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DEPARTMENT OF MATHEMATICS

Compactifications, Subordinations and Uniformities

by

S. Salbany

The University of Cape Town

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FOREWORD

This thesis relates hausdorff compactifications of spaces and other structures on the space.

Each chapter starts with an introduction, which describes its content, and ends with a collection of notes, where the references relevant to the chapter are given.

Most of the thesis is self contained.

The standard reference throughout is : General Topology, by Kelley. The notation is as in Kelley. We often use the abbreviations - s.t. for such that ; iff for if and only if; i.e. for that is.

This thesis arose from an attempt to prove the essential steps in the construction of υX, described by Aleksandrov in his survey : Some Results in the Theory of Topological Spaces, Obtained Within the Last Twenty-Five Years.

I want to thank Dr. H. Schlagbauer, my supervisor, for his encouragement, tremendous patience, and many helpful conversations. I cannot describe my debt, but it will be apparent to everyone who knows me.

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Chapter 1

Subordination and Generalised Topology

1.1. Generalised closure and neighbourhood spaces

In this section a one to one correspondence between generalised closure and neighbourhood structures on X is established .

Definition 1.1.1. Let X be a set and $u:p[X]\to p[X]$ a map from the family of subsets of X into itself , satisfying

U1 $u(\varphi) = \varphi$

U2 $A \subset u(A)$ for all $A \subset X$

U3 $u(A) \subset u(B)$ if $A \subset B$

u is called a generalised closure on X , and (X,u) a generalised closure space .

Definition 1.1.2. Let X be a set and $I:X\to p[p[X]]$ s.t.

I1 $I(x) \neq \varphi$ for each $x \in X$.

I2 $x \in V$ for all $V \in I(x)$

I3 If $V \in I(x)$ and $V \subset W$, then $W \in I(x)$.

I is called a neighbourhood function and (X,I) a neighbourhood space .

If (X,I) is a neighbourhood space , it is natural to specify a notion of closeness by – x is close to $A \subset X$ if $V \cap A \neq \varphi$ for all $V \in I(x)$; we write $x \in u(A)$ if x is close to A .

If (X,u) is a closure space , and V is to be a neighbourhood of x , then $x \notin u(X-V)$ since $V \cap X-V = \varphi$. These remarks motivate

<u>Definition 1.1.3.</u> Given a generalised closure u , let $I(x) = \{ V \mid V \subset X \text{ and } x \in X-u(X-V) \}$ We denote this I by $I[u]$.

For completeness we state

<u>Definition 1.1.4.</u> Given a neighbourhood space (X,I) , let $u:p[X]\to p[X]$ be defined by $u(A) = \{ x \mid V \cap A \neq \varphi \text{ for all } V \in I(x) \}$. We write $u[I]$ for this u .

<u>Proposition 1.1.1.</u> $I[u]$ is a neighbourhood function, for any generalised closure u .

We verify

I1 $x \in I(x)$ for all $x \in X$, since $X-u(X-X) = X-u(\varphi) = X$. Hence $I(x) \neq \varphi$ for all $x \in X$.

I2 If $V \in I(x)$, then $x \in X-u(X-V)$. Now $X-V \subset u(X-V)$,hence $X-u(X-V) \subset V$ so that $x \in V$ for all $V \in I(x)$.

I3 If $V \in I(x)$ and $V \subset W$, then $X-W \subset X-V$ and hence $u(X-W) \subset u(X-V)$, so that $X-u(X-W) \supset X-u(X-V)$. It follows that $x \in X-u(X-W)$ and so $W \in I(x)$, as required .

<u>Proposition 1.1.2.</u> $u[I]$ is a generalised closure for any neighbourhood function I .

We verify

U1 Assume $x \in u(\varphi)$ .Now $I(x) \neq \varphi$, so $\exists\, V \in I(x)$ and $V \cap \varphi \neq \varphi$ This is a contradiction , hence $u(\varphi) = \varphi$.

U2 If $x \in A$, then $x \in V \cap A$ for all $V \in I(x)$. Hence $A \subset u(A)$.

U3 If $A \subset B$ and $x \in u(A)$, then $V \cap B \supset V \cap A \neq \varphi$ for all $V \in I(x)$. Hence $x \in u(B)$, and so $u(A) \subset u(B)$ if $A \subset B$.

<u>Proposition 1.1.3</u> If I is a neighbourhood function and u the induced generalised closure , then $I = I[u]$.

To simplify the notation write I_1 for the neighbourhood function $I[u]$. If $V \in I_1(x)$, then $x \in X-u(X-V)$ so that $x \notin u(X-V)$. Hence $\exists\, W \in I(x)$ s.t. $W \cap X-V = \varphi$, so that $W \subset V$ and , by I3 , $V \in I(x)$. Conversely , if $V \in I(x)$, then $V \cap X-V = \varphi$ so that $x \notin u(X-V)$. It follows that $V \in I_1(x)$, and the proof is complete .

<u>Proposition 1.1.4.</u> If u is a generalised closure and I the induced neighbourhood function , then $u = u[I]$.

Denote $u[I]$ by u_1 . We first show that $u_1(A) \subset u(A)$ for all $A \subset X$. Let $x \notin u(A)$, then $X-A \in I(x)$ and hence $x \notin u_1(A)$, since $A \cap X-A = \varphi$. Conversely , to show that $u(A) \subset u_1(A)$, let $x \notin u_1(A)$. Then $\exists\, V \in I(x)$ s.t. $V \cap A = \varphi$ and so $A \subset X-V$. It follows that $u(A) \subset u(X-V)$, hence $X-u(X-V) \subset X-u(A)$. Now $x \in X-u(X-V)$, by definition of I , hence $x \in X-u(A)$. This completes the proof .

We have established

<u>Proposition 1.1.5.</u> There is a one to one correspondence between generalised closure operators and neighbourhood functions on X .

Often , a neighbourhood function is required to be intersective , in the sense that $V_1 \cap V_2 \in I(x)$ when $V_1, V_2 \in I(x)$. The corresponding requirement for u is that it be union preserving, i.e. $u(A \cup B) = u(A) \cup u(B)$.

If $u(u(A)) = u(A)$ for all $A \subset X$, u is said to be idempotent . The corresponding condition on I is that each neighbourhood of x contain an open neighbourhood of x - one which is a neighbourhood of each of its points . If such a neighbourhood function is also intersective , it is called topological .

A union preserving , idempotent , generalised closure is a Kuratowski closure , topological closure,or simply a closure on X , and will be denoted by u(a) or clA or A^- . When there is a need to specify that $A \subset X$, we write $u_X(A)$ or $cl_X A$ for the closure of A .

We now complement proposition 1.1.5.

<u>Proposition 1.1.6.</u> If u is an idempotent, generalised closure, then $I[u]$ has the open neighbourhood property .

Let $V \in I(x)$, then $x \in X-u(X-V)$. Let $W = X-u(x-V)$. Note that $W \subset V$ and $X-u(X-W) = X-u(X-(X-u(X-V)) = X-u(u(V)) = X-u(V) = W$. Hence , $y \in W$ implies $W \in I(y)$, and so W is a neighbourhood of each of its points, and $x \in W$.

Conversely

<u>Proposition 1.1.7.</u> If the neighbourhood function I has the open neighbourhood property , then $u = u[I]$ is idempotent .

Suppose $x \notin u(A)$, then $\exists\, V \in I(x)$ s.t. $V \cap A = \varphi$, and we may assume that $V \in I(y)$ for all $y \in V$. It follows that $V \cap u(A) = \varphi$, since $z \in V \cap u(A)$ implies $z \in V$ and $z \in u(A)$,so that $V \cap A \neq \varphi$ as $V \in I(z)$, this is impossible . Thus $u(u(A)) \subset u(A)$ for all $A \subset X$. Now u is a generalised closure , hence $u_\circ u = u$.

<u>Proposition 1.1.8.</u> If the neighbourhood function I is intersective , then $u = u[I]$ is union preserving .

If $x \notin u(A) \cup u(B)$, then $\exists\, V_1, V_2 \in I(x)$ s.t. $V_1 \cap A = V_2 \cap B = \varphi$, hence $(V_1 \cap V_2) \cap (A \cup B) = \varphi$, and so $x \notin u(A \cup B)$, since

$V_1 \cap V_2 \in I(x)$. Conversely , if $x \notin u(A \cup B)$, then $\exists\, V \in I(x)$ s.t. $V \cap (A \cup B) = \varphi$. It follows that $V \cap A = V \cap B = \varphi$, hence $x \notin u(A) \cup u(B)$, as required .

<u>Proposition 1.1.9.</u> If the generalised closure u is union preserving , then $I = I[u]$ is intersective .

Let $V_1, V_2 \in I(x)$. Now $X-u(X-(V_1 \cap V_2)) = X-u((X-V_1) \cup (X-V_2)) = (X-u(X-V_1)) \cap (X-u(X-V_2))$, since u preserves unions . Hence $x \in X-u(X-(V_1 \cap V_2))$, and so $V_1 \cap V_2 \in I(x)$ when $V_1, V_2 \in I(x)$.

For completeness , we note that a union preserving operator u is necessarily monotone , i.e. satisfies U3 . To prove this , suppose $A \subset B$, then $A \cup B = B$ and $u(A \cup B) = u(A) \cup u(B) = u(B)$, so that $u(A) \subset u(B)$. Thus, a set of axioms for Kuratowski closure is U1 and U2 , as for a generalised closure u ,

U3* $u(A \cup B) = u(A) \cup u(B)$

U4 $u(u(A)) = u(A)$ for all $A \subset X$.

1.2. <u>Separation ,Generalised Closure and Neighbourhood Spaces</u>

A generalised closure specifies which points are near to a given set , a proximity,or separation , specifies when two sets are near,or far . We shall write $A\delta B$ if A and B are near , and $A\bar{\delta}B$ if A is far from B , in the sense that $A\delta B$ is false .

There are certain natural requirements that $\bar{\delta}$ must satisfy -

S1 $\varphi \nabla A$ for all $A \subset X$

S2 If $A \nabla B$, then $B \nabla A$

S3 If $A \nabla B$, then $A \cap B = \varphi$

S4 If $A \subset B$ and $B \nabla C$, then $A \nabla C$.

<u>Definition 1.2.1.</u> ∇ is called a separation on X if it satisfies S1 - S4 , in which case (X,∇) is a separation space .

Every separation induces a notion of closeness in an obvious way :

<u>Definition 1.2.2.</u> Let (X,∇) be a separation space , define u by $x \in u(A)$ if $\{x\}\delta A$. We denote this u by $u[\nabla]$. We shall also write $x\delta A$ $(x\nabla A)$ for $\{x\}\delta A$ ($\{x\}\nabla A$) .

<u>Proposition 1.2.1.</u> u , defined above , is a generalised closure .

We verify

U1 $u(\varphi) = \varphi$, since $x\nabla\varphi$ for each $x \in X$,and hence $x\delta\varphi$ is false so that $x \notin u(\varphi)$ for all $x \in X$.

U2 If $x \notin u(A)$, then $x\nabla A$ so that $\{x\} \cap A = \varphi$. Thus $A \subset u(A)$.

U3 If $A \subset B$ and $x \notin u(B)$, then $x\nabla B$ andhence $x\nabla A$, so that $x \notin u(A)$. This shows that u is monotone .

However , this is not a general u - it satisfies $u(\{x\}) \subset V$ for all $V \in I(x)$, where $I = I[u]$.

Proposition 1.2.2. If u is induced by ∇, then $u(\{x\}) \subset V$ for all $V \in I(x)$, where $I = I[u]$.

$V \in I(x)$ iff $x \notin u(X-V)$ iff $x \nabla X-V$. So , if $V \in I(x)$ and $y \notin V$, then $y \in X-V$ so that $x \nabla y$. It follows that $y \notin u(\{x\})$, this completes the proof .

Only generalised closures with this property can arise from a separation . We show that any such closure does arise from a separation . A simple example shows that not all generalised closures have this property -

Example Let $X = \{a,b\}$ with $u(X) = X$, $u(\varphi) = \varphi$, $u(a) = X$ and $u(b) = \{b\}$. u is a generalised closure , in fact , a topological closure . Now $\{a\} \in I(a)$, since $\{a\} = X-\{b\} = X-u(\{b\}) = X-u(X-\{a\})$, but $u(\{a\}) = X \not\subset \{a\}$.

Suppose ∇ induces u . If $A \nabla B$, it is natural to ask how $u(A)$ and $u(B)$ are related . We note that $A \nabla B$ implies $x \nabla B$ for all $x \in A$, so that $A \cap u(B) = \varphi$. Similarly $B \cap u(A) = \varphi$. Hence $(A \cap u(B)) \cup (B \cap u(A)) = \varphi$ [*] . However, [*] does not imply $A \nabla B$, but it defines a separation ∇_0 which also induces u . Thus ∇_0 is the finest separation inducing u .
These remarks motivate

Definition 1.2.3. If u is a generalised closure , define ∇_0 by - $A \nabla_0 B$ if $(A \cap u(B)) \cup (B \cap u(A)) = \varphi$. We shall write $\nabla_0 = \nabla_0[u]$.

