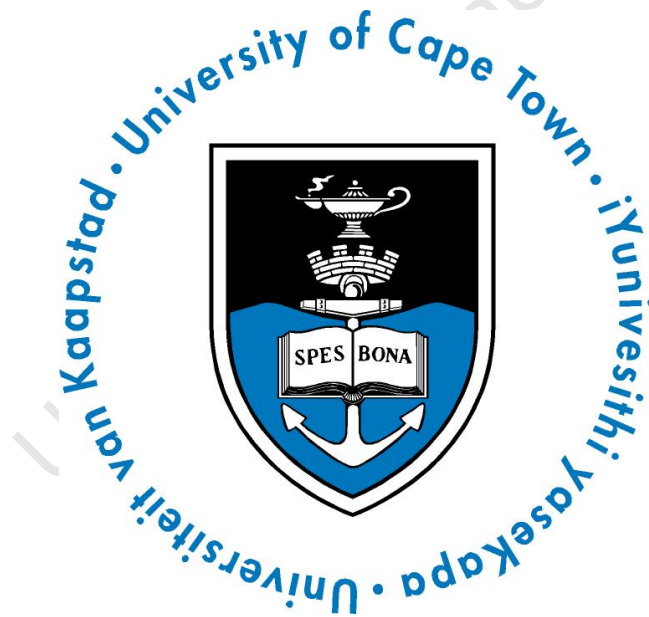


Topics in Gauge/Gravity Duality

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Doctoral thesis presented by

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*The effort to understand the universe
is one of the very few things that lifts human life
a little above the level of farce and gives it some of the grace of tragedy.*

Steven Weinberg

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How small the cosmos (a kangaroo's pouch would hold it), how paltry and puny in comparison to human consciousness, to a single individual recollection, and its expression in words!

Vladimir V. Nabokov, *Speak, Memory*

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Abstract

The gauge theory/gravity correspondence encompasses a variety of different specific dualities. We investigate various topics in the context of Super-Yang-Mills/type IIB string theory and superconformal Chern-Simons-matter/type IIA string theory dualities.

We carry out a rather extensive study of the type IIA $AdS_3 \times S^3 \times S^3 \times S^1$ Green-Schwarz superstring, up to quadratic order in fermions. We discuss issues related to fixing its κ -symmetry and show the one-loop finiteness of two-point functions of bosonic fields. We then perform a Hamiltonian analysis and compare $SU(2)$ string states with predictions from the conjectured Bethe equations. Furthermore, we show that, at least at tree-level, the two-body S-matrix is reflectionless.

We then concern ourselves with extending Mikhailov's construction of giant gravitons from holomorphic functions to include meromorphic functions, which lead to giants with non-trivial topologies in $AdS_5 \times S^5$. We explore what topological configurations giants, whose dynamics preserve a certain amount of supersymmetry, assume. We are particularly interested in solutions created by a localised modification of a set of intersecting spherical giant gravitons, as this seems the most tractable limit.

We finally explore some aspects of holographic particle-vortex duality, in particular its realisation in the ABJM model and a possible relation to Maxwell duality in AdS_4 . We formulate a symmetric version of the transformation that acts as a self-duality, show how to embed it as an abelian duality in the (2+1)-dimensional, $\mathcal{N} = 6$ super-Chern-Simons-matter theory that is the ABJM model, and speculate on a possible non-abelian extension.

Declaration

The work presented in this thesis is based on collaborations with

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- Jeffrey Murugan [*The Laboratory for Quantum Gravity & Strings, Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa*]
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- Andrea Prinsloo [*Department of Mathematics, University of Surrey, United Kingdom*]
- Jonathan P. Shock [*The Laboratory for Quantum Gravity & Strings, Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa*]
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- Linus Wulff [then *George P. & Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station*; now *Blackett Laboratory, Imperial College London, United Kingdom*]

The list below identifies chapters which are based on the listed publications:

- Chapter 2:** N. Rughoonauth, P. Sundin & L. Wulff, *Near-BMN dynamics of the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring*, JHEP **1207** (2012) 159, [arXiv:1204.4742 [hep-th]]
- Chapter 3:** M.C. Abbott, J. Murugan, A. Prinsloo & N. Rughoonauth, *Meromorphic Functions and the Topology of Giant Gravitons*, Phys.Lett. **B730** (2014) 215-220, [arXiv:1312.4900 [hep-th]]
- Chapter 4:** J. Murugan, H. Nastase, N. Rughoonauth & J.P. Shock, *Particle-vortex and Maxwell duality in the $AdS_4 \times CP^3/ABJM$ correspondence*, [arXiv:1404.5926 [math-ph]]

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

Signed by candidate

Nitin C. Rughoonauth

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Chapter 1

Introduction

Superstring theory has established itself as an amazingly elaborate conceptual framework that could today very well provide the best hints as to how to formulate and interpret a coherent picture of a possible unification of all fundamental interactions. Over its almost half a century of history, the many developments in this huge field of research have substantially and dramatically transformed our theoretical understanding of the nature of reality. Some of the most profound insights have been linked with the elucidation of a large set of symmetries underlying the mathematical structures of the physical theories. Many of those symmetries manifest themselves as dualities between markedly distinct, but mutually consistent, descriptions of some particular system or model.

Concretely, a particularly groundbreaking discovery in the late 1990s was the realisation [1, 2, 3, 4] that maximally supersymmetric, that is $\mathcal{N} = 4$, $SU(N)$ super-Yang-Mills (SYM) theory in four dimensions, which is a superconformal field theory, can be alternatively described in terms of type IIB string theory on the ten-dimensional $AdS_5 \times S^5$ spacetime. The *AdS/CFT* correspondence, while still a conjecture, has nevertheless been extensively tested and received substantial supporting evidence via the applications of gauge-gravity dualities. The proposed duality is of the strong-weak coupling type, in the sense that it maps the weakly coupled regime of the gauge theory to the strongly coupled sector of the string theory, and vice versa.

Let g_{YM} be the coupling constant of the $SU(N)$ SYM theory. Then, the 't Hooft coupling is

$$\lambda = g_{YM}^2 N .$$

On the string theory side, the characteristic length scale of Anti-de Sitter space is

$$R^4 = 4\pi g_s N \ell_s^4 ,$$

where, here,

- N is the *number of coincident D3-branes* sourcing the $AdS_5 \times S^5$ spacetime;
- $g_s = \frac{g_{YM}^2}{4\pi}$ is the *string coupling*; and
- $\ell_s = \sqrt{\alpha'}$ is the *string length*, α' being the *Regge slope*.

Therefore, we have that

$$\begin{aligned} R^4 &= g_{YM}^2 N \ell_s^4 \\ &= \lambda \ell_s^4, \end{aligned} \tag{1.1}$$

that is, when λ is large in the gauge theory, $1/R$ is small, which means that the string theory degrees of freedom are effectively weakly coupled. This duality is depicted in figure 1.1.

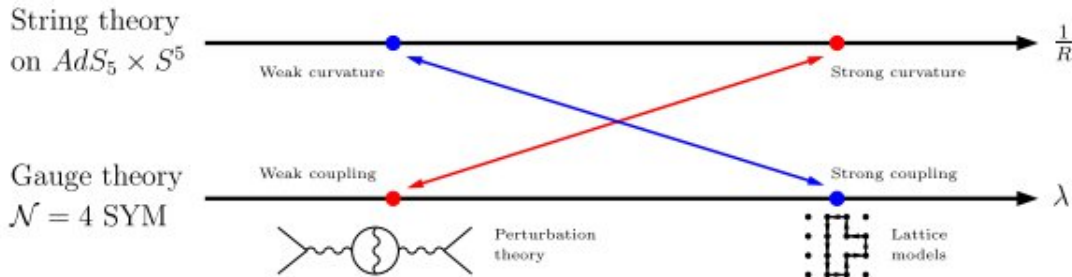


Figure 1.1: The hypothetical duality between four-dimensional $\mathcal{N} = 4$ super-Yang-Mills (SYM) and type IIB superstring theory on $AdS_5 \times S^5$, plotted as a function of the radius R of $AdS_5 \times S^5$, or equivalently the 't Hooft coupling constant λ of the gauge theory. The lines connect equivalent regions on both sides. The full story is more complex since both theories depend on two parameters, namely the number of colours, $N = N_c$, and the gauge coupling. Note that the number of colors N_c in the gauge theory corresponds to the number of coincident D-branes N in the string/gravitational theory. Furthermore, since lattice models are non-perturbative, they can be used for all values of λ .

Thus, the quantities that can be easily calculated in a perturbative expansion in the coupling constant in one theory become computationally intractable in the other. This is, on one hand, a blessing, since accepting the duality as a working hypothesis makes it possible for us to analytically investigate such gauge theory phenomena as heavy ion collisions [5] and strongly correlated condensed matter systems [6] that are usually only treatable via computer simulations, but on the other hand, a problem, because it makes a proof of the conjectured duality that much harder.

While a mathematically tight and rigorous demonstration of the validity of the correspondence remains elusive, much progress has been made in understanding the various ramifications of this conjecture. This dissertation is essentially an eclectic compendium of various theoretical investigations in the context of gauge/gravity duality carried out by the author, and that progressed to a publishable state. Admittedly, no attempt has been made to weave a common story that underlies the different topics addressed, although the unifying theme remains gauge/gravity duality. Instead, each of the three research projects is introduced and elaborated individually in separate chapters, with every effort having been made to make each chapter as comprehensive as necessary.

1.1 Overview of thesis

Our exploration begins in **Chapter 2** with a rather extensive analysis within the framework of AdS_3/CFT_2 correspondence. In particular, we focus on the type IIA

$AdS_3 \times S^3 \times S^3 \times S^1$ Green-Schwarz (GS) superstring action up to quadratic order in fermions, and discuss issues related to fixing its κ -symmetry. We subsequently derive the near-BMN expansion of the GS Lagrangian; as a first quantum consistency check, we show that the one-loop corrections to the two-point correlation functions for the bosonic fields are finite in dimensional regularisation – both the three- and four-vertex diagrams are separately divergent, but their summation is finite. We then concern ourselves with the classical, or tree-level, sector of the string theory, where we perform a Hamiltonian analysis and compare $SU(2)$ string states with predictions from a conjectured set of Bethe equations. We find perfect agreement for the rank-one sectors, while we do not fully understand how to match the product $SU(2) \times SU(2)$ sector. As it turns out, the string energies arising from the mixing sector exactly cancel between cubic and quartic interaction pieces. This means that, in order for the Bethe equations to reproduce the string calculation, the rank-one Bethe equations should decouple completely. We are not sure how to interpret this result, and further investigation is probably needed. Finally, in the last section, we look at $2 \rightarrow 2$ tree-level scattering processes for the bosonic fields on the worldsheet in an attempt to understand how to properly include the massless modes in the exact solution. We show that, at least at tree-level, the two-body S-matrix is reflectionless. This might be a useful finding if the Bethe equations have to be extended in order to incorporate the massless modes of the theory as fundamental excitations. *All of the work in this chapter is original and based on the paper [7]*

We switch gears and move to a completely different topic in **Chapter 3**, this time specifically in the context of AdS_5/CFT_4 correspondence. The primary motivation is to understand how different classical BPS states corresponding to 3-branes configurations with different topologies can be obtained from holomorphic surfaces via a method of construction, proposed by Mikhailov [103, 104], involving the introduction of a class of supersymmetric cycles in spacetimes of the form $AdS_m \times S^n$, which can be considered as generalisations of the giant gravitons. In the case where the spherical manifold is the five-sphere in $AdS_5 \times S^5$, branes wrapped on these cycles preserve 1/2, 1/4 or 1/8 of the supersymmetry, and constitute a class of configurations parametrised by the holomorphic curves in \mathbb{C}^3 . The starting point is the fact that those wrapped branes are giant gravitons, which are dual to subdeterminant operators and generalisations thereof known as Schur polynomials. Then comes the realisation that the worldvolume of a $D3$ -brane could, in principle, assume any possible topology $\mathbb{R} \times \mathcal{M}$, with \mathcal{M} being a closed 3-manifold. Such manifolds can have considerably complicated topology, and it would seem, via the gauge/gravity duality, to be a fascinating idea to ultimately understand the *emergence of topology* from the point of view of the dual super-Yang-Mills operators. Motivated by this, the goal in this chapter is to explore what topological configurations giants, whose dynamics preserve a certain amount of supersymmetry, assume. We are particularly interested in solutions created by a localised modification of a set of intersecting spherical giant gravitons, as this seems the most tractable limit. *The results in this chapter are original, and reported in [105].*

Finally, in **Chapter 4**, our attention turns to the study of some aspects of holographic particle-vortex duality. In particular, we focus on its realisation in the Aharony-Bergman-Jafferis-Maldacena (ABJM) model [8], and its possible relation to Maxwell duality in AdS_4 , via the AdS_4/CFT_3 correspondence. By combining

a path-integral version of particle-vortex duality with the Mukhi-Papageorgakis Higgs mechanism [126], we arrive at a symmetric version of the transformation that acts as a self-duality. We then proceed to show how to embed it as an abelian duality in the $(2 + 1)$ -dimensional, $\mathcal{N} = 6$ super-Chern-Simons-matter theory that is the ABJM model, and speculate on a possible non-abelian extension. Going to the gravity side of the correspondence, Maxwell duality in AdS_4 is found to reduce on the boundary to a particle-vortex duality acting on two independent gauge field sources and their associated currents. The main motivation for this work is two-fold: first, to understand whether particle-vortex duality is realised in the ABJM model with its rich solitonic spectrum and second, to see if the phenomenological work of [123] could be embedded in the concrete setting of the $AdS_4 \times CP^3$ /ABJM correspondence. *The work in this chapter is original and contained in [133].*

Having dispensed with all formalities, let us go exploring...

Chapter 2

Near-BMN dynamics of the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring

2.1 Introduction

As we mentioned previously, gauge/string dualities offer a fundamentally new perspective on how to understand strongly coupled systems [1, 2, 3, 4]. The best studied version remains the original example of AdS_5/CFT_4 [1, 3] which relates type IIB string theory on an $AdS_5 \times S^5$ background to $\mathcal{N} = 4$, $SU(N)$ super-Yang-Mills theory on the four-dimensional boundary of AdS_5 . Another more recent incarnation is AdS_4/CFT_3 , this time relating (in a certain limit) type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ to a three-dimensional Chern-Simons-matter theory [8]. A rather remarkable fact is that most of the theoretical tools developed for AdS_5/CFT_4 turn out to apply almost identically in the more recent AdS_4/CFT_3 duality. As is by now well known, the underlying reason for this similarity of seemingly different theories is the existence of *integrable structures*; or, in other words, the existence of an infinite set of conserved charges which, in principle, allows for an exact solution of the spectral problem.

The language of integrability allows for a reformulation of the spectral problem in terms of an abstract spin chain. The Hamiltonian acting on the spin chain can then be diagonalised using Bethe Ansatz (BA) techniques which allows us to arrive at the spectrum in a closed form (see [9] for a recent review on the subject.) In the $AdS_4 \times \mathbb{CP}^3$ case, there is one subtlety, however, which was not present in the $AdS_5 \times S^5$ case, and which is related to the fact that $AdS_4 \times \mathbb{CP}^3$ is not a maximally supersymmetric solution of the supergravity equations of motion. The standard proof of integrability of the string worldsheet theory [10] relies on a supercoset formulation. The supercoset sigma model [11, 12] can be obtained from the complete Green-Schwarz (GS) superstring [13] by (partial) gauge-fixing of the κ -symmetry. It turns out, however, that for certain configurations of the string, this gauge-fixing becomes inconsistent [11, 13], and the supercoset model is not capable of capturing all physical fermionic degrees of freedom of the string. This is the case, for example, when the string moves only in the AdS_4 subspace, or forms an instanton by wrapping $\mathbb{CP}^1 \subset \mathbb{CP}^3$ [14]. This suggests that a more general proof of integrability should be sought that does not rely on the supercoset

description. The first steps in this direction were taken in [15], where the classical integrability of the full GS superstring was demonstrated to quadratic order in fermions¹ (see also [16], and for a slightly different approach see [17])².

In [19], an analysis of the integrable structures of yet another duality, namely AdS_3/CFT_2 , was initiated. On the string theory side of the duality, we have either $AdS_3 \times S^3 \times T^4$ or $AdS_3 \times S^3 \times S^3 \times S^1$ supported by Ramond-Ramond (RR) flux. For the first background, the dual CFT_2 should be a two-dimensional sigma model on a moduli space built out of Q_1 instantons in a $U(Q_5)$ gauge theory on T^4 . This is somewhat natural since $AdS_3 \times S^3 \times T^4$ arises as the near-horizon limit of Q_1/Q_5 intersecting D_1/D_5 branes, [20, 21, 22, 23, 24, 25, 26]. On the other hand, the dual theory of $AdS_3 \times S^3 \times S^3 \times S^1$ remains largely unknown, mainly due to the fact that the supergravity approximation fails to be as useful as in the other examples (see [27]). Nevertheless, it is possible to write down a supercoset sigma model for this case whose classical equations of motion allow for a Lax representation which ensures classical integrability [19]. By integrating the Lax connection around a closed loop, one gets the monodromy matrix which can be used to generate an infinite tower of conserved charges. The finite-gap method can then be used to reformulate the equations of motion in terms of a set of integral equations [28, 29]. These integral equations in turn can be seen as the semiclassical limit of a set of conjectured quantum Bethe equations which diagonalises the exact S-matrix on the worldsheet [30, 31].

While the $AdS_3 \times S^3 \times M_4$ solutions of the supergravity equations of motion allow for pure Neveu-Schwarz–Neveu-Schwarz (NS-NS) flux, and incidentally allows for an exact solution using the representations of chiral algebras [20, 32, 33, 34], the string appearing in the AdS_3/CFT_2 considered here is supported by RR flux. This implies that the proper description is the GS superstring, which is more complicated. What is more, the superisometries of AdS_5/CFT_4 and AdS_4/CFT_3 have 32 and 24 supercharges respectively, while the duality at hand has only 16, making it even less symmetric than the higher-dimensional examples of integrable gauge/string dualities. Here, we shall work with the GS string action up to quadratic order in the fermions. We will see that the subtleties that appear in the $AdS_4 \times CP^3$ case are also present here. In this case, the κ -symmetry gauge-fixing which reduces the GS string to the supercoset model becomes inconsistent when the string moves only in $AdS_3 \subset AdS_3 \times S^3 \times S^3 \times S^1$ or in $AdS_3 \times S^3 \subset AdS_3 \times S^3 \times T^4$. However, we expect that, just as in $AdS_4 \times CP^3$ and $AdS_2 \times S^2 \times T^6$, it should be possible to prove the classical integrability of the full GS string to quadratic order in fermions also in this case along the lines of [15, 16], although we will not address this question here.

The (super)isometries of $AdS_3 \times S^3 \times S^3$ form two copies of $D(2, 1; \alpha)$, which is an exceptional supergroup with a free parameter $\alpha \in [0, 1]$ [35]³. The parameter α

¹The integrability was also shown to higher order in fermions in a truncated model

²A similar problem appears in $AdS_2 \times S^2 \times T^6$ except there, the supercoset model never describes all the physical fermions due to the low amount of supersymmetry; nevertheless the integrability of the GS action has been shown to hold to quadratic order in fermions [18, 16].

³The S^1 factor is not described by the supergroup and it has to be added by hand. One might be tempted to take it as a completely decoupled term in the Lagrangian, but this is, however, not the case in the supersymmetric formulation since the fermions couple to all transverse directions through the vielbeins.

enters the invariant bilinear form and can be related to the background geometry through the relation

$$\frac{1}{R_+^2} + \frac{1}{R_-^2} = \frac{1}{R^2} \quad (2.1)$$

where R_{\pm} are the radii of the three-spheres, and R is the AdS_3 radius. This allows for a trigonometric parametrisation as

$$\alpha = \frac{R^2}{R_+^2} = \cos^2 \phi, \quad \frac{R^2}{R_-^2} = \sin^2 \phi.$$

A few special cases are worth mentioning. If we take one of the S^3 radii to infinity, we effectively decompactify that part of the geometry. In other words, starting from the $AdS_3 \times S^3 \times S^3 \times S^1$, and sending one of the R_{\pm} with the S^1 radius (which is arbitrary) to infinity (or, equivalently, taking the $\alpha \rightarrow 0$ limit), we would end up with $AdS_3 \times S^3 \times T^4$. Hence, it should be possible to write down a sigma model parametrised by ϕ (or, equivalently, α) that can incorporate both backgrounds in one unified description. Indeed, this was done in [19]. What is more, the finite-gap method was used to propose a set of quantum Bethe equations [19] for $\alpha = \{0, \frac{1}{2}, 1\}$ (corresponding to the T^4 and equal S^3 radii cases). These were subsequently generalised to arbitrary α in [36]. One motivation for this work is to compare and augment the proposals of [19, 36] with explicit string calculations. While computations have been performed on the string theory side for the $AdS_3 \times S^3 \times T^4$ case [39, 40, 41, 42, 43, 44], very little has been explicitly computed for general α .⁴

In order to perform any worldsheet calculation, the string Lagrangian is needed, and we will use a near-BMN expansion up to quartic order in the number of fields (but only quadratic order in fermions). As an independent consistency check, we show that the theory is finite at one-loop order in dimensional regularisation. As a first test of the conjectured Bethe equations, we compare their predictions with the energies of string states. While this is only a tree-level computation, it is nevertheless an important consistency check to verify that the spectrum agrees. As we will show, we find at least partial agreement. For the rank-one sectors, agreement is perfect for arbitrary number of string oscillators (or equivalently, Bethe roots). However, looking at a product $SU(2) \times SU(2)$ sector, we find that the string energies in the mixing sector cancel between cubic and quartic contributions from the string Hamiltonian. This implies that, in order for the Bethe equations to reproduce the string calculation, the length parameter L of the Bethe equations should not mix the two sectors. This does not necessarily conflict with the results of [19], since these effects would be subleading in L . Thus, they should not change the semiclassical, $L \rightarrow \infty$, limit and the integral equations of [19] should still be reproduced.

We then set out to investigate some properties of the worldsheet S-matrix. In the string sigma model, there are heavy, light and massless modes. While the first two are incorporated by the Bethe equations as fundamental and composite excitations, the massless modes are absent. They do however appear as internal states (as intermediate lines in Feynman diagrams), but it is not possible to assign explicit excitation numbers to them. Thus, it might be desirable to extend the

⁴However, see the recent paper [45] where one-loop effects of spinning and folded string configurations are studied.

Bethe equations in a way so that this is possible. We address this question by showing that the reflection part of the worldsheet S-matrix is zero. This is a nice feature since it, in principle, makes it rather straightforward to add the massless modes by hand as a direct sum. We also collect all the remaining light-to-light bosonic scattering processes in the appendix.

Outline

We start out, in **section 2.2**, by writing down the GS superstring action to quadratic order in fermions using geometric quantities such as the vielbeins, the spin connection and the RR flux. We then discuss the κ -symmetry gauge-fixing of the action and show that in certain cases, the κ -symmetry fixing which would lead to the supercoset sigma model is not admissible, implying that for certain string configurations, the supercoset model is not able to describe all the physical fermionic degrees of freedom on the worldsheet. We use the standard light-cone-type κ -gauge, which does not suffer from this problem.

In **section 2.3**, we turn to a perturbative analysis, where we first fix the bosonic light-cone gauge [46, 48]. We then expand in transverse bosonic and fermionic string coordinates, and write down the theory up to quartic order in fields (but only quadratic order in fermions). As a first consistency check, we show that the theory is one-loop finite in dimensional regularisation.

We then provide an analysis of the Hamiltonian in **section 2.4** by comparing the Bethe equations of [36] with the string energies. Since the string Hamiltonian has both cubic and quartic interaction terms, the actual computation boils down to second-order perturbation theory. This, however, can be reformulated in terms of an equivalent first-order computation by utilising a canonical, or unitary, transformation of the Hamiltonian [48, 50]. The classical⁵ energies we compare with the Bethe equations, come from string states in an $SU(2)$ and $SU(2) \times SU(2)$ subsector. While we find complete agreement for the rank-one sector, we nevertheless observe that there are some issues with the product sector. In order for the Bethe equations to reproduce our findings, the length parameter L needs to be different for the two sectors. While we do not fully understand the implications of this, one possible explanation is that we simply have two disconnected spin chains.

In **section 2.5**, we show that the reflection piece of the bosonic worldsheet S-matrix is zero. We show this explicitly by computing $2 \rightarrow 2$ scatterings of bosonic fields on the string worldsheet. While we only present a tree-level computation here, we suspect this to be true in the quantum case also. In the appendix, we also compute the scattering and transmission part of the bosonic S-matrix. However, since the exact S-matrix is not known, we are not in a position to compare our findings with anything.

We end with a short summary and discussion about interesting future problems in **section 2.6**.

⁵By "classical", we mean that we ignore normal-ordering effects which, together with terms arising from the unitary transformation, should combine into finite-size effects, see [51, 52].

2.2 Green-Schwarz superstring in $AdS_3 \times S^3 \times S^3 \times S^1$

We begin by writing down the Green-Schwarz (GS) superstring action, up to quadratic order in fermions, using geometric quantities such as the vielbeins, the spin connection and the RR flux. We then discuss the κ -symmetry gauge-fixing of the action and show that in certain cases, the κ -symmetry fixing that would lead to the supercoset sigma model is not admissible, which implies that for certain string configurations, the supercoset model is not able to describe all the physical fermionic degrees of freedom on the worldsheet. We use the standard light-cone type κ -gauge, which does not suffer from this problem. For notational details, see appendix A.3.

2.2.1 GS superstring to quadratic order in fermions in a type II supergravity background

The GS superstring action in a type II supergravity background, with zero background fermionic fields and NSNS flux, and constant dilaton ϕ_0 , takes the following form, up to quadratic order in fermions [53, 54]⁶:

$$S = -g_s \int \left(\frac{1}{2} * e^A e_A + i * e^A \bar{\Theta} \Gamma_A \mathcal{D} \Theta - i e^A \bar{\Theta} \Gamma_A \hat{\Gamma} \mathcal{D} \Theta \right) \quad \text{where } \hat{\Gamma} = \begin{cases} \Gamma_{11} & \text{[IIA]} \\ \mathbb{1} \times \sigma^3 & \text{[IIB]} \end{cases} \quad (2.2)$$

The $e^A(X)$, $A = 0, 1, \dots, 9$, are worldsheet pullbacks of the vielbein one-forms of the purely bosonic part of the background, $*$ denotes worldsheet Hodge-dualisation, and the generalised covariant derivative acting on the fermions is given by

$$\mathcal{D} \Theta = \left(\nabla - \frac{1}{8} e^A \not{F} \Gamma_A \right) \Theta \quad \text{where } \nabla \Theta = \left(d - \frac{1}{4} \Omega^{AB} \Gamma_{AB} \right) \Theta, \quad (2.3)$$

where Ω^{AB} is the spin connection of the background spacetime. The coupling to the RR fields comes through the matrix

$$\not{F} = e^{\phi_0} \times \begin{cases} -\frac{1}{2} \Gamma^{AB} \Gamma_{11} F_{AB} + \frac{1}{4!} \Gamma^{ABCD} F_{ABCD} & \text{[IIA]} \\ i \sigma^2 \Gamma^A F_A - \frac{1}{3!} \sigma^1 \Gamma^{ABC} F_{ABC} + \frac{i}{2 \cdot 5!} \sigma^2 \Gamma^{ABCDE} F_{ABCDE} & \text{[IIB]} \end{cases} \quad (2.4)$$

in the type IIA and type IIB cases, respectively.

The two Majorana-Weyl spinors in the IIA case are described as one 32-component Majorana spinor Θ , and in the IIB case, as two 32-component Majorana spinors projected onto one chirality $\Theta^I = \frac{1}{2}(1 + \Gamma_{11}) \Theta^I$, $I = 1, 2$. The Pauli matrices σ^i , $i = 1, 2, 3$, act on the IIB $SO(2)$ indices $I, J = 1, 2$, which will be suppressed. The Majorana condition implies that the conjugate spinors satisfy

$$\bar{\Theta} = \Theta^\dagger \Gamma_0 = \Theta^t C \quad (2.5)$$

⁶The string coupling g_s is related to the background geometry as $g_s \sim \sqrt{\lambda} = \frac{R^2}{\alpha'}$, where R is the AdS curvature radius. How $\sqrt{\lambda}$ is to be defined in terms of the scale of the AdS space dual to CFT_2 is not yet known. In most equations, we set $g_s = 1$ for simplicity.

where C is the *charge conjugation matrix* (A.2) (when needed, we use the Γ -matrix representation given in appendix A.2). We now turn to the specific background of interest here, $AdS_3 \times S^3 \times S^3 \times S^1$.

2.2.2 GS string in type IIA $AdS_3 \times S^3 \times S^3 \times S^1$

There are two type II supergravity solutions of the form $AdS_3 \times S^3 \times S^3 \times S^1$ with RR flux. One is in type IIB and has F_3 flux, while the other is in type IIA and has F_4 flux. The type IIB solution arises as the near-horizon geometry of intersecting $D1$ - and $D5$ -branes. Both solutions preserve 16 supersymmetries, and they are easily seen to be related by a T-duality along the S^1 direction. Since the fermions in the type IIA case can be grouped into a single 32-component Majorana spinor, this case is slightly easier to work with, and since both backgrounds describe the same physics, we will work with this case.

The $AdS_3 \times S^3 \times S^3 \times S^1$ solution to type IIA supergravity is supported by RR four-form flux of the form

$$F_4 = 2 e^{-\phi_0} \left(\frac{1}{3!} e^c e^b e^a \epsilon_{abc} + \cos \phi \frac{1}{3!} e^{\hat{c}} e^{\hat{b}} e^{\hat{a}} \epsilon_{\hat{a}\hat{b}\hat{c}} + \sin \phi \frac{1}{3!} e^{c'} e^{b'} e^{a'} \epsilon_{a'b'c'} \right) e^9, \quad (2.6)$$

where we use units that set the AdS_3 radius to unity. The ten-dimensional space-time index $A = 0, \dots, 9$ splits up into four sets of indices: an AdS_3 index $a = 0, 1, 2$, the first and second set of S^3 indices $\hat{a} = 3, 4, 5$ and $a' = 6, 7, 8$, and the S^1 index 9. The vielbein $e^9 = dy$, where y is the S^1 coordinate.

On substituting (2.6) into (2.4), we obtain

$$\not{F} = 4\Gamma^* \Gamma^9 (1 - \mathcal{P}), \quad (2.7)$$

where $\Gamma^* = \frac{1}{3!} \Gamma^{cba} \epsilon_{abc} = \Gamma^{012}$, $(\Gamma^*)^2 = \mathbb{1}$ and \mathcal{P} is a projection matrix

$$\mathcal{P} = \frac{1}{2} (1 + \cos \phi \Gamma^* \Gamma^{345} + \sin \phi \Gamma^* \Gamma^{678}). \quad (2.8)$$

that singles out the 16 supersymmetries preserved by the background. To see this, one can look at the supersymmetry variation of the dilatino

$$\delta_\epsilon \lambda = \Gamma^A \not{F} \Gamma_A \epsilon = 8\Gamma^* \Gamma^9 (1 - \mathcal{P}) \epsilon.$$

which vanishes for the 16 supersymmetry parameters which satisfy $\epsilon = \mathcal{P}\epsilon$, and therefore correspond to those supersymmetries preserved by the background. Correspondingly, the 32 fermions Θ can be split into two sets of 16 each, by acting the projection operator on it in the following ways:

$$\vartheta = \mathcal{P}\Theta \quad \text{and} \quad \nu = (1 - \mathcal{P})\Theta.$$

The 16 ϑ 's correspond to the supersymmetries preserved by the background, and the 16 ν 's to the broken supersymmetries⁷.

⁷Those corresponding to the preserved supersymmetries are referred as *coset fermions*, while those corresponding to the broken supersymmetries are called *non-coset fermions* since a super-coset formulation only describes the fermions which correspond to unbroken supersymmetries.

Substituting (2.7) in the action (2.2), we find that the part of the Lagrangian quadratic in the fermionic fields takes the form (from now on we drop the worldsheet form notation)

$$\begin{aligned} \mathcal{L}_{(2,f)} = & i\gamma^{ij}e_i^A \bar{\Theta} \Gamma_A \nabla_j \Theta - i\epsilon^{ij}e_i^A \bar{\Theta} \Gamma_A \Gamma_{11} \nabla_j \Theta - \frac{i}{2}\gamma^{ij}e_i^A e_j^B \bar{\Theta} \Gamma_A \Gamma^{0129} (1 - \mathcal{P}) \Gamma_B \Theta \\ & + \frac{i}{2}\epsilon^{ij}e_i^A e_j^B \bar{\Theta} \Gamma_A \Gamma_{11} \Gamma^{0129} (1 - \mathcal{P}) \Gamma_B \Theta, \end{aligned}$$

where i, j are worldsheet indices and $\gamma^{ij} = \sqrt{-g}g^{ij}$ is the Weyl-invariant worldsheet metric satisfying $\det \gamma = -1$.

2.2.3 κ -symmetry gauge fixing

The GS superstring action is invariant under local fermionic transformations of the target space coordinates $Z^M = (X^M, \Theta^\mu)$ which take the form

$$\begin{aligned} \delta_\kappa Z^M \mathcal{E}_M^\alpha &= \frac{1}{2}(1 + \Gamma)^\alpha_\beta \kappa^\beta(\xi), \quad \alpha, \beta = 1, \dots, 32, \\ \delta_\kappa Z^M \mathcal{E}_M^A &= 0, \quad A = 0, 1, \dots, 9, \end{aligned}$$

where $\kappa^\beta(\xi)$ is an arbitrary 32-component spinor parameter, $(\mathcal{E}^A, \mathcal{E}^\alpha)$ are the background supervielbeins and $\frac{1}{2}(1 + \Gamma)^\alpha_\beta$ is a spinor projection matrix with

$$\Gamma = \frac{1}{2\sqrt{-\det g_{ij}}} \epsilon^{ij} \mathcal{E}_i^A \mathcal{E}_j^B \Gamma_{AB} \Gamma_{11}, \quad \text{and} \quad \Gamma^2 = 1,$$

$g_{ij} = \mathcal{E}_i^A \mathcal{E}_j^B \eta_{AB}$ being the induced metric on the worldsheet.

This κ -symmetry can gauge away 16 of the 32 fermions, but exactly which ones may depend on the motion of the string since Γ depends on this through the pullback of the supervielbeins \mathcal{E}_i^A . Let us consider a gauge-fixing of the form

$$M\Theta = 0,$$

where M is some 32×32 matrix which forces some n -dimensional projection of Θ to vanish ($n \leq 16$). By analysing a (linearised) κ -symmetry transformation of this gauge-fixing condition, using the fact that $\mathcal{E}_\mu^\alpha = \delta_\mu^\alpha + \mathcal{O}(\Theta^2)$, one finds that for the gauge-fixing to be admissible, there are essentially two possibilities⁸: either M coincides with the κ -symmetry projector $\frac{1}{2}(1 + \Gamma)$ in an n -dimensional subspace of the space it projects onto, *or* M is independent of Γ but⁹

$$\text{rank}([M, \Gamma]) \geq \frac{n}{2}.$$

(see also the discussion in section 3 of [55]). Let us now consider the implications of this fact for the present case.

