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A STUDY OF INTEGRABILITY CONDITIONS FOR IRROTATIONAL DUST SPACETIMES

by

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Abstract

This thesis examines consistency conditions for fluid solutions of the field equations of general relativity. The exact non-linear dynamic equations for a generic irrotational dust spacetime are consistent. To analyse conditions characterizing pure gravity waves, linearization instability in general relativity and consistency of the so-called "silent universes", further exact conditions are imposed locally on irrotational dust. These are classified into Class II conditions, which change evolution equations into constraint equations, and Class I and III conditions, which do not do so—rather they add a new constraint, leaving the propagation equations unchanged in form. Class I conditions are imposed on terms in the constraint equations, while Class II and III conditions are imposed on terms in the evolution equations.

In the Class I case it is shown that for irrotational dust spacetimes the divergence–free magnetic Weyl tensor and the divergence–free electric Weyl tensor (necessary conditions for gravity waves interacting with matter), both imply integrability conditions in the exact non-linear case. The integrability conditions for the divergence-free magnetic Weyl tensor are identically satisfied in the linearized perturbation case, but are non-trivial in the exact non-linear case. This leads to a linearization instability in these models. The integrability conditions for the divergence-free electric Weyl tensor are non-trivial in both the linear and non-linear cases.

The Class II case focuses on irrotational silent cosmological dust models characterized by vanishing magnetic Weyl tensor and vanishing electric Weyl tensor. In both these models there exist a series of integrability conditions that need to be satisfied. Integrability conditions for the zero magnetic Weyl tensor condition hold identically for linearized case, but are non-trivial in the exact non-linear case. Thus there is also a linearization instability. The zero electric Weyl tensor condition leads to a chain of non-trivial integrability conditions in both the linear and non-linear cases. Because of the complexity of the integrability conditions, it is highly unlikely that there is a large class of models in both the silent zero magnetic Weyl tensor case and the silent zero electric Weyl tensor case.

The Class III cases are of interest in relation to gravitational wave properties of solutions, for example in the exact non-linear theory the absence of gravity waves is associated with curl-free electric and magnetic Weyl tensors, but determining their consistency is more difficult than in the Class I and II. This still awaits completion.

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• Finally I would like to thank the cosmology group at the University of Cape Town for hospitality during the preparation of this work and the staff in the department of Applied Mathematics at the University of Fort Hare for support.

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1. **Therefore the parties agree to the following:**

1.1. The redevelopment of District Six will be strongly rooted on the aspirations of those who were removed.

1.2. It is recognised that there is a strong desire amongst claimants, as defined by the Act, to return to District Six.

1.3. In the interests of nation-building and reconciliation, it is imperative that non-negotiable fundamentals support the redevelopment process. These fundamentals include:

- the co-operation of primary parties;
- integrated and sustainable development, within District Six and the broader City of Cape Town;
- equity, transparency, inclusivity and representivity;
- cultural and religious tolerance and diversity, and gender sensitivity;
- affordability and economic viability ; and
- that the process be grounded in the needs of the dispossessed community, and the broader disadvantaged community.

1.4. The three parties shall agree to a process whereby an appropriate institutional framework and model for redevelopment will be identified. The redevelopment vehicle will be founded on the broader vision and fundamentals, as outlined above. Its roles shall include:

- managing a participative and consultative redevelopment process;
- developing a framework and creating mechanisms for planning and redevelopment, within the broader legal and statutory provisions;
- identifying resources, including financial, technical, architectural, environmental, human and community;
- representing stake-holders in the redevelopment process;
- monitoring and evaluating redevelopment and implementation.

1.5. The three primary parties will identify further role players which can contribute to the process.

1.6. The three parties will, in conjunction with other identified role players identify resources which could be committed to the planning and redevelopment of District Six.

1.7. Parties will jointly identify mechanisms and principles for a public land holding agreement.

2. **Terms and Conditions**

In order to facilitate and advance the redevelopment process an interim task team of the primary parties will be initiated, until a redevelopment vehicle is properly constituted. The following terms will serve as guidelines:

2.1. **Meetings**: The parties will meet at least every two weeks.

2.2. **Chairperson**: The venue and chair will rotate from within the three parties.

2.3. The interim task team will consist of designated members of the three parties. Parties shall also nominate alternates. The designated members are as follows:

- Trust: Anwah Nagia
- DLA: Terence Fife
- City: Nomaindia Mfeketo

2.4. **Logistical Co-ordination**: The Department shall fulfill logistical and secretarial duties, where appropriate. Duties shall include:

- the preparation of an agenda three days prior to the meeting;
- finalising minutes within three days of the meeting;
- ensuring the flow of information between members;
- dealing with correspondence and communication.

2.5. **Minutes**: The co-ordinator will keep a record of minutes and decisions. These will be circulated within three days of each meeting. Any changes to the draft minutes should be forwarded to the co-ordinator, within one day, in order for the minutes to be adopted at the next meeting.

2.6. **Proceedings at meetings**: Meetings will be conducted on the basis of trust, transparency, participation and respect. It is recognised that attendance and participation is voluntary. All discussions and decisions made during the process is privileged and without prejudice. Parties shall not disclose any information which came to their attention on a confidential basis.

2.7. **Decision-making**:

2.7.1. Decisions will be by consensus. Where this is not possible and , after thorough debate and discussion, a dispute arises, an independent party / ties acceptable to all, will be requested to assist.

2.7.2. Parties shall contain their differences to the structures which they have created for themselves. Only after pursuing all attempts to resolve their differences internally have failed, may they resort to pursuing respective options.

2.8. **Media**: All interaction with the media, in terms of this agreement, will be by consensus and in written form wherever possible.

2.9. **Support**: All parties will be requested to identify the resources, knowledge and skills which they can contribute to the process. All parties will agree to provide appropriate support to the claimant community wherever possible.

2.10. Parties will commit themselves to recognising **time frames.**

2.11. Each representative shall forward and represent the views of his/her organisation through regular report-back.

2.12. All parties to this record shall execute the necessary skill, care and diligence in the performance of their duties, in terms of this agreement.

2.13. Provided there is consensus between all parties to this agreement, clauses and time frames may be changed where absolutely necessary.

CHAPTER 1

Introduction

The dynamics of a system of particles having negligible non-gravitational interactions have been widely studied. Such a system may be modeled by a self-gravitating *collisionless* fluid often referred to as *dust* and presents a good approximation when studying the evolution of the late universe and the late stages of gravitational collapse.

The main current research problem seeks to explain the *formation of the large-scale structure* (galaxies, clusters, superclusters and voids) in the observed universe. The extreme degree of isotropy of the Planckian cosmic microwave background, and other observational and theoretical results, suggest that the overall structure of the observable universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) models of general relativity (Peebles *et al.* (1991) [78]). Perturbation theories assume that the observed large-scale structure of the universe evolved from initially small density perturbations in an FLRW background through the action of gravity. Such a process is called *gravitational instability.*

On the whole the general relativistic approach has been either a study of linear perturbations on the large scale or a study of symmetric solutions of Einstein's equations. Studies that focus on the non-linear case, with no symmetries present, are usually treated within the Newtonian theory. It is well-known that results of Newtonian theory are valid for regions with length scales small compared to Hubble's radius and large compared to the typical Schwarzschild radii of collapsing bodies (Peebles (1980) [77]).

In this chapter we first give a brief review of well-known literature on linear perturbation studies. Our focus, in the rest of the thesis, is on non-linear studies; particularly those first developed from a Newtonian point of view and later generalized to Einstein's general relativity theory. To maintain relevance to this thesis and to avoid a lengthy chapter we review only material that pertains to dust spacetimes.

The second part of this chapter gives an overview of the research undertaken in this thesis with some emphasis on its relevance to current research on non-linear irrotational dust spacetimes.

1.1 Linear perturbation studies

Various analytical approaches have been used to study the evolution of small-amplitude disturbances of an FLRW background. The most notable and well-known work on the *linear perturbation problem* on the large-scale includes:

(a) The *metric perturbation* approach, originally by Lifshitz (1946) [60], studies the growth of perturbations in relation to the problem of galaxy formation. In this approach physical results can be worked out only after a definite *gauge choice* (i.e., a correspondence between the real perturbed observable universe and the background FLRW spacetime) has been made. If this gauge freedom is not fully determined it leads to unphysical gauge modes in the evolution of the perturbation (Press and Vishniac (1980) [79]). Lifshitz and Khalatnikov (1963) [61] pointed out that the arbitrariness in the gauge choice is inherent from gauge-invariance of Einstein's linearized equations with respect to gauge transformations.

(b) The *covariant fluid flow* approach by Hawking (1966) [43] and later Olson (1976) [75] and Lyth *et al.* (1988, 1990) [62, 63], following the work by Ehlers (1961) [31], is based on curvature variables rather than the metric. The dynamics of the system is followed directly in terms of observable fluid variables (such as density, shear, vorticity, etc.) and other tensor quantities which describe the spacetime curvature (instead of the metric tensor), but the density perturbation is gauge-dependent. A comprehensive review of the covariant fluid flow approach is given by Ellis (1971, 1973) [34, 35].

(c) The *gauge-invariant* approach pioneered by Bardeen (1980) [2] following earlier work by Gerlach and Sengupta (1978) [40]. This is a linear cosmological theory that relates physical observables to quantities which are invariant under changes of the map between the physical (perturbed) and the background (unperturbed) spacetime. Bardeen variables do not have a simple geometric meaning, as they are defined with respect to a coordinate chart (Stewart (1990) [88]). This is a consequence of the approach to perturbations where any tensorial quantity T is split into a background part T_0 and a perturbation $\delta T : T = T_0 + \delta T$ in a particular family of coordinates and then made invariant under gauge transformations. The perturbation δT is a propagating field in the background metric and the splitting is initially defined in a given coordinate system.

(d) More recently, a fully gauge-invariant formulation has been presented by Ellis and Bruni (1989) [36] based on the lemma of Stewart and Walker (1974) [89]:

> *Perturbations to a background quantity, say T_0, is gauge-invariant if and only if T_0 either (a) is a constant scalar or (b) vanishes or (c) is a linear combination of products of Kronecker deltas with constant coefficients.*

The Ellis-Bruni approach is covariant and gauge-invariant with respect to an FLRW background. Some applications of the Ellis-Bruni theory have focused on the study of density fluctuations in perfect fluids [36], background radiation anisotropies and density inhomogeneities in imperfect fluids (Dunsby (1993) [26]).

1.2 Non-linear studies: Zeldovich approximation

In the real universe during gravitational collapse, a non-linear regime is reached where linear methods are inadequate. Non-linear evolution models using symmetries on dust spacetimes include:

(a) The Lemaître-Tolman-Bondi *spherically symmetric* model (Lemaître (1933) [55], Tolman (1934) [93]) and Bondi (1947) [8]) and more recent work by, *inter alia*, Bonnor (1985, 1987) [11, 12, 13]. Also the evolution of spherically symmetric inhomogeneities studied by Hendriksen and De Robertis (1980) [45].

(b) The *locally rotationally symmetric* dust models by Ellis (1967) [33]. This paper also treats the shear-free dust model.

In Newtonian theory the most influential method of examining non-linear structure formation and gravitational instability is *via* the Zeldovich approximation (see Zeldovich (1970) [106, 108, 107], and the review by Shandarin and Zeldovich (1989) [85]). The Zeldovich approximation is "an approximate non-linear theory" based on assumptions extrapolating the linear theory of gravitational instability.

The Newtonian-Zeldovich approximation is useful in interpreting results from red-shift surveys of galaxies. Buchert (1992) [19] presented a first order Lagrangian perturbative approximation based on Newtonian theory. This approach contained the Zeldovich approximation in a subclass and was extended to second order (Buchert and Ehlers (1993) [22]) and also to third order (Buchert (1994) [21]), giving some new information about self-gravitating systems. These and most other treatments (e.g., the N-Body codes) are Newtonian approaches and are thus valid only for perturbations on scales smaller than the horizon size (*i.e.*, the Hubble radius).

On super horizon scales, instead, one needs a relativistic approach. The pioneer was Lifshitz (1946) [60] with his linearized theory based on general relativity; which was extended to second order by Tomita (1967, 1971, 1972) [94, 95, 96]. In recent years there have been various attempts to improve upon the Zeldovich approximation to cover super horizon scales. The models used are the pressure-free, non-rotating solutions with vanishing magnetic tidal (Weyl) tensor. Inhomogeneous solutions with a purely electric Petrov type D Weyl tensor were found by Szekeres (1975) [91]. Parry, Salopek and Stewart (1994) [76, 81] presented a non-linear solution of the Hamilton-Jacobi equation for general relativity, using the spatial gradient expansion method developed in Salopek and Stewart (1992) [81] and reproduced the Zeldovich approximation. A systematic derivation of the general relativistic Zeldovich approximation is discussed in detail by Croudace *et al.* (1994) [24] and Salopek *et al.* (1994) [82]. They showed that the non-linear evolution of dust along world-lines contains the Szekeres line element as an exact but apparently unstable solution describing pancake collapse. Other attempts at improving the Zeldovich approximation include:

1. The extension to the adhesion approximation method (Kofman, Pogosyan and Shandarin (1990) [54], Gurbatov *et al.* (1989) [42]). The adhesion approximation is based on existing exact solutions of the generalized Burgers' equation for appropriately scaled peculiar-velocity. The main drawback in this approach is that the adhesion approximation is not based on a description of self-gravity and the momentum in this model is not conserved.

2. The truncated Zeldovich approximation by Coles, Mellot and Shandarin (1993) [23]. They compared N-body simulations for power law initial spectra of :

   (a) the linear and the chopped linear extrapolations of the initial perturbations;

   (b) the linear extrapolation with truncated spectrum on the non-linearity scale;

   (c) the lognormal model and the lognormal model with truncated spectrum,

   with the Zeldovich's approximation for initial perturbations and the Zeldovich's approximation with truncated spectrum; and found that the latter always yields the best results.

3. The higher order Lagrangian perturbation theory (Mellot, Buchert and Weiss (1995) [73]) follows from the Newtonian approach of Buchert (1992) [19].

4. Of relevance to this thesis is the Lagrangian relativistic approximation by Matarrese, Pantano and Saez (1994) [72, 71] based on fluid flow equations. They showed that on super horizon scales the condition of a vanishing magnetic Weyl tensor leads to the preferential collapse of fluid elements to spindle (filamentary) collapse in general, independent of the environmental conditions. A similar conclusion was reached by Bertschinger and Jain (1994) [7] and, Hui and Bertschinger (1995) [48], using the Lagrangian evolution approach for velocity and gravity gradients. (Pancake collapse on the other hand is the generic outcome of gravitational collapse of cold dust in the Newtonian Zeldovich approximation). Closely related to this is the tetrad-based Zeldovich approximation by Kasai (1992, 1993, 1995) [49, 50, 51] and Russ *et al.* (1996) [80].

5. Following this Kofman and Pogosyan (1995) [53] working on the Newtonian limit of the equations of general relativity showed that non-zero magnetic Weyl tensor arises as a post-Newtonian effect related to non-local gravitational interactions and more recently Ehlers and Buchert (1996) [32] argued that the magnetic Weyl tensor vanishes in the Newtonian limit.

1.3 Silent Universes

An important concept that emerges in the study of gravity waves is that of the so called 'silent' universe. We define *silent universes* to be spacetimes which are such that fluid elements evolves independently of each other so that once the initial conditions are satisfied, no further information is exchanged between the fluid elements (information on the initial matter distribution is exchanged when satisfying the initial conditions; once this has happened, this information is conserved and hence unchanging). This definition is satisfied if the covariant evolution equations are *ordinary differential equations.* So now if *all* the constraint equations are satisfied, fluid elements evolve silently until the formation of caustics when the one-to-one mapping between fluid elements and space points is lost. For irrotational dust spacetimes the silent criterion is equivalent to vanishing spatial rotation (*curl*) of the electric Weyl tensor E_{ab} and magnetic Weyl tensor H_{ab}.

The concept of silent models was first introduced by Matarrese *et al.* (1994) [72], in the study of irrotational dust spacetimes with a purely electric Weyl tensor; $E_{ab} \neq 0$ and $H_{ab} = 0$. More appropriately such models are called *purely electric* universes or *Newtonian-like* universes. (The latter is motivated by the role of the electric Weyl tensor E_{ab} as the relativistic generalization of the tidal tensor in Newtonian theory.)

There are other silent models, for example the *purely magnetic* universes. These are models with $H_{ab} \neq 0$ and $E_{ab} = 0$ and also satisfy the silent criterion (namely, covariant equations are ordinary differential equations). However since the magnetic Weyl tensor has no Newtonian analogue these spacetime are also referred to as *Anti-Newtonian* universes.

From a physical point of view a silent universe is one which does not contain gravity waves or sound waves. Such waves are the main source of interaction between neighbouring world-lines during evolution (and also the possibility of gravitational induction, see *e.g.* Bondi (1964) [9]).

The concept of silent universe depends on the assumption that the constraint equations are integrable and compatible with the evolution equations (Bertschinger and Hamilton (1994) [6]). This assumption forms part of the subject of this thesis.

1.4 Overview of thesis

1.4.1 Where it fits in

The definition and identification of silent universe models, as introduced in the previous section, can be fully achieved if accompanied by a consistency analysis. This involves determining conditions that render the dynamic equations integrable in the light of the constraints. It is however instructive to investigate integrability conditions from a slightly broader perspective. For this purpose we study integrability non-linear theory tends to shift from this analog particularly with reference to conditions leading to gravity waves. This then leads to an interesting question:

What are the conditions that define the gravity wave case in the exact theory?

The form of the wave equation requires that the curls of the electric Weyl tensor and the magnetic Weyl tensor must not vanish simultaneously if there are non-trivial waves present. To focus specifically on this property we set the divergence of both the electric E_{ab} and the magnetic H_{ab} Weyl tensors to zero. The set of dynamic equation introduces non-trivial integrability conditions. The Bianchi type II models satisfy the set criteria and have solutions that may be said to define "standing waves". The dual question to the one above is the following:

How does one fully 'switch-off' gravity waves?

In the linear theory this is clearly achieved by setting the magnetic part of the Weyl tensor to zero. Our analysis of the purely electric silent universes in the exact theory, is unfortunately not entirely conclusive. A detailed look at the equations shows that for spacetimes with an electric Weyl tensor of Petrov type O and type D gravity waves are not present and those spacetimes do evolve silently. However, because of the complexity of the integrability equations obtained, which appear to have no type I solutions, we conjecture that for irrotational dust, there exists no spatially inhomogeneous Petrov type I silent models. Of significance is that in the *linearized* theory the integrability conditions for the inhomogeneous Petrov type I case are trivially satisfied and the equations evolve consistently. This implies that there is linearization instability in the inhomogeneous Petrov type I silent models.

We conclude by showing that the purely magnetic silent universes in fact run into more restrictive integrability conditions. The integrability conditions form non-terminating chains that lead to inconsistencies in general. In particular in the linearized theory, the integrability conditions themselves lead to the vanishing of anisotropy and inhomogeneity, *i.e.*, to the FLRW case. There may be spatially homogeneous models of this type which satisfy the integrability conditions, but we have been unable to find examples.

1.4.2 Broad objectives

Irrotational dust spacetimes are an ideal arena for testing the connection of the Weyl tensor to gravitational waves. The choice of irrotational dust spacetime models as a test-bed, stems from the simplicity of the dynamic equations. In the cosmological context these models give a good description of the universe during the matter dominated era.

A convenient and useful property of irrotational dust spacetimes is that the equations for field variables can be clearly separated into evolution equations and constraint equations. The latter are equations which do not have any derivatives in the direction of the fluid four-velocity. The standard approach in studying such equations consists of the following:

1. Showing that the constraint equations are either consistent, *i.e.*, they are integrable as they stand (preserved in time), or that there are implied *integrability conditions*. Furthermore if there are implied integrability conditions the next step is to show that these are also consistent. To complete the full consistency analysis we include *spatial consistency* checks within the dynamic equations. *Spatial* differential operation (such as *grad*, *div* and *curl*) of one equation may lead to common spatial derivative terms appearing in more than one equation. Thus a check for spatial consistency then involves comparing such equations to determine whether these lead to new or existing equations.

2. Establishing the existence of cosmologically viable initial conditions satisfying the constraints and any other additional integrability conditions resulting from 1. The choice of initial conditions is guided by the algorithm employed

(Matarrese *et al.* (1993) [70]).

3. Finally developing algorithms to solve the evolution equations and as such obtaining the behaviour of the solutions.

Thus on the whole we set out to accomplish the following goals: First show that irrotational dust spacetimes are consistent generally (this follows the work by Maartens (1996) [65]). We then consider restrictions on the Weyl tensor terms appearing in the constraints, such as:

(a) divergence-free magnetic Weyl tensor,

(b) divergence-free electric Weyl tensor,

(c) and both (a) and (b),

that do not affect the evolution equations. We classify such assumptions as class I conditions. One only has to check the consistency of the new constraint; the already proved consistency of the rest remains unaffected. On the other hand shear and Weyl tensor restrictions, such as:

(d) zero shear tensor,

(e) zero magnetic Weyl tensor,

(f) zero electric Weyl tensor,

affect both the evolution and the constraint equations, indeed they convert an evolution equation into a new constraint. We classify such assumptions as class II conditions. We will show that the overall consistency for the generic case is *likely* to be disturbed for class II conditions. We use the word *'likely'* cautiously here as the zero shear case **(d)** is consistent here due to the condition of zero vorticity imposed on the dust matter [33] however, in the remaining two cases **(e)** and **(f)** there are extra constraints.

Conditions that affect the propagation equations but do not alter the nature of the system of dynamic equations are classified as Class III condition. Examples of such conditions are

(g) Curl-free magnetic Weyl tensor and

(h) Curl-free electric Weyl tensor.

The Class III conditions are not fully examined in this thesis but we will investigate the existence of spatially homogeneous models satisfying such conditions.

In this thesis we concern ourselves mainly with point 1. and where relevant hint on point 2. Point 3. will be referred to existing literature and is not within the scope of this research.

1.4.3 Layout and structure

The chapters in the thesis are organised as follows: In the preliminary chapter 2, we review the covariant fluid flow approach to cosmology as presented in papers by Ehlers (1961) [31], Hawking (1966) [43] and Ellis (1971, 1973) [34, 35]. This aims to set up the basic covariant notation and formulation needed in the later chapters.

In chapter 3 we discuss generic irrotational dust. This chapter reviews the paper by Maartens (1996) [65] on integrability conditions for generic irrotational dust. This work employs covariant tensorial identities which are introduced and greatly benefit the results in the later chapters.

Chapter 4 is devoted to conditions that are associated with the principal axes of the shear tensor. These conditions turn out to be among the Weyl tensor restrictions listed and classified above. Chapter 5 is on the Class I conditions and chapter 6 the Class II conditions. In chapter 7 we investigate Bianchi models that satisfy the various Weyl tensor conditions listed as **(a)**–**(h)** in the above section. Finally chapter 8 consolidates the definitions and results of this thesis in tabular form and formulates questions still open for further inquiry.

We conclude with comments on the originality of this thesis. The results of chapter 3 are based on material from Maartens (1996) [65]. The tetrad theorems in chapter 4 are reformulated from the material in Lesame, Ellis and Dunsby (1996) [59] and Lesame (1995) [56].

The class I results of chapter 5 appear in Maartens, Lesame and Ellis (1997) [66] (see section 5.4.) and some of the spacetime models used as examples are obtained from Maartens (1997) [65, 64]. The new and as yet unpublished results are in section 5.5 and section 5.6.

The first part of chapter 6 is based on Lesame, Dunsby and Ellis (1996). The material of section 6.4 leading to the conjecture in section 6.4.3 overlaps with the work by van Elst (1996) [98] and is contained in the collaborative paper by van Elst, Uggla, Lesame, Ellis and Maartens (1996) [102]. The approach used in van Elst *et al.* (1996) [102] is slightly outside the scope of this thesis and the results are reported briefly. Section 6.5 is a new result [67] and is based on the covariant identities in [64]. The Bianchi examples in chapter 7 are new and not yet published.

Thus chapter 5, chapter 6 and chapter 7 report original work, some carried out in collaboration with workers named here and some based on the derivation of the tetrad theorems reported in chapter 4. It should be noted that this thesis also corrects incorrect results previously published, namely Lesame, Ellis and Dunsby (1996) [59] erroneous because of an incorrect sign in an unnumbered equation above section 4.3 in Ellis (1971) [34] (the correct equation is given here in chapter 2 by equation (2.20)), and Lesame, Dunsby and Ellis (1996) [58] which contains a logical error, as pointed out in Bonilla *et al.* (1996) [10] and corrected in [102].

1.5 Notation and conventions

Throughout the thesis we use:

Units: $c = 8\pi G = 1$

Signature is $(-,+,+,+)$.

Latin indices assume values 0,1,2,3.

Greek indices assume values 1,2,3.

Round brackets enclosing indices denote symmetrization and square brackets enclosing indices denote anti–symmetrization.

CHAPTER 2

Covariant Approach to Cosmology

2.1 Introduction

In this chapter we will give a brief review of the covariant fluid flow approach to cosmology. This approach was systematized by Ehlers (1961) [31] and pursued further by Hawking (1966) [43] and Olson (1976) [75]. The lecture notes by Ellis (1971, 1973) [34, 35] give a complete and classic review of the covariant approach.

The purpose of this chapter is to introduce the basic formalism and notation used throughout this thesis. We aim to establish fundamental equations in a form that would be useful for later chapters.

2.2 Local physics

On the large scale the dominant force in the universe is gravity. The laws of physics which govern gravity on earth and the solar system were discovered by Newton and published in the celebrated Newton's *Principia* (1687) [74]. In the special theory of relativity, and later the general theory, Einstein introduced physical laws that cover a wider variety of applications. One such application is *relativistic cosmology* which aims to model the large scale structure of the physical universe.

In general relativity the metric tensor g_{ab} defines the line element

$$ds^2 = g_{ab}dx^a dx^b \ , \tag{2.1}$$

and obeys *Einstein's field equations*

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab} \ , \tag{2.2}$$

where x^a are general coordinates, the Ricci tensor R_{ab} is the contraction of the Riemann tensor: $R_{ab} \equiv R^c{}_{acb}$; the Ricci scalar R is the contraction of the Ricci tensor: $R \equiv R^a{}_a = R^{ca}{}_{ca}$ and Λ is the cosmological constant. The *Riemann* (curvature) tensor R_{abcd} represents the non-commutativity of the second covariant derivatives and thus for any vector field X^a:

$$X^a{}_{;dc} - X^a{}_{;cd} = R^a{}_{bcd} X^b \ . \tag{2.3}$$

Equations (2.3) are referred to as the Ricci identities. The tensor T_{ab} is the *energy-momentum* tensor of all the matter fields present. To make the field equations determinate further *auxiliary equations* determining the behaviour of the matter field must be specified.

The Riemann tensor satisfies the *Bianchi identities*

$$R_{ab[cd;e]} = 0 \Leftrightarrow R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0 \ . \tag{2.4}$$

Contracting these equations gives

$$R^a{}_{bcd;a} = R_{bd;c} - R_{bc;d} \ , \tag{2.5}$$

and contracting again yields

$$R^a{}_{c;a} = \tfrac{1}{2} R_{;c} \ , \tag{2.6}$$

from which it follows that the field equations (2.2) can be consistent only if the matter fields obey the conservation equations

$$T^{ab}{}_{;b} = 0 \ . \tag{2.7}$$

In fact (2.7) are the equations of conservation of energy and momentum.

2.2.1 The average velocity

The covariant approach depends on being able to find a unique four-velocity u^a. Assuming that the matter content of the universe can be described by a continuous fluid then at each point in spacetime there exists an average four-velocity u^a, which will be normalized:

$$u^a = \left.\frac{dx^a}{ds}\right|_{y^\nu=const} , \quad x^a = (y^\nu, s) , \quad u_a u^a = -1 , \tag{2.8}$$

where y^ν are comoving coordinates and s measures proper time along the fluid flow lines. In multi-fluid or imperfect fluid cases there are several competing definitions (from the average matter motion or eigenvector of the energy-momentum tensor) but each should determine a unique four-vector. Thus we have to choose between these definitions to determine a unique vector u^a at each point.

2.2.2 The projection tensor

A splitting of spacetime into {space + time} is determined at each point by u^a. So now at each point p there exists a subspace H_p of the tangent vector space T_p, which is orthogonal to u^a. The projection tensor h_{ab} is defined by

$$h_{ab} \equiv g_{ab} + u_a u_{ab} . \tag{2.9}$$

Effectively the tensor h_{ab} projects the tangent vector space T_p at each point onto the 3-dimensional subspace H_p orthogonal to u^a:

$$h^a{}_a = 3 , \quad h^a{}_b h^b{}_c = h^a{}_c , \quad h_{ab} u^b = 0 . \tag{2.10}$$

The 3-planes defined in each tangent space by h_{ab} do not in general mesh together to form 3-surfaces in spacetime. The condition that they do so is given in section 2.3.3. In terms of h_{ab} and u^a the spacetime metric (2.1) takes the form

$$ds^2 = g_{ab} dx^a dx^b = h_{ab} dx^a dx^b - (u_a dx^a)^2 . \tag{2.11}$$

The existence of a (*unique*) four-velocity at each point means that any tensor field can be split (*uniquely*) into time and space components. The following examples will be useful in the sequel.

The Energy-Momentum tensor

The energy-momentum tensor T_{ab} can be split with respect to a four-velocity field u_a and a projection tensor h_{ab}:

$$T_{ab} = \rho u_a u_b + p h_{ab} + q_a u_b + u_a q_b + \pi_{ab} \ , \tag{2.12}$$

where ρ is the total (relativistic) energy density of matter measured by an observer moving with four-velocity u^a; q_a is the energy flux relative to u^a:

$$q_a = -h_a{}^b T_{bc} u^c \ , \quad q_a u^a = 0 \ , \tag{2.13}$$

and represents processes such as diffusion and heat conduction; p is the isotropic pressure and finally π_{ab} is the anisotropic matter pressure (due to processes such as viscosity):

$$\pi_{ab} = h_a{}^c h_b{}^d T_{cd} - \tfrac{1}{3} h_{ab}(h_{cd} T^{cd}) \ , \quad \pi_{ab} u^b = 0 \ . \tag{2.14}$$

The scalars ρ, p, the vector q_a and the tensor π_{ab} represent *energy-momentum quantities* and are determined by the matter content.

The Weyl tensor

The *Weyl tensor* (or the conformal curvature tensor) C_{abcd} is defined as the trace-free part of the Riemann tensor by:

$$C^{ab}{}_{cd} \ = \ R^{ab}{}_{cd} - 2g^{[a}{}_{[c} R^{b]}{}_{d]} + \tfrac{R}{3} g^{[a}{}_{[c} g^{b]}{}_{d]} \ . \tag{2.15}$$

This definition implies that the Weyl tensor satisfies all the Riemann tensor symmetries:

$$\begin{aligned} R_{abcd} &= R_{[ab][cd]} \ , & C_{abcd} &= C_{[ab][cd]} \ , \\ R_{abcd} &= R_{cdab} \ , & C_{abcd} &= C_{cdab} \ , \\ R_{a[bcd]} &= 0 \ , & C_{a[bcd]} &= 0 \ , \end{aligned} \tag{2.16}$$

and in addition it is trace-free:

$$C^a{}_{bad} = 0 \ . \tag{2.17}$$

The Weyl tensor is split by a velocity field u^a into an "electric" part E_{ab} and a "magnetic" part H_{ab}:

$$E_{ab} = C_{agbh}u^g u^h \ , \quad H_{ab} = \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d \ . \tag{2.18}$$

We shall hereafter refer to E_{ab} as the electric Weyl tensor and H_{ab} as the magnetic Weyl tensor. Because of the symmetries in (2.16), (2.17) the tensors E_{ab} and H_{ab} are symmetric and trace-free in the rest spaces of u^a:

$$\begin{aligned} E_{ab} &= E_{(ab)} \ , \quad E^a{}_a = 0 \ , \quad E_{ab}u^b = 0 \ , \\ H_{ab} &= H_{(ab)} \ , \quad H^a{}_a = 0 \ , \quad H_{ab}u^b = 0 \ . \end{aligned} \tag{2.19}$$

The two tensors in turn completely determine the Weyl tensor as follows:

$$C_{abcd} = (g_{abij}g_{cdkl} - \eta_{abij}\eta_{cdkl})u^i u^k E^{jl} - (\eta_{abij}g_{cdkl} + g_{abij}\eta_{cdkl})u^i u^k H^{jl} \ , \tag{2.20}$$

where η_{abcd} is the total anti-symmetric tensor defined in the next section and g_{abcd} is defined by

$$g_{abcd} := g_{ac}g_{bd} - g_{ad}g_{bc} \ . \tag{2.21}$$

Equation (2.20) corrects a sign error in an unnumbered equation above section 4.3 in Ellis (1971) [34]. The tensors E_{ab} and H_{ab} are *curvature quantities.* Physically the meaning of E_{ab} is based on its Newtonian analog as follows: The quantity that may be regarded as analogous to the Riemann tensor in Newtonian theory is the second derivative $\Phi_{,\mu,\nu}$ of the gravitational potential Φ. This can be separated algebraically into its trace $\Phi^{,\nu}{}_{,\nu}$ and trace-free part $E_{\nu\mu}$:

$$E_{\nu\mu} := \Phi_{,\nu,\mu} - \tfrac{1}{3}h_{\nu\mu}\Phi^{,\tau}{}_{,\tau} \ . \tag{2.22}$$

This suggests that in general relativity the electric Weyl tensor E_{ab} is that part of the gravitational field which describes tidal interactions. On the other hand the magnetic Weyl tensor H_{ab} does not seem to have any direct analog in Newtonian theory. Bertschinger and Hamilton (1994) [6] proposed a Newtonian analog for magnetic Weyl tensor H_{ab} however this may be regarded as a general relativistic generalization of Newtonian theory (Ellis and Dunsby (1996) [37]). The physical relation of H_{ab} to the concepts of gravo-magnetic effects and pure gravitational waves is the subject of current research (see [32], [53]).

2.2.3 Volume elements

Volume elements in the instantaneous rest spaces of co-moving observers are given by the projected permutation tensor

$$\varepsilon_{abc} = \eta_{abcd}\, u^d \ , \qquad \epsilon_{123} = 1 = \epsilon^{123} \ , \tag{2.23}$$

where the total anti-symmetric (permutation) tensor η_{abcd} is defined by

$$\eta^{abcd} = \eta^{[abcd]} \ , \quad \eta^{0123} = (-g)^{-\frac{1}{2}} \ , \quad g := det(g_{ab}) \ . \tag{2.24}$$

This tensor is preserved under parallel transport:

$$\eta^{abcd}{}_{;e} = 0 \ , \tag{2.25}$$

and satisfies the identities:

$$\eta^{abcd}\eta_{efgh} = -4!\delta^{[a}_{e}\delta^{b}_{f}\delta^{c}_{g}\delta^{d]}_{h} \ . \tag{2.26}$$

2.2.4 Derivative operators

In this thesis we undertake a study of integrability conditions in which most calculations involve taking time derivatives of spatial constraints. The propagation of any tensor $S^{p...q}{}_{r...s}$ along the fluid flow is defined as the covariant derivative of this tensor along u^a:

$$\dot{S}^{p...q}{}_{r...s} \equiv S^{p...q}{}_{r...s\,;a} u^a \ . \tag{2.27}$$

Using the projection tensor $h_a{}^b$ we also write the projected covariant derivative as

$$D_a S^{p...q}{}_{r...s} \equiv h_a{}^b h^p{}_c \cdots h^q{}_d\, h_r{}^e \cdots h_s{}^f\, S^{c...d}{}_{e...f;b} \ . \tag{2.28}$$

In later chapters we will use the following notation: the covariant curl[1] of a tensor S_{ab} is written as

$$\mathrm{curl}\, S_{ab} := \varepsilon_{cd(a} D^d S_{b)}{}^c \,, \tag{2.29}$$

the projected, symmetric and trace-free part of S_{ab} is

$$S_{<ab>} = h_a{}^c h_b{}^d S_{(cd)} - \tfrac{1}{3} h_{ab} S_{cd} S^{cd} \,, \tag{2.30}$$

and the covariant divergence of S_{ab} is

$$(\mathrm{div}\, S)_a = D^b S_{ab} \,. \tag{2.31}$$

Further useful covariant properties and identities (derived by Maartens (1996) [64]) based on this notation are listed in Appendix A. 1.

2.3 Kinematic quantities

The various kinematic quantities are introduced by decomposing the first covariant derivative $u_{a;b}$ of the four-velocity u^a with respect to u_a and $h_a{}^b$ as follows:

$$\begin{aligned} u_{a;b} &= g_a{}^c (g_b{}^d + u_b u^d) u_{c;d} - (u_{a;d} u^d) u_b \\ &= D_b u_a - a_a u_b \,, \end{aligned} \tag{2.32}$$

where the vector a_a:

$$a_a \equiv \dot{u}_a = u_{a;b} u^b \,, \tag{2.33}$$

is called the *acceleration vector* and represents the combined effects of gravitational and inertial forces on the fluid. A fluid is said to be *inertial or geodesic* when $\dot{u}_a = 0$. Moreover the term $D_b u_a$ in (2.32) can be split into the symmetric and anti-symmetric parts:

$$\begin{aligned} D_b u_a &= \Theta_{ab} + \omega_{ab} \,, \\ \Theta_{ab} &= \Theta_{(ab)} \,, \quad \omega_{ab} = \omega_{[ab]} \,, \end{aligned} \tag{2.34}$$

[1] The definition in Maartens (1996) [65] is $\mathrm{curl}\, S_{ab} = \varepsilon_{cd(a} \mathrm{D}^c S_{b)}{}^d$ and differs from the one above by a sign.

where Θ_{ab} is split further into its trace and trace-free parts:

$$\Theta_{ab} = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} \; . \tag{2.35}$$

The scalar Θ and the tensors ω_{ab}, σ_{ab} are called *kinematic quantities* in fluid dynamics and are discussed below.

2.3.1 The volume expansion

The *volume expansion* Θ is the trace of $u_{a;b}$:

$$\Theta \equiv u^a{}_{;a} \; , \tag{2.36}$$

which represents the isotropic expansion of the fluid. The effect of Θ alone is to change a fluid sphere into another fluid sphere, with the same orientation but different volume. The fluid is said to be *volume-true* if $\Theta = 0$.

2.3.2 The shear

The *shear tensor* σ_{ab} is the trace-free symmetric part of the spatial projection of $u_{a;b}$:

$$\begin{aligned} \sigma_{ab} &\equiv h^c{}_a h^d{}_b u_{(c;d)} - \tfrac{1}{3}\Theta h_{ab} \; , \\ \sigma_{ab}u^b &= 0 \; , \quad \sigma^a{}_a = 0 \; . \end{aligned} \tag{2.37}$$

The effect of the shear alone is to cause a distortion, leaving volumes invariant. It distorts a fluid sphere into an ellipsoid of the same volume. The direction of any principal axis of shear is unchanged, but all other directions are changed by the action of the shear. The shear magnitude σ is defined by

$$\begin{aligned} \sigma^2 &= \tfrac{1}{2}\sigma_{ab}\sigma^{ab} \geq 0 \; , \\ \sigma &= 0 \Leftrightarrow \sigma_{ab} = 0 \; . \end{aligned} \tag{2.38}$$

The fluid is said to be *shear-free* if $\sigma = 0$. If the fluid is both volume-true and shear-free, ($\sigma = \Theta = 0$) all the distances to neighboring particles are constant in time and the fluid is said to be *rigid.*

2.3.3 The vorticity

The *vorticity tensor* ω_{ab} is the anti-symmetric part of the spatial projection of $u_{a;b}$:

$$\omega_{ab} \equiv h^c{}_a h^d{}_b u_{[c;d]} \ , \quad \omega_{ab}u^b = 0 \ . \tag{2.39}$$

The effect of vorticity alone is to give a change of orientation without a change of shape or volume. Its action on a fluid sphere is to change it to another fluid sphere of the same volume, but rotated with respect to local inertial axes. The axis of the rotation represented by ω_{ab} defines the *vorticity vector* ω_a by

$$\begin{aligned} \omega^a &\equiv \tfrac{1}{2}\eta^{abcd}u_b\omega_{cd} \Leftrightarrow \omega_{ab} = \eta_{abcd}\omega^c u^d \ , \\ \omega_a u^a &= 0 \ , \quad \omega_{ab}\omega^a = 0 \ . \end{aligned} \tag{2.40}$$

The magnitude of the vorticity is defined by:

$$\begin{aligned} \omega^2 &= \omega^a\omega_a = \tfrac{1}{2}\omega^{ab}\omega_{ab} \geq 0 \ , \\ \omega &= 0 \Leftrightarrow \omega_a = 0 \Leftrightarrow \omega_{ab} = 0 \ . \end{aligned} \tag{2.41}$$

Here also the fluid is said to be vortex-free (*irrotational*) if $\omega = 0$. The case of zero vorticity and its properties are important to our study of irrotational dust spacetimes. We state here that a fluid is vortex-free when there are spacelike hypersurfaces orthogonal to the matter world-lines, that is there exists a non-constant scalar t such that

$$\dot{t}u_a = -t_{,a} \quad (\Leftrightarrow h_a{}^b t_{,b} = 0) \ . \tag{2.42}$$

Thus the condition $\omega = 0$ is precisely the condition that the 3-planes defined in each tangent space by h_{ab} mesh together to form 3-surfaces in spacetime. The scalar t in (2.42) represents a local time coordinate referred to as *cosmic time* and can measure the *proper time* only in the case of inertial flows $\dot{u}_a = 0$. For such fluid flows (*i.e.*, inertial and vortex-free) we can set $\dot{t} = 1$ and it then follows from (2.42) that $u_a = t_{,a}$ and thus t is a solution of the Hamilton-Jacobi equation

$$t_{,a}t^{,a} = -1 \ , \tag{2.43}$$

for free particles in a gravitational field.