<u>Proposition 1.2.3.</u> ∇_0, defined above, is a separation .

We verify

S1 $\varphi \nabla_0 A$ for all $A \subset X$, since $(A \cap \varphi) \cup (u(A) \cap \varphi) = \varphi$, as $u(\varphi) = \varphi$.

S2 $A \nabla_0 B$ implies $B \nabla_0 A$ is obvious .

S3 If $A \nabla_0 B$, then $A \cap u(B) = \varphi$. Hence $A \cap B = \varphi$, since $B \subset u(B)$.

S4 If $A \subset B$ and $B \nabla_0 C$, then $(A \cap u(C)) \cup (C \cap u(A)) \subset (B \cap u(C)) \cup (C \cap u(B)) = \varphi$, since u is monotone and $A \subset B$. Hence $A \nabla_0 C$.

<u>Proposition 1.2.4.</u> Let $u_1 = u[\nabla[u]]$. $u(A) \subset u_1(A)$ for all $A \subset X$. Equality holds iff u satisfies the condition in proposition 1.2.2.

As a corollary we have

<u>Corollary</u> There is a one to one correspondence between separations on X and generalised closures satisfying - $u(\{x\}) \subset V$ for all $V \in I(x)$, where $I = I[u]$.

We now prove proposition 1.2.4. Let $x \notin u_1(A)$, then $x \nabla A$, so $\{x\} \cap u(A) = \varphi$. This proves $u(A) \subset U_1(A)$.

Suppose that $u(\{x\}) \subset V$ for all $V \in I(x)$. We show that $u_1(A) \subset u(A)$, by above , it follows that $u_1 = u$. If $x \notin u(A)$, then $X-A \in I(x)$ and so $u(\{x\}) \subset X-A$. Hence $(\{x\} \cap u(A)) \cup (A \cap u(\{x\})) = \varphi$, i.e. $x \nabla A$, so that $x \notin u_1(A)$.

Conversely , assume $u = u_1$. If $V \in I(x)$, then $x \notin u(X-V)$, and so $x\nabla(X-V)$, since $u = u_1$. Thus $y \in X-V$ implies $x\nabla y$, so that $y \notin u(\{x\})$. Hence $u(\{x\}) \subset V$, as required .

We now express proposition 1.2.4. in a slightly different form . For this we need

Definition 1.2.4. $A < B$ if $A\nabla X-B$. $<$ is called a subordination .

Proposition 1.2.5. If (X,I) is a neighbourhood space and $<$ the induced subordination , then $x < V$ implies $V \in I(x)$. Also , $x < V$ is equivalent to $V \in I(x)$ iff $u = u[I]$ has the property in proposition 1.2.2.

We note that this condition can be expressed in terms of I alone - for all $x,y \in X$, if $y \notin V$ for some $V \in I(x)$, then $\exists\, W \in I(y)$ s.t. $x \notin W$.

The proof of proposition 1.2.5. is similar to that of 1.2.4. and will be omitted .

u can be expressed in terms of $<$, as follows :

Proposition 1.2.6. If u is a generalised closure satisfying the condition in proposition 1.2.2. , then $u(A) = \cap\{ V \mid A < V \}$, where $<$ is any subordination that induces u .

If $\exists\, V$ s.t. $A < V$ and $x \notin V$, then $x\nabla A$, since $x \in X-V$ and $A\nabla X-V$. Hence $x \notin u(A)$, and so $u(A) \subset \cap\{V \mid A < V \}$. Conversely, if $x \notin u(A)$, then $x \nabla A$, so that $A < X-\{x\}$. But $x \notin X-\{x\}$, hence $x \notin \cap\{ V \mid A < V \}$ if $x \notin u(A)$. This completes the proof .

Corollary $i(A) = X-u(X-A) = \cup\{ V \mid V < A \}$.

1.3. Closure and Separation , Continued .

In this section T_1, T_2, regular and topological spaces are described in terms of u and ∇ .

The equivalence of i) and ii) in the following propositions is to mean : i) implies ii) with u a given generalised closure and ∇ any separation inducing u ; ii) implies i) with ∇ a given separation and u the induced closure .

We say that X is T_1, regular, T_2 if u (∇) satisfies i) (ii)) of propositions 1.3.1. - 1.3.3. , respectively .

Proposition 1.3.1. The following are equivalent

i) $u(\{x\}) = \{x\}$

ii) For all $x \neq y$, $x \nabla y$.

If i) and $x \neq y$, then $y \notin u(\{x\})$ so that $y \nabla x$. Conversely , assume ii) holds . If $x \neq y$, then $x \nabla y$ so that $y \notin u(\{x\})$. Hence $u(\{x\}) = \{x\}$, as $x \in u(\{x\})$.

Proposition 1.3.2. The following are equivalent

i) If $V \in I(x)$, then $\exists$ $U \in I(x)$ s.t. $u(U) \subset V$

ii) If $x \nabla y$, then $\exists$ U s.t. $x \nabla U$ and $V \subset \{y \mid y \nabla X-U \}$.

Assume i) and suppose $x \nabla V$. Then $x < X-V$, hence $X-V \in I(x)$, thus $\exists\, W \in I(x)$ s.t. $u(W) \subset X-V$. It follows that $y \nabla w$ if $y \in V$. Also $x < W$, hence $x \nabla X-W$, and so ii) is satisfied.

Conversely, assume ii). If $V \in I(x)$, then $x < V$ and so $x \nabla X-V$. Let $U = X-O$, O given by ii) s.t. $x \nabla O$ and $X-V \subset \{y \mid y \nabla X-O\}$, then $x < X-O$ so that $U \in I(x)$. Also, if $y \notin V$, then $y \nabla X-O = U$, hence $u(U) \subset V$. This completes the proof.

<u>Proposition 1.3.3.</u> The following are equivalent

i) If $x \neq y$, $\exists\, V_1 \in I(x)$ and $V_2 \in I(y)$ s.t. $V_1 \cap V_2 = \varphi$

ii) If $x \neq y$, $\exists\, G$ s.t. $x \nabla G$ and $y \nabla X-G$.

Assume i) and suppose $x \neq y$. Let V_1, V_2 be given by i), then $G = X-V_1$ is s.t. $x \nabla G$, since $x \notin u(X-V_1)$, and $y \nabla X-G$, since $y \notin u(V_1) = u(X-G)$. Thus ii) holds.

Conversely, if ii) holds and $x \neq y$, then let $V_1 = X-G$ and $V_2 = G$. It follows that $x < V_1$ and $y < V_2$ and that $V_1 \cap V_2 = \varphi$. Hence i).

We note that in the three preceding propositions the requirement - $u(\{x\}) \subset V$ for all $V \in I(x)$ - is superfluous, since it is satisfied for any u for which i) holds.

We recall that u is topological if

U1 $u(\varphi) = \varphi$

$$U2 \quad A \subset u(A)$$

$$U3^* \quad u(A \cup B) = u(A) \cup u(B)$$

$$U4 \quad u(u(A)) = u(A)$$

As remarked earlier, u is monotone if u satisfies U2 and U3*, and so $\nabla[u]$ is meaningful.

<u>Proposition 1.3.4.</u> If u is a generalised closure and $\nabla = \nabla[u]$, then the following are equivalent

i) $u(A \cup B) = u(A) \cup u(B)$

ii) $(A \cup B)\nabla C$ iff $A\nabla C$ and $B\nabla C$

We shall prove that if ∇ is any separation inducing u, then ii) implies i). Assume ii), we have only to prove that $u(A \cup B) \subset u(A) \cup u(B)$. If $x \notin u(A) \cup u(B)$, then $x\nabla A$ and $x\nabla B$ and so $x\nabla(A \cup B)$, as required. Conversely, assume i). S4 implies that $A\nabla C$ and $B\nabla C$ if $(A \cup B)\nabla C$; to prove the reverse implication, let $A\nabla C$ and $B\nabla C$. Then $(u(A) \cap C) \cup (A \cap u(C)) = \varphi$ and $(u(B) \cap C) \cup (B \cap u(C)) = \varphi$, so that $[(u(A) \cup u(B)) \cap C] \cup [(A \cup B) \cap u(C)] = \varphi$. Hence $[u(A \cup B) \cap C] \cup [(A \cup B) \cap u(C)] = \varphi$, i.e. $(A \cup B)\nabla C$.

<u>Proposition 1.3.5.</u> If u is a generalised closure and $\nabla = \nabla[u]$, then the following are equivalent

i) $u(u(A)) = u(A)$

ii) $x\nabla A$ implies $x < \nabla[A]$, where $\nabla[A] = \{y \mid y\nabla A\}$

As above, ii) implies i) even when ∇ is any separation inducing u. The proof of 1.3.5. will be omitted.

1.4 Separation and Completely Regular Spaces

We shall characterise completely regular spaces in terms of subordination . This provides an internal description of such spaces as opposed to an external description in terms of real valued,continuous functions . The two points of view are , of course , related : In this section we show that every c-subordination determines a ring $C_<$ of real valued,bounded,<-continuous functions , and these functions induce <,since $A < B$ iff $\exists\ f \in C_<$, $f:X\to[0,1]$ and $A \subset f^{-1}[0]$, $X-B \subset f^{-1}[1]$. In chapter 4 , we complete this result by showing that certain subrings R of the ring of real valued,bounded,continuous functions induce a c-subordination < , $A < B$ being defined by the condition above. Furthermore R is $C_<$, where < is the induced subordination .

This section differs from the previous ones in that it considers a family of subordinations inducing a given topology rather than the finest such subordination .

Definition 1.4.1. A relation < on p[X] is a c-subordination or a compact subordination , if

S1 $\varphi < A$, for all $A \subset X$

S2 If $A < B$, then $X-B < X-A$

S3 If $A < B$, then $A \subset B$

and

S4 If $A \subset B < C \subset D$, then $A < D$

S5 If $A_1, A_2 < B$, then $A_1 \cup A_2 < B$

S6 If $A < C$, $\exists$ B s.t. $A < B < C$

S7 If $x \neq y$, $\exists U,V$ s.t. $x < U$, $y < V$ and $U \cap V = \varphi$

We note that S1 - S4 are the axioms for a separation expressed in terms of $<$ instead of ∇ . S7 can be replaced by : $x \neq y$ implies $x \nabla y$, i.e. we require the separation to be T_1 .

We recall that $<$ induces a neighbourhood function I , by $V \in I(x)$ if $x < V$.

<u>Proposition 1.4.1.</u> A c-subordination induces a topological neighbourhood function .

We have only to verify that $x \nabla A$ implies $x \nabla \{y \mid y \delta A\}$. If $x \nabla A$, then $\exists$ C s.t. $x \nabla C$ and $X - C \nabla A$, so that $y \in X - C$ implies $y \nabla A$. Hence $\{y \mid y \delta A\} \subset C$. It follows that $x \nabla \{y \mid y \delta A\}$, as required .

The following propositions concern the restriction $<_0$, of $<$, to the family of open sets induced by $<$.

<u>Proposition 1.4.2.</u> If I,u are induced by a c-subordination $<$, then i) If $A < B$, then $X - u(B) < X - u(A)$

ii) If $A < C$, $\exists$ B, open, s.t. $A < B \subset B^- < C$.

iii) If $V \in I(x)$, $\exists$ U,open,s.t. $U \in I(x)$ and $U < V$.

iv) If $A < B_i$ i=1,2. , then $A < B_1 \cap B_2$

Proposition 1.2.6. is essential here .

i) If $A < C$, $\exists$ B s.t. $A < B < C$. Hence $A \subset A^- \subset B < C$. Now $A^- < B$, and so $X-B^- < X-A^-$ since $X-B^- \subset X-B < X-A^-$.

ii) If $A < C$,$\exists$ B_i i=1,2,3. s.t. $A < B_1 < B_2 < B_3 < C$. Now $B_1 \subset i(B_2)$ and $[i(B_2)]^- \subset B_2^- \subset B_3$. Hence $B = i(B_2)$ satisfies ii) .

iii) If $V \in I(x)$, then $x < V$, so that $\exists$ U,open, s.t. $x < U < V$. iii) follows since $x < U$ implies $U \in I(x)$.

iv) If $A < B_i$, then $X-B_i < X-A$. Hence $X-(B_1 \cap B_2) = (X-B_1) \cup (X-B_2) < X-A$, so that $A < B_1 \cap B_2$.