In order to describe the string as a supercoset sigma model, we choose the κ -symmetry gauge-fixing that removes the 16 non-coset fermions

$$v = (1 - \mathcal{P})\Theta = 0.$$

⁸In principle, intermediate cases could be considered, but they will not be relevant here.

⁹This is a necessary, but not always sufficient, condition ($M\frac{1}{2}(1 + \Gamma)$ still has to have rank n).

According to the above discussion, and using the form of \mathcal{P} in (2.8), we see that, for generic ϕ , this gauge choice is not possible if the string moves only in the AdS_3 subspace since, in that case,

$$[\mathcal{P}, \Gamma] = 0 \quad \Rightarrow \quad \text{rank}([\mathcal{P}, \Gamma]) = 0 < 8.$$

For the special case $\phi = 0$, that is $AdS_3 \times S^3 \times T^4$, the situation is worse, and the gauge-fixing is inconsistent if the string motion is in the $AdS_3 \times S^3$ subspace. The same holds, of course, for the opposite gauge-fixing, which sets the coset fermions to zero, namely $\mathcal{P}\Theta = 0$. We conclude from this that, for these string configurations, the set of physical fermions consists of *eight coset* fermions related to conserved supersymmetries and *eight non-coset* ones related to broken supersymmetries. Thus, not all physical fermions for these string configurations are captured by the supercoset sigma model. Essentially the same issue arises in the case of the $AdS_4 \times CP^3$ superstring [11, 13].

For that reason, we will avoid using the gauge that gives the supercoset model which was used in [19]. We will instead be interested in the BMN-expansion around a string moving along an S^1 in the first S^3 factor and an S^1 in the second S^3 so that the angle subtended between them is β . The case $\beta = 0$ corresponds to the string moving along only an S^1 in the first S^3 , while $\beta = \pi/2$ corresponds to the (essentially equivalent) case of the string moving only in the second S^3 . The κ -gauge we will impose is therefore the standard one involving the light-cone Γ -matrices adapted to the BMN geodesic:

$$\Gamma^+ \Theta = 0, \quad \text{where} \quad \Gamma^\pm = \frac{1}{2} \left[\Gamma^0 \pm \left(\cos \beta \Gamma^5 + \sin \beta \Gamma^8 \right) \right]. \quad (2.9)$$

The matrix M used in the κ -gauge fixing can be thought of as the projection matrix $-4\Gamma^-\Gamma^+$, and it is not hard to see that for this string configuration, it coincides with the κ -symmetry projection matrix $\frac{1}{2}(1 + \Gamma)$ in the 16-dimensional subspace of positive-chirality spinors. Since the chirality projector $\frac{1}{2}(1 + \Gamma_{11})$ commutes with the κ -symmetry projector, this gives only 8 instead of the 16 gauge conditions needed. It, therefore, appears that this standard gauge-fixing would be incomplete. The resolution of this puzzle is that, when we also fix the bosonic light-cone gauge, $x^+ \sim \tau$, x^- is fixed by the Virasoro conditions in terms of the other fields and this turns out to remove any would-be freedom to perform further κ -symmetry transformations. Therefore, consistency with the Virasoro conditions guarantees that the gauge-fixing is complete also in this case.

2.3 Light-cone BMN expansion of the action

We will study an expansion in transverse coordinates utilising a BMN-type expansion [56]. First, we consider the lowest-order quadratic theory with β , the angle the geodesic makes in the (5, 8)-plane, arbitrary, and then, when going to higher order in perturbation theory, we will consider only the case $\beta = \phi$ for simplicity.

We first fix the residual bosonic worldsheet symmetries. We impose a uniform light-cone gauge [46, 48], where we introduce the light-cone coordinates adapted

to the BMN geodesic

$$\begin{aligned} x^\pm &= \frac{1}{2} [t \pm (\cos \beta \varphi_5 + \sin \beta \varphi_8)] , \\ t &= x^+ + x^- , \\ v &= \sin \beta \varphi_5 - \cos \beta \varphi_8 , \\ \varphi_5 &= \cos \beta (x^+ - x^-) + \sin \beta v , \\ \varphi_8 &= \sin \beta (x^+ - x^-) - \cos \beta v , \end{aligned}$$

with φ_5 and φ_8 being the relevant angular coordinates of $S^3 \times S^3$. The use of light-cone gauge means that we align the worldsheet time coordinate with x^+ through

$$x^+ = \tau , \quad p^+ = \text{constant} ,$$

where p^+ is the conjugate worldsheet momentum density of x^- . In the near-BMN limit, the gauge-fixed Lagrangian has an expansion in the number of transverse fields as

$$\mathcal{L} = \mathcal{L}_2 + \frac{1}{\sqrt{g}} \mathcal{L}_3 + \frac{1}{g} \mathcal{L}_4 + \dots$$

where the subscripts denote the number of transverse coordinate in each term. To leading orders in perturbation theory, this gauge is also consistent with the conformal gauge, that is, a flat worldsheet metric. However, it fails to hold at quartic order in the transverse field expansion, and we need to add higher-order corrections to the worldsheet metric.

2.3.1 Quadratic Lagrangian with arbitrary β

Bosonic part

In the BMN limit, parametrised by the angle β , the bosonic terms in the Lagrangian (2.2) reduce, at quadratic order in fields and using conformal gauge $\gamma^{ij} = \eta^{ij}$, to (see appendix A.3 for the parametrisation)

$$\begin{aligned} \mathcal{L}_{(2,b)} &= -\frac{1}{2} \left[\left(\sum_{j=1}^4 \partial_i x_j \partial^i x_j \right) + \partial_i x_6 \partial^i x_6 + \partial_i x_7 \partial^i x_7 + \partial_i y_j \partial^i y_j + \partial_i v_j \partial^i v_j \right. \\ &\quad \left. - \left(x_1^2 + x_2^2 + \cos^2 \beta \cos^2 \phi (x_3^2 + x_4^2) + \sin^2 \beta \sin^2 \phi (x_6^2 + x_7^2) \right) \right] \end{aligned}$$

The spectrum consists of four pairs of bosons with masses

$$m = (1, \cos \beta \cos \phi, \sin \beta \sin \phi, 0) . \quad (2.10)$$

Fermionic part

We now turn to the fermionic terms. From (2.3) and (2.7), to leading order, the contributing pieces of the vielbein and \mathbb{F} are

$$\begin{aligned} e^A \Gamma_A &= \Gamma_+ dx^+, \\ \mathbb{F} &= 2\Gamma^{12+9} \left(1 + \cos \beta \cos \phi \Gamma^{1234} + \sin \beta \sin \phi \Gamma^{1267} \right) \\ &= 4\Gamma^{12+9} \sum_{i,j=\pm} m_{ij} P_{ij}, \end{aligned} \quad (2.11)$$

where Γ_+ is defined in (A.5) and

$$P_{\pm\pm} = \frac{1}{4} \left(1 \pm \Gamma^{1234} \right) \left(1 \pm \Gamma^{1267} \right), \quad m_{\pm\pm} = \frac{1}{2} \left(1 \pm \cos \beta \cos \phi \pm \sin \beta \sin \phi \right).$$

$P_{\pm\pm}$, which are products of two commuting projectors that project onto 16-dimensional subspaces, project onto an 8-dimensional subspace. On fixing the κ , light-cone and conformal gauges, the lowest order Lagrangian for the fermions becomes¹⁰

$$\mathcal{L}_{(2,f)} = i\bar{\Theta} \Gamma_+ \left(\partial_0 \Theta - \Gamma^{11} \partial_1 \Theta + \sum_{i,j=\pm} m_{ij} \Gamma^{129} P_{ij} \Theta \right). \quad (2.12)$$

Thus, we see that, for generic β and ϕ , there are four two-component fermions $\Theta_{\pm\pm} = P_{\pm\pm} \Theta$ of mass $\{m_{++}, m_{+-}, m_{-+}, m_{--}\}$. When $\beta = \phi$, we have four pairs with masses

$$m = \left(1, \cos^2 \phi, \sin^2 \phi, 0 \right), \quad (2.13)$$

which coincide with the bosonic mass spectrum (2.10), hence the maximum amount of worldsheet supersymmetry is preserved in this case. In order to simplify our analysis, we will only consider the $\beta = \phi$ case from now on.

2.3.2 Quadratic and cubic Lagrangians with $\beta = \phi$

We specialise to the $\beta = \phi$ case [19]. We first introduce new variables in terms of which (2.12) takes a nice two-dimensional form. This can be done using the explicit representation of Θ in appendix A.3, together with

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{2}}(x_1 - ix_2), & y_2 &= \frac{1}{\sqrt{2}}(x_3 - ix_4), & y_3 &= \frac{1}{\sqrt{2}}(x_6 - ix_7), & y_4 &= \frac{1}{\sqrt{2}}(v - ix_9), \\ \chi_1^\pm &= \cos \frac{\phi}{2} \theta_1^\pm + \sin \frac{\phi}{2} \theta_3^\pm, & \chi_2^\pm &= -\cos \frac{\phi}{2} \theta_2^\pm + \sin \frac{\phi}{2} \theta_4^\pm, \\ \chi_3^\pm &= \sin \frac{\phi}{2} \theta_2^\pm + \cos \frac{\phi}{2} \theta_4^\pm, & \chi_4^\pm &= \sin \frac{\phi}{2} \theta_1^\pm - \cos \frac{\phi}{2} \theta_3^\pm. \end{aligned}$$

¹⁰ $\Gamma^+ \Theta = 0$, $x^+ = \tau$, $\gamma^{ij} = \eta^{ij} = (+-)$, and $\varepsilon^{01} = 1$.

On rescaling the fermionic fields $\chi_i^\pm \rightarrow \frac{1}{2\sqrt{2}}\chi_i^\pm$, and using $\partial_\pm = (\partial_0 \pm \partial_1)$, we get

$$\begin{aligned} \mathcal{L}_{(2,f)} = \sum_{i=1}^4 \left\{ \frac{1}{2} (\partial_+ y_i \partial_- \bar{y}_i + \partial_- y_i \partial_+ \bar{y}_i) + i (\bar{\chi}_i^+ \partial_- \chi_i^+ + \bar{\chi}_i^- \partial_+ \chi_i^-) \right. \\ \left. - [m_i^2 y_i \bar{y}_i + m_i (\bar{\chi}_i^+ \chi_i^- + \bar{\chi}_i^- \chi_i^+)] \right\} \end{aligned} \quad (2.14)$$

where $m_i = (1, \cos^2 \phi, \sin^2 \phi, 0)$, $i = 1, \dots, 4$. Thus, all in all, we have $8_B + 8_F$ which come in pairs of equal masses.

The conformal and light-cone gauges are also compatible at cubic order, where \mathcal{F} is still effectively given by (2.11); expanding the Lagrangian (2.2) we get

$$\begin{aligned} \mathcal{L}_{(3,f)} = \frac{1}{2\sqrt{2}} \sin 2\phi \left[-\cos^2 \phi (\bar{\chi}_4^- \bar{\chi}_2^- - \bar{\chi}_1^- \bar{\chi}_3^- + \bar{\chi}_1^+ \bar{\chi}_3^+ - \bar{\chi}_4^+ \bar{\chi}_2^+) y_2 \right. \\ - i \sin^2 \phi (\chi_3^- \bar{\chi}_4^- + \chi_2^- \bar{\chi}_1^- + \chi_3^+ \bar{\chi}_4^+ + \chi_2^+ \bar{\chi}_1^+) y_3 \\ - 2 (\chi_2^- \bar{\chi}_3^+ + \chi_2^+ \bar{\chi}_3^-) y_1' + 2 (\chi_2^- \bar{\chi}_2^+ - \chi_3^+ \bar{\chi}_3^-) y_4 \\ + (\chi_3^- \bar{\chi}_4^+ - \chi_2^- \bar{\chi}_1^+) (y_3 + y_3') + (\chi_3^+ \bar{\chi}_4^- - \chi_2^+ \bar{\chi}_1^-) (y_3 - y_3') \\ \left. + i (\bar{\chi}_3^- \bar{\chi}_1^+ + \bar{\chi}_2^- \bar{\chi}_4^+) (y_2 + y_2') + i (\bar{\chi}_1^- \bar{\chi}_3^+ + \bar{\chi}_4^- \bar{\chi}_2^+) (y_2 - y_2') \right] \\ - \frac{1}{\sqrt{2}} \sin 2\phi (\cos^2 \phi |y_2|^2 - \sin^2 \phi |y_3|^2) y_4 + \text{h.c.}, \end{aligned} \quad (2.15)$$

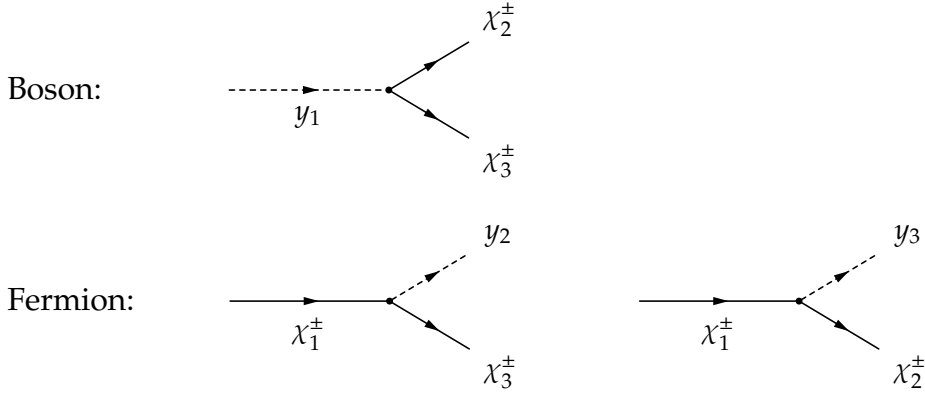
where Hermitian conjugation is defined in the usual way: $(\chi^- \bar{\chi}^+)^{\dagger} = \chi^+ \bar{\chi}^-$. Derivatives with respect to time and space coordinates are denoted by dots and primes, respectively. Also, note that when $\phi = 0, \pi/2$, both corresponding to the $AdS_3 \times S^3 \times T^4$ background, the entire cubic Lagrangian vanishes.

There are three obvious $U(1)$ charges: one $U(1)_{AdS}$ from the transverse AdS_3 and two $U(1)_{\pm}$ from $S^3 \times S^3$. Demanding that the cubic Lagrangian be neutral, we can easily read off the charges of the fields (see table (2.1)).

	y_1	y_2	y_3	y_4	χ_1^\pm	χ_2^\pm	χ_3^\pm	χ_4^\pm
$U(1)_{AdS}$	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$U(1)_+$	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$U(1)_-$	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Table 2.1: $U(1)$ charges

Before we end this section, let us point out one important property of the cubic Lagrangian. It is clear that the only decay processes possible for the heavy modes y_1 and χ_1^\pm are those via which they decay into two light ones:



This property was also observed for the $AdS_4 \times CP^3$ superstring, which exhibits a composite heavy mode [57]. While the above observation of heavy to light-light decay is certainly no proof of a composite heavy mode, it lends support to similar claims made in [19, 36]. This was investigated in [47].

2.3.3 Gauge-fixing the worldsheet metric

For technical reasons, it is easiest to fix the light-cone gauge by adding higher-order corrections to the worldsheet metric. As mentioned already, the uniform light-cone gauge

$$x^+ = \tau, \quad p^+ = \text{constant}$$

is compatible with the conformal gauge in the case of the cubic Lagrangian. However, with the quartic interactions included, the second condition, which is that the momentum conjugate to x^- is constant, fails to hold. In fact, the problem comes only from the purely bosonic part of the Lagrangian. The consistency of the gauge depends on the equation of motion for x^- (which we assume to be at least quadratic in the number of transverse coordinates) and the fermionic contribution comes from

$$-\frac{i}{8} \partial_i x^+ \partial_j x^- \bar{\Theta} \left[\gamma^{ij} (\Gamma^+ \not{\Gamma}^- + \Gamma^- \not{\Gamma}^+) - \varepsilon^{ij} (\Gamma^+ \Gamma^{11} \not{\Gamma}^- - \Gamma^- \Gamma^{11} \not{\Gamma}^+) \right] \Theta,$$

which vanishes in the κ -gauge $\Gamma^+ \Theta = 0$. Thus, the momentum conjugate to x^- contains no fermionic terms and therefore any modification of the conformal gauge will only contain the bosonic fields.

If we assume that the worldsheet metric receives quadratic corrections as $\gamma = \eta + \hat{\gamma}$, where $\hat{\gamma}$ is quadratic in fields, then we find that

$$\frac{\delta \mathcal{L}}{\delta \dot{x}^-} = -2 \left(\hat{\gamma}^{00} + |y_1|^2 - \cos^4 \phi |y_2|^2 - \sin^4 \phi |y_3|^2 \right), \quad \frac{\delta \mathcal{L}}{\delta x'^-} = -2 \hat{\gamma}^{01}$$

Thus, if we pick¹¹

$$\begin{aligned}\gamma^{00} &= 1 - |y_1|^2 + \cos^4 \phi |y_2|^2 + \sin^4 \phi |y_3|^2, \\ \gamma^{11} &= -1 - |y_1|^2 + \cos^4 \phi |y_2|^2 + \sin^4 \phi |y_3|^2, \text{ and} \\ \gamma^{01} &= 0\end{aligned}$$

we find that p^+ is constant, as required.

Given this choice of γ^{ij} , we are able to write down the full quartic Lagrangian (except of course for the quartic fermion terms) for arbitrary values of ϕ . The full Lagrangian is, of course, rather complicated and here we only present its purely bosonic part:

$$\begin{aligned}\mathcal{L}_{(4,b)} &= \frac{1}{4} \sin^2 2\phi (\cos^2 \phi |y_2|^2 - \sin^2 \phi |y_3|^2)^2 - \frac{1}{8} \sin^2 2\phi (\dot{y}_4^2 + \ddot{y}_4^2 - y_4'^2 - \ddot{y}_4'^2) (|y_2|^2 + |y_3|^2) \\ &\quad - |\dot{y}_4|^2 (|y_1|^2 - \cos 2\phi (\cos^2 \phi |y_2|^2 - \sin^2 \phi |y_3|^2)) + |\dot{y}_1|^2 (\cos^4 \phi |y_2|^2 + \sin^4 \phi |y_3|^2) \\ &\quad - (|\dot{y}_2|^2 + |\dot{y}_3|^2 + \sum_i^4 |y_i'|^2) (|y_1|^2 - \cos^4 \phi |y_2|^2 - \sin^4 \phi |y_3|^2) - |y_1'|^2 |y_1|^2 \\ &\quad - \cos^2 \phi |y_2|^2 (|\dot{y}_2|^2 - |y_2'|^2) - \sin^2 \phi |y_3|^2 (|\dot{y}_3|^2 - |y_3'|^2) \\ &\quad + \cos^2 \phi \sin^2 \phi |y_4'|^2 (|y_2|^2 + |y_3|^2).\end{aligned}\tag{2.16}$$

In the limiting cases $\phi = 0, \pi/2$, the pure $AdS_3 \times S^3$ piece is a direct truncation of the $AdS_5 \times S^5$ result given in [48]. For the remaining terms relevant for the one-loop computation in the next section, see appendix A.4.

2.3.4 One-loop finiteness

As a first probe of the quantum consistency of our action for the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring, we will show that the model is finite in dimensional regularisation. That is, if we consider one-loop corrected two-point functions for the bosonic coordinates y_i , we find that all $1/\epsilon$ terms cancel, where ϵ is the dimensional regularisation parameter. For arbitrary ϕ , we have both cubic and quartic interaction terms giving rise to bubble and tadpoles diagrams¹². The tadpoles built out of three-vertices are all zero, and the divergent terms arising from the bubble and tadpole diagrams cancel between each other.

The various loop diagrams encountered are regularised using the standard integral representation [49]

$$\begin{aligned}I_b^a(\Delta) &= \int d^d \ell \frac{(\ell^2)^a}{[\ell^2 + \Delta]^b} \\ &= \pi^{d/2} \frac{\Gamma(b - a - \frac{d}{2}) \Gamma(a + \frac{d}{2})}{\Gamma(b) \Gamma(\frac{d}{2})} \left(\frac{1}{\Delta}\right)^{b - a - \frac{d}{2}}\end{aligned}$$

evaluated at $d = 2 - \epsilon$. For the bubble diagrams, the divergent integrals are $I_2^1(\Delta)$, corresponding to a logarithmic divergence in hard cutoff. For the tadpoles, on

¹¹ γ^{11} is determined from the condition $\det(\gamma) = -1$.

¹²Actually, there are also three-vertex tadpoles. For the heavy and massless fields, these are trivially zero while for the light modes, they are zero due to cancellations between boson and fermion loops.

the other hand, we have both $I_1^1(\Delta)$ and $I_1^0(\Delta)$ integrals giving logarithmic and quadratic divergences for a hard cutoff.

In order to evaluate the contributing diagrams, we sum over all the terms arising from the cubic and quartic vertices, where the relevant terms for the latter are collected in (A.6). Since these constitute quite a large number of terms, the actual computation is rather involved, but after some effort, we find that the divergent contributions from the bubble and the tadpole diagrams are¹³

$$\mathcal{A}_B^i = \text{---} \xrightarrow{y_i} \text{---} \circ \text{---} \text{---} = \frac{1}{\epsilon} \frac{1}{2\pi} \sin^2 2\phi p_1^2 + \mathcal{O}(\epsilon^0)$$

and

$$\mathcal{A}_T^i = \text{---} \xrightarrow{y_i} \text{---} \text{---} \text{---} = -\frac{1}{\epsilon} \frac{1}{2\pi} \sin^2 2\phi p_1^2 + \mathcal{O}(\epsilon^0)$$

where i labels the bosonic fields. Thus, we see that the $1/\epsilon$ terms cancel exactly between the tadpoles and bubbles. In the limiting $\phi = 0, \pi/2$ cases, where there are no cubic terms, we see that the two-point functions are manifestly finite.

Before ending this section, we should note that, in order to determine the finite part of the spectrum, dimensional regularisation is not a suitable regulator for the loop integrals [58] (see also [59, 60, 61, 62, 63, 64, 65]). The reason is that, in order to maintain unitarity, one should choose a cutoff such that the decay processes, originating from the cubic Lagrangian, are energetically allowed [51] (see also [66]). The divergent terms, however, are not sensitive to these issues, but in order to determine the finite part unambiguously, one needs to regularise the theory properly¹⁴.

2.4 Hamiltonian analysis

In this section and the next, we will focus on the classical, or tree-level, sector of the string theory. We will start out by calculating energy shifts for a bosonic excitation of arbitrary length, and compare this calculation with a conjectured set of Bethe equations. In [19] and [36], Bethe equations for $\mathfrak{d}(2, 1; \alpha)$ were proposed; these are conjectured to predict the energies of string states for general values of ϕ . As was the case in $AdS_4 \times \mathbb{CP}^3$, the light modes are the fundamental excitations in the exact solution, and the heavier modes are described as composite states of two light modes. How the massless modes enter is not completely clear¹⁵. For certain

¹³These expressions are evaluated close to the bare pole, $p_0 = \sqrt{m_i^2 + p_1^2}$.

¹⁴There still seems to be a bit of uncertainty in how to regulate the $AdS_4 \times \mathbb{CP}^3$ string properly. Recently, the authors of [66] argued for using a regularisation method yielding a finite result different from that obtained using the unitarity based method.

¹⁵Recently, the authors of [37] showed how to incorporate massless modes [38] of pure-RR $AdS_3 \times S^3 \times T^4$ strings into the integrability machinery by presenting the complete all-loop S matrix for fundamental worldsheet excitations.

simplifying values of ϕ , the equations seem to capture the full critical spectrum, but in general, the situation seems to require further investigation [36].

2.4.1 Energy shifts

A very natural set of observables, from a worldsheet point of view, are energy corrections around a BMN vacuum [56]. The way to calculate these for closed strings in various AdS/CFT backgrounds is, by now, a rather well-established procedure [50, 52, 67, 68, 69, 70, 71, 72]. The starting point is the free quadratic BMN Lagrangian, which allows for an exact solution in terms of string oscillators. With the quadratic piece diagonalised, one then calculates the corresponding higher-order corrections to the energy perturbatively.

In order to obtain the energy shifts, we need the string Hamiltonian. We will calculate energy corrections to bosonic modes, mainly focusing on the light modes y_2 and y_3 . This has the nice advantage that we only need the purely bosonic Hamiltonian which we can immediately derive from (2.14), (2.15) and (2.16):

$$\begin{aligned} \mathcal{H} = & \sum_i^4 \left(|p_i|^2 + |y'_i|^2 + m_i^2 |y_i|^2 \right) + \frac{1}{\sqrt{2}} \sin 2\phi (p_4 + \bar{p}_4) \left(\cos^2 \phi |y_2|^2 - \sin^2 \phi |y_3|^2 \right) \\ & + 2|y_1|^2 |y'_1|^2 + \cos^2 \phi \left[\sin^2 \phi \left(|p_2|^2 + \cos^4 \phi |y_2|^2 \right) - \frac{1}{2} (3 + \cos 2\phi) |y_2'^2 \right] |y_2|^2 \\ & + \sin^2 \phi \left[\cos^2 \phi \left(|p_3|^2 + \sin^4 \phi |y_3|^2 \right) - \frac{1}{2} (3 - \cos 2\phi) |y_3'^2 \right] |y_3|^2 \\ & - \cos^4 \phi |y_2|^2 \left(|y_3'|^2 + |p_3|^2 \right) - \sin^4 \phi |y_3|^2 \left(|y_2'|^2 + |p_2|^2 \right) - 2 \cos^4 \phi \sin^4 \phi |y_2|^2 |y_3|^2 \\ & + \dots, \end{aligned} \quad (2.17)$$

where the ellipses indicate flavor-mixing terms, which will not contribute to our calculation.

The mode expansion that diagonalises the quadratic bosonic Hamiltonian is given by

$$y_i = \frac{1}{\sqrt{2\pi}} \int dp \frac{1}{2\omega^{(i)}(p)} \left(a(p)_i e^{-ip\sigma} + b(p)_i^\dagger e^{ip\sigma} \right), \quad \text{where} \quad \omega^{(i)}(p) = \sqrt{m_i^2 + p^2}.$$

We will calculate the energy corrections to several string states. First, we will consider states built out of one kind of string oscillator, $a(p_i)^\dagger$

$$|1_A\rangle = \prod_i^A a(p_i)_1^\dagger |0\rangle, \quad |2_A\rangle = \prod_i^A a(p_i)_2^\dagger |0\rangle, \quad |3_A\rangle = \prod_i^A a(p_i)_3^\dagger |0\rangle. \quad (2.18)$$

We shall also consider a more general state which takes values in both S^3 's. This subsector should constitute a closed $SU(2) \times SU(2)$ sector similar to that of the $AdS_4 \times CP^3$ string [70, 73]

$$|2_A, 3_B\rangle = \prod_i^B a(p_i)_3^\dagger \prod_j^A a(q_j)_2^\dagger |0\rangle, \quad (2.19)$$

where, for simplicity, all mode numbers are distinct, and $|0\rangle$ is the BMN vacuum annihilated by all lowering operators. Note that for both the single-flavour and

product states, switching oscillators a_i and b_i gives identical results. However, when the fermionic interaction terms are included, some of these states should mix since they are degenerate. The states above will not mix though, since it is not possible to construct other excitations with the same $U(1)$ charges and leading-order energy (see table 2.1).

Since we have cubic interactions for arbitrary ϕ , we need to make use of second-order perturbation theory, either by explicit calculation or by performing a unitary transformation so that the physical information of the cubic piece is rewritten in terms of quartic interactions (see [48, 50]). Evidently, both methods are completely equivalent and importantly, they give rise to terms that need to be regularised. Also, in the case of a nonvanishing cubic piece, the resulting quartic Hamiltonian is most probably *not* normal-ordered. In principle, this gives quadratic normal-ordering terms subject to some regularisation procedure. The cubic and quartic regularisation terms combine into quantum and finite-size effects. In the near-BMN limit, where the coupling is not strictly infinite, the finite-size effects correspond to the finite extension of the string worldsheet. For the $AdS_4 \times \mathbb{CP}^3$ string, these combined into Lüscher-like finite-size corrections (see [52]). We suspect that the same type of exponentially suppressed terms will appear also for the $AdS_3 \times S^3 \times S^3 \times S^1$ string.

Since not much is known about the quantum theory, we shall only consider the classical contribution to the spectrum. In other words, we shall simply ignore the terms that need to be regularised (see [50, 70] for details¹⁶). Nevertheless, the actual computation is still rather involved. What is more, the unitary transformation we will utilise depends on the massless coordinate p_4 . That is, even though the massless terms are not incorporated in the Bethe equations, they still appear as internal lines in Feynman diagrams. Or, as in this case, the massless modes appear as intermediate states in the unitary transformation. Let us explain how the procedure works. The unitary transformation $U = e^{iV}$ acts on the Hamiltonian as

$$e^{iV} \mathcal{H} e^{-iV} = -\mathcal{H}_3 + \text{induced quartic terms}$$

and thus, by construction, removes the cubic Hamiltonian at the cost of additional quartic terms. Here, we should note a small technical complication. Schematically, the unitary transformation is of the form

$$V = \frac{1}{\sqrt{g}} \sum_{r,s,t} \int dk dl dm \left[\frac{H_3(k, l, m)_{rst}^{+++}}{\omega^{(r)}(k) + \omega^{(s)}(l) + \omega^{(t)}(m)} + \frac{H_3(k, l, m)_{rst}^{+--}}{\omega^{(r)}(k) + \omega^{(s)}(l) - \omega^{(t)}(m)} + h.c \right]$$

where the r, s, t sums are over the four bosonic flavors, the labels \pm denote the number of creation/annihilation operators, and the integral is over mode numbers (see [48, 50] for details). Thus, for certain values of k, l and m , the denominator in the second term can be zero; this is an IR effect, and only happens when the mode number of the massless coordinate vanishes. In order to regularise this, one should introduce a small non-zero mass, m_4 , and only in the end send this to zero.

Using (2.17), together with the method described above, it is straightforward to compute the energy shifts for the states (2.18) and (2.19). A rather lengthy

¹⁶However, note that we expect the spectrum to be exact in the limiting $\phi = 0, \pi/2$ cases where the cubic Hamiltonian vanishes.

calculation gives (see [50] for details)

$$\begin{aligned}
\Delta E(p_A)_1 &= \frac{1}{4} \sum_{i \neq j}^A \frac{(p_i + p_j)^2}{\omega_i^{(1)} \omega_j^{(1)}} , \\
\Delta E(p_A)_2 &= - \sum_{i \neq j}^A \left\{ \left[\frac{\sin^2 2\phi (3 \cos^4 \phi + p_i^2 + p_i p_j + p_j^2 + \omega_i^{(2)} \omega_j^{(2)})}{16 \omega_i^{(2)} \omega_j^{(2)}} \right]_3 \right. \\
&\quad \left. + \left[\frac{\cos^2 \phi (\cos^2 \phi (p_i + p_j)^2 + p_i p_j \sin^2 \phi - 3 \cos^4 \phi \sin^2 \phi - \sin^2 \phi \omega_i^{(2)} \omega_j^{(2)})}{4 \omega_i^{(2)} \omega_j^{(2)}} \right]_4 \right\} \\
&= - \frac{\cos^2 \phi}{4} \sum_{i \neq j}^A \frac{(p_i + p_j)^2}{\omega_i^{(2)} \omega_j^{(2)}} , \\
\Delta E(p_A)_3 &= - \sum_{i \neq j}^A \left\{ \left[\frac{\sin^2 2\phi (3 \sin^4 \phi + p_i^2 + p_i p_j + p_j^2 + \omega_i^{(3)} \omega_j^{(3)})}{16 \omega_i^{(3)} \omega_j^{(3)}} \right]_3 \right. \\
&\quad \left. + \left[\frac{\sin^2 \phi (\sin^2 \phi (p_i + p_j)^2 + p_i p_j \cos^2 \phi - 3 \sin^4 \phi \cos^2 \phi - \cos^2 \phi \omega_i^{(3)} \omega_j^{(3)})}{4 \omega_i^{(3)} \omega_j^{(3)}} \right]_4 \right\} \\
&= - \frac{\sin^2 \phi}{4} \sum_{i \neq j}^A \frac{(p_i + p_j)^2}{\omega_i^{(3)} \omega_j^{(3)}} , \\
\Delta E(q_A, p_B)_{23} &= - \frac{\cos^2 \phi}{4} \sum_{i \neq j}^A \frac{(q_i + q_j)^2}{\omega_i^{(2)} \omega_j^{(2)}} - \frac{\sin^2 \phi}{4} \sum_{i \neq j}^B \frac{(p_i + p_j)^2}{\omega_i^{(3)} \omega_j^{(3)}} \\
&\quad - \frac{1}{2} \sum_i^A \sum_j^B \left\{ \frac{- [\cos^4 \phi q_j^2 + 2 \cos^4 \phi \sin^4 \phi + \sin^4 \phi p_i^2]_3}{\omega_i^{(2)} \omega_j^{(3)}} \right. \\
&\quad \left. + \frac{[\cos^4 \phi q_j^2 + 2 \cos^4 \phi \sin^4 \phi + \sin^4 \phi p_i^2]_4}{\omega_i^{(2)} \omega_j^{(3)}} \right\} \\
&= \Delta E(q_A)_2 + \Delta E(p_B)_3 , \tag{2.20}
\end{aligned}$$

where the subscript of the square bracket denotes whether the contribution originates from the cubic (3) or quartic (4) Hamiltonian. While both cubic and quartic contributions are rather complicated, the sum of the two simplifies. For the $SU(2) \times SU(2)$ sector, the mixing sector exactly cancels out, and the total energy is just the sum of the two distinct $SU(2)$ sectors¹⁷. Note that for the y_1 coordinate, the energy is, up to an overall sign, the same as the $SU(2)$ sector of $AdS_5 \times S^5$ [48]. Likewise, for $\phi = 0, \frac{\pi}{2}$, we see that, up to a sign, the $SL(2)$ result of [48] is reproduced.

¹⁷As can be seen in (A.7), this also happens for S-matrix processes mixing fields from the two $SU(2)$'s.

2.4.2 Bethe equations

The Bethe equations should encode the spectra of both the light and massive coordinates. However, since the heavy mode y_1 enters as a composite excitation in the exact solution, it can be rather involved to obtain its solution from the Bethe equations. For this reason, we will only try to reproduce the energies of the light excitations here.

The procedure is as follows. The starting point is the conjectured Bethe equations of [36]. These are expressed in terms of Zhukovsky variables x^\pm and the length L of the abstract spin-chain. The ground state of the spin-chain is related to the BMN vacuum which is proportional to $\sqrt{\lambda} \gg 1$. In order to reproduce the string spectrum, one needs to expand the Bethe equations at strong coupling and solve for the rapidity momentum p_k which parameterises x^\pm . Having obtained the (perturbative) solutions for p_k , one can then plug this into the magnon dispersion relation, which in turn gives a prediction for the energy which we match against the string calculation. For details of this procedure, we refer the reader to [48, 50, 71].

SU(2) sector

The energies (2.20) of the string states (2.18) should be reproducible from the equations of [36] reduced to a rank-one $SU(2)$ sector given by

$$\left(\frac{x_k^+}{x_k^-} \right)^L = \prod_{j \neq k}^A \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \left(\frac{1 - \frac{1}{x_k^+ x_j^-}}{1 - \frac{1}{x_k^- x_j^+}} \right) \sigma^2(x_k, x_j), \quad (2.21)$$

where L denotes the *length of the spin-chain*, and $\sigma(x_k, x_j)$ is a *dressing phase factor*. Since we are looking at BMN states, $L \sim g \sim \sqrt{\lambda}$.

