University of Cape Town

Topics in Categorical Algebra and Galois Theory

Author:
Jason Fourie

Supervisor:
Prof. George Janelidze
University of Cape Town

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Abstract

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by Jason Fourie

We provide an overview of the construction of categorical semidirect products and discuss their form in particular semi-abelian varieties. We then give a thorough description of categorical Galois theory, which yields an analogue for the fundamental theorem of Galois theory in an abstract category by making use of the notions of admissibility and effective descent. We show that the admissibility of a functor can be extended to the admissibility of the canonically induced functor on the associated category of pointed objects, and that an analogous extension can be made for effective descent morphisms. We use the extended notions of admissibility and effective descent to describe a new, pointed version of the categorical Galois theorem, and use this result to move the categorical formulation of the Galois theory of finite, separable field extensions into a non-unital context.

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Chapter 1

Introduction

The notion of the categorical semidirect product, which was introduced by Bourn and Janelidze [8], is a generalization of classical semidirect products of groups. The theoretical construction of categorical semidirect products can be obtained via internal object actions – which are essentially the algebra structures for a particular monad – in the framework of monoidal categories. Bourn and Janelidze also showed that the categorical semidirect product of groups coincides with its classical counterpart [8]. The same holds true for semidirect products of commutative rings.

An abstract Galois structure is a system that involves an adjunction between categories, and well-behaved classes of morphisms in those categories. Categorical Galois theory is the subject that describes how abstract Galois structures can be used to provide an analogue of the fundamental theorem of Galois theory in an abstract category (satisfying certain requisite conditions) [18]. This is done by presenting the Galois theorem as an equivalence between a category of particular morphisms in the category in question, and a category of (pre-)groupoid actions. The constructions in categorical Galois theory require the notions of admissible functors, which are functors that preserve pullbacks of a particular form, and effective descent morphisms, which are morphisms that allow one to deal with structures over an object by moving them into a more amenable "extension" of that object, and then "descending" back to the object in question once the appropriate calculations have been made. It can be shown that (in certain circumstances) the admissibility of a functor ensures the admissibility of the induced functor on the associated category of pointed objects. Further, we show that effective descent morphisms in a given category can be regarded as effective descent morphisms in the associated category of pointed objects whenever they can be regarded as morphisms of pointed objects.

The Galois theory of finite field extensions can be obtained as a categorical Galois theory in the opposite category of finite-dimensional unital K-algebras[20]. By using the result regarding the admissibility of the induced functor on the category of pointed objects, and categorical semidirect products, we show that this theory can be extended into the context of non-unital K-algebras.

Galois structures that involve an adjunction with the category of Stone spaces can be used to describe "infinite" Galois theories, and are known as Boolean Galois theories [9]. The Boolean Galois theory of commutative rings makes use of the Pierce representation. Each commutative ring has a Boolean algebra which underlies it, and each Boolean algebra corresponds to a unique Stone space under the Stone duality. The evident composite association from rings to Stone spaces can be extended to an admissible functor, which has a fully faithful right adjoint. This adjunction facilitates the description of the Boolean Galois theory of commutative rings.

Before the main body of work, it should be mentioned that the references used in the thesis have been chosen for their expedience, and that several of these are not the sources in which the respective results originally appeared.

1.1 Layout of the Paper

This thesis takes the following general form:

- Chapter (2) describes the notions of categories of families, connected objects in categories, and extensivity – and the interplay between them – and discusses separate induced adjunctions that will be used in the definition of the admissibility of a functor, and the extension of the notions of admissibility and effective descent from the contexts in which they are defined to the associated categories of pointed objects.

- Chapter (3) begins with a brief discussion of the context in which semidirect products arise, and of the form of particular categorical semidirect products. The chapter continues by giving a full description of how categorical semidirect products can be defined in terms of internal object actions, and concludes with a clear account of how the categorical semidirect products of groups and commutative rings coincide with their respective classical counterparts.

- Chapter (4) discusses and explains the various notions required to form the general Galois theorem in an abstract category. Throughout the chapter, a large emphasis is placed on the concept of effective descent. Further, it is shown how the notions of admissibility and effective descent can be induced in categories of pointed objects. Toward the end of the chapter, the conventional Galois theorem is presented, and then extended into a new, pointed formulation.

- Chapter (5) begins by describing how the Galois theory of finite field extensions can be seen as an instantiation of the Galois theorem in the opposite category of finite-dimensional unital commutative K-algebras, and provides an intuitive algebraic perspective for the theory. Following this, a new, explicit explanation of how to extend the Galois theory of finite field extensions into the context of non-unital K-algebras is given, and the form of the (trivial) coverings for the extended Galois theory is analysed.

- Chapter (6) provides a rigorous explanation of the details required to discuss the Boolean Galois theory of commutative rings (and algebras) via the Pierce representation, and neatly presents the results from Chapter (4) in the language of that representation.

- Chapter (7) formulates a concise conclusion of the thesis by providing closing summaries for each of its central chapters.

- Appendices (A.1) - (A.8) contain various supplementary materials for the thesis.

  - Appendix (A.1) recalls the definitions of both traditional and pointed G-sets, and notes the morphisms used to form their respective categories.
  - Appendix (A.2) describes the notion of a pure monomorphism, and emphasises the distinction between a morphism being a pure monomorphism and a module (over a commutative ring) being flat.
  - Appendix (A.3) expounds on the relationships between various types of epimorphism, and outlines the properties and behaviour of these epimorphisms in different categorical settings.
  - Appendix (A.4) recalls the definition of the category of points, and describes the inverse image functor between categories of points. Special mention is made for the inverse image functors induced by morphisms with the zero object as their domain.
  - Appendix (A.5) defines the notion of a protomodular category, makes mention of the fact that there are several equivalent formulations of this definition when the category in question is pointed, and recalls that protomodularity in varieties can be characterized by the existence of certain operations defined on the objects of the variety.
  - Appendix (A.6) provides a definition for regular categories, and emphasises that regular epimorphisms are particularly well behaved in regular categories.
  - Appendix (A.7) recalls the definition of an exact category (in the sense of Barr). Its contents shown that every reflexive pair in an exact category corresponds directly to an equivalence relation, and shares the coequalizer of that equivalence relation.
  - Appendix (A.8) defines the notion of a semi-abelian category, and notes the important result that a variety is semi-abelian if and only if it is both pointed and protomodular.

Chapter 2

Preliminaries

2.1 Abstract Families and Connected Objects in Categories

Suppose that $\mathbb{X}$ is a full subcategory (which is closed under finite limits) of Sets, and let $\mathbb{A}$ be any category. The category of families of objects in $\mathbb{A}$ is denoted by $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$, and is constituted as follows:

1. Objects in $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$ are given by families $A = (A_i)_{i \in I(A)}$ of objects in $\mathbb{A}$, indexed by sets $I(A)$ in $\mathbb{X}$,

2. Morphisms in $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$ are given by pairs $(A_i)_{i \in I(A)} \xrightarrow{(f,\alpha)} (B_j)_{j \in I(B)}$ in which

   a) $f : I(A) \to I(B)$ is a set-map (a morphism in $\mathbb{X}$),
   b) $\alpha \equiv \big(\alpha_i : A_i \to B_{f(i)}\big)_{i \in I(A)}$ is a family of morphisms in $\mathbb{A}$.

The association $(A_i)_{i \in I(A)} \mapsto I(A)$ extends to a functor

$$I : \mathsf{Fam}_{\mathbb{X}}(\mathbb{A}) \to \mathbb{X}$$

Moreover, if $\mathbb{A}$ has a terminal object 1, I will have a right adjoint

$$H : \mathbb{X} \to \mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$$

which sends each set X to the X-indexed family of terminal objects (i.e. the family $(1_x)_{x \in X}$ where $1_x = 1$ for each $x \in X$).

The following theorem [16] places the notion of connected objects in categories into context.

Theorem 2.1.1. *Let* Top *denote the category of topological spaces. For a topological space* $A \in \mathsf{Top}$*, the following conditions are equivalent:*

1. *A is connected as a topological space,*
2. *The functor* $\mathsf{Hom}_{\mathbb{C}}(A, -) : \mathbb{C} \to \mathsf{Sets}$ *preserves coproducts,*
3. *Any morphism from A into a coproduct in* Top *factors through one of the coproduct injections,*
4. *A is not initial in* Top*, and if $A \cong B + C$, then either B or C is initial in* Top*,*
5. *A is not initial in* Top*, and if $A \cong B + C$, then A is canonically isomorphic either to B or to C.*

While moving from Top to a more general category $\mathbb{C}$ (with coproducts) does not precisely preserve the contents of Theorem (2.1.1), one will still have the following implications [16]:

$$(2) \Rightarrow (3) \Rightarrow (5) \text{ and } (4) \Rightarrow (5) \tag{2.1}$$

Moreover, if $\mathbb{C}$ is extensive – like any categories of the form $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$ is[1] – then conditions $(2) - (5)$ will be equivalent in $\mathbb{C}$.

Moving into a more general setting, one should make the following definition of a connected object:

Definition 2.1.1 (Borceux and Janelidze [4])**.** *Let $\mathbb{C}$ be a category with coproducts and finite limits.*

1. *An object A in $\mathbb{C}$ is connected if:*

$$\mathsf{Hom}_{\mathbb{C}}(A, -) : \mathbb{C} \to \mathsf{Sets}$$

   preserves coproducts,
2. *$\mathbb{C}$ is connected if its terminal object 1 is connected,*
3. *$\mathbb{C}$ is locally connected if every object in $\mathbb{C}$ can be presented as a coproduct of connected objects in $\mathbb{C}$.*

For categories which have finite coproducts, there is an analogous notion of being finitely locally connected:

[1] See Section (2.2).

Definition 2.1.2. *Let $\mathbb{C}$ be a category with finite coproducts and finite limits. $\mathbb{C}$ is finitely locally connected if every object A in $\mathbb{C}$ is a finite coproduct of connected objects in $\mathbb{C}$.*

One can use the following intuition for the definition of connected objects: if A is a connected object in $\mathbb{C}$, then for each morphism $A \to \coprod_{i \in I} B_i$ from the connected object into a coproduct (which one might think of as a disjoint union) in $\mathbb{C}$, the image of A under this morphism should fall entirely within one of the components B_i of the coproduct, otherwise its image would be spread over several components, and would therefore be "disconnected". If one took $\mathbb{C}$ to be Top, and A to be a connected topological space, the mapping would be continuous if and only if A were mapped into precisely one of the components of the disjoint union $\coprod_{i \in I} B_i$.

The order of the implications in (2.1) supports the definition of connected objects in terms of hom-functors in more general categories. Indeed, in categories that are not extensive, calling an object connected if any of conditions (3), (4) or (5) from Theorem (2.1.1) held would lead to very strange results. For example, in the category of groups – which is not extensive – the hom-functors $\mathsf{Hom}(G, -)$ do not preserve coproducts. Concurrently, the coproduct in Groups is the free product of groups - which is always infinite. Therefore, no finite group can be presented as a non-trivial coproduct in Groups, even though each corresponding hom-functor will not preserve coproducts.

Proposition 2.1.1. *Let $\mathbb{C}$ be a category of the form $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$. Given an object $A \in \mathbb{C}$, the following statements are equivalent:*

1. *A is connected,*
2. *A cannot be presented as a non-trivial coproduct of objects in $\mathbb{C}$,*
3. *$I(A) = \{*\}$.*[2]

Proposition 2.1.2 (Borceux and Janelidze [4])**.** *In an arbitrary category $\mathbb{C}$, the following statements are equivalent:*

1. *$\mathbb{C}$ is of the form $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$,*
2. *$\mathbb{C}$ is of the form $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$, where $\mathbb{A}$ is the category of connected objects in $\mathbb{C}$,*
3. *$\mathbb{C}$ has $\mathbb{X}$-indexed coproducts, and every object A in $\mathbb{C}$ can be uniquely represented (up to isomorphism) as an $\mathbb{X}$-indexed coproduct of connected objects in $\mathbb{C}$.*

In other words, every category of the form $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$ is both connected and locally connected.

[2]i.e. $I(A)$ is the singleton set.

The opposite category of rings, Rings^{op}, and the opposite category of finite-dimensional[3] K-algebras, $\mathsf{K{-}Alg}^{op}$, can both be presented as categories of the form $\mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$. As both of these categories occur throughout the thesis, an outline of this fact will be provided below.

For the remainder of this section, let $\mathbb{C}$ denote the category $\mathsf{K{-}Alg}^{op}$.

Given a K-algebra A, any idempotent element $e \in A$ allows for the representation of A as the product:

$$A \cong eA \times \big(1-e\big)A$$

in $\mathbb{C}^{op}$. If one defines the following algebra homomorphisms:

- $\zeta(a) = \big(ea, (1-e)a\big)$,
- $\zeta^{-1}(ea_1, (1-e)a_2) = ea_1 + (1-e)a_2$,

and notes that $e(1-e) = 0$ for each idempotent, one can immediately calculate:

$$\zeta^{-1}\big(\zeta(a)\big) = \zeta^{-1}\big(ea, (1-e)a\big) = ea + (1-e)a = a,$$

$$\begin{aligned}\zeta\big(\zeta^{-1}(ea_1, (1-e)a_2)\big) &= \zeta\big(ea_1 + (1-e)a_2\big) \\ &= \big(e\big(ea_1 + (1-e)a_2\big), (1-e)\big(ea_1 + (1-e)a_2\big)\big) \\ &= \big(ea_1 + (1-e)a_2\big),\end{aligned}$$

which shows that ζ and ζ^{-1} are actually inverse to one another.

More generally, it is shown in Section (6.3) that every finite dimensional K-algebra A has a finite number $e_1, \ldots, e_n$ of idempotents which are orthogonal and sum to one, i.e:

1. $\displaystyle\sum_{i=1}^{n} e_i = 1$,
2. $i \neq j \Rightarrow e_i \cdot e_j = 0$.

With these idempotents, one can construct a more general isomorphism in $\mathbb{C}^{op}$:

$$A \underset{\zeta^{-1}}{\overset{\zeta}{\rightleftarrows}} \prod_{i=1}^{n} e_i A \tag{2.2}$$

where:

[3] Finite-dimensional as vector spaces.

- $\zeta(a) = \big(e_1 a, \ldots, e_n a\big)$
- $\zeta^{-1}(e_1 a_1, \ldots, e_n a_n) = \sum_{i=1}^{n} e_i a_i$

This means that:

$$A \cong \coprod_{i=1}^{n} e_i A$$

in $\mathbb{C}$. Of course, the same holds true in the (opposite) category of rings. Requiring that the objects in $\mathbb{C}$ be finite-dimensional as vector spaces is equivalent to requiring that the rings that underlie the algebras have a finite number of idempotents, and one should stipulate the latter condition when describing the situation for rings.

The proof of the following proposition makes use of concepts and results that are detailed in Section (6.3). Briefly, the idempotent elements in each ring A (and so each K-algebra) form a Boolean algebra $\mathsf{Idemp}(A)$, which has meet $a_1 \wedge a_2 = a_1 \cdot a_2$ given by multiplication in A, and join $a_1 \vee a_2 = a_1 + a_2 - 2a_1 a_2$. The set of atoms (non-zero minimal elements) in $\mathsf{Idemp}(A)$ will be denoted by $I(A)$. In fact, the atoms of A coincide with the elements $e_1, \ldots, e_n$ mentioned in Diagram (2.2).

Proposition 2.1.3. *Let $\mathbb{A}$ denote the opposite category of finite-dimensional K-algebras with no non-trivial idempotents, and let $\mathbb{X}$ be the category of finite sets. There is an equivalence of categories:*

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$$

Proof. Here, the functors are given by:

1. $F(A) = \big(e_i A\big)_{e_i \,\in\, I(A)}$
2. $G\big(C_i\big)_{i \in I(C)} = \coprod_{i=1}^{n} C_i$

For F to be well defined, it has to be shown that for every K-algebra A, and every non-zero minimal idempotent e_i in A, that $e_i A$ has no non-trivial idempotents. To see that this is indeed the case, note that any element $a \in \mathsf{Idemp}(A)$ in the Boolean algebra of idempotents in A can be presented as a finite join of the atoms in that Boolean algebra. From this, any idempotent $e_i a$ in $e_i A$ must be of the form:

$$e_i a = e_i \wedge \big(e_{j_1} \vee e_{j_2} \vee \cdots \vee e_{j_m}\big) = (e_i \wedge e_{j_1}) \vee \cdots \vee (e_i \wedge e_{j_m})$$

Since e_i is an atom, it can be identified with one of the e_{j_k}. Further, since the atoms are orthogonal (and in particular $i \neq j_k \Rightarrow e_i \wedge e_{j_k} = e_i \cdot e_{j_k} = 0$), the idempotent $e_i a$ will be given either by e_i or by 0, and therefore each $e_i A$ has no non-trivial idempotents.

Consider a morphism $\alpha : A \to B$ in $\mathbb{C}$. Since α is an algebra homomorphism from B to A, for each $e \in I(A) = \{e_1, e_2, \ldots, e_n\}$ one has the following composite in $\mathbb{C}^{op}$:

$$B \xrightarrow{\alpha} A \xrightarrow{\zeta} \prod_{e_i \in I(A)} e_i A \xrightarrow{\pi_e} eA$$

The elements in $I(B) = \{e'_1, e'_2, \ldots, e'_m\}$ have the properties that:

1. $\sum_{i=1}^{m} e'_i = 1$
2. $e'_i e'_j \neq 0 \Leftrightarrow i \neq j$

and this guarantees that for each $e \in I(A)$, only one atom in $e' \in I(B)$ is mapped to e under the composite. Since the composite preserves idempotents, it is clear that each $e' \in I(B)$ must be mapped either to 0 or to e in eA. If the composite maps no elements in $I(B)$ to e, the first condition (1.) on $I(B)$ would result in the composite mapping $1 \mapsto 0$, which is obviously impossible. If the composite mapped more than one element in $I(B)$ to e, the second condition (2.) on $I(B)$ would contradict the fact that the composite preserves multiplication.

Since the association of $e' \mapsto e$ is unique, it determines a set map:

$$I(\alpha) : I(A) \to I(B) \quad , \quad e \mapsto e'$$

If $e' \in I(B)$ is the unique element mapped to e under the composite, every element above e' in the Boolean algebra of idempotents of B will be mapped to e under the composite as well. Thus, one could equally well characterize $I(\alpha)$ as the mapping:

$$I(\alpha) : I(A) \to I(B) \quad , \quad e \mapsto min\{e' \in \mathsf{Idemp}(B) : e \leq \alpha(e')\}$$

Certainly, if $I(\alpha)(e_i) = e'_j$, then $\alpha(e'_j) = e_i$.[4] Therefore, for each e_i, one can restrict the domain of α to $I(\alpha)(e_i)B$, and so obtain the following family of morphisms in $\mathbb{A}^{op}$:

$$\big(\alpha_i : I(\alpha)(e_i)B \to e_i A\big)_{e_i \,\in\, I(A)}$$

[4] When α is considered as an algebra homomorphism from B to A.

which can be regarded as a family of morphisms in $\mathbb{A}$ in the obvious way.

Cumulatively, F sends every morphism $\alpha : A \to B$ in $\mathbb{C}$ to the pair $\big(I(\alpha), (\alpha_i)_{e_i \,\in\, I(A)}\big)$.

G is clearly well-defined on objects, and acts on morphisms via the universal property of the coproduct.

Verifying that the functors F and G are adjoint is straight-forward.

The object underlying the coproduct $\coprod_{i=1}^{n} C_i$ in $\mathbb{C}$ is the product $\prod_{i=1}^{n} C_i$ in $\mathbb{C}^{op}$. Multiplication in the product is computed component-wise, so its idempotents will be those elements whose components are all idempotent. Since each C_i has no non-trivial idempotents, the idempotents of the product $\prod_{i=1}^{n} C_i$ will be the elements which have either 0 or 1 (technically 0_{C_i} or 1_{C_i}) in each of their $I(C)$ components. Given this, it is clear that the non-zero minimal idempotents of the product are those elements which have a 1 in the i^{th} component, and 0's everywhere else. That is, elements of the form:

$$\delta_i := (0, \ldots, 1, \ldots, 0)$$

The assignment $\delta_i \mapsto i$ determines a bijection between the set of minimal idempotents of $\prod_{i=1}^{n} C_i$ and $I(C)$. This bijection, along with the (opposite of the) following family of presciently-named algebra isomorphisms:

$$\varepsilon_{C_i} : C_i \to \delta_i \prod_{j=1}^{n} C_j \quad , \quad c \mapsto (0, \ldots, c, \ldots, 0)$$

is the pair representing the counit for the adjunction. The unit is given by the (opposite of the) isomorphism $\zeta^{-1} : \prod_{e_i \in I(A)} e_i A \to A$ in Diagram (2.2). □

Proposition (2.1.2) now indicates that the connected objects in $\mathbb{C} = \mathsf{K\text{--}Alg}^{op}$ are those algebras with no non-trivial idempotents. Indeed:

Lemma 2.1.1 (Borceux and Janelidze [4]). *If a non-zero K-algebra A has no non-trivial idempotents, then the functor:*

$$\mathsf{Hom}_{\mathbb{C}}(A, -) : \mathbb{C} \to \mathsf{Sets}$$

preserves finite coproducts.

Proof. Fix an algebra $A \in \mathbb{C}$ with no non-trivial idempotents. To show that $\mathsf{Hom}_{\mathbb{C}}(A, -)$ preserves finite coproducts, one need only prove that it preserves trivial and binary coproducts.

There are no K-algebra homomorphisms which have the zero ring as their domain, so $\mathsf{Hom}_{\mathbb{C}}(A, -)$ trivially preserves the initial object/zero ring/empty coproduct in $\mathbb{C}$.

Since $\mathbb{C} \simeq \mathsf{Fam}(\mathbb{A})$ is extensive[5], showing that $\mathsf{Hom}_{\mathbb{C}}(A, -)$ preseves binary coproducts is equivalent to showing that each morphism of the form $\phi : A_1 \times A_2 \to B$ in $\mathbb{C}^{op}$ factors through one of the product projections $\pi_i : A_1 \times A_2 \to A_i$. Consider the idempotents $(1, 0)$ and $(0, 1)$ in $A_1 \times A_2$. Since they are orthogonal $((1, 0) \cdot (0, 1) = (0, 0))$ and sum to $(1, 1)$, the multiplicative identity of $A_1 \times A_2$, it is clear that if ϕ is an algebra homomorphism, it must map one of the two idempotents to 1_C, and map the other to 0 (which, by assumption, are the only idempotents in C). Explicitly, one must have either:

$$\big((1,0) \overset{\phi}{\mapsto} 1 \text{ and } (0,1) \overset{\phi}{\mapsto} 0\big) \quad \text{or} \quad \big((1,0) \overset{\phi}{\mapsto} 0 \text{ and } (0,1) \overset{\phi}{\mapsto} 1\big)$$

That is, ϕ must factor though one of the product projections in $\mathbb{C}^{op}$. □

2.2 Extensivity

Extensivity is a property that encapsulates favourable behaviour of the coproducts in a category [10], while simultaneously implying the existence of pullbacks along (binary) coproduct injections in that category. It is significantly more simple to show that a given functor is admissible[6] when the domain of that functor is extensive.

Definition 2.2.1. *A category with finite coproducts is extensive when the functor:*

$$+ : (\mathbb{C} \downarrow X) \times (\mathbb{C} \downarrow Y) \to (\mathbb{C} \downarrow (X+Y)) \quad , \quad \big((U, \upsilon), (V, \nu)\big) \mapsto (U + V, \upsilon + \nu)$$

is an equivalence of categories.

Of course, there is a natural analogue for the notion of an infinitely extensive category, where the "coproduct functor" $+$ is an equivalence over arbitrary (rather than binary) coproducts.

It is easy to show that if $+$ has a right adjoint, it will have the following association on objects:

[5] See Proposition (2.1.3) and Proposition (2.2.1).
[6] See Definition (4.1.1).

$$(W, \omega : W \to X + Y) \mapsto \big((X \times_{X+Y} W, \pi_1), (W \times_{X+Y} Y, \pi_2)\big)$$

where the objects are obtained by taking the pullbacks of ω along the first and second coproduct injections into $X + Y$, respectively.

Extensivity has a particularly intuitive diagrammatic presentation. If one fixes objects X and Y in $\mathbb{C}$, and considers Diagram (2.3):

$$\begin{array}{ccccc} U & \longrightarrow & W & \longleftarrow & V \\ {\scriptstyle \upsilon}\downarrow & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow[\iota_1]{} & X + Y & \xleftarrow[\iota_2]{} & Y \end{array} \tag{2.3}$$

one can see that if $\mathbb{C}$ is extensive, then the top line of Diagram (2.3) is a coproduct if and only if both squares in the diagram are pullbacks.

Extensivity is a reformulation of the notions of disjoint and universal coproducts, which were terms used by Grothendieck [16].

Definition 2.2.2. *Let $\mathbb{C}$ be a category with finite coproducts and pullbacks along coproduct injections.*

1. *$\mathbb{C}$ has disjoint coproducts if the pullback of the two injections of any given binary coproduct is the initial object in $\mathbb{C}$,*
2. *$\mathbb{C}$ has universal coproducts if the pullback of any morphism into a binary coproduct, along either of the coproduct injections, yields a coproduct diagram.*

Lemma 2.2.1 (Carboni, Lack, and Walters [10])**.** *Extensive categories have disjoint, universal coproducts.*

Proof. Take objects X, Y in an extensive category $\mathbb{C}$, and consider the following diagram:

$$\begin{array}{ccccc} X & = = = & X & \xleftarrow{\;!\;} & 0 \\ \| & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle !} \\ X & \xrightarrow[\iota_1]{} & X + Y & \xleftarrow[\iota_2]{} & Y \end{array} \tag{2.4}$$

Since the top line in Diagram (2.4) is a coproduct, it follows that both of the squares in the diagram are pullbacks. Therefore, the right-hand square exhibits the diagrammatic requirement for disjointness, and the left-hand square shows that ι_1 is a monomorphism (and of course the same holds for ι_2).

The alternative formulation of extensivity described via Diagram (2.3) shows precisely that coproducts will be universal in extensive categories. ☐

Since the coproduct injections in extensive categories are necessarily monomorphisms, if the morphisms in the category are set-maps, then one can think of the pullbacks of the inverse functor of $+$ as being the set-theoretic pre-image:

$$W \times_{X\,+\,Y} X \cong \omega^{-1}(X)$$

Proposition 2.2.1 (Borceux and Janelidze [4])**.** *Every category of families* $\mathbb{C} = \mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$ *is extensive.*

Proof. The functor $+$ has the following form in the infinite case:

$$\prod_{\theta\in\Theta}(\mathbb{C}\downarrow X_\theta) \xrightarrow{+} (\mathbb{C}\downarrow \coprod_{\theta\in\Theta} X_\theta) \quad , \quad \big(\upsilon_\theta : U_\theta \to X_\theta\big)_{\theta\in\Theta} \longmapsto \big(\coprod_{\theta\in\Theta} U_\theta \to \coprod_{\theta\in\Theta} X_\theta\big)$$

The inverse to $+$ can be constructed as follows:

As a left adjoint, I preserves coproducts. Thus, one has the following isomorphism in $\mathbb{X}$:

$$I\big(\coprod_{\theta\in\Theta} X_\theta\big) \cong \coprod_{\theta\in\Theta} I(X_\theta)$$

Further, one can use the following pullback in $\mathbb{X}$:

$$\begin{array}{ccc} I(\omega)^{-1}\big(I(X_\theta)\big) & \longrightarrow & I(X_\theta) \\ {\scriptstyle \subseteq}\big\downarrow & & \big\downarrow{\scriptstyle I(\iota_\theta)} \\ I(W) & \xrightarrow[I(\omega)]{} & \coprod_{\theta\in\Theta} I(X_\theta) \end{array}$$

to define a sub-family:

$$W_\theta := \big(W_i\big)_{i\,\in\, I(\omega)^{-1}(I(X_\theta))}$$

of $W = \big(W_i\big)_{i\in I(W)}$ for each $\theta \in \Theta$. One can also define $\omega_\theta : W_\theta \to X_\theta$ as the appropriate restriction of ω.

$(W, \omega) \mapsto \big(W_\theta, \omega_\theta\big)_{\theta\in\Theta}$ is the object association of the functor inverse to $+$.

Given $\big(\upsilon_\theta : U_\theta \to X_\theta\big)_{\theta\in\Theta}$, it is clear that:

- $I(\upsilon_\theta)^{-1}\big(I(X_\theta)\big) = I(U_\theta)$,
- One has the restriction $\Big(\coprod\limits_{\theta' \,\in\, \Theta} \upsilon_{\theta'}\Big)_\theta = \upsilon_\theta$.

It is therefore also clear that:

$$\Big(\coprod_{\theta' \,\in\, \Theta} U_{\theta'}\Big)_\theta = \big(U_i\big)_{i \,\in\, I(U_\theta)} = U_\theta \quad \forall\, \theta \in \Theta$$

Showing that the other composite is isomorphic to the identity functor is equally simple, if more difficult to write succinctly. ☐

2.3 Induced Adjunctions in Comma Categories

In the following section, a given adjunction $\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$ between categories will be used to induce adjunctions between various comma categories of $\mathbb{C}$ and $\mathbb{X}$, and these comma categories will be used throughout the chapters to follow.

Case 1:

If both $\mathbb{C}$ and $\mathbb{X}$ have pullbacks, and one takes an object $B \in \mathbb{C}$, one can induce:

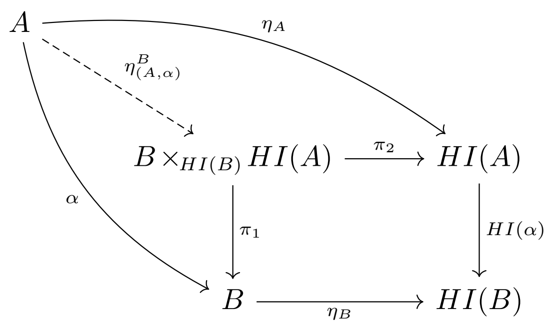
$$\big(\mathbb{C} \downarrow B\big) \underset{H_B}{\overset{I_B}{\rightleftarrows}} \big(\mathbb{X} \downarrow I(B)\big)$$

where:

1. $I_B(A, \alpha) = \big(I(A), I(\alpha) : I(A) \to I(B)\big)$,
2. $H_B(X, \varphi) = \big(B \times_{HI(B)} H(X), \pi_1 : B \times_{HI(B)} H(X) \to B\big)$, whose object is given by the pullback of $H(\varphi)$ along η_B, as in:

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{\pi_2} & H(X) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle H(\varphi)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array} \tag{2.5}$$

3. $H_B I_B(A,\alpha) = \big(B \times_{HI(B)} HI(A), \pi_1\big)$. Since the outer rectangle in Diagram (2.5) is a naturality square, it is clear that the unit $\eta^B_{(A,\alpha)} = \langle \alpha, \eta_A \rangle$ for $I_B \dashv H_B$ is determined by the universal property of the pullback, as in:



4. $I_B H_B(X,\varphi) = \big(I\big(B \times_{HI(B)} H(X)\big), I(\pi_1)\big)$, and so the obvious candidate for the counit $\varepsilon^B_{(X,\varphi)}$ of $I_B \dashv H_B$ is given by:

$$\varepsilon_X \circ I(\pi_2) : I\big(B \times_{HI(B)} H(X)\big) \to IH(X) \to X$$

Indeed, by pasting together the evident diagrams, the composite can easily be shown to be the universal arrow from I_B to (X,φ).

Adjunctions of the form $I_B \dashv H_B$ will play an important role in the definition of admissible functors,[7] and will be used to provide a setting in which to construct the fundamental theorem of Galois theory in an abstract category.[8]

Case 2:

Recall that if $\mathbb{C}$ is a category with a terminal object 1, then $\big(1 \downarrow \mathbb{C}\big)$ is known as category of the pointed objects of $\mathbb{C}$.

If one assumes that $\mathbb{C}$ and $\mathbb{X}$ have terminal objects and, further, that I preserves the terminal object (as a right adjoint, H always will), one can construct the following induced adjunction:

$$\big(1 \downarrow \mathbb{C}\big) \underset{(1 \downarrow H)}{\overset{(1 \downarrow I)}{\rightleftarrows}} \big(1 \downarrow \mathbb{X}\big)$$

where:

[7] See Definition (4.1.1).
[8] See Chapter (4).

1. $(1 \downarrow I)\big(A, a : 1 \to A\big) = \big(I(A), 1 \cong I(1) \to I(A)\big)$,
2. $(1 \downarrow H)\big(X, x : 1 \to X\big) = \big(H(X), 1 \cong H(1) \to H(X)\big)$.

If one takes $\big(A, a : 1 \to A\big) \in \big(1 \downarrow \mathbb{C}\big)$ and observes that the following is a naturality square for η:

$$\begin{array}{ccc} 1 & \xrightarrow[\cong]{\eta_1} & HI(1) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle HI(a)} \\ A & \xrightarrow[\eta_A]{} & HI(A) \end{array}$$

one sees that the unit for $(1 \downarrow I) \dashv (1 \downarrow H)$ is given by the unit of the original adjunction.

An analogous condition holds for the counit of $(1 \downarrow I) \dashv (1 \downarrow H)$.

These adjunctions allow one to extend the notions of admissibility and effective descent[9] from given categories to their categories of pointed objects, and this will facilitate the description of a new, pointed version of the fundamental theorem of Galois theory in an abstract category.[10]

[9] See Section (4.2.1).
[10] See Corollary (4.5.1).

Chapter 3

Categorical Semidirect Products

3.1 On the Form of Semidirect Products

This section provides an introduction to the notion of categorical semidirect products, discusses their definition in the context of monoidal categories, and shows that the categorical semidirect products of both groups and unital commutative rings coincide with their respective classical notions.

The section makes use of several notions from categorical algebra. If the reader is unfamiliar with the subject, they may wish to read briefly through [2], [3], or (more briefly still) Appendices (A.3) – (A.8).

If a category $\mathbb{C}$ has pullbacks (of split epimorphisms),[1] then every morphism $p : E \to B$ in $\mathbb{C}$ induces a functor:

$$p^* : \mathsf{Pt}(B) \to \mathsf{Pt}(E) \quad , \quad (A, f, s) \mapsto (E \times_B A, \pi_1, <1_E, s \circ p>),$$

which maps each point (A, f, s) to the pullback of f along p (together with the first projection and its canonical splitting).

The left adjoint of p^* will exist when $\mathbb{C}$ has pushouts of split monomorphisms. In such cases, it is given as follows:

$$p_! : \mathsf{Pt}(E) \to \mathsf{Pt}(B) \quad , \quad (D, g, t) \mapsto (D +_E B, [p \circ g, 1_B], \iota_2),$$

For each $p : E \to B$ in $\mathbb{C}$, one can describe the existence of p^* in a more general setting

[1] See Appendix (A.3).

[8] by defining the functors in terms of split pullbacks, rather than pullbacks of split epimorphisms.

Definition 3.1.1 (Bourn and Janelidze [8])**.** *Consider the following in an arbitrary category* $\mathbb{C}$*:*

1. *A diagram of the form:*

$$\begin{array}{ccc} D & \xrightarrow{\;q\;} & A \\ {\scriptstyle t}\uparrow\downarrow{\scriptstyle g} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ E & \xrightarrow[\;p\;]{} & B \end{array} \tag{3.1}$$

   is a split commutative square if $(A, f, s) \in \mathsf{Pt}(B)$*,* $(D, g, t) \in \mathsf{Pt}(E)$*,* $f \circ q = p \circ g$ *and* $s \circ p = q \circ t$*,*

2. *A split commutative square* (3.1) *is a split pullback if, for every split commutative square:*

$$\begin{array}{ccc} D' & \xrightarrow{\;q'\;} & A \\ {\scriptstyle t'}\uparrow\downarrow{\scriptstyle g'} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ E & \xrightarrow[\;p\;]{} & B \end{array}$$

   there exists a unique morphism $\delta : D' \to D$ *in* $\mathbb{C}$ *such that* $g \circ \delta = g'$*,* $\delta \circ t' = t$ *and* $q \circ \delta = q'$*,*

3. *A split commutative square* (3.1) *is a split pushout if, for every split commutative square:*

$$\begin{array}{ccc} D & \xrightarrow{\;q'\;} & A' \\ {\scriptstyle t}\uparrow\downarrow{\scriptstyle g} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} \\ E & \xrightarrow[\;p\;]{} & B \end{array}$$

   there exists a unique morphism $\alpha : A \to A'$ *in* $\mathbb{C}$ *such that* $f' \circ \alpha = f$*,* $\alpha \circ s = s'$ *and* $\alpha \circ q = q'$*,*

4. $\mathbb{C}$ *has split pullbacks if for every morphism* $p : E \to B$ *and every* $(A, f, s) \in \mathsf{Pt}(B)$*, there is a split pullback diagram of the form* (3.1)*,*

5. $\mathbb{C}$ *has split pushouts if for every morphism* $p : E \to B$ *and every* $(D, g, t) \in \mathsf{Pt}(E)$*, there is a split pushout diagram of the form* (3.1)*.*

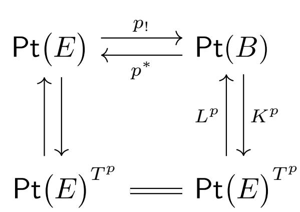
Bourn and Janelidze [8] note that while pullbacks of split epimorphisms are split pullbacks, there are categories which have split pullbacks that are not pullbacks of split epimorphisms. Therefore, in a category $\mathbb{C}$ with split pullbacks, there is a generalized notion of the definition of p^*, which maps any $(A, f, s) \in \mathsf{Pt}(B)$ to the split pullback of f along p. Accordingly, the left adjoints $p_!$ of the generalized functors p^* will exist if and only if $\mathbb{C}$ has split pushouts.

Bourn and Janelidze [8] also note that in a category which has split pullbacks, split pushouts, and pullbacks, the split pushouts coincide with pushouts of split monomorphisms.

Definition 3.1.2. *Let $\mathbb{C}$ be a category with split pullbacks. $\mathbb{C}$ has semidirect products if, for every morphism $p : E \to B$, the functor $p^* : \mathsf{Pt}(B) \to \mathsf{Pt}(E)$ is monadic.*

Bourn and Janelidze [8] introduced the notion of the categorical semidirect product, and showed that in the category of groups, the notions of the classical and categorical semidirect product coincide. It was shown by Borceux, Janelidze, and Kelly [5] that an analogue for the equivalence between the categories of split extensions and internal object actions[2] in Groups holds, more generally, for any semi-abelian variety.

If $\mathbb{C}$ has semidirect products, then the comparison functor K^p for the monad $T^p = p_! \circ p^*$ is an equivalence of categories.



For a given T^p-algebra $(X, h : T^p(X) \to X)$, one can define the semidirect product $(B, p) \ltimes (X, h)$ of (B, p) with (X, h) as the point in $\mathsf{Pt}(B)$ that corresponds to (X, h) under K^p. The above is short-hand notation, as $X \in \mathsf{Pt}(E)$.

What's more, when $\mathbb{C}$ is pointed, one can use morphisms of the form $i_B : 0 \to B$ to induce the monad $T^{i_B} = T^B = B\flat(-)$ on $\mathsf{Pt}(0) \cong \mathbb{C}$. The uniqueness of each i_B allows one to simplify the notation of semidirect products under $B\flat(-)$. One writes:

$$(B, i_B) \ltimes (X, h) = B \ltimes (X, h)$$

Although the form of categorical semidirect products in more general settings may be relatively complicated, they can be calculated explicitly in pointed contexts.

[2]See Section (3.3.2).

Theorem 3.1.1 (Bourn and Janelidze [8])**.** *Let $\mathbb{C}$ be an exact category. $\mathbb{C}$ has semidirect products if and only if it is protomodular and has pushouts of split monomorphisms.*

Proof.

($\Rightarrow$) : Suppose that $\mathbb{C}$ is exact and that it has semidirect products. By the latter statement, each functor p^* has a left adjoint $p_!$, and is monadic. Since $\mathbb{C}$ has split pullbacks, split pushouts, and finite limits, the existence of $p_!$ is equivalent to the existence of pushouts of split monomorphisms in $\mathbb{C}$.

If $\mathbb{C}$ is not protomodular, then there must exist some $p : E \to B$ in $\mathbb{C}$ such that p^* does not reflect isomorphisms. However, the Beck conditions for monadicity show that if p^* does not reflect isomorphisms, then it is not monadic (which would be impossible, as it would contradict the fact that $\mathbb{C}$ has semidirect products).

($\Leftarrow$) : Suppose that $\mathbb{C}$ is exact, protomodular, and has pushouts of split monomorphisms.

Since $\mathbb{C}$ is protomodular and has finite limits, it is Mal'cev [3].

By Theorem (A.7.1), every reflexive pair in $\mathbb{C}$ corresponds directly to an equivalence relation, and each pair has the same coequalizer as its corresponding equivalence relation. $\mathbb{C}$ being exact also guarantees that these coequalizers (regular epimorphisms) are pullback stable, i.e. that they will be preserved under p^*. Since pullbacks and coequalizers in comma categories are calculated as in the base category, the same conditions hold true in the categories of points of $\mathbb{C}$.

