

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

TRANSITIVE

QUASI-UNIFORM SPACES

by

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CONTENTS

	Introduction	(i)
	Acknowledgements	(iv)
	Notation	(v)
<u>CHAPTER 1</u>	1.1 Basic Contents	1
	1.2 Transitive Quasi-Uniform Spaces	6
<u>CHAPTER 2</u>	Compatible Transitive Quasi-Uniformities for Topological Spaces	10
<u>CHAPTER 3</u>	Compatible Transitive Quasi-Uniformities for Bitopological Spaces	22
<u>CHAPTER 4</u>	Quasi-Pseudometric Spaces	32
<u>CHAPTER 5</u>	Complements and Problems	41
<u>BIBLIOGRAPHY</u>		45

INTRODUCTION

The concept of a quasi-uniformity on a set X i.e. a structure satisfying the axioms of a uniformity with the exception of the symmetry axiom, is due to L. Nachbin (1948) who used the term "semi-uniformity" and D. Tamari (1954) who used the term "quasi-odoform base". It appears that the term "quasi-uniformity" was introduced by Császár (1960).

It was Császár who, in the same book, showed that every topological space admits a compatible quasi-uniformity. W.J. Pervin in 1962 gave a direct method of constructing a quasi-uniformity (which later turned out to be the same as that of Császár) for an arbitrary topological space. Pervin's quasi-uniformity has a base consisting of reflexive transitive relations.

P. Fletcher (1971) introduced another method of constructing compatible quasi-uniformities for a topological space. In general this method leads to several quasi-uniformities only one of which is that of Pervin. These too, have bases consisting of reflexive transitive relations.

Uniformities which have a base consisting of equivalence relations have been studied by, amongst others, A.F. Monna (1950), B. Banaschewski (1955), A.C.M. van Rooy (1970) and H.C. Reichel (1974). Such uniformities are called non-archimedean uniformities. Banaschewski has shown that the topology of every zero dimensional space can be induced by a non-archimedean uniformity.

It is a simple matter to show that the quasi-uniformities of Fletcher referred to above are functorial. In view of the characterization by G.C.L. Brümmer (1969) of the right inverses of the forgetful functor from quasi-uniform spaces to topological spaces it is natural to question the position of the right inverses of Fletcher in the ordered class of right inverses. The question is answered in Chapter 2.

In 1972 I.L. Reilly defined a notion of zero dimensionality for bitopological spaces. We ask whether the relationship between non-archimedean uniformities and zero dimensional topologies has a bitopological analogue. The major part of Chapter 3 is concerned with answering this question.

A survey of the contents of this thesis now follows.

Chapter 1 deals with basic properties of the category of quasi-uniform spaces and its full subcategory Qut of transitive quasi-uniform spaces.

Chapter 2 concerns Fletcher's construction. We extend the class of covers to which this construction may be applied and study the functoriality of the construction. The major result is that every right inverse of the forgetful functor Qut \rightarrow Top is obtainable by the extended Fletcher construction.

In Chapter 3 we characterize pairwise zero dimensional bitopologies as those admitting transitive quasi-uniformities. An initiality characterization of pairwise zero dimensional bitopological spaces suggested by Brümmer leads to a description of the coarsest right inverse of the forgetful functor.

In Chapter 4 we discuss countably based transitive quasi-uniformities, in that they relate to quasi-metrization. We elaborate on a result of Fletcher and Lindgren (1972) and obtain a bitopological analogue.

In Chapter 5 we bring together a number of topics which relate to our previous chapters and point to further questions.

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NOTATION

Let X be a non-empty set. $\Delta = \{ (x, x) : x \in X \}$

If $M \subset X \times X$ then $M^{-1} = \{ (x, y) : (y, x) \in M \}$

For subsets M, N of $X \times X$,

$$M \circ N = \{ (x, y) : \text{for some } z \in X, (x, z) \in N, (z, y) \in M \}$$

For $M \subset X \times X$, $A \subset X$, $x \in X$,

$$M[x] = \{ y \in X : (x, y) \in M \}$$

$$M[A] = \{ y \in X : \text{for some } a \in A, (a, y) \in M \}$$

If f maps a set X to a set Y we denote by $f^{-1}(V)$ the inverse image of $V \subset Y$ under f , the image of $A \subset X$ under f will be denoted by $f(A)$.

(I, u, l) and (D, u, l) denote the unit interval and the two point set with the upper and lower topologies. A base for u (respectively l) consists of the sets $\{x : x < a\}$ (respectively $\{x : x > a\}$) for some a .

uq denotes the upper quasi-uniformity on I or D a base for uq consists of the sets $\{(x, y) : y < x + r\} r > 0$.

cl_p denotes closure in the topology P .

We write $|\mathcal{A}|$ for the objects of a category \mathcal{A} .

CHAPTER 1

1.1 BASIC CONCEPTS

The concept of a quasi-uniformity on a set X was introduced by L. Nachbin (1948). Quasi-uniformities have since been studied by many authors. A useful source of facts on quasi-uniform spaces is the monograph of Murdeshwar and Nainpally (1966).

In this section we give certain basic definitions and quote a number of simple results.

Let X be a non-empty set and \mathcal{U} a family of subsets of $X \times X$. \mathcal{U} is called a quasi-uniformity on X if:

- (1) $\Delta = \{ (x, x) : x \in X \} \subset U$, for each $U \in \mathcal{U}$.
- (2) $U, V \in \mathcal{U}$ implies $U \cap V \in \mathcal{U}$
- (3) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.
- (4) If $U \in \mathcal{U}$ and $U \subset V$, then $V \in \mathcal{U}$.

The ordered pair (X, \mathcal{U}) is called a quasi-uniform space, and members of \mathcal{U} are called entourages.

Clearly, if \mathcal{U} is a quasi-uniformity on X , then $\mathcal{U}^{-1} = \{ U : U \in \mathcal{U} \}$

is also a quasi-uniformity on X . If $\mathcal{U} = \mathcal{U}^{-1}$ then \mathcal{U} is a uniformity.

Let \mathcal{U}, \mathcal{V} be quasi-uniformities on a set X . We say that \mathcal{U} is coarser than \mathcal{V} if $\mathcal{U} \subset \mathcal{V}$.

We shall often denote a quasi-uniform space by a single symbol X , in which case, $\text{ent } X$ will denote the family of entourages and $|X|$ the underlying set of X .

A family $\mathcal{B} \subset \mathcal{U}$ is a base for a quasi-uniformity \mathcal{U} if for each $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \subset U$. \mathcal{B} is a subbase for \mathcal{U} if the family of finite intersections of members of \mathcal{B} form a base for \mathcal{U} .

Not every family of relations on X , i.e. subsets of $X \times X$, constitute a base or even a subbase for some quasi-uniformity. We thus state without proof the following simple theorems.

Theorem 1.1.1 (Murdeswar and Naimpally 1966)

Let \mathcal{B} be a family of relations on X such that

- (1) $\Delta \subset B$, for each $B \in \mathcal{B}$.
- (2) For $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$.
- (3) For each $B \in \mathcal{B}$ there exists $A \in \mathcal{B}$ such that $A \circ A \subset B$.

Then there exists a unique quasi-uniformity on X for which \mathcal{B} is a base. \mathcal{U} is said to be generated by \mathcal{B} and may be defined as the family

$$\{U : U \supset B \text{ for some } B \in \mathcal{B}\}.$$

Theorem 1.1.2 (Murdeswar and Nainpally 1966)

Let \mathcal{B} be a family of relations on X such that

- (1) $\Delta \subset B$ for each $B \in \mathcal{B}$.
- (2) For each $B \in \mathcal{B}$, there exists $A \in \mathcal{B}$ such that $A \circ A \subset B$.

Then \mathcal{B} is a subbase for a unique quasi-uniformity.

If (X, \mathcal{U}) is a quasi-uniform space and $A \subset X$, then $(A, \mathcal{U}|A)$ is a quasi-uniform subspace of (X, \mathcal{U}) , where

$$\mathcal{U}|A = \{U \cap (A \times A) : U \in \mathcal{U}\} \text{ is a quasi-uniformity.}$$

Let (X, \mathcal{U}) and (Y, \mathcal{V}) be quasi-uniform spaces. A function $f: X \rightarrow Y$ is said to be quasi-uniformly continuous if for each V in \mathcal{V} , $(f \times f)^{-1}(V)$ is in \mathcal{U} .

Definition 1.1.3 A quasi-uniform space (X, \mathcal{U}) is initial with respect to a family of functions f_α from (X, \mathcal{U}) to arbitrary quasi-uniform spaces $(Y_\alpha, \mathcal{V}_\alpha)$ if \mathcal{U} is the coarsest quasi-uniformity on X such that all the f_α are quasi-uniformly continuous.

We denote by Qu the category of quasi-uniform spaces and quasi-uniformly continuous functions, by Top the category of topological spaces and continuous functions; and by Ens the category of sets and functions.

