# Pion Electromagnetic Form Factor in the Kroll-Lee-Zumino Model at Zero and Finite Temperature

Bernard Willers

Supervisors: Em. Prof. C.A. Dominguez
Prof. R.D. Viollier

Department of Physics

University of Cape Town

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## Abstract

The renormalizable Abelian quantum field theory model of Kroll, Lee and Zumino is used to calculate next-to-leading order corrections to the pion electromagnetic form factor in vector meson dominance. At zero temperature the predictions for both the form factor and electromagnetic radius are found to improve greatly over the tree level result, and are in good agreement with the experimental data. A calculation of the vertex and self energy functions in the Matsubara formalism at finite temperature do not agree with the results of the Gale & Kapusta calculations. The resulting prediction for the radius is found to increase with temperature, consistent with ideas about hadronic deconfinement.

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# CONTENTS

# INTRODUCTION

The electromagnetic form factor of the pion has been the subject of considerable study, and plays a key role in interactions such as the electron-positron annihilation process depicted in figure 1.1. The $e^+e^- \to \gamma$ vertex is very well described by QED, but the $\gamma \to \pi^+\pi^-$ is not as well understood.

Before the advent of QCD, many effective models of the interaction were proposed, the most successful of which is vector meson dominance (VMD), which is based on the experimental observation that the electromagnetic form factor of the pion is dominated by the $\rho$ meson, with a tiny contribution from the $\omega$ mixed in. Due to the inherent difficulty in solving the underlying QCD, the VMD principle remains of use in phenomenology.

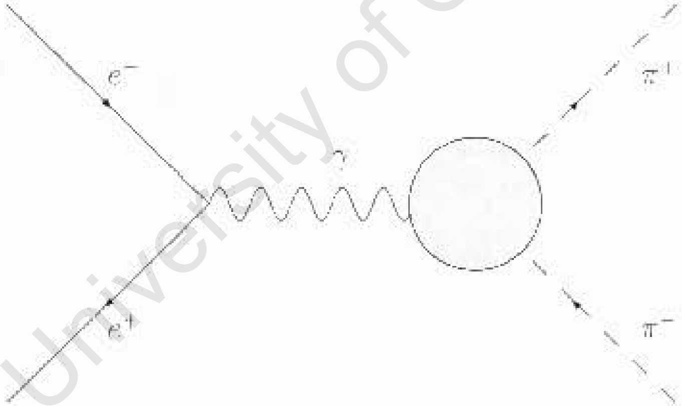

Figure 1.1: The general $e^+e^- \to \pi^+\pi^-$ process

The aim of this work is to study the effect of calculating next-to-leading order corrections to the VMD pion electromagnetic form factor within the framework of the Abelian renormalizable field theory model of Kroll, Lee and Zumino [1] at both zero and finite temperature. The results at zero temperature were published in Physical Review D [2], and the finite temperature results are presently being written up for publication. There is considerable overlap between this dissertation and those papers.

# VECTOR MESON DOMINANCE & THE KROLL-LEE-ZUMINO MODEL

## 2.1 Vector Meson Dominance

Discovered during the 1960's [3], the vector meson dominance principle states that, to a reasonable approximation, the hadronic electromagnetic current operator is equivalent to a sum of the known neutral vector meson operators. This can be thought of as a photon coupling to a vector meson, which in turn couples to the hadronic matter in question. The VMD principle only applies for an invariant mass below 1-2 GeV, beyond which the substructure of the hadrons becomes important.

The first seeds of VMD were sown by Nambu [4] who, in 1957, suggested that the charge distribution of nucleons could be explained by a heavy, neutral, isospin zero vector meson contributing to the nucleon form factor. Sakurai was the first to propose a theory of the strong interaction mediated by vector mesons [5] based on a non-Abelian field theory.

When considering hadrons involving only up and down quarks, the possible mediating mesons are the $\rho$, $\omega$ and their more massive resonances. Due to G-parity restrictions, the $\omega$ decays into three pions (neglecting $\rho - \omega$ mixing). For this reason the $\omega$ and radial excitations have been omitted from these calculations.

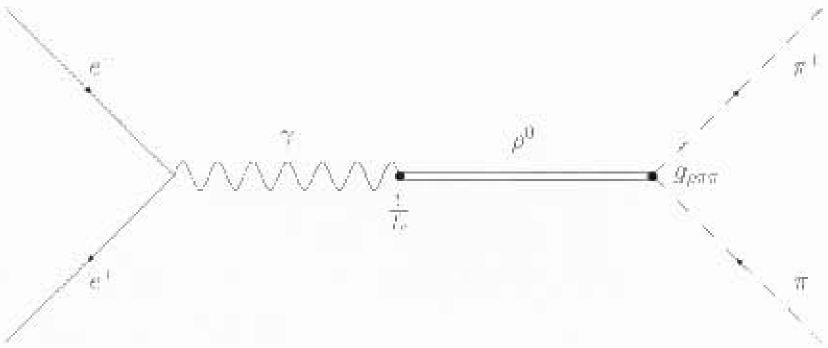

Figure 2.1: Vector meson dominance principle in the $e^+e^- \to \pi^+\pi^-$ process

Historically, there have been two Lagrangian formulations of VMD, differing in the nature of the $\gamma - \rho$ coupling. The VMD-1 Lagrangian, due to Kroll, Lee and Zumino [1] uses a coupling term of the form

$$\mathcal{L}_{\gamma\rho} = -\frac{e}{2f_\rho} F_{\mu\nu}\rho^{\mu\nu} \tag{2.1}$$

Converting to momentum space $\partial_\mu A_\nu \partial^\mu \rho^\nu$ is transformed into $-\partial_\mu \partial^\mu A_\nu \rho^\nu$ using integration by parts and derivative is sent to $\partial_\mu \rightarrow iq_\mu$ to yield

$$F_{\mu\nu}\rho^{\mu\nu} \rightarrow 2q^2 A_\mu \rho^\mu \tag{2.2}$$

This is insufficient as the photon and $\rho^0$ would decouple at $q^2 = 0$, so an explicit photon-hadron term is also introduced into the Lagrangian.

$$\mathcal{L}_{VMD1} = -\frac{e}{2f_\rho} F_{\mu\nu}\rho^{\mu\nu} - eA_\mu J^\mu + \ldots \tag{2.3}$$

The alternate formulation, VMD-2, was outlined by Sakurai and utilises a coupling term of the form

$$\mathcal{L}_{\gamma\rho} = -\frac{em_\rho^2}{f_\rho}\rho_\mu A^\mu \tag{2.4}$$

The appearance of $m_\rho^2$ in the coupling results in a mass term in the photon propagator, which is eliminated (see [6]) by the addition of an explicit photon mass term in the Lagrangian.

$$\mathcal{L}_{VMD2} = -\frac{em_\rho^2}{f_\rho}\rho_\mu A^\mu + \frac{1}{2}\left(\frac{em_\rho}{f_\rho}\right)^2 A_\mu A^\mu + \ldots \tag{2.5}$$

Despite the inelegance of a photon mass term in the Lagrangian, the second formulation is a popular formulation of vector meson dominance. In the universality limit, $g_{\rho\pi\pi} = f_\rho$, the two representations become equivalent.

## 2.2 The Pion Form Factor in Vector Meson Dominance

For VMD-2, the photon-rho equivalence becomes

$$\left\langle A \left| J_\mu^{EM} \right| B \right\rangle = e\left\langle B \left| \frac{m_\rho^2}{f_\rho}\rho_\mu \right| A \right\rangle, \tag{2.6}$$

applying this to the $\gamma \to \pi^{\pm}$ system gives, with $q^2 = (p_2 - p_1)^2$,

$$\langle \pi^+(p_2)\pi^-(p_1)\,|J_\mu^{EM}|\,0\rangle = e\frac{m_\rho^2}{f_\rho}\left\langle \pi^-(p_2)\pi^-(p_1)\left|\frac{g_{\rho\pi\pi}J_\mu^\pi}{m_\rho^2 - q^2}\right|0\right\rangle, \tag{2.7}$$

which reduces to

$$\begin{aligned}\langle \pi^+(p_2)\pi^-(p_1)\,|J_\mu^{EM}|\,0\rangle &= e\frac{-i}{f_\rho}\frac{m_\rho^2}{m_\rho^2 - q^2}\Gamma_{\rho\pi\pi}(p_1, p_2)_\mu \\ &= e\frac{g_{\rho\pi\pi}}{f_\rho}\frac{m_\rho^2}{m_\rho^2 - q^2}(p_1 - p_2)_\mu.\end{aligned} \tag{2.8}$$

Comparing this with the definition of the pion form factor

$$\langle \pi(p_2)\,|J_\mu^{EM}|\,\pi(p_1)\rangle = e(p_1 + p_2)_\mu F_\pi(q^2), \tag{2.9}$$

it is clear that the form factor in VMD-2 is given by

$$F_\pi(q^2) = \frac{g_{\rho\pi\pi}}{f_\rho}\frac{m_\rho^2}{m_\rho^2 - q^2}, \tag{2.10}$$

while, in contrast, VMD-1 leads to the expression

$$F_\pi(q^2) = \left[1 + \frac{q^2}{m_\rho^2 - q^2}\frac{g_{\rho\pi\pi}}{f_\rho}\right]. \tag{2.11}$$

Both of these result have inherit problems. The VMD-2 result does not correctly reproduce the normalisation of the form factor $F_\pi(0) = 1$ without assuming universality, and while the VMD-1 result does produce the correct normalisation, it becomes negative for large space-like $q^2$ if $g_{\rho\pi\pi} > f_\rho$.

The normalisation problem with the VMD-2 result can actually be corrected without universality by a suitable choice of renormalisation parameters at the one loop level. Nevertheless, to eliminate the problem of choosing a formulation the calculations that follow assume universality,

$$F_\pi(q^2) = \frac{m_\rho^2}{m_\rho^2 - q^2}, \tag{2.12}$$

which serves as a rough estimate but has an unsatisfactory fit to the available experimental data, with a chi-squared per degree of freedom of $\chi_F^2 \approx 5.0$.

## 2.3 Kroll-Lee-Zumino Model

As mentioned previously, Kroll, Lee and Zumino [1] constructed a Lagrangian formulation of vector meson dominance (VMD-1) in 1967. In the same paper they briefly discuss its application to a system of interacting $\rho^0$ and $\pi^\pm$ mesons. The model presented is an Abelian, renormalizable effective theory, neglecting the quark degrees of freedom. The Lagrangian for the model is

$$\mathcal{L}_{KLZ} = \partial_\mu\phi\partial^\mu\phi^* - m_\pi^2\phi^*\phi - \tfrac{1}{4}\rho_{\mu\nu}\rho^{\mu\nu} + \tfrac{1}{2}m_\rho^2\rho_\mu\rho^\mu + g_{\rho\pi\pi}\rho_\mu J^\mu_\pi + g^2_{\rho\pi\pi}\rho_\mu\rho^\mu\phi\phi^*. \quad (2.13)$$

Where $\rho_\mu$ is a vector field describing the $\rho^0$ meson; $\phi$ is a complex pseudo-scalar field describing the $\pi^\pm$ mesons; $\rho_{\mu\nu}$ the usual field strength tensor; $J^\mu_\pi$ the $\pi^+$ current

$$\rho_{\mu\nu} = \partial_\mu\rho_\nu - \partial_\nu\rho_\mu \qquad\qquad J^\mu_\pi = i\phi^*\overleftrightarrow{\partial_\mu}\phi; \quad (2.14)$$

and $m_\rho$ and $m_\pi$ are the physically measurable mass of the $\rho^0$ and $\pi^\pm$ mesons ($m_\rho \approx$ 775.80 MeV and $m_\pi \approx 139.57$ MeV according to [7]).

Despite the non-vanishing mass of the $\rho^0$ in the Lagrangian the theory remains renormalisable[1] as the vector meson field couples to a conserved current $\partial_\mu\rho^\mu = 0$. The existence of such a conserved current is due to the equivalence of the $\rho$ field and the electromagnetic current operator. A proof of renormalisability is presented in [1]

A non-Abelian extension of the Lagrangian, which is necessary to include charged $\rho^\pm$ mesons, is not renormalisable if the masses are included explicitly. Work is currently being done on extending the KLZ Lagrangian to the non-Abelian case through a Higgs-like mass generation scheme. Such an extension is necessary to facilitate the calculation of processes involving the full triplet of $\rho$ mesons, such as $\pi\pi \rightarrow \pi\pi$ scattering. However, the Abelian, renormalizable case is sufficient for the study of the pion form factor.

The remainder of the text investigates the single loop corrections that contribute to the form factor. It will be shown that the relatively mild coupling does not preclude perturbative calculations, as loop results are accompanied by a loop suppression factor of $1/(4\pi)^2$.

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[1] In general (see [8]), there are only two cases in which a massive vector field is renormalisable:

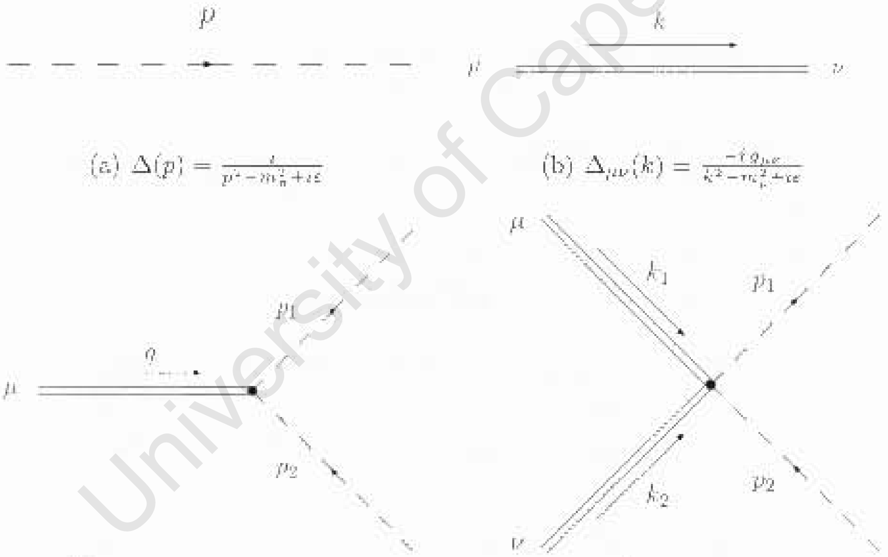
- Gauge theories in which mass is generated by spontaneous symmetry breaking;
- Theories with a massive vector boson coupled to a conserved current without additional self-interactions

## 2.4 Feynman Rules

Since the theory is gauge invariant, the usual approach of introducing an additional non-physical field and gauge fixing parameter $\xi$ applies. The general propagator for a massive vector boson is given by

$$\Delta_{\mu\nu}(k) = -i\frac{g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2-\xi m^2}}{k^2 - m^2 + i\varepsilon}. \tag{2.15}$$

Common gauge choices include the Landau gauge, $\xi = 0$, the unitary gauge $\xi \to \infty$ and the Feynman gauge $\xi = 1$. The preferred gauge for the calculations that follow is the Feynman gauge, which is characterised by the absence of the $k^\mu k^\nu$ term in the massive vector boson propagator, and leads to considerably simplified calculations. The Feynman rules for the theory are depicted in Fig. 2.2.



