

The copyright of this thesis vests in the author. No quotation from it or information derived from it is to be published without full acknowledgement of the source. The thesis is to be used for private study or non-commercial research purposes only.

Published by the University of Cape Town (UCT) in terms of the non-exclusive license granted to UCT by the author.

Robustness Analysis based on Weight Restrictions in Data Envelopment Analysis

Dieudonné Kabongo Kantu
Department of Statistical Sciences
University of Cape Town

This dissertation is prepared in fulfilment of the requirements for the degree of Master of Science in Operations Research at the University of Cape Town.

Supervisor: Professor T J Stewart

2006

Acknowledgements

I would like to express my profound gratitude to Professor Theo Stewart for your brilliant insights and guidance in writing this dissertation and to Dominique Katshunga and Guy Wilson for all your support and encouragement.

Je dédie cette thèse à ma chère épouse, Julie et à mes chères filles, Eunice et Lydia. Merci infiniment au Seigneur Jésus-Christ!

University of Cape Town

University of Cape Town

Synopsis

Evaluating the performance of organisations is essential to good planning and control. Part of this process is monitoring the performance of organisations against their goals. The comparative efficiency of organizations using common inputs and outputs makes it possible for organizations to improve their performance so that they can operate as the most efficient organizations. Resources and outputs can be very diversified in nature and it is complex to assess organizations using such resources and outputs. Data Envelopment Analysis models are designed to facilitate this type of assessment and aim to evaluate the relative efficiency of organisations.

Chapter 2 is dedicated to the basic Data Envelopment Analysis. We present the following:

- A review of the Data Envelopment Analysis models
- The properties and particularities of each model.

In chapter 3, we present our literature survey on weight restrictions. Data Envelopment Analysis is a value-free frontier analysis which has the advantage of yielding more objective efficiency measures. However, the complete freedom in the determination of weights for the factors (resources and products) relevant to the assessment of organisations has led to some problems such as: zero-weights and lack of discrimination between efficient organizations. Weight restriction methods were introduced in order to tackle these problems. The first part of chapter 3 gives in detail the motivations for weight restrictions while the second part presents the actual weight restriction methods.

Chapter 4 looks at the key questions considered in this study.

Chapter 5 deals with the structure and implementation of the present sensitivity analysis. In addition, it presents the data sets that will be analysed. The weight restriction methods were found to have some weaknesses. The weight determination process can be very subjective. In addition, it is sometimes difficult to specify weight bounds. The

current sensitivity analysis aims to deal with the uncertainty involved in the weight determination process. However our objective is not to provide exact weights but to monitor how weight changes affect the efficiency score of organisations. The need for stable and reliable efficiency measures motivated this sensitivity analysis which encompasses four methods.

Chapter 6 is dedicated to the results of the sensitivity analysis. This study was conducted on three data sets. For each data set, the four methods yield different results which are compiled in tables. Then, we compare the results from each method and draw some conclusions.

The final chapter presents our conclusions and recommendations. The following points were discussed:

- We consider what we can learn from this study and the results obtained. It is very interesting that the different methods look at the same problem from different angles. The rationale behind each method is also presented.
- We look at other sensitivity methods that have been performed in other studies to examine how the robustness of this analysis differs from them. The current sensitivity analysis uses 3 weight restriction methods. We briefly explore the other methods.
- We consider the possibility of conducting other sensitivity analyses which could give new insights into the efficiency of organisations and which could maximise objective information from the data. The impact of sensitivity analysis on the target levels is also analysed.

Contents

Acknowledgements	iii
Synopsis	v
1 Introduction	1
1.1 Background to the problem	1
1.2 Statement of the problem	2
1.3 Aims of the study	3
1.4 Limitation of the current research	3
1.5 Outline of the thesis	4
1.6 Terminology and Preliminary notation	4
2 Basic DEA Models	7
2.1 Introduction on DEA	7
2.2 Objectives of DEA	8
2.3 DEA and other efficiency methodologies	8
2.4 Some basic concepts	9
2.5 The Charnes, Cooper and Rhodes model (CCR model)	15
2.6 The Banker, Charnes and Cooper model (BCC model)	20
2.7 Returns to scale property and DEA models	26
2.8 Additive model	32

2.9	Multiplicative model	34
2.10	Model including exogenously fixed inputs or outputs	37
2.11	Weights based targets DEA model	43
2.12	The cost and revenue models	47
2.13	Models used in the experimented studies reported in this thesis	48
3	Literature Survey on Weight restrictions	49
3.1	Meaning of weights in DEA	50
3.2	Weights flexibility and restrictions	51
3.3	Effects of weight restrictions	54
3.4	Methods of weight restrictions	55
3.5	Common set of weights (CSW)	66
3.6	Cross-efficiency	69
3.7	Methods used in this study	73
4	Current Research Questions	75
4.1	Research questions	75
4.2	Sensitivity Analyses	76
4.3	Robustness analysis	79
5	Comparison of Methods for Robustness Analysis	81
5.1	Methods	81
5.2	Data sets	86
6	Results and Interpretation of results	89
6.1	Definition	89
6.2	Results and Interpretation	90
6.3	General Conclusion from numerical studies	105

7 Final Conclusions	123
7.1 Methods of robustness analysis	124
7.2 General Weight Restriction Methods	127
7.3 Sensitivity Analyses	128
7.4 Research questions addressed	130
7.5 Future research	131
7.6 A Final comment	133

University of Cape Town

University of Cape Town

Chapter 1

Introduction

1.1 Background to the problem

Every organisation exists for a purpose and sets its objectives accordingly. The extent to which an organisation reaches its objectives is determined by how well it performs. Performance or efficiency measures can redirect the practices within the organisation to ensure that it stays on track to succeed and reach its goals. Determination of firms which are technically efficient, provides the base for economic analysis (Thompson et al., 1990b).

It is difficult to compare organisations with different resources and/or outputs. However, DEA with its unique approach is able to tackle this question effectively. Efficiency measures are essential to the decision-maker. Where organisations have common resources and outputs, it is often interesting to compare their performance. An organisation having many branches or departments may well want to compare their performance. Such comparative efficiency measures should enable inefficient organisations to emulate the more efficient organisations and possibly set appropriate target levels that would render them more productive. Good planning cannot be done without a good assessment of the status quo. Inappropriate performance measures can be very misleading and even damaging for an organisation. It is therefore crucial to obtain reliable efficiency measures that can provide a sound basis for assessing the performance of an organisation. The efficiency measure is influenced by the operating conditions. A change in the operating conditions will normally affect an organisations efficiency. It is essential to comprehend the extent of the change that may occur.

Every organisation is assessed through its weights and factor levels, which yield an efficiency score. The weights are determined by how the organisation performs in terms

of their outputs vis a vis the resources employed and are intended to emphasise the efficiency of the organisation assessed. The rationale of DEA, which seeks to present each unit in its best possible light, is one which, at least initially, shuns subjective judgements about the importance of different factors contributing to the measurement performance (Stewart and Belton, 1999). Total flexibility in the determination of weights in DEA has led sometimes to inappropriate results and conclusions. Weight restriction methods were created to prevent extreme and unrealistic weights and they use preference information in order to set bounds on the weights. As we know preference information is subjective. Given that there is a degree of uncertainty involved in this method, there is a need to find out what happens to the efficiency measurement of an organisation when it deviates from the restrictions recommended by the stakeholders, analysts or decision-makers.

1.2 Statement of the problem

The DEA efficiency measure is a function of the weights. DEA is a method that derives the weights directly from the data. Many methods of weight restrictions have been developed within DEA to include value judgements to better assess organisations. We know that the weights are related to the operating conditions under which an organisation works. An efficiency measurement, which is only dependent on the present operating conditions, may not be viable for long term planning. Also, no one has total control on the exogenous factors affecting the economy. Many unexpected events can change the operating conditions and, as a result, affect the efficiency level of organisations e.g. a change of market prices, an increase in competition, etc. The fact is that the efficiency of an organisation may not survive in the face of changing conditions. In the same way a change of policies in the organisation or the introduction of new regulations is likely to affect the productivity of an organisation. The question is "how can we be certain that the efficiency measurement of an organisation is robust given changes in operating conditions?" Given the uncertainty in the weights, how can we be certain the efficiency is not the result of good luck or random behaviour? It would be interesting to identify organisations whose efficiency is relatively insensitive to major changes. Value judgements or preference information are not always available to help us in performing effective assessments of organisations. How we handle such cases and derive meaningful and objective information from the analyses is crucial.

Once an efficiency score has been determined for an organisation, we may be tempted to stop there. How much information does the efficiency measure convey? How much does it not convey? Does the efficiency measure reflect the strengths and weaknesses of the organisation? We would like to be able to explore the strengths and weaknesses inherent in an organisation.

Value judgements in the form of ratio bounds on the weights imply a degree of uncertainty in the mind of the decision maker as to what weights best reflects the relative importances of the individual inputs and outputs, and a single efficiency measure cannot capture the extent of this uncertainty (Stewart, 1996). How can we assess the degree of uncertainty in the mind of the decision maker concerning the relative value of factors?

All these answers should provide vital information for managers and decision-makers.

1.3 Aims of the study

This research work aims to achieve the following objectives:

- To go beyond the DEA efficiency measures that may conceal some weaknesses and strengths of the organisation assessed.
- To investigate the reliability and stability of DEA efficiency measures.
- To extract maximum objective information from DEA.
- To explore how changing operating conditions may affect an organisation's efficiency.
- To investigate the extent to which some inefficient organisations may perform better than efficient organisations.
- To get more insight from a DEA assessment in the absence of value judgements.
- To identify role model organisations which are efficient.

1.4 Limitation of the current research

- The three data sets analysed in this study were found in the literature. We assume that all the relevant resources and outputs were included. We also assume that there were no errors in the data.
- This study does not aim to obtain a single reliable efficiency measure. A range of efficiency scores is intended to show the impact of weight changes on efficiency.
- No value judgement was used in the weight restrictions imposed on each factor weight. The bounds are the same for all weight factors. No ranking of factors was used in the evaluation process.

- The Cross-efficiency method used in this sensitivity analysis is an existing method although it is specifically used for the purpose of testing the robustness of the efficiency measurement of organisations
- It is assumed that all marginal productivities are decreasing.

1.5 Outline of the thesis

Chapter 2 gives a wide review on the models used in DEA and the assumptions attached to each one. We look also at the specific characteristics of each model. The rationale behind each model together with their mathematical modelling is provided. We consider the primal formulation of the model, which is the envelopment approach, then we give the dual formulation, which is the value-based method.

Chapter 3 covers the literature review on weight restriction methods. We present various weight restriction methods with their specific traits. We start the chapter with the types of weight restrictions in DEA and the different motivations for weight restrictions. Some graphical illustrations are also provided for more clarity.

Chapter 4 presents the research questions considered in this study.

Chapter 5 gives a detailed presentation of the various methods used in this sensitivity analysis. In this study, we have used four methods, which we consider varied enough to draw solid conclusions. In the first part of chapter 5 we present the three data sets analysed.

Chapter 6 presents the results for each method. We apply the four methods to each data set and correlate the results obtained. An interpretation of the results is provided.

Chapter 7 contains the final conclusions and recommendations. Many additional ideas for possible further research are also included.

1.6 Terminology and Preliminary notation

- BCC model: DEA model developed by Banker, Charnes and Cooper for estimating the efficiency of DMUs. It discriminates between technical and scale efficiencies (Charnes et al., 1994).

- CCR model: DEA model developed by Charnes, Cooper and Rhodes for estimating the efficiency of DMUs. It is characterised by constant returns to scale (see page 15).
- Constant Returns to scale (CRS): a situation where the scale size of a DMU will not affect its productivity.
- Data Envelopment Analysis (DEA): a method for measuring the comparative efficiency of homogeneous Decision Making Units (DMUs).
- Decision Making Unit (DMU): An organisation that uses resources and produces products and/or services.
- Factor: input and/or output.
- Input: Resources used to generate products or services e.g. staff, capital assets, etc.
- Outputs: Products or services such as manufactured items, banking transactions, etc.
- Production Possibility Space (PPS): include all observed and feasible Decision-Making Units which function under a set of assumptions.
- Variable Returns to Scale (VRS): a situation where the scale size of a DMU will affect its productivity.

University of Cape Town

Chapter 2

Basic DEA Models

2.1 Introduction on DEA

2.1.1 Definition

Data Envelopment Analysis (DEA) is defined by Thanassoulis (1999) as a linear programming based method for assessing the performance of homogeneous organizational units, such as bank branches, schools, tax offices and hospitals. Yun et al. (2004) define Data Envelopment Analysis as a method to estimate the relative efficiency of decision making units (DMUs) performing similar tasks in a production system that consumes multiple inputs to produce multiple outputs. One can also define Data Envelopment Analysis as a method for measuring the comparative efficiency of homogeneous decision making units (DMUs). Homogeneous DMUs are those performing similar tasks and using the same type of inputs to produce the same type of outputs, although in varying amounts. The inputs include all resources and/or environmental factors that are transformed into the outputs. The outputs include all outcomes and/or environmental factors. In a production context, the resources are labour and capital and the outcomes are goods or services. DEA is a method which applies in a multiple inputs/outputs context but also in a single input or output context. A unit is characterized as a decision making unit when it is making decisions on its own in the process of transforming resources into outcomes.

2.1.2 Efficiency in DEA

The efficiency of a DMU using many inputs and outputs may be defined as weighted sum of its outputs divided by a weighted sum of its inputs (Charnes et al., 1978). The efficiency measure in DEA is not an absolute efficiency measure but it represents the

performance of a unit relative to the other units. Given the fact that DEA derives its efficiency measure from the actual observed data, the efficiency score is related to the context in which the DMU was assessed. If assessed with different DMUs, the efficiency of a DMU is likely to be different. The DMUs assessed should be able to improve their performance. We can identify two kinds of efficiencies in DEA: technical and allocative efficiency. Technical efficiency is the measure of conservation of the resources without prejudice to the outputs or output augmentation without detriment to the inputs. It represents the extent to which a DMU is efficient without taking into account the inputs or output prices. Technical efficiency may be split up into pure technical efficiency and scale efficiency which reveals whether a DMU is functioning at an efficient scale size or not (30) (Banker et al., 1984).

Allocative or price efficiency implies that the inputs and output prices are taken into consideration when assessing a DMU's performance. The prices are relative societal values for an organization's outputs and resource opportunity costs for its inputs. Price efficiency can be more important than technical efficiency (Sexton et al., 1986).

2.2 Objectives of DEA

Thanassoulis (2001), Golany and Roll (1988)

The objectives for carrying out DEA assessment are as follows:

- To identify the amount and sources of inefficiency of the DMUs.
- To rank the DMUs involved in the assessment.
- To identify the efficient DMUs that operate with the same working practices as the DMU under consideration so that the latter can emulate the former.
- To distinguish between managerial and program performance and evaluate them.
- To reallocate resources that are in excess in a DMU to the DMUs which are in need of them.

2.3 DEA and other efficiency methodologies

First of all, we would like to say that the different methods are not meant to exclude each other. Rather, they should complement each other.

DEA differs from other methodologies in the following ways (Charnes et al., 1985; Thanassoulis, 2001):

- Other performance methods like regression analysis are not able to identify the sources of inefficiencies of the DMUs.
- DEA does not need a prior specification of the underlying relations and possible connection between the inputs and outputs. In exchange for avoiding such specifications, DEA uses an optimizing principle and a model which utilizes all data on the DMUs and the inputs and outputs which are deemed to be pertinent to the desired evaluations (Charnes et al., 1985).
- DEA aims to optimize the performance of each individual DMU while other methodologies seek to improve the average performance of all the DMUs.
- DEA assumes that corrective action is possible for the inefficiency that is detected (Charnes et al., 1985). Hence, the identification of target input and output levels and the efficient peers.

2.4 Some basic concepts

2.4.1 Production Possibility Space

We assess N DMUs. We have m inputs to produce s outputs and we are assessing N DMUs. Let T be the Production Possibility Set.

Let

- X_j be the inputs vector for DMU j , $j = 1, \dots, N$
- Y_j be the outputs vector for DMU j , $j = 1, \dots, N$
- x_{ij} be the inputs of DMU j (i.e. elements of X_j)
- y_{rj} be the outputs of DMU j (i.e. elements of Y_j)
- x_{io} = observed value of input i for the target unit DMU 0
- y_{ro} = observed value of output r for the target unit DMU 0

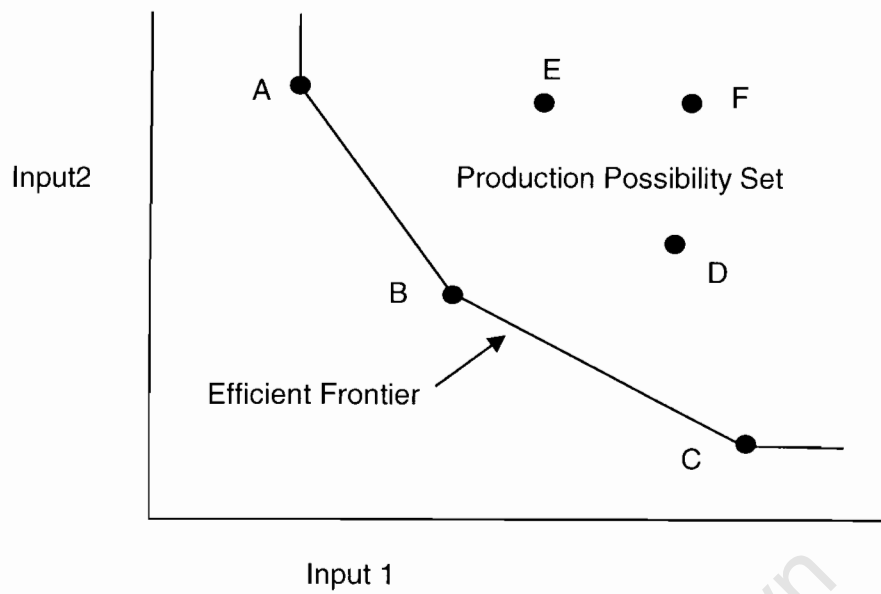


Figure 2.1: Production Possibility Set

- X is the $m \times n$ input matrix
- Y is the $s \times n$ output matrix
- v_i be the input weight of DMU 0
- u_r be the output weight of DMU 0
- u_{rj} : output weight for DMU j
- v_{ij} : input weight for DMU j

DMU j uses various inputs to produce various outputs and will be characterised by (X_j, Y_j) . The Production Possibility Set or Space (PPS) is the set of all input-output combinations (X_j, Y_j) which are observed and feasible in principle. The efficient frontier is defined by the part of the boundary of the PPS which is not dominated by any other element of the PPS (i.e. there exists no DMU in the PPS yielding more output for given input or using less input for given output) (Thanassoulis, 2001).

Empirically, the PPS may be estimated by the convex hull of the (X_j, Y_j) for $j = 1, \dots, N$. This is illustrated in figure 2.1, for the case of one output and two inputs. In this case, all factors may be scaled to a unitary output, so that the PPS can be displayed on the two input axes as shown in figure 2.1. The efficient frontier is represented by the solid line which includes the vertical line through A and the horizontal line through C. Banker et al. (1984) demonstrates that the boundary of the empirical PPS will in general be piecewise linear.

The Production Possibility Space T is based on the following assumptions (Thanassoulis, 2001; Banker et al., 1984)

- Inclusion of all observations (X_j, Y_j)
- Convexity: Interpolation between feasible input-output correspondences leads to input-output correspondences which are feasible in principle. If $(X_j, Y_j) \in T$ and $(X_j^*, Y_j^*) \in T$, then $(\lambda X_j + (1 - \lambda)X_j^*, \lambda Y_j + (1 - \lambda)Y_j^*) \in T$ for all $\lambda \in [0, 1]$
- Monotonicity: Inefficient production is possible. If $(X_j, Y_j) \in T$ and $X_j^* \geq X_j, Y_j^* \leq Y_j$, then $(X_j^*, Y_j^*) \in T$
- No output is possible unless some input is used
- Minimum extrapolation. If a Production Possibility Set T^* satisfies the above assumptions, then $T \subset T^*$. The Production Possibility Set T is the smallest set meeting the foregoing assumptions and containing all input-output correspondences observed.

2.4.2 Envelopment model

DMU 0 (X_o, Y_o) is said to be Pareto-efficient or technically efficient in the Production Possibility Set (PPS) if and only if there does not exist $(X_j, Y_j) \in \text{PPS}$ such that $(-X_j, Y_j) \geq (-X_o, Y_o)$ (Yun et al., 2004). The Production Possibility Set is made of all the observed and feasible DMUs that satisfy the set of the above assumptions.

For a DMU to be assessed in DEA, it is first of all projected onto the efficient frontier. This is illustrated in figure 2.2 where DMU D^* is the projection of DMU D on the efficient frontier. The envelopment model bases its assessment of the DMU's efficiency

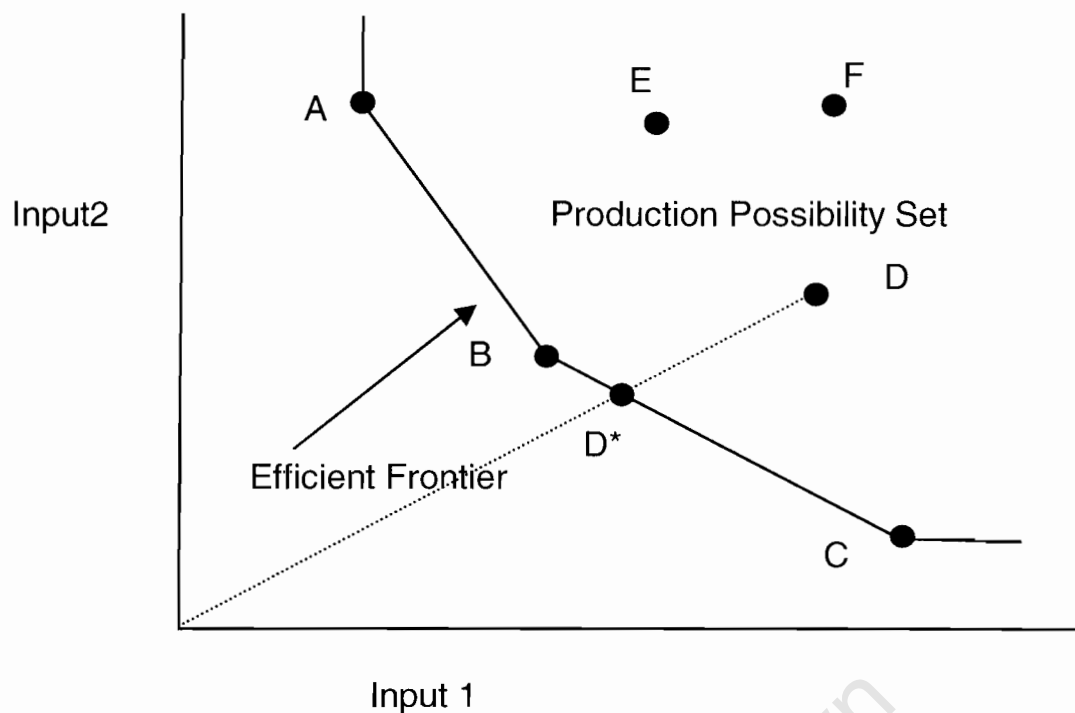


Figure 2.2: Projection of DMU on efficient frontier

on the distance from the DMU being evaluated to his projection on the efficient frontier. The closer a DMU is to the efficient frontier the more efficient it is. A DMU lying on the efficient frontier is said to be Pareto-efficient. Therefore DMUs A , B and C in figure 2.2 are Pareto-efficient whereas DMUs D , E , and F are inefficient. The efficient frontier envelops all the observed and feasible DMUs. Hence the name Data Envelopment Analysis.

This model is based on a dominance relationship between the DMUs. As said previously: a DMU is said to be Pareto-efficient or technically efficient in the Production Possibility Set (PPS) if and only if there does not exist $(X_j, Y_j) \in \text{PPS}$ such that $(-X_j, Y_j) \geq (-X_o, Y_o)$ (Yun et al., 2004). The efficient frontier or boundary of the efficient frontier is formed by all the DMUs or combinations of DMUs that dominate all other DMUs. A DMU is Pareto-efficient if it is not dominated by any other DMU or combination of DMUs. This translates into the following: DMU 0 is said to be Pareto-efficient (output orientation) if it is impossible to raise one of its output levels without raising at least one of its input levels and/or without lowering at least one of its output levels (Thanassoulis, 2001). There is no other DMU that can perform better on the outputs (maximize the output levels) than the Pareto-efficient DMU given its inputs levels. In the output-oriented model, the efficiency measure is related to the maximum

factor by which the output levels can be increased to make the target DMU efficient without increasing the inputs. We are interested in the radial or pro rata increase of the outputs. The efficiency score of the target unit is the inverse of the maximum factor by which the output can expand while keeping the same input levels. DMU 0 is also said to be Pareto-efficient (input orientation) if it is not possible to lower one of its input levels without lowering at least one of its output levels or without increasing at least one of its input levels (Thanassoulis, 2001). There is no other DMU that can perform better on the inputs (conserve or minimize the input levels) than the Pareto-efficient DMU given its outputs levels. In the input-oriented model, the efficiency measure is related to the minimum proportion of the current input levels that can be used to secure the same output levels. We are interested in the radial or pro rata decrease of the inputs that will make the target DMU efficient. The minimum proportion of the inputs that will render the target DMU efficient represents the efficiency score. In both model orientations, Cooper et al. (2000) put it this way: a DMU is fully efficient if it is not possible to improve any input or output levels without worsening some other input or output levels.

2.4.3 Value-based or weight-based model

The efficiency measure in DEA is the ratio of the weighted outputs to the weighted inputs (input-oriented efficiency) or the ratio of the weighted input to the weighted output (output-oriented efficiency). DEA maximizes the efficiency rating of a Decision Making Unit (DMU) subject to all the DMUs having an efficiency score not superior to one and all the weights being superior to ϵ , a non-Archimedean infinitesimal.

The input-oriented efficiency is given by the following linear fractional model (Charnes et al., 1994; Cooper et al., 2000):

$$\text{Max} \quad \frac{\sum_{r=1}^s u_r y_{ro}}{\sum_{i=1}^m v_i x_{io}} \quad (2.1)$$

s.t

$$\frac{\sum_{r=1}^s [u_r y_{rj}]}{\sum_{i=1}^m [v_i x_{ij}]} \leq 1, \quad \forall j \quad (2.2)$$

$$\frac{v_i}{\sum_{i=1}^m v_i x_{io}} \geq \epsilon \quad \forall i \quad (2.3)$$

$$\frac{u_r}{\sum_{i=1}^m v_i x_{io}} \geq \epsilon \quad \forall r \quad (2.4)$$

The output-oriented efficiency is given by the following linear fractional model (Charnes et al., 1994; Cooper et al., 2000):

$$\text{Min} \quad \frac{\sum_{i=1}^m v_i x_{io}}{\sum_{r=1}^s u_r y_{ro}} \quad (2.5)$$

s.t

$$\frac{\sum_{i=1}^m [v_i x_{ij}]}{\sum_{r=1}^s [u_r y_{rj}]} \geq 1, \quad \forall j \quad (2.6)$$

$$\frac{v_i}{\sum_{i=1}^m v_i x_{io}} \geq \epsilon \quad \forall i \quad (2.7)$$

$$\frac{u_r}{\sum_{i=1}^m v_i x_{io}} \geq \epsilon \quad \forall r \quad (2.8)$$

$\frac{v_i}{\sum_{i=1}^m v_i x_{io}}$ and $\frac{u_r}{\sum_{i=1}^m v_i x_{io}}$ are respectively the normalised input and output weights. The value-based model identifies the most favourable weights for DMU 0 i.e. the weights that maximize its efficiency score.

2.4.4 The meaning of efficiency: a few examples

- Value-free or production context(envelopment model)

If one is interested in assessing the efficiency of a bank branch, one would look at the resources it uses to generate the outputs. The resources or inputs are capital, labor, space, market size estimates and so forth. The outputs are loans, deposits and other revenue-generating financial products. The better the branch is at converting its resources into financial products, the higher its efficiency is (Thanassoulis, 1999).

- Value-laden context(value-based model)

If one wants to choose the best computer system between a set of alternative computer systems, the input would be the purchase price of each respective computer system while the outputs would be the set of multiple performance attributes: memory, capacity, speed, etc. The output weights would be the scores of each system on the performance attributes. It is therefore possible for DEA to assign to each computer system an efficiency rating that will be a proportion of its purchase price (Thanassoulis, 1997).