Top has the obvious forgetful functor

$$E : \underline{\text{Top}} \longrightarrow \underline{\text{Ens}}$$

The forgetful functor

$$T : \underline{\text{Qu}} \longrightarrow \underline{\text{Top}}$$

is given as follows: for each quasi-uniform space X and each $x \in |X|$, the neighbourhoods of x in TX are the sets $U[x]$ with $U \in \text{ent } X$, and T preserves functions. It is shown by Murdeshwar and Naimpally (Lemma 1.11) that the family $\{U[x] : U \in \text{ent } X\}$ does in fact constitute a neighbourhood system at x . It follows from the same lemma that if \mathcal{B} is a (sub)base for $\text{ent } X$ then the family $\{B[x] : B \in \mathcal{B}\}$ forms a (sub)base for the TX -neighbourhood system at x .

A bitopological space is a triple (X, P, Q) where X is a set and P, Q are topologies on X . The functions $f : (X_1, P_1, Q_1) \rightarrow (X_2, P_2, Q_2)$ such that $f : (X_1, P_1) \rightarrow (X_2, P_2)$ and $f : (X_1, Q_1) \rightarrow (X_2, Q_2)$ are both continuous are the morphisms for the category Bitop of bitopological spaces. We refer to these morphisms as bicontinuous maps.

The following definition is analagous to 1.1.3.

Definition 1.1.4 A topological space (bitopological space) $(X, P, (Q))$ is initial with respect to a family of functions f_α from $(X, P, (Q))$ to arbitrary topological (bitopological) spaces $(Y_\alpha, P_\alpha, (Q_\alpha))$ if $(P, (Q))$ is the coarsest topology (bitopology) on X such that the f_α are continuous (bicontinuous).

The forgetful functor

$$\bar{E} : \underline{\text{Bitop}} \rightarrow \underline{\text{Ens}}$$

Assigns the underlying sets and functions.

The forgetful functor

$$\bar{T} : \underline{\text{Qu}} \rightarrow \underline{\text{Bitop}}$$

is given by $X \rightsquigarrow (|X|, T(X), T(X^{-1}))$, functions being preserved.

Remark 1.1.5 The forgetful functors $T : \underline{\text{Qu}} \rightarrow \underline{\text{Top}}$ and $\bar{T} : \underline{\text{Qu}} \rightarrow \underline{\text{Bitop}}$ preserve initiality in the sense that the topology or bitopology induced by an initial quasi-uniformity is the initial topology or bitopology with respect to the same family of functions.

1.2 TRANSITIVE QUASI-UNIFORM SPACES

Any relation R on a set X is called transitive if $R \circ R \subset R$.

For an entourage U , the axiom $\Delta \subset U$, known as reflexivity, makes transitivity equivalent to $U \circ U = U$.

Lemma 1.2.1

- (a) If A_1 and A_2 are transitive relations on a set X , then $A_1 \cap A_2$ is a transitive relation on X .
- (b) If B is a transitive relation Y and $f : X \rightarrow Y$ is a function, then $(f \circ f)^{-1}(B)$ is a transitive relation on X .

Proof:

- (a) Suppose that A_1 and A_2 are transitive.

Let $(x, z) \in (A_1 \cap A_2) \circ (A_1 \cap A_2)$. Then there exists $y \in X$ such that $(x, y) \in (A_1 \cap A_2)$ and $(y, z) \in (A_1 \cap A_2)$. Thus $(x, y) \in A_1$ and $(y, z) \in A_1$, and so, by the transitivity of A_1 , $(x, z) \in A_1$. Similarly by the transitivity of A_2 , $(x, z) \in A_2$. Thus $(A_1 \cap A_2) \circ (A_1 \cap A_2) \subset (A_1 \cap A_2)$.

(b) Suppose that B is transitive. Let $(x, z) \in (f \circ f)^{-1}(B) \circ (f \circ f)^{-1}(B)$.

Then there exists a $y \in X$ such that $(x, y) \in (f \circ f)^{-1}(B)$ and

$(y, z) \in (f \circ f)^{-1}(B)$. Thus $(f(x), f(y)) \in B$ and $(f(y), f(z)) \in B$

and so, by the transitivity of B , $(f(x), f(z)) \in B$. Thus

$(x, z) \in (f \circ f)^{-1}(B)$.

A (sub)base \mathcal{B} for a quasi-uniformity is said to be transitive provided that each entourage in \mathcal{B} is transitive. By (a) above, if a quasi-uniformity has a transitive subbase then it has a transitive base. A quasi-uniformity with a transitive (sub)base is called a transitive quasi-uniformity.

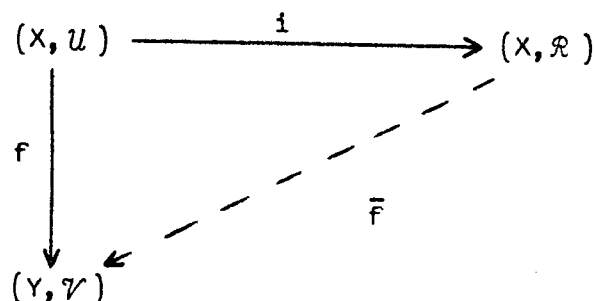
We denote by Qu_t the full subcategory of Qu whose objects are the transitive quasi-uniform spaces.

Theorem 1.2.2 Qu_t is monoreflective in Qu.

Proof: Let \mathcal{U} be a quasi-uniformity on a set X . We construct the transitive reflection by providing X with a transitive quasi-uniformity \mathcal{R} as follows:

Take as a base for \mathcal{R} those $U \in \mathcal{U}$ which are transitive. (By 1.2.1(a) they do constitute a base).

Consider the following diagram:



where \mathcal{V} is a transitive quasi-uniformity on a set Y , f is a quasi-uniformly continuous function and i is the identity set map. Then i is clearly quasi-uniformly continuous. Let $\bar{f} = f$ qua set maps. Let $V \in \mathcal{V}$. Then there exists a transitive $V' \in \mathcal{V}$ such that $V' \subset V$. Now $(f \circ f)^{-1}(V') \subset (f \circ f)^{-1}(V)$ and by 1.2.1(b) $(f \circ f)^{-1}(V')$ is transitive. Thus $(f \circ f)^{-1}(V) \in \mathcal{R}$. So \bar{f} is quasi-uniformly continuous. Clearly \bar{f} is unique.

Proposition 1.2.3 Given a set X , transitive quasi-uniform spaces $(Y_\alpha, \mathcal{V}_\alpha)$ and functions $f_\alpha : X \rightarrow Y$, there exists a transitive quasi-uniformity on X such that (X, \mathcal{U}) is initial with respect to the f_α to the $(Y_\alpha, \mathcal{V}_\alpha)$.

Proof: A subbase for the quasi-uniformity \mathcal{U} consists of the sets $(f_\alpha \times f_\alpha)^{-1}(V)$ where $f_\alpha : X \rightarrow (Y_\alpha, \mathcal{V}_\alpha)$, $V \in \mathcal{V}_\alpha$. \mathcal{U} is transitive by 1.2.1.

If $E_t : \underline{Qut} \rightarrow \underline{Ens}$

is the forgetful functor, then the above proposition may be restated in the terminology of (Brümmer 1971) as follows:

Proposition Qut is E_t - initiality complete

Remark 1.2.4 The concept of initiality completeness is self dual (Antoine 1962, Proposition 1) or (Roberts 1968, Proposition 4.6). Qut is thus co-initiality complete. Since Ens is complete and co-complete we have that Qut is complete and co-complete (Antoine 1966, Proposition 2) and thus Qut has products, sums and is closed for the taking of quotients. Qut is also closed for the taking of subspaces (a subspace is initial for its inclusion mapping).

CHAPTER 2

COMPATIBLE TRANSITIVE QUASI-UNIFORMITIES FOR TOPOLOGICAL SPACES

Let (X, \mathcal{J}) be a topological space and \mathcal{U} a quasi-uniformity on X .

\mathcal{U} is said to be compatible with \mathcal{J} if $\tau\mathcal{U} = \mathcal{J}$.

A Császár (1960) has shown that every topological space has a compatible quasi-uniformity. W.J. Pervin (1962) produced a much simpler proof of this fact by giving a direct method of constructing a compatible quasi-uniformity for an arbitrary topological space. G.C.L. Brümmer (1971) and, later, C. Votaw (1972) have shown that the quasi-uniformities of Császár and Pervin are, in fact, the same. Since the Pervin quasi-uniformity has a transitive base we deduce that every topological space has at least one compatible transitive quasi-uniformity.

In this chapter we discuss a method of constructing compatible transitive quasi-uniformities for an arbitrary topological space which was introduced by P. Fletcher (1971).

Let R be a reflexive transitive relation on X . Clearly R^{-1} is also a reflexive transitive relation on X . It is also clear from 1.1.2 that any family of reflexive transitive relations on a set X defines a subbase for a transitive quasi-uniformity on X . Now, given a topological space

(X, \mathcal{J}) and a family \mathcal{R} of reflexive transitive relations on X , two conditions are necessary and sufficient to ensure that the quasi-uniformities generated by the relations is compatible with \mathcal{J} :

(i) For each relation $R \in \mathcal{R}$ and each $x \in X$, $R[x]$ is a \mathcal{J} -neighbourhood of x . (The quasi-uniform topology is coarser than \mathcal{J}).

(ii) For each $x \in X$ and for each \mathcal{J} -neighbourhood A of x , there exists R_1, \dots, R_n such that $\bigcap_{k=1}^n (R_k[x]) \subset A$. (The quasi-uniform topology is finer than \mathcal{J}).