2.4 Matter description

2.4.1 Conservation of energy and momentum

We have seen how a timelike vector field u^a determines a decomposition of various tensor fields, including its own first covariant derivative $u_{a;b}$, into spacelike and timelike parts. One can similarly decompose tensor equations with respect to u^a. The conservation equation (2.7) can be split with respect to the four-velocity u^a and the projection tensor h_{ab} to give

$$\dot{\rho} + (\rho + p)\Theta + \pi_{ab}\sigma^{ab} + q^a{}_{;a} + \dot{u}_a q^a = 0 \,, \tag{2.44}$$

$$(\rho + p)\dot{u}_a + h_a{}^c(p_{,c} + \pi_c{}^b{}_{;b} + \dot{q}_c) + (\omega_a{}^b + \sigma_a{}^b + \tfrac{4}{3}\Theta h_a{}^b)q_b = 0 \,. \tag{2.45}$$

When auxiliary equations governing the behaviour of the matter are taken into account, it becomes apparent that (2.44) is the equation of conservation of (thermal) energy, and (2.45) is the equation of conservation of momentum.

2.4.2 Equations of state

The energy-momentum tensor T_{ab} in (2.12) is for generic matter. The physics of the fluid is introduced by specifying further properties of T_{ab}. Such properties are expressed in terms of equations restricting the energy-momentum quantities ρ, p, q_a and π_{ab}. Also one general restriction put on matter is that its energy density be positive. On writing Einstein's equation (2.2) as

$$R_{ab}u^a u^b = T_{ab}u^a u^b + \tfrac{1}{2}Tr(T_{ab}) \,, \tag{2.46}$$

it is assumed that for all physically reasonable classical matter the energy density is non-negative for all observers *i.e.*, $T_{ab}u^a u^b \geq 0$ for all four-velocities u^a. This assumption is known as the weak energy condition and is equivalent to

$$\rho \geq 0 \,; \quad \rho + p \geq 0 \,. \tag{2.47}$$

However, it is also physically reasonable that the stresses on matter will not become large and negative so as to make the right hand side of (2.46) negative *i.e.*,

$T_{ab}u^a u^b \geq -\frac{1}{2}Tr(T_{ab})$ for all observers. This assumption is known as the strong energy condition and is equivalent to (2.47) and

$$\rho + 3p \geq 0 \ . \tag{2.48}$$

The effect of the energy density restriction (2.47) will be seen in the following example of matter description. The strong energy condition (2.48) is violated by inflationary universes when quantum fields dominate the early expansion. A detailed discussion of the above energy conditions appears in Hawking and Ellis (1973) [44] and in Wald (1984) [105].

2.4.3 Perfect fluid

When a fluid has negligible heat conduction and the bulk viscosity vanishes:

$$q_a = \pi_{ab} = 0 \ , \tag{2.49}$$

it is called a *perfect fluid.* In this case the four-velocity vector u^a is *uniquely* defined as the eigenvector of the Ricci tensor R_{ab}. The energy-momentum tensor (2.12) becomes

$$T_{ab} = \rho u_a u_b + p h_{ab} \ , \tag{2.50}$$

and the conservation equations (2.44), (2.45) become

$$\dot{\rho} + (\rho + p)\Theta = 0 \ , \tag{2.51}$$

$$(\rho + p)\dot{u}_a + h_a{}^b p_{,b} = 0 \ . \tag{2.52}$$

The energy condition (2.47) used in (2.52) ensures that the acceleration $\dot{u}^a$ is determined by the projected spatial pressure gradient $h_a{}^b p_{,b}$ and is always away from a high-pressure region towards a neighbouring low-pressure region. For a perfect fluid, the fluid acceleration is only determined by pressure gradients so the restriction of vanishing pressure, $p = 0$, implies zero acceleration, $a^a = 0$. This means that each fluid element moves along a geodesic. A pressure-free perfect fluid is referred to as *dust.*

2.5 Kinematic equations

Kinematic equations can be clearly classified into two different sets of equations. The propagation (also called *evolution*) equations which describe the behaviour of kinematic quantities along the fluid flow and the *constraint* equations giving the spatial variation of the kinematic quantities.

2.5.1 Propagation equations

The propagation equations for the kinematical quantities defined in section 2.3 are obtained from the Ricci identity (2.3) written here in terms of the four-velocity vector u^a as

$$u^a{}_{;dc} - u^a{}_{;cd} = R^a{}_{bcd}u^b \ . \tag{2.53}$$

The equations are derived in Ellis (1971, 1973) [34, 35] and are as follows.

Raychaudhuri's equation

The propagation equation for the expansion scalar Θ obeys the *Raychaudhuri equation*:

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \dot{u}^a{}_{;a} + 2(\sigma^2 + \omega^2) + \tfrac{1}{2}\kappa(\rho + 3p) - \Lambda = 0 \ , \tag{2.54}$$

where $\kappa = 8\pi G$ is the gravitational constant. If we define a representative length scale l by the relation

$$\frac{\dot{l}}{l} = \tfrac{1}{3}\Theta =: H \ , \tag{2.55}$$

where H is the Hubble parameter and l represents completely the volume behaviour of the fluid, which varies as l^3 along the fluid flow lines, then (2.54) can be written in the form

$$\frac{3\ddot{l}}{l} = 2(\omega^2 - \sigma^2) + \dot{u}^a{}_{;a} - \tfrac{1}{2}\kappa(\rho + 3p) + \Lambda \ . \tag{2.56}$$

This shows how the evolution of the scale factor l is determined at each point by the matter density at that point, with the cosmological constant Λ term acting as a constant repulsive force (if and only if $\Lambda \geq 0$; we will assume $\Lambda = 0$ hereafter);

the vorticity ω tends to keep matter apart while the shear σ tends to cause contraction; acceleration also affects the average distance of the world-lines through its divergence; and finally pressure causes convergence directly. The latter is a general relativistic effect, contributing to the problem of gravitational collapse in general relativity. In the homogeneous and isotropic case where $\sigma_{ab} = 0$ and $\Theta = 3H$, equation (2.56) reduces to the equation:

$$3\dot{H} + 3H^2 + \tfrac{1}{2}\kappa\rho = 0 \, . \tag{2.57}$$

The vorticity propagation equation

The propagation equation for the vorticity is:

$$h^a{}_b(l^2\omega^b)^{\cdot} = \sigma^a{}_b(l^2\omega^b) + \tfrac{l^2}{2}\eta^{abcd}u_b\dot{u}_{c;d} \, . \tag{2.58}$$

It can be seen from (2.58) that for dust spacetimes, $\dot{u}_a = 0 = p$, if $\omega_a = 0$ then $\dot{\omega}_a = 0$ for all time.

The shear propagation equation

The propagation equation for the shear is given by:

$$\begin{aligned} h_a{}^f h_b{}^g \dot{\sigma}_{fg} - h_a{}^f h_b{}^g \dot{u}_{(f;g)} - \dot{u}_a\dot{u}_b + \sigma_{af}\sigma^f{}_b + \tfrac{2}{3}\Theta\sigma_{ab} \\ + h_{ab}(-\tfrac{1}{3}\omega^2 - \tfrac{2}{3}\sigma^2 + \tfrac{1}{3}\dot{u}^c{}_{;c}) - \tfrac{1}{2}\pi_{ab} + E_{ab} = 0 \, . \end{aligned} \tag{2.59}$$

Whereas in (2.54) the expansion Θ is affected directly by the matter at each point, in (2.59) matter directly affects the shear *via* π_{ab} and additionally the electric Weyl tensor E_{ab} affects the fluid flow by inducing shear in the flow lines; this shear then enters into the expansion equation (2.54) (tending to converge world-lines).

2.5.2 Constraint equations

The components of the Ricci identity (2.53) which are perpendicular to u^a on the d index do not involve time derivatives of the kinematic quantities and form two sets of constraint equations referred to as *Ricci constraints*.

The momentum-flux constraint

The first Ricci constraint follows by contracting on the indices a and c of the Ricci identity (2.53):

$$h^e{}_b\left(-h^d{}_c\,\omega^{bc}{}_{;d}-h^d{}_c\,\sigma^{bc}{}_{;d}+\tfrac{2}{3}\Theta^{;b}\right)+(\omega^e{}_b+\sigma^e{}_b)\dot{u}^b=q^e\,, \tag{2.60}$$

and is referred to as the momentum-flux constraint.

The H_{ab} constraint

The second Ricci constraint follows by multiplying the Ricci identity by $\eta^{cdef}u_e$ and symmetrizing on the indices a and d:

$$H_{ad}=-h^t{}_a h^s{}_d\left[\omega_{(t}{}^{b;c}+\sigma_{(t}{}^{b;c}\right]\eta_{s)fbc}u^f\,, \tag{2.61}$$

and is referred to as the "H_{ab}" constraint.

2.6 Bianchi identities

Additionally, the Bianchi identities (2.4) must be satisfied, as they are the integrability conditions for the Ricci equations. In terms of the Weyl tensor (2.15) the Bianchi identities (2.5) can be written as:

$$C^{abcd}{}_{;d}=R^{c[a;b]}-\tfrac{1}{6}g^{c[a}R^{;b]}\,. \tag{2.62}$$

We can split this equation with respect to u^a and h_{ab} obtaining equations for the electric Weyl tensor E_{ab} and the magnetic Weyl tensor H_{ab}. Assuming that the matter tensor is that of a perfect fluid these have the following forms (corresponding equations for general matter are given in Ellis (1971) [34]):

The $\dot{E}$ equation

$$\begin{aligned}h^m{}_a h^t{}_c\dot{E}^{ac}+h_a{}^{(m}\eta^{t)rsd}u_r H^a{}_{s;d}-2H_q^{(t}\eta^{m)bpq}u_b\dot{u}_p+h^{mt}\sigma^{ab}E_{ab}+\Theta E^{mt}&\\ -3E_s{}^{(m}\sigma^{t)s}-E_s{}^{(m}\omega^{t)s}=-\tfrac{1}{2}(\rho+p)\sigma^{tm}\,.&\end{aligned} \tag{2.63}$$

The $\dot{H}$ equation

$$h^m{}_a h^t{}_c \dot{H}^{ac} - h_a{}^{(m}\eta^{t)rsd}u_r E^a{}_{s;d} + 2E_q^{(t}\eta^{m)bpq}u_b\dot{u}_p + h^{mt}\sigma^{ab}H_{ab} + \Theta H^{mt}$$
$$-3H_s{}^{(m}\sigma^{t)s} - H_s{}^{(m}\omega^{t)s} = 0\ . \quad (2.64)$$

The Divergence of E equation

$$h^t{}_a h^d{}_s E^{as}{}_{;d} - \eta^{tbpq}u_b\sigma^d{}_q H_{dq} + 3H^t{}_s\omega^s = \tfrac{1}{3}h^t{}_b\rho^{;b}\ . \quad (2.65)$$

The Divergence of H equation

$$h^t_a H^{as}{}_{;d}h^d_s + \eta^{tbpq}u_b\sigma^d{}_p E_{qd} - 3E^t{}_s\omega^s = (\rho+p)\omega^t\ . \quad (2.66)$$

2.7 Analogy to Maxwell's equation

The Bianchi identities (2.63)—(2.66) are analogous to Maxwell's electromagnetic equation [34] in the following manner: The gradient $\frac{1}{3}h_a{}^b\rho_{,b}$ acts as a source of the divergence of the E_{ab} field in the "div E" constraint (2.65). For zero vorticity the H_{ab} field is source-free in the "div H" constraint (2.66). In the case of the propagation equations: curl E in (2.64) acts as a source of $\dot{H}$. On the other hand $\frac{1}{2}\rho\sigma^{tm}$ and curl H in (2.63) act as source terms of $\dot{E}$. It is these two curl terms that make gravitational waves possible (in direct analogy with the electromagnetic situation). One can take the time derivative of the $\dot{E}$ equation and substitute from the $\dot{H}$ equation to get a wave equation for E_{ab} (for details see Hawking (1966) [43] and Dunsby (1995) [27] in the linear case). Moreover we will show in chapter 4 that irrotational dust with a divergence-free magnetic Weyl tensor is consistent and (also in analogy to electromagnetic theory) is a characteristic of the presence of pure gravity waves.

This analog was further extended by Ellis and Hogan (1996) [38, 47] who examined the propagation of arbitrary information by electromagnetic and gravitational waves in spatially homogeneous and isotropic cosmological models. Both the electromagnetic equations and gravitational perturbation equations are shown to be integrable. The information carried by gravitational waves is characterized by divergence-free perturbations. A common result is that solutions of equations not describing gravitational waves correspond to a curl-free magnetic Weyl tensor. However the full extent of the analogy between the Bianchi identities and the electromagnetic equations is the subject of current research.

CHAPTER 3

Generic irrotational dust

3.1 Introduction

In constructing models for natural phenomena most theoretical physicists use, as a rule, some condition in order to reduce the system to a tractable one. The condition normally made is sufficiently reasonable so that the results produced give a good description of the phenomena being studied.

For our purpose here conditions made will be classified into two classes defined in Bruni, Matarrese and Pantano (1994) [15]. The first class are termed *exact conditions* and are special assumptions made to assist in obtaining exact solutions of the theory, for example assumptions on the matter content such as the perfect fluid assumption or the condition of vanishing shear. The second class are *conditions for general data*; and include all those approximate solutions of the equations of the theory which can be derived by making some *ansatz* under which the equations simplify, but still accept generic kinds of boundary conditions, for example asymptotic flatness or approximate homogeneity. All conditions made in this thesis are exact conditions and attempts shall be made to motivate them.

Recently, cosmologists' attention has been drawn to relativistic irrotational dust spacetimes. Such spacetimes have been widely studied, in particular as models for the late universe, and as arenas for the evolution of density perturbations and gravity wave perturbations. The assumption that the matter content of the late universe is described by a collisionless fluid (dust) is an *exact* condition. However, because of the high degree of non-linearity in Einstein's field equations, a number of additional physically motivated conditions are introduced to make the problem more tractable.

The additional assumption of zero vorticity, that is, $\omega_{ab} = 0$ is justified if we intend to give a description of the late universe on a large scale. The assumption $\omega_{ab} = 0$ is also an *exact* condition, as it is an exact assumption on the matter content. As a kinematical restriction on dust, $\omega_{ab} = 0$ implies that the flow is hypersurface orthogonal (Barnes and Rowlingson (1989) [3]) and there exists a normalized cosmic time g such that $u_a = g_{,a}$. Furthermore if spacetime is initially irrotational then it will remain irrotational in the absence of dissipative effects (Ellis (1971) [34]).

In this chapter we will focus solely on generic irrotational dust spacetimes. We show that the assumption of zero vorticity in dust spacetimes is consistent. The purpose of this discussion is to summarize the elegant covariant approach used by Maartens (1997) [65], based on covariant tensorial identities listed in Appendix A.1 (see also [64]). In later chapters we will adopt this covariant approach and only use the tetrad approach when it is convenient to do so.

3.2 Field variables

Following the definitions of chapter 2 the relativistic dynamics of irrotational dust are described by a pressure-free and non-rotating perfect fluid. The condition of vanishing pressure implies that the four-velocity vector u_a is geodesic *i.e.*, the acceleration vector $\dot{u}_a$ is zero. For dust the four-velocity vector u^a is unique and determines the expansion scalar Θ and the shear tensor σ_{ab}. Gravitational dynamic variables (due to the curvature tensor and hence the metric tensor g_{ab}) are the electric Weyl tensor E_{ab} and magnetic Weyl tensor H_{ab}. In addition the physics of the spacetime introduces the energy density ρ *via* the energy-momentum tensor T_{ab}. Other field variables such as the projection tensor h_{ab} are derivable from the above variables if the metric and the four-velocity are known. So now the set

$$\Phi = \{\rho, \Theta, \sigma_{ab}, E_{ab}, H_{ab}\} \ , \tag{3.1}$$

consists a maximal set of field variables covariantly representing the second derivatives of the metric and the first derivatives of the four-velocity. The aim of what follows is to see how much we can determine from putting conditions on these variables without explicitly introducing the metric, connection, or four-velocity components. We shall show below that the variables in the set Φ obey collectively two sets of equations that describe the behaviour of the solution: a set of propagation equations and a set of constraint equations that are specializations of those in section 2.5 and 2.6.

3.3 The case of zero rotation

When vorticity is zero, $\omega_{ab} = 0$, the four-velocity vector u^a of the fluid flow satisfies the relation

$$u_{[a}u_{b;c]} = 0 \Leftrightarrow u_{[a}u_{b,c]} = 0 \; . \tag{3.2}$$

This is the condition that there exists, locally, functions f, g such that $u_a = f\, g_{,a}$. Thus zero vorticity guarantees that at each point in spacetime there exists 3-surfaces (the surface $\{g = const\}$) orthogonal to u^a; the rest–spaces defined at each point by the projection tensor h_{ab} now mesh together to form a set of 3-surfaces in spacetime. These are surfaces of simultaneity and define the cosmic time coordinate g determined by the fluid flow. For irrotational dust spacetimes $\dot{u}_a = 0$ we can choose g to set $f = 1$ and thus the cosmic time coordinate can be locally normalized to measure *proper time* along each world-line.

Additionally for irrotational dust the curvature R^*_{ab} of the 3-spaces $S(t)$, $t = const$ follows from Gauss-Codacci equations, and may be written as

$$R^*_{ab} - \tfrac{1}{3}R^* h_{ab} = -\dot{\sigma}_{ab} - \Theta\sigma_{ab} \; , \tag{3.3}$$

together with the trace equation

$$R^* = -\tfrac{2}{3}\theta^2 + 2\rho + \sigma_{ab}\sigma^{ab} \; , \tag{3.4}$$

which is equivalent to the Friedmann equation. Both equations (3.3) and (3.4) are referred to as the Gauß equations. They lead to spatial Bianchi identities equivalent to the 'div E' equations. Hence equations (3.3, 3.4) do not define any new differential constraints. Throughout this thesis our analysis will focus on the full

4-space Bianchi identities, keeping implicit the Gauss-Codacci and the Friedmann equations. The question that will be answered first is the following:

Considering irrotational pressure-free spacetimes, are the covariant equations for the set of variables Φ generally consistent? Or do they require further (integrability) conditions to achieve consistency?

3.4 Propagation equations

For irrotational dust (*i.e.*, pressure-free perfect fluid) the propagation equations given in chapter 2 are as follows:
The conservation of energy and momentum $T^{ab}{}_{;b} = 0$ leads to only one equation;

(a) the continuity equation

$$\dot{\rho} = -\Theta\rho . \tag{3.5}$$

The propagation equations for the kinematic quantities follow from the Ricci identity (2.53) and are:

(b) the Raychaudhuri equation

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} - \tfrac{1}{2}\rho , \tag{3.6}$$

(c) and the kinematic propagation equation for the shear

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c<a}\sigma_{b>}{}^{c} - E_{ab} , \tag{3.7}$$

referred to as the "$\dot{\sigma}$" equation. The propagation equations for the Weyl tensors E_{ab} and H_{ab} due to the Bianchi identities (2.62) are:

(d) the "$\dot{E}$" propagation equation

$$\dot{E}_{ab} = -\Theta E_{ab} + 3\sigma_{c<a}E_{b>}{}^{c} - \operatorname{curl} H_{ab} - \tfrac{1}{2}\rho\sigma_{ab} , \tag{3.8}$$

(e) and the "$\dot{H}$" propagation equation

$$\dot{H}_{ab} = -\Theta H_{ab} + 3\sigma_{c<a}H_{b>}{}^{c} + \operatorname{curl} E_{ab} \ , \tag{3.9}$$

where the covariant "*curl*" derivative of a tensor is defined in (2.29). Note that the left hand side of each propagation equation consists of one and only one element of the set $\dot{\Phi} = \{\dot{\rho}, \dot{\Theta}, \dot{\sigma}_{ab}, \dot{E}_{ab}, \dot{H}_{ab}\}$. Consequently, if any element of $\dot{\Phi}$ vanishes e.g., $\dot{H}_{ab} = 0$, then the related propagation equation loses its time derivative term and becomes a constraint equation.

Besides the above propagation equations the variables must satisfy the constraint equations in the following section.

3.5 Constraint equations

Setting $p = \omega_{ab} = 0$ in the Ricci identity (2.53) gives as non-trivial constraints the "$(0,\nu)$" field equations

$$\mathcal{C}^{1}{}_{a} \ = \ \tfrac{3}{2}D^{b}\sigma_{ab} - \mathrm{D}_{a}\Theta = 0 \ , \tag{3.10}$$

and the "H_{ab}" constraint

$$\mathcal{C}^{2}{}_{ab} \ = \ H_{ab} + \operatorname{curl}\sigma_{ab} = 0 \ . \tag{3.11}$$

Constraint equations due to setting $p = \omega_{ab} = 0$ in the Bianchi identities (2.62) are the "$divE$" constraint

$$\mathcal{C}^{3}{}_{a} \ = \ \mathrm{D}^{b}E_{ab} - \varepsilon_{abc}\sigma^{b}{}_{d}H^{cd} - \tfrac{1}{3}\mathrm{D}_{a}\rho = 0 \ , \tag{3.12}$$

and the "$divH$" constraint

$$\mathcal{C}^{4}{}_{a} \ = \ \mathrm{D}^{b}H_{ab} + \varepsilon_{abc}\sigma^{b}{}_{d}E^{cd} = 0 \ . \tag{3.13}$$

The tensors σ_{ab}, E_{ab} and H_{ab} in equations (3.5–3.13) are all orthogonal to the fluid flow vector u_a and obey the property $S_{ab} = S_{<ab>}$ defined in (2.30). Consequently all tensorial identities listed in Appendix A. 1 are valid for σ_{ab}, E_{ab} and H_{ab}.

3.6 Consistency

Our approach to consistency analysis of irrotational dust spacetimes is two-fold:

- Firstly taking *spatial* differential operations (such as div, curl) of one constraint equation may lead to common spatial derivative terms appearing in other constraint equations. A check for spatial consistency then involves comparing such equations to determine whether or not this leads to new or existing constraint equations. The system of equations will be said to be *spatially consistent* if no new constraints arise from such comparisons. On the other hand if new constraint equations arise then such equations are spatial consistency conditions and enlarges the set of constraint equations. The process is then repeated and if it leads to a continuing chain of spatial consistency conditions that remains unsatisfied by existing conditions (except in special geometries; such as the Szekeres case and the locally rotationally symmetric case) then the equations are deemed inconsistent (except in those special geometries).

- The second is the usual interpretation of *integrability* conditions. The constraint equations (3.10–3.13) consist of purely spatial derivative terms. Thus the system (3.5–3.13) is consistent if the constraints remain preserved throughout the evolution of spacetime. This is checked by taking time derivatives of the constraint equations, and then using space derivatives of the propagation equations together with the Ricci identities to eliminate the time derivatives that occur in the resulting equations. This leads either to the identity $0 = 0$, or to new constraints, which must then also be checked for consistency. Additional conditions required to keep the constraints preserved are called integrability conditions. Full consistency is only achieved once the enlarged set of constraints and integrability conditions are preserved under time propagation. Here also the system is inconsistent (except in those special geometries) if on taking a series of time derivatives one gets a chain of integrability conditions that stays unsatisfied by existing conditions.

3.6.1 Spatial consistency

Constraint equations for irrotational dust are overdetermined since, from the "H_{ab}" constraint (3.11), one can in principle eliminate H_{ab}. However, this leads to second order derivatives in the propagation equations (3.8) and (3.9). It seems preferable to maintain H_{ab} as a basic field. Taking the divergence of (3.11) and comparing with (3.13), leads to a common *divH* term in the two equations. This confirms that for tensors the *div curl* is *not* zero unlike the Euclidean vector counterpart. In fact, the divergence of (3.11) reproduces (3.13), on using the (vector) *curl* of (3.10) and the identities (A.3), (A.14) and (A.23) (listed in appendix A. 1), this can be expressed as

$$\text{div (3.11) and curl (3.10)} \quad \rightarrow \quad (3.13)\,. \tag{3.14}$$

More specifically (3.14) may be written as

$$C^4{}_a = \tfrac{1}{2}\mathrm{curl}\, C^1{}_a - D^b C^2{}_{ab}\,. \tag{3.15}$$

Further differential relations amongst the propagation and constraint equations are

$$\text{curl (3.7) and (3.10) and (3.11) and (3.11)}^{\cdot} \quad \rightarrow \quad (3.9)\,,$$
$$\text{grad (3.6) and div (3.7) and (3.10) and (3.10)}^{\cdot} \text{ and (3.11)} \quad \rightarrow \quad (3.12)\,,$$

where the identities (A.13), (A.17), (A.21), (A.22) and (A.24), (in appendix A. 1), have been used. No further relations between spatial derivatives of the field variables can be obtained by taking the derivatives of the constraint equations. Thus the set of constraints is spatially consistent.

3.6.2 Integrability conditions

Consistency conditions may also arise through the necessity to preserve the constraint equations under propagation along u^a (see [47] and [59]). We now show that in the general case, *i.e.*, without imposing any assumptions on the Weyl tensor C_{abcd} or other quantities, the set of constraint equations (3.10) – (3.13) is consistent with the propagation equations (3.5)–(3.9). That is, there are no hidden integrability conditions in the covariant system of equations for a generic irrotational dust spacetime.

Following Maartens (1997) [65] we denote the constraint equations (3.10) – (3.13) by

$$\mathcal{C}^A = 0 \,, \tag{3.16}$$

where $\mathcal{C}^A = \{\mathcal{C}^{\mathbf{1}}{}_a, \mathcal{C}^{\mathbf{2}}{}_{ab}, \mathcal{C}^{\mathbf{3}}{}_a, \mathcal{C}^{\mathbf{4}}{}_a\}$ and $A = \mathbf{1}, \cdots, \mathbf{4}$. The evolution of $\mathcal{C}^A$ along u^a leads to a system of equations of form

$$\begin{aligned} \dot{\mathcal{C}}^A &= \mathcal{F}^A(\mathcal{C}^B) && (3.17)\\ &= \mathcal{F}^{Ab}\mathcal{C}^B{}_b + \mathcal{G}^A{}_b D^b \mathcal{C}^C \,, \quad A, B, C = 1, \cdots, 4 && (3.18) \end{aligned}$$

where $\mathcal{F}^A$ (and $\mathcal{F}^{Ab}$, $\mathcal{G}^A{}_b$) do not contain time derivatives, since these are eliminated *via* the propagation equations and tensorial identities and D^A in (3.18) is the spatial derivative operator defined in (2.28). The calculations (outlined in appendix B) follow the covariant method of Maartens (1997) [65] and avoid the lengthy tetrad calculations used in Lesame *et al.* [59]. Explicitly, after a considerable amount of calculation, we find that (3.17) have the form[1]

$$\begin{aligned} \dot{\mathcal{C}}^{\mathbf{1}}{}_a &= -\Theta \mathcal{C}^{\mathbf{1}}{}_a - 3\varepsilon_a{}^{bc}\sigma_b{}^d \mathcal{C}^{\mathbf{2}}{}_{cd} - \tfrac{3}{2}\mathcal{C}^{\mathbf{3}}{}_a \,, && (3.19)\\ \dot{\mathcal{C}}^{\mathbf{2}}{}_{ab} &= -\Theta \mathcal{C}^{\mathbf{2}}{}_{ab} + \varepsilon^{cd}{}_{(a}\sigma_{b)c}\mathcal{C}^{\mathbf{1}}{}_d \,, && (3.20)\\ \dot{\mathcal{C}}^{\mathbf{3}}{}_a &= -\tfrac{4}{3}\Theta \mathcal{C}^{\mathbf{3}}{}_a + \tfrac{1}{2}\sigma_a{}^b \mathcal{C}^{\mathbf{3}}{}_b - \tfrac{2}{9}\rho \mathcal{C}^{\mathbf{1}}{}_a \\ &\quad + \tfrac{2}{3}E_a{}^b \mathcal{C}^{\mathbf{1}}{}_b + \varepsilon_a{}^{bc}E_b{}^d \mathcal{C}^{\mathbf{2}}{}_{cd} + \tfrac{1}{2}\varepsilon_a{}^{bc}\mathrm{D}_b \mathcal{C}^{\mathbf{4}}{}_c \,, && (3.21)\\ \dot{\mathcal{C}}^{\mathbf{4}}{}_a &= -\tfrac{4}{3}\Theta \mathcal{C}^{\mathbf{4}}{}_a + \tfrac{1}{2}\sigma_a{}^b \mathcal{C}^{\mathbf{4}}{}_b \\ &\quad + \tfrac{2}{3}H_a{}^b \mathcal{C}^{\mathbf{1}}{}_b + \varepsilon_a{}^{bc}H_b{}^d \mathcal{C}^{\mathbf{2}}{}_{cd} - \tfrac{1}{2}\varepsilon_a{}^{bc}\mathrm{D}_b \mathcal{C}^{\mathbf{3}}{}_c \,, && (3.22) \end{aligned}$$

from which it follows that if the constraints (3.10–3.13) are satisfied, as in (3.16), in an open set U, then the *constraint evolution* equations (3.19)–(3.22) are also

[1] Equations (3.19)–(3.22) have different coefficients from those in Maartens (1997) [65] due to a sign difference in the definition of the *curl* operator (2.29) however structurally the two set of equation are equivalent.

identically satisfied in U. That is putting (3.16) into the right hand side of (3.19)–(3.22) gives

$$\dot{\mathcal{C}}^A = 0 \,, \tag{3.23}$$

which must be true if the solution is consistent, because (3.23) also follows on taking the direct time derivative of (3.16), calculated from the definitions of $\mathcal{C}^A$ without using any dynamical equations. Indeed in the generic case, if there exists a solution to the constraints on an initial spatial surface $\{t = t_0\}$, *i.e.*,

$$\mathcal{C}^A\Big|_{t_0} = 0 \,, \tag{3.24}$$

where t denotes proper time along the dust world-lines then by (3.19) – (3.22), it follows that the constraints are satisfied for all time, since $\mathcal{C}^A = 0$ is the unique solution for the given initial data. This follows from the two essential features of $\mathcal{F}^A(\mathcal{C}^B)$: namely

Linearity :

$$\mathcal{F}^A(a_1\mathcal{C}_1^B + a_2\mathcal{C}_2^B) = a_1\,\mathcal{F}^A(\mathcal{C}_1^B) + a_2\,\mathcal{F}^A(\mathcal{C}_2^B); \tag{3.25}$$

Maps zero to zero :

$$\mathcal{F}^A(0) = 0 \,, \tag{3.26}$$

which in fact is implied by (3.25) on setting $\mathcal{C}_1^B = \mathcal{C}_2^B$, $a_1 = -a_2$.

Thus this solution is necessarily consistent, and evolves consistently. This establishes that the covariant constraint equations are preserved under evolution. However, it does not prove existence of solutions to the constraints in the generic case — only that if solutions exist, then they evolve consistently. The question of existence is beyond the scope of this thesis and is a further topic currently under investigation. Seeking such solutions would involve an explicit construction of a metric from the given initial data in the covariant formalism. The issue is whether a metric and Christoffel symbols or rotation coefficients exist that are compatible with the imposed covariant conditions.

3.7 Conclusion

With no assumptions made on the curvature there are no hidden integrability conditions in the covariant system of equations for a generic irrotational dust spacetime. That is, the constraints are preserved under time evolution. We should emphasize that this does not however prove the existence of solutions in the generic case, only that if solutions exist, then they evolve consistently.

Although the result is not surprising, the proof is non-trivial and is the basis for tackling further useful assumptions that are made to give descriptions of the evolution of density perturbations and gravity waves. Such exact conditions will be introduced in the ensuing chapters wherein assumptions will be made on the Weyl tensor and hence on the curvature of spacetime.

CHAPTER 4

The tetrad formalism

4.1 Introduction

The main aim of this chapter is to identify, in irrotational dust spacetimes, conditions associated with a tetrad frame acting as principal axes to the shear tensor. All such conditions turn out to be conditions on the Weyl tensor and are hence curvature requirements. Furthermore, because of the physical significance of the curvature quantity as compared to the metric (see Hawking (1966) [43]), the resulting analysis contributes to the discussion of gravitational waves, the problem of linearization instability in general relativity and the consistency of *silent* universes.

In addition to the overall covariant approach, adopted and used in chapter 3, an orthonormal tetrad frame is used to prove some of the results in the later chapters. For this purpose we give here a description of the tetrad formalism following the notation adopted by Ellis (1967) [33]. A further specialization of the orthogonal tetrad frame as used by van Elst and Uggla (1996) [101] will feature in chapter 6.

4.2 Orthonormal tetrads

A set of vectors $\{\mathbf{e}_a\}$ that are *orthonormal* at each point is called a *tetrad.* The notation ∂_a is used to emphasize the action of these vectors as directional derivatives *e.g.*,

$$\partial_a f := \mathbf{e}_a(f) \, , \tag{4.1}$$

for any function f. Also if we let $\{x^i\}$ denote a local coordinate system, then the equations

$$\mathbf{e}_a = e^i{}_a \frac{\partial}{\partial x^i} \, , \tag{4.2}$$

define the functions $e^i{}_a$, $(det||e^i{}_a|| \neq 0)$, which are components of the tetrad vector $\mathbf{e}_a$ with respect to the basis $\partial/\partial x^i$, and are also the directional derivatives of the coordinate functions x^i:

$$e^i{}_a = \partial_a x^i \ . \tag{4.3}$$

All considerations are valid in some local neighbourhood of each point in spacetime.

For pressure-free, vortex-free matter it is useful to perform a series of specializations to the above tetrad frame. The first tetrad specialization sets the timelike vector $\mathbf{e}_0$ to be the unique four-velocity u^a of the matter fluid flow, that is,

$$e^j{}_0 = u^j \ , \quad u^a = \delta^a{}_0 \ , \quad u_a = -\delta_a{}^0 \ . \tag{4.4}$$

The tetrad basis $\{\mathbf{e}_a\}$ is then split into $\{\mathbf{u}, \mathbf{e}_\nu\}$ and this, together with the projection tensor h_{ab}, allows for a $\{$space + time$\}$ splitting of commutation relations and curvature variables. The spatial set of tetrad vectors $\{\mathbf{e}_\nu\}$ form a *triad* of orthonormal spacelike vectors spanning a set of 3-surfaces in spacetime called the *surfaces of simultaneity.* As pointed out in chapter 2 such surfaces exist because the vorticity vanishes. Since the tetrad vectors $\{\mathbf{e}_a\}$ are orthonormal, the metric has the tetrad components

$$g_{ab} := e_a{}^i e_{bi} = diag(-1, +1, +1, +1) \ , \tag{4.5}$$

with the metric components g^{ab} (inverse to the matrix g_{ab}) numerically equal to the components g_{ab} and the mixed components $g_a{}^b, g_i{}^j$ of the metric are respectively $\delta_a{}^b, \delta_i{}^j$.

The directional derivatives $\partial_a f$ in (4.1) with respect to the tetrad vectors do not commute in general and the commutation functions $\gamma^c{}_{ab}$ are defined by

$$\begin{aligned} [\mathbf{e}_a, \mathbf{e}_b] f &:= (\partial_a \partial_b - \partial_b \partial_a) f =: \gamma^c{}_{ab} \partial_c f \ , \\ \gamma^c{}_{ab} &= \gamma^c{}_{[ab]} \ . \end{aligned} \tag{4.6}$$

The rotation coefficients Γ_{abc} are defined by

$$\Gamma_{abc} := e_a{}^j \, e_{cj;i} \, e^i{}_b \ , \tag{4.7}$$

so $g_{ab;c} = 0$ implies

$$\Gamma_{abc} + \Gamma_{cba} = 0 \ . \tag{4.8}$$

Lowering and raising tetrad indices by g_{ab} and g^{ab}, it follows that

$$\gamma^c{}_{ab} = \Gamma^c{}_{ab} - \Gamma^c{}_{ba} \Leftrightarrow \Gamma_{abc} = \tfrac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca}) \ . \tag{4.9}$$

The Jacobi identities for $(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$ are given by

$$[\,[\mathbf{e}_a, \mathbf{e}_b\,], \mathbf{e}_c\,] + [\,[\mathbf{e}_b, \mathbf{e}_c\,], \mathbf{e}_a\,] + [\,[\mathbf{e}_c, \mathbf{e}_a\,], \mathbf{e}_b\,] = 0 \ , \tag{4.10}$$

and can also be written in terms of $\gamma^c{}_{ab}$ as follows

$$\begin{pmatrix} f \\ bcd \end{pmatrix} : \qquad \partial_{[d}\gamma^f{}_{cb]} + \gamma^g{}_{[dc}\gamma^f{}_{b]g} = 0 \ . \tag{4.11}$$

Now equation (4.6) gives each of the $\gamma^c{}_{bc}$ a direct geometrical interpretation as a component of the Lie derivative $L_{\mathbf{e}_a}\mathbf{e}_b$ and equation (4.7) gives each of the coefficients Γ_{abc} a direct geometrical interpretation as a component of a covariant derivative $\nabla_{\mathbf{e}_a}\mathbf{e}_b$. They determine the tetrad components of the covariant derivative of any tensor, for example:

$$\begin{aligned} T_{ab;c} &= T_{ij;k}e^i{}_a e^j{}_b e^k{}_c \\ &= (T_{ij}e^i{}_a e^j{}_b)_{;k}e^k{}_c - T_{ij}(e^i{}_{a;k})e^j{}_b e^k{}_c - T_{ij}e^i{}_a(e^j{}_{b;k})e^k{}_c \\ &= \partial_c T_{ab} - T_{fb}\Gamma^f{}_{ca} - T_{af}\Gamma^f{}_{cb} \ . \end{aligned} \tag{4.12}$$

Using the choice of $\mathbf{e}_0$ as the fluid flow velocity u^a we can, as in Ellis (1967) [33], relate tetrad components $T_{ab;c}$ with coordinate components $T_{ij;k}$. Equations (4.12) and (4.9) are used to relate the commutation functions $\gamma^a{}_{bc}$ to the kinematic quantities as follows

$$\gamma^0_{0\mu} = \dot{u}_\mu = 0 \ , \tag{4.13}$$

$$\gamma^0_{\nu\nu} = -\Theta_{\nu\nu} =: \Theta_\nu \quad \text{(no sum)} \ , \tag{4.14}$$

$$\gamma^0_{\mu\nu} = -2\epsilon_{\mu\nu\tau}\omega^\tau = 0 \ , \qquad (\text{for } \omega^\tau = 0) \ , \tag{4.15}$$

$$\gamma^\mu_{0\nu} = \mathbf{e}_\mu.\dot{\mathbf{e}}_\nu - \sigma_{\mu\nu} - \epsilon_{\mu\nu\tau}\omega^\tau \ , \qquad (\text{for } \mu \neq \nu) \ , \tag{4.16}$$

$$= \mathbf{e}_\mu.\dot{\mathbf{e}}_\nu - \sigma_{\mu\nu} \ , \qquad (\text{for } \omega^\tau = 0) \ , \tag{4.17}$$

where the terms Θ_ν are the expansion of the fluid in the directions $\mathbf{e}_\nu$ and $\mathbf{e}_\mu.\dot{\mathbf{e}}_\nu$ are the Fermi derivatives of the basis vectors along the world lines and may be represented by the quantity

$$\Omega^a = \eta^{abcd} u_b \mathbf{e}_c.\dot{\mathbf{e}}_d \,, \tag{4.18}$$

which can be interpreted as the local angular velocity, in the rest-frame of u^a, of a set of Fermi-propagated axes with respect to the triad $\{\mathbf{e}_\mu\}$.

In chapter 7 we will investigate spatially homogeneous dust. There the purely spatial commutators, $\gamma^\alpha{}_{\gamma\delta}$, are decomposed [1] into an object a_γ and a symmetric object $n_{\gamma\delta}$ as follows:

$$\gamma^\alpha{}_{\gamma\delta} := 2\, a_{[\gamma}\, \delta^\alpha{}_{\delta]} + \varepsilon_{\gamma\delta\epsilon}\, n^{\epsilon\alpha} \,, \tag{4.19}$$

where $\varepsilon_{\gamma\delta\epsilon}$ is the totally antisymmetric three-dimensional permutation tensor defined in (2.23).

4.3 Diagonality theorems

The tensorial quantities σ_{ab}, E_{ab} and H_{ab} representing the shear tensor, the electric Weyl tensor and the magnetic Weyl tensor respectively, are all trace-free tensors orthogonal to the fluid flow. Thus for generality, we introduce here an arbitrary tensor $S_{ab} = S_{<ab>}$ that is orthogonal to the fluid flow. On the 3-surfaces of simultaneity we may specialize the tetrad vectors $\{\mathbf{e}_\nu\}$ to be the principal axes of S_{ab} (S_{ab} is diagonalizable because h_{ab} is positive definite). Our first objective is to establish properties of the principal frame which simultaneously diagonalize S_{ab} and its covariant derivative $\dot{S}_{ab}$. We start with the assumption that there exists a triad $\{\mathbf{e}_\nu\}$ which acts as principal axes to both the tensors S_{ab} and $\dot{S}_{ab}$ in an *open set* U:

$$S_{ab} = \dot{S}_{ab} = 0 \,, \quad a \neq b \,. \tag{4.20}$$

[1]The decomposition (4.19) can be performed independently of the restriction to a spatially homogeneous spacetime.

The *covariant* consequence of (4.20) is that S_{ab} *commutes* with $\dot{S}_{ab}$ *i.e.*,

$$\varepsilon_{abc} S^{b}{}_{d} \dot{S}^{cd} = 0 . \tag{4.21}$$

Equation (4.21) is valid for both degenerate and non-degenerate cases. The *tetrad* consequence of (4.20) is as follows. Since S_{ab} is diagonal in an open set then

$$\partial_0 S_{ab} = 0 , \quad a \neq b , \tag{4.22}$$

so now using (4.20) in

$$\dot{S}_{ab} \equiv \partial_0 S_{ab} - S_{ap} \Gamma^{p}{}_{0b} - S_{pb} \Gamma^{p}{}_{0a} , \tag{4.23}$$

immediately yields

$$\Gamma^{a}{}_{0b} = 0 , \quad \text{for } a \neq b , \tag{4.24}$$

in the *non-degenerate* S_{ab}, $\dot{S}_{ab}$ case. The *degenerate* but non-zero case also yields (4.24) as follows. Suppose that both S_{ab} and $\dot{S}_{ab}$ are simultaneously diagonal and that S_{ab} is degenerate in an open set U. Then degeneracy of S_{ab} in U implies degeneracy of $\dot{S}_{ab}$ as seen in the following:

> If S_{ab} is diagonal and degenerate, say $S_{11} = S_{22}$, in a region U then $\partial_0 S_{11} = \partial_0 S_{22}$. Now, since S_{ab} is diagonal in U, it follows by (4.23) that
>
> $$\dot{S}_{11} = \partial_0 S_{11} - 2 S_{11} \Gamma^{1}{}_{01} = \partial_0 S_{11} , \tag{4.25}$$
>
> and similarly
>
> $$\dot{S}_{22} = \partial_0 S_{22} . \tag{4.26}$$
>
> Thus degeneracy of S_{ab} in U implies degeneracy of $\dot{S}_{ab}$.