While these equations have the same structural forms as those for the $SU(2)$ spin chain in $AdS_5 \times S^5$, the Zhukovsky map, however, is deformed to

$$x^\pm + \frac{1}{x^\pm} = x + \frac{1}{x} \pm \frac{i\omega_a}{2h}, \quad (2.22)$$

where

$$\omega_2 = 2 \cos^2 \phi, \quad \omega_3 = 2 \sin^2 \phi, \quad (2.23)$$

depending on the type of excitation, and $h = h(\lambda)$ is a function of the worldsheet coupling constant λ . If we use the notation x^\pm and y^\pm to denote excitations with mass $\cos^2 \phi$ and $\sin^2 \phi$ respectively, then a good parametrisation solving (2.22) is

[57]¹⁸

$$x^\pm(p_k) = \frac{\cos^2 \phi + \sqrt{\cos^4 \phi + 4h^2 \sin^2 \frac{p_k}{2}}}{2h \sin \frac{p_k}{2}} e^{\pm i \frac{p_k}{2}}, \quad p_k = \frac{p_k^0}{2g} + \frac{p_k^1}{(2g)^2} + \dots,$$

$$y^\pm(q_k) = \frac{\sin^2 \phi + \sqrt{\sin^4 \phi + 4h^2 \sin^2 \frac{q_k}{2}}}{2h \sin \frac{q_k}{2}} e^{\pm i \frac{q_k}{2}}, \quad q_k = \frac{q_k^0}{2g} + \frac{q_k^1}{(2g)^2} + \dots$$

The function $h = h(\lambda)$ has a leading-order strong-coupling expansion given by [19, 36]

$$h(\lambda) \approx \sqrt{\frac{\lambda}{2}} = \frac{g}{2\pi}, \quad \sqrt{\lambda}, g \gg 1.$$

For large values of h , $\sigma(x_i, x_j)$ is a slightly modified AFS phase [74, 36]

$$\sigma(x_k, x_l) = \frac{1 - \frac{1}{x_k^- x_l^+}}{1 - \frac{1}{x_k^+ x_l^-}} \left[\frac{\left(1 - \frac{1}{x_k^+ x_l^-}\right) \left(1 - \frac{1}{x_k^- x_l^+}\right)}{\left(1 - \frac{1}{x_k^- x_l^+}\right) \left(1 - \frac{1}{x_k^+ x_l^-}\right)} \right]^{\frac{i\hbar}{\omega_a} \left(x_k + \frac{1}{x_k} - x_l - \frac{1}{x_l}\right)}$$

Given a solution of (2.21), the corresponding total energy E and momentum P are given by

$$E = i\hbar \sum_k^A \left(\frac{1}{x_k^+} - \frac{1}{x_k^-} \right), \quad e^{iP} = \prod_k^A \frac{x_k^+}{x_k^-} \equiv 1, \quad (2.24)$$

where the first equation implies that the magnons have a dispersion relation given by

$$\epsilon_a(p_i) = \sqrt{\frac{\omega_a^2}{4} + 4h^2 \sin^2 \frac{p_i}{2}}, \quad (2.25)$$

where the masses ω_a are as given in (2.23).

In order to solve (2.21), we need to express the length L in terms of string theory variables such as the energy, angular momentum, and excitation number (A). For $\phi = 0, \frac{\pi}{2}$, the Bethe equations collapse to the rank-one equations of $PSU(2, 2|4)$ [36]. Furthermore, for these two values of ϕ , the cubic Lagrangian vanishes, and the relevant quartic terms are identical to the $AdS_5 \times S^5$ case [48, 71]. Thus, following [71], it becomes clear that L is expressed as

$$L = g + \frac{1}{2}A - \frac{1}{2}E, \quad \text{for } \phi = 0, \pi/2,$$

where E now denotes the leading-order piece of (2.24),

$$E = \sum_k^A \left(-\frac{\omega}{2} + \sqrt{\frac{\omega^2}{4} + m_k^2} \right) + \dots$$

¹⁸A comment on notation: what we call x_k^\pm correspond to $x_{3,k}^\pm$ or $x_{\bar{3},k}^\pm$, while y_k^\pm correspond to $x_{1,k}^\pm$ or $x_{\bar{1},k}^\pm$ in [36].

Focusing on the $\phi = 0$ case, we find that the equations collapse to

$$\left(\frac{x_k^+}{x_k^-}\right)^{g+\frac{1}{2}A} = \prod_{j \neq k}^A \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} + \mathcal{O}(g^{-2}), \quad (2.26)$$

and the solution to these equations nicely matches (2.20) [48]. In order to arrive at (2.26), we made use of the following nice identity for the AFS phase [71]:

$$\log\left(\frac{x_k^+}{x_k^-}\right)^{\beta E} \prod_{j \neq k}^A \frac{1 - \frac{1}{x_k^+ x_j^-}}{1 - \frac{1}{x_k^- x_j^+}} \sigma^2(x_k, x_j) = \frac{2\pi i}{\omega} \sum_j^A (-1 + \beta\omega) (-\omega/2 + \omega_j) p_k + \mathcal{O}(g^{-2}), \quad (2.27)$$

which vanishes for $\beta = 1/2$ and $\phi = 0$ ¹⁹.

In order to reproduce the energy shifts (2.20) for arbitrary ϕ , L has to equal

$$L = g + \frac{1}{2}A - \frac{1}{\omega}E. \quad (2.28)$$

It is important to stress that this relation is fixed uniquely, which is easy to see if one, for example, expands in small mode numbers. With this L , (2.27) is zero and the Bethe equations become

$$\left(\frac{x_k^+}{x_k^-}\right)^{g+\frac{1}{2}A} = \prod_{j \neq k}^A \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}, \quad \left(\frac{y_k^+}{y_k^-}\right)^{g+\frac{1}{2}B} = \prod_{j \neq k}^B \frac{y_k^+ - y_j^-}{y_k^- - y_j^+}, \quad (2.29)$$

and we have the momentum constraints

$$\prod_k \frac{x(p_k)^+}{x(p_k)^-} = 1 \quad \prod_k \frac{y(q_k)^+}{y(q_k)^-} = 1.$$

The dispersion relation (2.25) expands as

$$E_k^{(i)} = -\frac{\omega}{2} + \omega_k^{(i)} + \Delta E_k^{(i)}, \quad \Delta E_k^{(i)} = \frac{p_k}{8\pi\omega_k^{(i)}} p_k^1, \quad (2.30)$$

where we slightly abuse notation and denote p_k as the mode number of the oscillator state, and p_k^1 is the subleading piece of the magnon momentum which we solve for by using (2.29). Also note that unsubscripted ω refers to the masses (2.23). The index i is either 2 or 3 depending on the excitation. Using the explicit solution of p_k^1 immediately reproduces the energies of the rank-one sectors, $\Delta E(p_A)_2$ and $\Delta E(p_A)_3$, in (2.20).

$SU(2) \times SU(2)$ sector

We now want to reproduce the energy shift ΔE_{23} from the Bethe equations. The largest compact subalgebra of $\mathfrak{d}(2, 1, \alpha)$ is $\mathfrak{su}(2) \times \mathfrak{su}(2)$. At weak coupling, the spin-chain is that of two decoupled Heisenberg chains related only via the momentum

¹⁹The identity only holds when the momentum constraint (2.24) is satisfied.

constraint. At strong coupling, we expect the situation to be similar to the $AdS_4 \times \mathbb{CP}^3$ string, which also contains a closed $SU(2) \times SU(2)$ sector [70, 73].

From [19, 36], we deduce that the ΔE_{23} -shift should be encoded in the Bethe equations

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j \neq k}^A \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - \frac{1}{x_k^+ x_j^-}}{1 - \frac{1}{x_k^- x_j^+}} \sigma^2(x_k, x_j), \quad (2.31)$$

$$\left(\frac{y_k^+}{y_k^-}\right)^L = \prod_{j \neq k}^B \frac{y_k^+ - y_j^-}{y_k^- - y_j^+} \frac{1 - \frac{1}{y_k^+ y_j^-}}{1 - \frac{1}{y_k^- y_j^+}} \sigma^2(y_k, y_j),$$

augmented with

$$E = ih \left[\sum_k^A \left(\frac{1}{x_k^+} - \frac{1}{x_k^-} \right) + \sum_k^B \left(\frac{1}{y_k^+} - \frac{1}{y_k^-} \right) \right], \quad \prod_k^A \frac{x_k^+}{x_k^-} \prod_k^B \frac{y_k^+}{y_k^-} = 1.$$

The parameter L now relates the two equations, and following AdS_4/CFT_3 , it should be given by [50, 70, 71]

$$L = g + \frac{1}{2} \left(A + B - \frac{1}{\cos^2 \phi} \sum_k^A E(x_k^\pm) - \frac{1}{\sin^2 \phi} \sum_k^B E(y_k^\pm) \right).$$

If we impose that each subset of mode numbers are separately zero (and distinct),

$$\prod_k^A \frac{x_k^+}{x_k^-} = \prod_k^B \frac{y_k^+}{y_k^-} = 1,$$

then (2.31) become

$$\begin{aligned} \left(\frac{x_k^+}{x_k^-}\right)^{g+\frac{1}{2}A} &= \left(\frac{x_k^+}{x_k^-}\right)^{\frac{1}{2}(-B+\frac{1}{\sin^2 \phi} \sum_j^B E(y_j^\pm))} \prod_{j \neq k}^A \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}, \\ \left(\frac{y_k^+}{y_k^-}\right)^{g+\frac{1}{2}B} &= \left(\frac{y_k^+}{y_k^-}\right)^{\frac{1}{2}(-A+\frac{1}{\cos^2 \phi} \sum_j^A E(x_j^\pm))} \prod_{j \neq k}^B \frac{y_k^+ - y_j^-}{y_k^- - y_j^+}. \end{aligned} \quad (2.32)$$

Solving the above and using the solutions in (2.30), we find

$$\Delta E = \Delta E(q_A)_2 + \Delta E(p_B)_3$$

$$- \left[B \sum_k^A \frac{q_k^2}{\omega^{(2)}(q_k)} + A \sum_k^B \frac{p_k^2}{\omega^{(3)}(p_k)} \right] + \frac{1}{2} \sum_k^A \sum_j^B \frac{\frac{1}{\sin^2 \phi} q_k^2 [\omega^{(3)}(p_j)]^2 + \frac{1}{\cos^2 \phi} p_j^2 [\omega^{(2)}(q_k)]^2}{\omega^{(2)}(q_k) \omega^{(3)}(p_j)},$$

which does not reproduce (2.20) - the last line is not zero. Even in the limiting $\phi = \pi/4$ case, we still do not find agreement. We do not know the origin of

this mismatch. Perhaps this is a hint that the Bethe equations of [19, 36] actually describe two spin chains, completely unrelated in the $SU(2) \times SU(2)$ sector.

Indeed, we can reconcile the above with the string theory calculation if we assume the parameter L to be distinct for each $SU(2)$ factor. That is, taking

$$L_2 = g + \frac{1}{2}A - \frac{1}{\omega_2}E_2, \quad L_3 = g + \frac{1}{2}B - \frac{1}{\omega_3}E_3,$$

for each sector would reproduce the results of (2.20) since the first terms on the RHS of (2.32) vanish. We would like to stress that the expression for L written above is fairly unique. It is generally very hard to implement a mixing between the two sectors (for example by adding B and A excitations in L_2 and L_3 , respectively) without contradicting (2.20) or the S-matrix processes in (A.7). It would be very interesting to investigate this in more detail. For example, one could calculate the full worldsheet S-matrix and from there construct the (string) Bethe equations.

2.5 Tree-level scattering

In order to understand how to properly include the massless modes in the exact solution, we will study how they enter the S-matrix of worldsheet scattering processes. We will study some simple $2 \rightarrow 2$ scattering amplitudes for the bosonic particles. Since the exact S-matrix is not known, we are not able to explicitly compare the amplitudes, but we do however show that the S-matrix is completely reflectionless. If this is true for the all-loop case, this means the massless modes enter diagonally in the Bethe Ansatz, making it easier to generalise them for the full critical spectrum (see [9] and references therein).

That the S-matrix is reflectionless is a somewhat unusual property that was also observed in the case of AdS_4/CFT_3 duality [75, 76, 77]. Under the natural assumption that the S-matrix is also reflectionless at weak coupling, this could shed some light on the unknown CFT_2 dual of the $AdS_3 \times S^3 \times S^3 \times S^1$ string.

The worldsheet S-matrix can be separated into three parts:

$$\begin{array}{ll} \text{Scattering} & \mathbb{S} : \quad (yy \rightarrow yy) \\ \text{Transmission} & \mathbb{T} : \quad (y\bar{y} \rightarrow y\bar{y}) \\ \text{Reflection} & \mathbb{R} : \quad (y\bar{y} \rightarrow \bar{y}y). \end{array}$$

The S-matrix expands as

$$\mathbb{S} = \mathbb{1} + iS + \dots, \quad \mathbb{T} = \mathbb{1} + iT + \dots, \quad \mathbb{R} = iR + \dots,$$

where the contributing diagrams for each part are given by

$$\begin{aligned}
\mathbb{S} &= \mathbb{1} + \frac{i}{g}S + \dots = 1 + \begin{array}{c} 1 \longrightarrow 3 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 4 \\ (t) \end{array} + \begin{array}{c} 1 \longrightarrow 3 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 4 \\ (u) \end{array} + \begin{array}{c} 1 \longrightarrow 3 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 4 \\ (c) \end{array} \\
\mathbb{T} &= \mathbb{1} + \frac{i}{g}T + \dots = 1 + \begin{array}{c} 1 \longrightarrow 3 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 4 \\ (t) \end{array} + \begin{array}{c} 1 \longrightarrow 3 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 4 \\ (s) \end{array} + \begin{array}{c} 1 \longrightarrow 3 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 4 \\ (c) \end{array} \\
\mathbb{R} &= \frac{i}{g}R + \dots = \begin{array}{c} 1 \longrightarrow 4 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 3 \\ (t) \end{array} + \begin{array}{c} 1 \longrightarrow 4 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 3 \\ (s) \end{array} + \begin{array}{c} 1 \longrightarrow 4 \\ \longrightarrow \quad \longrightarrow \\ 2 \longrightarrow 3 \\ (c) \end{array}
\end{aligned}$$

Below, we will show that the \mathbb{R} piece vanishes for all bosonic $2 \rightarrow 2$ scatterings. We provide the light-to-light scattering and transmission components of \mathbb{S} and \mathbb{T} in appendix A.5.

2.5.1 Light-to-light reflections

Let us begin by considering light-to-light scattering processes. In two dimensions, the particles can either keep or exchange their momenta. Unless $\phi = \pi/4$, the masses of y_2 and y_3 are different, which means reflections of these coordinates are trivially zero due to energy conservation:

$$\omega^{(2)}(p_1) + \omega^{(3)}(p_2) \neq \omega^{(2)}(p_2) + \omega^{(3)}(p_1) \quad \text{when} \quad \phi \neq \pi/4.$$

Thus, the processes we consider are $y_i \bar{y}_i \rightarrow \bar{y}_i y_i$ for arbitrary ϕ , and the more general $y_i \bar{y}_j \rightarrow \bar{y}_k y_l$ case for $\phi = \pi/4$.

Ignoring the external leg and overall momentum delta-functions, we find

$$\begin{aligned}
\mathbb{R}[y_2 \bar{y}_2 \rightarrow \bar{y}_2 y_2] : & \quad \left[4 \cos^6 \phi \sin^2 \phi \right]_c - \frac{1}{2} \sin^2 2\phi \left[\cos^4 \phi - p_1 p_2 - \omega^{(2)}(p_1) \omega^{(2)}(p_2) \right]_t \\
& \quad - \frac{1}{2} \sin^2 2\phi \left[\cos^4 \phi + p_1 p_2 + \omega^{(2)}(p_1) \omega^{(2)}(p_2) \right]_s = 0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{R}[y_3 \bar{y}_3 \rightarrow \bar{y}_3 y_3] : & \quad \left[4 \cos^2 \phi \sin^6 \phi \right]_c - \frac{1}{2} \sin^2 2\phi \left[\sin^4 \phi - p_1 p_2 - \omega^{(3)}(p_1) \omega^{(3)}(p_2) \right]_t \\
& \quad - \frac{1}{2} \sin^2 2\phi \left[\sin^4 \phi + p_1 p_2 + \omega^{(3)}(p_1) \omega^{(3)}(p_2) \right]_s = 0,
\end{aligned}$$

and for the special case $\phi = \pi/4$, we have

$$\mathbb{R}[y_2 \bar{y}_3 \rightarrow \bar{y}_3 y_2] = \mathbb{R}[y_3 \bar{y}_2 \rightarrow \bar{y}_2 y_3] : \quad \frac{\left[\frac{1}{2}(p_1 + p_2)^2 \right]_c - \left[\frac{1}{2}(p_1 + p_2)^2 \right]_t}{1 - 4p_1 p_2 + 4\sqrt{\frac{1}{4} + p_1^2} \sqrt{\frac{1}{4} + p_2^2}} = 0,$$

where the subscripts s and t denote the relevant three-vertex diagrams, and c denotes the four-vertex contact contribution. We thus see that the reflection part of the S-matrix is indeed zero.

2.5.2 Light-to-massless reflection

The presence of the massless modes is a new feature of the AdS_3/CFT_2 duality. While they enter as normal excitations on the worldsheet, they are complicated to incorporate in the Bethe Ansatz equations since the finite-gap method fails to work.

In the limiting cases $\phi = 0, \pi/2$, new reflection processes are energetically allowed. For example, at $\phi = 0$, the $y_3 \bar{y}_4 \rightarrow \bar{y}_4 y_3$ process is not trivially zero. Of course, the same holds for the other case $\phi = \pi/2$, where this time the processes involve the particles y_2 and y_4 . For these special values of ϕ , the cubic piece (2.15) vanishes and we only have the contact terms. An easy calculation shows that

$$\begin{aligned} \mathbb{R} [y_3 \bar{y}_4 \rightarrow \bar{y}_4 y_3]_{\phi=0} &= 0, & \mathbb{R} [y_2 \bar{y}_4 \rightarrow \bar{y}_4 y_2]_{\phi=\pi/2} &= 0, \\ \mathbb{R} [y_1 \bar{y}_2 \rightarrow \bar{y}_2 y_1]_{\phi=0} &= 0, & \mathbb{R} [y_1 \bar{y}_3 \rightarrow \bar{y}_3 y_1]_{\phi=\pi/2} &= 0. \end{aligned}$$

With this we conclude that the S-matrix of the AdS_3/CFT_2 integrable system indeed seems to be reflectionless, at least at tree-level. Of course, to check also the S-matrix for the fermions, one would need the action to quartic order in fermions, but supersymmetry suggests that this property should also hold in the fermion sector.

2.6 Summary

We have carried out a rather extensive study of the type IIA $AdS_3 \times S^3 \times S^3 \times S^1$ Green-Schwarz (GS) superstring up to quadratic order in fermions, and have discussed issues related to fixing its κ -symmetry. We derived the near-BMN expansion of the GS Lagrangian with quadratic fermions up to quartic order in fields. As a first consistency check, we have shown that the one-loop corrections to the two-point functions, built out of the four complex coordinates y_i , are finite in dimensional regularisation - both the three- and four-vertex diagrams are separately divergent, but the sum of the two is finite. We then performed a Hamiltonian analysis and compared $SU(2)$ string states with predictions from the conjectured Bethe equations of [36]. For the rank-one sectors, we found perfect agreement, while we did not fully understand how to match the product $SU(2) \times SU(2)$ sector. As it turns out, the string energies arising from the mixing sector exactly cancel between cubic and quartic interaction pieces. This means that, in order for the Bethe equations to reproduce the string calculation, the rank-one Bethe equations should decouple completely. A natural way to achieve this is to assume L for each sector to have different subleading corrections. We are not sure how to interpret this result, and further investigation is probably needed. Note, however, that our result is not necessarily in conflict with [19, 36] since the subleading effects in L would not show up in the semiclassical limit, and hence the integral equations of [19, 36] should remain the same.

In the last section, we looked at $2 \rightarrow 2$ scattering processes for bosons on the worldsheet. We showed that, at least at tree-level, the two-body S-matrix is reflectionless; this somewhat odd property was also observed in the AdS_4/CFT_3

duality [75]. This might be a useful finding if the Bethe equations have to be extended in order to incorporate the massless modes as fundamental excitations.

A natural continuation of the present work would be to perform a proper quantum computation. While we verified that the theory is one-loop finite, it would definitely be interesting to compute the subleading term in (2.25) from the string theory side. This was, for example, done for the $AdS_4 \times \mathbb{CP}^3$ string in [58]. However, since the worldsheet fields come with different masses, one has to be very careful with the regularisation. We plan to return to this question in the future. Another interesting line of research would be to calculate one-loop corrections to the energy along the lines of [78].

It would also be interesting to verify the integrability of the full GS string (up to quadratic order in fermions) in this background as has been done for $AdS_4 \times \mathbb{CP}^3$ and $AdS_2 \times S^2 \times T^6$ [15, 18, 16] using similar techniques.

Chapter 3

Meromorphic Functions and the Topology of Giant Gravitons

3.1 Introduction

The basic premise of the AdS/CFT correspondence, and more generally of the *gauge-gravity duality*, which is that the dynamics of a d -dimensional gravitational theory could be exactly equivalent to the physics of some quantum field theory living in $(d - 1)$ -dimensional spacetime, has significantly transformed and expanded our fundamental understanding of the nature of spacetime. The best understood sector of the duality concerns the closed fundamental string of type IIB theory in asymptotically $AdS_5 \times S^5$, which, in its dual $\mathcal{N} = 4$, $SU(N)$ four-dimensional super-Yang-Mills (SYM) theory, correspond to single-trace operators of length much less than N [9].

Equation (1.1), which gives the relation between the AdS radius (measured in units of the string length ℓ_s), R , and the 't Hooft coupling, $\lambda = g_{YM}^2 N$, shows that, in the limit of small curvatures (where one could hope to recognise a familiar classical description of geometry), the quantum field theory is strongly coupled and calculations of relevant field theory observables are effectively intractable. Conversely, computing things perturbatively in the field theory necessarily means working in the small λ limit, where curvature corrections in the gravity dual are important and consequently, our usual notions of geometry break down. Fortunately enough, however, there is a way to extrapolate between these two coupling regimes, which is possible due to the large amount of supersymmetry enjoyed by the SYM theory. The states of the theory that are most protected from quantum corrections, preserving half of the maximal amount of supersymmetry, constitute the so-called $\frac{1}{2}$ -BPS sector

$\frac{1}{2}$ -BPS Sector

The field content of $\mathcal{N} = 4$ super-Yang-Mills theory includes six hermitean scalars, ϕ_i $i = 1, \dots, 6$, transforming in the adjoint representation of the gauge group.

These can be combined into three complex scalar fields

$$Z = \phi_1 + i\phi_2, \quad Y = \phi_3 + i\phi_4, \quad X = \phi_5 + i\phi_6,$$

forming the *Higgs sector* of the SYM theory. The $\frac{1}{2}$ -BPS chiral primary operators can be built from a single complex combination - we choose Z in what follows, by tracing over powers of Z . The most general $\frac{1}{2}$ -BPS operators take the form of multitrace operators, given by

$$O(Z) = \prod_{k=1}^n [\text{Tr}(Z^k)]^{v_k},$$

with the total number of Z fields

$$n = \sum_{k=1}^n k v_k.$$

These chiral operators have conformal weight $\Delta = J$, where J is a particular $U(1)$ charge in the \mathcal{R} -symmetry group. Thus, there is a one-to-one correspondence between these operators and $\frac{1}{2}$ -BPS representations of \mathcal{R} -charge n [89]. A beautiful argument, due to Berenstein [80], demonstrates the simplicity of the $\frac{1}{2}$ -BPS sector. Consider a time-slicing of $AdS_5 \times S^5$, which gives the hamiltonian

$$H = \frac{(\Delta - J) + \epsilon \Delta}{\epsilon}$$

where Δ is here the *dilatation operator*. In the $\epsilon \rightarrow 0$ limit, any state with $(\Delta - J) > 0$ will have a huge energy, and hence will decouple from the low-energy theory. This effectively decouples (a subspace of) the $\frac{1}{2}$ -BPS states of $\mathcal{N} = 4$ SYM; these low-lying states are protected by supersymmetry and will not be lifted from zero energy by interactions.

The $\frac{1}{2}$ -BPS sector of type IIB string theory on $AdS_5 \times S^5$ contains gravitons, strings, and D-branes. Apparently, all these objects are captured by the holomorphic sector of the quantum mechanics of a single complex matrix of size $N \times N$. As the \mathcal{R} -charge (J) of an operator in $\mathcal{N} = 4$ SYM is changed, its interpretation in the dual quantum gravity theory changes. This is a consequence of the *Myers effect* [81]: the background has a non-zero RR five-form field strength switched on, to which D3-branes can couple. Gravitons carry a D3 dipole charge and are hence polarised by the background flux [82]. As J is increased, the coupling to the background RR-flux increases and the gravitons expands to a radius

$$R = \sqrt{\frac{J}{N}} R_{AdS}, \quad \text{where} \quad R_{AdS}^2 = \sqrt{g_{YM}^2 N \alpha'}.$$

We consider the limit where N is very large, with g_{YM}^2 fixed and very small. Since the string coupling constant $g_s = g_{YM}^2$, this is the weak string coupling and small curvature limit in which we expect to be able to recognise the familiar objects of perturbative string theory. Hence, it is possible to construct a "*dictionary*" between the $\frac{1}{2}$ -BPS sector of the $\mathcal{N} = 4$ SYM theory and type IIB string theory on $AdS_5 \times S^5$ according to \mathcal{R} -charge:

- for \mathcal{R} -charge $J \sim \mathcal{O}(1)$, the gauge theory operator is dual to an object of zero size in string units, that is a *point-like graviton* [3];
- for \mathcal{R} -charge $J \sim \mathcal{O}(\sqrt{N})$, the gauge theory operator is dual to an object of fixed size in string units, that is a *string* [56];
- for \mathcal{R} -charge $J \sim \mathcal{O}(N)$, the gauge theory operator is dual to an object the size of which is of order R_{AdS} , that is a *giant graviton*;
- for \mathcal{R} -charge $J \sim \mathcal{O}(N^2)$, the size of objects to which the gauge operators are dual diverges, even when measured in units with $R_{\text{AdS}} = 1$ - this divergence is simply an indication that these operators do not have an interpretation in terms of new objects in $AdS_5 \times S^5$, and instead would correspond to new supergravity backgrounds. Such geometries should preserve an $R \times SO(4) \times SO(4)$ symmetry, and this ansatz is sufficiently specific that the general solution with these isometries have been written down by Lin, Lunin and Maldacena [83], and are subsequently known as *LLM geometries*.

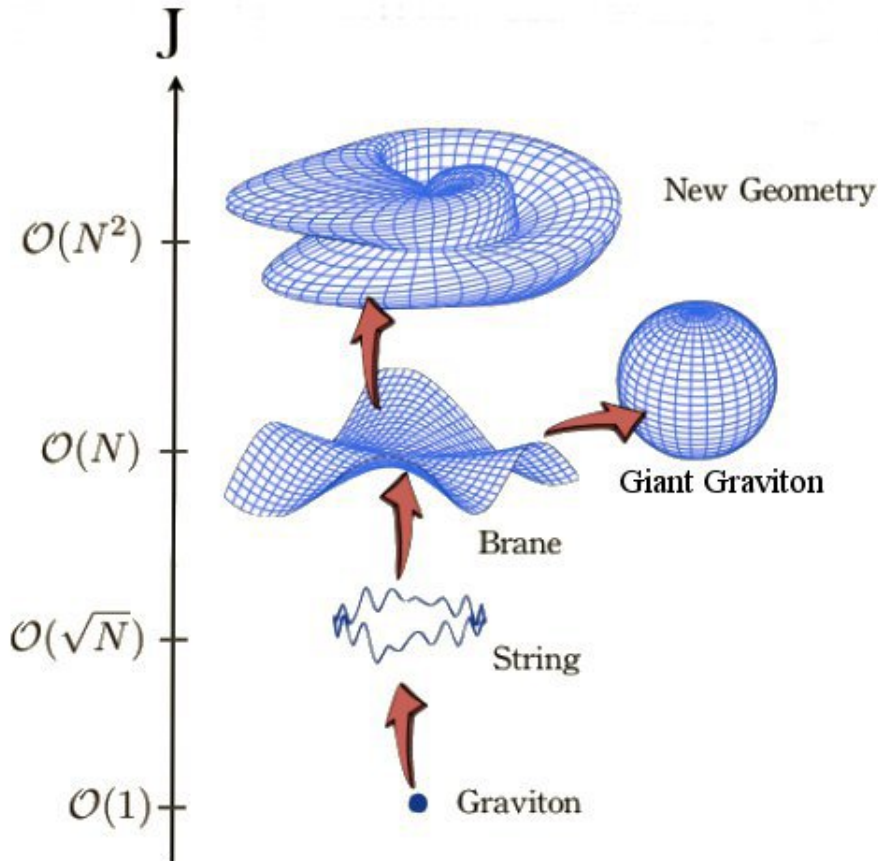


Figure 3.1: A schematic representation of the $\frac{1}{2}$ -BPS objects in gauge-gravity duality. Picture credit: Dino Giovannoni.

This organisation of the dictionary of the AdS/CFT correspondence in the $\frac{1}{2}$ -BPS sector according to \mathcal{R} -charge is shown in figure 3.1. This, together with subsequent studies (see for example [84, 85, 86, 87]), have led to a concrete proposal for the realisation of the idea that quantum gravity and spacetime itself are *emergent* phenomena encoded in the quantum interactions of a matrix model.

Giant Gravitons and Topology

The idea that spacetime, its *local* (geometrical) and *global* (topological) properties, are not fundamental but emerge in some "coarse-graining" limit of quantum gravity is not a new one, and is certainly not new to string theory. What string theory does bring to the table though is a concrete way to take such a limit via the AdS/CFT correspondence. This then begs the question of *how the geometry and topology of extended objects in the bulk theory are encoded in the gauge theory*.

Advances in Schur operator technology, starting with [89, 88] and more recently developed in the series of articles [91, 93], have substantially taken us closer to answering the question. For instance, it was convincingly argued in [94], and later verified in great detail in [91, 93], that the fact that the giant graviton world-volume is a compact space is encoded in the combinatorics of the Young diagrams that label the associated Schur operators. More precisely, any closed hypersurface (like the D3-brane worldvolume) must satisfy Gauss's law, thereby precisely constraining how open strings may be attached to the D-brane. In the gauge theory, attaching open strings translates into adding a word of length $\mathcal{O}(\sqrt{N})$ to the Schur polynomial corresponding to the giant or, equivalently, adding a box to a Young diagram. The *Littlewood-Richardson rules* that govern such additions precisely reproduce Gauss's law and consequently define the topology of the spherical giant.

On a more pragmatic level, one could very well argue that the claim that spacetime geometry and topology are emergent properties of the gauge theory at large N would be more convincing if said geometries and topologies were more interesting than just the sphere. For example, showing that Gauss's law is encoded in the combinatorics of the Young diagrams that label the Schur polynomials is an excellent step forward, but since it is a condition that must be satisfied by *any* compact worldvolume, it is, by itself, not a good characterisation of topology. An obvious next step would be to understand how a *topological invariant* such as *genus* is encoded in the gauge theory.

Until very recently, there were no known candidate operators dual to the topologically and geometrically nontrivial giant gravitons in the literature. The turnaround in this state of affairs came with the discovery of a new realisation of the AdS/CFT duality, this time between the type IIA superstring on $AdS_4 \times CP^3$ and a $\mathcal{N} = 6$, supersymmetric Chern-Simons theory on the 3-dimensional boundary of the AdS space - the so-called *ABJM model* [8, 95, 96, 97, 98, 99]. While this new AdS_4/CFT_3 duality shares much in common with its better known and understood higher-dimensional counterpart - a well-defined perturbative expansion, integrability, etc., it is also sufficiently different that the hope that it will provide just as invaluable a testing ground as AdS_5/CFT_4 is not without justification. In particular, in a recent study of spinning dual M2-branes in $AdS_4 \times S^7$ and, via orbifolding, their type IIA descendants [100], a new class of giant gravitons with large angular momentum and a D0-brane charge was discovered with a toroidal worldvolume. More importantly, with the gauge theory in this case nearly as controlled as $\mathcal{N} = 4$ SYM theory, a class of $\frac{1}{2}$ -BPS monopole operators has been proposed as the candidate duals to these giant torii in [101, 102] by matching the energy of the quadratic fluctuations about the monopole configuration to that of

the giant graviton. At this point, however, much work remains to be done to show how the full torus is recovered in the field theory.

The toroidal M2-brane worldvolume obtained in [100] arises as an analytical solution of the BPS equation derived using the ordinary *Bogomolnyi bound argument*, which is that the total lagrangian describing the wrapped brane is a total derivative that results from imposing a constraint obtained from the DBI part of the action on the BPS equation. This necessarily guarantees that the solution to the BPS equation satisfies the equation of motion obtained from the total lagrangian. In a beautiful paper [104], Mikhailov has shown, inspired by the realisation that brane polarisation provides a remarkable link between algebra and geometry, and following a lead [103] that spherical giant gravitons are representatives of the families of non-spherical solutions parametrised by holomorphic functions, that non-spherical giant gravitons (in the Penrose limit) in $AdS_7 \times S^4$ correspond to finite-dimensional representations of a nonlinear algebra of the coordinate functions on the spacial slice of the M2-brane in the matrix theory.

Nonspherical Giants à la Mikhailov

We will here be concerned with an alternative method, inspired from [103], of constructing D3-brane giant gravitons with nontrivial topologies in $AdS_5 \times S^5$. The worldsheet of any closed string state is a cylinder $\mathbb{R} \times S^1$; for D3-branes, however, any worldvolume with topology $\mathbb{R} \times \mathcal{M}$ is in principle possible, with \mathcal{M} being a closed 3-manifold. Of course, such manifolds can have considerably more complicated topology than closed 1-manifolds. The goal in this chapter is to explore what topological configurations giants, whose dynamics preserve a certain amount of supersymmetry, assume. We are particularly interested in solutions created by a localised modification of a set of intersecting spherical giant gravitons, as this seems the most tractable limit.

We start by recalling the map given by Mikhailov [103, 104]. Any analytic function $f : \mathbb{C}^3 \rightarrow \mathbb{C}$ defines a supersymmetric D3-brane solution in $\mathbb{R} \times S^5 \subset AdS_5 \times S^5$ as the surface

$$f(e^{-it}Z_1, e^{-it}Z_2, e^{-it}Z_3) = 0, \quad \sum_{i=1}^3 |Z_i|^2 = 1 \quad (3.1)$$

where $Z_i = r_i e^{i\phi_i}$ are the three complex embedding coordinates for S^5 . The amount of preserved supersymmetry is basically given by the number of arguments of the analytic function: $f(Z_1)$ gives a $\frac{1}{2}$ -BPS solution, while $f(Z_1, Z_2)$ and $f(Z_1, Z_2, Z_3)$ define $\frac{1}{4}$ - and $\frac{1}{8}$ -BPS solutions, respectively.