Let $<_0$ be a relation defined in the family of open sets of a hausdorff topological space . $<_0$ is called a c_0 subordination if

S1* $\varphi < A$, A any open set .

S2* If $A < B$, then $X-B^- < X-A^-$.

S3* If $A < B$, then $A \subset B$.

S4* If $A \subset B < C \subset D$, then $A < D$.

S5* If $A_i < B < C_i$ i=1,2. , then $A_1 \cup A_2 < B < C_1 \cap C_2$.

S6* If $A < C$, then $\exists$ B s.t. $A < B < C$.

S7* If $x \in V$,V open, then $\exists$ U s.t. $x \in U < V$.

Proposition 1.4.2. shows that any c-subordination is a c_0 - -subordination on the family of open sets that it induces .

Definition 1.4.2. If $<_0$ is a c_0 subordination on (X,u),define $<$ by

$$A < B \text{ if } \exists\ O_1, O_2 \text{ s.t. } A \subset O_1 <_0 O_2 \subset B .$$

Proposition 1.4.3. $<$, defined in 1.4.2. , is a c-subordination and induces the topology on X . Furthermore $<$,restricted to the family of open sets is $<_0$.

S2 is the only axiom for $<$ that is not obvious . To prove it,assume $A < B$. Then $\exists\, O_i$ $i=1,2,3.$,open,s.t. $A \subset O_1 <_0 O_2 <_0 O_3 \subset B$. Hence $X-B \subset X-O_3 \subset X-O_2^- <_0 X-O_1^- \subset X-O_1 \subset X-A$, since i) of proposition 1.4.2. still holds and $G <_0 H$ implies $G^- \subset H$. thus $X-B < X-A$, as required .

We now show that $<$ induces the topology on X .
Let V be open , then for each $x \in V$,$\exists$ O,open,s.t. $x \in O < V$, by S7*. Hence $x < V$, and so V is open in the topology induced by $<$. Conversely , let V be open in the topology induced by $<$. Then $x < V$ for all $x \in V$. Hence $\exists\, O_{ix}$ $i=1,2.$, open, s.t. $x \in O_{1x} <_0 O_{2x} \subset V$, and so $V = \bigcup_{x \in V} O_{1x}$ is open .

If $G <_0 H$, then $G < H$ by definition of $<$. Conversely , if $G < H$, G and H open , then $\exists\, O_1, O_2$, open, s.t. $G \subset O_1 <_0 O_2 \subset H$. Hence $G <_0 H$. This proves that the restriction of $<$ to the family of open sets is $<_0$.

Proposition 1.4.4. If $<_0$ is the restriction of $<$ to the family of open sets that it induces , then $<^* = <$, where $<^*$ is the subordination induced by $<_0$.

If $A <^* B$, then $\exists\, O_i$ $i=1,2.$,open,s.t. $A \subset O_1 <_0 O_2 \subset B$, i.e.

$A \subset O_1 < O_2 \subset B$. Hence $A < B$.
Conversely , if $A < B$, by proposition 1.4.2. ii) , $\exists\, O_i$ $i=1,2.$, s.t. $A < O_1 < O_2 < B$ and both O_i's are open . Hence $A <^* B$.

We have seen that a c-subordination gives rise to a topology on X . We show that this topology is completely regular. In fact slightly more is proved :

Proposition 1.4.6. $A < B$ iff $\exists\, f:X\to[0,1]$, s.t. f is <-continuous and $A \subset f^{-1}[o]$ and $X-B \subset f^{-1}[1]$.

Corollary If A is closed and $x \notin A$, $\exists f:X\to[0,1]$, continuous, s.t. $f(x) = 0$ and $A \subset f^{-1}[1]$. This is the usual definition of complete regularity .

Before proving proposition 1.4.6. , we show how continuity and <-continuity are related .

Definition 1.4.3. Let $f:(X,\nabla)\to(X^*,\nabla^*)$ be a map from a separation space into another . f is ∇-continuous if $f^{-1}[A]\nabla f^{-1}[B]$ whenever $A\nabla^* B$.
Also , $f:(X,u)\to(X^*,u^*)$ is u-continuous if $u(f^{-1}[A]) \subset f^{-1}[u^*(A)]$, $A \subset X^*$.

Proposition 1.4.7. Every ∇-continuous function is u-continuous with respect to $u_1 = u[\nabla_1]$ and $u_2 = u[\nabla_2]$.

To show that $u_1(f^{-1}[A]) \subset f^{-1}[u_2(A)]$, let $x \notin f^{-1}[u_2(A)]$. Then $f(x) \notin u_2(A)$ and hence $f(x)\nabla_2 A$. Now f is ∇-continuous, so $f^{-1}[f(x)]\ \nabla_1 f^{-1}[A]$, and so $x\nabla_1 f^{-1}[A]$. Thus $x \notin u_1(f^{-1}[A])$, and the proof is complete.

Conversely,

<u>Proposition 1.4.8.</u> If $f:(X_1,\mu_1)\to(X_2,u_2)$ is u-continuous, then f is ∇-continuous with respect to $\nabla_i=\nabla[u_i]$, $i = 1,2.$ •

Suppose $A\nabla_2 B$, then $(A \cap u_2(B)) \cup (B \cap u_2(A)) = \varphi$. Hence $(f^{-1}[A] \cap u_1(f^{-1}[B])) \cup (f^{-1}[B] \cap u_1(f^{-1}[A])) \subset (f^{-1}[A] \cap f^{-1}$ $f^{-1}[u_2(B)]) \cup (f^{-1}[B] \cap f^{-1}[u_2(A)]) = f^{-1}[\varphi] = \varphi$. It follows that $f^{-1}[A]\nabla_1 f^{-1}[B]$ •

However, if $f:(X,u)\to(X^*,u^*)$ is u-continuous, it does not follow that f is ∇-continuous with respect to ∇_1 and ∇_2. Where the ∇_i's are separations inducing u_i, but not the finest such separations. This is illustrated by the following example :

Let ω be the set of integers. Define $A\nabla B$ if $A \cap B = \varphi$ and not both sets are infinite. This separation induces the discrete topology on ω, and the function $f:\omega\to[0,1]$ defined by $f(n)=(-1)^n+1$ is continuous but not ∇-continuous since 0 and 2 are far in $[0,1]$ but $f^{-1}[0]\nabla f^{-1}[2]$ is false •

Thus ∇-continuity is a stronger concept than continuity.

We now return to proposition 1.4.6.

The proof of necessity rests on a construction due to Urysohn.

Let $G < H$. Then $\exists\, H_{\frac{1}{2}}$ s.t. $G < H_{\frac{1}{2}} < (H_{\frac{1}{2}})^- < H$. Also, $G < H_{\frac{1}{2}}$ and $(H_{\frac{1}{2}})^- < H$. Hence $\exists\, H_{\frac{1}{4}}$ and $H_{\frac{3}{4}}$ s.t. $G < H_{\frac{1}{4}} \subset (H_{\frac{1}{4}})^- < H_{\frac{1}{2}} \subset (H_{\frac{1}{2}})^- < H_{\frac{3}{4}} \subset (H_{\frac{3}{4}})^- < H$. Proceeding in this way, for each dyadic integer $t < 1$ ($t = \frac{m}{2^n}$ n,m are integers), we have a set H_t s.t. $G < H_{t_1} \subset H_{t_1}^- < H_{t_2} \subset H_{t_2}^- < H$ whenever $t_1 < t_2$.

For $x \in X$, let $f(x) = \inf\{t \mid x \in H_t\}$ if $x \in H_t$ for some t, and $f(x) = 1$ otherwise.

To prove that f is ∇-continuous, we note that two sets of real numbers A and B are far if $d(A,B) > 0$, where d is the usual metric for the reals. Thus $A < B \subset [0,1]$ iff $A^- \subset i(B)$, and so f is ∇-continuous if $f^{-1}[A] < f^{-1}[B]$ for all $A = A^- \subset i(B) = B$. This can be refined further.

If $A \subset B$, A closed and B open, then $\exists I_i, U_i$ $i=1,\ldots,n.$, s.t. $I_i \subset U_i$, I_i is a closed interval and U_i is an open interval, for each i, all having dyadic integers as end points, and s.t. $A \subset \bigcup_{i=1}^{n} I_i \subset \bigcup_{i=1}^{n} U_i \subset B$. This is a consequence of the Heine-Borel theorem and the fact that the dyadic integers are dense in the real line. Thus f is ∇-continuous if $f^{-1}[A] < f^{-1}[B]$, where $A = [s,t]$, $B = (r,u)$ and $r < s \leq t < u$ are dyadic integers. We now prove that this is the case for the function f defined above.

Suppose $[r,s] \subset (q,t) \subset [0,1]$. Let u,v be s.t. $q<u<r\leqslant s<v<t$ and $0<u<v<1$. Now $f^{-1}[[r,s]] \subset f^{-1}[(u,v)] = \{x| \ u<f(x)\} \cap \{x| f(x)<v\} \subset \{x| x \notin H_u \} \cap \{x| \ x \in H_v \} = X-H_u \cap H_v$. If a,b are dyadic integers s.t. $q<a<u<v<b<t$, then $X-H_a \cap H_b \subset \{x| \ a\leqslant f(x) \} \cap \{x| \ f(x) \leqslant b \} \subset f^{-1}[(q,t)]$. But $H_v < H_b$ and $H_b < H_u$, hence $X-H_u \cap H_v < X-H_a \cap H_b$, since $X-H_u < X-H_a$. Thus $f^{-1}[[r,s]] < f^{-1}[(q,t)]$,as required .

To prove the sufficient condition,assume $A,B \subset X$ s.t. $\exists$ f: $:X\to[0,1]$, f <-continuous , and $A \subset f^{-1}[0]$, $X-B \subset f^{-1}[1]$. Then $A\nabla X-B$, since 0 is far from 1 and f is <-continuous . Thus $A < B$, as required .

The following result completes our introduction to Chapter 2 .

<u>Proposition 1.4.9.</u> Let X be a hausdorff space with a c_0-subordination . X is compact iff every $<_0$-ultrafilter is convergent , i.e. has a non empty intersection .

The definition of a $<_0$-ultrafilter appears in Chapter 2 .

Note that $G^- = \cap\{V| \ G < V \}$, hence $\cap\{G| \ G \in \xi \} = \cap\{G^-| \ G \in \xi \}$, where ξ is a $<_0$ultrafilter . Necessity follows from compactness of X .

Conversely , if $\{F_\alpha\}$ is a filter of closed sets , then $\{V| \ \exists \ 0,\alpha$ s.t. $F_\alpha \subset 0 <_0 V \}$ is a $<_0$- filter . The result now follows since every $<_0$-filter can be included in a $<_0$-ultrafilter, which , by assumption, has a smaller and non-empty intersection .

Notes

Generalised closure was introduced by Hammer [H_1]. We show that the resulting spaces are the same as Fréchet's V-spaces [F_1]. The axioms S1-S3 for separation were given by Wallace [W_1] as axioms for a weak separation. They arise naturally when trying to characterise generalised closure spaces in terms of a notion of separation. Some of the propositions given here relating ∇ and u are similar to theorems in [W_1]. Axioms for separation were also considered by Krishna-Murti [KM_1] and Szymanski [Sz_1]. A more complete, though slightly different, treatment is found in Čech's book [C_1].

The axioms for a c-subordination are essentially the same as those given by Effremovič [E_1], who also proved that the topology induced is completely regular. Here we prove slightly more.

For technical reasons, it is convenient to restrict a subordination to the family of open sets that it induces, thus the notion of a c_0-subordination. The axioms for a c_0-subordination may be regarded as supplementing those given by Freudenthal [Fr_1]. With the set of axioms given in the text, one can establish a one to one correspondence between T_2 compactifications and c_0-subordinations. This is not possible with Freudenthal's axioms as remarked by Aleksandrov [A_1].

In the following chapters we shall use the term c-subordination or compact subordination for both a c-subordination and a c_0-subordination.

Chapter 2.

Subordinations and Compactifications

A subordination induces a generalised topology where V is a neighbourhood of x if $x < V$. For a compact subordination the neighbourhood system of x is a $<$-ultrafilter. The point of view of this section is that all $<$-ultrafilters have the same right to be called neighbourhoods of points. Thus, ideal elements are introduced that serve as points having $<$-ultrafilters as neighbourhood systems. This can be described as completing X with respect to a subordination. One would expect the resulting space to have the property that every $<$-ultrafilter converges, and, hence be compact. Conversely a compactification C induces, in a natural way, a subordination which gives rise to C.

Henceforth we shall refer to a compact subordination simply as a subordination.