Definition 2.1 If \mathcal{C} is a cover of a set X , then we define

$$R_{\mathcal{C}} = \{(x, y) \in X \times X : (\forall C \in \mathcal{C}) (x \in C \implies y \in C)\}$$

Clearly, $R_{\mathcal{C}}$ is a reflexive transitive relation on X .

Remark 2.2 We note that any family of subsets of X can be made into a cover of X simply by adding the whole set to the family. We could therefore equally well have chosen to generate reflexive transitive relations by means of families rather than covers.

Definition 2.3 If R is a reflexive relation on X we define a cover of X by

$$\mathcal{C}_R = \{ R[x] : x \in X \}$$

Proposition 2.4 If V is a reflexive transitive relation on X then

$$R_{\mathcal{C}_V} = V$$

Proof: Let $x \in X$.

$$\begin{aligned} \text{Then, } R_{\mathcal{C}_V}[x] &= \{y: (\forall z \in X) (x \in V[z] \implies y \in V[z])\} \\ &= \{y: (\forall z \in X) ((z, x) \in V \implies (z, y) \in V)\} \\ &= \{y: (x, y) \in V\} = V[x] \end{aligned}$$

Definition 2.5 A cover \mathcal{C} of a set X is a reduced cover if

$$\mathcal{C}_{R_{\mathcal{C}}} = \mathcal{C}$$

Proposition 2.6 There is a one to one correspondence between reflexive transitive relations and reduced covers on a set X .

Proof: If V is reflexive and transitive then $R_{\mathcal{C}_V} = V$

$$\text{Thus } \mathcal{C}_{R_{\mathcal{C}_V}} = \mathcal{C}_V$$

Definition 2.7 Let \mathcal{A} be a family of covers of X . Take

$\mathcal{B} = \{R_{\mathcal{C}}: \mathcal{C} \in \mathcal{A}\}$. Then \mathcal{B} is a subbase for a transitive quasi-uniformity

$\mathcal{U}_{\mathcal{A}}$ on X , called the covering quasi-uniformity generated by \mathcal{A} .

Theorem 2.8 A quasi-uniformity \mathcal{U} on X is transitive iff it is a covering quasi-uniformity.

Proof: Let \mathcal{B} be a transitive base for \mathcal{U} . Then by 2.4 $\{\mathcal{C}_B: B \in \mathcal{B}\}$ is the required family of covers.

Definition 2.9 A cover \mathcal{C} of a topological space (X, \mathcal{J}) is called a SN-cover of (X, \mathcal{J}) if for each $x \in X$, $R_{\mathcal{C}}[x]$ is a \mathcal{J} -neighbourhood of x .

Theorem 2.10 Let (X, \mathcal{J}) be a topological space and let \mathcal{A} be a collection of SN-covers of X such that if $x \in A \in \mathcal{J}$, then there exist covers

$\mathcal{C}_1, \dots, \mathcal{C}_n$ in \mathcal{A} such that $\bigcap_{i=1}^n (R_{\mathcal{C}_i}[x]) \in \mathcal{A}$. Then $\mathcal{U}_{\mathcal{A}}$ is a compatible transitive quasi-uniformity on X .

Proof: Clear from the above discussion.

Definition 2.11 (Fletcher 1971) An open cover \mathcal{C} of a topological space (X, \mathcal{J}) is called a Q-cover of (X, \mathcal{J}) if for each $x \in X$, $R_{\mathcal{C}}[x] \in \mathcal{J}$.

Remark 2.12 The transitivity of $R_{\mathcal{C}}$ implies that if $R_{\mathcal{C}}[x]$ is a neighbourhood of x for each $x \in X$ then $R_{\mathcal{C}}[x]$ is open for each $x \in X$. Thus the Q-covers are just the open SN-covers.

Definition 2.13 Let (X, \mathcal{J}) be a topological space and let \mathcal{A} be a collection of SN-covers of (X, \mathcal{J}) such that if $x \in A \in \mathcal{J}$, then there exist $\mathcal{C}_1, \dots, \mathcal{C}_n$ in \mathcal{A} such that $\bigcap_{i=1}^n (R_{\mathcal{C}_i}[x]) \subset A$. Then \mathcal{A} is

called an admissible collection of covers.

Theorem 2.14 (Fletcher 1971) Let (X, \mathcal{J}) be a topological space and let \mathcal{U} be a compatible quasi-uniformity for (X, \mathcal{J}) . Then \mathcal{U} is a transitive quasi-uniformity iff \mathcal{U} is generated by an admissible collection of reduced Q-covers of (X, \mathcal{J}) .

Proof: Suppose that \mathcal{U} is a compatible transitive quasi-uniformity for (X, \mathcal{J}) and let \mathcal{B} be a transitive base for \mathcal{U} . Let $B \in \mathcal{B}$. By 2.4, $R_{\mathcal{C}_B} [x] = B[x]$ which is a neighbourhood of x for each $x \in X$. Thus by 2.12, \mathcal{C}_B is a (reduced) Q-cover of X for each $B \in \mathcal{B}$. Clearly for each $x \in A \in \mathcal{J}$ there exists $B \in \mathcal{B}$ such that $R_{\mathcal{C}_B} [x] \subset A$. Thus \mathcal{A} is an admissible collection of covers where $\mathcal{A} = \{\mathcal{C}_B : B \in \mathcal{B}\}$. By 2.4 $R_{\mathcal{C}_B} = B$ for each $B \in \mathcal{B}$ and thus $\mathcal{U} = \mathcal{U}_{\mathcal{A}}$.

Corollary 2.15 Let (X, \mathcal{J}) be a topological space and let \mathcal{U} be a compatible quasi-uniformity for (X, \mathcal{J}) . Then the following statements are equivalent:

- (i) \mathcal{U} is a transitive quasi-uniformity.
- (ii) \mathcal{U} is generated by an admissible collection of SN-covers.
- (iii) \mathcal{U} is generated by an admissible collection of reduced Q-covers.

Lemma 2.16 If \mathcal{A} is the collection of all open finite (point finite, locally finite, Q -) covers of (X, \mathcal{J}) then \mathcal{A} is an admissible collection of Q -covers and $\mathcal{U}_{\mathcal{A}}$ is known as the open finite (point finite, locally finite, Q -) covering quasi-uniformity for (X, \mathcal{J}) .

Proof: Every cover referred to is surely a Q -cover. In addition, for each open set A , the cover $\{X, A\}$ belongs to each of the collections of covers.

Corollary 2.17 The Q -covering quasi-uniformity of a topological space (X, \mathcal{J}) is the finest compatible transitive quasi-uniformity for (X, \mathcal{J}) .

We now consider those transitive quasi-uniformities on a topological space (X, \mathcal{J}) which are finer than the Pervin quasi-uniformity.

Definition 2.18 (Fletcher 1970) Let (X, \mathcal{J}) be a topological space and let $A \in \mathcal{J}$. A fundamental cover of X about A is an open cover \mathcal{C} of X such that $A \in \mathcal{C}$ and if $B \in \mathcal{C}$ such that $B \cap A \neq \emptyset$ then $A \subset B$.

Theorem 2.19 (Fletcher 1970) Let (X, \mathcal{J}) be a topological space and let \mathcal{A} be a collection of Q -covers of X such that for each $A \in \mathcal{J}$, \mathcal{A} contains a fundamental cover of X about A . Then \mathcal{A} is an admissible collection of Q -covers.

Definition 2.20 (Fletcher 1970) Let (X, \mathcal{J}) be a topological space.

Let \mathcal{A} be a collection of \mathcal{Q} -covers of X such that for each $A \in \mathcal{J}$, \mathcal{A} contains a fundamental cover of X about A . Then $\mathcal{U}_{\mathcal{A}}$ is a fundamental covering quasi-uniformity for (X, \mathcal{J}) .

Let (X, \mathcal{J}) be a topological space. Let \mathcal{A} be a family of SN-covers of (X, \mathcal{J}) . Then the property:

* For each $A \in \mathcal{J}$ there is a cover $\mathcal{C} \in \mathcal{A}$ such that $A \in \mathcal{C}$.

is weaker than that required by Theorem 2.19.

Theorem 2.21 Let (X, \mathcal{J}) be a topological space and let \mathcal{U} be a compatible transitive quasi-uniformity for (X, \mathcal{J}) . Then the following statements are equivalent:

- (i) \mathcal{U} is finer than the Pervin quasi-uniformity \mathcal{P} .
- (ii) \mathcal{U} is a fundamental covering quasi-uniformity.
- (iii) \mathcal{U} is generated by a family of SN-covers of (X, \mathcal{J}) with property *.

Proof: Clearly (ii) \implies (iii).

(i) \implies (ii) The family of covers $\mathcal{A} = \{ \mathcal{C}_U : U \text{ is transitive and } U \in \mathcal{U} \}$ is a family of \mathcal{Q} -covers which generate \mathcal{U} . For each $A \in \mathcal{J}$, the subbasic Pervin entourage $S(A) = (A \times A) \cup ((X-A) \times X)$ is transitive and belongs to \mathcal{U} . Thus for each $A \in \mathcal{J}$, $\mathcal{C}_{S(A)} = \{ A, X \}$ is a fundamental cover of X about A which belongs to \mathcal{A} .

