A Lagrangian formulation of a theory of a scalar field superfluid dark matter

A thesis submitted in fulfilment of the requirements for the degree of **Master of Science**

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To Professor Saalih Allie

Abstract

In this thesis we discuss the dynamics of the relativistic Lagrangian of the theory of dark matter superfluidity. The second and third chapters of the thesis are a review. In the fourth chapter we show that a complex scalar field whose dynamics are dictated by such a Lagrangian, models dust in the background universe on cosmological scale. Prior to our calculations, the theory was shown to model dust on cosmological scale and a superfluid on galactic scale in the non-relativistic case [1]. This project, extends the non-relativistic theory, to include the relativistic background. We continued, using the relativistic Lagrangian, to investigate how perturbations of such a theory grow in a perturbed universe, and found that the density contrast (of the theory) is constant when the complex scalar field is not coupled to baryons in the weak-field limit.

Acknowledgements

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“Human beings are not the stars of a cosmic drama that has been planned from the beginning, but are part of a world governed by impersonal forces. A world in which accident plays a large role. A world in which we - ourselves, our intelligence, everything we hold dear, does not appear at a fundamental level. At the fundamental level there are mathematical equations. A rather cold, perhaps dispiriting view, so what sort of consolation can we find?” - Steven Weinberg, Of Beauty and Consolation.

Contents

Chapter 1

Introduction

The story of dark matter has too many heroes whom its discovery could be attributed to [7]. Fritz Zwicky and Vera Rubin are the most prominent of these heroes. In 1933, Zwicky calculated the mass-to-light ratio of the Coma cluster and found that it was not in agreement with the mass-to-light ratio of the Kapteyn system. Based on this calculation, he speculated that the cluster would be gravitationally anomalous unless it had additional mass. He called the additional mass, dark matter. In 1970, Vera Rubin and Kent Ford, as well as Kenneth Freeman were independently studying galaxy rotation curves [8]. Freeman discovered that the velocity maxima of NGC 300 and M33 occur at a larger radius than the one predicted using stellar photometry, he postulated in the appendix of his paper [9] that this shift in maxima might be resolved by undetected additional matter. Rubin and Ford were investigating the Andromeda nebula, when they too observed an anomaly in its velocity rotation curve [10]. The velocities remained constant as a function of the nebula's radius.

This is against the norm, for a gravitationally bound system like our solar system, the mass is concentrated at the center. Rotation velocities of the planets decrease as a function of the radius. This anomaly could be resolved by postulating that dark matter also exists within the Andromeda nebula. Zwicky, Rubin and Ford never ruled out a possibility that the gravitational anomalies they encountered in clusters and galaxies could be due to the inaccuracy of classical mechanics. Zwicky used the virial theorem to calculate the mass of the Coma cluster, Rubin and Ford relied upon Newton's law of gravity too to deduce their findings and so it is possible that this law needs to be modified. In 1983, Mordehai Milgrom solely took the first step in modifying the law. In his paper [11] he showed that Newton's 2nd law fails at very small accelerations. He came up with a modification of the law, known as MOND (Modified Newtonian Dynamics), which is just as successful as the dark matter paradigm at explaining the aforementioned gravitational anomalies in galaxies [12].

The situation has left physicists in a dilemma and caused division amongst them. There are two theories which solve the same problem but seem to not be compatible with one another. The conundrum is, which one is correct? Even though most physicists support the dark matter paradigm and only a few support MOND, it's hard to ignore the astronomical scaling relations that the latter explains, the Baryonic Tully-Fisher relation for instance [13]. What make a lot of physicists doubt MOND, is that it does not explain cosmological observations very well (at least for now). The dark matter paradigm on the other hand, could be incorporated into the parameterised big bang model of cosmology known as the ΛCDM model, which predicts the cosmic microwave background (CMB). In this model, dark matter is assumed to be a fluid of zero pressure.

In recent years, a long list of physicists have tried without success to unify MOND and ΛCDM. Lasha Berezhiani and Justin Khoury came up with a model which possibly marries the two [1]. In their paper [1], they state that dark matter could have two phases. It could be a fluid of zero pressure on cosmological scales in accord with ΛCDM. Such a fluid could undergo Bose-Einstein condensation and be in a superfluid phase on galactic scales. When in a superfluid phase, they have shown that, the dark matter fluid reproduces MOND. The idea of dark matter being a superfluid is not new. In the past, physicists were studying cosmological models in which dark matter was assumed to be a Bose-Einstein condensate [3,4]. What makes this model different is that the superfluid phase of dark matter has characteristics similar to MOND's.

1.1 Thesis outline

The thesis is divided into three main chapters excluding the Introduction. In chapter 2, we review the big bang model of cosmology. The review is carried out to introduce some of the scientific terms and concepts which will be predominant throughout our calculations. In chapter 3, we review MOND and Landau's model of superfluidity. We discuss both the non-relativistic and relativistic theories of MOND. The weak field limit of the relativistic theory (of MOND) has a mathematical formulation similar to that of the effective field theory of superfluids. This mathematical similarity has led the authors of [1] to conjecture that a superfluid in galaxies exerts a MOND-like force on baryons.

In the appendix of [1] the authors tidied up their conjecture by constructing a relativistic complex scalar field theory whose weak field limit reduces to the MOND-like theory of superfluids (mentioned in the above paragraph). In chapter 4, we study their relativistic theory, and demonstrate that in an FLRW universe, the complex scalar field (of their theory) describes a dust dark matter in harmony with ΛCDM. In addition to that, we extend their model, by investigating its scalar perturbation theory in the quasi-Newtonian limit.

1.2 Formal definitions and terminology

Proper Time

If we consider two events $(t_1,\, x_1,\, y_1,\, z_1)$ and $(t_2,\, x_2,\, y_2,\, z_2)$ in a Minkowski spacetime. The infinitesimal length of some arbitray path connecting the events is given by

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1.1}$$

When $ds^2 \geq 0$, equation (1.1) is said to be time-like and so we define

$$ds^2 = -c^2 d\tau^2 \tag{1.2}$$

where the quantity τ is called the proper time. We can rearrange (1.2) to get

$$\boxed{\tau = \int_{\mathrm{S}} \frac{ds}{c}}. \tag{1.3}$$

This suggests that the proper time is the time measured by an observer traversing a path S whose infinitesimal length is ds. When we substitute (1.2) into (1.1) we find that

$$d\tau^2 = dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} \tag{1.4}$$

which can be re-expressed as

$$\boxed{d\tau = \sqrt{1 - \frac{u_x}{c^2} - \frac{u_y}{c^2} - \frac{u_z}{c^2}}dt}. \tag{1.5}$$

The quantities $u_x = \frac{dx}{dt}$, $u_y = \frac{dy}{dt}$ and $u_z = \frac{dz}{dt}$ are the components of the observer's average speed along the path S. When these componets are zero $(u_x = u_y = u_z = 0)$, the proper time reduces to the coordinate time $(d\tau = dt)$. For the most general case when all the components are non-zero, the proper time is always smaller than the coordinate time $(u < c)$.

The flatness of galaxy rotation curves

A galaxy rotation curve is a plot of the rotation velocities of stars in a galaxy versus their radial distance from the galactic center. In the case of our solar system, a plot of the rotation velocities v of the planets versus their radial distance r from the Sun falls off in the manner $\propto \frac{1}{\sqrt{r}}$, see figure 1.1. For most (not all) galaxies the plot is constant or rather flat, see figure 1.2.

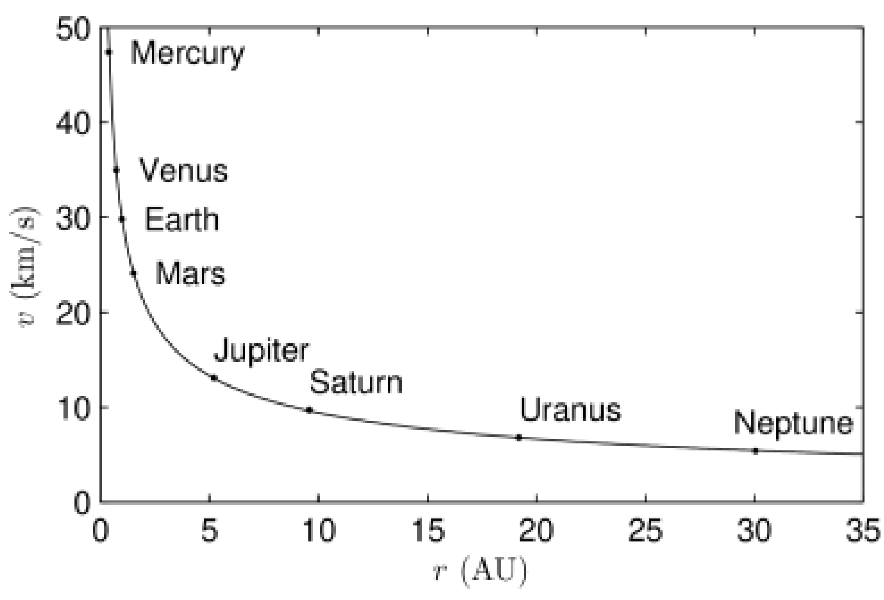

Figure 1.1: Rotation Curve of our solar system [29].

The inner planets, the ones closer to the Sun by distance, have bigger rotation velocities than the outer planets, the ones farther away from the Sun.

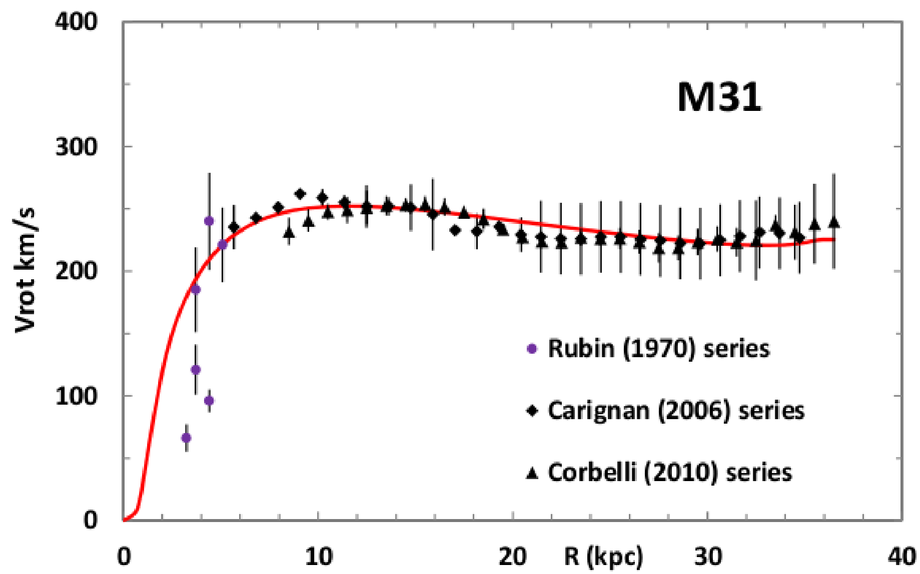

Figure 1.2: Rotation Curve of M31, with rotation velocity for the LN model (solid line) and observations with error bars [30].

The reason being that, most of the solar system's mass is concentrated at the center. As a result of that, the inner planets experience an intense gravitational pull than the outer ones. In galaxies, rotation velocities are almost the same for both the inner and the outer stars. This is an anomaly, the outer stars if they have the same rotation velocities as the inner ones, would cause the galaxy to fly apart unless there was additional mass distributed evenly throughout in the form of a halo to hold it together.

Mass-to-light ratio

The mass-to-light ratio $\Upsilon_\odot$, as its name suggests, is the ratio between the mass $M_\odot$ of an object and its luminosity $L_\odot$ (a measure of the amount of light the object emits) in solar mass and solar luminosity units:

$$\boxed{\Upsilon_\odot = \frac{M_\odot}{L_\odot}}. \tag{1.6}$$

The Sun for instance has $M_\odot = 1M_\odot$ and $L_\odot = 1L_\odot$. Its mass-to-light ratio is thus 1. If there was a galaxy which is made up of a thousand Suns its mass and luminosity would be $1000M_\odot$ and $1000L_\odot$ respectively, hence its mass-to-light ratio would still come out as 1. Many galaxies are made up of stars similar to the Sun and so it was thought that they too would have $\Upsilon_\odot = 1$. Instead, their mass-to-light ratio was found to be bigger than 1, leading to an anomaly which is resolved by imposing that there is "something" adding to the mass of the stars in those galaxies but gives off less light.

The Tully-Fisher relation

The Tully-Fisher relation is the relationship between the mass of the stars M and their flat rotation velocities v in a galaxy, expressed mathematically as a power law of the form:

$$\boxed{\mathrm{M} \propto \mathrm{v}^{\mathrm{n}}}. \tag{1.7}$$

It signifies that, galaxies rotate faster if they are bigger. Low mass galaxies are almost made of entirely gas and tend to not obey (1.7). The relation can however be recovered for all galaxies, irrespective of their mass by taking into account the mass of the stars + gas. The more fundamental relation is that, the rotation velocities correlate with the total mass, this is known as the Baryonic Tully-Fisher relation (BTFR) [31]. In the case of BTFR, the power law index n is determined as follows: if one applies log on both sides of (1.7) and replace M by $\mathrm{M}_{\mathrm{total}}$ they would get

$$\boxed{\log \mathrm{M}_{\mathrm{total}} = \mathrm{n}\log \mathrm{v} + \log \mathrm{W}} \tag{1.8}$$

where W is the constant of proportionality. Equation (1.8) has the form of an equation of a straight line $y = mx + c$, where the slope of the line is the power law index n. To determine n empirically, let us consider a log-log plot of the total mass of various galaxies against their rotation velocities as in figure 1.3. From the figure, the line of best-fit (the red line) of the data was found to have a slope of 3.6, which can be rounded off to 4. Thus n = 4 and the Baryonic Tully-Fisher relation takes the form:

$$\boxed{\mathrm{M}_{\mathrm{total}} \propto \mathrm{v}^4}. \tag{1.9}$$

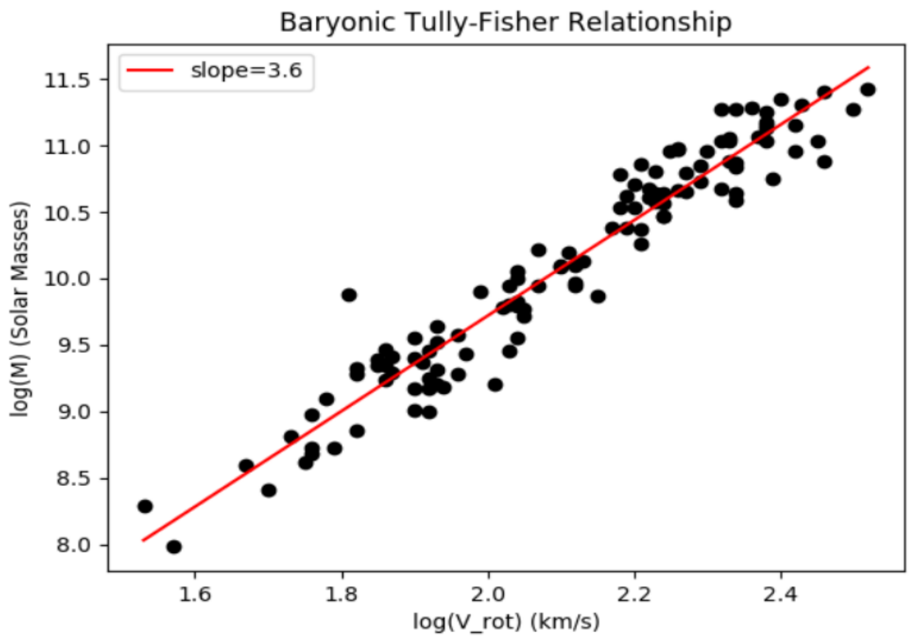

Figure 1.3: The Baryonic Tully-Fisher relation. Data retrieved from SPARC survey (without error bars) [32, 33].

Dark matter

In the introductory section, as well as from the definitions of "The flatness of galaxy rotation curves" and the "mass-to-light ratio", we have indirectly defined dark matter as the additional mass required to resolve the gravitational anomalies observed in galaxies. We have also mentioned that it has a very low luminosity, which is seemingly why it's called "dark". In section 2.2 a perfect fluid of negligible pressure will be used to model it. Based on this knowledge, we define dark matter as a fluid of zero pressure and zero luminosity. Typical fluids are made up of particles and dark matter is no exception. Due to its "dark" and pressureless character, the particles making it up are thought to be different from the ones constituting ordinary matter. Moreover, the dark matter fluid's particles are said to be moving slow, as a result of that, it is cold. Even though the scientific community (physicists to be specific) is aware of its existence, and its properties, it is unclear what a dark matter particle really is or how it looks like. There are various potential candidates for it, chosen based on its properties (some of them we have already mentioned): WIMPs[1] and Axions[2] to name a few. In this thesis we are more concerned about its behavior as a fluid. Its microscopic properties are subject to research in particle physics.