2.2. The Compactification Induced by a Subordination

Definition 2.2.1. A family ξ of open subsets of X s.t.

i) $\xi \neq \varphi$

ii) $\varphi \notin \xi$

iii) $G_1, G_2 \in \xi \Rightarrow G_1 \cap G_2 \in \xi$

iv) $G_1 \in \xi \Rightarrow \exists\ G_2$ s.t. $G_1 < G_2$

is a $<$-filter. A $<$-ultrafilter is a $<$ filter which is maximal with respect to set inclusion.

We now use Zorn's lemma to show that

<u>Proposition 2.2.1.</u> Every <-filter can be included in a <-ultrafilter.

If ξ_0 is a <-filter and C a chain of <-filters including ξ_0, then $\cup \{ \sigma \mid \sigma \in C \}$ is a <-filter including ξ_0. By Zorn's lemma the set of <-filters including ξ_0 has a maximal element.

<u>Definition 2.2.2.</u> $<X = \{ \xi \mid \xi \text{ is a } <\text{-ultrafilter} \}$.

For any open G , $G^* = \{ \xi \mid G \in \xi \}$.

<u>Proposition 2.2.2.</u> $(G_1 \cap G_2)^* = G_1^* \cap G_2^*$

If $\xi \in G_1^* \cap G_2^*$, then G_1 and $G_2 \in \xi$, hence $\exists\, G_3 , G_4 \in \xi$ s.t. $G_3 < G_1$ and $G_4 < G_2$ so that $G_1 \cap G_2 \in \xi$; since $G_3 \cap G_4 \in \xi$. Thus $G_1^* \cap G_2^* \subset (G_1 \cap G_2)^*$. Conversely, if $\xi \in (G_1 \cap G_2)^*$, then $G_1 \cap G_2 \in \xi$, now $G_1 \cap G_2 \in G_j$, $j = 1,2$, hence G_1 and $G_2 \in \xi$, so that $\xi \in G_1^* \cap G_2^*$. This completes the proof .

<u>Corollary</u> The G^*'s form a base [K_1] for open sets in $<X$.

Whenever we refer to $<X$ we mean $<X$ with the topology described above .

The following lemma is useful in establishing that $<X$ is compact and hausdorff .

<u>Lemma 2.2.3.</u> Let ξ be a <-ultrafilter , G open s.t. for any $H > G$, $H \cap A \neq \varphi$ for all $A \in \xi$; then any such $H \in \xi$.

In particular, if $G \cap A \neq \varphi$ for all $A \in \xi$, then $G < H$ implies $H \in \xi$.

Let $\zeta_0 = \{ A \cap H \mid A \in \xi \text{ and } H > G \}$. Note that $\xi \subset \zeta_0$ since $G < X$. Hence, if ζ_0 is a <-filter, then $\xi = \zeta_0$. Further-more $H \in \zeta_0$ if $H > G$, since $H = H \cap X$ and $X \in \xi$. To complete the proof we have only to show that ζ_0 is a <-filter.

i) $X \in \xi$ and $G < X$, hence $X \in \zeta_0$ and so $\zeta_0 \neq \varphi$.

ii) $A \cap H \neq \varphi$ for any $A \in \xi$ and $H > G$, hence $\varphi \notin \zeta_0$.

iii) If $A_j \cap H_j \in \zeta_0$ for $j = 1,\ldots,n.$, then $\bigcap_{j=1}^{n} (A_j \cap H_j) = \bigcap_{j=1}^{n} A_j \cap \bigcap_{j=1}^{n} H_j \in \zeta_0$, since $G < \bigcap_{j=1}^{n} H_j$ and ξ is a <-filter.

iv) If $A \cap H \in \zeta_0$, let $A_1 \in \xi$ and H_1 be s.t. $A_1 < A$ and $G < H_1 < H$. Then $A_1 \cap H_1 \in \zeta_0$ and $A_1 \cap H_1 < A \cap H$, so ζ_0 is a <- filter.

<u>Corollary 1.</u> If $G < H$ then $G^{*-} \subset H^*$, where $-$ denotes closure in $<X$.

If $\xi \in G^{*-}$ then $G^* \cap A^* \neq \varphi$, hence $G \cap A \neq \varphi$, for all $A \in \xi$. Hence $G < H$ implies $H \in \xi$ i.e. $\xi \in H^*$, as required.

We shall prove later that $G^{*-} = \cap \{ H^* \mid H > G \}$.

<u>Corollary 2.</u> If $G_j < H_j$ for $j=1,\ldots,n.$ and $\bigcup_{j=1}^{n} G_j = X$, then $\bigcup_{j=1}^{n} H_j^* = <X$.

By corollary 1, $G_j^{*-} \subset H_j^*$ for each j, so $\left(\bigcup_{j=1}^{n} G_j^{*-} \right) \subset \bigcup_{j=1}^{n} H_j^*$.

Let $\xi \in {<}X$, then for any $A \in \xi$ we have $A \cap G_j \neq \chi$ for some j, since $A \neq \varphi$, hence $A^* \cap G_j^* \neq \varphi$ and so $A^* \cap \bigcup_{j=1}^{n} G_j^* \neq \varphi$, thus $\xi \in \bigcup_{j=1}^{n} G_j^{*-} = (\bigcup_{j=1}^{n} G_j^*)^-$, this completes the proof .

<u>Proposition 2.2.4.</u> ${<}X$ is a hausdorff space .

Let $\xi \neq \zeta$ be elements of ${<}X$, $\exists\, A \in \xi - \zeta$. If $A_1 \in \xi$ is s.t. $A_1 < A$, then $A_1^{*-} \subset A^*$. Now $\zeta \notin A^*$ hence $\zeta \notin A_1^{*-}$, thus A_1^* and ${<}X - A_1^{*-}$ are disjoint open neighbourhoods of ξ and ζ, respec - tively .

<u>Proposition 2.2.5.</u> ${<}X$ is compact

We argue by contradiction. Suppose $\{ C_j^* \}$ is a family of basic open sets covering ${<}X$ such that no finite subfamily covers ${<}X$. Define $\{ B_i^* \mid B_i < C_j \text{ for some } j \} = \Omega$ .We verify that Ω is a cover and then exhibit an element of ${<}X$ which is not contained in any member of Ω , this is a contradiction and the compactness of ${<}X$ is established.

If $\xi \in {<}X$ then $\xi \in C_j^*$ for some j , so $C_j \in \xi$; hence $\exists\, B \in \xi$ s.t. $B < C_j$, it follows that $B^* \in \Omega$, also $\xi \in B^*$ hence Ω is an open cover of ${<}X$.

Let $\zeta_0 = \{ X\text{-cl } B_j \mid B_j^* \in \Omega \}$.

i) $\varphi \in \Omega$ so $X \in \zeta_0$, this shows that $\zeta_0 \neq \varphi$

ii) Suppose $X - \text{cl } B_j \in \zeta_0$ for $j=1,\dots,n$. and $\bigcap_{j=1}^{n} X - \text{Cl } B_j = \varphi$, then $\text{cl } (\bigcup_{j=1}^{n} B_j) = X$. Now the subordination is com- pact so $\exists\, D_j$ s.t. $B_j < D_j < C_j$, where $C_j^* \in$ original cover , and

$cl\ B_j \subset D_j$. We have , then , $\bigcup_{j=1}^{n} D_j = X$, hence $\bigcup_{j=1}^{n} C_j^* = {<}X$ which contradicts our choice of C_j^*'s .

iii) Given $B_j^* \in \Omega$, to find $B^* \in \Omega$ s.t. $X - cl\ B < X - cl\ B_j$, choose B s.t. $B_j < B < C_j$, where C_j is s.t. $B_j < C_j$.

The above shows that the finite intersections of members of ζ_0 constitute a <-filter . Let ζ be a <-ultrafilter including this <-filter , then $\zeta \notin B_j^*$ for any $B_j^* \in \Omega$ since $B_j \cap (X - cl\ B_j) = \varphi$ and $X - cl\ B_j \in \zeta$.

We now consider the embedding of X in <X .

<u>Definition 2.2.3.</u> $\Phi(x) = \{ O \mid O \text{ is open in } X \text{ and } x \in O \}$

<u>Proposition 2.2.6.</u> $\Phi(x)$ is a <-ultrafilter.

It is clear that $\Phi(x)$ is a <-filter. To show that it is maximal , assume $\exists\ \xi \in {<}X$ s.t. $\Phi(x) \subsetneqq \xi$. then $\exists\ G \in \xi$ s.t. $x \notin G$. Now $\exists\ G_1 \in \xi$ s.t. $G_1 < G$, hence $cl\ G_1 \subset G$ and so $x \notin cl G_1$ But this implies $X - cl\ G_1 \in \Phi(x) \subset \xi$, impossible since $G_1 \in \xi$ and $G_1 \cap X - cl\ G_1 = \varphi$.

<u>Proposition 2.2.7.</u> $\Phi : X \to {<}X$ is one-to-one .

Suppose $x \neq y$ are points of X. Now $X - \{ y \}$ is open and contains x , so $\exists$ O , open , s.t. $x \in O < X - \{ y \}$; hence $cl O \subset X - \{ y \}$ so $X - cl\ O \in \Phi(y)$. Noting that $O \in \Phi(x)$ we conclude $\Phi(x) \neq \Phi(y)$.

Proposition 2.2.8. For any open set G , $\Phi[G] = G^* \cap \Phi[X]$

For all $x \in X$, $\Phi(x) \in G^*$ iff $G \in \Phi(x)$ iff $x \in G$.

Corollary 1. Φ is an open map .

Corollary 2. Φ is continuous .

Note $\Phi^{-1}[G^*] = \Phi^{-1}[G^* \cap \Phi[X]] = \Phi^{-1}[\Phi[G]] = G$ by propositions 2.2. 7 and 8 . Continuity follows from the fact that the G^*'s form a base .

Corollary 3. $\Phi[X]$ is dense in <X .

Since any non empty basic open set G^* meets $\Phi[X]$ in $\Phi[G]$ which is not empty .

The following remarks will be useful later .

Proposition2.2.9. $(\bigcup_j G_j^*) \cap \Phi[X] = (\bigcup_j G_j)^* \cap \Phi[X]$

R.h.s. $= \Phi[\bigcup_j G_j] = \bigcup_j \Phi[G_j] = \bigcup_j (G_j^* \cap \Phi[X]) =$ l.h.s.

Corollary G^* is the largest open set meeting $\Phi[X]$ in $\Phi[G]$.

Proposition 2.2.10. For any open G , $G^{*-} = (\Phi[G])^-$

$\Phi[G]^- = (G^* \cap \Phi[X])^- = G^{*-}$ since G^* is open and $\Phi[X]$ is dense .

2.3. The Subordination Induced by a Compactification

Given a subordination $<$, we remarked that $G < H$ implies $G^{*-} \subset H^*$. Identifying X and $\Phi[X]$, we have $G < H$ implies $G^- \cap <X - H^* = \varphi$, where H^* is the largest open set in $<X$ that meets X in H.

When a compactification C is given, it is natural to look for a subordination inducing C. In view of the above remarks, a reasonable definition for $<$ is

Definition 2.3.1. Let C be a compactification of X. $G_1 < G_2$ if $G_1 \cap C - G_2^* = \varphi$, where G_1 and G_2 are open in X and G^* is the largest open subset of C intersecting X in G.

Proposition 2. 3.1. i) $A^* \cup B^* \subset (A \cup B)^*$

ii) $A \subset B$ implies $A^* \subset B^*$

iii) $A^* \cap B^* = (A \cap B)^*$

iv) $A^{*-} = A^-$

For any A, B open in X.

i) $(A^* \cup B^*) \cap X = (A^* \cap X) \cup (B^* \cap X) = A \cup B$, by definition it follows that i) is true.

ii) If $A \subset B$, then $A^* \subset A^* \cup B^* \subset (A \cup B)^* = B^*$.

iii) $(A^* \cap B^*) \cap X = A \cap B$, also $(A \cap B)^* \cap X = A \cap B \subset A$ Similarly, $(A \cap B)^* \subset X \subset B$, iii) follows.

iv) $A^{*-} = (A^* \cap X)^-$ since A^* is open and X is dense, now $(A^* \cap X)^- = A^-$, this completes the proof.

We can now prove that $<$ is a compact subordination.

<u>Proposition 2.3.2.</u> $<$, defined above, is a c-subordination .

S1* $\varphi < A$ is obvious .

S2* Let $A < B$, then $A^- \subset B^* \subset B^{*-} = B^-$. Hence $C-B^- \subset C-B^*$, and so $(C-B^-)^- \subset C-B^* \subset C-A^-$. Now $C-A^- \subset (X-clA)^*$, since $(C-A^-) \cap X = X-clA$. Hence $(X-clB)^- \subset (C-B^-)^- \subset (X-clA)^*$, so that $X-clB < X-clA$.

S3* If $A < B$, then $A^- \subset B^*$. Hence $A \subset A^- \cap X \subset B^* \cap X = B$.

