27}$$

In [6, 8] many applications of pseudo-fermions to physics have been discussed. The following result can be connected to a two-levels atom with damping as will be shown below.

In [20], an effective non self-adjoint hamiltonian describing a two level atom interacting with an electromagnetic field was analyzed in connection with pseudo-hermitian systems. In [10] it was shown that this model can be viewed in terms of pseudo-fermionic operators and the result of Theorem 6.2 appears to be connected with this specific physical model in mind, described in terms of $\mathbf{H}_{eff}$. We will now view this model in terms of the Pauli group, but it is interesting to check if these aspects hold in general for other models, i.e., in particular, if $\mathbb{Z}(4)$ always contains the physical constants of motion of any physical system described in terms of the Pauli group.

Proof of Theorem 6.2. If we start with the Schrödinger equation

$$i\dot{\Phi}(t) = \mathbf{H}_{eff}\Phi(t), \qquad \mathbf{H}_{eff} = \frac{1}{2}\begin{pmatrix} -i\delta & \overline{\omega} \\ \omega & i\delta \end{pmatrix}. \tag{6.28}$$

Here δ is a real quantity, related to the decay rates for the two levels, while the complex parameter ω characterizes the radiation-atom interaction. We refer to [20] for further details. It is clear that $\mathbf{H}_{eff} \neq \mathbf{H}_{eff}^\dagger$. Then, we introduce the operators

$$\mathbf{a} = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & -e^{-i\theta}(\Omega + i\delta) \\ e^{i\theta}(\Omega - i\delta) & |\omega| \end{pmatrix}, \quad \mathbf{b} = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & e^{-i\theta}(\Omega - i\delta) \\ -e^{i\theta}(\Omega + i\delta) & |\omega| \end{pmatrix}. \tag{6.29}$$

Here

$$\Omega = \sqrt{|\omega|^2 - \delta^2}, \tag{6.30}$$

which we will assume here to be real and strictly positive. A direct computation shows that $\{\mathbf{a}, \mathbf{b}\} = \mathbb{1}$, $\mathbf{a}^2 = \mathbf{b}^2 = 0$. Hence $\mathbf{a}$ and $\mathbf{b}$ are pseudo-fermionic operators. Moreover, $\mathbf{H}_{eff}$ can be written in terms of these operators as

$$\mathbf{H}_{eff} = \Omega\left(\mathbf{ba} - \frac{1}{2}\mathbb{1}\right). \tag{6.31}$$

Using (6.23) we can find the matrices that correspond to the elements of Γ_μ. They are

$$\begin{gathered} \mu_1 = \frac{1}{\Omega}\begin{pmatrix} -|\omega| & -i\delta e^{-i\theta} \\ -i\delta e^{i\theta} & |\omega| \end{pmatrix}, \quad \mu_2 = i\begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \\ \mu_3 = \frac{1}{\Omega}\begin{pmatrix} i\delta & -|\omega|e^{-i\theta} \\ -|\omega|e^{i\theta} & -i\delta \end{pmatrix}, \end{gathered} \tag{6.32}$$

which are a non-trivial representations of the generators of the Pauli group. In terms of these operators we have the following simple expression for $\mathbf{H}_{eff}$:

$$\mathbf{H}_{eff} = -\frac{\Omega}{2}\mu_3. \tag{6.33}$$

The elements u, xy and y in Lemma 5.1 can be computed and turns out to be

$$u = i\mu_3 = \frac{i}{\Omega}\begin{pmatrix} i\delta & -|\omega|e^{-i\theta} \\ -|\omega|e^{i\theta} & -i\delta \end{pmatrix}, \quad xy = i\mu_2 = -\begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \quad y = i\mathbb{1}. \tag{6.34}$$

This produces an interesting consequence for this model. Since y is (complexly) proportional to the identity element, and Q_8 only contains μ_2 and μ_3, we interpret the elements of $\mathbb{Z}(4)$ as the constants of motion of the physical system described by $\mathbf{H}_{eff}$, or by its generalized form

$$H'_{eff} = -\frac{\Omega}{2}\mu_3 + \alpha\mu_2, \tag{6.35}$$

for all possible real α. So this result shows us that going from a larger P_μ to a smaller group Q_8 does not affect at all the dynamical aspects (i.e., the generator of the time evolution) of the system, since these are all contained in Q_8. we can study this model at a simpler level of group. □

Observing the representations of our μ_j's in (6.32), we can recover the Pauli matrices in (5.1) by taking suitable limiting conditions on the parameters: so if $\theta, \delta \to 0$, then $u \to -iX$ and $xy \to -iY$. This shows how our result in Theorem 6.2 extends the work from Section 5.

7. Conclusions

Looking at the structure of $P = Q_8 \circ \mathbb{Z}(4)$, and noting the final part of the proof of Theorem 6.2 we see that going from a larger group of symmetries P_μ to a smaller group Q_8 does not actually affect the dynamics (i.e. the generator of the time evolution) of the system because the operators that contribute meaningfully are the ones that generate Q_8. So one might wonder what algebraic properties related to the structure of a group might allow us to extend Theorem 6.2 to more general models that feature pseudo-fermionic operators. And considering Remarks 3.17, 3.21, we believe that

Conjecture 7.1. *Groups of the form $A = Q_8 \circ B$, where B is an abelian group containing at most one element of order 2 may have a construction of the Hamiltonian as we made with $\boldsymbol{H}_{eff}$ in Theorem 6.2, producing the fact that going from a larger group A to a smaller group Q_8 all the dynamical aspects are not affected.*

Our intuition for the conjecture above can be justified by results outlined in Section 3.2 and what we have shown rigorously in Theorem 6.2. In fact another way to look at Theorem 6.2 would be that the constants of motion are unaffected in some sense by the operation of a central product, in our case where we have $P = Q_8 \circ \mathbb{Z}(4)$. It seems to us that we can expect a similar behaviour in such pseudo-fermionic (or indeed more general) models with an appropriate Hamiltonian with the underlying structure of a group $A = Q_8 \circ B$, where B can be any finite group acting freely on $\mathbb{S}^3$, for instance $\mathbb{Z}(2m)$ for any $m \geq 2$ (see [30]), and other noncyclic finite groups that are classified quite well in [47].

Due to [31, Corollary 9.59, (iv), The Sphere Group Theorem], the special property of $\mathbb{S}^3$ among all $\mathbb{S}^n$ leads us to believe that it is not reasonable to expect this interpretation of the constants of the motions for Hamiltonians (which are constructed similarly to our proof of Theorem 6.2) but with groups of the form $G = H \circ K$ for an arbitrary finite groups H and K. We think that it is possible for more general frameworks, not involving pseudo-fermionic operators, but then one could lose the physical relevance of such a description. However working in the direction of Conjecture 7.1 above could lead to significant ideas in the mathematical models of quantum mechanics with pseudo-fermions.

Anyway the question remains open in its generality:

Question 7.2. Given two arbitrary finite groups H and K, consider the group $G = H \circ K$. When can we introduce $\mathbf{H}_{eff}$ with the formalism of the pseudo-fermionic operators, producing a mathematical model where the generators of the time evolution are only depending on H?

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