(a) $\Delta(p) = \frac{i}{p^2 - m_\pi^2 + i\varepsilon}$ (b) $\Delta_{\mu\nu}(k) = \frac{-i\,g_{\mu\nu}}{k^2 - m_\rho^2 + i\varepsilon}$

(c) $\Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2) = i g_{\rho\pi\pi}(p_1 + p_2)^\mu$ (d) $\Gamma^{(0)}_{\rho\rho\pi\pi} = 2i g^2_{\rho\pi\pi} g^{\mu\nu}$

Figure 2.2: Feynman Rules at Zero Temperature

In order to facilitate dimensional regularisation the theory is treated in $d$ space-time dimensions at zero temperature. A renormalizable theory requires a dimensionless coupling (working in units with $\hbar = c = 1$), so a quantity $\mu$ with dimensions of mass is introduced

$$g_{\rho\pi\pi} \to \mu^{(2-\frac{d}{2})} g_{\rho\pi\pi}. \tag{2.16}$$

# ZERO TEMPERATURE

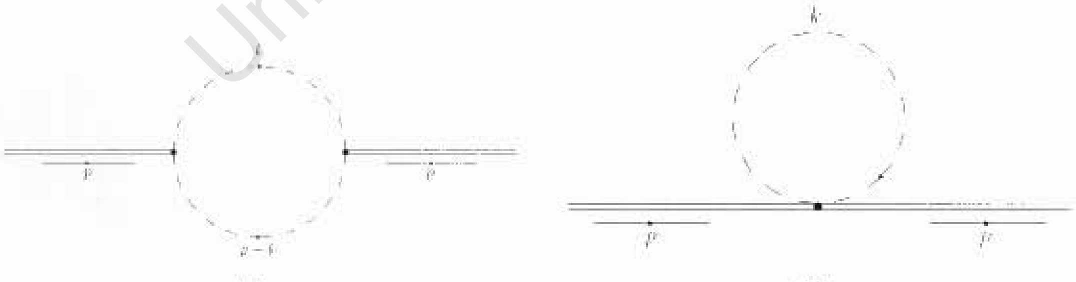
In order to compute the next-to-leading order corrections to tree-level vector dominance, it is necessary to consider the seven one loop diagrams:

- Two $\rho^0$ self energy diagrams,
- two $\pi^{\pm}$ self energy diagrams,
- the $\rho$ $\pi$ $\pi$ triangle diagram, and
- two seagull diagrams.

## 3.1 Rho Self Energy

There are two one-loop diagrams, Figs. 3.1(a) and 3.1(b), that contribute to the self energy of the $\rho^0$ meson, which is given by

$$i\Pi^{\mu\nu}(p) = g^2_{\rho\pi\pi}\left(\mu^2\right)^{\left(2-\frac{d}{2}\right)}\int\frac{d^dk}{(2\pi)^d}\left\{\frac{(2p-k)^{\mu}(2p-k)^{\nu}}{(k^2-m_\pi^2-i\varepsilon)[(k-p)^2-m_\pi^2+i\varepsilon]}-\frac{2g^{\mu\nu}}{k^2-m_\pi^2-i\varepsilon}\right\}. \tag{3.1}$$



(a) (b)

Figure 3.1: Rho Self Energy Diagrams

The $\rho^0$ self energy modifies the $\rho^0$ propagator, which in turn corrects the pion form factor, and is also important in fixing the renormalisation conditions. The self energy can be written in terms of both longitudinal and transverse components:

$$\Pi^{\mu\nu}(q) = \mathcal{F}(q)P_L^{\mu\nu} + \mathcal{G}(q)P_T^{\mu\nu}, \tag{3.2}$$

where $P_L$ and $P_T$ are the standard projection tensors

$$P_L^{\mu\nu} = \frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} - P_T^{\mu\nu}, \tag{3.3}$$

$$P_T^{00} = P_T^{0i} = P_T^{i0} = 0, \tag{3.4}$$

$$P_T^{ij} = \delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}. \tag{3.5}$$

The self energy modifies the $\rho^0$ propagator

$$\Delta^{\mu\nu}(k) = \frac{iP_L^{\mu\nu}}{k^2 - m_\rho^2 - \mathcal{F}(k)} + \frac{iP_T^{\mu\nu}}{k^2 - m_\rho^2 - \mathcal{G}(k)}. \tag{3.6}$$

To recast this into an alternate form, recall that the self energy is transverse, and that for any transverse two-point function

$$\Pi^{\mu\nu}(q) = \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\Pi(q^2), \tag{3.7}$$

multiplying by the metric, it is clear that

$$g_{\mu\nu}\Pi^{\mu\nu} = g_{\mu\nu}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\Pi(q^2) = -3\Pi(q^2). \tag{3.8}$$

Combining Eqs. 3.2 and 3.3 gives

$$\Pi^{\mu\nu}(q^2) = \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\mathcal{F}(q^2) + P_T^{\mu\nu}\Big(\mathcal{G}(q^2) - \mathcal{F}(q^2)\Big), \tag{3.9}$$

from which, together with Eq. 3.8, it follows that

$$\begin{aligned}
\Pi(q^2) &= -\frac{1}{3}\left[g_{\mu\nu}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\mathcal{F}(q^2) + g_{\mu\nu}P_T^{\mu\nu}\Big(\mathcal{G}(q^2) - \mathcal{F}(q^2)\Big)\right]\\
&= -\frac{1}{3}\Big[-3\mathcal{F}(q^2) + \left(g_{00}P_T^{00} + g_{ii}P_T^{ii}\right)\Big(\mathcal{G}(q^2) - \mathcal{F}(q^2)\Big)\Big]\\
&= -\frac{1}{3}\left[-3\mathcal{F}(q^2) - \delta_{ii}\left(\delta^{ii} - \frac{q^i q^i}{\mathbf{q}^2}\right)\Big(\mathcal{G}(q^2) - \mathcal{F}(q^2)\Big)\right]\\
&= \frac{1}{3}\Big[\mathcal{F}(q^2) + 2\mathcal{G}(q^2)\Big].
\end{aligned} \tag{3.10}$$

This factor modifies the $\rho^0$ propagator, which now becomes

$$\Delta_{\mu\nu}(k) \to \frac{-i\,g_{\mu\nu}}{k^2 - m_\rho^2 - \Pi(q^2)}. \tag{3.11}$$

When this correction is extended to the form factor, ignoring for a moment the vertex, the form factor becomes

$$F_\pi(q^2) = \frac{m_\rho^2 - \Pi(0,T)}{m_\rho^2 - q^2 + \Pi(q^2,T)}. \tag{3.12}$$

It now remains to determine $\mathcal{F}$ and $\mathcal{G}$. Since the diagrams under consideration are already in the literature [9], an in-depth calculation will not be shown here.

At zero temperature, only the vacuum polarisation contributes to $\Pi(q^2)$. It can be shown that the vacuum contributions to $\mathcal{F}$ and $\mathcal{G}$ are identical

$$\Pi(q^2) = \Pi_{vac}(q^2) = \mathcal{F}_{vac}(q^2) = \mathcal{G}_{vac}(q^2), \tag{3.13}$$

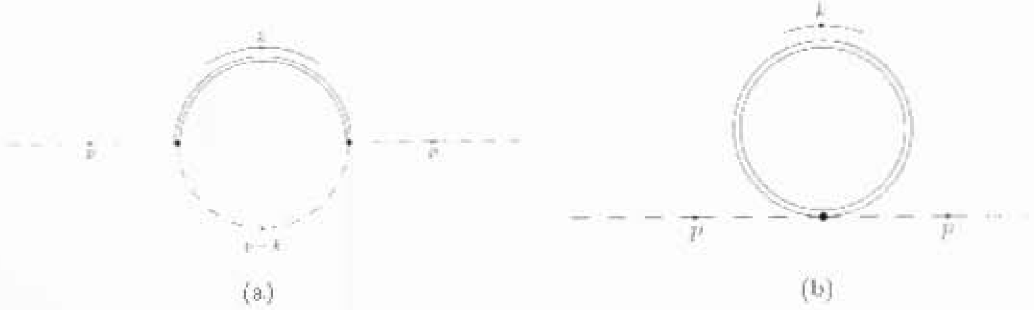
and has the form

$$\begin{aligned}\Pi_{vac}(q^2) = \frac{q^2}{3}\frac{g_{\rho\pi\pi}^2}{(4\pi)^2}\Bigg\{ \left(1 - \frac{4m_\pi^2}{q^2}\right)^{\frac{3}{2}} \left[ \ln\left|\frac{\sqrt{1-4m_\pi^2/q^2}+1}{\sqrt{1-4m_\pi^2/q^2}-1}\right| - i\pi\Theta(q^2-4m_\pi^2)\right] \\ - \left(1 - \frac{4m_\pi^2}{m_\rho^2}\right)^{\frac{3}{2}} \ln\left|\frac{\sqrt{1-4m_\pi^2/m_\rho^2}+1}{\sqrt{1-4m_\pi^2/m_\rho^2}-1}\right| - 8\frac{m_\pi^2}{m_\rho^2}\Bigg\} + \frac{8}{3}\frac{g_{\rho\pi\pi}^2}{(4\pi)^2}m_\pi^2.\end{aligned} \tag{3.14}$$

## 3.2 Pion Self Energy

There are two diagrams, Figs. 3.2(a) and 3.2(b), that contribute to the pion self energy, which is given by

$$i\Pi_0(p) = g_{\rho\pi\pi}^2 \left(\mu^2\right)^{\left(2-\frac{d}{2}\right)} \int \frac{d^dk}{(2\pi)^d}\left\{ \frac{(2p-k)^2}{(k^2-m_\rho^2+i\varepsilon)\left[(p-k)^2-m_\pi^2+i\varepsilon\right]} - \frac{d}{k^2-m_\rho^2+i\varepsilon}\right\} \tag{3.15}$$



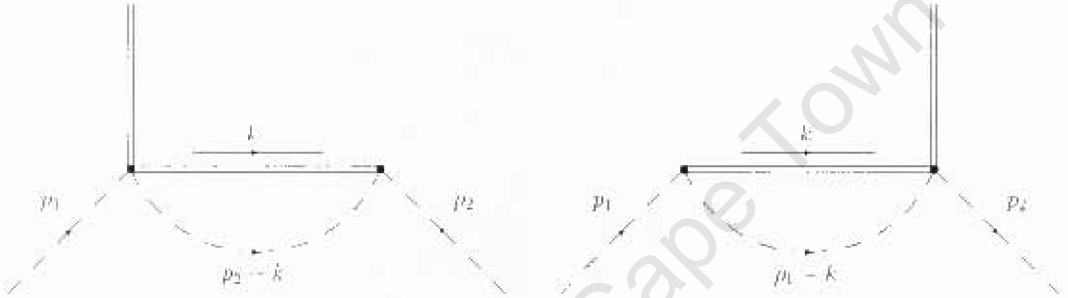
(a) (b)

Figure 3.2: Pion Self Energy Diagrams

The pion self energy does not contribute explicitly to the form factor, although it does play an important part in the renormalisation of the pion mass. As such, the definition $\Pi_0$ above suffices, and the full calculation is not relevant to the present work.

## 3.3 Seagull Diagrams

There are two more diagrams at the one loop level, the so-called 'seagull' diagrams depicted in Figs 3.3(a) and 3.3(b).



(a) (b)

Figure 3.3: Seagull Diagrams

A quick examination of the diagrams reveals that they have no $q^2$ dependence. As a result, these two terms will subtract off during renormalisation, and do not contribute to the form factor. However, they do contribute to the renormalisation constants in Eqs. 3.74 through 3.77.

Furthermore, they are necessary ensure the gauge invariance of the theory. The tadpole diagram in Fig. 3.1(b), which is proportional to $g_{\mu\nu}$, cancels with part of the seagull diagrams, resulting in a purely transverse function.

## 3.4 Rho-Pi-Pi Triangle Diagram

The $\rho\pi\pi$ interaction vertex is extended to include a single loop correction corresponding to the diagram in Fig. 3.4. Combining the relevant propagators the vertex function is given by

$$\begin{aligned}\widetilde{\Gamma}^{(1)}_{\rho\pi\pi}(p_1, p_2, q^2) &= g^3_{\rho\pi\pi}\left(\mu^{3}\right)^{2-\frac{d}{2}} \\ &\times \int \frac{d^dk}{(2\pi)^d} \frac{(p_1 - p_2 + 2k)^\mu\,(2p_1 + k)\cdot(2p_2 + k)}{[(p_1 + k)^2 - m_\pi^2 + i\varepsilon][(p_2 - k)^2 - m_\pi^2 + i\varepsilon](k^2 - m_\rho^2 + i\varepsilon)}. \quad (3.16)\end{aligned}$$

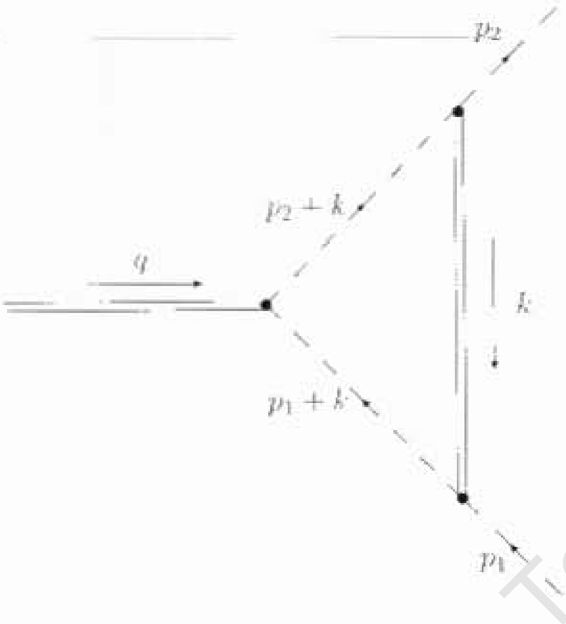


Figure 3.4: 1-Loop correction to $\rho\pi\pi$ vertex

Utilising the Feynman identity

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 dx_2 \cdots dx_n \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + x_2 A_2 + \cdots x_n A_n)^n}, \quad (3.17)$$

the denominator of the integrand becomes

$$\frac{1}{[(p_1+k)^2 - m_\pi^2 + i\varepsilon][(p_2+k)^2 - m_\pi^2 + i\varepsilon](k^2 - m_\rho^2 + i\varepsilon)} = \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) \frac{2}{D^3},$$

with $D$ given by

$$\begin{aligned} D &= x_1\left[(p_1+k)^2 - m_\pi^2 + i\varepsilon\right] + x_2\left[(p_2+k)^2 - m_\pi^2 + i\varepsilon\right] + x_3(k^2 - m_\rho^2 + i\varepsilon) \\ &= \underbrace{(x_1 + x_2 + x_3)}_{1}(k^2 + i\varepsilon) + 2k\cdot(x_1 p_1 + x_2 p_2) + x_1 \underbrace{({p_1}^2 - m_\pi^2)}_{0} + x_2 \underbrace{({p_2}^2 - m_\pi^2)}_{0} - m_\rho^2 x_3 \\ &= [k + (x_1 p_1 + x_2 p_2)]^2 - (x_1 p_1 + x_2 p_2)^2 - m_\rho^2 x_3 + i\varepsilon \\ &= [k + (x_1 p_1 + x_2 p_2)]^2 - ({x_1}^2 {p_1}^2 + {x_2}^2 {p_2}^2 + 2 x_1 x_2 p_1 \cdot p_2 + x_3 m_\rho^2) + i\varepsilon \\ &= [k + (x_1 p_1 + x_2 p_2)]^2 - ({x_1}^2 m_\pi^2 + {x_2}^2 m_\pi^2 + 2 x_1 x_2 p_1 \cdot p_2 + x_3 m_\rho^2) + i\varepsilon. \end{aligned} \quad (3.18)$$