S4* If $A \subset B < C \subset D$, then $A^- \subset B^- \subset C^* \subset D^*$. Hence $A < D$.

S5* If $A_i < B < C_i$, then $A_i^- \subset B^*$ and $B^- \subset C_i^*$. It follows that $(A_1 \cup A_2)^- = A_1^- \cup A_2^- \subset B^*$, and so $(A_1 \cup A_2) < B$. Also $B^- \subset C_1^* \cap C_2^* = (C_1 \cap C_2)^*$, hence $B < (C_1 \cap C_2)$.

S6* If $A < B$, then $A^- \subset B^*$. C is compact T_2 , hence normal [K_1], so $\exists$ O, open, s.t. $A^- \subset O \subset O^- \subset B^*$. Now $O \subset (O \cap X)^*$ and $(O \cap X)^- = O^-$, hence $A^- \subset (O \cap X)^* \subset (O \cap X)^{*-} = (O \cap X)^- = O^- \subset B^*$ .Hence $A < O \cap X < B$, and $O \cap X$ is open in X .

S7* If O is openin X and $x \in O$, then $x \in O^*$. Now **C** is regular, so $\exists$ V , open, s.t. $x \in V \subset V^- \subset O^*$. Thus $x \in V \cap X$, and $(X \cap V)^- = V^- \subset O^*$, so that $X \cap V < O$, as required .

2.4. <u>The correspondence between $<$'s and compactifications</u>

To show that every compactification arises from its induced subordination , we define a map $\pi: <X \to C$, C the given compactification , s.t. π is continuous, one to one, and onto, and hence a homeomorphism since all spaces involved are compact and T_2 .

To define π , we associate with each $<$-ultrafilter its unique adherence point , this is justified by

Proposition 2.4.1. If $\xi \in {<}X$, then $\cap\{ G^- \mid G \in \xi \} \neq \varphi$,
where $-$ = closure in C.

This follows immediately from compactness of C .

Proposition 2.4.2. Let $c \in \cap\{ G^- \mid G \in \xi \}$. $V \cap X \in \xi$
when V is open in C and $c \in V$.

Let U be open in C s.t. $c \in U \subset U^- \subset V$, then $(U \cap X)^- = U^- \subset V \subset (V \cap X)^*$, so $U \cap X < V \cap X$. Furthermore $(U \cap X) \cap \cap G = U \cap G \neq \varphi$ for all $G \in \xi$, hence $V \cap X \in \xi$, by lemma2.2.3.

Corollary If c , $d \in \cap\{ G^- \mid G \in \xi \}$ then $c = d$ .Assume $c \neq d$, C is a hausdorff space so $\exists$ U , V , open , s.t. $c \in U$, $d \in V$ and $U \cap V = \varphi$. By above $U \cap X$, $V \cap X \in \xi$; this is impossible.

Definition 2.4.1. For any $\xi \in {<}X$, let $\pi(\xi) = c$ where $\{c\} = \cap\{ G^- \mid G \in \xi \}$.

Proposition 2.4.3. π is one to one .

Let $\xi \neq \zeta \in {<}X$, then $\exists$ G_1^* , G_2^* , basic open subsets of ${<}X$, s.t. $\xi \in G_1^*$, $\zeta \in G_2^*$ and $G_1^* \cap G_2^* = \varphi$. Now $\pi(\xi) \in G_1$ and $\pi(\zeta) \in G_2$, hence $\pi(\xi) \neq \pi(\zeta)$ since $G_1 \cap G_2 = \varphi$.

Proposition 2.4.4. $\pi[\Phi(x)] = x$ for all $x \in X$.

$\pi[\Phi(x)] \in \cap\{ O^- \mid O$ open and $x \in O \}$; clearly , x belongs to the same set . By definition of π , we have $\pi[\Phi(x)] = x$.

To prove continuity of π , we use the following lemmas

<u>Lemma 2.4.5.</u> $\{ O^+ \mid O \text{ open in } X \}$ is a base for the topology on C , where O^+ is the largest open subset of C s.t. $O^+ \cap X = O$.

Note i) $O_1^+ \cap O_2^+ = (O_1 \cap O_2)^+$ by proposition 2.3.1.

ii) If V is an open neighbourhood of c , let U be open s.t. $c \in U \subset U^- \subset V$. Again by 2.3.1. , we have $U \subset (U \cap X)^+ \subset (U \cap X)^{+-} = (U \cap X)^- = U^- \subset V$, this completes the proof .

<u>Lemma 2.4.6 .</u> If $\{ F_\upsilon \}$ is a filter $[K_1]$ of closed subsets of a compact space C , and G is open and $\cap\{ F_\upsilon \} \subset G$, then $F_\upsilon \subset G$ for some υ .

We argue by contradiction . If $F_\upsilon \not\subset G$ for all υ , then $\{ F_\upsilon \cap (C-G) \}$ is a filter of closed sets, and so $\exists\, c \in \cap\{ F_\upsilon \cap (C-G) \} = \cap\{F_\upsilon\} \cap (C-G)$. This is impossible .

<u>Proposition 2.4.7 .</u> π is continuous .

By 2.45 it is sufficient to consider the sets $\pi^{-1}[O^+]$, O , open in X . $\pi^{-1}[O^+] = \{\xi \mid \cap \{ G^- \mid G \in \xi \} \subset O^+ \} = \{ \xi \mid \exists\, G \in \xi \text{ s.t. } G^- \subset O^+ \}$,by 2.4 6 . So $\pi^{-1}[O^+] = \{ \xi \mid \exists\, G \in \xi \text{ s.t. } G < O \} = \{ \xi \mid O \in \xi \} = O^*$, a basic open subset of $<X$.

<u>Proposition 2.4.8 .</u> π is onto .

$\pi[<X] \supset \pi[\Phi[X]] = X$. Also , $\pi[<X]$ is compact , hence closed in C , thus $\pi[<X] \supset X^{-} = C$ and the proof is complete .

As remarked above , we can now state

<u>Proposition 2.4.9 .</u> π is a homeomorphism leaving the points of X fixed in the sense that $\pi(\Phi(x)) = x$,for all $x \in X$.

A subordination on X determines a compactification $<X$ which ,in turn , induces a subordination on $\Phi[X]$. It is natural to ask how the two are related . Proposition 2.4.1o. shows that they are essentially the same .

<u>Proposition 2.4.10.</u> If $<$ is a subordination on X and $\prec$ the subordination induced by $<X$, then $G < H$ iff $\Phi[G] \prec \Phi[H]$.

We first prove sufficiency . Suppose $\Phi[G] \prec \Phi[H]$, then $(\Phi[G])^{-} \subset (\Phi[H])^{*}$, since , by the corollary to 2.2.9. , $(\Phi[H])^{*}$ is the largest open set in $<X$ meeting $\Phi[X]$ in $\Phi[H]$. Also $(\Phi[H])^{*} = H^{*}$, so that , for any $\xi \in (\Phi[G])^{-}$, $\xi \in H^{*}$. For any such ξ , $\exists\, A \in \xi$ s.t. $A < H$. Now $\xi \in A^{*}$ and $(\Phi[G])^{-}$ is compact , hence $\exists\, A_j$, $j = 1,\ldots,n.$ s.t. $(\Phi[G])^{-} \subset \bigcup_{j=1}^{n} A_j^{*}$ and $A_j < H$ for each j . Thus $\Phi[G] \subset \bigcup_{j=1}^{n} A_j^{*} \cap \Phi[X] = \Phi[\bigcup_{j=1}^{n} A_j]$, and , since Φ is one to one , we have $G \subset \bigcup_{j=1}^{n} A_j$. But $A_j < H$ for all j , hence $G < H$, as required .

Necessity - If $G < H$, then $(G^*)^- \subset H^*$, so $(\Phi[G])^- = (G^*)^- \subset [\Phi[H]]^* = H^*$. This shows that $\Phi[G] \prec \Phi[H]$.

<u>Corollary</u> After corollary 1 to 2.2.3. we remarked that $G^{*-} = \cap \{ H^* \mid G < H \}$. To prove this it is sufficient to show that $G^{*-} \subset H^*$ implies $G < H$. But this is $\Phi[G] \prec \Phi[H] \Rightarrow G < H$, which was proved above .

Propositions 2.4.10. and 2.4.9 . give

<u>Proposition 2.4.11.</u> There is a one to one correspondence between the subordinations on X and the compactifications of X .

2.5. Subordinations and Normal Bases

This section is a digression . The concept of normal base is introduced and shown to give rise to a compactification . The procedure is similar to the one described in section 2.2. but is simpler . Here each x in X is mapped into an ultrafilter of basic closed sets , rather than a <-ultrafilter of open sets .

The main object of this section is to show that a normal base gives rise to a subordination , and that the compactifications they induce are the same .

<u>Definition 2.5.I.</u> A family of closed sets is a normal family if

i) $\varphi, X \in \Gamma$, where Γ denotes the family

ii) $N_1, N_2 \in \Gamma \Rightarrow N_1 \cup N_2$, $N_1 \cap N_2 \in \Gamma$

iii) If F is closed and $x \notin F$ then $\exists$ $N \in \Gamma$ s.t. $x \in N$ and $N \cap F = \varphi$

iv) If $N_1, N_2 \in \Gamma$ and $N_1 \cap N_2 = \varphi$ then $\exists$ in Γ N_3, N_4 s.t. $N_j \subset X - N_{j+2}$, $j = 1,2$ and $X - N_3 \cap X - N_4 = \varphi$.

A normal family that is also a base for closed sets is a normal base.

<u>Definition 2.5.2.</u> A filter ξ s.t. $\xi \subset \Gamma$ is a Γ-filter . maximal Γ-filter is a Γ-ultrafilter .

<u>Definition 2.5.3.</u> $\Gamma X = \{ \xi \mid \xi$ is a Γ-ultrafilter on $X \}$. For any $N \in \Gamma$, let $N^* = \{ \xi \mid N \in \xi$ and $\xi \in \Gamma X \}$.

We show that the N^*'s form a base for closed sets in ΓX . This is an immediate consequence of

<u>Proposition 2.5.1.</u> $(N_1 \cup N_2)^* = N_1^* \cup N_2^*$ and $(N_1 \cap N_2)^* = N_1^* \cap N_2^*$, $N_1, N_2 \in \Gamma$.

$\xi \in (N_1 \cup N_2)^*$ iff $N_1 \cup N_2 \in \xi$ iff N_1 or $N_2 \in \xi$ iff $\xi \in N_1^* \cup N_2^*$

This proves the first equality in 2.5.1. , the proof of the second is similar .

<u>Proposition 2.5.2.</u> ΓX is compact .

Suppose $\xi^* = \{ N^*_\nu \}$ is an ultrafilter of basic closed subsets of ΓX . Let $\xi = \{ N_\nu \mid N^*_\nu \in \xi^* \}$. ξ is a Γ-filter since $(N_{\nu_1} \cap N_{\nu_2})^* = N^*_{\nu_1} \cap N^*_{\nu_2} \neq \varphi$ and so $N_{\nu_1} \cap N_{\nu_2} \neq \varphi$. To show that ξ is a Γ-ultrafilter , consider $N \in \Gamma$ s.t. $N \cap N_\nu \neq \varphi$ for all ν , then $N^* \cap N^*_\nu = (N \cap N_\nu)^* \neq \varphi$. Now ξ^* is an ultrafilter of basic sets , hence $N^* \in \xi^*$ and so $N \in \xi$, as required . Furthermore $N_\nu \in \xi$ for all ν , hence $\xi \in \cap \{ N^*_\nu \}$. This completes the proof .

<u>Proposition 2.5.3.</u> ΓX is hausdorff if Γ is a normal family .

Suppose $\xi_1 \neq \xi_2 \in \Gamma X$, then $\exists\, N_j \in \xi_j$ $j=1,2.$ s.t. $N_1 \cap N_2 = \varphi$, hence $\exists\, N_3, N_4 \in \Gamma$ s.t. $N_1 \subset X - N_3$, $N_2 \subset X - N_4$ and $X - N_3 \cap X - N_4 = \varphi$. So $N_1 \cap N_3 = \varphi$, hence $N_3 \notin \xi_1$ and thus $\xi_1 \in \Gamma X - N^*_3$. Similarly $\xi_2 \in \Gamma X - N^*_4$. These open sets are disjoint since $N^*_3 \cup N^*_4 = (N_3 \cup N_4)^* = X^* = \Gamma X$.

A partial converse is

<u>Proposition 2.5.4.</u> If Γ is a family of subsets of X satisfying ii) and iii) of definition 2.5.1. , then iv) also holds if ΓX is compact hausdorff .

We first remark that every Γ-filter can be included in a Γ-ultrafilter , it follows that $N^* \neq \varphi$ for any $N \in \Gamma$ s.t. $N \neq \varphi$.