Since $\mathbb{C}$ is protomodular, so is each of its categories of points.[3] Each p^* reflects isomorphisms, and one is able to conclude that each p^* is monadic by the reflexive form of Beck's Monadicity Theorem [25].

□

Of course, this means that every semi-abelian category has semidirect products.

When a category $\mathbb{C}$ has semidirect products, there is a correspondence between internal actions[4] and split extensions in $\mathbb{C}$. That is, if $\mathbb{C}$ has semidirect products, then each action $h : B\flat X \to X$ corresponds to the split extension (A, f, s, X) for which the following diagram commutes:

$$\begin{array}{ccccc} B\flat X & \xrightarrow{\kappa_{B,X}} & B+X & \underset{\iota_1}{\overset{\pi_{B,X}}{\rightleftarrows}} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle [s,k]} & & \| \\ X & \xrightarrow[k]{} & A & \underset{s}{\overset{f}{\rightleftarrows}} & B \end{array} \tag{3.2}$$

Here, $(A, f, s) \in \mathsf{Pt}(B)$, $X = Ker(f)$, $\pi_{B,X} = [1_B, 0]$, and $\kappa_{B,X} = ker(\pi_{B,X})$. Accordingly, given an action h, one can identify A in Diagram (3.2) as $B \ltimes (X, h)$.

[3]See Appendix (A.5.), or [3].
[4]See Section (3.3.2).

Inyangala [15] showed that in any variety of right Ω-loops,[5] the form of the categorical semidirect products is determined by the inherent operations of the variety.

Inyangala showed that in any variety which has operations satisfying the defining identities, one can use these operations to define bijective set maps φ and ψ which show that the categorical semidirect product $B \ltimes (X, h) = A$[6] has $B \times X$ as its underlying set. The two functions are defined as follows:

$$\varphi : B \times X \to A \quad , \quad (b, x) \mapsto s(b) + k(x)$$

$$\psi : A \to X \times B \quad , \quad a \mapsto k^{-1}\big(a - sf(a)\big)$$

Here, $a - sf(a)$ is an element in $X = Ker(f)$, as $f(a - sfa) = f(a) - f(a) = 0$. It is easy to verify that the maps are inverse to each other.

Cumulatively, this means that for any split extension (A, f, s, X) in a variety of right Ω-loops, one can easily see that the following diagram (of split extensions) commutes in the category of pointed sets:

$$\begin{array}{ccccc} X & \xrightarrow{k} & A & \underset{s}{\overset{f}{\rightleftarrows}} & B \\ \| & & \varphi \uparrow \; \downarrow \psi & & \| \\ X & \xrightarrow[<0,1>]{} & B \times X & \underset{<1,0>}{\overset{\pi_1}{\rightleftarrows}} & B \end{array} \tag{3.3}$$

Inyangala [15] also showed that in a semi-abelian variety which has binary operations $+$ and $-$, the maps φ and ψ induced by the operations will be bijections if and only if the operations obey the axioms of right Ω-loops.

This result was extended by Gray and Martins-Ferreira [13], who showed that varieties which have bijections φ and ψ also necessarily have operations $+$ and $-$ that satisfy the axioms of right Ω-loop. This shows that varieties of right Ω-loops are the only varieties in which the categorical semidirect products $B \ltimes (X, h)$ have the set-products $B \times X$ as their underlying sets.

Clementino, Montoli, and Sousa [11] extended these results further still, by using the characterization of semi-abelian varieties in terms of the operations $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \theta\}$[7] to explicitly describe the analogues of φ and ψ in the general semi-abelian context, and applying this to topological models in semi-abelian varieties (which goes far beyond the scope of the present work). They showed that if a semi-abelian variety contains given operations $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \theta\}$, then these operations can be used to construct set-maps:

[5] See Definition (A.5.3).
[6] Corresponding to an internal action $h : B\flat X \to X$ of B on X.
[7] See Theorem (A.8.1).

$$\varphi : X^n \times B \longrightarrow A \quad , \quad (x_1, x_2, \ldots, x_n, b) \overset{\varphi}{\longmapsto} \theta\big(x_1, x_2, \ldots, x_n, s(b)\big)$$

$$\psi : A \longrightarrow X^n \times B \quad , \quad a \overset{\psi}{\longmapsto} \big(\alpha_1(a, sfa), \ldots, \alpha_n(a, sfa), f(a)\big)$$

which generalize the maps given by Inyangala.

It should be noted that θ is actually a function of $k(x_i)$, and that the first n components of $\psi(a)$ are given by $k^{-1}\big(\alpha_i(a, sfa)\big)$. Since the form of elements in A need not match those in X, including k in the notation is often helpful (even if it can always be thought of as an inclusion). With this said, the notation will be suppressed in the context of general semi-abelian varieties, for the sake of legibility. The notation $X^n \times B$ used in [11] (which is mirrored here for ease of reference) with B as the last component in the product is ostensibly made for use in the generalized version of Diagram (3.3).

Further on this, in the general semi-abelian context, a small number of abuses of notation will be made for the sake of legibility:

1. $(\underline{x}, b) := (x_1, x_2, \ldots, x_n, b)$,
2. $\beta(\underline{x}) := \big(\beta(x_1), \beta(x_2), \ldots, \beta(x_n)\big)$ for any map β with X as its domain,
3. $\underline{\alpha}(z_1, z_2) := \big(\alpha_1(z_1, z_2), \alpha_2(z_1, z_2), \ldots, \alpha_n(z_1, z_2)\big)$ for all z_1, z_2.

It is clear that there is no obvious reason why φ and ψ should necessarily be bijections in a general semi-abelian variety.

Indeed, it can be shown [11] that one can restrict the domain of φ, and show that A is bijective to the set

$$\big\{(x_1, x_2, \ldots, x_n, b) \in X^n \times B \mid \underline{\alpha}\big(\theta\big(\underline{x}, s(b)\big), s(b)\big) = \underline{x}\big\}^{8} \tag{3.4}$$

From this, one can immediately deduce that for a given split extension (A, f, s, X) in a semi-abelian variety, A has $X^n \times B$ as its underlying object if and only if:

$$\underline{\alpha}\big(\theta(\underline{x}, y), y\big) = \underline{x} \text{ for all } \underline{x} \tag{3.5}$$

It is now clear that the semidirect products $B \ltimes (X, h)$ in varieties of right Ω-loops have $B \times X$ as their underlying sets because Equation (3.5) is satisfied by the third axiom[9] of right Ω-loops. Recall that Gray and Martins-Ferreira [13] showed that varieties of right Ω-loops are the only varieties with this property.

Of course, this observation prompts the question, "Can one find a semi-abelian variety that has operations $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \theta\}$, for $n \geq 2$, which satisfy Equation (3.5)?"

[8] Of course, this means that $\alpha_i\big(\theta\big(\underline{x}, s(b)\big), s(b)\big) = x_i \ \forall\ i \in \{1, 2, \ldots, n\}$.
[9] See Definition (A.5.3).

Interestingly, finding such a list of operations for $n \geq 2$ is possible - but only in varieties of right Ω-loops.

Theorem 3.1.2 (Clementino, Montoli, and Sousa [11])**.** *Let $\mathbb{C}$ be a semi-abelian variety with given operations $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \theta\}$:*

1. *For each split extension $X \xrightarrow{k} A \underset{s}{\overset{f}{\rightleftarrows}} B$ in $\mathbb{C}$, φ and ψ form a bijection between A and $X^n \times B$ if and only if Equation (3.5) is satisfied by the given operations.*
2. *If the operations $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \theta\}$ satisfy Equation (3.5), then they induce binary operations $+$ and $-$ which satisfy the identities defining right Ω-loops.*

Proof.

1. Given the definition of the set in Equation (3.4), the result follows immediately.
2. Given the list of operations $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \theta\}$ satisfying Equation (3.5), one can define binary operations $+$ and $-$ as follows:
   i) $x + y := \theta\big(\underline{\alpha}(x, 0), y\big)$,
   ii) $x - y := \theta\big(\underline{\alpha}(x, y), 0\big)$.

□

3.2 Monoidal Categories

Monoidal categories can be thought of as generalized categories with products [24]. Essentially, they are categories equipped with a "product" $\mathbb{C} \otimes \mathbb{C}$ that is associative, and has left and right unit (iso)morphisms, up to a natural isomorphism in $\mathbb{C}$. Full introductions to the subject can be found in [2], [5], and [24]. Several definitions from [5] are used in this chapter, in the interest of keeping the thesis as self-contained as possible.

Definition 3.2.1. *A monoidal category is a system $\big(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho\big)$, where:*

1. *$\mathbb{C}$ is a category,*
2. *$I \in \mathbb{C}$ is called the unit for $\otimes$,*
3. *$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a functor,*
4. *$\alpha \equiv (\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C)_{A,B,C \in \mathbb{C}}$ is a natural isomorphism,*
5. *$\lambda \equiv (\lambda_A : A \to I \otimes A)_{A \in \mathbb{C}}$ is a natural isomorphism,*
6. *$\rho \equiv (\rho_A : A \to A \otimes I)_{A \in \mathbb{C}}$ is a natural isomorphism,*

and where the above are such that the following diagrams commute:

$$\begin{array}{ccccc} A\otimes(B\otimes(C\otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & (A\otimes B)\otimes C)\otimes D)) \\ {\scriptstyle 1_A\otimes\alpha_{B,C,D}}\downarrow & & & & \uparrow{\scriptstyle \alpha_{A,B,C}\otimes 1_D} \\ A\otimes((B\otimes C)\otimes D)) & & \xrightarrow[\alpha_{A,B\otimes C,D}]{\qquad\qquad\qquad} & & (A\otimes(B\otimes C))\otimes D \end{array} \tag{3.6}$$

$$\begin{array}{ccc} A\otimes(I\otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A\otimes I)\otimes B \\ {\scriptstyle 1_A\otimes\lambda_B}\nwarrow & & \nearrow{\scriptstyle \rho_A\otimes 1_B} \\ & A\otimes B & \end{array}$$

As even the definition shows, using indices for α can be relatively cumbersome. As such, many of the morphisms $\alpha_{A,B,C}$ will simply be denoted by α.

Definition 3.2.2. *A monoidal category is strict if α, λ and ρ reduce to identity morphisms. That is, when:*

1. $A \otimes (B \otimes C) = (A \otimes B) \otimes C \ \forall \ A, B, C \in \mathbb{C}$,
2. $I \otimes A = A = A \otimes I \ \forall \ A \in \mathbb{C}$.

Monoidal categories form the setting required to define the notion of an internal monoid. One can circumvent the theory by simply defining an internal monoid as a triple (M, m, e) making the appropriate diagram commute in the category in question, but this still implicitly makes use of the fact that the category in question is monoidal.

As with adjunctions, examples of monoidal categories permeate mathematics. The following are only a few examples of this:

Example 3.2.1.

1. *The category of abelian groups $\mathsf{Ab} = (\mathsf{Ab}, \otimes, \mathbb{Z})$, with the familiar tensor product of groups, is the structure from which monoidal categories were generalized [5]. It is well known that the integers $\mathbb{Z}$ play the role of the unit for the tensor product of groups, and the natural isomorphisms taking the place of α, λ and ρ are well documented,*

2. *For a given unital commutative ring R, the category of R-modules $(\mathsf{R{-}Mod}, \otimes_R, R)$, with the tensor product over R as $\otimes$ and R itself as the unit, forms a monoidal category,*

3. *Any set monoid $M = (M, m, e)$ can be considered as a strict monoidal category. For this, one can regard the set M as a discrete category, with $m : M \times M \to M$ as $\otimes$, the one element set $\{*\}$ as I, and use the bijections $\{*\} \times M \cong M$ and $M \times \{*\} \cong M$ as λ and ρ respectively,*

4. *Any category $\mathbb{C}$ with finite products can be seen as a monoidal category by taking $\times = \otimes$, taking the terminal object (the empty product) 1 in $\mathbb{C}$ to be I, and taking α, λ and ρ to be the natural isomorphisms:*

$$\alpha_{A,B,C} : A \times (B \times C) \cong (A \times B) \times C,$$
$$\lambda_A : A \cong 1 \times A,$$
$$\rho_A : A \cong A \times 1,$$

induced by the properties of the categorical product,

5. *Dually, any category $\mathbb{C}$ with finite coproducts is a monoidal category $(\mathbb{C}, +, 0, \alpha, \lambda, \rho)$,*

6. *For any category $\mathbb{X}$, the category of endofunctors on $\mathbb{X}$ can be seen as a strict monoidal category. The order in which functors are composed is inconsequential to the result, and therefore if one takes the monoidal operation $\otimes$ to be the composition of functors $\circ$, and I to be the identity functor on $\mathbb{X}$, then $\mathsf{End}(\mathbb{X}) = (\mathsf{End}(\mathbb{X}), 1_{\mathbb{X}}, \circ)$ is a strict monoidal category.*

Having defined monoidal categories, one should also define the notions of "monoidal functor" and "monoidal natural transformation".

Definition 3.2.3. *Let $\big(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho\big)$ and $(\mathbb{C}', I', \otimes', \alpha', \lambda', \rho')$ be monoidal categories. A monoidal functor $(F, \theta, \phi) : \mathbb{C} \to \mathbb{C}'$ is comprised of:*

1. *A functor $F : \mathbb{C} \to \mathbb{C}'$,*

2. *A morphism $\theta : I' \to F(I)$ in $\mathbb{C}'$,*

3. *A natural transformation $\phi \equiv \big(\phi_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)\big)_{A,B \in \mathbb{C}}$*

such that the following diagrams commute:

$$\begin{array}{ccc}
F(A) \otimes' (F(B) \otimes' F(C)) & \xrightarrow{\alpha'} & (F(A) \otimes' F(B)) \otimes' F(C) \\
\downarrow {\scriptstyle 1_{F(A)} \otimes' \phi_{B,C}} & & \downarrow {\scriptstyle \phi_{A,B} \otimes' 1_{F(C)}} \\
F(A) \otimes' F(B \otimes C) & & F(A \otimes B) \otimes' F(C) \\
\downarrow {\scriptstyle \phi_{A,B \otimes C}} & & \downarrow {\scriptstyle \phi_{A \otimes B,C}} \\
F(A \otimes (B \otimes C)) & \xrightarrow[F(\alpha)]{} & F((A \otimes B) \otimes C)
\end{array} \tag{3.7}$$

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\lambda'} & I' \otimes' F(A) \\
{\scriptstyle F(\lambda)}\downarrow & & \downarrow{\scriptstyle \theta \otimes' 1_{F(A)}} \\
F(I \otimes A) & \xleftarrow[\phi_{I,A}]{} & F(I) \otimes' F(A)
\end{array}
\qquad
\begin{array}{ccc}
F(A) & \xrightarrow{\rho'} & F(A) \otimes' I' \\
{\scriptstyle F(\rho)}\downarrow & & \downarrow{\scriptstyle 1_{F(A)} \otimes' \theta} \\
F(A \otimes I) & \xleftarrow[\phi_{A,I}]{} & F(A) \otimes' F(I)
\end{array}
$$

Definition 3.2.4. *Let* $(F, \theta, \phi) : \mathbb{C} \to \mathbb{C}'$ *be a monoidal functor.*

1. (F, θ, ϕ) *is strong if both* θ *and all* $\phi_{A,B}$ *are isomorphisms,*
2. (F, θ, ϕ) *is strict if both* θ *and all* $\phi_{A,B}$ *are identity morphisms.*

A monoidal functor being strict immediately forces the identities:

1. $\alpha_{F(A),F(B),F(B)} = F(\alpha_{A,B,C})$,
2. $\lambda_{F(A)} = F(\lambda_A)$,
3. $\rho_{F(A)} = F(\rho_A)$,

to hold for all $A, B, C \in \mathbb{C}$.

The diagrams describing the general definition of a monoidal functor are relatively complicated, but restricting one's attention to either the strong or the strict monoidal functors yields a far simpler collection of diagrams (which, either implicitly or explicitly, operate in the same category).

Definition 3.2.5. *Let* $(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ *and* $(\mathbb{C}', I', \otimes', \alpha', \lambda', \rho')$ *be monoidal categories, and let* $(F_1, \theta_1, \phi), (F_2, \theta_2, \bar{\phi}) : \mathbb{C} \to \mathbb{C}'$ *be monoidal functors between them. A monoidal natural transformation* $\tau : (F_1, \theta_1, \phi) \to (F_2, \theta_2, \bar{\phi})$ *is a natural transformation* $\tau : F_1 \to F_2$ *for which the following diagrams commute:*

$$
\begin{array}{ccc}
 & \nearrow^{\theta_1} & F_1(I) \\
I & & \downarrow{\scriptstyle \tau} \\
 & \searrow_{\theta_2} & F_2(I)
\end{array}
\qquad
\begin{array}{ccc}
F_1(A) \otimes' F_1(B) & \xrightarrow{\phi_{A,B}} & F_1(A \otimes B) \\
{\scriptstyle \tau \otimes \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\
F_2(A) \otimes' F_2(B) & \xrightarrow[\bar{\phi}_{A,B}]{} & F_2(A \otimes B)
\end{array}
\tag{3.8}
$$

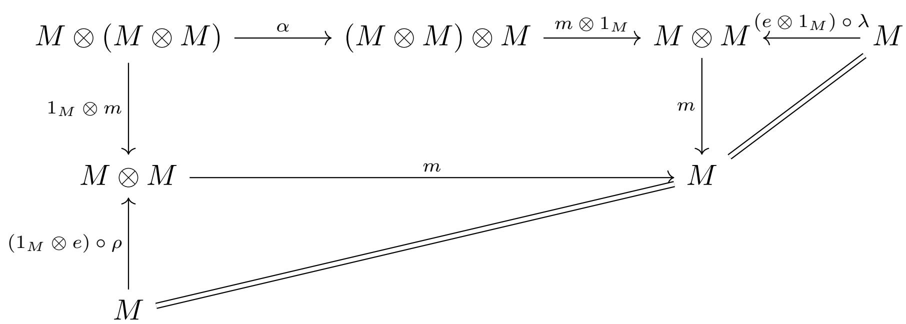
This illustrates that the collection of monoidal functors between two monoidal categories forms a category whose morphisms are monoidal natural transformations. Indeed, monoidal functors and monoidal natural transformations compose in the obvious way, and therefore the collection of monoidal categories forms a 2-category.

Definition 3.2.6. *Let* $(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ *be a monoidal category, and let* $\mathbf{1} = \{*\}$ *be the trivial set-monoid, considered as a monoidal category. An internal monoid in* $\mathbb{C}$ *is a monoidal functor* $(F, \theta, \phi) : \mathbf{1} \to \mathbb{C}$.

Internal monoids can be represented as a triples (M, m, e), where:

1. $M \in \mathbb{C}$,
2. $m : M \otimes M \to M$ is a morphism in $\mathbb{C}$,
3. $e : I \to M$ is a morphism in $\mathbb{C}$,

such that the following diagram commutes:



(3.9)

In more detail, since $(F, \theta, \phi) : \mathbf{1} \to \mathbb{C}$ is a monoidal functor:

- $M = F(*)$ is the only object in the image of F,
- $m = \phi_{*,*}$ is the only constituent map of the natural transformation ϕ,
- $e = \theta : I \to M$,
- Diagram (3.9) is obtained from the three diagrams in (3.7), which define the properties of a monoidal functor,
- $(1_M \otimes e) \circ \rho : M \to M \otimes I \to M \otimes M$ and $(e \otimes 1_M) \circ \lambda : M \to I \otimes M \to M \otimes M$ are the canonical left and right insertions of M into $M \otimes M$, respectively.

Moreover, if F_1 and F_2 represent respective monoids (M_1, m_1, e_1) and (M_2, m_2, e_2) in $\mathbb{C}$, a monoidal natural transformation $\tau : F_1 \to F_2$ is - by Definition (3.2.5) - such that $\tau(m_1) = m_2(\tau \otimes \tau)$ and $\tau(e_1) = e_2$, which is precisely the condition one would expect to find.

Definition 3.2.7. *Let (M_1, m_1, e_1) and (M_2, m_2, e_2) be internal monoids in $\mathbb{C}$. A morphism of internal monoids is a monoidal natural transformation $\tau : M_1 \to M_2$ between the functors the monoids represent.*

The collection of internal monoids (M, m, e) in a monoidal category $\mathbb{C}$ itself forms a category, which is denoted by $\mathsf{Mon}(\mathbb{C})$. Internal monoids in monoidal categories are generalizations of the familiar set-theoretic monoids, in that set-monoids are internal monoids in $(\mathsf{Sets}, \times, \{*\})$.

3.3 Category Actions and Internal Object Actions

At this point, two very natural, interlinked questions arise:

1. Given a monoidal category $\mathbb{C}$, is there a notion of a "monoidal action" of $\mathbb{C}$ on an arbitrary category $\mathbb{X}$?
2. Given an internal monoid (M, m, e) in a monoidal category $\mathbb{C}$, is there a notion of an "internal monoid action" of M on a given $A \in \mathbb{C}$?

Indeed, both of these notions exist.

For the notion of a monoidal action of $\mathbb{C}$, as soon as one recalls that for any $\mathbb{X}$, the category $\mathsf{End}(\mathbb{X})$ is a strict monoidal category, one is able to make the following definition:

Definition 3.3.1. *Let $(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ be a monoidal category, and let $\mathbb{X}$ be any category. A $\mathbb{C}$-action on $\mathbb{X}$ is a monoidal functor $(F, \theta, \phi) : \mathbb{C} \to \mathsf{End}(\mathbb{X})$.*

There is a convenient alternative formulation for $\mathbb{C}$-actions [5]:

Lemma 3.3.1. *A $\mathbb{C}$-action on $\mathbb{X}$ uniquely determines a triple $(\bullet, \theta, \gamma)$, where:*

1. $\bullet : \mathbb{C} \times \mathbb{X} \to \mathbb{X}$ *is a functor,*
2. $\theta \equiv (\theta_X : X \to I \bullet X)_{X \in \mathbb{X}}$ *is a natural transformation,*
3. $\gamma \equiv (\gamma_{A,B,X} : A \bullet (B \bullet X) \to (A \otimes B) \bullet X)_{A,B \in \mathbb{C}, X \in \mathbb{X}}$ *is a natural transformation.*

Proof. Given a monoidal functor $(F, \theta, \phi) : \mathbb{C} \to \mathsf{End}(\mathbb{X})$, one defines:

1. $A \bullet X = F(A)(X) = (F(A))(X)$,

2. $\theta_X : X \to F(I)(X)$ since $F(I)(X) = I \bullet X$,

3. $\gamma_{A,B,X} = \phi_{A,B}(X) : F(A)(F(B)(X)) \to F(A \otimes B)(X)$,

for all $A, B \in \mathbb{C}$ and for all $X \in \mathbb{X}$. □

This second formulation of actions by $\mathbb{C}$ on $\mathbb{X}$ very closely resembles the set-theoretic notion of a monoid action. Examples of category actions are wide-spread in mathematics. In particular, every monoidal category $(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ acts on itself canonically, with the action $(\bullet, \theta, \phi) := (\otimes, \lambda, \alpha)$.

3.3.1 Internal Monoid Actions

In order to properly discuss the notion of internal monoid actions (i.e. actions of internal monoids on objects in the given category), the idea of mapping internal monoids between monoidal categories (via a monoidal functor) must be made precise.

Suppose that $(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ and $(\mathbb{C}', I', \otimes', \alpha', \lambda', \rho')$ are monoidal categories, that $(F, \theta, \phi) : \mathbb{C} \to \mathbb{C}'$ is a monoidal functor between them, and that (M, m, e) is an internal monoid in $\mathbb{C}$. One has:

1. $\theta : I' \to F(I)$,

2. $\phi_{M,M} : F(M) \otimes' F(M) \to F(M \otimes M)$,

in $\mathbb{C}'$, and these hint at candidates for what the canonical monoid structure on $F(M)$ should be. It is clear that if one takes:

1. $e' = F(e) \circ \theta : I' \to F(I) \to F(M)$,

2. $m' = F(m) \circ \phi_{M,M} : F(M) \otimes' F(M) \to F(M \otimes M) \to F(M)$,

then $(F(M), m', e')$ satisfies the conditions in Definition (3.2.6). Indeed, the monoidal functor $(F, \theta, \phi) : \mathbb{C} \to \mathbb{C}'$ induces a functor:

$$\mathsf{Mon}(F) : \mathsf{Mon}(\mathbb{C}) \to \mathsf{Mon}(\mathbb{C}')$$

between the categories of internal monoids in $\mathbb{C}$ and $\mathbb{C}'$, and:

$$(M, m, e) \mapsto (F(M), m', e')$$

is the object-assignment of this functor.

In the special case where $\mathbb{C}'$ is taken to be the strict monoidal category $\mathsf{End}(\mathbb{X})$ – i.e. where $F : \mathbb{C} \to \mathsf{End}(\mathbb{X})$ is a $\mathbb{C}$-action on $\mathbb{X}$ – the induced functor:

$$\mathsf{Mon}(F) : \mathsf{Mon}(\mathbb{C}) \to \mathsf{Mon}(\mathsf{End}(\mathbb{X}))$$

sends each monoid (M, m, e) in $\mathbb{C}$ to:

$$(M \bullet (-), (m \bullet 1_X) \circ \gamma(-), (e \bullet 1_X) \circ \theta(-)) \tag{3.10}$$

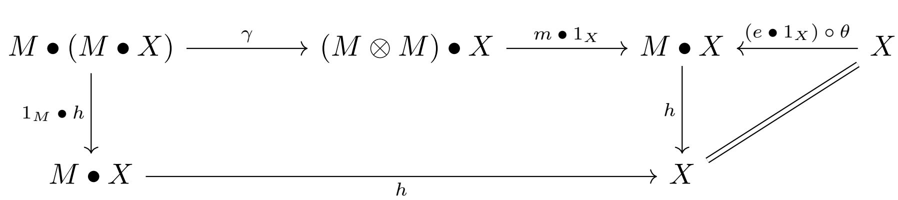
in $\mathsf{End}(\mathbb{X})$. For clarity, for each $X \in \mathbb{X}$, one can use Lemma (3.3.1) to see that:

1. $F(M)(X) = M \bullet X$,
2. $F(m) \circ \phi_{M,M}(X)$ is given by:
   $$((m \bullet 1_X) \circ \gamma_{M,M,X})(X) : M \bullet (M \bullet X) \to (M \otimes M) \bullet X \to M \bullet X,$$
3. $(F(e) \circ \theta)(X) : 1_{\mathbb{X}}(X) \to F(I)(X) \to F(M)(X)$ is given by:
   $$((e \bullet 1_X) \circ \theta)(X) : X \to I \bullet X \to M \bullet X.$$

Of course, internal monoids in $\mathsf{End}(\mathbb{X})$ are simply monads on $\mathbb{X}$.

The monoid structure $(M \bullet (-), (m \bullet 1_X) \circ \gamma(-), (e \bullet 1_X) \circ \theta(-))$ in $\mathsf{End}(\mathbb{X})$ allows for the following:

Definition 3.3.2. *Let $F : \mathbb{C} \to \mathsf{End}(\mathbb{X})$ be a $\mathbb{C}$-action on $\mathbb{X}$, let (M, m, e) be an internal monoid in $\mathbb{C}$, and take an object X in $\mathbb{X}$. An M-action on X is an action of the monad (3.10) on X, i.e. it is a morphism $h : M \bullet X \to X$ such that the following diagram commutes:*



The collection of all M-actions on objects in $\mathbb{X}$ forms a category, denoted by $\mathbb{X}^M$. Given two M-actions $h : M \bullet X \to X$ and $h' : M \bullet X' \to X'$, a morphism between the two is simply a morphism $\delta : X \to X'$ in $\mathbb{X}$ such that $\delta \circ h = h' \circ (1_M \bullet \delta)$.

3.3.2 Internal Object Actions

Recall that category $\mathbb{C}$ with finite coproducts is a monoidal category. In such categories, every object B in $\mathbb{C}$ has a unique internal monoid structure, with:

1. $m = [1_B, 1_B] : B + B \to B$,

2. $e = i_B : 0 \to B$.

The uniqueness of this association allows one to see that $\mathbb{C} = \mathsf{Mon}(\mathbb{C})$. This, in conjunction with the fact that every monoidal category acts canonically on itself, allows one to construct the functor:

$$F = \mathsf{Mon}(F) : \mathsf{Mon}(\mathbb{C}) \to \mathsf{Mon}\big(\mathsf{End}(\mathbb{C})\big),$$

which can be represented by the category action $(\bullet, \theta, \gamma)$:

1. $\bullet = \otimes = +$,
2. $\theta \equiv \big(\theta_X = \iota_2 : X \to 0 + X\big)_{X \in \mathbb{C}}$ is the family of canonical isomorphisms defined by the properties of 0 and the coproduct,
3. $\gamma = \alpha$.

To summarize, the canonical action of $\mathbb{C} = \big(\mathbb{C}, +, 0, \alpha, \lambda, \rho\big)$ on itself is obtained by substituting the above into Definition (3.2.6), and so assigning each object B in $\mathbb{C}$ to the unique monad:

$$\big(B + (-) : \mathbb{C} \to \mathbb{C}, ([1_B, 1_B] + 1_{(-)}) \circ \alpha, (e + 1_{(-)}) \circ \theta_{(-)}\big) = \big(B + (-), \mu^+, \iota\big).$$

Specifically, for each $X \in \mathbb{C}$, the unit $(e + 1_X) \circ \theta_X = \iota_{X,B}$ for the monad is the coproduct injection:

$$X \xrightarrow{\iota_{X,B}} B + X,$$

and the multiplication is given by the composite:

$$B + (B + X) \xrightarrow{\alpha} (B + B) + X \xrightarrow{[1_B, 1_B] + 1_X} B + X, \qquad \mu^+_X : B + (B + X) \to B + X.$$

In order to describe categorical semidirect products, one has to introduce a particular sub-monad of $\big(B + (-), \mu^+, \iota\big)$.

For the purposes of this construction, suppose that $\mathbb{C}$ is a pointed category with finite limits and finite coproducts. Now, $B + (-)$ can be regarded as a functor:

$$B + (-) : \mathbb{C} \to \mathsf{Pt}(B) \quad , \quad X \mapsto \big(B + X, \pi_{B,X}, \iota_{B,X}\big) \tag{3.11}$$

in which:

1. $\pi_{B,X} = [1_B, 0] : B + X \to X,$
2. $\iota_{B,X} : B \to B + X$ is the first coproduct injection,[10]

and this functor is left adjoint to:

$$Ker : \mathsf{Pt}(B) \to \mathbb{C} \quad , \quad (A, f, s) \mapsto Ker(f)$$

Recall that,[11] for each B, $Ker = i_B^*$ is the pullback functor induced by the morphism $i_B : 0 \to B$. If one relabels this adjunction as $F \dashv G$, one has:

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{Pt}(B)$$

with:

1. $F(X) = \big(B + X, \pi_{B,X}, \iota_{B,X}\big),$
2. $G(A, f, s) = Ker(f),$
3. $FG(A, f, s) = Ker(\pi_{B,X}) = B\flat X$, and $\eta^\flat_X : X \to B\flat X$ is the unique map induced by the property of the kernel $\kappa_{B,X} : B\flat X \to B + X$:

$$\begin{array}{ccc} X & \xrightarrow{\iota_{X,B}} & B + X \\ & {\scriptstyle \eta^\flat_X} \searrow \quad \nearrow {\scriptstyle \kappa_{X,B}} & \\ & B\flat X & \end{array} \tag{3.12}$$

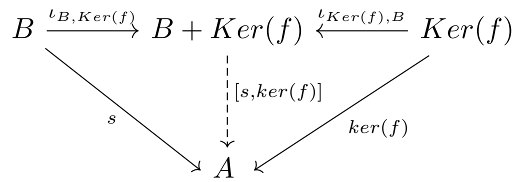
[10] This notation – also used in [5] – indicates the injection of B into the coproduct of B and X, and writing the objects as indices allows for various coproduct injections to be easily distinguished from one another.

[11] See Section (A.4).

4. $GF(X) = \big(B + Ker(f), \pi_{B,Ker(f)}, \iota_{B,Ker(f)}\big)$, and each

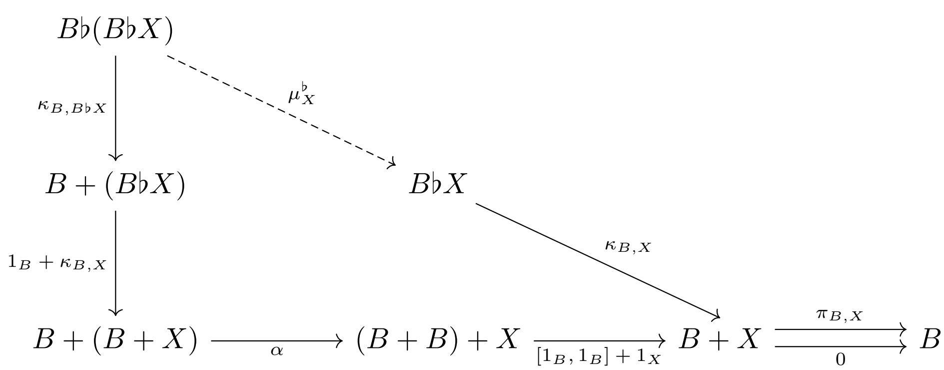
$$\varepsilon^{\flat}_{(A,f,s)} : \big(B + Ker(f), \pi_{B,Ker(f)}, \iota_{B,Ker(f)}\big) \to (A, f, s)$$

is given by $[s, ker(f)]$, as in:



One can use this adjunction to define a monad $GF(-) = T^B = B\flat(-)$ on $\mathbb{C}$.[12] For each $X \in \mathbb{C}$, the constituent parts of the monad are given as follows:

The unit is given by $\eta^{\flat}_X : X \to B\flat X$, and since the bottom composite[13] in Diagram (3.13) equalizes $\pi_{B,X}$ and 0, the multiplication $\mu^{\flat}_X = G\big(\varepsilon^{\flat}_{F(X)}\big)$ for the monad is the unique morphism making the diagram commute:



(3.13)

The collection of these monads $\big(B\flat(-), \mu^{\flat}, \eta^{\flat}\big)_{B\in\mathbb{C}}$ can be considered as a functor $\flat : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and so forms part of a category action of $\mathbb{C}$ on itself. In order to make this statement precise, one requires:

1. A family of morphisms $\big(\theta^{\flat}_X : X \to 0\,\flat X\big)_{X\in\mathbb{C}}$

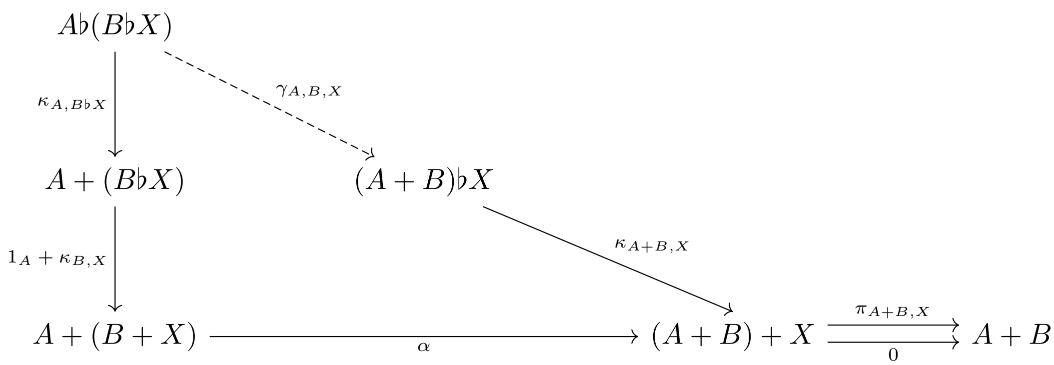
2. A family of morphisms $\big(\gamma_{A,B,X} : A\flat(B\flat X) \to (A + B)\flat X\big)_{A,B,X\in\mathbb{C}}$

The first family be obtained by taking $B = 0$ in Diagram (3.12), and writing

[12] See Section (3.1).
[13] $B\flat(B\flat X) \to B + (B\flat X) \to \cdots \to B + X$.

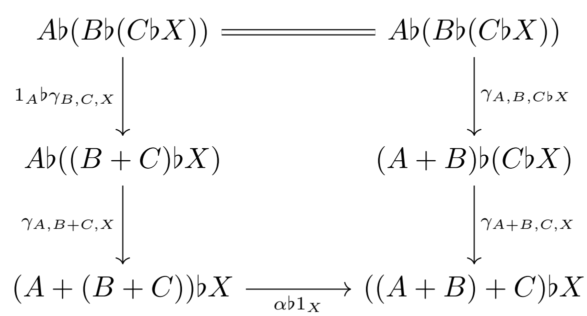
$$\theta^\flat_X = \eta^\flat_X : X \to 0 \flat X$$

As for the second family, consider the following. For each three objects A, B, X in $\mathbb{C}$, the bottom composite in Diagram (3.14) equalizes $\pi_{A+B,X}$ and 0, and therefore there exists a unique map $\gamma_{A,B,X}$ that makes the diagram commute:



(3.14)

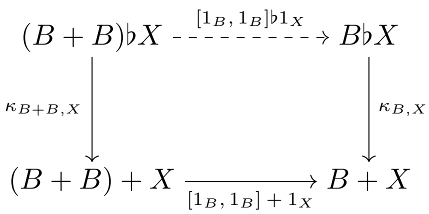
This family $(\gamma_{A,B,X})_{A,B,X\in\mathbb{C}}$ allows one to construct Diagram (3.15), which is precisely the diagram required by Lemma (3.3.1), once it has been reformulated into the language of $(\flat, \theta^\flat, \gamma)$.



(3.15)

Thus $(\flat, \theta^\flat, \gamma)$ is also a category action on $\mathbb{C}$.

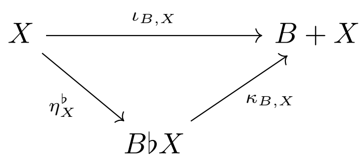
At this point, one might ask how exactly the category action $(\flat, \theta^\flat, \gamma)$ relates to $(+, \theta, \alpha)$. Since the composite $([1_B, 1_B] + 1_X) \circ \kappa_{B+B,X}$ equalizes $\pi_{B,X}$ and 0, there exists a unique induced morphism $([1_B, 1_B]\flat 1_X) : (B + B)\flat X \to B\flat X$ such that the following diagram commutes:



Filling these details into Diagram (3.13) - which amounts to pasting commutative diagrams together - allows one to see that for each $X \in \mathbb{C}$, the multiplication $\mu^\flat$ can be expressed as:

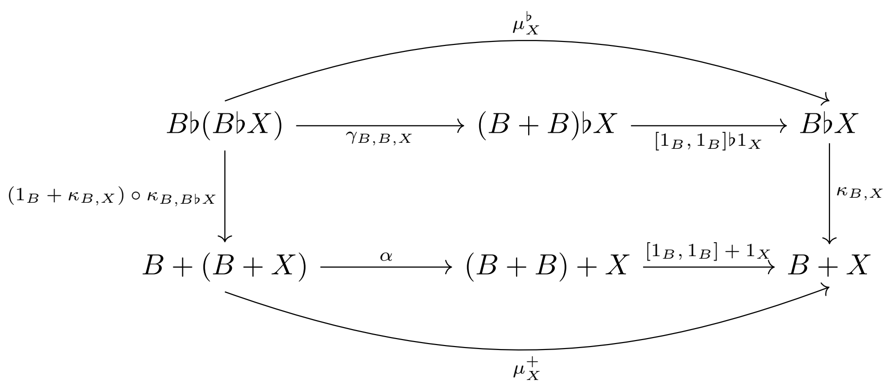
$$\mu^\flat_X = ([1_B, 1_B]\flat 1_X) \circ \gamma_{B,B,X} \tag{3.16}$$

This fact highlights the intuitive link between $(+, \theta, \alpha)$ and $(\flat, \theta^\flat, \gamma)$: for each object B in $\mathbb{C}$, the natural transformation $\kappa_{B,-} : B\flat(-) \to B + (-)$ constitutes a monad map. The proof of this requires only that the Diagram (3.17) and Diagram (3.18) commute:



(3.17)

Diagram (3.17) commutes by the definition of $\eta^\flat_X$, and Diagram (3.18) commutes by the definitions of $\mu^\flat_X$ and γ.



(3.18)

It is also straight-forward [5] to show that each $\kappa_{B,-} : B\flat(-) \to B + (-)$ is monomorphic as a monad map.