2.5 The Charnes, Cooper and Rhodes model (CCR model)

The CCR model adds an additional assumption to the standard Production Possibility Set assumptions mentioned earlier on, the Constant Returns to Scale assumption. This property means that the scale of operation for a DMU does not have an impact on its productivity. No matter how a DMU increases its scale size, its productivity will still be the same. CCR is a very strict model in the sense that it excludes from the efficient set not only DMUs which are technical inefficient but also those which are scale inefficient. In other words, if a DMU does not operate on an efficient scale it will not be rated efficient in the CCR model.

2.5.1 Envelopment model

The Production Possibility Set is constructed based on the assumptions mentioned earlier but with an additional assumption: constant returns to scale or ray unboundedness. If $(X_j, Y_j) \in \text{PPS}$ then $(k \times X_j, k \times Y_j) \in \text{PPS}$ for any $k > 0$ (Banker et al., 1984). Here, the Production Possibility Space is a conical hull and the envelopment surface is piecewise linear. In other words, productivity is not constant when the scale increases. This is illustrated in figure 2.3, for the case of one input and one output. The efficient frontier is given by the linear solid line going through A.

The CCR model ensures pre-emptive priority to optimization of the objective function by a two-stage procedure. In the first stage the objective function is maximised without taking into account the slack values. In the second stage the slack variables are maximized given the optimal value of the objective function found previously.

- Input orientation

The efficiency of DMU 0 is assessed with the following linear program (Thanassoulis, 2001):

[M1] (Charnes et al., 1994; Cooper et al., 2000)

$$\text{Min} \quad \theta_o - \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] \quad (2.9)$$

s.t

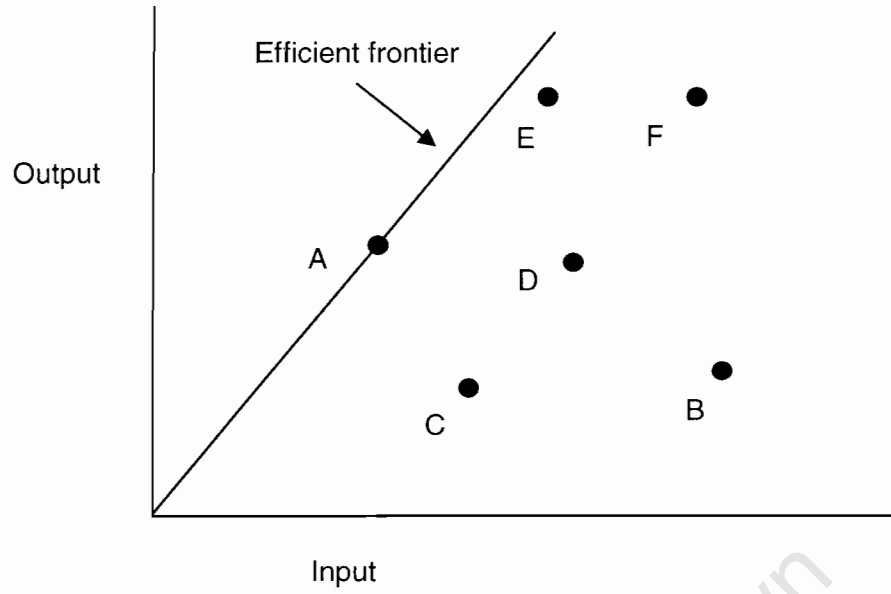


Figure 2.3: CCR-Production Possibility Set

$$\sum_{j=1}^N \lambda_j x_{ij} = \theta x_{io} - I_i \quad i = 1, \dots, m \quad (2.10)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = y_{ro} + O_r \quad r = 1, \dots, s \quad (2.11)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.12)$$

θ_o^* is the optimal value of the objective function

DMU 0 can achieve radial contraction of the inputs while maintaining its input mix. Radial contraction means that all input are contracted by the same percentage. The input mix refers to the ratio of the inputs. θ_o is the technical input efficiency of DMU 0 .

DMU 0 is said to be Pareto-efficient if and only $\theta_o = 1$ and $I_i = 0 \quad \forall i, \quad O_r = 0 \quad \forall r$. DMU 0 will be termed weakly efficient if some slacks are not zero and this type of inefficiency is called “mix inefficiency” (Cooper et al., 2000). The positive slacks have the benefit of improving the performance of DMU 0 without worsening the inputs or outputs.

- Output orientation

This model is given by:

[M2](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Max} \quad k_o + \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] \quad (2.13)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad (2.14)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = k_o y_{ro} + O_r \quad (2.15)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.16)$$

$$I_i, O_r \geq 0 \quad \forall i \quad \text{and} \quad r \quad (2.17)$$

k_o^* is the optimal value of the objective function

DMU 0 can achieve radial expansion of the outputs while maintaining its output mix. Radial expansion means that all outputs are expanded by the same percentage. The output mix refers to the ratio of the outputs. $\frac{1}{k_o}$ is the technical output efficiency of DMU 0.

DMU 0 is said to be Pareto-efficient if and only $\frac{1}{k_o} = 1$ and $I_i = 0 \quad \forall i, \quad O_r = 0 \quad \forall r$

2.5.2 Value-based Model

The value-based model seeks the input or output weights that maximize the efficiency rating of DMU 0. The weights u_r and v_i are the unknown in this model and they are respectively the output and the input weights. They are not assigned prior values and are viewed like imputed values on the inputs and outputs. The models presented in this section are the linearised form of the ratio form which is the original model. Charnes and Cooper developed a procedure that transforms the ratio form (the linear fractional model) into a linear program by equating $\sum_{i=1}^m v_i x_{io}$ to 1 in the original model without loss of generality (Charnes et al., 1994).

- Input orientation

This model is given by:

[M3](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Max} \quad \sum_{r=1}^s u_r y_{ro} \quad (2.18)$$

s.t

$$\sum_{i=1}^m v_i x_{io} = 1 \quad (2.19)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] \leq 0, \quad \forall j \quad (2.20)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (2.21)$$

The efficiency measure of DMU 0 is $\gamma_o = \sum_{r=1}^s u_r y_{ro}$ (Thanassoulis, 2001; Charnes et al., 1994).

The first constraint is the normalization constraint (or normalization constant). It is a scaling factor for the DEA weights and its value is the upper bound on the efficiency score. The normalization constant is arbitrarily set to 1 but it is advisable to set it to a greater value to avoid rounding errors in the computation of weights. The second constraint ensures that all efficiency scores will be less than

or equal to the upper bound on the efficiency rating.

DMU 0 is Pareto-efficient if and only γ_o equals 1 . If γ_o is less than 1, DMU 0 is inefficient.

- Output orientation

This model is given by:

[M4](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Min} \quad \sum_{i=1}^m v_i x_{io} \quad (2.22)$$

s.t

$$\sum_{i=1}^m u_i y_{ro} = 1 \quad (2.23)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] \leq 0, \quad \forall j \quad (2.24)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (2.25)$$

The objective function value being given by $z_o = \sum_{i=1}^m v_i x_{io}$, the efficiency measure of this model is $\frac{1}{z_o}$.

The first constraint is the normalizing constraint. It is a scaling factor for the DEA weights and it sets the upper bound on the efficiency score. It sets the value of the RHS to 1. The second constraint ensures that all efficiency scores will be less than or equal to 1.

DMU 0 is Pareto-efficient if and only $\frac{1}{z_o}$ equals 1 . If $\frac{1}{z_o}$ is less than 1, DMU 0 is inefficient.

- Weights

v_i is the input weight and u_r is the output weight. u_i and v_r also considered to be respectively the marginal imputed value of output r and the marginal imputed value of input i . They are DMU-specific (Thanassoulis, 1997). The magnitude of u_i and v_r expresses the value attached to the item by the DMU in question (Cooper et al., 2000).

The efficiency measure yielded by the value-based models is the ratio of total imputed value of its output levels to the total imputed value of its input levels (Thanassoulis, 2001). The product $u_r y_{rj}$ is called the virtual output. It is the relative contribution of output r to the efficiency score of the target DMU. The product $v_i x_{ij}$ is called the virtual input. It is the relative contribution of input i to the efficiency score (Cooper et al., 2000).

- Duality in the CCR model (Thanassoulis, 2001)

Model(1)(the input-oriented envelopment model) is dual to model(3)(the input-oriented value-based model). The optimal value of the objective function in model (1) is the same as the value of the objective function in model (3) by virtue of duality. As a result, the efficiency score yielded by model (1) is the same as the efficiency yielded by model (3) i.e $\theta_o = \gamma_o$

$$\theta_o - \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] = \sum_{r=1}^s u_r y_{ro} \quad (2.26)$$

Model(2)(the output-oriented envelopment model) is dual to (4)(the output-oriented value-based model). The optimal value of the objective function in model (2) is the same as the value of the objective function in model (4) by virtue of duality. As a result, the efficiency score yielded by model (2) is the same as the efficiency yielded by model (4) i.e $k_o = z_o$

$$k_o + \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] = \sum_{i=1}^m v_i x_{io} \quad (2.27)$$

2.6 The Banker, Charnes and Cooper model (BCC model)

2.6.1 Envelopment model

The BCC model is almost the same with the CCR model with the exception that an additional constraint $\sum_{j=1}^N \lambda_j = 1$ is present in the BCC model. This constraint is called convexity constraint and replaces the constant returns to scale assumption under CCR

model. It prevents any feasible DMU to be obtained by scaling down or up indefinitely an efficient DMU. In this case, the efficiency measure is free of the scale effect and it is called pure technical efficiency. The envelopment surface here is a convex hull. The model is said to have the variable returns to scale property given the constraint $\sum_{j=1}^N \lambda_j = 1$. In other words, productivity is not constant when the scale increases. This is illustrated in figure 2.4, for the case of one input and one output. The efficient frontier is now given by the piecewise linear solid line.

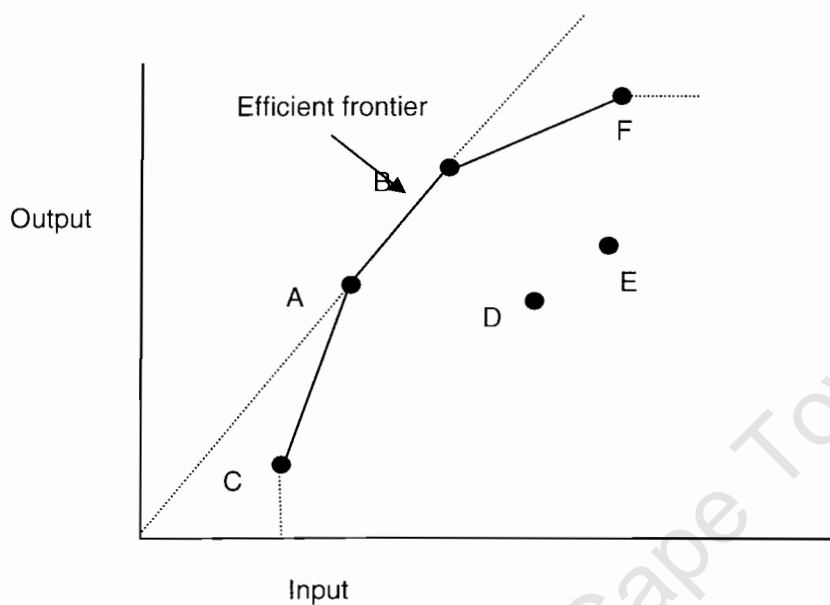


Figure 2.4: BCC-Production Possibility Set

- Input orientation

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M5](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Min} \quad \theta_o - \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] \quad (2.28)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = \theta x_{io} - I_i \quad (2.29)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = y_{ro} + O_r \quad (2.30)$$

$$\sum_{j=1}^N \lambda_j = 1 \quad (2.31)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.32)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.33)$$

DMU 0 can achieve radial contraction of the inputs while maintaining its input mix. The input mix refers to the ratio of the inputs. θ_o is the pure technical input efficiency of DMU 0. DMU 0 is said to be Pareto-efficient if and only $\theta_o = 1$ and $I_i = 0 \quad \forall i, \quad O_r = 0 \quad \forall r$. It is proved that a DMU that has a minimum input value for an input item or a maximum output value for an output item is BCC-efficient (Cooper et al., 2000).

θ_o^* is the optimal value of the objective function

- Output orientation

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M6](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Max} \quad k_o + \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] \quad (2.34)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad (2.35)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = k_o y_{ro} + O_r \quad (2.36)$$

$$\sum_{j=1}^N \lambda_j = 1 \quad (2.37)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.38)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.39)$$

k_o^* is the optimal value of the objective function

DMU 0 can achieve radial expansion of the outputs while maintaining its output mix. The output mix refers to the ratio of the outputs. $\frac{1}{k_o}$ is the pure technical output efficiency of DMU 0. DMU 0 is said to be Pareto-efficient if and only if $\frac{1}{k_o} = 1$ and $I_i = 0 \quad \forall i, \quad O_r = 0 \quad \forall r$

2.6.2 Value-based model

The BCC model differs from the CCR model by the inclusion in the BCC model of the variable ϕ which is dual to the convexity constraint. u_r and v_i are the output and input weights and they are respectively viewed like the imputed values on the outputs and inputs.

- Input orientation

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M7](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Max} \quad \sum_{r=1}^s u_r y_{ro} + \phi \quad (2.40)$$

s.t

$$\sum_{i=1}^m v_i x_{io} = 1 \quad (2.41)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] + \phi \leq 0, \quad \forall j \quad (2.42)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (2.43)$$

The efficiency measure of DMU 0 is determined by $\gamma_o = \sum_{r=1}^s u_r y_{ro}$.

The first constraint is the normalization constraint (or normalization constant). It is a scaling factor for the DEA weights and its value sets the upper bound on the efficiency score. The normalization constant is arbitrarily set to 1 but it is advisable to set it to a value greater than 1 to avoid rounding errors in the computation of weights. The second constraint ensures that all efficiency scores will be less than or equal to the upper bound of the efficiency rating.

DMU 0 is Pareto-efficient if and only if γ_o equals 1. If γ_o is less than 1, DMU 0 is inefficient.

- Output orientation

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M8](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Min} \quad \sum_{i=1}^m v_i x_{io} + \phi \quad (2.44)$$

s.t

$$\sum_{i=1}^m u_i y_{ro} = 1 \quad (2.45)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] - \phi \leq 0, \quad \forall j \quad (2.46)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (2.47)$$

The objective function value $z_0 = \sum_{i=1}^m v_i x_{io}$ of this model is found by the following linear program:

The first constraint is the normalizing constraint. It sets the value of the RHS to 1. The second constraint ensures that all efficiency scores will be less than or equal to 1.

DMU 0 is Pareto-efficient if and only if $\frac{1}{z_0}$ equals 1. If $\frac{1}{z_0}$ is less than 1, DMU 0 is inefficient.

- Rates of substitution and rates of transformation

Using the input and output weights, we can derive marginal weights of substitution between the inputs or outputs and the marginal weights of transformation of the inputs into the outputs. The ratios of the inputs to the outputs are the estimates of the marginal rates of transformation of the inputs into the outputs. The ratios of the inputs (or outputs) are the estimates of the marginal rates of substitution between the inputs (or outputs).

- Duality in the BCC model

The optimal value of the objective function in model (5)(input-oriented envelopment model) has the same value as the objective function in model (7) (input-oriented value-based model) by virtue of duality. The efficiency score yielded by model (5) has the same value as the efficiency yielded by model (7) i.e $\theta_o = \gamma_o$

The optimal value of the objective function in model (6)output-oriented envelopment model)has also almost the same value as the objective function in model (8) (output-oriented value-based model) by virtue of duality. The efficiency score yielded by model (6) is almost the same as the efficiency yielded by model (8) i.e $k_o = z_o$

- Comparison between CCR and BBC model

The CCR enables us to extrapolate the performance of the most efficient DMUs with efficient scale size (for the given input and output mixes) and identify any scale inefficiencies that may be reflected in the level of operations of other DMUs (Banker et al., 1984). This means that a DMU that is rated efficient in the CCR model must not only be technically efficient but also scale efficient. The CCR model

yields an overall efficiency that aggregates pure technical efficiency and scale efficiency (30). A DMU being technically efficient will be rated as inefficient because it does not operate under the most efficient scale size. The BBC model distinguishes between technical and scale inefficiencies by estimating pure technical efficiency at the given level of operations. It also identifies whether increasing, decreasing, or constant returns to scale are present for further exploitation (Charnes et al., 1994).

The Production Possibility Set under the CCR model is larger than the Production Possibility Set under BCC model. We can also observe that the number of inefficient DMUs obtained under the Constant Return to Scale model is at least the number of inefficient DMUs obtained under the Variable Returns to Scale model (Charnes et al., 1994). Therefore, we can say that CCR efficiency is more difficult to achieve than BCC efficiency.

The maximization and minimization procedure applied in the value-based procedure under the BBC and CCR models accords to DMU 0 the most favourable weighting that the constraints allow (Charnes et al., 1978). For both models, DMU 0 determines the weights that display it in the best light.

2.7 Returns to scale property and DEA models

The models and measures presented in this section help determine whether the scale size at which a DMU functions is optimum and how it affects its productivity. Corrective actions with respect to scale size are suggested in order to improve productivity.

2.7.1 Returns to scale property

The concept of returns to scale allows us to trace the change in the average productivity or marginal productivity resulting from a change in the production process scale size, under Pareto-efficient performance of DMUs. If the average productivity remains the same despite the change in scale size then we are dealing with a constant returns to scale (CRS) situation. In this case, the average productivity is not dependent on the scale size. If the proportion of increase in the average productivity is more important than the proportion of increase in the scale size, we are in an increasing returns to scale (IRS) situation. Conversely, if the proportion of increase in the average productivity is less important than the proportion of increase in the scale size, we are in decreasing returns to scale (DRS) situation (Cooper et al., 2000).

Let DMU 0 be Pareto-efficient and have input levels $X = (x_{ij}, i = 1..m)$ and output levels $Y = (y_{rj}, r = 1..s)$. Let us scale the input levels to $\alpha * X = (\alpha x_{ij}, i = 1..m)$, where $\alpha > 0$. Let the Decision Making Unit be capable in principle of becoming Pareto-efficient with output levels $\beta * Y = (\beta y_{rj}, r = 1..s)$, given its input levels. Finally let

$$\rho = \lim_{\alpha \rightarrow 1} \frac{\beta - 1}{\alpha - 1}.$$

Then

If $\rho > 1$ we have local IRS at (x, y) ;

If $\rho = 1$ we have local CRS at (x, y) ;

If $\rho < 1$ we have local DRS at (x, y) ;

(Thanassoulis, 2001)

2.7.2 Returns to scale measure

The estimation of returns to scale in the production possibility $(x_o, y_o) \in T$, T being the Production Possibility Space, can be done using Envelopment or value-based model.

Envelopment model

[M6](Banker and Thrall, 1992)

$$\text{Max} \quad k_o + \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] \quad (2.48)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad (2.49)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = k_o y_{ro} + O_r \quad (2.50)$$

$$\sum_{j=1}^N \lambda_j = 1 \quad (2.51)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.52)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.53)$$

We use the sum of the λ values to detect the type of returns to scale that characterize a DMU:

- If DMU 0 is technical and scale efficient (under CCR), it is proved that there is a solution with $\sum_{j=1}^N \lambda_j = 1$. Therefore, constant returns to scale prevails at DMU 0.

- If DMU 0 is technical and scale inefficient (under CCR) but technical efficient (under BCC) and if $\sum_{j=1}^N \lambda_j < 1$, then increasing returns to scale prevails at DMU 0.
- If DMU 0 is technical and scale inefficient (under CCR) but technical efficient (under BCC) and if $\sum_{j=1}^N \lambda_j > 1$, then decreasing returns to scale prevails at DMU 0.

It is worth noting that these rules hold for either input orientation or output orientation model.

Value-based model

We use the ϕ values to make our estimations about the kind of returns to scale that characterise a DMU.

- Input-orientation
[M7] (Banker and Thrall, 1992)

$$\text{Max} \quad \sum_{r=1}^s u_r y_{ro} + \phi_o \quad (2.54)$$

s.t

$$\sum_{i=1}^m v_i x_{io} = 1 \quad (2.55)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] + \phi_o \leq 0, \quad \forall j \quad (2.56)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (2.57)$$

- If DMU 0 is technical efficient and $\phi_o^* = 0$ for some optimal solutions, then constant returns to scale prevails at DMU 0.
- If DMU 0 is technical efficient and $\phi_o^* > 0$ for all optimal solutions, then increasing returns to scale prevails at DMU 0.
- If DMU 0 is technical efficient and $\phi_o^* < 0$ for all optimal solutions, then decreasing returns to scale prevails at DMU 0.

- Output-orientation

[M8](Banker and Thrall, 1992)

$$\text{Min} \quad \sum_{i=1}^m v_i x_{io} + \phi_o \quad (2.58)$$

s.t

$$\sum_{i=1}^m u_i y_{ro} = 1 \quad (2.59)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] - \phi_o \leq 0, \quad \forall j \quad (2.60)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (2.61)$$

- If DMU 0 is technical efficient and $\phi_o^* = 0$ for some optimal solutions, then constant returns to scale prevails at DMU 0.
- If DMU 0 is technical efficient and $\phi_o^* < 0$ for all optimal solutions, then increasing returns to scale prevails at DMU 0.
- If DMU 0 is technical efficient and $\phi_o^* > 0$ for all optimal solutions, then decreasing returns to scale prevails at DMU 0.

This is illustrated in figure 2.4 where it can be seen that:

- Increasing returns to scale prevails in the segment CA
- Constant returns to scale prevails in the segment AB
- Decreasing returns to scale prevails in the segment BF

Convexity constraint

The convexity constraint in DEA could be expressed as this: $\sum_{i=1}^N \lambda_i = 1$. In matrix notation it is equivalent to: $e\lambda = 1$. We can relax this condition like this: $L \leq e\lambda \leq U$

We can use this constraint to require that the observed DMUs would operate at an increasing or decreasing returns to scale size according to the type of returns to scale size that suits the situation. This requirement is going to dictate the choice of efficient peers and target levels. A DMU that is too small to operate most efficiently, should emulate the DMUs that are operating at a bigger scale size and try to reach higher levels of production. In the same way, a DMU that is too big for it to operate most efficiently, should emulate the DMUs that are operating at a smaller scale size and try to use less resources.

- Increasing returns to scale Model

If $L = 1$ and $U = \infty$, we are enforcing the increasing returns to scale property. In this case, only an increase of the scale is possible. This could be expressed mathematically as: $\frac{\Delta y}{y} \geq \frac{\Delta x}{x}$.

This model deals with DMUs whose scale of operations is too small and need to be increased.

- Decreasing returns to scale Model

If $L = 0$ and $U = 1$, we are enforcing the decreasing returns to scale property. In this case, only a decrease of scale is feasible. This could be expressed mathematically as: $\frac{\Delta y}{y} \leq \frac{\Delta x}{x}$.

This model deals with DMUs whose scale of operations is too big and need to be decreased (Banker and Thrall, 1992).

2.7.3 Scale efficiency

(Cooper et al., 2000)

Having the CCR efficiency score θ_{CCR}^* and the BCC efficiency score θ_{BCC}^* , we can obtain scale efficiency SE as the ratio of θ_{CCR}^* to θ_{BCC}^* .

$$SE = \theta_{CCR}^* / \theta_{BCC}^*$$

The scale efficiency reveals whether a DMU is functioning at an efficient scale size or not.

2.7.4 Most Productive Scale Size(MPSS)

A DMU that is too small should be able to increase its productivity by increasing its resources in order to operate at its full potential. A DMU that is too big should be able to increase its productivity by decreasing its resources in order to operate at its full potential because it is wasting some resources.

Solving the following LP will help us determine whether a DMU is a MPSS.

[M9](Thanassoulis, 2001; Cooper et al., 2000)

$$\text{Max} \quad \alpha/\beta \quad (2.62)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} \leq \alpha x_{io} \quad (2.63)$$

$$\sum_{j=1}^N \lambda_j y_{rj} \geq \beta y_{ro} \quad (2.64)$$

$$\sum_{j=1}^N \lambda_j = 1 \quad (2.65)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.66)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.67)$$

The Most Productive Scale Size is a production possibility that maximizes the average productivity ($= \beta/\alpha$) for its given input and output mix $(\alpha X_o, \beta Y_o)$. A production possibility $(X_o, Y_o) \in T$ is a MPSS if and only if it makes the DMU 0 both technical and scale efficient (Banker and Thrall, 1992). As a result, a production possibility is MPSS only if it has pure technical efficiency of 1 and if it is (radial) scale efficient (Scale efficiency =1). In other words, The DMU of interest must be Pareto-efficient under Constant returns to scale for it to be MPSS.

DMU 0 is a MPSS if and only if the optimal value of the objective function α/β is 1. Constant returns to scale prevails at the MPSS.

2.7.5 Constant and Variable returns to scale

The constant returns to scale postulate (or ray unboundedness postulate) enables us to extrapolate the performance of the most efficient DMUs with efficient scale sizes (for the given input or output mixes) and identify any scale inefficiencies that may be reflected in the level of operations of other DMUs (Banker et al., 1984). In other words, any inefficiency reflected in the efficiency score may not only be due to inefficient performance but also to inefficient scale size. By deleting the constant returns to scale postulate we can now restrict our attention strictly to production inefficiencies at the given level of operation for each DMU and in this way develop an efficiency measurement procedure that assigns an efficiency rating of one to a DMU if and only if the DMU lies on the efficient production surface, even when it may not be operating at the most efficient scale size. It is therefore possible to focus only on the evaluation of the performance of the DMU given the level of the operations (scale size). This identification of the efficient production surface will allow us to determine whether increasing, constant or decreasing returns to scale prevail in different segments of the production surface (Banker et al., 1984). Under variable returns to scale, the scale will affect the productivity of a DMU. In this way, the impact of the scale size is taken into account.

We have increasing returns to scale at the production possibility $(X_o, Y_o) \in T$ when the average productivity increases with increasing scale size. In this case, the scale size is too small and should be increased.

We have decreasing returns to scale at the production possibility $(X_o, Y_o) \in T$ when the average productivity decreases with decreasing scale size. In this case, the scale size is too big and should be decreased.

2.8 Additive model

2.8.1 Additive model

The additive model measures the distance from DMU 0 to the efficient frontier and the efficiency score yielded by this model conveys this measure. The additive model is translation invariant. A DEA model is translation invariant if changes to the origin of the co-ordinates system for the inputs or outputs does not change the optimal solution (Cooper et al., 2000). Changing the origin of the coordinates will not affect the efficiency evaluation by this model. This property allows the Additive model to handle negative data. Specifically, for a particular DMU 0, the primal problem (envelopment model) picks the most extreme of all combinations of DMUs with output levels $Y\lambda \geq Y_o$ and input levels $X\lambda \leq X_o$ (Charnes et al., 1994).

The value of the objective function of the additive model depends on the unit of measure i.e the scale with which the data are taken. In other words, this model is not scale-invariant. The remedy is to normalize the data.

There is no input nor output orientation for the additive model. The inputs and the outputs can improve simultaneously. DMU 0 is efficient if it is lying on the efficient frontier. The additive model operates under the variable returns to scale assumption as does the BCC model. That is why the efficient frontier in this model has the same form with the efficient frontier in the BCC model (20).

Envelopment model

This model is given by:

[M9](Charnes et al., 1994)

$$\text{Min } z_o = \left[\sum_{i=1}^m I_i + \sum_{r=1}^s O_r \right] \quad (2.68)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad (2.69)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = y_{ro} + O_r \quad (2.70)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.71)$$

The efficiency score measures the distance from DMU 0 to the efficient frontier. As a result, DMU 0 is Pareto-efficient if $z_o = 0$ i.e all the slack variables, I_i and O_r , are zero. DMU 0 is inefficient if z_o is not zero ,i.e, if any slack variable is positive.

2.8.2 Value-based model

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M10](Charnes et al., 1994)

$$\text{Max} \quad k_o = \sum_{r=1}^s u_r y_{ro} - \sum_{i=1}^m v_i y_{io} + u_o \quad (2.72)$$

s.t

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] + u_o \leq 0, \quad \forall j \quad (2.73)$$

$$v_i, u_r \geq 1 \quad \forall i, r \quad (2.74)$$

DMU 0 is Pareto-efficient if $k_o = 0$. DMU 0 is inefficient if k_o is not zero.

2.9 Multiplicative model

The BBC model was developed based on the assumptions of convexity and monotonicity. The Production Possibility Set and the efficient frontier had to be convex with this approach. The BBC model makes it possible to have increasing, decreasing or constant returns to scale in the production process. The BCC model also requires that the marginal productivity be negative i.e $(d/dx_i)(dy_r/dx_i) \leq 0$, where $y_r, r = 1, \dots, s$ are the outputs and $x_i, i = 1, \dots, m$ are the inputs. This situation correspond to the case where the marginal productivity decreases as the scale size increases. In other words, the gains in outputs are less than proportional to the increase in the inputs.