Defining a new momentum of the form

$$\boxed{l = k - (x_1 p_1 - x_1 p_2)}, \tag{3.19}$$

the parameter

$$s = (x_1 p_1 + x_1 p_2), \tag{3.20}$$

a new real variable $\Delta = \Delta(q^2)$

$$\begin{aligned} \Delta &= {x_1}^2 m_\pi^2 - {x_2}^2 m_\pi^2 + 2x_1 x_2 p_1 \cdot p_2 + m_\rho^2 x_3 \\ &= {x_1}^2 m_\pi^2 + {x_2}^2 m_\pi^2 + x_1 x_2 \left[{p_1}^2 - {p_2}^2 - (p_2 - p_1)\right]^2 + m_\rho^2 x_3 \\ &= {x_1}^2 m_\pi^2 + {x_2}^2 m_\pi^2 + x_1 x_2 (2m_\pi^2 - q^2) + m_\rho^2 x_3 \\ &= m_\pi^2 (x_1 - x_2)^2 - m_\rho^2 x_3 - x_1 x_2 q^2, \end{aligned} \tag{3.21}$$

and using the delta function to eliminate $x_3$

$$\int_0^1 dx^3 \delta \left(\sum_i x_i - 1\right) \frac{N(x_1, x_2, x_3)}{D(x_1, x_2, x_3)^3} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{N(x_1, x_2, 1 - x_1 - x_2)}{D(x_1, x_2, 1 - x_1 - x_2)^3}, \tag{3.22}$$

so that the expression in Eq. 3.21 becomes

$$\boxed{\Delta = m_\pi^2 (x_1 + x_2)^2 + m_\rho^2 (1 - x_1 - x_2) - x_1 x_2 q^2,} \tag{3.23}$$

the denominator becomes

$$\boxed{D = l^2 - \Delta + i\varepsilon.} \tag{3.24}$$

The numerator of the integral in Eq. 3.16 must also be reformulated in terms of the new momentum $l$. Clearly $k = l - s$, so

$$\begin{aligned} N =&(p_1 + p_2 + 2k)^\mu (2p_1 + k) \cdot (2p_2 + k) \\ =&(p_1 + p_2 + 2l - 2s)^\mu (l + 2p_1 - s) \cdot (l + 2p_2 - s) \\ =&(p_1 + p_2 + 2l - 2s)^\mu \left[l^2 + (2p_1 - s) \cdot (2p_2 - s) + 2l \cdot (p_1 + p_2 - s)\right], \end{aligned} \tag{3.25}$$

so Eq. 3.16 becomes

$$\begin{aligned} \bar{\Gamma}^{(1)}_{\mu\pi\pi}(p_1, p_2; q^2) =& 2\, g^3_{\rho\pi\pi} \, (\mu^3)^{2-\frac{d}{2}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^d l}{(2\pi)^d} \frac{(2l - 2s + p_1 - p_2)^\mu}{(l^2 - \Delta + i\varepsilon)^3} \\ & \times \left[l^2 - (2p_1 - s) \cdot (2p_2 - s) + 2l \cdot (p_1 + p_2 - s)\right]. \end{aligned} \tag{3.26}$$

This expression involves integrals of the forms

$$\begin{aligned}
&\int \frac{d^dl}{(2\pi)^d}\frac{l^{2s}}{(l^2-\Delta+i\varepsilon)^3} \qquad s=0,1 \qquad &&\int \frac{d^dl}{(2\pi)^d}\frac{(a\cdot l)}{(l^2-\Delta+i\varepsilon)^3}\\
&\int \frac{d^dl}{(2\pi)^d}\frac{l^{2s}l^\mu}{(l^2-\Delta+i\varepsilon)^3} \qquad s=0,1 \qquad &&\int \frac{d^dl}{(2\pi)^d}\frac{(a\cdot l)l^\mu}{(l^2-\Delta+i\varepsilon)^3} \; .
\end{aligned} \tag{3.27}$$

Two of these are identically zero, as they are integrals of odd functions over an interval symmetric about zero:

$$\begin{aligned}
&\int \frac{d^dl}{(2\pi)^d}\frac{l^{2s}l^\mu}{(l^2-\Delta+i\varepsilon)^3} = 0 \qquad\qquad s=0,1\\
&\int \frac{d^dl}{(2\pi)^d}\frac{(a\cdot l)}{(l^2-\Delta+i\varepsilon)^3} = a_\mu \int \frac{d^dl}{(2\pi)^d}\frac{l^\mu}{(l^2-\Delta+i\varepsilon)^3} = 0.
\end{aligned} \tag{3.28}$$

Defining

$$\boxed{I_{s,n}(\Delta) = \int \frac{d^dl}{(2\pi)^d}\frac{l^{2s}}{(l^2-\Delta+i\varepsilon)^n},} \tag{3.29}$$

and recalling that in a $d$-dimensional Minkowski space $g_{\mu\nu}g^{\mu\nu} = d$, the last integral is given by

$$\begin{aligned}
\int \frac{d^dl}{(2\pi)^d}\frac{(a\cdot l)l^\mu}{(l^2-\Delta+i\varepsilon)^3} &= a_\nu \int \frac{d^dl}{(2\pi)^d}\frac{l^\nu l^\mu}{(l^2-\Delta+i\varepsilon)^3}\\
&= a_\nu \frac{g^{\mu\nu}}{d}\int \frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta+i\varepsilon)^3} = \frac{a^\mu}{d} I_{1,3}(\Delta),
\end{aligned} \tag{3.30}$$

and Eq. 3.16 becomes

$$\begin{aligned}
&\widetilde{\Gamma}^{(1)}_{\rho\pi\pi}(p_1,p_2,q^2) = 2\, g^3_{\rho\pi\pi}\left(\mu^3\right)^{2-\frac{d}{2}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2\\
&\times\left\{(p_1+p_2-2s)^\mu\left[(2p_1-s)\cdot(2p_2-s)I_{0,3}(\Delta) + \left(1+\frac{4}{d}\right) I_{1,3}(\Delta)\right] + \frac{4}{d}s^\mu I_{1,3}(\Delta)\right\}.
\end{aligned} \tag{3.31}$$

Ultimately, the integral over the Feynman parameters $x_1$ and $x_2$ will need to be evaluated, so the coefficients in the previous expression are reformulated in terms of $x_1$ and $x_2$. Using ${p_1}^2 = {p_2}^2 = m_\pi^2$ and $q = p_2 - p_1$ it follows that

$$\begin{aligned}
2p_1\cdot p_2 &= 2m_\pi^2 - q^2 \qquad & s^2 &= m_\pi^2(x_1+x_2)^2 - q^2x_1x_2\\
2p_2\cdot s &= 2m_\pi^2(x_1+x_2) - q^2x_1 \qquad & 2p_1\cdot s &= 2m_\pi^2(x_1+x_2) - q^2x_2,
\end{aligned} \tag{3.32}$$

so

$$(2p_1 - s) \cdot (2p_2 - s) = m_\pi^2(x_1 + x_2 - 2)^2 - q^2(x_1x_2 - x_1 - x_2 + 2).$$

Inserting this into Eq. 3.31 yields

$$\widetilde{\Gamma}^{(1)}_{\rho\pi\pi}(p_1, p_2, q^2) = 2\, g^3_{\rho\pi\pi}\left(\mu^3\right)^{2-\frac{d}{2}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[(p_1+p_2)^\mu f_1(x_1, x_2) - 2s^\mu f_2(x_1, x_2)\right], \tag{3.33}$$

where

$$f_1(x_1, x_2) = \Big[m_\pi^2(x_1+x_2-2)^2 - q^2(x_1x_2 - x_1 - x_2 + 2)\Big] I_{0,3}(\Delta) + \Big[1 + \tfrac{4}{d}\Big] I_{1,3}(\Delta), \tag{3.34}$$

and

$$f_2(x_1, x_2) = \Big[m_\pi^2(x_1+x_2-2)^2 - q^2(x_1x_2 - x_1 - x_2 + 2)\Big] I_{0,3}(\Delta) + \Big[1 + \tfrac{2}{d}\Big] I_{1,3}(\Delta). \tag{3.35}$$

By noting that $f_2(x_1, x_2) = f_2(x_2, x_1)$ the $s$ can be factored out

$$\begin{aligned}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 f_2(x_1, x_2) s^\mu &= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 f_2(x_1, x_2)(x_1p_1 + x_2p_2)^\mu \\ &= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 f_2(x_2, x_1)(x_1p_1 + x_1p_2)^\mu \\ &= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 f_2(x_1, x_2) x_1(p_1 + p_2)^\mu,\end{aligned} \tag{3.36}$$

and so

$$\begin{aligned}\widetilde{\Gamma}^{(1)}_{\rho\pi\pi}(p_1, p_2, q^2) &= 2\, g^3_{\rho\pi\pi}\left(\mu^3\right)^{2-\frac{d}{2}} (p_1+p_2)^\mu \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[f_1(x_1, x_2) - 2x_1 f_2(x_1, x_2)\right] \\ &= \Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2) G_Z(q^2),\end{aligned} \tag{3.37}$$

where

$$\Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2) = i g_{\rho\pi\pi}\mu^{(2-\frac{d}{2})}(p_1+p_2)^\mu, \tag{3.38}$$

is the tree level vertex in $d$ dimensions, and

$$\begin{aligned}G_Z(q^2) = g^2_{\rho\pi\pi}\left(\mu^2\right)^{(2-\frac{d}{2})} \frac{2}{i} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \bigg\{ &\left[(1-2x_1) + \frac{4}{d}(1-x_1)\right] I_{1,3}(\Delta) \\ &+ (1-2x_1)\Big[m_\pi^2(x_1+x_2-2)^2 - q^2(x_1x_2 - x_1 - x_2 + 2)\Big] I_{0,3}(\Delta) \bigg\}.\end{aligned} \tag{3.39}$$

In 4 dimensions $I_{0,3}(\Delta)$ is convergent but $I_{1,3}(\Delta)$ is logarithmically divergent. The divergence can isolated by means of dimensional regularisation and then removed via renormalisation.

### 3.4.1 Dimensional Regularisation

Dimensional regularisation, first introduced by 't Hooft and Veltman [10], involves calculating loop integrals as an analytic function of the number of dimensions $d$. The limit $d \to 4$ yields the desired integral in four dimensions.

#### Evaluating the Momentum Integrals $I_s$

Recall from Eq. 3.29 the definition (in Minkowski space)

$$I_{s,n}(\Delta) \equiv \int \frac{d^d l}{(2\pi)^d} \frac{l^{2s}}{(l^2 - \Delta + i\varepsilon)^n}. \tag{3.40}$$

To evaluate this integral, consider the associated Euclidean integral

$$I^E_{s,n}(\Delta) = \int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^{2s}}{(l_E^2 + \Delta)^n}. \tag{3.41}$$

From the definition of Euler Gamma function

$$\int_0^\infty dt\, t^{n-1} e^{-ta} = \frac{\Gamma(n)}{a^n}, \tag{3.42}$$

it follows that

$$\frac{1}{({l_E}^2 + \Delta)^n} = \frac{1}{\Gamma(n)} \int_0^\infty dt\, t^{n-1} e^{-t\left({l_E}^2 + \Delta\right)}, \tag{3.43}$$

so the Euclidean integral becomes

$$\begin{aligned} I^E_{s,n}(\Delta) &= \frac{1}{\Gamma(n)} \int_0^\infty dt\, t^{n-1} \int \frac{d^d l_E}{(2\pi)^d} {l_E}^{2s} e^{-t\left({l_E}^2 + \Delta\right)} \\ &= \frac{1}{(n-1)!} \int_0^\infty dt\, t^{n-1} e^{-t\Delta} \int \frac{d^d l_E}{(2\pi)^d} {l_E}^{2s} e^{-t{l_E}^2} \\ &= \frac{(-1)^s}{(n-1)!} \int_0^\infty dt\, t^{n-1} e^{-t\Delta} \frac{d^s}{dt^s} \left( \int \frac{d^d l_E}{(2\pi)^d} e^{-t{l_E}^2} \right). \end{aligned} \tag{3.44}$$

The Gaussian integral is easily evaluated

$$\int \frac{d^d l_E}{(2\pi)^d} e^{-t{l_E}^2} = \frac{1}{(2\pi)^d} \left( \int_{-\infty}^\infty dx e^{-tx^2} \right)^d = (4\pi t)^{-\frac{d}{2}}, \tag{3.45}$$

yielding the derivative

$$\frac{d^s}{dt^s}\left(\int \frac{d^d l_E}{(2\pi)^d} e^{-t {l_E}^2}\right) = (4\pi)^{-\frac{d}{2}} \frac{d^s}{dt^s}\left(t^{-\frac{d}{2}}\right) = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^s}{2^s} \prod_{j=0}^{s-1}(d+2j)\, t^{-(s+\frac{d}{2})}, \tag{3.46}$$

so that Eq. 3.44 becomes

$$I^E_{s,n}(\Delta) = \frac{1}{(n-1)!\,(4\pi)^{\frac{d}{2}}} \left[\frac{1}{2^s}\prod_{j=0}^{s-1}(d+2j)\right] \int_0^\infty dt\, t^{(n-1-s-\frac{d}{2})} e^{-t\Delta}. \tag{3.47}$$

For notational convenience, it has been assumed that $s > 0$ in the previous expressions. In the case $s = 0$ the term in square brackets is to be replaced by unity in Eq. 3.47 and all subsequent results. The last integral can be evaluated using the simple parametrisation $z = \Delta t$

$$\begin{aligned} \int_0^\infty dt\, t^{(n-1-s-\frac{d}{2})} e^{-t\Delta} &= \Delta^{(s+\frac{d}{2}-n)} \int_0^\infty dz\, z^{(n-1-s-\frac{d}{2})} e^{-z} \\ &= \Delta^{(s+\frac{d}{2}-n)} \Gamma(n-s-\tfrac{d}{2}), \end{aligned} \tag{3.48}$$

yielding the value of the Euclidean integral

$$I^E_{s,n}(\Delta) = \frac{1}{(n-1)!\,(4\pi)^{\frac{d}{2}}} \left[\frac{1}{2^s}\prod_{j=0}^{s-1}(d+2j)\right] \Delta^{(s+\frac{d}{2}-n)} \Gamma\left(n-s-\tfrac{d}{2}\right). \tag{3.49}$$

In order to obtain the corresponding result in a Minkowski space, it is useful to note that Minkowski space is equivalent to a Euclidean space where the time coordinate is allowed to assume only imaginary values. Performing a Wick rotation ($l^0 \rightarrow i{l_E}^0$)

$$\int d^d l \equiv \int d^{d-1} l \int dl^0 = i \int d^{d-1} l_E \int d{l_E}^0 = i \int d^d l_E, \tag{3.50}$$

and

$$\begin{aligned} l^2 = \left(l^0\right)^2 - \vec{l}^{\,2} &= -\left({l_E}^0\right)^2 - \vec{l_E}^{\,2} \equiv -{l_E}^2 \\ \Rightarrow l^{2s} &= (-1)^s {l_E}^{2s}. \end{aligned} \tag{3.51}$$

Note that the $i\varepsilon$ term in the denominator is included by convention to shift the pole of the propagators into the complex plane. Since the primary area of interest for the vertex is below threshold, the $i\varepsilon$ term can be neglected. Thus the denominator of the integrand becomes

$$\left(l^2 - \Delta + i\varepsilon\right)^3 \to \left(-{l_E}^2 - \Delta\right)^3 = -\left({l_E}^2 + \Delta\right)^3, \tag{3.52}$$

combining these results gives

$$I_{s,n}(\Delta) = i(-1)^{s-1} I^E_{s,n}(\Delta), \tag{3.53}$$

and utilising the result in Eq. 3.49 it follows that

$$\boxed{I_{s,n}(\Delta) = \frac{(-1)^{(s-1)}}{(n-1)!}\frac{i}{(4\pi)^{\frac{d}{2}}}\left[\frac{1}{2^s}\prod_{j=0}^{s-1}(d+2j)\right]\Delta^{(s+\frac{d}{2}-n)}\Gamma\left(n-s-\tfrac{d}{2}\right).} \tag{3.54}$$

It is now clear that with $d = 4$ the $I_{1,3}(\Delta)$ divergence corresponds to the poles of the Euler Gamma function at $\Gamma(0)$.

### Isolating the Divergences as $d \to 4$

Before isolating the divergences in the $I_{s,n}$ integrals, recall from Eq. 3.39 that the expressions of interest are of the form

$$\frac{(\mu^2)^{(2-\frac{d}{2})}}{i} I_{s,n}. \tag{3.55}$$

In order to isolate the divergences in these integrals when $d = 4$, introduce

$$\varepsilon = 4 - d \Rightarrow \frac{d}{2} = 2 - \frac{\varepsilon}{2}, \tag{3.56}$$

(where $\varepsilon$ is not to be confused with the displacement of the poles in the propagators). Note that $\varepsilon$ is not restricted to integer values, and the integrals Eq. 3.40 are defined by the result Eq. 3.54 for non-integer values of $\varepsilon$.

