

A synthesis of cognitive science research on the *mental number line*, and
its relation to the pedagogical use of number lines in the teaching of
elementary arithmetic in the Foundation Phase of schooling

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Abstract

In this paper, I examine how biologically endowed computational systems help to frame our experiences as it relates to using the number line in school mathematics. This study is a synthesis of the research findings in cognitive science and mathematics education to elucidate the mechanisms that underpin the connection between the mental number line in humans and its relation to the growth of mathematical knowledge in young children.

This reported research drawn together here shows how we draw on core domain knowledge (the object tracking system (OTS), the approximate number line (ANS) and the mental number line (MNL) to construct mathematical ideas. The OTS allows us to track individual items, which provides us with the notion of exact number and the ANS creates an intuitive number sense from which we intuit that a collection can be assigned a cardinal value. Language mediates the integration of knowledge of the ANS and OTS to overcome the limits of the core systems and build the exact number system.

I review the literature to investigate spatial numerical associations and the properties of the internal number line and to clarify the relationship between cognitive development, cultural factors and education.

The synthesis concludes by using the ideas of representations and structure preservation to examine how closely the number line aligns with our natural intuitions about magnitude and number and the implications this has on education.

I argue that educators need to understand how biologically innate conceptions of number guide subsequent learning and provides us with a foundation for domains such as mathematics. This insight will enable educators to select teaching models that build on from our intuitive notions of number and understand why certain concepts are difficult for children to understand.

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List of abbreviations

ANS	Approximate number system
CAPS	Curriculum and assessment policy statements
DBE	Department of Basic Education
ENL	Empty Number Line
fMRI	functional Magnetic Resonance Imaging
hIPS	horizontal Intraparietal Sulcus
ICM	Integrated Causal Model
MNL	Mental Number Line
OM	Operational Momentum
OTS	Object Tracking System
PET	Positron Emission Topography
SNA	Spatial Numerical Associations
SNARC	Spatial numerical association response codes
SPM	Scalloped Power Model

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Glossary

approximate number system: an innate magnitude system that enables humans and non-humans to process large collections imprecisely.

cardinality: refers to the measure of a set's size

cardinality principal: to understand that the last word in the count represents the cardinality of a collection

isomorphism: defines a one-to-one correspondence between two or more sets or the parts that make up or constitute the set.

mental number line: an internal magnitude system

numerosity: refers to the number of elements in a set

number line: a representation of real numbers on a straight horizontal line where points are placed at equal intervals to correspond to a real number. Numbers are placed in ascending order where larger numbers are placed to the right of smaller numbers.

open/empty number line: a tool used by teachers and learners to model and develop strategies for addition and subtraction. Only the numbers needed for their calculation need to be reflected on the number line and the accuracy of the distance between numbers are inconsequential to the relationship between the numbers

ordinality: the sequential order of numbers derived from the successor function - by greater than/smaller than relations between numbers

object tracking system: an innate system-not exclusively a magnitude system

representations: involves setting up a correspondence between the appropriate physical objects in the situation and mathematical objects of interest, and then defining one or more operations over the mathematical objects that correspond to the operations over the physical objects.

successor function: defines how the natural numbers/integers are generated by adding one

Weber's law: is a mathematical equation that states that the ability to differentiate between unequal stimulus values varies proportionally as a function of the ratio difference

Chapter 1

Introduction

1.1 Research focus

Many researchers in the field of cognitive science posit the idea of the spontaneous—that is, untutored—development of a ‘mental number line’ in humans (e.g., Dehaene 2011), which one might refer to as a species property of creatures like us. The problem I explore in this project is that of how the existence of such a computational resource might relate to the use of number line representations of number and number operations in the elementary mathematics education in schooling.

To that end, I draw on cognitive science research as a critical resource for reading and assessing the mathematics education research on the use of the number line for facilitating the growth of number sense in young children. I use a synthesising approach to organise, integrate, and analyse the research on numerical cognition of relevance to my research problem. Specifically, I draw on research in cognitive and experimental psychology, behavioural studies of infants, brain imaging studies on the effects of lesions on numerical computation, neuroscience, as well as mathematics education to elucidate the mechanisms that underlie the connection between the purported existence of a MNL in humans and its relation to the growth of mathematical knowledge in young children.

1.2 Number sense

Berch (2005) distinguished between two kinds of number sense: (1) a ‘lower order’ skill set that is rooted in our biological endowment of quantity, and (2) a ‘high order’ skill set that develops with experience and enculturation. He observed that cognitive scientists generally refer to number sense as part of the core system of innate endowments, which limits the idea to intuitions about small numerosities and our ability to compare large numerosities. The innate biological components are connected to the ‘higher order’ perspective of number that enables us to understand our environment in terms of numerosities and magnitude.

1.2.1 Number sense in mathematics education

The pedagogical development number sense is one of the central goals of the South African Curriculum and Assessment Policy Statements (CAPS) for mathematics in Foundation Phase education (Grades R to 3). The CAPS state that each learner should have expanded their repertoire of skills in mathematics to include: “the meaning of different kinds of numbers, to understand relationships between different kinds of numbers, to understand the magnitude of numbers, to know

how numbers can be represented and to understand the effect of operations on numbers” (DBE, 2011a, p.9). Sixty-five percent of mathematics content taught in Grade 1 is devoted to developing number sense and operational fluency; 60% in Grade 2; and 58% in Grade 3. The notional time allocated to the topic Numbers, Operation and Relationships is significantly higher than that for other content areas in the mathematics curriculum, to ensure that learners are “competent and confident with numbers and calculations” (DBE, 2011a, p.10) once they exit the Foundation Phase.

There appears to be a reasonable consensus amongst mathematics education researchers concerning what children ought to do to demonstrate good number sense. In particular, children need to be able to work flexibly to solve number problems; choose appropriate methods that employ numerical and operative properties that display an understanding of numbers and numerical operations; and to have a good sense of numbers and their relationships (Anghilari 2006; Howden 1989, cited in Greeno, 1991; Sowder 1990). McIntosh, Reys and Reys (1992, p.4) identified three key components of number sense to include a knowledge of numbers, operations and its application in computational settings.

Several studies in mathematics education support the use of the number line to teach number sense to enhance children’s understandings of magnitude, number sequence and equivalence of number (Behr *et al.* 1983; Fueyo and Bushel 1998; Hannula 2003; Cramer, Ahrendt, Monson, Wyberg and Miller 2016; Bobis and Bobis 2005). Furthermore, success on number line estimation tasks has been found to be a good predictor of future mathematics achievement (Schneider *et al.* 2018; Sasanguie *et al.* 2012).