(iii) \Rightarrow (i) Let $A \in \mathcal{J}$ and let \mathcal{C} be a member of the generating family of covers which contains A . Now, for each $x \in A$, $R_{\mathcal{C}} [x] \subset A$ and therefore $R_{\mathcal{C}} \subset (A \times A) \cup ((X-A) \times X)$. Thus $\mathcal{P} \subset \mathcal{U}$.

Remark 2.22 The above result elaborates on Theorem 4 of (Fletcher 1970) in which the implication (ii) implies (i) is proved.

Theorem 2.23 (Fletcher 1971) Let (X, \mathcal{J}) be a topological space and let \mathcal{U} be the open finite covering quasi-uniformity for (X, \mathcal{J}) . Then \mathcal{U} is the Pervin quasi-uniformity \mathcal{P} .

Proof: By 2.21, $\mathcal{P} \subset \mathcal{U}$. Let $U \in \mathcal{U}$. Then there is a finite open cover $\mathcal{C} = \{A_1, \dots, A_n\}$ of X such that $U = R_{\mathcal{C}}$. Let $V = \bigcap_{i=1}^n \{(A_i \times A_i) \cup ((X-A_i) \times X)\}$. Then $U \subset V \in \mathcal{P}$ so that $\mathcal{U} \subset \mathcal{P}$.

Definition 2.24 Let \mathcal{A}_X be an admissible collection of SN-covers of X for each $X \in \underline{\text{Top}}$. Then the species $\{\mathcal{A}_X : X \in |\underline{\text{Top}}|\}$ is called a continuous species of covers for $\underline{\text{Top}}$ provided that if $\mathcal{C} \in \mathcal{A}_Y$ and $f: X \rightarrow Y$ is a continuous function then $\{f^{-1}(C) : C \in \mathcal{C}\} \in \mathcal{A}_X$.

We write $U_{\mathcal{A}_X} = U_{\mathcal{A}}(X)$. Let $U_{\mathcal{A}}(f) = f$ qua set maps.

Theorem 2.25 Let $\{\mathcal{A}_X : X \in |\underline{\text{Top}}|\}$ be a continuous species of covers. Then the association $U_{\mathcal{A}}(\) : \underline{\text{Top}} \rightarrow \underline{\text{Qu}}$ is a right inverse of the

forgetful functor $T : \underline{Qu} \rightarrow \underline{Top}$.

Proof: Let X, Y be topological spaces and let $f : X \rightarrow Y$ be a continuous map. We need only show that $f : U_{\mathcal{A}}(X) \rightarrow U_{\mathcal{A}}(Y)$ is quasi-uniformly continuous.

Let $R_{\mathcal{C}}$ be a subbasic entourage of $U_{\mathcal{A}}(Y)$. Let $x' \in X$.

$$\begin{aligned} \text{Then, } ((fxf)^{-1}(R_{\mathcal{C}})) [x'] &= f^{-1}\{R_{\mathcal{C}} [f(x')]\} \\ &= f^{-1}\{y \in Y : (\forall C \in \mathcal{C}) (f(x') \in C \Rightarrow y \in C)\} \\ &= \{x \in X : (\forall C \in \mathcal{C}) (f(x') \in C \Rightarrow f(x) \in C)\} \\ &= \{x \in X : (\forall C \in \mathcal{C}) (x' \in f^{-1}(C) \Rightarrow x \in f^{-1}(C))\} \\ &= R_{f^{-1}(\mathcal{C})} [x'] \end{aligned}$$

Thus, $(fxf)^{-1}(R_{\mathcal{C}}) = R_{f^{-1}(\mathcal{C})}$ which belongs to $U_{\mathcal{A}}(X)$ and f is quasi-uniformly continuous.

Definition 2.26 Let $\{\mathcal{A}_X : X \in \underline{Top}\}$ be a continuous species of covers.

Then $U_{\mathcal{A}}$ is called the covering functor generated by \mathcal{A} .

Remark 2.27 If \mathcal{A}_X is the class of all open finite (point finite, locally finite, Q -) covers of X , then the species $\{\mathcal{A}_X, X \in \underline{Top}\}$ is a continuous species of covers for \underline{Top} . $U_{\mathcal{A}}(\)$ is known as the finite (point finite, locally finite, Q -) covering functor.

Corollary 2.28 The finite covering functor is the Pervin functor and is thus the coarsest right inverse of T . (Brümmer 1969).

Let $T_t : \underline{Qut} \longrightarrow \underline{Top}$

be the restriction of the forgetful functor T .

Theorem 2.29 A functor $F : \underline{Top} \longrightarrow \underline{Qut}$ is a right inverse of T_t iff F is a covering functor.

Proof: For each $X \in |\underline{Top}|$ we let $\mathcal{A}_X =$

$\{\mathcal{C}_V : V \text{ is transitive and } V \in \text{ent } F(X)\}$. Then \mathcal{A}_X is an admissible collection of covers which generates the quasi-uniformity of $F(X)$.

Let $X, Y \in |\underline{Top}|$ and let $f : X \longrightarrow Y$ be continuous. Suppose that $V \in \text{ent } F(Y)$ and let $y \in Y$. Write $(f \circ f) = f_2$. We first show that if $z_1, z_2 \in f^{-1}(y)$ where $y \in Y$ then $(f_2^{-1}(V))[z_1] = (f_2^{-1}(V))[z_2]$.

$$\begin{aligned} x \in (f_2^{-1}(V))[z_1] &\implies (z_1, x) \in f_2^{-1}(V) \\ &\implies (f(z_1), f(x)) \in V \\ &\implies (f(z_2), f(x)) \in V \\ &\implies (z_2, x) \in f_2^{-1}(V) \\ &\implies x \in (f_2^{-1}(V))[z_2]: \end{aligned}$$

$$\begin{aligned}
\text{Thus, } f^{-1}(V[y]) &= \{ z : (y, f(z)) \in V \} \\
&= \{ z : (x, z) \in f_2^{-1}(V), x \in f^{-1}(y) \} \\
&= \bigcup \{ (f_2^{-1}(V))[x] : x \in f^{-1}(y) \} \\
&= (f_2^{-1}(V))[x] \text{ any } x \in f^{-1}(y).
\end{aligned}$$

If V is transitive then $f_2^{-1}(V)$ is transitive and $f_2^{-1}(V) \in \text{ent } F(X)$, so that $\{(f_2^{-1}(V))[x] : x \in X\} \in \mathcal{A}_X$. Thus $\{f^{-1}(V[y]) : y \in Y\} \in \mathcal{A}_X$ and $\{\mathcal{A}_X \in \underline{\text{Top}}\}$ is a continuous species of covers which generate F .

Corollary 2.30 The Q -covering functor is the finest right inverse of T_t .

We complete this chapter by deducing a further characterization of the right inverse of T_t from a theorem of Brümmer (1969).

Let \mathcal{L} be the class of those functors $F : \underline{\text{Top}} \rightarrow \underline{\text{Qu}}$ which preserve the underlying sets of the objects of $\underline{\text{Top}}$. Let D^* be the two point space with the upper topology.

Theorem 2.31 (Brümmer 1969) Let $F \in \mathcal{L}$. Then F is a right inverse of $T : \underline{\text{Qu}} \rightarrow \underline{\text{Top}}$ iff the following two conditions are satisfied.

$$(1) \quad TF(D^*) = D^*$$

(2) There exists a class \mathcal{B} of quasi-uniform spaces such that for each topological space X , $F(X)$ is initial in $\underline{\text{Qu}}$ for the continuous

mappings of X into members of \mathcal{B} .

Since we have already observed that Qut is initiality complete the following analogue of Theorem 2.31 may readily be proved by adapting the proof of 2.31 in the obvious way.

Theorem 2.32 Let $F \in \mathcal{L}$. Then F is a right inverse of $T_t : \underline{\text{Qut}} \rightarrow \underline{\text{Top}}$ iff the following two conditions are satisfied.

$$(1) \quad T_t F(D^*) = D^*$$

(2) There exists a class \mathcal{B} of transitive quasi-uniform spaces such that for each topological space X , $F(X)$ is initial in Qut for the continuous mappings of X into members of \mathcal{B} .

CHAPTER 3

COMPATIBLE TRANSITIVE QUASI-UNIFORMITIES FOR BITOPOLOGICAL SPACES

We have already noted that every topological space admits a compatible transitive quasi-uniformity. We now investigate those bitopological spaces which admit transitive quasi-uniformities.

Uniformities generated by equivalence relations were studied by A.F. Monna (1950) and B. Banaschewski (1955) under the name non-Archimedean uniformities. The relationship between these uniformities and zero dimensional topological space was first established by Banaschewski and later rediscovered and further investigated by N. Levine (1969).

A notion of zero dimensionality for bitopological spaces was introduced in (Reilly 1972). In the light of the above it is natural to investigate the relationship between transitive quasi-uniformities and pairwise zero dimensional bitopological spaces.

If V is a reflexive transitive relation on X we write $\mathcal{U}(V)$ for the quasi-uniformity generated by V . The closure of a subset A of X in the topology \mathcal{T} is written $\text{cl}_{\mathcal{T}} A$.

Definition 3.1 (Reilly 1972) Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological space.

\mathcal{T}_1 is zero dimensional with respect to \mathcal{T}_2 if \mathcal{T}_1 has a base of \mathcal{T}_2 closed

sets, that is, if for each point $x \in X$ and each \mathcal{J}_1 open set A containing x there is a \mathcal{J}_2 closed \mathcal{J}_1 open set G such that $x \in G \subset A$.