In the real world, the assumption of increasing marginal productivity may arise as explained by Banker in this way Banker and Maindiratta (1986):

Existence of a fixed input and gains from increasing specialization with larger scale sizes are the usual economic reasons to motivate such instances of increasing marginal products.

Let

\hat{x}_{ij} = the logarithm of the observed value of input i for DMU j

\hat{y}_{rj} = the logarithm of the observed value of output r for DMU j

\hat{x}_{io} = the logarithm of the observed value of input i for the target unit DMU 0

\hat{y}_{ro} = the logarithm of the observed value of output r for the target unit DMU 0

\hat{k}_o^* is the the logarithm of the optimal value of the objective function

\hat{I}_i = the logarithm of the slacks for input i for the target unit DMU 0

\hat{O}_r = the logarithm of the slacks for output o for the target unit DMU 0

The following are the postulates for the Production Possibility Set (Banker and Maindiratta, 1986):

Let

$(X_j, Y_j), j = 1, \dots, N$ be respectively the set of the observed inputs and outputs such as $X_j = (X_1, \dots, X_i, \dots, X_m)$ and $Y_j = (Y_1, \dots, Y_r, \dots, Y_s)$.

- Postulate 1: Geometric Convexity. If $(X_j, Y_j) \in T, j = 1, \dots, N$, and $\lambda_j \geq 0$ are nonnegative scalars such that $\sum_{j=1}^N \lambda_j = 1$, then $(Y_o, X_o) \in T$, where $Y_o = \sum_{j=1}^N \lambda_j Y_r^{j\lambda_j}$ and $X_o = \sum_{j=1}^N \lambda_j X_{ij}^{j\lambda_j}$
- Postulate 2: Monotonicity.
 - If $(Y_j, X_j) \in T$, and $\hat{X}_j \geq X_j$, then $(Y_j, \hat{X}_j) \in T$.
 - If $(Y_j, X_j) \in T$, and $0 < \hat{Y}_j < Y_j$, then $(\hat{Y}_j, X_j) \in T$
- Postulate 3: Inclusion of observations. All observed vectors $(Y_j, X_j) \in T, j = 1, \dots, N$
- Postulate 4: Minimum Extrapolation. T is the intersection of all sets \bar{T} satisfying postulates 1, 2 and 3.

Based on these assumptions, we can define the Production Possibility Set as T follows:

$$T = \{(X_j, Y_j) | \hat{X}_i \geq \sum_{j=1}^N \lambda_j X_{ij}^{j\lambda_j} > 0, i = 1, \dots, m; \\ 0 < \hat{Y}_r \leq \sum_{j=1}^N \lambda_j Y_{rj}^{j\lambda_j}, r = 1, \dots, s, \text{ for some } \lambda_j \text{ with } \sum_{j=1}^N \lambda_j = 1\} \quad (2.75)$$

Using the logarithms of the observed inputs and outputs, we define the set \hat{T} , a convex set \hat{T} as follows:

$$\hat{T} = \{(\hat{x}, \hat{y}) | \hat{x}_i \geq \sum_{j=1}^N \lambda_j \hat{x}_{ij}, \hat{y}_r \leq \sum_{j=1}^N \lambda_j \hat{y}_{rj} \text{ for some } \lambda_j \geq 0 \text{ with } \sum_{j=1}^N \lambda_j = 1\}$$

The assumption of negative marginal productivity in the BCC model is too restrictive and cannot always be true. We may encounter situations where a region of the efficient frontier is non-concave. As a result, the production possibility may not be convex and the efficient frontier may be S-shaped. The axiom of convexity of the production possibility set will be replaced by the geometric convexity axiom, where the frontier is estimated by a piecewise loglinear surface. The loglinear model deals with these kind of situations. If the marginal productivity is non-increasing, the loglinear model should also be used to measure the efficiency of DMUs. The efficiency measure will therefore represent a non-radial change in the input and output levels.

2.9.1 Envelopment model

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M11] (Banker and Maindiratta, 1986)

$$\text{Max} \quad \hat{k}_o + \varepsilon \left[\sum_{i=1}^m \hat{I}_i + \sum_{r=1}^s \hat{O}_r \right] \quad (2.76)$$

s.t

$$\sum_{j=1}^N \lambda_j \hat{x}_{ij} = \hat{x}_{io} - \hat{I}_i \quad i = 1, \dots, m \quad (2.77)$$

$$\sum_{j=1}^N \lambda_j \hat{y}_{rj} = \hat{k}_o + \hat{y}_{ro} + \hat{O}_r \quad r = 1, \dots, s \quad (2.78)$$

$$\sum_{j=1}^N \lambda_j = 1 \quad (2.79)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.80)$$

$$\hat{I}_i, \hat{O}_r \geq 0 \quad \forall i \text{ and } r \quad (2.81)$$

DMU 0 is said to be Pareto-efficient if and only if $\hat{k}_o = \hat{I}_i = \hat{O}_r = 0$

2.9.2 Value-based model

This model is given by:

[M12](Banker and Maindiratta, 1986)

$$\text{Min} \quad z_o = \sum_{r=1}^s v_i \hat{x}_{io} - \sum_{i=1}^s u_r \hat{y}_{ro} + u_o \quad (2.82)$$

s.t

$$\sum_{r=1}^s [v_i \hat{x}_{ij}] - \sum_{i=1}^m [u_r \hat{y}_{rj}] + u_o \leq 0, \quad \forall j \quad (2.83)$$

$$\sum_{r=1}^s u_r = 1 \quad (2.84)$$

$$v_i, u_r \leq \varepsilon \quad \forall i, r \quad (2.85)$$

DMU 0 is said to be Pareto-efficient if and only if $z_o = 0$

According to Charnes et al. (1994), the efficient frontier for the model above is piecewise Cobb-Douglas. Another variant of these models is obtained by suppressing the constraint on the λ values. Charnes et al. (1994). state that the efficient frontier in this case is piecewise loglinear.

2.10 Model including exogenously fixed inputs or outputs

We have two kinds of input/output variables included in this model: the discretionary and the non-discretionary variables. The discretionary variables are those whose levels are determined at the discretion of the managers of a DMU. The non-discretionary variables or exogenously fixed are those whose levels are fixed by some external decision maker. Management have no say on the levels of these factors. The sets of the inputs is partitioned into subsets E_D and E_F of discretionary and non-discretionary inputs. Similarly, the output set is partitioned in R_D and R_F . In the models developed below, the discretionary variables are allowed to contract or expand while the non-discretionary variables are not allowed to do so.

2.10.1 Envelopment model

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

- Input oriented model

[M13](Thanassoulis and Dyson, 1992)

$$\text{Min} \quad \theta_o - \varepsilon \left[\sum_{i \in E_D} I_i + \sum_{r=1}^s O_r \right] \quad (2.86)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = \theta_o x_{io} - I_i \quad i \in E_D \quad (2.87)$$

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad i \in E_F \quad (2.88)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = y_{ro} + O_r \quad r = 1, \dots, s \quad (2.89)$$

$$\sum_{j=1}^N \lambda_j = 1 \quad (2.90)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.91)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.92)$$

θ_o^* is the optimal value of the objective function

Note that, for the non-discretionary inputs, their slacks are not included in the objective function and they are not multiplied by the contracting factor θ_o^* .

- Output oriented model

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M14](Thanassoulis and Dyson, 1992)

$$\text{Max} \quad k_o + \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r \in R_D} O_r \right] \quad (2.93)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad i = 1, \dots, m \quad (2.94)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = k_o y_{ro} + O_r \quad r \in R_D \quad (2.95)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = y_{ro} + O_r \quad r \in R_F \quad (2.96)$$

$$\sum_{j=1}^N \lambda_j = 1 \quad (2.97)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.98)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.99)$$

k_o^* is the optimal value of the objective function

Note that, for the non-discretionary outputs, their slacks are not included in the objective function and they are not multiplied by the expanding factor k_o^* .

2.10.2 Value-based model

We present here the value-based models with the non-discretionary variables. They are the dual models of the envelopment models with the non-discretionary variables.

- Input-oriented model

Thanassoulis and Dyson explain that the non-discretionary inputs are treated here as 'negative outputs' (Thanassoulis and Dyson, 1992).

This model is given by:

[M15](Thanassoulis and Dyson, 1992)

$$\text{Max} \quad \frac{\sum_{r=1}^s u_r y_{ro} - \sum_{i \in E_F} [v_i x_{io}] - \phi}{\sum_{i \in E_D} v_i x_{io}} \quad (2.100)$$

s.t

$$\frac{\sum_{r=1}^s [u_r y_{rj}] - \sum_{i \in E_F} [v_i x_{ij}] - \phi}{\sum_{i \in E_D} [v_i x_{ij}]} \leq 1 \quad j = 1, \dots, N \quad (2.101)$$

$$v_i, u_r \geq \epsilon \quad \text{for } i \in E_D \quad \text{and } r = 1, \dots, s \quad (2.102)$$

$$v_i \geq 0 \quad \text{for } i \in E_F \quad (2.103)$$

- Output-oriented model

Thanassoulis and Dyson explain that the non-discretionary outputs are treated here as ‘negative inputs’ (Thanassoulis and Dyson, 1992).

This model is given by:

[M16](Banker and Morey, 1986; Thanassoulis, 2001)

$$\text{Min} \quad \frac{\sum_{i=1}^m v_i x_{io} - \sum_{r \in R_F} [u_r y_{ro}] - \phi}{\sum_{r \in R_D} u_r y_{ro}} \quad (2.104)$$

s.t

$$\frac{\sum_{i=1}^m [v_i x_{ij}] - \sum_{r \in R_F} [u_r y_{rj}] - \phi}{\sum_{r \in R_D} [u_r y_{rj}] - \phi} \geq 1 \quad j = 1, \dots, N \quad (2.105)$$

$$v_i, u_r \geq \epsilon \quad \text{for } i = 1, \dots, m \quad \text{and } r \in R_D \quad (2.106)$$

$$u_r \geq 0 \quad \text{for } r \in R_F \quad (2.107)$$

2.10.3 Envelopment model (CRS)

Under Constant Returns to Scale assumption, we need to modify the model so that the most productive scale size does not use more inputs or outputs than the non-discretionary input or output levels that are already fixed by the decision maker.

- Input oriented model

This model is given by:

[M17](Banker and Morey, 1986; Thanassoulis, 2001)

$$\text{Min} \quad \theta_o - \varepsilon \left[\sum_{i \in E_D} I_i + \sum_{r=1}^s O_r \right] \quad (2.108)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = \theta x_{io} - I_i \quad i \in E_D \quad (2.109)$$

$$\sum_{j=1}^N \lambda_j x_{ij} = \sum_{j=1}^N x_{io} - I_i \quad i \in E_F \quad (2.110)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = y_{ro} + O_r \quad r = 1, \dots, s \quad (2.111)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.112)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.113)$$

E_F and E_D are respectively the set of non-discretionary and discretionary inputs.

θ_o^* is the optimal value of the objective function

The non-discretionary inputs are not allowed to contract.

- Output oriented model

This model is given by:

[M18](Banker and Morey, 1986; Thanassoulis, 2001)

$$\text{Max} \quad k_o + \varepsilon \left[\sum_{i=1}^m I_i + \sum_{r \in R_D} O_r \right] \quad (2.114)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad i = 1, \dots, m \quad (2.115)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = k_o y_{ro} + O_r \quad r \in R_D \quad (2.116)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = \sum_{j=1}^N y_{ro} + O_r \quad r \in R_F \quad (2.117)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.118)$$

$$I_i, O_r \geq 0 \quad \forall i \text{ and } r \quad (2.119)$$

R_F and R_D are respectively the set of non-discretionary and discretionary outputs.

k_o^* is the optimal value of the objective function

The non-discretionary outputs are not allowed to expand.

2.10.4 Value-based model (CRS)

Under the Constant Returns to Scale assumption, we have the following value-based models that are the dual of the envelopment models with non-discretionary variables.

- Input-oriented model

This model is given by:

[M19](Banker and Morey, 1986; Thanassoulis, 2001)

$$\text{Max} \quad \frac{\sum_{r=1}^s u_r y_{ro}}{\sum_{i=1}^m v_i x_{io}} \quad (2.120)$$

s.t

$$\frac{\sum_{r=1}^s [u_r y_{rj}] - \sum_{i \in E_F} [v_i (x_{ij} - x_{io})]}{\sum_{i=1}^m [v_i x_{ij}]} \leq 1 \quad j = 1, \dots, N \quad (2.121)$$

$$v_i, u_r \geq \epsilon \quad \text{for } i \in E_D \text{ and } r = 1, \dots, s \quad (2.122)$$

$$v_i \geq 0 \quad \text{for } i \in E_F \quad (2.123)$$

E_F and E_D are respectively the set of non-discretionary and discretionary inputs.

The efficiency measure $\gamma_o = \sum_{r=1}^s u_r y_{ro}$ of DMU 0 is determined by (2.120)

- output-oriented model

This model is given by:

[M20] (Banker and Morey, 1986; Thanassoulis, 2001)

$$\text{Min} \quad \frac{\sum_{i=1}^m v_i x_{io}}{\sum_{r=1}^s u_r y_{ro}} \quad (2.124)$$

s.t

$$\frac{\sum_{i=1}^m [v_i x_{ij}] - \sum_{r \in R_F} [u_r (y_{rj} - y_{io})]}{\sum_{r=1}^s [u_r y_{rj}] - \phi} \geq 1 \quad j = 1, \dots, N \quad (2.125)$$

$$v_i, u_r \geq \epsilon \quad \text{for } i = 1, \dots, m \quad \text{and } r \in R_D \quad (2.126)$$

$$u_r \geq 0 \quad \text{for } r \in R_F \quad (2.127)$$

R_F and R_D are respectively the set of non-discretionary and discretionary outputs.

The objective function value $z_o = \sum_{i=1}^m v_i x_{io}$ of this model is given by (2.124)

2.11 Weights based targets DEA model

2.11.1 Target model

Weights are attached to the importance of selected inputs and/or outputs in seeking improvement to the performance of DMU 0. The weights are specified by the user or decision maker. The targets are compatible with the priorities over the improvement of input or output levels (input levels decrease, output levels increase). The inputs and outputs can improve simultaneously. This model does not yield an efficiency measure.

Nevertheless, it can tell you whether a DMU is efficient or not. The model partition factors(inputs/outputs) into those factors which should not worsen and then those which must be radially improved.

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M21](Thanassoulis and Dyson, 1992)

$$\text{Max} \quad \sum_{r \in R_r} W_r^+ k_r - \sum_{i \in E_i} W_i^- \theta_i + \varepsilon \left[\sum_{i \in E_{nw}} I_i + \sum_{r \in R_{nw}} O_r \right] \quad (2.128)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} = \theta_i x_{io} - I_i \quad i \in E_i \quad (2.129)$$

$$\sum_{j=1}^N \lambda_j x_{ij} = x_{io} - I_i \quad i \in E_{nw} \quad (2.130)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = k_r y_{ro} + O_r \quad r \in R_r \quad (2.131)$$

$$\sum_{j=1}^N \lambda_j y_{rj} = y_{ro} + O_r \quad r \in R_{nw} \quad (2.132)$$

$$\theta_i \leq 1 \quad \forall i \in E_i \quad (2.133)$$

$$k_r \geq 1 \quad \forall r \in R_r \quad (2.134)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.135)$$

$$I_i, O_r \geq 0 \quad \forall i \in E_{nw} \quad \text{and} \quad r \in R_{nw} \quad (2.136)$$

E_i and E_{nw} are respectively the set of inputs whose levels the decision maker needs to improve and the set of inputs whose levels must not worsen.

R_r and R_{nw} are respectively the set of outputs whose levels the decision maker needs to improve and the set of the outputs whose levels must not worsen.

This model supplies the targets levels (Thanassoulis and Dyson, 1992):

$$x'_{io} = \theta_i x_{io} - I_i \quad i \in E_i \quad (2.137)$$

$$x'_{io} = x_{io} - I_i \quad i \in E_{nw} \quad (2.138)$$

$$y'_{io} = k_r y_{ro} + O_r \quad r \in R_r \quad (2.139)$$

$$y'_{io} = y_{ro} + O_r \quad r \in R_{nw} \quad (2.140)$$

If the target levels are the same with the current levels, it is an indication that a DMU is efficient. It is understood that a DMU that is below the target levels should try to reach them in order to become Pareto-efficient

2.11.2 Ideal input-output levels based DEA approach

This approach developed by Thanassoulis is similar to goal programming. It uses a two-stage procedure. All weights used are specified by the user or decision maker. In the first stage, it identifies feasible target input and/or output levels that are as close as possible to the ideal input and/or output levels by using weights reflecting desired improvement over selected inputs and/or outputs. The weights attach a penalty to the deviations from the ideal targets which are input increase or output reduction. The more undesirable are the deviations, the heavier is the weight attached to them. The inputs and outputs can be improved simultaneously. The feasible targets are not necessarily efficient. At the second stage, efficient feasible targets which dominate the feasible targets are determined. In this way, the efficient feasible target input and/or output levels are compatible with the predetermined ideal target input and/or output levels (Thanassoulis and Dyson, 1992).

The efficiency measure of DMU 0 is determined by the objective function of the following linear program:

[M22](Thanassoulis and Dyson, 1992)

$$\text{Min} \quad \sum_{i=1}^m W_i^1 d_i^1 + \sum_{i=1}^m W_i^2 d_i^2 + \sum_{r=1}^s W_r^1 t_r^1 + \sum_{r=1}^s W_r^2 t_r^2 \quad (2.141)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} + d_i^1 - d_i^2 = x_i^i \quad i = 1, \dots, m \quad (2.142)$$

$$\sum_{j=1}^N \lambda_j y_{rj} + t_r^1 - t_r^2 = y_r^i \quad r = 1, \dots, s \quad (2.143)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.144)$$

$$d_i^1, d_i^2, t_r^1, t_r^2 \geq 0 \quad \forall i \quad \text{and} \quad r \quad (2.145)$$

This model supplies the following feasible targets levels :

$$x_i^f = x_i^i - d_i^1 + d_i^2 \quad i = 1, \dots, m$$

$$y_r^f = y_r^i - t_r^1 + t_r^2 \quad r = 1, \dots, s$$

Now, we are going to use the additive model to determine the Pareto-efficient target levels based on the feasible target levels obtained earlier on

[M23](Thanassoulis and Dyson, 1992)

$$\text{Max} \quad \sum_{i=1}^m d_i + \sum_{r=1}^s t_r \quad (2.146)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} + d_i = x_i^f \quad i = 1, \dots, m \quad (2.147)$$

$$\sum_{j=1}^N \lambda_j y_{rj} - t_r = y_r^f \quad r = 1, \dots, s \quad (2.148)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, N \quad (2.149)$$

$$d_i, t_r \geq 0 \quad \forall i \quad \text{and} \quad r \quad (2.150)$$

This model supplies the following Pareto-efficient feasible targets levels in accordance with the ideal levels specified earlier:

$$x_i^{ff} = x_i^i - d_i \quad i = 1, \dots, m$$

$$y_r^{ff} = y_r^i + t_r \quad r = 1, \dots, s$$

2.12 The cost and revenue models

The cost and revenue models are largely regular linear programs.

- Cost minimizing model

This model is given by:

[M24](Thanassoulis, 2001; Cooper et al., 2000)

$$MC = \text{Min} \quad \sum_{i=1}^m [IP_{io}x_{io}] \quad (2.151)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \quad (2.152)$$

$$\sum_{j=1}^N \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s \quad (2.153)$$

$$\begin{aligned} \lambda_j &\geq 0 \quad j = 1, \dots, N \\ x_{io} &\geq 0 \quad \forall i \end{aligned} \quad (2.154)$$

IP_{io} is the relative price of input i .

Given the price of the inputs, we would like to determine the amount of input that should be used in order to minimize the cost.

- Revenue maximizing model

This model is given by:

[M25](Thanassoulis, 2001; Cooper et al., 2000)

$$MR = \text{Max} \quad \sum_{i=1}^m [OP_{ro}y_{ro}] \quad (2.155)$$

s.t

$$\sum_{j=1}^N \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \quad (2.156)$$

$$\sum_{j=1}^N \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s \quad (2.157)$$

$$\begin{aligned} \lambda_j &\geq 0 \quad j = 1, \dots, N \\ y_{ro} &\geq 0 \quad \forall r \end{aligned} \quad (2.158)$$

OP_{ro} is the relative price of output r .

Given the price of the outputs, we would like to determine the amount of output that should be used in order to maximize the revenue.

2.13 Models used in the experimented studies reported in this thesis

As you have noticed, there is a variety of models in DEA. All of them explicitly (value-based form) or implicitly (dual of envelopment form) involve weights as factors to be determined in some way. The purpose of the experimental studies reported later is to evaluate the effects of different forms of restriction on these weights in terms of consequences for efficiencies.

For the purpose of this dissertation, we focus on the simplest form of model namely the CCR model since we do not have enough time to evaluate all DEA models. Later studies might examine similar properties for other models.

In the next chapter, before turning to the specific research questions, we review various issues regarding weight restrictions in DEA.

Chapter 3

Literature Survey on Weight restrictions

In the preceding chapter, we presented a number of models in DEA. However, we will mainly use the CCR model in this study. The CCR model (see page 15) is based on the constant returns to scale assumption which makes the productivity of a DMU indifferent to the scale size it takes. A DMU will be rated efficient in this case if and only if it is technically and scale efficient. Let v_i be the weight given to input i and u_r the weight given to output r .

The efficiency of DMU 0 is assessed with the following linear program (Thanassoulis, 2001):

[M3](Thanassoulis, 2001; Charnes et al., 1994; Cooper et al., 2000)

$$\text{Max} \quad \sum_{r=1}^s u_r y_{ro} \quad (3.1)$$

s.t

$$\sum_{i=1}^m v_i x_{io} = 1 \quad (3.2)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] \leq 0, \quad \forall j \quad (3.3)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (3.4)$$

The efficiency measure of DMU o is $\gamma_o = \sum_{r=1}^s u_r y_{ro}$ (Thanassoulis, 2001; Charnes et al., 1994).

Weight restrictions can be attached to each model in the form of constraints on the v_i and u_r . A wide variety of weight restriction methods have been suggested and are reviewed.

We deal first with the meaning and characteristics of weights in DEA. Then we discuss the reasons for weight restrictions and their effects and finally we present the methods of weight restrictions.

3.1 Meaning of weights in DEA

Weights in DEA may have a number of different meanings which we now describe.

1. Marginal productivity

Thanassoulis (2001) calls the weights v_i and u_r respectively the imputed value of the marginal unit of input i and the imputed value of the marginal unit of output r . The weights are related to the marginal productivities in the inputs or outputs. In this way $\frac{u_r}{u_l}$, being the rate of substitution between output r and l , indicates that the decrease of u_r units in output r may be compensated by an increase of u_l units in output l . In the same way $\frac{v_i}{v_k}$, being the rate of substitution between input i and k , indicates that the decrease of v_i units in input i may be compensated by an increase of v_k in input k (Charnes et al., 1978).

The ratios of weights are of interest in DEA, rather than their absolute values. A ratio of input weights defines the rate of substitution between these inputs, similarly for output weights defines the rate of substitution between those outputs. The rate of transformation between an input and an output is the ratio of their respective weights. However, ratios calculated from the DEA outputs may not be meaningful as the weights can sometimes take on the negligible value ε which is almost 0 (Thanassoulis, 2001).

2. Price information

In the single input-multiple output case, the input weights are related to price information (input price). An input price is the cost opportunity for that input

(Sexton et al., 1986). The input weights could also be considered to be the amount of resources consumed by DMU 0 in order to produce one unit of output. From a monetary point of view, the input is the cost incurred in producing one unit of output (Dyson and Thanassoulis, 1988). In the single output-multiple input case, the output weights could represent the amount of output or products which can be produced by one unit of input.

3. The relative importance attached to each item or factor

The product of a single input weight and its corresponding input ($v_i x_{io}$) or the product of an output weight and its corresponding output ($u_r y_{ro}$) are respectively called virtual input and virtual output for DMU 0. Total virtual input $\sum_{i=1}^m v_i x_{io}$ is the sum of all virtual inputs for DMU 0. Total virtual output is the sum of all virtual outputs for all DMUs.

The input efficiency score of DMU 0 is the ratio of total virtual output to total virtual input: $\gamma_o = \frac{\sum_{r=1}^s u_r y_{ro}}{\sum_{i=1}^m v_i x_{io}}$. The output efficiency is the ratio of total weighted input to total weighted output: $\Delta_0 = \frac{\sum_{i=1}^s v_i x_{io}}{\sum_{i=r}^m u_r y_{ro}}$.

The ratio of virtual input i to total virtual input $\frac{v_i x_{io}}{\sum_{i=1}^m v_i x_{io}}$ is called the proportional virtual input i and represents the importance attached to input i in assessing efficiency of DMU 0. From the output perspective, the ratio of virtual output r to total virtual output $\frac{u_r y_{ro}}{\sum_{i=1}^m u_r y_{ro}}$ is called proportional virtual output r and represents the importance attached to output r in assessing efficiency of DMU 0 (Sarrico and Dyson, 2004). Thus, a DEA assessment may result in ordinal relations between the inputs or outputs being established according to the importance accorded to each item (input or output). As we shall see, constraints can be placed on the virtual inputs and outputs to ensure that they represent the values of the decision-maker. In this way, DEA weights may be related to the importance attached by a DMU to an input or an output.

3.2 Weights flexibility and restrictions

The weights in the standard DEA models are characterised by a great flexibility that is manifested in two ways:

- No prior values are accorded to the weights. The only requirement is that they should be positive. The weights are free to be determined by a linear program.
- A single factor, input or output, will in general be weighted differently by different DMUs. The input or output weights are DMU-specific.

For real life problems, this flexibility may need to be restricted in some way. Applying DEA without including bounds on the weights gives to the DMUs the most favourable efficiency score. This feature is precisely the strength of the model. However, there are disadvantages to such weight flexibility, which have been included by many authors in different ways, as summarized below:

1. To ensure that the relative ordering or importance of factors is taken into account

The DMU assessed might perform well on a small number of inputs and outputs, but poorly on the most relevant inputs or outputs. With total flexibility in the assignment of values to the weights, a DMU which is rather inefficient on most inputs and/or outputs might still turn out to be efficient in the DEA assessment. In this way, the efficiency of a DMU might be the result of a judicious choice of weights rather than an inherently efficient practice (Thanassoulis et al., 1995). The concern here is then: is DEA efficiency a result of favourable weights or an inherent efficiency of the DMUs?

It may be found that the emphasis is placed on inputs or outputs considered to be less important to the decision maker by assigning them excessive weights. Such freedom in the assignment of weights is not appropriate especially where output quality measures are present in the model. Indeed it is often possible to have at least some ordinal information on the importance of factors (inputs or outputs). For example, in an assessment of perinatal care units in England, it was agreed that a very satisfied mother should receive a higher weight than a less satisfied mother since it is more preferable to get a higher level of satisfaction from patients (Thanassoulis et al., 1995). It may be agreed in comparing universities, that the weight attached to a postgraduate student should be higher than the weight attached to an undergraduate student.

There is a dilemma here: The efficiency assessment on the one hand should incorporate a general view on the relative importance of inputs and outputs, whilst on the other hand allowing differences from the general view for individual DMUs. The former would lead to a fixed set of weights for all DMUs, whilst the latter leads to the total weights flexibility of DEA (Dyson and Thanassoulis, 1988). Weight restrictions ensures a compromise in this case.

2. To ensure also that all DMUs are compared on the same basis

DEA can rate a DMU as efficient by assigning an effectively zero value to some output and/or input weights, thus ignoring them. In this case, the DMU being evaluated is not rated over the full set of inputs or outputs. In extreme cases, a DMU may be evaluated over a single input and a single output. A DMU may thus be rated to be efficient simply because its ratio on any output to any of one input is maximum. This is unacceptable since all the selected inputs and outputs are relevant to the assessment. The constraint that all the weights should be greater than ϵ does little to remedy to the problem of weight flexibility in DEA. When the ϵ value is attained for some weight to any degree, the relevant factor does not in fact influence the efficiency standing of that DMU (Golany, 1993). Different DMUs may still put an emphasis on different inputs or outputs. More realistic weight restrictions may help to remedy to this situation.

3. To relate the values of certain inputs or/and outputs

It is often deemed inappropriate to accord widely differing weights to the same factor, when assessing different DMUs. Cases in which the same factor receives drastically different weights across the DMUs, may be managerially unacceptable (Golany, 1993). In this case, weight restrictions can be used to ensure that the values given to the same weight factor by different DMUs do not vary widely.

4. To respect the economic notion of input (output) substitution

Allen et al. (1997) state the need for the rates of transformation of inputs into the outputs or the rates of substitution of the inputs or outputs to conform to economic theory. The use of weight restrictions in this case will ensure that these rates will correspond more closely to the economic theory.