1.3 The use of the number line in schools

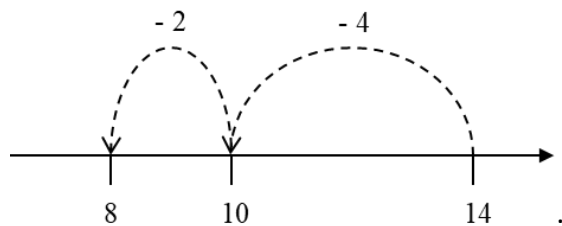
The number line was embraced as a teaching tool by the pedagogic constructivists because of its potential as a tool for thinking about numbers relationally. It was introduced into the Dutch school system as part of the Realistic Mathematics Education programme in the 1970s but was later abandoned because developers thought that the nature of the structured number line, with fixed markers for every natural number, impeded the progression from counting to more efficient and sophisticated computational methods (Klein *et al.* 1998).

A revised model of the number line was introduced by Treffers and De Moor in the 1990s. They argued that an empty number line was more suited to RME for several of the reasons outlined by Klein, Beishuizen and Treffers (1998, pp.446 - 7), viz.:

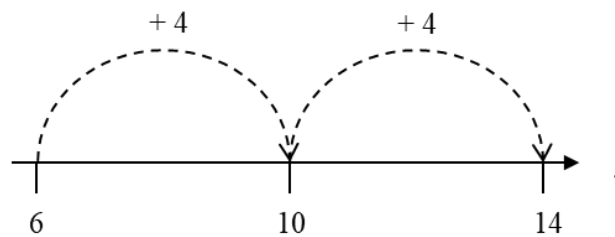
1. The ENL links up with children's untutored solution procedures because the number line can represent counting on one's fingers. The raising of fingers to signal addition can be represented as 'hopping' to the next number on the number line since here is a direct link between the fingers used and the hops on the line.

2. The ENL facilitates the use of more efficient counting strategies and increases the level of children's participation, since children are encouraged to develop their own solution strategies, allowing them to perform computations like, for example, $14 - 6$ in multiple ways:

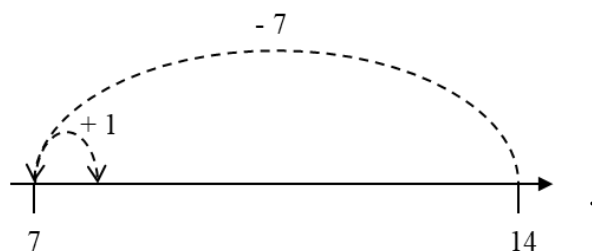
a) subtracting from 14 by counting backwards and using tens as a resource:



b) counting on from 6: $6 + 4 + 4 = 14$, again using tens:



c) using number facts (e.g., doubling or halving): $14 - 7 + 1 = 8$:



3. The ENL made implicit solution strategies clear because of its linear character.

4. The cognitive specificities of children's actions were made apparent since it was observed that children solved the computation task while simultaneously drawing jumps on the number line.

The enthusiasm for the use of the number line in RME appears to derive from an intelligent appreciation of its productive pedagogic effects in the teaching of elementary arithmetic but there is

little explicit engagement with the question of why the human brain might experience the number line as productive.

According to Ansari and Coch (2006) and De Smedt *et al.* (2011), a collaboration between empirical research in mathematics education and cognitive science could mutually benefit and inform future research projects in both disciplines with positive outcomes for curriculum development.

It is to the question of *why* such a pedagogical resource might be effective that I address from a cognitive scientific perspective, to which I turn in the chapters that follow.

1.5 Outline of the dissertation

I start by considering the contributions from cognitive science on how the brain represents number, discussing the nature of mathematical knowledge, starting with the twin ideas of *core* and *noncore domains of knowledge*. My focus is on how children construct mathematical ideas by drawing out the appropriate results of the *object tracking system* (OTS), the *approximate number system* (ANS), the *mental number line* (MNL) and *finger gnosis*.

Finally, I discuss how the afore-mentioned systems connect with the *exact number system* and *counting*. I conclude by discussing the ideas proposed by the *Realistic Mathematics Education* (RME) research literature that support the use of an *empty number line* (ENL)—also referred to as an *open* or *unstructured, number line*—to enhance number sense in young children.

Chapter 2

An account of approaches to the study of mind

2.1 Introduction

The search for an explanation into the origin of human cognition has fuelled many philosophical debates and motivated research through the ages. The debate is primarily polarized around rationalist and empiricist perspectives, both of which seek to discover accurate and true knowledge of the world. *Rationalism* refers to the idea that we are born with inherent knowledge and that our understanding of the world can be derived only through intellectual effort and deductive reasoning about the world. In contrast, empiricist beliefs privilege sensory perception and, to varying degrees, implicitly or explicitly, argue that our mind is a blank slate at birth and that we acquire knowledge through systematic observations of the world around us.

In this chapter I examine the two predominant perspectives surrounding human cognition and their implications for the teaching of mathematics as it pertains to my research on the use of the number line in elementary schooling. I will also elaborate on why I have chosen a rationalist position to proceed with in my analysis of research on the number line.

2.2 Empiricist perspectives

Approaches that originate from an empiricist paradigm assume an externalist view of mind, which presupposes that the mind must be studied with the environment in which it is situated (McGilvray, 2014: 32-33). A principal goal of empiricism is to provide an anti-nativist account of cognitive development and to reveal the nature and limits of reason, to demonstrate how the world shapes the mind. Prominent advocates of this approach were Locke, Hume, and Berkeley, who attempted to account for human knowledge by theorising how the senses provide initial knowledge and how combinatorial and associative principles could account for learning without adverting to innate ideas and knowledge (Butler-Bowden, 2013: 137-138; Palmer, 2002: 76; 90). Locke argued from a position that the brain starts out as a blank slate with some rudimentary cognitive structure. He proceeded to construct a theory to explain how perception and reflection combined to give us primitive ideas. Primitive ideas are combined to give us complex ideas and, through a process of repeating, comparing and uniting simple or complex ideas in infinite ways, new complex ideas are formed, and abstractions are generated (Palmer, 2005:188-192).

Hume followed this line of reasoning, attempting to show how our understanding of the world is tied to our observations. By appealing to associative principles of *connection, resemblance, contiguity and cause and effect*, he attempted to demonstrate how knowledge could be acquired entirely from experience (Hume, 1748/2007: 23). However, the cracks in empiricist thinking were starting to show as Hume struggled to explain how insufficient experience could lead to truth derived inductively. His theory was lacking since it could not explain one of the primary ways in which we form knowledge. Instead of conceding that not all knowledge could be underpinned by sensory and perceptual experience, he concluded that reasoning failed. He finally attributed the nature of inductive reasoning to the principles of imagination (Butler-Bowden, 2013: 138). Hume did concede that we were endowed with natural instincts, customs, and habits, in addition to the knowledge we gain from experience (Hume, 1748/2007). Commenting on the limits of creative thought, he argued that while it appeared that creative thought was unbounded and free, it was constrained by what we were able to conceive (Hume, 1748/2007: 13).