The $\frac{d}{2}$ powers in the coefficients of the Gamma function are reduced by noting that

$$(\mu^2)^{(2-\frac{d}{2})}\frac{\Delta^{(s+\frac{d}{2}-n)}}{(4\pi)^{\frac{d}{2}}} = \frac{\Delta^{(s+2-n)}}{(4\pi)^2}\left(\frac{\Delta}{4\pi\mu^2}\right)^{-\frac{\varepsilon}{2}}, \tag{3.57}$$

and expanding using $a^x = e^{x\ln a} = 1 + x\ln a + \mathcal{O}(x^2)$ to give

$$(\mu^2)^{(2-\frac{d}{2})}\frac{\Delta^{(s+\frac{d}{2}-n)}}{(4\pi)^{\frac{d}{2}}} = \frac{\Delta^{(s+2-n)}}{(4\pi)^2}\left[1 - \frac{\varepsilon}{2}\ln\left(\frac{\Delta}{4\pi\mu^2}\right)\right] + \mathcal{O}(\varepsilon^2). \tag{3.58}$$

In four dimensions, the poles in the Gamma functions appear only with $s > 0$ so it is possible to expand the product factor in terms of $\varepsilon$ for $s > 0$

$$\begin{aligned}
\left[\frac{1}{2^s}\prod_{j=0}^{s-1}(d+2j)\right] &= \left(\frac{1}{2^s}\prod_{j=0}^{s-1}(4-\varepsilon+2j)\right) = \prod_{j=0}^{s-1}\left(2-\frac{\varepsilon}{2}+j\right) \\
&= \left[2-\frac{\varepsilon}{2}\right]\left[3-\frac{\varepsilon}{2}\right]\cdots\left[(s+1)-\frac{\varepsilon}{2}\right] \\
&= (s+1)! - \frac{\varepsilon}{2}\left[\frac{(s+1)!}{2}+\frac{(s+1)!}{3}+\cdots+\frac{(s+1)!}{(s+1)}\right] + \mathcal{O}(\varepsilon^2) \\
&= (s+1)!\left[1-\frac{\varepsilon}{2}\sum_{j=1}^{s}\frac{1}{(1+j)}\right] + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{3.59}$$

Combining the results above gives

$$\begin{aligned}
(\mu^2)^{\left(2-\frac{d}{2}\right)} I_{s,n}(\Delta) = i(-1)^{s-1}\frac{(s+1)!}{(n-1)!}\frac{\Delta^{(s+2-n)}}{(4\pi)^2}&\left[1-\frac{\varepsilon}{2}\sum_{j=1}^{s}\frac{1}{(1+j)}\right] \\
\times\left[1-\frac{\varepsilon}{2}\ln\left(\frac{\Delta}{4\pi\mu^2}\right)\right]&\Gamma\left(n-2-s+\tfrac{\varepsilon}{2}\right)+\mathcal{O}(\varepsilon^2).
\end{aligned} \tag{3.60}$$

For the special case $s = 0$ the entire expression in Eq. 3.59 should be replaced by unity, as before. However, with $s = 0$, $(s+1)! = 1$, so it is sufficient to simply define

$$\left[\frac{\varepsilon}{2}\sum_{j=1}^{s}\frac{1}{(1+j)}\right] = 0 \qquad \text{for } s = 0. \tag{3.61}$$

Finally, the Gamma function is expanded[1] around the pole at $\Gamma(0)$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon), \tag{3.62}$$

where $\gamma \approx 0.5772157$ is the Euler-Mascheroni constant. Thus

$$\boxed{\begin{aligned}
\frac{(\mu^2)^{\left(2-\frac{d}{2}\right)} I_{0,3}(\Delta)}{i} &= -\frac{1}{2(4\pi)^2\Delta} \\
\frac{(\mu^2)^{\left(2-\frac{d}{2}\right)} I_{1,3}(\Delta)}{i} &= \frac{1}{(4\pi)^2}\left[\frac{2}{\varepsilon}-\ln\left(\frac{\Delta}{\mu^2}\right)-\frac{1}{2}-\gamma+\ln(4\pi)+\mathcal{O}(\varepsilon)\right].
\end{aligned}} \tag{3.63}$$

One final factor appearing in the integrand of Eq. 3.39 remains to be expressed in

---

[1] $\Gamma(x)$ has poles at negative real integer values of $x$, and the expansion about these poles is given by $\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon}+\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)-\gamma+\mathcal{O}(\varepsilon)\right]$

terms of $\varepsilon$:

$$\frac{4}{d} = \frac{4}{4-\varepsilon} = \frac{1}{1-\varepsilon/4} = 1 + \frac{\varepsilon}{4} + \mathcal{O}(\varepsilon^2). \tag{3.64}$$

Utilising the results above, $G_Z(q^2)$ becomes

$$G_Z(q^2) = -2\frac{g^2_{\rho\pi\pi}}{(4\pi)^2}\left(\mu^2\right)^{(2-\frac{d}{2})}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \Bigg\{(3x_1-2)\left[\frac{2}{\varepsilon} - \ln\left(\frac{\Delta}{\mu^2}\right) - \frac{1}{2}\right.$$
$$\left. - \gamma + \ln(4\pi)\right] + \frac{(1-2x_1)}{2\,\Delta}\Big[m_\pi^2(x_1+x_2-2)^2 - q^2(x_1x_2 - x_1 - x_2 + 2)\Big] + \mathcal{O}(\varepsilon)\Bigg\}. \tag{3.65}$$

Separating the terms involving divergences and constants from the rest of the expression gives

$$G_Z(q^2) = \widetilde{G}_Z(q^2) + A\left[\frac{2}{\varepsilon} - \frac{1}{2} - \gamma + \ln(4\pi)\right] + \mathcal{O}(\varepsilon), \tag{3.66}$$

where $\widetilde{G}_Z(q^2)$ is a $\frac{1}{\varepsilon}$ divergence free function of $q^2$

$$\widetilde{G}_Z(q^2) = -2\frac{g^2_{\rho\pi\pi}}{(4\pi)^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \Bigg\{(2-3x_1)\ln\left(\frac{\Delta}{\mu^2}\right)$$
$$+ \left(\frac{1-2x_1}{2\Delta}\right)\Big[m_\pi^2(x_1+x_2-2)^2 - q^2(x_1x_2 - x_1 - x_2 + 2)\Big]\Bigg\}. \tag{3.67}$$

The factor $A$ in Eq. 3.66 is an integral over $x_1$ and $x_2$, but does not depend on $q^2$. As a result, it is a constant that will be cancelled during renormalisation and there is no need to calculate it explicitly.

## 3.4.2 Renormalisation

### Separating the Lagrangian

The Lagrangian for the system, in terms of bare quantities, is given by

$$\mathcal{L}_0 = \partial_\mu\phi_0\partial^\mu\phi_0^* - m^2_{0\pi}\phi_0^*\phi_0 - \tfrac{1}{4}\rho^0_{\mu\nu}\rho_0^{\mu\nu} + \tfrac{1}{2}m^2_{0\rho}\rho^0_\mu\rho_0^\mu + ig_{0\rho\pi\pi}\rho_0^\mu\phi_0^*\overleftrightarrow{\partial_\mu}\phi_0 + g^2_{0\rho\pi\pi}\rho^0_\mu\rho_0^\mu\phi_0^*\phi_0. \tag{3.68}$$

Rescaling in order to eliminate the shift in field strength gives

$$\phi_0 = Z_\phi^{\frac{1}{2}}\phi \qquad\qquad \rho^0_\mu = Z_\rho^{\frac{1}{2}}\rho_\mu, \tag{3.69}$$

where $Z_\phi$ and $Z_\rho$ are the renormalisation constants associated with each field. Inserting these into the Lagrangian yields

$$\mathcal{L}_0 = Z_\phi \partial_\mu \phi \partial^\mu \phi^* - Z_\phi m_{0\pi}^2 \phi^* \phi - Z_\rho \tfrac{1}{4} \rho_{\mu\nu} \rho^{\mu\nu} + Z_\rho \tfrac{1}{2} m_{0\rho}^2 \rho_\mu \rho^\mu \\ + i Z_\phi Z_\rho^{\frac{1}{2}} g_{0\rho\pi\pi} \rho^\mu \phi^* \overleftrightarrow{\partial_\mu} \phi + Z_\phi Z_\rho g_{0\rho\pi\pi}^2 \rho_\mu^0 \rho_0^\mu \phi_0^* \phi_0. \quad (3.70)$$

Defining

$$\begin{aligned} \delta Z_\phi &= Z_\phi - 1 & \delta Z_\rho &= Z_\rho - 1 \\ \delta m_\pi^2 &= m_{0\pi}^2 Z_\phi - m_\pi^2 & \delta m_\rho^2 &= m_{0\rho}^2 Z_\rho - m_\rho^2 \\ g_{\rho\pi\pi} Z_g &= g_{0\rho\pi\pi} Z_\phi Z_\rho^{\frac{1}{2}} & \delta Z_g &= Z_g - 1 \\ g_{\rho\pi\pi}^2 Z_4 &= g_{0\rho\pi\pi}^2 Z_\phi Z_\rho & \delta Z_4 &= \tfrac{Z_g^2}{Z_\phi} - 1, \end{aligned} \quad (3.71)$$

where $m_\pi$, $m_\rho$, $g_{\rho\pi\pi}$ and $g_{\rho\pi\pi}^2$ are the physically measured mass of $\pi^\pm$, mass of the $\rho^0$ and couplings, the full Lagrangian now separates into terms involving only physically fields and a set of counter terms $\mathcal{L}_0 = \mathcal{L} + \Delta\mathcal{L}$

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m_\pi^2 \phi^* \phi - \tfrac{1}{4} \rho_{\mu\nu} \rho^{\mu\nu} + \tfrac{1}{2} m_\rho^2 \rho_\mu \rho^\mu + i g_{\rho\pi\pi} \rho^\mu \phi^* \overleftrightarrow{\partial_\mu} \phi + g_{\rho\pi\pi}^2 \rho_\mu \rho^\mu \phi^* \phi \quad (3.72)$$

$$\Delta\mathcal{L} = \delta Z_\phi \partial_\mu \phi \partial^\mu \phi^* - \delta m_\pi^2 \phi^* \phi - \tfrac{1}{4} \delta Z_\rho \rho_{\mu\nu} \rho^{\mu\nu} \\ + \tfrac{1}{2} \delta m_\rho^2 \rho_\mu \rho^\mu + i \delta Z_g g_{\rho\pi\pi} \rho^\mu \phi^* \overleftrightarrow{\partial_\mu} \phi + g_{\rho\pi\pi}^2 \delta Z_4 \rho_\mu \rho^\mu \phi^* \phi \quad (3.73)$$

The counter terms in $\Delta\mathcal{L}$ correspond to the diagrams shown in figure Eq. 3.5

### Renormalisation conditions

The six $\delta$ coefficients in the counter terms, combined with the fact that $Z_g^2 = Z_4 Z_\phi$, necessitate the definition of five renormalisation conditions. The traditional choice is to renormalise on the mass shell, using the conditions

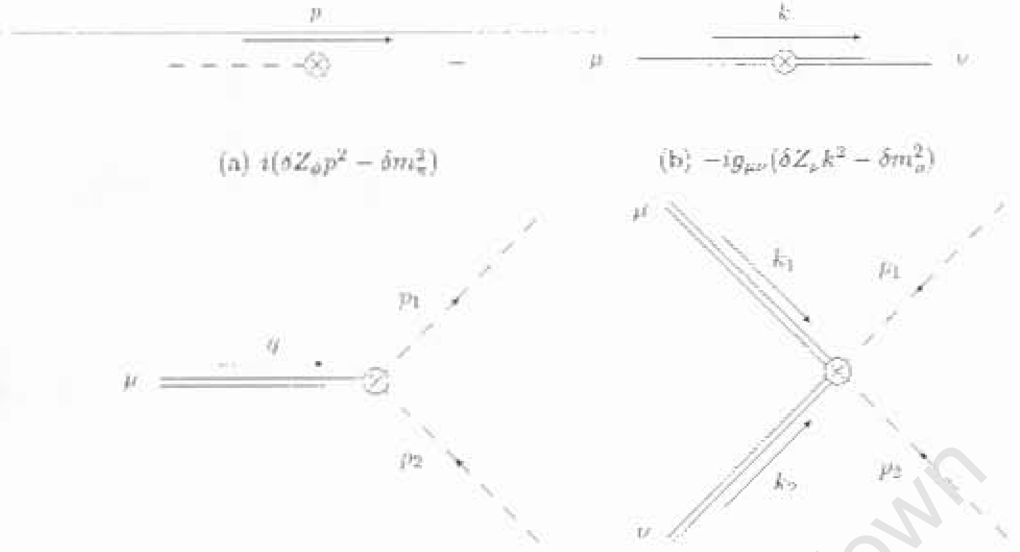
$$\Pi_0(p^2 = m_\pi^2) = 0 \quad (3.74)$$

$$\frac{d}{d(p^2)} \Pi_0(p^2)\Big|_{p^2 = m_\pi^2} = 0 \quad (3.75)$$

$$\Pi(p^2 = m_\rho^2) = 0 \quad (3.76)$$

$$\frac{d}{d(p^2)} \Pi(p^2)\Big|_{p^2 = m_\rho^2} = 0 \quad (3.77)$$

$$\Gamma_{\rho\pi\pi}^{(1)}(q^2 = m_\rho^2) = \Gamma_{\rho\pi\pi}^{(0)}. \quad (3.78)$$



(a) $i(\delta Z_\phi p^2 - \delta m_\pi^2)$ (b) $-ig_{\mu\nu}(\delta Z_\rho k^2 - \delta m_\rho^2)$

(c) $ig_{\rho\pi\pi}\delta Z_g(p_1 + p_2)^\mu$ (d) $2ig_{\rho\pi\pi}^2 \delta Z_1 g_{\mu\nu}$

Figure 3.5: Lagrangian Counter Terms

The vanishing self energy conditions Eq. 3.74 and Eq. 3.75 define the position and residue of the pole in pion propagator, precisely defining the mass of the $\pi^\pm$ mesons. Similarly, Eq. 3.76 and Eq. 3.77 precisely define the mass of the $\rho^0$ meson. Explicitly calculating $\delta m_\pi^2$, $\delta m_\rho^2$, $\delta Z_\phi$ and $\delta Z_\rho$ from these conditions is not necessary in order to investigate the pion form factor. The final condition Eq. 3.78 ensures that $g_{\rho\pi\pi}$ is the physical coupling measured on the mass shell ($g_{\rho\pi\pi} = 5.99 \pm 0.03$).

The full renormalised vertex function, in comparison with Eq. 3.37, becomes

$$\begin{aligned}\Gamma^{(1)}_{\rho\pi\pi}(p_1, p_2, q^2) &= \Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2) + \widetilde{\Gamma}^{(1)}_{\rho\pi\pi}(p_1, p_2) \\ &= \Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2)\left[1 + G_Z(q^2) + \delta Z_g\right],\end{aligned} \tag{3.79}$$

which, by Eq. 3.66 is

$$\Gamma^{(1)}_{\rho\pi\pi}(p_1, p_2, q^2) = \Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2)\left(1 + \widetilde{G}_Z(q^2) - A\left[\frac{2}{\varepsilon} - \frac{1}{2} - \gamma + \ln(4\pi)\right] + \delta Z_g\right). \tag{3.80}$$

The renormalisation condition Eq. 3.78 implies

$$\delta Z_g = -\widetilde{G}_Z(m_\rho^2) - A\left[\frac{2}{\varepsilon} - \frac{1}{2} - \gamma + \ln(4\pi)\right], \tag{3.81}$$

yielding the renormalised vertex function

$$\Gamma^{(1)}_{\rho\pi\pi}(p_1, p_2, q^2) = \Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2)\left[1 + \widetilde{G}_Z(q^2) - \widetilde{G}_Z(m_\rho^2)\right]. \tag{3.82}$$

While renormalising on the mass shell is the traditional choice to allow identification of $g_{\rho\pi\pi}$ with the measured coupling, this is unsuitable to analyse $F_\pi$ in the scattering ($q^2 < 0$) region.