The usual sphere giant graviton is given in this formalism by

$$f(Z_1) = Z_1 - \alpha. \quad (3.2)$$

Setting this function to zero completely constrains Z_1 , so that the worldsurface, at some arbitrary time, is the S^3 parameterised by Z_2 and Z_3 subject to the condition $|Z_2|^2 + |Z_3|^2 = 1 - \alpha^2$. Time-evolution of the giant is restricted to rotation in the Z_1 plane. The *maximal* giant graviton has $\alpha = 0$ and is thus stationary; in the opposite limit $\alpha \rightarrow 1$, the brane collapses to a point particle moving on a lightlike trajectory. A function $f(Z_1)$ with several zeros corresponds to a number of concentric spherical giants.

Outline

We shall hereafter refer to (3.2) as the case $(1, 0, 0)$, which is one Z_1 giant. The next section studies the effect of adding to this Z_2 terms, and then takes a limit in which these give n intersecting Z_2 giants, which we will refer to as cases $(1, n, 0)$. After that, we consider arbitrarily many intersecting Z_1 and Z_2 giants, cases $(m, n, 0)$ (section 3.3), and finally allow the addition Z_3 giants (section 3.4). We end by giving a concise statement of our findings in section 3.5.

3.2 Quarter-BPS Class $(1, n, 0)$

Single pole

To begin constructing topologically nontrivial solutions using Mikhailov's method, in this section, we add to the spherical giant's $f(Z_1)$ a meromorphic function of Z_2 . We thus first consider the function

$$f(Z_1, Z_2) = Z_1 - \alpha + \frac{\epsilon}{Z_2}. \quad (3.3)$$

For simplicity, we may assume $\alpha, \epsilon > 0$, and since the motion of the brane is rigid, we need only discuss its topology at time $t = 0$.

Let us parameterise the D3-brane worldvolume by the ϕ_3 circle and some region of the Z_2 -plane. This is possible since setting $f = 0$ allows us to write Z_1 in terms of Z_2 , and $\sum_i^3 r_i^2 = 1$ fixes r_3 . We can then easily deduce the topology of the brane from the topology of the area of the Z_2 -plane thus covered. Let Σ be the set of points on the Z_2 -plane for which $r_3 \geq 0$; the spherical giant graviton (3.2), for instance, clearly has the disk $|Z_2| \leq 1 - \alpha^2$ for Σ .

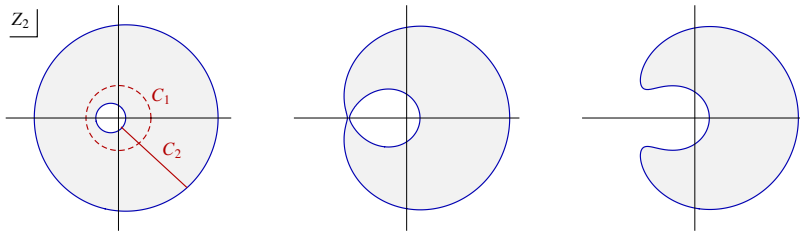


Figure 3.2: Plots showing the area Σ of the Z_2 plane which is covered by the D3-brane specified by (3.3). Increasing the residue ϵ , we progress from a torus $\mathcal{M} = S^2 \times S^1$ via the critical case (with the hole in the plane pinching off) to a deformed S^3 . Parameter values are $\alpha = 0.5$ and $\epsilon = 0.1, 0.1844, 0.2$. The dashed circle C_1 is an incontractible 1-cycle, while C_2 is an interval stretching radially between the boundaries of Σ as defined by (3.4).

Adding the meromorphic part opens up a hole in the base space Σ , thus increasing its genus (see figure 3.2). This may be understood by saying that in a neighbourhood of the pole, the term ϵ/Z_2 is so large that there are no solutions $|Z_1| \leq 1$. This obviously implies that the pole itself does not lie on the worldvolume of the brane.

To analyse this more carefully, it is easy to show, using (3.3), that the region Σ is given by the inequality

$$r_2^4 + r_2^2(\alpha^2 - 1) + \epsilon^2 \leq 2\epsilon\alpha r_2 \cos \phi_2. \quad (3.4)$$

When $\epsilon = 0$ and there is equality, we get $r_2 = 0, \sqrt{1 - \alpha^2}$ as solutions. Indeed, one could plot graphs of the left- and right-hand sides of (3.4) in terms of r_2^2 , so as to easily convince oneself that there are two intersections. Increasing ϵ , we notice that there is a range $0 < \epsilon < \epsilon_{\text{crit}}$ across which there are two intersections $r_2 > 0$ for all ϕ_2 , followed by a range $\epsilon_{\text{crit}} < \epsilon < \epsilon_{\text{max}}$ where there are two intersections at $\phi_2 = 0$ but none at $\phi_2 = \pi$. For larger ϵ , there are no intersections. This progression is shown in figure 3.2.

For $\epsilon < \epsilon_{\text{crit}}$, the topology of the brane is that of the product space $S^2 \times S^1$ - the incontractible 1-cycle C_1 (corresponding to the U(1) symmetry generated by ϕ_2) is the S^1 factor, while a radial line C_2 in Σ gives the S^2 factor - this is an interval over which is fibered the ϕ_3 1-cycle, which shrinks to zero at either end.

Single N -th order pole

About the simplest generalisation to the above case is to consider a higher-order pole:

$$f(Z_1, Z_2) = Z_1 - \alpha + \frac{\epsilon}{(Z_2)^N}. \quad (3.5)$$

When ϵ is sufficiently small, this leads to the same topology as for the single pole, but the geometry has an N -fold symmetry $Z_2 \rightarrow e^{i2\pi/N} Z_2$ corresponding to the order of the pole. Consequently, when $\epsilon_{\text{crit}} < \epsilon < \epsilon_{\text{max}}$, the brane degenerates and splits into N separate (deformed) 3-spheres. Figure 3.3 shows the case for which $N = 5$.

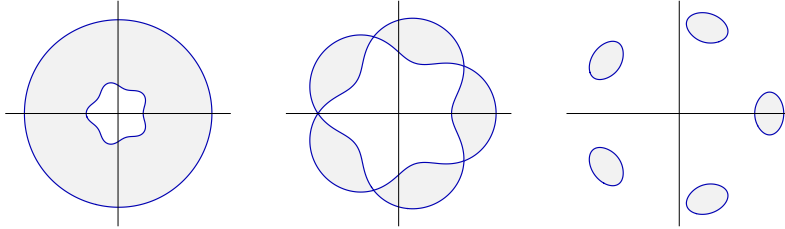


Figure 3.3: Plots showing Σ with a pole of order 5 (i.e. $N = 5$ in (3.5)), as evident from the 5-fold symmetry. By increasing the value of the residue ϵ , we pass from the torus on the left to a set of five disjoint 3-spheres on the right; the middle picture is when $\epsilon \approx \epsilon_{\text{crit}}$. Parameter values are $\alpha = 0.5$ and $\epsilon = 0.001, 0.03833, 0.2$.

n distinct poles

We can also consider the case with several distinct poles:

$$f(Z_1, Z_2) = Z_1 - \alpha + \sum_{j=1}^n \frac{\epsilon_j}{Z_2 - \beta_j}. \quad (3.6)$$

For small enough residues ϵ_j , the analysis very close to each pole will evidently be similar to that for one pole: expanding in $r_{2\beta} \equiv |Z_2 - \beta_j|$ gives us (3.4) plus terms higher order in $r_{2\beta}$. Thus, for any set of n poles located at β_j such that $\alpha^2 + |\beta_j|^2 < 1$, there exist residues $\epsilon_j \neq 0$ such that Σ is a disk with n holes. Cutting Σ along lines C_3 and C_4 in figure 3.4 so that each hole is isolated, we see that the resulting topology is a connected sum¹

$$\mathcal{M} = \#^n(S^2 \times S^1). \quad (3.7)$$

Note that all poles are evidently outside Σ , so that the function f is analytic everywhere on the worldvolume.

With multiple poles, the change in topology as we increase the residues ϵ_j can be quite complicated, and can produce several disjoint pieces. The case of three poles, with $\beta_i = \{-\frac{1}{2}, 0, \frac{1}{2}\}$, is shown in figure 3.4. Notice that the holes at ϵ_2/Z_2 and $\epsilon_3/(Z_2 - \beta_3)$ merge with each other in the middle picture. A similar effect can be produced by moving them together at fixed ϵ : when $\beta_3 \rightarrow 0$, these two approach (3.5). We study this kind of degeneration limit extensively below.

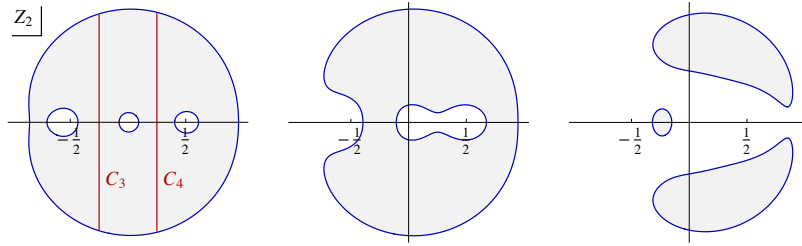


Figure 3.4: Plots showing Σ for the case $(1, 3, 0)$, using (3.6) with three poles at $\beta_1 = -\frac{1}{2}$, $\beta_2 = 0$ and $\beta_3 = \frac{1}{2}$. The residues are $\epsilon, \epsilon, -\epsilon$ respectively, with ϵ increasing from $\frac{1}{12}$ (left, with topology $\mathcal{M} = \#^3(S^2 \times S^1)$) to $\frac{1}{7}$ (centre, with topology $S^2 \times S^1$) to $\frac{1}{3}$ (right, $\sqcup_3 S^3$), and $\alpha = 0.5$. Notice that holes in Σ formed by residues of opposite signs attract, while those of the same sign repel. The red cuts C_3, C_4 in Σ each lift to a separating S^2 in \mathcal{M} .

Some interesting limits

Returning for a moment to our simplest case (3.3), there are two more distinct degeneration limits given by the two cycles shown in figure 3.2:

- As $\epsilon \rightarrow \epsilon_{\text{crit}}$, C_2 shrinks to a thin throat which is locally $S^2 \times \mathbb{R}$. The resulting geometry can effectively be interpreted as being due the effect of some strings with both ends attached to the brane and pinching it into a torus.

¹Recall that the notion of a connected sum is the following: If cutting a 3-manifold M along an S^2 separates the manifold into $M'_1 \sqcup M'_2$, and M_i is M'_i with a 3-ball glued to its boundary, then we write $M = M_1 \# M_2$. The 3-sphere is the identity in the sense $M = M \# S^3$. Every (oriented, closed, and connected) 3-manifold has a unique decomposition as a sum of prime manifolds, primeness meaning that every separating S^2 bounds a ball.

The connected sum of 2-manifolds, which we denote by \natural , is defined by similarly cutting along S^1 . This gives rise to the genus classification of surfaces $S^2, T^2, T^2 \natural T^2, \natural^g T^2$.

In the limit $\alpha = 0$, the brane becomes a thin torus, approaching the circular spinning string solution

$$Z_1 = \frac{1}{\sqrt{2}}e^{i(t+\sigma)}, \quad Z_2 = \frac{1}{\sqrt{2}}e^{i(t-\sigma)}, \quad Z_3 = 0.$$

The toroidal brane may thus be thought of as a blown-up circular string. However, note that all 3-branes given by (3.1) carry no worldsheet electric field F_{01} , and thus the string solution here is *not* an F-string.

- As $\epsilon \rightarrow 0$, C_1 wraps a small a throat locally $S^1 \times \mathbb{R}^2$. Here, it is useful to think of the $\epsilon \neq 0$ case of (3.3) not as resulting from the addition of a meromorphic term to (3.2), but (multiplying through by the denominator Z_2) as the addition of a small term to a factorised polynomial. In other words, $f = (Z_1 - \alpha)Z_2 + \epsilon$ describes exactly the same D3-brane as (3.3), but in the limit $\epsilon \rightarrow 0$, it more apparently approaches $f = (Z_1 - \alpha)Z_2$, which corresponds to a pair of intersecting sphere giants. If $\alpha = 0$, then it describes the intersection of a pair of maximal giants.

The effect of infinitesimal ϵ is localised near the intersection of the giants; at $Z_2 \neq 0$ and assuming $\alpha = 0$, it changes $Z_1 = 0$ to $Z_1 = \epsilon/Z_2$, perturbing the Z_1 giant smoothly away from maximality (and likewise for Z_2). But the effect close to $Z_1 = Z_2 = 0$ is not smooth, as $\sqcup_2 S^3$ is reconnected so as to give topologically $S^2 \times S^1$.

For the case of three poles in (3.6) (and taking the parameter values used in figure 3.4), the picture suggested by the limit $\epsilon \rightarrow 0$ is that of four intersecting branes, and hence we refer to this as the case $(1, 3, 0)$:

$$\begin{aligned} f(Z_1, Z_2) &= (Z_1 - \frac{1}{2})(Z_2 + \frac{1}{2}) Z_2 (Z_2 - \frac{1}{2}) + \epsilon (Z_2^2 - Z_2 - 1) \\ &\approx (Z_1 - \frac{1}{2}) Z_2 (Z_2^2 - \frac{1}{4}) + \epsilon'. \end{aligned}$$

The three Z_2 branes intersect the Z_1 brane at different places, and since $\epsilon \neq 0$ modifies the solution appreciably only near to the intersection, it is natural to think of the effect of several Z_2 branes as being simply due to the effect of one, (3.12). In the next section, we study more general cases in this limit, allowing also multiple Z_1 branes.

3.3 Class $(m, n, 0)$

$(3, 1, 0)$ case

Figure 3.4 above shows Σ for the case $(1, 3, 0)$. Let us now turn our attention to the case $(3, 1, 0)$, which must be an equivalent case. The simplest example is

$$f(Z_1, Z_2) = (Z_1^3 - \alpha^3) Z_2 + \epsilon. \quad (3.8)$$

Assuming $Z_2 \neq 0$, we solve for $Z_1 = \sqrt[3]{\alpha^3 - \epsilon/Z_2}$; if we again call the region of the Z_2 -plane covered by the solution Σ , this is now a three-sheeted Riemann surface, as shown in figure 3.5. Each sheet is a disk with one hole, and each branch

cut, as highlighted by the red line (from $Z_2 = 0$ to ϵ/α^3) runs from a point on the inner boundary of the disk to a point inside Σ . One can imagine implementing the following cut-and-glue process to better reveal the topology of the 3-manifold: cut across the branch cuts on the 3 sheets (and label each edge of each cut accordingly for identification), which leaves us with 3 disconnected pieces of the Riemann surface; then glue the pieces along their common S^2 's, as shown in the middle picture in figure 3.5. An alternative way of glueing is by making use of $S^2 \times I$ "2-handles", as illustrated in the right picture in figure 3.5, where a strip with red ends depicts such a handle with two 2-spheres attached at the extremities. It is clear then that Σ has three holes, thus we recover $\mathcal{M} = \#^3(S^2 \times S^1)$, as desired. The same procedure works equally well for branch cuts of any order.

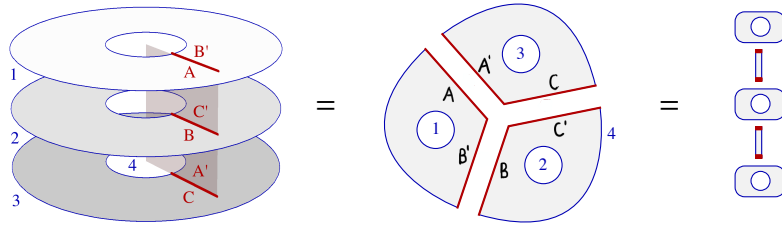


Figure 3.5: Branch cuts for the $(3, 1, 0)$ case (3.8). Each sheet of Σ on the left can be turned inside-out to give a wedge as shown in the middle picture; the numbers label boundary components. Glueing these back together, the result is a disk with three holes, drawn schematically on the right, and equivalent to figure 3.4. Note that the cuts labelled $A = A'$ etc. are *not* the S^2 glue lines of the connected sum.

Before moving on to a new case, let us emphasise the following point: as observed above, the *connected sum* of two 3-manifolds \mathcal{M}, \mathcal{N} can be achieved by connecting a 2-handle $S^2 \times I$ (i.e. S^3 with two punctures) between any two points on \mathcal{M} and \mathcal{N} :

$$\mathcal{M} \# \mathcal{N} = \mathcal{M} \# S^3 \# \mathcal{N} \quad \equiv \quad \text{two spheres} = \text{sphere with handle} = \text{two spheres with handle}$$

Attaching both ends of a 2-handle to the *same* manifold \mathcal{M} instead is equivalent to glueing a $S^2 \times S^1$ to that manifold:

$$\mathcal{M} + (S^2 \times I) = \mathcal{M} \# (S^2 \times S^1) \quad \equiv \quad \text{sphere with handle} = \text{sphere with hole}$$

Thus, in terms of the notational figures where $\text{circle} = S^2 \times S^1$ and $\text{strip} = S^2 \times I$ 2-handle, one can write

$$\text{circle} \text{---} \text{strip} \text{---} \text{circle} = \text{circle} \text{---} \text{circle} \text{---} \text{circle} = \#^3 \text{circle}.$$

$(2, 2, 0)$ case

Let us now consider the $(2, 2, 0)$ case:

$$f(Z_1, Z_2) = (Z_1^2 - \alpha^2)(Z_2^2 - \beta^2) + \epsilon. \quad (3.9)$$

It is easy to see that Σ is a two-sheeted Riemann surface, with two holes on each sheet, and both sheets are connected by a pair of branch cuts. To analyse the topology of this 3-manifold, we can first split each sheet along cuts C_i like those used in the previous section – see figure 3.6. Cutting along the branch cuts,

turning each of the four resulting pieces inside-out, and reconnecting along the C_i cuts and branch cuts by using a 2-handle, leads us to realise that the manifold has the same topology as the connectes sum $\sharp^5(S^2 \times S^1)$:

$$\mathcal{M} = \left[\sharp^2(S^2 \times S^1) \right] \sharp \left[\sharp^2(S^2 \times S^1) \right] + (S^2 \times I \text{ handle}) = \sharp^5(S^2 \times S^1).$$

Figure 3.6 illustrates this procedure. Instead of (3.9), we used $f(Z_1, Z_2) = (Z_1^2 - \alpha^2) + \epsilon/(Z_2 - \beta) + i\epsilon/(Z_2 + \beta)$, with $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ and $\epsilon = \frac{1}{9}$, to have a convenient arrangement of branch points. It also shows an alternative argument to verify that the choice of how we draw the branch cuts does not matter.

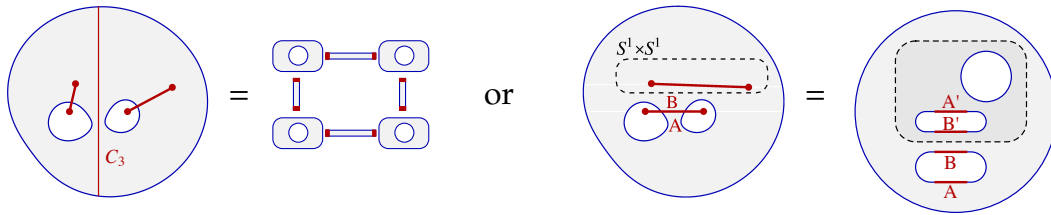


Figure 3.6: Branch cuts for the $(2, 2, 0)$ case (3.9). The diagram on the left shows, on the LHS, two square-root cuts each of which can be treated as in figure 3.5, giving the vertical connections shown on the RHS; the horizontal connections are from the glue line C_3 . Alternatively, the diagram on the right shows the same branch points pairwise-connected the other way. In this case we can pull the lower sheet of Σ through the cut to obtain the figure on the right - this happens within the dashed line, where the circular boundary component was the outer boundary of the lower sheet. Now Σ is a disk with three holes plus two 2-handles, giving the same topology.

$(m, n, 0)$ case

Generalising to the case $(m, n, 0)$, the topology of the corresponding manifold is

$$\mathcal{M} = \sharp^K(S^2 \times S^1), \quad \text{where } K = mn + (m - 1)(n - 1). \quad (3.10)$$

The counting comes from drawing a grid of $\textcircled{\square}$ and connecting horizontally (as in figure 3.4) and vertically (as in figure 3.5).

So far, for the most part, we have assumed that the $m + n$ intersecting branes are all at distinct positions, or in other words we considered only single poles. As we saw in the last section, allowing higher-order poles (3.5) did not necessarily change the topology, but this is no longer true here. We can investigate this by moving poles to coincide. There are two equivalent ways of doing this in (3.9), taking either $\alpha \rightarrow 0$ or $\beta \rightarrow 0$. Solving for Z_1 , the branch points are located at

$$Z_2 = \pm\beta, \pm\sqrt{\beta^2 + \epsilon/\alpha^2}.$$

When ϵ is sufficiently small, the second pair of branch points lie inside the region Σ , leading to the analysis above. However, if we hold ϵ fixed and take the limit

$\alpha \rightarrow 0$, those same points are moved infinitely far away from the first pair of branch points, leading to cuts that extend all the way across Σ . On the other hand, in the limit $\beta \rightarrow 0$, the holes in Σ merge into one (as in figure 3.4). Both situations are drawn in figure 3.7, and each leads to $\mathcal{M} = \sharp^3(S^2 \times S^1)$.

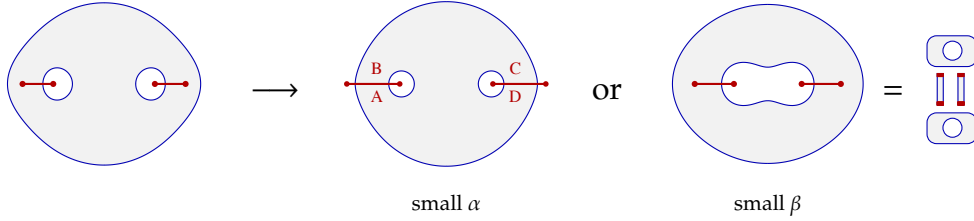


Figure 3.7: Degeneration limits of the $(2, 2, 0)$ case (3.9), drawing always the upper sheet of Σ . For the central picture (small α), we have two disks connected by four 2-handles (labelled A, \dots, D), while for the right-hand picture (small β), we have 2 tori connected by two 2-handles. The initial picture is with parameter values $\alpha = \beta = \frac{1}{2}$, $\epsilon = \frac{1}{10}$, and for each limit drawn, "small" means $\frac{1}{3}$.

We finally know how to treat any set of poles of arbitrary order for the $(m, n, 0)$ case. Let n be the number of distinct poles in Z_2 , and N be the sum of their orders – thus, two double poles give $n = 2$, $N = 4$, as do one single and one triple poles. Similarly, write $M \geq m$ for poles in Z_1 . If we draw an $m \times n$ grid of \odot , then it is clear that the number of vertical and horizontal connections (2-handles) is M and N , respectively – see figure 3.8. Then, counting the holes, we get that \mathcal{M} has topology

$$\mathcal{M} = \sharp^K(S^2 \times S^1), \quad \text{where } K = 1 + M(n - 1) + N(m - 1). \quad (3.14)$$

This change from (3.10) is a result of holes in Σ merging with each other as the branch points are brought close to each other. We learned in section 3.2 that a similar effect is observed when the residue ϵ is increased. Thus, we expect that, for a completely general $\frac{1}{4}$ -BPS giant, the topology will still be $\sharp^K(S^2 \times S^1)$ for some K .

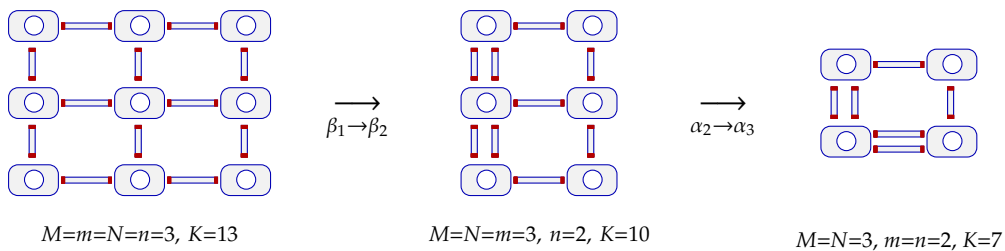


Figure 3.8: Degeneration of the $(3, 3, 0)$ case. Starting with three distinct single poles at $Z_1 = \alpha_i$ and three at $Z_2 = \beta_j$, we first allow two Z_2 poles to merge into a double pole, and then two Z_1 poles likewise. The prime decomposition of \mathcal{M} is given by (3.14) with the numbers shown.

3.4 Eighth-BPS Class

We now wish to include at least one intersecting Z_3 giant, and take a similar small- ϵ limit. Using what we have learned, we can immediately treat all $(1, n, 1)$ cases together. Consider the function

$$f(Z_1, Z_2, Z_3) = Z_1 Z_3 + \epsilon h(Z_2), \quad (3.11)$$

where h is a function with n poles. Setting $f = 0$ fixes $\phi_+ = \phi_1 + \phi_3$ and the product $r_1 r_3$ in terms of Z_2 , while $\phi_- = \phi_1 - \phi_3$ is unconstrained. We can solve for r_1 and r_3 as functions of Z_2 by writing $\sum_i r_i^2 = 1$ as

$$(r_1 \pm r_3)^2 = 1 - r_2^2 \pm 2 r_1 r_3 \equiv H_{\pm},$$

with $r_1 r_3 = \epsilon |h|$. This gives

$$(r_1, r_3) = \frac{1}{2} \left(\sqrt{H_+} \pm \sqrt{H_-}, \sqrt{H_+} \mp \sqrt{H_-} \right).$$

At a point Z_2 for which $H_- \geq 0$, there are two solutions (which coincide at $r_2 = \sqrt{1 - 2\epsilon |h|}$ when $H_- = 0$), while for $H_- < 0$ there are none. Notice that all points Z_2 , and all points sufficiently close to those, for which $h(Z_2)$ is singular, are excluded since, for such points, $H_- \rightarrow -\infty$ (also, when ϵ is small, each such hole will be small.) Define Σ to be two copies of the region of the Z_2 -plane for which $H_- \geq 0$, sewn up along the boundary. For the case of a single pole, the resulting topology is simply that of a torus, while for n poles, we have $\Sigma = \mathfrak{h}^n T^2$.

In the $\frac{1}{4}$ -BPS case, recall that we always have an S^1 fibred over Σ , shrinking to a point on the boundary $\partial\Sigma$. Now that there is no boundary, the $\phi_- = \phi_1 - \phi_3$ circle here never shrinks to a point. To see this, note that the metric is

$$ds^2 = \sum_{i=1}^3 \left(dr_i^2 + r_i^2 d\phi_i^2 \right) \sim (r_1^2 + r_3^2) d\phi_-^2 + \dots$$

Thus, for the length of the ϕ_- circle to vanish, we need $r_1 = r_3 = 0$, which implies $r_2 = 1$, but this is never part of Σ since, as we saw above, $r_2 \leq \sqrt{1 - 2\epsilon |h|}$. We therefore conclude that the topology is

$$\mathcal{M} = \left(\mathfrak{h}^n T^2 \right) \times S^1. \quad (3.12)$$

Note that all such three-manifolds \mathcal{M} are *prime*, that is they can be decomposed as connected sums only in the trivial way $\mathcal{M} = \mathcal{M} \# S^3$; the connected sum here is the two-dimensional one, and to canonically decompose \mathcal{M} , one must cut along T^2 's.

While we see that the region of the Z_2 -plane involved here is different to that for the $\frac{1}{4}$ -BPS case of section 3.2, we nevertheless observe that the essential point is similar in both cases, which is that each pole increases the genus of the base space Σ . It is natural to ask how much of our analysis of section 3.3 still holds. To begin to address this, we go back to the case $(1, 2, 1)$ case, where $f(Z_1, Z_2, Z_3) = Z_1 (Z_2^2 - \beta^2) Z_3 + \epsilon$. Solving for Z_2 when $f = 0$, we get

$$Z_2 = \pm \frac{\beta \sqrt{Z_1 Z_3 - \epsilon/\beta^2}}{\sqrt{Z_1 Z_3}}.$$

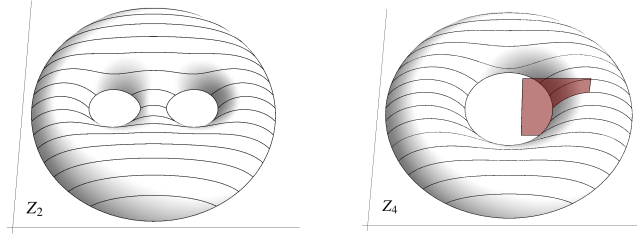


Figure 3.9: Plots for the $\frac{1}{8}$ -BPS case $(1, 2, 1)$. The picture on the left shows Σ drawn as two points fibered over Z_2 . The one on the right shows the upper of two sheets in the $Z_4 = Z_1 Z_3$ plane, which are connected by a branch cut drawn in red.

It is natural to think of this as having a branch cut in the $Z_4 = Z_1 Z_3$ plane. Fixing our position on this plane fixes Z_2 up to a choice of sheets, after which we still have (at a generic point) two solutions (r_1, r_3) . Figure 3.9 shows the upper-sheet part of Σ which, when glued along the branch cut drawn in red, gives a double torus, and thus the same topology as before. The angle in the S^1 factor is still $\phi_- = \phi_1 - \phi_3$.

One extension beyond (3.11) suggests itself fairly naturally. If we consider ²

$$f(Z_1, Z_2, Z_3) = 1 + \sum_{k=1}^m \frac{\epsilon'_k}{Z_1 Z_3 - \gamma_k} + \sum_{j=1}^n \frac{\epsilon_j}{Z_2 - \beta_j},$$

then we can invoke all the $\frac{1}{4}$ -BPS analysis. Just like we replaced Σ of figure 3.4 with two copies glued along their edges to get (3.12), similarly replace Σ of figures 3.6, 3.8 with their closed cousins. We get $(\mathbb{h}^K T^2) \times S^1$ with the same K as before. It is essential here, however, that f contains only the product $Z_1 Z_3$. The more natural class $(m, n, 1)$ of solutions

$$f(Z_1, Z_2, Z_3) = Z_3 + \sum_{i=1}^m \frac{\epsilon'_i}{Z_1 - \alpha_i} + \sum_{j=1}^n \frac{\epsilon_j}{Z_2 - \beta_j}$$

will break the S^1 symmetry in (3.12). Although we shall not further comment on this case, it would be nice to extend the analysis to this class of solutions, and ultimately to the completely general $\frac{1}{8}$ -BPS cases.

3.5 Conclusion

The main result of this work is the following:

Let $g(Z_1, Z_2)$ be a meromorphic function with m distinct poles at $Z_1 = \alpha_i$, and write M for the number of poles counting multiplicity. Similarly, let n and N count the poles and their multiplicities at $Z_2 = \beta_j$. Assume

²Note, as an aside, that we could likewise consider $\frac{1}{4}$ -BPS solutions of the form $f(Z_1, Z_4) = (Z_1 - \alpha)(Z_4 - \gamma) + \epsilon'$. For small γ , this can give a double-torus topology $\mathbb{h}^2 (S^2 \times S^1)$, but small ϵ' here does not guarantee that ϵ in (3.16) is small.

$m, n \geq 1$, and $|\alpha_i|^2 + |\beta_j|^2 \leq 1$ for all i, j .³ Consider the $\frac{1}{4}$ -BPS giant described by

$$f(Z_1, Z_2) = 1 + \epsilon g(Z_1, Z_2). \quad (3.13)$$

For sufficiently small ϵ , this has topology specified by the prime decomposition

$$\mathcal{M} = \#^K (S^2 \times S^1), \quad \text{where } K = 1 + M(n-1) + N(m-1). \quad (3.14)$$

As ϵ is increased, generically ⁴ K will decrease, and the brane may break up into several disjoint pieces. All resulting pieces are either 3-spheres or connected sums of $(S^2 \times S^1)$:

$$\mathcal{M} = \bigsqcup_i^L \#^{K_i} (S^2 \times S^1) \bigsqcup_j^{L'} S^3. \quad (3.15)$$

We found it convenient to deal with a function f with poles which in some sense repel the base space Σ , thus creating holes in the brane.⁵ The same solutions can equivalently be specified by polynomial functions of the form ⁶

$$f(Z_1, Z_2) = \prod_{i=1}^m (Z_1 - \alpha_i)^{\mu_i} \prod_{j=1}^n (Z_2 - \beta_j)^{\nu_j} + \epsilon \text{poly}(Z_1, Z_2). \quad (3.16)$$

Clearly, $\epsilon = 0$ gives a factorised f , which corresponds to a set of intersecting spherical giants (3.2). The effect of small ϵ is to suppress all but the simplest kind of interactions; the topology is unchanged when the last term here is replaced by a small constant. Nevertheless, what we have observed is that the effect of increasing ϵ is quite simple: the tori degenerate (thus reducing K), and ultimately split into disjoint spheres. Thus, we believe that (3.15) applies to generic polynomial functions $f(Z_1, Z_2)$.

In the case of $\frac{1}{8}$ -BPS geometries, we have more limited results. The generalisation which can be treated by borrowing much of the analysis from above is

$$f(Z_1, Z_2, Z_3) = 1 + \epsilon g(Z_4, Z_2), \quad \text{where } Z_4 = Z_1 Z_3$$

³We could weaken this condition to allow for cases where not every pair of branes intersect in the small- ϵ limit; one can be easily convinced that this has the effect of deleting some nodes from the corners of the lattice shown in figure 3.8, and thus reducing K , but not otherwise changing the topology.

⁴This is true if the residues of g are constants, in which case $\sum_i K_i \leq K$ and $L + L' \leq MN$ in (3.15). But if the numerator of g is of sufficiently high order, then K may increase.

⁵The poles are thus never on the worldvolume, so f is locally analytic, which is enough to guarantee a solution to the equations of motion from (3.1).

⁶Here $M = \sum_i \mu_i$ and $N = \sum_j \nu_j$.

with g defined as in (3.13). For small enough ϵ , the resulting topology is⁷

$$\mathcal{M} = \left[\mathfrak{h}^K (S^1 \times S^1) \right] \times S^1 ,$$

where K is as in (3.14). Notice that none of these topologies can occur in the $\frac{1}{4}$ -BPS case. Generalising this to allow other combinations of Z_3 branes (such as (3.16) with $\prod_k (Z_3 - \gamma_k)$ inserted) is an open problem. But it seems clear that the topology of \mathcal{M} will change, and in particular, will not have an overall S^1 factor.

While our focus in this work has been entirely on the classical membranes described by (3.1), a detailed quantisation of the moduli space of Mikhailov solutions was carried out in [106], and used to draw conclusions about the spectrum of $\frac{1}{8}$ -BPS states in $\mathcal{N} = 4$ SYM. It would be of great interest to pursue the relationship between those results and ours. This work forms part of a larger research program aimed at understanding how local and global properties of spacetime are encoded in gauge theory. For recent work in this direction, see [107, 108, 109] and [110] and references therein. In this context, more specifically in the AdS_5/CFT_4 correspondence, it would be nice to see how the topologies and topology changes studied here emerge and manifest themselves in operators dual to Mikhailov's giants in super-Yang-Mills theory. There too, perhaps our small- ϵ limit is likely to be the tractable one.