5. To incorporate prior views on efficient and inefficient DMUs

At times the DMUs rated efficient by DEA have been those judged inefficient by the experts. This was illustrated by Sun (1987) in a study aimed at managerial performance of banks. It was therefore imperative that such expert opinion would be taken into account in the assessment. A model was developed which could favour the DMUs unanimously agreed to be efficient by experts. In this way, weight restrictions ensured that prior judgements on DMUs were taken into account in the DEA assessment. In this method, the DMUs which performed similarly to the appraised DMUs were rated more efficient than those who didn't. As a result, the number of efficient DMUs was reduced. A similar model was developed by Charnes et al. (1990) to include information on the relative valuation of inputs or outputs by experts.

6. To ensure sensible ranges of input or output weights

In the single input-multiple output case, the input weights could be seen as the amounts of resources needed to produce one unit of output (Dyson and Thanassoulis, 1988). Given the complete flexibility in the computation of weights in DEA, some DMUs could put high weights on some outputs and at the same time very low weights on some other inputs in order to achieve efficiency. Such weights would be very unrealistic given an interpretation of the DEA weights as the amount of resources used for one unit of output. It might not be possible in reality to produce one unit of output with such a low amount of resources or the amount of output supposedly produced with one unit of input is beyond what can be reasonably achieved. For this reason, lower or upper bounds can be imposed on the weights to ensure that the weights are closer to reality.

7. To discriminate between Pareto-efficient units and to rank them

Sometimes, the basic DEA models may yield too many efficient DMUs. There may then be a need to discriminate between the Pareto-efficient DMUs by enforcing some restrictions on the weights. It may be agreed that all the Pareto-efficient DMUs do not have the same level of efficiency. The CCR, BCC and additive models do not allow the ranking of efficient units. Andersen and Petersen (1993) developed a procedure for the ranking of the Pareto-efficient DMUs. Thanassoulis (2001) also developed also a procedure that helps discriminate between the Pareto-efficient DMUs by looking at the number of times a DMU is referred to as an efficient peer for other DMUs or by looking at the importance of the contribution of a DMU in the target levels of other DMUs (Charnes et al., 1985; Boussofiane et al., 1991).

3.3 Effects of weight restrictions

Various authors have related weight restrictions to other aspects of the DEA problem. These include the following observations:

- Weight restrictions are equivalent to inserting unobserved DMUs into the Production Possibility Set as proved in (Roll et al., 1991; Podinovski, 2004b).
- Weight bounds modify the shape of the efficient frontier by the introduction of new points or DMUs into the Production Possibility Set which, in turn, results in the efficiencies of DMUs being changed.
- Setting weight bounds on a subset of weights has an effect on the remaining weight values. Setting an upper bound on a subset output weight imposes lower bounds

on the other output weights (Roll et al., 1991). Also, setting a lower bound on a subset of input weights limits the other input weights.

- Weight restrictions cause DMUs to have target inputs or outputs that are not a radial contraction or expansion of the current inputs or outputs (Allen et al., 1997).

3.4 Methods of weight restrictions

3.4.1 Direct weight restrictions

Given the perceived importance of the factors taking part in a production process, we may need to impose restrictions directly on the weights to ensure that the more important factors are accorded high weights.

Types of weight restrictions

In this section, we present various types of weight restrictions in DEA.

In matrix form, weight restrictions could in general be presented as follows (Halme and Korhonen, 2000):

$$-u^T B^y + v^T B^x \geq c \quad (3.5)$$

where

- $u \in R^p$ is the outputs vector
- $v \in R^m$ is the inputs vector
- $B^y \in R^{p \times r}$ is the output constraints matrix; r being the number of constraints
- $B^x \in R^{m \times r}$ is the input constraints matrix; r being the number of constraints

If $c = 0$, the weight restrictions are termed homogeneous, and may also be called relative weight restrictions, as they constrain relative but not absolute values of weight.

If $c \neq 0$, weight restrictions are non-homogeneous, i.e. linear inequalities with a non-zero constant on the right-hand side (Podinovski, 2004a). Non-homogeneous weight restrictions include the case of absolute weight restrictions where the range of the values of weights are bounded.

A summary of the different types of weight restrictions are given by the following:

1. $B_i v_i + B_k v_k \leq v_m$
2. $\alpha \leq v_i/v_k \leq \beta$
3. $\gamma \leq u_r/u_l \leq \delta$
4. $\zeta_i v_i \geq u_r$
5. $\alpha_i \leq v_i \leq \beta_i$
6. $\gamma_r \leq u_r \leq \delta_r$

Restrictions 1, 2 and 3 are called Assurance Region of type I. They relate different inputs or outputs to each other. Their aim is to limit the relative magnitude of the weights attached to a given input or output.

Restriction 4 is called Assurance Region of type II. It relates inputs to outputs. The aim is to ensure that we obtain reliable marginal rates of transformation of the inputs into the outputs.

Restrictions 5 and 6 are called absolute weight restrictions. They impose restrictions on the range of weights in an absolute sense i.e they do not take into account the other weights values for them to restrict the range of a given weight. They effectively determine a feasible interval respectively for each input or output.

All of the above restrictions are easily added the CCR value-based model.

Assurance region(AR)

Constraints on the weights limit the region of achievable weights to what may be called an assurance region (AR) (Cooper et al., 2000). In order for DMUs to be Assurance Regions (AR)-efficient (i.e. efficient subject to weight restricted to assurance region), they first need to be CCR-efficient. The AR concepts are defined in a value context, so that the most direct approach is to use value-based models but adding additional

constraints. Envelopment models can also be derived as the dual of the value based model with the additional constraints.

As an example, consider the case of two inputs and one output. All feasible input weights can be plotted on 2 dimensional axes.

Let

- $V1$ and $V2$ are the inputs weight axes
- \mathbf{v}_j is the normalised weight vector for efficient DMU j , $j = 1, \dots, 5$

Figure 3.1 illustrates a set of optimal weight needed for the 5 efficient DMUs, as determined by the CCR model.

Now, we impose the following weight restrictions:

$$\alpha_i v_1 \leq v_i \leq \beta_i v_1, \quad i = 2, \dots, m \quad (3.6)$$

These relations define AR, a restricted cone of feasible weights.

The optimal weight vector for DMU2 is located in the cone AR and DMU2 will thus still be efficient after the inclusion of weight restrictions in the CCR model. The other DMUs in general will have their weights outside the boundaries of the cone and, as a result, may not be efficient any longer. The above weight restrictions cause the feasible weights space for efficient DMUs in the CCR model to shrink to AR.

In defining the AR, it is possible to use the weight of one output or input as a standard, against which the other output or input weights may be compared (Golany, 1993). Thompson et al. (1990b) call the input or output used for comparison as the “input numeraire” or “output numeraire”. It is also possible to have different inputs or outputs used as “input numeraire” or “output numeraire” for different subsets of inputs or outputs.

If input k is defined as the “input numeraire” and output l the “output numeraire”, then the AR is defined by the following $m-1$ inequalities for the inputs and $s-1$ inequalities for the outputs.

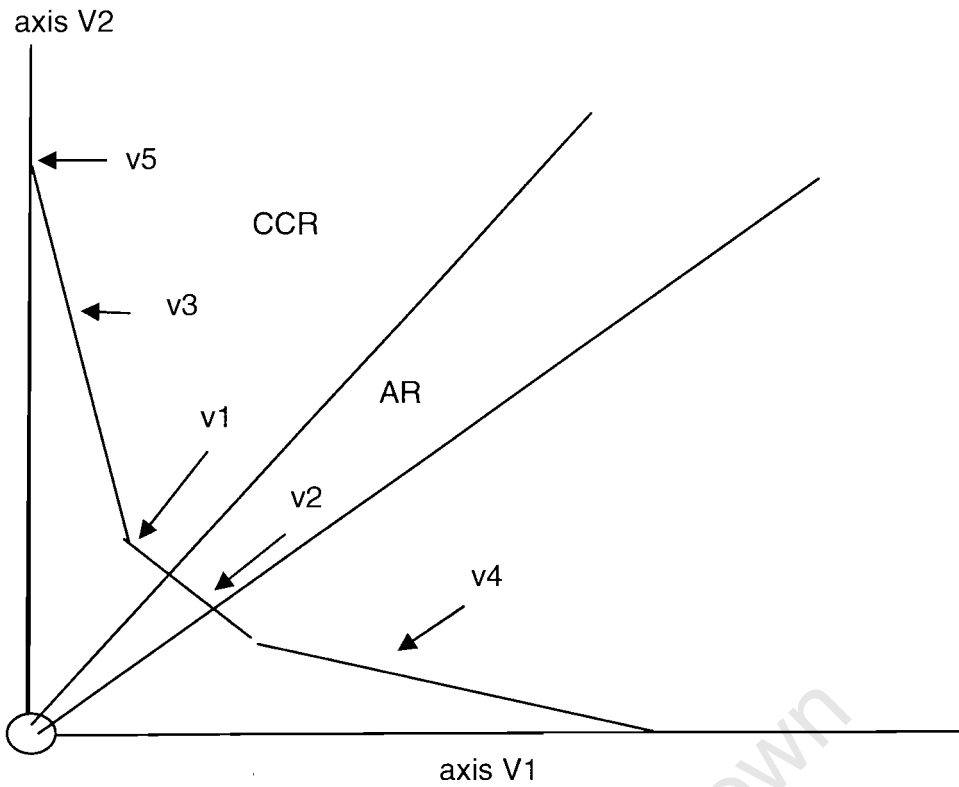


Figure 3.1: CCR and AR-weight space

For ease of interpretation, the constraints can be expressed as ratio bounds:

$$\frac{v_i}{v_k} \geq \alpha_i, \quad i = 1, \dots, k-1, k+1, \dots, m \quad (3.7)$$

$$\frac{v_i}{v_k} \leq \beta_i, \quad i = 1, \dots, k-1, k+1, \dots, m \quad (3.8)$$

$$\frac{u_r}{u_l} \geq \gamma_r, \quad r = 1, \dots, l-1, l+1, \dots, s \quad (3.9)$$

$$\frac{u_r}{u_l} \leq \delta_r, \quad r = 1, \dots, l-1, l+1, \dots, s \quad (3.10)$$

Note: Podinovski (2004b) discussed production trade-offs which are technologically realistic and shows that they are equivalent non-homogeneous weight restrictions which are exactly the same type of weight restrictions used in this approach.

Absolute weight restrictions

1. Regression analysis

This method can be used in a multiple output-single input or in the multiple input-single output case to impose absolute restrictions on the weights. For purpose of example, we will restrict attention to the multiple output-single input case. In this case, each output weight may be regarded as the amount of input or resource used to produce one unit of output (Dyson and Thanassoulis, 1988), so that [M3] can be expressed as:

$$\text{Max } h_o = \frac{\sum_{r=1}^s u_r y_{ro}}{x_o} \quad (3.11)$$

s.t

$$\sum_{r=1}^s u_r y_{rj} \leq x_j \quad j = 1 \cdots o \cdots N \quad (3.12)$$

$$u_r \geq \epsilon \quad r = 1 \cdots s \quad (3.13)$$

The key points of this method are as follows:

- We really seek the “best practice” weights i.e such that $\sum_{r=1}^s u_r y_{rj} =$ input needed to produce y_{rj} , $\forall j$
- Ideally, efficient DMUs should have $x_j = \sum_{r=1}^s u_r y_{rj}$ while inefficient DMUs have $x_j > \sum_{r=1}^s u_r y_{rj} = x_j$ i.e $x_j = \sum_{r=1}^s u_r y_{rj} + \beta_j$ for some $\beta_j > 0$
- A regression of x_j on $(y_{1j}, y_{2j}, \dots, y_{sj})$ will give average input needed per unit of each output as seen below:

$$x_j = \sum_{r=1}^s \alpha_r y_{rj}$$
- Best practice will be different, but not very different from most DMUs if they perform relatively well. A constraint is thus put on the output weight in the following way: $u_r \geq k\alpha_r$. This constraint prevents an output from using negligible input (Dyson and Thanassoulis, 1988).

2. Average weights

In a similar spirit to the use of regression, Golany (1992, 1993) suggests constrained weights which do not deviate too much from the norm defined by all other DMUs. The suggested procedure is as follows:

- First step:

We run the unbounded CCR model to get the optimal weights for each DMU. We compute average weights across all DMUs, say v_{av} and u_{av} for the inputs and outputs respectively. Robust estimation of averages obtained by eliminating outliers is advised.

- Second step:

An acceptable variation of weights for the same factor is specified i.e the ratio d of highest to lowest weight for the same factor.

- Third step:

We set the boundaries on the weights using :

$$\frac{2u_{av}}{1+d} \leq u_{rj} \leq \frac{2du_{av}}{1+d} \quad (3.14)$$

- Fourth step:

Re-run the CCR model with the inclusion of restrictions.

Note: we comment, however, that in the multiple input-multiple output case, the meaning of an average weight is arguable, as the weights are scaled differently when solving the CCR model for each DMU in turn.

3.4.2 Using unobserved DMUs to incorporate value judgments in DEA

Instead of specifying weight restrictions directly, an alternative is judgmentally to insert unobserved DMUs in the production possibility set. The procedure is as follows. We initially have a number of real DMUs with inputs x_{ij} and outputs y_{rj} for $j = 1, \dots, N$. We add to this initial set hypothetical DMUs which are obtained by re-scaling the inputs or outputs, for some DMUs to get x_{ij}^t and y_{rj}^t . Then we assess the efficiency for the observed DMUs within the aggregate set of observed and unobserved DMUs.

Since the DEA model may assign negligible weights to some factors, a DMU may not be fully enveloped by the efficient frontier. The unobserved DMUs approach attempt to remedy to this problem by inserting unobserved DMUs in the production possibility set in order to extend the efficient frontier. This process is illustrated by the following example based on 2 outputs and one input. It is then possible to re-scale the problem so that all DMUs use one unit of input, so that outputs can be compared as illustrated

in Figure 3.2

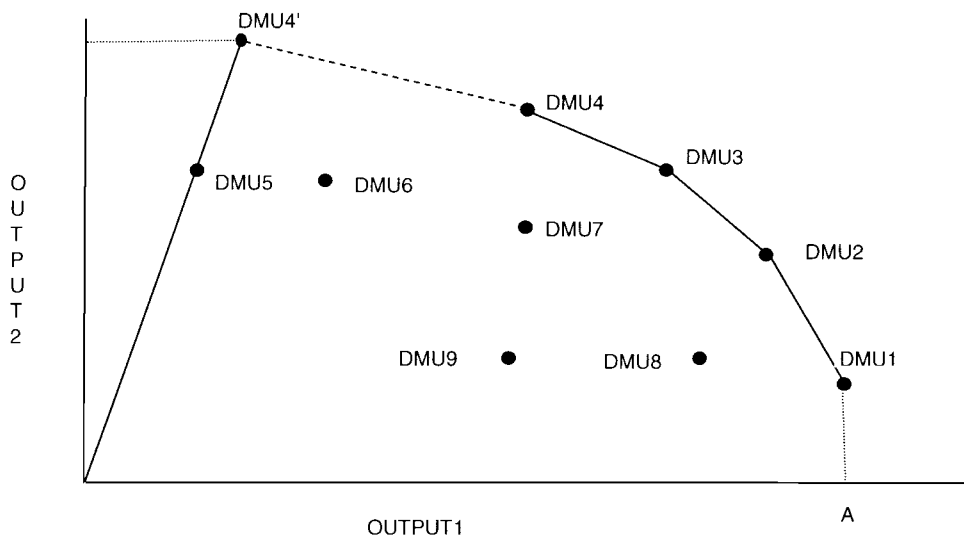


Figure 3.2: Anchor DMU

For the example in Figure 3.2, it can be seen that DMU5 is not fully enveloped by the efficient frontier defined by other DMUs. In scaling the DEA model for DMU5, the DMU can be attributed a high efficiency by ignoring output 1 (i.e. by giving it negligible weight). This may be practically unrealistic.

A decision maker may however judge that if DMU4 were adjusted to the same ratio of output 1 to output 2 as for DMU5, then its outputs would be represented by the point DMU4'. This hypothetical DMU extends the efficient frontier and provides a more realistic peer against which DMUs may be judged.

Thanassoulis (2001) proposed two sets of linear programs to identify which DMUs should be adjusted and how they are to be adjusted.

An equivalence between the assurance region and the unobserved DMUs approaches has been proved by Andersen and Petersen (1993) and by Thanassoulis and Allen (1998). For any set of weight restrictions, there exist a set of unobserved DMUs such that the efficiency assessed within the aggregate set of observed and unobserved DMUs will be the same as if it was assessed within the set of observed DMUs alone under weight restrictions.

3.4.3 Cone-Ratio method (CR) or DEA ratio model

The Cone-Ratio model translates experts opinion into explicit relative valuations of the inputs and outputs in order to provide a better assessment of the DMUs. The Cone-Ratio method assesses unobserved DMUs as in the unobserved DMUs approach. However, the adjusted input (output) levels derived from the observed input (output) levels are determined by a different procedure which is as follows:

1. Run the standard CCR model
2. Get expert appraisal to select DMUs judged as the best performers
3. Let u_{rk} and v_{ik} be the CCR weights for the selected DMUs, $k = 1, \dots, l$
4. We evaluate the efficiency of DMU 0 by using:

$$\text{Max} \quad \sum_{r=1}^s u_r y_{ro} \quad (3.15)$$

s.t

$$\sum_{i=1}^m v_i x_{io} = 1 \quad (3.16)$$

$$\sum_{r=1}^s [u_r y_{rj}] - \sum_{i=1}^m [v_i x_{ij}] \leq 0, \quad \forall j \quad (3.17)$$

$$u_r = \sum_{k=1}^l \alpha_k u_{rk} \quad r = 1, \dots, s \quad (3.18)$$

$$v_i = \sum_{k=1}^l \beta_k v_{ik} \quad i = 1, \dots, m \quad (3.19)$$

$$v_i, u_r \geq \epsilon \quad \forall i, r \quad (3.20)$$

In the value-based version of the cone-ratio, all weights are a linear combination of v_{ik} and u_{rk} . In this way, the CCR weights of the appraised DMUs restrict all weights and define the weights space as a cone.

An equivalent envelopment model can be extracted as the dual of the value-based model:

$$\text{Min } E_o \quad (3.21)$$

s.t

$$\sum_{j=1}^N \lambda_j \left[\sum_{i=1}^m v_{ik} x_{ij} \right] = E_o \sum_{i=1}^m v_{ik} x_{io} - I_i \quad i = 1, \dots, m \quad (3.22)$$

$$\sum_{j=1}^N \lambda_j \left[\sum_{r=1}^s u_{rk} y_{rj} \right] = \sum_{r=1}^s u_{rk} y_{ro} + O_r \quad r = 1, \dots, s \quad (3.23)$$

$$\begin{aligned} \lambda_j &\geq 0 \quad j = 1, \dots, N \\ I_i, O_r &\geq 0 \quad \forall i \text{ and } r \end{aligned} \quad (3.24)$$

In its envelopment version, the cone-ratio assess DMUs with transformed data. The transformed inputs (outputs) are the product of the CCR weights u_{rk} and v_{ik} of the DMUs appraised by experts and the original inputs/ outputs.

Assurance region and cone-ratio models

In the assurance region method, we use the bounds on the ratio of the CCR weights u_{rk} and v_{ik} of the appraised DMUs to restrict all the weights. In this case, the assurance region yield the same efficiency as the cone-ratio method using the same appraised DMUs to generate its weights space. In this way, the cone-ratio method is similar to the assurance region method (Thompson et al., 1990a).

The cone-ratio method serves more general purposes than the assurance region method. The CR model will further be used to achieve the following (Charnes et al., 1990):

- To highlight a given input or output which is believed to be more important than the others.
- To unveil weakly efficient DMUs i.e. the DMUs which appear efficient while using some slacks.

CCR and cone-ratio models

In the CCR model, an input cone (weights space) is spanned by the vectors of all input optimal weights. Similarly, an output cone is spanned by the vectors of all output optimal weights. In the Cone-Ratio model, the cone is spanned only by the vectors of the

optimal factor weights of the DMUs selected by experts. The unbounded weights in the CCR define a unrestricted cone that is in principle bigger than the constrained cone in the Cone-Ratio method. The weights in the CCR model that make a DMU appraised by experts are now used in the cone-ratio model to limit its weights space (Charnes et al., 1990).

A DMU will be rated efficient by the cone-ratio method, if firstly it rated efficient by the CCR model and secondly if the vectors of its optimal weights (in the CCR model) are in the constrained cone of the cone-ratio method. A DMU will be evaluated as inefficient by the cone-ratio CCR though assessed as efficient by the CCR model if the vectors of its optimal weights (in the CCR model) are not in the constrained cone of the Cone-Ratio model (Charnes et al., 1990).

3.4.4 Imposing virtual input and output restrictions

Until now we have been imposing direct restrictions on the weights or ratios of weights. At this point, restrictions are going to be placed on virtual inputs and outputs to effect weight restrictions. Recall that the virtual input (output) is the product of the input (output) and its corresponding weight, i.e:

- $u_r y_{rj}$ as the virtual output r for DMU j
- $\sum_{r=1}^s u_r y_{rj}$ as the total virtual output for DMU j
- $v_i x_{ij}$ as the virtual input i for DMU j
- $\sum_{i=1}^m v_i x_{ij}$ as the total virtual input for DMU j

The efficiency measure of DMU j is either the ratio of total virtual input to total virtual output or the inverse. Therefore:

- $\frac{v_i x_{ij}}{\sum_{i=1}^m v_i x_{ij}}$ be the proportion of the efficiency of DMU j attributed to input i . It represents the relative importance attached to input i in assessing the efficiency of DMU j .

- $\frac{u_r y_{rj}}{\sum_{r=1}^s u_r y_{rj}}$ be the proportion of the efficiency of DMU j attributed to output r . It represents the relative importance attached to output r in assessing the efficiency of DMU j .

We can then include virtual weight restrictions in model [M3] when assessing DMU 0's efficiency (Beasley, 1990; Wong and Beasley, 1990):

$$\text{Max} \quad \sum_{r=1}^s u_r y_{r0} \quad (3.25)$$

s.t

$$\sum_{i=1}^m v_i x_{i0} = 1 \quad (3.26)$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad \forall j \quad (3.27)$$

$$\alpha_r \leq \frac{u_r y_{rj}}{\sum_{r=1}^s u_r y_{rj}} \leq \beta_r \quad \forall r \quad (3.28)$$

$$\gamma_i \leq \frac{v_i x_{ij}}{\sum_{i=1}^m v_i x_{ij}} \leq \delta_i \quad \forall i \quad (3.29)$$

Restriction (3.28) keeps the relative importance of output r within specified bounds. Restriction (3.29) keeps the relative importance of input i within specified bounds.

Note: Given the interpretation of the above restrictions, we believe that they could be more meaningful to the decision maker than absolute weight restrictions. The decision maker may not be able to specify weight bounds especially in the presence of different scales for factors, but may at least be able to specify ordinal relationships between the factors which can be translated in virtual inputs (outputs) restrictions (Wong and Beasley, 1990).

3.5 Common set of weights (CSW)

In standard DEA, each DMU assigns its own weights to the inputs or outputs. Alternatively, a common set of weights could be used for evaluating all DMUs. Such a common set of weights is appropriate in a context where the operating conditions are similar for all the DMUs.

Many methods have been proposed as to how to determine a common set of weights, and a few of these are summarized below:

1. Central measure of weights

We run the CCR model for each DMU and retain some central measure of these weights of the inputs or outputs as common set of weights. Several central measures could be adopted in this context, e.g. the mean, the mode and the median of all weights (Roll et al., 1991).

2. Preferred order of factors

We start by ranking the various factors (inputs or outputs) in a descending order according to the order of importance. Then, we set ranges for each weight:

$$\alpha_i \leq v_i \leq \beta_i \tag{3.30}$$

$$\gamma_r \leq u_r \leq \delta_r \tag{3.31}$$

We will want to set wider ranges for the more important factors so that we are less restrictive on them.

Now we start with the most important factor, and maximize the value of its weight subject to the constraint that the efficiency of each DMU is bounded above by 1, and to 3.30 and 3.31. For the input weights this gives the model:

[M28](Golany, 1993)

$$\text{Max} \quad v_i \tag{3.32}$$

s.t

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, N \quad (3.33)$$

$$\alpha_i \leq v_i \leq \beta_i \quad (3.34)$$

$$\gamma_r \leq u_r \leq \delta_r \quad (3.35)$$

For the output weights, the model is as follows:

[M29](Golany, 1993)

$$\text{Max} \quad u_r \quad (3.36)$$

s.t

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, N \quad (3.37)$$

$$\alpha_i \leq v_i \leq \beta_i \quad (3.38)$$

$$\gamma_r \leq u_r \leq \delta_r \quad (3.39)$$

Once the solution is obtained, the factor weight is fixed at this level and the procedure is repeated for the next most important.

3. Maximum average efficiency score

The approach is based on an overall or average rating of all the DMUs being maximized. The common set of weights is selected as those which maximize the average efficiency of all DMUs. The set is determined by solving the following non linear program.

[M30] (Golany, 1993)

$$\text{Max} \quad 1/N \sum_{j=1}^N e_j \quad (3.40)$$

s.t

$$e_j = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}}, \quad j = 1, \dots, N \quad (3.41)$$

$$0 \leq e_j \leq 1, \quad j = 1, \dots, N \quad (3.42)$$

$$v_i \geq \varepsilon, u_r \geq \varepsilon, \quad \forall i, r \quad (3.43)$$

4. Weighting Technique

Let θ_j be the efficiency score of DMU j determined by the CCR model applied to DMU j .

A weighted average set of weights may be defined (Roll et al., 1991):

$$v_i = \frac{\sum_{j=1}^N \theta_j v_{ij}}{\sum_{j=1}^N \theta_j}$$

$$u_r = \frac{\sum_{j=1}^N k_j v_{rj}}{\sum_{j=1}^N k_j}$$

and used as the common set.

5. Maximising the number of efficient DMUs

In this approach we maximize the number of efficient DMUs by maximizing a set of binary variables which specify whether a DMU is efficient or not. The common set of weights is the set of the optimal weights that results from the solution to this program. It doesn't matter whether the overall efficiency rating across all the DMUs decreases or not. This model is formulated as a non linear mixed integer program.

[M31](Golany, 1993)

$$\text{Max} \quad \sum_{j=1}^N b_j \quad (3.44)$$

s.t

$$e_j = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}}, \quad j = 1, \dots, N \quad (3.45)$$

$$e_j - b_j \geq 0, \quad j = 1, \dots, N \quad (3.46)$$

$$0 \leq e_j \leq 1, \quad j = 1, \dots, N \quad (3.47)$$

$$v_i \geq \varepsilon, u_r \geq \varepsilon, \quad \forall i, r \quad (3.48)$$

$$b_j \in [0, 1], \quad j = 1, \dots, N \quad (3.49)$$

3.6 Cross-efficiency

In the standard DEA model, the efficiency of each DMU is assessed by choice of its own optimal weights. This type of efficiency is called standard or simple efficiency. The cross-efficiency approach makes possible for a DMU to be rated according to weights which are optimal for other DMUs. This type of efficiency is called cross-efficiency.

The simple efficiency of DMU k is defined as follows:

$$E_{kk} = \frac{\sum_{r=1}^s u_{rk} y_{rk}}{\sum_{i=1}^m v_{ik} x_{ik}} \quad (3.50)$$

where u_{rk} and v_{ik} are the optimal weights of DMU k . It can be seen from the above formula that the target DMU k rates itself with its own weights.

The cross-efficiency of DMU j is defined by:

$$E_{kj} = \frac{\sum_{r=1}^s u_{rk} y_{rj}}{\sum_{i=1}^m v_{ik} x_{ij}} \quad (3.51)$$

i.e. using the optimal weights for DMU k in order to assess DMU j .

In our discussion here, the simple efficiency and the cross-efficiency are computed on the basis of the CCR model, although generalisation to other models would be possible.

1. Aggressive and benevolent formulations

Each DMU uses in turn a linear programming (LP) to compute at once its own efficiency and the cross-efficiencies, constraining them to be less than one. In the end, every DMU will be rated by the other DMUs (rating by peers).

We can place the cross-efficiencies into a table such as the following cross-efficiency matrix, where the leading diagonal represents the self-rating of each DMU.

	1	2	3	...	n
1	E_{11}	E_{12}	E_{13}	...	E_{1n}
2	E_{21}	E_{22}	E_{23}	...	E_{2n}
3	E_{31}	E_{32}	E_{33}	...	E_{3n}
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
n	E_{n1}	E_{n2}	E_{n3}	...	E_{nn}
e_k	e_1	e_2	e_3	...	