On the other hand, if one considers an internal monoid $B = \big(B, [1_B, 1_B], i_B\big)$ in $\mathbb{C} = \big(\mathbb{C}, +, 0\big) = \mathsf{Mon}(\mathbb{C})$ and takes the image of this internal monoid under the monoidal functor that $(\flat, \theta^\flat, \gamma)$ represents,[14] one obtains precisely:

[14] See Definition (3.3.2).

$$\big(B\flat(-), ([1_B, 1_B]\flat 1_{(-)}) \circ \gamma_{B,B,(-)}, (i_B \flat 1_{(-)}) \circ \theta^\flat\big) = \big(B\flat(-), \mu^\flat, \eta^\flat\big).$$

Definition 3.3.3 (Borceux, Janelidze, and Kelly [5])**.** *Suppose that a pointed category $\mathbb{C}$ with finite limits and finite coproducts acts on itself via $(\flat, \theta^\flat, \gamma)$. An internal object action of B on X is a morphism $h : B\flat X \to X$ such that the following diagram commutes:*

$$\begin{array}{ccccccc}
B\flat(B\flat X) & \xrightarrow{\ \gamma\ } & (B+B)\flat X & \xrightarrow{[1_B,1_B]\flat 1_X} & B\flat X & \xleftarrow{\ \eta^\flat_X\ } & X \\
\downarrow{\scriptstyle 1_B\flat h} & & & & \downarrow{\scriptstyle h} & \swarrow\!\!= & \\
B\flat X & & \xrightarrow{\qquad\qquad h \qquad\qquad} & & X & &
\end{array}$$

Again, one writes $\mathbb{C}^B$ for the category of internal B-actions on objects in $\mathbb{C}$. It is clear that the category of internal object actions $\mathbb{C}^B$ is precisely the same as $\mathbb{C}^{B\flat(-)}$, the category of algebras for the monad $(B\flat(-), \mu^\flat, \eta^\flat)$.

With the notion of internal object actions, one can now define categorical semidirect products. Let $\mathbb{C} = \big(\mathbb{C}, +, 0, \alpha, \lambda, \rho\big)$ be a pointed category with binary coproducts and finite limits, and consider the adjunction (3.11). As noted in [5], if $\mathbb{C}$ also has coequalizers, one can use Beck's monadicity theorem to show that the comparison functor K of the monad $(GF, G\varepsilon_F, \eta) = (B\flat(-), \mu^\flat, \eta^\flat)$ is an equivalence of categories:

$$\mathbb{C}^B \underset{K}{\overset{L}{\rightleftarrows}} \mathsf{Pt}(B). \tag{3.19}$$

Explicitly, one has:

1. $K(A, f, s) = \big(Ker(f), G\varepsilon_{(A,f,s)}\big)$, where $G\varepsilon_{(A,f,s)}$[15] is the unique map induced by the universal property of the kernel[16] $(Ker(f), ker(f))$:

$$\begin{array}{ccc}
B\flat Ker(f) & \xrightarrow{\kappa_{B,Ker(f)}} & B + Ker(f) \\
{\scriptstyle G\varepsilon_{(A,f,s)}}\,\vdots & & \downarrow{\scriptstyle [s,kerf]} \\
Ker(f) & \xrightarrow[\ ker(f)\ ]{} & A
\end{array}$$

2. $L(X, h) = (B \ltimes (X, h), \pi'_{(X,h)}, \iota'_{(X,h)})$, which is given by the coequalizer of $F(h) = 1_B + h$ and $\varepsilon_{F(X)} = \varepsilon_{(B\,+\,X,\,\pi_{B,X},\,\iota_{B,X})} = [\iota_{B,X}, \kappa_{B,X}]$:

$$B + (B\flat X) \underset{1_B\,+\,h}{\overset{[\iota_{B,X},\,\kappa_{B,X}]}{\rightrightarrows}} B + X \xrightarrow{\ \sigma_{(X,h)}\ } B \ltimes (X, h).$$

[15] Recall that $\varepsilon_{(A,f,s)} = [s, kerf]$.

[16] Since the composite $f \circ [s, ker(f)] \circ \kappa_{B,Ker(f)}$ is equal to the appropriate zero morphism.

(a) Since $\pi_{B,X} \circ [\iota_{B,X}, \kappa_{B,X}] = \pi_{B,X} \circ (1_B + h)$, there exists a unique morphism $\pi'_{(X,h)} : B \ltimes (X, h) \to B$ such that $\pi'_{(X,h)} \circ \sigma_{(X,h)} = \pi_{B,X}$,

(b) $\iota'_{(X,h)}$ is given as the composite $\iota'_{(X,h)} = \sigma_{(X,h)} \circ \iota_{B,X}$.

This provides an explicit theoretical construction of the categorical semidirect product. As mentioned in Section (3.1), semi-abelian categories, and in particular semi-abelian varieties, satisfy the requisite conditions for this construction.

The following two sections will provide relatively detailed descriptions of how the categorical semidirect products of both groups and rings coincide with their classical counterparts.

3.4 Internal Actions and Semidirect Products of Groups

If B is a group, a classical group action (specifically, a classical B-action) on a group X is given by a group homomorphism $g : B \to Aut(X)$, and this data can be presented canonically as a B-group, which is a pair $(X, \bar{g} : B \times X \to X)$ in which:

1. $\bar{g}(0, x) = x$,
2. $\bar{g}(b_1 + b_2, x) = \bar{g}(b_1, \bar{g}(b_2, x))$.

This correspondence extends to an isomorphism of categories.

One can use any B-action (B, g) to define the classical semidirect product $B \ltimes_g X$ of B and X, by stipulating that:

1. The underlying set of $B \ltimes_g X$ is $B \times X$,
2. Addition in $B \ltimes_g X$ is given by $(b_1, x_1) + (b_2, x_2) := (b_1 + b_2, x_1 + \bar{g}(b_1)(x_2))$.

There is a well known result [5] showing that there is an equivalence between the category of split epimorphisms of groups, and the category of classical group actions:

$$\mathsf{SplEpi}(\mathsf{Groups}) \simeq \mathsf{ClassGrpAct} \tag{3.20}$$

in which:

1. Each split epimorphism (A, B, f, s) is sent to (B, X, φ), where $X = Ker(f)$ and:

$$\varphi(b)(x) = b * x = k^{-1}\big(s(b) + k(x) - s(b)\big) \;\forall\, b \in B, \forall\, x \in X,$$

2. Each group action (C, Y, g) is sent to $(C \ltimes_g Y, p, i)$, where p and i are the homomorphisms defined by $p(c, y) = c$ and $i(c) = (c, 0)$.

The equivalence in Diagram (3.20) can immediately be reduced to an equivalence between the category of split epimorphisms with codomain B (i.e $\mathsf{Pt}(B)$) and the category of classical B-actions:

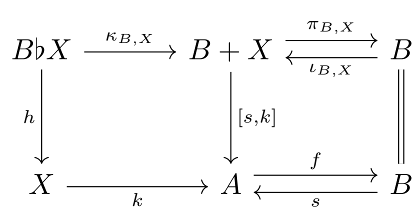
$$\mathsf{Pt}_{\mathsf{Groups}}(B) \simeq \mathsf{Class}{-}B{-}\mathsf{Act} \tag{3.21}$$

for any group B. With Diagram (3.19) in mind, this immediately shows that the notions (and categories) of classical and categorical actions of groups coincide. Showing explicitly that a given categorical action coincides with its classical counterpart amounts to composing equivalences of categories, and was initially shown in [8]. Longer descriptions of this are also given by Borceux and Bourn [3] and Borceux, Janelidze, and Kelly [5].

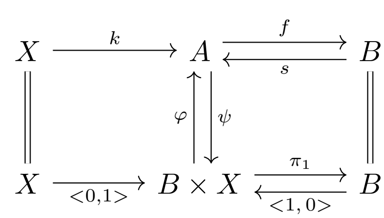
As mentioned in Section (3.1), the equivalence of categories $K \dashv L$ in Diagram (3.19) shows that each object action $h : B\flat X \to X$ corresponds to the split extension:

$$(A, f, s, X) = (B \ltimes (X, h), \pi'_{(X,h)}, \iota'_{(X,h)}, X)$$

as in:



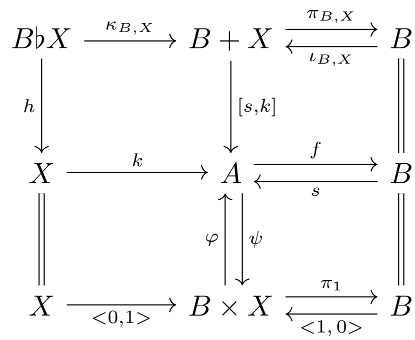
Further, recall that there are bijections φ and ψ[17] that show that A has $B \times X$ as its underlying set. That is, the following diagram commutes in the category of pointed sets:



(3.22)

[17] See Section (3.1).

These diagrams can be glued together to produce:



(3.23)

This shows that one can use $[s,k] \circ \kappa_{B,X}$ to calculate how h acts on elements in $B\flat X$, which is equivalent to finding $\psi \circ [s,k] \circ \kappa_{B,X} = \psi \circ k \circ h = <0,1> \circ h$. Diagram (3.23) also shows that, up to isomorphism, $\psi \circ [s,k]$ acts on elements as:

$$[<1_B, 0>, <0, 1_X>].$$

Now, since $\psi \circ k = <0, 1_X>$, one can deduce how h acts on elements in $B\flat X$ by inspecting the X-component of the semidirect product A.

At a glance, $B\flat X$ is simply the collection of the words $<b_1x_1b_2x_2 \cdots b_nx_n>$ in the free product $B + X$ such that $\sum_{i=1}^{n} b_i = 0_B$. However, one can use this property to show that the words of $B\flat X$ all take a very particular form.

To avoid possible ambiguity, in what follows, the juxtaposition of elements will denote words in $B\flat X$, while free-standing elements (and the explicit addition of these) will indicate elements in the groups (B or X) themselves. Now, consider the following:

- $b_1x_1 \in B\flat X \Rightarrow b_1 = 0$,
- $b_1x_1b_2 \in B\flat X \Rightarrow b_1 + b_2 = 0 \Leftrightarrow b_2 = -b_1$, and so the simplest words in $B\flat X$ take the form $<b_1x_1(-b_1)>$,[18]
- Similarly, $b_1x_1b_2x_2 \in B\flat X \Leftrightarrow b_2 = -b_1$, and so $b_1x_1b_2x_2 = b_1x_1(-b_1)x_2 = c_1x_1(-c_1)c_2x_2(-c_2)$, with $c_1 = b_1$ and $c_2 = 0$,
- $b_1x_1b_2x_2b_3 = b_1x_1\big((-b_1)b_1\big)b_2x_2\big((-b_2)(-b_1)b_1b_2\big)b_3$
  $= \big(b_1x_1(-b_1)\big)\big(b_1b_2x_2(-b_2)(-b_1)\big)b_1b_2b_3$
  $= \big(b_1x_1(-b_1)\big)\big(b_1b_2x_2(-b_2)(-b_1)\big)$
  $= \big(c_1x_1(-c_1)\big)\big(c_2x_2(-c_2)\big)$,
- Thus, by extending this process, any element of $B\flat X$ can be written as a product of words of the form $<bx(-b)>$.

[18] Of course $b_1 = 0$ is also allowed.

Under the correspondence in Diagram (3.21.), the classical action related to (A, f, s) is given by $b * x = k^{-1}\big(s(b) + k(x) - s(b)\big)$, and so addition in $B \times X$ – which underlies A – must be given by:

$$\begin{aligned}(b_1, x_1) + (b_2, x_2) &= \big(b_1 + b_2, x_1 + b_1 * x_2\big)\\ &= \Big(b_1 + b_2, x_1 + k^{-1}\big(s(b_1) + k(x_2) - s(b_1)\big)\Big).\end{aligned}$$

Since the way in which $h : B\flat X \to X$ acts on elements in $B\flat X$ is determined entirely by how it acts on words of the the form $<bx(-b)>$, one simply computes:

$$\begin{aligned}&[< 1_B, 0 >, < 0, 1_X >]\big(<bx(-b)>\big)\\ &= (b, 0) + (0, x) + (-b, 0)\\ &= \Big(b, k^{-1}\big(s(b) + k(x) - s(b)\big)\Big) + (-b, 0)\\ &= \Big(b - b, k^{-1}\big(s(b) + k(x) - s(b)\big) + k^{-1}\big(s(b) + k(0) - s(b)\big)\Big)\\ &= \Big(0, k^{-1}\big(s(b) + k(x) - s(b)\big)\Big)\\ &= \big(0, b * x\big).\end{aligned}$$

By placing this calculation back into the context of Diagram (3.23.), one can see that $h\big(<bx(-b)>\big) = b * x$, and this shows how the classical and categorical group actions, and semidirect products coincide.

3.5 Internal Actions and Semidirect Products of Rings

Definition 3.5.1 (Inyangala [15])**.** *A classical ring action of B on X is a collection (B, X, ζ, ϑ), where B and X are non-unital*[19] *rings, and $\zeta : B \times X \to X$ and $\vartheta : X \times B \to X$ are functions that form left and right non-unital B-algebra structures on X, and obey the following compatibility condition:*

$$\begin{array}{ccc} B \times X \times B & \xrightarrow{\zeta \times 1_B} & X \times B \\ {\scriptstyle 1_B \times \vartheta}\downarrow & & \downarrow{\scriptstyle \vartheta} \\ B \times X & \xrightarrow[\zeta]{} & X \end{array}$$

One can use any classical B-action of rings to define the classical semidirect product of rings, $B \ltimes_{(\zeta,\vartheta)} X$, in which:

1. The underlying set of $B \ltimes_{(\zeta,\vartheta)} X$ is $B \times X$,

[19] That is, not necessarily unital.

2. Addition is computed as in the underlying abelian group (point-wise),
3. Multiplication is computed as:

$$(b_1, x_1) \cdot (b_2, x_2) = \big(b_1 \cdot b_2, \zeta(b_1, x_2) + \vartheta(x_1, b_2) + x_1 \cdot x_2\big).$$

One can consider the collection of classical B-actions of rings as a category, by taking triples (X, ζ, ϑ) as objects, and choosing morphisms $\chi : (X_1, \zeta_1, \vartheta_1) \to (X_2, \zeta_2, \vartheta_2)$ to be ring homomorphisms $\chi : X_1 \to X_2$ that preseve both actions.

Let Rng denote the category of non-unital rings. As with Groups,[20] there is an equivalence – as mentioned by Inyangala [15] – between the category of classical B-actions of rings, and $\mathsf{Pt}_{\mathsf{Rng}}(B)$, under which:

1. Each split epimorphism (A, f, s) is sent to (X, ζ, ϑ), where:
   (a) $\zeta(b, x) = k^{-1}\big(s(b) \cdot k(x)\big) = s(b) \cdot x$,
   (b) $\vartheta(x, b) = k^{-1}\big(k(x) \cdot s(b)\big) = x \cdot s(b)$,
   (c) $X = Ker(f)$.
2. Each B-action (Y, ζ, ϑ) is sent to $(B \ltimes_{(\zeta,\vartheta)} Y, p, i)$, where:

$$p(b, y) = b \text{ and } i(b) = (b, 0).$$

If B has a multiplicative unit, and ζ and ϑ are (unital) B-algebra structures on X, then $B \ltimes_{(\zeta,\vartheta)} X$ will have a multiplicative identity, $(1, 0)$, which p and i will preserve. In this case, constructing the classical semidirect product of rings can be used as a method for adjoining a unit to a non-unital ring X. Further, if the rings in question are commutative, then the left and right algebra structures that are defined on them will be identical. The rings in the discussion to follow will all be assumed to be commutative.

$\mathbb{Z}$ is initial in the category of unital rings. For each ring A there is a unique ring homomorphism $s : \mathbb{Z} \to A$ that is determined by the mappings $0 \mapsto 0_A$ and $1 \mapsto 1_A$. Therefore, any unital ring homomorphism $f : A \to \mathbb{Z}$ is split by $s : \mathbb{Z} \to A$. Of course, saying that $\mathbb{Z}$ is initial in Rings is equivalent to saying that every unital ring A has a unique $\mathbb{Z}$-algebra structure defined on it, with:

$$\phi : \mathbb{Z} \times A \to A \quad , \quad (z, a) \mapsto s(z) \cdot a = z \cdot a.$$

Each ring A therefore has a canonical $\mathbb{Z}$-action (A, ϕ, ϕ) defined on it, and this action allows one to define $\mathbb{Z} \ltimes_{(\phi,\phi)} A = \mathbb{Z} \ltimes A$ the semidirect product of $\mathbb{Z}$ with A. In fact, since A has a multiplicative unit, the semidirect product $\mathbb{Z} \ltimes A$ coincides with the (categorical) product $\mathbb{Z} \times A$.[21] Specifically, one has the following commutative diagram:

[20] See Diagram (3.21).
[21] As emphasized below, one could replace $\mathbb{Z}$ with any commutative ring R with no non-trivial idempotents, and obtain an analogous result.

$$\begin{array}{ccc} \mathbb{Z}\ltimes A & \xrightarrow[\cong]{\theta} & \mathbb{Z}\times A \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & \mathbb{Z} & \end{array}$$

where $\theta(z,a) = (z, z+a)$ and $\theta^{-1}(z,a) = (z, z-a)$.

It is clear that both θ and θ^{-1} preserve the required identity elements and addition. To see that θ preserves multiplication, take $(z_1,a_1),(z_2,a_2) \in \mathbb{Z}\ltimes A$ and compute:

$$\begin{aligned} \theta\big((z_1,a_1)\cdot_{\mathbb{Z}\ltimes A}(z_2,a_2)\big) &= \theta\big(z_1z_2, z_1a_2+z_2a_1+a_1a_2\big) \\ &= \big(z_1z_2, z_1z_2+z_1a_2+z_2a_1+a_1a_2\big) \\ &= \big(z_1z_2,(z_1+a_1)(z_2+a_2)\big) \\ &= \big(z_1,z_1+a_1\big)\cdot_{\mathbb{Z}\times A}\big(z_2,z_2+a_2\big) \\ &= \theta\big(z_1,a_1\big)\cdot_{\mathbb{Z}\times A}\theta\big(z_2,a_2\big), \end{aligned}$$

and similarly for θ^{-1}. Both morphisms can easily be shown to be $\mathbb{Z}$-linear. No distinguishing property of $\mathbb{Z}$ is used to show that θ and θ^{-1} are ring isomorphisms, and there will be an analogous result for any ring R (i.e. when both R and A are rings with multiplicative units, $R\ltimes A \cong R\times A$).

Similarly, each non-unital ring U has an evident non-unital $\mathbb{Z}$-algebra structure defined on it, with:

$$\overline{\phi} : \mathbb{Z}\times U \to U \quad , \quad (z,u) \mapsto \sum_{i=1}^{z} u.$$

Of course, one will have $\overline{\phi}(z,u) = -\sum_{i=1}^{-z} u$ if $z<0$ and $\overline{\phi}(0,u) = 0$. These non-unital $\mathbb{Z}$-algebra structures allow for the construction of the classical semidirect product $\mathbb{Z}\ltimes U$ for any non-unital ring U.

The classical semidirect product of rings allows one to construct an equivalence of categories (3.24), which shows that non-unital rings can be recovered from unital rings (as ring homomorphisms into the integers):

$$\mathsf{Rng} \underset{K}{\overset{L}{\rightleftarrows}} \big(\mathsf{Rings}\downarrow\mathbb{Z}\big) \tag{3.24}$$

Here:

1. $L(U) = \big(\mathbb{Z}\ltimes U, \pi_1\big)$,
2. $K(A,f) = Ker(f)$,

3. Since $KL(U) = Ker(\pi_1)$, the unit of the adjunction is given by the isomorphism

$$\eta_U : U \to Ker(\pi_1) \quad , \quad u \mapsto (0, u)$$

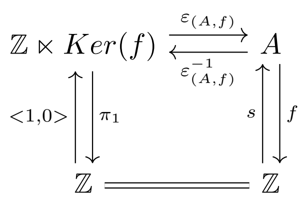
4. Since $LK(A, f) = \big(\mathbb{Z} \ltimes Ker(f), \pi_1\big)$, one has that:

$$\varepsilon_{(A,f)} : \big(\mathbb{Z} \ltimes Ker(f), \pi_1\big) \to (A, f) \quad , \quad (z, x) \mapsto s(z) + x = z \cdot 1_A + x$$

What is more, $\varepsilon_{(A,f)}$ has an inverse:

$$\varepsilon^{-1}_{(A,f)} : (A, f) \to \big(\mathbb{Z} \ltimes Ker(f), \pi_1\big) \quad , \quad a \mapsto \big(f(a),\ a - 1_R \cdot f(a)\big),$$

such that both the upwards and downwards paths of the following diagram commute:



The occurence of $\varphi = \varepsilon_{(A,f)}$ and $\psi = \varepsilon^{-1}_{(A,f)}$ from Section (3.1) in this context immediately leads one to suspect that the classical and categorical semidirect products of rings coincide. This follows as the maps (φ and ψ) that link split epimorphisms of right Ω-loops to their corresponding categorical semidirect products are also used to link the given split epimorphisms $\big(A, f : A \to \mathbb{Z}\big)$ to the corresponding classical semidirect products.

However, one cannot directly see the adjunction in Diagram (3.24) as a special case of the equivalence in Diagram (3.19) with $\mathbb{C} = \mathsf{Rings}$, since the category of (unital) rings is not pointed. One can circumvent this by recalling that unital rings can be seen as internal monoids in the monoidal category $(\mathsf{Ab}, \otimes, \mathbb{Z})$. In particular, Diagram (3.22) can be used to see that for a split extension $\big(A, f, s, X\big)$ of unital rings over $\mathbb{Z}$, there is an induced correspondence between the underlying group structures of $\big(A, f, s\big)$ and $(\mathbb{Z} \times X, \pi_1, <1, 0>)$, and one can use the maps φ and ψ to infer the multiplicative structure on $\mathbb{Z} \times X$.

These underlying group structures are necessarily abelian, and therefore the additive action for the semidirect product of rings must be trivial:

$$\begin{aligned}
(z_1, x_1) + (z_2, x_2) &= \big(z_1 + z_2, x_1 + z_1 * x_2\big) \\
&= \big(z_1 + z_2, x_1 + k^{-1}\big(s(z_1) + k(x_2) - s(z_1)\big)\big) \\
&= \big(z_1 + z_2, x_1 + k^{-1}\big(s(z_1) - s(z_1) + k(x_2)\big)\big) \\
&= \big(z_1 + z_2, x_1 + x_2\big).
\end{aligned}$$

Further, since φ and ψ are given by ring operations, they must convey the multiplicative structure of the rings. Therefore, one can obtain the form of the multiplicative operation in the semidirect product by mapping products of elements in $\mathbb{Z}\times X$ to A via φ, computing their product there, and then transporting the structure back via ψ:

$$\begin{aligned}(z_1,x_1)\cdot(z_2,x_2) &= \psi\big(\varphi\big((z_1,x_1)\cdot(z_2,x_2)\big)\big) = \psi\big(\varphi\big((z_1,x_1)\big)\cdot\varphi\big((z_2,x_2)\big)\big)\\ &= \psi\big(\big(k(x_1)+s(z_1)\big)\cdot\big(k(x_2)+s(z_2)\big)\big)\\ &= \psi\big(k(x_1)\cdot k(x_2)+s(z_1)\cdot k(x_2)+k(x_1)\cdot s(z_2)+s(z_1)\cdot s(z_2)\big).\end{aligned}$$

One can note the following to simplify the calculation:

1. $f(s(z)\cdot k(x)) = 0\ \forall\ z\in\mathbb{Z},\ \forall x\in X$, and therefore one might write $z\cdot x\in X$ for $k^{-1}\big(s(z)\cdot k(x)\big)$,
2. If $a = k(x_1)\cdot k(x_2)+s(z_1)\cdot k(x_2)+k(x_1)\cdot s(z_2)+s(z_1)\cdot s(z_2)$, then

$$sf(a) = s\big(0+0+0+f(s(z_1)\cdot s(z_2))\big) = s(z_1)\cdot s(z_2).$$

Now, one can (more easily) compute:

$$\begin{aligned}&(x_1,z_1)\cdot(x_2,z_2)\\ &= \psi\big(k(x_1)\cdot k(x_2)+s(z_1)\cdot k(x_2)+k(x_1)\cdot s(z_2)+s(z_1)\cdot s(z_2)\big)\\ &= \psi(a)\\ &= \big(f(a),k^{-1}\big(a-sf(a)\big)\big)\\ &= \big(f\big(s(z_1)\cdot s(z_2)\big),k^{-1}\big(k(x_1)\cdot k(x_2)+k(z_1\cdot x_2)+k(z_2\cdot x_1)\big)\\ &\qquad +s(z_1)\cdot s(z_2)-s(z_1)\cdot s(z_2)\big)\big)\\ &= \big(z_1\cdot z_2, z_1\cdot x_2+z_2\cdot x_1+x_1\cdot x_2\big).\end{aligned}$$

This is precisely the form of the classical semidirect product, and so the classical and categorical semidirect products of (non-unital) rings with $\mathbb{Z}$ coincide.

It should be noted that for any unital, commutative ring R, the equivalence in Diagram (3.24) can be generalized to an equivalence between the category of pointed[22] R-algebras and the slice category of R-algebras over R, as in:

$$\mathsf{R{-}Alg}_* \underset{L}{\overset{K}{\rightleftarrows}} \big(\mathsf{R{-}Alg}\downarrow R\big) \tag{3.25}$$

R-algebras can be seen as monoid objects in the monoidal category $\big(\mathsf{R{-}Mod},\otimes_R,R\big)$, and therefore one can use a suitably generalized version of Diagram (3.22) to see that in each split extension (A,f,s,X) of R-algebras, A has $R\times X$ as its underlying set, and one can extend the argument given above (and the operations and structure on $R\times X$) to show that the equivalence in Diagram (3.25) can also be seen as a special case of the equivalence in Diagram (3.19).

[22] Again, not necessarily unital.

Chapter 4

Galois Theory in Categories

The following chapter details the structures required for one to present the fundamental theorem of Galois theory in an abstract category. The notions of admissibility and effective descent both play crucial roles in the development of this theory. These notions are expounded upon, and then extended from the categories in which they are defined into the associated categories of pointed objects, and the chapter concludes by showing that these extended notions allow for the presentation of a new, pointed version of the Galois theorem in an abstract category.

As described in [4], [9] and [18], an adjunction:

$$\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$$

between categories with pullbacks constitutes a Galois theory in $\mathbb{C}$. What this means is that one can present an analogue of the fundamental theorem of Galois theory in $\mathbb{C}$ as an equivalence between a category of "special morphisms" in $\mathbb{C}$, and a category of internal (pre)groupoid actions on $\mathbb{X}$.

General categorical Galois structures do not make the relatively stringent requirement that $\mathbb{C}$ and $\mathbb{X}$ have pullbacks. Rather, they require that particular[1] classes of morphisms be specified in both categories, and that these classes behave well with respect to each other (the image of each class under either I or H should be contained in the other class).

With that, since there are several interesting cases in which the categories in question do have pullbacks, and since this assumption simplifies the full description of the adjunction considerably, the adjunctions $I \dashv H$ in the text to follow will all be taken to be between categories with pullbacks.

[1]The classes should contain all of the isomorphisms, and be closed under both pullbacks and composition.

4.1 Admissibility

In the context of an abstract Galois theory in a category with pullbacks, the admissibility of the functor I (with a suitably well-behaved right adjoint H) guarantees that I will preserve pullbacks of a very particular form. The preservation of these limits is crucial when establishing the equivalence that defines the Galois theorem in an abstract category.

Definition 4.1.1.

1. *An object B in $\mathbb{C}$ is admissible when H_B*[2] *is fully faithful (iff ε^B is an isomorphism),*

2. *I is admissible when every object in $\mathbb{C}$ is admissible.*

As mentioned above, when the functor H is fully faithful, the admissibility of I can be described in terms of its preservation of certain pullbacks.

Proposition 4.1.1. *If H is fully faithful, then for each object B in $\mathbb{C}$, the following conditions are equivalent:*

1. *B is admissible,*

2. *I preserves all pullbacks of the form (2.5).*

If H is fully faithful and I is admissible, the adjunction $I_B \dashv H_B$ presents $(\mathbb{X} \downarrow I(B))$ as a full reflective subcategory of $(\mathbb{C} \downarrow B)$, for every B in $\mathbb{C}$.

Carboni and Janelidze [9] describe a simple, general method for constructing adjunctions:

$$(I, H, \eta, \varepsilon) : \mathbb{C} \to \mathbb{X}$$

in which I is admissible. This is done by taking $\mathbb{X}$ to be the category of Sets, and considering a category $\mathbb{A}$ which has a terminal object 1. This induces the canonical adjunction $\mathbb{A} \rightleftarrows \mathbf{1}$ between $\mathbb{A}$ and the terminal/trivial category $\mathbf{1} = *\circlearrowleft$, which in turn induces an adjunction:

$$\mathbb{C} = \mathsf{Fam}_{\mathbb{X}}(\mathbb{A}) \rightleftarrows \mathsf{Fam}_{\mathbb{X}}(\mathbf{1}) \cong \mathbb{X}$$

because $\mathsf{Fam}_{\mathbb{X}} : \mathsf{Cat} \to \mathsf{Cat}$ is a 2-functor.

Of course, one could replace Sets with $\mathsf{FinSets}$, the category of finite sets, and obtain a similar result.

[2] See Diagram (2.5).

Any adjunction that is formed in this way will be admissible [9].

One can obtain the Galois theory of finite, separable field extensions ([17], [20]) by taking $\mathbb{X}$ to be $\mathsf{FinSets}$ and $\mathbb{A}$ to be the opposite category of finite-dimensional (as vector spaces), connected commutative K-algebras. The opposite category of finite-dimensional K-algebras $\mathbb{C} = \mathsf{Fam}_{\mathbb{X}}(\mathbb{A})$ is a finitely locally connected category.[3]

There are examples of adjunctions $I \dashv H$ in which $\mathbb{X}$ is not a full subcategory (closed under finite limits) of Sets. One can use the Galois theory of commutative rings[4] to see that the above construction will not necessarily provide meaningful descriptions of infinite Galois theories. Carboni and Janelidze [9] have described an approach to forming adjunctions that constitute categorical Galois theories in which $\mathbb{X}$ is the category of Stone spaces (compact, totally disconnected topological spaces).

4.1.1 Admissiblity in Categories of Pointed Objects

Proposition (4.1.2) shows that for an adjunction:

$$\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$$

in which $\mathbb{C}$ has a terminal object 1, I is admissible, and H is fully faithful, the induced functor $(1 \downarrow I)$ on the category of pointed objects $\big(1 \downarrow \mathbb{C}\big)$ will also be admissible. This result will contribute to the definition of a "pointed version" of the fundamental theorem of Galois theory in $\mathbb{C}$.[5]

Proposition 4.1.2. *Suppose that H is fully faithful and that $\mathbb{A}$ has a terminal object 1, which I preserves. If I is admissible, then so is $(1 \downarrow I)$.*

Proof. Recall that H is fully faithful if and only if $\varepsilon : IH \to 1_{\mathbb{X}}$ is an isomorphism. Since H is fully faithful, the admissibilty of I reduces to the fact that it preserves all pullbacks of the form (2.5). $I \dashv H$ and $(1 \downarrow I) \dashv (1 \downarrow H)$ have the same counit, and therefore $(1 \downarrow H)$ is also fully faithful. This means that $(1 \downarrow I)$ will be admissible if and only if it preserves the analogous pullbacks in $\big(1 \downarrow \mathbb{C}\big)$.

Consider the case where one has:

1. An object $(B, b : 1 \to B)$ in $\big(1 \downarrow \mathbb{C}\big)$,
2. An object $(X, x : 1 \to X)$ in $\big(1 \downarrow \mathbb{X}\big)$,
3. A morphism $\varphi : (X, x) \to (I(B), I(b))$ in $\big(1 \downarrow \mathbb{X}\big)$.

[3]See Proposition (2.1.3) and Proposition (2.2.1).
[4]See Chapter (6).
[5]See Section (4.4).

Since there is a canonical isomorphism $1 \times_1 1 \cong 1$ and I preserves 1, it is clear that the object underlying the pullback of $\eta_B : (B, b) \to (HI(B), HI(b))$ and $H(\varphi) : (H(X), H(x)) \to (HI(B), HI(b))$ in $\big(1 \downarrow \mathbb{C}\big)$ is given by the pullback $B \times_{HI(B)} H(X)$ of η_B and $H(\varphi)$ in $\mathbb{C}$. This, and the admissibility of I, will guarantee that $(1 \downarrow I)$ will preserve all the pullbacks required to satisfy admissibility. □

4.2 Descent Theory

4.2.1 A Brief Introduction to Effective Descent

Descent theory primarily has to do with seeking to more easily deal with 'problems' or identify 'structures' over a given object B by moving them to more amenable settings, dealing with the required calculations there, and then transporting the data back in a canonical way. One might think of this as transporting structure or data to an appropriate extension E of B, doing the calculations in a more conducive environment for problem-solving than B itself would be, and then transporting the data back ("descending") to B via a morphism $E \to B$. In the work that follows, the structure over B is relatively simple, in that it will always be taken to be an object in $\big(\mathbb{C} \downarrow B\big)$.

Certainly, not all morphisms will be suitable for the task of moving this data back and forth between E and B. Those morphisms that are suitable are termed effective descent morphisms (to be defined momentarily), and will be denoted by $p : E \to B$. These morphisms allow one to describe the form of objects in $\big(\mathbb{C} \downarrow B\big)$ in terms of the structure of objects in $\big(\mathbb{C} \downarrow E\big)$, by pulling back along p.

The reason one would be interested in talking about effective descent morphisms in the context of adjunctions $I \dashv H$ for admissible I is so that one can present a particular full subcategory of a slice category over $\mathbb{C}$ as being monadic over a slice category on $\mathbb{X}$. Under favourable circumstances, the category of algebras for the associated monad can be thought of as a category of G-sets, where G is a "Galois group(oid)," and establishes a categorical analogue for the fundamental theorem of Galois theory in $\mathbb{C}$.

Definition 4.2.1. *A morphism $p : E \to B$ in $\mathbb{C}$ is an effective descent morphism when:*

$$p^* : \big(\mathbb{C} \downarrow B\big) \to \big(\mathbb{C} \downarrow E\big) \quad , \quad (A, \alpha) \mapsto (E \times_B A, \pi_1)$$

is monadic.

When p is an effective descent morphism, one has the following:

1. p^* has left adjoint:

$$p_! : (\mathbb{C} \downarrow E) \to (\mathbb{C} \downarrow B) \quad , \quad (C, \gamma) \mapsto (C, p \circ \gamma),$$

2. $p_! \circ p^*(A, \alpha) = (E \times_B A, p \circ \pi_1)$,

3. $\varepsilon^p_{(A,\alpha)} = \pi_2 : (E \times_B A, p \circ \pi_1) \to (A, \alpha)$ is the counit for $p_! \dashv p^*$,

4. $p^* \circ p_!(C, \gamma) = (E \times_B C, \pi_1)$ is the pullback of $p \circ \gamma$ along p,

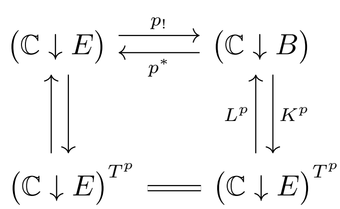
5. $\eta^p_{(C,\gamma)} = \langle\gamma, 1_C\rangle : (C, \gamma) \to (E \times_B C, \pi_1)$ is the unit for $p_! \dashv p^*$,

6. $T^p = p^* \circ p_!$ is the familiar monad on $(\mathbb{C} \downarrow E)$,

7. $\mu^p_{(A,\alpha)} = \langle\pi_1, \pi_3\rangle : (E \times_B E \times_B A, \pi_1) \to (E \times_B A, \pi_1)$ is the multiplication for T^p,

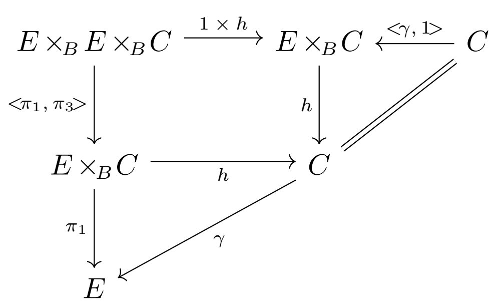
8. In Diagram (4.1), the unlabelled vertical arrows are the obvious functors between $(\mathbb{C} \downarrow E)$ and the category of T^p-algebras,

9. K^p is the comparison functor for T^p, and L^p is its left adjoint,



(4.1)

A T^p-algebra is a collection $((C, \gamma), h)$ where $(C, \gamma) \in (\mathbb{C} \downarrow E)$ and the structure map $h : E \times_B C \to C$ is a morphism in $\mathbb{C}$ which makes the following diagram commute:

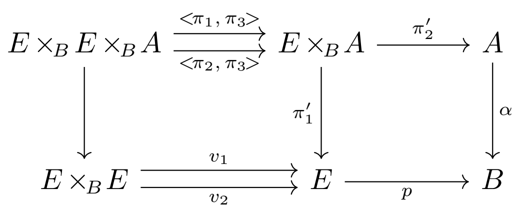


(4.2)

The top portion of Diagram (4.2) is comprised of the commutative segments required to define an algebra for a monad, and the bottom triangle is appended to indicate that $h : (E \times_B C, \pi_1) \to (C, \gamma)$ is a morphism in $(\mathbb{C} \downarrow E)$.

A pair of intuitive diagrammatic representations captures the notion of $p : E \to B$ being an effective descent morphism.

The first of these is illustrated in Diagram (4.3). Given any object (A, α) in $\big(\mathbb{C} \downarrow B\big)$, if one takes the pullback of α along p, and then takes respective kernel pairs, one finds that the top row of Diagram (4.3) must be a coequalizer.[6]



(4.3)

This is the case because $<\pi_2, \pi_3>$ is the structure map associated to (A, α) under the comparison functor K^p, and $<\pi_1, \pi_3>$ is the multiplication map for the monad T^p that induces K^p. Both morphisms are split by $<1_E, 1_E>\times_B 1_A$, so it is clear that $\big(<\pi_2, \pi_3>, <\pi_1, \pi_3>\big)$ is the canonical reflexive pair obtained under K^p. Now, since p^* is monadic, one has:

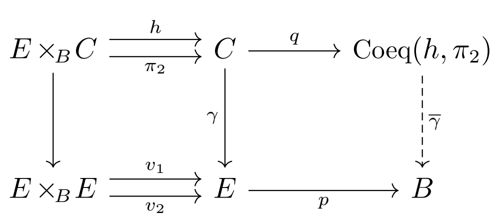
$$\mathrm{Coeq}\big(<\pi_2, \pi_3>, <\pi_1, \pi_3>\big) \cong L^p K^p(A, \alpha) \cong (A, \alpha)$$

For the second representation, consider a T^p-algebra $\big((C, \gamma),\ h : (E \times_B C, \pi_1) \to (C, \gamma)\big)$.

Since $\pi_2 : (E \times_B C, \pi_1) \to (C, \gamma)$ is the counit for $p_! \dashv p^*$ at (C, γ), it is clear that:

$$\mathrm{Coeq}(h, \pi_2) \cong L^p\big((C, \gamma), h\big)$$

From this, one can construct Diagram (4.4) by using the universal property of the kernel pair $\mathsf{Eq}(p)$ to induce the left hand vertical arrow, and the universal property of the coequalizer to induce the right hand vertical arrow. Since $\mathrm{Coeq}(h, \pi_2)$ is isomorphic to the image under L^p, it is clear that the pullback of $\overline{\gamma}$ along p is $K^p L^p\big((C, \gamma), h\big)$. Since p^* is monadic, $K^p L^p\big((C, \gamma), h\big) \cong C$, and so the right-hand square in Diagram (4.4) must be a pullback square.



(4.4)

Briefly, pullbacks along p yield coequalizers, and coequalizers over p yield pullbacks.

[6] As ever, the projections in Diagram (4.3) belong to their respective domains.

Proposition 4.2.1 (Borceux and Janelidze [4]). *The class of effective descent morphisms in a category $\mathbb{C}$ is closed under composition, and pullback-stable.*

The proof of this proposition makes use of the Beck-Chevalley condition, and the result itself is crucial to the proofs of several results in Section (4.4).

4.2.2 Characterizing Effective Descent Morphisms

When the category in question has the limits and colimits required for the constructions to be well defined, it is clear from the definition that effective descent morphisms will always be regular epimorphisms. However, finding additional necessary conditions to describe them fully in general settings can be laborious [20]. With this in mind, it has been shown [20] that in exact categories, the effective descent morphisms coincide exactly with the regular epimorphisms.

Effective descent morphisms are less easily characterized in more general categories because the properties that specify exactness are precisely those required to show the monadicity of p^*. Consider the following in a regular category (regularity is only a narrow generalization from exactness). To show the monadicity of p^* in this setting, one effectively has to show that it preserves coequalizers of equivalence relations.

Let $R \underset{r_2}{\overset{r_1}{\rightleftarrows}} X$ be an equivalence relation on an object X,[7] let $q = \mathsf{Coeq}(r_1, r_2)$ be the coequalizer of the given pair, and consider the image of this diagram under p^*, as in:

$$\left(R \underset{r_2}{\overset{r_1}{\rightleftarrows}} X \xrightarrow{q} Q\right) \overset{p^*}{\longmapsto} \left(E \times_B R \rightleftarrows E \times_B X \xrightarrow{\#} E \times_B Q\right)$$

In regular categories, regular epimorphisms are pullback stable, so $\#$ is also a regular epimorphism. What is more, p^* is a right adjoint, and will therefore preserve kernel pairs. Thus, if (r_1, r_2) is the kernel pair of its coequalizer q, the image of (r_1, r_2) under p^* will be the kernel pair of $\#$. In a regular category, there is no reason for (r_1, r_2) to be the kernel pair of q. However, equivalence relations in exact categories are effective, and so (r_1, r_2) will be the kernel pair of q, by definition, in this setting.