Berkeley argued that the senses and perceptions imprint ideas on the mind, which led him to question how the mind could conceive of an idea that it had not perceived (Pearce, 2017). Berkeley rejected Locke's doctrine of abstract ideas and said that it "was perfectly inconceivable" and "incomprehensible" (Pearce, 2017). Locke had argued that we could separate the primary qualities (qualities that inhere within objects, or the measurable qualities of an object) from its secondary qualities (essentially, qualities that we attach to objects) and Berkeley argued that the two were inextricably linked because when you 'prescind' (attempt to strip) secondary qualities from the primary qualities you are left with nothing. He reasoned that words allow us to signify, or name, ideas and produce general ideas. If we attended to ideas, he argued, we would realize that they are not abstract, but rooted in an idea that was originally perceived. He conceded that number was the only thing that could be abstracted since a number could be assigned to multiple objects and depended entirely on an individual's perspective (Bennett, 2017: 13- 4; 85- 6).

Despite failing to give an adequate account of all aspects of human cognition (especially, induction and abstract thought), empiricism has been the default epistemology since the seventeenth century (Margolis and Laurence, 2013: 704; McGilvray, 2014) and many theories of learning subsequently proceeded from blank slate assumptions. The strong appeal and widespread acceptance of empiricism appears to be that it offers a comprehensive theory that is rooted in direct experience and a commonsense view of human cognition (Carey, 2009; McGilvray, 2014: 132; Tooby and Cosmides, 1992). Its focus is on causal and temporal relationships between stimuli and behaviour without considering the internal workings of the mind and its representations.

Chomsky criticised empiricist approaches, which included behaviourism, to the study of the mind. He argued that one could not observe the causal relationship between stimuli and behaviour without considering the mind and its representations. He accused the empiricists of methodological dualism in that they were prepared to employ scientific inquiry methods to investigate the body, but not in the investigation of mind. Through his linguistic research, it appeared clear to him that there were more productive ways of accounting for cognitive development (Chomsky, 1959: 2757-58).

2.3 Rationalist perspectives

Rationalists¹ argue that the mind is not an empty slate waiting to be shaped by experiences. In fact, they argue that our experiences are insufficient to account for all learning and that it is therefore plausible to assume an innate cognitive capacity. This position can be traced back to Plato's theorizing on the nature of knowledge. He concluded that the mind held innate truths which the soul had collected during its sojourns in the ideal realm. Plato's theory of anamnesis proposed that not all learning was obtained through the senses and that some knowledge was known to us *a priori*.

Descartes' theory of cognition focused on the idea that the senses were adventitious in that they provided us with the external opportunity to form ideas that the mind was innately structured to receive and interpret (Chomsky, 2009: 100, 103). Leibniz's theory, purged of Plato's notion of a pre-existing, transmigrating soul, also developed the idea that "the senses prompt the actual perception of ideas but don't provide the ideas themselves" (Bennett, 2017: 34; Hunter and Inwood, 1984: 424, 432). In addition, Leibniz argued against the analogy of learning as remembering—as suggested by Plato—because it could not account for the creation of novel and original ideas (Hunter and Inwood, 1984: 433).

Kant argued that there were two kinds of cognitive faculties that yielded knowledge. First, intuition, which is induced from the ideas provided to us by our senses, and second, understanding, which results in concepts. Unlike Plato and Descartes, Kant rejected the idea that you could have one kind of knowledge without the other, arguing that "thoughts without content are empty and intuitions without concepts are blind" (Palmer, 1995: 105). Kant claimed that the senses simply conveyed representational content and was incapable of being judged correct or not. For him the mind was structured to enable us to mediate our experiences by imposing order and guiding our experience of

¹ Cartesian rationalists separate the existence of the mind and body. Contemporary rationalists are nativist and adopt a research strategy that proceeds from an internalist position, principally, the view of the mind as a biological organ.

our environment (Butler-Bowden, 2013: 158; McLear, 2016: 98-9; Palmer, 1995: 105). Kant, therefore, observed that the relationship between the two cognitive faculties were dynamic and bi-directional and that neither intuitions nor concepts should be privileged over the other.

The nature of innate structure is a point of divergence since not all rationalists agree on the form or origin of cognitive structures. An internalist approach presupposes that the mind is a biological system that develops according to the human genetic blueprint. This blueprint places constraints on the organism and affords it with certain abilities that ensures the phylogenetic development of the organism. It aids learning by directing attention and limiting the number of “admissible hypothesis” that we would have to contend with (Chomsky, 2016: 27-28; McGilvray, 2014: 80).

2.4 Empiricism vs Rationalism in research methodology

Empiricism and rationalism are distinct approaches one could adopt in a study concerning cognitive development, or any other phenomena. While empiricists adopt an externalist position and insist that the character of the mind is moulded through experience, rationalism proceeds from an internalist position, looking at how the structure of the brain generates and processes data, both sensory and non-sensory. Another difference is that empiricists conceive of the mind as a domain-neutral cognitive structure while rationalists view the mind as highly differentiated with domain-specific innate systems (Chomsky, 2009: 6,7; Laurence and Margolis, 2001: 219; McGilvray, 2014: 127).

Piaget’s theory of cognition is an interesting case since it does not conform strictly to either paradigm. Piaget claimed to have adopted a Kantian position towards cognition since they both rejected the rationalist claim that reason and thought were sufficient for cognition, that perception was the primary source of knowledge and that we derived knowledge from objects (Piaget, 1970: 16). But that is where the similarities end. Unlike Kant, who proposed that sensory data were innate ‘intuitive knowledge’ that allowed us to perceive the world as it conformed to our knowledge of it, Piaget emphasized that we constructed knowledge through the coordinated efforts of our actions (Piaget, 1970: 21). Kant proposed twelve innate concepts (categories for understanding), whereas Piaget minimized the role of innate structures and offered domain general processes of assimilation, accommodation, and equilibration to account for development across all domains of cognition (Dehaene, 2011: 31; Karmiloff-Smith, 1995: 7; McLear, 2015: 4; Palmer, 1996: 104). Another significant difference between Kant and Piaget was that while Kant proposed a dynamic and bi-directional relationship (essentially a union) between ‘intuitions and concepts about the world, Piaget argued that the crucial aspect of cognition was its operative and not its figurative aspect (Piaget, 1970: 14-19). Piaget appeared to have underestimated the issue of innatism, not only with respect to Kant, but also in his

debate with Chomsky (Piatelli-Palmarini, 1994: 317). He maintained that the innate hypothesis was unnecessary, since language did not appear until after sensorimotor intelligence was more or less achieved (Piaget, 1970: 47). According to Piatelli-Palmarini (1994), Piaget hoped that he and Chomsky would be able to reach a compromise, but Chomsky was uncompromising on the role of innate structures in cognition and an agreement was never reached. Chomsky's central thesis is that the mind should be studied as a biological organ with a specialised structure specifically tailored for cognition. He endorses nativism and therefore adopts an internalist position which is fundamentally very different to the attempted hybrid externalist-internalist position favoured by Piaget.