The entries e_k in the final row represents the average rating by peers (excluding the leading diagonal) for DMU k . We compute it as follows (Green and Doyle, 1994):

$$e_k = \frac{\sum_{j \neq k} E_{kj}}{n-1}$$

It can be proved that when a DMU is Pareto-efficient in DEA, the linear program may often have multiple optimal solutions and in this case, its weights are not unique. The proof of this can be found in Sexton et al. (1986). Since the weights that maximize simple efficiency are not unique, a secondary objective has been added to the linear program to ensure a unique solution for the target DMU. The cross-efficiency approach and the standard DEA have the same primary goal: to maximize simple efficiency. The secondary goal of the Cross-efficiency approach represents a quantity that is to be optimized only in the event of multiple optimal solutions in the original linear program (primary goal) and should be ignored if the linear program has a unique optimal solution (Sexton et al., 1986). The secondary goal can be formulated in two ways: an aggressive and a benevolent formulation.

- (a) Aggressive formulation (Green and Doyle, 1994; Sexton et al., 1986)

The aggressive formulation assumes that, given a choice among several alternate solutions that maximize a DMU's self-rated efficiency, the DMU chooses the one that will make the other DMUs as inefficient as possible (Sexton et al., 1986).

We define:

$$B_k = \sum_{r=1}^s (u_{rk} \sum_{j \neq k}^N y_{rk}) - \sum_{i=1}^m (v_{ik} \sum_{j \neq k}^N x_{ik}) \quad (3.52)$$

Then the following LP may be solved:

[M35](Sexton et al., 1986)

$$\text{Max} \quad E_{kk} - \delta B_k \quad (3.53)$$

s.t

$$\sum_{i=1}^m v_{ik} x_{ik} = 1 \quad (3.54)$$

$$\sum_{r=1}^s u_{rk} y_{rj} - \sum_{i=1}^m v_{ik} x_{ij} \leq 0, \quad j = 1 \cdots 0 \cdots N \quad (3.55)$$

$$u_{rk} \geq \epsilon \quad r = 1 \cdots s \quad (3.56)$$

$$v_{ik} \geq \epsilon \quad i = 1 \cdots m \quad (3.57)$$

Where $\delta \ll 1$ is a parameter selected by the user.

The solution to this linear program is used to compute E_{kj}

(b) Benevolent formulation (Sexton et al., 1986)

The aggressive formulation assumes that, given a choice among several alternate solutions which maximize a DMU's self-rated efficiency, the formulation will choose the solution which will maximize the DMU's rating of all the other DMUs using the same definition of B_k from 3.52.

[M35](Sexton et al., 1986)

$$\text{Max} \quad E_{kk} + \delta B_k \quad (3.58)$$

s.t

$$\sum_{i=1}^m v_{ik} x_{ik} = 1 \quad (3.59)$$

$$\sum_{r=1}^s u_{rk} y_{rj} - \sum_{i=1}^m v_{ik} x_{ij} \leq 0, \quad j = 1 \cdots 0 \cdots N \quad (3.60)$$

$$u_{rk} \geq \epsilon \quad r = 1 \cdots s \quad (3.61)$$

$$v_{ik} \geq \epsilon \quad i = 1 \cdots m \quad (3.62)$$

The solution to this linear program is used to compute E_{kj}

(c) Practical relevance of the aggressive and the benevolent formulations

The principle of the aggressive formulation seems to be that it forces DMUs to distinguish themselves from their peers as much as possible whenever they can achieve their first-priority goal which is efficiency. The aim is that a DMU will specialize so that it can be the best performer in a specific area. On the other hand, under the benevolent formulation, DMUs seem to distribute their weights across all inputs and outputs in order to raise the cross-efficiencies which they dispense (Thompson et al., 1990a). The benevolent seeks the best performance for the team.

2. Practical usefulness of cross-efficiency

There are two primary uses for the cross-efficiency measures:

(a) To rank DMUs

The e_k (average ranking by peers) can be used to rank the efficient DMUs. The higher the e_k , the higher the DMU's rating will be. An alternative measure is the maverick indicator M_k defined by:

$$M_k = (E_{kk} - e_k)/e_k \quad (3.63)$$

A maverick indicator is a measure of the discrepancy which exists between simple efficiency and cross-efficiency: The higher the maverick indicator, the lower the DMU's rating will be. A maverick DMU operates away from the crowd. Though it might be efficient (simple efficiency), it has a low e_k (average ranking by peers) because its optimal weights are so different from its peers.

- (b) To identify suitable paragon DMUs or cross-efficient peers

A cluster analysis can be applied to the columns of the cross-efficiencies, which represent the rating of DMU j by the other DMUs. DMUs can be assigned to clusters based on the correlation coefficients between the columns of the cross-efficiency matrix table. Each correlation coefficient between the columns of the cross-efficiencies matrix represents a distance between two DMUs and expresses how similarly the DMUs are rated by peers. Similar DMUs are then grouped into clusters.

It is also possible to use other distance measures for the cluster analysis. For example, vectors may use binary variables that will either be equal to one (if a positive weight is placed on the input or output) or zero (if no weight is placed on the input or output). The distance between two DMUs will be the number of inputs and outputs for which their weights do not agree. We know that the weights are scale dependent in DEA and the binary variable is used to remedy to this problem.

Having identified the clusters, we can identify the top DMU in each cluster. The DMU having the highest cross-efficiency score in a given cluster will be considered a paragon of efficiency (or role model for the other members of the cluster). It is the best appraised DMU within that cluster (Green and Doyle, 1994).

3. Cross-efficiency and Analysis of Covariance(ANCOVA)

An analysis of covariance can be used to investigate a relationship between the efficiency score and other variables not included in the set of inputs or outputs. In this case, the efficiency score will be the dependent variable and the new variable(s) will be the independent variable(s). The analysis of covariance has the advantage of handling both the continuous and the categorical variables. Once a relationship between the current inputs or outputs and other variables is established, it could lead us either to change the model by including those variables or use this information in planning future strategies (Sexton et al., 1986).

3.7 Methods used in this study

In this chapter, we have presented many different methods for introducing weight restrictions and others have been suggested such as the interactive approach to find value

efficiency (Halme and Korhonen, 2000; Joro et al., 2003), central values between bounds (Golany, 1993) and Golany method (Golany, 1988).

For the numerical studies later in this dissertation, we shall concentrate on comparing assurance region, virtual weight restrictions and cross-efficiency because:

- they are easily applied while the algorithms for the other methods are more difficult.
- the input required from the decision maker for these methods is more straightforward while the demands from the decision maker for the other methods are more difficult to express.

University of Cape Town

Current Research Questions

The previous chapter is based on weight restrictions. The aim of weight restrictions is to provide an answer to the problem of extreme or unrealistic weights. However, price or preference information are not always easily available and the element of subjectivity in the determination of weights are serious concerns. Therefore, a more objective method needs to be used. The need arises even more especially when no value judgements can be obtained. In this chapter, we will present the research questions then, we will introduce a new approach aimed at evaluating DMUs' robustness.

4.1 Research questions

At this point, we will look at the questions to be addressed in the experimented studies reported in this thesis.

We know that the DEA efficiency is the ratio of the weighted inputs to the weighted outputs for the output efficiency or the weighted outputs to the weighted inputs for the input efficiency. It is therefore clear that this efficiency measure is a function of the weights. The basic DEA is characterised by total weight flexibility which has a lot of drawbacks which have been explored in the weight restrictions literature review. One of the problems of this value-free approach is the extreme weights or zero-weights which cause some factors to be effectively ignored in the assessment process.

The input or output mix are related to the operating practices of DMUs or policies of a DMU (Thanassoulis, 2001). The input (or output) mix is the ratio of inputs (or outputs). It is also interesting to investigate the impact of changing operating practices on

a DMU's efficiency. The lack of change in the efficiency when the weights change testify to the strength of a DMU's efficiency standing.

In view of all the problems in use and interpretation of weights in DEA, the following questions have been formulated for further attention:

4.1.1 Questions concerning efficiency

1. How much information does the DEA efficiency convey?
2. To what extent does the efficiency measure reveal the strengths or weaknesses of a DMU?
3. Can we measure the degree of uncertainty or imprecision in the efficiency
4. How much credibility can we attach to an efficiency measure in view of all the uncertainties related to weight restrictions?
5. Can we be certain that the DEA efficiency measure for DMU0 is robust?
6. How sensitive is the efficiency to changes in the operating practices or values of the DMU ?
7. Can we, when no value judgement is available, identify the DMUs which are robustly efficient?

4.1.2 Questions of methodological development

- Can we develop a robustness analysis procedure that will not use value judgements to specify the weight bounds ?
- Can we develop a robustness analysis procedure that will allow a DMU to be judged by its own weights? (i.e in the original DEA sum)
- Given the pattern of efficiencies when the weights change, can we allocate DMUs to performance categories?

4.2 Sensitivity Analyses

The possibility of data errors has prompted investigations into the stability of DEA efficiency to such errors. Charnes et al. (1994) state as follows: In view of the possibility of erroneous or misleading data, some critics of DEA have questioned the validity and stability of measures of DEA efficiency. A robustness analysis was also conducted to

help refine a proposed input-output set (Thanassoulis, 2001).

Poor envelopment of DMUs by the efficient frontier, wrong model specification (models not suitable), returns to scale not properly specified, etc. are other issues which have led researchers to investigate the stability of the DEA efficiency measure.

Sensitivity analyses have been reported in a number of papers but with different approaches to ours. The terms 'robustness analysis' or 'sensitivity analysis' were not explicitly mentioned in some papers even though they were based on this idea. We briefly summarize a few of these approaches.

- The cross-efficiency approach makes it possible to test the robustness of a DMU's efficiency. A cross-efficiency for DMU₀ is the efficiency score obtained with the optimal set of weights for another DMU (Sexton et al., 1986). The cross-efficiencies are readily available for a DMU since it can be rated successively with the weights obtained for each of the other DMUs. The cross-efficiency approach may be seen as a process of rating by peers. The overall cross-efficiency score for DMU₀ is the mean of all its cross-efficiencies. It is obvious that a DMU cannot have a high overall cross-efficiency if in general most DMUs rate it lowly (most of its cross-efficiencies are low). For a DMU to have a high overall cross-efficiency, it must be in general highly rated (most of its cross-efficiencies must be high). In this approach, we investigate changes to the efficiency of a DMU when subjected to different sets of weights (the optimal weights of other DMUs). A DMU with a high overall cross-efficiency can be said to be robustly efficient. A maverick indicator (72) prevents a DMU which has a high (self) efficiency but a low cross-efficiency score from being highly ranked. A maverick DMU or a DMU with a low overall cross-efficiency are not both robustly efficient even if they have a high (self) efficiency score because their efficiency is dependent on a single set of weights (Sexton et al., 1986).
- Charnes et al. (1994) suggest another sensitivity analysis based on simultaneous data changes. This approach seeks to find how data changes affect the efficiency of DMUs. The authors were concerned about the reliability of efficiency measures in case of data error. Efficient DMUs which remain so despite all data variations are called SA (Stable always). They qualify to be extremely efficient DMUs or SCSC (Strongly Complementary Slackness Condition) solution. Some initially efficient DMUs become inefficient and are replaced in the basis of efficient DMUs by the inefficient DMUs which become efficient.
- Banker and Thrall (1992), the scale elasticity (30) measure is allowed to take different values. It then becomes possible to monitor changes in efficiency due to

changes in returns to scale.

- In Lang et al. (1995), an approach is developed which extends the efficient frontier in order to control how the efficient frontier envelops all the DMUs. Different envelopments yield different scores. Three types of envelopment methods are available: standard DEA, CEA (Controlled Envelopment Analysis) and CFA (Constrained Facet Analysis). Standard DEA makes the initial envelopment of the DMUs assessed with the requirement that the Production Possibility Set must be convex. If a DMU is not well enveloped, inefficient DMUs may be assessed as efficient. Standard DEA yields the upper bound on the efficiency measure. CEA aims to maximize the envelopment degree while CFA seeks to remove the slacks at each iteration while keeping past efficient peers. The envelopment degree is a measure of how well the efficient frontier covers or envelops the DMUs assessed and it is determined mainly by close observation of the production possibility set.
- In Kornbluth (1991), Multiple Objective Linear Fractional Programming (MOLFP) was used in DEA and a sensitivity analysis was also conducted. Many solutions determining the efficiency of DMUs were found. All solutions had to comply with the corporate policies of the industry. The solutions differed from one another by allowing for individual policies of firms that are not contrary to the corporate policies. Policies were incorporated in the solutions through weight restrictions. MOLFP ensures that the best solution is obtained for all DMUs simultaneously. A DMU is strongly efficient if it is efficient not only with its own policies but with other firms' policies. A strongly efficient DMU is efficient for many solutions. The efficiency of that DMU stands despite having been rated with different sets of weights. In other words, the efficiency of that DMU stands even when different sets of restrictions are placed on the weights. Different corporate policies can also be used to test the strength of the efficiency of DMUs.
- Another form of sensitivity analysis was conducted by Golany (1988). Different DEA models were used on a set of data and efficiency rankings across the models were obtained. DMUs were ranked according to their efficiency scores. A DMU has a robust efficiency standing if similar rankings are reached with different models. The strength of the efficiency standing of a DMU is tested as efficiency assessment underlying assumptions is allowed to vary under different models.
- A sensitivity analysis involving probabilities was also suggested by Stewart (1996). This is a Monte Carlo approach in which uniformly distributed weights are generated randomly. The efficiency score is generated from each set of weights generated, leading to a probability distribution for the range of values for each DMU. The upper bounds on the range of efficiency scores is obviously the DEA efficiency

measure because DEA yields the maximum efficiency score for a DMU. Simple box and whisker plots were used in conjunction with the Monte Carlo approach to represent the distributions over the weights. In this case, it was assumed that the weights followed the uniform distribution, but other distributions could have been used. The graphic showed the distributions of DMUs' weights with their probabilities. This method attaches a probability level with each possible efficiency score. In this way an efficiency solution (upper bound of 100%) may have most of its probability mass at a lower efficiency than another inefficient DMU. Such insights could not be obtained with the basic DEA.

4.3 Robustness analysis

DEA can yield an efficiency score that is the result of an extreme choice of weights intended only to maximize its efficiency score and not representing the real performance of the DMU in question. In this sense if care is not taken, the resulting efficiency measure can be misleading.

Weight restriction approaches aim to solve the problems caused by unconstrained and total weight flexibility. However, explicit weight bounds may not be available because price or preference information may be unobtainable (Sexton et al., 1986), which is why many techniques of weight restrictions have been developed. Problems with these techniques include their often subjective nature, absence of feasible solutions, especially for absolute weight restrictions.

In place of explicit bounds on weights, we propose structured robustness analysis in order to investigate the effects of weight changes on the efficiency. This also provides a test of the strength of a DMU's efficiency standing. In this analysis, the permissible weight ranges (or the weight space) for each factor will systematically and progressively be tightened until such weight change will affect the efficiency of the DMU. DMUs whose efficiency remains the same or changes only marginally will be considered robustly efficient. The bounds used in the restriction of the weight space are not based on value judgements. Bounds are initially set at large values, which are progressively tightened.

Our robustness analysis includes a few models designed to test the robustness of the efficiency standing of DMUs. The aim of this robustness analysis is not to try to obtain a single efficiency score for DMU0 but rather to evaluate the standard DEA measure, in order to determine whether the DMU's efficiency standing is robust. We should not expect that a single efficiency measure is sufficient to display the full extent of a DMU's situation. We gain more insight through a robustness analysis which generates a range of efficiency scores. In this way an inefficient DMU may have a more robust efficiency

standing than an efficient DMU.

In this chapter, we have presented the research questions and discussed previous sensitivity analyses. In addition, we have introduced robustness analysis. In the next chapter, we will discuss the implementation of robustness analysis.

University of Cape Town

Chapter 5

Comparison of Methods for Robustness Analysis

In the preceding chapter, we presented robustness analysis and what it aims to achieve. In this chapter we compare the methods of robustness analysis described below. Our research methodology is to apply the methods to 3 different data sets and to initially compare insights across data sets and methods. Therefore, we describe both the data sets and the methods in this chapter.

5.1 Methods

Four methods have been selected for comparison in this study. These are:

- Method 1: Systematic changes to assurance regions
- Method 2: Systematic changes to virtual factor restrictions
- Method 3: Systematic restrictions of virtual factors to values not too different from the population averages
- Method 4: Cross-efficiency

5.1.1 Method 1

This method is based on the principle of applied assurance regions method but applied in a sequential manner as we shall shortly describe. The assurance region is a direct

method of weight restriction in which intervals are ratios of input or output weights as follows:

$$\frac{v_i}{v_k} \leq \beta_{ik}, \quad \forall i, k; i \neq k \quad (5.1)$$

$$\frac{u_r}{u_l} \leq \delta_{rl}, \quad \forall r, l; r \neq l \quad (5.2)$$

Note that the above constraints do define intervals of the form

$$\frac{1}{\beta_{ki}} \leq \frac{v_i}{v_k} \leq \beta_{ik}, \quad \forall i, k; i \neq k \quad (5.3)$$

$$\frac{1}{\gamma_{lr}} \leq \frac{u_r}{u_l} \leq \gamma_{rl}, \quad \forall r, l; r \neq l \quad (5.4)$$

In the assurance regions, expert opinion is used to set up the bounds on the weights. however in place of expert opinion, we shall use systematic variation in the weight restrictions. We start the process by allowing wide intervals for the feasible weights which are then progressively tightened. For ease of application, we are going to use the same bounds for all input and output pairs i.e $\beta_{ik} = \gamma_{rl} = B > 1 \quad \forall i \neq k \quad \& \quad r \neq l$.

The upper bound on the weights will be B and the lower bound $\frac{1}{B}$. In order to avoid redundancy, it is sufficient to use the following restrictions by suppressing one of the two equations for respectively the inputs and outputs :

$$\frac{v_i}{v_k} \leq B, \quad \forall i, k; i \neq k \quad (5.5)$$

$$\frac{u_r}{u_l} \leq B, \quad \forall r, l; r \neq l \quad (5.6)$$

Intuitively a large value of B is chosen (implying no effective weight restrictions). In the subsequent runs, B is reduced progressively in the process of which the resulting changes in the efficiency standing of each DMU can be traced. In this way, we will test the robustness of the efficiency of DMUs. DMUs who stand the test are those whose efficiencies only change slowly as a result of the weight changes.

5.1.2 Method 2

This method is similar to Method 1, but is based on restricting virtual inputs and outputs, which restricts the importance attached by a DMU to a given factor (input or output).

We define:

$v_i x_{ij}$ as the virtual input i , and $u_r y_{rj}$ as the virtual output r , for DMU j . Then

$$V_{ij} = \frac{v_i x_{ij}}{\sum_{i=1}^m v_i x_{ij}} \quad (5.7)$$

is the proportional virtual input i for DMU j .

Similarly

$$U_{rj} = \frac{u_r y_{rj}}{\sum_{r=1}^s u_r y_{rj}} \quad (5.8)$$

is the proportional virtual output r for DMU j .

In principle, the virtual inputs and outputs for DMU0 can be constrained as follows:

$$\alpha_i \leq V_{io} \leq \beta_i \quad \forall i \quad (5.9)$$

$$\gamma_r \leq U_{ro} \leq \delta_r \quad \forall r \quad (5.10)$$

Once again, for ease of application, we use the same bounds for all inputs and outputs, i.e $\alpha_i = \gamma_r = \frac{1}{B}$ and $\beta_i = \delta_r = B \quad \forall i, r$

As in method 1, robustness is assessed by starting with a large value for B and progressively reducing it, tracing the resulting of DMU0 (for each DMU).

5.1.3 Method 3

This method extends method 2 by replacing fixed bounds on individual inputs and outputs by deviations from average values on all DMUs.

Let

$$V_{ij} = \frac{v_i x_{ij}}{\sum_{i=1}^m v_i x_{ij}} \text{ be the proportional virtual input } i \text{ for DMU } j \quad (5.11)$$

$$U_{rj} = \frac{u_r y_{rj}}{\sum_{r=1}^s u_r y_{rj}} \text{ be the proportional virtual output } r \text{ for DMU } j \quad (5.12)$$

The following steps should be followed:

- Run the standard DEA and work out the proportional virtual inputs V_{ij}^* and outputs U_{ij}^* using the optimal weights for inputs and outputs respectively.
- Find $V_{i;av}$ and $U_{r;av}$, the “averages” of all V_{ij}^* and U_{rj}^* respectively. The term “average” here refers not necessarily to the arithmetic mean but possibly to all other central measures such as median, weighted average, mode, etc.
- In principle, virtual inputs and outputs for DMU0 can be constrained as follows:

$$\alpha_i V_{i;av} \leq V_{i0} \leq \beta_i V_{i;av}, \quad \forall i \quad (5.13)$$

$$\delta_r U_{r;av} \leq U_{r0} \leq \gamma_r U_{r;av}, \quad \forall r \quad (5.14)$$

As in methods 1 and 2, we define $\alpha_i = \gamma_r = \frac{1}{B}$ and $\beta_i = \delta_r = B \forall i, r$, where B is progressively reduced toward 1 in order to assess robustness.

5.1.4 Method 4

Method 4 is simply the cross-efficiency approach as proposed and defined by Green and Doyle (1994). Let E_{kk} be the simple efficiency of DMU k , i.e as determined by the standard DEA.

Let u_{ij} and v_{rj} be the optimal weights obtained for DMU j . We can then define the cross-efficiency of DMU j E_{kj} as the ratio $\frac{\sum_{r=1}^s u_{rk} y_{rj}}{\sum_{i=1}^m v_{ik} x_{ij}}$ using optimal weights of DMU k .

In practice, however, solutions to the DEA LP may not be unique. To ensure a unique definition for cross-efficiency, therefore a secondary objective is added to the simple efficiency objective in the linear program. Two formulations are possible, termed aggressive and benevolent. The aggressive formulation, while maximizing the simple efficiency of DMU k , minimizes the cross-efficiencies of all DMUs. On the other hand, the benevolent formulation, maximizes both the simple efficiency of DMU k and all the cross-efficiencies. For purposes of the studies reported here, we use the aggressive formulation as it is expected to lead to a more discriminating solution than the benevolent formulation.

In the aggressive formulation, we define the expression:

$$B_k = \sum_{r=1}^s (u_{rk} \sum_{j \neq k}^N y_{rj}) - \sum_{i=1}^m (v_{ik} \sum_{j \neq j}^N x_{ij}) \quad (5.15)$$

The LP solved in order to assess the E_{kj} is then defined as follows:

[M39]

$$\text{Max} \quad E_{kk} - \delta B_k \quad (5.16)$$

s.t

$$\sum_{i=1}^m v_{ik} x_{ik} = 1 \quad (5.17)$$

$$\sum_{r=1}^s u_{rk} y_{rj} - \sum_{i=1}^m v_{ik} x_{ij} \leq 0, \quad j = 1 \cdots k \cdots N \quad (5.18)$$

$$\sum_{r=1}^s u_{rk} y_{rk} - E_{kk} = 0 \quad (5.19)$$

$$u_{rk} \geq \epsilon \quad r = 1 \cdots s \quad (5.20)$$

$$v_{ik} \geq \epsilon \quad i = 1 \cdots m \quad (5.21)$$

Where $\delta \ll 1$, δ is a model parameter.

As we know, each DMU is evaluated by all the other DMUs. This process can be compared to appraisal of DMU k by peers and it yields e_k (The average rating of DMU k by peers).

We present here two significant measures:

- The average rating of DMU k (or e_k ranking): represent the extent to which a DMU is appraised by other DMUs. In other words, it tells us how well a DMU performs in terms of its cross-efficiency. The higher the cross-efficiency is, the higher the efficiency standing will be.

e_k is computed in this way:

$$e_k = 1/(n-1) \sum_{j \neq k}^N E_{kj} \quad (5.22)$$

- Maverick ranking (or M_k): expresses the discrepancy between cross-efficiency and simple efficiency and it is defined by:

$$M_k = (E_{kk} - e_k)/e_k \quad (5.23)$$

The higher the discrepancy is, the lower the efficiency standing will be.

In this chapter we have presented all the methods used in this study and the data sets to which we will apply them. The next chapter will deal with the implementation of the methods and the results will be presented and discussed.

5.2 Data sets

For the purpose of comparing insights obtained from the different methods we present the raw data, as given by the authors, but for the experiments the data were normalized to make the factor weights comparable.

5.2.1 Example 1

In this hypothetical example, a study was conducted on the comparative efficiency of DMUs using 4 inputs and 3 outputs (Golany, 1992).

DMUs	(I)X1	(I)X2	(I)X3	(I)X4	(O)Y1	(O)Y2	(O)Y3
1	521	3130	1859	80	15500	460	0.85
2	747	5075	3491	44	13700	340	0.63
3	935	1483	2984	93	18000	1080	0.37
4	205	4583	1736	65	8900	490	0.56
5	177	2990	1823	87	10800	960	0.14
6	584	5467	1775	98	17300	890	0.47
7	634	7734	1700	58	21000	2930	0.91
8	456	6552	503	73	9500	240	0.78
9	471	1855	2528	42	9100	370	0.74
10	325	4579	818	51	6600	800	0.52
11	364	5713	1178	80	11800	610	0.87
12	585	4217	2012	84	26200	3600	0.41
13	343	4061	2957	91	11400	470	0.55
14	597	3242	665	73	7200	1350	0.39
15	1126	7658	1541	57	38000	2470	0.68

5.2.2 Example 2

In this example, a study was conducted on the comparative efficiency of the departments of the Science Faculty of the University of Cape Town, which was initially undertaken as an honours project by Behrman (2002).

The following factors were used by the DMUs:

- Inputs
 - X1: recurrent staffing cost
 - X2: Non-recurrent staffing costs
 - X3: Floor
 - X4: General expenses
- Outputs
 - Y1: Postgraduate student numbers
 - Y2: Research Output
 - Y3: Full Time equivalent (FTE) undergraduate head count
 - Y4: Number of undergraduates passed

DMUs	(I)X1	(I)X2	(I)X3	(I)X4	(O)Y1	(O)Y2	(O)Y3	(O)Y4
1	25.3552	3.2533	16.5541	20.6094	20.8696	17.3913	9.8771	11.4561
2	14.0627	12.6254	1.7734	2.8286	6.9565	30.4348	1.6141	1.8558
3	84.5454	11.2504	60.9127	100	65.2174	21.0145	14.6471	17.3091
4	53.0606	9.6426	39.9984	49.2444	60	48.5507	3.3486	3.1406
5	92.2176	39.661	100	76.7885	57.3913	38.4058	27.7523	32.0842
6	43.7472	30.5204	8.2405	58.4512	42.6087	11.5942	30.8119	37.5803
7	41.3394	7.7552	16.1637	47.1029	66.087	20.2899	14.2375	18.0228
8	65.4843	14.3401	33.4499	44.9291	51.3043	19.5652	4.9386	5.3533
9	100	100	14.5123	47.7831	72.1739	44.2029	85.5697	91.5061
10	66.8288	28.0182	43.8054	30.8876	25.2174	23.1884	34.6182	40.5782
11	23.0072	0.4301	8.1998	11.6405	25.2174	16.6667	2.7222	3.2477
12	67.3429	42.3374	6.2637	26.3751	33.913	42.7536	100	100
13	77.9325	12.0574	58.9278	66.5985	100	100	7.4199	7.4233

5.2.3 Example 3

In this example (Bowlin et al., 1985), a study was conducted on the comparative efficiency of a number of hospitals. All data are artificial.

The following factors were used by the DMUs:

- Inputs
 - X1: Staff utilized in terms of full-time equivalents
 - X2: Number of hospital bed days available/year
 - X3: Supplies in terms of dollar cost/year

- Outputs
 - Y1: Regular patient care/year
 - Y2: Severe patient care/year
 - Y3: Teaching of residents and interns/year

Hospitals	FTE	Bed days	Supply \$	Teach. Units	Reg.pats	Sev.pats
1	23.5	41050	130000	50	3000	2000
2	24.5	43160	140000	50	2000	3000
3	26	43160	150000	100	2000	3000
4	25	41050	140000	100	3000	2000
5	28.5	50530	160000	50	3000	3000
6	36	62105	210000	100	2000	5000
7	51.5	92630	270000	50	10000	2000
8	25	49475	140000	100	3000	2000
9	24.5	43160	165000	50	2000	3000
10	77	92630	340000	100	10000	2000
11	44.5	65260	265000	50	5000	3000
12	30	60000	170000	100	3000	3000
13	43.5	81110	245000	50	4000	5000
14	30	60000	170000	100	3000	3000
15	26.5	47370	160000	50	3000	2000

Chapter 6

Results and Interpretation of results

6.1 Definition

The first three methods (or methods 1, 2 and 3) lead to the classification of DMUs into four groups: robustly efficient DMUs, non-robustly efficient DMUs, inefficient DMUs with a robust efficiency standing and inefficient DMUs with a non-robust efficiency standing. The first and the third groups are made of DMUs highly rated with regard to their efficiency standing robustness whether they are efficient or not.