The tetrad consequence of (4.20) in the degenerate case follows from (4.23) and is given by

$$\Gamma^{1}{}_{03} = \Gamma^{2}{}_{03} = 0 . \tag{4.27}$$

The tetrad freedom of rotation in the $(\mathbf{e}_1, \mathbf{e}_2)$-plane through an angle ϕ;

$$\begin{aligned} \bar{e}^{i}_{1} &= e^{i}_{1} \cos\phi + e^{i}_{2} \sin\phi , \\ \bar{e}^{i}_{2} &= -e^{i}_{1} \sin\phi + e^{i}_{2} \cos\phi , \qquad \phi = \phi(x^0, x^i) , \end{aligned} \tag{4.28}$$

specifically preserves (4.20), (4.22) and degeneracy, while

$$\bar{\Gamma}^1{}_{02} = \Gamma^1{}_{02} - \partial_0 \phi \tag{4.29}$$

and this can then be used to set

$$\bar{\Gamma}^1{}_{02} = 0 \ , \tag{4.30}$$

so that equation (4.24) still holds.

The assumption (4.20) is made in an open set U and hence (4.24) is valid in U as well. The spatial tetrad vectors $\{\mathbf{e}_\mu\}$ satisfying (4.24) have vanishing Fermi derivatives *i.e.*, $\mathbf{e}_\nu.\dot{\mathbf{e}}_\mu = 0$ and are said to be *Fermi propagated*:

> **Theorem 1a:** *If a set of tetrad vectors $\{\mathbf{e}_\mu\}$ simultaneously diagonalizes a tensor S_{ab} and its covariant derivative $\dot{S}_{ab}$ then the tetrad vectors are Fermi propagated, or can be chosen so in the degenerate case.*

Conversely given a Fermi propagated tetrad, $\Gamma^a{}_{0b} = 0 \; ; a \neq b$, that diagonalizes S_{ab} in an open set U, it follows from (4.22), (4.23) that $\dot{S}_{ab}$ is also diagonal in U. One needs to be careful here as a Fermi tetrad that diagonalizes $\dot{S}_{ab}$ in U only diagonalizes $\partial_0 S_{ab}$ following (4.23) and not necessarily S_{ab}. Hence the converse of theorem 1a above is only valid in the following form:

> **Theorem 1b:** *A Fermi propagated tetrad frame, $\Gamma^a{}_{0b} = 0 \; ; a \neq b$, that acts as principal axes to S_{ab} in an open set U simultaneously diagonalizes the covariant derivative $\dot{S}_{ab}$.*

To relate *diagonalilty* and *commutativity* we use a result from linear algebra that two real symmetric $n \times n$ matrices are simultaneously diagonalizable if and only if they commute (Herstein (1975) [46]). This result is valid for both degenerate and non-degenerate matrices, thus:

Theorem 2: *The tensors S_{ab}, V_{ab} are simultaneously diagonalizable in an open set U if and only if they commute in U.*

4.4 The shear eigentetrad

We now apply the theorems of the previous section to irrotational dust spacetimes to establish additional kinematic properties that are associated with a tetrad frame that is also a shear eigenframe. We start from the assumption that the shear and its covariant time derivative are both diagonal in an open set U,

$$\sigma_{ab} = \dot{\sigma}_{ab} = 0 \,, \qquad a \neq b \,. \tag{4.31}$$

As in the previous section with S_{ab} replaced by σ_{ab} the covariant and tetrad consequence of (4.31) are

$$\varepsilon_{abc}\sigma^{b}{}_{d}\dot{\sigma}^{cd} = 0 \,, \qquad \text{and} \tag{4.32}$$

$$\Gamma^{a}{}_{0b} = 0 \,, \qquad \text{for } a \neq b \,, \tag{4.33}$$

respectively. Consequently the assumption (4.31) is equivalent to the condition that spacetime admits a Fermi propagated shear eigentetrad. If we now evolve equation (4.32) along the fluid flow vector u^a it follows that the second covariant derivative $\ddot{\sigma}_{ab}$ and all higher order covariant derivatives of σ_{ab} commute with σ_{ab}. This also follows if we use the tetrad approach for example, since $\partial_0\dot{\sigma}_{ab}$ is diagonal we get $\ddot{\sigma}_{ab} = (\dot{\sigma}_{aa} - \dot{\sigma}_{bb})\Gamma^{a}{}_{0b} = 0, \;\; a \neq b$.

In terms of a consistency analysis the series of equations expressing the condition that the shear commutes with all its higher order covariant time derivatives form the *first set of integrability conditions* resulting from the assumption (4.32) and we express this as follows

$$\varepsilon_{abc}\sigma^{b}{}_{d}\dot{\sigma}^{(n)dc} = 0 \,, \qquad n = 2, 3, 4 \ldots. \tag{4.34}$$

where a superscript (n) on $\dot{\sigma}_{ab}$ denotes the n^{th} covariant time derivative.

The electric Weyl tensor E_{ab} can be associated to the above shear eigentetrad by taking the shear propagation equation (3.7) as determining the electric Weyl tensor E_{ab}:

$$E_{ab} := -\dot{\sigma}_{ab} - \tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c<a}\sigma_{b>}{}^{c} . \tag{4.35}$$

It then follows that the assumption (4.32) now constrains the electric Weyl tensor E_{ab} to commute with σ_{ab} and establishes the *integrability condition*:

$$\varepsilon_{abc}\sigma^{b}{}_{d}E^{cd} \;=\; 0 . \tag{4.36}$$

Furthermore taking a series of covariant time derivatives of (4.36) also yield a chain of integrability conditions expressing the requirement the shear commutes with E_{ab} and all its covariant time derivatives:

$$\varepsilon_{abc}\sigma^{b}{}_{d}\dot{E}^{(n)dc} = 0 , \qquad n = 1,2,3\ldots . \tag{4.37}$$

In fact integrability conditions (4.34) and (4.37) can be stated in more general terms as follows:

> **Shear commutation property:** *For an arbitrary tensor A_{ab} if $\varepsilon_{abc}\sigma^{b}{}_{d}\dot{\sigma}_{cd} = 0$ and $\varepsilon_{abc}\sigma^{b}{}_{d}A_{cd} = 0$ then $\varepsilon_{abc}\sigma^{b}{}_{d}\dot{A}_{cd} = 0$ is the resulting integrability condition.*

Proof:

First given $\varepsilon_{abc}\sigma^{b}{}_{d}\dot{\sigma}_{cd} = 0$ and $\varepsilon_{abc}\sigma^{b}{}_{d}A_{cd} = 0$, it is clear that for $A_{ab} = \sigma_{ab}$ the two conditions coincide and taking a series of covariant time derivatives leads to (4.34). For tensors $A_{ab} \neq \sigma_{ab}$ two chains of integrability conditions are obtained. The first is identical to (4.34) and is due to the covariant time derivatives of $\varepsilon_{abc}\sigma^{b}{}_{d}\dot{\sigma}_{cd} = 0$. The second is identical to (4.37) with E replaced by A and is due to the covariant time derivatives of $\varepsilon_{abc}\sigma^{b}{}_{d}A_{cd} = 0$. For example the first time derivatives are as follows

$$\left[\varepsilon_{abc}\sigma^{b}{}_{d}\dot{\sigma}_{cd}\right]^{\bullet} \;=\; \varepsilon_{abc}\sigma^{b}{}_{d}\ddot{\sigma}_{cd} = 0 , \tag{4.38}$$

$$\left[\varepsilon_{abc}\sigma^{b}{}_{d}A_{cd}\right]^{\bullet} \;=\; \varepsilon_{abc}\dot{\sigma}^{b}{}_{d}A_{cd} + \varepsilon_{abc}\sigma^{b}{}_{d}\dot{A}_{cd} = \varepsilon_{abc}\sigma^{b}{}_{d}\dot{A}_{cd} = 0 , \tag{4.39}$$

where the term $\varepsilon_{abc}\dot{\sigma}^b{}_d A_{cd}$ vanishes in the Fermi propagated shear eigentetrad and since the commutator is a tensor, it vanishes in all frames. □

Now the assumption (4.32) further constrains the magnetic Weyl tensor H_{ab} *via* its *curl*, as follows. If we use the "$\dot{E}_{ab}$" propagation equation (3.8) in the integrability condition (4.37) with $n = 1$, that is, $\varepsilon_{abc}\sigma^b{}_d\dot{E}^{dc} = 0$ we obtain

$$\varepsilon_{abc}\sigma^b{}_d(\operatorname{curl} H)^{cd} \;=\; 0\,. \tag{4.40}$$

Equation (4.40) is clearly an *integrability condition* and is not identically satisfied (as pointed out by Sopuerta (1997) [87]). Furthermore a chain of integrability conditions arise from the evolution of (4.40) and they all require that the shear commutes with $(\operatorname{curl} H)_{ab}$ and its higher order time derivatives:

$$\varepsilon_{abc}\sigma^b{}_d(\operatorname{curl} H)^{\cdot(n)cd} \;=\; 0\,, \qquad n = 1, 2, 3, \ldots\,. \tag{4.41}$$

It also follows easily that if we start with the assumption that the shear and the electric Weyl tensor commute, as expressed by equation (4.36), then the shear propagation equation (3.7) yields (4.32) as an integrability condition. Thus (4.32) is equivalent to (4.36) and each acts as an integrability condition for the other. By virtue of the shear commutation property (4.32) and (4.36) introduces a chain integrability conditions expressed by (4.34), (4.37) and (4.41). We summarize the above results in the following theorem:

> **Theorem 3a:** *For irrotational dust spacetime the shear tensor σ_{ab} and its covariant time derivative $\dot{\sigma}_{ab}$ are diagonal in an open set U if and only if the electric Weyl tensor E_{ab} is diagonal in the same frame. Furthermore, this frame constrains the curl of the magnetic Weyl tensor $(curl H)_{ab}$ to be diagonal.*

We should stress that the property that the shear and its time derivative commute or, equivalently the shear and the electric Weyl tensor commute, is an assumption and must therefore be imposed directly or *via* local conditions made on spacetime. This assumption constrains the curl of the magnetic Weyl tensor to commute with the shear.

A close look at (4.35) and (3.8) indicates that in a Fermi tetrad, degeneracy of the tensors σ_{ab}, E_{ab} and $(\mathrm{curl}\, H)_{ab}$ is also simultaneous and occurs in the same plane:

> **Theorem 3b: Degeneracy.** *If the shear σ_{ab} and its covariant time derivative $\dot{\sigma}_{ab}$ are degenerate in an open set U then the electric Weyl tensor E_{ab} and the curl of the magnetic Weyl tensor $(curl\, H)_{ab}$ are constrained to be degenerate in the same plane as the shear.*

From these results it is clear that working with a shear eigentetrad reduces the number of independent variables considerably. For example, because of its trace-free, symmetric property, the shear tensor σ_{ab} has five independent components. In a shear eigentetrad, however, only two independent components remain. The same reduced number of independent components occurs for E_{ab} and $(\mathrm{curl}\, H)_{ab}$.

Finally if both the shear σ_{ab} and the electric Weyl tensor E_{ab} are simultaneously diagonalizable then from theorem 2 they commute and the Bianchi "div H" equation (3.13) gives $(div\, H)_a = h^t{}_a H^{as}{}_{;d} h^d{}_s = 0$. Conversely if $(div\, H)_a = 0$ then from the Bianchi "div H" constraint (3.13) we get $\varepsilon_{abc}\sigma^b{}_d E^{dc} = 0$, hence the tensors σ_{ab} and E_{ab} commute and, by theorem 2, are diagonal:

> **Theorem 4:** *Both the shear σ_{ab} and the electric Weyl tensor E_{ab} are simultaneously diagonalizable if and only if the divergence of the magnetic Weyl tensor is zero i.e., $(div H)_a = 0$.*

The results of theorem 3a and theorem 4 may be written in the commutation notation as follows:

$$(\mathrm{div}\, H)_a = 0 \quad \Leftrightarrow \quad \varepsilon_{abc}\sigma^b{}_d E^{dc} = 0 \quad \Leftrightarrow \quad \varepsilon_{abc}\sigma^b{}_d \dot{\sigma}^{dc} = 0\,. \tag{4.42}$$

Any one of the equations in (4.42) constrains the curl of the magnetic Weyl tensor to commute with the shear tensor.

We have thus set up conditions for irrotational dust to admit an orthonormal tetrad, associated with the matter four-velocity u^a, which is an eigentetrad for the shear σ_{ab} of the matter flow. In particular we note the pivotal role played by the condition $(divH)_a = 0$ in (4.42). In the ensuing chapters we will investigate the role of this condition in the context of non-linear irrotational dust spacetimes. For this purpose we state a series of conditions that imply $(divH)_a = 0$:

Theorem 5: *If*

(a) *the Weyl tensor is purely electric;* $H_{ab} = 0$ *or,*

(b) *the Weyl tensor is purely magnetic;* $E_{ab} = 0$ *or,*

(c) *the shear tensor and the electric Weyl tensor are simultaneously diagonalizable,*

then $(divH)_a = 0$.

The points **(a)**, **(b)** and **(c)** of theorem 5 all follow directly from the "div H" constraint (3.13). The result of point **(c)** was first obtained by Barnes and Rowlingson (1989) [3]. All four theorems in a slightly different form were obtained in Lesame (1995) [56] and Lesame, Ellis and Dunsby (1996) [59].

4.5 Conditions on the Weyl tensor

The theorems of the previous section give the basis of the direction to be taken in the rest of this thesis. From theorem 5 we see that the following conditions warrants separate further investigation:

(a) To relate non-linear theory to the results of linear theory the condition of vanishing divergence of the magnetic Weyl tensor, $(div\, H)_a = 0$, will be taken as an exact condition. This will test its role as a characteristic of gravity waves in the non-linear theory.

(b) The silent condition $H_{ab} = 0$ (see section 1.3), which implies the absence of gravity waves in Newtonian-like universes.

(c) The silent condition $E_{ab} = 0$ taken as an exact condition is expected to yield an FLRW universe. This is due to the role of E_{ab} which drives the shear σ_{ab}; which in turn affects the expansion Θ.

In addition other exact conditions will also be considered. In general special classes of irrotational dust spacetimes can arise from the vanishing of a covariant quantity (or quantities) $\mathcal{B}$ which occurs algebraically in the propagation and/or constraint equations. Such quantities would then be related to the field variables in the set $\Phi = \{\rho, \theta, \sigma_{ab}, E_{ab}, H_{ab}\}$, e.g., $\sigma_{ab} = 0$ or $(\operatorname{div} E)_a = 0$, but do not for example include $\mathrm{D}_a(\sigma_{bc}\sigma^{bc}) = 0$ nor $(E_{ab}H^{ab})^{\cdot} = 0$ which do not occur in these equations. Throughout the thesis an assumption;

$$\mathcal{B} = 0 \; , \tag{4.43}$$

is imposed *a priori* as a local condition, that is, as a property valid in some open set U in space-time. The conditions that arise from (4.43) are considered through the effect of (4.43) on the constraint and propagation equations in U, and the conclusions therefore hold in U. Three types of such classes may be identified, as has been mentioned before in section 1.4.2.:

Class I conditions: Here $\mathcal{B}$ appears algebraically in the constraint equations (3.10)–(3.13) and the new constraint $\mathcal{B} = 0$ does not affect the form of the propagation equations (3.5)–(3.9) as no element of $\dot{\Phi} = \{\dot{\rho}, \dot{\Theta}, \dot{\sigma}_{ab}, \dot{E}_{ab}, \dot{H}_{ab}\}$ vanishes.

The fact that the constraint evolution equations (3.19) – (3.22) are linear in $\mathcal{C}^A$ means that $\mathcal{B} = 0$ does not interfere with the already proven consistency of the constraints. Now with $\mathcal{B} = 0$ valid in an open set U this implies that $\dot{\mathcal{B}} = 0$ in U. Thus in Class I we need to check whether or not $\dot{\mathcal{B}} = 0$ introduces new conditions in the propagation/constraint equations. It is permissible to impose more than one condition in this class simultaneously.

Class II conditions: where $\mathcal{B} = 0$ occurs in the propagation equations such that an element of the set $\dot{\Phi} = \{\dot{\rho}, \dot{\Theta}, \dot{\sigma}_{ab}, \dot{E}_{ab}, \dot{H}_{ab}\}$ vanishes and the related propagation equation converts to a new constraint

$$\mathcal{C}^5 = 0 \,. \tag{4.44}$$

This explicitly disturbs the consistency proof already attained because the propagation/constraint structure of the general set of equations is altered. Although $\mathcal{B} = 0$ may also affect the constraint equations $\mathcal{C}^A$, $A = \mathbf{1}, \cdots, \mathbf{4}$, we still have

$$\dot{\mathcal{C}}^A = \mathcal{F}^A(\mathcal{C}^B) \quad A, B = \mathbf{1}, \cdots, \mathbf{4} \,, \tag{4.45}$$

and due to the linearity of $\mathcal{F}^A$ the old constraints $\mathcal{C}^A$, $A = \mathbf{1}, \cdots, \mathbf{4}$, remain consistent. The question is whether the new constraint (4.44) is consistent or not.

Class III conditions: where $\mathcal{B} = 0$ occurs in the propagation equations such that no element of the set $\dot{\Phi} = \{\dot{\rho}, \dot{\Theta}, \dot{\sigma}_{ab}, \dot{E}_{ab}, \dot{H}_{ab}\}$ vanishes and the related propagation equation does not convert to a new constraint. These conditions are not directly linked to a shear-eigentetrad and are not considered here. However this class includes mathematically interesting cases. For example the $(\operatorname{curl} E)_{ab} = 0$ condition implies that spatial derivatives are eliminated from the $\dot{H}$ equations. Consistency for this case is not investigated but we do give potential examples of spatially homogeneous models satisfying $(\operatorname{curl} E)_{ab} = 0$.

Our approach is in line with previous work, such as Ellis (1967) [33], wherein for the case of dust spacetimes with $\mathcal{B} = \sigma_{ab} = 0$, a tetrad approach is used to show that the consistency condition is $\omega_{ab}\Theta = 0$. The conditions are examined under the assumption that they are valid in an open set U, rather than being formulated in terms of an initial value problem and, as pointed out in Maartens (1997) [65], this approach cannot answer the question: *If $\mathcal{B}$ vanishes on $\{t = t_0\}$, does $\mathcal{B}$ vanish throughout spacetime?*

In the next chapters we will confine attention to the following physically motivated cases:

(I) $\mathcal{B} = (\operatorname{div} H)_a, (\operatorname{div} E)_a$ in chapter 5

(II) $\mathcal{B} = \sigma_{ab}, H_{ab}, E_{ab}$ in chapter 6.

(III) spatially homogeneous examples of (I) and (II) in chapter 7.

CHAPTER 5

Class I Conditions

5.1 Introduction

In this chapter we focus attention on exact conditions imposed on terms in the constraint equations that do not change the form of the propagation equation for irrotational dust spacetimes. We referred to such conditions as Class I conditions in section 1.4.2. A comparison with the results of the linearized theory is given and for this purpose we give the linear equations and highlight some physical significance of the conditions in section 2.

In section 3 we outline the general approach to this class of conditions. Here we emphasize the importance of checking the consistency of the new condition imposed – does it by itself imply new integrability conditions or not? In section 4 up to section 7 we identify integrability conditions for the various physically interesting conditions belonging to this class using the covariant tensorial identities of Maartens [65], [64] and the tetrad results of chapter 4.

5.2 Linearized equations

In the absence of pressure 3-gradients and their spatial variations that generate propagating sound waves within the perfect-fluid, gravitational waves are the only phenomena which convey information between adjacent world-lines. This is a dynamical interaction, for mathematically, it is represented in the Bianchi identities by terms that comprise of the *curls* of both the electric Weyl tensor and the magnetic Weyl tensor. Non-zero values of the *curls* of the Weyl curvature variables (E_{ab} and H_{ab}) generically induce temporal changes within the matter fluid (and vice versa), usually interpreted as propagating pure gravitational waves.

We will occasionally compare the results of the exact non-linear theory with those of the linearized theory. The aim here is to identify linearization instabilities that may arise in the equations. Linearization is taken relative to an FLRW model as follows. All the quantities that do not vanish in an FLRW model are treated as zero order quantities e.g., ρ, Θ and quantities that vanish in an FLRW model are treated as first order quantities e.g., σ_{ab}, E_{ab} and H_{ab}. Using this procedure we write the linearized propagation and constraint equations for irrotational dust respectively as:

$$\begin{aligned}
\dot{\rho} &= -\Theta\rho\,, && (5.1)\\
\dot{\Theta} &= -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\rho\,, && (5.2)\\
\dot{\sigma}_{ab} &= -E_{ab} - \tfrac{2}{3}\Theta\sigma_{ab}\,, && (5.3)\\
\dot{E}_{ab} &= -\Theta E_{ab} - \tfrac{1}{2}\rho\sigma_{ab} - (\mathrm{curl}\,H)_{ab}\,, && (5.4)\\
\dot{H}_{ab} &= -\Theta H_{ab} + (\mathrm{curl}\,E)_{ab}\,, && (5.5)
\end{aligned}$$

and,

$$\begin{aligned}
0 &= \tfrac{3}{2}D^b\sigma_{ab} - \mathrm{D}_a\Theta\,, && (5.6)\\
0 &= H_{ab} + (\mathrm{curl}\,\sigma)_{ab}\,, && (5.7)\\
0 &= \mathrm{D}^bE_{ab} - \tfrac{1}{3}\mathrm{D}_a\rho\,, && (5.8)\\
0 &= \mathrm{D}^bH_{ab}\,. && (5.9)
\end{aligned}$$

We note from (5.1)–(5.9) that if we set $H_{ab} = 0$ in an open set U, then $(\mathrm{curl}\,E)_{ab} = (\mathrm{curl}\,H)_{ab} = (\mathrm{curl}\,\sigma)_{ab} = 0$ and no information is exchanged between neighbouring particles, leading to the conclusion that gravity wave perturbations are characterised by non-zero H_{ab}. However in linear theory such gravity wave perturbations are accompanied by a divergence–free magnetic Weyl tensor *i.e.*, $(\mathrm{div}\,H)_a = 0$, $H \neq 0$, as seen in equation (5.9). Does this mean that $(\mathrm{div}\,H)_a = 0 \neq H$ is also a characteristic of non-linear pure gravity waves? It may be that $(\mathrm{div}\,H)_a = 0 \neq H$ is only a property of linearization in general and is not connected to gravity wave solutions. To obtain a certain answer to the role of $(\mathrm{div}\, H)_a = 0$ in solutions with $H_{ab} \neq 0$ we resort to the exact non-linear theory.

5.3 General approach

In chapter 3 we showed that constraint equations for generic irrotational dust remain preserved under evolution. We will first present a covariant argument which shows that in general this property is retained for Class I conditions. We first write the generic constraint equations as

$$\mathcal{C}^B = \mathcal{C}^{*B} + \mathcal{B}^B \,, \tag{5.10}$$

and consider the effect of introducing a Class I condition $\mathcal{B}^B = 0$ in an open set U. Putting this splitting into the time derivative of the constraint equations (3.17) we get

$$(\mathcal{C}^{*A} + \mathcal{B}^A)^{\cdot} = \mathcal{F}^A(\mathcal{C}^{*B} + \mathcal{B}^B) \,, \tag{5.11}$$

and so

$$(\mathcal{C}^{*A})^{\cdot} + (\mathcal{B}^A)^{\cdot} = \mathcal{F}^A(\mathcal{C}^{*B}) + \mathcal{F}^A(\mathcal{B}^B) \,, \tag{5.12}$$

by linearity of $\mathcal{F}^A$, (see (3.25)). Hence if we can consistently set

$$\mathcal{B}^A = 0 \,, \tag{5.13}$$

in an open set U, then by taking the time derivative of (5.13) (using Christoffel symbols is appropriate but not using any equations above except the definition of $\mathcal{B}^B$ - hence avoiding any danger of circular argument) we must also have

$$(\mathcal{B}^A)^{\cdot} = 0 \,, \tag{5.14}$$

in an open set U, independent of the Einstein field equations. Thus if equation (5.13) is true, then by equations (5.14) and (3.26), namely $\mathcal{F}^A$ maps zero to zero, equation (5.12) becomes

$$(\mathcal{C}^{*A})^{\cdot} = \mathcal{F}^A(\mathcal{C}^{*B}) \,, \tag{5.15}$$

because the Einstein field equations hold, and thus the overall set remains consistent because this is again of the form (3.17). But can we consistently set $\mathcal{B}^A = 0$ in U?

The splitting into 'constraint' and 'remainder' expressed in (5.10) is arbitrary. Thus if we can show that separately (5.15) and

$$(\mathcal{B}^A)^{\cdot} = \mathcal{F}^A(\mathcal{B}^B) , \tag{5.16}$$

are true for the original functional form in equation (3.18) then the linearity property (3.25), (and hence (3.26)) hold in both (5.15) and (5.16), and $\mathcal{B}^A = 0$ is consistent:

$$\mathcal{B}^A|_{t_o} = 0 \quad \Rightarrow \quad \mathcal{B}^A(t) = 0 , \tag{5.17}$$

and so is $\mathcal{C}^{*B} = 0$:

$$\mathcal{C}^{*A}|_{t_o} = 0 \quad \Rightarrow \quad \mathcal{C}^{*A}(t) = 0 . \tag{5.18}$$

We remark that if we know that (3.18) is true and set $\mathcal{B}^A = 0$ initially, this constraint will not remain true in general unless an equation like (5.16) is also true. Rather we have then that the right hand side of equation (5.12) will vanish when the constraints are initially satisfied and also $\mathcal{B}^A = 0$ initially, but that does not show that the two terms on the left hand side will separately vanish. Thus:

> *In Class I conditions the consistency of the existing set of equations remains undisturbed by imposing the new constraint $\mathcal{B}^A = 0$ in an open set U. However $\mathcal{B}^A = 0$ implies $\dot{\mathcal{B}}^A = 0$ in U, and it is the latter condition that needs to be investigated.*

In the next sections we apply this general consistency approach to physically interesting Class I irrotational dust spacetimes.

5.4 Spacetimes with $DivH = 0$

Suppose now that the magnetic Weyl tensor is divergence–free in an open set U

$$\mathcal{B}_a = \mathrm{D}^b H_{ab} = 0 , \quad H_{ab} \neq 0 . \tag{5.19}$$

In the **linear theory** the system of equations (5.1)–(5.9) represents spatially inhomogeneous perturbations of the FLRW spacetime and immediately satisfy the exact condition (5.19). This system has been shown to be consistent (see [47], [25]) as expected since the system (5.1)–(5.9) is a linearized form of the generic case whose consistency is proven in chapter 3. In particular, a consistency proof for linearized irrotational dust spacetime equations following the covariant approach of Maartens [65] confirms that there are no integrability conditions.

In the **non-linear exact** theory we look at the consistency of (5.19) using commutation theorems established in the previous chapter as follows. (See also Maartens, Lesame and Ellis (1997) [66]). Using $\varepsilon_{abc}S^b{}_dV^{cd}$ as the tensor notation for the covariant commutator of tensors we established in theorem 4 of chapter 4 (see equation (4.42)) that $\mathrm{D}^bH_{ab} = 0$ is equivalent to $\varepsilon_{abc}\sigma^b{}_dE^{cd} = 0$ which, in terms of the shear propagation equation and the shear commutation property, now *constrains* the shear to commute with its covariant time derivatives

$$\varepsilon_{abc}\sigma^b{}_d\dot{\sigma}^{(n)cd} = 0\ , \qquad n = 1, 2, 3, \ldots \tag{5.20}$$

In tetrad form (5.20) implies that the $(divH)_a = 0$ constrains the shear eigentetrad to be Fermi propagated (as shown by (4.33). Following the same argument as in leading to (4.40) we can show that the second set of integrability conditions is that the shear tensor commutes with curl of the magnetic Weyl tensor and its covariant time derivatives. This follows more directly as follows.

To determine the existence of other constraints it is more convenient to replace (5.19) by the equivalent condition

$$C^{4*}_a = \varepsilon_{abc}\sigma^b{}_dE^{dc} = 0\ . \tag{5.21}$$

In the **linearized** case (5.21) is identically satisfied since the left hand side is second order, and consistency is automatic:

$\boxed{\textit{The linearized } DivH = 0 \textit{ irrotational dust spacetimes are consistent}}$

In the **exact non-linear** case the evolution equation for (5.21) follows from the propagation equations (3.7) and (3.8) and gives

$$\dot{C}^{4*}_a = -\tfrac{5}{3}\Theta C^{4*}_a + \tfrac{1}{2}\sigma_a{}^b C^{4*}_b - \varepsilon_{abc}\sigma^b{}_d(\mathrm{curl}\, H)^{dc} \,. \tag{5.22}$$

Again we note that the constraint (5.21) is consistent provided

$$\varepsilon_{abc}\sigma^b{}_d(\mathrm{curl}\, H)^{dc} = 0 \,, \tag{5.23}$$

and using the shear propagation property equation (5.23) yields a chain of integrability conditions expressed (4.41).

Irrotational dust spacetimes with a divergence-free magnetic Weyl tensor are consistent provided the shear tensor commutes with (i) its covariant time derivatives and (ii) the curl of the magnetic Weyl tensor and its covariant time derivatives.

This corrects an earlier result, in Lesame *et al.* [59], reporting that $(\mathrm{div}\, H)_a = 0$ implied $H = 0$. The error in [59] is due to a sign error in the version of the Weyl tensor expression (2.20) given in [34], as pointed out by Maartens (1997) [65]. The identification of (5.23) as a constraint and not an identity was pointed out by Sopuerta (1997) [87]. In fact the covariant time derivative of (5.23) has the form

$$\begin{aligned}([\sigma, curl(H)]_a)^{\cdot} &= \varepsilon_{abc}\sigma^b{}_d D^2 E^{cd} - \frac{1}{2}\varepsilon_{abc}\sigma^{bd} D^c D_d \rho - \varepsilon_{abc}E^b{}_d(curl H)^{cd} \\ &-2\left(\sigma^{bc}H_{bc}\right) D_a\theta + 2H_a{}^b\sigma_b{}^c D_c\theta + \sigma_a{}^b H_b{}^c D_c\theta + 4\sigma^{bc}\sigma_b{}^d D_a H_{cd} - \\ &-2\sigma_a{}^b\sigma^{cd} D_b H_{cd} - \frac{1}{2}\sigma_a{}^b D_b\left(\sigma^{cd}H_{cd}\right) \,,\end{aligned} \tag{5.24}$$

which is not identically satisfied.

Degeneracy: The above results are based on the theorems in chapter 4 and are hence valid for the case of degeneracy in σ_{ab}, E_{ab} and $\mathrm{curl}\, H_{ab}$. In fact as in theorem 3b in chapter 4 degeneracy in the three tensors occurs in the same principal plane. Thus if $(\mathrm{div}\, H)_a = 0$ in the open set U and the shear σ_{ab} is degenerate then E_{ab}

is degenerate furthermore the shear eigenframe constrains $(\mathrm{curl}\, H)_{ab}$ to be diagonal and degenerate in the same plane.

Example. The above consistency analysis does not address the question of the existence of solutions with $(\mathrm{div}\, H)_a = 0$, $H \neq 0$ satisfying the integrability conditions. In [65] Maartens showed that such solutions do exist –as shown by *Bianchi type V spacetimes.* More details on Bianchi examples on this case are given in chapter 7. In short the Bianchi type V models are based on a triad of orthonormal spacelike vectors $\{\mathbf{e}_\mu\}$ which can be chosen to form a Fermi propagated shear eigenframe. The commutator functions defines the scalar $a := \gamma^\alpha{}_{0\alpha} \neq 0$. We obtain the following further properties of Bianchi type V spacetimes:

$$H_{ab} = \tfrac{1}{2}a(\sigma_{33} - \sigma_{22})\delta_{(a}{}^{2}\delta_{b)}{}^{3} \,. \tag{5.25}$$

$$\mathrm{D}^b H_{ab} = 0 \,, \tag{5.26}$$

$$(\mathrm{curl}\, H)_{ab} = \tfrac{1}{2}a^2(\sigma_{33} - \sigma_{22})\,\mathrm{diag}\,(0, 0, 1, -1) \,, \tag{5.27}$$

$$(\mathrm{curl}\,\mathrm{curl}\, H)_{ab} = 0 \,, \tag{5.28}$$

$$\mathrm{D}^b E_{ab} = a\sigma_{bc}\sigma^{bc}\delta_a{}^1 \,, \tag{5.29}$$

$$(\mathrm{curl}\, E)_{ab} = 0 \,. \tag{5.30}$$

The form of (5.27) verifies that the integrability condition (5.23) is satisfied. Both (5.23) and (5.20) and their subsequent covariant time derivatives are satisfied in a Fermi propagated frame. Although (5.26) is a necessary condition for gravity waves, it is *not sufficient*, and (5.28) and (5.30) confirm that the type V solutions cannot be interpreted as gravity waves. (This is seen from the identity (A.26) in appendix A which relates the Laplacian of a tensor $D^2 S_{ab}$ to the $(\mathrm{curl}\,\mathrm{curl}\, S)_{ab}$.) Nevertheless, these solutions do establish the existence of spacetimes with $(\mathrm{div}\, H)_a = 0 \neq H$.

5.5 Spacetimes with $DivE = 0$

The linear theory and Maxwell's electromagnetic theory are remarkably analogous. Ellis and Hogan (1996) [38, 47] showed that in the linear theory, propagation of arbitrary information leads to wave-like solutions with $(\operatorname{div} E)_a = 0$, $(\operatorname{curl} E)_{ab} \neq 0$ which can be interpreted as gravity waves. As in the previous section we show below that for the exact theory the divergence-free property on the electric Weyl tensor E_{ab} has integrability conditions.

Suppose that the electric Weyl tensor is divergence-free in an open set U, that is

$$\mathcal{B}_a = \mathrm{D}_b E^b{}_a = 0 \;, \qquad E_{ab} \neq 0 \;. \tag{5.31}$$

The propagation equations remain the same as in (3.5)–(3.8) and the only constraint equation that is affected is the Bianchi "$divE$" constraint (3.12); from which it follows that

$$(divE)_a = 0 \qquad \Leftrightarrow \qquad 0 \;=\; -\tfrac{1}{3}\mathrm{D}_a\rho - \varepsilon_{abc}\sigma^b{}_d H^{cd}. \tag{5.32}$$

Now assuming $(divE)_a = 0$ in an open set U implies $(divE)^{\cdot}_a = 0$. However as pointed out in section 5.3 it is the effect of $(divE)^{\cdot}_a = 0$ in the constraint/propagation equations that needs to be checked.

In the **linearized** case equation (5.32) becomes

$$(DivE)_a = 0 \qquad \Leftrightarrow \qquad 0 \;=\; \mathrm{D}_a\rho \;. \tag{5.33}$$

So now the effect of $(divE)^{\cdot}_a = 0$ in the constraint/propagation equation involves evolving

$$\mathcal{C}^5{}_a = \mathrm{D}_a\rho \;. \tag{5.34}$$

Using the linearized equations (5.1)–(5.2) of section (5.2) we first obtain the integrability condition

$$0 = \mathrm{D}_a\Theta \;, \tag{5.35}$$

and evolving (5.35) leads to the identity $0 = 0$. Furthermore (5.35) in (5.6)–(5.9) gives

$$D^b\sigma_{ab} = D^b E_{ab} = D^b H_{ab} = 0 \;, \tag{5.36}$$

which together with (5.34) and (5.35) means that there are no scalar or vector perturbations (see Hogan and Ellis [47]), thus:

> *In the linearized case irrotational dust spacetimes with $(DivE)_a = 0$ are consistent only for solutions with $D_a\Theta = 0$ and vanishing divergence of σ_{ab} and H_{ab}.*

In the **exact non-linear** case the effect of $(divE)_a^{\cdot} = 0$ in the propagation/constraint set involves evolving the equivalent condition

$$\mathcal{C}^5{}_a \;=\; -\tfrac{1}{3}\mathrm{D}_a\rho - \varepsilon_{abc}\sigma^b{}_d H^{cd}. \tag{5.37}$$

Taking the time derivative of (5.37) directly yields (see appendix B.5)

$$\begin{aligned}\dot{\mathcal{C}}^5_a \;=\;& -\frac{4}{3}\Theta\mathcal{C}^5_a - \frac{1}{3}\rho D_a\Theta \\ & -\frac{1}{3}\sigma_a{}^b D_b\rho - \frac{1}{2}\sigma_a{}^b\epsilon_{bcd}\sigma^c{}_p H^{dp} \\ & -\varepsilon_{abc}E^b{}_p H^{cp} - \varepsilon_{abc}\sigma^b{}_p(curlE)^{cp}\,.\end{aligned} \tag{5.38}$$

It then follows from (5.38) that $\dot{\mathcal{C}}^5_a = 0$ if

$$\begin{aligned}0 = J_a \;\equiv\;& -\frac{1}{3}\rho D_a\Theta - \frac{1}{3}\sigma_a{}^b D_b\rho - \frac{1}{2}\sigma_a{}^b\varepsilon_{bcd}\sigma^c{}_p H^{pd} \\ & -\varepsilon_{abc}E^b{}_p H^{cp} - \varepsilon_{abc}\sigma^b{}_p(curlE)^{cp}\,.\end{aligned} \tag{5.39}$$

The condition (5.39) is not identically satisfied and is the required integrability conditions.

> *For irrotational dust spacetime the divergence-free condition on the electric Weyl tensor introduces integrability conditions*

If we linearize (5.39) we obtain (5.35) as expected. We do not proceed with the further time derivatives of (5.39) however, unlike the $divH = 0$ integrability condition (5.23), further integrability conditions arising from (5.39) are not identically satisfied. But *the spatially homogeneous silent $H_{ab} = 0$ universes* consistently satisfy a divergence-free electric Weyl tensor and its subsequent integrability conditions. Since $\mathrm{D}_a\rho = 0$ by homogeneity, the Bianchi "div E" equation (3.12) shows that $(\mathrm{div}\,E)_a = 0$. But $E_{ab} \neq 0$, since otherwise $\sigma_{ab} = 0$ from the "$\dot{E}$" equation

(3.8). Note that from the "$\dot{H}$" equation (3.9) it follows that E_{ab} is *curl*–free, that is,

$$(divE)_a = 0 = \text{curl}\, E_{ab}\,, \quad E_{ab} \neq 0\,.$$

Hence the integrability condition (5.39) is satisfied but the solutions cannot be interpreted as gravity waves due to $(curl.curlE)_{ab} = 0$. (See chapter 7 for a full discussion of Bianchi examples for this case, where we also show that all subsequent time derivatives of (5.39) are also satisfied).

5.6 Divergence-free Weyl tensors

We consider now spacetimes with divergence-free electric and magnetic Weyl tensors,

$$(divH)_a = 0 = (divE)_a,\ H \neq 0,\ E_{ab} \neq 0\,. \tag{5.40}$$

Taken to be valid in an open set U, assumptions (5.40) do not change the form of the propagation equations (3.5)–(3.9) and therefore belong to the family of Class I conditions considered in this chapter. The "$(0,\nu)$" constraint (3.10) together with its evolution form (3.20) are identically satisfied. The only constraint equations affected by (5.40) are the Bianchi "div H" (3.12) and "div E" (3.13) constraints, from which it follows that

$$(divH)_a = 0 \;\Leftrightarrow\; \varepsilon_{abc}\sigma^b{}_d E^{cd} = 0\,, \tag{5.41}$$

$$(divE)_a = 0 \;\Leftrightarrow\; \tfrac{1}{3}D_a\rho + \varepsilon_{abc}\sigma^b{}_d H^{cd} = 0\,. \tag{5.42}$$

In the **linear** case evolving the $(\text{div}\, H)_a = 0$ condition introduces no new conditions. Evolving the $(\text{div}\, E)_a = 0$ condition, as in section 5.5, leads to integrability conditions restricting spacetime to $\mathrm{D}_a\Theta = 0$ models only.

In the **exact non-linear** case the time derivatives of (5.41), (5.42) are the same as in (5.22) and (5.38) and the consistency analysis is similar. Thus the integrability conditions introduced by (5.40) are those of $divH = 0$ namely, (5.20), (5.23) and those of $divE = 0$ namely, (5.39).