$(X, \mathcal{J}_1, \mathcal{J}_2)$ is pairwise zero dimensional if \mathcal{J}_1 is zero dimensional with respect to \mathcal{J}_2 and \mathcal{J}_2 is zero dimensional with respect to \mathcal{J}_1 .

Theorem 3.2 Let V be a reflexive transitive relation on X and let

$\mathcal{J}_1 = T(\mathcal{U}(V))$ and $\mathcal{J}_2 = T(\mathcal{U}(V^{-1}))$. Then,

- (i) V is open in $\mathcal{J}_2 \times \mathcal{J}_1$ and closed in $\mathcal{J}_1 \times \mathcal{J}_2$.
- (ii) V^{-1} is closed in $\mathcal{J}_2 \times \mathcal{J}_1$ and open in $\mathcal{J}_1 \times \mathcal{J}_2$.
- (iii) For all $A \subset X$, $\text{cl}_{\mathcal{J}_1}(A) = V^{-1}[A]$, $\text{cl}_{\mathcal{J}_2}(A) = V[A]$.
- (iv) For all $A \subset X$, A is \mathcal{J}_1 open iff A is \mathcal{J}_2 closed
 A is \mathcal{J}_2 open iff A is \mathcal{J}_1 closed.

Proof: (i) By corollary 1.44 of (Murdeswar and Naimpally 1966) the $\mathcal{J}_2 \times \mathcal{J}_1$ open members of $\mathcal{U}(V)$ form a base for $\mathcal{U}(V)$. Thus there is a $\mathcal{J}_2 \times \mathcal{J}_1$ open entourage of $\mathcal{U}(V)$ which is contained in V . Since V is contained in every member of $\mathcal{U}(V)$, V is $\mathcal{J}_2 \times \mathcal{J}_1$ open. Similarly, by Theorem 1.45 of (Murdeswar and Naimpally 1966) the $\mathcal{J}_1 \times \mathcal{J}_2$ closed members of $\mathcal{U}(V)$ form a base for $\mathcal{U}(V)$ and thus V is $\mathcal{J}_1 \times \mathcal{J}_2$ closed.

(ii) follows from (i)

(iii) $x \in \text{cl}_{\mathcal{J}_1}(A)$ iff $U[x]$ intersects A for each $U \in \mathcal{U}(V)$.

But $U[x]$ intersects A iff $x \in U^{-1}[A]$. Thus $\text{cl}_{\mathcal{J}_1}(A) =$

$$\cap \{U^{-1}[A] : U \in \mathcal{U}\} = V^{-1}[A]$$

Similarly $\text{cl}_{\mathcal{J}_2}(A) = V[A]$

(iv) A is \mathcal{J}_2 closed iff $A = \text{cl}_{\mathcal{J}_2}(A) = V[A]$

Since V is $\mathcal{J}_1 \times \mathcal{J}_1$ open, $V[A]$ is \mathcal{J}_1 open

i.e. A is \mathcal{J}_2 closed $\implies A$ is \mathcal{J}_1 open (a)

Similarly A is \mathcal{J}_1 closed $\implies A$ is \mathcal{J}_2 open (b)

If A is \mathcal{J}_1 open then $X-A$ is \mathcal{J}_1 closed. Thus $X-A$ is \mathcal{J}_2 open by

(b) and therefore A is \mathcal{J}_2 closed. Similarly A is \mathcal{J}_2 open $\implies A$ is \mathcal{J}_1 closed.

Theorem 3.3 If $(X, \mathcal{J}_1, \mathcal{J}_2)$ is a bitopological space in which every

\mathcal{J}_1 open set is \mathcal{J}_2 closed and every \mathcal{J}_2 open set is \mathcal{J}_1 closed then there

exists a unique reflexive transitive relation V on X such that $\mathcal{J}_1 =$

$\tau(U(V))$ and $\mathcal{J}_2 = \tau(U(V^{-1}))$.

Proof: Let $V = \{(x, y) : y \in \text{cl}_{\mathcal{J}_2}(x)\}$. Then V is clearly a reflexive

transitive relation on X . From the definition $V[x] = \text{cl}_{\mathcal{J}_2}(x)$. We show

that $V^{-1}[x] = \text{cl}_{\mathcal{J}_1}(x)$. If $y \in V^{-1}[x]$ then $x \in V[y]$, that is, $x \in \text{cl}_{\mathcal{J}_2}(y)$.

If $y \notin \text{cl}_{\mathcal{J}_1}(x)$ then there is a \mathcal{J}_1 closed set about x which does not contain

y . Its complement is a \mathcal{J}_2 closed set about y which does not contain x .

That is, $x \notin \text{cl}_{\mathcal{J}_2}(y)$. Contradiction.

Now suppose $y \in \text{cl}_{\mathcal{J}_1}(x)$. If $y \notin V^{-1}[x]$ then $x \notin V[y]$ and so $x \notin \text{cl}_{\mathcal{J}_2}(y)$.

Thus, by a similar argument to the above, $y \notin \text{cl}_{\mathcal{J}_1}(x)$. Contradiction.

We now show that $\mathcal{J}_1 = T(U(V))$. Suppose $x \in A \in \mathcal{J}_1$. Then A is \mathcal{J}_2 closed. By definition $V[x] = \text{cl}_{\mathcal{J}_2}(x)$ and so, $V[x] \subset A$. Thus A is $T(U(V))$ open.

Suppose $x \in A \in T(U(V))$. Then A is $T(U(V^{-1}))$ closed by 3.2 (iv). By 3.2 (iii) $V[x] = \text{cl}_{T(\mathcal{J}(V^{-1}))}(x)$ and so $V[x] \subset A$. Since $V[x]$ is \mathcal{J}_1 open, A is \mathcal{J}_1 open.

Using the fact that $V^{-1}[x] = \text{cl}_{\mathcal{J}_1}(x)$ it follows by a similar argument that $\mathcal{J}_2 = T(U(V^{-1}))$.

We now prove the uniqueness of V . Suppose $(X, \mathcal{J}_1, \mathcal{J}_2) = (X, T(U(W)), T(U(W^{-1})))$ for some reflexive transitive relation W on X . By definition we have that $V[x] = \text{cl}_{\mathcal{J}_2}(x)$ for each $x \in X$. But by Theorem 3.2 (iii) $\text{cl}_{\mathcal{J}_2}(x) = W[x]$ for each $x \in X$. Thus $V = W$.

Theorem 3.4 Let (X, U) be a transitive quasi-uniform space. Then $(X, T(U), T(U^{-1}))$ is pairwise zero-dimensional.

Proof: U has a base \mathcal{V} of reflexive transitive relations. For each $V \in \mathcal{V}$, $V[x]$ is $T(U(V))$ open and $T(U(V^{-1}))$ closed. Thus each $V[x]$ is $T(U)$ open and $T(U^{-1})$ closed and so the family $\{V[x] : V \in \mathcal{V}, x \in X\}$ is a base of $T(U^{-1})$ closed sets for $T(U)$. Similarly the family $\{V^{-1}[x] : V \in \mathcal{V}, x \in X\}$ is a base of $T(U)$ closed sets for $T(U^{-1})$.

Lemma 3.5 The bitopological space $(X, \mathcal{J}_1, \mathcal{J}_2)$ is pairwise zero dimensional iff \mathcal{J}_1 has a base \mathcal{B} of \mathcal{J}_2 closed sets such that the family

$\mathcal{B}' = \{ B' : (X - B') \in \mathcal{B} \}$ is a base of \mathcal{J}_1 closed sets for \mathcal{J}_2 .

Proof: Let \mathcal{C} be a base of \mathcal{J}_2 closed sets for \mathcal{J}_1 and \mathcal{D} a base of \mathcal{J}_1 closed sets for \mathcal{J}_2 . Put $\mathcal{B} = \{ B : B \in \mathcal{C} \text{ or } (X - B) \in \mathcal{D} \}$.

We now give a converse to Theorem 3.4.

Theorem 3.6 If $(X, \mathcal{J}_1, \mathcal{J}_2)$ is pairwise zero dimensional then $(X, \mathcal{J}_1, \mathcal{J}_2) = (X, \mathcal{T}(U), \mathcal{T}(U^{-1}))$ for some transitive quasi-uniformity U .

Proof: Let \mathcal{B} and \mathcal{B}' be bases as described in 3.5. For each $B \in \mathcal{B}$, let $\mathcal{J}(B) = \{ \emptyset, B, X \}$ and $\mathcal{J}'(B) = \{ \emptyset, X - B, X \}$. Then the bitopological space $(X, \mathcal{J}(B), \mathcal{J}'(B))$ satisfies the conditions of Theorem 3.3 and thus there is a reflexive transitive relation V_B such that $\mathcal{J}(B) = \mathcal{T}(V_B)$ and $\mathcal{J}'(B) = \mathcal{T}(V_B^{-1})$. Thus the quasi-uniformity generated by taking $\{ V_B : B \in \mathcal{B} \}$ as a subbase generates the bitopology having \mathcal{B} and \mathcal{B}' as respective bases.