- Robustly efficient DMU (RE): An initially efficient DMU whose efficiency score decreases minimally when the feasible weights interval is systematically reduced.
- Non-robustly efficient DMUs (NRE): An initially efficient DMU whose efficiency score decreases substantially when the feasible weights interval is systematically reduced.
- Inefficient DMUs with a robust efficiency standing (RI): An initially inefficient DMU whose efficiency score decreases minimally when the feasible weights interval is systematically reduced.
- Inefficient DMUs with a non-robust efficiency standing (NRI): An initially inefficient DMU whose efficiency score further decreases substantially as feasible weights interval is systematically reduced.

Interpretation of results from methods 1-3 is facilitated by graphically representing the changing efficiencies with changing weight intervals. However, given the number of DMUs, it will not be practical to display graphs for all DMUs. We rather have chosen to represent a few DMUs, each one belonging to a specific group. We know that each group depicts a specific pattern. In this way, the graph will prove to be a useful illustration of

the effects.

Method 4 leads to a different type of classification. Here also we will classify the DMUs in 4 groups: efficient DMUs highly appraised, efficient DMUs lowly appraised, inefficient DMUs highly appraised and inefficient DMUs lowly appraised.

- Efficient DMU highly appraised (HAE): An efficient DMU (self-evaluation) that has a high cross-efficiency.
- Efficient DMU lowly appraised (LAE): An efficient DMU (self-evaluation) that has a low cross-efficiency.
- inefficient DMUs highly appraised(HAI): An inefficient DMU (self-evaluation) that has a high cross-efficiency.
- inefficient DMUs lowly appraised(LAI): An inefficient DMU (self-evaluation) that has a low cross-efficiency.

Note:

- The classifications for the first three and the last methods respectively are quite similar.
- Some of the terms used in our definitions (such as minimal or substantial decrease in the efficiency score, low or high cross-efficiency) are not precisely defined at present, but we will elaborate on these later.
- All tables and figures have been included at the end of the chapter.

6.2 Results and Interpretation

6.2.1 Data set 1

6.2.1.1 Method 1

Table 6.1 (see end of chapter for all tables and figures) gives the efficiencies for each DMU for various values of B. The second last column contains the standard deviation on the efficiency scores for each DMU and the last column contains the ranks of the standard deviations such that the DMU with the smallest standard deviation (i.e the most robust) is ranked first.

The following classifications are then suggested:

- DMUs 7, 12 and 15 clearly belong to the RE group as their efficiency scores remain unchanged below $B = 1.5$ and even as low as $B = 1.1$ for DMU 7
- DMU 9 is more borderline as its efficiency score only starts changing from $B \leq 1.7$ and ends up not very low.
- DMUs 1,3, 4, 5, 8, 11, and 14 belong to the NRE group as their efficiency scores start to drop in some cases quite rapidly while B is relatively large.
- DMU 10 belongs to the RI group as its efficiency score though less than 1 doesn't drop quickly. DMU 10 doesn't lose much of its efficiency as demonstrated by the fact the standard deviation of its efficiency scores is relatively low.
- DMUs 2, 6 and 13 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.1 displays changes in the efficiency with changing B for one DMU from each group, namely:

- DMU 7 for the RE group
- DMU 9 for the RE—NRE group
- DMU 3 for the NRE group
- DMU 10 for the RI—NRI group
- DMU 2 for the NRI group

From the graph, we note the following:

DMU 3 though initially efficient ends up almost at the same efficiency level with DMU 2 which was initially inefficient. We can see that the efficiencies of DMUs 3 and 2 fall quickly and substantially. DMU 7 maintains its efficiency even as the bounds change. DMU 10 although inefficient doesn't lose much of its efficiency. DMU 9 remains efficient one most of the range and ends with a high efficiency score.

6.2.1.2 Method 2

Table 6.2 gives the efficiencies for each DMU for various values of B .

The following classifications are suggested:

- DMUs 7, 12 and 15 clearly belong to the RE group as their efficiency scores remain unchanged below $B = 1.5$ and even as $B = 1.1$
- DMU 3 is more borderline as its efficiency score only starts changing slowly from $B \leq 1.7$ and ends up not very low.
- DMUs 1, 4, 5, 8, 9, 11 and 14 belong to the NRE group as their efficiency scores start to drop in some cases quite rapidly while B is relatively large.
- DMU 10 belongs to the RI group as its efficiency score though less than 1 doesn't fall quickly. DMU 10 doesn't lose much of its efficiency as demonstrated by the fact that the standard deviation of its efficiency scores is relatively low.
- DMUs 2, 6 and 13 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.2 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 7 for the RE group
- DMU 3 for the RE—NRE group
- DMU 9 for the NRE group
- DMU 10 for the RI—NRI group
- DMU 2 for the NRI group

From the graph, we note the following:

DMU 9 though initially efficient ends up almost at the same efficiency level with DMU 2 which was initially inefficient. We can see that the efficiencies of DMUs 9 and 2 fall quickly and substantially. DMU 7 maintains its efficiency even as the bounds change. DMU 10 although inefficient doesn't lose much of its efficiency. DMU 3 remains efficient one most of the range and ends with a high efficiency score.

6.2.1.3 Method 3

Table 6.3 gives the efficiencies for each DMU for various values of B .

The following classifications are suggested:

- DMUs 4, 7, 12 and 15 clearly belong to the RE group as their efficiency scores remain unchanged below $B = 1.5$ and even as $B = 1.1$
- DMU 3 is more borderline as its efficiency score only starts changing slowly from $B \leq 1.7$ and ends up not very low.
- DMUs 1, 5, 8, 9, 11 and 14 belong to the NRE group as their efficiency scores start to drop in some cases quite rapidly while B is relatively large.
- DMUs 6 and 10 belong to the RI group as their efficiency scores though less than 1 don't fall quickly. DMUs 6 and 10 don't lose much of its efficiency as demonstrated by the fact that the standard deviation of its efficiency scores is relatively low.
- DMUs 2 and 13 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.3 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 7 for the RE group
- DMU 3 for the RE—NRE group
- DMU 8 for the NRE group
- DMU 6 for the RI—NRI group
- DMU 2 for the NRI group

From the graph, we note the following:

DMU 8 though initially efficient ends very low. We can see that the efficiencies of DMUs 8 and 2 fall quickly and substantially. DMU 7 maintains its efficiency even as the bounds change. DMU 6 although inefficient doesn't lose much of its efficiency. DMU 3 remains efficient one most of the range and ends with a high efficiency score.

6.2.1.4 Method 4

Cross-efficiencies for each DMU are reported in the rows of Table 6.4 (i.e the efficiency of DMU j as assessed by the weights for DMU k is given in row j , column k)

Table 6.5 gives for each DMU the self-evaluation, the average cross-efficiency, the rank based on the average cross-efficiency and the maverick indicator:

The following classifications are then suggested:

- DMUs 7, 12 and 15 belong to the HAE group as they form a cluster whose cross-efficiencies are higher than those of the rest of efficient DMUs (self-evaluation).
- DMUs 1, 8, 9 and 11 are more borderline since their cross-efficiencies are somewhat lower than those of the previous group.
- DMUs 3, 4, 5, and 14 belong to the LAE group as their cross-efficiencies are low and there is a wide gap between them and the HAE group.
- DMU 10 is more borderline since their cross-efficiencies are somewhat lower than the cross-efficiencies of the previous group.
- DMUs 2, 6 and 13 belong to the LAI group as their cross-efficiencies are low and there is a wide gap between them and the HAI group.

6.2.1.5 Summarized results for data set 1

Table 6.6 displays the categories to which a DMU is assigned according to each method:

The following observations may be made:

- Method 3 and 2 agree generally except for the evaluation of DMU 4 which method 2 regards as non-robustly efficient DMU while method 3 regards it as robustly efficient.
- Methods 1 and 2 agree in general except that they emphasize different borderline DMUs: DMU 9 for method 1 and DMU 3 for method 2. Method 1 regards DMU 3 as non-robustly efficient while method 2 regards DMU 9 as non-robustly efficient. Method 1 is dubious as to whether DMU 9 should belong to the robust or non-robust group while Method 2 is uncertain as to whether DMUs 3 should belong to the robust or non-robust group. There is no substantial disagreement between the two methods.
- Method 4 generally agrees with the other three methods although it is dubious about the status of a few DMUs: 1, 8, 9, 10, 11. In general, it seems more difficult to obtain a clear classification with Method 4.

The above observations indicate that the methods differ only slightly and yield similar results concerning the assignment of DMUs to the various categories.

6.2.2 Data set 2

6.2.2.1 Method 1

Table 6.7 gives the efficiencies for each DMU for various values of B . The second last column contains the standard deviation on the efficiency scores for each DMU and the last column contains the ranks of the standard deviations. The DMU with the smallest standard deviation (i.e. the most robust) is ranked first.

The following classifications are then suggested:

- DMUs 12 clearly belongs to the RE group as its efficiency scores remain unchanged above 1.3.
- DMU 2 is more borderline as its efficiency score only start changing from $B = 1.7$ and ends up not very low.
- DMUs 1, 6, 7, 9, 11 and 13 belong to the NRE group as their efficiency scores start to drop in some cases quite rapidly while B is relatively large.
- DMUs 5 and 10 belong to the RI group as their efficiency scores though less than 1 don't drop quickly. DMUs 5 and 10 don't lose much of their efficiencies as demonstrated by the fact that the standard deviations of their efficiency scores are relatively low.
- DMUs 3, 4 and 8 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.4 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 12 for the RE group
- DMU 2 for the RE—NRE group
- DMU 7 for the NRE group
- DMU 10 for the RI group
- DMU 4 for the NRI group

From the graph, we note the following:

DMUs 7 and 4 drop quickly and substantially. DMU 10 though inefficient doesn't lose much of its efficiency. DMU 2 remains efficient till near the end then drops quite slowly. DMU 12 keeps its efficiency all the way through.

6.2.2.2 Method 2

Table 6.8 gives the efficiencies for each DMU for various values of B.

The following classifications are then suggested:

- DMUs 11 and 12 clearly belongs to the RE group as their efficiency scores remain unchanged till the end.
- DMUs 1, 2, 6, 7, 9 and 13 belong to the NRE group as their efficiency scores start to drop in some cases quite rapidly while B is relatively large.
- DMUs 3, 5 and 10 belong to the RI group as their efficiency scores though less than 1 don't drop quickly. DMUs 3, 5 and 10 don't lose much of their efficiencies as demonstrated by the fact that the standard deviations of their efficiency scores are relatively low.
- DMUs 4 and 8 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.5 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 12 and 11 for the RE group
- DMU 2 for the NRE group
- DMU 10 for the RI group
- DMU 4 for the NRI group

From the graph, we note the following:

DMUs 2 and 4 drop quickly and substantially. DMU 2 experience such a severe loss of efficiency that it ends up almost at the same level with DMU 4. DMU 10 though inefficient doesn't lose much of its efficiency. DMUs 11 and 12 keep its efficiency all the way through.

6.2.2.3 Method 3

Table 6.9 gives the efficiencies for each DMU for various values of B .

The following classifications are suggested:

- DMUs 7, 11 and 12 clearly belongs to the RE group as their efficiency scores remain unchanged till the end.
- DMUs 1, 2, 6, 9 and 13 belong to the NRE group as their efficiency scores start to drop in some cases quite rapidly while B is relatively large.
- DMUs 3 and 5 belong to the RI group as their efficiency scores though less than 1 don't drop quickly. DMUs 3 and 5 don't lose much of their efficiencies as demonstrated by the fact that the standard deviations of their efficiency scores are relatively low.
- DMUs 4, 8 and 10 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.6 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 7 and 11 for the RE group
- DMU 2 for the NRE group
- DMU 3 for the RI group
- DMU 4 for the NRI group

From the graph, we note the following:

DMUs 2 and 4 drop quickly and substantially. DMU 2 experience such a severe loss of efficiency that it ends up just a bit above DMU 4. DMU 3 though inefficient doesn't lose much of its efficiency. DMUs 7 and 11 keep their efficiency all the way through.

6.2.2.4 Method 4

Cross-efficiencies for each DMU are reported in the rows of Table 6.10 (i.e the efficiency of DMU j as assessed by the weights for DMU k is given in row j , column k)

Table 6.11 gives for each DMU the self-evaluation, the average cross-efficiency, the rank based on the average cross-efficiency and the maverick indicator:

The following classifications are suggested:

- DMUs 7, 11 and 12 belong to the HAE group as they form a cluster whose cross-efficiencies are higher than those of the rest of efficient DMUs (self-evaluation). We note however that DMU 11 is far ahead of the rest of DMUs in its group.
- DMUs 1, 2, 6, 9 and 13 belong to the LAE group as their cross-efficiencies are low and there is a wide gap between them and the previous group.
- No DMU belongs to the HAI group as there is no inefficient DMU (self-evaluation) with high cross-efficiency.
- DMUs 3, 4, 5, 8 and 10 belong to the LAI group as their cross-efficiencies are low and there is a wide gap between them and the previous group.

6.2.2.5 Summarized results for data set 2

Table 6.12 displays each DMU with the group to which it belongs for each experience:

The following observations may be made:

Methods 1 and 2 agree on most DMUs except for DMU 3 which is regarded by method 1 as non-robust in terms of its efficiency while method 2 considers it to be robust. Method 3 agree with method 2 concerning DMU 3: it is a RE DMU. We note also that method 1 is dubious whether DMUs 2 and 11 belong to the robust or non-robust groups.

Method 3 agree with methods 1 and 2 in general but they disagree only concerning DMUs 7 and 10 which method 3 regard respectively as robustly efficient and non-robustly inefficient while method 1 and 2 regard them respectively as non-robustly efficient and robustly inefficient.

Methods 4 and 3 agree concerning DMU 7 which they consider to be a DMU with a robustly efficiency standing while method 1 and 2 view it as a DMU with a non-robust efficiency standing. Also, methods 4 consider DMU 9 to be an inefficient DMU while the rest of the methods consider it to be inefficient.

Following the observations above, we can say all the methods differ slightly and therefore they give in general similar results.

6.2.3 Data set 3

6.2.3.1 Method 1

Table 6.13 gives the efficiencies for each DMU for various values of B . The second last column contains the standard deviation on the efficiency scores for each DMU and the last column contains the ranks of the standard deviations. The DMU with the smallest standard deviation ranks first among all the DMUs.

The following classifications are suggested:

- DMUs 3 and 4 clearly belongs to the RE group as their efficiency scores remain unchanged till the end.
- DMU 6 is more borderline because it loses little of its efficiency score although a change in its efficiency score appears already at $B = 2$
- DMUs 1, 2, 5, 7, 10 and 13 belong to the NRE group as their efficiency scores start to drop quite rapidly while B is still large.
- DMU 8 belongs to the RI group as its efficiency score though less than 1 doesn't drop quickly.
- DMUs 12 and 14 are more borderline because they don't lose much of their efficiency scores as demonstrated by the fact that the standard deviations of their efficiency scores is relatively low.
- DMUs 9, 11 and 15 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.7 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 3 for the RE group
- DMU 6 for the RE—NRE group
- DMU 7 for the NRE group
- DMU 8 for the RI group
- DMU 14 for the RI—NRI group
- DMU 11 for the NRI group

From the graph, we note the following:

DMUs 3 keeps its efficiency all the way through. DMU 8 is slightly below DMU 3. DMUs 6 and 14 fall quite slowly and end up with a high efficiency score. DMU 7 though initially efficient ends up almost at the same efficiency level with DMU 11.

6.2.3.2 Method 2

Table 6.14 gives the efficiencies for each DMU for various values of B .

The following classifications are suggested:

- DMUs 3 and 4 clearly belongs to the RE group as their efficiency scores remain unchanged till the end.
- DMU 1 is more borderline as its efficiency start only for $B \leq 2$
- DMUs 1, 2, 5, 6, 7, 10 and 13 belong to the NRE group as their efficiency scores start to drop quite rapidly while B is still large.
- DMU 8 belongs to the RI group as its efficiency score though less than 1 doesn't drop quickly.
- DMUs 12 and 14 are more borderline because they don't lose much of their efficiency scores as demonstrated by the fact that the standard deviations of their efficiency scores is relatively low.
- DMUs 9, 11 and 15 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.8 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 3 for the RE group
- DMU 1 for the RE—NRE group
- DMU 7 for the NRE group
- DMU 8 for the RI group
- DMU 14 for the RI—NRI group
- DMU 11 for the NRI group

From the graph, we note the following:

The efficiencies of DMUs 3 and 8 drop slightly near the end. DMU 14 fall quite slowly and end up with a high efficiency score. DMU 1 is slightly below DMU 14. The efficiencies of DMUs 7 and 11 fall quickly and substantially.

6.2.3.3 Method 3

Table 6.15 gives the efficiencies for each DMU for various values of B .

The following classifications are suggested:

- DMUs 3 and 4 clearly belongs to the RE group as their efficiency scores remain unchanged above 1.3
- DMU 8, 12 and 14 are more borderline as they lose their efficiency minimally though they start losing it quite early on.
- DMUs 1, 2, 5, 6, 7, 9, 10 and 13 belong to the NRE group as their efficiency scores start to drop quite rapidly while B is still large.
- No DMU belongs to the RI group because all inefficient DMUs lose quickly their efficiency.
- DMUs 11 and 15 belong to the NRI group as their efficiency scores start to drop quite rapidly while B is relatively large.

As illustration of the effects observed, Figure 6.9 displays changes in the efficiency with B for each DMU from each group, namely:

- DMU 4 for the RE group
- DMU 8 for the RE--NRE group
- DMU 10 for the NRE group
- DMU 11 for the NRI group

From the graph, we note the following:

DMU 4 efficiency remains unchanged. DMU 8 efficiency drops slightly near then end. The efficiencies of DMUs 10 and 11 fall quickly and substantially.

6.2.3.4 Method 4

Cross-efficiencies for each DMU are reported in the rows of Table 6.16 (i.e the efficiency of DMU j as assessed by the weights for DMU k is given in row j , column k)

Table 6.17 gives for each DMU the self-evaluation, the average cross-efficiency, the rank based on the average cross-efficiency and the maverick indicator:

The following classifications are suggested:

- DMUs 3 and 4 belong to the HAE group as they form a cluster whose cross-efficiencies are higher than those of the rest of efficient DMUs (self-evaluation).
- DMU 1 is more borderline since its cross-efficiency is somewhat lower than the those of the previous group.
- DMUs 1, 2, 5, 6, 7 and 10 belong to the LAE group as their cross-efficiencies are low and there is a wide gap between them and the HAE group.

- DMU 8 belongs to the HAI group as as they form a cluster whose cross-efficiencies are higher than those of the rest of efficient DMUs (self-evaluation).
- DMUs 12 and 14 are more borderline since their cross-efficiencies are somewhat lower than those of the previous group.
- DMUs 9, 11, 15 and 13 belong to the LAI group as their cross-efficiencies are low and there is a wide gap between them and the HAI group.

6.2.3.5 Summarized results for Data Set 3

Table 6.18 displays each DMU with the group to which it belongs for each experience:

The following observations may be made:

With methods 1, 2 and 4, some DMUs (DMUs 8, 9, 12 and 14) lose their efficiency standing in the standard DEA i.e they become inefficient though they were assessed as efficient by the standard DEA. Method 3 on, his part, still assess them as efficient DMUs just as with the standard DEA.

Method 4 differ from the rest of the methods concerning DMU 13 which it assesses to be inefficient while the other methods consider it to be efficient.

DMUs 12 and 14 turn out to be more borderline in all the methods.

Following the observations above, we can say all the methods differ slightly and therefore they give in general similar results.

6.3 General Conclusion from numerical studies

The aim of the analysis in this chapter was to obtain:

- Partial ordering of DMUs: we should be able to distinguish between the better performing and the worst performing DMUs.
- Assessment of robustness for each DMU.

We noticed that in general there were only slight differences in the results given by the various methods and that they all agree to a large extent. Differences which exist arise because they look at the problems from different perspectives.

Methods 1, 2 and 3 are directly interested in the robustness analysis while method 4 looks at the rating of a DMU by other DMUs. Method 4 aims at a consensus between the DMUs while methods 1, 2 and 3 looks at characteristics which are intrinsic to each DMU. Although the first three methods are all concerned with robustness analysis, there is a different rationale behind each method. Method 1 restricts the range of the ratios of weights while method 2 limits the ratios of the virtual weights. Method 3 ensures that each DMU is not be too different from an “average DMU”. In all three case studies, the methods agree to a large extent

on which DMUs should be allocated to the first group (RE-HAE) because the robustly efficient DMUs are also likely to be highly appraised most other DMUs.

We have noticed that from a computational aspect, method 1 is simpler than methods 2 and 3. One can use the DEA solver LV (Cooper et al., 2000) for method 1 and find solutions for complex problems in a matter of a few minutes. Since all three methods yield essentially the same classification of DMUs, it makes sense to use the simplest version of the three methods which is method 1.

Method 4 is not directly concerned with the robustness analysis of DMUs and deals with the cross-efficiencies of DMUs. Method 4 distances itself from the other 3 methods because it is primarily interested in the appraisal of a DMU by other DMUs (peer evaluation). From the results, there seems to be in general more borderline DMUs with method 4 than with the rest of the methods. Differences between method 4 and the other methods can arise in two ways:

- Some DMUs are highly appraised by method 4 while lowly appraised by the other methods. This is the case with DMUs 4 and 7 in data set 2.
- Some DMUs are lowly appraised by method 4 while highly appraised by the other methods. This is the case with DMU 6 in data set 1 and DMUs 3, 5 and 10 in data set 2.

Overall, therefore, it is proposed that the user routinely apply both methods 1 and 4 in analysis of DMUs.

In using method 1, it is proposed that the following summaries of results be used:

- Individual trace plots for every DMU, displaying the pattern of the efficiency of DMUs: The non-robust DMUs are those whose efficiency is shown to fall drastically, on the graph, as the bounds are systematically decreased.
- Classification of DMUs to specific performance groups according to the rule:

Non-robust DMU if efficiency drops by 5 % while $B > 2$

Robust DMU if efficiency drops by more or less 7 % even when $B = 1.5$

Results of method 4 should be summarized by:

- The Cross-efficiency table.
- A summary table which gives for each DMU the self-evaluation, the average cross-efficiency, the rank based on the average cross-efficiency and the maverick indicator.

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	1.000	1.000	0.996	0.933	0.885	0.832	0.773	0.089	6
2	0.881	0.774	0.712	0.614	0.579	0.551	0.520	0.483	0.138	9
3	1.000	1.000	0.859	0.634	0.591	0.560	0.527	0.489	0.212	14
4	1.000	1.000	0.899	0.738	0.697	0.665	0.630	0.590	0.165	13
5	1.000	0.833	0.666	0.500	0.479	0.464	0.448	0.423	0.213	15
6	0.736	0.703	0.674	0.595	0.574	0.556	0.536	0.511	0.083	5
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
8	1.000	1.000	1.000	0.828	0.774	0.733	0.688	0.639	0.149	11
9	1.000	1.000	1.000	1.000	0.95	0.887	0.820	0.749	0.097	7
10	0.885	0.874	0.862	0.782	0.752	0.731	0.706	0.677	0.081	4
11	1.000	1.000	1.000	0.922	0.876	0.840	0.798	0.750	0.098	8
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.991	0.003	2
13	0.873	0.784	0.712	0.606	0.567	0.536	0.503	0.465	0.145	10
14	1.000	0.954	0.867	0.711	0.682	0.659	0.633	0.604	0.154	12
15	1.000	1.000	1.000	1.000	1.000	1.000	0.974	0.927	0.026	3

Table 6.1: Results of Data Set 1 Method 1

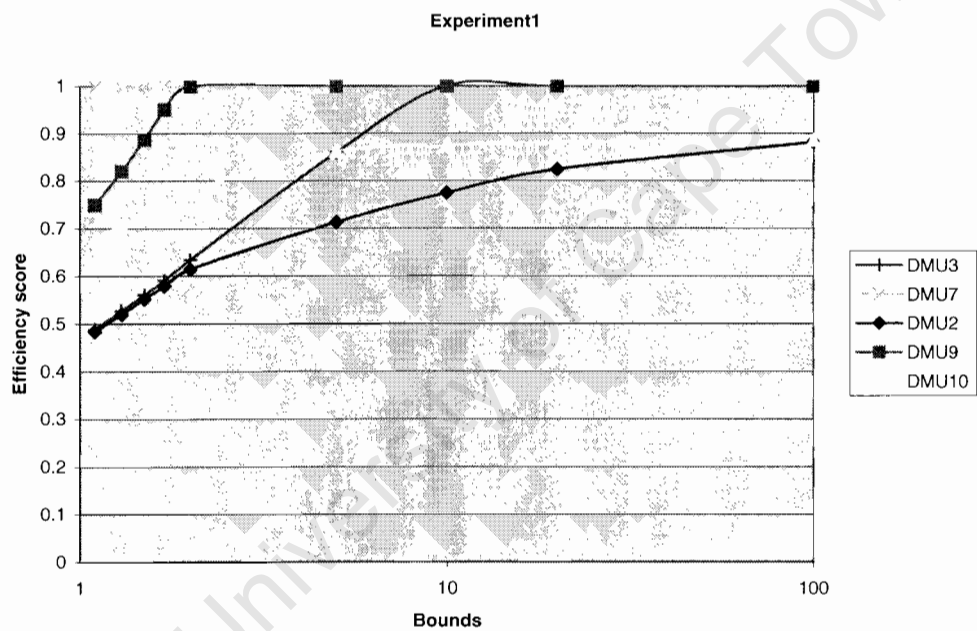


Figure 6.1: Graph of Data Set 1 Method 1

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	1.000	1.000	0.640	0.544	0.485	0.428	0.373	0.273	14
2	0.886	0.779	0.684	0.466	0.415	0.369	0.313	0.260	0.232	12
3	1.000	1.000	1.000	1.000	0.958	0.877	0.793	0.697	0.116	6
4	1.000	1.000	1.000	0.890	0.737	0.637	0.547	0.463	0.219	10
5	1.000	1.000	1.000	0.829	0.750	0.693	0.636	0.555	0.178	8
6	0.738	0.711	0.680	0.580	0.554	0.532	0.505	0.466	0.101	4
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
8	1.000	1.000	1.000	0.718	0.592	0.498	0.409	0.328	0.279	15
9	1.000	1.000	1.000	0.765	0.630	0.548	0.472	0.397	0.251	13
10	0.886	0.876	0.865	0.813	0.750	0.707	0.665	0.589	0.109	5
11	1.000	1.000	1.000	0.831	0.734	0.650	0.568	0.486	0.206	9
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.928	0.025	3
13	0.882	0.854	0.818	0.609	0.507	0.446	0.388	0.332	0.220	11
14	1.000	1.000	1.000	0.883	0.833	0.786	0.730	0.662	0.132	7
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.936	0.023	2

Table 6.2: Results of Data Set 1 Method 2

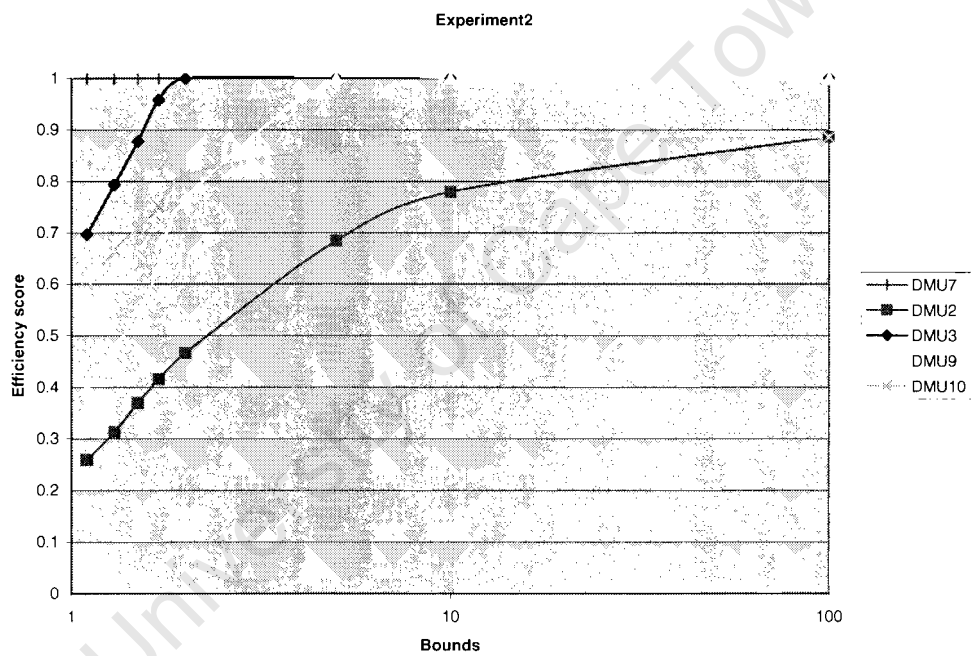


Figure 6.2: Graph of Data Set 1 Method 2

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	1.000	1.000	0.920	0.817	0.743	0.657	0.577	0.167	11
2	0.892	0.829	0.765	0.606	0.556	0.513	0.456	0.386	0.184	15
3	1.000	1.000	1.000	1.000	0.943	0.876	0.807	0.728	0.105	7
4	1.000	1.000	1.000	1.000	1.000	1.000	0.865	0.723	0.103	6
5	1.000	1.000	1.000	0.854	0.785	0.712	0.629	0.536	0.181	13
6	0.740	0.727	0.712	0.664	0.648	0.633	0.615	0.592	0.054	3
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
8	1.000	1.000	1.000	1.000	0.865	0.768	0.661	0.535	0.182	14
9	1.000	1.000	1.000	1.000	0.886	0.776	0.653	0.546	0.180	12
10	0.885	0.873	0.859	0.821	0.810	0.783	0.735	0.674	0.073	5
11	1.000	1.000	1.000	1.000	0.983	0.922	0.821	0.708	0.109	8
12	1.000	1.000	1.000	1.000	1.000	0.988	0.917	0.833	0.061	4
13	0.883	0.865	0.846	0.766	0.738	0.697	0.612	0.523	0.127	9
14	1.000	1.000	1.000	0.824	0.766	0.720	0.663	0.596	0.163	10
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.950	0.018	2