One implication of an internalist approach to understanding human cognition is that one would search for answers about human cognition by looking at the neurological structures of the brain and by asking how genetic endowment enables us to think in particular ways. Secondly, through comparative studies of species, and of people of various cultures, we can construct a more comprehensive and universal view of human cognition. Finally, an internalist approach acknowledges that the environment plays a crucial role as a trigger, or prompt, that is needed for a mental system to develop, but it does not shape the system (Laurence and Margolis, 2002; McGilvray/Chomsky, 2009: 15). Much of human development is time sensitive (as with vision and language) and if the infant does not receive minimal prompting from the environment, development could be, and often is, impaired (Dehaene, 2011:103; McGilvray/Chomsky, 2009: 14-16).

Karmiloff-Smith (1995) attempted to reconcile Piaget's constructivism with nativism. Drawing on cases of early brain damage, she argued that the brain is more malleable than was previously thought, possibly implying that the relationship between nature and nurture are much more fluid than either camp proposed. However, the presence of brain plasticity cannot serve as a basis for rejecting the core nativist propositions.

2.5 Number in the brain

2.5.1 An externalist research position

Piaget tried to demonstrate how mathematical logic was constructed during the sensorimotor stage when the infant explored their environment through their senses and learned to control their engagements with their surroundings through motor actions. He argued that coordinated actions were capable of being repeated and generalized and they became mental operations, which in turn constitute structures or schemas (Piaget, 1970: 43). Piaget reasoned that the notion of object permanence in the sensorimotor stage led to conservation at the operational level which he saw as integral in developing conceptions of number. He, therefore, argued that the concept of number only

developed once the individual passed a suitable conservation test. The conservation of number test gauges the presence or absence of one-to-one correspondence², which is a key component of exact equality.

2.5.2 An internalist research position

An internalist position attempts to identify which aspects of human cognition arise independently of experience and describes how *a priori* computational resources support early competences and structure the way we experience our environment. Internalist enquiry requires an interdisciplinary and comparative approach to human cognition and development that draws on empirical studies in a wide range of disciplines, like biology, anthropology, developmental psychology, neuroscience, genetics, and linguistics to contribute towards a more robust understanding of human cognition that can guide subsequent research and, in the case of those having an interest in education, pedagogy.

Comparative studies across species have revealed that we share core cognitive systems for number with other animals (Brannon and Merritt, 2011; Dehaene, 2011: 12-17; Nieder, 2016). Furthermore, the use of neuroimaging (positron emission topography (PET) and functional magnetic resonance imaging (fMRI)) have allowed researchers to view cerebral circuits in the brain, thereby providing rudimentary evidence in support of the argument for the brain as structured in a manner proposed by rationalist positions (Dehaene, 2011: 50, 238-256; Nieder, 2016). Such research backs up nativist theories that claim humans are equipped with fundamental knowledge systems that are shared by many non-human animal species and serve as the foundation for all of our knowledge systems. (Chomsky, 2015; Hauser, Chomsky and Fitch, 2002; Laurence and Margolis, 2001).

2.5.3 Domains of knowledge

Knowledge domains have been defined as, “a set of coherent principles that serve to capture the structure of a domain of knowledge” Gelman (2009: 247). Domains are distinguished from each other based on their origin: those that are innate and biologically endowed are referred to as *core domains* and those acquired later through socio-cultural activity, *noncore domains* (Gelman, 2009: 248-249).

Core domains are innate and rudimentary, but they allow infants to identify salient aspects of an environment and constrain reasoning about domain-related entities. This effectively reduces the scope of admissible hypotheses and directs attention (Gelman, 2009: 251; Gelman and Meck, 1986: 35-36; Karmiloff-Smith, 1995: 12; Spelke, 2003). In contrast, noncore domain systems are unique human

² One-to-one correspondence should be distinguished from 1-1 mappings. The former is necessarily both injective and surjective while the latter need only be injective.

constructions acquired through rote memorization and learning. They require many hours of practise in a specific area and are richly structured (Gelman, 2009; Wynn, 1990; Carey, 2009).

Research has shown that we are born with a limited number of core domain knowledge systems for acquiring language, representing objects, agents and their actions, numbers, places, and social partners (Spelke and Kinzler, 2007). Unlike core domains, the number of noncore domain areas can be as numerous and as varied as our interest and motivation to learn in specific areas. Noncore domains are not universal but rather dictated by culture, social needs, and individual interests.

The core domain system has five key signature properties that define it. Firstly, each system is limited. It is task-specific and encapsulated (Spelke, 2003; Gelman, 2009: 248), meaning that each system is constrained by its own principles and operates independently of the others. A core system for apprehending large approximate collections is qualitatively different from the system that tracks distinct objects (Carey and Spelke, 1996: 517).

Secondly, the core domain systems are shared by other animals. Spelke (2003: 289) has observed that, in some instances, non-human animals' performances in a specific domain equal or exceed the performance of humans. Classic examples of this would be the superior navigation systems exhibited by insects when compared to our ability to navigate using geographical or celestial markers (Spelke, 2003; Collett, Chittka and Collett, 2013).

Thirdly, the core domain systems are innate and emerge early in development, independent of any specific experience. Evidence for this comes from experiments with new-born infants. By using preferential looking paradigms, Izard demonstrated that new-born infants show a sensitivity to audio-visual-numerical correspondences. The infants oriented towards the visual arrays that corresponded to the sound sequences that they heard (Izard *et al*, 2009; Streri *et al*, 2013) illustrating that new-born infants can perceive proto-number within hours of birth.

Fourthly, the systems function throughout development even once they become elaborated and more refined (as in the case with the ANS, which improves with training and education) and are never overturned. They persist even after we have constructed new knowledge systems, and we continue to make estimations of large numerosities even once we have developed counting to process them in a more accurate way. Finally, they are universal across all cultures (Gelman, 2009; Spelke, 2003: 29; Spelke and Kinzler, 2007) and species (Tudusciuc and Nieder, 2007).

2.5.3.1 Two core systems of numerical cognition

From an internalist perspective, nonverbal and non-symbolic representations of quantities underlie young children's numerical abilities. This perspective contrasts with the externalist view that claims that pre-schoolers are perception-bound. Behavioural research and neuroscience data provide evidence that two separate mental systems exist for encoding numerical information. The first is a ratio-dependent magnitude system, referred to as the approximate number system (ANS) and responsible for processing large collections. The second is the object file system which is constrained by a strict set size limit for processing small collections. Each system is characterised by its own signature limits that have been identified across species, cultures, ages, and tasks (Cordes and Brannon, 2009; Coubart *et al.*, 2014; Hyde and Spelke, 2011; Dehaene, 2011; Hyde, 2011). The ANS activates distinct neural networks (Dehaene, 2011; Feigenson, 2011; Hyde, 2011; Spelke and Kinzler, 2007) and persists throughout development, even when a symbolic number system has been acquired.