⁷From the topologies written down here, it is trivial to obtain the homology groups. The Betti numbers are:

$$\begin{aligned} \frac{1}{4}\text{-BPS} : & \quad b_0 = b_3 = 1, & \quad b_1 = b_2 = K \\ \frac{1}{8}\text{-BPS} : & \quad b_0 = b_3 = 1, & \quad b_1 = b_2 = 2K + 1 \end{aligned}$$

For the $(1, 1, 0)$ case, the generators of H_1 and H_2 are cycles C_1 and C_2 in figure 3.2. It is easy to draw similar cycles in figure 3.4's case $(1, 3, 0)$. Similar cycles drawn in figure 3.9's case $(1, 2, 1)$ will all be 1-cycles.

Chapter 4

Particle-vortex and Maxwell duality in the $AdS_4 \times CP^3$ /ABJM correspondence

4.1 Introduction

Non-perturbative dualities in quantum field theories have delivered many profound insights over the past three or so decades. Most famous among these are the lessons that we have learned about the very nature of spacetime via the duality between strongly coupled quantum field theories and gravitational theories, as manifested via the AdS/CFT correspondence. Within the realm of quantum field theories alone, non-perturbative dualities rely on the fact that the generating functions of observables include an integration over the degrees of freedom. Consequently, the set of degrees of freedom in terms of which one would choose to describe some particular system may result in multiple possibilities.

In four dimensions, for example, the electromagnetic duality, manifest in the Maxwell equations, allows us to describe a system in terms of electric or magnetic fields and charges and exchanges fundamental particles for solitonic degrees of freedom. We therefore have a choice as to how we describe the system, and at the perturbative level, one or the other may be more appropriate depending on the problem at hand. This electric-magnetic duality (and its extension by Witten and Olive [111]) has had a powerful impact, not only on our understanding of the structure of gauge theories, but also on some of the deepest mathematical puzzles of our time [112].

A three-dimensional analogue of the four-dimensional duality above is one which exchanges fundamental particles with solitonic vortices but, defined only for abelian gauge theories, this *particle-vortex duality*, as well as its physical implications, is much less understood than its four-dimensional counterparts. It is by now well-known that (2+1)-dimensional electron systems have remarkable properties, as manifested in the quantum Hall effect and metal-insulator transitions, many features of which still resist theoretical explanation. The difficult part of describing these systems is that they involve strong correlations, and no small

parameters present themselves to help with the analysis. Two kinds of theoretical tools which have proven useful for analysing these kinds of strong-coupling problems are effective field theory techniques, and the exploitation of symmetries. Duality symmetries, in particular, are likely to be useful since they typically relate strongly-coupled degrees of freedom to weakly-coupled ones, and in (2+1)-dimensions, particles and vortices make natural candidates for dual partners. Indeed, particle-vortex duality has been used several times in the condensed matter literature to describe some aspects of both the quantum Hall effect and conductor-insulator transitions in superconducting films.

Particle-vortex duality and AdS/CFT correspondence

Like many concepts commonplace in high-energy theory, particle-vortex duality has its roots in the landscape of condensed matter; in this case in the theory of anyonic superconductivity [114]. After some limited further development in condensed matter physics, it was in the context of string theory that more development occurred, starting with Intriligator & Seiberg [115]. Since the anyon is usually thought of as a quasiparticle (obeying *fractional statistics*) in a strongly coupled system, and cannot be seen in the perturbative approach, it is interesting to see, within the framework of gauge-gravity duality, whether one can construct holographic duals of anyons of some strongly systems so that anyons can be realised as D-brane configurations. Motivated by the relation between anyons and (2+1)-dimensional Chern-Simons theory, it seems that the recently constructed ABJM theory [8] is a good starting point, since the ABJM theory is given as $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory in 2+1 dimensions. Furthermore, its gravity dual, type IIA supergravity in $AdS_4 \times CP^3$ background, is known.

The authors of [116], following a lead by Hartnoll [117], managed to construct the holographic anyons in ABJM theory from the gravity, CFT, and open string sides via AdS/CFT correspondence. The construction is more subtle than naively expected in all three aspects because it is the nontrivial generalisation of the usual anyon constructed in the $U(1)$ Chern-Simons effective theory. In the $U(1)$ case, one attaches the magnetic flux to the electron to make it anyon via the Chern-Simons coupling; in the ABJM case on the other hand, one attaches a *nonabelian 't Hooft operator* (a generalised Wilson loop) to the baryon in the CFT to make it anyonic. They find two types of holographic anyons as the *dressed baryons*: D0-D2 and D4-D6 bound states. For D4-D6, the anyonic phase is proportional to the 't Hooft coupling, and for D0-D2, its inverse. These bound states are related by T-duality¹. Moreover, by combining with level-rank duality, one can transform one anyonic phase to the other one. It is interesting to see whether the combination of D0/D4 duality and level-rank duality is related to the particle-vortex duality in the quantum Hall system. If this is the case, then D0-D2 and D4-D6 can be understood as the particle-vortex dual pair of collective modes of the CFT.

¹Interestingly, these two pairs are not related by the usual Hodge duality in ten dimensions, since it relates C_1 to C_7 and C_3 to C_5 , but in the relation above, the roles of the D0-brane and D4-brane are exchanged. It has been suggested that this relation can be understood as a kind of geometric duality inside CP^3 [118].

Particle-vortex duality à la Zee

To set the scene for what follows, we will first give an heuristic description of the particle-vortex duality elaborating on a discussion in the textbook of Zee [113], before embarking on a more technical treatment in the following section. Following [113] then, we start with an abelian Higgs model

$$\mathcal{L} = -\frac{1}{2} |(\partial_\mu - iqA_\mu)\phi|^2 - V(\phi^\dagger\phi) ,$$

with some well-behaved potential, $V(\phi^\dagger\phi)$, for the complex scalar field ϕ with $U(1)$ charge q . We shall ignore the potential term from now on, but presume that the theory exhibits vortex solutions (and consequently restrict our attention to three dimensions). Writing $\phi = |\phi| e^{i\theta}$ and restricting to the solution for which $|\phi| = v$ minimises the potential gives

$$\mathcal{L} = -\frac{1}{2} v^2 (\partial_\mu\theta - qA_\mu)^2 . \quad (4.1)$$

The spectrum of the lagrangian includes vortices and antivortices, located where $|\phi|$ vanishes. We can introduce an auxiliary field ξ_μ , so that the lagrangian takes the first-order form

$$\mathcal{L} = +\frac{1}{2v^2} \xi_\mu^2 - \xi^\mu (\partial_\mu\theta - qA_\mu) ,$$

The phase θ^2 , characterising the vortex is, in fact, singular at the origin for a vortex solution, allowing us to split it into a *smooth* part, and a *vortex* part:

$$\theta = \theta_{\text{smooth}} + \theta_{\text{vortex}} ,$$

where the *vortex monodromy* $\Delta\theta_{\text{vortex}} = 2\pi$ (the antivortex has monodromy $\Delta\theta_{\text{vortex}} = -2\pi$). Integrating over θ_{smooth} gives the constraint $\partial_\mu\xi^\mu = 0$, which implies that we can write ξ^μ as the curl of a vector field a_ρ

$$\xi^\mu = \epsilon^{\mu\nu\rho} \partial_\nu a_\rho .$$

Having integrated out θ_{smooth} and substituted in the new expression for ξ^μ , we get the following Lagrangian

$$\mathcal{L} = -\frac{1}{4v^2} f_{\mu\nu}^2 + \epsilon^{\mu\nu\rho} \partial_\nu a_\rho \partial_\mu \theta_{\text{vortex}} - A_\mu J^\mu , \quad (4.2)$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, and $J^\mu = q \epsilon^{\mu\nu\rho} \partial_\nu a_\rho$. Partially integrating the a_0 part of the second term in (4.2) picks up a factor of 2π via Stokes' theorem. Hence, $\frac{1}{2\pi} \epsilon^{ij} \partial_i \partial_j \theta_{\text{vortex}}$ measures the density of vortices, and is the time-component of some Lorentz-invariant vortex current j_{vortex}^μ , where

$$2\pi a_\mu j_{\text{vortex}}^\mu = a_\rho \epsilon^{\rho\mu\nu} \partial_\mu \partial_\nu \theta_{\text{vortex}} . \quad (4.3)$$

Thus, we have

$$\mathcal{L} = -\frac{1}{4v^2} f_{\mu\nu}^2 + 2\pi a_\mu j_{\text{vortex}}^\mu - A_\mu J^\mu . \quad (4.4)$$

²This is a pseudoscalar in the theory.

Note that (4.3) is a crucial step in this derivation. Since the partial derivatives are contracted with the antisymmetric tensor, by symmetry arguments, this expression will naively vanish. The only time this will not be the case is when θ is singular. Thus, θ_{vortex} is explicitly that part of θ which is not smooth and whose second derivative is related through this equation to the vortex current. In other words, *if there are no vortices, this expression will vanish and there will be no duality.*

The introduction of an auxiliary vector field ξ_μ leads to the coupling of the vortex current j_{vortex}^μ to the gauge field a_μ . We have thus gone from a description where the fundamental excitations are the particles associated to the field ϕ , to the vortex description where the fundamental degrees of freedom are the vortices associated to θ_{vortex} . However, to complete this description, we must have a field whose fundamental excitations themselves are vortices. To that end, we introduce a new complex scalar field Φ , which couples to a_μ precisely for this purpose. On adding this field, we can define an action which gives a dual description, with particle and vortex degrees of freedom swapped. The lagrangian

$$\mathcal{L} = -\frac{1}{4v^2} f_{\mu\nu}^2 - \frac{1}{2} |(\partial_\mu - i2\pi a_\mu)\Phi|^2 - W(|\Phi|^2) - A_\mu J^\mu, \quad (4.5)$$

then describes an abelian Higgs model for the vortex field Φ coupled to a_μ as opposed to the original field where ϕ was coupled to A_μ . The potential $W(|\Phi|^2)$ contains terms such as $\lambda(\Phi^\dagger\Phi)^2$ describing short-distance interaction of two vortices (or a vortex-antivortex pair)³. The action of the transformation

$$\partial_\mu \theta - qA_\mu = \xi_\mu = \epsilon_{\mu\nu\rho} \partial^\nu a^\rho$$

exchanges the scalar degree of freedom θ with the gauge field degree of freedom a_μ in the presence of the background gauge field A_μ . However, the necessity to introduce the new field Φ does not feel very satisfactory. We will see that there is a more complete way to formalise the duality⁴. The above transformation is also not strictly true in the presence of Φ .

Note, as an aside, that the above dual representation of the theory can, in turn, be dualised by requiring that Φ vanishes at points around which it picks a phase of 2π . We end up with a theory described by the original field ϕ , which carries the same electric charge, q , as the vortex. The vortex also carries some charge akin to a magnetic flux (the $\xi^0 = \epsilon^{ij} \partial_i a_j$ component of the auxiliary field), and from (4.5), we see that $2\pi a_i \rightarrow \partial_i \theta$, so that we have $2\pi \int d^2x \xi^0 = 2\pi \int d^2x \epsilon^{ij} \partial_i a_j = 2\pi \oint d\vec{x} \cdot \vec{a} = 2\pi$. The electric charge of the vortex is equal to $\int d^2x J^0 = q$, which is precisely the charge of the original field ϕ .

A supersymmetric generalisation of these ideas was proposed in [120] (see also [121]), where, however, a path integral transformation realising the particle-vortex duality could only be reduced to an unproven identity. Witten [122] later defined an $SL(2, \mathbb{Z})$ transformation on a conformal field theory by combining an S -transformation (which adds an $\epsilon B \partial A$ term to the lagrangian) with a

³The treatment so far has been completely heuristic, with the assumption that the coupling between θ_{vortex} and $|\phi|$ is small enough that it can be neglected.

⁴A more precise definition of particle-vortex duality, and an understanding of how it arises in a path integral formulation, was given by Burgess & Dolan in [119]. For completeness, we review their formulation in appendix B.1.

T -transformation (which adds a Chern-Simons term, $\epsilon A \partial A$). For example, starting with a charged scalar lagrangian of the form $\tilde{L}(\Phi, A)$, the TS -transformation maps

$$\tilde{L}(\Phi, A) \xrightarrow{TS} L(\Phi, A, B) = \tilde{L}(\Phi, A) + \epsilon^{ijk} B_i \partial_j A_k + \epsilon^{ijk} A_i \partial_j A_k .$$

The current-current two-point function of this three-dimensional conformal field theory is constrained by conformal symmetry to be of the form:

$$\langle J_i(k) J_j(-k) \rangle = (\delta_{ij} k^2 - k_i k_j) \frac{t}{2\pi \sqrt{k^2}} + \epsilon_{ijkl} k^l \frac{w}{2\pi} ,$$

where t and w combine to form a complex coupling $\tau = w + it$. The action of the TS -transformations of the $SL(2, \mathbb{Z})$ group on the complex parameter τ is then $\tau \rightarrow (a\tau + b)/(c\tau + d)$. Because this is an action on a conformal field theory, we can ask what the action of the transformation is on the gravity dual of this theory via the AdS/CFT correspondence. In this case the transformation acts on a $U(1)$ gauge field with a Maxwell action plus a topological θ -term.

Later, the constraints imposed on correlators in gauge theories by the existence of a particle-vortex duality were analysed in [123]. Note that when the theory is changed by the action of the duality, (i.e. the theory is *not* self-dual), the correlators are themselves transformed. The authors also analysed the $AdS_4 \times S^7$ gravity dual of the $\mathcal{N} = 8$ three-dimensional $SU(N)$ super-Yang-Mills theory in the large- N limit, and found that Maxwell duality in the bulk leads to the same type of constraints on correlators as would be obtained from a self-dual field theory. In abelian models, a similar relation was obtained, and a correspondence with $AdS_4 \times S^7$ was proposed as an implicit relation coming from large- N non-abelian gauge theories.

Today, the ABJM model [8] is understood as the correct description of the field theory living on M2-branes, and is dual (in the appropriate limit) to type IIA supergravity on $AdS_4 \times \mathbb{C}P^3$. This begs the question of whether the results of [123] can be reinterpreted from this point of view⁵.

The aim in this work is two-fold: first, we seek to provide a more precise definition of the particle-vortex duality at the level of a path integral transformation then, using this, we attempt to embed the duality transformation in the ABJM model.

Outline

The structure of the rest of this chapter is as follows. In **section 4.2**, we revisit the formulation of the particle-vortex duality by retaining some features of the relation of [119] (reviewed in **appendix B.1**) and defining it as an *action on the path integral* of the theory. In particular, we find that, by combining it with the *Mukhi-Papageorgakis Higgs mechanism* for three-dimensional Chern-Simons theories [126] (see also [127]), we can define it as a self-duality of abelian Chern-Simons theories. In **section 4.3**, we look explicitly at vortex solutions and the conditions under

⁵The ABJM theory is also known to admit a maximally supersymmetric mass deformation [124, 125], which not only allows us to go away from the conformal limit but also contains a rich spectrum of solitonic excitations.

which they exist in such theories. In **section 4.4**, we embed the particle-vortex duality in ABJM, showing that the abelian duality is part of the (large- N) non-abelian theory. Finally, in **section 4.5**, we show that the particle-vortex duality is naturally obtained as the boundary relation corresponding to Maxwell duality in the bulk, using the AdS/CFT prescription. Thus, as in [128, 129], we see that by using an abelian reduction of ABJM to an interesting non-conformal theory, we learn something about the structure of ABJM.

4.2 Abelian particle-vortex duality in the path integral

In this section, we shall extend the path integral formulation of [119] to provide a better definition of the particle-vortex duality in abelian theories. To this end, let us consider a path integral for an abelian Higgs model consisting of a complex scalar field $\Phi = \Phi_0 e^{i\theta}$ coupled to a U(1) gauge field a_μ . Any kinetic term for the gauge field will be no more than a spectator for the transformation, as in the Burgess-Dolan formulation described in appendix B.1, and we shall ignore any such term in what follows. Additionally, there will also be a potential term for Φ_0 , $V(\Phi_0^2)$, but this will also assume the role of a spectator, so we will also choose to omit it now. When we want to explicitly discuss vortex solutions however, the potential will be important and will be included. As long as we are not integrating over a_μ or Φ_0 , we do not need to consider those last terms we have mentioned.

The partition function for the theory is

$$\begin{aligned} Z &= \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\theta \exp \left[-\frac{i}{2} \int d^3x |(\partial_\mu - ie a_\mu) \Phi|^2 \right] \\ &= \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\theta \exp \left\{ -\frac{i}{2} \int d^3x \left[(\partial_\mu \Phi_0)^2 + (\partial_\mu \theta_{\text{smooth}} + \partial_\mu \theta_{\text{vortex}} + e a_\mu)^2 \Phi_0^2 \right] \right\}, \end{aligned}$$

where, as in the previous section, we have split the phase θ into a smooth part, and a topologically non-trivial and non-smooth vortex part. We define $\lambda_\mu = \partial_\mu \theta$, after which we promote it to an independent variable in a first-order formulation. $\lambda_\mu = \partial_\mu \theta$ follows from the constraint $\epsilon^{\mu\nu\rho} \partial_\nu \lambda_\rho = 0$, which can be imposed via a Lagrange multiplier b_μ , giving the path integral for the *master* action

$$\begin{aligned} Z &= \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}b_\mu \mathcal{D}\lambda_\mu \exp \left\{ -\frac{i}{2} \int d^3x \left[(\partial_\mu \Phi_0)^2 + (\lambda_{\mu,\text{smooth}} + \lambda_{\mu,\text{vortex}} + e a_\mu)^2 \Phi_0^2 \right. \right. \\ &\quad \left. \left. + \frac{2}{e} \epsilon^{\mu\nu\rho} b_\mu \partial_\nu \lambda_\rho \right] \right\}. \end{aligned}$$

Integrating over b_μ takes us back to the original formulation for the partition function, establishing the self-consistency of our procedure. If, however, we integrate over λ_μ first, we obtain the equation of motion

$$(\lambda_{\mu,\text{smooth}} + \lambda_{\mu,\text{vortex}} + e a_\mu) e \Phi_0^2 = -\epsilon^{\mu\nu\rho} \partial_\nu b_\rho, \quad (4.6)$$

which, on substitution back into the action produces the path integral for the *dual* action,

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}b_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{4e^2 \Phi_0^2} f_{\mu\nu}^b f^{b\mu\nu} + \epsilon^{\mu\nu\rho} b_\mu \partial_\nu a_\rho - \frac{2\pi}{e} j_{\text{vortex}}^\mu(t) b_\mu + \frac{1}{2} (\partial_\mu \Phi_0)^2 \right] \right\},$$

where $j_{\text{vortex}}^\mu(t)$ is the vortex current in (B.5), that is,

$$j_{\text{vortex}}^\mu(t) = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho \theta = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho \omega = \sum_a N_a \dot{y}_a^\mu \delta[x - y_a(t)],$$

and is associated with the existence of vortex boundary conditions for θ in the original action with vortices positioned at $\vec{y}_a(t)$ in the two-dimensional space (see equation (B.4) for a definition of ω). In the dual action, this current appears as an explicit source term. Here, the summation is over all vortex positions labelled by the index a . Also note that, as in (B.10), $j_\mu = e \Phi_0^2 \partial_\mu \theta$ is a scalar current, and we then have the duality relation between the vortex current and the scalar current:

$$\boxed{j_{\text{vortex}}^\mu(t) = \frac{1}{2\pi e \Phi_0^2} \epsilon^{\mu\nu\rho} \partial_\nu j_\rho}. \quad (4.7)$$

Notice that, here, Φ_0 has the interpretation of a coupling constant for the field b_μ dual to θ , which itself becomes a dynamical Maxwell gauge field. In this sense, this duality maps particles to vortices, hence the name *particle-vortex duality*.

4.2.1 The Mukhi-Papageorgakis Higgs mechanism

There is a striking similarity between the particle-vortex duality described here and a version of the Higgs mechanism for three-dimensional Chern-Simons theories discovered by Mukhi and Papageorgakis in [126] in the context of ABJM theories, but valid more generally (see also [127] for more details about its implementation). The statement analogous to the usual Higgs mechanism statement that a massless gauge field eats a scalar and becomes massive, is now that a Chern-Simons gauge field (with no dynamical degrees of freedom) eats a scalar and becomes dynamical, i.e. of Maxwell (or Yang-Mills) form with one dynamical degree of freedom.

The mechanism itself goes as follows. We start with an action for a complex scalar, Ψ , coupled to a Chern-Simons gauge field, a_μ ,

$$S = - \int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{a}_\rho + \frac{1}{2} |(\partial_\mu - ie a_\mu) \Psi|^2 + V(|\Psi|^2) \right], \quad (4.8)$$

with a vacuum solution $\Psi = b$, and \tilde{a} is an auxiliary field. We can then expand the scalar degrees of freedom around the ground state

$$\Psi = (b + \delta\psi) e^{-i\delta\theta}, \quad \text{where} \quad \delta\theta = \theta_{\text{smooth}} + \theta_{\text{vortex}},$$

and plug it back in the action to find

$$S = - \int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{a}_\rho + \frac{1}{2} (\partial_\mu \delta\psi)^2 + \frac{1}{2} (\partial_\mu \theta_{\text{smooth}} + \partial_\mu \theta_{\text{vortex}} + e a_\mu)^2 b^2 + \dots \right].$$

The omitted terms include the $\delta\psi$ self-interaction in $V(|\Psi|^2)$ and the $\delta\theta$ - $\delta\psi$ interaction. Note that, for the purposes of making a comparison, we have allowed for the possibility that $\delta\theta$ contains a singular vortex piece θ_{vortex} . The mechanism by which the Chern-Simons vector eats the scalar and becomes a dynamical Maxwell vector happens through exactly the same redefinition as in the the usual Higgs mechanism. Here we write

$$\partial_\mu \theta_{\text{smooth}} + \partial_\mu \theta_{\text{vortex}} + e a_\mu = e a'_\mu, \quad (4.9)$$

where a'_μ is the Maxwell gauge field. We then trivially integrate out θ , and add a boundary term to the action to obtain

$$S = - \int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\rho} a'_\mu \partial_\nu \tilde{a}_\rho + \frac{1}{2} (\partial_\mu \delta\psi)^2 + \frac{1}{2} (e a'_\mu)^2 b^2 - \frac{k}{e} j_{\text{vortex}}^\mu \tilde{a}_\mu + \dots \right].$$

Solving for a'_μ gives

$$a^\mu + \frac{1}{e} \partial^\mu \delta\theta = a'^\mu = -\frac{k}{2\pi b^2 e^2} \epsilon^{\mu\nu\rho} \partial_\nu \tilde{a}_\rho,$$

which is similar to (4.6) for the particle-vortex duality. Defining $\tilde{f}_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$, we find

$$S = - \int d^3x \left[\frac{k^2}{16\pi^2 b^2 e^2} \tilde{f}_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{1}{2} (\partial_\mu \delta\psi)^2 - \frac{k}{e} j_{\text{vortex}}^\mu \tilde{a}_\mu + \dots \right], \quad (4.10)$$

where, again, the terms alluded above have been omitted. We close this section with two remarks. Firstly, adding a term proportional to $-\epsilon^{\mu\nu\rho} \tilde{a}_\mu \partial_\nu b_\rho$ to either (4.8) or (4.10) does not change anything since the transformations do not act on either of the fields \tilde{a} or b . Secondly, assuming vortex boundary conditions in the initial action (that is, assuming θ has a singular part) leads to a vortex current coupling in the final action. Again, this is as in the case of particle-vortex duality, although here we can assume regular boundary conditions and thus avoid the vortex current j_{vortex}^μ .

4.2.2 A symmetric duality

As described in the previous section, particle-vortex duality is not a self-duality – it effectively maps the original action to a manifestly different action. In particular, it does so by dualising the phase angle θ to the gauge field b_μ . For our purposes of embedding the duality in the ABJM model, it will be useful to ‘*symmetrise*’ this duality. As we demonstrate now, this may be achieved by adding a gauge field and a real scalar, and dualising them to a single complex scalar field. This implies that the original and final action will end up looking the same. As before, we may also add vortex currents. We will also omit a possible kinetic term for a_μ , and

explicitly write the self-interactions of the scalars Φ and χ . Our starting point, again, will be the path integral

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\theta \mathcal{D}\tilde{b}_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{2} |(\partial_\mu - ie a_\mu) \Phi_0 e^{-i\theta}|^2 + \frac{1}{2} (\partial_\mu \chi_0)^2 \right. \right. \\ \left. \left. + \frac{1}{4 e^2 \chi_0^2} f_{\mu\nu}^{(\tilde{b})} f^{(\tilde{b})\mu\nu} + \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{b}_\rho - \frac{2\pi}{e} \tilde{b}_\mu \tilde{j}_{\text{vortex}}^\mu(t) + V(\Phi_0^2) + V(\chi_0^2) \right] \right\}, \quad (4.11)$$

where $\tilde{j}_{\text{vortex}}^\mu(t)$ is a source term that, in the dual version, will be associated to vortex boundary conditions for the dual scalar. \tilde{b}_μ and χ_0 are our new gauge and scalar fields. It is the addition of these two that will lead to a self-dual action. We again write a first-order formulation for $\lambda_\mu = \partial_\mu \theta$, and then impose this relation as the constraint $\epsilon^{\mu\nu\rho} \partial_\nu \lambda_\rho = 0$ through a Lagrange multiplier b_μ . Conversely, one can define $\tilde{\lambda}_\mu$ via a tilde version of (4.6), namely

$$(\tilde{\lambda}_{\mu,\text{smooth}} + \tilde{\lambda}_{\mu,\text{vortex}} + e a_\mu) e \chi_0^2 = -\epsilon_\mu^{\nu\rho} \partial_\nu \tilde{b}_\rho,$$

and then introduce $\tilde{\lambda}_\mu$ in the action such that one ends up with the above expression as its equation of motion. Either way, we obtain the path integral for the *master* action

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\lambda_\mu \mathcal{D}b_\mu \mathcal{D}\tilde{\lambda}_\mu \mathcal{D}\tilde{b}_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{2} (\partial_\mu \Phi_0)^2 + \frac{1}{2} (\partial_\mu \chi_0)^2 \right. \right. \\ \left. \left. + \frac{1}{e} \epsilon^{\mu\nu\rho} (b_\mu \partial_\nu \lambda_\rho + \tilde{b}_\mu \partial_\nu \tilde{\lambda}_\rho) + \frac{1}{2} (\lambda_\mu + \lambda_{\mu,\text{vortex}} + e a_\mu)^2 \Phi_0^2 + \frac{1}{2} (\tilde{\lambda}_\mu + \tilde{\lambda}_{\mu,\text{vortex}} + e \tilde{a}_\mu)^2 \chi_0^2 \right. \right. \\ \left. \left. + V(\Phi_0^2) + V(\chi_0^2) \right] \right\}.$$

Now, repeating the same procedure with the fields with tilde replaced with unaccented fields (or, equivalently, integrating over λ_μ and \tilde{b}_μ , to write $\tilde{\lambda}_\mu = \partial_\mu \tilde{\theta}$), we obtain the path integral for the dual action

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\tilde{\theta} \mathcal{D}b_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{2} |(\partial_\mu - ie a_\mu) \chi_0 e^{-i\tilde{\theta}}|^2 + \frac{1}{2} (\partial_\mu \Phi_0)^2 \right. \right. \\ \left. \left. + \frac{1}{4 e^2 \Phi_0^2} f_{\mu\nu}^{(b)} f^{(b)\mu\nu} + \epsilon^{\mu\nu\rho} a_\mu \partial_\nu b_\rho - \frac{2\pi}{e} b_\mu j_{\text{vortex}}^\mu(t) + V(\Phi_0^2) + V(\chi_0^2) \right] \right\}. \quad (4.12)$$

Assuming that a_μ has no kinetic term, we can now actually integrate it out in both the original and dual actions. Indeed, the terms containing a_μ in the lagrangian (4.11) are

$$\mathcal{L}_{(a)} = -\frac{1}{2} e^2 a_\mu^2 \Phi_0^2 - a^\mu (j_\mu + J_\mu),$$

where $j_\mu = -\frac{ie}{2} (\Phi \partial_\mu \Phi^* - \Phi^* \partial_\mu \Phi) = e \Phi_0^2 \partial_\mu \theta$ is the scalar current, and $J^\mu = \epsilon^{\mu\nu\rho} \partial_\nu \tilde{b}_\rho$ the topological (vortex-like) current. Solving for a_μ we obtain

$$a_\mu = -\frac{1}{e^2 \Phi_0^2} (j_\mu + J_\mu),$$

and substituting back into $\mathcal{L}_{(a)}$, produces an extra contribution

$$\mathcal{L}_{\text{extra}} = +\frac{1}{2e^2\Phi_0^2}(j_\mu + J_\mu)^2 = -\frac{1}{4e^2\Phi_0^2}(f_{\mu\nu}^{\tilde{b}} - \epsilon_{\mu\nu\rho}j^\rho)^2.$$

Having thus eliminated a_μ from the picture, we are now in a position to realise the duality as a map from

$$Z = \int \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\theta \mathcal{D}\tilde{b}_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{2} |\partial_\mu(\Phi_0 e^{-i\theta})|^2 + \frac{1}{2} (\partial_\mu \chi_0)^2 + \frac{1}{4e^2\chi_0^2} f_{\mu\nu}^{(\tilde{b})} f^{(\tilde{b})\mu\nu} \right. \right. \\ \left. \left. + \frac{1}{4e^2\Phi_0^2} (f_{\mu\nu}^{(\tilde{b})} - \epsilon_{\mu\nu\rho}j^\rho)^2 - \frac{2\pi}{e} \tilde{b}_\mu \tilde{j}_{\text{vortex}}^\mu(t) + V(\Phi_0^2) + V(\chi_0^2) \right] \right\},$$

into

$$Z = \int \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\tilde{\theta} \mathcal{D}b_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{2} |\partial_\mu(\chi_0 e^{-i\tilde{\theta}})|^2 + \frac{1}{2} (\partial_\mu \Phi_0)^2 + \frac{1}{4e^2\Phi_0^2} f_{\mu\nu}^{(b)} f^{(b)\mu\nu} \right. \right. \\ \left. \left. + \frac{1}{4e^2\chi_0^2} (f_{\mu\nu}^{(b)} - \epsilon_{\mu\nu\rho}\tilde{j}^\rho)^2 - \frac{2\pi}{e} b_\mu j_{\text{vortex}}^\mu(t) + V(\Phi_0^2) + V(\chi_0^2) \right] \right\},$$

that furnishes a formulation of the particle-vortex duality with an explicitly self-dual action.

Of course, since our aim is to embed the particle-vortex duality into the ABJM model, where we have only scalars and a Chern-Simons gauge field at our disposal, we will need to combine the symmetric form of the duality above with the Mukhi-Papageorgakis Higgs mechanism of the previous section. Moreover, in order for the duality to be nontrivial, we need to retain the vortex boundary conditions only in the original scalar, not the one that gets Higgsed. Starting from the path integral

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\theta \mathcal{D}\tilde{b}_\mu \mathcal{D}\chi \mathcal{D}\chi^* \mathcal{D}\mathcal{A}_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{2} |(\partial_\mu - ie a_\mu) \Phi_0 e^{-i\theta}|^2 \right. \right. \\ \left. \left. + \frac{1}{2} |(\partial_\mu - ie \mathcal{A}_\mu) \chi_0 e^{-i\phi}|^2 + \epsilon^{\mu\nu\rho} \left(\frac{1}{e} \mathcal{A}_\mu \partial_\nu \tilde{b}_\rho + a_\mu \partial_\nu \tilde{b}_\rho \right) + V(\phi_0^2) + V(\chi_0^2) \right] \right\}, \quad (4.13)$$

we first implement the Mukhi-Papageorgakis Higgs mechanism by shifting $\mathcal{A}_\mu \rightarrow \mathcal{A}'_\mu$ as in equation (4.9), thus absorbing ϕ in the process, and performing the (now trivial) path integral over ϕ . Subsequently, we integrate over \mathcal{A}'_μ using the equation of motion

$$\partial_\mu \phi + \partial_\mu \phi_{\text{vortex}} + e \mathcal{A}_\mu \equiv e \mathcal{A}'_\mu = \frac{1}{e^2 \chi_0^2} \epsilon_\mu{}^{\nu\rho} \partial_\nu \tilde{b}_\rho, \quad (4.14)$$

and get exactly the path integral in (4.11) which, as we saw previously, is dual to (4.12). We now undo the Mukhi-Papageorgakis Higgs mechanism by writing a first-order formalism for $f_{\mu\nu}^{(b)}$ in terms of a field $\tilde{\mathcal{A}}'_\mu$, then introducing a trivial path integration over a variable $\tilde{\phi}$ and shifting \mathcal{A}'_μ by

$$\partial_\mu \tilde{\phi} + \partial_\mu \tilde{\phi}_{\text{vortex}} + e \tilde{\mathcal{A}}'_\mu \equiv e \tilde{\mathcal{A}}'_\mu = \frac{1}{e^2 \Phi_0^2} \epsilon_\mu{}^{\nu\rho} \partial_\nu b_\rho,$$

so that we finally arrive at the path integral

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\chi_0 \mathcal{D}\tilde{\theta} \mathcal{D}b_\mu \mathcal{D}\Phi \mathcal{D}\Phi^* \mathcal{D}\tilde{\mathcal{A}}_\mu \exp \left\{ -i \int d^3x \left[\frac{1}{2} |(\partial_\mu - ie a_\mu) \chi|^2 + \frac{1}{2} |(\partial_\mu - ie \tilde{\mathcal{A}}_\mu) \Phi|^2 + \epsilon^{\mu\nu\rho} \left(\frac{1}{e} \tilde{\mathcal{A}}_\mu \partial_\nu b_\rho + a_\mu \partial_\nu b_\rho \right) + V(\phi_0^2) + V(\chi_0^2) \right] \right\} ,$$

where now, $\chi = \chi_0 e^{-i\tilde{\theta}}$ and $\Phi = \Phi_0 e^{-i\tilde{\phi}}$. Naively, it would seem that (4.14) undoes the duality transformation but it does not, since the interpretation is different. In the Higgs mechanism, we solve for \mathcal{A}_μ and ϕ , while retaining \tilde{b}_μ in the theory. In the particle-vortex duality, we exchange \tilde{b}_μ for $\tilde{\theta}$, and similarly for quantities with tilde and unaccented exchanged.

4.3 Vortex solutions

Let us recapitulate what we have done so far. We have formulated a manifest duality in the path integral formalism and argued that such a duality should exchange particles with vortices. Obviously, for this to be true, we need to have vortex solutions in the theory. Up to now, we have simply *presumed* the existence of such vortices in the field theories under investigation. Clearly, this will not be the case for all field theories of the form we have been discussing. Here, therefore, we devote some time to elaborating on the constraints that determine the form of the potential which will lead to such solutions. Thus, we consider the action in the path integral (4.13). In order to do this, one first writes down the full equations of motion, and *only afterwards* sets $\chi = \tilde{b}_\mu = \mathcal{A}_\mu = 0$ (which is itself a solution of these equations). The remaining equations of motion then become

$$\begin{aligned} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho &= 0 , \\ \Phi(D_\mu \Phi)^\dagger - \Phi^\dagger D_\mu \Phi &= 0 , \end{aligned} \quad (4.15)$$

where D_μ is the gauge covariant derivative, and the equation of motion for Φ , which depends on the potential $V(|\Phi|^2)$, is

$$D_\mu D^\mu \Phi = \frac{dV}{d|\Phi|^2} , \quad (4.16)$$

Note that the first of equations (4.15) implies that a_μ is pure gauge, while the second equation means that

$$D_\mu \theta = 0 \Rightarrow \partial_\alpha \theta = a_\alpha , \quad (4.17)$$

where α is the polar angle in the complex plane, and θ is the phase angle of Φ , that is $\Phi = |\Phi|e^{i\theta}$. In particular, this relation is valid at infinity. This gives the usual charge quantisation condition $\oint d\alpha a_\alpha = \oint d\theta$, or equivalently $\theta = N\alpha$, where N is proportional to the winding number of vector field around the vortex core. From the 0-component of (4.17), we get for static solutions that $a_0 = 0$.