Table 6.3: Results of Data Set 1 Method 3

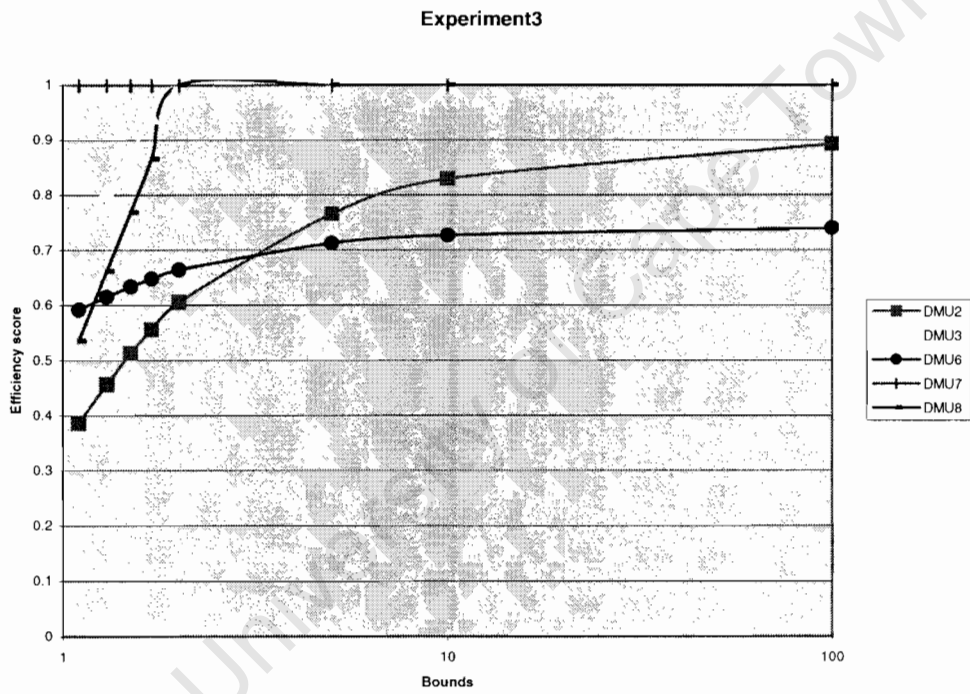


Figure 6.3: Graph of Data Set 1 Method 3

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1.000	0.464	0.505	0.558	0.223	0.421	0.657	0.676	1.000	0.610	0.778	0.511	0.484	0.610	0.529
2	0.708	0.794	0.378	0.543	0.206	0.395	1.000	0.626	1.000	0.611	0.670	0.555	0.414	0.383	1.000
3	0.522	0.288	1.000	0.222	0.365	0.332	0.316	0.172	0.521	0.177	0.242	0.654	0.298	0.257	0.526
4	0.667	0.351	0.180	1.000	0.332	0.342	0.617	0.684	0.642	0.648	0.934	0.343	0.614	0.287	0.283
5	0.449	0.270	0.328	0.676	1.000	0.474	0.767	0.317	0.322	0.450	0.537	1.000	0.493	0.332	0.590
6	0.929	0.495	0.510	0.778	0.616	0.700	1.000	0.873	0.678	0.758	1.000	1.000	0.589	0.556	1.000
7	0.216	0.243	0.273	0.222	0.247	0.233	1.000	0.158	0.297	0.376	0.245	0.876	0.160	0.405	0.866
8	0.355	0.144	0.109	0.252	0.075	0.215	0.434	1.000	0.231	0.475	0.547	0.200	0.149	0.440	0.374
9	0.725	0.345	0.598	0.346	0.157	0.254	0.360	0.338	1.000	0.329	0.431	0.333	0.371	0.349	0.287
10	0.674	0.308	0.263	0.650	0.338	0.449	1.000	1.000	0.514	0.878	1.000	0.741	0.390	0.649	0.617
11	0.708	0.333	0.213	0.672	0.216	0.393	0.750	1.000	0.562	0.775	1.000	0.402	0.410	0.452	0.445
12	0.318	0.160	0.648	0.204	0.385	0.254	0.534	0.123	0.384	0.277	0.224	1.000	0.210	0.515	0.470
13	1.000	0.570	0.532	1.000	0.857	0.690	0.921	0.727	0.863	0.706	0.997	1.000	0.829	0.453	0.809
14	0.182	0.075	0.214	0.184	0.294	0.293	0.937	0.335	0.111	0.543	0.332	0.967	0.108	1.000	0.867
15	0.379	0.475	0.335	0.279	0.226	0.316	0.672	0.276	0.426	0.289	0.315	0.575	0.240	0.224	1.000
e_k	0.559	0.323	0.363	0.470	0.324	0.362	0.712	0.522	0.539	0.502	0.589	0.654	0.352	0.422	0.619

Table 6.4: Cross-efficiencies (Method 4) for Data Set 1

	E_{kk}	e_j	Ranks	M_k
DMU1	1.000	0.559	5	0.788
DMU2	0.794	0.323	15	1.459
DMU3	1.000	0.363	11	1.754
DMU4	1.000	0.470	9	1.126
DMU5	1.000	0.324	14	2.086
DMU6	0.700	0.362	12	0.937
DMU7	1.000	0.712	1	0.405
DMU8	1.000	0.522	7	0.916
DMU9	1.000	0.539	6	0.854
DMU10	0.878	0.502	8	0.749
DMU11	1.000	0.589	4	0.697
DMU12	1.000	0.654	2	0.529
DMU13	0.829	0.352	13	1.356
DMU14	1.000	0.422	10	1.368
DMU15	1.000	0.619	3	0.616

Table 6.5: Summarized results from Method 4 for Data Set 1

Data Set 1	Method 1	Method 2	Method 3	Method 4
DMU1	NRE	NRE	NRE	HAE—LAE
DMU2	NRI	NRI	NRI	LAI
DMU3	NRE	RE—NRE	RE—NRE	LAE
DMU4	NRE	NRE	RE	LAE
DMU5	NRE	NRE	NRE	LAE
DMU6	RI	RI	RI	LAI
DMU7	RE	RE	RE	HAE
DMU8	NRE	NRE	NRE	HAE—LAE
DMU9	RE—NRE	NRE	NRE	HAE—LAE
DMU10	RI	RI	RI	HAI—LAI
DMU11	NRE	NRE	NRE	HAE—LAE
DMU12	RE	RE	RE	HAE
DMU13	NRI	NRI	NRI	LAI
DMU14	NRE	NRE	NRE	LAE
DMU15	RE	RE	RE	HAE

Table 6.6: Summary Table for Data Set 1

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1		
1	1.000	0.908	0.837	0.740	0.668	0.615	0.559	0.498	0.176	12
2	1.000	1.000	1.000	1.000	0.981	0.900	0.815	0.721	0.107	4
3	0.568	0.503	0.482	0.422	0.369	0.332	0.295	0.257	0.109	5
4	0.832	0.818	0.806	0.743	0.647	0.578	0.505	0.429	0.154	8
5	0.530	0.507	0.485	0.407	0.368	0.340	0.310	0.277	0.095	3
6	1.000	0.873	0.808	0.623	0.576	0.543	0.507	0.468	0.194	13
7	1.000	1.000	1.000	0.928	0.820	0.744	0.666	0.585	0.165	10
8	0.629	0.605	0.583	0.457	0.405	0.368	0.329	0.286	0.133	6
9	1.000	0.914	0.850	0.681	0.653	0.632	0.611	0.588	0.157	9
10	0.582	0.535	0.517	0.464	0.443	0.426	0.408	0.387	0.068	2
11	1.000	1.000	1.000	1.000	0.922	0.821	0.720	0.619	0.149	7
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
13	1.000	1.000	1.000	0.968	0.845	0.756	0.663	0.564	0.172	11

Table 6.7: Results of Data Set 2 Method 1

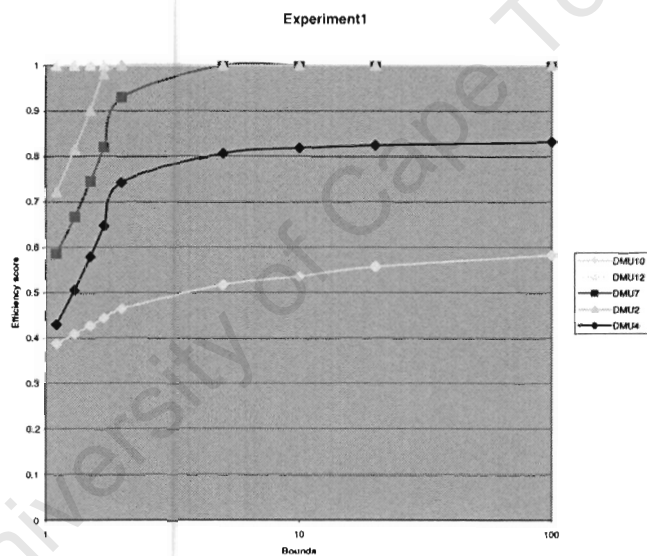


Figure 6.4: Graph of Data Set 2 Method 1

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	1.000	1.000	0.944	0.908	0.876	0.839	0.792	0.080	4
2	1.000	1.000	1.000	0.410	0.333	0.284	0.238	0.194	0.372	13
3	0.581	0.565	0.548	0.489	0.454	0.429	0.401	0.364	0.080	5
4	0.828	0.769	0.631	0.316	0.243	0.198	0.157	0.118	0.289	11
5	0.533	0.521	0.507	0.454	0.428	0.397	0.352	0.293	0.085	6
6	1.000	1.000	0.954	0.657	0.580	0.531	0.478	0.394	0.249	10
7	1.000	1.000	1.000	0.945	0.874	0.820	0.755	0.605	0.143	7
8	0.628	0.577	0.489	0.302	0.250	0.208	0.169	0.132	0.193	9
9	1.000	1.000	0.966	0.753	0.693	0.655	0.618	0.580	0.178	8
10	0.588	0.574	0.558	0.502	0.488	0.476	0.462	0.439	0.056	3
11	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
13	1.000	1.000	0.990	0.545	0.428	0.348	0.275	0.207	0.344	12

Table 6.8: Results of Data Set 2 Method 2

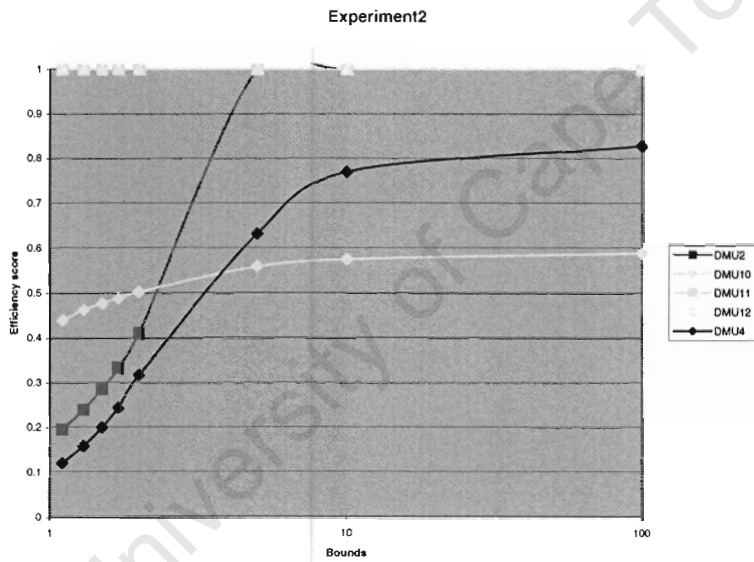


Figure 6.5: Graph of Data Set 2 Method 2

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	1.000	1.000	0.900	0.843	0.797	0.748	0.699	0.121	9
2	1.000	1.000	1.000	0.553	0.474	0.420	0.366	0.310	0.306	13
3	0.582	0.572	0.561	0.529	0.515	0.505	0.494	0.480	0.038	4
4	0.830	0.805	0.768	0.493	0.390	0.326	0.264	0.206	0.256	11
5	0.533	0.524	0.514	0.484	0.470	0.457	0.441	0.422	0.041	5
6	1.000	1.000	1.000	0.911	0.877	0.850	0.819	0.779	0.088	8
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.001	2
8	0.630	0.609	0.575	0.450	0.409	0.349	0.289	0.231	0.151	10
9	1.000	1.000	1.000	0.948	0.926	0.906	0.883	0.852	0.058	6
10	0.588	0.577	0.548	0.496	0.486	0.478	0.427	0.371	0.074	7
11	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.908	0.032	3
13	1.000	1.000	1.000	0.739	0.654	0.546	0.443	0.345	0.264	12

Table 6.9: Results of Data Set 2 Method 3

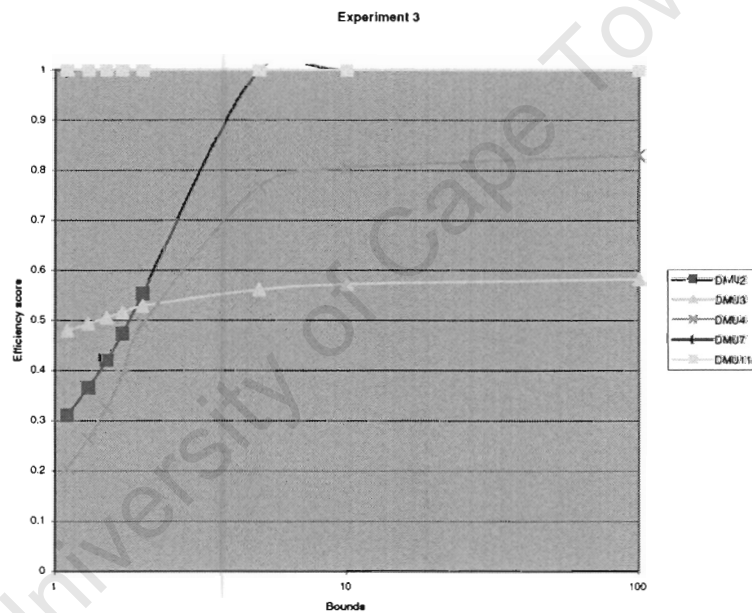


Figure 6.6: Graph of Data Set 2 Method 3

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1.000	0.064	0.444	0.135	0.277	0.431	0.728	0.133	0.327	0.488	1.000	0.822	0.232
2	0.067	1.000	0.023	0.075	0.025	0.105	0.085	0.038	0.206	0.035	0.127	0.456	0.104
3	0.906	0.193	0.539	0.546	0.381	0.552	1.000	0.372	0.463	0.506	1.000	1.000	0.711
4	0.651	0.837	0.477	0.820	0.449	0.642	1.000	0.501	0.586	0.348	0.804	0.601	1.000
5	0.740	1.000	0.471	0.797	0.492	0.758	1.000	0.481	0.778	0.488	0.826	1.000	1.000
6	0.335	0.765	0.273	0.388	0.159	1.000	0.947	0.372	0.939	0.166	0.718	1.000	0.444
7	0.408	0.269	0.325	0.446	0.166	0.485	1.000	0.394	0.317	0.165	1.000	0.425	0.526
8	0.654	0.712	0.467	0.777	0.463	0.599	1.000	0.606	0.747	0.420	1.000	0.735	0.935
9	0.553	0.931	0.393	0.625	0.318	0.684	1.000	0.543	0.915	0.343	1.000	1.000	0.746
10	0.997	0.158	0.462	0.314	0.339	0.523	0.846	0.224	0.419	0.562	1.000	1.000	0.466
11	0.116	0.011	0.102	0.110	0.026	0.025	0.151	0.063	0.014	0.017	1.000	0.017	0.148
12	0.056	0.192	0.026	0.025	0.025	0.272	0.095	0.025	0.406	0.061	0.058	1.000	0.032
13	0.653	0.567	0.269	0.680	0.227	0.163	0.487	0.248	0.166	0.221	1.000	0.340	1.000
e_k	0.511	0.475	0.311	0.410	0.238	0.437	0.695	0.283	0.014	0.018	0.083	0.028	0.083

Table 6.10: Cross-efficiencies (Method 4) for Data Set 2

	E_{kk}	e_j	Ranks	M_k
DMU1	1.000	0.511	5	0.956
DMU2	1.000	0.475	6	1.106
DMU3	0.539	0.311	10	0.732
DMU4	0.820	0.410	9	1.002
DMU5	0.492	0.238	13	1.067
DMU6	1.000	0.437	8	1.290
DMU7	1.000	0.695	3	0.439
DMU8	0.606	0.283	11	1.142
DMU9	0.915	0.447	7	1.045
DMU10	0.562	0.271	12	1.069
DMU11	1.000	0.794	1	0.259
DMU12	1.000	0.700	2	0.430
DMU13	1.000	0.529	4	0.892

Table 6.11: Summarized results from Method 4 for Data Set 2

Data Set 2	Method 1	Method 2	Method 3	Method 4
DMU1	NRE	NRE	NRE	LAE
DMU2	RE—NRE	NRE	NRE	LAE
DMU3	NRI	RI	RI	LAI
DMU4	NRI	NRI	NRI	HAI
DMU5	RI	RI	RI	LAI
DMU6	NRE	NRE	NRE	LAE
DMU7	NRE	NRE	RE	HAE
DMU8	NRI	NRI	NRI	LAI
DMU9	NRE	NRE	NRE	LAI
DMU10	RI	RI	NRI	LAI
DMU11	RE—NRE	RE	RE	HAE
DMU12	RE	RE	RE	HAE
DMU13	NRE	NRE	NRE	LAE

Table 6.12: Summary Table for Data Set 2

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	0.992	0.930	0.820	0.799	0.783	0.765	0.745	0.103	9
2	1.000	0.994	0.953	0.849	0.827	0.809	0.789	0.765	0.095	7
3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
5	1.000	0.990	0.921	0.804	0.780	0.761	0.739	0.713	0.115	11
6	1.000	1.000	1.000	0.950	0.930	0.914	0.897	0.877	0.050	4
7	1.000	1.000	1.000	0.708	0.662	0.629	0.593	0.555	0.197	15
8	0.999	0.988	0.978	0.955	0.949	0.944	0.939	0.932	0.025	3
9	0.999	0.965	0.918	0.806	0.783	0.766	0.745	0.722	0.107	10
10	1.000	1.000	0.931	0.692	0.656	0.630	0.602	0.571	0.184	14
11	0.986	0.877	0.792	0.645	0.618	0.598	0.576	0.551	0.160	13
12	0.999	0.986	0.958	0.904	0.892	0.884	0.874	0.862	0.053	5
13	1.000	0.965	0.895	0.745	0.714	0.689	0.660	0.627	0.145	12
14	0.999	0.986	0.958	0.904	0.892	0.884	0.874	0.862	0.053	5
15	0.886	0.857	0.798	0.699	0.681	0.667	0.652	0.635	0.098	8

Table 6.13: Results of Data Set 3 Method 1

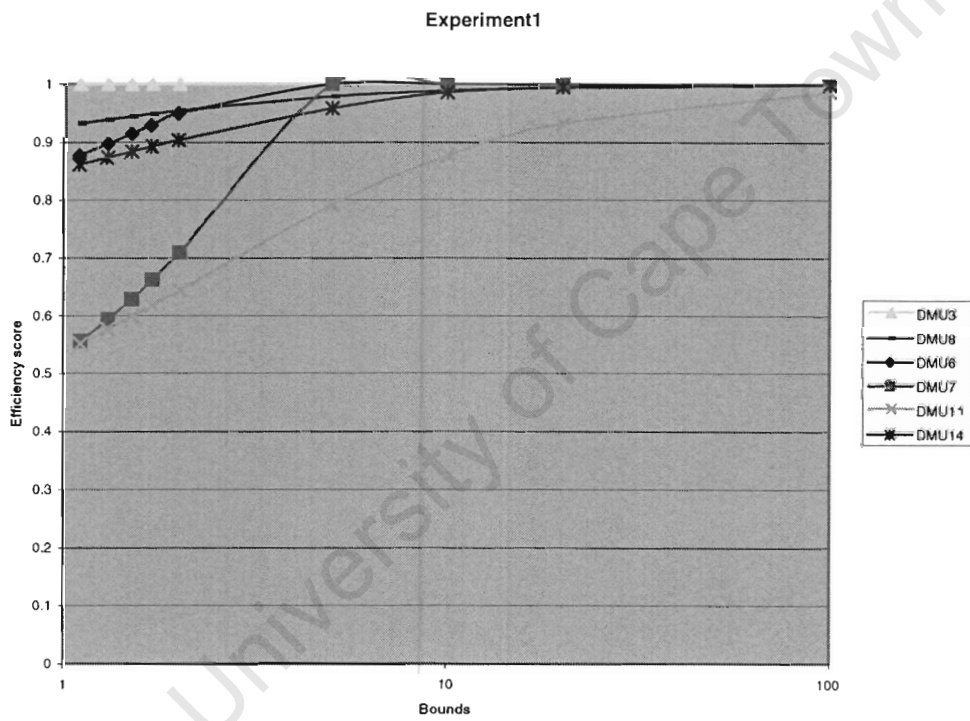


Figure 6.7: Graph of Data Set 3 Method 1

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	1.000	1.000	0.910	0.885	0.864	0.838	0.805	0.078	6
2	1.000	1.000	0.999	0.907	0.882	0.837	0.786	0.738	0.102	8
3	1.000	1.000	1.000	1.000	1.000	1.000	0.967	0.923	0.028	3
4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
5	1.000	1.000	0.988	0.868	0.837	0.812	0.773	0.726	0.108	9
6	1.000	1.000	1.000	0.977	0.899	0.837	0.780	0.725	0.110	10
7	1.000	1.000	1.000	0.715	0.611	0.543	0.485	0.436	0.243	15
8	0.999	0.994	0.989	0.972	0.967	0.962	0.956	0.948	0.019	2
9	0.999	0.995	0.982	0.879	0.850	0.806	0.754	0.702	0.114	11
10	1.000	1.000	1.000	0.784	0.675	0.605	0.544	0.488	0.215	14
11	0.993	0.930	0.869	0.712	0.667	0.632	0.594	0.551	0.165	12
12	0.999	0.995	0.989	0.961	0.950	0.940	0.926	0.894	0.037	4
13	1.000	0.983	0.916	0.739	0.698	0.655	0.605	0.553	0.175	13
14	0.999	0.995	0.989	0.961	0.950	0.940	0.926	0.894	0.037	4
15	0.886	0.884	0.875	0.788	0.763	0.742	0.718	0.689	0.079	7

Table 6.14: Results of Data Set 3 Method 2

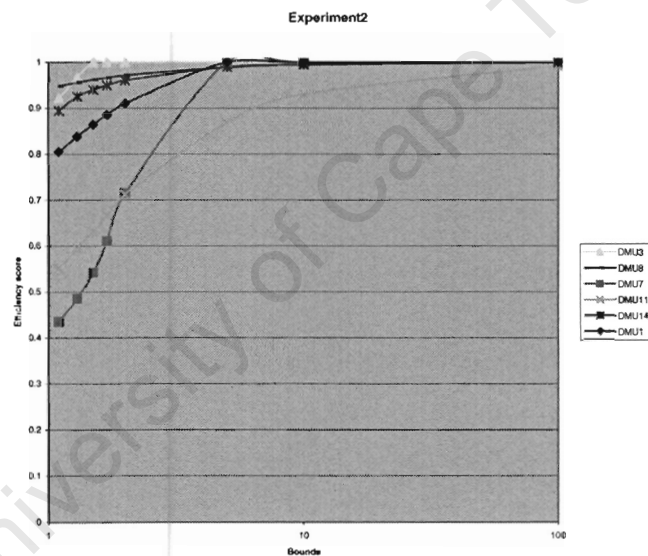


Figure 6.8: Graph of Data Set 3 Method 2

DMUs	B								St. Dev.	Ranks
	100	10	5	2	1.7	1.5	1.3	1.1		
1	1.000	1.000	1.000	0.961	0.942	0.927	0.907	0.882	0.046	6
2	1.000	1.000	1.000	0.957	0.939	0.924	0.878	0.817	0.066	8
3	1.000	1.000	1.000	1.000	1.000	1.000	0.984	0.930	0.025	5
4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.000	1
5	1.000	1.000	1.000	0.932	0.909	0.889	0.865	0.824	0.068	9
6	1.000	1.000	1.000	0.999	0.952	0.897	0.807	0.704	0.111	11
7	1.000	1.000	1.000	0.798	0.710	0.623	0.547	0.485	0.212	14
8	1.000	0.997	0.994	0.985	0.982	0.980	0.977	0.973	0.010	2
9	1.000	0.996	0.991	0.931	0.911	0.893	0.848	0.784	0.077	10
10	1.000	1.000	1.000	0.768	0.680	0.602	0.529	0.455	0.224	15
11	0.995	0.949	0.901	0.757	0.723	0.695	0.663	0.625	0.140	13
12	1.000	0.997	0.994	0.985	0.978	0.972	0.965	0.947	0.018	3
13	1.000	0.991	0.968	0.831	0.797	0.769	0.727	0.667	0.128	12
14	1.000	0.997	0.994	0.985	0.978	0.972	0.965	0.947	0.018	3
15	0.887	0.884	0.882	0.835	0.818	0.803	0.785	0.761	0.049	7

Table 6.15: Results of Data Set 3 Method 3

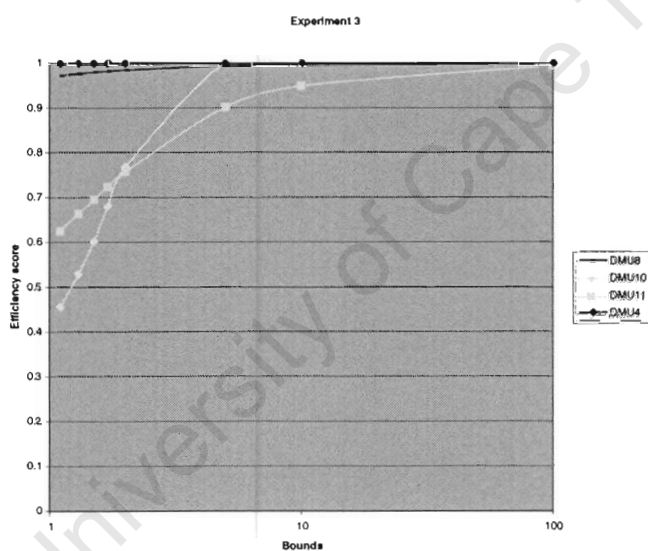


Figure 6.9: Graph of Data Set 3 Method 3

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.991	0.993	0.702	0.824	0.992	0.976	0.992	0.883
2	0.986	1.000	0.983	0.969	0.996	1.000	0.977	0.960	0.859	0.800	0.759	0.973	0.983	0.973	0.805
3	0.581	0.602	1.000	1.000	0.523	0.779	0.292	0.841	0.598	0.468	0.411	0.737	0.396	0.737	0.503
4	0.538	0.521	0.960	1.000	0.453	0.695	0.270	0.840	0.517	0.467	0.357	0.707	0.309	0.707	0.465
5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989	0.992	0.705	0.825	0.990	0.976	0.990	0.882
6	0.663	0.879	0.900	0.698	0.768	1.000	0.352	0.689	0.869	0.268	0.513	0.783	0.800	0.783	0.584
7	0.763	0.572	0.630	0.812	0.654	0.501	1.000	0.798	0.497	0.841	0.597	0.689	0.571	0.689	0.627
8	0.568	0.554	0.970	1.000	0.486	0.727	0.294	0.991	0.550	0.355	0.324	0.842	0.346	0.842	0.501
9	0.993	1.000	0.992	0.986	0.998	1.000	0.990	0.977	0.993	0.693	0.821	0.983	0.978	0.983	0.877
10	0.826	0.638	0.750	0.942	0.710	0.586	1.000	0.805	0.627	1.000	0.772	0.697	0.597	0.697	0.712
11	0.926	0.835	0.906	1.000	0.867	0.816	1.000	0.847	0.825	0.997	0.883	0.796	0.782	0.796	0.800
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989	0.938	0.745	0.804	0.990	0.976	0.990	0.857
13	0.975	0.977	1.000	1.000	0.972	0.988	0.955	0.984	0.847	0.800	0.747	0.978	0.943	0.978	0.800
14	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989	0.938	0.745	0.804	0.990	0.976	0.990	0.857
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990	0.992	0.704	0.825	0.991	0.976	0.991	0.882
e_k	0.844	0.827	0.935	0.958	0.816	0.864	0.795	0.906	0.789	0.664	0.670	0.868	0.760	0.868	0.725

Table 6.16: Cross-efficiencies (Method 4) for Data Set 3

	E_{kk}	e_k	Ranks	M_k
DMU1	1.000	0.844	7	0.185
DMU2	1.000	0.827	8	0.209
DMU3	1.000	0.935	2	0.070
DMU4	1.000	0.958	1	0.044
DMU5	1.000	0.816	9	0.225
DMU6	1.000	0.864	6	0.158
DMU7	1.000	0.795	10	0.258
DMU8	0.991	0.906	3	0.093
DMU9	0.993	0.789	11	0.259
DMU10	1.000	0.664	15	0.507
DMU11	0.883	0.670	14	0.318
DMU12	0.990	0.868	4	0.141
DMU13	0.943	0.760	12	0.240
DMU14	0.990	0.868	4	0.141
DMU15	0.882	0.725	13	0.216

Table 6.17: Summarized results from Method 4 for Data Set 3

Data Set 3	Method 1	Method 2	Method 3	Method 4
DMU1	NRE	RE—NRE	NRE	LAE
DMU2	NRE	NRE	NRE	LAE
DMU3	RE	RE	RE	HAE
DMU4	RE	RE	RE	HAE
DMU5	NRE	NRE	NRE	LAE
DMU6	RE—NRE	NRE	NRE	HAE—LAE
DMU7	NRE	NRE	NRE	LAE
DMU8	RI	RI	RE—NRE	HAI
DMU9	NRI	NRI	NRE	LAI
DMU10	NRE	NRE	NRE	LAE
DMU11	NRI	NRI	NRI	LAI
DMU12	RI—NRI	RI—NRI	RE—NRE	HAI—LAI
DMU13	NRE	NRE	NRE	LAI
DMU14	RI—NRI	RI—NRI	RE—NRE	HAI—LAI
DMU15	NRI	NRI	NRI	LAI

Table 6.18: Summary Table for Data Set 3

University of Cape Town

Final Conclusions

The efficiency score in DEA is an expression of the performance of a DMU and it is a function of the input and output weights. A change in the weights leads to a different efficiency score. The scope of changes in an efficiency score of a DMU as a result of weight changes reveals the efficiency robustness of this DMU.