The ANS is an evolved system for representing large collections or ensembles. It is an imprecise, nonverbal, and non-symbolic system that is available to preverbal infants, older children, adults, and other animal species (Mussolin *et al.*, 2014; Izard, Sann, Spelke, Streri, 2009; Libertus and Brannon, 2010, Xu and Spelke, 2000; Cantlon and Brannon, 2006, Meck and Church, 1983, Nieder and Dehaene, 2009). Humans and non-humans can represent many objects as a collection with an approximate aggregate (Feigenson, Carey and Hauser, 2002: 150; Hyde, 2011: 1) whereas the object tracking system (OTS), or object-file system, represents each object as a distinct object file without representing it as making up the cardinality of a collection. Object files enter into judgements of numerical equivalence when attempts are made to establish one-to-one correspondences between spatially or temporally distinct collections, thereby enabling the detection of changes or differences in arrays (Feigenson *et al.*, 2002: 150; Hyde, 2011: 1; LeCorre and Carey, 2007: 398). The ANS contributes something to the notion of a collection, and we get the idea of discrete individuals from the OTS. The natural number system combines the two ideas to give us collections of individuals.

The OTS allows us to keep track of distinct objects up to a set size limit of three in infants and four in adults and non-human primates (Antell and Keating, 1983; Butterworth, Reeve, Reynolds and Lloyd, 2008; Feigenson, Carey and Hauser, 2002; McCrink and Wynn, 2004; Spelke, 2003; Wood and Spelke, 2005; Xu, 2003). Each distinct object in an array is represented by a distinct file and infants have an exact one-to-one corresponding representation for each object (since there is one file per object). This makes it possible to track each object precisely as it moves in or out of a set.

Although working memory restricts our attention to three or four visual items at a time, what counts as an item is flexible. Short-term working memory is hierarchically organised so that objects, entire collections, and ensembles can function as items in working memory thereby allowing quantification of larger numerosities. We can hold three collections in working memory and aggregate up to three objects in a collection (Feigenson, 2011).

The ANS does not have an upper limit, but precision of the ANS is characterised by Weber's law³ which describes the ratio dependence in number discrimination of two collections. It becomes more difficult to discriminate between the numerosities of two collections as the ratio approaches one. One explanation for this phenomenon is that numerosity is represented on a linearly and the noise surrounding each mental magnitude increase in proportion to that magnitude. The alternate explanation is that numerosity is represented on a logarithmic scale which results in larger numbers being compressed into a smaller space yielding an imprecise representation (Gallistel and Gelman, 1992: 56, 57; Dehaene *et al.*, 2008; Cantlon *et al.*, 2009; Brannon and Merritt, 2011: 210). The ratio-dependent effect can be observed in human adults who have received formal education and in those who have not (Pica, Lemer, Izard and Dehaene, 2004; Frank *et al.*, 2008), in preschool children (Lipton and Spelke, 2005), in new-born infants (Izard, Strerri, Spelke and Sann, 2009) and in non-human animals (Dehaene, 2011). Although the ANS is present throughout development, ratio effects change and its acuity improves from infancy to adulthood (Coubert *et al.*, 2014: 11; Feigenson, Libertus and Halberda, 2013: 75). Studies with new-borns using preferential looking paradigms conclude that new-born infants discriminate numerosities when the ratio is 3:1 (12 vs. 4, 18 vs. 6) but not for a smaller ratio of 2:1 (Izard *et al.*, 2009; Coubert *et al.*, 2014: 15), and not across visual and auditory modalities (Coubert *et al.*, 2014: 16). Izard (2009) also found that new-borns looked longer at congruent associations between sounds and visual objects (12 syllables matched with 12 objects) than non-congruent associations. By the age of 6 months, infants are adept at discriminating between dot arrays when the ratio is 2:1 (8 vs. 4), and by 9-months infants discriminate 3:2 ratios (12 vs. 8) (Coubert *et al.*, 2014: 11-13; Brannon and Merritt, 2011; Izard *et al.*, 2009; Wood and Spelke, 2005; Xu, 2003). Acuity continues to improve throughout childhood and older children and adults are able to discriminate at progressively smaller ratios (Lipton and Spelke, 2003; Halberda, Mazzocco and Feigenson, 2008). However, number estimates remain approximate. Number representations are abstract since they apply across sensory modalities as varied as an array of dots, sequence of tones

³ According to Weber's law, the ability to differentiate between two magnitudes is regulated by the proportional difference, not the absolute difference (Gallistel, 2011).

and sequence of actions, like the jumps of a puppet (Wynn, 1996; Dehaene *et al.*, 1998; Lipton and Spelke, 2003).

The Weber fraction, (w) is predominantly used to measure ANS acuity. It measures the discriminability of two numerosities, quantifying the smallest numerical change that can be made to a collection for it to be reliably perceived (Halberda and Feigenson, 2008: 1457). More accurate estimates are reflected in a smaller Weber fraction. (See Appendix 1 for an alternate way to measure ANS acuity.)

The ANS facilitates rough estimates of collection sizes and rudimentary calculations such as approximate addition and subtraction (McCrink and Wynn, 2004; Barth *et al.*, 2005) in addition to ordering numerical quantities (Coubert *et al.*, 2014: 11; Feigenson *et al.*, 2013: 74; Pica *et al.*, 2004; Barth *et al.*, 2005).

The OTS has also been recruited to compare small numerosities. Cordes and Brannon (2009) report at least four separate studies that tested and confirmed infants' (ranging from 5 to 8 months) discrimination of small collections across various perceptual arrays (visual, auditory, and tactile) and using a variety of techniques (preferential looking, where expectations were violated, and where infants had to search) with precision. Cheung and LeCorre (2017) concluded from their tests that infants as young as two years compared collections by using one to one correspondence based on numerosity.

LeCorre and Carey (2007) have proposed an additional system that represents number which they refer to as the 'set based quantificational system'. This system uses determiners like, 'a' and 'some' to quantify sets. Anthropological studies of peoples with limited vocabulary for number words exhibit this counting behaviour. The Mundurukú, for example, rely on approximate quantifiers like *some*, *many* and *small* quantity to refer to collections. They have a set of number names to five but use them approximately (Pica *et al.*, 2004). The Pirahã have an even smaller lexicon of number words and they do not express exact numbers, not even 'one'. The word 'hoi' translates to 'roughly one' and the terms to express two and many are used approximately (Frank, Everett, Fedorenko and Gibson, 2008). These counting behaviours are evident in subset knowers⁴, who do not associate the cardinal values with all the number terms in their count list. Children's number processing changes around the age of three, and they learn that a count word corresponds to a precise numerosity (Wynn, 1990).