Bose-Einstein condensation

In classical mechanics, a gas is modelled as a collection of distinguishable particles in thermal equilibrium. In quantum mechanics, the particles are indistinguishable, and are classified as either Bosons or Fermions. By definition, Fermions are particles which obey Pauli's exclusion principle, Bosons are those that do not. This implies that Bosons can occupy the same energy state. When the particles of a quantum gas are Bosons, and are all occupying the ground state (the state with the lowest energy), the gas forms a Bose-Einstein condensate. The process by which a Bosonic gas changes from a gaseous phase to a Bose-Einstein condensate is called Bose-Einstein condensation. It is achieved by cooling the gas to almost absolute zero. Temperature is directly proportional to the kinetic energy of particles [3]. By cooling the particles, we decrease their energy, hence the ground state corresponds to the coldest temperature possible. Bose-Einstein condensation is not restricted to gases. Let us consider the liquid phase of Helium-4 (an isotope of Helium). The particles (or atoms) constituents of the liquid are Bosons. When it is cooled to temperatures close to absolute zero, the Bosons settle in the ground state, and the liquid exhibits "odd" properties. For instance, under such conditions the liquid has zero viscosity, and is referred to as a superfluid. Liquid Helium-3 also forms a superfluid when cooled to almost absolute zero, even though it is made up of Fermions. In that event, a Fermion particle forms a pair with another Fermion particle of opposite spin resulting in an overall spin of either zero or one. The pair thus forms a composite Boson since they have a combined integer spin[4]. These composite Bosons when they are at their lowest energy state, make liquid Helium-3 behave like a superfluid. In addition to particles of a gas being Bosons and occupying the ground state, there are two most general criteria for a gas or a liquid to undergo Bose-Einstein condensation:

[1] Weakly interacting massive particles.

[2] Hypothetical low mass particles proposed to solve the strong CP problem.

[3] The Kinetic theory of gases.

[4] Bosons have integer spin.

- The de Broglie wavelength $\lambda = \frac{1}{mv}$ (where the Planck constant is normalized to 1) of the particles in a gas or liquid must be larger than their mean interparticle separation $l = \left(\frac{m}{\rho}\right)^{\frac{1}{3}}$
- The particles' interaction rate must be larger than the dynamical time for particles in the gas or liquid

Many "ordinary" fluids do not meet these requirements, making them unlikely to form a superfluid or a Bose-Einstein condensate when cooled to very low temperatures. In the next section we investigate whether dark matter can.

1.3 Dark matter condensation

The condition that the de Broglie wavelength of particles must be larger than their mean interparticle separation translates to

$$m \lesssim \left(\frac{\rho}{v^3}\right)^{\frac{1}{4}}. \tag{1.10}$$

From the standard gravitational collapse theory, the density at virialization[5] is

$$\rho_{\text{vir}} = (1 + z_{\text{vir}})^3 \frac{\delta\rho}{\rho} \rho^0_{\text{DM}} \tag{1.11}$$

where $\rho^0_{\text{DM}} = 3 \times 10^{-30} \frac{g}{cm^3}$ is the present dark matter cosmological density, z_{vir} is the redshift at virialization and $\frac{\delta\rho}{\rho} = 180$ in order for virialization to occur. The velocity is given by [34]

$$v_{\text{vir}} = 127 \left(\frac{M}{10^{12}h^{-1}M_\odot}\right)^{\frac{1}{3}} \sqrt{1 + z_{\text{vir}}} \text{ km/s} \tag{1.12}$$

where $h \approx 0.7$ is the fundamental constant "Little h" and M is the mass of an object, be it a halo. Inserting (1.11) and (1.12) into (1.10) we find that

$$\boxed{m \lesssim 2.3\,(1 + z_{\text{vir}})^{\frac{3}{8}} \left(\frac{M}{10^{12}h^{-1}M_\odot}\right)^{-\frac{1}{4}} \text{ eV}}. \tag{1.13}$$

[5]When the dark matter halo is formed and the collapse ceased.

The interaction rate is defined as [35]

$$\Gamma = \mathcal{N} v \rho_{\rm vir} \frac{\sigma}{m}, \tag{1.14}$$

where σ is the scattering cross-section, and the Bose enhancement factor $\mathcal{N}$ is determined as follows:

$$\mathcal{N} = \frac{\rho_{\rm vir} \left(2\pi\right)^3}{m \frac{4\pi}{3} \left(mv\right)^3} = 10^3 \left(1 + z_{\rm vir}\right)^{\frac{3}{2}} \left(\frac{m}{\rm eV}\right)^{-4} \frac{10^{12} h^{-1} M_\odot}{M}. \tag{1.15}$$

The dynamical time on the other hand is given by

$$t_{\rm dyn} = \frac{1}{\sqrt{G_N \rho_{\rm vir}}}. \tag{1.16}$$

Substituting (1.14) and (1.16) into the second condition $\Gamma t_{\rm dyn} \gtrsim 1$ results in

$$\boxed{\frac{\sigma}{m} \gtrsim \left(1 + z_{\rm vir}\right)^{-\frac{7}{2}} \left(\frac{m}{\rm eV}\right)^4 \left(\frac{M}{10^{12} h^{-1} M_\odot}\right)^{\frac{2}{3}} 52 \frac{\rm cm^2}{\rm g}}. \tag{1.17}$$

The dark matter particles undergo Bose-Einstein condensation when (1.13) and (1.17) are satisfied. The critical temperature at which the condensation occurs is computed from $K_B T_c = \frac{1}{3} m v_c^2$ by assuming equipartition where v_c saturates. It was found to be

$$T_c = 6.5 \left(\frac{\rm eV}{m}\right)^{\frac{5}{3}} \left(1 + z_{\rm vir}\right)^2 \ \text{mK}. \tag{1.18}$$

The temperature in a halo (in the units of T_c) is thus given by

$$\frac{T}{T_c} = \frac{0.1}{1 + z_{\rm vir}} \left(\frac{m}{\rm eV}\right)^{\frac{8}{3}} \left(\frac{M}{10^{12} h^{-1} M_\odot}\right). \tag{1.19}$$

At sub-critical temperatures, the system is modelled as mixture of the condensate and the normal dark matter fluid. The fraction of the condensed particles (when interactions are ignored) in such a system is given by [36]

$$\frac{N_{\rm condensed}}{N} = 1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}}. \tag{1.20}$$

Substituting (1.19) into (1.20) we get

$$\frac{N_{\text{condensed}}}{N} = 1 - \frac{0.03}{(1+z_{\text{vir}})^{\frac{3}{2}}}\left(\frac{m}{\text{eV}}\right)^4 \frac{M}{10^{12}h^{-1}M_\odot}. \tag{1.21}$$

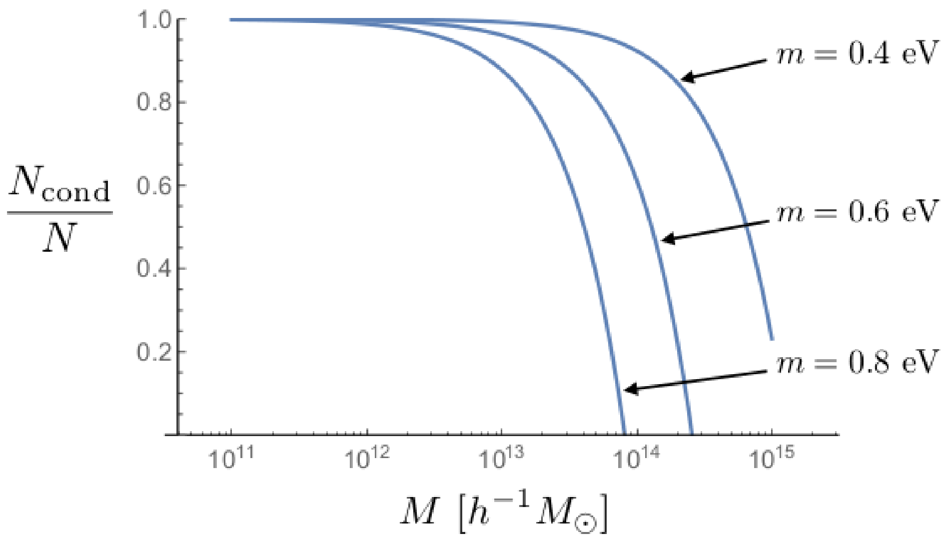

Figure 1.4: The fraction of condensed particles of dark matter versus the halo mass M when $z_{\text{vir}} = 0$, and $m = 0.4$, 0.6 and 0.8 eV [1].

From figure 1.4: we can see that, since galactic halos weigh $M \lesssim 10^{12}h^{-1}M_\odot$, then the dark matter in galaxies is comprised of a large number of particles in a condensed form. In galaxy clusters, the mass range of the halo is $10^{14}h^{-1}M_\odot \lesssim M \lesssim 10^{15}h^{-1}M_\odot$ suggesting that dark matter exists in its normal fluid form.

1.4 An overview of the superfluid dark matter model

The mass discrepancy in the universe has not been solved by the standard model of cosmology (ΛCDM) or modified Newtonian dynamics (MOND) paradigms so far. The problems and solutions of either scenario are mutually exclusive on cosmological and galactic scales. It has recently been proposed, by assuming that dark matter is a superfluid, MOND-like effects can be achieved on galactic scales while preserving the success of ΛCDM on cosmological scales [1]. The most obvious definition of superfluidity is the abilitiy of a liquid to flow through narrow channels without apparent friction. It was discovered in 1936-1937, through experiments involving Helium-4 that below the lambda point (2.17 Kelvins) Helium-4 possessed properties different from any other substance known at the time. In particular, the thermal conductivity of the low temperature phase, now known as Helium-II is very large, which suggests a convention mechanism, but with anomalously low viscosity. Today, superfluidity is something we can directly observe in Helium isotopes and in ultra-cold atomic gases. It is conjectured to occur in extra-terrestrial systems, such as Neutron stars, and there is circumstantial evidence supporting its existence in other terrestrial systems, such as excitons, which are bound electron-hole pairs found in semiconductors.

Several dark matter-MOND hybrid theories have been put forth over the years, but such proposals generally face two important challenges. First, there is the potential drawback of having two a priori unrelated ingredients: a dark matter-like component and a modified gravity component. Second, the theory must be adjusted such as to avoid the phenomenology.

This thesis examines a unified framework for the dark matter and MOND phenomena, based on dark matter superfluidity, which brings together concepts of condensed matter physics, cold atom physics and astrophysics. In this new approach, the dark matter and MOND components represent different phases of single underlying material, unified through the rich and well-studied physics of superfluidity. This model was first proposed in [1]. In the model, the MOND empirical law emerges from the superfluid phase of dark matter. As in ΛCDM, the model assumes dark matter particles, which behave as a cold, collisionless fluid on cosmological scales. As nonlinear structures form, the increase in dark matter density triggers a phase transition, causing dark matter to condense into a superfluid phase. This requires dark matter particles to be sufficiently light (mass of order eV), such that their de Broglie wavelengths overlap, and have self-interactions. The superfluid nature of dark matter dramatically changes its macroscopic properties in galaxies. Instead of behaving as individual collisionless particles, the dark matter is more aptly described as collective excitations, which at low energies are phonons. Phonons play a key role by coupling to ordinary matter, and thereby mediate an additional force (beyond Newtonian gravity) between baryons. For a particular choice of the superfluid equation of state, the dark matter superfluid reproduces the MOND law in galaxies.

The mathematical description of the underlying model is derived from a non-relativistic Lagrangian of superfluid phonons [2] which the authors of [1] conjectured to be the non-relativistic Lagrangian of a theory of MOND. The conjecture unifies MOND and the theory of superfluid phonons. The Lagrangian is modified by adding a free parameter Λ to it and an additional parameter α which couples the superfluid phonons to baryons. When the parameter α is $0.86\left(\frac{\Lambda}{\mathrm{meV}}\right)^{-\frac{2}{3}}$ or rather when α is set to order 1 and consequently Λ to meV on galactic scales the model hints that superfluid phonons exert a MOND-like force on baryons. When α is much smaller than $2.4\times10^{-4}\left(\frac{m}{\mathrm{eV}}\right)^{2}$ on cosmological scales the model describes dust, a cosmological name of cold dark matter in its usual phase. The parameters α and Λ are different for either phase leading to a speculation that they are dependent on temperature. One drawback of the model is that such temperature dependance is not well understood. A further point to note is that the theory of superfluid phonons is an effective field theory, and so it has to break down at some point. The theory breaks down when the velocity of the superfluid exceeds the critical velocity of the Bose-Einstein condensate [1].

The velocity of the superfluid is directly proportional to the gradient of the field. When the gradient of the field gets larger the velocity of the superfluid gets larger as well, and it could eventually exceed the critical velocity of the Bose-Einstein condensate. In galaxies it can be shown that such gradients are very small except in the galaxy's central regions (of order kiloparsec) [1]. In the solar system the Sun creates a huge gradient on the superfluid profile (figure 1.5) which violates this effective field theory bound. The effective field theory is no longer valid (in the solar system), and what you have is just the normal dark matter particles whizzing around [1].

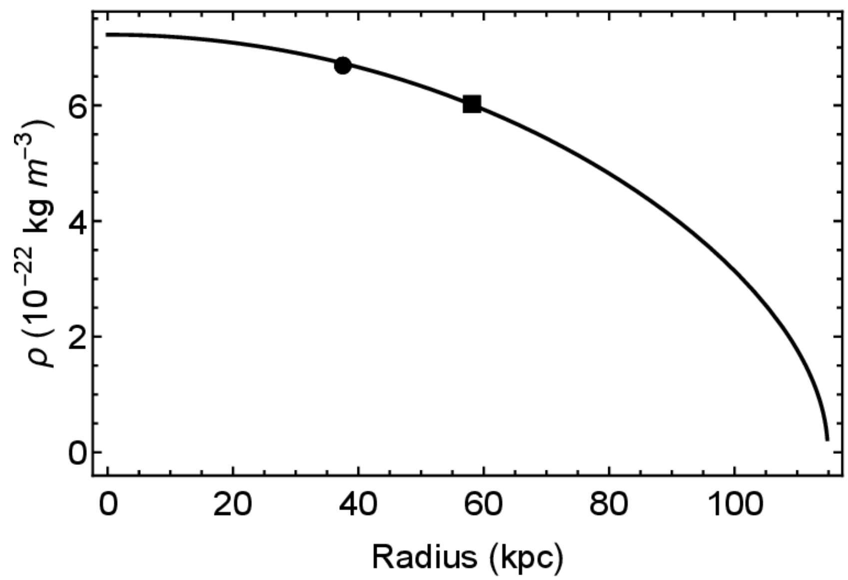

Figure 1.5: The density profile of a superfluid dark matter system (no baryon interactions) with central gravitational potential $\Phi_0 = -1012\,\mathrm{m}^2\mathrm{s}^{-2}$ and $\Lambda = 0.3 \times 10^{-3}\,\frac{\mathrm{eV}^4}{m^3c^8}$ [37].

Chapter 2

Cosmology

2.1 Dynamics of a homogeneous universe

2.1.1 Friedmann-Lemaitre-Robertson-Walker metric

The line element of spacetime in general relativity is given by

$$ds^2 = g_{\mu\nu}\left(t, \mathbf{x}\right) dx^\mu dx^\nu \tag{2.1}$$

where $g_{\mu\nu}\left(t, \mathbf{x}\right)$ is the metric tensor, t is the time and $\mathbf{x}$ represents the 3-dimensional spatial coordinates defined locally on the spacetime. If we consider the case when the coordinates $\mathbf{x}$ are Cartesian, the line element (2.1) in synchronous gauge takes the form

$$ds^2 = -c^2 dt^2 + g_{11}\left(t, \mathbf{x}\right) dx^2 + g_{22}\left(t, \mathbf{x}\right) dy^2 + g_{33}\left(t, \mathbf{x}\right) dz^2 \tag{2.2}$$

where the indices $\mu, \nu = 0, 1, 2, 3$ and the signature of the metric is (-,+,+,+). There are various symmetries one can consider to make the line element (2.2) simpler. Suppose it is homogeneous, in that case, the coefficients in front of the coordinates depend on t alone

$$ds^2 = -c^2 dt^2 + g_{11}\left(t\right) dx^2 + g_{22}\left(t\right) dy^2 + g_{33}\left(t\right) dz^2. \tag{2.3}$$

If it is also isotropic, then $g_{11}\left(t\right) = g_{22}\left(t\right) = g_{33}\left(t\right) = a^2(t)$ and equation (2.3) becomes

$$ds^2 = -c^2 dt^2 + a^2\left(t\right)\left(dx^2 + dy^2 + dz^2\right). \tag{2.4}$$

Equation (2.4) describes a homogeneous and isotropic spacetime with zero curvature. In spherical coordinates the line element reduces to

$$ds^2 = -c^2dt^2 + a^2(t)\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{2.5}$$

If one generalizes it to not just a flat spacetime but to a spacetime with arbitrary curvature K, equation (2.5) takes the form

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{2.6}$$

On large scales, in accord with the cosmological principle, the universe is homogeneous (it looks the same everywhere) and isotropic (it looks the same in every direction) hence its geometry is best described by equation (2.6), which is known as the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. When that is the case, the function $a(t)$ is called the scale-factor and the curvature K takes three possible values: $K = 0$ for a flat universe, $K = 1$ for a spherical (closed) universe and $K = -1$ for a hyperbolic (open) universe.