DEA can yield an efficiency score that is the result of an extreme choice of weights intended only to maximize its efficiency score and not representing the real performance of the DMU in question. In this sense if care is not taken, the resulting efficiency measure can be misleading.

Weight restriction approaches aim to solve the problems caused by unconstrained and total weight flexibility. However, explicit weight bounds may not be available because price or preference information may be unobtainable (Sexton et al., 1986), which is why many techniques of weight restrictions have been developed. Problems with these techniques include their often subjective nature, absence of feasible solutions, especially for absolute weight restrictions.

In place of explicit bounds on weights, we have proposed a structured robustness analysis in order to investigate the effects of weights changes on the efficiency. This also provides a test of the strength of a DMU's efficiency standing. In this analysis, the permissible weight ranges (or the weight space) for each factor is systematically and progressively tightened until the weight changes affect the efficiency of the DMU. DMUs whose efficiency remains the same or changes only marginally are considered robustly efficient.

Our robustness analysis includes 4 methods designed to test the robustness of the efficiency standing of DMUs. The aim of this robustness analysis is not to try to obtain

a single efficiency score for DMU0 but rather to evaluate the standard DEA measure, in order to determine whether the DMU's efficiency standing is robust. We should not expect that a single efficiency measure is sufficient to display the full extent of a DMU's situation. We gain more insight through a robustness analysis which generates a range of efficiency scores. In this way an inefficient DMU may have a more robust efficiency standing than an efficient DMU.

We briefly review each method and its practical meaning. Thereafter, we discuss the results and look at the insights from the literature review. Finally, further areas of investigation are suggested.

7.1 Methods of robustness analysis

7.1.1 Objectives of the methods used

The following methods were used in the robustness analysis:

- Method 1: Systematic changes to assurance regions
- Method 2: Systematic changes to virtual factor restrictions
- Method 3: Systematic restrictions of virtual factors to values not too different from the population averages
- Method 4: Cross-efficiency

1. Systematic changes to assurance regions:

The weight ratios express trade-offs between inputs or outputs which reflect the operating conditions prevailing at a DMU. The ability to make trade-offs is essential for economic purposes such as resources allocation, maximization of profit, etc. The ratio of input or output weights is called the input or output mix.

In this approach, we monitor how changes in operating conditions modifies the efficiency. The DMUs that are indifferent to this change have a robust efficiency standing. In other words, robustly efficient DMUs can function efficiently under diverse operating conditions.

2. Systematic changes to virtual weights restrictions:

The virtual input i (output r) is the product of input i (output r) weight and the input i (output r) level. Total virtual input (output) is the sum all virtual inputs

(outputs). The ratio of virtual input i (output r) to the total virtual input (output) conveys the importance of input i (output r). This approach places bounds on these ratios. A change in virtual weights represents a change of values or policies in the DMU. We seek to find how a change in such values or policies in the DMU can affect the DMU's efficiency. DMUs that are more or less insensitive to those changes are believed to have a robust efficiency standing.

3. Systematic restrictions of virtual factors to values not too different from the population averages:

This approach is similar to that of method 2, but restrictions are expressed in terms of deviations of the ratios from the population averages or norm. Again here the DMUs whose efficiency survive the changes in values or policies are considered robustly efficient.

4. Cross-Efficiency:

This approach is based on a concept of peer-appraisal, in which each DMU is assessed relative to other DMUs i.e its efficiency score is calculated from other DMUs' weights (Green and Doyle, 1994).

7.1.2 Pattern of efficiency

For the first three methods, the different patterns of a DMU's efficiencies when faced with weight changes categorised into the following quadrants:

- Robustly efficient DMU (RE): An initially efficient DMU whose efficiency score decreases minimally when the feasible weights interval is systematically reduced.
- Non-robustly efficient DMUs (NRE): An initially efficient DMU whose efficiency score decreases substantially when the feasible weights interval is systematically reduced.
- Inefficient DMUs with a robust efficiency standing (RI): An initially inefficient DMU whose efficiency score decreases minimally when all feasible weight intervals are systematically reduced.
- Inefficient DMUs with a non-robust efficiency standing (NRI): An initially inefficient DMU whose efficiency score decreases substantially when all feasible weight intervals are systematically reduced.

In case of the cross-efficiency analyses, DMUs may be categorized into the following four groups:

- Efficient DMU, highly appraised (HAE): An efficient DMU (self-evaluation) that has a high cross-efficiency.
- Efficient DMU, lowly appraised (LAE): An efficient DMU (self-evaluation) that has a low cross-efficiency.
- Inefficient DMUs, highly appraised(HAI): An inefficient DMU (self-evaluation) that has a high cross-efficiency.
- Inefficient DMUs, lowly appraised(HAI): An inefficient DMU (self-evaluation) that has a low cross-efficiency.

7.1.3 Discussion of the results

We noticed that in general there were only slight differences in the results given by the various methods and that they agreed to a large extent.

Methods 1, 2 and 3 are directly interested in the robustness analysis while method 4 looks at the rating of a DMU by other DMUs. Method 4 aims at a consensus between the DMUs while methods 1, 2 and 3 look at characteristics which are intrinsic to each DMU. Although the first three methods are all concerned with robustness analysis, there is a different rationale behind each method. Method 1 restricts the range of the ratios of weights while method 2 limits the ratios of the virtual weights. Method 3 ensures that each DMU is not too different from an “average DMU”. In all three case studies, the methods essentially agree on which DMUs should be allocated to the first group (RE-HAE) because the robustly efficient DMUs are also likely to be highly appraised by most other DMUs.

We have noticed that from a computational aspect, method 1 is simpler than methods 2 and 3. One can use the DEA solver LV (Cooper et al., 2000) for method 1 and find solutions for complex problems in a matter of a few minutes. Since all three methods yield essentially the same classification of DMUs, it makes sense to use the simplest version of the three methods, which is method 1.

Method 4 is not directly concerned with the robustness analysis of DMUs and deals with the cross-efficiencies of DMUs. Method 4 distances itself from the other 3 methods because it is primarily interested in the appraisal of a DMU by other DMUs (peer evaluation). From the results, there seem to be, in general, more borderline DMUs with method 4 than with the rest of the methods.

7.2 General Weight Restriction Methods

Our robustness analysis is based on concepts of weight restrictions. Weight restrictions followed the application of DEA into real world problems. Some DMUs were rated efficient because they put high weight values on less important factors while attaching low weight values to more important factors. Weight restrictions were therefore introduced to avoid extreme and unrealistic weights. Weight restriction methods operate either by direct modification of the Production Possibility Set or by restricting the ranges of relative valuation of inputs or outputs on the performance valuation side (Charnes et al., 1990). Three weight restriction methods have been employed in this study: direct weight restrictions, virtual weight restrictions and cross-efficiency. However, additional weight restriction methods exist, and many of these were presented in chapter 3.

Some of the more interesting methods of weight restriction in the literature included the following:

1. Direct Weight Restrictions

Direct relative or absolute weight restrictions can be placed on the weight factors, usually specifying a range of possible values rather than a single value. However, absolute weight restrictions are not easily determined and can lead to unfeasible problems. Relative weight restrictions, also called assurance regions, constrain the ratio of weight factors, i.e. a rate of substitution which demonstrates how a loss of one unit in one factor can be compensated by an increase of a number of units in another factor (Allen et al., 1997; Cooper et al., 2000). The efficiency of a DMU may then be optimized given the bounds on the proportional virtual inputs or outputs.

2. Virtual weight restrictions

Restrictions may also be applied to virtual weights (Sarrico and Dyson, 2004). The ratio of a virtual input or output to the total virtual input or output reveals the relative importance attached to that input or output and they are called respectively proportional virtual input or output weight. In this approach, bounds will be set on the proportional virtual weights to limit the importance taken by a factor. The efficiency of a DMU will be optimized given the bounds on the proportional virtual inputs or outputs.

3. Using unobserved DMUs to incorporate value judgments in DEA

Instead of specifying weight restrictions directly, unobserved DMUs may be inserted into the production possibility set in order to extend the efficient frontier. An

equivalence between the assurance region and the unobserved DMUs approaches has been proved by Andersen and Petersen (1993) and by Thanassoulis and Allen (1998). For any set of weight restrictions, there exist a set of unobserved DMUs such that the efficiency assessed within the aggregate set of observed and unobserved DMUs will be the same as if it was assessed within the set of observed DMUs alone under weight restrictions (Andersen and Petersen, 1993; Thanassoulis and Allen, 1998).

4. Transformed data

We use expert opinion to identify those DMUs that are functioning efficiently. The weights of those DMUs are used to limit the weight space in such a way that the DMUs whose weight vectors are outside the restricted weight space are rated inefficient. This defines a primal problem. The corresponding dual problem corresponds to assessing DMUs in the basic DEA but with transformed data. The transformed inputs and outputs are the product of the weights of the appraised DMUs with respectively the observed inputs and outputs of DMUs (Charnes et al., 1990; Cooper et al., 2000).

In future research, robustness analysis may be extended to others of the above weight restriction methods.

7.3 Sensitivity Analyses

Our robustness analysis can be viewed as a form of sensitivity analysis. Other methods of sensitivity analyses have been reported in a number of papers which we briefly summarized in section 4.2. For emphasis, we will present them again in this section. The terms ‘robustness analysis’ or ‘sensitivity analysis’ were not explicitly mentioned in some papers even though they were based on this idea.

1. Simultaneous data changes approach

Charnes et al. (1994) suggest a sensitivity analysis based on simultaneous data changes. This approach seeks to find how data changes affect the efficiency of DMUs. The authors were concerned about the reliability of efficiency measures in case of data error. Efficient DMUs which remain so despite all data variations are called SA (Stable always). They qualify to be extremely efficient DMUs or SCSC (Strongly Complementary Slackness Condition) solution. Some initially efficient DMUs become inefficient and are replaced in the basis of efficient DMUs by the

inefficient DMUs which become efficient.

2. Envelopment approach

In Lang et al. (1995), an approach is developed which extends the efficient frontier in order to control how the efficient frontier envelops all the DMUs. Different envelopments yield different scores. Three types of envelopment methods are available: standard DEA, CEA (Controlled Envelopment Analysis) and CFA (Constrained Facet Analysis). Standard DEA makes the initial envelopment of the DMUs assessed with the requirement that the Production Possibility Set must be convex. If a DMU is not well enveloped, inefficient DMUs may be assessed as efficient. Standard DEA yields the upper bound on the efficiency measure. CEA aims to maximize the envelopment degree while CFA seeks to remove the slacks at each iteration while keeping past efficient peers. The envelopment degree is a measure of how well the efficient frontier covers or envelops the DMUs assessed and it is determined mainly by close observation of the production possibility set.

3. Multiple objective linear fractional programming

In Kornbluth (1991), multiple objective linear fractional programming (MOLFP) was used in DEA for sensitivity analysis. Many solutions determining the efficiency of DMUs were found. All solutions had to comply with the corporate policies of the industry. The solutions differed from one another by allowing for individual policies of firms that are not contrary to the corporate policies. Policies were incorporated in the solutions through weight restrictions. MOLFP ensures that the best solution is obtained for all DMUs simultaneously. A DMU is strongly efficient if it is efficient not only with its own policies but with other firms' policies. A strongly efficient DMU is efficient for many solutions. The efficiency of that DMU stands despite having been rated with different sets of weights. In other words, the efficiency of that DMU stands even when different sets of restrictions are placed on the weights. Different corporate policies can also be used to test the strength of the efficiency of DMUs.

4. Models approach

Another form of sensitivity analysis was conducted by Golany (1988). Different DEA models were used on a set of data and efficiency rankings across the models were obtained. DMUs were ranked according to their efficiency scores. A DMU has a robust efficiency standing if similar rankings are reached with different models. The strength of the efficiency standing of a DMU is tested as efficiency assessment

underlying assumptions are allowed to vary under different models.

5. Monte Carlo methods and visual displays

A sensitivity analysis involving probabilities was also suggested by Stewart (1996). This is a Monte Carlo approach in which uniformly distributed weights are generated randomly. The efficiency score is generated from each set of weights generated, leading to a probability distribution for the range of values for each DMU. The upper bounds on the range of efficiency scores is obviously the DEA efficiency measure because DEA yields the maximum efficiency score for a DMU. Simple box and whisker plots were used in conjunction with the Monte Carlo approach to represent the distributions over the weights. In this case, it was assumed that the weights followed the uniform distribution, but other distributions could have been used. This method attaches a probability level with each possible efficiency score. In this way an efficient solution (upper bound of 100%) may have most of its probability mass at a lower efficiency than another (inefficient) DMU. Such insights could not be obtained with the basic DEA.

Future research might compare insights from the above methods with those from the four methods examined in detail in the dissertation.

7.4 Research questions addressed

In section 4.1 a number of research questions were posed, some related to efficiency in DEA and others to methodology issues.

7.4.1 Questions concerning efficiency

The questions related to efficiency which received the main attention in this dissertation were questions 5 to 7. These have been addressed directly by the numerical studies. The answers are discussed at length in sections 6.2 and 6.3. Although questions 1 to 4 were less directly addressed, the discussion of the numerical results in sections 6.2 and 6.3 have produced many insights.

7.4.2 Questions concerning methodology

The questions related to methodology have all been addressed. We have been able to use a DMUs own weights to assess its efficiency robustness and assign the DMUs to a number of categories which represent different patterns of efficiency. By using a wide range of weight ratios for each factor, we have been able to take into account various weight restrictions that could have been specified by decision makers.

7.5 Future research

The following represent possible areas for future research:

7.5.1 Other approaches for robustness analysis

The robustness analysis conducted in this study could be improved by taking into account the following issues:

- The robustness analysis has assumed that inputs and outputs are not directly related. Sometimes, inputs and outputs may be related, in which case, it could be important to include ratios of inputs to outputs in the robustness analysis.
- Only direct cardinal measures were used for all inputs and outputs. We suggest that the robustness analysis could be extended to make use of the indices or ordinal data. In many problems involving efficiency analysis using DEA, certain factors may only be measurable on an ordinal scale. It may be only possible to rank DMUs according to an ordinal factor rather than being able to assign a specific numerical value to that factor. Qualitative measures are sometimes indispensable to an evaluation process. We must resist the temptation of undervaluing quality inputs for the purposes of computing efficiency (Sexton et al., 1986).
- One of the innovations of DEA is that it can handle exogenously fixed factors. Exogenously fixed factors are the factors whose levels are not determined at the discretion of the management of a DMU. The levels of these factors cannot be expanded or contracted. If we treat these factors as the factors under managerial control, we will reach wrong conclusions about the DMU's efficiency. We suggest that another robustness analysis approach be developed to handle exogenously fixed DMUs.
- Unobserved DMUs can progressively be inserted in the Production Possibility Set and efficiency changes traced. The insertion of Unobserved DMUs in the Production Possibility Space is equivalent to weight restrictions (Andersen and Petersen,

1993; Thanassoulis and Allen, 1998). A progressive insertion of unobserved DMUs corresponds to weight being progressively restricted. In this way, efficiency changes can be monitored.

7.5.2 Possible use of role model DMUs

We could view the DMUs selected as robustly efficient by the robustness analysis as the role models. These DMUs can be used to improve the cross-efficiency and cone-ratio approaches in the following:

- In the standard Cross-Efficiency, each DMU is evaluated by all the other DMUs. Some DMUs which do have a rather non-robust efficiency standing are allowed to participate in the process. As a consequence, the efficiency rating that they attach to other DMUs is likely to be unstable and we may question retaining those DMUs as assessors. We suggest that they be excluded from the assessment process and only the DMUs with a strong efficiency, either efficient or inefficient DMUs, be the only ones to take part in the Cross-efficiency evaluation. It could also be possible to use the DMUs with a high overall cross-efficiency score as assessors. We believe that using such DMUs in the evaluation process could lead to more reliable cross-efficiency scores. In this way, the average cross-efficiency could be more meaningful.
- The Cone-Ratio model is one of the methods of weight restrictions. Experts or decision makers are allowed to select the DMUs which they judged to be the best performers. The optimal weights of those DMUs are used to restrict all the weights. Thus all DMUs are assessed through the lenses of those DMUs highly appraised by experts or decision makers. This type of efficiency is very interesting and could be closer to reality. We suggest that the robustly efficient DMUs be used to rate the other DMUs in a Cone-Ratio model. This procedure could be very useful especially when no preference information is available.

7.5.3 Further investigation

The following issues need further attention:

- Target levels are very important to a DMU because they reveal the potential for improvement within that DMU. If a DMU reaches all his target levels, it is going to become efficient. It could be interesting to trace how successive weight restrictions affect the target levels for a DMU. A DMU whose target levels are the current levels through successive weight restrictions are certainly strongly efficient.

- Value judgements can be used to rank order factors. The weight restrictions in our robustness analysis treats all weight factors the same by imposing the same bounds on each one of them. A robustness analysis which takes into account value judgements concerning the importance of the factors can thus be conducted.

7.6 A Final comment

DEA is a very promising field with much potential that has yet to be uncovered. Initially, DEA was used to assess non-profit organizations. Then, the applicability was expanded to include banks, financial services, hospitals, police services, regulation of water, allocation of industrial sites, etc. We believe that DEA can address a wider range of problems with its large arsenal of tools and techniques and be applied to new areas.

DEA can also be used in conjunction with other methods such as MCDA, regression analysis, multivariate statistical analysis, non-parametric tests, etc. in order to enhance its ability to handle complex problems. Initial results suggest that cluster analysis can guide and inform the process of assigning the DMUs to their respective groups in method 4. While the initial emphasis of DEA was on the assessment of efficiency of DMUs, it can be applied to problems of choice which are a strong focus of MCDA (Stewart and Belton, 1999; Stewart, 1996). We should explore every avenue to find a more efficient combination of DEA with other existing methods.

The wide range of future research topics suggested here indicates that many possible achievements lie within the reach of this powerful method, DEA. In this study, our intention was not to replace the use of preference information in DEA. Rather, we wanted to show that objective information derived directly from DEA can complement and even monitor the use of value judgements in DEA especially in cases where there is little or no preference information. DEA practitioners should seek to learn from the extensive experience of MCDA analysts and researchers in eliciting and working with value judgements (Stewart and Belton, 1999).

University of Cape Town

Bibliography

- R. Allen, A. Athanassopoulos, R. G. Dyson, and E. Thanassoulis. Weights restrictions and value judgements in data envelopment analysis: Evolution, development and future directions. *Annals of operations research*, 73:13–34, 1997.
- P. Andersen and N. C. Petersen. A procedure for ranking efficient units in data envelopment analysis. *Management Science*, 39:1261–1264, 1993.
- R. D. Banker and A. Maindiratta. Piecewise loglinear estimation of efficient production surfaces. *Management Science*, 32:126–135, 1986.
- R. D. Banker and R. C. Morey. Efficiency analysis for exogenously fixed inputs and outputs. *Operations Research*, 34:513–521, 1986.
- R. D. Banker and R. M. Thrall. Estimation of returns to scale using data envelopment analysis. *European Journal of Operational Research*, 62:74–84, 1992.
- R. D. Banker, A. Charnes, and W. W. Cooper. Some models for estimating technical & scale inefficiency in data envelopment analysis. *Management Science*, 30:1078–1092, 1984.
- J. E. Beasley. Comparing university departments. *OMEGA*, 18:171–183, 1990.
- E. Behrman. Data envelopment analysis (DEA): Efficiency analysis of the departments of science faculty of the university of cape town a further investigation into robustness through weights restrictions in DEA. Technical report, Department of Mathematics, University of Cape Town, 2002. Honours project report.
- A. Boussofiane, R. G. Dyson, and E. Thanassoulis. Applied data envelopment analysis. *European Journal of Operational Research*, 52:1–15, 1991.
- W. F. Bowlin, A. Charnes, W. W. Cooper, and H. D. Sherman. Data envelopment analysis and regression approaches to efficiency estimation and evaluation. *Annals of Operations Research*, 2:113–138, 1985.

- A. Charnes, W. W. Cooper, and E. Rhodes. Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2:429–444, 1978.
- A. Charnes, C. T. Clark, W. W. Cooper, and B. Golany. A development study of data envelopment analysis in measuring the efficiency of maintenance units in the u.s. air forces. *Annals of Operations Research*, 2:95–112, 1985.
- A. Charnes, W. W. Cooper, and Z. M. Huang. Polyhedral cone-ratio DEA models with an illustrative application to large commercial banks. *Journal of Econometrics*, 46: 73–91, 1990.
- A. Charnes, W. W. Cooper, and A. Y. Lewin. *Data Envelopment Analysis: Theory, Methodology, and Application*. Kluwer Academic Publishers, Massachusetts, 1994.
- W. W. Cooper, L. M. Seiford, and K. Tone. *DATA ENVELOPMENT ANALYSIS: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*. Kluwer Academic Publishers, Massachusetts, 2000.
- R. G. Dyson and E. Thanassoulis. Reducing weight flexibility in data envelopment analysis. *Journal of the Operational Research Society*, 39:563–576, 1988.
- B. Golany. A note on including ordinal relations among multipliers in data envelopment analysis. *Management Science*, 34:1029–1033, 1988.
- B. Golany. Alternate methods of treating factor weights in DEA. *OMEGA*, 21:99–109, 1992.
- B. Golany. Alternate methods of treating factor weights in DEA. *OMEGA*, 21:99–109, 1993.
- B. Golany and Y. Roll. An application procedure for data envelopment analysis. *Omega*, 17:237–250, 1988.
- R. H. Green and J. Doyle. Efficiency and cross-efficiency in DEA: derivations, meanings and uses. *Journal of the Operational Research Society*, 45:567–578, 1994.
- M. Halme and P. Korhonen. Restricting weights in value efficiency analysis. *European Journal of Operational Research*, 126:175–188, 2000.
- T. Joro, P. Korhonen, and S. Zionts. An interactive approach to improve estimates of value efficiency in data envelopment analysis. *European Journal of Operational Research*, 149:688–699, 2003.
- J. S. H. Kornbluth. Analysing policy effectiveness using cone restricted data envelopment analysis. *Journal of the Operational Research Society*, 42:1097–1104, 1991.
- P. Lang, O. R. Yolalan, and O. Kettani. Controlled envelopment by face extension in DEA. *Journal of the Operational Research Society*, 46:473–491, 1995.

- V. V. Podinovski. Suitability and redundancy of non-homogeneous weight restrictions for measuring the relative efficiency in DEA. *European Journal of Operational Research*, 154:380–395, 2004a.
- V. V. Podinovski. Production trade-offs and weight restrictions in data envelopment analysis. *Journal of the Operational Research Society*, 55:1311–1322, 2004b.
- Y. Roll, W. D. Cook, and B. Golany. Controlling factor weights in data envelopment analysis. *IIE transactions*, 23:2–9, 1991.
- C. S. Sarrico and R. G. Dyson. Restricting virtual weights in data envelopment analysis. *European Journal of Operational Research*, 159:17–34, 2004.
- T. R. Sexton, R. H. Silkman, and A. J. Hogan. *Measuring efficiency: an assessment of data envelopment analysis*. Jossey-Bass, San Francisco, 1986.
- T. J. Stewart. Relationship between data envelopment analysis and multicriteria decision analysis. *Journal of the Operational Research Society*, 47:654–665, 1996.
- T. J. Stewart and V. Belton. DEA and MCDA: competing or complementary approaches? In N. Meskens and M. Roubens, editors, *Advances in Decision Analysis*, pages 87–104. Kluwer Academic Publishers, Dordrecht, 1999.
- D. B. Sun. Evaluation of managerial performance of large commercial banks by data envelopment analysis. Technical report, Graduate School of Business, University of Texas, 1987. Unpublished Ph.D. dissertation.
- E. Thanassoulis. *Theory and Application of Data Envelopment Analysis*. Kluwer Academic Publishers, Massachusetts, 2001.
- E. Thanassoulis. Duality in data envelopment analysis under constant returns to scale. *IMA Journal of Mathematics Applied in Business & Industry*, 8:253–266, 1997.
- E. Thanassoulis. Data envelopment analysis and its use in banking. *Interfaces*, 29:1–13, 1999.
- E. Thanassoulis and R. Allen. Simulating weights restrictions in data envelopment analysis by means of unobserved DMUs. *Management Science*, 44:586–594, 1998.
- E. Thanassoulis and R. G. Dyson. Estimating preferred target input-output levels using data envelopment analysis. *European Journal of Operational Research*, 56:80–97, 1992.
- E. Thanassoulis, A. Boussofiane, and R. G. Dyson. Exploring output quality in the provision of perinatal care in england using data envelopment analysis. *European Journal of Operational Research*, 80:588–607, 1995.
- R. G. Thompson, L. N. Langemeier, C. T. Lee, E. Lee, and R. M. Thrall. The role of multiplier bounds in efficiency analysis with application to kansas farming. *Journal of Econometrics*. 46:93–108, 1990a.

- R. G. Thompson, L. N. Langemeier, C. T. Lee, and R. M. Thrall. The role of multiplier bounds in efficiency analysis with application to kansas farming. *Journal of Econometrics*, 46:93–108, 1990b.
- Y. H. B. Wong and J. E. Beasley. Restricting weight flexibility in data envelopment analysis. *Journal of the Operational Research Society*, 41:829–835, 1990.
- Y. B. Yun, H. Nakayama, and T. Tanino. A generalized model for data envelopment analysis. *European Journal of Operational Research*, 2004.

University of Cape Town