This result would, however, imply that $|\Phi(r=0)| = 0$, consistent with the vortex ansatz. This, in turn, means that the second equation in (4.15) is automatically

satisfied at $r = 0$, without necessarily having $a_\alpha = N$ at $r = 0$. That would be good, since substituting $a_\alpha = N$, requires that

$$\epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho \theta \propto j_{\text{vortex}}^\mu \propto \delta(r),$$

so the first of equations (4.15) would be satisfied everywhere except at $r = 0$ which would, in turn, imply a discontinuous a_μ at $r = 0$, necessitating some kind of regularisation at this point. In fact, as we shall soon demonstrate, in order to have a solution, we need $|\Phi| \neq 0$ at $r = 0$. Consequently, the solution, as it stands, will be valid everywhere except at $r = 0$. It remains now to satisfy the $|\Phi|$ equation of motion in order to determine the vortex profile. We know already from (4.16) that any vortex solution must satisfy

$$\frac{|\Phi|''}{|\Phi|} = \frac{dV}{d|\Phi|^2},$$

where the prime denotes derivation with respect to R ; from general considerations about vortices, the one-vortex solution should behave like $|\Phi| \sim Ar$ as $r \rightarrow 0$. If, in addition, we consider the most general renormalisable potential in three dimensions, namely the sextic, $V = C_1|\Phi|^6 + \lambda|\Phi|^4 + m^2|\Phi|^2$ for which $\frac{dV}{d|\Phi|^2} = m^2 + 2\lambda|\Phi|^2 + 3C_1|\Phi|^4$, several cases of interest for the asymptotic (large r) behaviour of these solutions present themselves. They are (in no particular order):

- $m \neq 0$ and $\lambda \neq 0$:

In this case, $V = C_1|\Phi|^6 + \lambda|\Phi|^4 + m^2|\Phi|^2$. Near the origin, we take as an ansatz for the field

$$|\Phi| \sim Ar + Cr^p + \dots$$

This reduces the equation of motion in this region to

$$\frac{p(p-1)Cr^{p-2}}{Ar} = m^2,$$

which requires that $p = 3$ and $C = Am^2/6$. Therefore, the small- r form of the field is

$$|\Phi| \sim Ar \left(1 + \frac{m^2}{6} r^2 + \dots \right), \quad \text{for small } r.$$

Clearly we could analytically go to any order if needed. Taking the other asymptotic limit $r \rightarrow \infty$, and choosing $|\Phi| \sim \tilde{A}/r^n$, we find that there is an inconsistency for non-zero n , as $|\Phi|''/|\Phi| \sim 1/r^2 \rightarrow 0$, whereas $dV/d|\Phi|^2 = m^2 + \dots$. To avoid this, we choose instead

$$|\Phi| \sim \tilde{A} + \frac{\tilde{B}}{r^n} + \dots$$

where $\tilde{A} \neq 0$. With this ansatz, the equation of motion reduces to

$$\frac{\tilde{B}}{\tilde{A}} \frac{n(n+1)}{r^{n+2}} = \left[m^2 + \tilde{A}^2 (2\lambda + 3C_1 \tilde{A}^2) \right] + \frac{4\tilde{A}\tilde{B}}{r^n} (\lambda + 3C_1 \tilde{A}^2) + \frac{2\tilde{B}^2}{r^{2n}} (2\lambda + 9C_1 \tilde{A}^2),$$

and we see that we need $n = 2$ to satisfy the radial behaviour, along with the separate vanishing of the coefficients of the r^0 and r^{-n} terms. From the first of these, we find

$$m^2 + 2\lambda\tilde{A}^2 + 3C_1\tilde{A}^4 = 0,$$

which says that $|\Phi| = \tilde{A}$ is the nontrivial vacuum of the theory (as it should be), satisfying $dV/d|\Phi|^2 = 0$. From the second parenthesis, we have $\lambda + 3C_1\tilde{A}^2 = 0$, which, taken together with the first, give

$$\tilde{A}^2 = -\frac{m^2}{\lambda}, \quad C_1 = \frac{\lambda^2}{3m^2}.$$

C_1 is a constraint on the potential, allowing for only a certain class of sixth-order potentials with non-zero quadratic and quartic terms to lead to vortex solutions. This tells us that the potential must be truly sextic, since by assumption $\lambda \neq 0$. Equating the r^{-4} terms in the equation of motion then gives

$$\tilde{B} = \frac{3}{\tilde{A}(2\lambda + 9C_1\tilde{A}^2)},$$

so that

$$|\Phi| \sim \tilde{A} + \frac{3}{\tilde{A}(2\lambda + 9C_1\tilde{A}^2)} \frac{1}{r^2} + \dots, \quad \text{for large } r.$$

Clearly, as $m \rightarrow 0$, this solution vanishes.

• $m = 0$ and $\lambda \neq 0$:

In this case, $V = C_1|\Phi|^6 + \lambda|\Phi|^4$. As before, we take the asymptotics close to the vortex origin to be

$$|\Phi| \sim Ar + Cr^p + \dots$$

The equation of motion is now

$$\frac{p(p-1)Cr^{p-2}}{Ar} \simeq 2\lambda|\Phi|^2 \simeq 2\lambda A^2 r^2,$$

which gives $p = 5$ and $C = \frac{\lambda A^3}{10}$ so that

$$|\Phi| \sim Ar \left(1 + \frac{\lambda A^2}{10} r^4 + \dots \right), \quad \text{for small } r.$$

Far away from the vortex, we take $|\Phi| \sim \frac{\tilde{A}}{r^n}$, which reduces the equation of motion to

$$\frac{n(n+1)}{r^2} \simeq 2\lambda|\Phi|^2 = 2\lambda \frac{\tilde{A}^2}{r^{2n}}.$$

This fixes $n = 1$ and $\tilde{A} = \frac{1}{\sqrt{\lambda}}$, meaning that

$$|\Phi| \sim \frac{1}{\sqrt{\lambda} r} + \dots, \quad \text{for large } r.$$

Note that $\tilde{A} + \tilde{B}/r^n$ leads to a contradiction in the equations of motion and thus the leading term must be $\sim \frac{1}{r}$. In contrast to the first case above, there is no constraint on the potential.

- $m = 0$ and $\lambda = 0$:

Here, $V = C_1|\Phi|^6$, a purely sextic potential. At $r = 0$, as above, we find

$$\frac{p(p-1)Cr^{p-2}}{Ar} = 3C_1|\Phi|^4 \simeq 3C_1A^4r^4,$$

which gives $p = 7$ and $C = C_1A^5/14$, so

$$|\Phi| \sim Ar \left(1 + \frac{C_1A^4}{14}r^6 + \dots \right), \quad \text{for small } r.$$

At infinity, with $|\Phi| \sim \tilde{A}/r^n$, the equation of motion is

$$\frac{n(n+1)}{r^2} = \frac{3C_1\tilde{A}^4}{r^{4n}},$$

which gives $n = 1/2$ and $\tilde{A}^4 = \frac{1}{4C_1}$, so that

$$|\Phi| \sim \frac{1}{(4C_1)^{1/4} \sqrt{r}}, \quad \text{for large } r.$$

To summarise: in order to find a non-trivial solution in the case of a massive (sextic) potential, the constraint $C_1 = \lambda^2/(3m^2)$, with $m^2/\lambda < 0$, must be satisfied, whereas for the two massless scenarios, there are always solutions. In the next section, however, we shall show that the constraint is *not* satisfied in the case of the *massive ABJM* model, which would imply that an embedding of the particle-vortex duality into massive ABJM is not possible; within *massless ABJM*, on the other hand, only the *purely sextic* potential will be relevant.

4.3.1 Pure sextic potential

It turns out that in the pure sextic case, $V = C_1|\Phi|^6$, we can arrive at an explicit solution using some simple considerations. The equation of motion is

$$|\Phi|'' = 3C_1|\Phi|^5,$$

which we write in terms of $v = |\Phi|'$ as

$$v \, dv = 3C_1|\Phi|^5 \, d|\Phi|,$$

the solutions of which are

$$v^2 = C_1|\Phi|^6 + K_1 \quad \Rightarrow \quad d|\Phi| = \pm \sqrt{C_1|\Phi|^6 + K_1} \, dr. \quad (4.18)$$

The general solution is then

$$r + \frac{K_2}{\sqrt{C_1}} = \pm \int \frac{d|\Phi|}{\sqrt{C_1|\Phi|^6 + K_1}}.$$

Note however that, if $|\Phi| \sim Ar$ for small r and $|\Phi| \sim A/r^n$, $n > 0$ for large r , there must exist at least one intermediate r for which $v = |\Phi|' = 0$, that is $C_1|\Phi|^6 + K_1 = 0$, or where $|\Phi|_{\text{int}} = (-K_1/C_1)^{\frac{1}{6}}$. This, in turn, means that⁶ $K_1/C_1 < 0$. If we assumed $K_1 < 0, C_1 > 0$ for the branch connected to $r = 0$, we would immediately find from (4.18) that $|\Phi|'$ is imaginary, so we would be tempted to choose $K_1 > 0, C_1 < 0$ instead. However, this is inconsistent, as we would have a runaway potential with no stable vacuum.

We must therefore choose $C_1 < 0$ and $K_1 > 0$. This choice is, if anything, worse since it implies that the potential is *negative-definite*. In fact, even if the vacuum were stable in this case, there would be a problem because $|\Phi|' = +\sqrt{K_1 - |C_1||\Phi|^6}$ until we reach $|\Phi|_{\text{int}}$, where $|\Phi|'$ vanishes, and thereafter $|\Phi|' = -\sqrt{K_1 - |C_1||\Phi|^6}$. This means that we would reach $|\Phi| = 0$ with nonzero derivative, $|\Phi|' = -\sqrt{K_1}$. Since $|\Phi| \geq 0$, this results in a singularity at this point, as $|\Phi|'$ would jump discontinuously.

In other words, there is *no smooth solution for the vortex*. This, however, does not constitute a problem per se, as the smoothness constraint is not required. We saw that, in any case, the solution is not valid at $r = 0$ itself, so we can ignore the constraint that $|\Phi| = 0$ there. With a little more thought, it is clear that, with $C_1 > 0$ as it should be, the only solution that makes sense (which goes to zero at infinity) is one with $K_1 = 0$, since if $K_1 < 0$, $|\Phi|'$ must become imaginary before reaching $r = \infty$, and if $K_1 > 0$, $|\Phi|'$ must remain finite as $|\Phi| = 0$, which means it is again reached before $r = \infty$. Then the solution is⁷

$$\sqrt{C_1}r + K_2 = - \int \frac{d|\Phi|}{|\Phi|^3} = \frac{1}{2|\Phi|^2},$$

so that

$$|\Phi| = \frac{1}{\sqrt{2(\sqrt{C_1}r + K_2)}},$$

which has

$$|\Phi|'(0) = -\frac{\sqrt{C_1}}{(2K_2)^{\frac{3}{2}}},$$

which is finite and real for $K_2 > 0$; nevertheless, as we remarked, we must excise and regularise an infinitesimal region around $r = 0$. To summarise this section: there is a strong constraint on the form of the massive sextic potential which leads to a vortex solution, while for a purely sextic potential, there will be non-smooth solutions which, modulo excision of the singular core, will correspond to vortices.

4.4 Embedding particle-vortex duality in ABJM

In order to formulate the particle-vortex duality within ABJM, we must be able to find an abelian reduction of the ABJM model which can both be mapped to the

⁶Note that we can (and in general, should) glue different branches of the solution at the point where $v = 0$.

⁷One can easily see, from the form of the solution, that the *+ve* sign in front of the integral does not make sense.

path integral in equation (4.13) as well as shown to fulfill the constraint which leads to vortex solutions. We will show below that, while the mass-deformed ABJM theory has the appropriate mapping to the self-dual action with non-zero mass, the vortex constraints on the potential are not fulfilled, and thus we can only get a self-dual theory with vortices in the massless case. See Appendix B.2 for a brief overview of the ABJM formalism.

4.4.1 Constructing a self-dual abelian reduction of ABJM

For the two *bifundamental scalars*, Φ and χ , of ABJM, we split the N -dimensional matrix space into two (block-diagonal) $N/2$ -dimensional subspaces. Subsequently, we will be able to use each of the sub-spaces to construct a self-duality via the particle-vortex transformation. In the first subspace, we take the ansatz

$$\begin{aligned} A_\mu &= a_\mu^{(1)} \mathbf{1}_{N/2 \times N/2} , \\ \hat{A}_\mu &= \hat{a}_\mu^{(1)} \mathbf{1}_{N/2 \times N/2} , \\ Q^1 &= \phi G_{N/2 \times N/2}^1 , \\ Q^2 &= \phi G_{N/2 \times N/2}^2 , \\ R^\alpha &= 0 , \end{aligned} \tag{4.19}$$

where:

- A_μ and \hat{A}_μ are fields associated with the two unitary groups making up the $U(N) \times U(N)$ gauge symmetry of ABJM;
- Q^α and R^α , $\alpha = 1, 2$, are the two (first sub-space) bifundamental scalars;
- G^α , $\alpha = 1, 2$, are $N/2 \times N/2$ matrices that satisfy the relations (B.15).

The combination of Q^α and R^α , often labeled as N^α , can be shown with this choice of R^α to vanish while the other combination will be non-zero. The *covariant derivative* on the scalar Q^α is given by

$$D_\mu Q^\alpha = G^\alpha \left[\partial_\mu + i(a_\mu^{(1)} - \hat{a}_\mu^{(1)}) \right] \phi ,$$

where the $N/2 \times N/2$ subscript, for notational convenience, is now taken to be implicit for G^α . This leads to the kinetic terms

$$\text{Tr} \left[|D_\mu Q^\alpha|^2 \right] = 2 \frac{N}{2} \left(\frac{N}{2} - 1 \right) \left| \left[\partial_\mu + i(a_\mu^{(1)} - \hat{a}_\mu^{(1)}) \right] \phi \right|^2 .$$

The contribution to the mass deformed potential comes from

$$M^\alpha = \mu Q^\alpha + \frac{2\pi}{k} (Q^\alpha Q_\beta^\dagger Q^\beta - Q^\beta Q_\beta^\dagger Q^\alpha) = G^\alpha \left(\mu + \frac{2\pi}{k} \phi^2 \right) \phi ,$$

and thus, the full potential is

$$\begin{aligned} V = \text{Tr} \left[|M^\alpha|^2 \right] &= \frac{N}{2} \left(\frac{N}{2} - 1 \right) |\phi|^2 \left| \mu + \frac{2\pi}{k} \phi^2 \right|^2 \\ &= \frac{N}{2} \left(\frac{N}{2} - 1 \right) |\phi|^2 \left(\mu + \frac{2\pi}{k} |\phi|^2 \right)^2 . \end{aligned} \tag{4.20}$$

With this ansatz, the Chern-Simons terms reduce to

$$\frac{k}{4\pi} \frac{N}{2} \epsilon^{\mu\nu\rho} \left(a_\mu^{(1)} \partial_\nu a_\rho^{(1)} - \hat{a}_\mu^{(1)} \partial_\nu \hat{a}_\rho^{(1)} \right) = \frac{k}{4\pi} \frac{N}{2} \epsilon^{\mu\nu\rho} \left(a_\mu^{(1)} + \hat{a}_\mu^{(1)} \right) \partial_\nu \left(a_\rho^{(1)} - \hat{a}_\rho^{(1)} \right),$$

so we obtain the first half of the required action, with the additional identification $\Phi \rightarrow \phi$, $a_\mu \rightarrow \left(a_\mu^{(1)} - \hat{a}_\mu^{(1)} \right)$ and $\tilde{b}_\mu \rightarrow \left(a_\mu^{(1)} + \hat{a}_\mu^{(1)} \right)$. The other half, for χ and \mathcal{A}_μ , is obtained from the second $N/2$ -dimensional subspace, now with the constraint $\tilde{b}_\mu = \left(a_\mu^{(1)} + \hat{a}_\mu^{(1)} \right) = \left(a_\mu^{(2)} + \hat{a}_\mu^{(2)} \right)$.

4.4.2 Vortex constraints on the ABJM potential

We are now in a position where we can construct a duality in this constrained sector of ABJM, by mapping the action to a known self-dual one. However, to prove that this is a particle-vortex duality, we first need to show that there is enough freedom in the sextic potential to lead to vortex solutions. In the massive case, there is a constraint on the potential, which is that $C_1 = \lambda^2/3m^2$, in order to have a vortex, that is

$$\begin{aligned} V(|\phi|^2) &= \frac{|\phi|^2}{3m^2} (\lambda^2 |\phi|^4 + 3m^2 \lambda |\phi|^2 + 3m^4) \\ &= \frac{|\phi|^2}{3m^2} \left[\left(\lambda |\phi|^2 + \frac{3m^2}{2} \right)^2 + \frac{3m^4}{4} \right], \end{aligned}$$

in order to have solitons. Comparing to (4.20), we see that this does not agree with the form of the potential for the mass-deformed ABJM model. Therefore, the embedding does not work when the ABJM mass parameter $\mu \neq 0$. However, at $\mu = 0$, that is, in the case of the purely sextic potential, vortex solutions do actually exist, as shown in the previous section. This, along with the field identifications in section (4.4.1), suffice to demonstrate that, at $\mu = 0$, we can construct a reduction of ABJM which exhibits a particle-vortex self-duality.

4.4.3 Toward a non-abelian extension

To close this section, we speculate on a possible extension of the particle-vortex duality to non-abelian vortices, starting with the observation that, with the embedding of the particle vortex duality, we can write it *on the reduction ansatz* in the invariant form

$$\frac{1}{2} \text{Tr} \left[Q_\alpha^\dagger D^\mu Q^\alpha - Q^\alpha (D^\mu Q^\alpha)^\dagger \right] = \frac{1}{e} \epsilon^{\mu\nu\rho} \partial_\nu \text{Tr} \left(A_\rho + \hat{A}_\rho \right) = \frac{1}{2} \text{Tr} \left[\tilde{Q}_\alpha^\dagger \tilde{D}^\mu \tilde{Q}^\alpha - \tilde{Q}^\alpha (\tilde{D}^\mu \tilde{Q}^\alpha)^\dagger \right],$$

where the trace is taken over one of the $N/2 \times N/2$ diagonal blocks of the matrix. With the caveat that we have not been able to prove that this holds in general (that is, not on the reduction ansatz), it is tempting to think that one could write a nonabelian generalisation of the type

$$\frac{1}{2} \left[Q_\alpha^\dagger D^\mu Q^\alpha - Q^\alpha (D^\mu Q^\alpha)^\dagger \right] = \frac{1}{e} \epsilon^{\mu\nu\rho} \partial_\nu \left(A_\rho + \hat{A}_\rho \right) = \frac{1}{2} \left[\tilde{Q}_\alpha^\dagger \tilde{D}^\mu \tilde{Q}^\alpha - \tilde{Q}^\alpha (\tilde{D}^\mu \tilde{Q}^\alpha)^\dagger \right],$$

for half the matrix space, and a similar one for the other half. Showing that this is indeed the case in general would go a long way towards generalising the (self-dual) particle-vortex duality.

4.5 Particle-vortex duality from Maxwell duality in the bulk, via the AdS/CFT correspondence

Having established a framework to understand particle-vortex duality in (at least a reduction of) the ABJM model, we now relate the duality with Maxwell duality in the bulk, via the AdS/CFT correspondence. The partition function in a three-dimensional conformal field theory for a gauge field with a source is generically (in Euclidean signature)

$$Z_{CFT}[a_i] = \int \mathcal{D}\phi \exp\left(-S[\phi] + \int d^3x J^i a_i\right), \quad \text{with } i = 1, 2, 3, \quad (4.21)$$

where ϕ represents all of the fields in the gauge theory, J^i is the U(1) current that couples to the source a_i , which is itself the *boundary value for the bulk gauge field* A_μ .

The corresponding supergravity partition function in the bulk (in Euclidean signature) is given by the bulk Maxwell action in an AdS geometry

$$Z_{SUGRA}[a_i] = \int \mathcal{D}\Phi \exp\left[-\int d^4x \sqrt{-g} \left(+\frac{1}{4g^2} F_{\mu\nu}^2\right)\right],$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the bulk field strength, and Φ encompasses all dynamical fields in the bulk. We work in the radial gauge $A_z = 0$, so $A_i \rightarrow a_i$ on the boundary. We define the four-dimensional Maxwell duality,

$$\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (4.22)$$

in terms of which we can rewrite the partition function as

$$Z_{SUGRA}[a_i] = Z_{SUGRA}[\tilde{a}_i] = \int \mathcal{D}\Phi \exp\left[-\int d^4x \sqrt{-g} \left(+\frac{1}{4g^2} \tilde{F}_{\mu\nu}^2\right)\right],$$

where $\tilde{F}^{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$, and $\tilde{A}_i \rightarrow \tilde{a}_i$ on the boundary. The question is how to relate this bulk duality to the particle-vortex duality we have already found, in a theory with a known gravity dual. The field theory dual to the self-dual Maxwell theory in the bulk can itself be rewritten, after defining a particle-vortex – type duality for currents similar to (4.7)

$$\boxed{J^i = \frac{1}{2} \epsilon^{ijk} \partial_j \tilde{j}_k}, \quad (4.23)$$

as

$$\begin{aligned} Z_{CFT}[a_i] &= \int \mathcal{D}\phi \exp\left(-S[\phi] + \int d^3x \frac{1}{2} \epsilon^{ijk} a_i \partial_j \tilde{j}_k\right) \\ &= \int \mathcal{D}\phi \exp\left[-S[\phi] + \int d^3x \tilde{j}^i \left(\frac{1}{2} \epsilon^{ijk} \partial_j a_k\right)\right], \end{aligned}$$

so that, if we take

$$\tilde{a}^i = \frac{1}{2} \epsilon^{ijk} \partial_j a_k ,$$

$Z_{\text{CFT}}[\tilde{a}_i]$ would be written in exactly the form to match $Z_{\text{SUGRA}}[\tilde{a}_i]$, thus relating particle-vortex duality (4.23) in the CFT with Maxwell duality (4.22) in the bulk.

4.5.1 Maxwell duality in AdS₄

Having identified a link between a generic $d + 1$ -dimensional Maxwell duality and a d -dimensional particle-vortex-like duality, we now turn specifically to the case of Maxwell duality in AdS₄. In Poincaré coordinates, the AdS metric element is

$$ds^2 = \frac{-dt^2 + dx^2 + dy^2 + dz^2}{z^2} ,$$

such that the Maxwell duality (4.22) becomes

$$\tilde{F}_{01} = -F_{23} ; \quad \tilde{F}_{23} = -F_{01} , \dots ,$$

that is, electric and magnetic field components, including the radial direction, are exchanged via the duality. In the radial gauge, $A_3 = \tilde{A}_3 = 0$, we have $F_{23} = -\partial_3 A_2$ and $\tilde{F}_{23} = -\partial_3 \tilde{A}_2$, so

$$\tilde{F}_{01}(z=0) = \partial_z A_2(z=0) ; \quad F_{01}(z=0) = \partial_z \tilde{A}_2(z=0) ,$$

where $z = 0$ is the boundary of AdS. Expanding the bulk gauge fields in power series near that boundary

$$\begin{aligned} A_i &= a_i + z \bar{a}_i + \frac{z^2}{2} a_i^{(2)} + \frac{z^3}{3!} a_i^{(3)} + \dots , \\ \tilde{A}_i &= \tilde{a}_i + z \tilde{\bar{a}}_i + \frac{z^2}{2} \tilde{a}_i^{(2)} + \frac{z^3}{3!} \tilde{a}_i^{(3)} + \dots , \end{aligned}$$

we find that the above Maxwell duality relations give

$$\tilde{f}_{ij} = \frac{1}{2} \epsilon_{ijk} \bar{a}^k ; \quad f_{ij} = \frac{1}{2} \epsilon_{ijk} \tilde{\bar{a}}^k , \quad (4.24)$$

where f_{ij} corresponds to the field strength coming only from the leading term in the above expansion, that is, $f_{ij} = \partial_i a_j - \partial_j a_i$, as well as

$$\begin{aligned} \tilde{\tilde{f}}_{ij} &= -\frac{1}{2} \epsilon_{ij}{}^k \partial_1^2 a_k = \frac{1}{2} \epsilon_{ij}{}^k a_k^{(2)} , \\ \tilde{f}_{ij} &= -\frac{1}{2} \epsilon_{ij}{}^k \partial_1^2 \tilde{a}_k = \frac{1}{2} \epsilon_{ij}{}^k \tilde{a}_k^{(2)} , \end{aligned}$$

and so on, where, again, $\tilde{f}_{ij} = \partial_i \tilde{a}_j - \partial_j \tilde{a}_i$. This result is obtained from two applications of the duality transformations. For the first equality, we first write the duality for \tilde{a}_i in terms of f_{ij} , and then take a derivative. For the second, we look at the order z term in the duality for \tilde{F}_{ij} versus A_k . Equating the two results gives $\partial_1^2 a_k = -a_k^{(2)}$. We will see shortly that this follows from the Maxwell equations.

Normally, in $d \neq 4$, one should be able to give only the a_i as boundary condition, but not \bar{a}_i , the subleading term in the z expansion. In $d = 4$ however, as a result of the Maxwell duality, we can specify both a_i and \bar{a}_i , or equivalently, both a_i and the dual, \tilde{a}_i . In Poincaré coordinates, the Maxwell equations are

$$\partial_\rho \left(\sqrt{g} g^{\rho\mu} g^{\sigma\nu} \partial_{[\mu} A_{\nu]} \right) = 0 ,$$

and since $g_{\mu\nu} = \frac{1}{z^2} \delta_{\mu\nu}$, this reduces to

$$\partial_\mu \partial_{[\mu} A_{\sigma]} = 0 . \quad (4.25)$$

Notice that all factors of z have cancelled out in the equation; this happens only in four dimensions. More generally in AdS _{d} , there will be an extra contribution of $\frac{(4-d)}{z} \partial_z A_\sigma$, which, in particular, implies that the leading term in this equation is of order $1/z$, namely, for $\sigma = i$, it is $\frac{(4-d)}{z} \bar{a}_i$, implying that $\bar{a}_i = 0$. In the radial gauge $A_z = 0$, we have $\partial_i \partial_z A_i = 0$ for $\sigma = z$. This constrains $\partial_i \bar{a}_i = 0$, and $\partial_i a_i^{(n)} = 0$ for $n \geq 2$, leaving only $\partial_i a_i$ possibly nonvanishing. However, since it is consistent to set it to zero, we shall do so. This is equivalent to the usual radiation gauge with time replaced by z , $a_z = 0$ and $\partial_i a_i = 0$. For $\sigma = i$, (4.25) becomes

$$\left(\partial_j^2 + \partial_z^2 \right) A_i - \partial_i \left(\partial_j A_j \right) = 0 ,$$

which when expanded in z (and taking into account the conditions above for $\sigma = z$), results in the system of equations

$$\begin{aligned} \partial_j^2 a_i + a_i^{(2)} - \partial_i (\partial_j a_j) &= 0 , \\ \partial_j^2 \bar{a}_i + a_i^{(3)} &= 0 , \\ \partial_j^2 a_i^{(n)} + a_i^{(n+2)} &= 0 . \end{aligned}$$

Note that the first relation also implies $\partial_i a_i^{(2)} = 0$, as it should. Thus, in the radiation gauge for a_i , we have $a_i^{(n+2)} = -\partial_j^2 a_i^{(n)}$. Specifying a_i and \bar{a}_i (or equivalently, \tilde{a}_i) then completely fixes the solution to the Maxwell equation in AdS₄.

Returning to the gauge theory side of the correspondence, we need to specify a_i and \bar{a}_i as sources for the path integral (4.21), or exchange a_i with \tilde{a}_i and \bar{a}_i with $\tilde{\bar{a}}_i$. As claimed earlier, this exchange of a_i with \tilde{a}_i would correspond to a particle-vortex duality exchanging dual currents as in (4.23). These currents, however, need to be currents of global symmetries that can couple to the gravity dual gauge fields. We need to have *two currents*, one for particles and one for vortices, that can be replaced by their corresponding particle-vortex dual currents. According to our embedding of particle-vortex duality in ABJM (4.19), the scalar ϕ appears in half of the U(N) space and χ in the other half. With this ansatz. $j^\mu = \hat{j}^\mu$ from (B.14), but splits into two currents (for each of the two $N/2$ -dimensional subspaces) of \tilde{J} type in (4.23), $\tilde{J}_i^{(1)}$ and $\tilde{J}_i^{(2)}$, that couple to \bar{a}_k and $\tilde{\bar{a}}_k$ respectively.

4.6 Conclusions

In this chapter, we have explored some aspects of holographic particle-vortex duality. In particular, we have focused on its realisation in the ABJM model

and a possible relation to Maxwell duality in AdS_4 via the AdS/CFT correspondence. By combining a path integral version of particle-vortex duality with the Mukhi-Papageorgakis Higgs mechanism, we have formulated a symmetric version of the transformation that acts as a self-duality. We then proceeded to show how to embed it as an abelian duality in the $(2 + 1)$ -dimensional, $\mathcal{N} = 6$ super-Chern-Simons-matter theory that is the ABJM model, and speculated on a possible non-abelian extension. Going to the gravity side of the correspondence, Maxwell duality in AdS_4 is found to reduce on the boundary to a particle-vortex duality acting on two independent gauge field sources \bar{a} and \tilde{a} and their associated currents $\tilde{j}^{(1)}$ and $\tilde{j}^{(2)}$.

Our main motivation for this work was two-fold: first, we wanted to understand whether particle-vortex duality is realised in the (mass-deformed) ABJM model with its rich solitonic spectrum and second, we wanted to see if the phenomenological work of [123] could be embedded in the concrete setting of the $AdS_4 \times CP^3$ /ABJM correspondence. This work paves the way for both these directions, but there remains much to be done. Among the possible extensions of this work are

- the further elaboration of our speculations on a non-abelian version of the particle-vortex duality. To the best of our knowledge, the duality, thus far, has been formulated only for vortices of the conventional *Nielsen-Olesen type* exhibited by the abelian Higgs model and variants thereof. Vortices, however, come in many different guises, such as non-abelian as well as semi-local kinds. It would be of great interest to understand if and how the duality applies to these;
- an understanding of the manifestation of the *full* particle-vortex duality on the gravity side of the correspondence. In particular, having established here that the duality can actually be embedded into (at least some reduction of) the ABJM model, we are naturally led to ask the important question of how precisely it acts on state spectrum of type IIA superstring on $AdS_4 \times CP^3$;
- the extraction of the phenomenological results for quantum critical transport uncovered in [123]; and
- a more complete understanding of how the particle-vortex duality we have looked at relates to *level-rank duality* and its generalisations discovered by Kutasov and collaborators in recent years.

It is quite clear that particle-vortex duality should be of great interest to both the holographic condensed matter as well as the more formal string theory communities, and we hope that this work will stimulate further work in this area.

Appendix A

A.1 The $\mathfrak{d}(2, 1; \alpha)$ superalgebra

The bosonic subalgebra of $\mathfrak{d}(2, 1; \alpha)$, with $\alpha \neq \{-1, 0, \infty\}$, consists of three commuting $\mathfrak{sl}(2)$'s. The supercharges are in their tri-spinor representation. We are interested in the real form of $\mathfrak{d}(2, 1; \alpha)$, where one of the $\mathfrak{sl}(2)$'s is non-compact and the other two are compact, so that the bosonic subalgebra is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$. With this choice of the real form, $\alpha \in [0, 1]$, and it is convenient to introduce the trigonometric parameterization $\alpha = \cos^2 \phi$, with $\phi \in [0, \pi/2]$.

A.1.1 Generators and their (anti-)commutations

We denote the $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{su}(2)_+$, and $\mathfrak{su}(2)_-$ generators by S_μ ($\mu = 0, 1, 2$), L_n ($n = 3, 4, 5$), and $R_{\dot{n}}$ ($\dot{n} = 6, 7, 8$), respectively. The supercharges are $Q_{a\alpha\dot{\alpha}}$, where $a, \alpha, \dot{\alpha} \in \{+, -\}$.

To describe the action of the $\mathfrak{sl}(2)$ generators on the supercharges, we introduce three sets of Pauli matrices:

$$\gamma^\mu = (i\sigma^2, \sigma^1, \sigma^3), \quad \gamma^n = (\sigma^1, \sigma^2, \sigma^3), \quad \gamma^{\dot{n}} = (\sigma^1, \sigma^2, \sigma^3). \quad (\text{A.1})$$

The (anti-)commutation relations of $\mathfrak{d}(2, 1; \cos^2 \phi)$ then read:

$$\begin{aligned} [S_\mu, S_\nu] &= \epsilon_{\mu\nu\lambda} S^\lambda \\ [L_m, L_n] &= \epsilon_{mnp} L^p \\ [R_{\dot{m}}, R_{\dot{n}}] &= \epsilon_{\dot{m}\dot{n}\dot{p}} R^{\dot{p}} \end{aligned}$$

$$[S_\mu, Q_{a\alpha\dot{\alpha}}] = -\frac{1}{2} Q_{b\alpha\dot{\alpha}} \gamma_{\mu a}^b$$

$$[L_m, Q_{a\alpha\dot{\alpha}}] = -\frac{i}{2} Q_{a\beta\dot{\alpha}} \gamma_m^\beta \alpha$$

$$[R_{\dot{m}}, Q_{a\alpha\dot{\alpha}}] = -\frac{i}{2} Q_{a\alpha\dot{\beta}} \gamma_{\dot{m}}^{\dot{\beta}} \dot{\alpha}$$

$$\begin{aligned} \{Q_{a\alpha\dot{\alpha}}, Q_{b\beta\dot{\beta}}\} &= s_1 (\epsilon\gamma^\mu)_{ab} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} S_\mu + s_2 \epsilon_{ab} (\epsilon\gamma^m)_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} L_m \\ &+ s_3 \epsilon_{ab} \epsilon_{\alpha\beta} (\epsilon\gamma^{\dot{m}})_{\dot{\alpha}\dot{\beta}} R_{\dot{m}}, \end{aligned}$$

where

- $\epsilon_{012} = \epsilon_{345} = \epsilon_{678} = 1$;
- ε is the *charge conjugation matrix*;
- the vector indices are raised and lowered by $\eta_{\mu\nu} = \text{diag}(-++)$, δ_{nm} , and $\delta_{\dot{m}\dot{n}}$;
- $s_i = \{i, -\cos^2 \phi, -\sin^2 \phi\}$, $i = 1, 2, 3$, and $s_1 + is_2 + is_3 = 0$ is imposed by the *generalised Jacobi identity*.