Effective Descent Morphisms of Modules and Algebras

Categories that are not exact - like the opposite category of rings, or the opposite category of algebras over a ring - require more explicit descriptions for their effective descent morphisms. It was shown by Janelidze and Tholen [21] that pure monomorphisms[8] of

[7] These should be considered as objects in $(\mathbb{C} \downarrow B)$, but the additional structure is inconsequential here, so the pairs are represented by their object-parts.

[8] See Definition (A.2.1).

modules can be used to characterize the effective descent morphisms in the aforementioned categories.

Theorem 4.2.1. *Given a homomorphism of commutative rings $p : B \to E$, the functor:*

$$E \otimes_B (-) : \mathsf{B\text{--}Mod} \to \mathsf{E\text{--}Mod}$$

is comonadic if and only if p is a pure monomorphism of B-modules.

Given a homomorphism $\alpha : B \to A$ of commutative rings, the tensor product $E \otimes_B A$ is the pushout of p along α in the category of commutative rings, and thus it is also the object that underlies the pullback $E \times_B A$ in the opposite category. Therefore, $E \otimes_B (-)$ is comonadic on $\mathsf{B\text{--}Mod}$ if and only if $p^* = E \times_B (-)$ is monadic on the opposite category of B-modules.

Characterizing the effective descent morphisms of rings in terms of pure monomorphisms is a significant generalization from the previous attempt to do so [21]. This previous attempt – due to Grothendieck [14] – included only the sufficient condition that p make E a faithfully flat B-module to show that p is an effective descent morphism in the opposite category of rings.

It can also shown [21] that the above result holds for algebras as well as modules. This is the case because the multiplication from an algebra structure will not affect the calculation of the required (co)equalizers in the underlying abelian groups of the modules in question. This means that irrespective of the algebra structures considered on B and E, Theorem (4.2.1) provides a full characterization of the effective descent morphisms in the opposite category of algebras over these rings.

This allows for a reformulation – also given in the paper – directly into the language of the present work:

Corollary 4.2.1. *Let R be a ring. An R-algebra homomorphism $p : B \to E$ will be an effective descent morphism*[9] *in the opposite category of algebras over R if and only if it is a pure monomorphism of B-modules.*

It is well known that split monomorphisms satisfy all of the conditions of their 'weaker' counterparts.[10] Specifically, split monomorphisms of B-modules will be pure monomorphisms, and will therefore be effective descent morphisms in Rings^{op}. Note, however, that if these homomorphisms were split as morphisms of rings, but not as morphisms of B-modules, then the conditions in Theorem (4.2.1) would not be met, and the homomorphisms in question would not be effective descent morphisms in Rings^{op}.

Proposition 4.2.2 (Borceux and Janelidze [4])**.** *Every morphism of rings $p : B \to E$ that has a B-linear retraction $s : E \to B$ is an effective descent morphism in Rings^{op}.*

[9] When considered as a morphism $E \mapsto B$ in the opposite category of algebras.
[10] See Section (A.3).

Corollary 4.2.2 (Borceux and Janelidze [4]). *Every field homomorphism $p : K \to L$ is an effective descent morphism both in Rings^{op} and in $\left(\mathsf{K{-}Alg}\right)^{op}$.*

Proof. L can be considered as a vector space over K, and p can be considered as an injective linear transformation from K into L. This means that L can be presented as a direct sum of vector spaces $L = K \oplus M$, where $K \cap M = \{0\}$. The projection $K \oplus M \to K$ will be the K-linear retraction of p required in Proposition (4.2.2). □

Recall that field homomorphisms are morphisms of rings between fields.

Corollary (4.2.2) shows that for every Galois extension of fields $K \subseteq E$, the inclusion of K into E can be regarded as an effective descent morphism in Rings^{op} and, further, the corollary will also be used to propose the form in which effective descent morphisms of non-unital rings should be considered.[11]

4.3 Covering Morphisms

The work in the following section takes place in the context of an adjunction:

$$\left(I, H, \eta, \varepsilon\right) : \mathbb{C} \to \mathbb{X}$$

in which I is admissible and H is fully faithful. As mentioned at the beginning of the current chapter, $I \dashv H$ determines a Galois theory, and this Galois theory yields an analogue for the fundamental theorem of Galois theory in $\mathbb{C}$ - which takes the form of an equivalence between a category of “special morphisms” in $\mathbb{C}$, and a category of internal (pre)groupoid actions on $\mathbb{X}$. Specifically, each effective descent morphism $p : E \to B$ in $\mathbb{C}$ will determine a (special) class of morphisms – known as covering morphisms – and if one takes the image of the kernel pair of an effective descent morphism $p : E \to B$ under I, one will obtain a (pre)groupoid that acts on the objects of $\mathbb{X}$.

Definition 4.3.1. *An object (A, α) in $\left(\mathbb{C} \downarrow B\right)$ is a trivial covering of B if any of the following equivalent conditions holds:*

1. $(A, \alpha) \cong H_B I_B(A, \alpha) = \left(B \times_{HI(B)} HI(A), \pi_1\right)$,[12]
2. $(A, \alpha) \cong H_B(X, \varphi)$ *for some* (X, φ) *in* $\left(\mathbb{X} \downarrow I(B)\right)$,
3. *The following square is a pullback in* $\mathbb{C}$*:*

[11] See Sections (5.2) and (5.2.1).

[12] i.e. $\eta^{B}_{(A,\alpha)}$ is an isomorphism.

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle HI(\alpha)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

From this, one can see that the admissibility of I guarantees that it will preserve all pullbacks along trivial coverings.

Example 4.3.1. *Recall from Section (2.1) that there is an adjunction:*

$$\big(I, H, \eta, \varepsilon\big) : \big(\mathsf{K{-}Alg}\big)^{op} \to \mathsf{FinSets}$$

where:

i. $I(A)$ *is the set of non-zero minimal idempotents of the Boolean algebra* $\mathsf{Idemp}(A)$,

ii. $H(X)$ *is an* X*-indexed coproduct of copies of* K *in* $\big(\mathsf{K{-}Alg}\big)^{op}$ *i.e.*

$$H(X) = \coprod_{x \in X} K_x, \text{ with } K_x = K \text{ for every } x \in X.$$

Recall, further, that I *is admissible, and that* H *is fully faithful. In the language of the current section, this allows one to consider the following situation:*

1. $\mathbb{C} = \big(\mathsf{K{-}Alg}\big)^{op}$,
2. $\mathbb{X} = \mathsf{FinSets}$,
3. $\alpha : A \to B$ *is a trivial covering in* $\mathbb{C}$,
4. B *is a connected algebra* $($*i.e.* $I(B) = \{1\}$, $HI(B) \cong K)$.

If one proceeds in $\mathbb{C}^{op}$ for the sake of clarity, and writes K^n for $\displaystyle\coprod_{e \in I(A)} K_e = HI(A)$,[13] then one obtains the following pushout:[14]

$$\begin{array}{ccc} K & \xrightarrow{\Delta_K} & K^n \\ {\scriptstyle\rho_B}\downarrow & & \downarrow{\scriptstyle\iota_2} \\ B & \xrightarrow[\iota_1]{} & B \otimes_K K^n \end{array} \tag{4.5}$$

[13] Recall from Section (2.1) that each $K_e = K$ and that $HI(A)$ is an $I(A)$-indexed coproduct of copies of K in $\mathbb{C}$.

[14] This diagram is opposite to the form of the pullback in Diagram (2.5).

Since the tensor product distributes over the direct sum here, one has:

$$(A, \alpha) \cong \big(B \otimes_K K^n, \iota_1\big) \cong \big(B^n, \Delta_B\big) \text{ in } \big(B \downarrow \mathbb{C}^{op}\big) = \big(\mathbb{C} \downarrow B\big)^{op},$$

where Δ denotes a diagonal morphism into the product.

Thus, trivial coverings (A, α) of connected algebras are particular diagonal morphisms in $\mathbb{C}^{op}$. Of course, a trivial covering of an algebra D that is not connected will be given by the tensor product over (the object underlying) $HI(D)$ in $\mathbb{C}^{op}$.

Definition 4.3.2. *An object (A, α) in $\big(\mathbb{C} \downarrow B\big)$ is split by a morphism $p : E \to B$ when the unit for $I_E \dashv H_E$ is an isomorphism at $p^*(A, \alpha) = (E \times_B A, \pi_1)$. That is, when:*

$$\eta^E_{E \times_B A} : E \times_B A \to E \times_{HI(E)} HI\big(E \times_B A\big)$$

is an isomorphism.

Definition 4.3.3.

1. *Let $\mathsf{Cov}(B)$ denote the category of covering morphisms - the full subcategory of $\big(\mathbb{C} \downarrow B\big)$ comprised of the objects (A, α) that are split by an effective descent morphism in $\mathbb{C}$.*
2. *Let $p : E \to B$ be a given effective descent morphism in $\mathbb{C}$. One writes $\mathsf{Spl}_B(E, p)$ for the full subcategory of $\mathsf{Cov}(B)$, comprised of the objects (A, α) in $\big(\mathbb{C} \downarrow B\big)$ that are split by p.*
3. $\mathsf{TrivCov}(B) = \mathsf{Spl}_B(B, 1_B)$.

Claiming that an object (A, α) is a trivial covering if and only if it is split by $(B, 1_B)$ follows from the statement that $(A, \alpha) \cong (B \times_B A, \pi_1)$, which itself follows from the fact that the pullback projections of $B \times_B A$ are jointly monomorphic.

One immediately has:

$$\mathsf{TrivCov}(B) \subseteq \mathsf{Cov}(B) = \bigcup_{(E,p)} \mathsf{Spl}_B(E, p)$$

where the union is taken over all effective descent morphisms p with codomain B.

Further, note that an object (A, α) in $\mathbb{C}$ is a covering morphism of B when there exists some effective descent morphism $p : E \to B$ for which either of the following equivalent conditions hold:

1. $(A, \alpha) \in \mathsf{Spl}_B(E, p)$ $\big($i.e. $\eta^E_{E \times_B A} = <\pi_1, \eta_{E \times_B A}>$ is an isomorphism$\big)$,

2. The following square is a pullback in $\mathbb{C}$:

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\eta_{E \times_B A}} & HI\big(E \times_B A\big) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle HI(\pi_1)} \\ E & \xrightarrow[\eta_E]{} & HI(E) \end{array}$$

This formulation makes the interplay between splittings and trivial coverings clear, as can be seen in Corollary (4.3.1).

Corollary 4.3.1. *Suppose that $p : E \to B$ is an effective descent morphism in $\mathbb{C}$. For an object $(A, \alpha) \in \big(\mathbb{C} \downarrow B\big)$, the following conditions are equivalent:*

1. $(A, \alpha) \in \mathsf{Spl}_B(E, p)$,
2. $p^*(A, \alpha) = (E \times_B A, \pi_1) \in \mathsf{TrivCov}(E)$.

That is, $\alpha : A \to B$ is a covering morphism when there is an effective descent morphism $p : E \to B$ for which the pullback projection $\pi_1 : E \times_B A \to A$ is a trivial covering of E, and conversely.

Example 4.3.2. *Consider the situation where:*

- $\mathbb{C} = \big(\mathsf{K{-}Alg}\big)^{op}$,
- $\mathbb{X} = \mathsf{FinSets}$,
- *$\alpha : A \to B$ is a covering morphism in $\mathbb{C}$,*
- *B and E are a connected algebras.*

Just as in Example (4.3.1), one can form a pushout diagram to find:

$$\big(E \otimes_B A, \iota_1\big) \cong \big(E^n, \Delta_E\big) \text{ in } \big(B \downarrow \mathbb{C}^{op}\big). \tag{4.6}$$

Lemma 4.3.1. *The class of covering morphisms in a category $\mathbb{C}$ is pullback-stable.*[15]

The proof of Lemma (4.3.1) follows directly from Proposition (4.2.1). Further, Lemma (4.3.1) guarantees that:

$$\mathsf{Cov}(B) = \bigcup_{(E,p)} \mathsf{Spl}_B(E, p) \tag{4.7}$$

is a directed union [4]. This fact can be used to see if the union in Equation (4.7) has a largest ("universal")[16] element, then the union reduces to that largest element.[17]

[15] See Proposition 6.6.2 in [4] for further detail.
[16] See Definition (4.4.4).
[17] See Proposition (4.4.3).

4.3.1 Coverings and Descent in Categories of Pointed Objects

Recall that if a category $\mathbb{C}$ has a terminal object 1, then $\left(1 \downarrow \mathbb{C}\right)$ is known as the category of the pointed objects of $\mathbb{C}$. Objects in $\left(1 \downarrow \mathbb{C}\right)$ will be represented by pairs such as $(A, a : 1 \to A)$.

In order to be able to construct a pointed version of the Galois theorem, the notions of covering morphisms and effective descent must be extended from the category $\mathbb{C}$ into the category of pointed objects $\left(1 \downarrow \mathbb{C}\right)$. To this end, consider the following.

Lemma 4.3.2. *Let (A, a) and (B, b) be objects in $\left(1 \downarrow \mathbb{C}\right)$. A morphism $\alpha : A \to B$ is a trivial covering in $\mathbb{C}$ if and only if $\alpha : (A, a) \to (B, b)$ is a trivial covering in $\left(1 \downarrow \mathbb{C}\right)$.*

Proof. In order to be precise, the statement of the lemma should read as follows. A trivial covering (A, α) in $\mathbb{C}$ can be considered as a trivial covering in $\left(1 \downarrow \mathbb{C}\right)$ whenever there are morphisms $a : 1 \to A$ and $b : 1 \to B$ making $\alpha : (A, a) \to (B, b)$ a morphism in $\left(1 \downarrow \mathbb{C}\right)$, and – conversely – if $\alpha : (A, a) \to (B, b)$ is any covering morphism in $\left(1 \downarrow \mathbb{C}\right)$, then $\alpha : A \to B$ will also be a covering morphism in $\mathbb{C}$.

The forward implication is trivial, since $1 \times_1 1 \cong 1$. The converse implication holds because diagrams of the form:

$$\begin{array}{ccc} (A, a) & \xrightarrow{\eta_A} & \big(HI(A), HI(a)\big) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle HI(\alpha)} \\ (B, b) & \xrightarrow[\eta_B]{} & \big(HI(B), HI(b)\big) \end{array}$$

in $\left(1 \downarrow \mathbb{C}\right)$ do not make use of the morphisms $a : 1 \to A$, $b : 1 \to B$ in the construction of the pullback. □

Moreover, there is an obvious analogue of Lemma (4.3.2) for covering morphisms.

Lemma 4.3.3. *Let (B, b) and (E, e) be objects in $\left(1 \downarrow \mathbb{C}\right)$. If $p : E \to B$ is an effective descent morphism in $\mathbb{C}$, then $p : (E, e) \to (B, b)$ is an effective descent morphism in $\left(1 \downarrow \mathbb{C}\right)$.*

The forward implication follows immediately, since the relevant calculations can effectively be done without the morphisms $b : 1 \to B$ and $e : 1 \to E$.

The converse statement does not hold in general. For an effective descent morphism $p : (E, e) \to (B, b)$ in $\big(1 \downarrow \mathbb{C}\big)$, the functor $p^* : \big(\mathbb{C} \downarrow B\big) \to \big(\mathbb{C} \downarrow E\big)$ will exhibit the required property only for those objects (A, α) in $\big(\mathbb{C} \downarrow B\big)$ that have a morphism $a : 1 \to A$ such that $\alpha : (A, a) \to (B, b)$ is a morphism in $\big(1 \downarrow \mathbb{C}\big)$.

If $p : (E, e) \to (B, b)$ is an effective descent morphism in $\big(1 \downarrow \mathbb{C}\big)$ then for each covering morphism $\alpha : (A, a) \to (B, b)$ in $\big(1 \downarrow \mathbb{C}\big)$, the pullback $(E \times_B A, {<}e, a{>})$ yields:[18]

$$\pi_2 : (E \times_B A, {<}e, a{>}) \to (E, e)$$

as a coequalizer in $\big(1 \downarrow \mathbb{C}\big)$, and so $\pi_2 : E \times_B A \to A$ as a coequalizer in $\mathbb{C}$ (since coequalizers in comma categories are calculated as in the base category). However, for morphisms $\alpha' : A' \to B$ that cannot be considered as morphisms in $\big(1 \downarrow \mathbb{C}\big)$, there is no way to show that $\pi_2 : E \times_B A' \to A'$ is a coequalizer in $\mathbb{C}$.

Thus, if $p : (E, e) \to (B, b)$ is an effective descent morphism in $\big(1 \downarrow \mathbb{C}\big)$ for which $p : E \to B$ is an effective descent morphism in $\mathbb{C}$, it follows that taking pullbacks along p can be done independently of the existence of morphisms $a : 1 \to A$ and $b : 1 \to B$, and therefore one has the following convenient formulation on objects:

$$\mathsf{Spl}_{(B,b)}\big((E, e), p\big) = \Big\{\big((A, a), \alpha\big) \in \Big(\big(1 \downarrow \mathbb{C}\big) \downarrow (B, b)\Big) : (A, \alpha) \in \mathsf{Spl}_B(E, p)\Big\}$$

In general, one has:

$$\begin{array}{ccc} \mathsf{Spl}_{(B,b)}\big((E, e), p\big) & \neq & \big(1 \downarrow \mathsf{Spl}_B(E, p)\big) \\ {\scriptstyle\subseteq}\big\downarrow & & \big\downarrow{\scriptstyle\subseteq} \\ \Big(\big(1 \downarrow \mathbb{C}\big) \downarrow (B, b)\Big) & & \Big(\big(1, id_1\big) \downarrow \big(\mathbb{C} \downarrow B\big)\Big) \end{array}$$

For each effective descent morphism $p : E \to B$ in $\mathbb{C}$, the Galois theorem in $\mathbb{C}$ manifests as an equivalence between $\mathsf{Spl}_B(E, p)$ and a category of (pre)groupoid actions on $\mathbb{X}$. In order to describe a pointed version of the Galois theorem in $\mathbb{C}$ – i.e. to describe the Galois theorem in $\big(1 \downarrow \mathbb{C}\big)$ – one should use the Galois theory that $(1 \downarrow I) \dashv (1 \downarrow H)$ determines and an effective descent morphism $p : (E, e) \to (B, b)$ in $\big(1 \downarrow \mathbb{C}\big)$ to show that $\mathsf{Spl}_{(B,b)}\big((E, e), p\big)$ is equivalent to a category of pointed (pre)groupoid actions on $\mathbb{X}$.[19] One should not consider $\big(1 \downarrow \mathsf{Spl}_B(E, p)\big)$ for this purpose.

[18] See Section (4.2.1).
[19] That is, a category of (pre)groupoid actions on pointed sets.

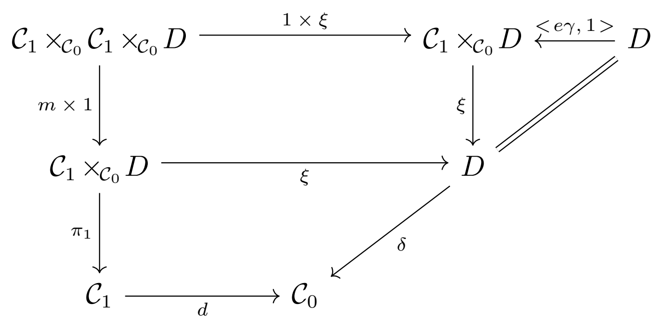
4.4 The Galois Theorem

4.4.1 Internal Category Actions and The Galois Groupoid

The following section details the properties of particular internal categories and category actions that will be used to describe the equivalence that provides the aforementioned analogue for the fundamental theorem of Galois theory in a category $\mathbb{C}$.

The definition of an internal category mirrors the definition of a "conventional" category almost exactly. A full description of the notions of internal categories and internal groupoids can be found in the appendices of [3].

Definition 4.4.1. *Let $\mathbb{C}$ be a category with pullbacks, and let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, c, d, e, m)$ be an internal category in $\mathbb{C}$. An internal category action of $\mathcal{C}$ is given by a pair $\big((D, \delta), \xi\big)$ where $\delta : D \to \mathcal{C}_0$ and $\xi : \mathcal{C}_1 \times_{\mathcal{C}_0} D \to D$ are morphisms in $\mathbb{C}$ such that the following diagram commutes:*



(4.8)

Definition 4.4.2. *A category (internal or otherwise) $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, c, d, e, m)$ is a groupoid if each of its morphisms is an isomorphism.*

The information required for a category to be a groupoid can be codified by appending a morphism $\sigma : \mathcal{C}_1 \to \mathcal{C}_1$ that has the following properties [3]:

$$d \circ \sigma = c, \quad c \circ \sigma = d, \quad m \circ < 1_{\mathcal{C}_1}, \sigma >= s \circ c, \quad m \circ < \sigma, 1_{\mathcal{C}_1} >= s \circ d$$

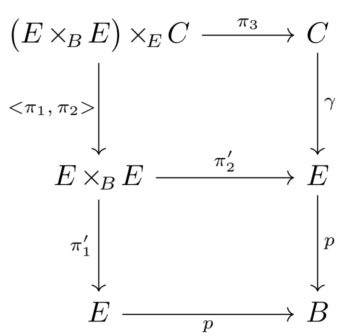
The first two identities coordinate the domain and codomain of inverse morphisms, and the latter identities ensure that for each arrow $f : X \to Y$ in $\mathcal{C}_1$, $\sigma(f)$ is actually inverse to f.

Theorem 4.4.1. *In a category $\mathbb{C}$ with pullbacks, if $p : E \to B$ is a morphism in $\mathbb{C}$ and $(C, \gamma) \in (\mathbb{C} \downarrow E)$, then the morphism:*

$$\overline{\gamma} = {<}\pi_1, \pi_3{>} : (E \times_B E) \times_E C \to E \times_B C$$

is an isomorphism.

Proof. Diagram (4.9) clearly shows that $E \times_B C$ is isomorphic to the iterated pullback:



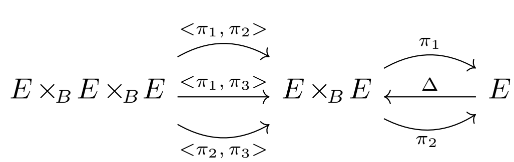
(4.9)

More directly, if one denotes the pullback projections of $E \times_B C$ as π_1'' and π_2'', it is easy to verify that:

$${<}{<}\pi_1'', \gamma \circ \pi_2''{>}, \pi_2''{>} : E \times_B C \to (E \times_B E) \times_E C$$

is inverse to $\overline{\gamma} = {<}\pi_1, \pi_3{>}$. ☐

Example 4.4.1. *If $p : E \to B$ is an effective descent morphism in $\mathbb{C}$, then $\mathsf{Eq}(p)$ – the kernel pair of p – is an internal groupiod in $\mathbb{C}$.*



Since

$$(E \times_B E) \times_E (E \times_B E) \cong E \times_B E \times_B E$$

one can think of ${<}\pi_1, \pi_2{>}$ and ${<}\pi_2, \pi_3{>}$ as the first and second projections from the pullback that defines the composition of internal morphisms in $\mathsf{Eq}(p)$.

Proposition 4.4.1. *Let $\mathbb{C}$ be a category with pullbacks. For a morphism $p : E \to B$ in $\mathbb{C}$, the category of algebras for the monad T^p is isomorphic to the category of actions of the internal groupoid $\mathsf{Eq}(p)$ on $\mathbb{C}$. That is, there is an isomorphism of categories:*

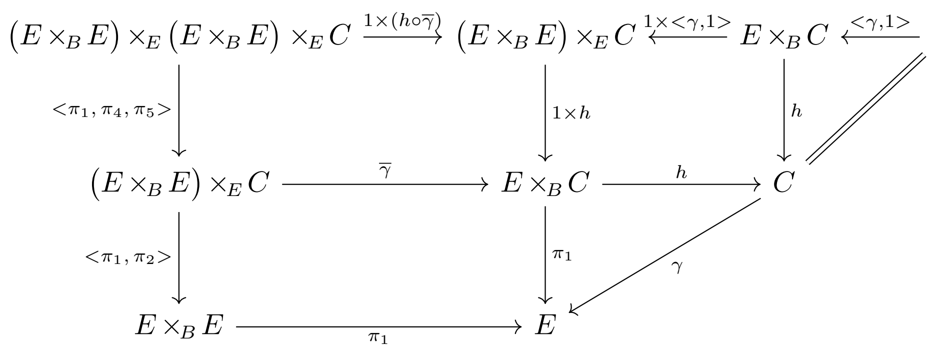
$$\big(\mathbb{C} \downarrow E\big)^{T^p} \cong \mathbb{C}^{\mathsf{Eq}(p)}$$

Proof. If one considers Diagram (4.2), and substitutes the following isomorphisms into the diagram:

1. $E \times_B C \cong \big(E \times_B E\big) \times_E C,$

2. $$\begin{aligned} E \times_B E \times_B C &\cong \big(E \times_B E\big) \times_E \big(E \times_B C\big) \\ &\cong \big(E \times_B E\big) \times_E \big(E \times_B E\big) \times_E C, \end{aligned}$$

one has:



(4.10)

Here:

1. Given the isomorphisms mentioned directly above, the following morphisms correspond directly to one another:
   - $\mu^p_{(C,p\circ\gamma)} = <\pi_1, \pi_3> : E \times_B E \times_B C \to E \times_B C$ in Diagram (4.2),[20]
   - $<\pi_1, \pi_4, \pi_5> : (E \times_B E) \times_E (E \times_B E) \times_E C \to (E \times_B E) \times_E C$ in Diagram (4.10),

   Moreover, $<\pi_1, \pi_4, \pi_5>$ is obviously identical to:

   $\mu^p_{(E,p)} \times_E 1_C : \big(E \times_B E \times_B E\big) \times_E C \to \big(E \times_B E\big) \times_E C,$

2. $1_E \times h : E \times_B E \times_B C \to E \times_B C$ in Diagram (4.2) corresponds directly to $1_{E \times_B E} \times (h \circ \overline{\gamma})$ in Diagram (4.10),

3. $(\overline{\gamma})^{-1} \circ <\gamma, 1_C> = (1_E \times <\gamma, 1_C>) \circ <\gamma, 1_C> = <<1_E, 1_E> \circ \gamma, 1_C>,$

[20] Which is not the same as the isomorphism $\bar{\gamma} = <\pi_1, \pi_3> : (E \times_B E) \times_E C \to E \times_B C$ given in Diagram (4.9).

4. The composite $\big(E \times_B E\big) \times_E C \to E \times_B E \to E$ in the bottom left corner of Diagram (4.10) corresponds to the first projection $\pi_1 : E \times_B C \to E$,

5. The equalities:

   - $h \circ <\gamma, 1_C> = 1_C$,
   - $\gamma \circ h = \pi_1$,
   - $h \circ (1_E \times h) = h \circ \mu$,

   are obtained from the definition of $\big((C, \gamma), h\big)$.

Therefore, if one writes $\xi := h \circ \overline{\gamma}$, Diagram (4.8) shows that ξ is precisely the morphism required to make Diagram (4.10) the description of an internal category action of $\mathsf{Eq}(p)$.

Similarly, transforming $\mathsf{Eq}(p)$-actions into T^p-algebras is a matter of exploiting canonical isomorphisms, and showing that the 'transformations' are inverse to each another is straight-forward. □

Proposition 4.4.2 (Janelidze [16]). *Let $\mathbb{C}$ and $\mathbb{X}$ be categories, let $I : \mathbb{C} \to X$ be a functor between them, and let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, c, d, e, m)$ be an internal category in $\mathbb{C}$. If the following canonical morphisms:*

$$I\big(\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1\big) \to I\big(\mathcal{C}_1\big) \times_{I(\mathcal{C}_0)} I\big(\mathcal{C}_1\big)$$

$$I\big(\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1\big) \to I\big(\mathcal{C}_1\big) \times_{I(\mathcal{C}_0)} I\big(\mathcal{C}_1\big) \times_{I(\mathcal{C}_0)} I\big(\mathcal{C}_1\big)$$

are isomorphisms in $\mathbb{X}$, then the following conditions hold:

1. *$I(\mathcal{C}) = (I(\mathcal{C}_0), I(\mathcal{C}_1), I(c), I(d), I(e), I(m))$ is an internal category in $\mathbb{X}$,*

2. *If $\mathcal{C}$ is an internal groupoid in $\mathbb{C}$, then $I(\mathcal{C})$ is an internal groupoid in $\mathbb{X}$.*

Lemma 4.4.1. *Let $\mathbb{C}$ be a category with pullbacks. An object $\mathcal{C}$ is an internal group in $\mathbb{C}$ if and only if it is an internal groupoid $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, c, d, e, m, \sigma)$ in which $\mathcal{C}_0$ is the terminal object 1 in $\mathbb{C}$.*

That is, internal groups are internal groupoids that have the terminal object as their object of objects.

In order to be able to describe the Galois theorem in a category, one requires the notion of a morphism of Galois descent.

Definition 4.4.3. *An effective descent morphism $p : E \to B$ is a morphism of Galois descent when, for every $(X, \varphi) \in \big(\mathbb{X} \downarrow I(E)\big)$:*

$$(p_! \circ H_E)(X, \varphi) = \big(E \times_{HI(E)} H(X), p \circ \pi_1\big) \in \mathsf{Spl}_B(E, p)$$

Effectively, a morphism of Galois descent is an effective descent morphism $p : E \to B$ with the property that composing any trivial covering of E with p yields a covering morphism of B.

If one foregoes the general assumption that H is fully faithful, then one should include the requirement that H_E[21] be fully faithful in the definition of a morphism of Galois descent.

Lemma 4.4.2. *Let $p : E \to B$ be a morphism of Galois descent. If $(C, \gamma) \in \mathsf{TrivCov}(E)$, then $T^p(C, \gamma) = (p^* \circ p_!)(C, \gamma) = p^*(C, p \circ \gamma) = \big(E \times_B C, \pi_1\big) \in \mathsf{TrivCov}(E)$.*

Proof. Since H_E is fully faithful, one knows that there exists some $(X, \varphi) \in \big(\mathbb{X} \downarrow I(E)\big)$ such that $(C, \gamma) \cong H_E(X, \varphi) = \big(E \times_{HI(E)} H(X), \pi_1\big)$. Now, since p is a morphism of Galois descent, $p_!(C, \gamma) \cong \big(E \times_{HI(E)} H(X), p \circ \pi_1\big) \in \mathsf{Spl}_B(E, p)$, which is the case if and only if $(p^* \circ p_!)(C, \gamma) \in \mathsf{TrivCov}(E)$. □

Corollary 4.4.1. *If $p : E \to B$ is a morphism of Galois descent, then $(E, 1_E) \in \mathsf{TrivCov}(E)$.*

Definition 4.4.4.

1. *An object E in $\mathbb{C}$ is Galois closed if it has no non-trivial coverings i.e. $\mathsf{Cov}(E) = \mathsf{TrivCov}(E)$,*
2. *An effective descent morphism $p : E \to B$ is normal if it splits itself i.e. $(E, p) \in \mathsf{Spl}_B(E, p)$,*
3. *A morphism $p : E \to B$ is a universal covering of B if it is an effective descent morphism that has a Galois closed domain E.*

If an effective descent morphism $p : E \to B$ is normal, then the pullback projection $\pi_1 : E \times_B E \to E$ is a trivial covering of E. Note that this means that I will preserve pullbacks along $\pi_1 : E \times_B E \to E$. Therefore, if one takes $\mathcal{C}_1 = E \times_B E$ in Proposition (4.4.2), the canonical morphisms mentioned in the proposition will indeed be isomorphisms.

[21] Where E is the domain of p.

Definition 4.4.5. *Let $p : E \to B$ be a normal effective descent morphism in $\mathbb{C}$. The Galois groupoid $\mathsf{Gal}[p]$ of p is defined as the following internal groupoid in $\mathbb{X}$:*

$$I\big(E \times_B E \times_B E\big) \begin{array}{c} \xrightarrow{I(<\pi_1,\pi_2>)} \\ \xrightarrow{I(<\pi_1,\pi_3>)} \\ \xrightarrow[I(<\pi_2,\pi_3>)]{} \end{array} I\big(E \times_B E\big) \begin{array}{c} \xrightarrow{I(\pi_1)} \\ \xleftarrow{I(\Delta)} \\ \xrightarrow[I(\pi_2)]{} \end{array} I(E)$$

Again, the Galois groupoid of p is obtained by taking the the kernel pair $\mathsf{Eq}(p)$ of p, considering it as an internal groupoid in $\mathbb{C}$, and then transforming it into an internal groupoid in $\mathbb{X}$ by taking its image under I.

Example 4.4.2. *Let $\mathbb{C} = \big(\mathsf{K{-}Alg}\big)^{op}$, and consider a finite field extension $K \subseteq E$ as a morphism $p : E \to K$ in $\mathbb{C}$. E is Galois closed in $\mathbb{C}$ and, moreover, p is a universal covering of K in $\mathbb{C}$ if and only if $K \subseteq E$ is a Galois extension. See [16] for further details.*[22]

Proposition 4.4.3. *Given a universal covering morphism $p : E \to B$ in $\mathbb{C}$, the following conditions hold:*

1. *(E, p) is normal,*
2. *$\mathsf{Cov}(B) = \mathsf{Spl}_B(E, p)$.*

Proof. Since *(2)* $\Rightarrow$ *(1)*, it only has to be shown that $\mathsf{Cov}(B) = \mathsf{Spl}_B(E, p)$. To this end, take an arbitrary $(A, \alpha) \in \mathsf{Cov}(B)$, and consider the pullback of α along p:

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

Since the class of covering morphisms is pullback-stable, π_1 is also a covering morphism. Since E is Galois closed, π_1 must be a trivial covering of E, and therefore $(A, \alpha) \in \mathsf{Spl}_B(E, p)$. $\square$

[22] Note that the definition of a universal covering in the current text is not the the same as the definition used in [16].

4.4.2 The Galois Theorem via Monadicity and Internal Category Actions

Lemma 4.4.3. *If $p : E \to B$ is a morphism of Galois descent in $\mathbb{C}$, then the restriction:*

$$p^* : \mathsf{Spl}_B(E, p) \to \mathsf{TrivCov}(E)$$

is monadic.

Proof. Being able to use the notation p^* for both the functor

$$p^* : \big(\mathbb{C} \downarrow B\big) \to \big(\mathbb{C} \downarrow E\big)$$

and for its restriction

$$p^* : \mathsf{Spl}_B(E, p) \to \mathsf{TrivCov}(E)$$

is convenient, and since there are few situations in which the context will not make it clear which of the two meanings is being used, they shall not be differentiated. The current proof will be the only exception to this.

It should also be mentioned that the coequalizers in the various comma categories (and their full subcategories) have been represented only by their object parts. The unwritten morphisms can be induced in the obvious way from the morphisms – say α' in (A', α') – of the objects that precede them in the various coequalizer diagrams.

It has to be checked that the restriction $p^* : \mathsf{Spl}_B(E, p) \to \mathsf{TrivCov}(E)$ satisfies Beck's monadicity criteria.

1. The restriction of p^* has a left adjoint, as:

   $$(C, \gamma) \in \mathsf{TrivCov}(E) \Rightarrow p_!(C, \gamma) = (C, p \circ \gamma) \in \mathsf{Spl}_B(E, p)$$

2. Since p is an effective descent morphism, $p^* : \big(\mathbb{C} \downarrow B\big) \to \big(\mathbb{C} \downarrow E\big)$ reflects isomorphisms. Since this property will hold trivially for restrictions to full subcategories of $\big(\mathbb{C} \downarrow E\big)$, the restriction of p^* will reflect isomorphisms from $\mathsf{TrivCov}(E)$.

3. Lastly, for any pair of parallel morphisms $(A, \alpha) \overset{\delta}{\underset{\sigma}{\rightrightarrows}} (A', \alpha')$ in $\mathsf{Spl}_B(E, p)$ such that $\mathrm{Coeq}\big(p^*(\delta), p^*(\sigma)\big)$ is a split (and therefore absolute) coequalizer in $\mathsf{TrivCov}(E)$:

$$p^*(A,\alpha) \underset{p^*(\sigma)}{\overset{p^*(\delta)}{\longleftarrow\!\!\!\rightrightarrows}} p^*(A',\alpha') \underset{s}{\overset{q}{\rightleftarrows}} \mathrm{Coeq}\big(p^*(\delta),p^*(\sigma)\big)$$

one has immediately that $\mathrm{Coeq}\big(p^*(\delta),p^*(\sigma)\big)$ is a split coequalizer in $\big(\mathbb{C}\downarrow E\big)$. Next, since p^* is monadic, the coequalizer of δ and σ in $\big(\mathbb{C}\downarrow B\big)$ exists, and it is preserved by $p^* : \big(\mathbb{C}\downarrow B\big) \to \big(\mathbb{C}\downarrow E\big)$. It is the case that $p^*\big(\mathrm{Coeq}(\delta,\sigma)\big) \in \mathsf{TrivCov}(E)$, and therefore $\mathrm{Coeq}\big(p^*(\delta),p^*(\sigma)\big) \cong p^*\big(\mathrm{Coeq}(\delta,\sigma)\big) \in \mathsf{TrivCov}(E)$.

□

Lemma 4.4.4. *If one restricts the domain of the functor I_E from $\big(\mathbb{C}\downarrow E\big)$ to the full subcategory $\mathsf{TrivCov}(E)$, $I_E \dashv H_E$ becomes an equivalence of categories:*

$$\mathsf{TrivCov}(E) \underset{H_E}{\overset{I_E}{\rightleftarrows}} \big(\mathbb{X}\downarrow I(E)\big)$$

Proof. The counit for the adjunction is an isomorphism by assumption (H is fully faithful and I is admissible), and the unit is an isomorphism at each object in $\mathsf{TrivCov}(E)$ by definition. □

Corollary 4.4.2. *If $p : E \to B$ is a morphism of Galois descent in $\mathbb{C}$, then the composite:*

$$\mathsf{Spl}_B(E,p) \xrightarrow{p^*} \mathsf{TrivCov}(E) \xrightarrow{I_E} \big(\mathbb{X}\downarrow I(E)\big) \quad , \quad (A,\alpha) \longmapsto \big(I(E\times_B A), I(\pi_1)\big)$$

is monadic.

In this context, the functor $I_E \circ p^*$ has a left adjoint:

$$\big(\mathbb{X}\downarrow I(E)\big) \xrightarrow{H_E} \mathsf{TrivCov}(E) \xrightarrow{p_!} \mathsf{Spl}_B(E,p) \quad , \quad (X,\varphi) \longmapsto \big(E\times_{HI(E)} H(X), p\circ\pi_1\big)$$

which is well defined because p is a morphism of Galois descent.