The existence of examples satisfying divergence-free Weyl tensor conditions (5.40) follows from Bianchi models of class A. For these models σ_{ab}, E_{ab} and H_{ab} have a common Fermi propagated eigenframe. Thus from (3.14), (3.15) it can be seen that $(\mathrm{div}\, H)_a = (\mathrm{div}\, E)_a = 0$. Here the *curl.curl* of the tensor quantities σ_{ab}, E_{ab} and H_{ab} do not necessarily vanish. For example using equations (7.12)–(7.14) the *Bianchi type II* spacetimes yields

$$E_{aa} = \tfrac{2}{3}\rho + 2a^2 - \tfrac{1}{2}n_1^2 - \Theta\Theta_{aa} + (\Theta_{aa})^2 \; ; \quad (\text{no sum}) . \tag{5.43}$$

and by replacing S_{ab} with σ_{ab}, E_{ab} and H_{ab} we obtain the non-zero *curls* of the three tensors from:

$$\begin{aligned}
(curlS)_{11} &= -\tfrac{3}{2}nS_{11} \; ; \\
(curlS)_{22} &= -\tfrac{1}{2}n(S_{33} - S_{11}) \; ; \\
(curlS)_{33} &= \tfrac{1}{2}n(S_{11} - S_{22}) \; ; \\
(curlcurlS)_{11} &= -\tfrac{9}{4}n^2 S_{11} \; ; \\
(curlcurlS)_{22} &= -\tfrac{1}{4}n^2(4S_{11} - S_{22}) \; ; \\
(curlcurlS)_{33} &= \tfrac{1}{4}n^2(S_{33} - 4S_{11}) \, .
\end{aligned} \tag{5.44}$$

The integrability condition (5.20) is satisfied by the Fermi eigenframe and using (5.44) and the shear commutation property (see page 48) we can also that the integrability conditions (5.23) and (5.39) and their further time derivatives are satisfied. Clearly the *curl.curl* of the tensorial quantities do not necessarily vanish and since the spacetime is spatially homogeneous, we may think of the pure gravitational waves as "standing" waves.

CHAPTER 6

Class II Conditions

6.1 Introduction

In this chapter we consider local conditions which affect the form of propagation equations of irrotational dust spacetime by converting them to constraint equations– we refer to these conditions as Class II conditions We focus mainly on irrotational dust spacetimes with (a) vanishing magnetic Weyl tensor, $H_{ab} = 0$ and (b) vanishing electric Weyl tensor, $E_{ab} = 0$.

In the first part of the chapter we show that for irrotational dust spacetimes the shear-free condition $\sigma_{ab} = 0$ assumed valid in an open set U leads to consistent solutions; these are just the FLRW universes. The converse to this is given by irrotational dust spacetimes with $E_{ab} = H_{ab} = 0$; these conditions also lead FLRW models, as they imply $\sigma_{ab} = 0$. However if $E_{ab} = H_{ab} = 0$ initially then these conditions remain valid at later times. Consequently irrotational dust with $E_{ab} = H_{ab} = 0$ describe an initial value formulation (unlike the $\sigma_{ab} = 0$ version which must be assumed valid in an open set).

An important concept that emerges in the rest of the chapter is that of the silent universe. These have been defined in section 1.3; the idea is expressed by Matarrese *et al.* (1994) [72] as follows:

> *Silent universes are spacetimes which are such that fluid elements evolves independently of each other so that no information, which was not already present in the initial conditions, is exchanged between the fluid elements.*

One criterion for a silent universe is that the propagation equations become ordinary differential equations. Consequently if *all* the constraint equations are satisfied, fluid elements evolves silently because no information, which was not already present in the initial conditions, is exchanged between the fluid elements.

In irrotational dust spacetimes $\dot{u}_a = 0$ and the silent universe condition is equivalent to setting the curls of the electric Weyl tensor E_{ab} and magnetic Weyl tensor H_{ab} to zero. These conditions are equivalent to the absence of any gravity waves. However there are difficulties in establishing consistency of $curlH = curlE = 0$ conditions and this may be an indication that there are very few silent irrotational dust universes. Clearly these conditions will be satisfied in any universe in which either $H_{ab} = 0$ or $E_{ab} = 0$. This is what we now consider in the second part of the chapter.

Taken as an exact condition $H_{ab} = 0$ restricts irrotational dust to the following spacetimes, using the algebraic classification of the Weyl tensor: (a) The completely general Petrov type I, (b) the degenerate type D and, (c) the conformally flat type O, characterised by $H_{ab} = 0 = E_{ab}$.

The type D spacetimes have a gravitational field which is purely Coulombian (Szekeres (1965) [90]) as in the Schwarzschild or Kerr solutions. The solutions are known (Barnes and Rowlingson (1989) [3]) and are due to Szekeres (1975) [91] (see also [41]), as well as the dust subclass of locally rotationally symmetric (LRS) class II spacetime geometries which include the Lemaître-Tolman-Bondi model and the orthogonally spatially homogeneous (OSH) Kantowski-Sachs model (see [100]). The type O spaces, which are the FLRW universes, are degenerate cases of the type D solutions, obtained from that class by choosing special initial conditions. The type I solutions are characterized by a superposition of purely Coulombian and transverse fields (Szekeres (1975) [91]). A Petrov type I example of a silent $H = 0$ model is provided by the orthogonally spatially homogeneous Bianchi Type-I dust solution (see [39]).

When $H_{ab} = 0$, the dynamic equations consist of a set of six ordinary differential propagation equations, namely the Raychaudhuri equation and the continuity equation, together with two pairs of equations for the independent components of

σ_{ab} and E_{ab} (Matarrese, Pantano and Saez (1993) [70]). Consequently spacetime evolves silently. There are no sound waves in dust, and when $H_{ab} = 0$ there can be no gravitational radiation. In addition to these propagation equations various constraint equations have to be satisfied. The generic constraints remain consistent as shown in chapter 3. However an additional constraint is introduced by the "$\dot{H}$" Bianchi identity. Physically this can be thought of as limiting the *curl* of the electric tidal field E_{ab}.

We show that for a Weyl tensor of Petrov type O and type D spacetimes, the equations are fully integrable. For the general Weyl tensor of Petrov type I we show that there exists a series of integrability conditions which hold identically in the linearized case but not in the non-linear exact case. Thus there is linearization instability. However due to the complexity of the non-linear integrability equations in the non-linear exact case we *conjecture* that there are no consistent spatially inhomogeneous solutions with a Weyl tensor of Petrov type I.

We conclude the chapter by showing that the purely magnetic spacetimes *i.e.*, $E_{ab} = 0, H_{ab} \neq 0$ run into similar integrability problems as the purely electric case. Since $E_{ab} = 0$ also implies no gravitational radiation [47], the anti-Newtonian models are also silent. Here we show that for irrotational dust anti-Newtonian spacetimes are subject to integrability conditions that are even more restrictive than in the $H_{ab} = 0$ case. The integrability conditions again form a non-terminating chain that leads to inconsistencies in general. There are two primary integrability conditions, each of which produces an indefinite chain of further conditions after differentiation. There may be spatially homogeneous models of this type which satisfy the integrability conditions, but we have been unable to find examples. However, a further result indicates that there are unlikely to be any consistent exact solutions. This result is that the only linearized irrotational dust solutions with $E_{ab} = 0$ are exactly FLRW (with $H_{ab} = 0$) – thus there are no linearized anti-Newtonian models $E_{ab} = 0$, $H_{ab} \neq 0$. Indeed, in the linearized theory, the primary conditions themselves lead to the vanishing of anisotropy and inhomogeneity.

6.2 Spacetimes with $\sigma_{ab} = 0$

We first consider the simpler case of irrotational dust with vanishing shear, $\sigma_{ab} = 0$, in an open set U. This is a Class II condition since the "$\dot{\sigma}$" propagation equation (3.7) is reduced to the constraint $E_{ab} = 0$. The "H_{ab}" constraint (3.11) becomes $H_{ab} = 0$ and using this it follows from the "$\dot{E}$" and "$\dot{H}$" equations (3.8) and (3.9) that $\dot{E}_{ab} = 0$ and $\dot{H}_{ab} = 0$ and thus both new constraints are consistent. The remaining variables describe FLRW spacetimes. Propagation equations are the continuity equation, $\dot{\rho} = -\Theta\rho$, and the Raychaudhuri equation $\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}\rho$. The two non-trivial constraints $D_a\Theta = 0$ and $D_a\rho = 0$ follow from the '$(0,\mu)$' field equations (3.10) and the Bianchi "div E" equation (3.12) respectively. Using the identity (A.13) we can show that both constraints are consistent *i.e.*, $(D_a\Theta)^\bullet = 0 = (D_a\rho)^\bullet$ and the FLRW spacetimes are consistent. The FLRW models have been discussed extensively in the literature (see Ellis (1971) [34] and further references therein).

(**Remark**: The shear-free rotating dust spacetimes are treated in Ellis (1967) [33], wherein an orthonormal tetrad approach is used to show that the consistency condition is $\omega\Theta = 0$. Using a covariant approach the same result was regained by Szekeres (1996) [92] and then generalized in [84] from dust to vanishing fluid four-acceleration.)

6.3 Spacetimes with $E_{ab} = 0 = H_{ab}$

The converse to the above shear-free case is given by irrotational dust spacetimes with vanishing electric Weyl tensor and magnetic Weyl tensor, that is $E_{ab} = 0 = H_{ab}$ (see Trümper [97]). This is also a Class II condition for two of the generic propagation are now reduced to one constraint equation: the "$\dot{H}$" equation (3.9) is identically satisfied and the '$\dot{E}$' equation (3.8) becomes the constraint $0 = \rho\sigma_{ab}$. For dust spacetimes $\rho \neq 0$ and thus the constraint arising from the "$\dot{E}$" equation is precisely $\sigma_{ab} = 0$. This constraint is consistent as seen from the "$\dot{\sigma}$" (3.7) equation and the "H_{ab}" equation (3.11). Consequently spacetimes with $E_{ab} = 0 = H_{ab}$ are

FLRW spacetimes and are consistent as shown in the above section. Furthermore this is an initial value formulation (unlike the $\sigma_{ab} = 0$ case, in the above section, which must be assumed valid in an open set U) for if $E_{ab} = H_{ab} = 0$ initially then from the "H_{ab}" constraint (3.11), the "$\dot{E}$" equation (3.8) and the "$\dot{H}$" equation (3.9) the variables σ_{ab}, E_{ab} and H_{ab} remain zero at later times.

6.4 Spacetimes with $H_{ab} = 0$

In linear theory the cosmological context of $H_{ab} = 0$ can be visualized as "switching off" pure gravitational waves and all gravo-magnetic effects. In this section we investigate whether this scenario extends to the exact theory. To this effect we consider irrotational dust spacetimes with non-zero electric Weyl tensor but the magnetic Weyl tensor vanishing in an open set U

$$\mathcal{B}^{\mathbf{1}}{}_{ab} = H_{ab} = 0 \,, \qquad E_{ab} \neq 0 \,. \tag{6.1}$$

We then use (6.1) in the generic equations (3.5)–(3.13) in chapter 3 to write the propagation equations as

$$\dot{\rho} = -\Theta\rho \,, \tag{6.2}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\rho - \sigma_{ab}\sigma^{ab} \,, \tag{6.3}$$

$$\dot{\sigma}_{ab} = -E_{ab} - \tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c<a}\sigma_{b>}{}^{c} \,, \tag{6.4}$$

$$\dot{E}_{ab} = -\Theta E_{ab} + 3\sigma_{c<a}E_{b>}{}^{c} - \tfrac{1}{2}\rho\sigma_{ab} \,, \tag{6.5}$$

and the constraint equations as

$$\mathcal{C}^{\mathbf{1}}{}_{a} = \tfrac{3}{2}D^b\sigma_{ab} - \mathrm{D}_a\Theta \,, \tag{6.6}$$

$$\mathcal{C}^{\mathbf{2}}{}_{ab} = \mathcal{C}^{\mathbf{2}*} + \mathcal{B}^{\mathbf{1}}{}_{ab} \,, \tag{6.7}$$

$$\mathcal{C}^{\mathbf{3}}{}_{a} = \mathcal{C}^{\mathbf{3}*} + \mathcal{B}^{\mathbf{2}}{}_{a} \,, \tag{6.8}$$

$$\mathcal{C}^{\mathbf{4}}{}_{a} = \mathcal{C}^{\mathbf{4}*} + \mathcal{B}^{\mathbf{3}}{}_{a} \,, \tag{6.9}$$

where

$$\begin{aligned} \mathcal{C}^{2*}{}_{ab} &:= \operatorname{curl}\sigma_{ab} , & (6.10)\\ \mathcal{C}^{3*}{}_{a} &:= \mathrm{D}^b E_{ab} - \tfrac{1}{3}\mathrm{D}_a\rho , & (6.11)\\ \mathcal{C}^{4*}{}_{a} &:= \varepsilon_{abc}\sigma^b{}_d E^{cd} ; & (6.12)\end{aligned}$$

and the vanishing terms are

$$\mathcal{B}^1{}_{ab} = H_{ab} , \quad \mathcal{B}^2{}_a = -\varepsilon_{abc}\sigma^b{}_d H^{cd} , \quad \mathcal{B}^3{}_a = \mathrm{D}^b H_{ab} . \tag{6.13}$$

The constraint (6.6) is the same as in the generic case whereas the constraints $\mathcal{C}^A$, $A = 2,3,4$ are algebraically modified and, as in the previous chapter, their spatial consistency does not add any new constraints to the system $\mathcal{C}^A$. Furthermore, if the constraints are satisfied, the evolution equations of the constraints $\mathcal{C}^A$ along u^a do not introduce any new constraints. However an additional constraint arises from the generic "$\dot{H}_{ab}$" propagation equation (3.9) namely, that the curl of the electric Weyl tensor vanishes:

$$\mathcal{C}^5{}_{ab} = \operatorname{curl} E_{ab} . \tag{6.14}$$

As in the previous chapter we will first use the covariant identities of Maartens [64] to analyse the consistency of equation (6.14). We will later adopt the tetrad notation of chapter 4 and its modified form as formulated in the $1+3$ orthonormal frame used by van Elst *et al.* [98], [101].

Spatial consistency of $\mathcal{C}^5$: From (A.21), (A.23) and (6.8), (6.9) we get

$$\mathrm{D}^b\mathcal{C}^5{}_{ab} = -\tfrac{1}{2}\operatorname{curl}\mathcal{C}^{3*}{}_a + \tfrac{1}{3}\Theta\mathcal{C}^{4*}{}_a + \sigma_a{}^b\mathcal{C}^{4*}{}_b , \tag{6.15}$$

thus no consistency condition arises from the divergence, and $\operatorname{div}\mathcal{C}^5$ is determined by $\mathcal{C}^A$, where $A = \mathbf{3}*$, $\mathbf{4}*$. Also as in equation (3.15) it follows that $\mathcal{C}^{4*}{}_a = \frac{1}{2}\operatorname{curl}\mathcal{C}^1{}_a - D^b\mathcal{C}^{2*}{}_{ab}$. The spatial consistency for the generic constraints, established in chapter 3, remains unchanged.

Integrability of $\mathcal{C}^5$. Using (A.17), (A.24) and (6.5), (6.7) we get

$$\begin{aligned} \dot{\mathcal{C}}^5{}_{ab} &= -\tfrac{4}{3}\Theta\mathcal{C}^5{}_{ab} - \tfrac{3}{2}\varepsilon^{cd}{}_{(a}E_{b)c}\mathcal{C}^1{}_d \\ &\quad - \tfrac{1}{2}\rho\mathcal{C}^{2*}{}_{ab} - \tfrac{3}{2}\varepsilon^{cd}{}_{(a}\sigma_{b)c}\mathcal{C}^{3*}{}_d + \tfrac{3}{2}\mathcal{A}_{ab} , \end{aligned} \tag{6.16}$$

where

$$\begin{aligned}\mathcal{A}_{ab} &= \varepsilon_{cd(a}\left\{E_{b)}{}^{c}\mathrm{D}^{e}\sigma^{d}{}_{e} + \mathrm{D}^{c}\left[\sigma_{b)e}E^{de}\right]\right.\\ &\quad \left. + \mathrm{D}^{c}\left[E_{b)e}\sigma^{de}\right] + \sigma_{b)}{}^{c}\mathrm{D}^{e}E^{d}{}_{e} - \tfrac{2}{3}\sigma^{c}{}_{e}\mathrm{D}^{e}E_{b)}{}^{d}\right\}\,. \qquad (6.17)\end{aligned}$$

By (6.13), $\varepsilon_{abc}\sigma^{b}{}_{d}E^{cd} = 0$ so that $\mathrm{curl}\,(\sigma E) = \mathrm{curl}\,(E\sigma)$, and we can rewrite (6.17) as

$$\begin{aligned}\mathcal{A}_{ab} &= \varepsilon_{cd(a}\left\{\mathrm{D}^{e}\left[E_{b)}{}^{c}\sigma^{d}{}_{e}\right] + 2\mathrm{D}^{c}\left[\sigma_{b)e}E^{de}\right]\right.\\ &\quad \left. + \sigma_{b)}{}^{c}\mathrm{D}^{e}E^{d}{}_{e} + \tfrac{1}{3}\sigma^{c}{}_{e}\mathrm{D}^{e}E_{b)}{}^{d}\right\}\,. \qquad (6.18)\end{aligned}$$

From (6.16) we see that a necessary condition for consistency in silent universes is the covariant condition

$$\mathcal{A}_{ab} = 0\,. \qquad (6.19)$$

It follows from (6.18) that (6.19) is satisfied in the linearized case, *i.e.*,

In linearized purely electric irrotational dust spacetimes the constraint equations evolve consistently.

This confirms the result of linear theory where $H_{ab} = 0$ consistently ensures that there are no gravitational waves (see (5.1)–(5.9)). In the following sections we show that a further analysis of (6.18), in a shear eigen-tetrad frame, indicates that in the non-linear case, the condition (6.19) is non–trivial, *i.e.*, not all silent $H = 0$ solutions are consistent.

6.4.1 Tetrad approach

A direct conversion of the propagation equations (6.2)–(6.5) into a tetrad system, which is an eigenframe for both the shear tensor and the electric Weyl tensor (see theorem 3a and theorem 4, chapter 4), yields the following propagation equations

(Barnes and Rowlingson (1989) [3]);

$$
\begin{aligned}
\dot{\rho} &= -\rho\Theta\,, \\
\dot{\Theta} &= -\tfrac{1}{3}\Theta^2 - ({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2) - \tfrac{1}{2}\rho\,, \\
\dot{\sigma}_\mu &= -(\sigma_\mu)^2 - \tfrac{2}{3}\Theta\sigma_\mu + \tfrac{1}{3}({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2) - E_\mu\,, \\
\dot{E}_\mu &= -\Theta E_\mu - \tfrac{1}{2}\rho\sigma_\mu + 3\sigma_\mu E_\mu - (\sigma_1 E_1 + \sigma_2 E_2 + \sigma_3 E_3)\,,
\end{aligned} \tag{6.20}
$$

where the first two equations are the tetrad forms of (6.2), (6.3) respectively, and the last two equations are the diagonal parts of (6.4) and (6.5) respectively. The non-diagonal parts of (6.4) and (6.5) (the propagation equations for $\dot{E}_{ab}$ and $\dot{\sigma}_{ab}$ with $a \neq b$) introduce two additional conditions;

$$0 = (\sigma_\nu - \sigma_\mu){\Gamma^\nu}_{0\mu}\,, \quad \mu \neq \nu\,, \tag{6.21}$$

$$0 = (E_\nu - E_\mu){\Gamma^\nu}_{0\mu}\,, \quad \mu \neq \nu\,, \tag{6.22}$$

where the Ricci rotation coefficients Γ_{abc} are defined in (4.7) and one can raise and lower indices in the first place as usual by using the metric.

The tetrad used here is an orthonormal tetrad, as discussed in section 4.2. As noted there, if we denote the local coordinate system by $\{x^i\}$ and the tetrad by $\{\mathbf{e}_a\}$ then the equations (Ellis (1967) [33]) $\mathbf{e}_a = e^i_a(\partial/\partial x^i)$ define the functions e^i_a, which are components of the tetrad vectors $\mathbf{e}_a$ with respect to the basis $\partial/\partial x^i$, and are also directional derivatives of the coordinate functions x^i as $e^i_a = \partial_a(x^i)$.

This choice of a tetrad simultaneously diagonalizes the shear tensor, the electric Weyl tensor and their covariant time derivatives (see equation (6.20)). So now using the results of theorem 1a in chapter 4 the tetrad vectors are Fermi propagated:

$${\Gamma^\nu}_{0\mu} = 0; \quad \mu \neq \nu\,. \tag{6.23}$$

Also it follows trivially from the propagation equations (6.20) that $E_\mu = 0$ if and only if $\sigma_\mu = 0$ (FLRW model).

Constraint equations

The constraint equations give spatial restrictions on the tetrad form of the dynamical variables. The $(0,\nu)$ field equations (6.6) are:

$$\tfrac{2}{3}\partial_1\theta = \partial_1\sigma_1 + (\sigma_1-\sigma_2)\Gamma^2{}_{21} + (\sigma_1-\sigma_3)\Gamma^3{}_{31}\,, \tag{6.24}$$

$$\tfrac{2}{3}\partial_2\theta = \partial_2\sigma_2 + (\sigma_2-\sigma_1)\Gamma^1{}_{12} + (\sigma_2-\sigma_3)\Gamma^3{}_{32}\,, \tag{6.25}$$

$$\tfrac{2}{3}\partial_3\theta = \partial_3\sigma_3 + (\sigma_3-\sigma_1)\Gamma^1{}_{13} + (\sigma_3-\sigma_2)\Gamma^2{}_{23}\,. \tag{6.26}$$

The "H_{ab}" constraint (6.7) for vanishing H_{ab} takes the form:

$$\Gamma^1{}_{32}(\sigma_2-\sigma_1) = \Gamma^1{}_{23}(\sigma_3-\sigma_1)\,, \tag{6.27}$$

$$\Gamma^2{}_{31}(\sigma_1-\sigma_2) = \Gamma^2{}_{13}(\sigma_3-\sigma_2)\,, \tag{6.28}$$

$$\Gamma^3{}_{12}(\sigma_2-\sigma_3) = \Gamma^3{}_{21}(\sigma_1-\sigma_3)\,, \tag{6.29}$$

$$\partial_1(\sigma_3-\sigma_2) = \Gamma^3{}_{31}(\sigma_1-\sigma_3) - \Gamma^2{}_{21}(\sigma_1-\sigma_2)\,, \tag{6.30}$$

$$\partial_2(\sigma_3-\sigma_1) = \Gamma^3{}_{32}(\sigma_2-\sigma_3) - \Gamma^1{}_{12}(\sigma_2-\sigma_1)\,, \tag{6.31}$$

$$\partial_3(\sigma_2-\sigma_1) = \Gamma^2{}_{23}(\sigma_3-\sigma_2) - \Gamma^1{}_{13}(\sigma_3-\sigma_1)\,. \tag{6.32}$$

The "div E" constraint (6.8) takes the form:

$$\tfrac{1}{3}\partial_1\rho = \partial_1E_1 + (E_1-E_2)\Gamma^2{}_{21} + (E_1-E_3)\Gamma^3{}_{31}\,, \tag{6.33}$$

$$\tfrac{1}{3}\partial_2\rho = \partial_2E_2 + (E_2-E_1)\Gamma^1{}_{12} + (E_2-E_3)\Gamma^3{}_{32}\,, \tag{6.34}$$

$$\tfrac{1}{3}\partial_3\rho = \partial_3E_3 + (E_3-E_1)\Gamma^1{}_{13} + (E_3-E_2)\Gamma^2{}_{23}\,. \tag{6.35}$$

The "div H" constraint equations (6.9) are identically satisfied, indeed it is they that allowed us to simultaneously diagonalize the shear tensor and the electric Weyl tensor (as shown in theorem 4, chapter 4).

Finally the new constraint (6.14), due to the dynamical restriction $\dot{H}_{\mu\nu} = 0$, becomes

$$\Gamma^1{}_{32}(E_2-E_1) = \Gamma^1{}_{23}(E_3-E_1)\,, \tag{6.36}$$

$$\Gamma^2{}_{31}(E_1-E_2) = \Gamma^2{}_{13}(E_3-E_2)\,, \tag{6.37}$$

$$\Gamma^3{}_{12}(E_2-E_3) = \Gamma^3{}_{21}(E_1-E_3)\,, \tag{6.38}$$

$$\partial_1(E_3 - E_2) = \Gamma^3{}_{31}(E_1 - E_3) - \Gamma^2{}_{21}(E_1 - E_2)\,, \tag{6.39}$$

$$\partial_2(E_3 - E_1) = \Gamma^3{}_{32}(E_2 - E_3) - \Gamma^1{}_{12}(E_2 - E_1)\,, \tag{6.40}$$

$$\partial_3(E_2 - E_1) = \Gamma^2{}_{23}(E_3 - E_2) - \Gamma^1{}_{13}(E_3 - E_1)\,. \tag{6.41}$$

The tetrad form of $\mathcal{A}_{ab}$ in (6.18), (6.19), arising from the constraint evolution equation (6.16), becomes

$$0 = \Gamma^1{}_{23}(E_3 - E_1)(\sigma_2 - \sigma_3)\,, \tag{6.42}$$

$$0 = \Gamma^2{}_{31}(E_1 - E_2)(\sigma_1 - \sigma_3)\,, \tag{6.43}$$

$$0 = \Gamma^3{}_{12}(E_2 - E_3)(\sigma_1 - \sigma_2)\,, \tag{6.44}$$

for $\mu = \nu$ and

$$\begin{aligned} 0 &= (E_2 - E_3)\partial_1\sigma_3 + (\sigma_2 - \sigma_3)\partial_1 E_2 + \tfrac{1}{3}\Gamma^3{}_{31}(\sigma_1 - \sigma_3)(5E_1 + 4E_3) \\ &- \tfrac{1}{3}\Gamma^2{}_{21}(E_1 - E_2)(5\sigma_1 + 4\sigma_2)\,, \end{aligned} \tag{6.45}$$

$$\begin{aligned} 0 &= (E_3 - E_1)\partial_2\sigma_1 + (\sigma_3 - \sigma_1)\partial_2 E_3 + \tfrac{1}{3}\Gamma^1{}_{12}(\sigma_2 - \sigma_1)(5E_2 + 4E_1) \\ &- \tfrac{1}{3}\Gamma^3{}_{32}(E_2 - E_3)(5\sigma_2 + 4\sigma_3)\,, \end{aligned} \tag{6.46}$$

$$\begin{aligned} 0 &= (E_1 - E_2)\partial_3\sigma_2 + (\sigma_1 - \sigma_2)\partial_3 E_1 + \tfrac{1}{3}\Gamma^2{}_{23}(\sigma_3 - \sigma_2)(5E_3 + 4E_2) \\ &- \tfrac{1}{3}\Gamma^1{}_{13}(E_3 - E_1)(5\sigma_3 + 4\sigma_1)\,. \end{aligned} \tag{6.47}$$

for $\mu \neq \nu$.

To pursue the analysis further consider the following specializations based on the Petrov type O, D and I classification of the Weyl tensor with $H_{ab} = 0$ (see [58]).

6.4.2 Specializations

Petrov type O

For this FLRW class, considered in sections 6.2 and 6.3, $E_\nu = 0 \Leftrightarrow \sigma_\nu = 0$, all constraint equations are trivially satisfied except for the Friedmann equation which controls the dynamics, and the evolution is that of the FLRW models because we are assuming $\rho \neq 0$ for dust.

Petrov type D

Without loss of generality we set $E_1 = E_2 = \mathcal{E} \neq E_3$. The following tetrad properties are valid:

1. From equations (6.41) and (6.36, 6.37) we obtain:

$$\Gamma^1{}_{13} = \Gamma^2{}_{23} , \tag{6.48}$$

$$\Gamma^1{}_{23} = \Gamma^2{}_{13} = 0 . \tag{6.49}$$

2. From either (6.27) or (6.28) we write:

$$(\sigma_2 - \sigma_1)\Gamma^1{}_{32} = 0 , \tag{6.50}$$

which has the following two subcases:

(a) For $\sigma_2 = \sigma_1 = \Sigma \neq \sigma_3$ the tetrad is free by a rotation in the e_1, e_2 plane. As pointed out in section 4.3, a rotation in that plane can be performed so that equation (6.23) remains valid. This leaves the tetrad arbitrary by an initial rotation in a surface $x^0 = const$. To determine the tetrad up to an initial angle ϕ_0 we perform a further rotation of e_1 , e_2 which preserves (6.23), where the value of $\partial\phi/\partial x^3$ is determined from the requirement that $\Gamma^1{}_{32} = 0$ in a surface $x^0 = const$. From the Jacobi identities (listed in Appendix A.2);

$$\begin{pmatrix} 1 \\ 023 \end{pmatrix} : \quad \partial_0\gamma^1{}_{32} = \gamma^1{}_{32}(\theta_1 - \theta_2 - \theta_3) , \tag{6.51}$$

$$\begin{pmatrix} 2 \\ 031 \end{pmatrix} : \quad \partial_0\gamma^2{}_{13} = \gamma^2{}_{13}(-\theta_1 + \theta_2 - \theta_3) , \tag{6.52}$$

$$\begin{pmatrix} 3 \\ 012 \end{pmatrix} : \quad \partial_0\gamma^3{}_{12} = \gamma^3{}_{12}(-\theta_1 - \theta_2 + \theta_3) , \tag{6.53}$$

it follows that

$$\Gamma^1{}_{32} = 0 , \tag{6.54}$$

everywhere. The tetrad vectors can hence be chosen to be hypersurface orthogonal. This result was also obtained in Barnes and Rowlingson (1989) [3]. For this class integrability conditions are consistently satisfied as follows:

i. Constraints (6.30), (6.31) and (6.39), (6.40) are written respectively as

$$\begin{aligned} \partial_1 \Sigma &= -\Sigma {\Gamma^3}_{31} \ ; \quad \partial_2 \Sigma = -\Sigma {\Gamma^3}_{32} \ , && (6.55) \\ \partial_1 \mathcal{E} &= -\mathcal{E} {\Gamma^3}_{31} \ ; \quad \partial_2 \mathcal{E} = -\mathcal{E} {\Gamma^3}_{32} \ , && (6.56) \end{aligned}$$

with constraints (6.32) and (6.41) identically satisfied due to equation (6.48).

ii. First we recall that the time propagation of constraints (6.27)–(6.32) are identically satisfied. Also the time propagation equations (6.42)–(6.44) and (6.47) are identically satisfied in this class. On the other hand time propagation equations (6.45), (6.46) take the forms;

$$\begin{aligned} 0 &= -2\mathcal{E}\partial_1 \Sigma + \Sigma \partial_1 \mathcal{E} - \Sigma \mathcal{E} {\Gamma^3}_{31} \ ; && (6.57) \\ 0 &= E\partial_2 \Sigma + 2\Sigma \partial_2 \mathcal{E} + \Sigma \mathcal{E} {\Gamma^3}_{32} \ , && (6.58) \end{aligned}$$

which are identically satisfied on using (6.55) and (6.56).

(b) For $\sigma_2 \neq \sigma_1$: We note that if $\sigma_2 \neq \sigma_1 = \sigma_3$ then from (6.20) we get $E_1 = E_3$. Similarly $\sigma_1 \neq \sigma_2 = \sigma_3$ implies $E_2 = E_3$. Both cases fall into the specified Type D class. So for non-vanishing shear, the eigenvalues are distinct. Furthermore from the time propagation equation (6.20) we get

$$0 = \dot{E}_2 - \dot{E}_1 = (\sigma_2 - \sigma_1)(3\mathcal{E} - \tfrac{1}{2}\rho) \ , \qquad (6.59)$$

from which it follows that

$$\mathcal{E} = \tfrac{1}{6}\rho \ . \qquad (6.60)$$

Taking the time derivative of (6.60) gives

$$\mathcal{E}(\sigma_1 + \sigma_2) = 0 \ , \qquad (6.61)$$

from which we get $\mathcal{E} = 0$ (if and only if $\Sigma = 0$) or

$$(\sigma_1 + \sigma_2) = 0 \ . \tag{6.62}$$

By a series of three further time derivatives of (6.62) one may show that the eigenvalues E_i of the Weyl tensor vanish (*i.e.*, it is type O rather than type D), and leads to an FLRW solution.

This proves the following:

> *For irrotational dust with a purely electric type Weyl tensor that is* **degenerate**, *the shear is also degenerate in the same plane (see also theorem 3b in chapter 4), furthermore, the constraint equations are consistent.*

Petrov type I

In this case, $E_1 \neq E_2 \neq E_3 \neq E_1$ and the Fermi tetrad has the following additional properties:

1. From the propagation equations (6.20) it follows that the shear eigenvalues are also distinct. For if say $\sigma_1 = \sigma_2$ then from (6.20) we get $E_1 = E_2$ which contradicts the requirements of this class.

2. The tetrad vectors are uniquely determined.

3. From the new constraint (6.42)–(6.44), that is the time development of the constraint equations (6.36)–(6.38), we note that the spatial tetrad vectors are hypersurface orthogonal [3] *i.e.*,

$$\Gamma^1{}_{23} = \Gamma^2{}_{31} = \Gamma^3{}_{12} = 0 \ . \tag{6.63}$$

A previous analysis of this case in Lesame *et al.* (1995) [58] presented an erroneous argument as pointed out by Bonilla *et al.* (1996) [10]. However the latter paper does not carry out a consistency check for the Weyl type I class. This problem is revisited in van Elst, Uggla, Lesame, Ellis and Maartens (1997) [102],[101] by further specializing the tetrad frame. This specialization is suitable for extensive usage of computer algebra packages in the analysis and overlaps with the work done by van Elst (1996) [98]. In particular the computer algebra package **REDUCE** has been used to obtain the results of [102] which are summarized below:

We use, for both tensors σ_{ab} and E_{ab}, the tracefree-adapted irreducible frame components (here: $A_{ab}\,u^b = 0$, $A^a{}_a = 0$, $A^a{}_b\,A^b{}_a \geq 0$)

$$\begin{aligned} &A_+ := -\tfrac{3}{2}\,A_{11} \;\; = \tfrac{3}{2}\,(A_{22} + A_{33})\;, \quad A_- := \tfrac{\sqrt{3}}{2}\,(A_{22} - A_{33})\;, \\ &A_1 := \sqrt{3}\,A_{23}\;, \quad A_2 := \sqrt{3}\,A_{31}\;, \qquad A_3 := \sqrt{3}\,A_{12}\;, \end{aligned} \tag{6.64}$$

and this implies that

$$0 = A_1 = A_2 = A_3 \quad \Rightarrow \quad A^2 = \tfrac{1}{3}\,[\,(A_+)^2 + (A_-)^2\,]\,. \tag{6.65}$$

All the tetrad equations obtained in the above section can be written in terms of the $1+3$ orthonormal frame approach [101, 102, 98] by representing the commutation coefficients $\gamma^a{}_{bc}$ and related Christoffel symbols $\Gamma^a{}_{bc}$ by the quantities $n_{\alpha\beta}, a_\beta$ given in (4.13) (see also chapter 7). As was shown by Barnes and Rowlingson [3], it follows from (6.63) that $n_{\alpha\alpha} = 0$ (no sum). The tetrad commutators become

$$[\,\mathbf{e}_0, \mathbf{e}_\alpha\,] \;=\; -\left(\tfrac{1}{3}\Theta\delta^\beta{}_\alpha + \sigma^\beta{}_\alpha\right)\mathbf{e}_\beta\;, \tag{6.66}$$

$$[\,\mathbf{e}_\alpha, \mathbf{e}_\beta\,] \;=\; \left(2a_{[\alpha}\delta^\tau{}_{\beta]} + \varepsilon_{\alpha\beta\sigma}n^{\sigma\tau}\right)\mathbf{e}_\tau\;. \tag{6.67}$$

In particular the necessary condition, $\mathcal{A}_{ab} = 0$, in (6.18), (6.19) for consistency in silent $H = 0$ universes may be written in the $1 + 3$ orthonormal frame as the conditions:

$$\begin{aligned} 0 &= E_- \, \mathbf{e}_1(\Theta) + \tfrac{1}{2}\sigma_- \, \mathbf{e}_1(\mu) - 2\,(a_1\,\sigma_- + \sqrt{3}\,n_{23}\,\sigma_+)\,E_+ \\ &\quad - 2\,(a_1\,\sigma_+ + \tfrac{1}{\sqrt{3}}n_{23}\,\sigma_-)\,E_- \,, \qquad (6.68) \\ 0 &= (E_+ - \tfrac{1}{\sqrt{3}}E_-)\,\mathbf{e}_2(\Theta) + \tfrac{1}{2}\,(\sigma_+ - \tfrac{1}{\sqrt{3}}\sigma_-)\,\mathbf{e}_2(\mu) \\ &\quad + 2\,(a_2 - n_{31})\,(\sigma_+ + \tfrac{1}{\sqrt{3}}\sigma_-)\,E_+ \\ &\quad + \tfrac{2}{\sqrt{3}}a_2\,(\sigma_+ - \sqrt{3}\,\sigma_-)\,E_- - \tfrac{2}{\sqrt{3}}n_{31}\,(\sigma_+ + \tfrac{5}{\sqrt{3}}\sigma_-)\,E_- \,, \qquad (6.69) \\ 0 &= (E_+ + \tfrac{1}{\sqrt{3}}E_-)\,\mathbf{e}_3(\Theta) + \tfrac{1}{2}\,(\sigma_+ + \tfrac{1}{\sqrt{3}}\sigma_-)\,\mathbf{e}_3(\mu) \\ &\quad + 2\,(a_3 + n_{12})\,(\sigma_+ - \tfrac{1}{\sqrt{3}}\sigma_-)\,E_+ \\ &\quad - \tfrac{2}{\sqrt{3}}a_3\,(\sigma_+ + \sqrt{3}\,\sigma_-)\,E_- - \tfrac{2}{\sqrt{3}}n_{12}\,(\sigma_+ - \tfrac{5}{\sqrt{3}}\sigma_-)\,E_- \,. \qquad (6.70) \end{aligned}$$

These equations are equivalent to the tetrad expressions (6.45)–(6.47) derived above and can be interpreted as expressions for the spatial 3-gradients of the fluid rate of expansion, $\mathbf{e}_\alpha(\Theta)$. For the Petrov type I case ($E_- \neq 0 \Rightarrow \sigma_- \neq 0$), contrary to what was claimed by Lesame *et al.*, (1995) [58], equations (6.68)–(6.70) do *not* vanish identically[1], but constitute a *new* set of constraints. However it follows easily that:

Equations (6.68)–(6.70) are trivially satisfied, if a Petrov type I spacetime geometry is of orthogonal spatially homogeneous Bianchi Type–I ($\mathbf{e}_\alpha(f) = 0$, $0 = a_\alpha = n_{\alpha\beta}$).

In the non-homogeneous case, propagating equations (6.68)–(6.70) along the fluid flow vector u^a and resubstituting from known relations, one obtains algebraic expressions for the 3-gradients of the fluid's total energy density, $\mathbf{e}_\alpha(\mu)$, in the form

$$\mathbf{e}_1(\mu) = f_1[\, a_1, n_{23}, \sigma_+, \sigma_-, E_+, E_-, \mu \,] \,, \qquad (6.71)$$

[1] This was demonstrated by Bonilla *et al.* [10].

$$\mathbf{e}_2(\mu) = g_1[\, a_2, n_{31}, \sigma_+, \sigma_-, E_+, E_-, \mu \,]\,, \tag{6.72}$$

$$\mathbf{e}_3(\mu) = h_1[\, a_3, n_{12}, \sigma_+, \sigma_-, E_+, E_-, \mu \,]\,. \tag{6.73}$$

Here, f_1, g_1 and h_1 are multivariate polynomial expressions of the variables indicated. Each individual term therein is linear in either a_α or $n_{\alpha\beta}$, and contains a factor of either a power of σ_- or a power of E_-.