Remark 3.7 Theorem 4.6 associates with each $B \in \mathcal{B}$ the entourages $V_B = \{ (x, y) : y \in \text{cl}_{\mathcal{J}'(B)}(x) \}$. Now for each $x \in B$, $\text{cl}_{\mathcal{J}'(B)}(x) = B$ and for each $x \in X - B$, $\text{cl}_{\mathcal{J}'(B)}(x) = X$. Thus the quasi-uniformity U of 3.6 is simply the Pervin quasi-uniformity on X with respect to the base \mathcal{B} .

Example 3.8 A pairwise zero dimensional space for which the Pervin quasi-uniformity on the first topology is not compatible.

Let X be an infinite set and let \mathcal{J}_1 be the discrete topology on X . For $A \subset X$, let $S(A) = (A \times A) \cup ((X-A) \times X)$. Let \mathcal{J} be the transitive quasi-uniformity with subbase entourages $S(\{x\})$, $x \in X$. Then $(X, T(\mathcal{J}), T(\mathcal{J}^{-1})) = (X, \mathcal{J}, T(\mathcal{J}^{-1}))$ is pairwise zero dimensional. Now $T(\mathcal{J}^{-1})$ has subbasic open sets of the form $X - \{x\}$, $x \in X$, and $T(\mathcal{P}^{-1}(\mathcal{J}_1))$ has subbasic open sets of the form $X - A$, $A \in \mathcal{J}_1$. Thus $T(\mathcal{J}^{-1})$ is strictly coarser than $T(\mathcal{P}^{-1}(\mathcal{J}_1))$. (If A is an infinite set then $X - A$ does not belong to $T(\mathcal{J}^{-1})$).

In view of Theorems 3.4 and 3.6 the relationship between pairwise zero dimensionality and the notions of total disconnectedness introduced by J. Swart (1971) is of interest.

Definition 3.9 (Swart 1971)

Let $(X, \mathcal{J}_1, \mathcal{J}_2)$ be a bitopological space. A pair of non-empty disjoint sets A and B such that A is \mathcal{J}_1 open and B is \mathcal{J}_2 open and $X = A \cup B$ is called a disconnection of X . We write $X = A/B$.

$(X, \mathcal{J}_1, \mathcal{J}_2)$ is said to be totally disconnected if for every two distinct points x and y there exists a disconnection $X = A/B$ with $x \in A$ and $y \in B$.

$(X, \mathcal{J}_1, \mathcal{J}_2)$ is said to be weakly totally disconnected if for each two distinct points, there exists a disconnection $X = A/B$ such that the one point belongs to A and the other point to B - the role of the points need not be interchangeable.

In (Murdeswar and Naimpally 1966) and (Fletcher, Hoyle and Patty 1969) the term pairwise T_0 is used to describe different axioms. We distinguish between these notions by writing MN pairwise T_0 and FHP pairwise T_0 respectively.

Definition 3.10 A bitopological space $(X, \mathcal{J}_1, \mathcal{J}_2)$ is MN pairwise T_0 if for each pair of distinct points of X there is a set which is either \mathcal{J}_1 open or \mathcal{J}_2 open containing one of the points but not the other.

$(X, \mathcal{J}_1, \mathcal{J}_2)$ is FHP pairwise T_0 if for each pair x, y of distinct points of X there is either a \mathcal{J}_1 open set A such that $x \in A$ and $y \notin A$ or a \mathcal{J}_2 open set B such that $y \in B$ and $x \notin B$.

Theorem 3.11 (Reilly 1972) If $(X, \mathcal{J}_1, \mathcal{J}_2)$ is FHP pairwise T_0 and pairwise zero dimensional then it is totally disconnected.

Proof: Let x and y be distinct points in X. Then there is

(i) a \mathcal{J}_1 open set A such that $x \in A, y \notin A$

or (ii) a \mathcal{J}_2 open set B such that $x \notin B, y \in B$

Since \mathcal{J}_1 is zero dimensional with respect to \mathcal{J}_2 , in case

(i) there is a \mathcal{J}_1 open \mathcal{J}_2 closed set G such that $x \in G \subset A$. Then $G/X-G$ is a disconnection of X with $x \in G, y \in X-G$ so that $(X, \mathcal{J}_1, \mathcal{J}_2)$ is totally disconnected. Since \mathcal{J}_2 is zero dimensional with respect to \mathcal{J}_1 case (ii) follows similarly.

Theorem 3.12 (Reilly 1972) If $(X, \mathcal{J}_1, \mathcal{J}_2)$ is MN pairwise T_0 and pairwise zero dimensional then it is weakly totally disconnected.

Proof: The proof is similar to that of 3.11 except that there are four cases to consider.

A further interesting characterization of those bitopological spaces which admit a compatible transitive quasi-uniformity, which was suggested by G.C.L. Brümmer, follows from the next theorem.

We denote by D the set having two points 0 and 1, by u and l the upper and lower topologies on D and by uqu the upper quasi-uniformity on D .

Theorem 3.13 $(X, \mathcal{J}_1, \mathcal{J}_2)$ is pairwise zero dimensional iff it is initial for its bicontinuous maps into (D, u, l) .

Proof: Suppose $(X, \mathcal{J}_1, \mathcal{J}_2)$ is pairwise zero dimensional.

Let A be a basic \mathcal{J}_1 open \mathcal{J}_2 closed set. We define $f : (X, \mathcal{J}_1, \mathcal{J}_2) \rightarrow (D, u, 1)$

as follows:

$$\begin{aligned} f(x) &= 0 & x \in A \\ f(x) &= 1 & \text{otherwise} \end{aligned}$$

Then $f^{-1}(0) = A \in \mathcal{J}_1$

$$f^{-1}(1) = (X-A) \in \mathcal{J}_2$$

and hence f is bicontinuous.

Thus A is the preimage of $\{0\}$ under a bicontinuous map. Similarly, we show that if B is a basic \mathcal{J}_2 open, \mathcal{J}_1 closed set then B is the preimage under a bicontinuous map of $\{1\}$. Suppose $(X, \mathcal{J}_1, \mathcal{J}_2)$ is initial for its continuous maps into $(D, u, 1)$. Subbasic \mathcal{J}_1 open sets are of the form $f^{-1}(0)$ where f is a bicontinuous map. Now $X - f^{-1}(0) = f^{-1}(1)$ which is \mathcal{J}_2 open. Thus $f^{-1}(0)$ is \mathcal{J}_2 closed and \mathcal{J}_1 has a subbase of \mathcal{J}_1 open \mathcal{J}_2 closed sets. Similarly \mathcal{J}_2 has a subbase of \mathcal{J}_2 open \mathcal{J}_1 closed sets.

Let POd be the category of pairwise zero dimensional bitopological spaces and bicontinuous maps. Let

$$\bar{T}_t : \underline{\text{Qut}} \rightarrow \underline{\text{POd}} \quad \text{be the forgetful functor.}$$

Theorem 3.13 leads to a characterization of the coarsest right inverse of \bar{T}_t .

CHAPTER 4

QUASI-PSEUDOMETRIC SPACES

In this chapter we obtain generalizations of some metrization theorems in the literature. Of particular interest is the role played by transitive quasi-uniformities in the proofs of these theorems.

We begin by recalling a number of definitions.

A quasi-pseudometric d on a set X is a real valued function defined on $X \times X$ and satisfying

- (1) $d(x, y) \geq 0$ for all x, y in X .
- (2) $d(x, x) = 0$ for all x in X .
- (3) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z in X .

(3) is often referred to as the triangle inequality.

If d also satisfies

- (4) $x \neq y$ implies $d(x, y) > 0$

then d is a quasi-metric (J.C. Kelly 1963).

If d satisfies the stronger triangle inequality

- (5) $d(x, y) \leq \max \{ d(x, z), d(z, y) \}$ for all x, y, z in X ,

then d is a non-archimedean quasi-(pseudo)metric.

A quasi-pseudometric d for which $\{d(x,y) : (x,y) \in X \times X\} \subset \{0,1\}$ will be termed an e-quasi-pseudometric (Levine 1969).

An e-quasi-pseudometric is clearly non-archimedean.

If d is a quasi-(pseudo)metric on X then so is its conjugate d^* which is defined by $d^*(x,y) = d(y,x)$.

Every family \mathcal{D} of quasi-pseudometrics induces a topology $\mathcal{J}(\mathcal{D})$, where the subbasic open neighbourhoods of a point x are the sets $\{y:d(x,y) < r\}$, $d \in \mathcal{D}$ and $r > 0$. Hence every family \mathcal{D} of quasi-pseudometrics induces a bitopological space $(X, \mathcal{J}(\mathcal{D}), \mathcal{J}(\mathcal{D}^*))$.

A (bi)topological space is said to be quasi-pseudometrizable if its (bi)topology can be induced by a single quasi-pseudometric in the above manner.

A family \mathcal{D} of quasi-pseudometrics generates a quasi-uniformity \mathcal{U} if the sets $\{(x,y):d(x,y) < r\}$ $d \in \mathcal{D}$ and $r > 0$, form a subbase for \mathcal{U} . We note that if a quasi-pseudometric d generates a quasi-uniformity $\mathcal{U}(d)$ then $T\mathcal{U}(d) = \mathcal{J}(d)$. (Murdeswar and Nainpally 1966, p30).

The following result is a simple generalization of a result of N. Levine (1969) for Uniform Spaces. (I have recently found that S. Salbany has also known this result for some time).