⁴ Subset knowers might demonstrate understanding of small cardinal number terms like one or two, but when asked to handover three items, they respond in ways inconsistent with knowledge of cardinality.

2.5.3.2 Neuronal underpinnings for evidence of core domain systems for number

Numerous studies using fMRI, PET scans and studies of patients with brain lesions, provide supporting evidence for the existence of two number systems. The horizontal segment of the intraparietal sulcus (hIPS) has been identified as the cortical region associated with arithmetic tasks, including number comparison, approximation, approximating addition, and subtraction operations. This area responds to all modalities, spoken or written, dots or symbols, and varies according to the size of the number and the distance (close, or distant), in accord with the distance effect (Nieder and Dehaene, 2009; Dehaene, 2011: 239). The distance effect refers to the discriminability of two numerosities. Participants took longer to compare two numerosities that are close together, like 8 and 9, than when numerosities are further apart such as 1 and 8. In other words, as the numerosities become closer, our reaction time increases and accuracy decreases. In addition to locating a neural substrate for the ANS, Hyde and Spelke (2011) found neural signatures that could support the behavioural data for two separate number systems. They discovered two distinct neural pathways that respond to small sets as discrete objects in the occipital temporal brain region and to large sets as approximate quantities found over parietal areas.

Brain imaging also indicates a shift in activation influenced by training and education. When we are still learning arithmetic facts, the hIPS and intraparietal areas in both hemispheres are more active, but with training and practise, arithmetic facts are stored in our memory and hIPS activity decreases while activity in the left hemisphere, most notably the angular gyrus, increases (Dehaene, 2011: 241-242; Butterworth, 2011: 251). A distinct area for retrieval of subitising facts was confirmed by Dehaene (2011: 257). He observed that participants not only responded faster when they were asked to recognize one, two and three numerosities than when they had to identify collections of ten, twenty and thirty, but their responses were also more accurate.

The existence of specialised cerebral circuits for processing number disproves the hypothesis that the brain is a blank slate at birth and clearly points to brain networks that support reasoning in mathematics. This fact has obvious implications for education if we want children to be successful at mathematics. We can design pedagogies that are grounded in cognitive scientific research, with the aim of rationally connecting school mathematics with the rich set of core knowledge and skills that children possess.

2.6 Counting numbers: The exact number system

Humans have the cognitive capacity to go beyond representing number approximately, and to develop a symbolic system to represent numbers exactly, seriate a count list (ordinality), and count collections of discrete or abstract entities. Numerous studies have been conducted to describe how humans acquire natural numbers and we are beginning to understand how such acquisition connects with the two non-verbal core systems for numerical cognition. Two contrasting hypotheses have emerged from the literature (Carey, 2009; LeCorre and Carey, 2007), namely, the *continuity* and *discontinuity* hypotheses, which have implications for early childhood education.

2.6.1 The continuity thesis

Continuity theorists⁵ (e.g., Gallistel and Gelman, 1992: 65; Gelman, 2009; Karolis and Butterworth, 2016) propose that there are structural similarities between the preverbal counting system and acquisition of natural numbers that initially guide mapping between the two systems. The mappings from analogue magnitudes to corresponding numbers conform to the three ‘how-to-count’ principles central to counting (Gallistel and Gelman, 1992: 51). The counting principles require (a) that the objects to be counted be placed in one-to-one correspondence and assigned a unique tag; (b) ensure the tags have a consistent order in which they are applied across trials; and (c) that the cardinal value of the collection is denoted by the last number in a count (Gelman and Meck, 1986; Gallistel and Gelman, 1992). The thesis proposes that children will understand that each number word refers to a distinct cardinal value once they connect the list of number words to their own number tags (Piazza, 2011: 267; Wynn, 1992: 221). One of the challenges continuity theorists encounter is that of explaining how the ANS, which represents magnitude as continuous, could support the notion of exact values and serve as a basis for explaining the successor function. While the counting principles characterise counting, they do not express the capacity to generate all other positive integers, which is crucial in natural whole number system.

Chomsky (2012), Leslie *et al.* (2008) and Hauser *et al.* (2002) propose that the origin of exact number is innate. An essential property of language is its generative function which yields a recursive principle. Recursion is the “unconstrained capacity to produce a discrete infinity” (McGilvray/Chomsky, 2009:32,34,85), which is unique to our species. Recursion is not restricted to language use and operates over a range of cognitive elements including numbers. An example is the successor function, which generates the natural numbers. If we begin with concept of the one (since its innate) and one adds the next cardinal number ($n + 1$), it gives two; and when one adds one to that,

⁵Some scholars refer to the continuity hypothesis as the “principles-before” hypothesis (Wynn, 1990).

it yields three—with each item discernibly distinct from each other—and so forth. Leslie *et al.* (2008) go into detail to explain why the next number is an integer value and not a real number. Further, they ask whether the number words are constrained to integer values because we typically count discrete physical objects. To investigate that question, Shipley and Shepperson (1990) reported on research showing that when some objects in an array were cut in half, children counted each item (such as toy cars or forks) as individual items, rather than the parts of the whole. Young children (3- and 4-year-olds) made errors when they were asked to count properties or kinds of things, and Shipley and Shepperson (1990) concluded that children's counting routines were strongly biased towards discrete objects.

2.6.2 The discontinuity thesis

LeCorre and his collaborators (2006, 2007) present arguments against the continuity thesis. They cite studies where children who knew how to 'count' by rote to seven could not give a cardinal response, where children did not use counting to construct arrays of 1 to 4, and where children repeated the last word when counting a collection, rather than demonstrating understanding of the cardinal principle (Wynn, 1990: 157, 161-4). When children demonstrate knowledge of the cardinal principle, they can indicate the numerosity of the items counted, which indicates an understanding of the counting process as it relates to cardinality.

The studies by Wynn (1992) and Carey (2009) provide additional evidence in support of the discontinuity thesis. Wynn has argued that the mapping of numerosities onto number words require that children memorise the sequence of number words and develop an understanding that number words signify the properties of a collection of items and not to the individual item or label for that item. She argued that children first learn how to associate count words with objects in a collection and learn the relevant meanings of the count before they understand that we count to establish the cardinal value of a set. She proposed that children first map the number word *one*, then *two*, then *three* with each numerosity without using a counting strategy. They contrast their understanding of one with many before they understand that number words refer to a specific numerosity. She called this the *subset knower stage* since children do not yet understand the cardinality of all the numbers in their count list.

An abrupt change in their development comes when children understand the cardinal number *four* because this presumes that they understand how counting determines the numerosity of a collection. At this stage, children make two related inductions: the first is that every word in the count list labels a distinct cardinal number and the second, that each cardinal number greater than one can be

constructed by recursively adding one to generate a list of positive integers (the successor function). Once children can apply the cardinal notion to *all* the numbers within their counting range simultaneously they become *cardinal principle knowers*. From her experiments, Wynn suggests that the process is a slow, protracted affair, since a year elapses between the time that a child knows the cardinal meaning for one until they become cardinal principle knowers (Wynn,1992; LeCorre and Carey, 2007: 654).