Examination of Eq. 3.67 indicates that the integrand, and by extension $\widetilde{G}_Z(q^2)$ has a pole when $\Delta(q^2) = 0$. It can be shown that $\Delta$ is non-zero in the range $q \in (-\infty, 4m_\pi^2)$, but for larger values of $q^2$ (notice that $q^2 = 4m_\pi^2$ is the threshold value for production of two pions) the integrand has poles within the integration region and $\widetilde{G}_Z(q^2)$ becomes complex. On the mass shell, $q^2 = m_\rho^2$, $\widetilde{G}_Z(q^2)$ becomes purely imaginary, as one would intuitively expect. As a result, the $\widetilde{G}_Z(m_\rho^2)$ term in Eq. 3.82 cannot be evaluated easily.

This problem can be circumvented by choosing a different renormalisation condition to replace Eq. 3.78. As an alternative it can be required that the theory must reproduce the unitary normalisation of the form factor,

$$F_\pi(0) = 1, \tag{3.83}$$

in which case the vertex function is given by

$$\Gamma^{(1)}_{\rho\pi\pi}(p_1, p_2, q^2) = \Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2)\left[1 + \widetilde{G}_Z(q^2) - \widetilde{G}_Z(0)\right] \tag{3.84}$$

The subtraction of $\widetilde{G}_Z(0)$ serves to cancel the $\mu^2$ term in Eq. 3.67, and gives

$$\Delta G_Z(q^2) = \widetilde{G}_Z(q^2) - \widetilde{G}_Z(0) = -2\frac{f_\rho^2}{(4\pi)^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left\{(2-3x_1)\ln\left(\frac{\Delta(q^2)}{\Delta(0)}\right)\right.$$
$$\left. + \frac{(1-2x_1)}{2}\left[m_\pi^2(x_1+x_2-2)^2\left(\frac{1}{\Delta(q^2)} - \frac{1}{\Delta(0)}\right) - \frac{q^2}{\Delta(q^2)}(x_1x_2 - x_1 - x_2 + 2)\right]\right\}. \tag{3.85}$$

In the scattering region ($q^2 < 0$), none of the $\Delta$ terms have a zero within the integration region, and $\Delta G(q^2)$ is well defined. Evaluating the integral in $\Delta G(q^2)$ analytically is not possible, but it can be computed numerically without great difficulty.

## 3.5 Form Factor

As previously discussed, the $\rho^0$ self energy modifies the propagator of the $\rho^0$, resulting in a modified expression for the form factor

$$F_\pi(q^2) = \frac{m_\rho^2 + \Pi(0)}{m_\rho^2 - q^2 + \Pi(q^2)}. \tag{3.86}$$

The one loop vertex introduces a correction to the tree level which has the form

$$F_\pi(q^2) = \frac{m_\rho^2}{m_\rho^2 - q^2}\left[1 + \widetilde{G}_Z(q^2) - \widetilde{G}_Z(0)\right]. \tag{3.87}$$

Combining these, and discarding the $\mathcal{O}(g^4_{\rho\pi\pi})$ term which is a two loop correction, we obtain the next-to-leading order form factor

$$F_\pi(q^2) = \frac{m_\rho^2 + \Pi(0)}{m_\rho^2 - q^2 + \Pi(q^2)} + \frac{m_\rho^2}{m_\rho^2 - q^2}\left[\widetilde{G}_Z(q^2) - \widetilde{G}_Z(0)\right]. \tag{3.88}$$

### 3.5.1 Space-like Region

In the space-like region, Eq. 3.88 is in excellent agreement with the available experimental data, as shown in Figs. 3.6 and 3.7 below.

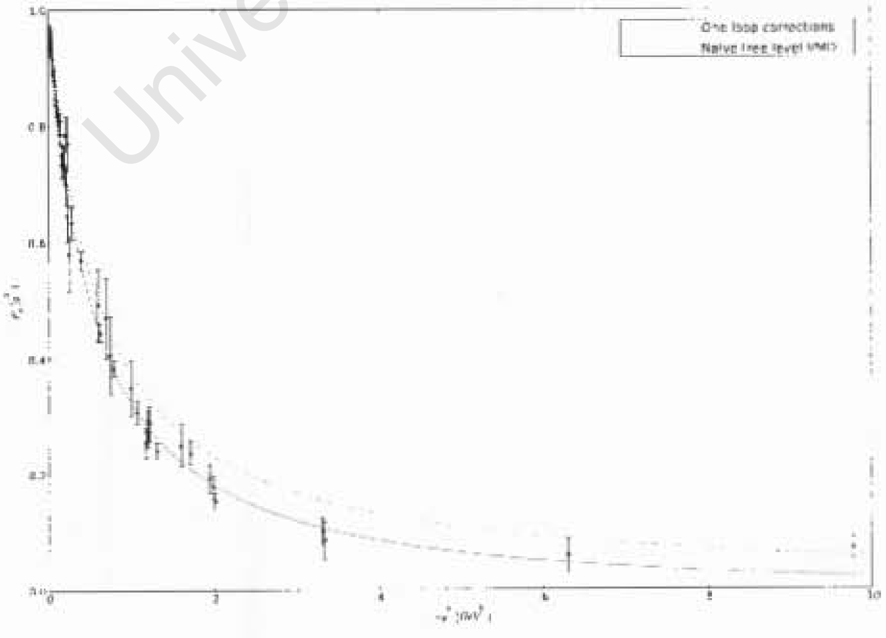


Figure 3.6: Space-like Form Factor at Zero Temperature

Tree level VMD suffers from a poor chi-square per degree of freedom $\chi^2_F = 5.0$, largely due to the low $q^2$ region (depicted in Fig. 3.7). The next-to-leading order correction improves hugely over this result, with $\chi^2_F = 1.1$.

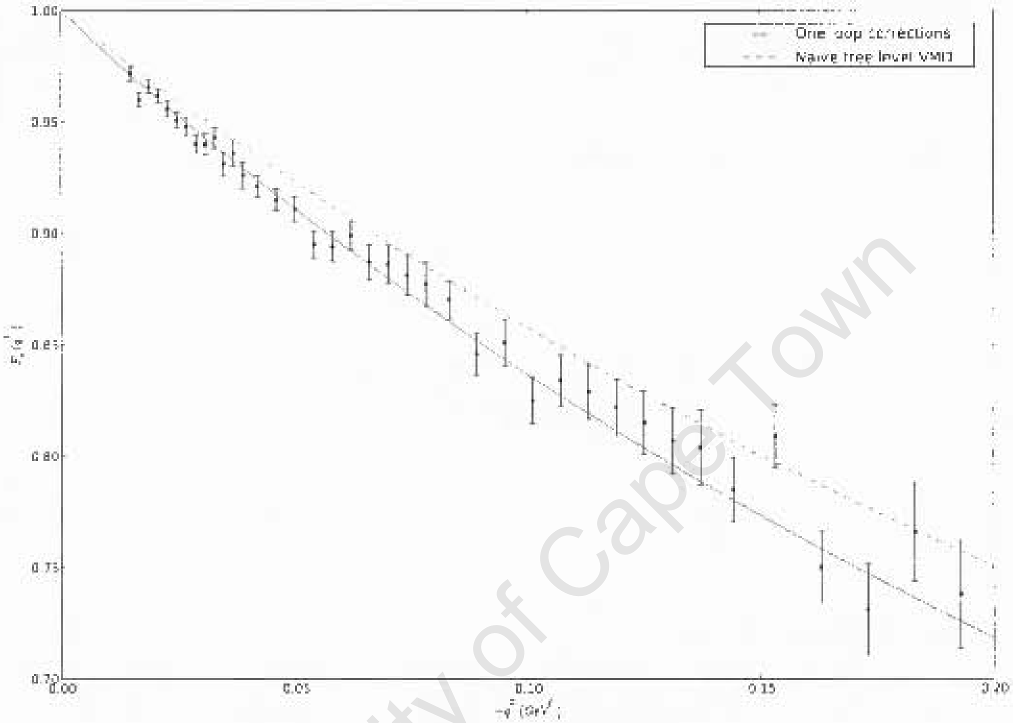


Figure 3.7: Space-like Form Factor at low $q^2$ at Zero Temperature

### 3.5.2 Time-like Region

In the time-like region the vacuum polarisation dominates, and the vertex correction contribution is small. The form factor does not fit the data as well as it does in the space-like region, but still provides a reasonable approximation, as shown in Fig. 3.8.

From Eq. 3.14 it is clear that in the time-like region

$$\mathrm{Im}\,\Pi(q^2) = -\frac{g^2_{\rho\pi\pi}}{48\pi}\frac{(q^2 - 4m_\pi^2)^{\frac{3}{2}}}{\sqrt{q^2}} \tag{3.89}$$

so at the $\rho$ peak it follows that the hadronic width, defined [12] as

$$\Gamma_\rho \equiv \Gamma_\rho(m_\rho^2) = -\frac{1}{m_\rho}\mathrm{Im}\Pi(m_\rho^2), \tag{3.90}$$

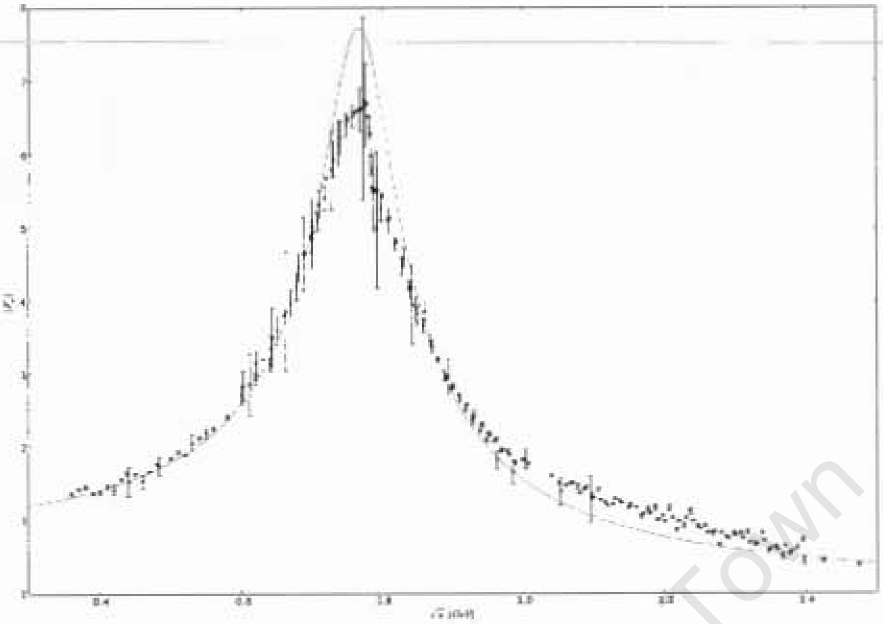


Figure 3.8: Time-like Form Factor at Zero Temperature

is given by

$$\Gamma_\rho = \frac{g^2_{\rho\pi\pi}}{48\pi} \frac{(m_\rho^2 - 4m_\pi^2)^{\frac{3}{2}}}{m_\rho^2} , \tag{3.91}$$

which is the standard kinematical relation between the width and coupling of a vector and two pseudo-scalar particles. Note how this result follows automatically in the KLZ model, it has not been imposed as a constraint.

Computing the $s$-dependent width from Eqs. 3.14 and 3.91 produces the momentum dependent Gounaris-Sakurai width

$$\Gamma_\rho(s) = \frac{m_\rho \Gamma_\rho}{\sqrt{s}} \left[ \frac{s - 4m_\pi^2}{m_\rho^2 - 4m_\pi^2} \right]^{\frac{3}{2}} , \tag{3.92}$$

though it should be pointed out that the Gounaris-Sakurai formula is only reproduced exactly at the $\rho$ pole.

## 3.6 Electromagnetic Radius

The quadratic radius associated with a particular current is defined by the coefficient of the $q^2$ term in the Taylor expansion in the relevant form factor around $q^2 = 0$

$$F(q^2) = F(0) \left[ 1 + \tfrac{1}{6} \langle r^2 \rangle q^2 + \mathcal{O}(q^4) \right] . \tag{3.93}$$

Vector dominance deals with the electromagnetic current $J_{EM}$, so the electromagnetic quadratic radius of the pion is defined as

$$\langle r_\pi^2 \rangle = 6 \left[ \frac{d}{d(q^2)} \frac{F_\pi(q^2)}{F_\pi(0)} \right]_{q^2=0} , \tag{3.94}$$

which, at the one loop level in the KLZ model, reduces to

$$\langle r_\pi^2 \rangle = 6 \left[ \frac{1 - \frac{d\Pi_{vac}(q^2)}{d(q^2)}}{m_\rho^2 - \Pi(0)} - \frac{d\widetilde{G}_Z(q^2)}{d(q^2)} \right]_{q^2=0} . \tag{3.95}$$

Differentiating the vertex function yields

$$\frac{d\widetilde{G}_Z(q^2)}{d(q^2)} = -\frac{2g_{\rho\pi\pi}^2}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{\Delta(q^2)} \Bigg\{ (2-3x_1) \frac{d}{d(q^2)} \Delta(q^2)$$
$$- \frac{(1-2x_1)}{2} \left[ m_\pi^2 (x_1+x_2-2)^2 \frac{\frac{d}{d(q^2)}\Delta(q^2)}{\Delta(q^2)} + (x_1 x_2 - x_1 - x_2 + 2) \left( 1 - q^2 \frac{\frac{d}{d(q^2)}\Delta(q^2)}{\Delta(q^2)} \right) \right] \Bigg\}, \tag{3.96}$$

which simplifies to

$$\frac{d\widetilde{G}_Z(q^2)}{d(q^2)} = \frac{2g_{\rho\pi\pi}^2}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{\Delta(q^2)} \Bigg\{ \frac{(1-2x_1)(x_1 x_2 - x_1 - x_2 + 2)}{2}$$
$$+ \frac{x_1 x_2}{2} \left[ 4 - 6x_1 - \frac{(1-2x_1)}{\Delta(q^2)} \left[ m_\pi^2 (x_1+x_2-2)^2 - q^2 (x_1 x_2 - x_1 - x_2 + 2) \right] \right] \Bigg\}, \tag{3.97}$$

and after evaluating at zero gives

$$\frac{d\widetilde{G}_Z(q^2)}{d(q^2)} = \frac{2g_{\rho\pi\pi}^2}{(4\pi)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{\Delta(0)} \Bigg\{ \frac{(1-2x_1)(x_1 x_2 - x_1 - x_2 + 2)}{2}$$
$$+ \frac{x_1 x_2}{2} \left[ 4 - 6x_1 - \frac{m_\pi^2 (1-2x_1)(x_1+x_2-2)^2}{\Delta(0)} \right] \Bigg\}. \tag{3.98}$$

Differentiating the rho self energy $\Pi(q^2)$ below the pion production threshold gives

$$
\begin{aligned}
\frac{d\Pi_{vac}(q^2)}{d(q^2)} = -\frac{g^2_{\rho\pi\pi}}{3(4\pi)^2}\Bigg\{ & \frac{8m_\pi^2}{m_\rho^2} + \frac{(m_\rho^2-4m_\pi^2)^{\frac{3}{2}}}{m_\rho^3}\ln\left(\frac{m_\rho+\sqrt{m_\rho^2-4m_\pi^2}}{m_\rho-\sqrt{m_\rho^2-4m_\pi^2}}\right) \\
& + \left(1-\frac{4m_\pi^2}{q^2}\right)^{\frac{3}{2}}\ln\left|\frac{\sqrt{1-\frac{4m_\pi^2}{q^2}}+1}{\sqrt{1-\frac{4m_\pi^2}{q^2}}-1}\right| + \frac{8m_\pi^2\left(2m_\pi^2-m_\rho^2+m_\rho\sqrt{m_\rho^2-4m_\pi^2}\right)}{q^4m_\rho^2\left(\sqrt{1-\frac{4m_\pi^2}{m_\rho^2}}-1\right)^2\left(\sqrt{1-\frac{4m_\pi^2}{q^2}}-1\right)^2}\Bigg\{ \\
& 3\left[q^2\left(1-\sqrt{1-\frac{4m_\pi^2}{q^2}}\right)+2m_\pi^2\left(\sqrt{1-\frac{4m_\pi^2}{q^2}}-2\right)\right]\ln\left|\frac{\sqrt{1-\frac{4m_\pi^2}{q^2}}+1}{\sqrt{1-\frac{4m_\pi^2}{q^2}}-1}\right| \\
& + (q^2-4m_\pi^2)\left(\frac{\sqrt{1-\frac{4m_\pi^2}{q^2}}-1}{\sqrt{1-\frac{4m_\pi^2}{q^2}}+1}\right)\operatorname{sgn}\left(\frac{\sqrt{1-\frac{4m_\pi^2}{q^2}}+1}{\sqrt{1-\frac{4m_\pi^2}{q^2}}-1}\right)\Bigg\}\Bigg\}. \qquad (3.99)
\end{aligned}
$$