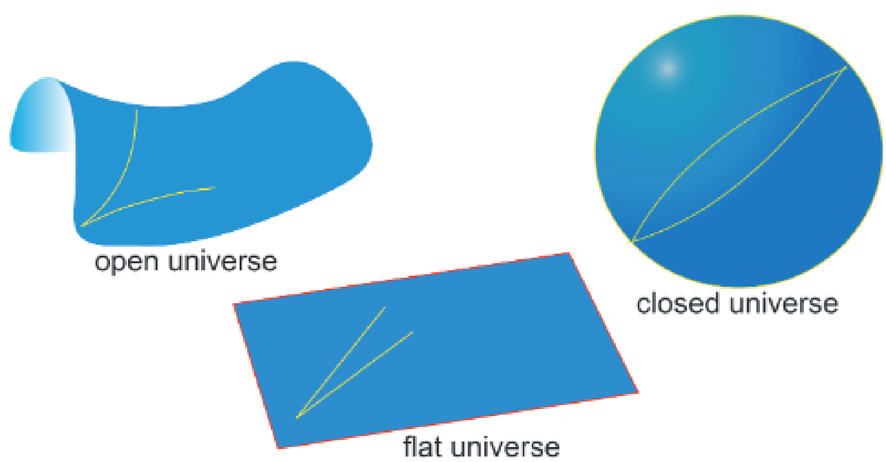

Figure 2.1: The three possible shapes of the universe [26].

2.1.2 The energy-momentum tensor of a perfect fluid

The FLRW metric tells us everything we need to know about the geometry of spacetime, but it does not say anything about the matter content within it. The matter content of the universe is described by a perfect fluid of energy-momentum tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu + P\left(u_\mu u_\nu + g_{\mu\nu}\right). \tag{2.7}$$

The fluid is characterized by its isotropic pressure p, its density ρ and its 4-velocity u_μ. The velocity is measured to be $u_\mu = (-1, 0, 0, 0)$ by observers who see the fluid at rest. Such observers are said to be comoving. If observers in the universe were not comoving, the condition of isotropy would no longer hold. The energy-momentum tensor (2.7) is constrained by the conservation laws

$$\nabla_\mu T^\mu_\nu = 0 \tag{2.8}$$

which can be expanded as

$$\partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda_\nu - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda = 0 \tag{2.9}$$

where the Γs are the Christoffel symbols. If we set $\nu = 0$ and substituting in for the relevant components of the energy-momentum tensor and the Christoffel symbols, equation (2.9) becomes the general relativity analogue of the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + P\right) = 0. \tag{2.10}$$

For a barotropic fluid whose equation of state is given by $P = w\rho$, it follows that the solution to equation (2.10) is

$$\rho \propto a^{-3(1+w)}. \tag{2.11}$$

For dust, when $w = 0$, we get $\rho \propto a^{-3}$. In the case of radiation, $w = 1/3$, which implies that $\rho \propto a^{-4}$. Finally, for dark energy, when $w = -1$, we get $\rho \propto a^0$. Equation (2.11) describes the evolution of mass densities of various kinds of fluids in the universe as a function of the scale-factor. When the scale-factor is very small, radiation dominates over matter and dark energy until at the matter-radiation crossover. After that, matter dominates both radiation and dark energy. As the scale factor gets very large we will reach a point where matter density equals the dark energy density (matter-dark energy crossover), after which dark energy dominates both matter and radiation. This divides the universe into three eras: radiation dominated, matter dominated and dark energy dominated eras, see figure 2.2.

2.1.3 The field equations

If one substitutes the metric (2.6) and the energy-momentum tensor (2.7) into Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.12}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature, they would get

$$\frac{\dot{a}^2 + K}{a^2} = \frac{8\pi G\rho}{3} \tag{2.13}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(P + \rho\right). \tag{2.14}$$

These are known as Friedmann's equations. We shall assume a flat universe ($K = 0$) and solve for the scale-factor a from equation (2.13). In the case of matter, the solution is

$$\boxed{a\left(t\right) \propto t^{\frac{2}{3}}} \tag{2.15}$$

for radiation, we find that

$$\boxed{a\left(t\right) \propto t^{\frac{1}{2}}} \tag{2.16}$$

and for dark-energy, the solution is

$$\boxed{a\left(t\right) \propto e^{Ht}} \tag{2.17}$$

where $H = \frac{\dot{a}}{a}$ is called the Hubble parameter. Equations (2.15) and (2.16), suggest that, if we go back in time, the scale-factor gets smaller and eventually at $t = 0$ it goes to zero. The scale-factor multiplies the spatial dimensions of the FLRW metric. When it is zero, space ceases to exist. We cannot do physics without space and so at $t = 0$ we have a singularity. If we go forward in time, the scale-factor gets bigger. This implies that, right after the singularity - which is at the very beginning, the universe started expanding. It does so faster during the matter dominated era than during the radiation dominated era and exponentially so during the dark-energy dominated era, which occurs in the far future when equation (2.17) becomes relevant.

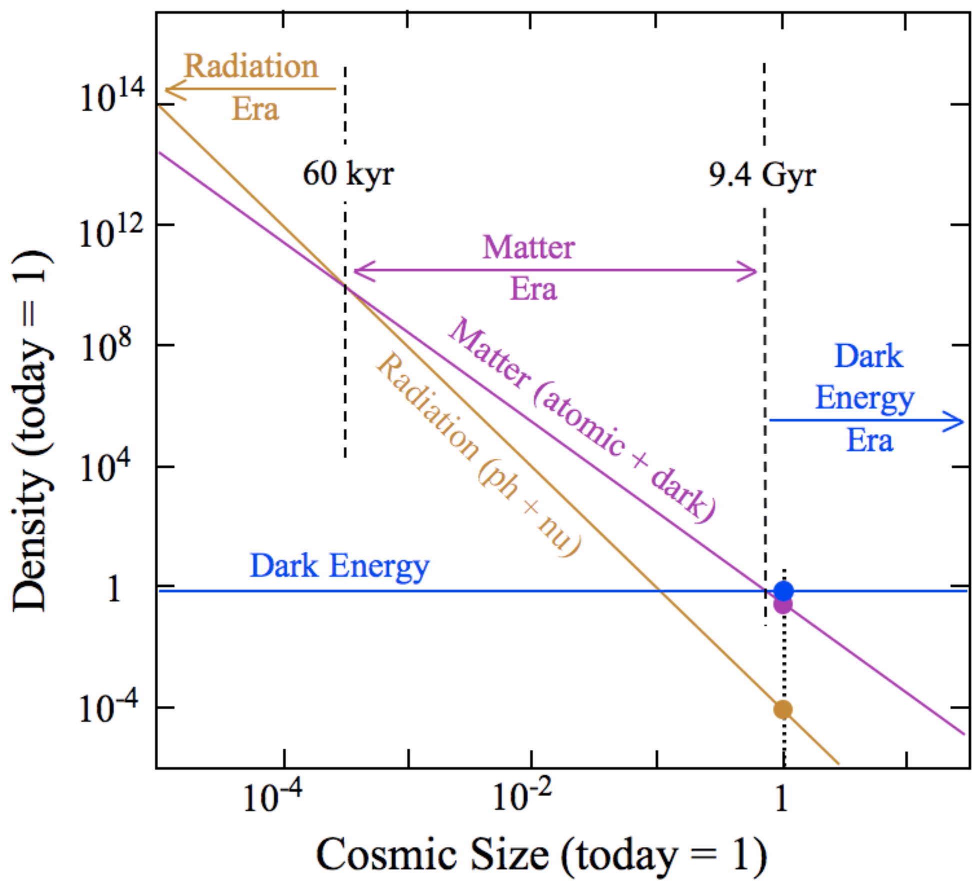

Figure 2.2: Evolution of energy densities of radiation, matter, dark energy and the total (matter + radiation + dark energy) in the universe [28].

2.2 The LambdaCDM model

We can rewrite the first Friedmann equation in terms of the Hubble parameter

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}. \tag{2.18}$$

$K = 0$ corresponds to the critical density

$$\rho_{\text{critical}} = \frac{3H^2}{8\pi G}. \tag{2.19}$$

If we use the subscript '0' to denote cosmological parameters evaluated today i.e at $t = t_0$, then the critical density takes the form

$$\rho_{\text{critical},0} = \frac{3H_0^2}{8\pi G}. \tag{2.20}$$

Equation (2.20) serves as a benchmark for defining dimensionless density parameters evaluated today

$$\Omega_{I,0} = \frac{\rho_{I,0}}{\rho_{\text{critical},0}} \tag{2.21}$$

where I represents either radiation, matter, curvature K or the cosmological constant Λ. When expressed in terms of these dimensionless density parameters, the Friedmann equation becomes

$$H^2\left(a\right) = H_0^2\left[\Omega_{r0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{K,0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}\right]. \tag{2.22}$$

By convention: $a_0 = 1$ and we can drop the subscript '0' on the density parameters to get a familiar redefined form of equation (2.22)

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda \tag{2.23}$$

where Ω_r, Ω_m, Ω_Λ and $\Omega_K = -\frac{K}{H_0^2}$ are the dimensionless density parameters of radiation, matter, cosmological constant and curvature respectively, as evaluated today. Observations on cosmological scales show that these quantities (radiation, matter, cosmological constant and curvature) are available in the universe, in the following amounts:

$$|\Omega_K| \leq 0.01,\ \Omega_r = 9.4 \times 10^{-5},\ \Omega_m = 0.32,\ \Omega_\Lambda = 0.68. \tag{2.24}$$

The matter density parameter Ω_m could be split into

$$\Omega_m = \Omega_b + \Omega_c \tag{2.25}$$

where Ω_b and Ω_c are dimensionless density parameters of baryons and cold-dark matter respectively

$$\Omega_b = 0.05,\ \Omega_c = 0.27. \tag{2.26}$$

This suggests that the universe is made up of 5% baryons and 27% cold-dark matter. The cosmological constant has an equation of state similar to that of dark energy ($w_\Lambda = -1$) and makes up about 68% of the universe.

2.3 Dynamics of a perturbed universe

Even on the largest scales, the universe is far from being completely homogeneous. The model that we have considered so far is ideal, the real universe consists of fluctuations, tiny or large depending on the scale of observation. In this section, we will expand the metric and the energy-momentum tensor of the homogeneous universe using the Taylor expansion, up to 1st order, to get a better approximation of the real universe. By that, we mean that we will add first order linear perturbations to the metric and the energy-momentum tensor and study how those perturbations grow in the expanding universe.

2.3.1 Newtonian perturbation theory

We first investigate the evolution of perturbations in terms of Newton's theory. This Newtonian description is valid for non-relativistic fluids whose dynamics are dictated by

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0 \\ \rho \frac{\partial u}{\partial t} + \rho \left(u \cdot \nabla \right) u &= -\nabla p - \rho \nabla \Phi \\ \nabla^2 \Phi &= 4\pi G \rho \end{aligned} \tag{2.27}$$

where ρ is the density of the fluid, p is the fluid's pressure, u is the fluid's velocity and Φ is the gravitational potential. In the expanding universe, the perturbed form [1] of equations (2.27) is

$$\begin{aligned} \frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot \left[\left(1 + u \right) u \right] &= 0 \\ \frac{\partial u}{\partial t} + Hu + \frac{1}{a} \left(u \cdot \nabla \right) u &= -\frac{1}{a} \nabla \Phi - \frac{1}{a\bar{\rho}} \nabla \left(\delta p \right) \\ \nabla^2 \Phi &= 4\pi G \bar{\rho} a^2 \delta \end{aligned} \tag{2.28}$$

where $\delta = \frac{\delta\rho}{\bar{\rho}}$ is called the density contrast, it quantifies the fluctuations. Equations (2.28) when combined and transformed to Fourier space reduce to

$$\frac{\partial^2 \delta}{\partial t^2} + 2H \frac{\partial \delta}{\partial t} = 4\pi G \bar{\rho} \delta - \frac{c_s^2 k^2}{a^2} \delta \tag{2.29}$$

[1] $\rho = \bar{\rho} + \delta\rho$ and $p = \bar{p} + \delta p$ for $\delta\rho \ll \bar{\rho}$ and $\delta p \ll \bar{p}$.

which is the evolution equation of 1st order linear perturbations in the Newtonian framework. $c_s^2 = \frac{\delta p}{\delta \rho}$ (adiabatic perturbation theory) is the speed of sound. For matter, we have that $c_s^2 = 0$ and $H = \frac{2}{3t}$, thus equation (2.29) becomes

$$\frac{\partial^2}{\partial t^2}\delta_m + \frac{4}{3t}\frac{\partial}{\partial t}\delta_m - \frac{2}{3t^2}\delta_m = 0 \tag{2.30}$$

where we have used Friedmann's 1st equation $4\pi G\bar{\rho}_m = \frac{3}{2}H^2$ to eliminate the gravitational constant term. The solutions to equation (2.30) are

$$\delta_m \propto \begin{cases} t^{-1} \propto a^{-\frac{3}{2}} \\ t^{\frac{3}{2}} \propto a. \end{cases}$$

This implies that during the matter-dominated era, matter fluctuations grow proportional to the scale-factor a. The solution $\delta_m \propto a^{-\frac{3}{2}}$ is decaying, it is of no relevance to structure formation, and so we discard it. During radiation dominated era equation (2.29) takes the form [24]

$$\frac{\partial^2}{\partial t^2}\delta_m + 2H\frac{\partial}{\partial t}\delta_m - 4\pi G\sum_I \bar{\rho}_I\delta_I = 0 \tag{2.31}$$

the sum in (2.31) is over radiation and matter. The author of [24] claims that, radiation perturbations in the non-relativistic limit oscillate as sound waves and that their time averaged density contrast is zero. This means that, since $H = \frac{1}{2t}$, the evolution equation for matter fluctuations during radiation dominated era is

$$\frac{\partial^2}{\partial t^2}\delta_m + \frac{1}{t}\frac{\partial}{\partial t}\delta_m - 4\pi G\bar{\rho}_m\delta_m = 0. \tag{2.32}$$

On cosmological timescales and when $\bar{\rho}_r \gg \bar{\rho}_m$, we have that

$$\frac{\partial^2}{\partial t^2}\delta_m \sim H^2\delta_m \sim \frac{8\pi G}{3}\bar{\rho}_r\delta_m \gg 4\pi G\bar{\rho}_m\delta_m \tag{2.33}$$

and so we can ignore the last term in equation (2.32) to get

$$\frac{\partial^2}{\partial t^2}\delta_m + \frac{1}{t}\frac{\partial}{\partial t}\delta_m = 0. \tag{2.34}$$

Equation (2.34) is solved by

$$\delta_m \propto \begin{cases} \text{constant} \\ \ln t \propto \ln a. \end{cases}$$

During the radiation-dominated era when $\delta_m \propto \ln a$, matter fluctuations grow slower than during the matter-dominated phase. In the case when $\delta_m =$ constant matter fluctuations do not grow. Dark energy does not cluster by definition and so the evolution equation for matter perturbations during dark energy dominated era ($H^2 \gg 4\pi G\bar{\rho}_m\delta_m$) where H is a constant, derived from similar arguments as equation (2.31), except in this case the sum is over matter and dark energy is given by

$$\frac{\partial^2}{\partial t^2}\delta_m + 2H\frac{\partial}{\partial t}\delta_m = 0 \tag{2.35}$$

and has a solution

$$\delta_m \propto \begin{cases} \text{constant} \\ e^{-2Ht} \propto a^{-2} \end{cases}$$

which implies that matter fluctuations do not grow during dark energy dominated era (δ_m is a constant). The other solution ($\delta_m \propto a^{-2}$) hints that matter fluctuations decay very fast, and so we neglect it (based on the same reason we neglected $\delta_m \propto a^{-\frac{3}{2}}$ when the universe was matter-dominated). In the next subsection, we consider how these matter perturbations grow when relativistic effects are turned on.

2.3.2 The perturbed Friedmann-Lemaitre-Robertson-Walker metric

The line element of the perturbed FLRW metric in Newtonian gauge is

$$ds^2 = a^2(\eta)\left[-(1+2\Phi)\,d\eta^2 + (1-2\Psi)\,\delta_{ij}dx^i dx^j\right] \tag{2.36}$$

where η is the conformal time and is related to the proper time t by $dt^2 = a^2 d\eta^2$, Φ is the gravitational potential and Ψ is called the curvature perturbation. If the anisotropic stress is zero, we have that $\Phi = \Psi$ and the metric tensor of the line element (2.36) is given by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2\mathrm{diag}\left(-1-2\Phi, 1-2\Phi, 1-2\Phi, 1-2\Phi\right). \tag{2.37}$$

The perturbed components of the Einstein tensor associated with (2.37) are

$$\begin{aligned} \delta G_{00} &= 2\nabla^2\Phi - 6\mathcal{H}\Phi' \\ \delta G_{0i} &= 2\partial_i\left(\Phi' + \mathcal{H}\Phi\right) \\ \delta G_{ij} &= \left[2\Phi'' + 6\mathcal{H}\Phi' + 4\left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi\right]\delta_{ij} \end{aligned} \tag{2.38}$$

where $\mathcal{H}$ is the Hubble parameter in conformal time, the primes indicate differentiation with respect to conformal time and δ_{ij} is the Kronecker delta.