For illustrative purposes, we explicitly give the exact form of Eq. (6.71). We have that

$$\begin{aligned} \mathbf{e}_1(\mu) &= f_1[\, a_1, n_{23}, \sigma_+, \sigma_-, E_+, E_-, \mu \,] \\ &= 4\,[\, a_1\, A_1 + \tfrac{1}{2\sqrt{3}}\, n_{23}\, B_1 \,] \,/\, C_1\,, \end{aligned} \tag{6.74}$$

where

$$\begin{aligned} A_1 \;:=\;& 6\,\sigma_+\,\sigma_-\,E_+\,E_- - (\sigma_-)^2\,\mu\,E_+ \\ &+(\sigma_-)^2\,(E_+)^2 - (\sigma_-)^2\,(E_-)^2 + 2\,E_+\,(E_-)^2\,, \end{aligned} \tag{6.75}$$

$$\begin{aligned} B_1 \;:=\;& 3\,(\sigma_+)^2\,\mu\,E_- + 24\,(\sigma_+)^2\,E_+\,E_- - 6\,\sigma_+\,\sigma_-\,\mu\,E_+ + 6\,\sigma_+\,\sigma_-\,(E_+)^2 \\ &- 6\,\sigma_+\,\sigma_-\,(E_-)^2 - (\sigma_-)^2\,\mu\,E_- + 4\,(\sigma_-)^2\,E_+\,E_- \\ &+ 6\,(E_+)^2\,E_- + 2\,(E_-)^3\,, \end{aligned} \tag{6.76}$$

$$C_1 \;:=\; 3\,\sigma_+\,\sigma_-\,E_- - (\sigma_-)^2\,\mu + (\sigma_-)^2\,E_+ + 2\,(E_-)^2\,. \tag{6.77}$$

The right hand sides of the remaining conditions (6.72) and (6.73) at this and all subsequent levels of differentiation can be obtained from a transformation rule related to a cyclic permutation of the axes of the spatial frame $\{\, \mathbf{e}_\alpha \,\}$. This rule is given by making the substitutions

$$1\ (23) \rightarrow 2\ (31) \rightarrow 3\ (12) \rightarrow 1\,(23),$$

$$A_+ \rightarrow -\tfrac{1}{2}\,(A_+ + \sqrt{3}\,A_-) \rightarrow -\tfrac{1}{2}\,(A_+ - \sqrt{3}\,A_-) \rightarrow A_+\,,$$

and

$$A_- \to \tfrac{1}{2}(\sqrt{3}\,A_+ - A_-) \to -\tfrac{1}{2}(\sqrt{3}\,A_+ + A_-) \to A_- .$$

Now propagating equation (6.71)–(6.73) along the fluid flow vector u^a and re-substituting from known relations, one obtains algebraic expressions for the spatial commutation functions a_α in the form

$$\begin{aligned} a_1 &= n_{23}\, f_2[\,\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu\,]\,, && (6.78)\\ a_2 &= n_{31}\, g_2[\,\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu\,]\,, && (6.79)\\ a_3 &= n_{12}\, h_2[\,\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu\,]\,, && (6.80)\end{aligned}$$

where again f_2, g_2 and h_2 are multivariate polynomial expressions of the variables indicated, with each individual term containing a factor of either a power of σ_- or a power of E_-. The exact form of (6.78) is as $f_2 = \frac{(Numf)_2}{(Denf)_2}$ where

$$\begin{aligned} (Numf)_2 \;:=\; & 44\sqrt{3}E_-^4\sigma_- - 84\sqrt{3}E_-^3E_+\sigma_+ + 160\sqrt{3}E_-^3\sigma_-^2\sigma_+ + 4\sqrt{3}E_-^3\sigma_-^2\Theta \\ & -48\sqrt{3}E_-^2E_+^2\sigma_- - 16\sqrt{3}E_-^2E_+\sigma_-^3 - 312\sqrt{3}E_-^2E_+\sigma_-\sigma_+^2 \\ & -12\sqrt{3}E_-^2E_+\sigma_-\sigma_+\Theta - 14\sqrt{3}E_-^2\mu\sigma_-^3 - 24\sqrt{3}E_-^2\mu\sigma_-\sigma_+^2 \\ & +144\sqrt{3}E_-^2\sigma_-^3\sigma_+^2 - 96\sqrt{3}E_-E_+^2\sigma_-^2\sigma_+ + 42\sqrt{3}E_-E_+\mu\sigma_-^2\sigma_+ \\ & -288\sqrt{3}E_-E_+\sigma_-^2\sigma_+^3 - 36\sqrt{3}E_-\mu\sigma_-^4\sigma_+ - 36\sqrt{3}E_-\mu\sigma_-^2\sigma_+^3 \\ & -24\sqrt{3}E_+^3\sigma_-^3 + 24\sqrt{3}E_+^2\mu\sigma_-^3 - 16\sqrt{3}E_+^2\sigma_-^5 - 96\sqrt{3}E_+^2\sigma_-^3\sigma_+^2 \\ & +20\sqrt{3}E_+\mu\sigma_-^5 + 84\sqrt{3}E_+\mu\sigma_-^3\sigma_+^2 - 4\sqrt{3}\mu^2\sigma_-^5 + 12\sqrt{3}\mu^2\sigma_-^3\sigma_+^2\,, \end{aligned} \tag{6.81}$$

and

$$\begin{aligned} (Denf)_2 \;:=\; & 54E_-^4\sigma_+ + 78E_-^3E_+\sigma_- + 48E_-^3\sigma_-^3 + 216E_-^3\sigma_-\sigma_+^2 \\ & +18E_-^3\sigma_-\sigma_+\Theta - 414E_-^2E_+^2\sigma_+ - 120E_-^2E_+\sigma_-^2\sigma_+ - 6E_-^2E_+\sigma_-^2\Theta \\ & -120E_-^2E_+\sigma_+^3 - 48E_-^2E_+\sigma_+^2\Theta - 3E_-^2\mu\sigma_-^2\sigma_+ - 69E_-^2\mu\sigma_+^3 \\ & -6E_-^2\mu\sigma_+^2\Theta + 72E_-^2\sigma_-^4\sigma_+ + 216E_-^2\sigma_-^2\sigma_+^3 + 18E_-E_+^3\sigma_- + 48E_-E_+^2\sigma_-^3 \\ & -552E_-E_+^2\sigma_-\sigma_+^2 + 6E_-E_+^2\sigma_-\sigma_+\Theta - 39E_-E_+\mu\sigma_-^3 + 3E_-E_+\mu\sigma_-\sigma_+^2 \end{aligned}$$

$$+12E_-E_+\mu\sigma_-\sigma_+\Theta + 24E_-E_+\sigma_-^5 - 288E_-E_+\sigma_-^3\sigma_+^2 - 216E_-E_+\sigma_-\sigma_+^4$$
$$-24E_-\mu\sigma_-^5 - 36E_-\mu\sigma_-^3\sigma_+^2 - 108E_-\mu\sigma_-\sigma_+^4 - 192E_+^3\sigma_-^2\sigma_+$$
$$+6E_+^3\sigma_-^2\Theta + 192E_+^2\mu\sigma_-^2\sigma_+ - 6E_+^2\mu\sigma_-^2\Theta - 120E_+^2\sigma_-^4\sigma_+ - 72E_+^2\sigma_-^2\sigma_+^3$$
$$+132E_+\mu\sigma_-^4\sigma_+ + 36E_+\mu^2\sigma_-^2\sigma_+^3 - 12\mu^2\sigma_-^4\sigma_+ + 36\mu^2\sigma_-^2\sigma_+^3 \,. \quad (6.82)$$

Next, propagating equations (6.78)–(6.80) along the fluid flow vector u^a and resubstituting from known relations, one obtains purely algebraic constraints of the form

$$0 = n_{23}\, f_3[\,\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu\,]\,, \quad (6.83)$$
$$0 = n_{31}\, g_3[\,\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu\,]\,, \quad (6.84)$$
$$0 = n_{12}\, h_3[\,\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu\,]\,, \quad (6.85)$$

where f_3, g_3 and h_3 are high-order multivariate polynomial expressions of the variables indicated, with each individual term containing a factor of either a power of σ_- or a power of E_-. (The exact form of f_3 is given in Appendix C). At this stage, in the $1+3$ covariant terms one has taken the fourth fully orthogonally projected covariant time derivative of the condition that the electric Weyl tensor needs to be $curl - free$ (6.14). Further differentiation leads to further equations of ever-increasing complexity.

The above analysis is based on tetrad methods. In particular we obtain **algebraic** conditions on tensorial objects thus they reflect the covariant property that the constraint are not identically satisfied and do not become compatible after repeated differentiation. The set of equations (6.68)–(6.70); (6.71)–(6.73); (6.78)–(6.80) and (6.83)–(6.85), and the further equations obtained on taking the further derivatives of (6.83)–(6.85), form a chain of integrability conditions which must be satisfied. Moreover if equations (6.68)–(6.70) are identically satisfied then all subsequent equations (6.71)–(6.85) should also be identically satisfied. However, in the general case there are no equations that can be used to reduce equations (6.68)–(6.70) to $0 = 0$ identities. Furthermore, from (6.71) and onwards we have algebraic restrictions which are non-trivial and lead to higher derivatives and new constraints. From equations

(6.83) - (6.85) one can attempt to satisfy these conditions in four different ways (modulo a cyclic permutation of the axes of the spatial frame $\{ \mathbf{e}_\alpha \}$), depending on the number of non-zero $n_{\alpha\beta}$ variables:

(i) $0 = n_{23} = n_{31} = n_{12}$; which implies from (6.78)–(6.80) that $a_\alpha = 0$, and this case corresponds to the orthogonal spatially homogeneous dust models of Bianchi Type–I,

(ii) $0 = n_{31} = n_{12}$, $n_{23} \neq 0$, which implies $0 = a_2 = a_3$ and $f_3 = 0$,

(iii) $n_{12} = 0$, $n_{23} \neq 0 \neq n_{31}$, which implies $a_3 = 0$ and $0 = f_3 = g_3$,

(iv) all of n_{23}, n_{31} and n_{12} are non-zero, which implies $0 = f_3 = g_3 = h_3$.

Due to the particular structure of f_3, g_3 and h_3, case **(iv)** could be solved, e.g., by $\sigma_- = 0 \Leftrightarrow E_- = 0$, which just reproduces the Petrov type D situation discussed above. The interesting case is to see whether other non-trivial solutions to $0 = f_3 = g_3 = h_3$ in the variables Θ, σ_+, σ_-, E_+, E_-, and μ could be found, which would establish the existence of spatially inhomogeneous silent models with a Weyl tensor of algebraic Petrov type I. Because of their complexity, we have been unable to determine if there is such a solution to these equations. However we doubt that other solutions exist.

Similarly we have been unable to find non-trivial solutions in cases **(ii)** and **(iii)**, each of which involves

(a) solving an equation of the same complexity as those in **(iv)**,

(b) additionally satisfying equations (6.71)–(6.80), and

(c) then showing that the time derivatives of this solution are consistent (note that we have not concluded the set of time derivatives needed to prove the consistent result generically, rather we ceased pursuing the consistency conditions beyond equations (6.83)–(6.85) because of the number of terms involved).

So the cases **(ii)** - **(iv)** each require further investigation to establish a conclusive result.

6.4.3 Conjecture

On the basis of the analysis given in the previous section we conclude, as in [102], with the following:

> **Conjecture:** *There are no spatially inhomogeneous irrotational silent $H = 0$ dust models, whose Weyl curvature tensor is of algebraic Petrov type I.*

If this is correct, the silent assumption $H_{ab} = 0$ for irrotational dust spacetimes would reproduce already known classes of spatially inhomogeneous spacetime geometries [3]. This supports the conjecture of Matarrese *et al.* [72] that realistic gravitational collapse scenarios should involve a non-zero magnetic Weyl tensor, $H_{ab} \neq 0$.

In the linearized silent $H = 0$ models there are no extra constraints and such models are consistent. On the other hand the non-linear silent models have extra conditions. Consequently this leads to a linearization instability in irrotational silent $H = 0$ cosmological models. In other words, there would be consistent linearized solutions, which do *not* correspond to consistent exact solutions. These results together cast serious doubt on the validity and usefulness of pursuing exact $H_{ab} = 0$ solutions as realistic models of the late universe or of gravitational instability. It appears that realistic general relativistic models involve non-vanishing magnetic Weyl tensor [6]. However, as shown in the following section, a *purely* magnetic Weyl tensor field also leads to severe restrictions.

6.5 Spacetimes with $E_{ab} = 0$

We now proceed to show that an analogous situation arises in the anti-Newtonian case $E_{ab} = 0 \neq H_{ab}$. When $E_{ab} = 0$, the curl term in the '$\dot{H}$' propagation equation (3.9) falls away, and the '$\dot{E}$' propagation equation (3.8) is converted into a new constraint equation, which we investigate below. The set of propagation equations now reduce to a coupled system of ordinary differential evolution equations again reflecting the *silent* nature of $E = 0$ dust spacetimes.

However, in the $E = 0$ case the propagation equations are not independent. Propagation equation (3.9) for $\dot{H}$ must be consistent with the curl of propagation equation (3.7) for $\dot{\sigma}_{ab}$, by virtue of the constraint equation (3.11). Using identity (A.24), we can rewrite equation (3.9) as

$$\operatorname{curl}\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\operatorname{curl}\sigma_{ab} - \sigma_e{}^c\varepsilon_{cd(a}\mathrm{D}^e\sigma_{b)}{}^d = 0\,,$$

and we find that the curl of equation (3.7) becomes

$$\operatorname{curl}\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\operatorname{curl}\sigma_{ab} + \varepsilon_{cd(a}\sigma_{b)}{}^d\mathrm{D}_e\sigma^{ce} + \varepsilon_{cd(a}\mathrm{D}^c\left[\sigma^{de}\sigma_{b)e}\right] = 0\,,$$

where we also used identity (A.22) and the constraint equation (3.10). The difference between these equations leads to the condition

$$\varepsilon_{cd(a}\left\{\mathrm{D}^c\left[\sigma^{de}\sigma_{b)e}\right] + \mathrm{D}^e\left[\sigma_{b)}{}^d\sigma^c{}_e\right]\right\} = 0\,.$$

which is an identity (satisfied by any tracefree symmetric tensor, (B.21), (see also Maartens, 1997 [65]). Thus we can ignore the propagation equation (3.9) for H_{ab}, since it follows from the shear propagation equation (3.7) and the constraint equation (3.10), using covariant identities.

The dynamic equations for irrotational dust spacetimes with a vanishing electric Weyl tensor

$$\mathcal{B}_{ab} = E_{ab} = 0\,, \tag{6.86}$$

are then described by the following coupled system of ordinary differential evolution equations

$$\dot{\rho} = -\Theta\rho\,, \tag{6.87}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\rho - \sigma_{ab}\sigma^{ab}\,, \tag{6.88}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c<a}\sigma_{b>}{}^c\,, \tag{6.89}$$

(6.90)

and the constraint equations;

$$\begin{aligned}
\mathcal{C}^{1}{}_{a} &= \mathrm{D}^{b}\sigma_{ab} - \tfrac{2}{3}\mathrm{D}_{a}\Theta = 0\ , && (6.91)\\
\mathcal{C}^{2}{}_{ab} &= \mathrm{curl}\,\sigma_{ab} + H_{ab} = 0\ , && (6.92)\\
\mathcal{C}^{3}{}_{a} &= \tfrac{1}{3}\mathrm{D}_{a}\rho + \varepsilon_{abc}\sigma^{b}{}_{d}H^{cd} = 0\ , && (6.93)\\
\mathcal{C}^{4}{}_{ab} &= \mathrm{D}^{b}H_{ab} = 0\ . && (6.94)
\end{aligned}$$

With $\mathcal{B} = E = 0$ in an open set, we see that the generic constraints $\mathcal{C}^{A}$ in chapter 3 are algebraically modified, but still remain consistent. The additional constraint

$$\mathcal{C}^{5}{}_{ab} = -\mathrm{curl}\,H_{ab} - \tfrac{1}{2}\rho\sigma_{ab}\ , \qquad (6.95)$$

arises from the $\dot{E}$ propagation equation (3.8).

If we eliminate H_{ab} via the constraint equation (6.92) then constraint equation (6.93) becomes

$$\varepsilon_{abc}\sigma^{b}{}_{d}\mathrm{curl}\,\sigma^{cd} = -\tfrac{1}{3}\mathrm{D}_{a}\rho\,. \qquad (6.96)$$

Second-order derivatives arising from the constraint equation (6.95) may be rewritten as

$$\mathrm{D}^{2}\sigma_{ab} = \left(\tfrac{1}{2}\rho - \tfrac{1}{3}\Theta^{2} + \sigma_{cd}\sigma^{cd}\right)\sigma_{ab} - \Theta\sigma^{c}{}_{(a}\sigma_{b)c} + \tfrac{3}{2}\mathrm{D}_{(a}\mathrm{D}^{c}\sigma_{b)c}\,, \qquad (6.97)$$

after using identity (A.25) for the curl of the curl of a tensor. The constraint equation (6.97) is a nonlinear generalization of the covariant Helmholtz equation. It may also be deduced as a special case of the nonlinear wave equation for the shear that is derived in Maartens [65].

Using the covariant identities by Maartens [64] we establish the following consistency conditions (to appear in Maartens, Lesame and Ellis (1997) [67]).

Spatial consistency: We first consider whether or not the divergence of the new tensor constraint $\mathcal{C}^{5}$, in equation (6.95), leads to an additional vector constraint. By the identity (A.23) and the constraints (6.91), (6.94), we find

$$D^{b}\mathcal{C}^{5}_{ab} = -\tfrac{1}{2}\rho D^{b}\mathcal{C}^{1}_{a} + \tfrac{1}{2}\Theta D^{b}\mathcal{C}^{3}_{a} - \tfrac{1}{2}\mathrm{curl}\,\mathcal{C}^{4}_{a} + \tfrac{1}{9}\mathcal{E}_{a}\ , \qquad (6.98)$$

where

$$\mathcal{E}_a \equiv \Theta \mathrm{D}_a \rho - 3\rho \mathrm{D}_a \Theta - \tfrac{3}{2}\sigma_a{}^b \mathrm{D}_b \rho \,. \tag{6.99}$$

Thus by (6.98) we conclude that:

A necessary condition for spatial consistency in irrotational dust spacetimes with $E = 0$ is the covariant condition;

$$\rho \mathrm{D}_a \Theta = \tfrac{1}{3}\Theta \mathrm{D}_a \rho - \tfrac{1}{2}\sigma_a{}^b \mathrm{D}_b \rho \,. \tag{6.100}$$

This can be interpreted as an algebraic relation between the spatial gradients of ρ and Θ, or we can use the constraint equations (6.91)–(6.93) to rewrite it as an algebraic condition on the div and curl of the shear:

$$9\sigma_{ab}\varepsilon^{bcd}\sigma_c{}^e \mathrm{curl}\,\sigma_{de} - 6\Theta\varepsilon_{abc}\sigma^b{}_d \mathrm{curl}\,\sigma^{cd} - 4\rho \mathrm{D}^b \sigma_{ab} = 0 \,. \tag{6.101}$$

Equation (6.100) is a primary integrability condition, whose successive derivatives must also be satisfied. There is at least one special case where this condition is identically satisfied. If $\mathrm{D}_a \rho = 0$, as for example in spatially homogeneous models, then by the energy conservation equation (6.87) and the identity (A.13), we find $\mathrm{D}_a \Theta = 0$. It follows that (6.100) is identically zero. However, the new constraint equation (6.95) itself is not necessarily identically satisfied when $\mathrm{D}_a \rho = 0$ – only its divergence vanishes identically, as seen from equation (6.98). We have been unable to find spatially homogeneous solutions that satisfy equation (6.95) when $H_{ab} \neq 0$, and the existence of such solutions remains an open question. (Clearly FLRW solutions, with $H_{ab} = 0 = \sigma_{ab}$, satisfy equation (6.95) identically.)

In general, $\mathrm{D}_a \rho$ is nonzero and the condition (6.100) is not trivial. The evolution of the integrability condition (6.100) along u^a produces a further integrability condition. Using identity (A.13) to commute time and space derivatives, propagation equations (6.87)–(6.89) to eliminate time derivative terms, and condition (6.100) to eliminate $\mathrm{D}_a\Theta$, we get

$$\rho \mathrm{D}_a \left[\left(\sigma^2\right)_b{}^b\right] = \left[-\tfrac{1}{3}\rho + \tfrac{5}{12}\left(\sigma^2\right)_b{}^b\right] \mathrm{D}_a\rho - \tfrac{1}{3}\Theta\sigma_a{}^b\mathrm{D}_b\rho - \tfrac{1}{4}\left(\sigma^2\right)_{\langle a}{}^{b\rangle}\mathrm{D}_b\rho\,, \tag{6.102}$$

where $(\sigma^2)_{ab} = \sigma_a{}^c\sigma_{bc}$ is the tensor product. In general, with $\mathrm{D}_a\rho \neq 0$, condition (6.102) is not automatically satisfied. A further time derivative gives

$$\begin{aligned} \rho \mathrm{D}_a \left[\left(\sigma^3\right)_b{}^b\right] &= \left[\tfrac{1}{36}\rho\Theta - \tfrac{1}{8}\Theta\left(\sigma^2\right)_b{}^b + \tfrac{7}{16}\left(\sigma^3\right)_b{}^b\right] \mathrm{D}_a\rho + \tfrac{1}{8}\left(\sigma^2\right)_c{}^c\sigma_a{}^b\mathrm{D}_b\rho \\ &\quad - \tfrac{1}{8}\Theta\left(\sigma^2\right)_{\langle a}{}^{b\rangle}\mathrm{D}_b\rho - \tfrac{3}{16}\left(\sigma^3\right)_{\langle a}{}^{b\rangle}\mathrm{D}_b\rho\,, \end{aligned} \tag{6.103}$$

where we used the conditions (6.100) and (6.102). Clearly the n-th derivative leads to an integrability condition of the form

$$\rho \mathrm{D}_a \left[\left(\sigma^{n+1}\right)_b{}^b\right] = \alpha_{(n+1)}\mathrm{D}_a\rho + \alpha_{(n)}\sigma_a{}^b\mathrm{D}_b\rho + \cdots \alpha_{(0)}\left(\sigma^{n+1}\right)_{\langle a}{}^{b\rangle}\mathrm{D}_b\rho\,, \tag{6.104}$$

where $\alpha_{(m)}$ involves in general ρ, Θ and $(\sigma^i)_b{}^b$, $i = 0, 1, \cdots, m$.

Thus there is an indefinite chain of derived integrability conditions all of which must be satisfied. At each level, the condition does not follow automatically from lower-level conditions. Since each such equation involves only the initial data $\{\rho, \Theta, \sigma_{ab}\}$, it is clear that in general the chain of conditions will lead to inconsistencies. Thus:

Conjecture: *The new constraint equation (6.95) and the integrability conditions (6.100), (6.102), ... that follow from it are only consistent in the FLRW case;* $\sigma_{ab} = H_{ab} = E_{ab} = 0$.

Integrability conditions: The conjecture is that there are no consistent $E = 0$ models is reinforced by the existence of a further chain of integrability conditions. This arises from the *time* evolution of (6.95), which must also be satisfied. Using the identities (A.22) and (A.24), the propagation equations (6.87) and (6.89), and the constraint equations (6.91) and (6.92), we find that

$$\dot{\mathcal{C}}^5{}_{ab} = -\tfrac{4}{3}\Theta\mathcal{C}^5{}_{ab} - \tfrac{3}{2}\varepsilon^{cd}{}_{(a}H_{b)c}\mathcal{C}^1{}_d + \mathcal{E}_{ab}\,, \tag{6.105}$$

where

$$\begin{aligned}\mathcal{E}_{ab} &= \tfrac{1}{6}\rho\Theta\sigma_{ab} + \tfrac{1}{2}\rho\sigma_{c(a}\sigma_{b)}{}^{c} + 3H_{c(a}H_{b)}{}^{c} + 3\mathrm{curl}\left[\sigma^{c}{}_{(a}H_{b)c}\right] \\ &\quad + \tfrac{3}{2}\varepsilon_{cd(a}H_{b)}{}^{c}\mathrm{D}^{e}\sigma^{d}{}_{e} - \sigma_{e}{}^{c}\varepsilon_{cd(a}\mathrm{D}^{e}H_{b)}{}^{d}\,. \end{aligned} \tag{6.106}$$

It follows that *a necessary condition for consistent evolution of the constraints in anti-Newtonian universes is the covariant condition*

$$\mathcal{E}_{ab} = 0\,. \tag{6.107}$$

Note that, in contrast to the previous integrability condition (6.100), this condition involves second derivatives of the shear, given that $H_{ab} = \mathrm{curl}\,\sigma_{ab}$. Clearly this condition is identically satisfied in the case of $\sigma_{ab} = 0$, in line with the conjecture stated above. The evolution of this integrability condition must also be satisfied. As before, the propagation equations can be used to eliminate time derivatives and arrive at a chain of derived integrability conditions. We have not finalized the very complicated conditions arising from $\dot{\mathcal{E}}_{ab} = 0$ and further time derivatives.

The point that we wish to highlight is that key role of the condition (6.107) emerges in the linearized case. The linearization about an FLRW background of (6.106) gives

$$\Theta\sigma_{ab} = 0\,, \tag{6.108}$$

implying either $\Theta = 0$ or $\sigma = 0$. The linearized form of the other primary integrability condition (6.100) is automatically satisfied if $\Theta = 0$, and also holds if $\sigma_{ab} = 0$, since in that case the spacetime is FLRW. For non-zero shear $\Theta = 0$ implies, *via* the linearization of propagation equation (6.88) that $\rho = 0$, and we have ruled out this vacuum case by our definition of *dust* universes. Thus *for $\sigma \neq 0$ there are no linearized anti-Newtonian universes.* The implication of this result is that is difficult to see how any consistent exact solution with $E = 0$ can exist. Such a solution would need to be shear-free on linearization. This reinforces the Conjecture made above.

CHAPTER 7

Bianchi examples

7.1 Introduction

In this chapter we seek orthogonal spatially homogeneous dust models (these spacetimes are also geodesic and irrotational) admitting additional conditions of Class I and Class II. A consistency analysis of these classes has been carried out in chapters 5 and 6. We also highlight problems encountered in Class III cases due to additional integrability conditions. Consistency of Class III conditions has not been proven and the potential examples obtained here may not be consistent.

The focus of this chapter is on Bianchi spatially homogeneous dust models with non–vanishing shear tensor, and is largely based on the paper by Ellis and MacCallum (1969) [39]. Some relevant equations and theorems are introduced in section 2 and are used to obtain the various solutions using the Bianchi Class A (in section 3) and Class B (in section 4) classifications. Note that because these universes are all spatially homogeneous, their evolution (when described in a suitable tetrad formalism) is in all cases given by a set of ordinary differential equations, whether or not the covariant equations satisfy the definition of 'silent' given previously. Thus spatial homogeneity is an alternative route to situations which may be described as 'silent'. However our aim here is to investigate the covariant relations that are the focus of the rest of the thesis, in the context of the spatially homogeneous models. Some of these spatially homogeneous models are not silent in the covariant sense, because the curls of both E and H are non-zero. In section 5 we identify those solutions that are covariantly silent.

7.2 Orthogonal Bianchi models

7.2.1 Tetrad formalism and equations

Following the work by Schücking, Kundt and Behr, the spatially homogeneous spacetimes were comprehensively analysed and classified by Ellis and MacCallum (1969) [39]. They used an invariant orthonormal tetrad[1] $\{\mathbf{e}_0 = \mathbf{u}, \mathbf{e}_\nu\}$ whose commutator functions are dependent only on the preferred time parameter t, and the purely spatial commutators $\gamma^\alpha{}_{\gamma\delta}$ introduced in (4.13) are used to separate $\varepsilon^{\beta\gamma\delta}\Gamma^\alpha{}_{\gamma\delta}$ into a symmetric object $n_{\gamma\delta}$ and an vector a_γ:

$$\begin{aligned} \gamma^\alpha{}_{\gamma\delta} &:= 2\,a_{[\gamma}\,\delta^\alpha{}_{\delta]} + \varepsilon_{\gamma\delta\epsilon}\,n^{\epsilon\alpha}\;, \quad \gamma^\alpha{}_{\gamma\delta} = \gamma^\alpha{}_{\gamma\delta}(t)\;; && (7.1)\\ n^{\alpha\beta} &= \tfrac{1}{2}\gamma^{(\alpha}{}_{\gamma\delta}\varepsilon^{\beta)\gamma\delta}\;, \qquad a_\gamma = \tfrac{1}{2}\gamma^\delta{}_{\gamma\delta}\;. && (7.2)\end{aligned}$$

In the rest space of u^a the Fermi-derivatives $\mathbf{e}_\mu.\dot{\mathbf{e}}_\nu$ may be represented by the local angular velocity vector Ω^a (see also (4.19)):

$$\begin{aligned} \Omega^a &= \eta^{abcd}u_b\mathbf{e}_c.\dot{\mathbf{e}}_d\;, \\ \gamma^\mu_{0\nu} &= -\Theta_{\mu\nu} + \varepsilon_{\mu\nu\tau}\Omega^\tau\;; \quad u_a\Omega^a = 0\;. && (7.3)\end{aligned}$$

So now a *Fermi propagated* tetrad is characterized by

$$\Omega^\mu = 0 \Leftrightarrow \mathbf{e}_\mu.\dot{\mathbf{e}}_\nu = 0\;. \tag{7.4}$$

The spatial subset of tetrad vectors $\{\mathbf{e}_\mu\}$ form a *triad* of orthonormal spacelike vectors spanning a family of 3-surfaces in spacetime called the *surfaces of homogeneity.* The Jacobi identities for $(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$ are

$$0 = \partial_a\gamma^d{}_{bc} + \partial_c\gamma^d{}_{ab} + \partial_b\gamma^d{}_{ca} + \gamma^d{}_{af}\gamma^f{}_{bc} + \gamma^d{}_{cf}\gamma^f{}_{ab} + \gamma^d{}_{bf}\gamma^f{}_{ca}\;. \tag{7.5}$$

Applying the identities (7.5) to $(\mathbf{e}_0, \mathbf{e}_\mu, \mathbf{e}_\nu)$ gives the time-derivatives of a_γ and $n_{\gamma\delta}$ as

$$0 = \partial_0 a_\gamma + \sigma_{\gamma\beta}a^\beta + \tfrac{1}{3}\Theta a_\gamma + \varepsilon_{\gamma\beta\delta}a^\beta\Omega^\delta\;; \tag{7.6}$$

[1] The orthonormal tetrad frame chosen is invariant under the translations induced by the generators of the simply transitive G_3 of the various Bianchi types, such that all geometrically defined variables only depend on the cosmic proper time t.

$$0 = \partial_0 n_{\alpha\beta} + 2n^{\delta}{}_{(\alpha}\varepsilon_{\beta)\delta\epsilon}\Omega^{\epsilon} - 2n_{\delta(\alpha}\sigma_{\beta)}{}^{\delta} + \tfrac{1}{3}n_{\alpha\beta}\Theta \,. \tag{7.7}$$

Also applying the identities (7.5) to $(\mathbf{e}_\sigma, \mathbf{e}_\mu, \mathbf{e}_\nu)$ gives the spacelike Jacobi identities

$$n_{\beta\gamma}a^{\gamma} = 0 \,. \tag{7.8}$$

The triad $\{\mathbf{e}_\mu\}$ is chosen so that $n_{\gamma\delta} = n_{\gamma\delta}(t)$ is diagonal and $a_\beta = a_\beta(t)$ is along $\mathbf{e}_1$:

$$n_{\gamma\delta} = diag(n_1, n_2, n_3) \,, \quad a_\gamma = (a, 0, 0) \,, \tag{7.9}$$

the Jacobi identities (7.8) then become

$$a n_1 = 0 \,. \tag{7.10}$$

The two equations (7.9) and (7.10) lead to two distinct classes *viz.*, the group Class A $(a = 0)$ and the group Class B $(a \neq 0)$. Throughout this analysis we will use the tetrad specialization (7.9) and the Bianchi-Behr classification of groups G_3 (Behr, (1962) [4]) which when $n_2 n_3 \neq 0$, introduces the time-invariant quantity

$$h = \frac{a^2}{n_2 n_3} \,. \tag{7.11}$$

In terms of the tetrad choice (7.9) the (div $S)_\alpha$ and (curl $S)_{\alpha\beta}$ terms have the explicit form (see [98])

$$(\mathrm{div}\, S)_\alpha \equiv \mathrm{D}^{\beta} S_{\alpha\beta} = \partial_\beta S_\alpha{}^{\beta} - 3a^{\beta}S_{\alpha\beta} - \varepsilon_{\alpha\beta\gamma}n^{\beta\delta}S^{\gamma}{}_{\delta} \,, \tag{7.12}$$

$$\begin{aligned} \mathrm{curl}\, S_{\alpha\beta} &= \varepsilon_{\gamma\delta(\alpha}\partial^{\delta}S_{\beta)}{}^{\gamma} - \varepsilon_{\gamma\delta(\alpha}S_{\beta)}{}^{\delta}a^{\gamma} + \tfrac{1}{2}S_{\alpha\beta}n^{\delta}{}_{\delta} \\ &\quad -3n^{\gamma}{}_{(\alpha}S_{\beta)\gamma} + \delta_{\alpha\beta}S_{\gamma\delta}n^{\gamma\delta} \,. \end{aligned} \tag{7.13}$$

For later use we write down the Weyl tensor components obtained in Ellis and MacCallum (1969) [39] as

$$E_{\alpha\beta} = \partial_0\sigma_{\alpha\beta} + \sigma_{\alpha\delta}\sigma^{\delta}{}_{\beta} + \tfrac{2}{3}\Theta\sigma_{\alpha\beta} - \tfrac{2}{3}\delta_{\alpha\beta}\sigma^2 \,, \tag{7.14}$$

$$H_{\alpha\beta} = (\sigma_{\gamma\delta}n^{\gamma\delta})\delta_{\alpha\beta} + \tfrac{2}{3}n_{\gamma}{}^{\gamma}\sigma_{\alpha\beta} - 3\sigma_{\delta(\alpha}n^{\delta}{}_{\beta)} - \varepsilon^{\gamma\delta}{}_{(\alpha}\sigma_{\beta)\delta}a_{\gamma} \,, \tag{7.15}$$

where (7.14) is the shear propagation equation and (7.15) is the "$H_{\alpha\beta}$" equation. The continuity equation (3.5) becomes

$$\partial_0\rho = -\Theta\rho \,. \tag{7.16}$$

The Raychaudhuri equation (3.6) becomes

$$\partial_0\Theta = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\rho - \sigma_{\alpha\beta}\sigma^{\alpha\beta} . \tag{7.17}$$

The '$(0,\mu)$' equation (3.10) may be written, using (7.12), as

$$3a^{\beta}\sigma_{\alpha\beta} - \varepsilon_{\alpha\beta\gamma}n^{\beta}{}_{\delta}\sigma^{\delta\gamma} = 0 , \tag{7.18}$$

and represents a divergence-free shear tensor. Also useful for later is the trace-free part of the (μ,ν) equations given by

$$\begin{aligned}\partial_0\sigma_{\mu\nu} \;&=\; -\Theta\sigma_{\mu\nu} + 2\sigma^{\gamma}{}_{(\mu}\varepsilon_{\nu)\delta\gamma}\Omega^{\delta} + 2\varepsilon_{\gamma\delta(\mu}n_{\nu)}{}^{\gamma}a^{\delta} - 2n_{\mu\gamma}n^{\gamma}{}_{\nu} + Tr(n)n_{\mu\nu} \\ &\quad + \tfrac{1}{3}\delta^{\mu\nu}\left[2n_{\gamma\delta}n^{\gamma\delta} - (Tr(n))^2\right] ,\end{aligned} \tag{7.19}$$

where $Tr(n) := n_1 + n_2 + n_3$.

The following theorems and lemmas taken from Ellis and MacCallum (1969) [39] are useful in obtaining a systematic classification of class B models:

(a) The vector a^{β} is a shear eigenvector if and only if $\sigma_{12} = \sigma_{13} = 0$.

(b) The vector a^{β} is not a shear eigenvector if and only if $\sigma_{12}\sigma_{13} \neq 0$.

(c) Lemma 5.2: *A spacetime of Class B has $n_2 = n_3$ on an open neighborhood if and only if there is a group of type V.*

Theorem 5.1: *The only Class B spacetime in which a^{β} is not necessarily a shear eigenvector is that with a group of type VI_h with $h = -1/9$.*

Using the above properties the Class B orthogonal spatially homogeneous models are further classified into:

Case Ba models with $n_2 = n_3$ on an open set $\Leftrightarrow$ there is a group of type V. Here the tetrad may be chosen such that $n_{\alpha} = 0$, $\sigma_{\alpha\beta} = 0 : \;\alpha \neq \beta$.

Case Bb models with $n_3 - n_2 \neq 0$ are further subdivided into

Case Bbi models with $\sigma_{12} = \sigma_{13} = 0$, $\sigma_{23} \neq 0$.

Case Bbii models with $\sigma_{12}\sigma_{13} \neq 0 \;\Leftrightarrow$ group type VI_h, $h = -1/9$.

7.2.2 Useful formulae

In the following sections we attempt to identify spatially homogeneous solutions of dust which admit the class I conditions:

$$\begin{aligned}
&\textbf{(a)} \qquad (\operatorname{div} H)_\alpha &=& \; 0 \Leftrightarrow \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta E^{\delta\gamma} = 0, && (7.20)\\
&\textbf{(b)} \qquad (\operatorname{div} E)_\alpha &=& \; 0 \Leftrightarrow \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta H^{\delta\gamma} = 0, && (7.21)\\
&\textbf{(c)} \qquad (\operatorname{div} H)_\alpha &=& \; (\operatorname{div} E)_\alpha = 0 \Leftrightarrow \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta E^{\delta\gamma} = 0 = \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta H^{\delta\gamma}. && (7.22)
\end{aligned}$$

The equivalent conditions in (7.20)–(7.22) arise from the Bianchi "div H" constraint (3.13) and "div E" constraint (3.12). In chapter 5 we obtained integrability conditions (5.20) (implying the existence of a Fermi propagated frame) and (5.23) associated with (7.20) and, using (7.4), these have the spatially homogeneous form:

$$0 = \Omega^\mu \qquad (7.23)$$

$$0 = \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\sigma(\operatorname{curl} H)^{\gamma\sigma} . \qquad (7.24)$$

The integrability condition (5.39) associated with (7.21) has the spatially homogeneous form:

$$0 = \varepsilon_{\alpha\beta\gamma}E^\beta{}_\sigma H^{\gamma\sigma} + \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\sigma(\operatorname{curl} E)^{\sigma\gamma} . \qquad (7.25)$$

We will also investigate spatially homogeneous models admitting Class III conditions;

$$\textbf{(d)} \qquad (CurlH)_{\alpha\beta} = 0\,, \qquad (7.26)$$

$$\textbf{(e)} \qquad (CurlE)_{\alpha\beta} = 0\,. \qquad (7.27)$$

Both conditions have integrability conditions whose consistency has not been proven. We will investigate potential spatially homogeneous solutions admitting **(d)** and **(e)**. Finally a similar analysis of Class II conditions;

$$\textbf{(f)} \qquad H_{\alpha\beta} = 0 \Leftrightarrow (Curl\sigma)_{\alpha\beta} = 0, \quad \text{(Newtonian case)} \qquad (7.28)$$

$$\textbf{(g)} \qquad E_{\alpha\beta} = 0, \quad \text{(Anti-Newtonian case)} \qquad (7.29)$$

turns out to be problematic. This is due mainly to severe integrability conditions associated to the two cases. Both conditions (7.28) and (7.29) satisfy the covariant silent criterion $(\mathrm{curl}\, H)_{\alpha\beta} = (\mathrm{curl}\, E)_{\alpha\beta} = 0$. We will confirm the proof, in section 6.4, that the integrability conditions are satisfied for the spatially homogeneous Newtonian-like models **(f)**. We also conjectured, in section 6.5, that the anti-Newtonian case **(g)** is inconsistent and we have been unable to find any spatially homogeneous solutions for this case.

All the items **(a)**-**(c)** requires the tetrad form of the commutator $J_\alpha := \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta S^{\delta\gamma}$, written here as:

$$J_1 = \sigma_{23}(S_{33} - S_{22}) + S_{23}(\sigma_{22} - \sigma_{33}) + \sigma_{12}S_{13} - \sigma_{13}S_{12}\ ; \tag{7.30}$$

$$J_2 = \sigma_{13}(S_{11} - S_{33}) + S_{13}(\sigma_{33} - \sigma_{11}) + \sigma_{23}S_{12} - \sigma_{12}S_{23}\ ; \tag{7.31}$$

$$J_3 = \sigma_{12}(S_{22} - S_{11}) + S_{12}(\sigma_{11} - \sigma_{22}) + \sigma_{13}S_{23} - \sigma_{23}S_{13}\ , \tag{7.32}$$

where $S_{\alpha\beta}$ is respectively $E_{\alpha\beta}$ and $H_{\alpha\beta}$. Also useful for later is the tetrad form of $(\mathrm{div}\ S)_\alpha$ expanded here as

$$(\mathrm{div}\ S)_1 = -3aS_{11} + S_{23}(n_3 - n_2)\ ; \tag{7.33}$$

$$(\mathrm{div}\ S)_2 = -3aS_{12} + S_{13}(n_1 - n_3)\ ; \tag{7.34}$$

$$(\mathrm{div}\ S)_3 = -3aS_{13} + S_{12}(n_2 - n_1)\ . \tag{7.35}$$

It is also useful, for the various consistency checks performed throughout, to write (7.6, 7.7) in terms of the tetrad specialization (7.9):

$$\partial_0 a = -\Theta_1 a\ , \tag{7.36}$$

$$\begin{aligned}
\partial_0 n_1 &= (\Theta_1 - \Theta_2 - \Theta_3)n_1\ ;\\
\partial_0 n_2 &= (\Theta_2 - \Theta_3 - \Theta_1)n_2\ ;\\
\partial_0 n_3 &= (\Theta_3 - \Theta_1 - \Theta_2)n_3\ .
\end{aligned} \tag{7.37}$$

The remaining items use the tetrad form of $K_{\alpha\beta} := (\mathrm{curl}\ S)_{\alpha\beta}$ which on expanding (7.13) becomes

$$K_{11} = -\tfrac{3}{2}n_1 S_{11} + \tfrac{1}{2}(n_2 - n_3)(S_{22} - S_{33})\ ; \tag{7.38}$$

$$K_{22} = -\tfrac{3}{2}n_2 S_{22} + \tfrac{1}{2}(n_3 - n_1)(S_{33} - S_{11}) + aS_{23}\ ; \tag{7.39}$$
$$K_{33} = -\tfrac{3}{2}n_3 S_{33} + \tfrac{1}{2}(n_1 - n_2)(S_{11} - S_{22}) - aS_{23}\ ; \tag{7.40}$$
$$K_{12} = \tfrac{1}{2}S_{12}(n_3 - 2n_2 - 2n_1) + \tfrac{1}{2}aS_{13}\ ; \tag{7.41}$$
$$K_{13} = \tfrac{1}{2}S_{13}(n_2 - 2n_1 - 2n_3) - \tfrac{1}{2}aS_{12}\ ; \tag{7.42}$$
$$K_{23} = \tfrac{1}{2}S_{23}(n_1 - 2n_3 - 2n_2) - \tfrac{1}{2}a(S_{22} - S_{33})\ . \tag{7.43}$$

Throughout the chapter we associate to the tensor $S_{\alpha\beta}$ tensorial properties common to the tensors $\sigma_{\alpha\beta}$, $E_{\alpha\beta}$ and $H_{\alpha\beta}$ namely, $S_{\alpha\beta}$ is orthogonal and trace-free. So now the above formulas may be specialized to either those of $\sigma_{\alpha\beta}$, $E_{\alpha\beta}$ or $H_{\alpha\beta}$. For example the "$H_{\alpha\beta}$" equations (7.15) may be obtained in full by replacing $S_{\alpha\beta}$ with $\sigma_{\alpha\beta}$ in (7.38-7.43):

$$\begin{aligned}
H_{11} &= -\tfrac{3}{2}n_1\sigma_{11} + \tfrac{1}{2}(n_2 - n_3)(\sigma_{22} - \sigma_{33})\ ;\\
H_{22} &= -\tfrac{3}{2}n_2\sigma_{22} + \tfrac{1}{2}(n_3 - n_1)(\sigma_{33} - \sigma_{11}) + a\sigma_{23}\ ;\\
H_{33} &= -\tfrac{3}{2}n_3\sigma_{33} + \tfrac{1}{2}(n_1 - n_2)(\sigma_{11} - \sigma_{22}) - a\sigma_{23}\ ;\\
H_{12} &= \tfrac{1}{2}\sigma_{12}(n_3 - 2n_2 - 2n_1) + \tfrac{1}{2}a\sigma_{13}\ ;\\
H_{13} &= \tfrac{1}{2}\sigma_{13}(n_2 - 2n_1 - 2n_3) - \tfrac{1}{2}a\sigma_{12}\ ;\\
H_{23} &= \tfrac{1}{2}\sigma_{23}(n_1 - 2n_3 - 2n_2) - \tfrac{1}{2}a(\sigma_{22} - \sigma_{33})\ .
\end{aligned} \tag{7.44}$$

The $E_{\alpha\beta}$ components following from (7.14), (7.19) are written in full as

$$\begin{aligned}
E_{12} &= \sigma_{13}\sigma_{23} - \Theta_3\sigma_{12}\ ;\\
E_{13} &= \sigma_{12}\sigma_{23} - \Theta_2\sigma_{13}\ ;\\
E_{23} &= \sigma_{12}\sigma_{13} - \Theta_1\sigma_{23} - a(n_2 - n_3)\ ;\\
E_{11} &= \tfrac{2}{3}\rho + 2a^2 - \tfrac{1}{2}n_1^2 + \tfrac{1}{2}(n_2 - n_3)^2 - \Theta\Theta_1 + \Theta_1^2 + \sigma_{12}^2 + \sigma_{13}^2\ ;\\
E_{22} &= \tfrac{2}{3}\rho + 2a^2 - \tfrac{1}{2}n_2^2 + \tfrac{1}{2}(n_3 - n_1)^2 - \Theta\Theta_2 + \Theta_2^2 + \sigma_{23}^2 + \sigma_{12}^2\ ;\\
E_{33} &= \tfrac{2}{3}\rho + 2a^2 - \tfrac{1}{2}n_3^2 + \tfrac{1}{2}(n_1 - n_2)^2 - \Theta\Theta_3 + \Theta_3^2 + \sigma_{13}^2 + \sigma_{23}^2\ .
\end{aligned} \tag{7.45}$$

Also useful is the difference

$$\begin{aligned}
E_{\mu\mu} - E_{\nu\nu} &= -\tfrac{1}{2}(n_\mu - n_\nu)(n_\mu + n_\nu) + \tfrac{1}{2}(n_\tau - n_\nu)^2 - \tfrac{1}{2}(n_\tau - n_\mu)^2\\
&\quad + \Theta_\tau(\sigma_{\mu\mu} - \sigma_{\nu\nu}) + {\sigma_{\mu\tau}}^2 - {\sigma_{\tau\nu}}^2\ , \quad \text{(no sum)}
\end{aligned} \tag{7.46}$$

where $\mu \neq \nu \neq \tau \neq \mu$ (see also Ellis and MacCallum [39]).