Since the superalgebras defined by the triplets $\lambda s_1, \lambda s_2, \lambda s_3$ ($\lambda \in \mathbb{C}$) are isomorphic, one can set $s_2/is_1 = \alpha$ and $s_3/is_1 = 1 - \alpha$.

The invariant bilinear form on $\mathfrak{d}(2, 1; \cos^2 \phi)$ is given by

$$\begin{aligned} \text{Str}(S_\mu S_\nu) &= -\frac{1}{4is_1} \eta_{\mu\nu} , \\ \text{Str}(L_m L_n) &= -\frac{1}{4s_2} \delta_{mn} , \\ \text{Str}(R_{\dot{m}} R_{\dot{n}}) &= -\frac{1}{4s_3} \delta_{\dot{m}\dot{n}} , \\ \text{Str}(Q_{a\alpha\dot{\alpha}} Q_{b\beta\dot{\beta}}) &= \frac{i}{2} \varepsilon_{ab} \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} , \end{aligned}$$

and is consistent with the $(- + \dots +)$ signature in the target space.

A.2 Gamma matrices

We pick the following representation for the 10d Dirac matrices:

$$\begin{aligned} \Gamma^\mu &= \sigma^1 \otimes \sigma^2 \otimes \gamma^\mu \otimes \mathbb{1} \otimes \mathbb{1} , & \mu &= 0, 1, 2 , \\ \Gamma^n &= \sigma^1 \otimes \sigma^1 \otimes \mathbb{1} \otimes \gamma^n \otimes \mathbb{1} , & n &= 3, 4, 5 , \\ \Gamma^{\dot{n}} &= \sigma^1 \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma^{\dot{n}} , & \dot{n} &= 6, 7, 8 , \\ \Gamma^9 &= -\sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} , \end{aligned}$$

where the 3d gamma-matrices γ^i are as in (A.1).

In this basis,

$$\begin{aligned} \Gamma^{012} &= \sigma^1 \otimes \sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} , \\ \Gamma^{345} &= i\sigma^1 \otimes \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} , \\ \Gamma^{678} &= i\sigma^1 \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} , \\ \Gamma^{012345} &= \mathbb{1} \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} , \\ \Gamma^{012678} &= -\mathbb{1} \otimes \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} , \\ \Gamma &= \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} , \end{aligned}$$

and

$$\begin{aligned}\Gamma^{\mu\nu} &= -\epsilon^{\mu\nu\lambda} \mathbb{1} \otimes \mathbb{1} \otimes \gamma_\lambda \otimes \mathbb{1} \otimes \mathbb{1}, \\ \Gamma^{mn} &= i\epsilon^{mnp} \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma_p \otimes \mathbb{1}, \\ \Gamma^{\dot{m}\dot{n}} &= i\epsilon^{\dot{m}\dot{n}\dot{p}} \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma_{\dot{p}}.\end{aligned}$$

The charge conjugation matrix is

$$C = i\sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2. \quad (\text{A.2})$$

A.3 Notation and parameterisation

The $AdS_3 \times S^3 \times S^3 \times S^1$ metric is given by

$$ds^2 = ds^2_{(AdS_3)} + \frac{1}{\cos^2 \phi} ds^2_{(S^3)} + \frac{1}{\sin^2 \phi} ds^2_{(S^3)} + ds^2_{(S^1)} \quad (\text{A.3})$$

where the "sin" and "cos" factors are there to ensure the *triangle identity* between the curvature radii (2.1) and the AdS_3 radius is set to one. We choose the following metric elements in global coordinates [79]

$$ds^2_{(AdS_3)} = -\left(\frac{1 + \frac{1}{4}x_i^2}{1 - \frac{1}{4}x_i^2}\right)^2 dt^2 + \frac{1}{\left(1 - \frac{1}{4}x_i^2\right)^2} dx_i^2, \quad ds^2_{(S^3)} = \left(\frac{1 - \frac{1}{4}x_i^2}{1 + \frac{1}{4}x_i^2}\right)^2 d\varphi_i^2 + \frac{1}{\left(1 + \frac{1}{4}x_i^2\right)^2} dx_i^2$$

where φ_5, φ_8 are the S^3 angles which we single out, and $\{x_1, x_2\}, \{x_3, x_4\}, \{x_6, x_7\}, x_9$ are the transverse coordinates. In order to have a smooth interpolation between different values of ϕ , we will also scale the S^3 coordinates as

$$(\varphi_5, x_3, x_4) \rightarrow \cos \phi (\varphi_5, x_3, x_4), \quad (\varphi_8, x_6, x_7) \rightarrow \sin \phi (\varphi_8, x_6, x_7),$$

which allows for nice T^4 limits when $\phi = 0$ or $\phi = \pi/2$.

The vielbeins can be read off immediately from (A.3) and (??). The spin connection of the background is also needed and can be computed from the vanishing of the torsion

$$de^A + e^B \Omega_B^A = 0. \quad (\text{A.4})$$

One finds the non-zero components

$$\begin{aligned}\Omega^{01} &= -\frac{x_1}{1 - \frac{1}{4}x_i^2} dt, & \Omega^{02} &= -\frac{x_2}{1 - \frac{1}{4}x_i^2} dt, & \Omega^{12} &= -\frac{1}{2}(x_2 e^1 - x_1 e^2), \\ \Omega^{35} &= -\cos^2 \phi \frac{x_3}{1 + \frac{\cos^2 \phi}{4}x_i^2} d\varphi_5, & \Omega^{45} &= -\cos^2 \phi \frac{x_4}{1 + \frac{\cos^2 \phi}{4}x_i^2} d\varphi_5, & \Omega^{34} &= \frac{\cos^2 \phi}{2}(x_4 e^3 - x_3 e^4), \\ \Omega^{68} &= -\sin^2 \phi \frac{x_6}{1 + \frac{\sin^2 \phi}{4}x_i^2} d\varphi_8, & \Omega^{78} &= -\sin^2 \phi \frac{x_7}{1 + \frac{\sin^2 \phi}{4}x_i^2} d\varphi_8, & \Omega^{67} &= \frac{\sin^2 \phi}{2}(x_7 e^6 - x_6 e^7),\end{aligned}$$

where $x_i^2 = \{(x_1^2 + x_2^2), (x_3^2 + x_4^2), (x_6^2 + x_7^2)\}$ in the first, second, and third rows, respectively.

Working in light-cone coordinates, we define

$$e^\pm = \frac{1}{2} \left[e^0 \pm (\cos \beta e^5 + \sin \beta e^8) \right], \quad e^v = \sin \beta e^5 - \cos \beta e^8,$$

where the angle β gives the direction in the (5,8)-plane of the geodesic we are interested in.

We will use the Γ matrix notation of [19] with the light-cone combinations defined as

$$\Gamma_\pm = \frac{1}{2} [\Gamma_0 \pm (\cos \beta \Gamma_5 + \sin \beta \Gamma_8)], \quad \Gamma_v = \sin \beta \Gamma_5 - \cos \beta \Gamma_8, \quad \Gamma_{11} = \prod_{i=0}^9 \Gamma_i. \quad (\text{A.5})$$

which satisfy the relations

$$\{\Gamma_+, \Gamma_-\} = 2\eta_{+-} = -4, \quad \{\Gamma_\pm, \Gamma_v\} = 0, \quad \Gamma_v^2 = 1.$$

The spinor Θ satisfying (2.5) and subject to the gauge-fixing condition (2.9) can be decomposed as

$$\Theta = \begin{pmatrix} -i(\sin \beta \theta_1^+ - \cos \beta \theta_3^+) \\ i(\sin \beta \theta_2^+ + \cos \beta \theta_4^+) \\ -(\sin \beta \theta_2^+ + \cos \beta \theta_4^+) \\ -(\sin \beta \theta_1^+ - \cos \beta \theta_3^+) \\ \theta_3^+ \\ \theta_4^+ \\ -i\theta_4^+ \\ i\theta_3^+ \end{pmatrix} \oplus \begin{pmatrix} -i(\cos \beta \theta_1^+ + \sin \beta \theta_3^+) \\ -i(\cos \beta \theta_2^+ - \sin \beta \theta_4^+) \\ \cos \beta \theta_2^+ - \sin \beta \theta_4^+ \\ -(\cos \beta \theta_1^+ + \sin \beta \theta_3^+) \\ \theta_1^+ \\ \theta_2^+ \\ -i\theta_2^+ \\ i\theta_1^+ \end{pmatrix} \oplus \begin{pmatrix} \theta_3^- \\ \theta_4^- \\ i\theta_4^- \\ -i\theta_3^- \\ i(\sin \beta \theta_1^- - \cos \beta \theta_3^-) \\ -i(\sin \beta \theta_2^- + \cos \beta \theta_4^-) \\ -(\sin \beta \theta_2^- + \cos \beta \theta_4^-) \\ -(\sin \beta \theta_1^- - \cos \beta \theta_3^-) \end{pmatrix} \oplus \begin{pmatrix} \theta_1^- \\ \theta_2^- \\ i\theta_2^- \\ -i\theta_1^- \\ i(\cos \beta \theta_1^- + \sin \beta \theta_3^-) \\ i(\cos \beta \theta_2^- - \sin \beta \theta_4^-) \\ \cos \beta \theta_2^- - \sin \beta \theta_4^- \\ -(\cos \beta \theta_1^- + \sin \beta \theta_3^-) \end{pmatrix}$$

A.4 Relevant piece of quartic lagrangian

Here, we provide the piece of the quartic lagrangian needed for demonstrating one-loop finiteness¹

$$\begin{aligned}
\mathcal{L}_{(4,b+f)} = & \frac{i}{4} \left\{ \sum_{i=1}^4 (\dot{\chi}_i^+ \bar{\chi}_i^+ + \chi_i^- \bar{\chi}_i^-) |y_1|^2 - \cos^4 \phi \left[\sum_{i=1}^4 (\dot{\chi}_i^+ \bar{\chi}_i^+ + \chi_i^- \bar{\chi}_i^-) - 4i \sin^2 \phi (\chi_2^- \bar{\chi}_2^+ - \chi_3^- \bar{\chi}_3^+) \right] |y_2|^2 \right. \\
& \left. - \sin^4 \phi \left[\sum_{i=1}^4 (\dot{\chi}_i^+ \bar{\chi}_i^+ + \chi_i^- \bar{\chi}_i^-) + 4i \cos^2 \phi (\chi_2^- \bar{\chi}_2^+ - \chi_3^- \bar{\chi}_3^+) \right] |y_3|^2 \right\} \\
& - \frac{i}{4} \left\{ \left[(\chi_1^- \bar{\chi}_1^- + \chi_2^- \bar{\chi}_2^- - \chi_3^- \bar{\chi}_3^- - \chi_4^- \bar{\chi}_4^-) - (\chi_1^+ \bar{\chi}_1^+ + \chi_2^+ \bar{\chi}_2^+ - \chi_3^+ \bar{\chi}_3^+ - \chi_4^+ \bar{\chi}_4^+) \right] y_1 (\dot{y}_1 - \dot{y}'_1) \right. \\
& + \cos^2 \phi \left[\left(\sum_{i=1}^4 \chi_i^- \bar{\chi}_i^- \right) y_2 (\dot{y}_2 - \cos^2 \phi \dot{y}'_2) + \left(\sum_{i=1}^4 \chi_i^+ \bar{\chi}_i^+ \right) y_2 (\dot{y}'_2 - \cos^2 \phi \dot{y}_2) \right] \\
& + \sin^2 \phi \left[\left(\chi_1^- \bar{\chi}_1^- - \chi_2^- \bar{\chi}_2^- - \chi_3^- \bar{\chi}_3^- + \chi_4^- \bar{\chi}_4^- \right) y_3 (\dot{y}_3 - \sin^2 \phi \dot{y}'_3) \right. \\
& \left. + \left(\chi_1^+ \bar{\chi}_1^+ - \chi_2^+ \bar{\chi}_2^+ - \chi_3^+ \bar{\chi}_3^+ + \chi_4^+ \bar{\chi}_4^+ \right) y_3 (\dot{y}'_3 - \sin^2 \phi \dot{y}_3) \right] \left. \right\} \\
& - \frac{1}{2} \left[\left(\chi_1^- \bar{\chi}_1^+ + \cos^2 \phi \chi_2^- \bar{\chi}_2^+ + \sin^2 \phi \chi_3^+ \bar{\chi}_3^- \right) \dot{y}_1 y'_1 - \left(\cos^2 \phi \chi_1^+ \bar{\chi}_1^- + \chi_2^+ \bar{\chi}_2^- + \sin^2 \phi \chi_4^+ \bar{\chi}_4^- \right) \dot{y}_2 y'_2 \right. \\
& \left. - \left(\sin^2 \phi \chi_1^+ \bar{\chi}_1^- + \chi_3^- \bar{\chi}_3^+ + \cos^2 \phi \chi_4^+ \bar{\chi}_4^- \right) \dot{y}_3 y'_3 - \left(\sin^2 \phi \chi_2^- \bar{\chi}_2^+ + \cos^2 \phi \chi_3^+ \bar{\chi}_3^- + \chi_4^- \bar{\chi}_4^+ \right) \dot{y}_4 y'_4 \right] \\
& + \text{h.c.} + \dots, \tag{A.6}
\end{aligned}$$

where the ellipses account for terms not relevant for the computation.

A.5 Light-to-light scattering

Here we collect the *scattering* (\mathbb{S}) and *transmission* (\mathbb{T}) pieces of the *light-to-light S-matrix*:

22 \rightarrow 22:

$$\begin{aligned}
\mathbb{S} = & \frac{1}{2} \sin^2 2\phi \left[-\cos^4 \phi + p_1 p_2 + \omega^{(2)}(p_1) \omega^{(2)}(p_2) \right]_u + \frac{1}{2} \sin^2 2\phi \left[p_1^2 + p_2^2 \right]_t \\
& + \cos^2 \phi \left[2 \cos^4 \phi \sin^2 \phi + 2 \cos^2 \phi (p_1^2 + p_1 p_2 + p_2^2) + 2 p_1 p_2 - 2 \sin^2 \phi \omega^{(2)}(p_1) \omega^{(2)}(p_2) \right]_c \\
= & 2 \cos^2 \phi (p_1 + p_2)^2,
\end{aligned}$$

$$\begin{aligned}
\mathbb{T} = & \frac{1}{2} \sin^2 2\phi \left[-\cos^4 \phi - p_1 p_2 - \omega^{(2)}(p_1) \omega^{(2)}(p_1) \right]_s + \frac{1}{2} \sin^2 2\phi \left[p_1^2 + p_2^2 \right]_t \\
& + \cos^2 \phi \left[2 \cos^4 \phi \sin^2 \phi + 2 \cos^2 \phi (p_1^2 - p_1 p_2 + p_2^2) - 2 p_1 p_2 + 2 \sin^2 \phi \omega^{(2)}(p_1) \omega^{(2)}(p_1) \right]_c \\
= & 2 \cos^2 \phi (p_1 - p_2)^2,
\end{aligned}$$

¹For notational convenience, and to keep the expression as compact as possible, we denote the action of ∂_+ with a *dot* and that of ∂_- with a *prime*.

33→ 33:

$$\begin{aligned} \mathbb{S} &= \frac{1}{2} \sin^2 2\phi \left[-\sin^4 \phi + p_1 p_2 + \omega^{(3)}(p_1) \omega^{(3)}(p_2) \right]_u + \frac{1}{2} \sin^2 2\phi \left[p_1^2 + p_2^2 \right]_t \\ &\quad + \sin^2 \phi \left[2 \sin^4 \phi \cos^2 \phi + 2 \sin^2 \phi (p_1^2 + p_1 p_2 + p_2^2) + 2p_1 p_2 - 2 \cos^2 \phi \omega^{(3)}(p_1) \omega^{(3)}(p_2) \right]_c \\ &= 2 \sin^2 \phi (p_1 + p_2)^2 , \end{aligned}$$

$$\begin{aligned} \mathbb{T} &= \frac{1}{2} \sin^2 2\phi \left[-\sin^4 \phi - p_1 p_2 - \omega^{(3)}(p_1) \omega^{(3)}(p_1) \right]_s + \frac{1}{2} \sin^2 2\phi \left[p_1^2 + p_2^2 \right]_t \\ &\quad + \sin^2 \phi \left[2 \sin^4 \phi \cos^2 \phi + 2 \sin^2 \phi (p_1^2 - p_1 p_2 + p_2^2) - 2p_1 p_2 + 2 \cos^2 \phi \omega^{(3)}(p_1) \omega^{(3)}(p_1) \right]_c \\ &= 2 \sin^2 \phi (p_1 - p_2)^2 , \end{aligned}$$

23→ 23:

$$\mathbb{S} = \mathbb{T} = -2 \left[\cos^4 \phi p_2^2 + \sin^4 \phi p_1^2 \right]_t + 2 \left[\cos^4 \phi p_2^2 + \sin^4 \phi p_1^2 \right]_c = 0 ,$$

32→ 32:

$$\mathbb{S} = \mathbb{T} = -2 \left[\cos^4 \phi p_1^2 + \sin^4 \phi p_2^2 \right]_t + 2 \left[\cos^4 \phi p_1^2 + \sin^4 \phi p_2^2 \right]_c = 0 . \quad (\text{A.7})$$

Note that we have neglected the overall delta-functions and external leg factors. As was the case in the hamiltonian computation, the various contributions tend to cancel among each other.

As a final comment here, we would like to mention that care has to be taken when evaluating the t-channel contributions. Naively, one gets 0/0 expressions and, in order to obtain the correct result, one should symmetrise over the in- and outgoing momenta before enforcing overall energy and momentum conservations².

²We would like to thank Kostya Zarembo for bringing these subtleties to our attention.

Appendix B

B.1 Particle-vortex duality à la Burgess & Dolan

In this appendix, we review the duality of [119], ignoring some terms that are not essential to our argument.

B.1.1 First derivation

The starting point is (4.1), the action of an abelian Higgs system of constant modulus, with a weak external electromagnetic gauge field A_μ . One also introduces a *statistical* (Chern-Simons) *gauge field* a_μ ¹:

$$\mathcal{L}(\phi, \xi, a, A) = -\frac{\kappa}{2} \left[(\partial_\mu - q_\phi (a + A)_\mu) \phi \right]^2 - \frac{\pi}{2\theta} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \mathcal{L}_p(\xi, a + A) + \dots, \quad (\text{B.1})$$

where

- q_ϕ is the (nonzero) *electric charge* of the complex order parameter, with *phase* ϕ . That $q_\phi \neq 0$ means the U(1) gauge group is *spontaneously broken*, with the scale of the symmetry-breaking expectation value of order of the scale of the parameter κ ;
- ϕ is the *Goldstone boson* for the assumed symmetry-breaking, the couplings of which with $(a + A)_\mu$ are dictated by gauge invariance;
- $\theta = 2\pi n$, n integer, corresponds to bosonic particles, while $\theta = (2n + 1)\pi$ describes fermionic particles;

and

$$\mathcal{L}_p(\xi, a + A) = \sum_k \left(\frac{m}{2} \dot{\xi}_k^\mu \dot{\xi}_{k,\mu} + q_k \dot{\xi}_k^\mu (a + A)_\mu \right) \delta(x - \xi_k(t)) \quad (\text{B.2})$$

is the (*first-quantised*²) *particle lagrangian*, in the absence of other interactions. Here, q_k is the particle charge, normalised so that $q_k = -1$ for fermions, and $\xi_k^\mu(t)$ is the

¹This field, which arises from the combinatorics of the charged particles, has no dynamical degrees of freedom. It ensures that the interchange of two particles in the theory produces the phase $e^{i\theta}$

²A first-quantised representation is chosen because this makes the duality between particles and vortices most transparent.

position of the k th particle at time t . Finally, \dots represent all of the other *effective interactions*, involving inverse powers of the higher-energy scales (such as κ), obtained when all higher degrees of freedom are integrated out, which turn out to be irrelevant for the present purposes compared to those explicitly displayed.

As is standard for dualities in the path integral formulation, this action can be lifted to a *master action* through the coupling of ϕ to a new gauge field, \mathcal{A}_μ , constrained by a Lagrange-multiplier field b_μ (which is the dual of the Goldstone field ϕ) to be pure gauge³:

$$\begin{aligned} \mathcal{L}(\phi, \xi, a, b, A, \mathcal{A}) &= -\frac{\kappa}{2} \left[(\partial_\mu - q_\phi(a + A + \mathcal{A})_\mu) \phi \right]^2 - \frac{\pi}{2\theta} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \\ &+ \mathcal{L}_p(\xi, a + A) + \epsilon^{\mu\nu\rho} b_\mu \partial_\nu \mathcal{A}_\rho + \dots \end{aligned} \quad (\text{B.3})$$

Indeed, integrating over b_μ , we find $\partial_{[\nu} \mathcal{A}_{\rho]} = 0$, and then integrating over \mathcal{A}_μ is equivalent to putting it to zero⁴. On the other hand, integrating first over ϕ , and then over \mathcal{A}_μ , will lead to a dual action in terms of the Lagrange multiplier b_μ .

To do that, care must be taken about the periodicity of ϕ in the presence of vortices for the original complex scalar field Φ . We have

$$\phi(\vartheta + 2\pi) = \phi(\vartheta) + 2\pi \sum_a N_a,$$

where $N_a \in \mathbb{Z}$ labels the *vorticity* (or *winding number*) of vortex a , and ϑ is the polar angle extended from the vortex position and taken at spatial infinity. It is convenient to write $\phi = \omega + \varphi$, where φ satisfies periodic boundary conditions, $\varphi(\vartheta + 2\pi) = \varphi(\vartheta)$, and $\omega(x)$ is an explicit vortex solution that has the same boundary condition as ϕ :

$$\omega(x) = \sum_a N_a \arctan\left(\frac{x^1 - y_a^1}{x^2 - y_a^2}\right) \equiv \sum_a N_a \theta_a, \quad (\text{B.4})$$

where $y_a^i(t)$, $i = 1, 2$ are the coordinates of vortex positions, and

$$\left(\frac{x^1 - y_a^1}{x^2 - y_a^2}\right) = \tan \theta_a$$

defines the angle of rotation around vortex a . In the notation of [113] described in the introduction, ω corresponds to θ_{vortex} , and φ to θ_{smooth} .

The gauge potential defined by v_μ , where

$$\begin{aligned} v_\mu &= \partial_\mu \omega = \sum_a N_a \frac{1}{\sec^2 \theta_a} \partial_\mu \tan \theta_a \\ &= \sum_a N_a \partial_\mu \theta_a \end{aligned}$$

³We want the associated field strength $\mathcal{F}_{\mu\nu} = \partial_{[\mu} \mathcal{A}_{\nu]}$ to vanish at $|x| = \infty$.

⁴Performing the integration over b_μ produces a functional delta function which enforces the constraint $\epsilon^{\mu\nu\rho} \partial_\nu \mathcal{A}_\rho = 0$; this, together with the gauge fixing condition, implies that integrating over \mathcal{A}_μ is equivalent to setting $\mathcal{A}_\mu = 0$.

has δ -function singularities (it vanishes everywhere else). This means that

$$\begin{aligned}\epsilon^{\mu\nu\rho}b_\mu\partial_\nu v_\rho &= b_\mu \sum_a N_a \epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho \theta_a \\ &= 2\pi b_\mu \sum_a N_a \dot{y}_a^\mu \delta(x - y_a(t)) \\ &= 2\pi b_\mu j^\mu(t),\end{aligned}$$

where

$$j^\mu(t) = j_{\text{vortex}}^\mu(t) = \sum_a N_a \dot{y}_a^\mu \delta(x - y_a(t)) \quad (\text{B.5})$$

is the vortex current. Since $\epsilon^{ij}\partial_i\partial_j\theta_a = 2\pi\delta^2(x)$, we can indeed verify the above formula for static $y_a^i(t) = y_a^i$, when $\dot{y}_a^0 = 1$ and 0 otherwise, giving

$$\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\theta_a = \delta^{\mu 0}\epsilon^{ij}\partial_i\partial_j\theta_a = 2\pi\delta^{\mu 0}\delta^2(x - y_a).$$

Note now that (B.3) has a gauge invariance

$$\delta A_\mu = \partial_\mu \lambda; \quad \delta\phi = q_\phi \lambda,$$

which we can gauge-fix by putting $\varphi = 0$ (that is, $\phi = \omega$), thus making the path integration over ϕ trivial. We are thus left with only the path integral over \mathcal{A}_μ , and since $\partial_\mu\phi = \partial_\mu\omega = v_\mu$, the path integral we need to determine is

$$\int \mathcal{D}\mathcal{A} \exp \left\{ i \int d^3x \left[-\frac{\kappa}{2} (v_\mu - q_\phi(a + A + \mathcal{A})_\mu)^2 + \epsilon^{\mu\nu\rho} \partial_\mu b_\nu \mathcal{A}_\rho \right] \right\},$$

and of course, we still have the particle action and the statistical gauge field part of the action outside the path integral. Then, defining

$$J^\rho \equiv \epsilon^{\mu\nu\rho} \partial_\mu b_\nu,$$

and completing the square in the lagrangian, we get the path integral

$$\begin{aligned}& \int \mathcal{D}\mathcal{A} \exp \left\{ i \int d^3x \left[-\frac{\kappa q_\phi^2}{2} \left((a + A + \mathcal{A})_\mu - \frac{v_\mu}{q_\phi} - \frac{J_\mu}{\kappa q_\phi^2} \right)^2 - J^\mu \left((a + A)_\mu - \frac{v_\mu}{q_\phi} \right) + \frac{J_\mu^2}{2\kappa q_\phi^2} \right] \right\} \\ &= \mathcal{N} \exp \left\{ -i \int d^3x \left[J^\mu \left((a + A)_\mu - \frac{v_\mu}{q_\phi} \right) - \frac{J_\mu^2}{2\kappa q_\phi^2} \right] \right\}\end{aligned}$$

where

$$\mathcal{N} = \int \mathcal{D}\mathcal{A} \exp \left\{ i \int d^3x \left[-\frac{\kappa q_\phi^2}{2} \left((a + A + \mathcal{A})_\mu - \frac{v_\mu}{q_\phi} - \frac{J_\mu}{\kappa q_\phi^2} \right)^2 \right] \right\}.$$

Given

$$\int J^\mu v_\mu = \int \epsilon^{\mu\nu\rho} \partial_\nu b_\rho v_\mu = \int \epsilon^{\mu\nu\rho} b_\nu \partial_\rho v_\mu = 2\pi \int b_\mu j^\mu(t)$$

where $j^\mu(t)$ is as in (B.5), and

$$J_\mu^2 = 2 \delta_{\mu\nu}^{\rho\sigma} \partial^\mu b^\nu \partial_\rho b_\sigma = \frac{1}{2} f_{\mu\nu}^{(b)2},$$

where $f_{\mu\nu}^{(b)} = \partial_{[\mu} b_{\nu]}$, we have as the *dual action*

$$\tilde{\mathcal{L}}(\xi, y, a, b, A) = -\frac{1}{4\kappa q_\phi^2} f_{\mu\nu}^{(b)2} - \epsilon^{\mu\nu\rho} b_\mu \partial_\nu (a + A)_\rho - \frac{\pi}{2\theta} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \mathcal{L}_p(\xi, a + A) + \mathcal{L}_v(y, b). \quad (\text{B.6})$$

where \mathcal{L}_v describes the dynamics of the vortices and their couplings to b_μ ,

$$\mathcal{L}_v(y, b) = \sum_a \left[\frac{M}{2} \dot{y}_a^\mu \dot{y}_{a\mu} - \frac{2\pi N_a}{q_\phi} \dot{y}_a^\mu b_\mu \right] \delta(x - y_a(t)), \quad (\text{B.7})$$

M being the *vortex mass*. Note that the kinetic term in (B.7) does not itself follow directly from dualising, and is fixed quite generally from symmetry arguments. As we can observe, besides the dualisation from the field ϕ to the field b_μ , we have also obtained an explicit action for moving vortices, with positions $y_a^\mu(t)$. Therefore, we explicitly see that the dualisation of ϕ to b_μ also exchanges particles with vortices, deserving the name of particle-vortex duality, that is (B.1) and (B.6) describe exactly the same physics. The lagrangians (B.2) and (B.7) are very similar, and suggest that the duality involves the following interchanges:

particle \leftrightarrow vortex $\xi_k^\mu \leftrightarrow y_a^\mu$ $(a + A)_\mu \leftrightarrow b_\mu$ $q_k \leftrightarrow \frac{2\pi N_a}{q_\phi}$ $m \leftrightarrow M$	(B.8)
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B.1.2 Second derivation

We now review a second derivation from [119], which is closer to what we use in the bulk of the chapter. We start with an abelian Higgs action where the complex scalar field Φ is coupled to a Chern-Simons gauge field a_μ and an external gauge field A_μ , with an arbitrary scalar potential depending only on $|\Phi|$,

$$S[\Phi, a, A] = -\frac{1}{2} \int d^3x \left[[(i\partial_\mu - e\tilde{a}_\mu) \Phi]^\dagger [(i\partial^\mu - e\tilde{a}^\mu) \Phi] + \frac{\pi e^2}{\theta} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \right] + S_{\text{int}} [|\Phi|^2], \quad (\text{B.9})$$

where $\tilde{a} \equiv (a + A)$, and S_{int} is the interaction part of the action, the form of which we will ignore in our discussion. We rewrite (B.9) as

$$S[\Phi, a, A] = -\frac{1}{2} \int d^3x \left[(\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) + e^2 |\Phi|^2 \tilde{a}_\mu \tilde{a}^\mu - 2 \tilde{a}_\mu j^\mu + \frac{\pi e^2}{\theta} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \right] + S_{\text{int}} [|\Phi|^2],$$

where the scalar current is

$$\begin{aligned} j_\mu &= \frac{ie}{2} \left[\Phi^\dagger \partial_\mu \Phi - (\partial_\mu \Phi^\dagger) \Phi \right] \\ &= e |\Phi|^2 \partial_\mu \theta. \end{aligned} \quad (\text{B.10})$$

As before, we split Φ into a vortex part v and a smooth part,

$$\Phi(\vec{r}) = \Phi_0(\vec{r}) e^{-i\theta(\vec{r})} v(\vec{r}),$$

where

$$v(\vec{r}) = \exp \left[\frac{2\pi i}{q_\phi} \sum_a N_a \arctan \left(\frac{x^1 - y_a^1}{x^2 - y_a^2} \right) \right].$$

Then we have

$$\begin{aligned} S_a[\Phi_0, \theta, a, A] &= -\frac{1}{2} \int d^3x \left[(\partial_\mu \Phi_0)^2 + (\partial_\mu \theta - i v^* \partial_\mu v - e \tilde{a}_\mu)^2 \Phi_0^2 \right] + S_{\text{CS}}[a] + S_{\text{int}}[|\Phi|^2] \\ &= -\frac{1}{2} \int d^3x \left[(\partial_\mu \Phi_0)^2 + e^2 \Phi_0^2 \tilde{a}_\mu \tilde{a}^\mu + \frac{1}{e^2 \Phi_0^2} j_\mu j^\mu - 2 \tilde{a}_\mu j^\mu \right] \\ &\quad - \frac{\pi e^2}{2\theta} \int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + S_{\text{int}}[\Phi_0^2], \end{aligned}$$

where the current is

$$j_\mu = e \Phi_0^2 (\partial_\mu \theta + i v^* \partial_\mu v).$$

We now define $\lambda_\mu = \partial_\mu \theta$; this is the $\partial_\mu \varphi$ from the last subsection, whereas $-i v^* \partial_\mu v$ is the $\partial_\mu \omega$ from there. We substitute the integration over θ with integration over λ_μ , subject to the constraint $\epsilon^{\mu\nu\rho} \partial_\nu \lambda_\rho = 0$ imposed with a Lagrange multiplier \tilde{b}_μ , that is

$$\begin{aligned} &\int \mathcal{D}\theta \exp \left[-\frac{i}{2} \int d^3x (\partial_\mu \theta + i v^* \partial_\mu v - e \tilde{a}_\mu)^2 \Phi_0^2 \right] \\ &= \int \mathcal{D}\lambda_\mu \mathcal{D}\tilde{b}_\mu \exp \left\{ -\frac{i}{2} \int d^3x \left[(\lambda_\mu + i v^* \partial_\mu v - e \tilde{a}_\mu)^2 \Phi_0^2 + \epsilon^{\mu\nu\rho} \tilde{b}_\mu \partial_\nu \lambda_\rho \right] \right\}. \end{aligned}$$

Integrating over λ_μ first, we obtain

$$\begin{aligned} S_b[\Phi_0, A, a, \tilde{b}] &= \int d^3x \left[-\frac{1}{4e^2 \Phi_0^2} \tilde{f}_{\mu\nu}^{(b)} \tilde{f}^{(b)\mu\nu} + \tilde{j}^\mu \tilde{b}_\mu - \epsilon^{\mu\nu\rho} \tilde{a}_\mu \partial_\nu \tilde{b}_\rho - \frac{\pi e^2}{2\theta} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \right] \\ &\quad - \frac{1}{2} \int d^3x \partial_\mu \Phi_0 \partial^\mu \Phi_0 + S'_{\text{int}}[\Phi_0^2], \end{aligned} \tag{B.11}$$

where as usual, $\tilde{f}_{\mu\nu}^{(b)} = \partial_{[\mu} \tilde{b}_{\nu]}$, and

$$\tilde{j}^\mu = \frac{i}{e} \epsilon^{\mu\nu\rho} \partial_\nu v^* \partial_\rho v$$

is the *vortex current*.

The duality between the actions S_a and S_b is exactly the same as that of the last subsection, with the kinetic term for Φ_0 and the interaction term being spectators, but here it was derived from an abelian Higgs action by path integration.

Note that the classical solution for λ_μ is

$$\partial_\mu \theta \equiv \lambda_\mu = -i v^* \partial_\mu v + e \tilde{a}_\mu + \frac{i}{e \Phi_0^2} \epsilon^{\mu\nu\rho} \partial_\nu \tilde{b}_\rho,$$

which matches with the duality transform of Zee (from the introduction) with $v = \text{constant}$, $\tilde{a} = 0$.

B.2 Review of ABJM and its massive deformation

The ABJM model [8] is obtained as the low-energy limit of the theory of N coincident M2-branes in a $\mathbb{C}^4/\mathbb{Z}_k$ background. It is a supersymmetric $\mathcal{N} = 6$ $U(N) \times U(N)$ (or $SU(N) \times SU(N)$) Chern-Simons (CS) gauge theory, with bifundamental scalars Y^I and fermions $\psi_I, I = 1, \dots, 4$, in the fundamental of the $SU(4)_R$ symmetry group, and the two CS gauge fields, A_μ and \hat{A}_μ , have equal and opposite levels k and $-k$. Its action is

$$\begin{aligned}
S = \int d^3x & \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right. \\
& - \text{Tr} \left(D_\mu Y_I^\dagger D^\mu Y^I + i \psi^{I\dagger} \gamma^\mu D_\mu \psi_I \right) + \frac{4\pi^2}{3k^2} \text{Tr} \left(Y^I Y_I^\dagger Y^J Y_J^\dagger Y^K Y_K^\dagger \right. \\
& + Y_I^\dagger Y^I Y_J^\dagger Y^J Y_K^\dagger Y^K + 4 Y^I Y_J^\dagger Y^K Y_I^\dagger Y^J Y_K^\dagger - 6 Y^I Y_J^\dagger Y^J Y_I^\dagger Y^K Y_K^\dagger Y^K \\
& + \frac{2\pi i}{k} \text{Tr} \left(Y_I^\dagger Y^I \psi^J + \psi_J - \psi^{J\dagger} Y^I Y_I^\dagger \psi_J - 2 Y_I^\dagger Y^J \psi^{I\dagger} \psi_J + 2 \psi^{J\dagger} Y^I Y_I^\dagger \psi_J \right. \\
& \left. \left. + \epsilon^{IJKL} Y_I^\dagger \psi_J Y_K^\dagger \psi_L - \epsilon_{IJKL} Y^I \psi^{J\dagger} Y^K \psi^{L\dagger} \right) \right]. \tag{B.12}
\end{aligned}$$

Here the covariant derivative acts like

$$D_\mu Y^I = \partial_\mu Y^I + i \left(A_\mu Y^I - Y^I \hat{A}_\mu \right).$$

The action (B.12) has $SU(4) \times U(1)$ R-symmetry associated with the $\mathcal{N} = 6$ supersymmetries.