2.3.3 The perturbed energy-momentum tensor

The components of the perturbed energy-momentum tensor $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$ of a perfect fluid are given by

$$\begin{aligned} \delta T_{00} &= a^2 \left(\delta\rho + 2\bar{\rho}\Phi\right) \\ \delta T_{0i} &= -a^2 \left(\bar{\rho} + \bar{p}\right) \partial_i V \\ \delta T_{ij} &= a^2 \left(\delta p - 2\bar{p}\Phi\right) \delta_{ij}. \end{aligned} \tag{2.39}$$

We assume irrotational flow and so the peculiar velocity v_i [2] of the fluid can be written as $v_i = \partial_i V$, where V is the velocity potential. The perturbed form of (2.8) is

$$\delta\left(\nabla_\mu T^\mu_\nu\right) = 0 \tag{2.40}$$

which when expanded becomes

$$\partial_\mu \delta T^\mu_\nu + \bar{\Gamma}^\mu_{\mu\alpha} \delta T^\alpha_\nu + \bar{T}^\alpha_\nu \delta\Gamma^\mu_{\mu\alpha} - \bar{\Gamma}^\alpha_{\mu\nu} \delta T^\mu_\alpha - \bar{T}^\mu_\alpha \delta\Gamma^\alpha_{\mu\nu} = 0. \tag{2.41}$$

The continuity equation ($\nu = 0$) is given by

$$\delta' + 3\left(c_s^2 - w\right)\mathcal{H}\delta = (1 + w)\left(3\Phi' - \nabla^2 V\right) \tag{2.42}$$

and the Euler equation ($\nu = i$) takes the form

$$V' + \left(1 - 3c_s^2\right)\mathcal{H}V = -\Phi - \frac{c_s^2}{1 + w}\delta, \tag{2.43}$$

where $w = \frac{\bar{p}}{\bar{\rho}}$. Equations (2.42) and (2.43) are the relativistic analogue of the first two equations in (2.28).

2.3.4 The perturbed field equations

The pertubed field equations $\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$ are

$$\begin{aligned} \nabla^2\Phi &= 4\pi G a^2 \delta\rho + 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) \\ \Phi' &= -4\pi G a^2 \bar{\rho}\,(1 + w)\,V - \mathcal{H}\Phi \\ \Phi'' &= 4\pi G a^2 \delta p - 3\mathcal{H}\Phi' - \left[4\mathcal{H}' + (2 + 3w)\,\mathcal{H}^2\right]\Phi. \end{aligned} \tag{2.44}$$

[2] For irrotational flow, the curl of the peculiar velocity is zero, thence it can be expressed as the gradient of the velocity potential V.

The first equation in the set (2.44) is the relativistic form of the perturbed Poisson's equation, the 3rd equation in (2.28). These equations describe the evolution of the gravitational potential Φ in the perturbed universe. If we substitute $\delta p = c_s^2 \delta\rho$ and $\mathcal{H}' = -\frac{1}{2}(1+3w)\mathcal{H}^2$ into the third equation of the set and combine it with the first we get the Bardeen equation

$$\Phi'' + 3\left(1 + c_s^2\right)\mathcal{H}\Phi' + 3\left(c_s^2 - w\right)\mathcal{H}^2\Phi = c_s^2\nabla^2\Phi \tag{2.45}$$

which during the matter and radiation dominated eras (in Fourier space), takes the form

$$\Phi'' + 3\left(1 + c_s^2\right)\mathcal{H}\Phi' + c_s^2 k^2\Phi = 0 \tag{2.46}$$

where we have set $c_s^2 = w$, which is the case for adiabatic perturbation theory. Just as before, during the matter dominated era $c_s^2 = \frac{\delta p}{\delta\rho} = 0$ and so (2.46) becomes

$$\Phi'' + \frac{6}{\eta}\Phi' = 0 \tag{2.47}$$

which is solved by

$$\Phi \propto \begin{cases} \text{constant} \\ \eta^{-5} \propto a^{-\frac{5}{2}}. \end{cases}$$

The growing mode solution is constant, implying that the gravitational potential is frozen during the matter dominated era on all scales.

2.4 Structure formation

Let us define the quantity

$$\mathcal{R} = -\Phi - \frac{2}{3(1+w)\mathcal{H}}\left(\Phi' + \mathcal{H}\Phi\right) \tag{2.48}$$

and call it the comoving curvature perturbation. It can be shown (to be shown in the appendix) that when $k \ll \mathcal{H}$ (super-Hubble scales), equation (2.48) becomes

$$\mathcal{R} = -\frac{(5+3w)}{(3+3w)}\Phi. \tag{2.49}$$

For dust on super-Hubble scales during the matter dominated era, $w = 0$ and equation (2.49) reduces to

$$\Phi_{\text{matter}} = -\frac{3}{5}\mathcal{R}. \tag{2.50}$$

We have mentioned that Φ_{matter} is constant on all scales and so (2.50) holds even on sub-Hubble scales. The relativistic analogue of Poisson's equation can be written as

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\Delta_m \tag{2.51}$$

where $\Delta_m = \delta_m + \frac{3\mathcal{H}^2\Phi}{4\pi G a^2\bar{\rho}}$. On sub-Hubble scales $\Delta_m \simeq \delta_m$ and so (2.51) reduces to

$$k^2\Phi = -4\pi G a^2 \bar{\rho}\delta_m \tag{2.52}$$

inserting $\frac{3}{2}\mathcal{H} = 4\pi G a^2\bar{\rho}$ into equation (2.52) we find that

$$\delta_m = \frac{2}{5}\left(\frac{k}{\mathcal{H}}\right)^2 \mathcal{R}. \tag{2.53}$$

During the matter dominated epoch $\mathcal{H} = H_0 a^{-\frac{1}{2}}$ and so

$$\delta_m \propto a. \tag{2.54}$$

The density contrast for dust on sub-Hubble scales grows just as before.

2.4.1 The transfer function

By combining (2.49) with the relativistic Poisson's equation, we find that, on super-Hubble scales, radiation fluctuations evolve as follows:

$$\delta_{\text{rad}} = \frac{4}{9}\left(\frac{k}{\mathcal{H}}\right)^2 \mathcal{R}. \tag{2.55}$$

At the horizon, $k = \mathcal{H}$, thus equation (2.55) reduces to:

$$\delta_{\text{rad}} = \frac{4}{9}\mathcal{R}. \tag{2.56}$$

The radiation mass density splits into the mass density of photons ρ_γ and the mass density of the neutrinos ρ_ν. We are assuming primordial adiabatic perturbations and therefore the relations : δ_c[3] $= \frac{3}{4}\delta_{\text{rad}}$ and $\delta_\gamma = \delta_{\text{rad}}$ hold on super-Hubble scales. At the horizon entry, we find that

$$\delta_c = \frac{1}{3}\mathcal{R}. \tag{2.57}$$

[3]The density contrast of cold dark matter.

If we also assume that δ_c remains constant until the universe becomes matter-dominated $\eta = \eta_{\text{eq}}$, then

$$\delta_c\left(\eta_{\text{eq}}\right) = \frac{1}{3}\mathcal{R}. \tag{2.58}$$

When $\eta > \eta_{\text{eq}}$, δ_c grows just like in the case of matter i.e $\propto \left(\frac{k}{\mathcal{H}}\right)^2$

$$\delta_c\left(\eta\right) = \frac{1}{3}\left(\frac{k_{\text{eq}}}{\mathcal{H}}\right)^2\mathcal{R} \tag{2.59}$$

for as long as the universe stays matter-dominated. Let us define the transfer function $T\left(\eta, k\right)$ by

$$\delta_m\left(\eta\right) = \frac{2}{5}\left(\frac{k}{\mathcal{H}}\right)^2 T\left(\eta, k\right)\mathcal{R}. \tag{2.60}$$

Thus by definition, $T\left(\eta, k\right) = 1$ when $k \ll k_{\text{eq}}$. During the matter-dominated era $k \gg k_{\text{eq}}$, we have that

$$T\left(\eta, k\right) = \frac{1}{3} \times \frac{5}{2}\left(\frac{k_{\text{eq}}}{k}\right)^2 = \frac{5}{6}\left(\frac{k_{\text{eq}}}{k}\right)^2. \tag{2.61}$$

Once we are well into the era, we can drop the time dependence on the transfer function to get

$$T\left(k\right) \propto \begin{cases} 1 & k \ll k_{\text{eq}} \\ \left(\frac{k_{\text{eq}}}{k}\right)^2 & k \gg k_{\text{eq}}. \end{cases}$$

2.4.2 The power spectrum

The Fourier transform of the matter density contrast

$$\delta_k = \sum \delta_m e^{-ik\cdot r} \tag{2.62}$$

lead us to

$$P\left(k\right) = \left\langle\left|\delta_k\right|\right\rangle^2 = k^n \tag{2.63}$$

where $P\left(k\right)$ is called the power spectrum. When $n = 1$ we get Harrison and Zeldovich's power spectrum $P\left(k\right)_{\text{primordial}} = k$, the power spectrum of fluctuations at the horizon entry. The matter power spectrum is defined as

$$P\left(k\right)_{\text{matter}} = T\left(k\right)^2 P_{\text{primordial}} \tag{2.64}$$

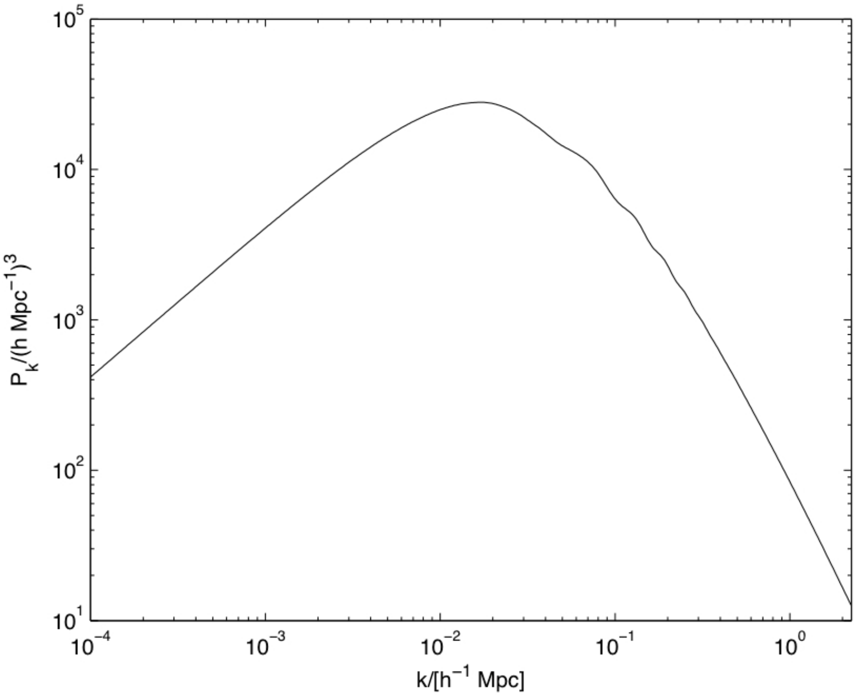

Figure 2.3: A plot of the power spectrum of matter with the power spectral density on the y-axis and the wave-number k on the x-axis [28].

which results in

$$P\left(k\right)_{\text{matter}} \propto \begin{cases} k & \text{when}\, k \ll k_{\text{eq}} \\ k^{-3} & \text{when}\, k \gg k_{\text{eq}}. \end{cases}$$

See figure 2.3 for the plot of the matter power spectrum, which matches the power spectrum of the distribution of matter displayed in figure 2.4.

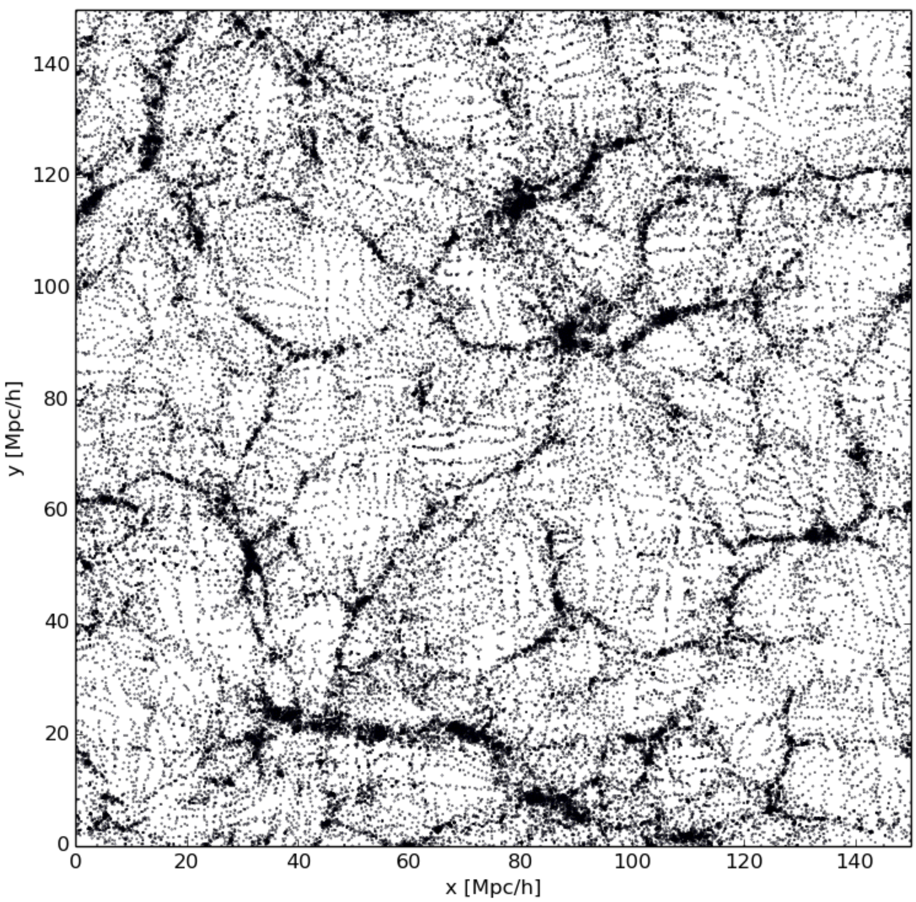

Figure 2.4: The cosmic web, as determined from cosmological N-body simulations. The dark spots are galaxy clusters connected by dark-lines filaments. The large white portions are voids. x and y are dimensions of the simulation box [27].

Chapter 3

A theory of a MOND-like superfluid

In this chapter we discuss MOND, the theory of superfluidity and how the two are unified by the idea that dark matter is a superfluid. Our discussion of MOND is largely based on [6]. The study of superfluidity is centred around [17]. A possible unification of the two stems from [1].

3.1 Modified Newtonian Dynamics

The mass within galaxies is determined from the virial relation $v^2 = \frac{MG}{r}$, where v is the velocity of stars in the galaxy, r is the size of the galaxy, M is the unknown mass and G is the gravitational constant. This relation according to [11] is based on three assumptions: (1) The force which governs the dynamics is gravity. (2) The gravitational force on a star depends on the mass of the star and the distribution of mass which produces the force. (3) Newton's second law holds. The mass determined from the virial relation is an overestimate of the true gravitational mass which arises when assumptions (1)-(3) are disobeyed [11]. As a result of that, there might not be much missing mass in the universe. It is said that (1)-(3) are based on evidence from the solar system and laboratory experiments. Since parameters like mass, distance and acceleration of galaxies differ by many orders of magnitude from the ones in the solar system and laboratories, it might be the case that (1)-(3) need to be modified to account for galactic dynamics. Perhaps it is particularly Newton's second law that needs to be modified [11]. A proposed modification of the law is as follows

$$F = m_{\mathrm{g}}\mu\left(\frac{g}{a_0}\right)g \tag{3.1}$$

where $\mu(x \gg 1) = 1$, $\mu(x \ll 1) = x$ for $x = \frac{g}{a_0}$, m_{g} is the gravitational mass upon which the force field F acts on and a_0 is the acceleration scale which marks the point where Newtonian mechanics collapse. This means that for accelerations larger than the acceleration constant $a_0 = 1.2 \times 10^{-10}\mathrm{ms}^{-2}$ we have that $\mu(x) = 1$ and Newton's second law is restored. For those smaller than a_0

$$g = \sqrt{g_{\mathrm{N}} a_0} \tag{3.2}$$

which is the case for galaxies when $\mu(x) = x$, the law fails but takes a new form. Let us suppose we have a test mass orbiting a galactic center of mass at a distance r from the center and moving at a velocity v with the mass enclosed by the orbit being M . The test mass' acceleration could be determined from

$$a = \frac{v^2}{r^2}. \tag{3.3}$$

The Newtonian gravitational acceleration g_{N} of the test mass due to M is given by $g_N = \frac{GM}{r^2}$ of which when substituted into equation (3.2) we get

$$g = \sqrt{\frac{GM}{r^2} a_0}. \tag{3.4}$$

Equating (3.4) to (3.3) results in

$$v^4 = a_0 GM \tag{3.5}$$

which implies that $v^4 \propto M$ - this is the Baryonic Tully-Fisher relation. Also, note that since M, G and a_0 are constants, the velocity v in (3.5) is also a constant, this explains why rotation curves of some galaxies are flat. It was pointed out in [21], that equation (3.1) does not conserve linear momentum. As a result of that, the equation does not describe a complete theory but rather a phenomenology for which an underlying theory is needed. In the following sections we consider various possible theories (relativistic and non-relativistic) of this MOND phenomenology.

3.1.1 Lagrangian formulation of Newton's theory of gravity

The Lagrangian for Newton's theory of gravity is given by

$$L_{\mathrm{N}}(\phi_{\mathrm{N}}, \nabla\phi_{\mathrm{N}}) = -\int \left[\rho\phi_{\mathrm{N}} + \frac{(\nabla\phi_{\mathrm{N}})^2}{8\pi G}\right] d^3r. \tag{3.6}$$

When (3.6) is substituted into the Euler-Lagrange equations, the resulting equation of motion is Poisson's equation

$$\nabla^2 \phi_{\rm N} = 4\pi G\rho. \tag{3.7}$$

Equation (3.7) tells us that, the mass density ρ is the source of the gravitational potential $\phi_{\rm N}$, which gives rise to gravity. The constant G is Newton's gravitational constant.