Theorem 4.1 A quasi-uniform space (X, \mathcal{U}) is transitive iff \mathcal{U} is generated by a family of e-quasi-pseudometrics.

Proof: Let \mathcal{U} be transitive and let U be a basic transitive entourage.

We define

$$d(U)(x,y) = \begin{cases} 0 & \text{if } (x,y) \in U \\ 1 & \text{otherwise} \end{cases}$$

Let $x, y, z \in X$.

If $d(U)(x,y) = 0$, then $d(U)(x,y) = d(U)(x,z) + d(U)(z,y)$

If $d(U)(x,z) = 1$, then $d(U)(x,y) \leq d(U)(x,z) + d(U)(z,y)$

since $d(U)(x,z) = 1$ or $d(U)(z,y) = 1$ by the transitivity of U .

Thus the triangle inequality is satisfied and $d(U)$ is an e-quasi-pseudometric. Clearly $U = \{(x,y) : d(x,y) < 1\}$.

Suppose that \mathcal{D} is a family of e-quasi-pseudometrics which generates \mathcal{U} . Then for each $d \in \mathcal{D}$, $U(d) = \{(x,y) : d(x,y) < 1\}$ is transitive and the family $\{U(d) : d \in \mathcal{D}\}$ is a transitive subbase for \mathcal{U} .

Corollary 4.2 A bitopological space $(X, \mathcal{J}_1, \mathcal{J}_2)$ is pairwise zero dimensional iff $(X, \mathcal{J}_1, \mathcal{J}_2)$ is induced by a family of e-quasi-pseudometrics.

Proof: $(X, \mathcal{J}, \mathcal{J}_e)$ is pairwise zero dimensional iff it has a compatible transitive quasi-uniformity (Theorems 3.4 and 3.6).

Corollary 4.3 A transitive quasi-uniform space (X, \mathcal{U}) has a countable base iff it is generated by a countable family of e -quasi-pseudometrics.

Proof: Clearly if \mathcal{U} has a countable base then it has a countable transitive base.

We now discuss a theorem proved in (Fletcher and Lindgren 1972).

Two definitions are required.

Definition 4.4 (Fletcher and Lindgren 1972)

Let (X, \mathcal{J}) be a topological space and let \mathcal{C} be a collection of open sets such that if $x \in X$, then $\{C \in \mathcal{C} : x \in C\} \in \mathcal{J}$. Then \mathcal{C} is a Q-collection.

Definition 4.5 (Fletcher and Lindgren 1972)

Let (X, \mathcal{J}) be a topological space and let \mathcal{B} be a base for \mathcal{J} .

If there exists a sequence $\{\mathcal{B}_i\}_{i=1}^{\infty}$ of Q-collections such that $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, then \mathcal{B} is a σ -Q-base for \mathcal{J} .

Theorem 4.6 (Fletcher and Lindgren 1972)

Let (X, \mathcal{J}) be a T_1 topological space. Then the following are equivalent

- (i) There exists a σ - \mathcal{Q} -base for \mathcal{J} .
- (ii) (X, \mathcal{J}) is generated by a non-archimedean quasi-metric ρ .
- (iii) (X, \mathcal{J}) has a compatible transitive quasi-uniformity with a countable base.

Since our interest extends also to bitopological spaces we examine firstly those aspects of the above theorem which can be presented in a quasi-uniform setting rather than in a topological one.

Theorem 4.7 A transitive quasi-uniform space (X, \mathcal{U}) has a countable base iff \mathcal{U} is generated by a non-archimedean quasi-pseudometric ρ .

Proof: Let $\{U_i\}_{i=1}^{\infty}$ be a transitive base for \mathcal{U} . Let $x, y, z \in X$.

If $(x, y) \in U_j$ for each j we define $\rho(x, y) = 0$. Otherwise we let

$\rho(x, y) = \frac{1}{2^j}$ where j is the least positive integer such that $(x, y) \notin U_j$.

If $\rho(x, y) \leq \frac{1}{2^i}$ and $\rho(y, z) \leq \frac{1}{2^i}$, then for $1 \leq k < i$, $(x, y) \in U_k$ and

$(y, z) \in U_k$. By the transitivity of the U_k , $(x, z) \in U_k$ for $1 \leq k < i$ and

so $\rho(x, z) \leq \frac{1}{2^i}$. Thus for $x, y, z \in X$, $\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}$.

ρ is compatible since for each i , $\bigcap_{j=1}^i U_j \subset \{(x, y) : \rho(x, y) < \frac{1}{2^i}\} \subset U_i$.

Suppose \mathcal{U} is generated by a non-archimedean quasi-pseudometric ρ

For each positive integer n , let $U_n = \{(x, y) \in X \times X : \rho(x, y) < \frac{1}{n}\}$.

Then $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ a transitive base for \mathcal{U} .

Remark 4.8 We now recall our earlier remark that any family of subsets of a set X can be made into a cover of X by adding the set X to the family.

Theorem 4.9 Let (X, \mathcal{J}) be a topological space. Then (X, \mathcal{J}) has a compatible transitive quasi-uniformity with a countable base iff there exists a σ - Q -base for \mathcal{J} .

Proof: Let $\{U_i\}_{i=1}^{\infty}$ be a transitive base for a compatible quasi-uniformity for (X, \mathcal{J}) . As in the proof of Theorem 2.14, it follows that

for each positive integer i , $\mathcal{C}_{U_i} = \{U_i[x] : x \in X\}$ is a Q -cover of X .

Thus $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{C}_{U_i}$ is a σ - Q -base for \mathcal{J} .

Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ be a σ - Q -base for \mathcal{J} . By our remark above each \mathcal{B}_i defines a Q -cover of (X, \mathcal{J}) . Since \mathcal{B} is a base for \mathcal{J} we have that for each $A \in \mathcal{J}$ and $x \in X$, there exists a Q collection \mathcal{B}_i such that $\bigcap_{B \in \mathcal{B}_i} B[x] \subset A$. Thus the \mathcal{B}_i define an admissible collection of Q -covers of (X, \mathcal{J}) and there exists a compatible transitive quasi-uniformity with a countable base for (X, \mathcal{J}) .

Remark 4.10 Since a quasi-pseudometrizable topological space is quasi-metrizable iff the topology is T_1 , we recover the Fletcher-Lindgren Theorem 4.5 from Theorems 4.9 and 4.6.

A result analogous to Theorem 4.9 can be obtained for pairwise zero dimensional bitopological spaces. The following definitions are necessary.

Definition 4.11 A pair $(\mathcal{C}, \tilde{\mathcal{C}})$ of covers of a set X is called a complementary paircover of X if $\tilde{\mathcal{C}} = \{X - A : A \in \mathcal{C}\}$. Recalling the notation of 2.1, we have:

Lemma 4.12 Let $(\mathcal{C}, \tilde{\mathcal{C}})$ be a complementary paircover of a set X .

Then, $R_{\tilde{\mathcal{C}}}^{-1} = R_{\mathcal{C}}^{-1}$

Proof: $R_{\mathcal{C}}^{-1} = \{(x, y) : (y, x) \in R_{\mathcal{C}}\}$
 $= \{(x, y) : (\forall C \in \mathcal{C})(y \in C \implies x \in C)\}$
 $= \{(x, y) : (\forall C \in \mathcal{C})(x \in (X - C) \implies y \in (X - C))\}$
 $= \{(x, y) : (\forall C \in \tilde{\mathcal{C}})(x \in C \implies y \in C)\}$
 $= R_{\tilde{\mathcal{C}}}$

Definition 4.13 Let $(X, \mathcal{J}_1, \mathcal{J}_2)$ be a bitopological space. If \mathcal{C} is an SN cover for \mathcal{J}_1 and $\tilde{\mathcal{C}}$ is an SN cover for \mathcal{J}_2 then $(\mathcal{C}, \tilde{\mathcal{C}})$ is called an SN paircover for $(X, \mathcal{J}_1, \mathcal{J}_2)$.

A pair base for $(X, \mathcal{J}_1, \mathcal{J}_2)$ is a pair $(\mathcal{B}, \tilde{\mathcal{B}})$ such that \mathcal{B} is a base for \mathcal{J}_1 and $\tilde{\mathcal{B}}$ is a base for \mathcal{J}_2 .

Let $(X, \mathcal{J}_1, \mathcal{J}_2)$ be a pairwise zero dimensional space and let $(\mathcal{B}, \tilde{\mathcal{B}})$ be a pairbase for $(X, \mathcal{J}_1, \mathcal{J}_2)$. If there exists a sequence $\{\mathcal{B}_i, \tilde{\mathcal{B}}_i\}_{i=1}^{\infty}$ of complementary SN paircovers such that $\{\mathcal{B}, \tilde{\mathcal{B}}\} = \{\bigcup_{i=1}^{\infty} \mathcal{B}_i, \bigcup_{i=1}^{\infty} \tilde{\mathcal{B}}_i\}$, then $(\mathcal{B}, \tilde{\mathcal{B}})$ is a σ -SN pairbase for $(X, \mathcal{J}_1, \mathcal{J}_2)$.

Theorem 4.14 A bitopological space $(X, \mathcal{J}_1, \mathcal{J}_2)$ has a compatible transitive quasi-uniformity \mathcal{U} with a countable base iff $(X, \mathcal{J}_1, \mathcal{J}_2)$ has a σ -SN pairbase.