7.3 Class A models

For this class the shear eigenvectors are Fermi propagated (or may be chosen as such) *i.e.,* $\Omega^\alpha = 0$ and $\Theta_{\mu\nu} = diag(\Theta_1, \Theta_2, \Theta_3)$. From the $(0, \mu)$ equation (7.18) it follows that $\sigma_{\alpha\beta}$ and $n_{\alpha\beta}$ commute;

$$\varepsilon_{\alpha\beta\gamma} n^\beta{}_\delta \sigma^{\delta\gamma} = 0 \, , \tag{7.47}$$

for all class A models and hence $\sigma_{\alpha\beta} = 0$; $\alpha \neq \beta$.

Divergence-free tensors

If we use (7.47) in equations (7.14) and (7.15) we obtain

$$0 = \varepsilon_{\alpha\beta\gamma} n^\beta{}_\delta E^{\delta\gamma}, \tag{7.48}$$

$$0 = \varepsilon_{\alpha\beta\gamma} n^\beta{}_\delta H^{\delta\gamma}, \tag{7.49}$$

and thus the tensors $E_{\alpha\beta}$ and $H_{\alpha\beta}$ commute simultaneously with $\sigma_{\alpha\beta}$. But equations (7.48), (7.49) are precisely the conditions equivalent to $(\mathrm{div}\, H)_\alpha = (\mathrm{div}\, E)_\alpha = 0$ as expressed in (7.22). The tetrad frame is Fermi propagated thus the integrability condition (7.23) is satisfied. The tensors $\sigma_{\alpha\beta}$, $E_{\alpha\beta}$ and $H_{\alpha\beta}$ and their subsequent higher order time derivatives are diagonal and therefore they are mutually commutative. Also from (7.38–7.43) both the $(curlE)_{\alpha\beta}$ and the $(curlH)_{\alpha\beta}$ are diagonal and their further time derivatives are diagonal. Thus the integrability conditions (7.24) and (7.25) and their time derivatives are satisfied. This, taken together with the divergence-free shear prescribed by the $(0, \mu)$ equation (7.18), leads to the following theorem (see also [98]):

> *All Bianchi Class A spacetimes are such that the shear tensor, the electric Weyl tensor and the magnetic Weyl tensor are divergence-free.*

Curl-free tensors

Class A cases with a trace-free and curl-free tensor $S_{\alpha\beta}$ which commute with $n_{\alpha\beta}$ *i.e.*, $\varepsilon_{\alpha\beta\gamma}n^{\beta}{}_{\delta}S^{\delta\gamma}=0$ may be obtained as follows: Without loss of generality we set

$$
\begin{aligned}
n_1 &= n, \quad n_2 = \kappa n, \quad n_3 = \lambda n \,, &(7.50)\\
S_{11} &= S, \quad S_{22} = \tau S, \quad S_{33} = -(1+\tau)S \,, &(7.51)
\end{aligned}
$$

where $\kappa = \kappa(t)$, $\lambda = \lambda(t)$, $\tau = \tau(t)$. Then the $(\operatorname{curl} S)_{\alpha\beta}$ equations (7.38–7.43) become

$$
\begin{aligned}
K_{11} &= \frac{1}{2}nS\left[-3+(\kappa-\lambda)(1+2\tau)\right],\\
K_{22} &= \frac{1}{2}nS\left[-3\kappa\tau+(1-\lambda)(2+\tau)\right],\\
K_{33} &= \frac{1}{2}nS\left[3\lambda(1+\tau)+(1-\kappa)(1-\tau)\right], \qquad (7.52)
\end{aligned}
$$

with $K_{\alpha\beta}=0$; $\alpha\neq\beta$. Clearly $K_{\alpha\alpha}=(\operatorname{curl} S)_{\alpha\alpha}=0$, (no sum), in the group type I case $(n=0)$.

For the remaining $n\neq 0$ cases we proceed as follows. For $(\operatorname{curl} S)_{\alpha\beta}=0\neq S_{\alpha\beta}$ it follows from (7.52) that $K_{\alpha\alpha}=0\neq S$, (no sum), if the terms inside the square brackets in (7.52) are zero:

$$
\begin{aligned}
0 &= -3+(\kappa-\lambda)(1+2\tau),\\
0 &= -3\kappa\tau+(1-\lambda)(2+\tau),\\
0 &= 3\lambda(1+\tau)+(1-\kappa)(1-\tau). \qquad (7.53)
\end{aligned}
$$

Solving for κ,λ gives

$$
\kappa=\frac{(2+\tau)^2}{(1+2\tau)^2};\quad \lambda=\frac{(1-\tau)^2}{(1+2\tau)^2};\quad \tau\neq-\tfrac{1}{2}. \qquad (7.54)
$$

Both κ and λ are non-negative and $\kappa\neq\lambda$, thus the models under further consideration are of group type VII_0 $(\lambda=0)$ and type IX. The case $\tau=-\frac{1}{2}$ corresponds to $S_{22}=S_{33}$ and from (7.52) we see that $K_{11}=0$ leads to $S=0$.

Most importantly though, equations (7.54) are additional conditions on space-time and need to be checked for consistency. First taking the time derivatives of

(7.50) and using (7.37) gives two equations;

$$\partial_0 \kappa = 2(\sigma_{22} - \sigma_{11})\kappa \ , \quad \partial_0 \lambda = 2(\sigma_{33} - \sigma_{11})\lambda \ ; \tag{7.55}$$

which are not integrability condition but just equations for $\partial_0 n_\alpha$ from (7.37) written in terms of κ and τ. Secondly, taking the time derivatives of the new constraints (7.54) gives

$$\partial_0 \kappa = -6\frac{(2+\tau)}{(1+2\tau)^3}\partial_0 \tau \ , \quad \partial_0 \lambda = -6\frac{(2-\tau)}{(1+2\tau)^3}\partial_0 \tau \ . \tag{7.56}$$

So now using equations (7.54), (7.55) in (7.56) yield two separate equations for $\partial_0 \tau$, namely

$$\partial_0 \tau = -\tfrac{1}{3}(\sigma_{22} - \sigma_{11})(1+2\tau)(2+\tau) \ , \tag{7.57}$$

$$\partial_0 \tau = -\tfrac{1}{3}(\sigma_{33} - \sigma_{11})(1+2\tau)(1-\tau) \ , \tag{7.58}$$

which lead to the integrability condition

$$\sigma_{22} = \tau \sigma_{11}. \tag{7.59}$$

This is a particularly interesting result for taken together with (7.51) it means that spatially homogeneous dust spacetimes consistently admit a curl-free, trace-free tensor $S_{\alpha\beta}$ provided the ratio of the independent components of $S_{\alpha\beta}$ and the shear tensor $\sigma_{\alpha\beta}$ are equal i.e,

$$\tau = \frac{\sigma_{22}}{\sigma_{11}} = \frac{S_{22}}{S_{11}}. \tag{7.60}$$

A: $(\mathrm{curl}\, H)_{\alpha\beta} = 0$ case

Specializing $S_{\alpha\beta}$ to $H_{\alpha\beta}$ leads to the question, are there functions $\tau(t)$ that satisfy (7.60)? The required $\tau(t)$ follows from the "$H_{\alpha\beta}$" equations (7.44), on using $H_{22} = \tau H_{11}$ and $\sigma_{22} = \tau\sigma_{11}$, and are given by the solutions of

$$\tau\left[\tau(1+2\tau)^3 - (1-\tau)(5+7\tau)\right] = 0 \ . \tag{7.61}$$

Hence the values (τ, κ, λ) are independent of the time parameter and $\partial_0\tau = \partial_0\kappa = \partial_0\lambda = 0$. Furthermore equation (7.56) is satisfied. However, the consistency conditions (7.57, 7.58) lead to $\sigma_{22} - \sigma_{11} = 0$ and $\sigma_{33} - \sigma_{11} = 0$. This together with the trace-free property leads to $\sigma_{ab} = 0$. Thus there are no class A spatially homogeneous dust models admitting $(\mathrm{curl}\, H)_{\alpha\beta} = 0 \neq H$.

B: $(\mathbf{curl}\, E)_{\alpha\beta} = 0$ case

If we specialize $S_{\alpha\beta} = E_{\alpha\beta}$ and use $E_{22} = \tau E_{11}$ and $\sigma_{22} = \tau\sigma_{11}$ in the "$E_{\alpha\beta}$" equations (7.45), together with (7.54), we find that τ must satisfy

$$\begin{aligned} 0 \;=\; & (\tfrac{2}{3}\rho + 2a^2 - \tfrac{2}{9}\Theta^2 + \sigma^2\tau)(\tau - 1) \\ & -\tfrac{1}{2}n^2\left[\tau^2 - 2\tau\tfrac{(2+\tau)^2}{(1+2\tau)^2} + \tfrac{(2+\tau)^4}{(1+2\tau)^4} - \tfrac{3\tau}{(1+2\tau)} + \tfrac{9\tau^2(2+\tau)^2}{(1+2\tau)^4}\right] . \end{aligned} \tag{7.62}$$

Now the involvement of ρ and Θ strongly indicates that the solutions of (7.62) are dependent on the time parameter *i.e.*, $\tau = \tau(t)$. Such solutions define new integrability conditions and must be tested for consistency. However, we do not proceed with this further. We do show that solutions of (7.62) exist. First it follows directly from (7.13) that any group type I model will satisfy $(curlE)_{\alpha\beta} = 0$. Secondly a time-independent solution of (7.62) is given by $\tau = 1$, which is the case when $n_{\alpha\beta}$, $\sigma_{\alpha\beta}$ and $E_{\alpha\beta}$ are simultaneously degenerate. The set $(\tau, \kappa, \lambda) = (1, 1, 0)$ satisfies equations (7.55)–(7.58) and we state:

> *The group type I (n=0) and the LRS group type VII_0 ($\tau = 1 \Rightarrow \lambda = 0, \kappa = 1$) models for spatially homogeneous dust spacetimes admit $(curl\, E)_{\alpha\beta} = 0 \neq E_{\alpha\beta}$.*

The group type IX case arising from (7.62) is still open for further investigation.

C: $(\mathbf{curl}\, \sigma)_{\alpha\beta} = 0$ case

If we specialize $S_{\alpha\beta}$ to be $\sigma_{\alpha\beta}$ then the consistency condition (7.60) is immediately satisfied. It is useful to note that the assumption $(\mathrm{curl}\, \sigma)_{\alpha\beta} = 0$ in U is equivalent to $H_{\alpha\beta} = 0$ in U. Furthermore it follows from (7.21) that $(divE)_\alpha = 0$. In chapter 6 equation (6.14) we showed that the "$\dot{H}$" propagation equation becomes the constraint $(\mathrm{curl}\, E)_{\alpha\beta} = 0$. The latter confirms that the integrability condition

(7.25) associated with $(divE)_\alpha = 0$ is satisfied. We showed in chapter 6 that if the constraint $(\mathrm{curl}\, E)_{\alpha\beta} = 0$ in (6.14) then the integrability conditions (6.16)–(6.18) yield the following:

(a) They are satisfied in Petrov type D spacetime geometries. In the homogeneous case the solutions have $\Theta_2 = \Theta_3$ and are the LRS Class A solutions and the type VI_0 with $n^\beta{}_\beta = 0$. (See [39] table 3 for a list of LRS models and a discussion of isotropic class A group types).

(b) They are trivially satisfied in the Petrov type I for Bianchi type I, and thus we state:

> *The group Class A spatially homogeneous dust spacetimes which admit $(curl\,\sigma)_{\alpha\beta} = (divE)_\alpha = (curl\,E)_{\alpha\beta} = 0$ with E and σ non-zero are the group type I (n=0), group type VI_0 and the all the LRS Class A models.*

7.4 Class B models

7.4.1 Case Ba

The Class Ba models are the group type V spacetimes and the tetrad may be chosen such that $n_\beta = \Omega_\beta = \sigma_{\alpha\beta} = 0$; $\alpha \neq \beta$. The $(0, \mu)$ equation (7.18) leads to

$$\sigma_{11} = 0\ , \tag{7.63}$$

and from (7.19) we get $\partial_0 \sigma_{11} = 0$ and thus (7.63) is satisfied everywhere. The "$H_{\alpha\beta}$" equation (7.15) and the "$E_{\alpha\beta}$" equation (7.45) give

$$\begin{aligned} H_{\alpha\beta} &= \tfrac{1}{2} a(\sigma_{33} - \sigma_{22})\delta_{(a}{}^2 \delta_{\beta)}{}^3\ , && (7.64)\\ E_{\alpha\beta} &= 0;\quad \alpha \neq \beta. && (7.65) \end{aligned}$$

Divergence-free tensors

A: $(\mathrm{div}\, H)_\alpha = 0$ case

The $(\mathrm{div}\, H)_\alpha = 0$ condition is immediately satisfied on using (7.64) in (7.33–7.35). The tetrad is Fermi propagated and from (7.38-7.43) it follows that $(curlH)_{\alpha\beta} = 0$. The integrability conditions (7.23) and (7.24) are satisfied. Hence

All group type V spacetimes are such that $(divH)_\alpha = 0 \neq H$

.

B: $(\mathrm{div}\, E)_\alpha = 0$ case

The condition $(\mathrm{div}\, E)_\alpha = 0$ may be treated as follows. Using the equivalent condition (7.21) expanded as in (7.30-7.32) with S replaced by H and together with (7.64) we get

$$
\begin{aligned}
(\mathrm{div}\, E)_\alpha = 0 \Leftrightarrow 0 &= \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta H^{\delta\gamma} && (7.66)\\
&= H_{23}(\sigma_{22} - \sigma_{33}) && (7.67)\\
&= a(\sigma_{22} - \sigma_{33})^2\,, && (7.68)
\end{aligned}
$$

and thus $(\mathrm{div}\, E)_\alpha = 0$ if and only if $\sigma_{22} = \sigma_{33}$. However the latter condition, taken together with (7.63) and the trace-free property for the shear tensor, leads to $\sigma_{\alpha\beta} = 0 \Rightarrow H_{\alpha\beta} = E_{\alpha\beta} = 0$. Thus there are no group type V models with $(\mathrm{div}\, E)_\alpha = 0 \neq E_{\alpha\beta}$ (see also Maartens (1997) [65]). In fact using (7.12) we may write

$$(\mathrm{div}\, E)_\alpha = -3aE^1{}_1\delta^1{}_\alpha \neq 0\,. \qquad (7.69)$$

Curl-free tensors

For the following cases we specialize $S_{\alpha\beta}$ in the curl equations (7.38-7.43) to the tensors $H_{\alpha\beta}$, $E_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ respectively.

A: $(\mathbf{curl}\, H)_{\alpha\beta} = 0$ case

For group type V cases the "$H_{\alpha\beta}$" equations (7.64) gives $H_{11} = H_{22} = H_{33} = 0$. So now from (7.44) we see that $K_{22} = K_{33} = 0$ if and only if $H_{23} = 0$. Thus there are no group type V solutions with $(\mathrm{curl}\, H)_{\alpha\beta} = 0 \neq H_{\alpha\beta}$.

B: $(\mathbf{curl}\, E)_{\alpha\beta} = 0$ case

From (7.45) we see that the electric Weyl tensor is diagonal $E_{\alpha\beta} = 0;\ \alpha \neq \beta$ and from equations (7.38-7.42) all $K_{\alpha\beta} = 0$ except for K_{23}. In particular $K_{23} = 0$ if $E_{22} = E_{33} = E$ and from (7.14) $E_{33} - E_{22} = 0$ if $\Theta_1(\sigma_{33} - \sigma_{22}) = 0$. This leads to the following cases:

case ci : the degenerate case $\sigma_{33} = \sigma_{22}$, taken together with (7.63) and the trace-free property of the shear gives $\sigma_{\alpha\beta} = 0 \Rightarrow E_{\alpha\beta} = 0$.

case cii: $\Theta_1 = 0$ taken with (7.63) gives

$$\Theta = 0\,. \tag{7.70}$$

Now taking the time derivative of (7.70) and using (7.17) gives

$$\tfrac{1}{2}\rho = -{\sigma_{22}}^2 - {\sigma_{33}}^2\,, \tag{7.71}$$

and a further time derivative of (7.71), using (3.5) and (7.63) leads to $0 = 0$. However we exclude this case since from the evolution equations we see that all the variables are time-independent scalars, that is $\partial_0\rho = \partial_0 a = \partial_0\sigma_{22} = \partial_0\sigma_{33} = \partial_0 H_{23} = \partial_0 E_{\alpha\beta} = 0$ and spacetime is not shearing.

Thus there are no group type V solutions admitting $(\mathrm{curl}\, E)_{\alpha\beta} = 0 \neq E_{\alpha\beta}$.

C: $(\mathbf{curl}\, \sigma)_{\alpha\beta} = 0$ case

These solutions are the silent $H_{\alpha\beta} = 0$ cases. However from (7.43) $K_{23} = 0$ if $\sigma_{22} = \sigma_{33}$ and this taken together with (7.63) and the trace-free property leads to $\sigma_{\alpha\beta} = 0$. Thus there are no group type V solutions admitting $(\mathrm{curl}\, \sigma)_{\alpha\beta} = 0 \neq \sigma_{\alpha\beta}$.

7.4.2 Case Bbi

Conditions characterizing group Class Bbi spacetimes are $\sigma_{12} = \sigma_{13} = 0 \neq \sigma_{23}$ with $n_2 - n_3 \neq 0$. If we substitute the latter conditions in equations (7.14) and (7.15) we get $H_{12} = H_{13} = 0 = E_{12} = E_{13}$. The $(0, \mu)$ constraint (7.18) may be taken to define σ_{23}:

$$\sigma_{23} := \frac{-3a\sigma_{11}}{(n_2 - n_3)} \neq \ 0 \Leftrightarrow \ \sigma_{11} \neq 0. \tag{7.72}$$

For this case we may set $n_1 = 0$ (as in Ellis and MacCallum [39]) and thus

$$\Omega_1 = \tfrac{(n_2+n_3)}{(n_3-n_2)}\sigma_{23} \ ; \quad \Omega_2 = \Omega_3 = 0 \ . \tag{7.73}$$

We find the following substitution to be useful. Let

$$n_3 = n \neq 0, \quad n_2 = \kappa n, \quad n_1 = 0; \quad \kappa = \kappa(t) \ ; \tag{7.74}$$

$$\sigma_{11} = \sigma \neq 0, \quad \sigma_{22} = \tau\sigma, \quad \sigma_{33} = -(1+\tau)\sigma; \ \ \tau = \tau(t), \tag{7.75}$$

where $n_2 - n_3 \neq 0$ requires that $\kappa \neq 1$. We consider the various conditions on spacetime below.

Divergence-free tensors

A: $(\mathbf{div}\, H)_\alpha = 0$ case

Are there Case Bbi group types that satisfy $(\mathrm{div}\, H)_\alpha = 0$ or, equivalently, $J_\alpha = \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta E^{\delta\gamma} = 0$? From equations (7.31, 7.32) with S replaced by E it is clear that $J_2 = J_3 = 0$ and J_1 becomes

$$\begin{aligned} J_1 &= \sigma_{23}(E_{33} - E_{22}) + E_{23}(\sigma_{22} - \sigma_{33}) \\ &= (n_2 - n_3)\left[\sigma_{23}(n_2 + n_3) - a(\sigma_2 - \sigma_3)\right] \\ &= 3a(n_2 + n_3)(\sigma_{22} + \sigma_{33}) - a(n_2 - n_3)(\sigma_{22} - \sigma_{33}) \ , \end{aligned} \tag{7.76}$$

where to obtain the last step we used (7.72). From (7.76) it can be seen by inspection that particular solutions for $J_1 = 0$ are the LRS Bianchi type III solutions $\sigma_{22} = \sigma_{33}$; $(n_2 + n_3) = 0$ (see Ellis and MacCallum [39]. We show below that this solutions are the only type III solutions satisfying $divH = 0$ and its integrability conditions.

If we use (7.74) and (7.75) in (7.76), we get

$$J_1 = -an\sigma\left[3(1+\kappa) - (1-\kappa)(1+2\tau)\right] \,, \tag{7.77}$$

and it becomes clear that for $\sigma \neq 0$ we have $J_1 = 0$ if and only if

$$\tau = \frac{1+2\kappa}{1-\kappa}; \quad \kappa \neq 1. \tag{7.78}$$

Significantly (7.78) only those special solutions with shear components

Now J_2 and J_3 are always zero in Case Bbi, thus it follows that (7.78) is equivalent to $(\operatorname{div} H)_\alpha = 0$. Hence in a given eigenframe *i.e.*, $(n_2, n_3) = n(\kappa, 1)$, only those solutions with shear components satisfying

$$\begin{aligned}
\sigma_{11} &= \sigma, \quad \sigma_{22} = \frac{1+2\kappa}{1-\kappa}\sigma, \\
\sigma_{33} &= -(\sigma_{11}+\sigma_{22}), \quad \sigma_{23} = \frac{3a\sigma}{n(1-\kappa)} \,,
\end{aligned} \tag{7.79}$$

admit the condition $(\operatorname{div} H)_\alpha = 0$. In this sense (7.78) is an additional constraint on spacetime. Furthermore the tensor $H_{\alpha\beta}$ is not necessarily zero. This is seen if we use (7.78) in the "$H_{\alpha\beta}$" equations (7.44) to get

$$\begin{aligned}
H_{11} &= -\frac{3}{2}n\sigma(1+\kappa)\,, \quad H_{23} = -\frac{9}{2}a\sigma\frac{(1+\kappa)}{(1-\kappa)}\,, \\
H_{22} &= -\frac{3}{2}n\sigma\frac{(2\kappa^2+\kappa+1)}{(1-\kappa)} + \frac{3a^2\sigma}{n(1-\kappa)}\,, \\
H_{33} &= \frac{3}{2}n\sigma\frac{(\kappa^2+\kappa+2)}{(1-\kappa)} - \frac{3a^2\sigma}{n(1-\kappa)}\,.
\end{aligned} \tag{7.80}$$

To analyse the consistency of (7.79) we first solve (7.78) for κ:

$$\kappa = \frac{\tau-1}{\tau+2}; \quad \text{and } \tau \neq -2\,, \tag{7.81}$$

and take the time derivative of (7.81) to obtain

$$\partial_0\kappa = \frac{3}{(2+\tau)^2}\partial_0\tau\,. \tag{7.82}$$

Now $\partial_0\kappa$ may be obtained by taking the time derivative of $n_{\alpha\beta}$ in (7.74) and using (7.37) and $\partial_0\tau$ obtained by taking the time derivative of $\sigma_{\mu\nu}$ in (7.75) and using

(7.19). In both cases a further use of (7.78) together with $h\kappa = \frac{a^2}{n^2}$ from (7.11) yield

$$\partial_0\kappa = 2\sigma(1+2\tau)\kappa\ ; \tag{7.83}$$

$$\partial_0\tau = -18\sigma\frac{h\kappa(1+\kappa)}{(1-\kappa)^3}\ . \tag{7.84}$$

For the group type IV solutions, ($\kappa = 0$), the constraint equation (7.81) gives $\tau = 1$ and from (7.75) this is the new constraint $\sigma_{11} = \sigma_{22}$. Taking the time derivative of the latter constraint gives $\sigma_{23} = 0$ and this contradicts (7.72).

For the group type VII_h and type VI_h we substitute (7.83, 7.84) into (7.82) to get

$$h =: \frac{a^2}{\kappa n^2} = -1\ . \tag{7.85}$$

and, recalling that h is time invariant, no further integrability conditions arise from (7.85). The group type VII_h is excluded for $h < 0$ and this leaves the group type VI_{-1} which is equivalent to the Bianchi type III. In fact by virtue of (7.73) the LRS type III case $n_2 + n_3 = 0$, $\sigma_{22} = \sigma_{33}$ is the only type III case satisfying the integrability condition (7.23). But for this case substituting $\kappa = -1$, $\tau = -1/2$, and $a^2 = n^2$ from (7.85) into the $H_{\alpha\beta}$ equations (7.80) gives $H_{\alpha\beta} = 0$. Thus in group Class Bbi dust spacetimes there are no solutions that satisfy the condition $(\operatorname{div} H)_\alpha = 0 \neq H$.

B: $(\operatorname{div} E)_\alpha = 0$ case

On the other hand, case Bbi solutions which satisfy $(\operatorname{div} E)_\alpha = 0$ or, equivalently, $J_\alpha = \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta H^{\delta\gamma} = 0$ turn out to be problematic. Here also, using (7.30-7.32) with S replaced by H, we see that $\sigma_{12} = \sigma_{13} = 0 \Rightarrow J_2 = J_3 = 0$. So now $(\operatorname{div} E)_\alpha = 0$ if and only if $J_1 = 0$ and from (7.30) we write

$$J_1 = \sigma_{23}(H_{33} - H_{22}) + H_{23}(\sigma_{22} - \sigma_{33}) + \sigma_{12}H_{13} - \sigma_{13}H_{12}\ . \tag{7.86}$$

Clearly $J_1 = 0$ in the silent case $H_{\alpha\beta}$ discussed below. However we have been unable to obtain Case Bbi solutions with $divE = 0, E \neq 0, H \neq 0$.

Curl-free tensors

We now seek Case Bbi models admitting a trace-free, symmetric and curl-free tensor $S_{\alpha\beta}$. In order to make $S_{\alpha\beta}$ comparable with $H_{\alpha\beta}$, $E_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ we also require that $S_{12} = S_{13} = 0$. From equation (7.38) $K_{11} = 0$ if

$$S_{22} = S_{33} = S. \tag{7.87}$$

If we use the (7.87) in (7.43) we get $K_{23} = 0$ if $(n_2 + n_3)S_{23} = 0$. This leads to two cases.

(a) The $S_{23} = 0$ case. Here it follows immediately from (7.39,7.40) that K_{22} and K_{33} vanish provided that $S(n_3 - n_2) = 0$. This leads to $S_{\alpha\beta} = 0$. Thus there are no Case Bbi models with $S_{23} = 0$ admitting $(\mathrm{curl}\, S)_{\alpha\beta} = 0 \neq S$.

(b) The $n_2 = -n_3 = n > 0$ case. These are the group type VI_h solutions. Taking the time derivative of $n_3 + n_2 = 0$ yields

$$\sigma_{22} = \sigma_{33} \tag{7.88}$$

and the time derivative of (7.88) is identically satisfied. So now $(n_3 + n_2) = 0$ introduces only one integrability condition (7.88) which is consistent and as in [39] solutions belong to the LRS Bianchi type III as shown in **C** below.

From (7.39, 7.40) we see that $K_{22} = K_{33} = 0$ if

$$S_{23} = \frac{3nS}{a}. \tag{7.89}$$

We now specialize $S_{\alpha\beta}$ to the tensors $H_{\alpha\beta}, \sigma_{\alpha\beta}$ and $E_{\alpha\beta}$ respectively as follows.

A: $(\mathrm{curl}\, H)_{\alpha\beta} = 0$ case

From equations (7.38) $K_{11} = 0 \Rightarrow H_{33} = H_{22} = H$, $H_{11} = -2H$. So now if we use $H_{33} - H_{22} = 0$ in (7.15) we get

$$\sigma_{23} = -\frac{3n}{2a}\sigma_{11}. \tag{7.90}$$

On comparing (7.90) with the $(0,\mu)$ equation (7.72) we get

$$\frac{a^2}{n^2} = 1 \Leftrightarrow n = a. \tag{7.91}$$

From (7.39, 7.40) (also (7.89) using (7.91) it follows that $K_{22} = K_{33} = 0$ if

$$H_{23} = \frac{3nH}{a} \tag{7.92}$$

$$= 3H.. \tag{7.93}$$

However the "$H_{\alpha\beta}$" equations (7.44) gives

$$H_{23} = 3H = \tfrac{1}{2}a(\sigma_{33} - \sigma_{22}), \tag{7.94}$$

$$-H_{11} = 2H = a(\sigma_{33} - \sigma_{22}), \tag{7.95}$$

from which it follows that $3H = H$. Thus there are no group Class Bbi spacetimes admitting $(\operatorname{curl} H)_{\alpha\beta} = 0 \neq H$.

B: $(\operatorname{curl} E)_{\alpha\beta} = 0$ case

Here equation (7.87) for $K_{11} = 0$ becomes $E_{22} - E_{33} = 0$ and is identically satisfied using (7.46) and we write $E_{22} = E_{33} = E$, $E_{11} = -2E$. From (7.39, 7.40) we see that $K_{22} = K_{33} = 0$ if

$$E_{23} = \frac{3nE}{a}\,, \tag{7.96}$$

and we have been unable to conclude the consistency of the new constraint (7.96). If (7.96) is consistent then solutions admitting $\operatorname{curl} E_{\alpha\beta} = 0 \neq E$ in the group Class Bbi are of group type VI_h.

C: $(\operatorname{curl} \sigma)_{\alpha\beta} = 0$ case

For $K_{11} = 0$ it follows from (7.87) that $\sigma_{22} = \sigma_{33} = \sigma$; $\sigma_{11} = -2\sigma$. This is line with (7.88). The $(0,\mu)$ equation takes the form

$$\sigma_{23} = \frac{3a\sigma}{n}, \tag{7.97}$$

and from (7.39, 7.40) $K_{22} = K_{33} = 0$ if

$$\sigma_{23} = \frac{3n\sigma}{a}\,, \tag{7.98}$$

which is an additional constraint. If we use the $(0,\mu)$ constraint (7.97) then (7.98) gives

$$h = -1\,, \tag{7.99}$$

which is time-invariant and no further conditions are introduced. This confirms the result obtained by Ellis and MacCallum [39] that if the shear is degenerate then the solutions are of group type VI_{-1} (equivalent to Bianchi LRS type III). Here also the magnetic Weyl tensor vanishes $H_{\alpha\beta} = 0$, the electric Weyl tensor is of Petrov type D due to (7.88) (see theorem 3b in section 4.4) and the $(\mathrm{curl}\, E)_{\alpha\beta} = 0$ constraint arising from the $\dot{H}$ equations is consistent as proven in the inhomogeneous case (see section 6.4.1). Finally $(divE)_\alpha = 0$ and its associated integrability condition (7.25) are identically satisfied. Thus

> *The only group Class Bbi models admitting the silent condition $(curl\sigma)_{\alpha\beta} = 0$ are the LRS Bianchi type III spacetimes.*

7.4.3 Case Bbii

Here $\sigma_{12}\sigma_{13} \neq 0$ and only the group type VI_h spacetimes with $h = -1/9$ belong to this case. The tetrad form of the $(0,\mu)$ equation (7.18) becomes

$$\sigma_{23}(n_2 - n_3) = -3a\sigma_{11}, \tag{7.100}$$

$$n_2 = 3a\frac{\sigma_{13}}{\sigma_{12}}, \tag{7.101}$$

$$n_3 = -3a\frac{\sigma_{12}}{\sigma_{13}}. \tag{7.102}$$

Furthermore we write Ω^α as

$$\Omega^1 = \tfrac{(n_2+n_3)}{(n_3-n_2)}\sigma_{23}\ ; \quad \Omega^2 = -\sigma_{13}\ ; \quad \Omega^3 = \sigma_{12}\ . \tag{7.103}$$

The various conditions on spacetime are treated as follows.

Divergence-free tensors

A: $(\mathrm{div}\, H)_\alpha = 0$ case

We show that there are no group Class Bbii dust spacetimes which admit $(\mathrm{div}\, H)_\alpha = 0 \Leftrightarrow J_\alpha = \varepsilon_{\alpha\beta\gamma}\sigma^\beta{}_\delta E^{\delta\gamma} = 0$. If we use the "$E_{\alpha\beta}$" equation (7.45) in (7.30-7.32) with S replaced by E we get

$$\begin{aligned} J_1 &= (n_2 - n_3)\left[\sigma_{23}(n_2 + n_3)a(\sigma_{22} - \sigma_{33})\right], && (7.104)\\ J_2 &= (n_2 - n_3)\left[-\sigma_{13}n_2 + a\sigma_{12}\right], && (7.105)\\ J_3 &= (n_2 - n_3)\left[-\sigma_{12}n_3 - a\sigma_{13}\right]. && (7.106)\end{aligned}$$

From (7.105, 7.106) it follows that $J_2 = J_3 = 0$ if

$$n_2 = a\tfrac{\sigma_{12}}{\sigma_{13}}; \quad n_3 = -a\tfrac{\sigma_{13}}{\sigma_{12}}. \qquad (7.107)$$

If we first write $X = \frac{\sigma_{13}}{\sigma_{12}}$ and then compare (7.101, 7.102) with (7.107) we get

$$3X^2 = 1; \quad 3 = X^2, \qquad (7.108)$$

from which we see that there are no Case Bbii models satisfying $(\mathrm{div}\ H)_\alpha = 0$.

Additionally the integrability condition (7.23) cannot be satisfied as these solutions do not permit Fermi propagation $\Omega_2 = -\sigma_{13} \neq 0$ and $\Omega_3 = -\sigma_{12} \neq 0$.

B: $(\mathrm{div}\, E)_\alpha = 0$ case

There are also no class Bbii models satisfying $(\mathrm{div}\, E)_\alpha = 0$. This is seen in the following. Replacing S with H in (7.31, 7.32) we get $J_1 = 0$ and

$$\begin{aligned} J_2 &= \sigma_{13}(H_{11} - H_{33}) + H_{13}(\sigma_{33} - \sigma_{11}) + \sigma_{23}H_{12} - \sigma_{12}H_{23}\\ &= -\tfrac{3}{2}\sigma_{22}\sigma_{13}n_3 + \tfrac{1}{2}\sigma_{12}\sigma_{23}(2n_3 - n_2) + \tfrac{3}{2}a\sigma_{13}\sigma_{23} - \tfrac{3}{2}a\sigma_{12}\sigma_{33}; \qquad (7.109)\end{aligned}$$

$$\begin{aligned} J_3 &= \sigma_{12}(H_{22} - H_{11}) + H_{12}(\sigma_{11} - \sigma_{22}) + \sigma_{13}H_{23} - \sigma_{23}H_{13}\\ &= \tfrac{3}{2}\sigma_{33}\sigma_{12}n_2 - \tfrac{3}{2}\sigma_{23}\sigma_{13}n_2 - \tfrac{3}{2}a\sigma_{22}\sigma_{13} - \tfrac{3}{2}a\sigma_{12}\sigma_{23}\,, \qquad (7.110)\end{aligned}$$

and substituting n_2, n_3 from the $(0, \mu)$ equations (7.101, 7.102) into (7.109, 7.110) we get

$$\begin{aligned} J_2 &= \tfrac{3}{2}a\sigma_{12}\left[3\sigma_{22} - \sigma_{33} - 2\sigma_{23}\tfrac{\sigma_{12}}{\sigma_{13}}\right], &(7.111)\\ J_3 &= \tfrac{3}{2}a\sigma_{13}\left[3\sigma_{33} - \sigma_{22} - 3\sigma_{23}\tfrac{\sigma_{13}}{\sigma_{12}} - \sigma_{23}\tfrac{\sigma_{12}}{\sigma_{13}}\right]. &(7.112)\end{aligned}$$

Now if we substitute n_2, n_3 from (7.101, 7.102) into (7.100) we get

$$\sigma_{23} = -\frac{\sigma_{11}\sigma_{12}\sigma_{23}}{(\sigma_{12}^2 + \sigma_{13}^2)}, \tag{7.113}$$

and if we multiply (7.113) by $\frac{\sigma_{12}}{\sigma_{13}}$ and its inverse we obtain

$$\begin{aligned} \sigma_{23}\frac{\sigma_{12}}{\sigma_{13}} &= -\sigma_{11}C, &(7.114)\\ \sigma_{23}\frac{\sigma_{13}}{\sigma_{12}} &= -\sigma_{11}D, &(7.115)\end{aligned}$$

where we set

$$C = \frac{\sigma_{12}^2}{(\sigma_{12}^2 + \sigma_{13}^2)}; \quad D = \frac{\sigma_{13}^2}{(\sigma_{12}^2 + \sigma_{13}^2)}. \tag{7.116}$$

We then substitute (7.114, 7.115) into (7.111, 7.112) to obtain

$$\begin{aligned} J_2 &= \tfrac{3}{2}a\sigma_{12}\left[\sigma_{22}(3 - 2C) - \sigma_{33}(1 + 2C)\right], &(7.117)\\ J_3 &= \tfrac{3}{2}a\sigma_{13}\left[\sigma_{33}(3 - C - 3D) - \sigma_{22}(1 + C + 3D)\right], &(7.118)\end{aligned}$$

from which it is clear that J_2 and J_3 vanish simultaneously if

$$\begin{aligned} 0 &= \sigma_{22}(3 - 2C) - \sigma_{33}(1 + 2C), &(7.119)\\ 0 &= -\sigma_{22}(1 + C + 3D) + \sigma_{33}(3 - C - 3D). &(7.120)\end{aligned}$$

Non-trivial solutions of (7.119, 7.120) follow from the requirement that the determinant of the two equations vanish and this gives

$$\begin{aligned} (1 + C + 3D)(1 + 2C) &= (3 - 2C)(3 - C - 3D) &(7.121)\\ \Rightarrow C &= -D, &(7.122)\end{aligned}$$

and using (7.116) in (7.122) leads to

$$\sigma_{12}^2 + \sigma_{13}^2 = 0, \tag{7.123}$$

which is not allowed for $\sigma_{12}\sigma_{13} \neq 0$.

On the other hand the trivial solution of (7.119, 7.120) is $\sigma_{22} = \sigma_{33} = 0$ and this taken together with the trace-free property for $\sigma_{\alpha\beta}$ gives $\sigma_{11} = 0$. In addition, from the $(0, \mu)$ equation (7.100) we get $\sigma_{23} = 0$. Now taking the time derivative of $\sigma_{23} = 0$, using (7.19) and (7.103) leads to $0 = \sigma_{12}\sigma_{13}$. Thus there are no class Bbii models admitting $(\operatorname{div} E)_\alpha = 0$.

Curl-free tensors

Consider a trace-free, curl-free tensor $S_{\alpha\beta}$. Here, as in the Case Bbi, from equation (7.38) $K_{11} = 0 \Rightarrow S_{22} = S_{33} = S$ and from (7.43) $K_{23} = 0 \Rightarrow (n_2 + n_3)S_{23} = 0$. The latter gives the following two cases:

(a) The $S_{23} = 0$ case. From (7.39, 7.40) we obtain that $K_{22} = K_{33} = 0$ if $(n_2 - n_3)S = 0$ and hence $S_{11} = S_{22} = S_{33} = 0$. Furthermore from (7.41, 7.42) we see that K_{12}, K_{13} vanishes if

$$0 = \frac{1}{2}\left[(n_3 - 2n_2)S_{12} + aS_{13}\right], \tag{7.124}$$

$$0 = \frac{1}{2}\left[-aS_{12} + (n_2 - 2n_3)S_{13}\right]. \tag{7.125}$$

One may solve the above system of two equations for S_{12} and S_{13} provided the determinant of the coefficient vanishes, that is,

$$(n_3 - 2n_2)(n_2 - 2n_3) + a^2 = 0\ . \tag{7.126}$$

Equation (7.126) is an additional constraint on spacetime. However taking the time-derivative of (7.126) and using (7.36, 7.37) leads once more to (7.126) and thus no new consistency conditions emerge. Vanishing components of $S_{\alpha\beta}$ are also additional constraints on spacetime and for further analysis we now specialize $S_{\alpha\beta}$ to $\sigma_{\alpha\beta}$, $H_{\alpha\beta}$ and $E_{\alpha\beta}$ respectively.