It is now an easy consequence of iii) that $N^*_1 \cup N^*_2 = \Gamma X$ implies $N_1 \cup N_2 = X$.

To prove iv), let $N_1, N_2 \in \Gamma$ be disjoint. Then $N_1^* \cap N_2^* = \varphi$, and, since ΓX is compact and hausdorff, $\exists$ basic closed sets N_3^* and N_4^* s.t. $N_1^* \subset \Gamma X - N_3^*$ and $N_2^* \subset \Gamma X - N_4^*$ and also $N_3^* \cup N_4^* = \Gamma X$

By a remark above $N_3 \cup N_4 = X$, hence $X - N_3 \cap X - N_4 = \varphi$. To complete the proof note that $N_1^* \cap N_3^* = \varphi$, hence $N_1 \cap N_3 = \varphi$, similarly $N_2 \cap N_4 = \varphi$. Thus iv) holds.

The Embedding of X in ΓX

Definition 2.5.4. $\Phi(x) = \{ N \mid N \in \Gamma \text{ and } x \in N \}$.

Proposition 2.5.5. If Γ satisfies iii) of D.2.5.1., then $\Phi(x)$ is a Γ-ultrafilter.

It is clear that $\Phi(x)$ is a Γ-filter. If $N \in \Gamma$ and $x \notin N$, then $\exists N_1 \in \Gamma$ s.t. $x \in N_1$ and $N_1 \cap N = \varphi$. Hence $N \notin \Phi(x)$, this completes the proof.

A partial converse is

Proposition 2.5.6. If Γ is a base for closed sets and $\Phi(x)$ is a Γ-ultrafilter then iii) of D.2.5.1. is satisfied.

Suppose $x \in X$ and $F = F^- \subset X$ s.t. $x \notin F$, then $\exists N \in \Gamma$ s.t. $x \notin N$ and $N \supset F$. Thus $N \notin \Phi(x)$ and so $\exists N_1 \in \Phi(x)$ s.t. $N_1 \cap N = \varphi$, as required.

Proposition 2.5.7. If Γ satisfies iii) of D.2.5.1. then Φ is one to one.

If $x \neq y$, then $\exists\, N_1 \in \Gamma$ s.t. $x \in N$ and $y \notin N_1$, by iii) . Again by iii) , $\exists\, N_2 \in \Gamma$ s.t. $y \in N_2$ and $N_1 \cap N_2 = \varphi$. Hence $\Phi(x) \neq \Phi(y)$.

<u>Proposition 2.5.8.</u> $\Phi[N] = N^* \cap \Phi[X]$, for all $N \in \Gamma$.

$x \in N$ iff $N \in \Phi(x)$ iff $\Phi(x) \in N^*$ .This completes the proof .

<u>Corollary</u> Φ is an open map .

This follows from the above proposition and the fact that Φ is one to one .

<u>Proposition 2.5.9.</u> $\Phi^{-1}[N^*] = N$, for any $N \in \Gamma$.

$\Phi^{-1}[N^*] = \Phi^{-1}[N^* \cap \Phi[X]] = \Phi^{-1}[\Phi[N]] = N$, since Φ is 1 to 1 .

<u>Corollary</u> Φ is continuous .

<u>Proposition 2.5.10.</u> $\Phi[X]$ is dense in ΓX .

Let $\xi \in \Gamma X$ and $\Gamma X - N^*$ be a basic open neighbourhood of ξ, then $N \notin \xi$. Now $N \neq X$, so , for any $x \notin N$, we have $N \notin \Phi(x)$ and hence $\Phi(x) \in \Gamma X - N^*$, as required .

Thus ΓX is a compactification of X .

<u>The subordination induced by a normal base .</u>

<u>Definition 2.5.5.</u> Let G , H be open in X . Define $G < H$ if $\exists\, N_1, N_2 \in \Gamma$ s.t. $G \subset N_1 \subset X - N_2 \subset H$.

<u>Proposition 2.5.11</u> $<$ is a subordination on X .

S1* $\varphi < G$, G open — is true since $\varphi \subset \varphi \subset X-X \subset G$, for any G .

S2* Let $G < H$, then $\exists\ N_1, N_2 \in \Gamma$ s.t. $G \subset N_1 \subset X-N_2 \subset H$, so that $clG \subset N_1 \subset X-N_2 \subset clH$, since the N's are closed . Hence $X-clH \subset N_2 \subset X-N_1 \subset X-clG$, so $X-clH < X-clG$.

S3* $A < B$ implies $A \subset B$, by definition .

S5* If $A_i < B < C_i$, then $A_1 \cup A_2 < B < C_1 \cap C_2$, since Γ is closed under finite unions and intersections .

S6* Let $G < H$. We show that $\exists\ N \in \Gamma$ s.t. $G < X-N < H$. $\exists N_1, N_2 \in \Gamma$ s.t. $G \subset N_1 \subset X-N_2 \subset H$ .Now $N_1 \cap N_2 = \varphi$, hence $\exists\ N_3, N_4 \in \Gamma$ s.t. $N_1 \subset X-N_3 \subset N_4 \subset X-N_2$. Hence $G < X-N_3 < H$, as required .

S7* If $x \in V$, V open , then $x \notin X-V$ so that $\exists\ N_1, N_2 \in \Gamma$ s.t. $x \in N_1 \subset X-N_2 \subset V$. Hence $\exists\ N_3, N_4 \in \Gamma$ s.t. $N_1 \subset X-N_3 \subset N_4 \subset X-N_2 \subset V$. Thus $x \in X-N_3 < V$.

S4* It is obvious that $G_1 \subset G_2 < H_2 \subset H_1$ implies $G_1 < H_1$.

Let < denote the subordination induced by Γ . We now show that $<X$ and ΓX are homeomorphic , the homeomorphism leaving the points of X fixed . To simplify the notation we denote the image of $x \in X$ in $<X$ and ΓX by x .

<u>Proposition 2.5.12.</u> $\{N \mid N \in \Gamma$ and $\exists\ G \in \xi$ s.t. $G \subset N\}$ can be included in a unique Γ-ultrafilter , when ξ is a <-ultrafilter .

Suppose $\exists \zeta_1 \neq \zeta_2 \in \Gamma$ including this set .$\exists N_i \in \zeta_i$ s.t. $N_1 \cap N_2 = \varphi$, hence $\exists N_3, N_4$ in Γ s.t. $N_1 \subset X-N_3 \subset N_4 \subset X-N_2$. Thus $X-N_3 < X-N_2$.

Now, for any $G \in \xi$, $X-N_3 \cap G \supset N_1 \cap G \supset G_1 \cap G$ for some $G_1 \in \xi$. Hence $X-N_3 \cap G \neq \varphi$ for all $G \in \xi$. By lemma 2.2.3. it follows that $X-N_2 \in \xi$, this contradicts $N_2 \in \zeta_2$ and completes the proof .

<u>Definition 2.5.6.</u> In the notation of the above proposition, let $\tau(\xi) = \zeta$.

<u>Proposition 2.5.13.</u> τ is a one to one map .

If $\xi_1 \neq \xi_2 \in \mathfrak{X}$, then $\exists\ G_j \in \xi_j$, $j=1,2.$ s.t. $G_1^{*-} \cap G_2^{*-} = \varphi$, since $<X$ is compact hausdorff . It follows that $G_1 < X-clG_2$, hence $\exists\ N \in \Gamma$ s.t. $G_1 < X-N < X-clG_2$, so that $X-N \in \xi$ and $G_2 \subset N$. Thus $N \in \tau(\xi_2)$ and $N \notin \tau(\xi_1)$, so $\tau(\xi_1) \neq \tau(\xi_2)$.

<u>Proposition 2.5.14</u> $\tau(x) = x$, for all $x \in X$.

This follows from proposition 2.5.12 and the fact that $\{ N \mid N \in \Gamma \text{ and } x \in N \} \supset \{ N \mid N \supset G \text{ for some } G \text{ s.t. } x \in G \}$.

<u>Proposition2.5.15</u> $\tau^{-1}[N^*] = X-(X-N)^*$ for all $N \in \Gamma$.

Let $\xi \in (X-N)^*$, then $X-N \in \xi$ and so $N \notin \tau(\xi)$. This shows that $\tau^{-1}[N^*] \subset X-(X-N)^*$. Conversely, if $N \notin \tau(\xi)$, then $\exists$ $N_1 \in \tau(\xi)$ and $N_2, N_3 \in \Gamma$ s.t. $N_1 \subset X-N_2 \subset N_3 \subset X-N$ so that $\exists\ G \in \xi$, $G < X-N$. Thus $X-N \in \xi$ i.e. $\xi \in (X-N)^*$. The proof is complete.

<u>Corollary</u> τ is continuous .

Proposition 2.5.16 τ is onto .

The proof is the same as that of proposition 2.4.8.

We have now shown that $<X$ and ΓX are homeomorphic , the homeomorphism leaving the points of X fixed .
To complete this digression we remark that it is not known if a subordination $<$ induces a normal base which , in turn , gives rise to $<$.

Notes

Wallman[Wa_1] obtained a compactification of an arbitrary T_1 space by embedding it in the space of all maximal closed filters on the space . A similar technique had been used before by Stone [S_1],and resembles the embedding of a partially ordered space in a complete lattice . The construction of ΓX given here,is in [Fk_1] and is a modification of Wallman's construction . Aleksandrov [A_2] obtained βX ,also by a modification of Wallman's method. The constructions in Shirota [Sh_1],Horne [H_1],de Vries [Vr_1] and the one used here , all embed X in a space of R-ultrafilters,where R is a relation in p[X]. Freudenthal [Fr_1] considered introducing a subordination with the aim of compactifying a topological space. As remarked earlier,he did not obtain a one to one correspondence between his subordinations and compactifications of the space . This was achieved by Smirnov [Sm_1]in terms of proximities on X, and his result was reformulated in terms of a subordination by Aleksandrov and Ponomarev [AP_1] ,who axiomatised the notion of a subordination of a closed set to an open set . Here we give axioms for a subordination of an open set to an open set,as Freudenthal had done,and we obtain a one to one correspondence between these subordinations and compactifications .

Chapter 3

Uniformities and Compactifications

3.1. Just as a metric space can be completed by the introduction of ideal points that are limits of Cauchy sequences, so can a uniform space be completed by adjoining limits to Cauchy filters. The completion of a precompact metric space is compact, and, similarly, the completion of a precompact uniform space is compact. In fact, every compactification arises in this way. It is the purpose of this chapter to establish a one to one correspondence between the precompact uniformities on a completely regular space X and the subordinations on X.

3.2. Definition and some Properties of Uniformities.

Definition 3.2.1. A family Σ of subsets of $X \times X$ is a uniformity on X if

i) $\Delta \subset U$ for all $U \in \Sigma$, where $\Delta = \{(x,x) \mid x \in X\}$

ii) If $U \in \Sigma$, then $U^{-1} \in \Sigma$, where $U^{-1} = \{(x,y) \mid (y,x) \in U\}$

iii) For each $U \in \Sigma$, $\exists\, V \in \Sigma$ s.t. $V \circ V \subset U$, where $V \circ V = \{(x,y) \mid \exists z \text{ s.t. } (x,z), (z,y) \in V\}$

iv) If $U, V \in \Sigma$, then $\exists\, W \in \Sigma$ s.t. $W \subset U \cap V$

v) If $U \in \Sigma$ and $U \subset V$, then $V \in \Sigma$.

vi) The uniformity is separated, i.e. $\cap\{U \mid U \in \Sigma\} = \Delta$.

A topological space is uniformisable if $\exists$ a uniformity Σ which is compatible with the topology, in the sense that $U[x] =$

$\{ y \mid (y,x) \in \Sigma \}$, $U \in \Sigma$, is a base for the neighbourhood system of x . Σ is then said to be an admissible uniformity . We shall say (X,Σ) is a uniform space , when Σ is an admissible uniformity .

<u>Proposition 3.2.1.</u> Let (X,Σ) be a uniform space , the family of symmetric open neighbourhoods of Δ is a base for Σ , in the sense that if $U,V \in \Sigma$, then $\exists\, W \in \Sigma$ s.t. $W \subset U \cap V$, $W = W^{-1}$, and W is open in $X \times X$.

A proof can be found in $[K_1]$.

<u>Proposition 3.2.2.</u> If $U[A] \cap B = \varphi$ and V is a symmetric subset of $X \times X$ s.t. $V^2 = V \circ V \subset U$, then $V[A] \cap V[B] = \varphi$. Where $W[A] = \bigcup_{a \in A} W[a]$ for any $W \subset X \times X$ and $A \subset X$.