Since this expression cannot be explicitly evaluated at $q^2 = 0$ it is necessary to evaluate the limit $q^2 \to 0$. Performing an expansion around zero gives

$$
\begin{aligned}
\left.\frac{d\Pi_{vac}}{d(q^2)}\right|_{q^2=0} = \frac{g^2_{\rho\pi\pi}}{3(4\pi)^2}\Bigg\{ & \frac{8}{3} - \frac{8m_\pi^2}{m_\rho^2} + \frac{(m_\rho^2-4m_\pi^2)^{\frac{3}{2}}}{m_\rho^3}\ln\left(\frac{m_\rho+\sqrt{m_\rho^2-4m_\pi^2}}{m_\rho-\sqrt{m_\rho^2-4m_\pi^2}}\right) \\
& + q^2\left[\frac{2m_\pi^2-m_\rho^2+m_\rho\sqrt{m_\rho^2-4m_\pi^2}}{5m_\rho^2m_\pi^4\left(\sqrt{1-\frac{4m_\pi^2}{m_\rho^2}}\right)^2} - \frac{1}{10m_\pi^2}\right] + \mathcal{O}\left(q^4\right)\Bigg\}, 
\end{aligned} \qquad (3.100)
$$

simplifying to

$$
\left.\frac{d\Pi_{vac}}{d(q^2)}\right|_{q^2=0} = \frac{g^2_{\rho\pi\pi}}{3(4\pi)^2}\left\{\frac{8}{3} - \frac{8m_\pi^2}{m_\rho^2} + \frac{(m_\rho^2-4m_\pi^2)^{\frac{3}{2}}}{m_\rho^3}\ln\left(\frac{m_\rho+\sqrt{m_\rho^2-4m_\pi^2}}{m_\rho-\sqrt{m_\rho^2-4m_\pi^2}}\right)\right\}. \qquad (3.101)
$$

Evaluating these expressions numerically gives

$$
\langle r_\pi^2\rangle_{KLZ} = 0.456 \text{ fm}^2, \qquad (3.102)
$$

which overshoots the experimental value of $\langle r_\pi^2\rangle_{exp} = 0.439 \pm 0.008$ fm$^2$, but is 65% closer than the basic tree level VMD result $\langle r_\pi^2\rangle_{tree} = 0.39$ fm$^2$.

# FINITE TEMPERATURE

## 4.1 Matsubara Formalism

Computing perturbative expansions for particle interactions at a finite temperature $T = \frac{1}{\beta}$ has additional complexities not present at zero temperature. In particular, the time-like component of the momenta are quantized in order to make bosonic fields periodic and fermion fields anti-periodic in time, as required by the definition of the partition function.

The allowable values, called Matsubara frequencies, are $i\omega_n^B = \frac{2\pi i n}{\beta}$ for bosons, and $i\omega_n^F = \frac{(2n+1)\pi i}{\beta}$ for fermions (with $n \in \mathbb{Z}$). So, to move from the zero temperature results above to finite temperature the momenta must be transformed as $p \rightarrow (i\omega_n, \mathbf{p}) = \left(\frac{2\pi i n}{\beta}, \mathbf{p}\right)$, and integrals over the time-like components must be replaced by sums over the Matsubara frequencies, i.e.

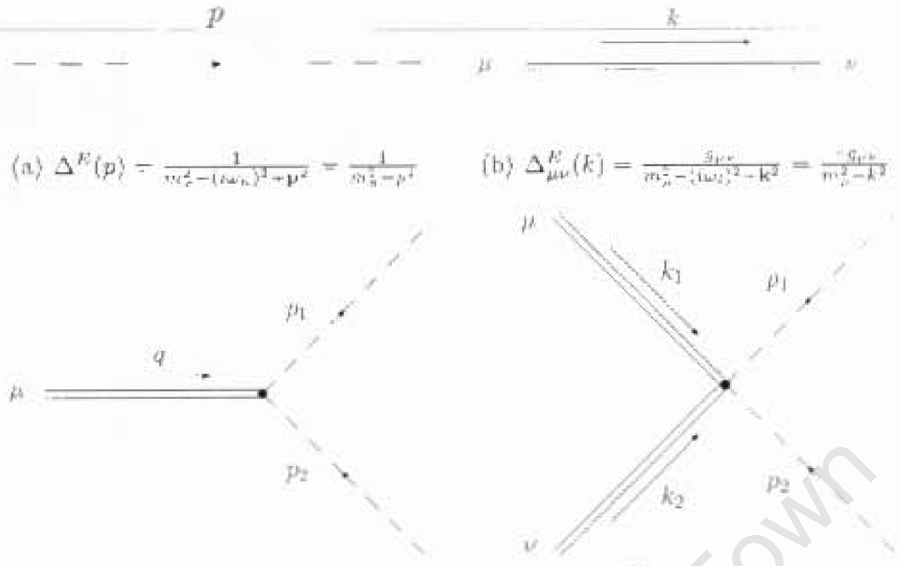
$$\int \frac{d^4k}{(2\pi)^4} \rightarrow \frac{i}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3}. \tag{4.1}$$

Calculations are now done entirely in Euclidean space, and the Feynman rules for the KLZ model are depicted in Fig. 4.1. When diagrams are computed in this formalism, they include both temperature dependent and temperature independent terms.

The terms without a thermal dependency correspond to the zero temperature diagrams computed using standard means. The terms with a thermal dependency correspond to the thermal corrections to the zero temperature case, and contribute no additional divergences. In other words, a theory renormalized at zero temperature remains renormalized at finite temperature.

There has been a great deal of work in developing methods to evaluate sums over Matsubara frequencies. A brief review of the method used is presented below, for more detail see [13].

Suppose a function $f(z)$, which has no singularities along the imaginary axis, needs



(a) $\Delta^F(p) = \frac{1}{m_\pi^2 - (i\omega_n)^2 + \mathbf{p}^2} = \frac{1}{m_\pi^2 - p^2}$ (b) $\Delta^F_{\mu\nu}(k) = \frac{g_{\mu\nu}}{m_\rho^2 - (i\omega_l)^2 + \mathbf{k}^2} = \frac{-g_{\mu\nu}}{m_\rho^2 - k^2}$

(c) $\Gamma^{(0)}_{\rho\pi\pi}(p_1, p_2) = g_{\rho\pi\pi}(p_1 + p_2)^\mu$ (d) $\Gamma^{(0)}_{\rho\rho\pi\pi} = 2g^2_{\rho\pi\pi} g^{\mu\nu}$

Figure 4.1: Feynman Rules at Finite Temperature

to be summed over the (bosonic) Matsubara frequencies $i\omega_n = \frac{2\pi i n}{\beta}$, i.e.

$$S = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} f(i\omega_n), \tag{4.2}$$

multiplying $f$ by an auxiliary function with simple poles with residue $\frac{1}{\beta}$ at $z = i\omega_n$, such as (notice the use of Bose statistics, if the sum were over fermionic frequencies this function would be unsuitable)

$$n(z) = \frac{1}{e^{\beta z} - 1}, \tag{4.3}$$

makes it possible to rewrite the sum, using Cauchy's integral theorem, as

$$S = \frac{1}{2\pi i} \oint_C f(z) n(z) dz, \tag{4.4}$$

where $C$ is the contour in Fig. 4.2(a). Since this contour can be deformed into the contour shown in Fig. 4.2(b) the sum can now be written as

$$S = -\frac{1}{2\pi i} \int_{-i\infty-\epsilon}^{i\infty-\epsilon} f(z) n(z) dz + \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} f(z) n(z) dz. \tag{4.5}$$

By using the contours in Fig. 4.2(c) and Fig. 4.2(d) these integrals can be rewritten as

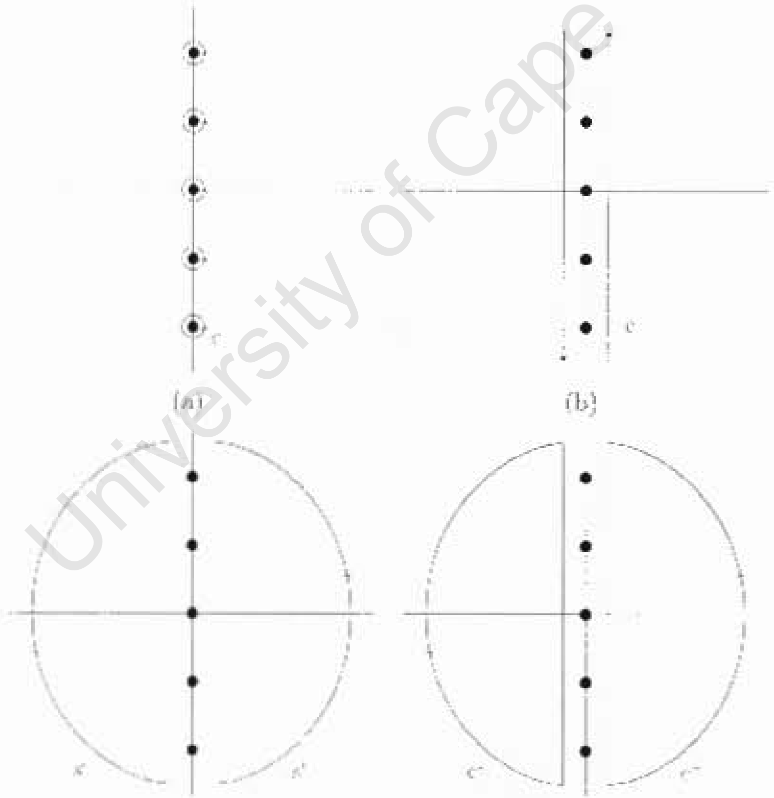
$$\int_{i\infty-\epsilon}^{-i\infty-\epsilon} = \oint_{C^-} + \int_{S^-} \quad \text{and} \quad \int_{-i\infty+\epsilon}^{i\infty+\epsilon} = \int_{S^+} - \oint_{C^+} \tag{4.6}$$

but if $f(z)n(z)$ falls off fast enough as $|z| \to \infty$ the integrals over $S^+$ and $S^-$ will vanish, so the sum becomes

$$S = -\frac{1}{2\pi i}\oint_{C^-} f(z)n(z)dz - \frac{1}{2\pi i}\oint_{C^+} f(z)n(z)dz. \tag{4.7}$$

Once more utilising Cauchy's relation, combined with the knowledge that the poles of $n(z)$ are along the imaginary axis and thus outside the integration contours, it follows that

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} f(z = i\omega_n) = -\sum_{\text{poles of } f(z)} \text{Res}\Big\{f(z)n(z)\Big\}. \tag{4.8}$$



(a) (b) (c) (d)

Figure 4.2: Integration Contours

## 4.2 Thermal Corrections to the Rho-Pi-Pi Vertex

Using the thermal propagators in Eq. 4.1, and a similar algebraic approach as in the zero temperature case, it can be shown that

$$G_F(q^2,T) = g^2_{\rho\pi\pi}\frac{2}{i}\int_0^1 dx_1\int_0^{1-x_1} dx_2\Big\{(2-3x_1)I_1+ \\ (1-2x_1)\left[m_\pi^2(x_1+x_2-2)^2 - q^2(x_1x_2-x_1-x_2+2)\right] I_0\Big\}. \quad (4.9)$$

However, as discussed above, the integral over the time-like momentum component is replaced by a frequency sum, so the integrals $I_s$ become

$$I_s = -\frac{i}{\beta}\sum_m \int \frac{d^3l}{(2\pi)^3}\frac{(-1)^s(\omega_m^2+\mathbf{l}^2)^s}{(\omega_m^2+\mathbf{l}^2+\Delta)^3}, \quad (4.10)$$

so that

$$I_0 = -i\int\frac{d^3l}{(2\pi)^3}S_0 \qquad\qquad I_1 = i\int\frac{d^3l}{(2\pi)^3}\Big[S_1+\mathbf{l}^2S_0\Big], \quad (4.11)$$

where the frequency sums have been defined as

$$S_s = \frac{1}{\beta}\sum_m \frac{\omega_m^{2s}}{(\omega_m^2+E_l^2)^3} \qquad \text{where} \qquad E_l(q^2) \equiv \sqrt{\mathbf{l}^2+\Delta(q^2)}. \quad (4.12)$$

Since these sums satisfy the condition that the function to be summed must have no poles alone the imaginary axis as low as $q^2$ is below threshold, they may be computed using the method described above. For the particular sums in question it is clear that $f(z)$ has the form

$$f(z) = -i^{2s}\frac{z^{2s}}{(z^2-E_l^2)^3} = (-1)^{s+1}\frac{z^{2s}}{(z+E)^3(z-E)^3}, \quad (4.13)$$

and has two poles of third order at $z=-E_l$ and $z=E_l$. Notice that for $s<3$ $f(z)$ falls off rapidly as $|z|\to\infty$ as required for Eq. 4.8 to be valid.