Dehaene (2011: 260) claims that our understanding of natural numbers is the product of combining the ANS and the OTS. From the object tracking system, we track individual items giving us the notion of exact number; and the ANS creates an intuitive number sense from which we intuit that a collection can be assigned a cardinal value. Dehaene argues that these two systems “snap” together around the age of 3 or 4, which gives children access to engagement with mathematical ideas.

Carey (2009) would argue that there is no “snapping” together of separate systems. She argues that there is a conceptual discontinuity between the verbal and nonverbal representation of number. Her idea of *bootstrapping* shows how the natural numbers is a culturally constructed system that transcends both core number systems. In the first instance, she argues that, structurally, the ANS accumulator method⁶ represents numerosity in a different format than the counting system we use (Carey, 2009: 43). The counting system is an iterative process that recruits the successor function, contrasted with the accumulator, and that “it is the entire fullness of the accumulator, not the final increment alone, that represents the numerosity of the items counted. In this way number is inherently embodied in the structure of the representation, which is itself a magnitude value” (Carey, 2009; Wynn,1992: 228).

Secondly, the object tracking system cannot represent cardinal sets and it is limited to representing three individual items in infants and four in older children and adults. Number is only implicitly represented in the system. For these reasons she argues that natural numbers cannot be rooted in such preverbal systems.

LeCorre and Carey propose an enriched parallel individuation system that combines parallel individuation (which they say is not strictly the OTS, but similar) with set base quantification to map meanings of the small count words (LeCorre and Carey, 2008: 656-8) to support the construction of

⁶ The accumulator method (Meck & Church, 1983) describes how magnitudes can be represented by numerosity by imagining a container being filled by cupfuls. One cup represents an item, or an event and the accumulated cups can be compared to a magnitude which is stored as a memory.

the numeral list counting process. The count list acts as a placeholder structure and the child slowly associates the number meanings with number names through inductive reasoning. Carey argues that the successor function is induced (Cheung et al, 2017).

However, when Lipton and Spelke (2006) tested five-year-old children on the meanings of count words beyond their count list, they found that children understood that removing objects from a collection changed the cardinal value of the collection. They concluded that children understood symbolic number concepts before they mastered the verbal count routine. More studies challenging the count list as a placeholder theory found that even though ‘one-’ and ‘two-knowers’ were not able to label the cardinality of large collections, they were able to infer that unknown words in their count list suggested collections with different aggregates (Condry and Spelke, 2008). More recently, Posid and Cordes (2018) found that when subset-knowers received a five-minute intervention session in labelling and counting, they were able to identify collections of six items and improved on the ⁷Give-N tasks. They suggest that the counting behaviours exhibited by children demonstrate that they are not merely mimicking the counting procedure.

2.6.2.1 The role of language

Most researchers appear to agree that language is critical in the acquisition of natural number notions, but there is contention over how language is able to do so. For Gallistel and Gelman (1992) it is a question about how “language permits human thought to transcend the limitations imposed by the preverbal representations” (1992: 72). Similarly, Spelke and others have argued that language mediates the integration of knowledge of the ANS and OTS to overcome the limits of the core systems and build new systems of knowledge (Izard et al., 2009: 10383; Nieder, 2011: 115; Pinker, 1997: 360; Spelke, 2003: 296, 301; Spelke, 2011: 303;).

For Carey (2009) and other discontinuity theorists, the use of language to quantify collections precisely is grounded in the words we use to signify and refer to things. They use the research on the Pirahã and Mundurukú to argue that the lexicon of number words limits the ability to develop number concepts and reason about numbers. Contributing further towards this idea is the argument of Frank *et al.* (2008) that number words serve as an important ‘cognitive technology’.

⁷ Give-N-task: This task is used to assess children’s number knowledge (i.e., whether children understand the cardinal meaning of number words). Children are presented with a set of objects and asked to give the experimenter a subset of this set. Children are then classified as a non-knower’ ‘one-knower’ or ‘two-knowers’ based on their performance.

However, there are studies of Australian indigenous people who, despite having a limited lexicon of number words (a classifier language), are still able to enumerate objects in their environment (Butterworth *et al.*, 2008). They concluded that conceptual development determined the acquisition of counting words and that counting words were useful, but not necessary (Spelke and Kinzler, 2007). Finally, Condry and Spelke (2008) examined the role of language and concluded that natural number concepts develop in conjunction with number rather than prior to, or after the acquisition of number.

The current debates to understand the extent and limits of numerical knowledge will continue to contribute toward a better understanding of how children construct mathematics in a school context and inform teaching and intervention practises.

2.7 Concluding remarks

This chapter outlined reasons for preferring a rationalist approach in my research of the pedagogical use of the number line. Firstly, a rationalist approach is interdisciplinary, exploiting studies across a wide range of disciplines to provide us with a more accurate description of human cognition. In addition, it argues that humans are born with a small set of core domain knowledge systems that guide subsequent learning and provides us with a foundation for domains such as mathematics.

Furthermore, it argues that rationalism considers how biologically endowed computational systems serve to structure our experiences whereas empiricism places a premium on experiences. With an increasing awareness and understanding of the common architecture of the brain, educators will be able to adopt teaching strategies to optimize learning. The following chapter examines the use of the number line as an example of a resource that is compatible with the ANS.

Chapter 3

Space, quantity, and the mental number line

3.1 Introduction

The 1990s saw a resurgence in the use of the number line in mathematics education. It was reintroduced as an empty number line in Dutch schools primarily to improve number sense and mental arithmetic and to represent number relations graphically. The approach proved successful in schools across the Netherlands (Beishuizen, 1997) and the concept of the empty number line has since been incorporated into curricula across the world, including Australia, England, South Africa, and Indonesia (van den Heuvel-Panhuizen, 2020: 7-8). Locally, the Curriculum and Assessment Policy Statement (CAPS) also promotes the use of the structured and the empty number line as calculation resources for mathematics and as enabling of a problem-solving technique in the Foundation Phase (DBE, 2011). Several researchers have endorsed the use of the number line and agree that it supports procedural arithmetic operations (e.g., Fosnot & Dolk, 2001; Fosnot, Dolk, Cameron and Hirsch, 2004; Klein, Beishuizen, Treffers, 1998). It is claimed that the visual nature of the number line not only represents mathematical ideas but allows children to model their mathematical thinking (Klein *et al.*, 1998), and that it supports the sequential order and magnitude of number. Its adaptive and flexible nature induces children's mental activity by encouraging learners to choose and label their methods, represent their solution strategies on the number line, and share their thinking with their peers through such representations. Such are the most frequently cited reasons for choosing the number line as a pedagogic tool ahead of Dienes blocks, unifix cubes and the hundreds chart (Klein *et al.*, 1998). In recent studies, number line training games have been used successfully to improve number development in children with dyscalculia (Kucian *et al.*, 2011; Siegler and Ramani., 2009).