A: $(\mathrm{curl}\, H)_{\alpha\beta} = 0$ case

For this case the only non-vanishing $H_{\alpha\beta}$ components are H_{12}, H_{13}. From the "$H_{\alpha\beta}$" equation (7.15) we see that if $H_{11} = 0$ then $\sigma_{22} = \sigma_{33}$, and if $H_{23} = 0$ then $\sigma_{23} = 0$. From the $(0, \mu)$ equation (7.100) we get $\sigma_{11} = 0$ and using the trace-free property we get $\sigma_{22} = \sigma_{33} = 0$. But taking the time derivative of $\sigma_{23} = 0$, using (7.19) and (7.103) leads to $0 = \sigma_{12}\sigma_{13}$. Thus there are no class Bbii models admitting $(\mathrm{curl}\, H)_{\alpha\beta} = 0$.

B: $(\mathrm{curl}\, E)_{\alpha\beta} = 0$ case

Here only the components E_{12}, E_{13} are non-zero and from (7.45) we write

$$E_{12} = \sigma_{13}\sigma_{23} - \Theta_3\sigma_{12}, \quad E_{13} = \sigma_{12}\sigma_{23} - \Theta_2\sigma_{13}. \tag{7.127}$$

However what still remains is a consistency analysis on the additional constraints $E_{11} = E_{22} = E_{33} = 0 = E_{23}$. A series of time derivative are obtained but we have not been able to show that consistency is achieved.

C: $(\mathrm{curl}\, \sigma)_{\alpha\beta} = 0$ case

Here the only non-vanishing shear components are σ_{12}, σ_{13}. Again taking the time derivative of $\sigma_{23} = 0$, using (7.19) and (7.103) leads to $0 = \sigma_{12}\sigma_{13}$. Thus there are no class Bbii models admitting $(\mathrm{curl}\, \sigma)_{\alpha\beta} = 0$.

(b) The $n_2 = -n_3 = n > 0$ case. From (7.39, 7.40) we get that $K_{22} = K_{33} = 0$ if

$$S_{23} = -\tfrac{3nS}{a}\,, \tag{7.128}$$

and from (7.41, 7.42) we see that $K_{12} = K_{13} = 0$ if

$$S_{12}{}^2 = S_{13}{}^2\,. \tag{7.129}$$

We recall also that $K_{11} = 0$ if $S_{22} = S_{33} = S$. Additionally $n_2 + n_3 = 0$ is now a constraint on spacetime and taking the time derivative of this constraint yields

$$\sigma_{12}{}^2 = \sigma_{13}{}^2 \,. \tag{7.130}$$

The time derivative of (7.130) gives

$$\sigma_{22} = \sigma_{33} \,, \tag{7.131}$$

and a further time derivative of (7.131) gives (7.130) and no further integrability conditions are introduced. Specializing $S_{\alpha\beta}$ gives the following.

A: (curl $H)_{\alpha\beta} = 0$ case

A similar result to case (a) above is obtained if we specialize $S_{\alpha\beta}$ to $H_{\alpha\beta}$. Here $K_{11} = 0$ if $H_{22} = H_{33}$ and using the latter equality in the "H_{ab}" equations (7.44) gives $\sigma_{23} = 3n\sigma_{11}/2a$. A comparison with (7.100) leads to $a^2 = n^2$ and thus $h = -1$ and there are no class Bbii solutions admitting $(\mathrm{curl}\, H)_{\alpha\beta} = 0$.

B: (curl $E)_{\alpha\beta} = 0$ case

In the case $S_{ab} = E_{ab}$ the $(\mathrm{curl}\, E)_{ab}$ is zero provided

$$E_{22} = E_{33} = E; \quad E_{12}{}^2 = E_{13}{}^2 \,; \quad E_{23} = -\tfrac{3nE}{a} \,. \tag{7.132}$$

Here the condition $n_3 + n_2 = 0$ introduces only two consistency conditions (7.130, 7.131) which do not add any new conditions. Now equations (7.132) are the new constraints on spacetime and are purely dependent of the components of E_{ab}. Again the first two conditions in (7.132) are identically satisfied using (7.131) and (7.45) and the third is identical to (7.96). Here also we have been unable to show that the latter new constraint is consistent.

C: (curl $\sigma)_{\alpha\beta} = 0$ case

If we specialize S_{ab} to σ_{ab} then equation (7.128) for σ_{23} compared to the $(0, \mu)$ equation (7.100), using $\sigma_{11} = -2\sigma$, yields $\sigma_{23} = -\frac{3n\sigma}{a} = \frac{3a\sigma}{n}$ from which it follows that $a^2 = n^2$ and hence $h = -1$. Thus there are no class Bbii solutions admitting $(\text{curl }\sigma)_{\alpha\beta} = 0$.

7.5 Silent cases

In the above analysis we showed that, in the silent $(\text{curl}\,\sigma)_{\alpha\beta} = 0 \Leftrightarrow H_{\alpha\beta} = 0$ case, spatially homogeneous solutions of Petrov type I are of group type I, and spatially homogeneous solutions of Petrov type D are all the LRS class A solutions, the type VI_0 solutions with $n^\beta{}_\beta = 0$ and the LRS Bianchi type III solutions.

The silent $E_{\alpha\beta} = 0$ case on the other hand requires the following

$$E_{\alpha\beta} = 0 \Rightarrow (\text{div}\,H)_\alpha = (\text{div}\,E)_\alpha = 0; \tag{7.133}$$

$$\dot{E}_{\alpha\beta} = 0 \Rightarrow (\text{curl}\,H)_{\alpha\beta} = \tfrac{1}{2}\rho\sigma_{\alpha\beta}. \tag{7.134}$$

The first condition (7.133) is characteristic of all group Class A spacetimes. The second condition on the other hand defines a new constraint that we conjectured to be inconsistent [67]. So far we have been unable to find models which satisfy the integrability conditions introduced by this constraint.

CHAPTER 8

Final Remarks

Starting from the generic consistency of irrotational dust spacetimes the choice of a shear eigentetrad serves as a useful geometric framework in the analysis of integrability conditions for the various exact Weyl tensor conditions imposed on spacetime. The analysis presented in this thesis is conducted *via* either covariant or tetrad methods based on tensorial objects. The tetrad formalism used does not alter the underlying covariant nature of the problems tackled. The covariant methods are used as a first choice due to the simplified calculations involved.

The results of this thesis highlights the importance of irrotational dust spacetimes in the on-going research in the following areas:

(a) **Pure gravity waves**: We distinguish between solutions that contain some gravity waves and those that contain only gravity waves; the latter are 'pure' gravity waves (see Ellis and Hogan (1996) [38], [47]) and are characterized by the absence of vector and scalar modes in the variation of the electric and the magnetic Weyl tensor. In irrotational dust spacetimes pure gravity waves are characterized by divergence-free electric and magnetic Weyl tensors with non-zero curls.

The generic consistency of irrotational dust spacetimes facilitates a scenario where any additional condition imposed on spacetime may be studied separately. In particular in the linearized case the condition $(divH)_a = 0$ is consistent and is necessary for the presence of pure gravity waves. This result does not extend to the exact non-linear theory where for irrotational dust the $(divH)_a = 0$ condition is a necessary condition for pure gravity waves, but it is not sufficient and furthermore the curl of the magnetic Weyl tensor is constrained to commute with the shear. Similarly the $(divE)_a = 0 \neq E_{ab}$ condition is also a necessarily condition for gravity waves and introduces integrability conditions.

The orthogonally spatially homogeneous solutions satisfying the $(divH)_a = 0$ condition and the $(divE)_a = 0$ condition are listed in table 8.3 and the associated consistency conditions are listed in table 8.2 below. Senovilla and Vera (1997) [83] presented two inhomogeneous Petrov type I dust solutions satisfying $(divH)_a = 0$, $H \neq 0$. Both solutions belong to the Class B(ii) of Wainwright's classification (1981) [103]. The integrability conditions for these solutions are currently under investigation.

(b) **Purely electric universes, $H_{ab} = 0$:** Also known as Newtonian-like universes, these spacetimes are 'silent' and are inconsistent in general and unlikely to extend beyond the known spatially homogeneous or Szekeres examples. Furthermore, they are subject to a linearization instability (see chapter 6). This supports the argument that there is no straightforward relationship between general relativistic universes and Newtonian universes.

(c) **Purely magnetic universes, $E_{ab} = 0$:** These are also silent universes. Because the magnetic Weyl tensor $H_{ab} \neq 0$ has no Newtonian analogue and the analogue of the Newtonian tidal force term vanishes in these universes, they are called anti-Newtonian universes [67]. These models are also subject to severe integrability conditions and are inconsistent in general. We showed that there are no anti-Newtonian spacetimes that are linearized perturbations of Robertson-Walker universes (chapter 6). The $E_{ab} = 0 \neq H_{ab}$ solution in [1] is not a dust solution, and it is kinematically Gödel-like but dynamically unphysical as shown in [67].

Some open questions and summary

We have shown that the gravity-wave condition $(\mathrm{div}\, H)_a = 0$, taken over from linearized theory into the nonlinear theory, leads to a chain of integrability conditions which admit solutions because of this simplicity. In this sense, they are unlike the integrability conditions (6.18–6.19) in the cases of $H_{ab} = 0$ and integrability conditions (6.100), (6.106–6.107) in the case of $E_{ab} = 0 \neq H_{ab}$, whose time derivatives lead to indefinite chains of conditions of high complexity. The importance of such investigations flows precisely from the physical motivation, which mean that the results have potentially important implications for the dynamics of matter and interacting gravity waves in general relativity. We list here some open questions identified in the thesis:

- The Class III set of conditions (*viz,* $CurlH_{ab} = 0$ and $CurlE_{ab} = 0$) introduced. The only potential spatially homogeneous solutions obtained are the group type I and type VII_0 admitting $(\mathrm{curl}\, E)_{ab} = 0$. The latter is also appears as an integrability conditions for silent $H_{ab} = 0$ solutions. It is clear from this study that for gravity waves the crucial conditions that must be satisfied are $curlH_{ab} \neq 0$ and $curlE_{ab} \neq 0$. A full consistency analysis of these conditions in both the homogeneous and the inhomogeneous cases has to be given.

- In this analysis we succeeded in identifying Bianchi models admitting the various conditions. However current research interest is focused on inhomogeneous solutions. In particular Senovilla and Vera [83] identified two dust G_2 inhomogeneous solutions satisfying $H_{ab} \neq 0$, $(divH)_a = 0$. An interesting question is then; what other inhomogeneous solutions admit the various conditions on irrotational dust spacetimes?

- All conditions on spacetime considered in the thesis are conditions on the Weyl tensor. A complete treatment of integrability conditions on irrotational dust spacetime will include other conditions such as existence of symmetries.

We conclude with a summary of the results of the thesis as given in tables 1 to 3 below.

Table 8.1: A list of some important definitions for matter content restricted to dust.

Concept	Property	Implications
Dust Section (2.4.3)	Pressure-free $P = 0$ with	$\Rightarrow \dot{u}_a = 0$ non-zero energy density $\rho \neq 0$
Irrotation Section (2.3.3)	zero vorticity $\omega = 0$	$\Rightarrow$ existence of cosmic time g: $u_a = f\, g_{,a}$
Irrotational dust Section (3.3)	$\rho \neq 0$, $P = 0 = \omega = 0$	$\Leftrightarrow$ there exists normalized cosmic time t: $u_a = t_{,a}$
Silent dust universe Section (1.3)	Independent evolution of fluid elements along u^a	$\Leftrightarrow$ Covariant evolution equations are ODE's $\Leftrightarrow$ $(\mathrm{curl}\, E)_{ab} = (\mathrm{curl}\, H)_{ab} = 0$
Purely electric universe (also known as Newtonian-like) universe. Section (6.1)	$H_{ab} = 0 \neq E_{ab}$	$\Rightarrow$ silent for dust
Purely magnetic universe (also known as anti-Newtonian) universe. Section (6.1)	$E_{ab} = 0 \neq H_{ab}$	$\Rightarrow$ silent for dust
Gravity waves Section (1.4.1)	$(\mathrm{curl}\, E)_{ab} \neq 0$ $(\mathrm{curl}\, H)_{ab} \neq 0$	
Pure gravity waves Section (1.4.1)	Gravity waves with $(\mathrm{div}\, H)_a = 0 = (\mathrm{div}\, E)_a = 0$	$\Rightarrow$ no scalar or vector modes in the variation of E_{ab} and H_{ab}

Table 8.2: Consistency of the various exact conditions imposed on irrotational dust spacetimes.

Generic Irrotational Dust

Exact conditions: Pressure-free and vorticity-free perfect fluid, Ref.[65].
The variable set $\Phi = \{\rho, \Theta, \sigma_{ab}, E_{ab}, H_{ab}\}$ gives a complete description of the covariant dynamics.
Evolution of the constraint equations does not introduce new conditions and the dynamical equations are consistent with the constraint equations.

Class I Conditions			**Class II Conditions**	
$(\mathrm{div}\,H)_a = 0$ Ref.[66] Section (5.4)	$(\mathrm{div}\,E)_a = 0$ Section (5.5)	$(\mathrm{div}\,H)_a = (\mathrm{div}\,E)_a = 0$ Section (5.6)	$H_{ab} = 0$ Ref.[58, 102] Section (6.4)	$E_{ab} = 0$ Ref.[67] Section (6.5)
Linear Case consistent spatially inhomogeneous perturbations of FLRW	**Linear Case** consistent only for solutions with $D_a\Theta = D_a\rho = 0$	**Linear Case** consistent only for solutions with $D_a\Theta = D_a\rho = 0$	**Linear Case** consistent spatially inhomogeneous perturbations of FLRW	**Linear Case** consistent provided $\sigma_{ab}\Theta = 0$ $\Rightarrow$ FLRW
Non-linear Case Integrability conditions: $\varepsilon_{abc}\sigma^b{}_p\dot{\sigma}^{(n)cp} = 0$ and $\varepsilon_{abc}\sigma^b{}_p(\mathrm{curl}\,H)^{(\cdot n)cp} = 0$	**Non-linear Case** Integrability conditions: Equation (5.39)	**Non-linear Case** Integrability conditions: Equation (5.39) $\varepsilon_{abc}\sigma^b{}_p\dot{\sigma}^{(n)cp} = 0$ $\varepsilon_{abc}\sigma^b{}_p(\mathrm{curl}\,H)^{(\cdot n)cp} = 0$	**Non-linear Case** Equations unlikely to be consistent except for Petrov type D and O, and homogeneous type I	**Non-linear Case** Equations unlikely to be consistent

Table 8.3: Summary of spatially homogeneous vortex-free dust models admitting the conditions in table 2. We use X to denote no solutions exist, Y to denote solutions exist and ? denotes we've been unable to find solutions.

	Class A	**Class B**		
	Section (7.3)	Ba Section (7.4.1)	Bbi Section (7.4.2)	Bbii Section (7.4.3)
Conditions		$\Leftrightarrow$ Type V		$\Leftrightarrow$ Type $VI_{-1/9}$
$DivH = 0 \neq DivE$	X	Y	X	X
$DivE = 0 \neq DivH$	X	X	X	X
$DivH = DivE = 0$	Y All group types	X	X	X
$CurlH = 0 \neq H$	X	X	X	X
$CurlE = 0 \neq E$	Y group type I type IX ?? typeVII_0	X	?	?
$Curl\sigma = 0 \neq \sigma$ $\Leftrightarrow H = 0$	Y group type I typeVII_0 and all LRS solutions	X	Y Bianchi type III	X
$E = 0$	?	?	?	?

APPENDIX A

Identities

A.1 Covariant tensorial identities

In this appendix we list a set of covariant tensorial identities derived by Maartens (1996) [64] and used successfully in various other papers including [65] and [69]. For all identities, $S_{ab} = S_{<ab>}$ and $V_{ab} = V_{<ab>}$.

Using the properties of η_{abcd} and h_{ab}, one can derive

$$\varepsilon^{abc}\varepsilon_{def} = 3!\, h^{[a}{}_{d}h^{b}{}_{e}h^{c]}{}_{f} \,, \tag{A.1}$$

$$\varepsilon_{abc}S^{b}{}_{d}V^{cd} = \tfrac{1}{2}\varepsilon_{abc}\,[S,V]^{bc} \,, \tag{A.2}$$

$$\varepsilon^{abc}\varepsilon_{dec} = h^{a}{}_{d}h^{b}{}_{e} - h^{a}{}_{e}h^{b}{}_{d} \,, \tag{A.3}$$

and

$$\varepsilon_{abc}S^{b}{}_{p}S^{c}{}_{q}V^{pq} = 0 \,, \tag{A.4}$$

$$\varepsilon_{abc}S^{b}{}_{p}S^{p}{}_{q}V^{cq} = -S_{ab}\varepsilon^{bcd}S_{c}{}^{p}V_{dp} \,, \tag{A.5}$$

$$\mathrm{curl}\,(S^2)_{ab} = \varepsilon_{cd(a}\mathrm{D}^{e}\left\{S_{b)}{}^{d}S^{c}{}_{e}\right\} \,, \tag{A.6}$$

where $[S,V]$ is the commutator, $(S^2)_{ab} \equiv S_{a}{}^{c}S_{cb}$, and (A.5) and (A.6) are easiest to prove by using a tetrad that diagonalizes S_{ab}. We also have

$$\mathrm{D}_a h_{bc} = 0 = \mathrm{D}_a\varepsilon_{bcd}\,, \qquad \dot{h}_{ab} = 0 = \dot{\varepsilon}_{abc}\,. \tag{A.7}$$

The curvature tensor R_{abcd} can be written in terms of the Weyl tensor C_{abcd} as

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \tfrac{1}{6}Rg_{abcd} \,, \tag{A.8}$$

where

$$g_{abcd} \equiv g_{ac}g_{bd} - g_{ad}g_{bc},$$

and

$$C_{abcd} = (g_{abpq}g_{cdrs} - \eta_{abpq}\eta_{cdrs})u^p u^r E^{qs} - (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^p u^r H^{qs} . \tag{A.9}$$

The Weyl tensor C_{abcd} may also be written as

$$\frac{1}{2}C_{scba} = u_s u_{[b}h_{a]t}E^t{}_c + u_c u_{[a}h_{b]t}E^t{}_s + E_{a[c}h_{s]b} + E_{b[s}h_{c]a} + u_{[b}H_{]a}{}^t\varepsilon_{tsc} + u_{[c}H_{]s}{}^t\varepsilon_{tba} . \tag{A.10}$$

Using field equations for dust, the Ricci tensor is

$$R_{ab} \equiv R^c{}_{acb} = T_{ab} - \frac{1}{2}Tg_{ab} = \tfrac{1}{2}\rho(u_a u_b + h_{ab}) , \tag{A.11}$$

and, together with (A.9), in (A.8) gives

$$R_{abcd} = 2\left(h_{a[c} + u_a u_{[c}\right)E_{d]b} + 2E_{a[c}\left(h_{d]b} + u_{d]}u_b\right) + 2\varepsilon_{abe}u_{[c}H_{d]}{}^e + 2\varepsilon_{cde}u_{[a}H_{b]}{}^e + \tfrac{1}{3}\rho\left(h_{a[c}u_{d]}u_a - h_{b[c}u_{d]}u_a + 2h_{a[c}h_{d]b}\right) . \tag{A.12}$$

The Ricci identities, $f_{[;ab]} = 0$, and

$$2Y_{a;[bc]} = R^d{}_{abc}Y_d ,$$

$$2W_{ab;[cd]} = R^e{}_{acd}W_{eb} + R^e{}_{bcd}W_{ae} ,$$

together with (A.12) and

$$u_{a;b} = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} , \quad \dot{u}_a = 0 = \omega_{ab} ,$$

lead to the following identities:

$$(\mathrm{D}_a f)^{\cdot} = f_{;ab}u^b = f_{;ba}u^b = \mathrm{D}_a\dot{f} - \tfrac{1}{3}\Theta\mathrm{D}_a f - \sigma_a{}^b\mathrm{D}_b f , \tag{A.13}$$

with

$$\mathrm{D}_{[a}\mathrm{D}_{b]}f = 0 . \tag{A.14}$$

A similar set of calculations done for (A.13) leads to

$$\begin{aligned}
(\mathrm{D}_a S_b)^{\cdot} &= \mathrm{D}_a\dot{S}_b - \tfrac{1}{3}\Theta\mathrm{D}_a S_b - \sigma_a{}^c\mathrm{D}_c S_b + H_a{}^d\varepsilon_{dbc}S^c , && \text{(A.15)}\\
(\mathrm{D}_a S_{bc})^{\cdot} &= \mathrm{D}_a\dot{S}_{bc} - \tfrac{1}{3}\Theta\mathrm{D}_a S_{bc} - \sigma_a{}^d\mathrm{D}_d S_{bc} + 2H_a{}^d\varepsilon_{de(b}S_{c)}{}^e , && \text{(A.16)}\\
\left(\mathrm{D}^b S_{ab}\right)^{\cdot} &= \mathrm{D}^b\dot{S}_{ab} - \tfrac{1}{3}\Theta\mathrm{D}^b S_{ab} - \sigma^{bc}\mathrm{D}_c S_{ab} + \varepsilon_{abc}H^b{}_d S^{cd} , && \text{(A.17)}
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{D}_{[a}\mathrm{D}_{b]}S_c &= \left(\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}\rho\right) S_{[a}h_{b]c} - \sigma_{c[a}\sigma_{b]d}S^d \\
&\quad + S_{[b}\left(E_{c]a} - \tfrac{1}{3}\Theta\sigma_{c]a}\right) + h_{c[a}\left(E_{b]d} - \tfrac{1}{3}\Theta\sigma_{b]d}\right) S^d , && \text{(A.18)}\\
\mathrm{D}_{[a}\mathrm{D}_{b]}S^{cd} &= 2\left(\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}\rho\right) S_{[a}{}^{(c}h_{b]}{}^{d)} - 2\sigma_{[a}{}^{(c}\sigma_{b]e}S^{d)e} \\
&\quad - 2S_{[a}{}^{(c}\left\{E_{b]}{}^{d)} - \tfrac{1}{3}\Theta\sigma_{b]}{}^{d)}\right\} \\
&\quad + 2h_{[a}{}^{(c}\left\{E_{b]e} - \tfrac{1}{3}\Theta\sigma_{b]e}\right\} S^{d)e} . && \text{(A.19)}
\end{aligned}$$

We now define the curl of a tensor V_{ab} by

$$(\operatorname{curl} V)_{ab} = \varepsilon_{pq(a}D^q V^p{}_{b)} . \tag{A.20}$$

Then (A.13)–(A.19) imply the further identities

$$\begin{aligned}
\varepsilon_{abc}S^b{}_p \operatorname{curl} V^{cp} &= \tfrac{1}{2}S_{ab}\mathrm{D}_c V^{bc} - 2S^{bc}\mathrm{D}_{[a}V_{b]c} , && \text{(A.21)}\\
\operatorname{curl}\,(fS_{ab}) &= f\operatorname{curl} S_{ab} + \varepsilon_{cd(a}S_{b)}{}^d\mathrm{D}^c f , && \text{(A.22)}\\
\mathrm{D}^b \operatorname{curl} S_{ab} &= \tfrac{1}{2}\varepsilon_{abc}\mathrm{D}^c\left(\mathrm{D}_d S^{bd}\right) + \varepsilon_{abc}S^b{}_d\left(E^{cd} - \tfrac{1}{3}\Theta\sigma^{cd}\right) \\
&\quad + \sigma_{ab}\varepsilon^{bcd}\sigma_{ce}S^e{}_d , && \text{(A.23)}\\
(\operatorname{curl} S_{ab})^{\cdot} &= \operatorname{curl}\dot{S}_{ab} - \tfrac{1}{3}\Theta\operatorname{curl} S_{ab} \\
&\quad + \sigma_e{}^c\varepsilon_{cd(a}\mathrm{D}^e S_{b)}{}^d - 3H_{c<a}S_{b>}{}^c , && \text{(A.24)}\\
(\operatorname{curl} fS)_{ab} &= f(\operatorname{curl} S)_{ab} + \varepsilon_{cd(a}S^c{}_{b)}D^d f , && \text{(A.25)}\\
(\operatorname{curl}\operatorname{curl} S)_{ab} &= -D^2 S_{ab} + \frac{3}{2}D_{<a}D^c S_{b>c}\left(\rho - \frac{1}{3}\Theta^2\right) S_{ab} \\
&\quad +3S_{c<a}\left[E_{b>}{}^c - \frac{1}{3}\Theta\sigma_{b>}{}^c\right] + \sigma_{cd}S^{cd}\sigma_{ab} \\
&\quad -S^{cd}\sigma_{ca}\sigma_{db} + \sigma^{cd}\sigma_{c(a}S_{b)d} , && \text{(A.26)}
\end{aligned}$$

A.2 Jacobi Identities

The set of orthonormal tetrad vectors $\{\mathbf{e}_a, \mathbf{e}_b\}$ satisfies the following commutation

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c{}_{ab}\mathbf{e}_c \ , \qquad a, b, c = 0 \cdots 3 \ , \tag{A.27}$$

where the structure constants $\gamma^c{}_{ab}$ are related to the Ricci coefficients $\Gamma^c{}_{ab}$ by:

$$\gamma^c{}_{ab} = \Gamma^c{}_{ab} - \Gamma^c{}_{ba} \ , \tag{A.28}$$

and inversely by:

$$\Gamma_{abc} = \tfrac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca}) \ . \tag{A.29}$$

As in Ellis (1967) [33] the Jacobi identities

$$[\mathbf{e}_b, [\mathbf{e}_c, \mathbf{e}_d]] + [\mathbf{e}_d, [\mathbf{e}_b, \mathbf{e}_c]] + [\mathbf{e}_c, [\mathbf{e}_d, \mathbf{e}_b]] = 0 \ , \tag{A.30}$$

can be written in terms of $\gamma^c{}_{ab}$ as:

$$\begin{pmatrix} f \\ bcd \end{pmatrix} : \qquad \partial_{[d}\gamma^f{}_{cb]} + \gamma^g{}_{[dc}\gamma^f{}_{b]g} = 0 \ . \tag{A.31}$$

That is:

$$\begin{pmatrix} 1 \\ 123 \end{pmatrix} : \ \partial_1\gamma^1{}_{32} - \partial_2\gamma^1{}_{31} + \partial_3\gamma^1{}_{21} \ = \ -\gamma^2{}_{32}\gamma^1{}_{21} - \gamma^3{}_{23}\gamma^1{}_{31} + \gamma^1{}_{32}(\gamma^2{}_{12} - \gamma^3{}_{13}) \ , \tag{A.32}$$

$$\begin{pmatrix} 2 \\ 123 \end{pmatrix} : \ -\partial_1\gamma^2{}_{32} + \partial_2\gamma^2{}_{13} - \partial_3\gamma^2{}_{12} \ = \ -\gamma^1{}_{31}\gamma^2{}_{12} + \gamma^3{}_{13}\gamma^2{}_{32} + \gamma^2{}_{13}(\gamma^3{}_{23} + \gamma^1{}_{21}) \ , \tag{A.33}$$

$$\begin{pmatrix} 3 \\ 123 \end{pmatrix} : \ -\partial_1\gamma^3{}_{23} + \partial_2\gamma^3{}_{13} - \partial_3\gamma^3{}_{12} \ = \ \gamma^1{}_{21}\gamma^3{}_{13} - \gamma^2{}_{12}\gamma^3{}_{23} - \gamma^3{}_{12}(\gamma^1{}_{31} + \gamma^2{}_{32}) \ , \tag{A.34}$$

$$\begin{pmatrix} \mu \\ 0\mu\nu \end{pmatrix} : \ \partial_0 \gamma^{\mu}{}_{\nu\mu} = -\partial_\nu \theta_\mu - \theta_\nu \gamma^{\mu}{}_{\nu\mu} \ , \tag{A.35}$$

$$\begin{pmatrix} 1 \\ 023 \end{pmatrix} : \ \partial_0 \gamma^{1}{}_{32} = \gamma^{1}{}_{32}(\theta_1 - \theta_2 - \theta_3) \ , \tag{A.36}$$

$$\begin{pmatrix} 2 \\ 031 \end{pmatrix} : \ \partial_0 \gamma^{2}{}_{13} = \gamma^{2}{}_{13}(-\theta_1 + \theta_2 - \theta_3) \ , \tag{A.37}$$

$$\begin{pmatrix} 3 \\ 012 \end{pmatrix} : \ \partial_0 \gamma^{3}{}_{12} = \gamma^{3}{}_{12}(-\theta_1 - \theta_2 + \theta_3) \ . \tag{A.38}$$

APPENDIX B

Sample calculations of time derivatives

In this appendix we give sample calculation of time-derivatives of constraint equations first in the generic irrotational dust case, and in the case of $DivE = 0$ irrotational dust spacetimes (section B.5).

The generic propagation equation are:

$$\dot{\rho} + \Theta\rho = 0\,, \tag{B.1}$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 = -\sigma_{ab}\sigma^{ab} - \tfrac{1}{2}\rho\,, \tag{B.2}$$

$$\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\sigma_{ab} + \sigma_{c<a}\sigma_{b>}{}^{c} = -E_{ab}\,, \tag{B.3}$$

$$\dot{E}_{ab} + \Theta E_{ab} - 3\sigma_{c<a}E_{b>}{}^{c} = -\mathrm{curl}\, H_{ab} - \tfrac{1}{2}\rho\sigma_{ab}\,, \tag{B.4}$$

$$\dot{H}_{ab} + \Theta H_{ab} - 3\sigma_{c<a}H_{b>}{}^{c} = \mathrm{curl}\, E_{ab}\,. \tag{B.5}$$

In the ensuing calculations we denote the constraint equations (B.6) – (B.9) by $\mathcal{C}^A = 0$, where

$$\mathcal{C}^A = \left(\mathrm{D}^b\sigma_{ab} - \tfrac{2}{3}\mathrm{D}_a\Theta\,,\ \mathrm{curl}\,\sigma_{ab} + H_{ab}\,,\ \cdots\right)\,,$$

and $A = \mathbf{1}, \cdots, \mathbf{4}$. Written explicitly the generic constraint equations are:

$$\mathcal{C}^{\mathbf{1}}{}_a = \tfrac{3}{2}D^b\sigma_{ab} - \mathrm{D}_a\Theta\,; \tag{B.6}$$

$$\mathcal{C}^{\mathbf{2}}{}_{ab} = H_{ab} + \mathrm{curl}\,\sigma_{ab}\,; \tag{B.7}$$

$$\mathcal{C}^{\mathbf{3}}{}_a = \mathrm{D}^b E_{ab} - \varepsilon_{abc}\sigma^b{}_d H^{cd} - \tfrac{1}{3}\mathrm{D}_a\rho\,; \tag{B.8}$$

$$\mathcal{C}^{\mathbf{4}}{}_a = \mathrm{D}^b H_{ab} + \varepsilon_{abc}\sigma^b{}_d E^{cd}\,. \tag{B.9}$$

We recall that this set is said to be satisfied if $\mathcal{C}^A = 0;\ A = 1, 2, 3, 4$.

We now show that, in the general case, the evolution of $\mathcal{C}^A$ along u^a leads to a system of equations

$$\dot{\mathcal{C}}^A = \mathcal{F}^A(\mathcal{C}^B)\,, \tag{B.10}$$

where $\mathcal{F}^A$ do not contain time derivatives, since these are eliminated via the propagation equations and tensorial identities (see appendix A.1). The calculations are outlined in the following sections.

B.1 The shear constraint

First consider the shear constraint (B.6) as

$$\mathcal{C}^1 = \mathrm{D}^b\sigma_{ab} - \tfrac{2}{3}\mathrm{D}_a\Theta \; . \tag{B.11}$$

In (B.11) the time derivative of the first term on the right hand side, using (A.17) and (B.3), becomes

$$\begin{aligned} (D^b\sigma_{ab})^{\cdot} &= D^b\dot{\sigma}_{ab} - \tfrac{1}{3}\Theta D^b\sigma_{ab} - \sigma^{bc}D_c\sigma_{ab} - \varepsilon_{abc}\sigma^b{}_d H^{dc} \\ &= -\sigma_a{}^c D^b\sigma_{bc} - 2\sigma^{bc}D_b\sigma_{ac} - \tfrac{2}{3}\sigma_a{}^b D_b\Theta \\ &\quad -\Theta D^b\sigma_{ab} + \tfrac{2}{3}\sigma^{bc}D_a\sigma_{bc} - D^b E_{ab} - \varepsilon_{abc}\sigma^b{}_d H^{dc} \; , \end{aligned} \tag{B.12}$$

and that of the second term, using (A.13) and (B.2), becomes

$$\begin{aligned} \tfrac{2}{3}(D_a\Theta)^{\cdot} &= \tfrac{2}{3}\left[D_a\dot{\Theta} - \tfrac{1}{3}\Theta D_a\Theta - \sigma_a{}^b D_b\Theta\right] \\ &= \tfrac{2}{3}\left[-\theta D_a\Theta - \tfrac{1}{2}D_a\rho - 2\sigma^{bc}D_a\sigma_{bc} - \sigma_a{}^b D_b\Theta\right] \; . \end{aligned} \tag{B.13}$$

The evolution of the constraint (B.11) then becomes

$$\begin{aligned} \dot{\mathcal{C}}^1{}_a &= -\Theta(D^b\sigma_{ab} - \tfrac{2}{3}D_a\Theta) - \tfrac{2}{3}\left(D_b E^b{}_a - \tfrac{1}{3}D_a\rho - \varepsilon_{abc}\sigma^b{}_d H^{dc}\right) \quad ? \\ &\quad -\tfrac{2}{3}\left(2\varepsilon_{abc}\sigma^b{}_d H^{dc} + 2\sigma^{bc}\mathrm{D}_a\sigma_{bc} - 2\sigma^{bc}\mathrm{D}_a\sigma_{bc} + \sigma_{ac}\mathrm{D}_b\sigma^{bc}\right) \; . \end{aligned} \tag{B.14}$$

Now using the identity (A.21) in equation (B.14) yields

$$\dot{\mathcal{C}}^1{}_a = -\Theta\mathcal{C}^1{}_a - \tfrac{3}{2}\mathcal{C}^3{}_a - 3\varepsilon_a{}^{bc}\sigma_b{}^d\mathcal{C}^2{}_{cd} \; , \tag{B.15}$$

and thus the shear constraint is consistent if the constraints $\mathcal{C}^1$,$\mathcal{C}^2$ and $\mathcal{C}^3$ are satisfied.

B.2 The H_{ab} equation

The H_{ab} constraint is written as

$$\mathcal{C}^2 = \mathrm{curl}\,\sigma_{ab} + H_{ab}\,. \tag{B.16}$$

A similar procedure, as in the above section, is used to obtain the time evolution of (B.16). We obtain first using (B.5)

$$\dot{H}_{ab} = -\Theta H_{ab} + 3\sigma_{c<a}H_{b>}{}^{c} + \mathrm{curl}\,E_{ab}\,, \tag{B.17}$$

and using (A.16), (A.24) and (B.3) we obtain

$$\begin{aligned}[(\mathrm{curl}\,\sigma)_{ab}]^{\cdot} = {} & \tfrac{1}{2}\varepsilon_{cda}\Big[-\sigma_{be}D^d\sigma^{ec} - \sigma_{ec}D^d\sigma^{be} - \tfrac{2}{3}\sigma_b{}^cD^d\Theta - \Theta D^d\sigma_b{}^c \\ & +\tfrac{2}{3}h^c{}_b\sigma^{ef}D^d\sigma_{ef} - D^dE_b{}^c - \sigma^{ed}D_e\sigma_b{}^c + H^{qd}\sigma^{hc}\varepsilon_{hbq} \\ & + H^{qd}\sigma^h{}_b\varepsilon_h{}^c{}_q\Big] \\ & +\tfrac{1}{2}\varepsilon_{cdb}\Big[-\sigma_{ae}D^d\sigma^{ec} - \sigma_{ec}D^d\sigma^{ae} - \tfrac{2}{3}\sigma_a{}^cD^d\Theta - \Theta D^d\sigma_a{}^c \\ & +\tfrac{2}{3}h^c{}_a\sigma^{ef}D^d\sigma_{ef} - D^dE_a{}^c - \sigma^{ed}D_e\sigma_a{}^c + H^{qd}\sigma^{hc}\varepsilon_{haq} \\ & + H^{qd}\sigma^h{}_a\varepsilon_h{}^c{}_q\Big]\,. \end{aligned} \tag{B.18}$$

From (B.17) and (B.18) we then obtain

$$\begin{aligned}\dot{\mathcal{C}}^2{}_{ab} = {} & -\Theta\mathcal{C}^2{}_{ab} + \varepsilon_{cd(a}\sigma_{b)}{}^d\mathcal{C}^{1c} - \varepsilon_{cd(a}\sigma_{b)e}D^d\sigma^{ce} + h_{ab}\sigma^{dc}H_{dc} - 3\sigma_{d(a}H_{b)}{}^d \\ & -\sigma^{ec}\varepsilon_{cd(a}D^d\sigma_{b)e} - \sigma^{ed}\varepsilon_{cd(a}D^d\sigma_{b)}{}^c - \varepsilon_{cd(a}\sigma_{b)}{}^cD_e\sigma^{ed} \\ & +\tfrac{1}{2}H^{qd}\sigma^{hc}\left(\varepsilon_{cda}\varepsilon_{hbq} + \varepsilon_{cdb}\varepsilon_{haq}\right) + \tfrac{1}{2}H^d{}_q\left(\sigma_{hb}\varepsilon_{cda}\varepsilon^{hcq} + \sigma_{ha}\varepsilon_{cdb}\varepsilon^{hcq}\right)\,, \end{aligned} \tag{B.19}$$

and using identities (A.3), (A.5), propagation equations (B.3) and (B.5), and the constraints (B.6) and (B.7), one then gets

$$\dot{\mathcal{C}}^2{}_{ab} = -\Theta\mathcal{C}^2{}_{ab} + \varepsilon_{cd(a}\sigma_{b)}{}^d\mathcal{C}^{1c}\,, \tag{B.20}$$

where the identity (A.6) is used to eliminate the following term from the right hand side of (B.20)

$$\varepsilon_{cd(a}\left\{\mathrm{D}^c\left[\sigma_{b)}{}^e\sigma^d{}_e\right] + \mathrm{D}^e\left[\sigma_{b)}{}^d\sigma^c{}_e\right]\right\} = 0\,. \tag{B.21}$$

Thus (B.20) is consistent provided the constraints (B.6) and (B.7) are satisfied.

B.3 The $DivE$ equation

Also for the Bianchi "Div E" constraint (B.8)

$$\mathcal{C}^3 = \mathrm{D}^b E_{ab} - \varepsilon_{abc}\sigma^b{}_d H^{cd} - \tfrac{1}{3}D_a\rho\ , \tag{B.22}$$

we obtain the following time derivatives; using the identities (A.5) and (A.16), the propagation equations (B.1) and (B.3)–(B.5), and the constraint (B.9):

$$\begin{aligned}\left(D^b E_{ab}\right)^{\cdot} &= -\tfrac{4}{3}\Theta D^b E_{ab} - \tfrac{1}{2}\rho D^b\sigma_{ab} - \tfrac{1}{2}\sigma_{ab}D^b\rho + E^{bc}D_d\sigma_{ac} + \sigma_{ab}D_c E^{bc} + \tfrac{1}{2}E_{ab}D_c\sigma^{bc} \\ &\quad +\sigma^{bc}D_b E_{ac} - D_a(\sigma^{bc}E^{bc}) - \tfrac{1}{3}\Theta\varepsilon_{abc}\sigma^b{}_d H^{cd} - \sigma_{ab}\varepsilon^{bcd}\sigma_{ce}H_d{}^e\ ,\end{aligned} \tag{B.23}$$

and

$$\begin{aligned}\left[\varepsilon_{abc}\sigma^b{}_d H^{cd} + \tfrac{1}{3}D_a\rho\right]^{\cdot} &= -\tfrac{4}{3}\Theta D^b E_{ab} - \tfrac{1}{2}\rho D^b\sigma_{ab} + \varepsilon_{abc}\sigma^b{}_d(\mathrm{curl}\,E)^{cd} - \tfrac{1}{3}\Theta\varepsilon_{abc}\sigma^b{}_d H^{cd} \\ &\quad -\sigma_{ab}D_c E^{bc} + \tfrac{1}{2}\sigma_{ab}\varepsilon^{bcd}\sigma_{ce}H_d{}^e - \varepsilon_{abc}E^b{}_d H^{cd}\ .\end{aligned} \tag{B.24}$$

So now (B.23) and (B.24), on using the identity (A.21), yield

$$\begin{aligned}\dot{\mathcal{C}}^3_a &= -\tfrac{4}{3}\Theta\mathcal{C}^3{}_a + \tfrac{1}{2}\sigma_a{}^b\mathcal{C}^3{}_b - \tfrac{2}{9}\rho\mathcal{C}^1{}_a \\ &\quad + \tfrac{2}{3}E_a{}^b\mathcal{C}^1{}_b + \varepsilon_a{}^{bc}E_b{}^d\mathcal{C}^2{}_{cd} + \tfrac{1}{2}\varepsilon_a{}^{bc}\mathrm{D}_b\mathcal{C}^4{}_c\ .\end{aligned} \tag{B.25}$$

By virtue of the constraints (B.6)–(B.9), no new conditions arise from propagating the first Bianchi constraint.

B.4 The $DivH$ equation

Finally taking the time derivative of the Bianchi "Div H" constraint (B.9),

$$\mathcal{C}^4_a = \mathrm{D}^b H_{ab} + \varepsilon_{abc}\sigma^b{}_d E^{cd}$$

and using the propagation equations (B.3)–(B.5), the constraints (B.6), (B.8) and (B.9), and the identities (A.1), (A.5) and (A.16), one gets

$$\begin{aligned}\dot{\mathcal{C}}^4_a &= \left(\mathrm{D}^b H_{ab}\right)^{\cdot} + \left[\varepsilon_{abc}\sigma^b{}_d E^{cd}\right]^{\cdot} \\ &= -\tfrac{4}{3}\Theta\mathcal{C}^4{}_a + \tfrac{1}{2}\sigma_a{}^b\mathcal{C}^4{}_b \\ &\quad + \tfrac{2}{3}H_a{}^b\mathcal{C}^1{}_b + \varepsilon_a{}^{bc}H_b{}^d\mathcal{C}^2{}_{cd} - \tfrac{1}{2}\varepsilon_a{}^{bc}\mathrm{D}_b\mathcal{C}^3{}_c\ ,\end{aligned} \tag{B.26}$$

so that no condition arises from propagating the Bianchi "Div H" constraint in the general case.