3.1.2 Lagrangian formulation of modified Newtonian dynamics

If we now consider the following modification of Lagrangian (3.6)

$$L = -\int \left[\rho\phi + \frac{a_0^2}{8\pi G} F\left(\frac{(\nabla\phi)^2}{a_0^2}\right)\right] d^3r \tag{3.8}$$

and suppose that $\mu(x) = F'(x^2)$, where $x = \frac{\nabla\phi}{a_0}$. The resulting equation of motion obtained when (3.8) is inserted into the Euler-Lagrange equations is

$$\mu\left(\frac{g}{a_0}\right) g = g_{\rm N}. \tag{3.9}$$

When $\mu\left(\frac{g}{a_0}\right) = 1$, which is the case when $g \gg a_0$, we recover Newton's theory of gravity

$$g = g_{\rm N}. \tag{3.10}$$

When $g \ll a_0$ then $\mu\left(\frac{g}{a_0}\right) = \frac{g}{a_0}$ and (3.9) reduces to

$$\frac{g^2}{a_0} = g_{\rm N} \tag{3.11}$$

just as we have seen before, particularly from equation (3.2). Lagrangian (3.8) thus describes a non-relativistic theory of MOND.

3.1.3 A possible relativistic theory of MOND

The weak-field limit of Einstein's general theory of relativity is Newton's theory of gravity. If one is to construct a relativistic theory of MOND, it has to reduce to MOND in the weak-field limit. MOND is non-Newtonian and so its relativistic counterpart has to be non-Einsteinian. Such a theory does not obey the strong equivalence principle. To accomodate the violation of the strong equivalence principle in their relativistic theory, the authors in [11] proposed an extra dynamical degree of freedom, a scalar field ψ, in addition to the metric $g_{\mu\nu}$ of Einstein's theory. Their theory has two equivalent formulations: in one case, the metric is a solution to Einstein's field equations but particles do not move along the geodesics of the metric tensor $g_{\mu\nu}$. For the other case, particles do move along the geodesics (of the metric tensor $\tilde{g}_{\mu\nu}$) but the field equations are different from the ones in Einstein's theory. They claim that the former of such formulations corresponds to the usage of gravitational units while the latter corresponds to the usage of atomic units. The field equations in atomic units are determined from the action

$$S = S_{\mathrm{g}} + S_{\psi} + S_{\mathrm{m}}. \tag{3.12}$$

The gravitational action S_{g} is defined as

$$S_{\mathrm{g}} = c^4 \left(16\pi G_0\right)^{-1} \int e^{-\frac{2\psi}{c^2}} \left[R\left(\tilde{g}_{\mu\nu}\right) + 6c^{-4}\psi_{,\alpha}\psi_{,}^{\alpha}\right]\sqrt{-\tilde{g}}d^4x \tag{3.13}$$

where G_0 is the gravitational constant in atomic units. The scalar field action S_{ψ} is given by

$$S_{\psi} = \frac{-a_0^2\beta\left(1+\beta\right)^2}{8\pi G_0} \int e^{-\frac{4\psi}{c^2}} F\left(\frac{e^{\frac{2\psi}{c^2}}\psi,_{\mu}\psi,^{\mu}}{a_0^2\left(1+\beta\right)^2}\right)\sqrt{-\tilde{g}}d^4x. \tag{3.14}$$

The function $F\left(X\right)$ in (3.14) is by definition

$$F\left(X\right) \approx \frac{2}{3}X^{\frac{3}{2}} \tag{3.15}$$

when $X \ll 1$ and

$$F\left(X\right) \approx X \tag{3.16}$$

when $X \gg 1$, where $X = \frac{e^{\frac{2\psi}{c^2}}\psi,_{\mu}\psi,^{\mu}}{a_0^2(1+\beta)^2}$. The particle action S_{m} is defined as

$$S_{\mathrm{m}} = -mc^2 \int d\tau \tag{3.17}$$

where τ in (3.17) represents the proper time. It is said that this relativistic theory satisfies the requirements for MOND in the weak-field limit when $X \ll$

1 in a static universe [11]. The strong field limit when $X \gg 1$ corresponds to the Brans-Dicke scalar-tensor theory when $\beta = 2\omega + 3$ where ω is the parameter of such a theory. In the next section we will show how (3.15) is crucial in unifying MOND and the theory of superfluidity.

3.2 Superfluidity

In section 1.3 we analysed how dark matter condense into a superfluid. The analysis was gleaned from experimental apprehension. In this section, we give a theoretical description of superfluids based on Landau's model, and link it to Bose-Einstein condensation of dark matter via [1], [2] and [14].

3.2.1 Landau's model

Suppose you have a complex scalar field ϕ whose dynamics are to be determined from the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\Phi|^2 - \mathcal{J} |\phi|^4 \tag{3.18}$$

where $\mathcal{J}$ is the coupling constant, which is always positive and m is the mass of the scalar field. The Lagrangian is invariant under global $\mathrm{U}(1)$ transformations

$$\phi \to e^{-i\alpha}\phi. \tag{3.19}$$

The transformations (3.19) are global in a sense that α is a constant, it is independent of space-time coordinates. The complex scalar field ϕ can be expressed in terms of its modulus ρ and phase ψ [15]

$$\phi = \frac{\rho}{\sqrt{2}} e^{i\psi}. \tag{3.20}$$

This redefines Lagrangian (3.18) to

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{2}\left(\partial_\mu \psi \partial^\mu \psi - m^2\right) - \frac{\mathcal{J}}{4}\rho^4. \tag{3.21}$$

If one substitutes (3.21) into the Euler-Lagrange equations for ρ, they would get

$$\Box \rho = \rho\left(\sigma^2 - m^2 - \mathcal{J}\rho^2\right) \tag{3.22}$$

where $\sigma = \sqrt{\partial_\mu \psi \partial^\mu \psi}$ and $\Box = \partial^\mu \partial_\mu$ is the d'Alembertian. Similarly, they would get an equation of motion for ψ from the Euler-Lagrange equations for ψ

$$\partial_\mu \left(\rho^2 \partial^\mu \psi \right) = 0. \tag{3.23}$$

Equations (3.22) and (3.23) determine the dynamics of the field ϕ.

Spontaneous symmetry breaking

For the case when ρ and $\partial^\mu \psi$ are constant, we can rewrite (3.21) as

$$\mathcal{L} = -\mathrm{U}. \tag{3.24}$$

The Lagrangian (3.24) is still invariant under U(1) transformations. U is called the tree-level potential

$$\mathrm{U} = -\frac{\rho^2}{2} \left(\sigma^2 - m^2 \right) + \frac{\mathcal{J}}{4} \rho^4. \tag{3.25}$$

The extremal points of (3.25) with respect to ρ are determined from

$$\frac{\partial \mathrm{U}}{\partial \rho} = 0. \tag{3.26}$$

Equation (3.26) has the solution $\rho = 0$ or

$$\rho^2 = \frac{\sigma^2 - m^2}{\mathcal{J}}. \tag{3.27}$$

See figure 3.1 for the plot of U versus ϕ when $\sigma^2 < m^2$, $\sigma^2 = m^2$ and $\sigma^2 > m^2$, which corresponds to the symmetrical phase, the transition point and the non-symmetrical phase respectively. In other words, Landau's model mimics a phase transition from a symmetrical to a non-symmetrical phase.

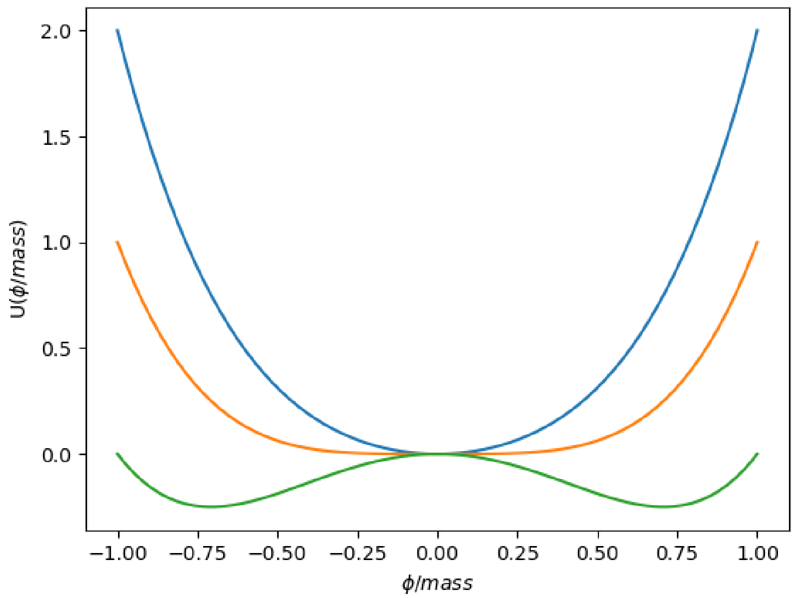

Figure 3.1: A plot of U versus ϕ. The green, blue and orange curves represent U when $\sigma^2 > m^2$, $\sigma^2 < m^2$ and $\sigma^2 = m^2$ respectively.

From our plot we can see that when $\sigma^2 > m^2$ we have two (in reality they are infinite) minima with the same potential. This means that the ground state of the non-symmetrical phase is degenerate. Particles which are known to allow degenerate states are Bosons, they do not obey Pauli's exclusion principle. When these Bosons are in their degenerate ground states they behave collectively or in unison and such a collection is often referred to as a superfluid or a Bose-Einstein condensate. Based on that, our complex scalar field ϕ behaves like a superfluid when it is in one of the minima of the non-symmetrical phase. When the field settles into one of the minima, the potential U is no longer U(1) invariant. The symmetry gets broken, hence the name "non-symmetrical phase".

Effective description of superfluid phonons

Remember that, the modulus ρ of the field as determined by equation (3.27) is fixed, only the phase ψ varies. We have already stated that the field's ground state is degenerate when $\sigma^2 > m^2$. This alludes, that the phase ψ is a Nambu-Goldstone mode in accord with Goldstone's theorem. It was also declared that the mode behaves like a superfluid by choosing to settle in one of the minima of the non-symmetrical phase. This means that $\psi = \mu t + \phi$, where μ is the chemical potential associated with the superfluid and ϕ are excitations of ψ known as phonons (not to be confused with the field itself). The effective field theory of these superfluid phonons at low energies is described by a Lagrangian of the form

$$\mathcal{L} = P\left(X\right) \tag{3.28}$$

where $X = \mu + \dot{\phi} - m\Phi - \frac{(\nabla\phi)^2}{2m}$ [2] [14] and Φ is the gravitational potential. Lagrangian (3.28) is invariant under Galilean transformations $\implies$ the theory of superfluid phonons is non-relativistic. In [19] it was shown that $P\left(X\right)$ represents the pressure of the superfluid. Its number density is determined by

$$n = \frac{\partial P}{\partial X}. \tag{3.29}$$

From the non-relativistic relation $\rho = mn$, one can calculate the mass density ρ by substituting in for n determined by equation (3.29) into the relation.

3.3 A MOND-like superfluid

In the previous section, $P\left(X\right)$ could be any function of X. In [1] it was conjectured to be

$$\mathcal{L} = P\left(X\right) = \frac{2\Lambda\left(2m\right)^{\frac{3}{2}}}{3} X\sqrt{|X|} \tag{3.30}$$

which is of the same form as equation (3.15), the Lagrangian that defines MOND. If we set the gravitational potential and the excitations to zero, the pressure of the MOND-like superfluid is given by

$$P\left(\mu\right) = \frac{2\Lambda}{3}\left(2m\mu\right)^{\frac{3}{2}}. \tag{3.31}$$

Resorting to (3.29), the number density of the MOND-like superfluid is found to be

$$n = \frac{\partial P}{\partial \mu} = \Lambda\left(2m\right)^{\frac{3}{2}}\mu^{\frac{1}{2}} \tag{3.32}$$

and thus we have that

$$P = \frac{\rho^2}{12\Lambda^2 m^6}. \tag{3.33}$$

From (3.33), we can see that the MOND-like superfluid is a polytrope of $n = \frac{1}{2}$ (n in this case is an index not the number density).

3.3.1 Baryon interactions

We can couple the complex scalar field via $\mathcal{L}_{\text{int}}$

$$\mathcal{L}_{\text{int}} = -\alpha \frac{\Lambda}{\mathrm{M}_{\mathrm{Pl}}} \phi \rho_{\mathrm{b}} \tag{3.34}$$

where M_{Pl} is the Planck mass and ρ_{b} is the mass density of baryons to the Lagrangian (3.30). The Lagrangian of a theory of the MOND-like superfluid becomes

$$\mathcal{L} = \frac{2\Lambda \left(2m\right)^{\frac{3}{2}}}{3} X \sqrt{|X|} - \alpha \frac{\Lambda}{\mathrm{M}_{\mathrm{Pl}}} \phi \rho_{\mathrm{b}}. \tag{3.35}$$

In the static, spherically symmetric approximation we have that $\phi \equiv \phi\left(r\right)$. The equation of motion determined from (3.35) is

$$\vec{\nabla} \cdot \left(\sqrt{2m\left|X\right|} \nabla \phi\right) = \frac{\alpha \rho_{\mathrm{b}}}{2\mathrm{M}_{\mathrm{Pl}}}. \tag{3.36}$$

Equation (3.36) when intergrated reduces to

$$\sqrt{2m\left|X\right|} \nabla \phi = \frac{\alpha \mathrm{M}_{\mathrm{b}}}{8\pi \mathrm{M}_{\mathrm{Pl}} r^2}. \tag{3.37}$$

Let $\kappa = \frac{\alpha \mathrm{M}_{\mathrm{b}}}{8\pi \mathrm{M}_{\mathrm{Pl}} r^2}$, the solution to (3.37) for $X < 0$[1] is

$$\phi'\left(r\right) = \sqrt{m} \left(\hat{\mu} + \sqrt{\hat{\mu}^2 + \frac{\kappa^2}{m^2}}\right)^{\frac{1}{2}} \tag{3.38}$$

where $\hat{\mu} = \mu - m\Phi$ and ϕ' is the derivative of ϕ with respect to r. When $\kappa \gg \hat{\mu}$[2] we get

$$\phi'\left(r\right) = \sqrt{\kappa} = \sqrt{\frac{\alpha \mathrm{M}_{\mathrm{b}}}{8\pi \mathrm{M}_{\mathrm{Pl}} r^2}}. \tag{3.39}$$

The scalar field acceleration is determined by

$$a_{\phi}\left(r\right) = \alpha \frac{\Lambda}{\mathrm{M}_{\mathrm{Pl}}} \phi' = \sqrt{\frac{\alpha^3 \Lambda^2 G \mathrm{M}_{\mathrm{b}}}{r^2 \mathrm{M}_{\mathrm{Pl}}}}. \tag{3.40}$$

[1]It was shown that $X > 0$ does not give MOND in the appendix of [1].

[2]This corresponds to the MONDian regime.

In order for (3.40) to reproduce MOND, one has to set

$$\alpha^3 \Lambda = \sqrt{a_0 \mathrm{M_{Pl}}} = 0.8\,\mathrm{meV} \tag{3.41}$$

or rather

$$\alpha = 0.86 \left(\frac{\Lambda}{\mathrm{meV}}\right)^{-\frac{2}{3}}. \tag{3.42}$$

3.3.2 A relativistic theory of the MOND-like superfluid

As in [1], the relativistic theory of the MOND-like superfluid is based on a complex scalar field ϕ whose dynamics are to be determined from the U (1) invariant Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left(|\partial_\mu \phi|^2 + m^2 |\phi|^2\right) - \frac{\Lambda^4}{6\left(\Lambda_c^2 + |\phi|^2\right)^6}\left(|\partial_\mu \phi|^2 + m^2 |\phi|^2\right)^3. \tag{3.43}$$

The constant Λ_c is introduced to ensure that the theory admits a $\phi = 0$ vacuum [1]. Let us define $\mathcal{L}_{\text{free}}$ as

$$\mathcal{L}_{\text{free}} = g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi^* + m^2 \phi\phi^*. \tag{3.44}$$

When that is the case, equation (3.43) reduces to

$$\mathcal{L} = -\frac{1}{2}\mathcal{L}_{\text{free}} - \frac{\Lambda^4}{6\left(\Lambda_c^2 + |\phi|^2\right)^6}\left(\mathcal{L}_{\text{free}}\right)^3. \tag{3.45}$$

In the weak-field limit, we have that

$$g^{\mu\nu} = \mathrm{diag}\left(-1 + 2\Phi, 1 + 2\Phi, 1 + 2\Phi, 1 + 2\Phi\right) \tag{3.46}$$

where Φ is the Newtonian gravitational potential. If we substitute (3.46) and a field redefinition $\phi = \rho e^{i\lambda}$ into (3.44) we get

$$\begin{aligned}\mathcal{L}_{\text{free}} = {} & -\dot{\rho}^2 + 2\Phi\dot{\rho}^2 + (\nabla\rho)^2 + 2\Phi\,(\nabla\rho)^2 - \rho^2\dot{\lambda}^2 + 2\Phi\rho^2\dot{\lambda}^2 \\ & + \rho^2\,(\nabla\lambda)^2 + 2\Phi\rho^2\,(\nabla\lambda)^2 + m^2\rho^2.\end{aligned} \tag{3.47}$$

The phase λ spontaneously breaks the U (1) symmetry when

$$\lambda = mt + \phi. \tag{3.48}$$

Remember that $\dot{f} \ll \nabla f$ in the weak-field limit for any f, using that fact, together with (3.48), equation (3.47) becomes

$$\mathcal{L}_{\text{free}} = (\nabla\rho)^2 - 2\rho^2 m\left(-m\Phi - \frac{(\nabla\phi)^2}{2m}\right). \tag{3.49}$$

Furthermore, in the weak-field limit $X = -m\Phi - \frac{(\nabla\phi)^2}{2m}$ and so (3.49) reduces to

$$\mathcal{L}_{\text{free}} = (\nabla\rho)^2 - 2m\rho^2 X. \tag{3.50}$$

Equation (3.45) takes the form

$$\mathcal{L} = -\frac{1}{2}\left[(\nabla\rho)^2 - 2m\rho^2 X\right] - \frac{\Lambda^4}{6\left(\Lambda_c^2 + \rho^2\right)^6}\left[(\nabla\rho)^2 - 2m\rho^2 X\right]^3. \tag{3.51}$$

To leading order in the derivative expansion we can set $(\nabla\rho)^2$ to zero, taking the MOND limit $\rho \gg \Lambda_c$ (this condition is necessary if we want the theory to reproduce MOND on galactic scales [1]), equation (3.51) reduces to

$$\mathcal{L} = m\rho^2 X + \frac{8\Lambda^4}{6}m^3\rho^{-6}X^3. \tag{3.52}$$

The Euler-Lagrange equation with respect to ρ is

$$\frac{\partial\mathcal{L}}{\partial\rho} = 0 \tag{3.53}$$

which results in the equation of motion

$$\rho^2 = \Lambda\sqrt{2m\,|X|}. \tag{3.54}$$

Substituting (3.54) into (3.51) we would get back Lagrangian (3.30). Thus, the relativistic Lagrangian reduces to a theory of the MOND-like superfluid in the weak-field limit.