If one relabels the functors as $F = p_! \circ H_E$ and $U = I_E \circ p^*$, one has:

$$\begin{array}{ccc}
\big(\mathbb{X}\downarrow I(E)\big) & \underset{U}{\overset{F}{\rightleftarrows}} & \mathsf{Spl}_B(E,p) \\
\uparrow\downarrow & & {\scriptstyle L'}\uparrow\downarrow{\scriptstyle K'} \\
\big(\mathbb{X}\downarrow I(E)\big)^T & = & \big(\mathbb{X}\downarrow I(E)\big)^T
\end{array}$$

in which:

1. $F(X,\varphi) = \big(E \times_{HI(E)} H(X), p \circ \pi_1\big)$,

2. $U(A,\alpha) = \big(I(E \times_B A), I(\pi_1)\big)$,

3. $UF(X,\varphi) = \big(I\big(E \times_B E \times_{HI(E)} H(X)\big), I(\pi_1)\big)$. Since p is a morphism of Galois descent, $\big(E \times_{HI(E)} H(X), p \circ \pi_1\big)$ is a covering of B, and therefore the pullback $E \times_B \big(E \times_{HI(E)} H(X)\big)$ will be preserved by the admissible functor I. Further, H_E is fully faithful, so $\big(I\big(E \times_{HI(E)} H(X)\big), I(\pi_1)\big) \cong (X,\varphi)$, and therefore one has that:

$$\begin{aligned}\big(I\big(E \times_B E \times_{HI(E)} H(X)\big), I(\pi_1)\big) &\cong \big(I(E) \times_{I(B)} I\big(E \times_{HI(E)} H(X)\big), \pi_1\big)\\ &\cong \big(I(E) \times_{I(B)} X, \pi_1\big),\end{aligned}$$

where the latter object is the pullback of $I(p) \circ \varphi$ along p. Therefore, the unit

$$\overline{\eta}_{(X,\varphi)} : (X,\varphi) \to \big(I(E) \times_{I(B)} X, \pi_1\big)$$

for $F \dashv U$ is given, up to isomorphism, by the unique morphism $<\varphi, 1_X>$,

4. $FU(A,\alpha) = \big(E \times_{HI(E)} HI\big(E \times_B A\big), p \circ \pi_1\big)$. However, since $(A,\alpha) \in \mathsf{Spl}_B(E,p)$, one knows that:

$$\big(E \times_{HI(E)} HI\big(E \times_B A\big), \pi_1\big) \cong \big(E \times_B A, \pi_1\big).$$

Therefore, the counit

$$\overline{\varepsilon}_{(A,\alpha)} : \big(E \times_{HI(E)} HI\big(E \times_B A\big), p \circ \pi_1\big) \to (A,\alpha)$$

for $F \dashv U$ is given, up to isomorphism, by the second pullback projection:

$$\varepsilon^p_{(A,\alpha)} = \pi_2 : \big(E \times_B A, p \circ \pi_1\big) \to (A,\alpha),$$

5. $T = U \circ F$ is the associated monad on $\big(\mathbb{X} \downarrow I(E)\big)$,

6. Again, since $(A,\alpha) \in \mathsf{Spl}_B(E,p)$, one has:

$$\begin{aligned}UFU(A,\alpha) &= \Big(I\big(E \times_B E \times_{HI(E)} HI\big(E \times_B A\big)\big), I(\pi_1)\Big)\\ &\cong \Big(I\big(E \times_B E \times_B A\big), I(\pi_1)\Big)\\ &\cong \Big(I\big(E \times_B E\big) \times_{I(E)} I\big(E \times_B A\big), I(\pi_1)\Big)\end{aligned}$$

Further, since $\varepsilon_{(A,\alpha)}$ is given by the second projection $\pi_2 : \big(E \times_B A, p \circ \pi_1\big) \to (A, \alpha)$, one can see that $U(\varepsilon_{(A,\alpha)})$ is given, up to isomorphism, by $I(\mu^p_{(A,\alpha)})$,[23] which provides a groupoid structure on $I\big(E \times_B A\big)$. Since Diagram (4.11) commutes:

$$\begin{array}{ccc} I\big(E \times_B E\big) \times_{I(E)} I\big(E \times_B A\big) & \xrightarrow{I(\mu^p_{(A,\alpha)})} & I\big(E \times_B A\big) \\ \Big\downarrow{\scriptstyle I(<\pi_1,\pi_2>)} & & \Big\downarrow{\scriptstyle I(\pi_1)} \\ I\big(E \times_B E\big) & \xrightarrow[I(\pi_1)]{} & I(E) \end{array} \tag{4.11}$$

the structure map is well-defined. Thus, the comparison functor K associated to T is given by:

$$K(A,\alpha) = \Big(I\big(E \times_B A\big), I(\mu^p_{(A,\alpha)}) : I\big(E \times_B E\big) \times_{I(E)} I\big(E \times_B A\big) \to I\big(E \times_B A\big)\Big).$$

Theorem 4.4.2 (Borceux and Janelidze [4])**.** *Given an adjunction* $\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$ *where* I *is admissible and* H *is fully faithful, and a morphism of Galois descent* $p : E \to B$ *in* $\mathbb{C}$*, one has the following equivalence of categories:*

$$\mathsf{Spl}_B(E,p) \simeq \big(\mathbb{X} \downarrow I(E)\big)^{\mathsf{Gal}[p]}$$

If E is connected, then $I(E) = \{*\}$, so $\big(\mathbb{X} \downarrow I(E)\big) = \big(\mathbb{X} \downarrow 1\big) \cong \mathbb{X}$. This means that $\mathsf{Gal}[p]$ is actually a group and $\mathbb{X}^{\mathsf{Gal}[p]}$ is a full subcategory of the category of $\mathsf{Gal}[p]$-sets.

Thus, when $p : E \to B$ is a universal covering[24] (and B is connected), the adjunction in Theorem (4.4.2) becomes:

$$\mathsf{Cov}(B) \simeq \mathbb{X}^{\mathsf{Gal}[p]} \tag{4.12}$$

Lemma 4.4.5 (Borceux and Janelidze [4])**.** *Let* $B \in \mathbb{C}$ *be a connected object. If* $p : E \to B$ *is a universal covering of* B*, then the following conditions hold:*

1. *For every pair of coverings* $(A_1, \alpha_1), (A_2, \alpha_2) \in \mathsf{Cov}(B)$ *– where neither* A_1 *nor* A_2 *is initial in* $\mathbb{C}$ *– the pullback* $A_1 \times_B A_2$ *is not initial in* $\mathbb{C}$*,*
2. *For every covering* $(A, \alpha) \in \mathsf{Cov}(B)$ *– where* A *is not initial in* $\mathbb{C}$ *– there exists a morphism* $(E, p) \to (A, \alpha)$ *in* $\big(\mathbb{C} \downarrow B\big)$*, and* (A, α) *is an effective descent morphism.*

[23] See Section (4.2.1).
[24] See Proposition (4.4.3).

Lemma 4.4.6 (Borceux and Janelidze [4])**.** *Let $B \in \mathbb{C}$ be a connected object, and let $p : E \to B$ be a universal covering of B. For every $(A, \alpha) \in \mathsf{Cov}(B)$, the following conditions are equivalent:*

1. *(A, α) is a universal covering of B,*
2. *A is not the initial object in $\mathbb{C}$, and there exists a morphism $(A, \alpha) \to (E, p)$ in $\big(\mathbb{C} \downarrow B\big)$,*
3. *A is not the initial object in $\mathbb{C}$, and for every covering $(A', \alpha') \in \mathsf{Cov}(B)$ in which A' is not initial in $\mathbb{C}$, there exists a morphism $(A, \alpha) \to (A', \alpha')$ in $\big(\mathbb{C} \downarrow B\big)$,*
4. *(A, α) corresponds directly to a free (non-empty) $\mathsf{Gal}[p]$-set under the equivalence in Diagram (4.12).*

Corollary 4.4.3. *If an object $B \in \mathbb{C}$ is connected and admits a universal covering morphism, then there exists a unique universal covering $p : E \to B$ where E is connected.*

With a full understanding of the Galois theorem, one can use the extended notions of admissibility and effective descent – presented earlier in the current chapter – to construct a pointed version of the Galois theorem in $\mathbb{C}$, a Galois theorem in $\big(1 \downarrow \mathbb{C}\big)$.

4.5 The Pointed Galois Theorem

Below, $\mathbb{X}_*$ is used as a short-hand notation for the category of pointed objects $\big(1 \downarrow \mathbb{X}\big)$. For example, if $\mathbb{X} = \mathsf{Sets}$, then $X_* = \big(1 \downarrow \mathbb{X}\big)$ will be the category of pointed sets.

A slightly more nuanced use of the notation will be made for examples resembling $\mathbb{C} = \big(\mathsf{K{-}Alg}\big)^{op}$. In such cases, $\mathbb{C}_*$ will be used to denote the opposite category of non-unital commutative K-algebras, while $\big(1 \downarrow \mathbb{C}\big)$ will denote the actual slice category $\big((K, 1_K) \downarrow \big(\mathsf{K{-}Alg}\big)^{op}\big)$. As shown in Section (5.2), these categories are equivalent.

Recall from Proposition (4.1.2) and the arguments in Section (2.3) that for an adjunction

$$\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$$

where I is admissible and H is fully faithful, the adjunction:

$$\big((1 \downarrow I), (1 \downarrow H), \eta, \varepsilon\big) : \big(1 \downarrow \mathbb{C}\big) \to \big(1 \downarrow \mathbb{X}\big)$$

is such that $(1 \downarrow I)$ is admissible and $(1 \downarrow H)$ is fully faithful. Further, Lemma (4.3.3) ensures that any effective descent morphism $p : E \to B$ in $\mathbb{C}$ that can be regarded as a morphism $p : (E, e) \to (B, b)$ in $\big(1 \downarrow \mathbb{C}\big)$ will be an effective descent morphism in the category of pointed objects.

This means that one can use Theorem (4.4.2) to obtain:

Corollary 4.5.1. *Given a connected object E in $\mathbb{C}$, an adjunction: $\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$ where I is admissible and H is fully faithful, and a morphism of Galois descent $p : (E, e) \to (B, b)$ in $\big(1 \downarrow \mathbb{C}\big)$, one has the following equivalence of categories:*

$$\mathsf{Spl}_{(B,b)}\big((E, e), p\big) \simeq \mathbb{X}_*^{\mathsf{Gal}[p]}$$

This presents $\mathsf{Spl}_{(B,b)}\big((E, e), p\big)$ as being equivalent to a full subcategory of the category of pointed $\mathsf{Gal}[p]$-sets.[25]

It should be noted that:[26]

$$\mathsf{Spl}_{(B,b)}\big((E, e), p\big) \neq \big(1 \downarrow \mathsf{Spl}_B(E, p)\big)$$

in general, and that one should use Corollary (4.5.1) to construct the pointed Galois theorem, rather than constructing the equivalence in the convention Galois theorem $\mathsf{Spl}_B(E, p) \simeq \mathbb{X}^{\mathsf{Gal}[p]}$ and then finding the pointed objects in the respective categories.

Corollary (4.5.1) provides an important generalization of the conventional Galois theorem, and will be used in the Chapter (5) to expand the Galois theory of finite, separable field extensions into a non-unital context.

[25]See Appendix (A.1).
[26]See the final remarks in Section (4.3.1).

Chapter 5

Galois Theory of Finite Unital and Non-Unital Commutative K-Algebras

5.1 Galois Theory of Finite Field Extensions

The work in this section makes use of the Pierce representation of commutative rings [27]. Essentially, each commutative ring (or algebra) A has an underlying Boolean algebra $\mathsf{Idemp}(A)$ whose operations are induced by the operations of the ring it is contained in. The minimal non-zero elements of each $\mathsf{Idemp}(A)$ are known as its atoms, and if A has a finite number of idempotents, then any element in the corresponding Boolean algebra can be seen as a finite join of these atoms. What is more, for each atom e_i of $\mathsf{Idemp}(A)$, e_iA is a connected ring (algebra), and the collection of these connected rings (algebras) can be used to form a decomposition of A into the product:

$$A \cong \prod_{i \in I(A)} e_i A$$

Although each e_iA is a ring (algebra) in its own right, they will not be subrings (subalgebras) of A in general, as they will not have the same multiplicative identity as A (unless A is connected). For further detail, see Section (6.3).

Detailed accounts of the following can be found in [17] and [20]. A prime example of an adjunction $\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$ where I is admissible and H is fully faithful can be obtained by taking $\mathbb{C}$ to be the opposite category of finite-dimensional unital commutative K-algebras over a fixed field K, and $\mathbb{X}$ to be the category of finite sets.

The opposite category of connected unital commutative K-algebras $\mathbb{A}$ forms a full subcategory of $\mathbb{C}$.[1] Since K is the terminal object in $\mathbb{A}$, there is an adjunction of the

[1] See Proposition (2.1.3).

form $\mathbb{A} \rightleftarrows \mathbf{1}$ that extends to the adjunction:

$$\mathbb{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathbb{X}$$

in which:

1. $I(A) = \{e_iA : e_i \in \mathsf{Atom}\big(\mathsf{Idemp}(A)\big)\}$

   Recall from Section (2.1) that for a morphism $\alpha : A \to B$ in $\mathbb{C}$, I acts as follows. Since α is an algebra homomorphism from B to A, for each $e \in I(A) = \{e_1, e_2, \ldots, e_n\}$ one has the following composite in $\mathbb{C}^{op}$:

   $$B \xrightarrow{\alpha} A \xrightarrow{\zeta} \prod_{e_i \in I(A)} e_iA \xrightarrow{\pi_e} eA$$

   Now since the elements in $I(B) = \{e'_1, e'_2, \ldots, e'_m\}$ have the properties that:

   (a) $\sum_{i=1}^{m} e'_i = 1$,

   (b) $e'_ie'_j \neq 0 \Leftrightarrow i \neq j$,

   it can be shown that for each $e \in I(A)$, only one atom in $I(B)$ is mapped to e under the composite. Since the composite preserves idempotents, it is clear that each $e' \in I(B)$ must be mapped either to 0 or to e in eA. If the composite mapped no elements in $I(B)$ to e, (a) would result in the composite mapping $1 \mapsto 0$, which would be a contradiction. If the composite mapped more than one element in $I(B)$ to e, (b) would contradict the fact that the composite preserves multiplication.

   Since the association of $e' \mapsto e$ is unique, it determines a set map

   $$I(\alpha) : I(A) \to I(B) \quad , \quad e \mapsto e'$$

   If e' is the unique element in $I(B)$ mapped to e under the composite, every element above e' in $\mathsf{Idemp}(B)$ – the Boolean algebra of idempotents of B – will be mapped to e under the composite as well. Thus, one could equally well characterize $I(\alpha)$ as the mapping:

   $$I(\alpha) : I(A) \to I(B) \quad , \quad e \mapsto min\{e' \in \mathsf{Idemp}(B) : e \leq \alpha(e')\}$$

2. $H(X) = \coprod_{x \in X} K_x$ is the coproduct of "X copies of K" [17] in $\mathbb{C}$,

3. If one works in $\mathbb{C}^{op}$ for convenience, it is clear that $H(X)$ has $\prod_{x \in X} K_x$ as its underlying object. Thus, if one writes $\rho_i : K_i \to e_iA$ for the given actions of K on the respective algebras, one obtains:

$$
\begin{array}{ccccc}
\prod_{i \in I(A)} K_i & \overset{<\rho_i>}{\dashrightarrow} & \prod_{i \in I(A)} e_i A & \underset{\zeta}{\overset{\zeta^{-1}}{\rightleftarrows}} & A \\
\Big\downarrow {\scriptstyle \pi'_i} & & \Big\downarrow {\scriptstyle \pi_i} & & \\
K_i & \xrightarrow[\rho_i]{} & e_i A & &
\end{array}
$$

Thus, the unit of $I \dashv H$ is determined by $\eta_A = \zeta^{-1} \circ <\rho_i>$ in $\mathbb{C}^{op}$, as in:

$$\eta_A : \prod_{i \in I(A)} K_i \to A \quad , \quad (k_1, k_2, \ldots, k_n) \mapsto k_1 e_1 + k_2 e_2 + \cdots + k_n e_n$$

4. Since $H(X)$ has $\prod_{x \in X} K_x$ as its underlying object, one knows that

$$IH(X) = I(\prod_{x \in X} K_x).$$

Since each $K_x = K$ is connected, the minimal idempotents of $\prod_{x \in X} K_x$ are simply the elements with 1_K in the x^{th} position and a 0 in every other. If one writes $\delta_x = (0, \ldots, 1, \ldots, 0)$ for each of these, it is clear that the counit for the adjunction is given by the bijection:

$$\varepsilon_X : IH(X) \overset{\cong}{\to} X \quad , \quad \delta_x \mapsto x$$

As shown in the Example (4.3.1), trivial coverings (A, α) of connected algebras $B \in \mathbb{C}$ have finite products of B as their underlying objects:

$$(A, \alpha) \cong (B^n, \Delta_B) \text{ in } \big(B \downarrow \mathbb{C}^{op}\big).$$

Similarly, if $p : E \to B$ is an effective descent morphism in $\mathbb{C}$, $(A, \alpha) \in \mathsf{Spl}_B(E, p)$, and E is connected, then one has the following:

$$\big(E \otimes_B A, \iota_1\big) \cong \big(E^n, \Delta_E\big) \text{ in } \big(B \downarrow \mathbb{C}^{op}\big). \tag{5.1}$$

Moreover, if $B = K$ is a field, $K \subseteq E$ is a Galois extension, and p is taken to be the inclusion homomorphism from K into E, considered as a morphism in $\mathbb{C}$, then one can show that (5.1) is equivalent either to the statement that A is isomorphic to a product $A_1 \times A_2 \times \cdots \times A_n$ of field extensions of K, or that $A \cong A_1 \times A_2 \times \cdots \times A_n$ and each A_i is a subextension of $K \subseteq E$ [20].

5.1.1 An Algebraic Interpretation

Continuing in the case where $\mathbb{C}$ is the opposite of the category of finite-dimensional commutative K-algebras, if one considers a Galois extension of fields $B = K \subseteq E$ as a universal covering $p : E \to K$ in $\mathbb{C}$ and writes $G = \mathsf{Gal}[p]$, then the equivalence:

$$\mathsf{Cov}\big(K\big) \simeq \mathbb{X}^G \tag{5.2}$$

can be interpreted in an "uncategorical," but very intuitive, algebraic way. To this end, a large portion of this section will refer to the algebras themselves (rather than to the objects in the opposite category) and talk freely about (5.2) as being a contravariant equivalence:

$$\mathsf{Cov}\big(K\big)^{op} \simeq \mathbb{X}^G \tag{5.3}$$

In this context, one can think of both functors in the equivalence as contravariant hom-functors.

Definition 5.1.1. *Let X be a G-set, and take $x \in X$.*

1. $\mathrm{O}_x = \{gx : g \in G\}$ *is the orbit of x,*
2. $G_x = \{g \in G : gx = x\}$ *is the stabilizer of x.*

Note that each stabilizer is completely determined by its respective orbit. In fact, there is a bijection:

$$G/G_x \cong \mathrm{O}_x \quad , \quad gG_x \mapsto gx$$

In this sense, each orbit in X corresponds directly to a subgroup of G. By their definition – and, implicitly, by the properties of G – orbits are either disjoint or coincide completely. What's more, each G-set $X = \coprod_{x \in X} G/G_x$ is the disjoint union of its orbits (the orbits partition X), and no orbit can be expressed as a non-trivial disjoint union.

In $\mathbb{X}^G$ – the category of finite G-sets – and therefore in $\mathsf{Cov}(K)$, there are connected objects, and every object in the respective categories can be presented as a coproduct of these connected objects. In $\mathsf{Cov}(K)$, each object (A, α) is a coproduct of subextensions of $K \subseteq E$. Of course, this means that the algebra underlying each (A, α) can be split into a product of subextensions.

E is the largest (universal) connected object in $\mathsf{Cov}(K)$, in the sense that it will have a morphism into every other connected object.[2] Intuitively, these morphisms in $\mathsf{Cov}(K)$ correspond to inclusion homomorphisms from subextensions $K \subseteq A \subseteq E$ into E in $\mathsf{Cov}(K)^{op}$.

In order to discuss the correspondence between the subfields of E and the subgroups of G, it will be helpful to introduce the following notation:

[2] See Lemma (4.4.5).

1. $E^H = \{e \in E : g \in H \Rightarrow ge = e\}$ for any subgroup $H \leq G$,
2. $G_A = \{g \in G : a \in A \Rightarrow ga = a\}$ for any subfield $A \subseteq E$.

In this context, H will always be taken to be a subgroup of the form $G_x \leq G$, and A will always be a subextension $K \subseteq A \subseteq E$.

It is clear that G_A is the automorphism group of A, in the usual Galois-theoretic sense.

The above constructions are "inverse" to each another, in that for each stabilizer G_x (for some $x \in X$), one has:

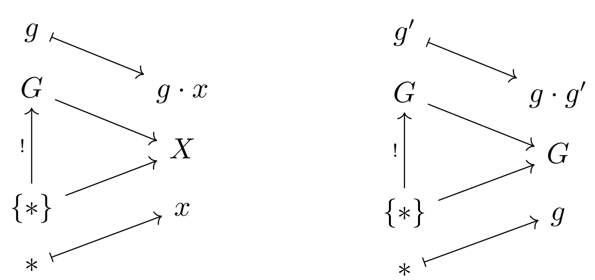
$$\begin{aligned} G_{E^{G_x}} &= \{g \in G : e \in E^{G_x} \Rightarrow ge = e\} \\ &= \{g \in G : e \in \{e \in E : g \in G_x \Rightarrow ge = e\} \Rightarrow ge = e\} \\ &= G_x, \end{aligned}$$

and for each subextension $K \subseteq A \subseteq E$, one finds:

$$\begin{aligned} E^{G_A} &= \{e \in E : g \in G_A \Rightarrow ge = e\} \\ &= \{e \in E : g \in \{g \in G : a \in A \Rightarrow ga = a\} \Rightarrow ge = e\} \\ &= A. \end{aligned}$$

Moreover, the inclusion of one subgroup in another will correspond to an inclusion of field extensions in the opposite direction. It is trivial to check that in the language of these constructions, E corresponds to the trivial subgroup $\{*\} \leq G$, which in turn corresponds to the orbit $G/\{*\} \cong G$. Similarly, K corresponds the entire group G.[3]

The left-hand triangle in Diagram (5.4) shows what it means for any (X, γ) to be a G-set. It also shows that G is the free G-set on one element, as it exhibits the required universal property. The right-hand triangle shows that when one considers G – with its own multiplication – as a G-set, choosing an element in G (by assigning $*$ to some $g \in G$) is equivalent to choosing an automorphism on G, and so G must be its own automorphism group (when considered as a G-set).



(5.4)

Since automorphism groups are preserved under equivalences, this shows that G is also the automorphism group of E.

[3] K corresponds to G when the latter is considered as a subgroup of itself, rather than as a G-set.

Given this fact, one can consider E as a G-set, where the crossed multiplication is given by evaluation $(g, e) \mapsto g(e)$ at points in E.

If two small abuses of notation are permitted:

1. $\mathsf{Hom}_{\mathbb{X}^G}\big(-, E\big) = \mathsf{Hom}_G\big(-, E\big)$,
2. $\mathsf{Hom}_{\mathsf{Cov}(K)^{op}}\big(-, E\big) = \mathsf{Hom}_K\big(-, E\big)$,

one can quite easily show that the duality in Diagram (5.3) can be represented by the given pair of contravariant hom-functors.

Lemma 5.1.1 (Lastaria [23])**.** *For each subgroup $H \leq G$, evaluation at $[H]$ is a K-algebra isomorphism:*

$$\mathbf{ev}_{[H]} : \mathsf{Hom}_{\mathsf{G}}\big(G/H, E\big) \to E^H \quad , \quad f \mapsto f([H])$$

Proof. It first has to be checked that the morphism is well-defined. To this end, take $f \in \mathsf{Hom}_{\mathsf{G}}\big(G/H, E\big)$, let $e := \mathbf{ev}_{[H]}(f) = f([H])$, and take $h \in H$. Now, one easily calculates:

$$he = h(e) = h(f([H])) = f(h[H]) = f([H]) = e$$

Therefore, $\mathbf{ev}_{[H]}(f) \in E^H$ for each $f \in \mathsf{Hom}_{\mathsf{G}}\big(G/H, E\big)$. The evaluation $\mathbf{ev}_{[H]}$ is certainly a homomorphism, and it is injective because each $f \in \mathsf{Hom}_{\mathsf{G}}\big(G/H, E\big)$ commutes with elements of G. Finally, $\mathbf{ev}_{[H]}$ is surjective, since any $e \in E^H$ can be seen as the image of a G-set homomorphism that has the association $[gH] \mapsto ge$. □

Lemma 5.1.2 (Lastaria [23])**.** *For each subgroup $H \leq G$, there is a G-set isomorphism:*

$$\lambda : G/H \to \mathsf{Hom}_K\big(E^H, E\big)$$

Proof. Under λ, each coset $[gH]$ is mapped to the function:

$$\lambda_{[gH]} : E^H \to E \quad , \quad e \mapsto ge$$

The properties required for λ to well defined are inherited from the G-set structure on E, and λ is injective because the cosets are either equal or disjoint. What is more, λ is also surjective, because every $t \in \mathsf{Hom}_{\mathsf{Alg}}\big(E^H, E\big)$ extends to an automorphism on the algebraic closure of K, and restricting this automorphism to E yields an element $g' := \bar{t}|_E$ of G, since E is a Galois extension of K. Of course, g' can be used to construct $\lambda_{[g'H]}$, which shows that λ is surjective. □

Since E is connected,

$$\mathsf{Hom}_G\big(-,E\big):\mathbb{X}^G\to\mathsf{Cov}\big(K\big)^{op}$$

will take coproducts to products, and, similarly,

$$\mathsf{Hom}_K\big(-,E\big):\mathsf{Cov}\big(K\big)^{op}\to\mathbb{X}^G$$

will take products to coproducts.

Now, if one writes A_i for the connected components of an algebra (A,α) in $\mathsf{Cov}\big(K\big)^{op}$, one finds:

$$\begin{aligned}
\mathsf{Hom_G}\big(\mathsf{Hom_K}(A,E),E\big) &\cong \mathsf{Hom_G}\big(\mathsf{Hom_K}(\prod_{i\in I(A)}A_i,E),E\big)\\
&\cong \mathsf{Hom_G}\big(\coprod_{i\in I(A)}\mathsf{Hom_K}(A_i,E),E\big)\\
&\cong \mathsf{Hom_G}\big(\coprod_{i\in I(A)}\mathsf{Hom_K}(E^{G_{A_i}},E),E\big)\\
&\cong \mathsf{Hom_G}\big(\coprod_{i\in I(A)}G/G_{A_i},E\big)\\
&\cong \prod_{i\in I(A)}\mathsf{Hom_G}\big(G/G_{A_i},E\big)\\
&\cong \prod_{i\in I(A)}E^{G_{A_i}}\\
&\cong \prod_{i\in I(A)}A_i\\
&\cong A
\end{aligned}$$

Likewise, if one writes G/G_x for each of the orbits of a G-set X, one has:

$$\begin{aligned}
\mathsf{Hom_K}\big(\mathsf{Hom_G}(X,E),E\big) &\cong \mathsf{Hom_K}\big(\mathsf{Hom_G}(\coprod_{x\in X}G/G_x,E),E\big)\\
&\cong \mathsf{Hom_K}\big(\prod_{x\in X}\mathsf{Hom_G}(G/G_x,E),E\big)\\
&\cong \mathsf{Hom_K}\big(\prod_{x\in X}E^{G_x},E\big)\\
&\cong \coprod_{x\in X}\mathsf{Hom_K}\big(E^{G_x},E\big)\\
&\cong \coprod_{x\in X}G/G_x\\
&\cong X
\end{aligned}$$

which (up to few technical details) neatly proves the desired result.

5.2 Galois Theory of Finite Field Extensions in a Non-unital Context

Although speaking of "non-unital" field extensions would be a contradiction in terms, considering field extensions as structures of rings or algebras that are not necessarily unital – as will be done in this section – is perfectly reasonable.

As in Section (5.1), the following makes use of the Pierce representation of rings, as detailed in Section (6.3).

It was shown in Section (3.5) that non-unital commutative rings can be recovered from unital commutative rings as morphisms into the integers. Further, if R is a unital, commutative ring, then an analogous relationship holds for R-algebras. In particular, when $R = K$ is a field, one has the following equivalence of categories:

$$\mathsf{K\text{--}Alg}_* \underset{G}{\overset{F}{\rightleftarrows}} \big(\mathsf{K\text{--}Alg} \downarrow K\big)$$

If one assesses this in the context of the now-familiar adjunction:

$$\big(\mathsf{K\text{--}Alg}\big)^{op} = \mathbb{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathbb{X} = \mathsf{FinSets} \tag{5.5}$$

one can use the admissibility of I and Proposition (4.1.2) to guarantee the admissibility of $(1 \downarrow I)$ and, further, to extend the adjunction in Diagram (5.5) to:

$$\mathbb{C}_* \underset{G^{op}}{\overset{F^{op}}{\rightleftarrows}} \big(1 \downarrow \mathbb{C}\big) \underset{(1\downarrow H)}{\overset{(1\downarrow I)}{\rightleftarrows}} \big(1 \downarrow \mathbb{X}\big) \cong \mathbb{X}_* \tag{5.6}$$

where the top composite is admissible, and the bottom composite is fully faithful. If one relabels these composites as I' and H', respectively, one finds:

1. $I'(U) = \big(I(K \ltimes U), e_U\big)$, where $e_U = (k, u)$ is the smallest element in the Boolean algebra $\mathsf{Idemp}\big(K \ltimes U\big)$ satisfying $\pi_1(k, u) = 1$.

   In fact, it can be shown that e_U is an atom in $\mathsf{Idemp}\big(K \ltimes U\big)$, and that it is the only atom that satisfies this property. By Proposition (6.3.3), there is a finite number of idempotent elements $(k_1, u_1), (k_2, u_2), \ldots, (k_n, u_n) \in K \ltimes U$ such that:

   (a) $\displaystyle\sum_{i=1}^{n}(k_i, u_i) = (1, 0)$,

   (b) $(k_i, u_i)(k_j, u_j) = (0, 0) \Leftrightarrow i \neq j$.

Given this, one can use the fact that addition in the semidirect product is computed point-wise to see that at least one of the (k_i, u_i) satisfies $\pi_1(k_i, u_i) = 1$, and one can use (b) to verify that no more than one of them will do so.

If $e_U = (1, u)$, one can think of u as a "generalized -1" in U, in the sense that if U has a multiplicative unit 1_U, then u will be equal to -1_U.

2. The object underlying the image of any pointed set (X, x) under $(1 \downarrow H)$ is the projection:

$$\prod_{y \in X} K_y \xrightarrow{\pi_x} K_x \text{ in } \mathbb{C}^{op},$$

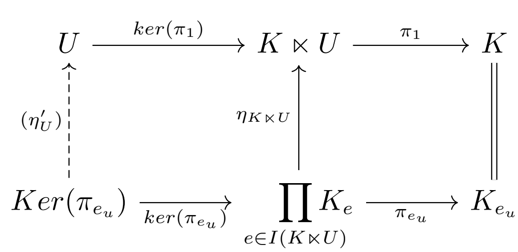
and the image of this morphism under G^{op} is its kernel in $\mathbb{C}_*^{op}$, considered as the cokernel in $\mathbb{C}_*$. That is,

$$H'(X, x) = Coker(\pi_x) \text{ in } \mathbb{C}_* \tag{5.7}$$

and its underlying object is given by:

$$Ker(\pi_x) \quad \cong \prod_{y \in X \setminus \{x\}} K_y \text{ in } \mathbb{C}_*^{op}. \tag{5.8}$$

3. If one continues working in $\mathbb{C}_*^{op}$, one sees that $H'I'(U) = Ker(\pi_{e_u})$, and that η'_U is determined by the universal property of the kernel $U \cong Ker(\pi_1)$:



4. Since $F \dashv G$ is an equivalence of categories, the counit of $I' \dashv H'$ is given, up to isomorphism, by the counit ε of $(1 \downarrow I) \dashv (1 \downarrow H)$.

Given that I' is admissible and H' is fully faithful, a (slightly modified) version of Corollary (4.5.1) holds in this context. In general, it is convenient to represent the objects in $\mathbb{C}_*$ in terms of their images in $\big(1 \downarrow \mathbb{C}\big)$, so for a morphism of Galois descent $p : (E, e) \to (B, b)$ in $\big(1 \downarrow \mathbb{C}\big)$, the pointed version of the Galois theorem in this context will still take the form:

$$\mathsf{Spl}_{(B,b)}\big((E, e), p\big) \simeq \mathbb{X}_*^{\mathsf{Gal}[p]} \tag{5.9}$$

As shown in Lemma (4.3.3), effective descent morphisms in $\mathbb{C}$ can always be considered as effective descent morphisms in $\big(1 \downarrow \mathbb{C}\big)$. Thus, any Galois extension of fields $K \subseteq E$, considered as a morphism of Galois descent in $\mathbb{C}$, will also be a morphism of Galois descent in $\mathbb{C}_* \simeq \left(1 \downarrow \mathbb{C}\right)$. This shows that field extensions $K \subseteq E$ still play a significant role in the Galois theory of this pointed context.

It can be shown that the form of the trivial coverings of K in $\mathbb{C}$ closely resembles the form of the trivial coverings of $\left(K, 1_K\right)$ in $\mathbb{C}_*$. Again with reference to a Galois extension $K \subseteq E$, if one restricts the equivalence:

$$\mathsf{Spl}_K\left(E\right) \simeq \mathbb{X}^{\mathsf{Gal}[p]}$$

to the category of trivial coverings of K, one has:

$$\mathsf{TrivCov}_{\mathbb{C}}\left(\mathsf{K}\right) \simeq \mathbb{X}$$

Similarly, if one restricts the equivalence in Diagram (5.9) to the trivial coverings of $(K, 1_K)$, one obtains:

$$\mathsf{TrivCov}_{\mathbb{C}_*}\left(K, 1_K\right) \simeq \mathbb{X}_*$$

Since $Ker(\pi_x) \cong \prod_{y \in X \setminus \{x\}} K_y$ in $\mathbb{C}_*^{op}$, (2) may leave one with the impression that the trivial coverings of K in $\mathbb{C}_*$ are identical to those in $\mathbb{C}$. Indeed, inspecting $I' \dashv H'$ to see how the adjunction acts on non-unital K-algebras shows that the objects in $\mathsf{TrivCov}_{\mathbb{C}_*}(K, 1_K)$ actually coincide with those in $\mathsf{TrivCov}_{\mathbb{C}}(K)$. The algebras that underlie the trivial coverings of $(K, 1_K)$ in $\mathbb{C}_*$ are finite products of copies of K, and as such have multiplicative identities, even though this condition was never required in the construction of the extended adjunction in Diagram (5.6). The difference between the categories is that in $\mathsf{TrivCov}_{\mathbb{C}_*}(K, 1_K)$, morphisms are not required to preserve these multiplicative identities.

Note that the multiplicative identity of each $Ker(\pi_x)$ is not the same as identity of the product $\prod_{x \in X} K_x$ in which it is contained. Therefore, even though each $Ker(\pi_x)$ can be considered as ring, it will not be a subring of the product.

Concurrently, the following correspondence:

$$\begin{array}{ccc} \mathsf{TrivCov}_{\mathbb{C}}(K) & \simeq & \mathsf{FinSets} \\ \subseteq \big\downarrow & & \big\downarrow \subseteq \\ \mathsf{TrivCov}_{\mathbb{C}_*}(K, 1_K) & \simeq & \mathsf{FinSets}_* \end{array} \tag{5.10}$$

may also seem incongruous. When one moves from $\mathsf{FinSets}$ to $\mathsf{FinSets}_*$, one has to add structure (choose base points) and remove morphisms (consider only those functions that preserve base points). Whereas when one moves from $\mathsf{TrivCov}_{\mathbb{C}}(K)$ to

$\mathsf{TrivCov}_{\mathbb{C}_*}(K, 1_K)$, one has to keep the same objects, and add morphisms (one will no longer require that they preserve a multiplicative identity).

If, however, one recalls that the category of pointed sets is isomorphic to the category of partial sets, it is clear that paradox presented by Diagram (5.10) is only an equivocation.

5.2.1 Effective Descent Morphisms of Non-unital K-Algebras

In seeking to understand the behaviour of effective descent morphisms in $\mathbb{C}_*$, one might wish to consider a Galois extension of unital K-algebras as an extension in $\mathbb{C}_*$ directly. That is, one might consider the situation in which:

1. $(B, b) = (K, 1_K)$,
2. (E, e) is such that is $K \subseteq E$ is a Galois extension of K, and E is equipped with an algebra homomorphism $e : E \to K$, considered as an object in $\mathbb{C}_* \simeq \big(1 \downarrow \mathbb{C}\big)$,
3. p is the inclusion $K \to E$, considered as a morphism in $\mathbb{C}_*$.

However, no such Galois extension exists: E is a field, and all algebra homomorphisms between fields are injective (i.e. they are monomorphisms). This means that there are no split epimorphisms between fields that are not isomorphisms. Any homomorphism $e : E \to K$ would necessarily admit the inclusion p as a section in $\mathbb{C}^{op}$, and would therefore be an isomorphism (and so the extension $K \subseteq E$ would be trivial).

Another simple example that suggests itself is as follows:

1. $(B, b) = (K, 1_K)$,
2. $(E, e) = (K \ltimes L, \pi_1)$, where $K \subseteq L$ is a Galois extension of K, and (E, e) is considered as an object in $\mathbb{C}_*$,
3. p is the inclusion homomorphism $<1, 0> : K \to K \ltimes L$, considered as a morphism in $\mathbb{C}_*$.

It is clear that $p = <1, 0>$ has a section $s = \pi_1$ in the $\mathbb{C}_*$.

For any covering morphism $\alpha : (A, a) \to (B, b)$ in $\mathsf{Spl}_{(B,b)}\big((E, e), p\big)$, one knows that pulling α back along p will yield a trivial covering of (E, e), and that pulling this trivial covering back along any morphism will yield a second trivial covering.

$$\begin{array}{ccccc} A & \longrightarrow & \bullet & \longrightarrow & A \\ {\scriptstyle \sharp\sharp}\downarrow & & \downarrow{\scriptstyle \sharp} & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow[s]{} & E & \xrightarrow[p]{} & B \end{array}$$

In particular, if one pulls α back along p to find $\sharp$ as a trivial covering, and then pulls $\sharp$ back along the section s, one will find that $\sharp\sharp = \alpha$. This shows that all of the coverings $\big((A,a),\alpha\big)$ that are split by p in $\mathbb{C}_*$ are necessarily trivial. However, if one recalls that $(K, 1_K)$ corresponds to the zero ring under the equivalence $\mathbb{C}_* \simeq \big(1 \downarrow \mathbb{C}\big)$, and that there is no interesting Galois theory of the zero ring, it becomes clear that Galois extensions in the context of $\mathbb{C}_*$ should be considered as follows:

- $(B, b) = (K \ltimes K, \pi_1)$,
- $(E, e) = (K \ltimes L, \pi_1)$, where $K \subseteq L$ is a Galois extension of K,
- p is the inclusion $B \to E$, considered as a morphism in $\mathbb{C}_*$.

B and E can be considered as 2- and n-dimensional vector spaces over K respectively, and therefore one can use Proposition (4.2.2) to see that the inclusion p will be an effective descent morphism in $\mathbb{C}$, and so too in $\mathbb{C}_*$.

This example illustrates the form in which one should consider effective descent morphisms (and field extensions in particular) of non-unital rings.

The chapter to follow provides a full description of the Boolean Galois theory of commutative rings, which can be used, among more general pursuits, to describe infinite field extensions. Although the form of the adjunction constituting a Boolean Galois theory deviates slightly from the form of the adjunctions used in the present chapter, they will still facilitate the construction of a pointed version of the Galois theorem.

Chapter 6

The (Boolean) Galois Theory of Commutative Rings and Algebras

The Stone duality, in one of its general forms, is a contravariant equivalence between the category of Boolean algebras and the category of Stone spaces.[1] By noting that each commutative ring has an underlying Boolean algebra, one can make use of the Stone duality to form an adjunction between the category of commutative rings and the category of Stone spaces. This adjunction will mimic the adjunctions

$$\big(I, H, \eta, \varepsilon\big) : \mathbb{C} \to \mathbb{X}$$

of Chapter (4), but now with $\mathbb{X}$ as the category of Stone spaces. For a description of the general construction of Boolean Galois theories (those theories determined by adjunctions with the category of Stone spaces, rather than sets), see [9].

The majority of the following chapter builds on the description of the Pierce representation given in [4].

6.1 Filters in Boolean Algebras

Filters play an important role in the presentation of Boolean algebras as topological spaces. The following definition, and successive lemmas, will provide a foundation with which to describe the Stone duality, and the spectrum of a commutative ring.

[1] Compact, totally disconnected topological spaces.

Definition 6.1.1. *Suppose that B is a Boolean algebra. A filter F on B is a subset of B such that:*

1. $1 \in F$,
2. $\big(x \in F$ *and* $y \in F\big) \Rightarrow x \wedge y \in F$,
3. $\big(x \in F$, $y \in B$ *and* $x \leq y\big) \Rightarrow y \in F$.

Example 6.1.1. *The simplest example of a filter is that of a principal filter: a filter of the form* $F = \uparrow x = \{b \in B : x \leq b\}$.

The collection of proper ($F \neq B$) filters on a Boolean algebra B forms a partially ordered set (when ordered by inclusion), and is often denoted by $\mathcal{F}(B)$.

Definition 6.1.2. *Let B be a Boolean algebra, and let $F \in \mathcal{F}(B)$ be a filter on B.*

1. *The maximal elements in $\mathcal{F}(B)$ are known as ultrafilters. The collection of ultrafilters on B is referred to as the spectrum of B, and is denoted by* $\mathsf{Spec}(B)$.
2. *The elements $F \in \mathcal{F}(B)$ with the property that:*

$$x \vee y \in F \Rightarrow (x \in F \text{ or } y \in F)$$

   are known as prime filters.

Interestingly, for a given Boolean algebra B, the prime filters and ultrafilters on B coincide.

Theorem 6.1.1. *Let B be a Boolean algebra. For a filter $F \in \mathcal{F}(B)$, the following conditions are equivalent:*

1. *F is a prime filter,*
2. *F is an ultrafilter,*
3. $\forall\, b \in B : b \in F \Leftrightarrow \neg b \notin F$,
4. $\exists$ *a Boolean algebra homomorphism* $f : B \to \mathbf{2} \mid f^{-1}(1) = F$.

Proposition 6.1.1. *Given a Boolean algebra B, the following conditions hold:*

1. *Every proper filter in B is contained in an ultrafilter on B,*
2. *Every non-zero element of B belongs to an ultrafilter on B,*
3. *For* $x, y \in B : x \not\leq y \Rightarrow \exists\, F \in \mathsf{Spec}(B) \mid x \in F$ *and* $y \notin F$,
4. *Every filter in B is the intersection of the ultrafilters that contain it.*

Proof.