B.5 Consistency of $DivE = 0$ spacetimes

Suppose that the electric Weyl tensor is divergence-free in an open set U, that is

$$\mathcal{B}_a = \mathrm{D}_b E^b{}_a = 0 \,, \qquad E_{ab} \neq 0 \,. \tag{B.27}$$

The propagation equations for irrotational dust spacetimes remain the same as in (B.1)–(B.4) and the only constraint equation that is affected is the Bianchi "*divE*" constraint (B.8):

$$\begin{aligned} \mathcal{C}^3 &= \mathcal{C}^{3*} + \mathcal{B} \,, & \text{(B.28)} \\ \mathcal{C}^{3*}{}_a &= -\tfrac{1}{3}\mathrm{D}_a\rho - \varepsilon_{abc}\sigma^b{}_d H^{cd} \,, & \text{(B.29)} \end{aligned}$$

from which it follows that

$$DivE = 0 \qquad \Leftrightarrow \qquad 0 \;=\; -\tfrac{1}{3}\mathrm{D}_a\rho - \varepsilon_{abc}\sigma^b{}_d H^{cd}. \tag{B.30}$$

Now using (B.3), (B.5), (B.6), (B.7) and (A.13) in the evolution of (B.27) we obtain

$$\begin{aligned} \dot{\mathcal{C}}^3_a &= -\frac{4}{3}\Theta\mathcal{C}^3_a - \frac{1}{3}\rho D_a\Theta \\ &\quad -\frac{1}{3}\sigma_a{}^b D_b\rho - \frac{1}{2}\sigma_a{}^b \epsilon_{bcd}\sigma^c{}_p H^{dp} \\ &\quad -\varepsilon_{abc}E^b{}_p H^{cp} - \varepsilon_{abc}\sigma^b{}_p (curlE)^{cp} \,, \end{aligned} \tag{B.31}$$

It then follows that $\dot{\mathcal{C}}^3_a = 0$ if $J_a = 0$ where

$$\begin{aligned} J_a \;\equiv\; & -\frac{1}{3}\rho D_a\Theta - \frac{1}{3}\sigma_a{}^b D_b\rho - \frac{1}{2}\sigma_a{}^b \varepsilon_{bcd}\sigma^c{}_p H^{pd} \\ & -\varepsilon_{abc}E^b{}_p H^{cp} - \varepsilon_{abc}\sigma^b{}_p (curlE)^{cp} \,. \end{aligned} \tag{B.32}$$

APPENDIX C

Integrability of $H = 0$

Preservation of 'silent' constraint?: $f_3 = 0$

$$
\begin{aligned}
f_3 \; := \;& 2376E_-^9\sigma_+ - 2376E_-^8E_+\sigma_- - 3520E_-^8\sigma_-^3 + 25164E_-^8\sigma_-\sigma_+^2 \\
& +432E_-^8\sigma_-\sigma_+\Theta - 14256E_-^7E_+^2\sigma_+ - 19656E_-^7E_+\sigma_-^2\sigma_+ - 432E_-^7E_+\sigma_-^2\Theta \\
& -13056E_-^7E_+\sigma_+^3 - 1248E_-^7E_+\sigma_+^2\Theta - 4140E_-^2\mu\sigma_-\sigma_+^2 - 6600E_-^7\mu\sigma_+^3 \\
& -264E_-^7\mu\sigma_+^2\Theta - 29088E_-^7\sigma_-^4\sigma_+ - 1536E_-^7\sigma_-^4\Theta + 102480E_-^7\sigma_-^2\sigma_+^3 \\
& +4200E_-^7\sigma_-^2\sigma_+^2\Theta + 72E_-^7\sigma_-^2\sigma_+\Theta^2 + 14256E_-^6E_+^3\sigma_- + 26172E_-^6E_+^2\sigma_-^3 \\
& -89352E_-^6E_+^2\sigma_-\sigma_+^2 - 96E_-^6E_+^2\sigma_-\sigma_+\Theta + 4140E_-^6E_+\mu\sigma_-^3 + 31224E_-^6E_+\mu\sigma_-\sigma_+^2 \\
& +528E_-^6E_+\mu\sigma_-\sigma_+\Theta + 16160E_-^6E_+\sigma_-^5 - 31696E_-^6E_+\sigma_-^3\sigma_+^2 + 8912E_-^6E_+\sigma_-^3\sigma_+\Theta \\
& -72E_-^6E_+\sigma_-^3\Theta^2 - 96768E_-^6E_+\sigma_-\sigma_+^4 - 15216E_-^6E_+\sigma_-\sigma_+^3\Theta - 384E_-^6E_+\sigma_-\sigma_+^2\Theta^2 \\
& +132E_-^6\mu^2\sigma_-\sigma_+^2 + 4096E_-^6\mu\sigma_-^5 - 17420E^6\mu\sigma_-^3\sigma_+^2 + 256E_-^6\mu\sigma_-^3\sigma_+\Theta \\
& -50940E_-^6\mu\sigma_-\sigma_+^4 - 2460E_-^6\mu\sigma_-\sigma_+^3\Theta - 48E_-^6\mu\sigma_-^2\sigma_+\Theta^2 + 4224E_-^6\sigma_-^7 \\
& -86912E_-^6\sigma_-^5\sigma_+^2 - 4672E_-^6\sigma_-^5\sigma_+\Theta - 128E_-^6\sigma_-^5\Theta^2 + 203808E_-^6\sigma_-^3\sigma_+^4 \\
& +9744E_-^6\sigma_-^3\sigma_+^3\Theta + 876E_-^6\sigma_-^3\sigma_+^2\Theta^2 + 21384E_-^5E_+^4\sigma_+ + 33840E_-^5E_+^3\sigma_-^2\sigma_+ \\
& +1344E_-^5E_+^3\sigma_-^2\Theta - 1152E_-^5E_+^3\sigma_+^3 + 3744E_-^5E_+^3\sigma_+^2\Theta - 17808E_-^5E_+^2\mu\sigma_-^2\sigma_+ \\
& +72E_-^5E_+\mu\sigma_+^3\Theta^2 + 25472E_-^5E_+\sigma_-^6\sigma_+ - 1216E_-^5E_+\sigma_-^6\Theta + 74944E_-^5E_+\sigma_-^4\sigma_+^3 \\
& +30128E_-^5E_+\sigma_-^4\sigma_+^2\Theta + 184E_-^5E_+\sigma_-^4\sigma_+\Theta^2 - 269376E_-^5E_+\sigma_-^2\sigma_+^5 - 37488E_-^5E_+\sigma_-^2\sigma_+^4\Theta \\
& -4416E_-^5E_+\sigma_-^2\sigma_+^3\Theta^2 + 2694E_-^5\mu^2\sigma_-^4\sigma_+ + 4134E_-^5\mu^2\sigma_-^2\sigma_+^3 - 276E_-^5\mu^2\sigma_-^2\sigma_+^2\Theta \\
& +4554E_-^5\mu^2\sigma_+^5 + 396E_-^5\mu^2\sigma_+^4\Theta + 20944E_-^5\mu\sigma_-^6\sigma_+ + 1120E_-^5\mu\sigma_-^6\Theta \\
& -22456E_-^5\mu\sigma_-^4\sigma_+^3 - 140E_-^5\mu\sigma_-^4\sigma_+^2\Theta - 4E_-^5\mu\sigma_-^4\sigma_+\Theta^2 - 142104E_-^5\mu\sigma_-^2\sigma_+^5 \\
& -6360E_-^5\mu\sigma_-^2\sigma_+^4\Theta - 552E_-^5\mu\sigma_-^2\sigma_+^3\Theta^2 - 264E_-^5E_+^2\mu\sigma_-^2\Theta - 17496E_-^5E_+^2\mu\sigma_+^3
\end{aligned}
$$

$$+792E_-^5E_+^2\mu\sigma_+^2\Theta + 44368E_-^5E_+^2\sigma_-^4\sigma_+ + 712E_-^5E_+^2\sigma_-^4\Theta - 224448E_-^5E_+^2\sigma_-^2\sigma_+^3$$
$$-31008E_-^5E_+^2\sigma_-^2\sigma_+^2\Theta + 336E_-^5E_+^2\sigma_-^2\sigma_+\Theta^2 + 13248E_-^5E_+^2\sigma_+^5 + 10368E_-^5E_+^2\sigma_+^4\Theta$$
$$+576E_-^5E_+^2\sigma_+^3\Theta^2 - 264E_-^5E_+\mu^2\sigma_-^2\sigma_+ - 252E_-^5E_+\mu^2\sigma_+^3 - 3640E_-^5E_+\mu\sigma_-^4\sigma_+$$
$$-256E_-^5E_+\mu\sigma_-^4\Theta + 160320E_-^5E_+\mu\sigma_-^2\sigma_+^3 + 3204E_-^5E_+\mu\sigma_-^2\sigma_+^2\Theta + 96E_-^5E_+\mu\sigma_-^2\sigma_+\Theta^2$$
$$+18864E_-^5E_+\mu\sigma_+^5 + 2088E_-^5E_+\mu\sigma_+^4\Theta + 12096E_-^5\sigma_-^8\sigma_+ - 111360E_-^5\sigma_-^6\sigma_+^3 - 3552E_-^5\sigma_-^6\sigma_+^2\Theta$$
$$-192E_-^5\sigma_-^6\sigma_+\Theta^2 + 199872E_-^5\sigma_-^4\sigma_+^5 + 6624E_-^5\sigma_-^4\sigma_+^4\Theta + 1152E_-^5\sigma_-^4\sigma_+^3\Theta^2 - 21384E_-^4E_+^5\sigma_-$$
$$-26472E_-^4E_+^4\sigma_-^3 + 162540E_-^4E_+^4\sigma_-\sigma_+^2 - 3600E_-^4E_+^4\sigma_-\sigma_+\Theta - 6816E_-^4E_+^3\mu\sigma_-^3$$
$$+18216E_-^4E_+^3\mu\sigma_-\sigma_+^2 - 1584E_-^4E_+^3\mu\sigma_-\sigma_+\Theta + 1200E_-^4E_+^3\sigma_-^5 + 107520E_-^4E_+^3\sigma_-^3\sigma_+^2$$
$$+3792E_-^4E_+^3\sigma_-^3\sigma_+\Theta + 48E_-^4E_+^3\sigma_-^3\Theta^2 - 42624E_-^4E_+^3\sigma_-\sigma_+^4 + 46800E_-^4E_+^3\sigma_-\sigma_+^3\Theta$$
$$-576E_-^4E_+^3\sigma_-\sigma_+^2\Theta^2 + 132E_-^4E_+^2\mu^2\sigma_-^3 + 360E_-^4E_+^2\mu^2\sigma_-\sigma_+^2 - 15804E_-^4E_+^2\mu\sigma_-^5$$
$$-8976E_-^4E_+^2\mu\sigma_-^3\sigma_+^2 - 564E_-^4E_+^2\mu\sigma_-^3\sigma_+\Theta - 48E_-^4E_+^2\mu\sigma_-^3\Theta^2 - 89316E_-^4E_+^2\mu\sigma_-\sigma_+^4$$
$$+3492E_-^4E_+^2\mu\sigma_-\sigma_+^3\Theta - 72E_-^4E_+^2\mu\sigma_-\sigma_+^2\Theta^2 + 12288E_-^4E_+^2\sigma_-^7 + 13504E_-^4E_+^2\sigma_-^5\sigma_+^2$$
$$+12336E_-^4E_+^2\sigma_-^5\sigma_+\Theta + 92E_-^4E_+^2\sigma_-^5\Theta^2 - 372800E_-^4E_+^2\sigma_-^3\sigma_+^4 - 92224E_-^4E_+^2\sigma_-^3\sigma_+^3\Theta$$
$$+568E_-^4E_+^2\sigma_-^3\sigma_+^2\Theta^2 + 72096E_-^4E_+^2\sigma_-\sigma_+^6 + 31296E_-^4E_+^2\sigma_-\sigma_+^5\Theta + 5856E_-^4E_+^2\sigma_-\sigma_+^4\Theta^2$$
$$-2694E_-^4E_+\mu^2\sigma_-^5 - 27762E_-^4E_+\mu^2\sigma_-^3\sigma_+^2 + 552E_-^4E_+\mu^2\sigma_-^3\sigma_+\Theta - 6228E_-^4E_+\mu^2\sigma_-\sigma_+^4$$
$$+36E_-^4E_+\mu^2\sigma_-\sigma_+^3\Theta - 13584E_-^4E_+\mu\sigma_-^7 - 56520E_-^4E_+\mu\sigma_-^5\sigma_+^2 - 9176E_-^4E_+\mu\sigma_-^5\sigma_+\Theta$$
$$+4E_-^4E_+\mu\sigma_-^5\Theta^2 + 287104E_-^4E_+\mu\sigma_-^3\sigma_+^4 + 10856E_-^4E_+\mu\sigma_-^3\sigma_+^3\Theta + 1240E_-^4E_+\mu\sigma_-^3\sigma_+^2\Theta^2$$
$$+89448E_-^4E_+\mu\sigma_-\sigma_+^6 + 8220E_-^4E_+\mu\sigma_-\sigma_+^5\Theta + 1068E_-^4E_+\mu\sigma_-\sigma_+^4\Theta^2 + 4032E_-^4E_+\sigma_-^9$$
$$-32896E_-^4E_+\sigma_-^7\sigma_+^2 - 1856E_-^4E_+\sigma_-^7\sigma_+\Theta - 64E_-^4E_+\sigma_-^7\Theta^2 + 232896E_-^4E_+\sigma_-^5\sigma_+^4$$
$$+23712E_-^4E_+\sigma_-^5\sigma_+^3\Theta + 768E_-^4E_+\sigma_-^5\sigma_+^2\Theta^2 - 334080E_-^4E_+\sigma_-^3\sigma_+^6 - 26208E_-^4E_+\sigma_-^3\sigma_+^5\Theta$$
$$-5760E_-^4E_+\sigma_-^3\sigma_+^4\Theta^2 - 42E_-^4\mu^3\sigma_-^3\sigma_+^2 - 72E_-^4\mu^3\sigma_-\sigma_+^4 - 1728E_-^4\mu^2\sigma_-^7$$
$$+2099E_-^4\mu^2\sigma_-^5\sigma_+^2 - 52E_-^4\mu^2\sigma_-^5\sigma_+\Theta + 23938E_-^4\mu^2\sigma_-^3\sigma_+^4 - 460E_-^4\mu^2\sigma_-^3\sigma_+^3\Theta$$
$$-8E_-^4\mu^2\sigma_-^3\sigma_+^2\Theta^2 + 21975E_-^4\mu^2\sigma_-\sigma_+^6 + 1578E_-^4\mu^2\sigma_-\sigma_+^5\Theta + 96E_-^4\mu^2\sigma_-\sigma_+^4\Theta^2$$
$$-4032E_-^4\mu\sigma_-^9 + 39664E_-^4\mu\sigma_-^7\sigma_+^2 + 1712E_-^4\mu\sigma_-^7\sigma_+\Theta + 64E_-^4\mu\sigma_-^7\Theta^2$$

$$-8256E_-^4\mu\sigma_-^5\sigma_+^4 - 816E_-^4\mu\sigma_-^5\sigma_+^3\Theta - 336E_-^4\mu\sigma_-^5\sigma_+^2\Theta^2 - 172656E_-^4\mu\sigma_-^3\sigma_+^6$$
$$-4896E_-^4\mu\sigma_-^3\sigma_+^5\Theta - 720E_-^4\mu\sigma_-^3\sigma_+^4\Theta^2 + 8640E_-^4\sigma_-^9\sigma_+^2 - 51840E_-^4\sigma_-^7\sigma_+^4$$
$$+77760E_-^4\sigma_-^5\sigma_+^6 - 45576E_-^3E_+^5\sigma_-^2\sigma_+ - 144E_-^3E_+^5\sigma_-^2\Theta - 21204E_-^3E_+^4\mu\sigma_-^2\sigma_+$$
$$+792E_-^3E_+^4\mu\sigma_-^2\Theta - 99744E_-^3E_+^4\sigma_-^4\sigma_+ + 960E_-^3E_+^4\sigma_-^4\Theta + 335088E_-^3E_+^4\sigma_-^2\sigma_+^3$$
$$-10440E_-^3E_+^4\sigma_-^2\sigma_+^2\Theta + 72E_-^3E_+^4\sigma_-^2\sigma_+\Theta^2 + 36E_-^3E_+^3\mu^2\sigma_-^2\sigma_+ - 28368E_-^3E_+^3\mu\sigma_-^4\sigma_+$$
$$-180E_-^3E_+^3\mu\sigma_-^4\Theta + 38304E_-^3E_+^3\mu\sigma_-^2\sigma_+^3 - 10476E_-^3E_+^3\mu\sigma_-^2\sigma_+^2\Theta - 72E_-^3E_+^3\mu\sigma_-^2\sigma_+\Theta^2$$
$$-59840E_-^3E_+^3\sigma_-^6\sigma_+ + 784E_-^3E_+^3\sigma_-^6\Theta + 268672E_-^3E_+^3\sigma_-^4\sigma_+^3 - 28768E_-^3E_+^3\sigma_-^4\sigma_+^2\Theta$$
$$+496E_-^3E_+^3\sigma_-^4\sigma_+\Theta^2 - 106176E_-^3E_+^3\sigma_-^2\sigma_+^5 + 113232E_-^3E_+^3\sigma_-^2\sigma_+^4\Theta - 1344E_-^3E_+^3\sigma_-^2\sigma_+^3\Theta^2$$
$$+26958E_-^3E_+^2\mu^2\sigma_-^4\sigma_+ - 276E_-^3E_+^2\mu^2\sigma_-^4\Theta + 18162E_-^3E_+^2\mu^2\sigma_-^2\sigma_+^3 - 1656E_-^3E_+^2\mu^2\sigma_-^2\sigma_+^2\Theta$$
$$-7448E_-^3E_+^2\mu\sigma_-^6\sigma_+ - 764E_-^3E_+^2\mu\sigma_-^6\Theta + 115408E_-^3E_+^2\mu\sigma_-^4\sigma_+^3 + 24032E_-^3E_+^2\mu\sigma_-^4\sigma_+^2\Theta$$
$$-800E_-^3E_+^2\mu\sigma_-^4\sigma_+\Theta^2 - 145848E_-^3E_+^2\mu\sigma_-^2\sigma_+^5 - 1932E_-^3E_+^2\mu\sigma_-^2\sigma_+^4\Theta - 1992E_-^3E_+^2\mu\sigma_-^2\sigma_+^3\Theta^2$$
$$-1280E_-^3E_+^2\sigma_-^8\sigma_+ - 224E_-^3E_+^2\sigma_-^8\Theta + 113024E_-^3E_+^2\sigma_-^6\sigma_+^3 + 16576E_-^3E_+^2\sigma_-^6\sigma_+^2\Theta$$
$$+320E_-^3E_+^2\sigma_-^6\sigma_+\Theta^2 - 455424E_-^3E_+^2\sigma_-^4\sigma_+^5 - 69024E_-^3E_+^2\sigma_-^4\sigma_+^4\Theta - 1536E_-^3E_+^2\sigma_-^4\sigma_+^3\Theta^2$$
$$+130176E_-^3E_+^2\sigma_-^2\sigma_+^7 + 23616E_-^3E_+^2\sigma_-^2\sigma_+^6\Theta + 7488E_-^3E_+^2\sigma_-^2\sigma_+^5\Theta^2 + 84E_-^3E_+\mu^3\sigma_-^4\sigma_+$$
$$+270E_-^3E_+\mu^3\sigma_-^2\sigma_+^3 + 16462E_-^3E_+\mu^2\sigma_-^6\sigma_+ + 52E_-^3E_+\mu^2\sigma_-^6\Theta - 97724E_-^3E_+\mu^2\sigma_-^4\sigma_+^3$$
$$+716E_-^3E_+\mu^2\sigma_-^4\sigma_+^2\Theta + 16E_-^3E_+\mu\sigma_-^4\sigma_+\Theta^2 - 30954E_-^3E_+\mu^2\sigma_-^2\sigma_+^5 - 1104E_-^3E_+\mu^2\sigma_-^2\sigma_+^4\Theta$$
$$-228E_-^3E_+\mu^2\sigma_-^2\sigma_+^3\Theta^2 - 5408E_-^3E_+\mu\sigma_-^8\sigma_+ + 400E_-^3E_+\mu\sigma_-^8\Theta - 114544E_-^3E_+\mu\sigma_-^6\sigma_+^3$$
$$-12752E_-^3E_+\mu\sigma_-^6\sigma_+^2\Theta - 160E_-^3E_+\mu\sigma_-^6\sigma_+\Theta^2 + 209280E_-^3E_+\mu\sigma_-^4\sigma_+^5 + 10608E_-^3E_+\mu\sigma_-^4\sigma_+^4\Theta$$
$$+3072E_-^3E_+\mu\sigma_-^4\sigma_+^3\Theta^2 + 143568E_-^3E_+\mu\sigma_-^2\sigma_+^7 + 7632E_-^3E_+\mu\sigma_-^2\sigma_+^6\Theta + 1440E_-^3E_+\mu\sigma_-^2\sigma_+^5\Theta^2$$
$$+5760E_-^3E_+\sigma_-^{10}\sigma_+ - 51840E_-^3E_+\sigma_-^8\sigma_+^3 + 155520E_-^3E_+\sigma_-^6\sigma_+^5 - 155520E_-^3E_+\sigma_-^4\sigma_+^7$$
$$-861E_-^3\mu^3\sigma_-^6\sigma_+ + 1230E_-^3\mu^3\sigma_-^4\sigma_+^3 + 204E_-^3\mu^3\sigma_-^4\sigma_+^2\Theta - 4653E_-^3\mu^3\sigma_-^2\sigma_+^5$$
$$-198E_-^3\mu^3\sigma_-^2\sigma_+^4\Theta - 2168E_-^3\mu^2\sigma_-^8\sigma_+ - 176E_-^3\mu^2\sigma_-^8\Theta - 16372E_-^3\mu^2\sigma_-^6\sigma_+^3$$

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\begin{aligned}
&-188E_-^3\mu^2\sigma_-^6\sigma_+^2\Theta - 16E_-^3\mu^2\sigma_-^6\sigma_+\Theta^2 + 48288E_-^3\mu^2\sigma_-^4\sigma_+^5 + 312E_-^3\mu^2\sigma_-^4\sigma_+^4\Theta \\
&+192E_-^3\mu^2\sigma_-^4\sigma_+^3\Theta^2 + 35676E_-^3\mu^2\sigma_-^2\sigma_+^7 + 1476E_-^3\mu^2\sigma_-^2\sigma_+^6\Theta + 144E_-^3\mu\sigma_-^2\sigma_+^5\Theta^2 \\
&-5760E_-^3\mu\sigma_-^{10}\sigma_+ + 25920E_-^3\mu\sigma_-^8\sigma_+^3 - 77760E_-^3\mu\sigma_-^4\sigma_+^7 - 20772E_-^2E_+^6\sigma_-^3 \\
&+20484E_-^2E_+^5\mu\sigma_-^3 - 35616E_-^2E_+^5\sigma_-^5 + 75600E_-^2E_+^5\sigma_-^3\sigma_+^2 - 5184E_-^2E_+^5\sigma_-^3\sigma_+\Theta \\
&-72E_-^2E_+^5\sigma_-^3\Theta^2 - 144E_-^2E_+^4\mu^2\sigma_-^3 + 38556E_-^2E_+^4\mu\sigma_-^5 - 109908E_-^2E_+^4\mu\sigma_-^3\sigma_+^2 \\
&+4428E_-^2E_+^4\mu\sigma_-^3\sigma_+\Theta + 72E_-^2E_+^4\mu\sigma_-^3\Theta^2 - 18144E_-^2E_+^4\sigma_-^7 + 25984E_-^2E_+^4\sigma_-^5\sigma_+^2 \\
&-640E_-^2E_+^4\sigma_-^5\sigma_+\Theta - 104E_-^2E_+^4\sigma_-^5\Theta^2 + 171456E_-^2E_+^4\sigma_-^3\sigma_+^4 + 24240E_-^2E_+^4\sigma_-^3\sigma_+^3\Theta \\
&-948E_-^2E_+^4\sigma_-^3\sigma_+^2\Theta^2 - 3330E_-^2E_+^3\mu^2\sigma_-^5 - 16578E_-^2E_+^3\mu^2\sigma_-^3\sigma_+^2 + 1620E_-^2E_+^3\mu^2\sigma_-^3\sigma_+\Theta \\
&+20184E_-^2E_+^3\mu\sigma_-^7 - 140576E_-^2E_+^3\mu\sigma_-^5\sigma_+^2 + 1544E_-^2E_+^3\mu\sigma_-^5\sigma_+\Theta + 112E_-^2E_+^3\mu\sigma_-^5\Theta^2 \\
&+17448E_-^2E_+^3\mu\sigma_-^3\sigma_+^4 - 42720E_-^2E_+^3\mu\sigma_-^3\sigma_+^3\Theta + 816E_-^2E_+^3\mu\sigma_-^3\sigma_+^2\Theta^2 - 896E_-^2E_+^3\sigma_-^9 \\
&-51456E_-^2E_+^3\sigma_-^7\sigma_+^2 + 3616E_-^2E_+^3\sigma_-^7\sigma_+\Theta + 64E_-^2E_+^3\sigma_-^7\Theta^2 + 123008E_-^2E_+^3\sigma_-^5\sigma_+^4 \\
&-44672E_-^2E_+^3\sigma_-^5\sigma_+^3\Theta + 512E_-^2E_+^3\sigma_-^5\sigma_+^2\Theta^2 - 33792E_-^2E_+^3\sigma_-^3\sigma_+^6 + 77280E_-^2E_+^3\sigma_-^3\sigma_+^5\Theta \\
&+1344E_-^2E_+^3\sigma_-^3\sigma_+^4\Theta^2 - 42E_-^2E_+^2\mu^3\sigma_-^5 - 252E_-^2E_+^2\mu^3\sigma_-^3\sigma_+^2 - 3009E_-^2E_+^2\mu^2\sigma_-^7 \\
&+33910E_-^2E_+^2\mu^2\sigma_-^5\sigma_+^2 + 260E_-^2E_+^2\mu^2\sigma_-^5\sigma_+\Theta - 8E_-^2E_+^2\mu^2\sigma_-^5\Theta^2 + 57111E_-^2E_+^2\mu^2\sigma_-^3\sigma_+^4 \\
&-3696E_-^2E_+^2\mu^2\sigma_-^3\sigma_+^3\Theta + 132E_-^2E_+^2\mu^2\sigma_-^3\sigma_+^2\Theta^2 - 720E_-^2E_+^2\mu\sigma_-^9 - 28064E_-^2E_+^2\mu\sigma_-^7\sigma_+^2 \\
&-5248E_-^2E_+^2\mu\sigma_-^7\sigma_+\Theta - 80E_-^2E_+^2\mu\sigma_-^7\Theta^2 + 217584E_-^2E_+^2\mu\sigma_-^5\sigma_+^4 + 34272E_-^2E_+^2\mu\sigma_-^5\sigma_+^3\Theta \\
&-576E_-^2E_+^2\mu\sigma_-^5\sigma_+^2\Theta^2 - 76608E_-^2E_+^2\mu\sigma_-^3\sigma_+^6 - 7200E_-^2E_+^2\mu\sigma_-^3\sigma_+^5\Theta - 4464E_-^2E_+^2\mu\sigma_-^3\sigma_+^4\Theta^2 \\
&+960E_-^2E_+^2\sigma_-^{11} - 28800E_-^2E_+^2\sigma_-^9\sigma_+^2 + 155520E_-^2E_+^2\sigma_-^7\sigma_+^4 - 259200E_-^2E_+^2\sigma_-^5\sigma_+^6 \\
&+77760E_-^2E_+^2\sigma_-^3\sigma_+^8 + 861E_-^2E_+\mu^3\sigma_-^7 + 4338E_-^2E_+\mu^3\sigma_-^5\sigma_+^2 - 408E_-^2E_+\mu^3\sigma_-^5\sigma_+\Theta \\
&+5373E_-^2E_+\mu^3\sigma_-^3\sigma_+^4 - 18E_-^2E_+\mu^3\sigma_-^3\sigma_+^3\Theta + 1176E_-^2E_+\mu^2\sigma_-^9 + 42316E_-^2E_+\mu^2\sigma_-^7\sigma_+^2 \\
&+1832E_-^2E_+\mu^2\sigma_-^7\sigma_+\Theta + 16E_-^2E_+\mu^2\sigma_-^7\Theta^2 - 92688E_-^2E_+\mu^2\sigma_-^5\sigma_+^4 - 624E_-^2E_+\mu^2\sigma_-^5\sigma_+^3\Theta \\
&-384E_-^2E_+\mu^2\sigma_-^5\sigma_+^2\Theta^2 - 58500E_-^2E_+\mu^2\sigma_-^3\sigma_+^6 - 2520E_-^2E_+\mu^2\sigma_-^3\sigma_+^5\Theta - 720E_-^2E_+\mu^2\sigma_-^3\sigma_+^4\Theta^2 \\
&-1920E_-^2E_+\mu\sigma_-^{11} + 23040E_-^2E_+\mu\sigma_-^9\sigma_+^2 - 77760E_-^2E_+\mu\sigma_-^7\sigma_+^4 + 51840E_-^2E_+\mu\sigma_-^5\sigma_+^6 \\
&+77760E_-^2E_+\mu\sigma_-^3\sigma_+^8 - 12E_-^2\mu^4\sigma_-^5\sigma_+^2 + 36E_-^2\mu^4\sigma_-^3\sigma_+^4 + 440E_-^2\mu^3\sigma_-^9
\end{aligned}
$$

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\begin{aligned}
&-1406E_-^2\mu^3\sigma_-^7\sigma_+^2 - 92E_-^2\mu^3\sigma_-^7\sigma_+\Theta + 5356E_-^2\mu^3\sigma_-^5\sigma_+^4 + 440E_-^2\mu^3\sigma_-^5\sigma_+^3\Theta \\
&+16E_-^2\mu^3\sigma_-^5\sigma_+^2\Theta^2 - 15294E_-^2\mu^3\sigma_-^3\sigma_+^6 - 492E_-^2\mu^3\sigma_-^3\sigma_+^5\Theta - 48E_-^2\mu^3\sigma_-^3\sigma_+^4\Theta^2 \\
&+960E_-^2\mu^2\sigma_-^{11} - 720E_-^2\mu^2\sigma_-^9\sigma_+^2 - 19440E_-^2\mu^2\sigma_-^7\sigma_+^4 + 32400E_-^2\mu^2\sigma_-^5\sigma_+^6 \\
&+19440E_-^2\mu^2\sigma_-^3\sigma_+^8 - 27216E_-E_+^6\sigma_-^4\sigma_+ - 72E_-E_+^5\sigma_-^4\Theta + 21528E_-E_+^5\mu\sigma_-^4\sigma_+ \\
&+468E_-E_+^5\mu\sigma_-^4\Theta - 57792E_-E_+^5\sigma_-^6\sigma_+ + 144E_-E_+^5\sigma_-^6\Theta + 139968E_-E_+^5\sigma_-^4\sigma_+^3 \\
&-8208E_-E_+^5\sigma_-^4\sigma_+^2\Theta - 72E_-E_+^5\sigma_-^4\sigma_+\Theta^2 + 5058E_-E_+^4\mu^2\sigma_-^4\sigma_+ - 396E_-E_+^4\mu^2\sigma_-^4\Theta \\
&+58680E_-E_+^4\mu\sigma_-^6\sigma_+ + 168E_-E_+^4\mu\sigma_-^6\Theta - 117432E_-E_+^4\mu\sigma_-^4\sigma_+^3 + 3492E_-E_+^4\mu\sigma_-^4\sigma_+^2\Theta \\
&+36E_-E_+^4\mu\sigma_-^4\sigma_+\Theta^2 - 41408E_-E_+^4\sigma_-^8\sigma_+ + 160E_-E_+^4\sigma_-^8\Theta + 167168E_-E_+^4\sigma_-^6\sigma_+^3 \\
&-11168E_-E_+^4\sigma_-^6\sigma_+^2\Theta - 64E_-E_+^4\sigma_-^6\sigma_+\Theta^2 - 52800E_-E_+^4\sigma_-^4\sigma_+^5 + 44160E_-E_+^4\sigma_-^4\sigma_+^4\Theta \\
&-1536E_-E_+^4\sigma_-^4\sigma_+^3\Theta^2 - 18E_-E_+^3\mu^3\sigma_-^4\sigma_+ + 5640E_-E_+^3\mu^2\sigma_-^6\sigma_+ - 516E_-E_+^3\mu^2\sigma_-^6\Theta \\
&-16632E_-E_+^3\mu^2\sigma_-^4\sigma_+^3 + 3924E_-E_+^3\mu^2\sigma_-^4\sigma_+^2\Theta + 36E_-E_+^3\mu^2\sigma_-^4\sigma_+\Theta^2 + 52944E_-E_+^3\mu\sigma_-^8\sigma_+ \\
&-192E_-E_+^3\mu\sigma_-^8\Theta - 158304E_-E_+^3\mu\sigma_-^6\sigma_+^3 + 15168E_-E_+^3\mu\sigma_-^6\sigma_+^2\Theta - 192E_-E_+^3\mu\sigma_-^6\sigma_+\Theta^2 \\
&-1584E_-E_+^3\mu\sigma_-^4\sigma_+^5 - 43776E_-E_+^3\mu\sigma_-^4\sigma_+^4\Theta + 576E_-E_+^3\mu\sigma_-^4\sigma_+^3\Theta^2 - 9600E_-E_+^3\sigma_-^{10}\sigma_+ \\
&+63360E_-E_+^3\sigma_-^8\sigma_+^3 - 120960E_-E_+^3\sigma_-^6\sigma_+^5 + 51840E_-E_+^3\sigma_-^4\sigma_+^7 - 7200E_-E_+^2\mu^3\sigma_-^6\sigma_+ \\
&+204E_-E_+^2\mu^3\sigma_-^6\Theta - 4536E_-E_+^2\mu^3\sigma_-^4\sigma_+^3 + 630E_-E_+^2\mu^3\sigma_-^4\sigma_+^2\Theta - 7044E_-E_+^2\mu^2\sigma_-^8\sigma_+ \\
&-60E_-E_+^2\mu^2\sigma_-^8\Theta - 18216E_-E_+^2\mu^2\sigma_-^6\sigma_+^3 - 3672E_-E_+^2\mu^2\sigma_-^6\sigma_+^2\Theta + 288E_-E_+^2\mu^2\sigma_-^6\sigma_+\Theta^2 \\
&+42012E_-E_+^2\mu^2\sigma_-^4\sigma_+^5 - 540E_-E_+^2\mu^2\sigma_-^4\sigma_+^4\Theta + 864E_-E_+^2\mu^2\sigma_-^4\sigma_+^3\Theta^2 + 14400E_-E_+^2\mu\sigma_-^{10}\sigma_+ \\
&-86400E_-E_+^2\mu\sigma_-^8\sigma_+^3 + 129600E_-E_+^2\mu\sigma_-^6\sigma_+^5 + 24E_-E_+\mu^4\sigma_-^6\sigma_+ - 72E_-E_+\mu^4\sigma_-^4\sigma_+^3 \\
&-4780E_-E_+\mu^3\sigma_-^8\sigma_+ + 92E_-E_+\mu^3\sigma_-^8\Theta + 11080E_-E_+\mu^3\sigma_-^6\sigma_+^3 - 328E_-E_+\mu^3\sigma_-^6\sigma_+^2\Theta \\
&-32E_-E_+\mu^3\sigma_-^6\sigma_+\Theta^2 + 9780E_-E_+\mu^3\sigma_-^4\sigma_+^5 + 156E_-E_+\mu^3\sigma_-^4\sigma_+^4\Theta + 96E_-E_+\mu^3\sigma_-^4\sigma_+^3\Theta^2 \\
&-4320E_-E_+\mu^2\sigma_-^{10}\sigma_+ + 21600E_-E_+\mu^2\sigma_-^8\sigma_+^3 - 12960E_-E_+\mu^2\sigma_-^6\sigma_+^5 - 38880E_-E_+\mu^2\sigma_-^4\sigma_+^7 \\
&+126E_-\mu^4\sigma_-^8\sigma_+ - 756E_-\mu^4\sigma_-^6\sigma_+^3 + 1134E_-\mu^4\sigma_-^4\sigma_+^5 - 480E_-\mu^3\sigma_-^{10}\sigma_+ \\
&+1440E_-\mu^3\sigma_-^8\sigma_+^3 + 4320E_-\mu^3\sigma_-^6\sigma_+^5 - 12960E_-\mu^3\sigma_-^4\sigma_+^7 - 5040E_+^7\sigma_-^5 \\
&+9936E_+^6\mu\sigma_-^5 - 9408E_+^6\sigma_-^7 + 23328E_+^6\sigma_-^5\sigma_+^2 - 1584E_+^6\sigma_-^5\sigma_+\Theta \\
&-36E_+^6\sigma_-^5\Theta^2 - 4968E_+^5\mu^2\sigma_-^5 + 20208E_+^5\mu\sigma_-^7 - 42336E_+^5\mu\sigma_-^5\sigma_+^2
\end{aligned}
$$

$$
\begin{aligned}
&+2700E_+^5\mu\sigma_-^5\sigma_+\Theta + 72E_+^5\mu\sigma_-^5\Theta^2 - 5952E_+^5\sigma_-^9 + 29312E_+^5\sigma_-^7\sigma_+^2 \\
&-992E_+^5\sigma_-^7\sigma_+\Theta - 64E_+^5\sigma_-^7\Theta^2 - 9024E_+^5\sigma_-^5\sigma_+^4 + 7008E_+^5\sigma_-^5\sigma_+^3\Theta \\
&-384E_+^5\sigma_-^5\sigma_+^2\Theta^2 + 72E_+^4\mu^3\sigma_-^5 - 12420E_+^4\mu^2\sigma_-^7 + 15156E_+^4\mu^2\sigma_-^5\sigma_+^2 \\
&-702E_+^4\mu^2\sigma_-^5\sigma_+\Theta - 36E_+^4\mu^2\sigma_-^5\Theta^2 + 13824E_+^4\mu\sigma_-^9 - 61344E_+^4\mu\sigma_-^7\sigma_+^2 \\
&+1872E_+^4\mu\sigma_-^7\sigma_+\Theta + 144E_+^4\mu\sigma_-^7\Theta^2 + 8928E_+^4\mu\sigma_-^5\sigma_+^4 - 13680E_+^4\mu\sigma_-^5\sigma_+^3\Theta \\
&+720E_+^4\mu\sigma_-^5\sigma_+^2\Theta^2 - 1280E_+^4\sigma_-^{11} + 8640E_+^4\sigma_-^9\sigma_+^2 - 17280E_+^4\sigma_-^7\sigma_+^4 \\
&+8640E_+^4\sigma_-^5\sigma_+^6 + 1632E_+^3\mu^3\sigma_-^7 + 3816E_+^3\mu^3\sigma_-^5\sigma_+^2 - 414E_+^3\mu^3\sigma_-^5\sigma_+\Theta \\
&-9972E_+^3\mu^2\sigma_-^9 + 35832E_+^3\mu^2\sigma_-^7\sigma_+^2 - 768E_+^3\mu^2\sigma_-^7\sigma_+\Theta - 96E_+^3\mu^2\sigma_-^7\Theta^2 \\
&+7596E_+^3\mu^2\sigma_-^5\sigma_+^4 + 6336E_+^3\mu^2\sigma_-^5\sigma_+^3\Theta - 288E_+^3\mu^2\sigma_-^5\sigma_+^2\Theta^2 + 3200E_+^3\mu\sigma_-^{11} \\
&-20160E_+^3\mu\sigma_-^9\sigma_+^2 + 34560E_+^3\mu\sigma_-^7\sigma_+^4 - 8640E_+^3\mu\sigma_-^5\sigma_+^6 - 12E_+^2\mu^4\sigma_-^7 \\
&+36E_+^2\mu^4\sigma_-^5\sigma_+^2 + 2226E_+^2\mu^3\sigma_-^9 - 4556E_+^2\mu^3\sigma_-^7\sigma_+^2 - 112E_+^2\mu^3\sigma_-^7\sigma_+\Theta \\
&+16E_+^2\mu^3\sigma_-^7\Theta^2 - 6366E_+^2\mu^3\sigma_-^5\sigma_+^4 + 336E_+^2\mu^3\sigma_-^5\sigma_+^3\Theta - 48E_+^2\mu^3\sigma_-^5\sigma_+^2\Theta^2 \\
&-2640E_+^2\mu^2\sigma_-^{11} + 15120E_+^2\mu^2\sigma_-^9\sigma_+^2 - 19440E_+^2\mu^2\sigma_-^7\sigma_+^4 - 6480E_+^2\mu^2\sigma_-^5\sigma_+^6 \\
&-126E_+\mu^4\sigma_-^9 + 756E_+\mu^4\sigma_-^7\sigma_+^2 - 1134E_+\mu^4\sigma_-^5\sigma_+^4 + 800E_+\mu^3\sigma_-^{11} \\
&-4320E_+\mu^3\sigma_-^9\sigma_+^2 + 4320E_+\mu^3\sigma_-^7\sigma_+^4 + 4320E_+\mu^3\sigma_-^5\sigma_+^6 - 80\mu^4\sigma_-^{11} + 720\mu^4\sigma_-^9\sigma_+^2 \\
&-2160\mu^4\sigma_-^7\sigma_+^4 + 2160\mu^4\sigma_-^5\sigma_+^6 \,. \qquad \text{(C.1)}
\end{aligned}
$$

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