3.4 A very short summary

In this chapter, we started by reviewing Modified Newtonian Dynamics or MOND in short. We have shown that this MOND phenomenology explains why the rotation curves of some galaxies are flat. We have also mentioned that the Baryonic Tully-Fisher relation, the fact that the total mass in a galaxy is proportional to the fourth power of the galaxy's rotation velocity, which is a known empirical fact, is by nature incorporated into MOND. We studied both the relativistic and non-relativistic theories of MOND and realized that in the weak-field limit, the relativistic theory can be expressed in terms of a Lagrangian density whose kinetic energy is raised to the power three halves. In [1], such a Lagrangian density was conjectured to be the function $P(X)$ of the effective field theory of superfluids. This conjecture unifies MOND and the theory of superfluidity. It conveys that a superfluid is responsible for all the MOND phenomenon observed in galaxies.

Chapter 4

Structure formation with scalar field dark matter

4.1 The field approach

The last two chapters set the stage for what we are trying to achieve in this thesis. In this chapter, we will discuss the cosmology of the theory of dark matter superfluidity, based on the non-relativistic Lagrangian (3.35), as it was discussed in [1]. In section 4.1.2, we will study the cosmological implications of the relativistic Lagrangian (3.43), for comparisons with the ones deduced from the non-relativistic Lagrangian.

4.1.1 Background cosmology from the non-relativistic Lagrangian

We can make the non-relativistic Lagrangian (3.35) of the theory of superfluidity diffeomorphism invariant if we multiply it by $\sqrt{-g}$

$$\mathcal{L} \to \mathcal{L}\sqrt{-g}. \tag{4.1}$$

In an FLRW universe, we have that $\sqrt{-g} = a^3$ (in proper time coordinates) and so the theory takes the form

$$\mathcal{L} = \frac{2\Lambda_0 (2m)^{\frac{3}{2}}}{3} a^3 \left(\dot{\psi} - m\Phi - \frac{(\nabla\psi)^2}{2m}\right)^{\frac{3}{2}} - \alpha_0 \frac{\Lambda_0}{\mathrm{M_{Pl}}} a^3 \psi \rho_b \tag{4.2}$$

where Λ_0, α_0 are parameters of the theory on cosmological scales and $\psi = \mu + \phi$ for the case when the scalar field behaves like a superfluid. Varying (4.2) with respect to ψ results in

$$\frac{d}{dt}\left[(2m)^{\frac{3}{2}}\, a^3 \dot{\psi}^{\frac{1}{2}}\right] = -\frac{\alpha}{\mathrm{M_{Pl}}} a^3 \rho_b. \tag{4.3}$$

The term $a^3\rho_b$ is a constant [1] and so integrating (4.3) with respect to time leads to

$$(2m)^{\frac{3}{2}}\, \dot{\psi}^{\frac{1}{2}} = \frac{\alpha_0}{\mathrm{M_{Pl}}}\rho_b t + \frac{C}{a^3} \tag{4.4}$$

where C is the constant of integration. The mass density of the superfluid is determined by

$$\rho_m = m\Lambda_0\, (2m)^{\frac{3}{2}}\, \dot{\psi}^{\frac{1}{2}} \tag{4.5}$$

and so (4.4) reduces to

$$\rho_m = -\frac{\alpha_0 \Lambda_0}{\mathrm{M_{Pl}}} m t \rho_b + \rho_{\mathrm{dust}} \tag{4.6}$$

where $\rho_{\mathrm{dust}} = \frac{m\Lambda_0 C}{a^3}$. This implies that, the scalar field ψ behaves like dust when

$$\frac{\alpha_0 \Lambda_0}{\mathrm{M_{Pl}}} m t_0 \frac{\rho_b}{\rho_{\mathrm{dust}}} \lesssim 1. \tag{4.7}$$

If we substitute $t_0 = 13.9 \times 10^9$ yrs $\simeq 6 \times 10^{32}\mathrm{eV}^{-1}$ (the age of the universe) and $\frac{\rho_{\mathrm{dust}}}{\rho_b} = 6$ into (4.7) we get

$$\alpha_0 \ll 2.4 \times 10^{-5}\, \frac{\mathrm{eV}^2}{\Lambda_0 \mathrm{m}} \tag{4.8}$$

which is different from the MONDian value $\alpha = 0.86\left(\frac{\Lambda}{\mathrm{meV}}\right)^{-\frac{2}{3}}$ in (3.42). That is to say, Lagrangian (3.35) describes a complex scalar field theory which reduces to a theory of superfluidity when (3.42) holds but reduces to LambdaCDM when (4.8) is satisfied.

4.1.2 Background cosmology from the relativistic Lagrangian

In subsection 3.3.2, we have shown that the relativistic Lagrangian (3.43) when interactions are ignored reduces to the free Lagrangian (3.44). This free Lagrangian will be our starting point. In an FLRW universe, the free Lagrangian in conformal time takes the form

$$-2a^2\bar{\mathcal{L}}_{\text{free}} = -\left(\bar{\rho}'\right)^2 + \left(\bar{\rho}\right)^2\left[a^2m^2 - \left(\bar{\lambda}'\right)^2\right]. \tag{4.9}$$

The volume element in this case is given by $\sqrt{-g} = a^4$ and so to couple the theory to gravity, we rewrite (4.9) as

$$\sqrt{-g}\bar{\mathcal{L}}_{\text{free}} = -\frac{a^2}{2}\left[-\left(\bar{\rho}'\right)^2 + \bar{\rho}^2\left(a^2m^2 - \left(\bar{\lambda}'\right)^2\right)\right]. \tag{4.10}$$

Inserting (4.10) into the Euler-Lagrange equation of $\bar{\lambda}$ yields

$$\frac{d}{d\eta}\left(\bar{\lambda}'a^2\bar{\rho}^2\right) = 0. \tag{4.11}$$

Thus for a constant Q of dimensions mass^3 we have that

$$\bar{\lambda}'a^2\bar{\rho}^2 = Q. \tag{4.12}$$

Equation (4.11) is a conservation law and Q is the conserved charge. The Euler-Lagrange equation with respect to $\bar{\rho}$ reads

$$\bar{\rho}'' + 2\mathcal{H}\bar{\rho}' + \left[a^2m^2 - \left(\bar{\lambda}'\right)^2\right]\bar{\rho} = 0. \tag{4.13}$$

If we suppose that $\bar{\lambda}' = ma\left(\eta\right)$, equation (4.13) simplifies to

$$\bar{\rho}'' + 2\mathcal{H}\bar{\rho}' = 0. \tag{4.14}$$

During the matter-dominated era we have that $\mathcal{H} = H_0a^{-\frac{1}{2}}$. If we do a coordinate transformation $\eta \to a$ then (4.14) becomes

$$\frac{d^2\bar{\rho}}{da^2} + \frac{5}{2a}\frac{d\bar{\rho}}{da} = 0. \tag{4.15}$$

Equation (4.15) has the solution

$$\bar{\rho} \propto a^{-\frac{3}{2}}. \tag{4.16}$$

Since this solution has to satisfy equation (4.12) as well, it is precisely

$$\bar{\rho} = \sqrt{\frac{Q}{m}}a^{-\frac{3}{2}}. \tag{4.17}$$

The energy-momentum tensor of the complex scalar field of the theory is given by

$$\bar{T}_{\mu\nu} = \partial_\mu \bar{\rho} \partial_\nu \bar{\rho} + \bar{\rho}^2 \partial_\mu \bar{\lambda} \partial_\nu \bar{\lambda} - \frac{\bar{g}_{\mu\nu}}{2} \mathcal{L}_{\text{free}}. \tag{4.18}$$

Its components are

$$\begin{aligned} \bar{T}_{00} &= a^2 \bar{\mu} = \frac{1}{2} \left(\bar{\rho}'\right)^2 + a^2 m^2 \bar{\rho}^2 \\ \bar{T}_{ij} &= a^2 \bar{P} \delta_{ij} = \frac{1}{2} \left(\bar{\rho}'\right)^2 \delta_{ij}. \end{aligned} \tag{4.19}$$

In the matter-dominated universe we have that

$$\begin{aligned} \bar{\mu} &= \frac{9QH_0^2}{8m} a^{-6} + mQa^{-3} \\ \bar{P} &= \frac{9QH_0^2}{8m} a^{-6}. \end{aligned} \tag{4.20}$$

As the universe expands, $\bar{\mu} \simeq mQa^{-3}$ ($\bar{P} \propto a^{-6} \ll \bar{\mu}$), implying that the scalar field behaves like dust without requiring strong assumptions on its mass or fine-tuning of initial conditions. The constraint $\bar{P} \ll \bar{\mu}$ for $a > a_{eq}$ leads to

$$\left(\frac{m}{H_0}\right)^2 \gg \left(1 + z_{eq}\right)^3 \simeq 10^9. \tag{4.21}$$

In general, when interactions are taken into account, varying Lagrangian (3.43) with respect to ϕ^* for the case when $\Lambda_c \gg \phi\phi^*$ (the opposite of the MOND limit $\Lambda_c \ll \phi\phi^* = \rho$ should place us within the realms of cosmology) leads to

$$\begin{aligned} \frac{1}{2}\Box\phi + \frac{1}{2}\frac{\Lambda^4 \Box\phi}{\Lambda_c^{12}} \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* + m^2\phi\phi^*\right)^2 &= -\frac{1}{2}m^2\phi \\ -\frac{1}{2}\frac{\Lambda^4 m^2\phi}{\Lambda_c^{12}} \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* + m^2\phi\phi^*\right)^2 &+ \frac{\Lambda^4\phi}{\Lambda_c^{14}} \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* + m^2\phi\phi^*\right)^3. \end{aligned} \tag{4.22}$$

The imaginary part of (4.22) is

$$\left(2\dot{\bar{\rho}}\dot{\bar{\lambda}} + \bar{\rho}\ddot{\bar{\lambda}} + 3H\bar{\rho}\dot{\bar{\lambda}}\right)\left[1 + \frac{\Lambda^4}{\left(\Lambda_c^2 + \bar{\rho}^2\right)^6}\left(-\dot{\bar{\rho}}^2 - \bar{\rho}^2\dot{\bar{\lambda}}^2 + m^2\bar{\rho}^2\right)^2\right] = 0 \tag{4.23}$$

where we have expressed the complex scalar field ϕ in terms of its modulus $\bar{\rho}$ and phase $\bar{\lambda}$. The real part of (4.22) is given by

$$\frac{1}{2}\left(\ddot{\bar{\rho}}-\bar{\rho}\dot{\bar{\lambda}}^2+3H\dot{\bar{\rho}}\right)+\frac{1}{2}\frac{\Lambda^4}{(\Lambda_c^2+\bar{\rho}^2)^6}\left(\ddot{\bar{\rho}}-\bar{\rho}\dot{\bar{\lambda}}^2+3H\dot{\bar{\rho}}\right)\left(-\dot{\bar{\rho}}^2-\bar{\rho}^2\dot{\bar{\lambda}}^2+m^2\bar{\rho}^2\right)^2$$
$$=\frac{1}{2}m^2\bar{\rho}-\frac{\Lambda^4 m^2\bar{\rho}}{(\Lambda_c^2+\bar{\rho}^2)^6}\left(-\dot{\bar{\rho}}^2-\bar{\rho}^2\dot{\bar{\lambda}}^2+m^2\bar{\rho}^2\right)^2$$
$$+\frac{\Lambda^4}{(\Lambda_c^2+\bar{\rho}^2)^7}\left(-\dot{\bar{\rho}}^2-\bar{\rho}^2\dot{\bar{\lambda}}^2+m^2\bar{\rho}^2\right)^3. \quad (4.24)$$

Equation (4.23) holds when

$$2\dot{\bar{\rho}}\dot{\bar{\lambda}}+\bar{\rho}\ddot{\bar{\lambda}}+3H\bar{\rho}\dot{\bar{\lambda}}=0 \quad (4.25)$$

or

$$1+\frac{\Lambda^4}{(\Lambda_c^6+\bar{\rho}^2)^6}\left(-\dot{\bar{\rho}}^2-\bar{\rho}^2\dot{\bar{\lambda}}^2+m^2\bar{\rho}^2\right)^2=0. \quad (4.26)$$

Substituting $\dot{\bar{\lambda}}=m$ and $\Lambda_c \gg \bar{\rho}$ into equation (4.26) we get

$$\frac{\Lambda^4\dot{\bar{\rho}}^4}{\Lambda_c^{12}}=-1. \quad (4.27)$$

The solution to equation (4.27) does not result in a dust-like scalar field and so we discard it. Equation (4.25) on the other hand, can be re-expressed as a separable differential equation

$$\frac{d\bar{\rho}}{\bar{\rho}}+\frac{1}{2}\frac{d\dot{\bar{\lambda}}}{\dot{\bar{\lambda}}}+\frac{3}{2}\frac{da}{a}=0. \quad (4.28)$$

Equation (4.28) is solved by

$$\ln\left(\bar{\rho}^2\dot{\bar{\lambda}}a^3\right)=\mathrm{C} \quad (4.29)$$

where C is the integration constant. From (4.29) we get

$$\bar{\rho}=\sqrt{\frac{Q}{m}}a^{-\frac{3}{2}} \quad (4.30)$$

where the phase is chosen such that $\lambda = mt + \text{constant}$ (as in [1]). Remarkably, this is the same solution we had for the interactions free theory. The reason why this is the case will be explained under "Parameterising the background" subsubsection. The (00) component of the energy-momentum tensor determined from the Lagrangian (3.43) leads to

$$\bar{\mu} = \frac{1}{2}\dot{\bar{\rho}}^2 + \frac{1}{2}\bar{\rho}^2\dot{\bar{\lambda}}^2 + \frac{1}{2}m^2\bar{\rho}^2 + \frac{\bar{\Lambda}^4\left(\dot{\bar{\rho}}^2 + \bar{\rho}^2\dot{\bar{\lambda}}^2\right)}{\left(\Lambda_c^2 + \bar{\rho}^2\right)^6}\left(-\dot{\bar{\rho}}^2 - \bar{\rho}^2\dot{\bar{\lambda}}^2 + m^2\bar{\rho}^2\right)^2 + \frac{\Lambda^4}{6\left(\Lambda_c^2 + \bar{\rho}^2\right)^6}\left(-\dot{\bar{\rho}}^2 - \bar{\rho}^2\dot{\bar{\lambda}}^2 + m^2\bar{\rho}^2\right)^3. \quad (4.31)$$

When we substitute $\dot{\bar{\lambda}} = m$ in it and using the fact that $\Lambda_c \gg \bar{\rho}$, equation (4.31) reduces to

$$\bar{\mu} = \frac{1}{2}\dot{\bar{\rho}}^2 + m^2\bar{\rho}^2 + \frac{\Lambda^4}{\Lambda_c^2}\bar{\rho}^2 m^2\dot{\bar{\rho}}^4 + \frac{5\Lambda^4\dot{\bar{\rho}}^6}{6\Lambda_c^{12}}. \quad (4.32)$$

The pressure of the scalar field on the other hand is given by

$$\bar{P} = \frac{\dot{\bar{\rho}}^2}{2} + \frac{\bar{\rho}^2\dot{\bar{\lambda}}^2}{2} - \frac{1}{2}m^2\bar{\rho}^2 - \frac{\Lambda^4}{6\Lambda_c^{12}}\left(-\dot{\bar{\rho}}^2 - \bar{\rho}^2\dot{\bar{\lambda}}^2 + m^2\bar{\rho}^2\right)^3 \quad (4.33)$$

which reduces to

$$\bar{P} = \frac{1}{2}\dot{\bar{\rho}}^2 + \frac{\Lambda^4}{6\Lambda_c^{12}}\dot{\bar{\rho}}^6. \quad (4.34)$$

In order for the scalar field to behave like dust in the general case, the term

$$m^2\bar{\rho}^2 = mQa^{-3} \quad (4.35)$$

should dominate all the terms in (4.32) and (4.34): $m^2\bar{\rho}^2 \gg \frac{1}{2}\dot{\bar{\rho}}^2$ leads us to the constraint

$$\boxed{\left(\frac{m}{H_0}\right)^2 \gg \frac{9}{8}\left(1 + z_{eq}\right)^3} \quad (4.36)$$

$m^2\bar{\rho}^2 \gg \frac{\Lambda^4}{\Lambda_c^{12}} m^2\bar{\rho}^2\dot{\bar{\rho}}^4$ corresponds to

$$\boxed{\left(\frac{m}{H_0}\right)^2 \gg \frac{81}{16}\frac{\Lambda^4}{\Lambda_c^{12}}Q^2 H_0^2\left(1 + z_{eq}\right)^{12}} \quad (4.37)$$

and finally when $m^2\bar{\rho}^2 \gg \frac{5\Lambda^4}{6\Lambda_c^{12}}\dot{\bar{\rho}}^6$ we get

$$\boxed{\left(\frac{m}{H_0}\right)^2 \gg \frac{729}{64}\frac{5\Lambda^4}{6\Lambda_c^{12}}\frac{Q^2}{m^2}H_0^4\left(1+z_{eq}\right)^{15}}. \tag{4.38}$$

For now (subject to change), the scalar field ϕ behaves like dust in the general relativistic case when (4.36),(4.37) and (4.38) are all satisfied.