1. The collection of proper filters that contain a given filter $F \in \mathcal{F}(B)$ forms a nested partially ordered set, and therefore the first condition holds by Zorn's lemma.

2. For a given non-zero $b \in B$, one can take $F = \uparrow b$ and apply (*1.*) to see that the second condition holds.

3. Take $x, y \in B$ such that $x \not\leq y$. One can use (*2.*) to see that $x \wedge \neg y$ is contained in an ultrafilter F. Clearly $x, \neg y \in F$, and since F is an ultrafilter, $y \notin F$.

4. The trivial filter $G = B$ (i.e. the filter containing 0) is not contained in any ultrafilter - so the intersection of these is empty, and is therefore given by the entire Boolean algebra B.

For a proper filter $G \in \mathcal{F}(B)$, it is clear that every element of G is in the intersection of the ultrafilters that contain G. If one takes an element x in B that is not in G, one can generate a filter on the set $G \cup (B \setminus \{x\})$, and then find an ultrafilter F that contains this filter. Since $B \setminus \{x\} \subset F$, it must be the case that $x \notin F$. That is, for every element $x \notin G$ there exists an ultrafilter F that contains G, but not x. Therefore, the intersection of the ultrafilters containing G is G itself.

□

For every finite Boolean algebra B, there is a direct correspondence between the non-zero minimal elements of B, and the ultrafilters defined on B.

Definition 6.1.3. *In a Boolean algebra B, the non-zero minimal elements are known as atoms. That is, an element $a > 0$ is an atom when:*

$$\forall\, b \in B : 0 \leq b \leq a \Rightarrow \big(b = 0 \text{ or } b = a\big)$$

Further, the collection of atoms in a Boolean algebra is denoted by $\mathsf{Atom}(B)$

Lemma 6.1.1. *For a finite Boolean algebra B, there is a bijection* $\mathsf{Atom}(B) \cong \mathsf{Spec}(B)$.

Proof. Suppose that $F \in \mathsf{Spec}(B)$ is an ultrafilter, and define:

$$a_F := \bigwedge_{b \in F} b$$

It is clear that $b \in F \Leftrightarrow a_F \leq b$. Since F is an ultrafilter, a_F must be a minimal non-zero element in B.

If a_F were not an atom, the set $\{b \in B : 0 < b < a_F\}$ would be non-empty, and one could use its smallest element c (which would have to exist, since F is finite) to find the filter generated by $F \cup \{c\}$. This filter would contain both F and $\{b \in B : 0 < b < a_F\}$, and would therefore violate the maximality of F.

For the opposite association, suppose that $a \in B$ is an atom, and consider the principal filter $\uparrow a$ it determines. Next, take any $b \in B$. Since a is an atom, $a \wedge b$ is either a or 0. In particular:

$$a \wedge b = \begin{cases} a & \text{if and only if } a \leq b, \\ 0 & \text{if and only if } a \leq \neg b \end{cases}$$

And so, $\uparrow a$ is an ultrafilter (because a is an atom).

What is more, it is clear that the associations:

$$F \longmapsto \bigwedge_{b \in F} b \longmapsto \uparrow \Big(\bigwedge_{b \in F} b\Big)$$

$$a \longmapsto \uparrow a \longmapsto \bigwedge (\uparrow a)$$

are identities, and so form a bijection. □

Every finite Boolean algebra can be presented uniquely as a powerset (on the set of ultrafilters/atoms of the Boolean algebra in question). If one defines:

$$B \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} \mathcal{P}\big(\mathsf{Spec}(B)\big)$$

with $\Phi(b) = \big\{F \in \mathsf{Spec}(B) \mid b \in F\big\}$ and $\Phi^{-1}\big(F\big) = \bigcap F = \bigcap \{x \in F\}$, it is trivial to show that Φ is an isomorphism of Boolean algebras.

For infinite Boolean algebras, however, this presentation is not necessarily unique [12], as there are infinite, non-isomorphic Boolean algebras $B \ncong B'$ such that $\mathcal{P}\big(\mathsf{Spec}(B)\big) \cong \mathcal{P}\big(\mathsf{Spec}(B')\big)$. This means that, without additional information, one cannot recover a Boolean algebra from its image under Φ. The additional information required to make the distinction between such (non-isomorphic) Boolean algebras from their images under Φ comes in the form of topologies on $\mathsf{Spec}(B)$ and $\mathsf{Spec}(B')$. In fact, it can be shown that $\Phi(B)$ forms a base for a topology on $\mathsf{Spec}(B)$, which allows one to do precisely this.

As will be shown briefly, open sets in the topology generated by the elements of $\Phi(B)$ take a very particular form.

Definition 6.1.4 (Borceux and Janelidze [4]). *Let B be a Boolean algebra, and for every filter $G \in \mathcal{F}(B)$, define:*

$$\mathcal{O}_G = \{F \in \mathsf{Spec}(B) \mid G \nsubseteq F\}$$

Lemma 6.1.2 (Borceux and Janelidze [4]). *For a Boolean algebra B, the following equalities hold:*

1. $\mathcal{O}_{\{1\}} = \emptyset$ *and* $\mathcal{O}_B = \mathsf{Spec}(B)$
2. $\mathcal{O}_{F_1 \cap F_2} = \mathcal{O}_{F_1} \cap \mathcal{O}_{F_2} \; \forall \; F_1, F_2 \in \mathcal{F}(B)$
3. $\mathcal{O}_{\langle \bigcup_{i \in I} F_i \rangle} = \bigcup_{i \in I} \mathcal{O}_{F_i}$

Thus, the set $\{\mathcal{O}_G : G \in \mathcal{F}(B)\}$ forms a topology τ_B on $\mathsf{Spec}(B)$.

Definition 6.1.5. *For a Boolean algebra B, $(\mathsf{Spec}(B), \tau_B)$ is the dual space of B.*

Lemma 6.1.3 (Borceux and Janelidze [4]). *Let B be a Boolean algebra, and for each $b \in B$ define:*

$$\mathcal{O}_b := \mathcal{O}_{\uparrow b} = \{F \in \mathsf{Spec}(B) \mid \uparrow b \nsubseteq F\} = \{F \in \mathsf{Spec}(B) \mid b \notin F\}$$

$(\mathcal{O}_b)_{b \in B}$ *forms a clopen base for the topology τ_B on $\mathsf{Spec}(B)$*

Proof. It has to be shown that $(\mathcal{O}_b)_{b \in B}$ is closed under finite intersections and covers $\mathsf{Spec}(B)$, and that each $\mathcal{O}_b$ is clopen.

It can easily be shown that for each filter $G \in \mathcal{F}(B)$ on B:

$$G = \bigcup_{b \in G} \uparrow b$$

The inclusion $G \subseteq \bigcup_{b \in G} \uparrow b$ is trivial. For the opposite inclusion, one need only notice that:

$$x \in \bigcup_{b \in G} \uparrow b \Leftrightarrow \exists \; b' \in G \mid b' \leq x$$

which, since G is a filter, means that $x \in G$.

One can now use (*3.*) in Lemma (6.1.2) to see that $\mathcal{O}_G = \bigcup_{b \in G} \mathcal{O}_b$.

Next, by (*3.*) of Theorem (6.1.1), given two elements $b, b' \in B$ and an ultrafilter $F \in \mathsf{Spec}(B)$, it is clear that $b \vee b' \notin F$ if and only if neither b nor b' is an element of F. From this, one has:

$$\begin{aligned} \mathcal{O}_b \cap \mathcal{O}_{b'} &= \big\{F \in \mathsf{Spec}(B) \mid b \notin F \text{ and } b' \notin F\big\} \\ &= \big\{F \in \mathsf{Spec}(B) \mid b \vee b' \notin F\big\} \\ &= \mathcal{O}_{b \,\vee\, b'} \end{aligned}$$

Therefore $\big(\mathcal{O}_b\big)_{b \in B}$ is closed under finite intersections.

Finally, one can use Theorem (6.1.1) to see that

$$\begin{aligned} \mathcal{O}_{\neg b} &= \big\{F \in \mathsf{Spec}(B) \mid \neg b \notin F\big\} \\ &= \big\{F \in \mathsf{Spec}(B) \mid b \in F\big\} \\ &= \mathsf{Spec}(B) \backslash \mathcal{O}_b \\ &= \neg \mathcal{O}_b \end{aligned}$$

And this shows that each $\mathcal{O}_b$ is clopen in $\big(\mathsf{Spec}(B), \tau_B\big)$. □

Using clopen sets of the form

$$\mathcal{U}_b := \mathcal{O}_{\neg b} = \big\{F \in \mathsf{Spec}(B) \mid b \in F\big\}$$

will be convenient in many of the proofs to follow. Lemma (6.1.4) is a concise formulation of the relationships between elements in B and elements of the base of $\big(\mathsf{Spec}(B), \tau_B\big)$.

Lemma 6.1.4 (Borceux and Janelidze [4])**.** *In a Boolean algebra B, the following conditions hold:*

1. $\mathcal{U}_0 = B$ *and* $\mathcal{U}_1 = \emptyset$
2. $b \leq b' \Rightarrow \mathcal{U}_b \subseteq \mathcal{U}_{b'}$
3. $b \neq b' \Rightarrow \mathcal{U}_b \neq \mathcal{U}_{b'}$
4. $\mathcal{U}_{b \,\wedge\, b'} = \mathcal{U}_b \cap \mathcal{U}_{b'}$

6.2 The Stone Duality

"The Stone Duality" is an overarching correspondence between certain partially-ordered sets and particular topological spaces. In one of its simpler forms, it is a contravariant equivalence between the category of Boolean algebras and the category of Stone spaces. One can form a well-behaved topology on the set of ultrafilters of any given Boolean algebra, and the collection of clopen sets of any Stone space forms a Boolean algebra.

As the word "duality" suggests, there are functors in the following section that are contravariant. The convention of writing the functors in covariant form, but describing the morphisms in the respective categories as either continuous functions or Boolean algebra homomorphisms, rather than as morphisms in either of the opposite categories, will be adhered to throughout the chapter.

Proposition 6.2.1. *If B is a Boolean algebra, then the dual space $\big(\mathsf{Spec}(B), \tau_B\big)$ of B is a Stone space.*

Proof. It has to be shown that $\big(\mathsf{Spec}(B), \tau_B\big)$ is compact and totally disconnected.

To see that the dual space is compact, recall from Lemma (6.1.3) that $\big(\mathcal{U}_b\big)_{b\in B}$ forms a clopen base for $\big(\mathsf{Spec}(B), \tau_B\big)$, which effectively means that any open cover of $\mathsf{Spec}(B)$ is of the form $\big(\mathcal{U}_a\big)_{a\in A}$ for some subset $A \subseteq B$.

If $\big(\mathcal{U}_a\big)_{a\in A}$ is an open cover of $\mathsf{Spec}(B)$, then one knows that:

$$\mathsf{Spec}(B) = \bigcup_{a\in A} \mathcal{U}_a$$

and therefore has that:

$$\forall\ F \in \mathsf{Spec}(B)\ \exists\ a \in A \mid F \in \mathcal{U}_a$$

This is clearly equivalent to the fact that:

$$\forall\ F \in \mathsf{Spec}(B)\ \exists\ a \in A \mid a \in F \tag{6.1}$$

Further, if $\big(\mathcal{U}_a\big)_{a\in A}$ has no finite subcover, then for any finite list of elements $a_1, \dots, a_n$ in A, there must exist $F_{a_1,\dots,a_n} \in \mathsf{Spec}(B)$ such that:

$$F_{a_1,\dots,a_n} \notin \mathcal{U}_{a_1} \cup \cdots \cup \mathcal{U}_{a_n} = \mathcal{U}_{a_1 \vee \cdots \vee a_n} \tag{6.2}$$

Note that the following two conditions are equivalent to the statement in (6.2):

1. $a_1 \vee \cdots \vee a_n \notin F_{a_1,\dots,a_n}$
2. $\neg(a_1 \vee \cdots \vee a_n) = \neg a_1 \wedge \cdots \wedge \neg a_n \in F_{a_1,\dots,a_n}$

Since each $F_{a_1,\ldots,a_n}$ is an ultrafilter, there is no finite list of elements in A such that $\neg a_1 \wedge \cdots \wedge \neg a_n = 0$. Therefore, if one takes G to be the smallest filter in B that contains the set $\{\neg a : a \in A\}$:

$$G = \{b \in B | \; \exists \; a_1, \ldots, a_n \in A : \neg a_1 \wedge \cdots \wedge \neg a_n \leq b\}$$

one knows that G is a proper filter ($0 \notin G$). By (*1.*) in Proposition (6.1.1), there is an ultrafilter F that contains G, and one has that:

$$\{\neg a : a \in A\} \subseteq G \subseteq F$$

Since F is an ultrafilter that contains $\{\neg a : a \in A\}$, it cannot contain any elements of A. This, however, contradicts the assumption that $\big(\mathcal{U}_a\big)_{a \in A}$ is an open cover.[2] Therefore, $\big(\mathcal{U}_a\big)_{a \in A}$ does have a finite subcover, and $\big(\mathsf{Spec}(B), \tau_B\big)$ is compact.

To see that the dual space is totally disconnected, consider two ultrafilters $F \neq F' \in \mathsf{Spec}(B)$. As F is an ultrafilter, $F \nsubseteq F'$, and $\exists \; b \in F \backslash F'$. Since $b \in F$ and $b \notin F'$, one must have that $F \in \mathcal{U}_b$ and $F' \notin \mathcal{U}_b$. Since $\mathcal{U}_b$ is clopen, this shows that $\big(\mathsf{Spec}(B), \tau_B\big)$ is totally disconnected. □

Corollary 6.2.1 (Borceux and Janelidze [4])**.** *Let B be a Boolean algebra, and take $M \subseteq \mathsf{Spec}(B)$. M is clopen in the dual space of B if and only if $\exists \; b \in B \mid M = \mathcal{U}_b$.*

Proof. It has already been shown that all $\mathcal{U}_b = \mathcal{O}_{\neg b}$ are clopen in $\mathsf{Spec}(B)$.

If, conversely, one assumes that a subset $M \subseteq \mathsf{Spec}(B)$ is clopen, then – as a closed subset of a compact space – M is compact.

Since M is open, and $\big(\mathcal{U}_b\big)_{b \in B}$ forms a base for the topology on $\mathsf{Spec}(B)$, one can write:

$$M = \bigcup_{b_i \, \in \, B} \mathcal{U}_{b_i}$$

for some collection of elements $\big(b_i\big)_{i \in I}$ in B. By compactness, this covering of M must have a finite subcover i.e. there must exist $b_1, b_2, \ldots, b_n \in B$ such that:

$$M = \bigcup_{i=1}^{n} \mathcal{U}_{b_i} = \mathcal{U}_{b_1 \, \vee \, b_2 \, \vee \, \cdots \, \vee \, b_n}$$

□

Lemma 6.2.1. *If X is a Stone space, then the clopen elements $\mathsf{Cl}(X)$ of X form a Boolean algebra.*

[2] See Equation (6.1).

Proof. Clopen sets are closed under intersection, union and complements. It is also clear that X and $\emptyset$ play the roles of top and bottom elements in the lattice of clopen subsets of X. Moreover, because of the set-theoretic properties of the intersection and union, the lattice is distributive. □

Lemma 6.2.2. *The association* $X \mapsto \mathsf{Cl}(X)$ *induces a covariant functor:*

$$\mathsf{Cl} : \mathsf{Stone} \to \mathsf{Bool}^{\mathsf{op}}$$

Proof. It has already been shown that $\mathsf{Cl}(X)$ is a Boolean algebra. Given a continuous function $g : X_1 \to X_2$ between Stone spaces X_1 and X_2, one can define:

$$\mathsf{Cl}(g) : \mathsf{Cl}(X_2) \to \mathsf{Cl}(X_1) \quad , \quad M \mapsto g^{-1}(M)$$

in Bool. Since every element in the domain of $\mathsf{Cl}(g)$ is clopen, and g is continuous, $\mathsf{Cl}(g)$ is well defined. Further, the favourable properties of taking preimages ensure that $\mathsf{Cl}(g)$ preserves the top and bottom elements of $\mathsf{Cl}(X_2)$, and that $\mathsf{Cl}(g)\big(X_2\big)$ is closed under intersections and unions. Cumulatively, this shows that $\mathsf{Cl}(g)$ is a Boolean algebra homomorphism.

□

Lemma 6.2.3. *The association* $B \mapsto \big(\mathsf{Spec}(B), \tau_B\big)$ *induces a covariant functor:*

$$\mathsf{Spec} : \mathsf{Bool}^{\mathsf{op}} \to \mathsf{Stone}$$

Proof. It is clear that the object association is well defined. All that remains is to define how Spec acts on morphisms. For each morphism $f : B_1 \to B_2$ in Bool, define:

$$\mathsf{Spec}(f) : \mathsf{Spec}(B_2) \to \mathsf{Spec}(B_1) \quad , \quad F \mapsto f^{-1}(F)$$

In Stone. Now,

1. Since f is a Boolean algebra homomorphism, $f(1_{B_1}) = 1_{B_2}$, and so $1_{B_1} \in f^{-1}(F)$,[3]
2. $b, b' \in f^{-1}(F) \Rightarrow f(b) \wedge f(b') = f(b \wedge b') \in F \Rightarrow b \wedge b' \in f^{-1}(F)$,
3. $\big(b \in f^{-1}(F), b' \in B_1, b \leq b'\big) \Rightarrow \big(f(b) \leq f(b')\big) \Rightarrow f(b') \in F \Rightarrow b' \in f^{-1}(F)$,

[3]Note that this point breaks the convention of denoting the identity morphism of an object B by "1_B", and instead refers to the top elements of the respective Boolean algebras.

4. $b \notin f^{-1}(F) \Leftrightarrow f(b) \notin F \Leftrightarrow \neg f(b) = f(\neg b) \in F \Leftrightarrow \neg b \in f^{-1}(F)$.

This shows that $f^{-1}(F) \in \mathsf{Spec}(B_1)$, and that $\mathsf{Spec}(f)$ is well defined.

In order to see that $\mathsf{Spec}(f)$ is continuous, take an open set $\mathcal{U}_b$ in $\mathsf{Spec}(B_2)$, and consider:

$$\begin{aligned}\mathsf{Spec}(f)^{-1}(\mathcal{U}_b) &= \big\{F \in \mathsf{Spec}(B_2) \mid b \in \mathsf{Spec}(f)\big(F\big)\big\} \\ &= \big\{F \in \mathsf{Spec}(B_2) \mid b \in f^{-1}(F)\big\} \\ &= \big\{F \in \mathsf{Spec}(B_2) \mid f(b) \in F\big\} \\ &= \mathcal{U}_{f(b)}.\end{aligned}$$

□

Theorem 6.2.1. $\big($*The Stone Duality*$\big)$ *The following is an equivalence of categories:*

$$\mathsf{Bool}^{\mathsf{op}} \underset{\mathsf{Cl}}{\overset{\mathsf{Spec}}{\rightleftarrows}} \mathsf{Stone}$$

Proof. Both functors are well defined on objects and morphisms. It remains to be shown that the unit and counit for the adjunction are natural isomorphisms that obey the triangle identities.

For each Boolean algebra B, the counit for the equivalence takes the form:

$$\varepsilon_B : B \to \mathsf{Cl}\big(\mathsf{Spec}(B)\big) \quad , \quad b \mapsto \mathcal{U}_b$$

in Bool. Lemma (6.1.4) ensures that ε_B is a Boolean algebra homomorphism, and Corollary (6.2.1) shows that ε_B is surjective. To show its injectivity, suppose that $\mathcal{U}_b = \mathcal{U}_{b'}$ for some $b, b' \in B$. One immediately has that:

$$b \in \uparrow b' \text{ and } b' \in \uparrow b.$$

This will be the case if and only if both $b' \leq b$ and $b \leq b'$. Of course, this means that $b = b'$, and therefore that each ε_B is an isomorphism.

$$\begin{array}{ccc} B & \xrightarrow{\varepsilon_{B_1}} & \mathsf{Cl}\big(\mathsf{Spec}(B_1)\big) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{Cl}(\mathsf{Spec}(f))} \\ B_2 & \xrightarrow[\varepsilon_{B_2}]{} & \mathsf{Cl}\big(\mathsf{Spec}(B_2)\big) \end{array} \tag{6.3}$$

The following identities prove the commutativity of Diagram (6.3), which expresses the naturality of ε for a given morphism $f : B_1 \to B_2$ in Bool:

$$\begin{aligned} \mathsf{Cl}\big(\mathsf{Spec}(f)\big)\big(\mathcal{U}_b\big) &= \big(\mathsf{Spec}(f)\big)^{-1}\big(\mathcal{U}_b\big) \\ &= \big\{F \in \mathsf{Spec}(B_2) \mid b \in \mathsf{Spec}(f)(F)\big\} \\ &= \big\{F \in \mathsf{Spec}(B_2) \mid b \in f^{-1}(F)\big\} \\ &= \big\{F \in \mathsf{Spec}(B_2) \mid f(b) \in F\big\} \\ &= \mathcal{U}_{f(b)}. \end{aligned}$$

The unit for the adjunction takes the form:

$$\eta_X : X \to \mathsf{Spec}(\mathsf{Cl}(X)) \quad , \quad x \mapsto \big\{M \in \mathsf{Cl}(X) \mid x \in M\big\}$$

for each Stone space X. Recall that a continuous bijection from a compact space into a Hausdorff space[4] is necessarily a homeomorphism [4].

A trivial succession of facts:

1. $x \in X$, and $x \notin \emptyset$,
2. The intersection of the clopen sets containing x will be a clopen set that contains x,
3. A clopen set containing a clopen set that contains x, itself contains x,

ensures that $\big\{M \in \mathsf{Cl}(X) \mid x \in M\big\}$ is indeed a proper filter on $\mathsf{Cl}(X)$. Further, it is also clear that for each $N \in \mathsf{Cl}(X)$, and for each $y \in X$, either $y \in N$ or $y \in X \backslash N$.

This shows that $\eta_X(x) = \big\{M \in \mathsf{Cl}(X) \mid x \in M\big\}$ is an ultrafilter on $\mathsf{Cl}(X)$, i.e. that η_X is well defined.

Now, since X is totally disconnected, if one takes distinct points $x, y \in X$, then:

$$\exists\ N \in \mathsf{Cl}(X) \mid x \in N \text{ and } y \in \neg N$$

i.e. $N \in \eta_X(x)$ and $N \notin \eta_X(y)$. That is, $x \neq y \Rightarrow \eta_X(x) \neq \eta_X(y)$, so η_X is injective.

To show that η_X is surjective, take an ultrafilter $F \in \mathsf{Spec}\big(\mathsf{Cl}(X)\big)$, and consider $\bigcap F = \bigcap\big\{M \in F\big\}$. Because X is compact, $\bigcap F$ can be represented as a finite intersection of clopen sets, and this finite intersection must be non-empty, because F is a proper filter on $\mathsf{Cl}(X)$. Thus, there must exist $x \in \bigcap F$. It is certainly true that:

[4]Stone spaces have both compact and Hausdorff.

$$F \subseteq \{M \in \mathsf{Cl}(X) \mid x \in M\} = \eta_X(x).$$

However, since F is an ultrafilter, it is also the case that $\eta_X(x) \subseteq F$, and so they must be equal.

All that remains to be shown is the continuity of η_X. Recall[5] that any open set in $\mathsf{Spec}(\mathsf{Cl}(X))$ must take the form $\mathcal{U}_N = \{F \in \mathsf{Spec}(\mathsf{Cl}(X)) \mid N \in F\}$, for some $N \in \mathsf{Cl}(X)$. From this one can find the pre-image of any $\mathcal{U}_N$ under η_X:

$$\begin{aligned} \eta_X^{-1}(\mathcal{U}_N) &= \Big\{x \in X \mid \eta_X(x) \in \mathcal{U}_N\Big\} \\ &= \Big\{x \in X \mid \eta_X(x) \in \{F \in \mathsf{Spec}(\mathsf{Cl}(X)) \mid N \in F\}\Big\} \\ &= \Big\{x \in X \mid N \in \eta_X(x)\Big\} \\ &= \Big\{x \in X \mid N \in \{M \in \mathsf{Cl}(X) \mid x \in M\}\Big\} \\ &= \{x \in X \mid x \in N\} \\ &= N \end{aligned} \tag{6.4}$$

Therefore, as a continuous bijection from a compact space into a Hausdorff space, each η_X is a homeomorphism.

The inverse for each η_X is given by:

$$\eta_X^{-1} : \mathsf{Spec}(\mathsf{Cl}(X)) \to X \quad , \quad F \mapsto \bigcap \{M \in \mathsf{Cl}(X) \mid M \in F\} \tag{6.5}$$

The fact that X is a Stone space guarantees that the intersection will be a singleton.

To show that η is a natural transformation, it has to be shown that the following diagram commutes for every morphism $g : X_1 \to X_2$ in Stone:

$$\begin{array}{ccc} X_1 & \xrightarrow{\eta_{X_1}} & \mathsf{Spec}(\mathsf{Cl}(X_1)) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathsf{Spec}(\mathsf{Cl}(g))} \\ X_2 & \xrightarrow[\eta_{X_2}]{} & \mathsf{Spec}(\mathsf{Cl}(X_2)) \end{array} \tag{6.6}$$

Since one has that:

$$\begin{aligned} \mathsf{Spec}(\mathsf{Cl}(g))(F) &= \mathsf{Cl}(g)^{-1}(F) \\ &= \{N \in \mathsf{Cl}(X_2) \mid \mathsf{Cl}(g)(N) \in F\} \\ &= \{N \in \mathsf{Cl}(X_2) \mid g^{-1}(N) \in F\}, \end{aligned}$$

[5] See Proposition (6.2.1).

for every ultrafilter $F \in \mathsf{Spec}(\mathsf{Cl}(X))$, the commutativity of Diagram (6.6) follows from:

$$\begin{aligned}
\mathsf{Spec}\big(\mathsf{Cl}(g)\big)\big(\eta_{X_1}(x)\big) &= \Big\{N \in \mathsf{Cl}(X_2) \mid g^{-1}(N) \in \eta_{X_1}(x)\Big\} \\
&= \Big\{N \in \mathsf{Cl}(X_2) \mid g^{-1}(N) \in \big\{M \in \mathsf{Cl}(X_1) \mid x \in M\big\}\Big\} \\
&= \big\{N \in \mathsf{Cl}(X_2) \mid x \in g^{-1}(N)\big\} \\
&= \big\{N \in \mathsf{Cl}(X_2) \mid g(x) \in N\big\} \\
&= \eta_{X_2}\big(g(x)\big).
\end{aligned}$$

The triangle identities for the adjunction are given in Diagram (6.7):

$$\begin{array}{ccc}
\mathsf{Cl}(X) & \xrightarrow{\varepsilon_{\mathsf{Cl}(X)}} & \mathsf{Cl}\Big(\mathsf{Spec}\big(\mathsf{Cl}(X)\big)\Big) \\
& \searrow\!\!= & \downarrow{\scriptstyle \mathsf{Cl}(\eta_X)} \\
& & \mathsf{Cl}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{Spec}\Big(\mathsf{Cl}\big(\mathsf{Spec}(B)\big)\Big) & \xrightarrow{\mathsf{Spec}(\varepsilon_B)} & \mathsf{Spec}(B) \\
{\scriptstyle \eta_{\mathsf{Spec}(B)}}\uparrow & =\!\!\nearrow & \\
\mathsf{Spec}(B) & &
\end{array}
\tag{6.7}$$

For a given $N \in \mathsf{Cl}(X)$, one has:

$$\big(\mathsf{Cl}(\eta_X) \circ \varepsilon_{\mathsf{Cl}(X)}\big)(N) \quad = \quad \eta_X^{-1}(\mathcal{U}_N).$$

Therefore, one can use Equation (6.4) to see that the left-hand triangle in Diagram (6.7) commutes.

Lastly, for any ultrafilter $F \in \mathsf{Spec}(B)$, one can calculate:

$$\begin{aligned}
\big(\mathsf{Spec}(\varepsilon_B) \circ \eta_{\mathsf{Spec}(B)}\big)(F) \quad &= \quad \varepsilon_B^{-1}\Big(\big\{M \in \mathsf{Cl}\big(\mathsf{Spec}(B)\big) \mid F \in M\big\}\Big) \\
&= \quad \big\{b \in B \mid F \in \varepsilon_B(b)\big\} \\
&= \quad \big\{b \in B \mid F \in \mathcal{U}_b\big\} \\
&= \quad \big\{b \in B \mid b \in F\big\} \\
&= \quad F,
\end{aligned}$$

to show that the right-hand triangle in Diagram (6.7) commutes. □

6.3 Pierce Representation of Commutative Rings

As mentioned in the introduction to the chapter, the Pierce representation is obtained by finding the Boolean algebra of idempotents that underlies each commutative ring, and then using the Stone duality to obtain the corresponding dual space to that Boolean algebra. This composite association extends to an admissible functor $\mathsf{Rings}^{op} \to \mathsf{Stone}$ with a fully faithful right adjoint, and therefore allows one to reproduce the theory in Chapter (4) in this context. Moreover, for any connected commutative ring R,[6] there is a similar adjunction between the opposite category of R-algebras and the category of Stone spaces.[7]

In order to define the spectrum of a ring, one must first describe the Boolean algebra of idempotents contained in each unital commutative ring.

Proposition 6.3.1. *If R is a unital commutative ring, then the set of idempotents $\{r \in R \mid r^2 = r\}$ in R forms a Boolean algebra, with operations defined by:*

$$r_1 \wedge r_2 = r_1 \cdot r_2 \qquad , \qquad r_1 \vee r_2 = r_1 + r_2 - r_1 \cdot r_2 \qquad , \qquad \neg r_1 = 1 - r_1$$

Proof. The commutativity of R and the idempotence of r_1 and r_2 guarantee that $\wedge$, $\vee$ and $\neg$ are closed on $\{r \in R \mid r^2 = r\}$. It is also clear that 1 and 0 in R play the roles of top and bottom elements in the lattice $\{r \in R \mid r^2 = r\}$ determined by $\wedge$ and $\vee$, respectively.

The following statement:

$$r_1 \wedge r_2 = r_1 \cdot r_2 = r_1 \Leftrightarrow r_2 = r_2 + r_1 - r_1 = r_1 + r_2 - r_1 \cdot r_2 = r_1 \vee r_2,$$

defines the canonical partial order relation:

$$r_1 \leq r_2 \Leftrightarrow r_1 \wedge r_2 = r_1 \Leftrightarrow r_1 \vee r_2 = r_2,$$

on $\{r \in R \mid r^2 = r\}$.

Moreover, the lattice structure that $\wedge$ and $\vee$ form on $\{r \in R \mid r^2 = r\}$ is distributive, since:

$$\begin{aligned} r_1 \wedge (r_1 \vee r_2) &= r_1 \cdot (r_1 + r_2 - r_1 \cdot r_2) = r_1 + r_1 \cdot r_2 - r_1 \cdot r_2 = r_1 \\ &= r_1 + r_1 \cdot r_2 - r_1 \cdot (r_1 \cdot r_2) = r_1 \vee (r_1 \wedge r_2), \end{aligned}$$

for all $r_1, r_2 \in \{r \in R \mid r^2 = r\}$.

Lastly, given $r_1 \in \{r \in R \mid r^2 = r\}$, it is clear that:

$$r_1 \wedge (1 - r_1) = r_1 - r_1 = 0 \quad \text{and} \quad r_1 \vee (1 - r_1) = r_1 + (1 - r_1) - 0 = 1.$$

Therefore, the complement in $\{r \in R \mid r^2 = r\}$ is given by the operation $\neg$ defined above. □

[6] See Proposition (2.1.3) and Lemma (2.1.1).

[7] The connectedness of R will ensure that the right adjoint of the admissible functor is fully faithful.

The operations of the Boolean ring determined by the Boolean algebra structure on $\{r \in R \mid r^2 = r\}$ are as follows:

i) $r_1 \bar{\cdot}\, r_2 = r_1 \wedge r_2 = r_1 \cdot r_2$

$$\begin{aligned} \textit{ii)}\ r_1 \bar{+}\, r_2 &= \big(r_1 \wedge (1 - r_2)\big) \vee \big(r_2 \wedge (1 - r_1)\big) \\ &= \big(r_1 \cdot (1 - r_2)\big) \vee \big(r_2 \cdot (1 - r_1)\big) \\ &= r_1(1 - r_2) + r_2(1 - r_1) + r_1 r_2 (1 - r_1)(1 - r_2) \\ &= r_1 + r_2 - 2r_1 r_2. \end{aligned}$$

Here, the " $^-$ " is used to differentiate the operations in the Boolean ring from those in R.

Proposition 6.3.2. *The association* $R \mapsto \{r \in R \mid r^2 = r\}$ *induces a covariant functor:*

$$\mathsf{Idemp} : \mathsf{Rings} \to \mathsf{Bool}$$

Proof. Any morphism $h : R_1 \to R_2$ of rings will preserve idempotents, and because the constructions of $\wedge$, $\vee$ and $\neg$ are all defined in terms of the inherent ring operations, h will preserve these operations as well. Therefore, Idemp should "act trivially" on morphisms:

$$\mathsf{Idemp}(h) : \mathsf{Idemp}(R_1) \to \mathsf{Idemp}(R_2) \quad , \quad r \mapsto h(r)$$

□

Definition 6.3.1. *The Pierce spectrum of a ring* R *is its image under the composite* $\mathsf{Sp} = \mathsf{Spec} \circ \mathsf{Idemp}$*:*

$$\mathsf{Sp} : \mathsf{Rings}^{\mathsf{op}} \xrightarrow{\mathsf{Idemp}} \mathsf{Bool}^{\mathsf{op}} \xrightarrow{\mathsf{Spec}} \mathsf{Stone} \quad , \quad R \mapsto \mathsf{Spec}\Big(\{r \in R \mid r^2 = r\}\Big) = \mathsf{Sp}(R)$$

Although Idemp has been defined on the category of rings, rather than its opposite category, introducing and maintaining the more accurate $(-)^{op}$ notation would cause a significant amount of clutter, without providing very much clarity. For this reason, the convention of writing adjunctions in covariant form while making the relevant explanations in the categories of rings (or spaces) will be maintained throughout this section.

Since Idemp acts trivially on morphisms, one can show that for any $h : R_1 \to R_2$ in Rings, $\mathsf{Sp}(h)$ is the morphism defined by:

$$\mathsf{Sp}(h) : \mathsf{Sp}(R_2) \to \mathsf{Sp}(R_1) \quad , \quad F \mapsto \mathsf{Idemp}(h)^{-1}(F)$$

where:

$$\mathsf{Idemp}(h)^{-1}(F) = \big\{r \in \mathsf{Idemp}\big(R_1\big) \mid h(r) \in F\big\}.$$

Recall from Corollary (6.2.1) that any clopen set M in the spectrum of a Boolean algebra B must be of the form $M = \mathcal{U}_b$, for some $b \in B$. In particular, every clopen set in $\mathsf{Sp}(R) = \mathsf{Spec}(\mathsf{Idemp}(R))$ must be of the form $M = \mathcal{U}_r$, for some $r \in \mathsf{Idemp}(R)$.

Proposition 6.3.3 (Borceux and Janelidze [4]). *For a commutative, unital ring R, any partition of a clopen set $M = \mathcal{U}_r$ in $\mathsf{Sp}(R)$ must take the form:*

$$M = \mathcal{U}_r = \mathcal{U}_{r_1 \vee r_2 \vee \cdots \vee r_n} = \bigcup_{i=1}^{n} \mathcal{U}_{r_i}$$

for a finite number of non-zero idempotents $r_1, r_2, \ldots, r_n \in \mathsf{Idemp}(R)$, in such a way that:

1. $\sum_{i=1}^{n} r_i = r$,
2. $i \neq j \Rightarrow r_i \cdot r_j = 0$.

Proof. As a clopen set in $\mathsf{Sp}(R)$, $M = \mathcal{U}_r$ is compact, which means that every partition of $\mathcal{U}_r$ must be finite. Therefore, any such partition yields a representation:

$$\mathcal{U}_r = \mathcal{U}_{r_1} \cup \mathcal{U}_{r_2} \cup \cdots \cup \mathcal{U}_{r_n} = \mathcal{U}_{r_1 \vee r_2 \vee \cdots \vee r_n}$$

where each r_i is a non-zero element in $\mathsf{Idemp}(R)$. This shows that

$$r = r_1 \vee r_2 \vee \cdots \vee r_n$$

Since each clopen set $\mathcal{U}_{r_i}$ covers a distinct portion of $\mathcal{U}_r$, they must be pairwise disjoint. Specifically, $i \neq j \Rightarrow \emptyset = \mathcal{U}_{r_i} \cap \mathcal{U}_{r_j} = \mathcal{U}_{r_i \wedge r_j} = \mathcal{U}_0$, and so:

$$i \neq j \Rightarrow r_i \wedge r_j = r_i \cdot r_j = 0.$$

With this fact, it is trivial to verify that:

$$\begin{aligned} r &= r_1 \vee r_2 \vee \cdots \vee r_n \\ &= \sum_{i=1}^{n} r_i - \sum_{i=1}^{n-1} r_i \big(\sum_{j=i+1}^{n} r_j \big) + \sum_{i=1}^{n-2} r_i \big(\sum_{j=i+1}^{n-1} r_j \big(\sum_{k=j+1}^{n} r_k \big)\big) - \cdots + (-1)^{n+1} \sum_{i=1}^{1} r_i \big(\cdots \sum_{z=n}^{n} r_z \big) \\ &= \sum_{i=1}^{n} r_i - \sum_{i=1}^{n-1} r_i \big(\sum_{j=i+1}^{n} r_j \big) + \sum_{i=1}^{n-2} r_i \big(\sum_{j=i+1}^{n-1} r_j \big(\sum_{k=j+1}^{n} r_k \big)\big) - \cdots + (-1)^{n+1} \prod_{i=1}^{n} r_i \\ &= \sum_{i=1}^{n} r_i \end{aligned}$$

All but the first sum in the above expression are factorizations of products that contain elements $r_i \cdot r_j \cdots$ for $i \neq j$, and therefore all the sums but the first reduce to a value of 0. □

Proposition (6.3.4) shows that Sp has a fully faithful right adjoint, $\mathsf{Hom}_{\mathsf{Top}}(-, \mathbb{Z})$. The reason $\mathbb{Z}$ is used in this context is because it is the initial object in:[8]

$$\mathsf{Rings} \cong \mathbb{Z}\text{--}\mathsf{Alg}.$$

Further, the only property of $\mathbb{Z}$ that is used to prove that the right adjoint is fully faithful is the fact that $\mathbb{Z}$ has no non-trivial idempotents (i.e. it is connected). Therefore Proposition (6.3.4) will have a direct analogue for any connected ring R - as Proposition (6.3.5) makes clear.

Proposition 6.3.4 (Borceux and Janelidze [4]). *The functor:*

$$\mathsf{Sp} : \mathsf{Rings}^{\mathsf{op}} \to \mathsf{Stone}$$

is left adjoint to:

$$\mathsf{Hom}_{\mathsf{Top}}(-, \mathbb{Z}) : \mathsf{Stone} \to \mathsf{Rings}^{\mathsf{op}}$$

and, moreover, $\mathsf{Hom}_{\mathsf{Top}}(-, \mathbb{Z})$ *is fully faithful.*

Proof. Both functors are well-defined on objects and morphisms. The following proof has been broken up into subsections for legibility.

<u>Form of the Unit:</u>

The expression of the unit for the adjunction is relatively complicated. It must be a natural transformation:

[8] And therefore the terminal object in the opposite category.

$$\Big(\eta_R : \mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R), \mathbb{Z}\big) \to R\Big)_{R \,\in\, \mathsf{Ring}}$$

For each $f \in \mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R), \mathbb{Z}\big)$, the collection of inverse images $\big(f^{-1}(z)\big)_{z\in\mathbb{Z}}$ of the singletons in $\mathbb{Z}$ covers $\mathsf{Sp}(R)$, and this cover must have a finite subcover. That is, only finitely many integers have non-empty inverse images under f. Moreover, since f is continuous, it must be constant on each of these non-empty pre-images $f^{-1}(z)$. Therefore, each $f \in \mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R), \mathbb{Z}\big)$ is locally constant.