The residue of a function $h(z)$ with a pole of order $n$ at $z=c$ is given by

$$\text{Res}\Big(h(z)\Big)_{z=c} = \frac{1}{(n-1)!}\lim_{z\to c}\left(\frac{d}{dz}\right)^{n-1}\Big[h(z)(z-c)^n\Big]. \quad (4.14)$$

The residues needed to compute the sums $S_s$ are given by

$$\text{Res}\Big(f(z)n(z)\Big)_{z=\pm E_l} = \frac{1}{2}\lim_{z\to\pm E_l}\frac{d^2}{dz^2}\Big[f(z)n(z)(z\mp E_l)^3\Big]. \quad (4.15)$$

Defining

$$g_s^{\pm} = f(z)n(z)(z \mp E_l)^3 = (-1)^{s+1}\frac{z^{2s}n(z)}{(z \pm E_l)^3}, \tag{4.16}$$

and noting that

$$\frac{dn(z)}{dz} = \beta n(z)n(-z) \qquad \text{and} \qquad n(-z) = -n(z) - 1 \tag{4.17}$$

the first derivative becomes

$$\frac{dg_s^{\pm}}{dz} = (-1)^{s+1}\left[\frac{n(z)}{(z \pm E_l)^3}2s \cdot z^{2s-1} - \frac{3n(z)z^{2s}}{(z \pm E_l)^4} + \beta\frac{z^{2s}n(z)n(-z)}{(z \pm E_l)^3}\right], \tag{4.18}$$

which simplifies to

$$\frac{dg_s^{\pm}}{dz} = (-1)^{s+1}\frac{n(z)z^{2s-1}}{(z \pm E_l)^3}\left[2s - \frac{3z}{(z \pm E_l)} + \beta n(-z)z\right]. \tag{4.19}$$

Computing and simplifying the second derivative yields

$$\begin{aligned}\frac{d^2g_s^{\pm}}{dz^2} = (-1)^{s+1}\frac{n(z)z^{2s-2}}{(z \pm E_l)^3}\Bigg[&2s(2s-1) + 2s \cdot z\left(-\frac{6}{(z \pm E_l)} + 2\beta n(-z)\right)\\ &+ z^2\left(\frac{12}{(z \pm E_l)^2} - \beta\frac{6n(-z)}{(z \pm E_l)} + \beta^2 n^2(-z) - \beta^2 n(z)n(-z)\right)\Bigg],\end{aligned} \tag{4.20}$$

so that the residues become

$$\begin{aligned}\mathrm{Res}\Big(g_s^{\pm}(z)\Big)_{z=\pm E_l} = \mp(-1)^s\frac{1}{2}\frac{n(\pm E_l){E_l}^{2s-2}}{8{E_l}^3}\Big\{&2s(2s-1) - 6s\\ &+ 3 \pm (4s-3)\beta E_l n(\mp E_l) + \beta^2{E_l}^2 n^2(\mp E_l) - \beta^2{E_l}^2 n(E_l)n(-E_l)\Big\}.\end{aligned} \tag{4.21}$$

Subsequently, for the case $s = 0$ it can be shown that

$$\mathrm{Res}\Big(g_0^{\pm}(z), \pm E_l\Big) = -\beta\frac{n(E_l)[n(E_l)+1]}{16{E_l}^4}\left\{3 + \beta E_l\Big[2n(E_l) + 1\Big]\right\} \mp \frac{3n(\pm E_l)}{16{E_l}^5}, \tag{4.22}$$

from which it follows that the sum $S_0$ is given by

$$S_0 = \beta\frac{n(E_l)[n(E_l)+1]}{8{E_l}^4}\left\{3 + \beta E_l\Big[2n(E_l) + 1\Big]\right\} + \frac{3n(E_l)}{8{E_l}^5} + \frac{3}{16{E_l}^5}. \tag{4.23}$$

Similarly, for the case $s = 1$ it can be shown that

$$\mathrm{Res}\Big(g_1^{\pm}(z), \pm E_l\Big) = \beta \frac{n(E_l)[n(E_l)+1]}{16{E_l}^2} \Bigg\{ 1 - \beta E_l \Big[ 2n(E_l) + 1 \Big] \Bigg\} \mp \frac{n(\pm E_l)}{16{E_l}^3}, \tag{4.24}$$

which implies a value for $S_1$ of

$$S_1 = -\beta \frac{n(E_l)[n(E_l)+1]}{8{E_l}^2} \Bigg\{ 1 - \beta E_l \Big[ 2n(E_l) + 1 \Big] \Bigg\} + \frac{n(E_l)}{8{E_l}^3} + \frac{1}{16{E_l}^3}. \tag{4.25}$$

The final term in both these expressions is clearly independent of temperature ($\beta$), and corresponds to the unrenormalised zero temperature case calculated in the previous section. Since it is the thermal corrections that are currently of interest, the zero temperature terms are discarded, yielding

$$\boxed{\begin{aligned} S_0 &= \beta \frac{n(E_l)[n(E_l)+1]}{8E_l^4} \Bigg\{ \beta E_l \Big[ 2n(E_l) + 1 \Big] + 3 \Bigg\} + \frac{3n(E_l)}{8E_l^5} \\ S_1 &= -\beta \frac{n(E_l)[n(E_l)+1]}{8E_l^2} \Bigg\{ \beta E_l \Big[ 2n(E_l) + 1 \Big] - 1 \Bigg\} + \frac{n(E_l)}{8E_l^3}. \end{aligned}} \tag{4.26}$$

Since the sums depend only on the magnitude $\mathbf{l}^2$ of the momentum being integrated over in $I_s$, the integration coordinates can be changed to spherical polars, so the integrals $I_s$ become

$$I_0 = -\frac{i}{2\pi^2} \int_0^\infty \mathbf{l}^2 d|\mathbf{l}| \frac{n(E_l)}{8E_l^5} \Bigg\{ \beta E_l [n(E_l) + 1] \Big( \beta E_l [2n(E_l) + 1] + 3 \Big) + 3 \Bigg\}, \tag{4.27}$$

and

$$\begin{aligned} I_1 = \frac{i}{(4\pi)^2} \int_0^\infty \mathbf{l}^2 d|\mathbf{l}| \frac{n(E_l)}{{E_l}^5} \Bigg\{ & \beta^2 {E_l}^2 [n(E_l) + 1][2n(E_l) + 1][\mathbf{l}^2 - {E_l}^2] \\ & + \left(3\mathbf{l}^2 + {E_l}^2\right) \Big( 1 + \beta E_l [n(E_l) + 1] \Big) \Bigg\}. \end{aligned} \tag{4.28}$$

As with the zero temperature case, the integrals must be computed numerically. The full vertex correction function is now simply the sum of the zero and finite temperature corrections

$$G(q^2, T) = G_Z(q^2) + G_F(q^2, T). \tag{4.29}$$

## 4.3 Thermal Corrections to Self Energy

At finite temperature the matter contribution becomes non-zero, so the modification to the propagator now has the form

$$\Pi(q^2,T) = \Pi_{vac}(q^2) + \Pi_{mat}(q^2,T). \tag{4.30}$$

### 4.3.1 Gale & Kapusta Result

The Gale & Kapusta paper presents results dependant on both $q^2$ and $\mathbf{q}$. The results presented are unsuitable for analysis of the form factor, and could not be confirmed by independent calculation. The longitudinal and transverse matter components presented by Gale & Kapusta are given by [9]

$$\begin{aligned}\mathcal{F}_{mat}(q^2,T) = \left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2 \frac{q^2}{\mathbf{q}^2}\int_0^\infty \frac{dpp^2}{\omega} n(\omega)\Bigg\{ & \frac{4\omega^2+E^2}{2p|\mathbf{q}|}\left(\ln|a| - i\pi\xi\right) \\ & -4 + \frac{2\omega E}{p|\mathbf{q}|}\left(\ln|b| + i\pi\xi\right)\Bigg\},\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\mathcal{G}_{mat}(q^2,T) = \left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2 \int_0^\infty \frac{dpp^2}{\omega} n(\omega)\Bigg\{ & \frac{2(E^2+\mathbf{q}^2)}{\mathbf{q}^2} - \frac{E\omega q^2}{p|\mathbf{q}^3|}\left(\ln|b| + i\pi\xi\right) \\ & + \frac{\mathbf{q}^2(4p^2-\mathbf{q}^2+2E^2) - E^2(E^2+4\omega^2)}{4p|\mathbf{q}|^3}\left(\ln|a| - i\pi\xi\right)\Bigg\},\end{aligned} \tag{4.32}$$

where the following have been introduced

$$\begin{aligned} \omega &= \sqrt{p^2+m_\pi^2} & E &= q_0 = \sqrt{q^2+\mathbf{q}^2} \\ a &= \frac{(q^2+2p|\mathbf{q}|)^2 - 4\omega^2E^2}{(q^2-2p|\mathbf{q}|)^2 - 4\omega^2E^2} & b &= \frac{(q^2)^2 - 4\left(p|\mathbf{q}| + E\omega\right)^2}{(q^2)^2 - 4\left(p|\mathbf{q}| - E\omega\right)^2} \\ \xi &= \Theta\left(E\sqrt{1-\frac{4m_\pi^2}{q^2}} + |\mathbf{q}| - 2p\right) & &\times\Theta\left(2p - \left|E\sqrt{1-\frac{4m_\pi^2}{q^2}} - |\mathbf{q}|\right|\right).\end{aligned} \tag{4.33}$$

Recall that the modification to the form factor is a linear combination of $\mathcal{F}$ and $\mathcal{G}$, $\Pi_{mat}(q^2,T) = \frac{1}{3}\left[\mathcal{F}_{mat}(q^2,T) + 2\mathcal{G}_{mat}(q^2,T)\right]$. $\Pi_{mat}$ is actually independent of $b$ when the longitudinal and transverse contributions are combined, and has the form

$$\Pi_{mat}(q^2,T) = \frac{1}{3}\left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2 \int_0^\infty \frac{dpp^2}{\omega} n(\omega)\frac{1}{2p|\mathbf{q}|^3}\Bigg\{ -8p|\mathbf{q}|q^2 + 8p|\mathbf{q}|(E^2+\mathbf{q}^2)$$
$$+ (\ln|a| - i\pi\xi)\Big[q^2(4\omega^2+E^2) + \mathbf{q}^2(4p^2-\mathbf{q}^2+2E^2) - E^2(E^2+4\omega^2)\Big]\Bigg\}, \quad (4.34)$$

which simplifies to

$$\Pi_{mat}(q^2,T) = \frac{1}{3}\left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2 \int_0^\infty \frac{dpp}{\omega} n(\omega)\left\{ 8p + (\ln|a| - i\pi\xi)\frac{q^2-4m_\pi^2}{2|\mathbf{q}|}\right\}. \quad (4.35)$$

The derivative at $q^2 = 0$, required for analysing the radius, is given by

$$\left.\frac{d\Pi_{mat}}{d(q^2)}\right|_{q^2=0} = \frac{4m_\pi}{3\mathbf{q}^2}\left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2 \int_0^\infty dp\frac{n(\omega)}{\omega}\frac{p^2}{(m_\pi+2p)}. \quad (4.36)$$

These results are unsuitable for analysing the thermal dependence of either the form factor or the radius. Choosing the frame at rest, $\mathbf{q} = 0$, immediately gives divergent results.

### 4.3.2 Independent Calculation

Performing an independent calculation, the self energy is given by

$$\Pi^{\mu\nu} = \frac{g^2_{\rho\pi\pi}}{\beta}\sum_m \int \frac{d^3k}{(2\pi)^3}\left\{\frac{(2k-q)^\mu(2k-q)^\nu}{(m_\pi^2-k^2)[m_\pi^2-(k-q)^2]} + \frac{g^{\mu\nu}}{m_\pi^2-k^2}\right\}. \quad (4.37)$$

The first integral is rewritten using the Feynman parametrisation to give

$$\frac{(2k-q)^\mu(2k-q)^\nu}{(m_\pi^2-k^2)[m_\pi^2-(k-q)^2]} = \int dx_1dx_2\delta(x_1+x_2-1)\frac{N_\rho}{D_\rho^2}, \quad (4.38)$$

with

$$\begin{aligned} D_\rho =& x_1(m_\pi^2-k^2) + x_2[m_\pi^2-(k-q)^2] \\ =& m_\pi^2 - k^2 + 2x_2k\cdot q - x_2q^2 \\ =& m_\pi^2 - (k-x_2q)^2 - x_2(1-x_2)q^2 \\ =& \Delta_\rho - l_\rho^2 \end{aligned}, \quad (4.39)$$

where

$$l_\rho = k - x_2 q \qquad\qquad \Delta_\rho = m_\pi^2 - x_2(1-x_2)q^2, \tag{4.40}$$

and

$$\begin{aligned} N_\rho &= (2k-q)^\mu(2k-q)^\nu = [2l_\rho - (2x_2-1)q]^\mu[2l_\rho - (2x_2-1)q]^\nu \\ &= 4l_\rho^\mu l_\rho^\nu + 2(2x_2-1)(q^\mu l_\rho^\nu + l_\rho^\mu q^\nu) + (2x_2-1)^2 q^\mu q^\nu \end{aligned} . \tag{4.41}$$

Single powers of $l_\rho$ vanish in integration, so the first integral becomes

$$\int \frac{d^3k}{(2\pi)^3}\frac{(2k-q)^\mu(2k-q)^\nu}{(m_\pi^2-k^2)[m_\pi^2-(k-q)^2]} = \int_0^1 dx \int \frac{d^3l_\rho}{(2\pi)^3}\frac{4l_\rho^2 g^{\mu\nu} + (2x_2-1)^2 q^\mu q^\nu}{(l_\rho^2-\Delta_\rho)^2}, \tag{4.42}$$

which allows the self energy to be written as

$$i\Pi^{\mu\nu}(q^2) = g_{\rho\pi\pi}^2\left\{\int_0^1 dx\,\left[g^{\mu\nu}I_{1,2}(\Delta_\rho) + (1-2x_1)^2 q^\mu q^\nu I_{0,2}(\Delta_\rho)\right] - 2g^{\mu\nu}I_{0,1}(m_\pi^2)\right\}. \tag{4.43}$$

The fact that the self energy is transverse

$$q_\mu \Pi^{\mu\nu}(q^2) = 0, \tag{4.44}$$

implies

$$\int_0^1 dx\left[q^\nu I_{1,2}(\Delta_\rho) + (1-2x)^2 q^2 q^\nu I_{0,2}(\Delta_\rho)\right] - 2q^\nu I_{0,1}(m_\pi^2) = 0, \tag{4.45}$$

so the self energy becomes

$$\Pi^{\mu\nu}(q^2) = g_{\rho\pi\pi}^2\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)q^2\int_0^1 dx(1-2x)^2 I_{0,2}(\Delta_\rho). \tag{4.46}$$

It follows that

$$\Pi(q^2) = -g_{\rho\pi\pi}^2 q^2 \int_0^1 dx(1-2x)^2 I_{0,2}(\Delta_\rho). \tag{4.47}$$

The momentum integral at finite temperature

$$I_{0,2} = \frac{1}{\beta}\sum_m \int \frac{d^3l_\rho}{(2\pi)^3}\frac{1}{(\omega_m^2 + \mathbf{l}_\rho^2 + \Delta_\rho)^2}, \tag{4.48}$$

can be calculated (below threshold) using Eq. 4.8 and the function

$$f(z) = \frac{1}{(z^2 - E_\rho^2)^2} \qquad \text{with} \qquad E_\rho^2 = \mathbf{l}_\rho^2 + \Delta_\rho. \tag{4.49}$$

The residues needed to compute the sums are given by

$$\mathrm{Res}\Big(f(z)n(z)\Big)_{z=\pm E_\rho} = \lim_{z\to\pm E_\rho}\frac{d}{dz}\Big[f(z)n(z)(z\mp E_\rho)^2\Big]. \qquad (4.50)$$

Defining

$$g_\rho^\pm = f(z)n(z)(z\mp E_\rho)^2 = \frac{n(z)}{(z\pm E_\rho)^2}, \qquad (4.51)$$

and differentiating

$$\frac{dg_\rho^\pm}{dz} = -\frac{2n(z)}{(z\pm E_\rho)^3} + \beta\frac{n(z)n(-z)}{(z\pm E_\rho)^2} = -\frac{n(z)}{(z\pm E_\rho)^3}\Big[2-\beta n(-z)(z\pm E_\rho)\Big], \qquad (4.52)$$

the residues become

$$\mathrm{Res}\Big(f(z)n(z)\Big)_{z=+E_\rho} = \frac{n(\pm E_\rho)}{4E_\rho^3}\Big[\mp 1+\beta E_\rho n(\mp E_\rho)\Big]. \qquad (4.53)$$

So using Eq. 4.8 the sum becomes

$$I_{0,2} = \int\frac{d^3l_\rho}{(2\pi)^3}\left\{\frac{n(E_\rho)}{4E_\rho^3}\Big[1+\beta E_\rho[n(E_\rho)+1]\Big]+\frac{n(E_\rho)+1}{4E_\rho^3}\Big[1+\beta E_\rho n(E_\rho)\Big]\right\}, \qquad (4.54)$$

which, after discarding the zero temperature result and simplifying

$$I_{0,2} = \int\frac{d^3l_\rho}{(2\pi)^3}\frac{n(E_\rho)}{2E_\rho^3}\left\{1+\beta E_\rho\Big[n(E_\rho)+1\Big]\right\}. \qquad (4.55)$$

and switching to spherical polars gives

$$I_{0,2} = \frac{1}{(2\pi)^2}\int_0^\infty d|\mathbf{l}|\,\mathbf{l}^2\frac{n(E_\rho)}{E_\rho^3}\left\{1+\beta E_\rho\Big[n(E_\rho)+1\Big]\right\}. \qquad (4.56)$$

so finally

$$\Pi_{mat}(q^2,T) = -q^2\left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2\int_0^1 dx(1-2x)^2\int_0^\infty d|\mathbf{l}|\,\mathbf{l}^2\frac{n(E_\rho)}{E_\rho^3}\left\{1+\beta E_\rho\Big[n(E_\rho)+1\Big]\right\}. \qquad (4.57)$$

The difference between this and Eq. 4.35 is clear. Unfortunately Gale & Kapusta simply present their results without showing any details of their calculation, which makes a step by step comparison of the calculation impossible.