Proof: Suppose \mathcal{U} has a countable transitive base $\{U_i\}_{i=1}^{\infty}$. Let $\mathcal{B}_i = \{B : B = U_i[x] \text{ some } x \text{ in } X \text{ or } B = X - U_i^{-1}[x] \text{ some } x \text{ in } X\}$ and $\tilde{\mathcal{B}}_i = \{B : B = U_i^{-1}[x] \text{ some } x \text{ in } X \text{ or } B = X - U_i[x] \text{ some } x \text{ in } X\}$. Since each $U_i^{-1}[x]$ is \mathcal{J}_1 closed (see proof of Theorem 3.4), $(X - U_i^{-1}[x])$ is \mathcal{J}_1 open for each x in X . Hence \mathcal{B}_i is a \mathcal{Q} cover for \mathcal{J}_1 . Similarly $\tilde{\mathcal{B}}_i$ is a \mathcal{Q} cover for \mathcal{J}_2 . Clearly $(\mathcal{B}_i, \tilde{\mathcal{B}}_i)$ is a complementary pairbase for each i . Hence $(X, \mathcal{J}_1, \mathcal{J}_2)$ has a σ -SN pairbase.

Let $(\mathcal{B}, \tilde{\mathcal{B}}) = (\bigcup_{i=1}^{\infty} \mathcal{B}_i, \bigcup_{i=1}^{\infty} \tilde{\mathcal{B}}_i)$ be a σ -SN pairbase for $(X, \mathcal{J}_1, \mathcal{J}_2)$. As in the proof of Theorem 4.9 the \mathcal{B}_i form an admissible collection of SN-covers for \mathcal{J}_1 and the $\tilde{\mathcal{B}}_i$ form an admissible collection of SN-covers for \mathcal{J}_2 . Since $R_{\mathcal{B}_i}^{-1} = R_{\tilde{\mathcal{B}}_i}$ the family $\{R_{\mathcal{B}_i}\}_{i=1}^{\infty}$ is a countable transitive base for a compatible quasi-uniformity for $(X, \mathcal{J}_1, \mathcal{J}_2)$.

Corollary 4.15 A bitopological space is non-archimedeanly quasi-pseudometrizable iff it has a σ -SN pairbase.

Proof: Clear, from Theorem 4.7.

It follows from Lemma 3.5 that a bitopological space which has a σ -SN pairbase is pairwise zero dimensional. We thus obtain as a corollary a theorem of Reilly.

Corollary 4.16 (Reilly 1972)

If a bitopological space is non-archimedeanly quasi-pseudometrizable then it is pairwise zero dimensional.

CHAPTER 5

COMPLEMENTS AND PROBLEMS

In this chapter we present a number of topics complementary to, or originating from, our preceding discussion and point to various problems which arise.

(1) Previously we have written $\bar{T} : \underline{Qu} \rightarrow \underline{Bitop}$. In fact the range of the forgetful functor \bar{T} is the full subcategory \underline{Pcr} of \underline{Bitop} whose objects are the pairwise completely regular spaces (Lane (1967) or Brümmer (1970)). Recall the forgetful functor $T : \underline{Qu} \rightarrow \underline{Top}$. We define a forgetful functor $K : \underline{Pcr} \rightarrow \underline{Top}$ by $(X, \mathcal{P}, Q) \rightsquigarrow (X, \mathcal{P})$. Clearly $K\bar{T} = T$.

Salbany (1971) proved that K has a unique right inverse, which we denote by Q , and asked: If F is any right inverse of T , does there exist a right inverse \bar{F} of \bar{T} such that $\bar{F}Q = F$? In a paper which is in preparation Brümmer answers Salbany's question in the affirmative by giving a method of constructing \bar{F} from F . Since, clearly, $Q = \bar{T}P$ where P is the Pervin functor we remark that the range of Q is contained in \underline{POd} . Not every object in \underline{POd} is reached by Q however, as is shown by example 3.8. The question of uniqueness of the above factorization has not been settled.

(2) Let us now recall the manner in which the coarsest right inverse $\bar{\mathcal{C}}_t$ of $\bar{T}_t : \underline{Qu} \rightarrow \underline{POd}$ was constructed. For each $X \in |\underline{POd}|$, $\bar{\mathcal{C}}_t(X)$ is initial on X for the set of all bicontinuous maps $X \rightarrow (D, u, l)$ with respect to (D, uqu) . Similarly the coarsest right inverse $\bar{\mathcal{C}}^*$ of $\bar{T} : \underline{Qu} \rightarrow \underline{Pcr}$ associates with each $X \in |\underline{Pcr}|$ the initial quasi-uniformity on X for the bicontinuous maps $X \rightarrow (I, u, l)$ with respect to (I, uqu) . Brümmer (1969) showed that the Pervin quasi-uniformity on a topological space X is initial for the set of all continuous maps $X \rightarrow (D, u)$ with respect to (D, uqu) . Thus we may think of the right inverses $\bar{\mathcal{C}}_t$, $\bar{\mathcal{C}}^*$ and \mathcal{P} as being "induced" by the single objects (D, uqu) , (I, uqu) and (D, uqu) respectively. Right inverses which may be induced by single objects in this manner are called simple (Brümmer 1971). A formal discussion of such right inverses may be found in (Brümmer 1971).

It was also shown in the above mentioned thesis that the finest right inverses of the forgetful functors $T : \underline{Qu} \rightarrow \underline{Top}$ and $\bar{T} : \underline{Qu} \rightarrow \underline{Pcr}$ respectively are non-simple and the question was raised as to whether these are the only non-simple right inverses of the respective functors. In a private communication to Brümmer, Fletcher has pointed out that an example of Kofner (1973) does not have a transitive fine quasi-uniformity, thus answering a question posed in (Fletcher 1972). As has been observed by Brümmer, this implies that the finest right inverse of T_t is another example of a non-simple right inverse of T . The proof follows easily from the following two facts. (a) For each infinite cardinal m there

exists a right inverse F_m of T_t such that $m < n$ implies $F_m < F_n$.

(Proof as in Brümmer 1969). (b) For each simple right inverse G of T_t , there exists m such that $G \leq F_m$. (We need only take m sufficiently large).

(3) Fletcher has also pointed out that Kofner's example leaves open the problem of giving a purely topological characterization of those space for which the fine quasi-uniformity is transitive.

(4) We have not specifically discussed methods of constructing compatible transitive quasi-uniformities for pairwise zero dimensional bitopological spaces. In Chapter 4, however, we did discuss the construction of compatible transitive quasi-uniformities with a countable base. The method used there may easily be generalized to give a bitopological analogue of the construction of Fletcher, discussed in Chapter 2, as follows: a family \mathcal{A} of complementary SN paircovers of $(X, \mathcal{P}, \mathcal{Q})$ is said to be admissible if the families $\mathcal{A}(\mathcal{C}) = \{ \mathcal{C} : (\mathcal{C}, \tilde{\mathcal{C}}) \in \mathcal{A} \}$ & $\mathcal{A}(\tilde{\mathcal{C}}) = \{ \tilde{\mathcal{C}} : (\mathcal{C}, \tilde{\mathcal{C}}) \in \mathcal{A} \}$ are admissible collections of SN covers for (X, \mathcal{P}) & (X, \mathcal{Q}) respectively. Then the family $\{ R_{\mathcal{C}} : (\mathcal{C}, \tilde{\mathcal{C}}) \in \mathcal{A} \}$ is a subbase for a compatible transitive quasi-uniformity on $(X, \mathcal{P}, \mathcal{Q})$. Let \mathcal{A}_X be an admissible collection of complementary SN paircovers of $(X, \mathcal{P}, \mathcal{Q})$ for each $(X, \mathcal{P}, \mathcal{Q}) \in |\text{POd}|$. Then the species $\{ \mathcal{A}_X : X \in |\text{POd}| \}$ is called a continuous species of

paircovers for POd provided that the species $\{ \mathcal{A}_X(\mathcal{C}) : X \in |\text{POd}| \}$ and $\{ \mathcal{A}_X(\tilde{\mathcal{C}}) : X \in |\text{POd}| \}$ are continuous pspecies of covers for Top. It can be shown, using the methods of Theorems 2.29 and 4.14 that every right inverse of \bar{T}_t can be generated by a continuous species of paircovers.

(5) Salbany (1973) defined a completion (X^*, U^*) for quasi-uniform spaces (X, U) as follows: Let X^* be the set of all $U \vee U^{-1}$ -Cauchy filters on X and U^* the quasi-uniformity on X with base elements $U^* = \{ (\alpha, \beta) \in X^* \times X^* : (\exists A \in \alpha, B \in \beta) (A \times B \subset U) \text{ for all } U \in U \}$.

We show Qut is closed for the taking of completions.

Suppose (X, U) is transitive and consider U (transitive) $\in U$.

We show U^* is transitive.

Let $(\alpha, \beta) \in U^*$ and $(\beta, \gamma) \in U^*$. Then there exist $A \in \alpha$ and $B \in \beta$ such that $A \times B \subset U$ and there exist $B' \in \beta$ and $C \in \gamma$ such that $B' \times C \subset U$. Let $B \cap B' = B''$ say, then $B'' \in \beta$. Thus $A \times B'' \subset U$ and $B'' \times C \subset U$. Thus $(\alpha, \gamma) \in U^*$.

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