The fact that $\mathbb{Z}$ is taken with the discrete topology guarantees that each $f^{-1}(z)$ is clopen in $\mathsf{Sp}(R)$. Moreover, for distinct integers $z, z' \in \mathbb{Z}$, $f^{-1}(z) \cap f^{-1}(z') = \emptyset$ (supposing otherwise would contradict the fact that f is a function). This means that $\mathsf{Sp}(R) = \bigcup\limits_{z\in\mathbb{Z}} f^{-1}(z)$ is a (clopen) partition, and so one can use Proposition (6.3.3) to see that there exists a finite number of non-zero idempotents $r_1, r_2, \ldots, r_n \in \mathsf{Idemp}(R)$ such that:

1. $\sum\limits_{i=1}^{n} r_i = 1,$
2. $i \neq j \Rightarrow r_i \cdot r_j = 0.$

and these idempotents allow one to write:

$$\mathsf{Sp}(R) = \bigcup_{z\in\mathbb{Z}} f^{-1}(z) = \mathcal{U}_1 = \bigcup_{i=1}^{n} \mathcal{U}_{r_i}$$

Since f must be constant on each $\mathcal{U}_{r_i} = \big\{F \in \mathsf{Sp}(R) \mid r_i \in F\big\}$, it is clear that f is completely determined by its values on each of these clopen sets. If one takes $\big\{z_i : 1 \leq i \leq n\big\}$ to be the n values f takes in $\mathbb{Z}$, each with the property that:

$$\big(r_i \in F \Rightarrow f(F) = z_i\big)\ \forall\ F \in \mathsf{Sp}(R)$$

then one can define:

$$\eta_R(f) := \sum_{i=1}^{n} z_i \cdot r_i$$

for each $f \in \mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R), \mathbb{Z}\big)$. Further, this representation is unique, in that it is not affected by the choice of partition of $\mathsf{Sp}(R)$. To see this, begin by recalling that f takes a constant value of z_i on each $\mathcal{U}_{r_i}$, and since $i \neq j \Leftrightarrow z_i \neq z_j$, any partition of $\mathsf{Sp}(R)$ must preserve this, i.e. any partition must be a refinement of $\big(\mathcal{U}_{r_i}\big)_{i=1}^{n}$.

A refinement of $\big(\mathcal{U}_{r_i}\big)_{i=1}^{n}$ will take the form $\Big(\big(\mathcal{U}_{r_{ij}}\big)_{j=1}^{n_i}\Big)_{i=1}^{n}$, because each $\mathcal{U}_{r_i}$ is clopen. And so – again by Proposition (6.3.3) – for each i one will have that:

$$\mathcal{U}_{r_i} = \bigcup_{j=1}^{n_i} \mathcal{U}_{r_{ij}} = \mathcal{U}_{r_{i1} \vee r_{i2} \vee \cdots \vee r_{in_i}}$$

with the properties that $\sum_{j=1}^{n_i} r_{ij} = r_i$ and $F \in \mathcal{U}_{r_{ij}} \Rightarrow f(F) = z_i \ \forall \ j \in \{1, 2, \dots n_i\}$. But this means that:

$$\sum_{i=1}^{n}\Big(\sum_{j=1}^{n_i} z_i r_{ij}\Big) = \sum_{i=1}^{n} z_i \cdot \Big(\sum_{j=1}^{n_i} r_{ij}\Big) = \sum_{i=1}^{n} z_i r_i$$

and shows that the definition of η_R is independent of the partition chosen on $\mathsf{Sp}(R)$.

Next, it has to be shown that each η_R is a ring homomorphism.

Given $f, g \in \mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R), \mathbb{Z}\big)$, addition and multiplication in the ring of homomorphisms are given by:

1. $(f + g)(F) = f(F) + g(F) \ \forall \ F \in \mathsf{Sp}(R)$,
2. $(f \cdot g)(F) = f(F) \cdot g(F) \ \forall \ F \in \mathsf{Sp}(R)$.

Therefore, if one takes $f, g \in \mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R), \mathbb{Z}\big)$, both functions determine finite partitions of $\mathsf{Sp}(R)$, given by, say $\big(\mathcal{U}_{u_j}\big)_{j=1}^{n_f}$ and $\big(\mathcal{U}_{v_k}\big)_{k=1}^{n_g}$, respectively. From these, one can create a new partition by intersecting each $\mathcal{U}_{u_j}$ with every $\mathcal{U}_{v_k}$ it overlaps with in $\mathsf{Sp}(R)$. Specifically, and more formally, one can consider:

$$\big\{\mathcal{U}_r = \mathcal{U}_{u_j} \cap \mathcal{U}_{v_k} = \mathcal{U}_{u_j \cdot v_k} \mid j \in \{1, 2, \dots, n_f\} \text{ and } k \in \{1, 2, \dots, n_g\}\big\}.$$

The non-empty elements of this set will form a finite partition $\big(\mathcal{U}_{r_i}\big)_{i=1}^{n}$ of $\mathsf{Sp}(R)$ with the property that both f and g will be constant on each $\mathcal{U}_{r_i}$.

Write z_i and z_i' for the values that f and g take on $\mathcal{U}_{r_i}$ respectively.[9] So:

$$F \in \mathcal{U}_{r_i} \Rightarrow f(F) = z_i \text{ and } g(F) = z_i'.$$

The collection of idempotents $\{r_1, r_2, \dots, r_n\}$ has the now-familiar properties that:

[9]Note that f and g can repeat values, i.e. both f and g can send filters from different regions to the same integer value, on the constructed partition.

1. $\sum_{i=1}^{n} r_i = 1$,
2. $i \neq j \Rightarrow r_i \cdot r_j = 0$.

From this, and the definitions given above, one has:

i) $\eta_R(f) + \eta_R(g) = \sum_{i=1}^{n} z_i r_i + \sum_{i=1}^{n} z_i' r_i = \sum_{i=1}^{n}(z_i + z_i')r_i = \eta_R(f+g)$,

ii)
$$\begin{aligned}
\eta_R(f) \cdot \eta_R(g) &= \big(\sum_{i=1}^{n} z_i \cdot r_i\big) \cdot \big(\sum_{i=1}^{n} z_i' \cdot r_i\big) \\
&= \big(z_1 r_1 + z_2 r_2 + \cdots + z_n r_n\big) \cdot \big(z_1' r_1 + z_2' r_2 + \cdots + z_n' r_n\big) \\
&= z_1 r_1 \big(z_1' r_1 + z_2' r_2 + \cdots + z_n' r_n\big) + \cdots + z_n r_n \big(z_1' r_1 + z_2' r_2 + \cdots + z_n' r_n\big) \\
&= z_1 r_1 \cdot z_1' r_1 + z_1 r_1 \big(z_2' r_2 + \cdots + z_n' r_n\big) + \cdots + z_n r_n \cdot z_n' r_n + z_n r_n \big(z_1' r_1 + \cdots + z_{n-1}' r_{n-1}\big) \\
&= z_1 \cdot z_1' \cdot r_1 + z_2 \cdot z_2' \cdot r_2 + \ldots + z_n \cdot z_n' \cdot r_n \\
&= \sum_{i=1}^{n} (z_i \cdot z_i') \cdot r_i \\
&= \eta_R(f \cdot g).
\end{aligned}$$

The cancellations made in *ii)* follow from the property that $i \neq j \Rightarrow r_i \cdot r_j = 0$.

What is more, η_R will take the constant functions that act as additive and multiplicative identities in $\mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R), \mathbb{Z}\big)$ to 0 and 1 in R, respectively (the proof of the former is trivial, and that of the latter requires only that $\sum_{i=1}^{n} r_i = 1$).

Naturality of the Unit:

For every morphism $h : R_1 \to R_2$ of rings, $\mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(h), \mathbb{Z}\big)$ is just pre-composition with $\mathsf{Sp}(h)$. Therefore, showing that η is a natural transformation is equivalent to showing that the following diagram is commutative:

$$\begin{array}{ccc}
\mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R_1), \mathbb{Z}\big) & \xrightarrow{\eta_{R_1}} & R_1 \\
\Big\downarrow {\scriptstyle (-)\circ \mathsf{Sp}(h)} & & \Big\downarrow {\scriptstyle h} \\
\mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R_2), \mathbb{Z}\big) & \xrightarrow[\eta_{R_2}]{} & R_2
\end{array} \tag{6.8}$$

for each $h : R_1 \to R_2$.

For a given ring R, every clopen subset of $\mathsf{Sp}(R) = \mathsf{Spec}(\mathsf{Idemp}(R))$ takes the form $\mathcal{U}_r$, for some idempotent r in R.[10] With this in mind, the preimage of a clopen set $\mathcal{U}_r$ in $\mathsf{Sp}(R_1)$ under $\mathsf{Sp}(h)$ can be calculated as:

$$\begin{aligned}\mathsf{Sp}(h)^{-1}(\mathcal{U}_r) &= \{F \in \mathsf{Sp}(R_2) \mid \mathsf{Sp}(h)(F) \in \mathcal{U}_r\} \\ &= \{F \in \mathsf{Sp}(R_2) \mid r \in \mathsf{Sp}(h)(F)\} \\ &= \{F \in \mathsf{Sp}(R_2) \mid h(r) \in F\} \\ &= \mathcal{U}_{h(r)}.\end{aligned}$$

Further, recall that each $f \in \mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(R_1), \mathbb{Z})$ yields a partition $(\mathcal{U}_{r_i})_{i=1}^{n}$ of $\mathsf{Sp}(R_1)$ with the familiar properties on $\{r_1, \ldots, r_n\}$. Therefore, one has the following:

$$\begin{aligned}\Big(\mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(h), \mathbb{Z})(f)\Big)(\mathcal{U}_{h(r_i)}) &= (f \circ \mathsf{Sp}(h))(\mathcal{U}_{h(r_i)}) \\ &= f\Big(\mathsf{Sp}(h)(\mathcal{U}_{h(r_i)})\Big) \\ &= f(\mathcal{U}_{r_i}) \\ &= \mathcal{U}_{f(r_i)}\end{aligned}$$

for all $f \in \mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(R_1), \mathbb{Z})$, and for each $\mathcal{U}_{r_i}$ in the clopen partition of $\mathsf{Sp}(R_1)$.

One can use this to see that:

$$\eta_{R_1}(f) = \sum_{i=1}^{n} z_i r_i \;\Rightarrow\; \eta_{R_2}(f \circ \mathsf{Sp}(h)) = \sum_{i=1}^{n} z_i h(r_i) \tag{6.9}$$

for each $f \in \mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(R_1), \mathbb{Z})$. Equation (6.9) allows one to write:

$$\eta_{R_2}(f \circ \mathsf{Sp}(h)) = \sum_{i=1}^{n} z_i h(r_i) = h\Big(\sum_{i=1}^{n} z_i r_i\Big) = h\Big(\eta_{R_1}(f)\Big)$$

for each $f \in \mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(R), \mathbb{Z})$, and for every $h : R_1 \to R_2$. This is the equational formulation of the commutativity of Diagram (6.8). Since the arguments above hold for every morphism of rings $h : R_1 \to R_2$, η is a natural transformation.

Form of the Counit:

Diagram (6.10) will provide context for the discussion to follow:

$$\mathsf{Rings}^{\mathsf{op}} \underset{\mathsf{Hom}_{\mathsf{Top}}(-,\mathbb{Z})}{\overset{\mathsf{Sp}}{\rightleftarrows}} \mathsf{Stone} \underset{\mathsf{Spec}}{\overset{\mathsf{Cl}}{\rightleftarrows}} \mathsf{Bool}^{\mathsf{op}} \tag{6.10}$$

The counit for $\mathsf{Sp} \dashv \mathsf{Hom}_{\mathsf{Top}}(-, \mathbb{Z})$ should take the form:

[10] See Corollary (6.2.1).

$$\big(\varepsilon_X : \mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big) \to X\big)_{X \in \mathsf{Stone}}$$

Since $\mathsf{Sp} = \mathsf{Spec} \circ \mathsf{Idemp}$, and $\mathsf{Cl} \circ \mathsf{Spec} \cong 1_{\mathsf{Stone}}$, Diagram (6.10) shows that one can make use of the Stone Duality to describe each ε_X by finding the form of $\mathsf{Cl}(\varepsilon_X)$, for each Stone space X.

Since:

$$\begin{aligned}\mathsf{Cl}\big(\mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big)\big) &= \big(\mathsf{Cl} \circ \mathsf{Spec} \circ \mathsf{Idemp}\big)\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big) \\ &\cong \mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big),\end{aligned}$$

it is clear that, up to an isomorphism in Bool, each $\mathsf{Cl}(\varepsilon_X)$ has the form:

$$\mathsf{Cl}(\varepsilon_X) : \mathsf{Cl}(X) \to \mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big)$$

If one inspects the elements in the ring of continuous functions $\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})$, it is clear that – because of the way multiplication is defined in the ring – for a function to be idempotent, all the elements in its image must be idempotent. Of course 0 and 1 in $\mathbb{Z}$ are the only such elements in this case.

Given this, the only choice for the image of a clopen set $M \in \mathsf{Cl}(X)$ is an indicator function for the set. Explicitly, one defines:

$$\mathsf{Cl}(\varepsilon_X) : \mathsf{Cl}(X) \to \mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big) \quad , \quad M \mapsto I_M$$

where $I_M(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases}$

It has to be checked that $\mathsf{Cl}(\varepsilon_X)$ is a Boolean algebra homomorphism. To this end, take $M, M' \in \mathsf{Cl}(X)$ and consider the following:

$$\begin{aligned} I_{M \cap M'}(x) = 1 &\Leftrightarrow x \in M \cap M' \\ &\Leftrightarrow I_M(x) = I_{M'}(x) = 1 \\ &\Leftrightarrow \big(I_M \cdot I_{M'}\big)(x) = I_M(x) \cdot I_{M'}(x) = I_M(x) \cap I_{M'}(x) = 1. \end{aligned}$$

This shows that $\mathsf{Cl}(\varepsilon_X)$ preserves meets.

Similarly, $I_{M \cup M'}(x) = 1 \Leftrightarrow x \in M \cup M'$. This is the case if and only if:

i) $x \in M$ and $x \notin M' \Leftrightarrow I_M(x) + I_{M'}(x) - I_M(x) \cdot I_{M'}(x) = 1 + 0 - 0 = 1,$

ii) $x \notin M$ and $x \in M' \Leftrightarrow I_M(x) + I_{M'}(x) - I_M(x) \cdot I_{M'}(x) = 0 + 1 - 0 = 1,$

iii) $x \in M \cap M' \Leftrightarrow I_M(x) + I_{M'}(x) - I_M(x) \cdot I_{M'}(x) = 1 + 1 - 1 = 1.$

Thus $I_{M \cup M'} = I_M + I_{M'} - I_M \cdot I_{M'}$ and $\mathsf{Cl}(\varepsilon_X)$ preserves joins. Given this, and the fact that $\mathsf{Cl}(\varepsilon_X)$ clearly takes X and $\emptyset$ to the constant functions whose values are 1 and 0, $\mathsf{Cl}(\varepsilon_X)$ must also preserve $\neg$, and is therefore a Boolean algebra homomorphism.

Since $\mathbb{Z}$ has a discrete topology, each function in $\mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}(X, \mathbb{Z})\big)$ must have a value of 1 on some clopen subset of X, and 0 on its complement. This immediately shows that $\mathsf{Cl}(\varepsilon_X)$ is surjective. Since it is trivially injective, $\mathsf{Cl}(\varepsilon_X)$ is a Boolean algebra isomorphism.

Naturality of the Counit:

For any continuous function $g : X_1 \to X_2$ between Stone spaces, it is clear that $\mathsf{Hom}_{\mathsf{Top}}\big(g, \mathbb{Z}\big)$ is just pre-composition with g. Now, because Idemp acts trivially on morphisms,[11] $\mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}\big(g, \mathbb{Z}\big)\big)$ is given by pre-composition (of the idempotent functions from X_2 to $\mathbb{Z}$) with g. Thus, it has to be shown that the following diagram commutes:

$$\begin{array}{ccc}
\mathsf{Cl}(X_2) & \xrightarrow{\mathsf{Cl}(\varepsilon_{X_2})} & \mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}\big(X_2, \mathbb{Z}\big)\big) \\
{\scriptstyle \mathsf{Cl}(g)}\big\downarrow & & \big\downarrow{\scriptstyle (-)\circ g} \\
\mathsf{Cl}(X_1) & \xrightarrow[\mathsf{Cl}(\varepsilon_{X_1})]{} & \mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}\big(X_1, \mathbb{Z}\big)\big)
\end{array}$$

for every $g : X_1 \to X_2$.

That is, it has to be shown that for each $N \in \mathsf{Cl}(X_2)$:

$$\Big(\mathsf{Cl}(\varepsilon_{X_1}) \circ \mathsf{Cl}(g)\Big)(N) = I_{g^{-1}(N)} \text{ equals } \Big(\big((-) \circ g\big) \circ \mathsf{Cl}(\varepsilon_{X_2})\Big)(N) = I_N \circ g.$$

However, this follows immediately from:

$$\begin{aligned}
I_{g^{-1}(N)}(x) = 1 &\Leftrightarrow x \in g^{-1}(N) \\
&\Leftrightarrow g(x) \in N \\
&\Leftrightarrow I_N(g(x)) = \big(I_N \circ g\big)(x) = 1
\end{aligned}$$

Thus, $\mathsf{Cl}(\varepsilon)$ is a natural isomorphism.

By recalling the definition of Sp and again taking advantage of the Stone duality, one can use the above calculations to find the form of each ε_X by inspecting $\mathsf{Spec}\big(\mathsf{Cl}(\varepsilon_X)\big)$:

[11] See Lemma (6.3.2).

$$\mathsf{Spec}\big(\mathsf{Cl}(\varepsilon_X)\big) : \mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big) \to \mathsf{Spec}\big(\mathsf{Cl}(X)\big) \quad , \quad F \mapsto \big\{M \in \mathsf{Cl}(X) \mid I_M \in F\big\}$$

where the mapping is obtained as follows:

$$\begin{aligned}\big(\mathsf{Cl}(\varepsilon_X)\big)^{-1}(F) &= \big\{M \in \mathsf{Cl}(X) \mid \mathsf{Cl}(\varepsilon_X)(M) \in F\big\}\\ &= \big\{M \in \mathsf{Cl}(X) \mid I_M \in F\big\}.\end{aligned}$$

One can now use the inverse morphism of the unit for the Stone duality to see that each ε_X is given by:

$$\varepsilon_X : \mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big) \to X \quad , \quad F \mapsto \bigcap \big\{M \in \mathsf{Cl}(X) \mid I_M \in F\big\}$$

What is more, since $\mathsf{Cl}(\varepsilon_X)$ is an isomorphism, its image under Spec will be an isomorphism as well. Therefore, the Stone Duality guarantees that ε_X is an isomorphism, whose inverse is given as follows:

$$\varepsilon_X^{-1} : X \to \mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big) \quad , \quad x \mapsto \big\{I_M \in \mathsf{Hom}\big(X,\mathbb{Z}\big) \mid M \in \mathsf{Cl}(X), x \in M\big\}$$

Thus, each ε_X is an isomorphism, which proves the $\mathsf{Hom}_{\mathsf{Top}}\big(-,\mathbb{Z}\big)$ is fully faithful.

Triangle Identities:

The first of the triangle identities is given by:

$$\begin{array}{ccc} \mathsf{Hom}_{\mathsf{Top}}\Big(\mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big),\mathbb{Z}\Big) & \xleftarrow{(-)\circ\varepsilon_X} & \mathsf{Hom}_{\mathsf{Top}}\big(X,\mathbb{Z}\big) \\ \Big\downarrow {\scriptstyle \eta_{\mathsf{Hom}(X,\mathbb{Z})}} & \diagup\!\!\diagup & \\ \mathsf{Hom}_{\mathsf{Top}}\big(X,\mathbb{Z}\big) & & \end{array} \tag{6.11}$$

By an argument presented near the beginning of this proof, for each $f \in \mathsf{Hom}_{\mathsf{Top}}\big(X,\mathbb{Z}\big)$, only finitely many integers $z_1,\ldots,z_n$ have non-empty inverse images under f, and:

$$X = \bigcup_{i=1}^{n} f^{-1}(z_i).$$

Naturally, this partition means that $f = \sum_{i=1}^{n} z_i I_{f^{-1}(z_i)}$. One can extend this further in an obvious way by noting that:

$$\left(\varepsilon_X^{-1}\left(f^{-1}(z_i)\right)\right)_{i=1}^{n}$$

is a clopen partition of $\mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big)$.[12] Now, by considering Corollary (6.2.1) and the definition of Sp,[13] for each $i \in \{1, 2, \dots, n\}$ one has that:

$$\varepsilon_X^{-1}\left(f^{-1}(z_i)\right) = \mathcal{U}_{f_i}$$

for some $f_i \in \mathsf{Idemp}\big(\mathsf{Hom}(\mathsf{X},\mathbb{Z})\big)$.

Let $\overline{\varepsilon}$ denote the counit for the Stone Duality.[14] The fact that:

$$\overline{\varepsilon}_{\mathsf{Idemp}(\mathsf{Hom}(X,\mathbb{Z}))} : \mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big) \to \mathsf{Cl}\Big(\mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(X,\mathbb{Z})\big)\Big) \quad , \quad f \mapsto \mathcal{U}_f$$

is an isomorphism guarantees each $f_i = I_{f^{-1}(z_i)}$. This allows one to write:

$$\begin{aligned} \eta_{\mathsf{Hom}(\mathsf{X},\mathbb{Z})}\left(f \circ \varepsilon_X\right) &= \sum_{i=1}^{n} z_i f_i \\ &= \sum_{i=1}^{n} z_i I_{f^{-1}(z_i)} \\ &= f, \end{aligned}$$

and therefore proves the commutativity of Diagram (6.11).

The second triangle identity can be expressed as the commutativity of the following diagram:

$$\begin{array}{ccc} \mathsf{Sp}(R) & \xrightarrow{\mathsf{Sp}(\eta_R)} & \mathsf{Sp}\Big(\mathsf{Hom}_{\mathsf{Top}}\big(\mathsf{Sp}(R),\mathbb{Z}\big)\Big) \\ & \searrow\!\!= & \Big\downarrow \varepsilon_{\mathsf{Sp}(R)} \\ & & \mathsf{Sp}(R) \end{array} \tag{6.12}$$

If one recalls that $\mathsf{Cl} \circ \mathsf{Sp} \cong \mathsf{Idemp}$ and considers the following naturality square for $\overline{\varepsilon}$:

$$\begin{array}{ccc} \mathsf{Idemp}\big(\mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(R),\mathbb{Z})\big) & \xrightarrow{\mathsf{Idemp}(\eta_R)} & \mathsf{Idemp}(R) \\ \overline{\varepsilon}_{\mathsf{Idemp}(\mathsf{Hom}(\mathsf{Sp}(R),\mathbb{Z}))} \Big\downarrow \cong & & \cong \Big\downarrow \overline{\varepsilon}_{\mathsf{Idemp}(R)} \\ \mathsf{Cl}\Big(\mathsf{Sp}\big(\mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(R),\mathbb{Z})\big)\Big) & \xrightarrow[\mathsf{Cl}(\mathsf{Sp}(\eta_R))]{} & \mathsf{Cl}\big(\mathsf{Sp}(R)\big) \end{array}$$

[12] ε_X^{-1} represents the pre-image under ε_X, not the inverse morphism.
[13] See Definition (6.3.1).
[14] See Theorem (6.2.1).

one can see that the composite $(\overline{\varepsilon}_{\mathsf{Idemp}(R)})^{-1} \circ \mathsf{Cl}(\mathsf{Sp}(\eta_R)) \circ \overline{\varepsilon}_{\mathsf{Idemp}(\mathsf{Hom}(\mathsf{Sp}(R),\mathbb{Z}))}$ is equivalent to $\mathsf{Idemp}(\eta_R)$.

Therefore, the commutativity of Diagram (6.12) is equivalent to the commutativity of:

$$\begin{array}{ccc} \mathsf{Idemp}(R) & \xleftarrow{\mathsf{Idemp}(\eta_R)} & \mathsf{Idemp}(\mathsf{Hom}_{\mathsf{Top}}(\mathsf{Sp}(R),\mathbb{Z})) \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \mathsf{Cl}(\varepsilon_{\mathsf{Sp}(R)})} \\ & & \mathsf{Cl}(\mathsf{Sp}(R)) \\ & & \uparrow{\scriptstyle \overline{\varepsilon}_{\mathsf{Idemp}(R)}} \\ & & \mathsf{Idemp}(R) \end{array} \tag{6.13}$$

One easily calculates:

$$\left(\mathsf{Cl}(\varepsilon_{\mathsf{Sp}(R)}) \circ \overline{\varepsilon}_{\mathsf{Idemp}(R)}\right)(r) = \mathsf{Cl}(\varepsilon_{\mathsf{Sp}(R)})(\mathcal{U}_r) = I_{\mathcal{U}_r}$$

Therefore, for the entire composite, one has:

$$\begin{aligned} \left(\mathsf{Idemp}(\eta_R) \circ \mathsf{Cl}(\varepsilon_{\mathsf{Sp}(R)}) \circ \overline{\varepsilon}_{\mathsf{Idemp}(R)}\right)(r) &= \mathsf{Idemp}(\eta_R)(I_{\mathcal{U}_r}) \\ &= \eta_R(I_{\mathcal{U}_r}) \\ &= 1 \cdot r + 0 \\ &= r \end{aligned}$$

Thus Diagram (6.13) commutes, and this concludes the proof. □

As mentioned above, the only distinguishing property of $\mathbb{Z}$ that is used to show that the right adjoint of the spectrum functor is fully faithful is the fact that $\mathbb{Z}$ is connected. Thus, for any connected ring R, there is a generalized version of Proposition (6.3.4).

Proposition 6.3.5 (Borceux and Janelidze [4])**.** *For any ring R, the following conditions are equivalent:*

1. *R is connected,*

2. *The functor* $\mathsf{Hom}_{\mathsf{Top}}(-,R) : \mathsf{Stone} \to (\mathsf{R{-}Alg})^{op}$ *is fully faithful.*

Proof. To show that *(1)* $\Rightarrow$ *(2)*, one need only show that if R has no non-trivial idempotents, then for each Stone space X, the morphism:

$$\varepsilon_X : \mathsf{Sp}(\mathsf{Hom}_{\mathsf{Top}}(X,R)) \to X \quad , \quad F \mapsto \bigcap \{M \in \mathsf{Cl}(X) \mid I_M \in F\}$$

is a homeomorphism. Since ε_X is a continuous function from a compact space into a Hausdorff space, this reduces to showing that it is bijective. Of course, when $X = \emptyset$ is the empty Stone space, $\mathsf{Hom}_{\mathsf{Top}}(X, R)$ is the zero ring – which has no proper filters – and $\mathsf{Sp}(\mathsf{Hom}_{\mathsf{Top}}(X, R)) = \emptyset$, so ε_X is trivially a homeomorphism.

If $X \neq \emptyset$, one recalls that for any continuous function $g : X \to R$ to be idempotent, it must map all of the elements in X to either 1 or 0 in R. Since R has a discrete topology, this means that g must have a value of 1 on some clopen set $M \in \mathsf{Cl}(X)$, and a value of 0 on its complement $\neg M$, so $g = I_M$. Given this, it is clear that the association:

$$\mathsf{Idemp}(\mathsf{Hom}_{\mathsf{Top}}(X, R)) \to \mathsf{Cl}(X) \quad , \quad g = I_M \mapsto M$$

is a Boolean algebra isomorphism. The image of this isomorphism under Spec will trivially be an isomorphism, and its inverse morphism will be ε_X, up to an isomorphism in Stone.

For the converse implication, assume that each ε_X is a homeomorphism. In particular, when $X = \{*\}$, one has that $\mathsf{Hom}_{\mathsf{Top}}(\{*\}, R) \cong R$, and therefore:

$$\mathsf{Sp}(R) \cong \mathsf{Sp}(\mathsf{Hom}_{\mathsf{Top}}(\{*\}, R)) \cong \{*\}$$

which shows precisely that R has no non-trivial idempotents. □

Corollary 6.3.1. *If R is connected, then the functor:*

$$\mathsf{Sp} : (\mathsf{R\text{–}Alg})^{op} \to \mathsf{Stone}$$

has a fully faithful right adjoint:

$$\mathsf{Hom}_{\mathsf{Top}}(-, R) : \mathsf{Stone} \to (\mathsf{R\text{–}Alg})^{op}$$

As will be clarified in the following section, when R is a connected ring, adjunctions of the form $\mathsf{Sp} \dashv \mathsf{Hom}_{\mathsf{Top}}(-, R)$ can be used to describe "infinite" Galois theories. In particular, when $R = K$ is a field, one can describe the Galois theory of infinite field extensions over K.

6.4 Galois Theory of Commutative Rings and Algebras

The following is a reformulation of the contents of Chapter (4), in the language of the current chapter.

Including the "Top" subscript for the various hom-functors provides a simple marker for context in complicated expressions. However, since the work to follow makes use of the induced adjunctions described in Section (2.3) – which themselves require distinguishing subscripts – these functors will simply be written as $\mathsf{Hom}(-, B)$ from this point onwards.

One can use a sheaf-theoretic argument made in [27] to show that the opposite category of commutative rings is extensive, and use this result to more easily prove that the spectrum functor $\mathsf{Sp} : \big(\mathsf{B{-}Alg}\big)^{op} \to \mathsf{Stone}$ is admissible, for each commutative ring B. Therefore, whenever B is connected,[15] the adjunction:

$$\big(\mathsf{Sp}, \mathsf{Hom}(-, B), \eta, \varepsilon\big) : \big(\mathsf{B{-}Alg}\big)^{op} \to \mathsf{Stone}$$

can be seen as an extended example of the theory developed in Chapter (4). As such, B will be taken to be a connected algebra throughout the rest of this section.

If one considers $\big(\mathsf{Sp}, \mathsf{Hom}(-, B), \eta, \varepsilon\big) : \big(\mathsf{B{-}Alg}\big)^{op} \to \mathsf{Stone}$, along with a B-algebra E, and calculates the induced adjunction described in Section (2.3), one obtains:

$$\big(\mathsf{E{-}Alg}\big)^{op} \simeq \Big(\big(\mathsf{B{-}Alg}\big)^{op} \downarrow E\Big) \underset{\mathsf{Hom}(-,\,B)_E}{\overset{\mathsf{Sp}_E}{\rightleftarrows}} \big(\mathsf{Stone} \downarrow \mathsf{Sp}(E)\big)$$

in which:

1. If $\gamma : E \to C$ is an E-algebra structure on C, one has:[16]

$$\mathsf{Sp}(C) = \mathsf{Sp}_E(C, \gamma : C \to E) = \big(\mathsf{Sp}(C), \mathsf{Sp}(\gamma) : \mathsf{Sp}(C) \to \mathsf{Sp}(E)\big),$$

2. The image of an object $\big(X, \varphi : X \to \mathsf{Sp}(E)\big)$ under $\mathsf{Hom}(-, B)_E$ is determined by the pushout[17] of $\mathsf{Hom}(\varphi, B) : \mathsf{Hom}\big(\mathsf{Sp}(E), B\big) \to \mathsf{Hom}\big(X, B\big)$ along $\eta_E : \mathsf{Hom}\big(\mathsf{Sp}(E), B\big) \to E$ in $\mathsf{B{-}Alg}$.

[15] See Corollary (6.3.1).
[16] Formally, one considers $\gamma : E \to C$ as a B-algebra homomorphism and takes the image of its opposite morphism under Sp.
[17] Considered as a pullback in the opposite category.

As in the finite case,[18] when E is connected (i.e. has no non-trivial idempotents), the objects in the image of $\mathsf{Hom}(-, B)_E$ – effectively, the trivial coverings of E[19] – have a simplified form:

Lemma 6.4.1. *If B is a ring and E is a connected B-algebra, then there is an isomorphism of E-algebras:*

$$E \otimes_B \mathsf{Hom}(X, B) \cong \mathsf{Hom}(X, E)$$

for each Stone space X.

Proof. Since E is connected, one knows that $\mathsf{Sp}(E) = \{*\}$, and so $\big(\mathsf{Stone} \downarrow \mathsf{Sp}(E)\big) = \big(\mathsf{Stone} \downarrow \{*\}\big) \cong \mathsf{Stone}$, under which each Stone space X is identified with the continuous function $\varphi : X \to \{*\}$. E being connected also ensures that $\mathsf{Hom}\big(\mathsf{Sp}(E), B\big) = \mathsf{Hom}\big(\{*\}, B\big) \cong B$, where each function $\{*\} \to B$ is identified with the unique element in its image. Thus, $\mathsf{Hom}(\varphi, B)$ effectively acts on elements in B, and assigns each $b \in B$ to the constant function:

$$Const_b : X \to B \quad , \quad x \mapsto b$$

Further, since each function $f : \{*\} \to B$ in $\mathsf{Hom}\big(\mathsf{Sp}(E), B\big)$ yields $\mathsf{Sp}(E) = \{*\} = f^{-1}(r) = \mathcal{U}_1$, one can identify $\eta_E : \mathsf{Hom}\big(\mathsf{Sp}(E), B\big) \to E$ with the action $\rho_E : B \to E$ that makes E a B-algebra. Therefore, $\mathsf{Hom}(X, B)_E$ is determined by the following pushout in $\mathsf{B{-}Alg}$:

$$\begin{array}{ccc} B & \xrightarrow{Const_{(-)}} & \mathsf{Hom}(X, B) \\ {\scriptstyle \rho_E}\big\downarrow & & \big\downarrow{\scriptstyle \iota_2} \\ E & \xrightarrow[\iota_1]{} & E \otimes_B \mathsf{Hom}(X, B) \end{array}$$

Finally, since the adjunction $\big(\mathsf{Sp}, \mathsf{Hom}(-, E), \eta, \varepsilon\big) : \big(\mathsf{E{-}Alg}\big)^{op} \to \mathsf{Stone}$ is identical to:

$$\big(\mathsf{E{-}Alg}\big)^{op} \simeq \Big(\big(\mathsf{B{-}Alg}\big)^{op} \downarrow E\Big) \underset{\mathsf{Hom}(-,B)_E}{\overset{\mathsf{Sp}_E}{\rightleftarrows}} \mathsf{Stone} \tag{6.14}$$

one can see that $E \otimes_B \mathsf{Hom}(X, B) \cong \mathsf{Hom}(X, E)$. □

[18] See Example (4.3.1) and Chapter (5).
[19] See Definition (4.3.1).

Definition 6.4.1. *An E-algebra C is a trivial covering of E in $\left(\mathsf{B{-}Alg}\right)^{op}$ when:*

$$\eta_C : \mathsf{Hom}\left(\mathsf{Sp}_E(C), B\right)_E \to C$$

is an isomorphism of E-algebras.

Note that the ring B is mentioned in Definition (6.4.1) because η_C is a component of the unit for the adjunction in Diagram (6.14). One writes $\mathsf{TrivCov}(E)$ for the category of trivial coverings of E in $\left(\mathsf{B{-}Alg}\right)^{op}$.

In what follows, a morphism (whether a homomorphism or its opposite) will be referred to as an "effective descent morphism" ("morphism of Galois descent") if it is an effective descent morphism (resp. morphism of Galois descent) when considered as a morphism in Rings^{op}.

Corollary (4.2.2) shows that every morphism $p : B \to E$ between fields in Rings will be an effective descent morphism in both Rings^{op} and $\left(\mathsf{B{-}Alg}\right)^{op}$.

It is clear that $\left(\left(\mathsf{B{-}Alg}\right)^{op} \downarrow (B, 1_B)\right) \cong \mathsf{B{-}Alg}^{op}$ and $\left(\left(\mathsf{B{-}Alg}\right)^{op} \downarrow E\right) \cong \left(\mathsf{E{-}Alg}\right)^{op}$, and further, that if a ring homomorphism $p : B \to E$ is an effective descent morphism in Rings^{op}, then in the adjunction:

$$\left(\mathsf{B{-}Alg}\right)^{op} \underset{p_!}{\overset{p^*}{\rightleftarrows}} \left(\mathsf{E{-}Alg}\right)^{op}$$

p^* sends each B-algebra structure $B \xrightarrow{\alpha} A$ to the pushout of α along p, and $p_!$ sends each E-algebra structure $E \xrightarrow{\gamma} C$ to the B-algebra $B \xrightarrow{p} E \xrightarrow{\gamma} C$, both considered as objects in their respective opposite categories.

Thus, one can see that Diagram (6.15) provides the context in which to define covering morphisms in $\left(\mathsf{B{-}Alg}\right)^{op}$:

$$\left(\mathsf{B{-}Alg}\right)^{op} \underset{p_!}{\overset{p^*}{\rightleftarrows}} \left(\mathsf{E{-}Alg}\right)^{op} \underset{\mathsf{Hom}(-,B)_E}{\overset{\mathsf{Sp}_E}{\rightleftarrows}} \left(\mathsf{Stone} \downarrow \mathsf{Sp}(E)\right) \tag{6.15}$$

It is noted again that B (and therefore E) are be assumed to be connected, as this ensures that the right adjoint in $\mathsf{Sp} \dashv \mathsf{Hom}(-, B)$ is fully faithful.

The following two results neatly translate the relationships between the coverings of B and the trivial coverings of E – as originally discussed in Section (4.3) – into the language of the current chapter.

Definition 6.4.2. *Let $p : B \to E$ be an effective descent morphism, let η^E denote the unit of the adjunction $\mathsf{Sp}_E \dashv \mathsf{Hom}(-, B)_E$ in Diagram (6.15) and let A be a B-algebra. A is split by p when:*

$$\eta^E_{E\otimes_B A} : \mathsf{Hom}\big(\mathsf{Sp}(E\otimes_B A), B\big)_E \to E\otimes_B A$$

is an isomorphism of E-algebras.

Again, one will write $A \in \mathsf{Spl}_B(E,p)$ for all objects in $\big(\mathsf{B{-}Alg}\big)^{op}$ split by p.

Lemma 6.4.2. *Given an E-algebra C, the following conditions are equivalent:*

1. *$C \in \mathsf{TrivCov}(E)$,*
2. *C is split by 1_E,*
3. *$C \cong \mathsf{Hom}\big(X, B\big)_E$ for some $(X,\varphi) \in \big(\mathsf{Stone} \downarrow \mathsf{Sp}(E)\big)$.*

Lemma 6.4.3. *A B-algebra A is split by p if and only if the E-algebra $E\otimes_B A$ is split by 1_E.*

Definition 6.4.3. *A homomorphism of rings $p : B \to E$ is a morphism of Galois descent when:*

1. *p is an effective descent morphism in Rings^{op},*
2. *$\forall\ (X,\varphi) \in \big(\mathsf{Stone} \downarrow \mathsf{Sp}(E)\big)$, the B-algebra given by the composite:*

$$B \xrightarrow{p} E \xrightarrow{\iota_1} E \otimes_{\mathsf{Hom}(\mathsf{Sp}(E),B)} \mathsf{Hom}\big(X, B\big)$$

is split by p.

Example 6.4.1. *Every Galois extension of fields $K \subseteq E$ yields the inclusion $K \to E$ as a morphism of Galois descent.*

The following definition hails from classical Galois theory, and can be used to show that the notion of an algebra being split by an effective descent morphism in the categorical sense – i.e. in the sense of Definition (6.4.2) – actually relates to the splitting of polynomials [20], as evidenced by Proposition (6.4.2).

Definition 6.4.4. *Let $K \subseteq E$ be a field extension, and let A be a K-algebra. E splits A as a K-algebra when:*

1. *A is algebraic over K,*
2. *The minimal polynomial $p(X) \in K[X]$ of each element in A has a factorization into linear polynomials with distinct roots in $E[X]$.*

Corollary 6.4.1. *Let $K \subseteq E$ be a Galois extension, and let A be a K-algebra. If A is split by E as an algebra, then there is an isomorphism of E-algebras:*

$$E \otimes_K A \cong E^n$$

This provides an obvious link between the K-algebras that are split by E and the covering morphisms of $B = K$.

The correspondence between the notions of categorical splitting and the splitting of polynomials is further corroborated by the fact that every Galois extension of fields (universal covering) splits itself as an algebra. This mimics the result in Proposition (4.4.3) directly.

Proposition 6.4.1. *Given a field extension $K \subseteq E$, the following conditions are equivalent:*

1. *$K \subseteq E$ is a Galois extension,*
2. *E splits itself as a K-algebra.*

Proposition 6.4.2 (Borceux and Janelidze [4])**.** *Let $K \subseteq E$ be a Galois extension (considered as a morphism p of Galois descent). Every finite-dimensional K-algebra A that is split by E as a K-algebra is split by p.*

Proof. It has to be shown that, for each finite-dimensional K-algebra A, the following (homo)morphism:

$$\eta^E_{E\otimes_K A} : \mathsf{Hom}\big(\mathsf{Sp}\big(E \otimes_K A\big), E\big) \to E \otimes_K A$$

is an isomorphism of E-algebras.