Parameterising the background

The Friedmann constraint

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2\bar{\mu} \tag{4.39}$$

can be reduced to

$$\boxed{mQ = \frac{3}{8\pi}H_0^2\mathrm{M}_{\mathrm{Pl}}^2 = 1.2\times 10^{-47}\mathrm{GeV}^4} \tag{4.40}$$

which suggests that for $\Lambda_c = 1\mathrm{eV}$ (the value chosen by the authors of [1] on cosmological scales) we get

$$\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12}} = \frac{m^2Q^2}{\Lambda_c^8} = \frac{1.44\times 10^{-94}}{\Lambda_c^8}\mathrm{GeV}^8 \ll 1 \tag{4.41}$$

where Λ and Λ_c are of the same order [1]. Furthermore, equation (4.40) can be rearranged to

$$Q = \frac{1.2\times 10^{-47}}{m}\mathrm{GeV}^4. \tag{4.42}$$

This implies that

$$\frac{Q^2H_0^4}{m^2} = \frac{1.44\times 10^{-262}}{m^4}\mathrm{GeV}^{12} \tag{4.43}$$

and

$$Q^2H_0^2 = \frac{1.44\times 10^{-178}}{m^2}\mathrm{GeV}^{10}. \tag{4.44}$$

From equation (4.36), it can be shown that the following holds

$$m \gg 1.52\times 10^{-37}\mathrm{GeV}. \tag{4.45}$$

Substituting this m into (4.43) and (4.44), we find that

$$\frac{81\Lambda^4}{16\Lambda_c^{12}}Q^2H_0^2\left(1+z_{eq}\right)^{12} \ll 1 \tag{4.46}$$

and
$$\frac{729}{64}\frac{5\Lambda^4}{6\Lambda_c^{12}}\frac{Q^2H_0^4}{m^2}\left(1+z_{eq}\right)^{15}\ll 1. \tag{4.47}$$

This implies that the scalar field behaves like dust when (4.36) holds (just as it was the case for the free theory). Any $\left(\frac{m}{H_0}\right)^2$ value that satisfies (4.36) automatically satisfies the other 2 constraints, (4.37) and (4.38).

4.1.3 Perturbation theory in the weak-field limit of the relativistic Lagrangian

The perturbed form of equation (4.22) is given by

$$\begin{aligned}
&\frac{1}{2}\bar{g}^{\mu\nu}\nabla_\mu\nabla_\nu\delta\phi+\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\nabla_\nu\bar{\phi}+\frac{1}{2}\frac{\Lambda^4}{\Lambda_c^{12}}\bar{g}^{\mu\nu}\nabla_\mu\nabla_\nu\delta\phi\left(\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}^*+m^2\bar{\phi}\bar{\phi}^*\right)^2\\
&\quad+\frac{1}{2}\frac{\Lambda^4}{\Lambda_c^{12}}\bar{g}^{\mu\nu}\nabla_\mu\nabla_\nu\bar{\phi}\delta\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^*+m^2\phi\phi^*\right)^2+\\
&\quad\frac{1}{2}\frac{\Lambda^4}{\Lambda_c^{12}}\delta g^{\mu\nu}\nabla_\mu\nabla_\nu\bar{\phi}\left(\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}^*+m^2\bar{\phi}\bar{\phi}^*\right)^2=-\frac{1}{2}m^2\delta\phi\\
&-\frac{1}{2}\frac{\Lambda^4}{\Lambda_c^{12}}m^2\delta\phi\left(\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}^*+m^2\bar{\phi}\bar{\phi}^*\right)^2-\frac{1}{2}\frac{\Lambda^4}{\Lambda_c^{12}}m^2\bar{\phi}\delta\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^*+m^2\phi\phi^*\right)^2\\
&+\frac{\Lambda^4}{\Lambda_c^{14}}\bar{\phi}\delta\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^*+m^2\phi\phi^*\right)^3+\frac{\Lambda^4}{\Lambda_c^{14}}\delta\phi\left(\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}^*+m^2\bar{\phi}\bar{\phi}^*\right)^3.
\end{aligned} \tag{4.48}$$

When (4.48) is expressed in terms of the field's modulus $\bar{\rho}$ and phase $\bar{\lambda}$ and substituting in it the background solution, it too splits into two equations:

the real part equation

$$
\begin{aligned}
&\left[1+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\right]\frac{d^2\delta\rho}{da^2}+\\
&\frac{5}{2a}\left[1+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}-\frac{36\Lambda^4 Q^2 m^2}{5\Lambda_c^{12} a^9}-\frac{36\Lambda^4 m Q^3}{5\Lambda_c^{14} a^9}+\frac{12\Lambda^4 Q^2 m^2}{5\Lambda_c^{14} a^6}+\frac{24\Lambda^4 Q^2 m^2}{5\Lambda_c^{12} a^6}\right]\frac{d\delta\rho}{da}\\
&\quad+\left[-\frac{2m\sqrt{\frac{Q}{m}}}{H_0 a}\left(1+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\right)-\frac{12\Lambda^4 Q^2 H_0 m}{\Lambda_c^{12} a^{10}}\sqrt{\frac{Q}{m}}+\frac{12\Lambda^4 m^3 Q^2}{H_0\Lambda_c^{12} a^7}\right]\frac{d\delta\lambda}{da}\\
&\quad-\frac{a}{H_0^2}\left[-k^2+2m^2+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\left(-k^2+m^2\right)-\frac{\Lambda^4 m^4 Q^2}{\Lambda_c^{12} a^6}+\frac{6\Lambda^4 m^3 Q^3}{\Lambda_c^{12} a^9}\right]\delta\rho-\\
&\Phi\sqrt{\frac{Q}{m}}\left[\left(\frac{3}{a^{\frac{7}{2}}}-\frac{2}{a^{\frac{1}{2}}}\left(\frac{m}{H_0}\right)^2\right)\left(1-\frac{10\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\right)+\frac{4\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^{\frac{13}{2}}}\left(\frac{m}{H_0}\right)^2-\frac{12\Lambda^4 m^3 Q^3}{\Lambda_c^{14} H_0^2 a^{\frac{19}{2}}}\right]\\
&\qquad= 0
\end{aligned}
\tag{4.49}
$$

and the imaginary part equation

$$
\begin{aligned}
&\left[1+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\right]\frac{d^2\delta\lambda}{da^2}-\frac{1}{2a}\left[1+\frac{2\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^5}-\frac{12\Lambda^4 Q^2 m^2}{\Lambda_c^{12} a^6}\right]\frac{d\delta\lambda}{da}\\
&\quad-\frac{a}{H_0^2}\left[-k^2+2m^2+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\left(-k^2+m^2\right)+\frac{\Lambda^4 m^4 Q^2}{\Lambda_c^{12} a^4}\right]\delta\lambda\\
&\quad+\left[\frac{2m}{H_0\sqrt{\frac{Q}{m}}}a^2\left(1+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\right)+\frac{18\Lambda^4 m^2 Q H_0}{\Lambda_c^{12} a^8}\sqrt{\frac{Q}{m}}\right]\frac{d\delta\rho}{da}\\
&+\left[\frac{3m}{H_0\sqrt{\frac{Q}{m}}}a\left(1+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\right)\right]\delta\rho+6\Phi\frac{m}{H_0}a^{-\frac{1}{2}}\left(1+\frac{\Lambda^4 m^2 Q^2}{\Lambda_c^{12} a^6}\right)=0
\end{aligned}
\tag{4.50}
$$

where we have performed transformations $\eta \to a$, $\nabla^2 \to -k^2$ and used the fact that $mQa^{-3} \gg m^2$. The gravitational potential Φ could be determined from Poisson's equation (in Fourier space) in the quasi-Newtonian limit [16]

$$
\Phi = -\frac{4\pi G}{k^2}a^2\delta\mu \tag{4.51}
$$

where $\delta\mu$ is determined from the perturbed energy-momentum tensor $\delta T_{\mu\nu}$ as follows:

$$\begin{aligned}
\delta T_{\mu\nu} = \delta\left(\partial_\mu\phi\partial_\nu\phi^*\right) + \frac{\Lambda^4}{\Lambda_c^{12}}\delta\left(\partial_\mu\phi\partial_\nu\phi^*\right)\left(\bar{g}^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\bar{\phi}^* + m^2\bar{\phi}\bar{\phi}^*\right)^2 \\
+ \frac{\Lambda^4}{\Lambda_c^{12}}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}^*\delta\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi^* + m^2\phi\phi^*\right)^2 \\
- \frac{1}{2}\delta g_{\mu\nu}\left(\bar{g}^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\bar{\phi}^* + m^2\bar{\phi}\bar{\phi}^*\right) - \frac{1}{2}\bar{g}_{\mu\nu}\delta\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi^* + m^2\phi\phi^*\right) \\
- \frac{\Lambda^4}{6\Lambda_c^{12}}\delta g_{\mu\nu}\left(\bar{g}^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\bar{\phi}^* + m^2\bar{\phi}\bar{\phi}^*\right)^3 - \frac{\Lambda^4}{\Lambda_c^{12}}\bar{g}_{\mu\nu}\delta\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi^* + m^2\phi\phi^*\right)^3 .
\end{aligned} \tag{4.52}$$

The (00) component of (4.52) when the background solution of $\bar{\rho}$ and $\bar{\lambda}$ is substituted in it, is given by:

$$\begin{aligned}
\delta T_{00} = \frac{3QH_0}{a^{\frac{3}{2}}}\frac{d}{da}\delta\lambda - \frac{9}{2}\frac{H_0^2}{a^{\frac{3}{2}}}\sqrt{\frac{Q}{m}}\frac{d}{da}\delta\rho + 2m^2\sqrt{\frac{Q}{m}}a^{\frac{1}{2}}\delta\rho + \frac{3\Phi mQ}{a} \\
+ \frac{\Lambda^4Q^2m^2}{a^6\Lambda_c^{12}}\left(-\frac{3H_0^2}{a^{\frac{7}{2}}}\sqrt{\frac{Q}{m}}\frac{d}{da}\delta\rho + \frac{2QH_0}{a^{\frac{7}{2}}}\frac{d}{da}\delta\lambda + \frac{2m^2}{a^{\frac{1}{2}}}\sqrt{\frac{Q}{m}}\delta\rho\right) \\
+ \frac{2\Lambda^4Qm}{\Lambda_c^{12}a}\left(\frac{3H_0^2}{a^{\frac{7}{2}}}\sqrt{\frac{Q}{m}}\frac{d}{da}\delta\rho - \frac{2QH_0}{a^{\frac{7}{2}}}\frac{d}{da}\delta\lambda + \frac{2\Phi Qm}{a^3}\right) + \frac{\Lambda^4m^3Q^3}{3\Lambda_c^{12}a^7}\Phi \\
+ \frac{3\Lambda^4m^2Q^2}{\Lambda_c^{12}a^6}\left(\frac{3H_0^2}{a^{\frac{7}{2}}}\sqrt{\frac{Q}{m}}\frac{d}{da}\delta\rho - \frac{2QH_0}{a^{\frac{7}{2}}}\frac{d}{da}\delta\lambda + \frac{2\Phi Qm}{a^3}\right)
\end{aligned} \tag{4.53}$$

where $a^{-2}\delta T_{00} = \delta\mu$. From equation (4.41), we can see that the interaction term $\frac{\Lambda^4m^2Q^2}{\Lambda_c^{12}a^6}$ in (4.49), (4.50) and (4.53) is much smaller than one for any value of the scale-factor between equality and today and so we neglect it. Based on the same argument, we neglect the terms $\frac{\Lambda^4m^2Q^2}{\Lambda_c^{12}a^5}$, $\frac{\Lambda^4m^2Q^2}{\Lambda_c^{12}a^4}$, $\frac{\Lambda^4m^4Q}{\Lambda_c^{12}a^4}$ and $\frac{\Lambda^4m^2QH_0}{\Lambda_c^{12}a^8}\sqrt{\frac{Q}{m}}$ as well. This implies that our perturbation theory is interactions free, just as it was the case for the background.

Structure formation

When these interactions (terms with a factor of Λ in 4.49 and 4.50) are neglected, equation (4.49) takes the form

$$\frac{d^2}{da^2}\delta\rho + \frac{5}{2a}\frac{d}{da}\delta\rho - \frac{2m\sqrt{\frac{Q}{m}}}{H_0}\frac{d}{da}\delta\lambda - \frac{a}{H_0^2}\left(2m^2 - k^2\right)\delta\rho - \Phi\sqrt{\frac{Q}{m}}\left[\frac{3}{a^{\frac{7}{2}}} - \frac{2}{a^{\frac{1}{2}}}\left(\frac{m}{H_0}\right)^2\right] = 0 \quad (4.54)$$

whereas equation (4.50) becomes

$$\frac{d^2}{da^2}\delta\lambda - \frac{1}{2a}\frac{d}{da}\delta\lambda - \frac{a}{H_0^2}\left(2m^2 - k^2\right)\delta\lambda + \frac{2m}{H_0\sqrt{\frac{Q}{m}}}a^2\frac{d}{da}\delta\rho + \frac{3m}{H_0\sqrt{\frac{Q}{m}}}a\delta\rho + 6\Phi\frac{m}{H_0}a^{-\frac{1}{2}} = 0. \quad (4.55)$$

The perturbed mass density of the scalar field in that case reduces to

$$\delta\mu = a^{-2}\delta T_{00} = \frac{3QH_0}{a^{\frac{7}{2}}}\frac{d}{da}\delta\lambda - \frac{9}{2}\frac{H_0^2}{a^{\frac{7}{2}}}\sqrt{\frac{Q}{m}}\frac{d}{da}\delta\rho + \frac{2m^2}{a^{\frac{3}{2}}}\sqrt{\frac{Q}{m}}\delta\rho + \frac{3\Phi mQ}{a^3}. \quad (4.56)$$

Substituting (4.56) into (4.51) for the case when $k \ll m$, we get

$$\boxed{\Phi = \frac{3}{2}\frac{H_0^2}{mQ}\sqrt{\frac{Q}{m}}a^{-\frac{1}{2}}\frac{d}{da}\delta\rho - \frac{H_0}{m}a^{-\frac{1}{2}}\frac{d}{da}\delta\lambda - \frac{2}{3}\left(\frac{Q}{m}\right)^{-\frac{1}{2}}a^{\frac{3}{2}}\delta\rho}. \quad (4.57)$$

Equation (4.54) when (4.57) is substituted in it becomes

$$\frac{d^2}{da^2}\delta\rho + \frac{5}{2a}\frac{d}{da}\delta\rho - \frac{10}{3}\left(\frac{m}{H_0}\right)^2 a\delta\rho\left[1 + \frac{3}{5a}\frac{H_0}{m}\sqrt{\frac{Q}{m}}\frac{1}{\delta\rho}\frac{d}{da}\delta\lambda + \frac{3}{5a^2}\frac{H_0}{m}\sqrt{\frac{Q}{m}}\frac{1}{\delta\rho}\frac{d}{da}\delta\lambda\right] = 0. \quad (4.58)$$

The last two terms in the square brackets are much smaller than 1 and so we ignore them, this reduces (4.58) to

$$\frac{d^2}{da^2}\delta\rho + \frac{5}{2a}\frac{d}{da}\delta\rho - \frac{10}{3}\left(\frac{m}{H_0}\right)^2 a\delta\rho = 0. \quad (4.59)$$

The growing mode solution to (4.59) is

$$\boxed{\delta\rho\left(a\right)=\frac{Ae^{\frac{2}{3}\sqrt{\frac{10}{3}}\frac{m}{H_0}a^{\frac{3}{2}}}}{a^{\frac{3}{2}}}}. \tag{4.60}$$

Similarly, (4.55) can be reduced to

$$\frac{d^2}{da^2}\delta\lambda-\frac{13}{2a}\frac{d}{da}\delta\lambda-2\left(\frac{m}{H_0}\right)^2a\delta\lambda=0. \tag{4.61}$$