## 4.4 Form Factor

The finite temperature form factor is given to $\mathcal{O}(g^2_{\rho\pi\pi})$ by

$$F_\pi(q^2,T) = \frac{m_\rho^2 \cdot \Pi(0,T)}{m_\rho^2 - q^2 - \Pi(q^2,T)} + \frac{m_\rho^2}{m_\rho^2 - q^2}\left[G(q^2,T) - G(0,T)\right] \tag{4.58}$$

Where $\Pi(q^2,T)$ and $G(q^2,T)$ are as given in equations 4.29 and 4.30 respectively. Note that, as in the zero temperature case, the self energy and vertex function corrections are not combined as this would constitute a higher order correction.

The resulting form factor is shown for different temperature values in figure 4.3 below. It is clear that increasing temperature causes the form factor to fall off faster.

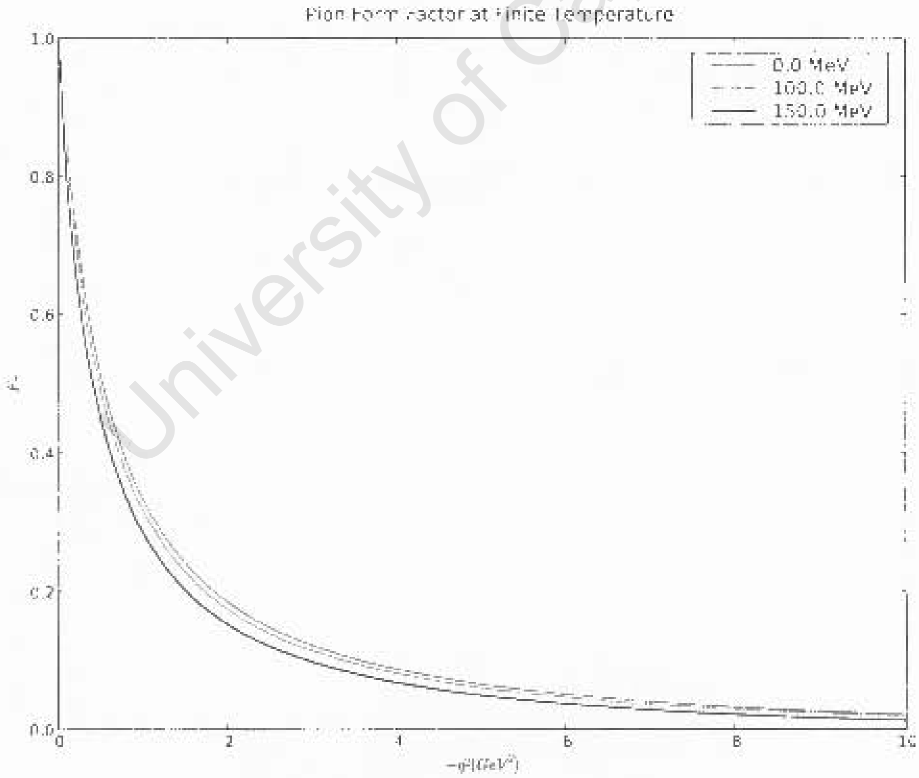


Figure 4.3: Temperature dependence of $F_\pi$

## 4.5 Electromagnetic Radius

In order to evaluate the electromagnetic radius at finite temperature it is necessary to calculate the first derivative of both the thermal vertex correction $G_F(q^2,T)$ and the self energy matter contribution $\Pi_{mat}(q^2,T)$.

Differentiating the thermal vertex correction gives

$$\frac{dG_F}{d(q^2)} = g^2_{\rho\pi\pi}\frac{2}{i}\int_0^1 dx_1\int_0^{1-x_1} dx_2\Big\{-(1-2x_1)(x_1x_2-x_1-x_2+2)I_0+ \\ (2-3x_1)\frac{dI_1}{d(q^2)}+(1-2x_1)\left[m_\pi^2(x_1+x_2-2)^2-q^2(x_1x_2-x_1-x_2+2)\right]\frac{dI_0}{d(q^2)}\Big\}. \tag{4.59}$$

Changing variables

$$\frac{dE_l}{d(q^2)} = \frac{1}{2E_l}\frac{d\Delta}{d(q^2)} = -\frac{x_1x_2}{2E_l} \qquad \text{so} \qquad \frac{d}{d(q^2)} = -\frac{x_1x_2}{2E_l}\frac{d}{dE_l}, \tag{4.60}$$

the derivatives of the momentum integrals become

$$\begin{aligned}\frac{dI_0}{d(q^2)} &= -\frac{i}{2\pi^2}\int_0^\infty d|\mathbf{l}|\;|\mathbf{l}|^2\frac{x_1x_2}{2E_l}\frac{dS_0}{dE_l}\\ \frac{dI_1}{d(q^2)} &= -\frac{i}{2\pi^2}\int_0^\infty d|\mathbf{l}|\;|\mathbf{l}|^2\left[\frac{x_1x_2}{2E_l}\frac{dS_1}{dE_l}+|\mathbf{l}|^2\frac{x_1x_2}{2E_l}\frac{dS_0}{dE_l}\right].\end{aligned} \tag{4.61}$$

Differentiating and simplifying the sums gives

$$\frac{dS_0}{dE_l} = -\frac{\beta n(E_l)}{8E_l^5}\left[n(E_l)+1\right]\Big\{\left(\beta E_l[2n(E_l)+1]+3\right)^2 \\ +2\beta^2E_l^2n(E_l)\left[n(E_l)+1\right]+6\Big\} - \frac{15n(E_l)}{8E_l^6}, \tag{4.62}$$

and

$$\frac{dS_1}{dE_l} = -\beta\frac{n(E_l)[n(E_l)+1]}{8E_l^3}\left\{\beta^2E_l^2\left([2n(E_l)+1]^2+2n^2(E_l)[n(E_l)+1]\right)+3\right\} - \frac{3n(E_l)}{8E_l^4}. \tag{4.63}$$

It follows that the derivatives of the integrals take the form

$$\frac{dI_0}{d(q^2)} = -\frac{ix_1x_2}{(4\pi)^2}\int_0^\infty \mathbf{l}^2 d|\mathbf{l}|\,\frac{1}{2E_l}\frac{n(E_l)}{E_l^6}\Bigg\{15 + \beta E_l\big[n(E_l)-1\big]\Bigg[\Big(\beta E_l[2n(E_l)+1]+3\Big)^2 + 2\beta^2 {E_l}^2 n(E_l)\Big[n(E_l)+1\Big] - 6\Bigg]\Bigg\}, \quad (4.64)$$

and

$$\frac{dI_1}{d(q^2)} = \frac{ix_1x_2}{(4\pi)^2}\int_0^\infty \mathbf{l}^2 d|\mathbf{l}|\frac{1}{2E_l}\frac{n(E_l)}{E_l^6}\Bigg\{15\mathbf{l}^2 + 3{E_l}^2 + \beta E_l\big[n(E_l)-1\big]\Bigg[\mathbf{l}^2(6-9{E_l}^2\beta^2) - {E_l}^2 + \beta^2{E_l}^2\Big([2n(E_l)+1]^2(\mathbf{l}^2+{E_l}^2) + 2n^2(E_l)\big[n(E_l)+1\big]\big[\mathbf{l}^2+n(E_l){E_l}^2\big]\Big) + 6\beta E_l[2n(E_l)+1]\mathbf{l}^2\Bigg]\Bigg\}. \quad (4.65)$$

Differentiating the matter contribution gives

$$\frac{d\Pi_{mat}}{d(q^2)} = -\left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2\int_0^1 dx(1-2x)^2\int_0^\infty d|\mathbf{l}|\,\mathbf{l}^2\Bigg\{\frac{n(E_\rho)}{E_\rho^3}\Big(1+\beta E_\rho\Big[n(E_\rho)+1\Big]\Big) - q^2\frac{d}{d(q^2)}\left(\frac{n(E_\rho)}{E_\rho^3}\Big[1-\beta E_\rho[n(E_\rho)+1]\Big]\right)\Bigg\}. \quad (4.66)$$

Switching the derivative to $E_\rho$

$$\frac{d}{d(q^2)} = -x(1-x)\frac{d}{dE_\rho}, \quad (4.67)$$

the matter derivative becomes

$$\frac{d\Pi_{mat}}{d(q^2)} = -\left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2\int_0^1 dx(1-2x)^2\int_0^\infty d|\mathbf{l}|\,\mathbf{l}^2\Bigg\{\frac{n(E_\rho)}{E_\rho^3}\Big(1+\beta E_\rho\Big[n(E_\rho)+1\Big]\Big) - x(1-x)q^2\frac{d}{dE_\rho}\left(\frac{n(E_\rho)}{E_\rho^3}\Big[1+\beta E_\rho[n(E_\rho)+1]\Big]\right)\Bigg\}. \quad (4.68)$$

which finally simplifies to

$$\frac{d\Pi_{cont}}{d(q^2)} = -\left(\frac{g_{\rho\pi\pi}}{2\pi}\right)^2 \int_0^1 dx(1-2x)^2 \int_0^\infty d|l|\, l^2 \Bigg\{ \frac{n(E_\rho)}{E_\rho^3}\left(1+\beta E_\rho\Big[n(E_\rho)+1\Big]\right)$$
$$+ x(1-x)q^2 \frac{n(E_\rho)}{E_\rho^4}\left(3+2\beta E_\rho\Big[n(E_\rho)+1\Big]+\beta^2 E_\rho^2\Big[n(E_\rho)+1\Big]\Big[2n(E_\rho)+1\Big]\right)\Bigg\}. \tag{4.69}$$

The radius is now given by

$$\langle r_\pi^2 \rangle = 6\left[\frac{1-\frac{d\Pi(q^2,T)}{d(q^2)}}{m_\rho^2+\Pi(0)} + \frac{dG(q^2,T)}{d(q^2)}\right]_{q^2=0} \tag{4.70}$$

By evaluating the integrals numerically, the thermal dependence of the electromagnetic radius can be investigated. As is clear from Fig. 4.4, the radius increases steadily with temperature. This is conceptually compatible with the idea of hadrons dissolving into a quark-gluon plasma. However, the KLZ model remains an effective theory, so the thermal results are far from 'definitive' proof of a deconfinement phase.

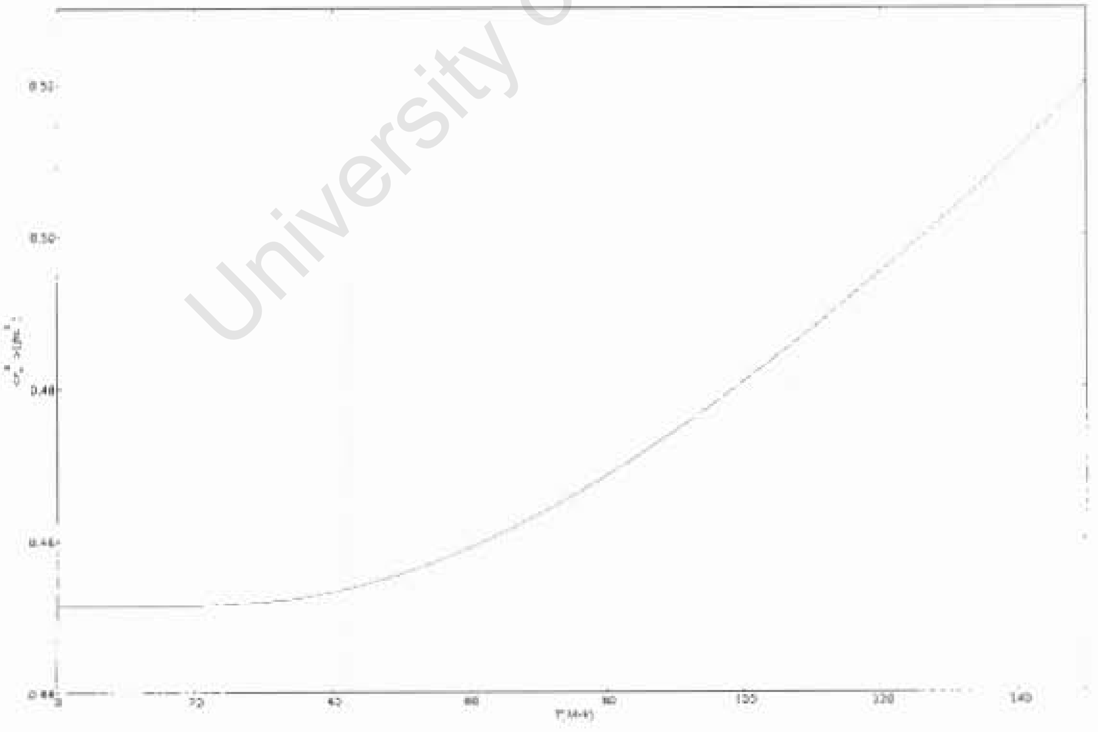


Figure 4.4: Temperature dependence of $\langle r_\pi^2 \rangle$

# COMPARISON OF RESULTS

At zero temperature, the next-to-leading order corrections were found to improve hugely over the tree level VMD result, with a chi-square per degree of freedom of $\chi^2_F \approx 1.1$ compared to the tree level result of $\chi^2_F \approx 5$ in the space-like region.

This excellent agreement between theory and experiment is comparable to that found in dual large $N_C$ QCD ($\mathrm{QCD}_\infty$) [14], which gives $\chi^2_F \approx 1.2$. Dual-$\mathrm{QCD}_\infty$, after unitarization in the time-like region, like KLZ, generates a correction to naive VMD. However, it does this by considering an infinite set of vector meson radial excitations.

The results also compare favourably with modern approaches such as holographic QCD [15], and light front dynamics and the AdS/QCD correspondence [16]. However, unlike the other approaches mentioned, the (comparatively simple) KLZ model contains no free parameters, which makes the excellent agreement with data a remarkable result.

At finite temperature, the primary comparison must be made with the Gale & Kapusta results [9]. As was mentioned previously, the finite temperature results presented in this paper could not be confirmed by independent calculation. This is a matter of some concern as the Gale & Kapusta result is widely cited. A closer inspection is warranted.

# CONCLUSION

Next-to-leading order corrections to the electromagnetic form factor of the pion were calculated using the Kroll-Lee-Zumino model as a quantum field theory basis for vector meson dominance. At zero temperature, these corrections were found to considerably improve the agreement between the model and experimental data for $F_\pi(q^2)$ in both the space-like and time-like region.

At finite temperature, increasing temperature values were found to make the form factor fall off faster in the space-like region. The electromagnetic radius was found to increase with temperature, with an increase of 14% around the expected deconfinement value $T = 150$ MeV. While these results are intuitively consistent with ideas of hadronic deconfinement, they certainly do not prove the existence of a quark gluon plasma.

The KLZ model has been shown to be a reasonable framework to calculate systematic corrections to vector dominated processes in perturbation theory. The relatively mild coupling does not preclude perturbative calculations, as single loop results are accompanied by a loop suppression factor of $1/(4\pi)^2$. Increasing orders in perturbation theory are expected to include higher powers of this suppression factor.

The success of the model in calculating the electromagnetic form factor suggest application to other processes dominated by the $\rho$ meson, including

- Performing an analytic extension of the vertex and matter contributions past threshold will enable a fuller comparison with the Gale & Kapusta result, as well as enabling the study of the thermal dependence of the width of the $\rho$ meson.

- Extending the KLZ Lagrangian to the non-Abelian case via a Higgs-like mass generation mechanism will facilitate the study of $\pi\pi$ scattering dynamics. This is currently being investigated by C.A. Dominguez, M. Loewe, K. Schilcher and H. Spiesburger.

- The scalar radius of the pion, important in fixing the $\bar{\ell}_4$ parameter in chiral perturbation theory, can be calculated in the KLZ model. A next-to-leading order calculation has already been submitted to and approved by Physical Review D [17], but computing the two loop result is an interesting challenge.

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