Since $K \subseteq E$ is a Galois extension, E splits itself as a K-algebra, and one can use Lemma (6.4.1) to see that:

$$E \otimes_K \mathsf{Hom}(X, K) \cong \mathsf{Hom}(X, E)$$

Therefore, one has:

$$\begin{aligned} A \otimes_K \mathsf{Hom}\big(X, E\big) &\cong A \otimes_K E \otimes_K \mathsf{Hom}\big(X, K\big) \\ &\cong E^n \otimes_K \mathsf{Hom}\big(X, K\big) \\ &\cong \big(E \otimes_K \mathsf{Hom}\big(X, K\big)\big)^n \\ &\cong \mathsf{Hom}\big(X, E\big)^n. \end{aligned}$$

Further, the spectrum functor:

$$\mathsf{Sp} : \big(\mathsf{K}\text{–}\mathsf{Alg}\big)^{op} \to \mathsf{Stone}$$

is a left adjoint, and will thus preserve colimits. Therefore, when considered as a contravariant functor, it will take limits to colimits. With this, one can calculate:

$$\begin{aligned}
\mathsf{Hom}\Big(\mathsf{Sp}\big(A \otimes_K \mathsf{Hom}(X, E)\big), E\Big) &\cong \mathsf{Hom}\Big(\mathsf{Sp}\big(\mathsf{Hom}(X, E)^n\big), E\Big) \\
&\cong \mathsf{Hom}\Big(\coprod_{i=1}^{n} \mathsf{Sp}\big(\mathsf{Hom}(X, E)\big), E\Big) \\
&\cong \mathsf{Hom}\Big(\coprod_{i=1}^{n} X, E\Big) \\
&\cong \prod_{i=1}^{n} \mathsf{Hom}(X, E) \\
&\cong A \otimes_K \mathsf{Hom}(X, E)
\end{aligned}$$

for each Stone space X. If one chooses $X = \{*\}$, one has that $\mathsf{Hom}(X, E) \cong E$, and therefore that the above isomorphism reduces to:

$$\eta^E_{E \otimes_K A} : \mathsf{Hom}\big(\mathsf{Sp}(E \otimes_K A), E\big) \to E \otimes_K A$$

which shows precisely that $A \in \mathsf{Spl}_K(E, p)$. □

There is an analogous result for infinite-dimensional K-algebras, which would certainly be of interest, given that the context develops a framework in which to discuss infinite Galois theory. However, the proof of this result requires more work from classical Galois theory than can be included in this thesis.

The following succession of results is required in order to construct the Galois theorem of commutative rings.

Lemma 6.4.4. *If one restricts the domain of the functor* Sp_E *from* $\big(\mathsf{E}\text{–}\mathsf{Alg}\big)^{op}$ *to the full subcategory* $\mathsf{TrivCov}(E)$*, then* $\mathsf{Sp}_E \dashv \mathsf{Hom}(-, B)_E$ *becomes an equivalence of categories:*

$$\mathsf{TrivCov}(E) \underset{\mathsf{Hom}(-,B)_E}{\overset{\mathsf{Sp}_E}{\rightleftarrows}} \big(\mathsf{Stone} \downarrow \mathsf{Sp}(E)\big)$$

Lemma 6.4.5. *If a ring homomophism $p : B \to E$ is a morphism of Galois descent, then the restriction:*

$$p^* : \mathsf{Spl}_B(E, p) \to \mathsf{TrivCov}\big(E\big)$$

is monadic.

Corollary 6.4.2. *If a homomorphism of rings $p : B \to E$ is a morphism of Galois descent, then the composite:*

$$\mathsf{Spl}_B(E, p) \xrightarrow{\mathsf{Sp}_E \circ p^*} \big(\mathsf{Stone} \downarrow \mathsf{Sp}\big(E\big)\big) \quad , \quad (A, \alpha) \mapsto \big(\mathsf{Sp}\big(E \otimes_B A\big), \mathsf{Sp}(\iota_1)\big)$$

is monadic.

Lemma 6.4.6. *If a homomorphism of rings $p : B \to E$ is a morphism of Galois descent, then the functor:*

$$\mathsf{Sp} : \big(\mathsf{B\text{–}Alg}\big)^{op} \to \mathsf{Stone}$$

carries the cokernel pair of p – viewed as an internal groupoid/kernel pair in $\big(\mathsf{B\text{–}Alg}\big)^{op}$ – to an internal groupoid in Stone.

The cokernel pair of p is the cogroupoid:

$$E \otimes_B E \otimes_B E \;\; \begin{array}{c} \xleftarrow{[\iota_1, \iota_2]} \\ \xleftarrow{[\iota_1, \iota_3]} \\ \xleftarrow[{[\iota_2, \iota_3]}]{} \end{array} \;\; E \otimes_B E \;\; \begin{array}{c} \xleftarrow{\iota_1} \\ \xrightarrow{[1,1]} \\ \xleftarrow[\iota_2]{} \end{array} \;\; E \tag{6.16}$$

in $\mathsf{B\text{–}Alg}$.

Definition 6.4.5. *Let $p : B \to E$ be a morphism of Galois descent. The Galois groupoid $\mathsf{Gal}[p]$ of p is the following internal groupoid in Stone:*

$$\mathsf{Sp}\big(E \otimes_B E \otimes_B E\big) \;\; \begin{array}{c} \xrightarrow{\mathsf{Sp}([\iota_1, \iota_2])} \\ \xrightarrow{\mathsf{Sp}([\iota_1, \iota_3])} \\ \xrightarrow[\mathsf{Sp}([\iota_2, \iota_3])]{} \end{array} \;\; \mathsf{Sp}\big(E \otimes_B E\big) \;\; \begin{array}{c} \xrightarrow{\mathsf{Sp}(\iota_1)} \\ \xleftarrow{\mathsf{Sp}([1,1])} \\ \xrightarrow[\mathsf{Sp}(\iota_2)]{} \end{array} \;\; \mathsf{Sp}\big(E\big)$$

Theorem 6.4.1. *If $p : B \to E$ is a homomorphism of rings that is a morphism of Galois descent in $\mathsf{B}\text{–}\mathsf{Alg}^{op}$, then there is an equivalence of categories:*

$$\mathsf{Spl}_B(E, p) \simeq \mathsf{Stone}^{\mathsf{Gal}[p]}$$

Since E is assumed to be connected (as is the case when p is the inclusion homomorphism corresponding to a Galois extension $B = K \subseteq E$ of fields), one knows that $\mathsf{Sp}(E) = \{*\}$, $\big(\mathsf{Stone} \downarrow \mathsf{Sp}(E)\big) \simeq \mathsf{Stone}$, and $\mathsf{Gal}[p]$ is actually an internal group in the category of Stone spaces.

Since the spectrum functor Sp is admissible, and $\mathsf{Hom}(-, B)$ is fully faithful when B is connected, one can use Proposition (4.1.2) and Lemma (4.3.3) to describe an instantiation of Corollary (4.5.1) – that is, of a pointed version of the Galois theorem – in the current context. Concretely, one can consider the case where $\mathbb{C} = \mathsf{K}\text{–}\mathsf{Alg}^{op}$, and use Proposition (4.1.2) to show that the functor $(1 \downarrow \mathsf{Sp})$ is admissible. One can also use Lemma (4.3.3) to show that any homomorphism of rings $p : B \to E$ that is a morphism of Galois descent in $\mathbb{C}$ will also be a morphism of Galois descent in $\big(1 \downarrow \mathbb{C}\big)$, whenever it can be considered as a morphism in that category. Thus, if (B, b) and (E, e) are objects $\big(1 \downarrow \mathbb{C}\big)$ such that $p : (E, e) \to (B, b)$ is a morphism in $\big(1 \downarrow \mathbb{C}\big)$, one can apply Corollary (4.5.1) to find that:

$$\mathsf{Spl}_{(B,b)}\big((E, e), p\big) \simeq \mathsf{Stone}_*^{\mathsf{Gal}[p]}$$

This is an example of the fact that the new, pointed version of the Galois theorem – as described in Section (4.5) – can be applied in a wide variety of situations.

Chapter 7

Conclusion

In Chapter (4), the fundamental theorem of Galois theory in an abstract category was presented via the central notions of admissibility and effective descent. It was shown that any admissible functor that possesses a fully faithful right adjoint necessarily induces an admissible functor on the associated category of pointed objects. In the same vein, it was proven that any effective descent morphism in a category will also be an effective descent morphism in the associated category of pointed objects, provided that it can be regarded as a morphism in that category. The extension of these notions to the category of pointed objects facilitated the construction a new, pointed version of the Galois theorem in an abstract category.

In Chapter (5), the theory of finite, separable field extensions was presented as a Galois theory in the opposite category of finite-dimensional unital commutative K-algebras, and an intuitive algebraic interpretation of this realization was provided. Further, this Galois theory was expanded into the context of non-unital algebras by means of the extended notions of admissibility and effective descent, as described in Chapter (4), and categorical semidirect products, as detailed in Chapter (3). The form of the functors constituting the new, extended Galois theory was discussed at length, and the specific version of the pointed Galois theorem that these functors produce was described. Finally, an appropriate form in which to consider effective descent morphisms in the non-unital context was proposed.

In Chapter (6), a detailed construction of the Boolean Galois theory of commutative rings was given by means of the Pierce representation, which uses the Stone duality to associate the Boolean algebra underlying every commutative ring to the Stone space of ultrafilters on that Boolean algebra. This Galois theory was used to reconstruct the material given in Chapter (4), and yielded a version of the categorical Galois theorem of commutative rings able to describe infinite Galois extensions. Under the equivalence this Galois theorem provided, each covering morphism of rings was seen to be directly associated to a group(oid) structure with a naturally endowed Stone topology. Finally, it was shown that the pointed version of the Galois theorem can be instantiated in the context of Boolean Galois theories. The author hopes that this small extension of the reaches of categorical Galois theory will lead to further interesting developments and formulations.

Appendix A

Supplementary Material

A.1 G-Sets

Let G be a group. Recall that a G-set is a pair (X, ϕ), where X is a set and $\phi : G \times X \to X$ is a function – often written as $\phi(g, x) = g \cdot x$ – satisfying:

1. $1_G \cdot x = x \quad \forall\, x \in X$,
2. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \forall\, g_1, g_2 \in G,\ \forall\, x \in X$.

A morphism $f : (X_1, \phi_1) \to (X_2, \phi_2)$ between G-sets is a set-function f that commutes with elements of G acting on X, i.e. a function f such that:

$$f(g \cdot x) = g \cdot f(x)\ \forall\, g \in G,\ \forall\, x \in X$$

Of course the collection of G-sets and their morphisms forms a category. One can also consider actions of a group on pointed sets, to obtain pointed G-sets:

Definition A.1.1 (Muhiuddin [26])**.** *Let G be a group. A pointed G-set is a pair $((X, \overline{x}), \phi)$ where $(X, \overline{x})$ is a pointed set and $\phi : G \times X \to X$ is a function such that:*

1. $1_G \cdot x = x \quad \forall\, x \in X$,
2. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \forall\, g_1, g_2 \in G,\ \forall\, x \in X$,
3. $g \cdot \overline{x} = \overline{x}\ \forall\, g \in G$.

That is, a pointed G-set can be thought of as a pointed set with a (conventional) G-set structure that acts trivially on the distinguished point $\overline{x}$. Morphisms of pointed G-sets are morphisms of G-sets that preserve distinguished points.

A.2 Pure Monomorphisms of Modules

Definition A.2.1. *Let R be a unital commutative ring, and let A and B be R-modules. A monomorphism $m : A \to B$ is a pure monomorphism of R-modules when, for every R-module C, the following canonical morphism:*

$$m \otimes 1_C : A \otimes_R C \to B \otimes_R C$$

is a monomorphism.

There is a significant difference between the notions of a module being flat and a monomorphism being pure [2]. Recall that an R-module C is flat when – for every monomorphism $m : A \to B$ – the canonical morphism:

$$m \otimes 1_C : A \otimes_R C \to B \otimes_R C$$

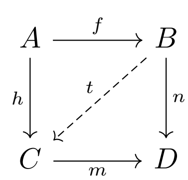
is a monomorphism.

A.3 On the Types of Epimorphism

It is well known that a set function is surjective if and only if a list of equivalent conditions holds (the function being right-cancellable, having a right inverse, acting as a quotient map, having a canonical image factorization, etc). Epimorphisms, and their split, regular, strong and extremal variations, are generalizations of these various conditions.

Definition A.3.1. *A morphism $f : A \to B$ in a category $\mathbb{C}$ is:*

1. *an epimorphism if it is right-cancellable $\big($that is, $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$ for all morphisms $h_1, h_2 : B \to C$ in $\mathbb{C}\big)$,*
2. *an extremal epimorphism if whenever f can be factorized as $f = m \circ h$, where h is any morphism and m is a monomorphism, m is necessarily an isomorphism,*
3. *a strong epimorphism if, given any commutative square*



where m is a monomorphism, there exists a unique morphism $t : B \to C$ such that the diagram commutes,

4. *a regular epimorphism when f is the coequalizer of some pair of morphisms in $\mathbb{C}$,*

5. *a split epimorphism if there exists a morphism $s : B \to A$ in $\mathbb{C}$ such that $f \circ s = 1_B$.*

A well known list of conditions (that exactly mirrors conditions that hold for surjective set functions) holds for the collection of epimorphisms in any category $\mathbb{C}$.

Proposition A.3.1.

1. *If a morphism is an isomorphism, then it is both a monomorphism and an epimorphism,*

2. *The composite of two epimorphisms is an epimorphism,*

3. *If a composite $f \circ g$ is an epimorphism, then f is an epimorphism.*

Although there are analogues of Proposition (A.3.1) for the various types of epimorphism, these analogues do not necessarily hold in arbitrary categories. Split epimorphisms form an exception to this, in that they too obey the conditions in Proposition (A.3.1) in any category. Split epimorphisms, which capture the property of having a right inverse, encapsulate the strongest notion of epimorphism, in the sense that they possess the properties of all the others. Split epimorphisms also interact well with pullbacks.

Proposition A.3.2. *In any category $\mathbb{C}$, whenever it exists, the pullback of a split epimorphism will be a split epimorphism.*

It can be shown[1] that this property does not always hold for more general types of epimorphism. In particular, even in categories with finite limits, neither the results from Proposition (A.3.1) nor Proposition (A.3.2) hold for regular epimorphisms. These conditions will hold for regular epimorphisms, however, in regular categories, as will be discussed in Section (A.6).

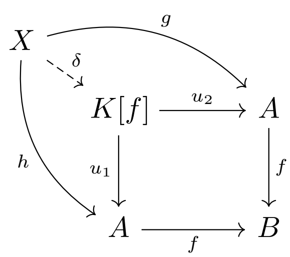
Regular epimorphisms capture the notion of a morphism being a quotient map. It is clear that any split epimorphism $f : A \to B$ (with splitting s) is regular: one can show that $f = \text{Coeq}(1_A, s \circ f)$, as any morphism $g : A \to B'$ such that $g \circ 1_A = g \circ (s \circ f)$ induces the unique $g \circ s : B \to B'$, which satisfies the requisite condition.

In the presence of finite limits, regular epimorphisms have many favourable properties.

[1]See A.4.14 in [3].

Lemma A.3.1 (Borceux and Bourn [3])**.** *In a category* $\mathbb{C}$ *with finite limits, every regular epimorphism is the coequalizer of its kernel pair.*

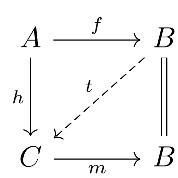
Proof. Suppose that $g, h : X \to A$ are morphisms in $\mathbb{C}$, let $f = \mathrm{Coeq}(g, h)$ be a regular epimorphism, and let (u_1, u_2) be its kernel pair. By the definition of f, one has $f \circ g = f \circ h$. Thus, by the property of the pullback, there exists a unique map δ such that the following diagram commutes:



Therefore, any morphism $f' : A \to B'$ for which $f' \circ u_1 = f' \circ u_2$ certainly yields $f' \circ h = f' \circ u_1 \circ \delta = f' \circ u_2 \circ \delta = f' \circ g$. Thus, since $f = \mathrm{Coeq}(g, h)$, any such f' determines a unique $\gamma : B \to B'$ such that $\gamma \circ f = f'$. And so, f must be the coequalizer of its kernel pair (u_1, u_2). □

Strong epimorphisms, like extremal epimorphisms, capture the property of image factorization. Extremal epimorphisms are a minor generalization of strong epimorphisms, in that the two coincide in any category with pullbacks.

In any setting, every strong epimorphism is an extremal epimorphism. If $f : A \to B$ is a strong epimorphism, and one considers a factorization $f = m \circ h$ where m is a monomorphism, then one finds:



Therefore, there exists a unique $t : B \to C$ such that $t \circ f = h$ and $m \circ t = 1_B$. Thus, as it is both a monomorphism and a split epimorphism, m is an isomorphism (and so f must be an extremal epimorphism).

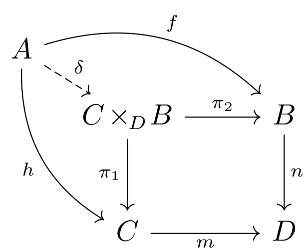
Proposition A.3.3 (Borceux and Bourn [3])**.** *In a category* $\mathbb{C}$ *with pullbacks, strong and extremal epimorphisms coincide.*

Proof. Every strong epimorphism is an extremal epimorphism.

For the converse, suppose that $f : A \to B$ is an extremal epimorphism, and consider any commutative diagram of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle n} \\ C & \xrightarrow[m]{} & D \end{array} \tag{A.1}$$

where m is a monomorphism. If one takes the pullback of h along m, one has:



Thus, there exists a unique $\delta : A \to C \times_D B$ such that $\pi_1 \circ \delta = h$ and $\pi_2 \circ \delta = f$. Since π_2 is a monomorphism and f is extremal, π_2 must be an isomorphism.

Thus, there exists a unique morphism $t = \pi_1 \circ \pi_2^{-1} : B \to C$ that makes Diagram (A.1) commute. To see this, note that

- $t \circ f = \left(\pi_1 \circ \pi_2^{-1}\right) \circ \left(\pi_2 \circ \delta\right) = \pi_1 \circ \delta = h$,
- $m \circ t \circ \pi_2 = m \circ \left(\pi_1 \circ \pi_2^{-1}\right) \circ \pi_2 = m \circ \pi_1 = n \circ \pi_2 \Rightarrow n = m \circ t$, since π_2 is an epimorphism.

□

Thus, in a context where categories are assumed to have pullbacks or finite limits, the notions of strong and extremal epimorphisms are often taken to be synonymous, and are simply referred to as "strong" [3].

Proposition A.3.4. *In a category $\mathbb{C}$ with pullbacks:*

1. *A morphism is an isomorphism if and only if it is a monomorphism and a strong epimorphism,*
2. *The composite of two strong epimorphisms is a strong epimorphism,*

3. If a composite $f \circ g$ is a strong epimorphism, then f is a strong epimorphism.

Proof.

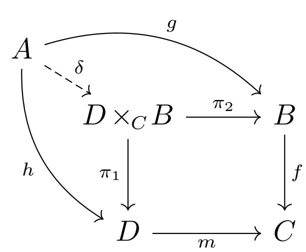
1. Suppose that f is an isomorphism that can be decomposed as $f = m \circ g$, where m is a monomorphism. If m were an isomorphism, an obvious candidate for its inverse would be $g \circ f^{-1}$. If one notes that:

   - $m \circ g \circ f^{-1} = f \circ f^{-1} = 1$,
   - $m \circ \left(g \circ f^{-1} \circ m\right) = f \circ f^{-1} \circ m = 1 \circ m \Rightarrow g \circ f^{-1} \circ m$, since m is a monomorphism,

   one can immediately see that m is an isomorphism, and therefore that f is both a monomorphism and a strong epimorphism.

   For the converse, if f is both a monomorphism and a strong epimorphism, one can use the fact that $f = f \circ 1$ (where f is regarded as a monomorphism) to see that f must be an isomorphism.

2. Let f and g be strong epimorphisms whose composite $f \circ g = m \circ h$ can be factorized as a morphism h, followed by a monomorphism m, and let $D \times_C B$ be the pullback of f along m.

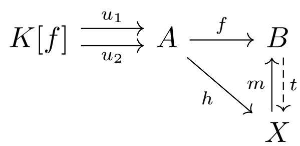
   Since the outer square in the above diagram commutes, there exists a unique $\delta : A \to D \times_C B$ such that $\pi_1 \circ \delta = h$ and $\pi_2 \circ \delta = g$. Now, since g is a strong epimorphism and π_2 is a monomorphism, π_2 must be an isomorphism. This means that one can write $f = m \circ \pi_1 \circ \pi_2^{-1}$, and because f is strong, m must be an isomorphism.

3. Suppose that $f \circ g$ is a strong epimorphism, and that $f = m \circ h$ can be factorized as a morphism h, followed by a monomorphism m. This means that $f \circ g = m \circ h \circ g$, and so m is an isomorphism, and f is strong.

□

Proposition A.3.5 (Borceux and Bourn [3]). *In a category $\mathbb{C}$ with pullbacks, every regular epimorphism is a strong epimorphism.*

Proof. Suppose that $f : A \to B$ is a regular epimorphism that can be factorized as $f = m \circ h$, where m is a monomorphism, and let (u_1, u_2) be the kernel pair of f, as in:



It is clear that $m \circ h \circ u_1 = f \circ u_1 = f \circ u_2 = m \circ h \circ u_1 \Rightarrow h \circ u_1 = h \circ u_2$, since m is a monomorphism. Now, since f is the coequalizer of its kernel pair, there exists a unique $t : B \to X$ such that $t \circ f = h$.

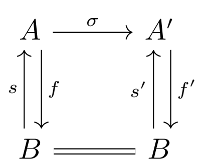
One can use this to write $m \circ t \circ f = m \circ h = f = 1_B \circ f \Rightarrow m \circ t = 1_B$, since f is an epimorphism. As it is both a monomorphism and a split epimorphism, m is an isomorphism, and f is a strong epimorphism. □

It will be shown in Section (A.6) that the converse implication in Proposition (A.3.5) – and therefore an analogue of Proposition (A.3.4) involving regular epimorphisms – holds when $\mathbb{C}$ is a regular category.

Protomodular, Regular, Exact and Semi-Abelian Categories

A.4 The Category of Points

One can write $\mathsf{Pt}_{\mathbb{C}}(B) = \mathsf{Pt}(B)$ for the category of points of B in $\mathbb{C}$, the category whose objects are the triples (A, f, s) in which $f : A \to B$ and $s : B \to A$ are morphisms in $\mathbb{C}$ such that $f \circ s = 1_B$.[2] Effectively, $\mathsf{Pt}(B)$ is the category of split epimorphisms f in $\mathbb{C}$ – each with given splitting s – that have B as their codomain. Morphisms $(A, f, s) \to (A', f', s')$ between objects in $\mathsf{Pt}(B)$ are given by morphisms $\sigma : A \to A'$ in $\mathbb{C}$ for which both the upward and downward squares in the following diagram commute:



If $\mathbb{C}$ has pullbacks (of split epimorphisms), then any morphism $p : E \to B$ induces a functor:

$$\mathsf{Pt}(B) \xrightarrow{p^*} \mathsf{Pt}(E) \quad , \quad \big(A \underset{s}{\overset{f}{\rightleftarrows}} B\big) \mapsto \big(E \times_B A \underset{<1,\, sp>}{\overset{\pi_1}{\rightleftarrows}} E\big) \tag{A.2}$$

which maps each point (A, f, s) to the pullback of f along p (with the first projection and its canonical splitting).

If $\mathbb{C}$ is pointed, there is a canonical isomorphism of categories $\mathbb{C} \cong \mathsf{Pt}(0)$, and for each $B \in C$, the morphism $i_B : 0 \to B$ from the zero object into B induces the functor $i_B^* = Ker(-)$:

[2]This modifies the notation of Bourn [3] slightly.

$$\mathsf{Pt}(B) \xrightarrow{i_B^*} \mathsf{Pt}(0) \cong \mathbb{C} \quad , \quad \big(A \underset{s}{\overset{f}{\rightleftarrows}} B\big) \mapsto \big(Ker(f) \rightleftarrows 0\big)$$

A.5 Protomodular Categories

Protomodular categories were first introduced by Bourn [6]. Protomodularity characterizes which categories have a meaningful notion of normal subobject.

Definition A.5.1. *A category $\mathbb{C}$ is protomodular if:*

1. *$\mathbb{C}$ has pullbacks of split epimorphisms,*
2. *For each morphism $p : E \to B$ in $\mathbb{C}$, the functor:*

$$p^* : \mathsf{Pt}(B) \to \mathsf{Pt}(E)$$

reflects isomorphisms.

Lemma A.5.1 (Borceux and Bourn [3])**.** *If $\mathbb{C}$ and $\mathbb{E}$ are categories with pullbacks of split epimorphisms, and $U : \mathbb{C} \to \mathbb{E}$ is a functor that preserves pullbacks of split epimorphisms and reflects isomorphisms, then if $\mathbb{E}$ is protomodular, so is $\mathbb{C}$.*

The forgetful functors:

$$\big(\mathbb{C} \downarrow B\big) \to \mathbb{C} \quad , \quad (A, f) \mapsto A$$

$$\big(B \downarrow \mathbb{C}\big) \to \mathbb{C} \quad , \quad (C, g) \mapsto C$$

satisfy the conditions in Lemma (A.5.1), and therefore the slice categories and the categories of points of a given protomodular category are also protomodular.

When $\mathbb{C}$ is pointed, there is a further, convenient characterization of protomodularity.

Definition A.5.2. *Let $\mathbb{C}$ be a pointed category. The split short five lemma holds in $\mathbb{C}$ when for any diagram of the form:*

$$\begin{array}{ccccc}
Ker(f) & \xrightarrow{ker(f)} & A & \underset{s}{\overset{f}{\rightleftarrows}} & B \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
Ker(f') & \xrightarrow[ker(f')]{} & A' & \underset{s'}{\overset{f'}{\rightleftarrows}} & B'
\end{array}$$

if both β and k are isomorphisms, then α is an isomorphism as well.

Theorem A.5.1 (Borceux and Bourn [3])**.** *For a pointed category $\mathbb{C}$ with pullbacks of split epimorphisms, the following conditions are equivalent:*

1. *For each morphism $p : E \to B$ in $\mathbb{C}$, the functor: $p^* : \mathsf{Pt}(B) \to \mathsf{Pt}(E)$ reflects isomorphisms,*

2. *For each object B in $\mathbb{C}$, the functor: $i_B^* = Ker(-) : \mathsf{Pt}(B) \to \mathbb{C}$ reflects isomorphisms,*

3. *The split short five lemma holds in $\mathbb{C}$.*

Proof.

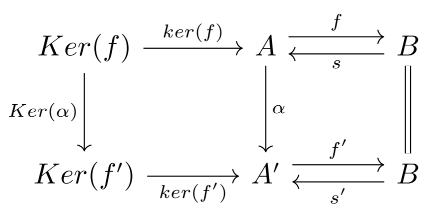
(*1*⇔*2*): The forward implication is trivial. Conversely, since $\mathbb{C}$ is pointed, for every morphism $p : E \to B$ one has that $i_B = p \circ i_E$. This means that there is an isomorphism of functors $i_B^* \cong i_E^* \circ p^*$.[3] Thus, both these functors must reflect isomorphisms. Since i_E^* preserves isomorphisms (as every functor does), one can see that the composite $i_E^* \circ p^*$ reflects isomorphisms if and only if p^* reflects them, since for any morphism $\alpha : (A, f, s) \to (A', f', s')$ between points:

$$p^*(\alpha) \text{ iso} \Leftrightarrow \big(i_E^* \circ p^*\big)(\alpha) \text{ iso} \Leftrightarrow \alpha \text{ iso.}$$

(*2*⇔*3*): Suppose that (A, f, s) and (A', f', s') are two points over an object B, and that $\alpha : (A, f, s) \to (A', f', s')$ is such that

$$i_B^*(\alpha) = Ker(\alpha) : Ker(f) \to Ker(f')$$

is an isomorphism. If (*2.*) holds – that is, if $Ker(-)$ reflects isomorphisms – then α must be an isomorphism as well. Therefore, in the following diagram:



$Ker(\alpha)$ being an isomorphism implies that α is an isomorphism. This is nothing but the statement that the split short five lemma holds (up to an isomorphism on B). The converse follows similarly.

□

[3] See, for example, Proposition 7.7 in [3].

Further, it was shown by Bourn and Janelidze [7] that protomodularity in varieties of universal algebras can be characterized by particular operations in those varieties. Readers unfamiliar with universal algebra or, specifically, the notions of algebraic theories and varieties, can find a straight-forward description of these concepts in Chapter 3 of [2].

Theorem A.5.2. *Given an algebraic theory $\mathcal{T}$, the variety $\mathbb{C} = \mathsf{Sets}^{\mathcal{T}}$ corresponding to that theory is protomodular if and only if for some natural number $n \in \mathbb{N}$, the variety contains n constant terms $\{e_1, e_2, \ldots, e_n\}$, n binary operations $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and one $(n+1)$-ary operation θ such that:*

$$\alpha_i(x, x) = e_i \text{ for each } i \in \{1, 2, \ldots, n\}, \text{ and for each } x$$

$$\theta\big(\alpha_1(x, y), \alpha_2(x, y), \ldots, \alpha_n(x, y), y\big) = x \text{ for all } x \text{ and } y$$

One can think of the n binary terms as analogues for division (or subtraction), and θ as being an '$(n+1)$-ary multiplication' (or addition).

A variety that is mentioned repeatedly in Chapter (3) is that of right Ω-loops. Groups, rings and many other familiar structures satisfy the right Ω-loops axioms.

Definition A.5.3. *A pointed variety of universal algebras is a variety of right Ω-loops if it contains, amongst others, binary operations $+$ and $-$ that satisfy the following identities:*

1. $x + 0 = x$,
2. $0 + x = x$,
3. $(x + y) - y = x$,
4. $(x - y) + y = x$.

Example A.5.1. *The categories of groups, rings, rings without unit, modules and algebras over a given ring, loops, right Ω-loops, Heyting semilattices*[4] *and Heyting algebras, amongst many others, are protomodular.*

For groups (or rings, modules or algebras), one could write $n = 1$, with $e = e_1 = 0$, $\alpha = \alpha_1 = -$ and $\theta = +$.

[4] See [22].

A.6 Regular Categories

The conditions defining a regular category $\mathbb{C}$ essentially guarantee that the internal relations in $\mathbb{C}$ are well-behaved, and that there is a coherent notion of image factorization in $\mathbb{C}$.

Definition A.6.1. *A category $\mathbb{C}$ is regular when:*

1. *$\mathbb{C}$ has finite limits,*
2. *Every kernel pair in $\mathbb{C}$ has a coequalizer,*
3. *Regular epimorphisms in $\mathbb{C}$ are pullback-stable.*

Lemma A.6.1 (Borceux and Bourn [3])**.** *In a regular category $\mathbb{C}$, if $f : A \to B$ is a regular epimorphism and $g : B \to C$ is any morphism, then*

$$f{\times_C}f : A{\times_C}A \to B{\times_C}B$$

is an epimorphism.

Proof. Let

- $\big(B{\times_C}B, a, b\big)$ be the pullback of g along itself,
- $\big(A{\times_C}B, c, d\big)$ be the pullback of gb along gf,
- $\big(B{\times_C}A, e, h\big)$ be the pullback of gf along ga,
- $\big(A{\times_C}A, i, j\big)$ be the pullback of gfh along gfc,

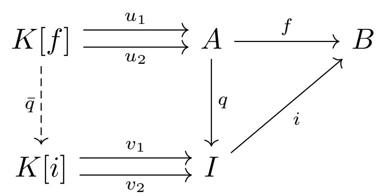
as in

$$\begin{array}{ccccc}
A{\times_C}A & \xrightarrow{\;j\;} & B{\times_C}A & \xrightarrow{\;h\;} & A \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle f} \\
A{\times_C}B & \xrightarrow{\;d\;} & B{\times_C}B & \xrightarrow{\;b\;} & B \\
\Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle g} \\
A & \xrightarrow[\;f\;]{} & B & \xrightarrow[\;g\;]{} & C
\end{array}$$

Since regular epimorphisms are pullback-stable in $\mathbb{C}$ and f is a regular epimorphism, it is clear that d, e, i and k are also regular epimorphisms. Therefore, as the composite of (regular) epimorphisms, $f \times_C f = e \circ j = d \circ i$ is itself an epimorphism. □

Theorem A.6.1 (Borceux and Bourn [3])**.** *In a regular category $\mathbb{C}$, every morphism can be factorized uniquely (up to isomorphism) as a regular epimorphism, followed by a monomorphism.*

Proof. Let f be any morphism in $\mathbb{C}$, let (u_1, u_2) be its kernel pair, take q to be the coequalizer of (u_1, u_2), and consider the following diagram:

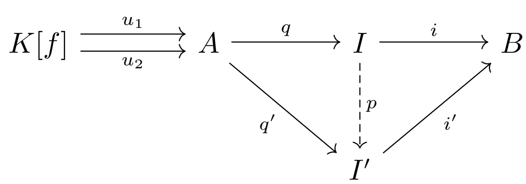


By definition, $f \circ u_1 = f \circ u_2$, and since i is the coequalizer of (u_1, u_2), there exists a unique $i : I \to B$ such that $f = i \circ q$. If one takes (v_1, v_2) to be the kernel pair of i, one has that $i \circ q \circ u_1 = f \circ u_1 = f \circ u_2 = i \circ q \circ u_2$ and therefore there exists a unique $\bar{q} : K[f] \to K[i]$ such that $v_1 \circ \bar{q} = q \circ u_1$ and $v_2 \circ \bar{q} = q \circ u_2$, because $K[i]$ is a pullback.

Of course $\bar{q} : K[f] \to K[i]$ is identical to $q \times_B q : A \times_B A \to I \times_B I$ and one can use Lemma (A.6.1) to see that $\bar{q} = q \times_B q$ is an epimorphism.

Thus, $v_1 \circ \bar{q} = q \circ u_1 = q \circ u_2 = v_2 \circ \bar{q} \Rightarrow v_1 = v_2$. This can be the case if and only if i is a monomorphism, and thus $f = i \circ q$ is a sufficient factorization.

To show that $f = i \circ q$ is a unique factorization, assume that f has a second factorization $f = i' \circ q'$, and consider the following diagram:



As i' is a monomorphism, $i' \circ q' \circ u_1 = f \circ u_1 = f \circ u_2 = i' \circ q' \circ u_2 \Rightarrow q' \circ u_1 = q' \circ u_2$. Again, since i is the coequalizer of (u_1, u_2), there exists a unique $p : I \to I'$ such that $q' = p \circ q$.

Since q' is a regular epimorphism, it is also a strong epimorphism. Thus, p must be a strong epimorphism.[5]

Lastly, since q is a (regular) epimorphism, $i' \circ p \circ q = i' \circ q' = f = i \circ q \Rightarrow i' \circ p = i$. Since i is a monomorphism, so is p.

Cumulatively, this shows that p is both a monomorphism and a strong epimorphism, and therefore an isomorphism. □

The above factorization is known as "the image factorization of f".

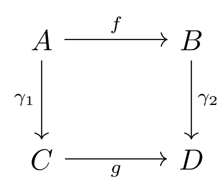
This image factorization allows for one to create an analogue for Proposition (A.3.4). One only needs:

Lemma A.6.2 (Borceux and Bourn [3])**.** *In a regular category $\mathbb{C}$, regular and strong epimorphisms coincide.*

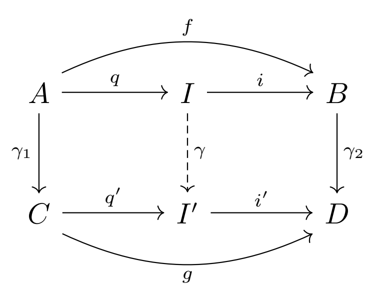
Proof. Since $\mathbb{C}$ has finite limits, regular epimorphisms are strong in $\mathbb{C}$.

For the converse, suppose that f is a strong epimorphism, and that it can be factorized as $f = i \circ q$, where i is a monomorphism and q is a regular epimorphism. Now, since both f and q are strong epimorphisms, i must be a strong epimorphism as well. This means that f is given, up to the isomorphism i, by a regular epimorphism, q. □

It can also be shown [3] that the image factorizations in a regular category are functorial, i.e. that for any given pair of morphisms (f, g) in $\mathbb{C}$ and any commutative square



involving f and g, one has a unique "diagram of factorizations" [3]:



[5]See Proposition (A.3.4).

This means that every functor that has a regular category $\mathbb{C}$ as its domain will preserve the factorization of morphisms as well.

All of the slice and coslice categories of a regular category $\mathbb{C}$ are regular as well. It is well known that if a category has finite (co)limits, then these are inherited by all of its slice and coslice categories [2]. Thus, all slice and coslice categories of $\mathbb{C}$ have finite limits, and the remaining conditions required for them to be regular follow from the fact that in any slice or coslice category of $\mathbb{C}$, kernel pairs and their coequalizers, as well as all pullbacks, are calculated as they would be in $\mathbb{C}$. Of course, this means for each $B \in \mathbb{C}$, $\mathsf{Pt}(B)$ is a regular category.

A.7 Exact Categories

Through the course of this text, 'exact' is taken to mean Barr-exact, that is:

Definition A.7.1 (Barr [1])**.** *A category* $\mathbb{C}$ *is exact when:*

1. $\mathbb{C}$ *is regular,*
2. *Every equivalence relation in* $\mathbb{C}$ *is effective.*[6]

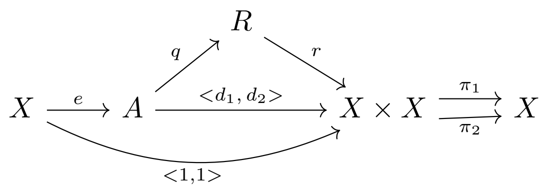
Exact categories encapsulate the "non-additive" behaviour of abelian categories, in that a category is abelian if and only if it is both additive and exact [2].

It can easily be shown that equivalence relations in (co)slice categories are given by the same relation in the base category.[7] This, in combination with the fact that (co)slice categories of a regular category are regular, allows one to see that all the slice and coslice categories of an exact category inherit exactness from the base category.

Theorem A.7.1 (Borceux and Bourn [3])**.** *If a category* $\mathbb{C}$ *is both Mal'cev and exact, then every reflexive pair in* $\mathbb{C}$ *has a coequalizer.*

Proof. Consider a reflexive pair $\left(A \underset{d_2}{\overset{d_1}{\underset{\longleftarrow e}{\rightrightarrows}}} X\right)$ in $\mathbb{C}$, and suppose that the factorization of $<d_1, d_2> : A \to X \times X$ is given by a regular epimorphism q, followed by a monomorphism r. Given the following diagram:

[6] That is, every equivalence relation in $\mathbb{C}$ is a kernel pair relation.
[7] See Example A.5.13 in [3].



one can see that the composite $<d_1, d_2> \circ e = <1_X, 1_X>$, as $d_1 \circ e = d_2 \circ e = 1_X$. If one writes $r_1 = \pi_1 \circ r$ and $r_2 = \pi_2 \circ r$, and observes that

$$\pi_1 \circ <d_1, d_2> \circ e = r_1 \circ q \circ e = r_2 \circ q \circ e = \pi_2 \circ <d_1, d_2> \circ e = 1_X,$$

one can see that the reflexive pair $\left(R \;\substack{\xrightarrow{r_1} \\ \xleftarrow{qe} \\ \xrightarrow{r_2}}\; X\right)$ is a reflexive relation. Since $\mathbb{C}$ is Mal'cev, this reflexive pair must be an equivalence relation, and further, it must be a kernel pair relation, because $\mathbb{C}$ is exact. Since $\mathbb{C}$ is regular, (r_1, r_2) has a coequalizer u. However, since q is an epimorphism, a map will equalize r_1 and r_2 precisely when it equalizes $r_1 \circ q = \pi_1 \circ r \circ q = d_1$ and $r_2 \circ q = \pi_2 \circ r \circ q = d_2$.

Cumulatively, this means that $\left(A \;\substack{\xrightarrow{d_1} \\ \xleftarrow{e} \\ \xrightarrow{d_2}}\; X\right)$ has a coequalizer u, given by $u = \mathrm{Coeq}(r_1, r_2)$.

□

A.8 Semi-Abelian Categories

Semi-abelian categories were proposed by Janelidze, Márki, and Tholen [19]. These categories capture and generalize the properties of groups or group-like structures, in the same way abelian categories generalize the properties of the category of abelian groups. Semi-abelian categories derive their name from the fact that a semi-abelian category will be abelian if and only if the categorical semidirect products in that category coincide with its categorical products.

Definition A.8.1. *A category $\mathbb{C}$ is semi-abelian when:*

1. *$\mathbb{C}$ is pointed,*
2. *$\mathbb{C}$ is protomodular,*
3. *$\mathbb{C}$ is exact,*
4. *$\mathbb{C}$ has binary coproducts.*

It can be shown [3] that all semi-abelian categories have coequalizers, and – since they also have binary coproducts – have all finite colimits as well.

It was shown by Janelidze, Márki, and Tholen [19] that a variety is semi-abelian if and only if it is pointed and protomodular. The following characterization of semi-abelian varieties emerges from this, as noted by Bourn and Janelidze [7]:

Theorem A.8.1. *For a given algebraic theory $\mathcal{T}$, the variety $\mathbb{C} = \mathsf{Sets}^{\mathcal{T}}$ is semi-abelian if and only if it contains a unique constant 0, a list of n binary operations*[8] $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, *and an $(n+1)$-ary operation θ such that:*

1. $\alpha_i(x, x) = 0$ *for all x and for each i,*
2. $\theta\big(\alpha_1(x, y), \alpha_2(x, y), \ldots, \alpha_n(x, y), y\big) = x$ *for all x, y.*

[8] For some $n \in \mathbb{N}$.

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