Its growing mode solution is

$$\boxed{\delta\lambda\left(a\right)=Be^{\frac{2}{3}\sqrt{2}\frac{m}{H_0}a^{\frac{3}{2}}}\left[8\left(\frac{m}{H_0}\right)^2a^3-18\sqrt{2}\frac{m}{H_0}a^{\frac{3}{2}}+27\right]} \tag{4.62}$$

where A and B in (4.60) and (4.62) are integration constants. Substituting (4.60) and (4.62) into (4.56) results in an exponentially growing density contrast, which is not the result we expected in the quasi-Newtonian limit and so we discard it. Let us now consider $\boxed{k\gg m}$. In that case, equation (4.54) takes the form

$$\frac{d^2}{da^2}\delta\rho+\frac{5}{2a}\frac{d}{da}\delta\rho+\frac{k^2a}{H_0^2}\delta\rho=0. \tag{4.63}$$

It has the solution

$$\delta\rho\left(a\right)=Ae^{\frac{2ika^{\frac{3}{2}}}{3H_0}}a^{-\frac{3}{2}}. \tag{4.64}$$

The real part of (4.64) is

$$\boxed{\delta\rho\left(a\right)=Aa^{-\frac{3}{2}}\cos\left(\frac{2k}{3H_0}a^{\frac{3}{2}}\right)}. \tag{4.65}$$

Equation (4.55) becomes

$$\frac{d^2}{da^2}\delta\lambda-\frac{1}{2a}\frac{d}{da}\delta\lambda+\frac{k^2a}{H_0^2}\delta\lambda=0. \tag{4.66}$$

The solution to (4.66) is

$$\boxed{\delta\lambda\left(a\right)=B\sin\left(\frac{2ka^{\frac{3}{2}}}{3H_0}\right)+C\cos\left(\frac{2ka^{\frac{3}{2}}}{3H_0}\right)} \tag{4.67}$$

which can be written as

$$\boxed{\delta\lambda\left(a\right) = D\text{cos}\left(\frac{2ka^{\frac{3}{2}}}{3H_0} - p\right)} \tag{4.68}$$

where $D = \sqrt{C^2 + B^2}$ and $p = \tan\left(\frac{C}{B}\right)$. Substituting these solutions into (4.56) for the case when $\boxed{k \gg \frac{12\pi GmQ}{a}}$ we find that

$$\begin{aligned}\delta\mu = &\frac{27}{4}H_0^2\sqrt{\frac{Q}{m}}Aa^{-\frac{11}{2}}\cos\left(\frac{2k}{3H_0}a^{\frac{3}{2}}\right)\\ &+\frac{9}{2}\sqrt{\frac{Q}{m}}H_0kAa^{-\frac{9}{2}}\text{sin}\left(\frac{2ka^{\frac{3}{2}}}{3H_0}\right) - 3Qka^{-3}D\text{sin}\left(\frac{2ka^{\frac{3}{2}}}{3H_0} - p\right)\\ &+ 2m^2\sqrt{\frac{Q}{m}}A\text{cos}\left(\frac{2k}{3H_0}a^{\frac{3}{2}}\right).\end{aligned} \tag{4.69}$$

The density contrast by definition is $\delta = \frac{\delta\mu}{\bar{\mu}}$ where $\bar{\mu} = mQa^{-3}$, this lead us to

$$\begin{aligned}\delta = &\frac{27}{4}\frac{H_0^2}{mQ}\sqrt{\frac{Q}{m}}Aa^{-\frac{5}{2}}\text{cos}\left(\frac{2k}{3H_0}a^{\frac{3}{2}}\right)\\ &+\frac{9}{2}\sqrt{\frac{Q}{m}}\frac{H_0}{mQ}kAa^{-\frac{3}{2}}\text{sin}\left(\frac{2ka^{\frac{3}{2}}}{3H_0}\right) - \frac{3k}{m}D\text{sin}\left(\frac{2ka^{\frac{3}{2}}}{3H_0} - p\right)\\ &+ 2a^{-3}\sqrt{\frac{m}{Q}}A\text{cos}\left(\frac{2k}{3H_0}a^{\frac{3}{2}}\right).\end{aligned} \tag{4.70}$$

Analysing the density contrast term by term: we can see that all the terms in (4.70) decay except for $\frac{3k}{m}D\text{sin}\left(\frac{2ka^{\frac{3}{2}}}{3H_0} - p\right)$ and so

$$\boxed{\delta = \frac{3k}{m}D\text{sin}\left(\frac{2ka^{\frac{3}{2}}}{3H_0} - p\right)}. \tag{4.71}$$

Taking the time average of (4.71) we get

$$\langle\delta\rangle = \frac{3k}{m}\frac{D}{\Delta T}\int_0^{\Delta T}\sin\left(\frac{2k}{3H_0}a^{\frac{3}{2}} - p\right)da. \tag{4.72}$$

Remember that the period of the sine function is $\Delta T = 2\pi$. If we suppose that $\frac{k}{H_0} \gg p$ then the integral in (4.72) has the value

$$\langle \delta \rangle = \frac{3k}{m}\frac{D}{2\pi} \times \text{“a very small number”}. \tag{4.73}$$

Equation (4.73) suggests that, on average, the density contrast is a constant. This is in agreement with the results obtained by the authors in [16] when the baryon coupling term is neglected. By “a very small number” we mean that the number is almost zero but not zero. If we choose $\frac{k}{H_0} = 10^{10}$ for instance, then

$$\int_0^{2\pi} \sin\left(\frac{2k}{3H_0}a^{\frac{3}{2}}\right) da = 2.207 \times 10^{-7} = \text{“a very small number”}. \tag{4.74}$$

Estimating the relevance of the neglected terms

In the “structure formation” subsubsection, we have assumed that the term $\frac{2H_0 m}{k^2 a}\sqrt{\frac{Q}{m}}\frac{1}{\delta\rho}\frac{d}{da}\delta\lambda$ is negligible. Let us prove if it is the case. The derivative of $\delta\lambda$ with respect to a is

$$\frac{d\delta\lambda}{da} = -D\frac{k}{H_0}a^{\frac{1}{2}}\sin\left(\frac{2ka^{\frac{3}{2}}}{3H_0} - p\right). \tag{4.75}$$

When $k \gg m$, equation (4.75) implies that

$$\boxed{\frac{2H_0 m}{k^2 a}\sqrt{\frac{Q}{m}}\frac{1}{\delta\rho}\frac{d}{da}\delta\lambda = -\frac{D}{A}\frac{2ma}{k}\sqrt{\frac{Q}{m}}\frac{\sin\left(\frac{2ka^{\frac{3}{2}}}{3H_0} - p\right)}{\cos\left(\frac{2k}{3H_0}a^{\frac{3}{2}}\right)} \ll 1}. \tag{4.76}$$

Similarly, the terms: $\frac{2mH_0 a}{k^2\sqrt{\frac{Q}{m}}}\frac{1}{\delta\lambda}\frac{d}{da}\delta\rho$ and $\frac{3mH_0}{k^2\sqrt{\frac{Q}{m}}}\frac{\delta\rho}{\delta\lambda}$ could be proved to be negligible. This justifies the approximations we have made to reduce (4.49) to (4.59),(4.50) to (4.61) and (4.53) to (4.56).

Chapter 5

Discussion and Conclusion

5.1 Discussion

The non-relativistic theory in [1] is constructed on the basis of two parameters: Λ and α. Throughout the thesis, we asserted that these parameters take different values on galactic and cosmological scales. We have also mentioned that, along with the scale varying parameters follows a phase transition. The two must thus be connected. Since phase transitions occur when the temperature of a thermodynamic system varies, the authors of [1] concluded that Λ and α must be temperature dependent. In fact, they claim that the parameter Λ as a function of temperature obeys the relation

$$\boxed{\Lambda\left(T\right) = \frac{\Lambda_0}{1+\kappa_\Lambda\left(\frac{T}{T_c}\right)^{\frac{1}{4}}}} \tag{5.1}$$

where $\kappa_\Lambda \simeq 10^4$, T_c is the temperature at the transition point and Λ_0 is the value of Λ on cosmological scales. The parameter α on the other hand obeys

$$\boxed{\alpha\left(T\right) = \alpha_0\left(1+\kappa_\alpha\left(\frac{T}{T_c}\right)^{\frac{1}{4}}\right)} \tag{5.2}$$

where $\kappa_\alpha = \kappa_\Lambda$ and α_0 is the value of α on cosmological scales. It is not clear how equations (5.1) and (5.2) were derived. It seems as if they were put in by hand, and so we will have to take them as being true although a thorough investigation needs to be done. From these relations, it can be shown that, at sub-critical temperature $T \ll T_c$: $\Lambda = \Lambda_0$ and $\alpha = \alpha_0$, which correspond to cosmological scales and the dust dark matter phase. Beyond the critical temperature $T \gg T_c$, we get the values of Λ and α which coordinate with the ones deduced on galactic scales when the non-relativistic theory in [1] models a superfluid. In essence, dark matter undergoes Bose-Einstein condensation in galaxies to form a superfluid but remains in its usual phase on cosmological scales. From our brief calculations in chapter 3, distinctly from section 3.3 onwards, we have emphasized that, superfluidity can be consolidated with MOND. To wit, the superfluid dark matter explains why rotation curves are flat, why the mass-to-light ratio of some galaxy clusters are bigger than 1 and ultimately, why the power law index of BTFR is 4.

5.2 Conclusion

The standard model of cosmology (ΛCDM) and MOND separately successfully predict physics on two different scales: cosmological and galactic scales respectively. A superfluid model that could mediate these scales was first proposed in [1]. The authors adopted a non-relativistic and non-perturbative approach in their work. In this thesis, we extended the formulation of [1] to include, variously, the perturbation theory and relativity. Our study of linear perturbations of a quasi-Newtonian perturbative formulation shows that when baryon couplings are neglected, the density contrast of matter is constant. A constant density contrast implies no growth in structure. Baryon couplings can yield different results. It was found in [16], when examining linear perturbations of a quasi-Newton model, that the introduction of baryon coupling led to a density contrast scaling as $a^{2.5}$, where a is the scale factor. This growth is faster than that obtained in the analysis of the standard ΛCDM model. The analysis in [16] was based on the non-relativistic Lagrangian (3.35). In contrast, we examined the relativistic Lagrangian (3.43). We were able to recover the results in [1] when we examined the case using a relativistic Lagrangian where $\left(\frac{m}{H_0}\right)^2 \gg \frac{9}{8}\left(1+z_{eq}\right)^3 \simeq 10^9$ and $\Lambda_c \gg \phi\phi^*$ are chosen in line with $\alpha_0 \ll 2.4 \times 10^{-5}\,\frac{\mathrm{eV}^2}{\Lambda_0 m}$ as given in [1]. Our analyses recover the galactic and cosmological scales results in [1].

5.2.1 New physics

In contrast to the non-relativistic Lagrangian (3.35), the complex scalar field of the relativistic model of the superfluid dark matter (3.43) is not coupled to baryons. In the background case, the relativistic theory was able to yield the same results as the non-relativistic theory without a need to couple the scalar field to baryons. In addition to that, when we studied the perturbations of the relativistic theory in the quasi-Newtonian limit, we were able to reproduce the non-relativistic perturbation theory in [16] when baryon couplings to the complex scalar field were neglected. Not studied in this thesis but something worth investigating is the effect on structure growth when baryons couple to a complex scalar field. This holds the potential for new physics and will be examined in future.

Appendices

Appendix A

Chapter 2 (3 and 4) derivations and calculations

This Appendix is meant to give a detailed calculation and derivation of some of the equations and assumptions we took for granted in chapters 2, 3 and 4.

A.1 The field redefinition

Let us suppose that $\phi = \frac{\rho}{\sqrt{2}}e^{i\psi}$. The terms $\partial_\mu\phi^*$ and $\partial^\nu\phi$ are computed as follows: starting with $\partial_\mu\phi^*$, we find that

$$\partial_\mu\phi^* = \frac{1}{\sqrt{2}}\partial_\mu\rho e^{-i\psi} - i\frac{\rho}{\sqrt{2}}\partial_\mu\psi e^{-i\psi} \tag{A.1}$$

whereas $\partial^\nu\phi$ takes the form

$$\partial^\nu\phi = \frac{1}{\sqrt{2}}\partial^\nu\rho e^{i\psi} + i\frac{\rho}{\sqrt{2}}\partial^\nu\psi e^{i\psi}. \tag{A.2}$$

This lead us to

$$\partial_\mu\phi^*\partial^\nu\phi = \frac{1}{2}\partial_\mu\rho\partial^\nu\rho + \frac{\rho^2}{2}\partial_\mu\psi\partial^\nu\psi. \tag{A.3}$$

The term $|\phi|^2$ on the other hand was found to be

$$|\phi|^2 = \frac{\rho^2}{2} \tag{A.4}$$

and so $|\phi|^4$ is given by

$$|\phi|^4 = \frac{\rho^4}{4}. \tag{A.5}$$

When we substitute (A.3), (A.4) and (A.5) into (3.18) we get (3.21).

A.2 The comoving curvature perturbation

On super-Hubble scales $k \ll \mathcal{H}$, equation (2.46) reduces to

$$\Phi'' + 3\left(1 + c_s^2\right)\mathcal{H}\Phi' = 0 \tag{A.6}$$

which implies that the gravitational potential is constant, and so $\Phi' = 0$. As a result of that, equation (2.48) becomes

$$\mathcal{R} = -\Phi - \frac{2}{3\,(1+w)}\Phi \tag{A.7}$$

which can be rewritten as

$$\mathcal{R} = -\Phi\left(1 + \frac{2}{3+3w}\right). \tag{A.8}$$

Equation (A.8) when oversimplified takes the form

$$\mathcal{R} = -\Phi\left(\frac{3+3w}{3+3w} + \frac{2}{3+3w}\right) \tag{A.9}$$

and finally

$$\boxed{\mathcal{R} = -\frac{5+3w}{3+3w}\Phi}. \tag{A.10}$$

A.3 Solving Friedmann's equation

The Friedmann equation (2.13) when $K = 0$ reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{A.11}$$

For dust $\rho \propto a^{-3}$, and so equation (A.11) becomes

$$\dot{a} \propto a^{-\frac{1}{2}} \tag{A.12}$$

which can be expressed as

$$a^{\frac{1}{2}}\,da \propto dt. \tag{A.13}$$

Integrating both sides of (A.13) we get

$$\frac{2}{3}a^{\frac{3}{2}} \propto t \tag{A.14}$$

which results in

$$\boxed{a \propto t^{\frac{2}{3}}}. \tag{A.15}$$

For radiation $\rho = a^{-4}$, and so (A.11) takes the form

$$\dot{a} \propto a^{-1} \tag{A.16}$$

whose solution is

$$\boxed{a \propto t^{\frac{1}{2}}}. \tag{A.17}$$

Lastly, for dark energy $\rho = a^0 = \rho_{\text{const}}$, and so

$$\dot{a} = \sqrt{\frac{8\pi G}{3}\rho_{\text{const}}}\;\; a \tag{A.18}$$

which has the solution of the form

$$a \propto e^{\sqrt{\frac{8\pi G}{3}\rho_{\text{const}}}\;\; t}. \tag{A.19}$$

Setting $\rho = \rho_{\text{const}}$ in (A.11), we find that

$$\rho_{\text{const}} = \frac{3H^2}{8\pi G} \tag{A.20}$$

where $H = $ constant. Substituting (A.20) into (A.19) lead us to

$$\boxed{a \propto e^{Ht}}. \tag{A.21}$$

A.4 The evolution equation of $\delta\lambda$

Equation (4.55) when $k \ll m$ reduces to

$$\frac{d^2}{da^2}\delta\lambda - \frac{1}{2a}\frac{d}{da}\delta\lambda - 2\left(\frac{m}{H_0}\right)^2 a\delta\lambda + \frac{2m}{H_0\sqrt{\frac{Q}{m}}}a^2\frac{d}{da}\delta\rho + \frac{3m}{H_0\sqrt{\frac{Q}{m}}}a\delta\rho + 6\Phi\frac{m}{H_0}a^{-\frac{1}{2}} = 0. \tag{A.22}$$

The gravitational potential in (4.57) for the case $m \gg H_0$ reduces to [1]

$$\boxed{\Phi = -\frac{2}{3}\left(\frac{Q}{m}\right)^{-\frac{1}{2}} a^{\frac{3}{2}}\delta\rho} \tag{A.23}$$

and so, equation (A.22) when (A.23) is substituted in it becomes

$$\frac{d^2}{da^2}\delta\lambda - \frac{1}{2a}\frac{d}{da}\delta\lambda - 2\left(\frac{m}{H_0}\right)^2 a\delta\lambda + \frac{2m}{H_0\sqrt{\frac{Q}{m}}}a^2\frac{d}{da}\delta\rho - \frac{m}{H_0\sqrt{\frac{Q}{m}}}a\delta\rho = 0. \tag{A.24}$$

Equation (A.24) can be expressed as

$$\frac{d^2}{da^2}\delta\lambda - \frac{1}{2a}\frac{d}{da}\delta\lambda - 2\left(\frac{m}{H_0}\right)^2 a\delta\lambda\left[1 + \frac{H_0}{m\sqrt{\frac{Q}{m}}}a\frac{1}{\delta\lambda}\frac{d}{da}\delta\rho - \frac{2H_0}{m\sqrt{\frac{Q}{m}}}\frac{\delta\rho}{\delta\lambda}\right] = 0. \tag{A.25}$$

Based on the same reasons as in (4.76), we consider the 2nd and 3rd terms in the square brackets of (A.25) insignificant, and so

$$\boxed{\frac{d^2}{da^2}\delta\lambda - \frac{1}{2a}\frac{d}{da}\delta\lambda - 2\left(\frac{m}{H_0}\right)^2 a\delta\lambda = 0}. \tag{A.26}$$

[1] See the subsubsection "Estimating the relevance of the neglected terms".

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