If $y \in V[A] \cap V[B]$, then (y,a) , $(y,b) \in V$ for some $a \in A$ and $b \in B$. Hence $(a,b) \in V^2$, since $V = V^{-1}$, and so $b \in U[a] \subset U[A]$, a contradiction .

<u>Corollary</u> If $(X;\Sigma)$ is a uniform space and $V \in \Sigma$, then $A^- \subset V[A]$ for any $A \subset X$.

By 3.2.1. $\exists\, U \in \Sigma$, U a symmetric open set including Δ s.t. $U^2 \subset V$. If $x \notin V[A]$, then $U[x] \cap U[A] = \varphi$, by 3.2.2. . Now $U[x]$ is open and $A \subset U[A]$, hence $x \notin A^-$. The proof is complete.

3.3. Uniformity and Subordination.

Definition 3.3.1. $G < H$ if $\exists$ $U \in \Sigma$ s.t. $U[G] \subset H$.

Proposition 3.3.1. $<$, defined above , is a subordination .

$S1^*$ $\varphi < H$ is obviously true .

$S2^*$ Let $A < B$, then $\exists$ $U \in \Sigma$ s.t. $U[A] \subset B$ so that $X-B \cap U[A] = \varphi$. By 3.2.2. $V[A] \cap V[X-B] = \varphi$ for any symmetric open V s.t. $V^2 \subset U$, hence $V[X-B^-] \subset V[X-B] \subset X-V[A] \subset X-A^-$ as $X-B^- \subset X-B$ and $A^- \subset V[A]$.

$S3^*$ It is clear that $A_1 \subset A < B \subset B_1$ implies $A_1 < B_1$.

$S4^*$ If $A < B$, it follows that $A \subset U[A] \subset B$, as required

$S5^*$ If $A_i < A$, then $\exists$ $U_i \in \Sigma$ s.t. $U_i[A_i] \subset A$, $i=1,2$. . Let $U = U_1 \cap U_2$, then $U[A_1 \cup A_2] \subset A$, hence $A_1 \cup A_2 < A$ as $U \in \Sigma$.

$S6^*$ It is clear that $A < A_i$ $i=1,2$, implies $A < A_1 \cap A_2$.

$S7^*$ If $x \in V$, V open, then $\exists$ $U \in \Sigma$ s.t. $U^2[x] \subset V$. Hence $x \in U[x]$, $U[x]$ is open and $U[x] < V$.

As remarked earlier, $<X$ is a compact space , hence $\exists$ a unique uniform structure $\Sigma_{<X}$ on $<X$ consisting of all neighbourhoods of the diagonal $\Delta_{<X}[K_1]$. The restriction of this uni-

formity to X is compatible with the topology on X. So $\Sigma_X = \{ U \cap X\times X \mid U \in \Sigma_{<X} \}$ is a uniformity on X . We shall also denote Σ_X by $\Sigma_<$.

A base for the uniformity on <X consists of all open subsets of X×X including Δ ; since <X is compact , one can select a base $\Sigma_0 = \{ \bigcup_{j=1}^{n} O_j \times O_j \mid O_j$ open in <X and $\bigcup_{j=1}^{n} O_j = {<X} \}$. The restriction Σ_{0X} of Σ_0 to X×X is a base for Σ_X . This base can be described in terms of X and < alone –

<u>Proposition 3.3.2.</u> $\Sigma_{0X} = \{ \bigcup_{j=1}^{n} H_j \times H_j \mid \exists\, G_j$ s.t. $G_j < H_{i_j}$ for some i_j, and $\bigcup_{j=1}^{m} G_j = X \}$.

Suppose $\{ H_i \}$ is a finite cover of X with $\{ G_j \}$ s.t. $G_j < H_{i_j}$ for some i_j and $\bigcup_{j=1}^{m} G_j = X$. By corollary 2 to 2.2.3. we have $\bigcup_i H_i^* = {<X}$, hence $\bigcup_i H_i^* \times H_i^* \in \Sigma_0$ and so $\bigcup_i H_i \times H_i = \bigcup_i H_i^* \times H_i^* \cap X\times X \in \Sigma_{0X}$.

Conversely , let $U \in \Sigma_{<X}$. Then $\exists\, V \in \Sigma_{<X}$, V symmetric and open in <X × <X s.t. $V^2 \subset U$. Now $\mathfrak{a} = \{ V[x] \mid x \in X \}$ is an open cover of <X , since $\xi \in {<X}$ implies $\exists\, x \in X \cap V[\xi]$ as X is dense and $V[\xi]$ is open and so $\xi \in V[x]$ as $V = V^{-1}$.
Since <X is regular , $\exists$ a basic open family $\mathfrak{a}_0 = \{ G_j^* \}$, covering <X and s.t. for each $j, G_j^* \subset V[x]$ for some $x \in X$. Hence

$G_j < V[x]$. Since $<X$ is compact , $\exists$ a finite subfamily of α_0 that covers $<X$ and , hence , α , when restricted to X , has a finite subordinate refinement . Furthermore $V[x] \times V[x] \subset V^2 \subset U$, by our choice of V . This completes the proof .

We now describe the relationship between a $<$ on X and the subordination induced by $\Sigma_<$; and , conversely , show that Σ on X is the same as $\Sigma_<$, where $<$ is the subordination induced by Σ .

<u>Proposition 3.3.3.</u> Let $<$ be a subordination on X , and Σ the induced uniformity . If $<_0$ is the subordination induced by Σ , then $< = <_0$.

Suppose $G_1 <_0 G_2$, then $\exists$ $U \in \Sigma$ s.t. $U[G_1] \subset G_2$. By 3.3.2 $\exists$ $C_1,\ldots,C_n$ s.t. $C = \bigcup_{i=1}^{n} C_i \times C_i \subset U$ and $\bigcup_{i=1}^{n} C_i^* = <X$. Now $C[G_1] = \{ \bigcup_{j=1}^{k} C_{i_j} \mid C_{i_j} \cap G_1 \neq \varphi \} \subset G_2$, also , if $\xi \in G_1^-$ then ,for some i , $\xi \in C_i^*$ so $C_i \cap G_1 \neq \varphi$ and hence $G_1^- \subset \{ \bigcup_j C_{i_j}^* \mid C_{i_j} \cap G_1 \neq \varphi \}$, But $\bigcup C_{i_j}^* \subset (\bigcup C_{i_j})^* \subset G_2^*$, so $G_1^- \subset G_2^*$ and hence $G_1 < G_2$.

Conversely , suppose $G_1 < G_2$. Then $G_1^- \subset G_2^*$ and so $\exists$ x_1---$x_n \in G_1^-$ and U_1---$U_n \in \Sigma$ s.t. $G_1^- \subset \bigcup_{i=1}^{n} U_i[x_i] \subset \bigcup_{i=1}^{n} U_i^2[x_i] \subset G_2^*$. Let $V = \bigcap_i U_i$, then $V[G_1] \subset G_2^*$,since $y \in V[G_1]$ implies $y \in V[g]$ for some $g \in G_1$ and $g \in U_i[x_i]$ for some i ,so that $y \in V \circ U_i[x_i] \subset U_i^2[x_i] \subset G_2^*$. It follows that $G_1 <_0 G_2$.

<u>Proposition 3.3.4.</u> Let (X,Σ) be a uniform space and $<$ the induced subordination . If Σ_0 is the uniformity induced by $<$, then Σ_0 is the finest precompact uniformity coarser than Σ . Thus, $\Sigma_0 = \Sigma$ iff Σ is precompact .

As an immediate consequence of propositions 3.3.3. and 3.3.4. we have

<u>Proposition 3.3.5.</u> There is a one to one correspondence between subordinations on X , and hence compactificatons of X , and precompact uniformities on X .

To prove proposition 3.3.4. , we show that

i) $\Sigma_0 \subset \Sigma$

ii) If $\Sigma_1 \subset \Sigma$ and Σ_1 is precompact , then $\Sigma_1 \subset \Sigma_0$.

i) If $U \in \Sigma_0$, then , by proposition 3.3.2. , $\exists\, G_i, H_i$, open , $G_i < H_i$, $\bigcup_{i=1}^{n} G_i = X$ and $\bigcup_{i=1}^{n} G_i \times G_i \subset U$. We now show that $\bigcup_{i=1}^{n} G_i \times G_i \in \Sigma$. $G_i < H_i$, hence $\exists\, U \in \Sigma$, U symmetric , s.t. $U[G_i] \subset H_i$ for $i = 1,\ldots,n$. Now , if $(x,y) \in U$ then $(x,y) \in H_i \times H_i$, for some i, since $x \in G_i$ for some i and so $y \in U[x] \subset U[G_i] \subset H_i$, noting that $G_i \subset H_i$ we get $(x,y) \in \bigcup_{i=1}^{n} H_i \times H_i$ and hence $U \subset \bigcup_{i=1}^{n} H_i \times H_i$. The proof of i) is now complete .

ii) Let Σ_1 be a precompact uniformity coarser than Σ.
If $U \in \Sigma_1$, choose $U1 \in \Sigma_1$ s.t. $U_1^3 = U_1 \circ U_1 \circ U_1 \subset U$ and U_1 symmetric. To show that $U \in \Sigma_0$, note that $\exists A_i$, $i = 1,\ldots,n.$ s.t. $A_i \times A_i \subset U_1$ and $\bigcup_{i=1}^{n} A_i = X$, since Σ_1 is precompact. Now $A_i < U_1[A_i]$, by definition of $<$, hence $\bigcup_{i=1}^{n} U_1[A_i] \times U_1[A_i] \in \Sigma_0$, by proposition 3.3.2. Also, if $(x,y) \in U_1[A_i] \times U_1[A_i]$ for some i, then $\exists a_1, a_2 \in A_i$ s.t. $(x,a_1) \in U_1$ and $(y,a_2) \in U_1$, but $(a_1,a_2) \in A_i \times A_i \subset U_1$ and so $(x,y) \in U_1^3 \subset U$. This shows that $U \in \Sigma_0$.

3.4. <u>Compactification and Completion</u>

We have seen that every precompact uniformity Σ_0 on X determines a compactification $<X$ of X, where $<$ is the subordination induced by Σ_0. Now the completion of a uniform space is essentially unique, and the completion of a precompact uniformity is easily seen to be compact, it follows that $<X$ is $(X;\Sigma_0^*)$, the completion of (X,Σ_0).

<u>Definition 3.4.1.</u> A filter of subsets of X is a Cauchy filter if, for every $U \in \Sigma$, $\exists$ a member of the filter, A, s.t. $A \times A \subset U$.

(X,Σ) is called complete if $\cap\{ A^- \mid A \in \xi \} \neq \varphi$ for all Cauchy filters ξ.

<u>Proposition 3.4.1.</u> Every uniform space can be embedded, as a dense subspace, in a complete uniform space. This space is unique up to uniform isomorphism leaving the points of X fixed.

For a proof we refer to [K_1].

<u>Proposition 3.4.2.</u> If (X,Σ_0) is a precompact uniform space and $<$ the induced subordination, then $<X$ is uniformly isomorphic to the completion of (X,Σ_0).

This follows from the above proposition, and the fact that $<X$ is compact and hence a complete uniform space, and

<u>Lemma</u> i) $\tau : (X,\Sigma_0) \to {<X}$ is uniformly continuous

ii) $\tau^{-1}: X \subset {<X} \to X$ is also uniformly continuous.

Where $\tau(x) = x$ for all $x \in X$.

A map $f: (X,\Sigma) \to (Y,\Omega)$ is uniformly continuous if $f_*^{-1}[V] \in \Sigma$ for all $V \in \Omega$, where $f_*: (x,y) \to (f(x),f(y))$.

i) Given $U \in \Sigma_{<X}$, $\exists\, G_i^*, H_i^*$ s.t. $G_i^{*-} \subset H_i^*$ and $\bigcup_{i=1}^{n} G_i^* = {<X}$ and $V = \bigcup_{i=1}^{n} H_i^* \times H_i^* \subset U$, by compactness of $<X$. Now $\tau_*^{-1}[V] = \bigcup_{i=1}^{n} H_i \times H_i$, and by proposition 3.3.2. $\tau_*^{-1}[V] \in \Sigma_0$. Thus τ is uniformly continuous.

ii) is clear.

Notes

The results of this chapter were first obtained by Samuel [Sa_1] and later,but independently,by Shirota [Sh_1] . They obtain the compactifications by completing the precompact uniformities,for this,X is embedded in a space of ultrafilters or equivalence classes of ultrafilters. However,once <X has been constructed there is no need todo this .

Given <X , the exposition in chapter 3 is a natural one . It is also suggested in Aleksandrov's survey [A_1] when discussing the work of Smirnov , to whom proposition 3.3.2. is due . I do not know Smirnov's proof.

Gal [G_1] offers a direct proof of proposition3.3.2. .