B.2.1 Massive ABJM

There exists a unique supersymmetry-preserving massive deformation of the model [130], parametrised by μ , that breaks the R-symmetry down to $SU(2) \times SU(2) \times U(1)_A \times U(1)_B \times \mathbb{Z}_2$ as a consequence of splitting the scalars as

$$Y^I = (Q^\alpha, R^\alpha), \quad \alpha = 1, 2.$$

The \mathbb{Z}_2 action interchanges Q^α and R^α , each $SU(2)$ factor acts on only one of the doublets Q^α and R^α , and the $U(1)_A$ symmetry rotates Q^α with a charge +1 and R^α with a charge -1. The mass deformation gives mass to the fermions and changes the potential of the theory. The bosonic part of the action in the mass deformed case is

$$\begin{aligned}
\mathcal{L}_{\text{bosonic}} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} & \left[A_\mu \partial_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda + \frac{2i}{3} \left(A_\mu A_\nu A_\lambda - \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right] \\
& - \text{Tr} |D_\mu Q^\alpha|^2 - \text{Tr} |D_\mu R^\alpha|^2 - V. \tag{B.13}
\end{aligned}$$

The sextic scalar potential in (B.13) is

$$V = \text{Tr} \left(|M^\alpha|^2 + |N^\alpha|^2 \right),$$

where

$$M^\alpha = \mu Q^\alpha + \frac{2\pi}{k} \left(2 Q^{[\alpha} Q_\beta^\dagger Q^{\beta]} + R^\beta R_\beta^\dagger Q^\alpha - Q^\alpha R_\beta^\dagger R^\beta + 2 Q^\beta R_\beta^\dagger R^\alpha - 2 R^\alpha R_\beta^\dagger Q^\beta \right),$$

$$N^\alpha = -\mu R^\alpha + \frac{2\pi}{k} \left(2 R^{[\alpha} R_\beta^\dagger R^{\beta]} + Q^\beta Q_\beta^\dagger R^\alpha - R^\alpha Q_\beta^\dagger Q^\beta + 2 R^\beta Q_\beta^\dagger Q^\alpha - 2 Q^\alpha Q_\beta^\dagger R^\beta \right).$$

The equations of motion of the bosonic fields are

$$D_\mu D^\mu Q^\alpha = \frac{\partial V}{\partial Q_\alpha^\dagger}, \quad D_\mu D^\mu R^\alpha = \frac{\partial V}{\partial R_\alpha^\dagger},$$

$$F_{\mu\nu} = \frac{2\pi}{k} \epsilon_{\mu\nu\lambda} J^\lambda, \quad \hat{F}_{\mu\nu} = \frac{2\pi}{k} \epsilon_{\mu\nu\lambda} \hat{J}^\lambda,$$

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + i[A_\mu, A_\nu]$, and the two gauge currents J^μ and \hat{J}^μ , expressed as

$$J^\mu = i \left[Q^\alpha (D^\mu Q^\alpha)^\dagger - (D^\mu Q^\alpha) Q_\alpha^\dagger + R^\alpha (D^\mu R^\alpha)^\dagger - (D^\mu R^\alpha) R_\alpha^\dagger \right],$$

$$\hat{J}^\mu = -i \left[Q_\alpha^\dagger (D^\mu Q^\alpha) - (D^\mu Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger (D^\mu R^\alpha) - (D^\mu R^\alpha)^\dagger R^\alpha \right],$$

are covariantly conserved, that is, $\nabla_\mu J^\mu = \nabla_\mu \hat{J}^\mu = 0$. The trace parts of those gauge currents yields two abelian currents j^μ and \hat{j}^μ corresponding to the global $U(1)_A$ and $U(1)_B$ invariances

$$j^\mu = \text{Tr } J^\mu, \quad \hat{j}^\mu = \text{Tr } \hat{J}^\mu, \quad (\text{B.14})$$

which are ordinarily conserved, that is, $\partial_\mu j^\mu = \partial_\mu \hat{j}^\mu = 0$. The gauge choice $A_0 = \hat{A}_0 = 0$ implies that the energy density is given by

$$H = \text{Tr} \left[(\partial_0 Q^\alpha)^\dagger (\partial_0 Q^\alpha) + (D_i Q^\alpha)^\dagger (D_i Q^\alpha) + (\partial_0 R^\alpha)^\dagger (\partial_0 R^\alpha) + (D_i R^\alpha)^\dagger (D_i R^\alpha) + V \right].$$

Since this is a Chern-Simons theory, varying with respect to A_0 and \hat{A}_0 gives the Gauss law constraints

$$F_{12} = \frac{2\pi i}{k} J^0 = \frac{2\pi i}{k} \left[Q^\alpha (\partial^0 Q^\alpha)^\dagger - (\partial^0 Q^\alpha) Q_\alpha^\dagger + R^\alpha (\partial^0 R^\alpha)^\dagger - (\partial^0 R^\alpha) R_\alpha^\dagger \right],$$

$$\hat{F}_{12} = \frac{2\pi i}{k} \hat{J}^0 = -\frac{2\pi i}{k} \left[Q_\alpha^\dagger (\partial^0 Q^\alpha) - (\partial^0 Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger (\partial^0 R^\alpha) - (\partial^0 R^\alpha)^\dagger R^\alpha \right].$$

Note as an aside that the gauge choice does not uniquely specify the hamiltonian. Choosing A_0 and \hat{A}_0 different from zero introduces an extra term in the hamiltonian, $\epsilon^{\mu\nu\lambda} \text{Tr}[A_\mu A_\nu A_\lambda - \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda]$. In the abelianisation ansatz of [129], this vanishes anyway since it is proportional to $\epsilon^{\mu\nu\lambda} a_\mu^{(i)} a_\nu^{(j)} a_\lambda^{(k)}$ and there are only two $a_\mu^{(i)}$'s. So in the abelian case, the hamiltonian is the same even away from the gauge $A_0 = \hat{A}_0 = 0$.

The mass deformed theory has fuzzy sphere ground states given by⁵

$$R^\alpha = cG^\alpha; \quad Q^\alpha = 0 \quad \text{and} \quad Q_\alpha^\dagger = cG^\alpha; \quad R^\alpha = 0,$$

⁵General vacuum configurations could also be direct sums of these irreducible solutions.

where $c \equiv \sqrt{\frac{\mu k}{2\pi}}$, and the matrices G^α , $\alpha = 1, 2$, satisfy the equations [124, 125]

$$G^\alpha = G^\alpha G_\beta^\dagger G^\beta - G^\beta G_\beta^\dagger G^\alpha. \quad (\text{B.15})$$

In [131, 132], it was shown that this solution corresponds to a fuzzy 2-sphere, not a 3-sphere as originally thought.

An explicit solution of these equations, which is the unique irreducible one up to a $U(N) \times U(N)$ gauge transformation, is given by

$$\begin{aligned} (G^1)_{m,n} &= \sqrt{m-1} \delta_{m,n}, \\ (G^2)_{m,n} &= \sqrt{(N-m)} \delta_{m+1,n}, \\ (G_1^\dagger)_{m,n} &= \sqrt{m-1} \delta_{m,n}, \\ (G_2^\dagger)_{m,n} &= \sqrt{(N-n)} \delta_{n+1,m}. \end{aligned}$$

In particular, $G^1 = G_1^\dagger$. In the case of pure ABJM, instead of a fuzzy sphere ground state, there is a *fuzzy funnel BPS solution* with c replaced by

$$c(s) = \sqrt{\frac{k}{4\pi s}}.$$

Here s is one of the two spatial coordinates of the ABJM model. The matrices G^α are bifundamental under $U(N) \times U(N)$, implying that $G^1 G_1^\dagger$ and $G^2 G_2^\dagger$ are in the adjoint of the first $U(N)$, and that $G_1^\dagger G^1$ and $G_2^\dagger G^2$ are in the adjoint of the second $U(N)$.

Bibliography

- [1] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, [arXiv:hep-th/9711200].
- [2] S.S. Gubser, I.R. Klebanov & A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys.Lett. **B428** (1998) 105-114, [arXiv:hep-th/9802109].
- [3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, [arXiv:hep-th/9802150].
- [4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri & Y. Oz, *Large-N field theories, string theory and gravity*, Phys. Rept. **323** (2000) 183, [arXiv:hep-th/9905111].
- [5] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal & U.A. Wiedemann, *Gauge/String Duality, Hot QCD and Heavy Ion Collisions*, Cambridge University Press (2014), [arXiv:1101.0618 [hep-th]].
- [6] S.A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, Class.Quant.Grav. **26** (2009) 224002, [arXiv:0903.3246 [hep-th]].
- [7] N. Rughoonauth, P. Sundin & L. Wulff, *Near-BMN dynamics of the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring*, JHEP **1207** (2012) 159, [arXiv:1204.4742 [hep-th]].
- [8] O. Aharony, O. Bergman, D.L. Jafferis & J. Maldacena, *$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **0810** (2008) 091, [arXiv:0806.1218 [hep-th]].
- [9] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov & R.A. Janik *et al.*, *Review of AdS/CFT Integrability: An Overview*, Lett. Math. Phys. **99** (2012) 3, [arXiv:1012.3982 [hep-th]].
- [10] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, Phys. Rev. D **69** (2004) 046002, [arXiv:hep-th/0305116].
- [11] G. Arutyunov and S. Frolov, *Superstrings on $AdS_4 \times CP^3$ as a Coset Sigma-model*, JHEP **0809** (2008) 129, [arXiv:0806.4940 [hep-th]].
- [12] B. Stefanski Jr, *Green-Schwarz action for Type IIA strings on $AdS_4 \times CP^3$* , Nucl. Phys. B **808** (2009) 80, [arXiv:0806.4948 [hep-th]].
- [13] J. Gomis, D. Sorokin & L. Wulff, *The Complete $AdS_4 \times CP^3$ superspace for the type IIA superstring and D-branes*, JHEP **0903** (2009) 015, [arXiv:0811.1566 [hep-th]].

- [14] A. Cagnazzo, D. Sorokin & L. Wulff, *String instanton in $AdS_4 \times \mathbb{CP}^3$* , JHEP **1005** (2010) 009, [arXiv:0911.5228 [hep-th]].
- [15] D. Sorokin & L. Wulff, *Evidence for the classical integrability of the complete $AdS_4 \times \mathbb{CP}^3$ superstring*, JHEP **1011** (2010) 143, [arXiv:1009.3498 [hep-th]].
- [16] A. Cagnazzo, D. Sorokin & L. Wulff, *More on integrable structures of superstrings in $AdS_4 \times \mathbb{CP}^3$ and $AdS_2 \times S^2 \times T^6$ superbackgrounds*, JHEP **1201** (2012) 004, [arXiv:1111.4197 [hep-th]].
- [17] D.V. Uvarov, *Kaluza-Klein gauge and minimal integrable extension of $OSp(4|6)/(SO(1,3) \times U(3))$ sigma-model*, Int.J.Mod.Phys. **A27** (2012) 1250118, [arXiv:1203.3041 [hep-th]].
- [18] D. Sorokin, A. Tseytlin, L. Wulff & K. Zarembo, *Superstrings in $AdS_2 \times S^2 \times T^6$* , J. Phys. A **A44** (2011) 275401, [arXiv:1104.1793 [hep-th]].
- [19] A. Babichenko, B. Stefanski Jr. & K. Zarembo, *Integrability and the AdS_3/CFT_2 correspondence*, JHEP **1003** (2010) 058, [arXiv:0912.1723 [hep-th]].
- [20] S. Elitzur, O. Feinerman, A. Giveon & D. Tsabar, *String theory on $AdS_3 \times S^3 \times S^3 \times S^1$* , Phys. Lett. B **449** (1999) 180, [arXiv:hep-th/9811245].
- [21] J.P. Gauntlett, R.C. Myers & P.K. Townsend, *Supersymmetry of rotating branes*, Phys. Rev. D **59** (1999) 025001, [arXiv:hep-th/9809065].
- [22] P.M. Cowdall & P.K. Townsend, *Gauged supergravity vacua from intersecting branes*, Phys. Lett. B **429** (1998) 281, [Erratum-ibid. B **434** (1998) 458] [arXiv:hep-th/9801165].
- [23] H.J. Boonstra, B. Peeters & K. Skenderis, *Brane intersections, anti-de Sitter spacetimes and dual superconformal theories*, Nucl. Phys. B **533** (1998) 127, [arXiv:hep-th/9803231].
- [24] G. Papadopoulos, J.G. Russo & A.A. Tseytlin, *Curved branes from string dualities*, Class. Quant. Grav. **17** (2000) 1713, [arXiv:hep-th/9911253].
- [25] A. Giveon & A. Pakman, *More on superstrings in $AdS_3 \times N$* , JHEP **0303** (2003) 056, [arXiv:hep-th/0302217].
- [26] J. de Boer, A. Pasquinucci & K. Skenderis, *AdS/CFT dualities involving large 2D $\mathcal{N} = 4$ superconformal symmetry*, Adv. Theor. Math. Phys. **3** (1999) 577, [arXiv:hep-th/9904073].
- [27] S. Gukov, E. Martinec, G.W. Moore & A. Strominger, *The Search for a holographic dual to $AdS_3 \times S^3 \times S^3 \times S^1$* , Adv. Theor. Math. Phys. **9** (2005) 435, [arXiv:hep-th/0403090].
- [28] V.A. Kazakov, A. Marshakov, J.A. Minahan & K. Zarembo, *Classical/quantum integrability in AdS/CFT* , JHEP **0405** (2004) 024, [arXiv:hep-th/0402207].
- [29] N. Beisert, V.A. Kazakov, K. Sakai & K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$* , Commun. Math. Phys. **263** (2006) 659, [arXiv:hep-th/0502226].

- [30] M. Staudacher, *The factorized S-matrix of CFT/AdS*, JHEP **0505** (2005) 054, [arXiv:hep-th/0412188].
- [31] N. Beisert, *The SU(2|2) dynamic S-matrix*, Adv. Theor. Math. Phys. **12** (2008) 945, [arXiv:hep-th/0511082].
- [32] J.M. Maldacena & H. Ooguri, *Strings in AdS₃ and SL(2, R) WZW model, I: The Spectrum*, J. Math. Phys. **42** (2001) 2929, [arXiv:hep-th/0001053].
- [33] J.M. Maldacena, H. Ooguri & J. Son, *Strings in AdS₃ and the SL(2, R) WZW model, II: Euclidean black hole*, J. Math. Phys. **42** (2001) 2961, [arXiv:hep-th/0005183].
- [34] J.M. Maldacena & H. Ooguri, *Strings in AdS₃ and the SL(2, R) WZW model, III: Correlation functions*, Phys. Rev. D **65** (2002) 106006, [arXiv:hep-th/0111180].
- [35] L. Frappat, P. Sorba & A. Sciarrino, *Dictionary on Lie superalgebras*, [arXiv:hep-th/9607161].
- [36] O.O. Sax & B. Stefanski Jr., *Integrability, spin-chains and the AdS₃/CFT₂ correspondence*, JHEP **1108** (2011) 029, [arXiv:1106.2558 [hep-th]].
- [37] R. Borsato, O.O. Sax, A. Sfondrini & B. Stefanski, *All-loop worldsheet S-matrix for AdS₃ × S³ × T⁴*, [arXiv:1403.4543 [hep-th]].
- [38] R. Borsato, O.O. Sax, A. Sfondrini & B. Stefanski, *The complete AdS₃ × S³ × T⁴ worldsheet S-matrix*, [arXiv:1406.0453 [hep-th]].
- [39] J.R. David & B. Sahoo, *Giant magnons in the D1-D5 system*, JHEP **0807** (2008) 033, [arXiv:0804.3267 [hep-th]].
- [40] J.R. David & B. Sahoo, *S-matrix for magnons in the D1-D5 system*, JHEP **1010** (2010) 112, [arXiv:1005.0501 [hep-th]].
- [41] O. Lunin & S.D. Mathur, *Rotating deformations of AdS₃ × S³, the orbifold CFT and strings in the pp-wave limit*, Nucl. Phys. B **642** (2002) 91, [arXiv:hep-th/0206107].
- [42] J. Gomis, L. Motl & A. Strominger, *PP-wave/CFT₂ duality*, JHEP **0211** (2002) 016, [arXiv:hep-th/0206166].
- [43] E. Gava & K.S. Narain, *Proving the PP-wave/CFT₂ duality*, JHEP **0212** (2002) 023, [arXiv:hep-th/0208081].
- [44] L. Sommovigo, *Penrose limit of AdS₃ × S³ × S³ × S¹ and its associated sigma model*, JHEP **0307** (2003) 035, [arXiv:hep-th/0305151].
- [45] V. Forini, V.G.M. Puletti & O.O. Sax, *Generalized cusp in AdS₄ × CP³ and more one-loop results from semiclassical strings*, J.Phys. **A46** (2013) 115402, [arXiv:1204.3302 [hep-th]].
- [46] G. Arutyunov & S. Frolov, *Uniform light-cone gauge for strings in AdS₅ × S⁵: Solving SU(1|1) sector*, JHEP **0601** (2006) 055, [arXiv:hep-th/0510208].
- [47] P. Sundin & L. Wulff, *Classical integrability and quantum aspects of the AdS₃ × S³ × S³ × S¹ superstring*, JHEP **1210** (2012) 109, [arXiv:1207.5531 [hep-th]].

- [48] S. Frolov, J. Plefka & M. Zamaklar, *The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations*, J.Phys. **A39** (2006) 13037, [arXiv:hep-th/0603008].
- [49] M. Srednicki, *Quantum Field Theory*, Cambridge, UK: Univ.Pr. (2007) 660 pages
- [50] P. Sundin, *The $AdS_4 \times CP^3$ string and its Bethe equations in the near-plane-wave limit*, JHEP **0902** (2009) 046, [arXiv:0811.2775 [hep-th]].
- [51] D. Astolfi, G. Grignani, E. Ser-Giacomi & A.V. Zayakin, *Strings in $AdS_4 \times CP^3$: finite-size spectrum vs Bethe Ansatz*, [arXiv:1111.6628 [hep-th]].
- [52] D. Astolfi, V.G.M. Puletti, G. Grignani, T. Harmark & M. Orselli, *Finite-size corrections for quantum strings on $AdS_4 \times CP^3$* , JHEP **1105** (2011) 128, [arXiv:1101.0004 [hep-th]].
- [53] A.A. Tseytlin, *On dilaton dependence of type II superstring action*, Class. Quant. Grav. **13** (1996) L81, [arXiv:hep-th/9601109].
- [54] M. Cvetič, H. Lu, C.N. Pope & K.S. Stelle, *T-duality in the Green-Schwarz formalism, and the massless/massive IIA duality map*, Nucl. Phys. B **573** (2000) 149, [arXiv:hep-th/9907202].
- [55] P.A. Grassi, D. Sorokin & L. Wulff, *Simplifying superstring and D-brane actions in $AdS_4 \times CP^3$ superbackground*, JHEP **0908** (2009) 060, [arXiv:0903.5407 [hep-th]].
- [56] D.E. Berenstein, J.M. Maldacena & H.S. Nastase, *Strings in flat space and pp-waves from $N = 4$ super-Yang-Mills*, JHEP **0204** (2002) 013, [arXiv:hep-th/0202021].
- [57] K. Zarembo, *Worldsheet spectrum in AdS_4/CFT_3 correspondence*, JHEP **0904** (2009) 135, [arXiv:0903.1747 [hep-th]].
- [58] M.C. Abbott & P. Sundin, *The Near-Flat-Space and BMN Limits for Strings in $AdS_4 \times CP^3$ at One Loop*, J.Phys. **A45** (2012) 025401, [arXiv:1106.0737 [hep-th]].
- [59] N. Gromov & V. Mikhaylov, *Comment on the Scaling Function in $AdS_4 \times CP^3$* , JHEP **0904** (2009) 083, [arXiv:0807.4897 [hep-th]].
- [60] T. McLoughlin & R. Roiban, *Spinning strings at one-loop in $AdS_4 \times IP^3$* , JHEP **0812** (2008) 101, [arXiv:0807.3965 [hep-th]].
- [61] L.F. Alday, G. Arutyunov & D. Bykov, *Semiclassical Quantization of Spinning Strings in $AdS_4 \times CP^3$* , JHEP **0811** (2008) 089, [arXiv:0807.4400 [hep-th]].
- [62] C. Krishnan, *AdS_4/CFT_3 at One Loop*, JHEP **0809** (2008) 092, [arXiv:0807.4561 [hep-th]].
- [63] T. McLoughlin, R. Roiban & A.A. Tseytlin, *Quantum spinning strings in $AdS_4 \times CP^3$: Testing the Bethe Ansatz proposal*, JHEP **0811** (2008) 069, [arXiv:0809.4038 [hep-th]].
- [64] M.A. Bandres & A.E. Lipstein, *One-Loop Corrections to Type IIA String Theory in $AdS_4 \times CP^3$* , JHEP **1004** (2010) 059, [arXiv:0911.4061 [hep-th]].

- [65] I. Shenderovich, *Giant magnons in AdS_4/CFT_3 : dispersion, quantization and finite-size corrections*, [arXiv:0807.2861 [hep-th]].
- [66] C. Lopez-Arcos & H. Nastase, *Eliminating ambiguities for quantum corrections to strings moving in $AdS_4 \times \mathbb{CP}^3$* , Int.J.Mod.Phys. **A28** (2013) 1350058, [arXiv:1203.4777 [hep-th]].
- [67] C.G. Callan Jr., T. McLoughlin & I. Swanson, *Holography beyond the Penrose limit*, Nucl. Phys. B **694** (2004) 115, [arxiv:hep-th/0404007].
- [68] C.G. Callan Jr., T. McLoughlin & I. Swanson, *Higher impurity AdS/CFT correspondence in the Near-BMN limit*, Nucl. Phys. B **700** (2004) 271, [arXiv:hep-th/0405153].
- [69] T. McLoughlin & I. Swanson, *N-impurity superstring spectra near the pp-wave limit*, Nucl. Phys. B **702** (2004) 86, [arXiv:hep-th/0407240].
- [70] D. Astolfi, V.G.M. Puletti, G. Grignani, T. Harmark & M. Orselli, *Finite-size corrections in the $SU(2) \times SU(2)$ sector of type IIA string theory on $AdS_4 \times \mathbb{CP}^3$* , Nucl. Phys. B **810** (2009) 150, [arXiv:0807.1527 [hep-th]].
- [71] A. Hentschel, J. Plefka & P. Sundin, *Testing the nested light-cone Bethe equations of the $AdS_5 \times S^5$ superstring*, JHEP **0705** (2007) 021, [arXiv:hep-th/0703187].
- [72] P. Sundin, *On the worldsheet theory of the type IIA $AdS_4 \times \mathbb{CP}^3$ superstring*, JHEP **1004** (2010) 014, [arXiv:0909.0697 [hep-th]].
- [73] J.A. Minahan & K. Zarembo, *The Bethe ansatz for superconformal Chern-Simons*, JHEP **0809** (2008) 040, [arXiv:0806.3951 [hep-th]].
- [74] G. Arutyunov, S. Frolov & M. Staudacher, *Bethe ansatz for quantum strings*, JHEP **0410** (2004) 016, [arxiv:hep-th/0406256].
- [75] C. Ahn & R.I. Nepomechie, *$\mathcal{N} = 6$ super Chern-Simons theory S-matrix and all-loop Bethe ansatz equations*, JHEP **0809** (2008) 010, [arXiv:0807.1924 [hep-th]].
- [76] C. Ahn & R.I. Nepomechie, *Two-loop test of the $\mathcal{N} = 6$ Chern-Simons theory S-matrix*, JHEP **0903** (2009) 144, [arXiv:0901.3334 [hep-th]].
- [77] C. Ahn, P. Dorey and R.I. Nepomechie, *On the absence of reflection in AdS_4/CFT_3* , JHEP **1001** (2010) 129, [arXiv:0910.5584 [hep-th]].
- [78] M.C. Abbott, I. Aniceto & D. Bombardelli, *Quantum Strings and the AdS_4/CFT_3 Interpolating Function*, JHEP **1012** (2010) 040, [arXiv:1006.2174 [hep-th]].
- [79] G. Arutyunov & S. Frolov, *Foundations of the $AdS_5 \times S^5$ Superstring. Part I*, J.Phys. **A42** (2009) 254003, [arXiv:0901.4937 [hep-th]].
- [80] D. Berenstein, *A Toy model for the AdS/CFT correspondence*, JHEP **0407** (2004) 018, [arXiv:hep-th/0403110].
- [81] R.C. Myers, *Dielectric branes*, JHEP **9912** (1999) 022, [arXiv:hep-th/9910053].
- [82] J. McGreevy, L. Susskind & N. Toumbas, *Invasion of the giant gravitons from Anti-de Sitter space*, JHEP **0006** (2000) 008, [arXiv:hep-th/0003075].

- [83] H. Lin, O. Lunin & J.M. Maldacena, *Bubbling AdS space and 1/2-BPS geometries*, JHEP **0410** (2004) 025, [arXiv:hep-th/0409174].
- [84] D. Berenstein, *Large-N BPS states and emergent quantum gravity*, JHEP **0601** (2006) 125, [arXiv:hep-th/0507203].
- [85] D. Berenstein & R. Cotta, *Aspects of emergent geometry in the AdS/CFT context*, Phys.Rev. **D74** (2006) 026006, [arXiv:hep-th/0605220].
- [86] M.M. Sheikh-Jabbari, *Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture*, JHEP **0409** (2004) 017, [arXiv:hep-th/0406214].
- [87] M.M. Sheikh-Jabbari & M. Torabian, *Classification of all 1/2-BPS solutions of the tiny graviton matrix theory*, JHEP **0504** (2005) 001, [arXiv:hep-th/0501001].
- [88] V. Balasubramanian, M. Berkooz, A. Naqvi & M.J. Strassler, *Giant gravitons in conformal field theory*, JHEP **0204** (2002) 034, [arXiv:hep-th/0107119].
- [89] S. Corley, A. Jevicki & S. Ramgoolam, *Exact correlators of giant gravitons from dual N=4 SYM theory*, Adv.Theor.Math.Phys. **5** (2002) 809-839, [arXiv:hep-th/0111222].
- [90] R. de Mello Koch & J. Murugan, *Emergent Spacetime*, Foundations of Space and Time: Reflections on Quantum Gravity (CUP) (2009) 164-184, [arXiv:0911.4817 [hep-th]].
- [91] R. de Mello Koch, J. Smolic & M. Smolic, *Giant Gravitons - with Strings Attached (I)*, JHEP **0706** (2007) 074, [arXiv:hep-th/0701066].
- [92] R. de Mello Koch, J. Smolic & M. Smolic, *Giant Gravitons - with Strings Attached (II)*, JHEP **0709** (2007) 049, [arXiv:hep-th/0701067].
- [93] D. Bekker, R. de Mello Koch & M. Stephanou, *Giant Gravitons - with Strings Attached (III)*, JHEP **0802** (2008) 029, [arXiv:0710.5372 [hep-th]].
- [94] V. Balasubramanian, D. Berenstein, B. Feng & M.-x. Huang, *D-branes in Yang-Mills theory and emergent gauge symmetry*, JHEP **0503** (2005) 006, [arXiv:hep-th/0411205].
- [95] J. Bagger & N. Lambert, *Modeling Multiple M2's*, Phys.Rev. **D75** (2007) 045020, [arXiv:hep-th/0611108].
- [96] A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl.Phys. **B811** (2009) 66-76, [arXiv:0709.1260 [hep-th]].
- [97] J. Bagger & N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, Phys.Rev. **D77** (2008) 065008, [arXiv:0711.0955 [hep-th]].
- [98] J. Bagger & N. Lambert, *Comments on multiple M2-branes*, JHEP **0802** (2008) 105, [arXiv:0712.3738 [hep-th]].
- [99] A. Gustavsson, *One-loop corrections to Bagger-Lambert theory*, Nucl.Phys. **B807** (2009) 315-333, [arXiv:0805.4443 [hep-th]].

- [100] T. Nishioka & T. Takayanagi, *Fuzzy Ring from M2-brane Giant Torus*, JHEP **0810** (2008) 082, [arXiv:0808.2691 [hep-th]].
- [101] D. Berenstein & J. Park, *The BPS spectrum of monopole operators in ABJM: Towards a field theory description of the giant torus*, JHEP **1006** (2010) 073, [arXiv:0906.3817 [hep-th]].
- [102] M.M. Sheikh-Jabbari & J. Simon, *On Half-BPS States of the ABJM Theory*, JHEP **0908** (2009) 073, [arXiv:0904.4605 [hep-th]].
- [103] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, JHEP **0011** (2000) 027, [arXiv:hep-th/0010206].
- [104] A. Mikhailov, *Nonspherical giant gravitons and matrix theory*, [arXiv:hep-th/0208077].
- [105] M.C. Abbott, J. Murugan, A. Prinsloo & N. Rughoonauth, *Meromorphic Functions and the Topology of Giant Gravitons*, Phys.Lett. **B730** (2014) 215-220, [arXiv:1312.4900 [hep-th]].
- [106] I. Biswas, D. Gaiotto, S. Lahiri & S. Minwalla, *Supersymmetric states of N=4 Yang-Mills from giant gravitons*, JHEP **0712** (2007) 006, [arXiv:hep-th/0606087].
- [107] J. Pasukonis & S. Ramgoolam, *From counting to construction of BPS states in N=4 SYM*, JHEP **1102** (2011) 078, [arXiv:1010.1683 [hep-th]].
- [108] R. de Mello Koch & S. Ramgoolam, *A double coset ansatz for integrability in AdS/CFT*, JHEP **1206** (2012) 083, [arXiv:1204.2153 [hep-th]].
- [109] J. Pasukonis & S. Ramgoolam, *Quantum states to brane geometries via fuzzy moduli spaces of giant gravitons*, JHEP **1204** (2012) 077, [arXiv:1201.5588 [hep-th]].
- [110] D. Berenstein, *Giant gravitons: a collective coordinate approach*, Phys.Rev.D **12** (2013) 126009, [arXiv:1301.3519 [hep-th]].
- [111] E. Witten & D.I. Olive, *Supersymmetry Algebras That Include Topological Charges*, Phys.Lett. **B78** (1978) 97.
- [112] A. Kapustin & E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program*, Commun.Num.Theor.Phys. **1** (2007) 1-236, [arXiv:hep-th/0604151].
- [113] A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press (2010).
- [114] D.H. Lee & M.P.A. Fisher, *Anyon superconductivity and charge-vortex duality*, Int.J.Mod.Phys. **B5** (1991) 2675-2699.
- [115] K.A. Intriligator & N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, Phys.Lett. **B387** (1996) 513-519, [arXiv:hep-th/9607207].
- [116] S. Kawamoto & F.-L. Lin, *Holographic Anyons in the ABJM Theory*, JHEP **1002** (2010) 059, [arXiv:0910.5536 [hep-th]].

- [117] S.A. Hartnoll, *Anyonic strings and membranes in AdS space and dual Aharonov-Bohm effects*, Phys.Rev.Lett. **98** (2007) 111601, [arXiv:hep-th/0612159].
- [118] C.-S. Park, *Comments on Baryon-like Operators in $\mathcal{N} = 6$ Chern-Simons-matter theory of ABJM*, [arXiv:0810.1075 [hep-th]].
- [119] C.P. Burgess & B.P. Dolan, *Particle vortex duality and the modular group: Applications to the quantum Hall effect and other 2-D systems*, Phys.Rev. **B63** (2001) 155309, [arXiv:hep-th/0010246].
- [120] A. Kapustin & M.J. Strassler, *On mirror symmetry in three-dimensional Abelian gauge theories*, JHEP **9904** (1999) 021, [arXiv:hep-th/9902033].
- [121] S. Sachdev & X. Yin, *Quantum phase transitions beyond the Landau-Ginzburg paradigm and supersymmetry*, Annals Phys. **325** (2010) 2-15.
- [122] E. Witten, *$SL(2, \mathbb{Z})$ action on three-dimensional conformal field theories with Abelian symmetry*, [arXiv:hep-th/0307041].
- [123] C.P. Herzog, P. Kovtun, S. Sachdev & D.T. Son, *Quantum critical transport, duality, and M-theory*, Phys.Rev. **D75** (2007) 085020, [arXiv:hep-th/0701036].
- [124] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk & H. Verlinde, *A Massive Study of M2-brane Proposals*, JHEP **0809** (2008) 113, [arXiv:0807.1074 [hep-th]].
- [125] S. Terashima, *On M5-branes in $N=6$ Membrane Action*, JHEP **0808** (2008) 080, [arXiv:0807.0197 [hep-th]].
- [126] S. Mukhi & C. Papageorgakis, *M2 to D2*, JHEP **0805** (2008) 085, [arXiv:0803.3218 [hep-th]].
- [127] X. Chu, H. Nastase, B.E.W. Nilsson & C. Papageorgakis, *Higgsing M2 to D2 with gravity: $N=6$ chiral supergravity from topologically gauged ABJM theory*, JHEP **1104** (2011) 040, [arXiv:1012.5969 [hep-th]].
- [128] A. Mohammed, J. Murugan & H. Nastase, *Towards a Realization of the Condensed-Matter/Gravity Correspondence in String Theory via Consistent Abelian Truncation*, Phys.Rev.Lett. **109** (2012) 181601, [arXiv:1205.5833 [hep-th]].
- [129] A. Mohammed, J. Murugan & H. Nastase, *Abelian-Higgs and Vortices from ABJM: towards a string realization of AdS/CMT*, JHEP **1211** (2012) 073, [arXiv:1206.7058 [hep-th]].
- [130] K. Hosomichi, K.-M. Lee & S. Lee, *Mass-Deformed Bagger-Lambert Theory and its BPS Objects*, Phys.Rev. **D78** (2008) 066015, [arXiv:0804.2519 [hep-th]].
- [131] H. Nastase, C. Papageorgakis & S. Ramgoolam, *The Fuzzy S^2 structure of M2-M5 systems in ABJM membrane theories*, JHEP **0905** (2009) 123, [arXiv:0903.3966 [hep-th]].
- [132] H. Nastase & C. Papageorgakis, *Bifundamental fuzzy 2-sphere and fuzzy Killing spinors*, SIGMA **6** (2010) 058, [arXiv:1003.5590 [math-ph]].

- [133] J. Murugan, H. Nastase, N. Rughonauth & J.P. Shock, *Particle-vortex and Maxwell duality in the $AdS_4 \times CP^3$ /ABJM correspondence*, [arXiv:1404.5926 [math-ph]].

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