Chapter 4

Morphisms and Compactifications

4.1. We recall

<u>Definition</u> $f:(X_1,<_1)\to(X_2,<_2)$ is $(<_1,<_2)$ continuous , or simply $<$-continuous , if $f^{-1}[G] <_1 f^{-1}[H]$ whenever $G <_2 H$.

<u>Definition</u> $f:(X_1,\Sigma_1)\to(X_2,\Sigma_2)$ is uniformly continuous if $f_*^{-1}[V] \in \Sigma_1$ for all $V \in \Sigma_2$, where $f_*(x,y)=(f(x),f(y))$

<u>Definition 4.1.1.</u> $f:(X_1,\Gamma_1)\to(X_2,\Gamma_2)$ is Γ-continuous if $f^{-1}[N] \in \Gamma_1$ for all $N \in \Gamma_2$, where Γ is a normal base for X .

We now relate these concepts

<u>Proposition4.1.1.</u> Let $f:(X,<_1)\to(Y,<_2)$, then f is uniformly continuous when regarded as a map from $(X,\Sigma_{<_1})$ to $(Y,\Sigma_{<_2})$.

Let $U \in \Sigma_2$, then $\exists\, G_i < H_i$, $i=1,\dots,n.$, open in Y s.t. $Y = \bigcup_{i=1}^{n} G_i$ and $\bigcup_{i=1}^{n} H_i\times H_i \subset U$, by proposition 3.3.2. . We show that $f_*^{-1}[\bigcup_{i=1}^{n} H_i\times H_i] \in \Sigma_1$. f is <-continuous so $f^{-1}[G_i] < f^{-1}[H_i]$, also $\bigcup_{i=1}^{n} f^{-1}[G_i] = X$, since $\bigcup_{i=1}^{n} G_i = Y$, hence $\bigcup_{i=1}^{n} f^{-1}[H_i]\times f^{-1}[H_i] \in \Sigma_1$, again by proposition 3.3.2.

But $\bigcup_{i=1}^{n} f^{-1}[H_i]\times f^{-1}[H_i] = \bigcup_{i=1}^{n} f_*^{-1}[H_i\times H_i] = f_*^{-1}[\bigcup_{i=1}^{n} H_i\times H_i] \subset f_*^{-1}[U]$

and so $f_*^{-1}[U] \in \Sigma_1$. This completes the proof .

<u>Proposition 4.1.2.</u> If $f:(X,\Sigma_1)\to(Y,\Sigma_2)$ is uniformly continuous and $<_1, <_2$ are the induced subordinations, then f is $(<_1,<_2)$ continuous .

Let $G <_2 H$, then $\exists\ V \in \Sigma_2$ s.t. $V[G] \subset H$. Let $U = f_*^{-1}[V]$, then $U[f^{-1}[G]] \subset f^{-1}[H]$, since $x \in U[f^{-1}[G]]$ implies $(x,y) \in U$ for some y s.t. $f(y) \in H$, now $f_*(x,y) \in V$ by the choice of U so $f(x) \in V[G] \subset H$, and hence $x \in f^{-1}[H]$. Thus $f^{-1}[G] <_1 f^{-1}[H]$, as required .

<u>Proposition 4.1.3.</u> If $f:(X_1,\Gamma_1)\to(X_2,\Gamma_2)$ is Γ-continuous and $<_1, <_2$ are the induced subordinations , then f is $(<_1,<_2)$ continuous .

Let $G <_2 H$, then $\exists\ N_1,N_2 \in \Gamma_2$ s.t. $G \subset N_1 \subset X_2 - N_2 \subset H$, so that $f^{-1}[G] \subset f^{-1}[N_1] \subset X_1 - f^{-1}[N_2] \subset f^{-1}[H]$. Now f is Γ-continuous so $f^{-1}[N_i] \in \Gamma_1$, $i=1,2.$, hence $f^{-1}[G] <_1 f^{-1}[H]$; the proof is complete .

4.2. <u>Extension of maps</u>

In this section we show that the bounded, $<$-continuous, real valued, functions on X are the ones having an extension to $<X$. This can be proved directly , but it is a consequence of a

similar extension theorem expressed in terms of uniformities rather than subordinations .

Definition 4.2.1. $f:X \to Y$ is an extension of $g:A \subset X \to Y$, if $f(a)=g(a)$ for all $a \in A$.

Proposition 4.2.1. Let $f:A \subset X \to Y$, X a uniform space and Y a complete uniform space . f has an extension to A^- iff f is uniformly continuous .

A proof can be found in [K_1] .

Corollary 1 If $f:(X,<_1) \to (Y,<_2)$ is <-continuous , then f has an extension, f^-, to a map from <X to <Y . In particular , if $f:(X,<) \to [0,1]$, then f has an extension to <X iff f is <continuous .

Corollary 2 If $f:(X,\Gamma_1) \to (Y,\Gamma_2)$ is Γ-continuous , then f has an extension , f^-, to a map from ΓX to ΓY . Thus $f:(X,\Gamma) \to [0,1]$ has an extension to ΓX iff f is Γ-continuous , where [0,1] has as normal base the family of all closed subsets of [0,1] .

These follow from propositions 4.1.2. and 4.1.3. and the one above .

4.3. Morphisms and <X .

We now show that $<X$ can be characterised by its $<$-continuous bounded real valued functions : any compactification C of X, s.t. any bounded, $<$-continuous, real valued function on X has an extension to C, is homeomorphic to $<X$, the homeomorphism leaving the points of X fixed .

We need

<u>Definition 4.3.1.</u> Let $\{X_\alpha\}$ be a family of topological spaces, the product PX_α is the set $=\{ f \mid f(\alpha) \in X_\alpha$, for all $\alpha \}$ topologised by taking $\pi_\alpha^{-1}[G_\alpha] = \{ f \in PX_\alpha \mid \pi_\alpha(f)=f(\alpha) \in G_\alpha \}$ as a subbase for open sets , where G_α is open in X_α .

If $f_\alpha : X \to X_\alpha$ is a family of maps , then $\exists$ e , $e : X \to PX_\alpha$, s.t. $\pi_\alpha \circ e = f_\alpha$, in fact this relation defines e .

<u>Proposition 4.3.1.</u> $e : X \to PX_\alpha$, each X_α a topological space

i) e is continuous iff $\pi_\alpha \circ e$ is continuous for all α .

ii) e is one to one if for any $x \neq y$ in X , $\exists\ \alpha$ s.t. $f_\alpha(x) \neq f_\alpha(y)$.

iii) e is open if, for any $x \in X$ and $A \subset X$ s.t. $x \notin \mathrm{cl}A$, $\exists\ \alpha$ s.t. $f_\alpha(x) \notin (f_\alpha[A])^-$

The proof can be found in $[K_1]$.

<u>Proposition 4.3.2.</u> The product of hausdorff spaces is a hausdorff space.

The product of compact spaces is compact.

Let $C_<$ denote the family of real valued, bounded, $<$-continuous functions on X, and let $e:X\to P_{f\in C_<}(f[X])^-$ be s.t. $\pi_f \circ e = f$.

<u>Proposition 4.3.3.</u> If C is any compactification of X s.t. any $f \in C_<$ has an extension to C, then C is homeomorphic to $(e[X])^-$.

<u>Corollary</u> Any such compactification is homeomorphic to $<X$.

This follows from the fact that any $f \in C_<$ has an extension to $<X$, by corollary 1 to proposition 4.2.1., and so $<X$ and $(e[X])^-$ are homeomorphic.

We now prove proposition 4.3.3.

If $f \in C_<$, let f^* denote its continuous extension to C. Now $f^*[C] = f^*[X^-] \subset (f^*[X])^- = (f[X])^-$, so that $f^*[C] \subset \pi_f[(e[X])^-]$ for each f. Thus we can define $\pi:C\to(e[X])^-$ by $\pi_f(\pi(c)) = f^*(c)$ for each $c \in C$.

We verify that π is a homeomorphism of C onto $(e[X])^-$.

i) $\pi_f \circ \pi = f^*$, so $\pi_f \circ \pi$ is continuous for each f and hence π is continuous.

ii) If $c_1 \neq c_2 \in C$, then $\exists$ f^*, continuous, s.t. $f^*:C\to[0,1]$ and $f^*(c_1) \neq f^*(c_2)$ and so $\pi_f(\pi(c_1)) \neq \pi_f(\pi(c_2))$, where f is the restriction of f^* to X .

iii) for $x \in X$, $\pi_f(\pi(x)) = f^*(x) = f(x0 = \pi_f(e(x))$ for all $f \in C_<$, hence $\pi|X = e|X$,where $g|X$ denotes the restriction of g to X . Thus π leaves the points of X fixed , if x and $e(x)$ are identified .

iv) π is onto , since $\pi[C] \supset \pi[X] = e[X]$ and $\pi[C]$ is compact hausdorff and , hence , closed,so that $(e[X])^- \subset \pi[C]$. But $\pi[C] \subset (e[X])^-$, by definition of π , so the proof is complete .

The result now follows , since all spaces are compact hausdorff and ,hence,continuous ,one to one, onto maps are homeomorphisms .

In the same way ,

<u>Proposition 4.3.4.</u> The compactification C of (X,Γ) s.t. any Γ-continuous,bounded ,real valued function has an extension to C,is unique up to homeomorphism leaving the points of X fixed .

<u>Proposition4.3.5.</u> The compactification C of a precompact uniform space (X,Σ_0) ,s.t. any uniformily continuous, bounded, real valued function has an extension to C , is homeomorphic to $\Sigma_0 X = <_{\Sigma_0} X$ the completion of (X,Σ_0)

4.4. Rings of continuous functions and compactifications

Proposition 4.3.2. shows that any family of real valued, bounded, continuous functions satisfying iii), gives rise to a compactification of X. Conversely, a compactification C of X determines a family R of real valued, bounded, continuous functions which satisfies iii) - the restrictions to X of all real valued continuous functions on C. These functions form a ring [K_I], which contains all constant, real valued functions on X, and is closed in the norm topology for $C^*(X)$; we call such rings c_X-rings or, simply, c-rings. Furthermore proposition 4.3.3. shows that $e: X \to \underset{f \in R}{P} (f[X])^-$ is a homeomorphism, since R is the family of those functions in $C^*(X)$ which have an extension to C. Conversely

<u>Proposition 4.4.1.</u> If R is a c-ring, then R gives rise to a compactification C of X s.t. R is the family of all functions in $C^*(X)$ that have an extension to C, X being regarded as a dense subspace of C.

From this and what was said above, we have

<u>Corollary</u> There is a one to one correspondence between c-rings and compactifications of X.

To prove 4.4.1. , we note that R gives rise to a compactification $C = (e[X])^{-} \subset \underset{f \in R}{P} (f[X])^{-}$. We now identify x and e(x), and note that π_f is an extension of f , since π_f is continuous and $f(x) = \pi_f \circ e(x) = \pi_f(e(x)) = \pi_f(x)$ for all $x \in X$.

Furthermore, the extensions to C of the functions in R form a ring , and it is easy to see that it is a closed subring of the ring of real valued,continuous functions on C . Now this ring separates the points of C , since $c_1 \neq c_2 \in C$ implies $\exists f \in R$ s.t. $\pi_f(c_1) \neq \pi_f(c_2)$ and π_f is the extension of $f \in R$, hence it is the ring of real vaued continuous functions on C, by the Stone-Weierstrass theorem [S_2] .

The above proposition,together with earlier results , shows that there is a one to one correspondence between c_X-rings and subordinations on X . This can be obtained directly as follows. A subordination < gives rise to a c_X-ring , viz the ring of real valued,bounded,<-continuous functions . Conversely any c_X-ring induces a subordination < by: G < H if $\exists$ f in the given ring s.t. $G \subset f^{-1}[0]$ and $X-H \subset f^{-1}[1]$ and $f:X \to [0,1]$. Furthermore c_X induced by < gives < back again , and < induced by a c_X-ring , gives rise to the same ring .

Concerning spaces with a normal base , we have the partial result that Γ determines a c_X-ring - the ring of Γ-continuous , real vaued,bounded functions .

Notes

This chapter makes use of the representation of $<X$, ΓX and (Σ_0^*, X) as subspaces of products of lines . For completely regular spaces this idea is due to Tychonoff [T_1] . The final section of this chapter is a consequence of this point of view .

In the language of categories, this section shows that $X \to <X$, $X \to \Gamma X$ and $(X, \Sigma_0) \to (X, \Sigma_0^*)$ are reflections [Freyd] in the categories of <spaces and <-continuous maps ; semi-normal spaces and Γ-continuous maps ; Uniform spaces and uniformly continuous maps , respectively .

We wish to remark that definition 4.1.1. is not the one given by Frink . The two are equivalent , but his definition is not what one would call category-theoretic .

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