My goal in this chapter is to show how the linear nature of the number line supports our intuitions about number and space, making it a particularly efficacious tool for growing number sense. The research referred to in what follows confers many advantages on education, but perhaps the most valuable is that, as teachers, we should appreciate that numerical cognition is rooted in our biological cognitive systems that connect us with other animals. Knowing how the human brain is predisposed to acquiring a number system and understanding how culture and experience influence our nascent ideas adds a rich and insightful knowledge that can contribute to teaching practice and the design of productive intervention strategies.

3.2 Spatial numerical associations: The mental number line

The MNL hypothesis posits that the internal representation of number is primarily spatial in nature. The origin and characterisation of the internal number line has been the subject of many empirical studies and while the work of a growing number of researchers investigating this phenomenon add to the canon of knowledge, it also raises more questions and opens debate about the nature of spatio-numerical associations. Two main questions driving research in this area is to explain how the brain associates space and number, and to clarify the relationship between cognitive development, cultural factors and education.

3.2.1 Behavioural paradigms

A wide range of behavioural paradigms have reported on the close associations between magnitude and space exhibited by humans and non-humans. Each paradigm affords researchers a unique perspective on how numbers are conceptualised in the brain.

Number-space mappings have been reported since Galton (1880), who learned through participants' introspective accounts that they perceived numbers vividly and precisely on a number line. His findings, while questionable at the time, form part of a more acceptable study of synaesthesia⁸ today. While spatio-numerical mappings are implicit for most people, synesthetes are consciously aware of their number-spatial representations (Kadosh and Gertner, 2011: 124-5). This provides valuable information and supports some hypotheses about how number and space are represented in the human brain.

Dehaene and his colleagues (Dehaene, Bossini and Giraux, 1993) described how the MNL impinges on our response in certain tasks. While conducting a number comparison task, they noticed that participants who had to compare a number to the reference number, pressed a key to their left to indicate 'smaller' responses and a key to their right to indicate 'larger' responses faster and more accurately than participants who were forced to respond in the opposite way (Dehaene *et al*, 1993; Dehaene, 2011:69). This led Dehaene to explore similar comparison situations where number magnitude was not relevant. In parity judgement tasks (deciding whether a number was odd or even), Dehaene and colleagues (1993) noted that adults responded faster to small digits on the left-hand side whereas larger digits led to faster responses on the right-hand side. He interpreted this pattern of responses as evidence that number was spatially organised in an analogical format on a MNL. The digits were organised along a left/small to right/large continuum according to the magnitude they

⁸ Synesthesia is a neurological condition that describes cross-modal associations. An experience in one modality or percept triggers an experience in an unrelated modality.

presented. He named this phenomenon the spatial-numerical association of response codes (SNARC) effect. The SNARC effect has also been confirmed for synesthetes (Cohen Kadosh and Limor, 2011:125; Cohen Kadosh, Tzelgov and Henik, 2006) and it has been observed in many other studies (Wood, Willmes, Nuerk & Fischer, 2008). This confirms that the presence of a distance effect, a size effect and SNARC reflect the basic features of a MNL. Spatial associations are attached to the internal representation of number which is subjected to the psychophysical reactions that obey Weber's law governing sensory modalities. So, despite numerical digits allowing us to represent discrete numbers with precision, our evolutionary brain spontaneously retrieves an analogical representation for magnitude (Dehaene, 2011:64-5; de Hevia and Spelke, 2009; Fias, 2004; Moyer and Landauer, 1976:15-20; Mussolin *et al.*, 2012).

Furthermore, the number line bisection tasks⁹ have been used successfully to support the argument for universal spatial numerical associations in humans that emerge prior to and independently of instruction. Participants across various age cohorts (five-year-olds, 7-year-olds, and adults) were asked to bisect a physical line that had an array of dots on either side. De Hevia and Spelke (2009) showed that 5-year olds' performances on a number line bisection task converged with the performance of 7-year-old children and adults. Analysis of their results showed the same robust pattern of spatial numerical bias toward larger numerosities.

Moreover, Zorzi, Priftis and Umiltà (2002) tested patients with hemispatial neglect on their ability to bisect a physical line and compared it with their performance on a number interval bisection task¹⁰. Their results revealed that neglect patients systematically displaced the middle of the line segment further to the right and failed to attend to the left space. A consistent error pattern was observed in bisecting numerical intervals orally that resembled that of the number line bisection task. Neglect patients often overestimated the midpoint number consistently (e.g., choosing 17 as the midpoint between 11 and 19). These results prompted Zorzi and colleagues (2002) to propose a functional isomorphism between the MNL and physical lines (see Fias, van Dijck and Gevers, 2011 for an alternate view).

Fischer (2001) also reported that digit string bisection tasks automatically activated a spatial response replicating the SNARC effect. Digit strings composed of small numbers induced a leftward bias whereas digit strings composed of larger magnitudes induced a rightward bias. Furthermore, Fischer

⁹ Children are asked to bisect a line between two non-symbolic numbers.

¹⁰ Participants are asked to identify the midpoint of a number interval e.g., 11-19.

and colleagues (2003) findings' support the neuro-imaging studies for partially overlapping regions in the parietal cortex. They report that spatial biases in numerical tasks can be extended to saccadic movement and are therefore not limited to motor responses. See appendix 2 for additional examples of how number influences attention.

Some researchers have argued that the presence of a SNARC effect is not sufficient evidence to support the MNL hypothesis. Núñez (2011), for example, has argued that the MNL is a feature of a higher order cognitive mechanism, viz. conceptual mapping, that is culturally mediated. Santens and Gevers (2008) observed a SNARC effect when participants judged whether a number was close to, or far from a target number and Itto and Hatta (2004) demonstrated that making greater than and less than judgements also solicited a vertical SNARC effect.

Empirical studies that attempt to establish links between spatial-numerical associations and finger gnosis (to be able to mentally represent each finger distinctly) or finger counting (manumerical cognition) offer yet another hypothesis about how directional spatial bias might develop. Finger counting has a long cultural tradition and can be traced back to the late palaeolithic age. Cave paintings at Grottoe Cosquer (see Figure 3.1) provide interesting evidence of finger counting that predates reading and writing. These paintings show a fixed order for enumeration of numbers (see Figure 3.2) from a thumb extended; thumb and index finger extended, and systematically progresses to where all the fingers are extended (Overmann, 2014; Rouillon, 2006, as cited in Göbel *et al.*, 2011).



Figure 3.1: Preserved hand stencils found at Grotto Cosquer near Marseille in France.
(Source:<https://www.bradshawfoundation.com/cosquer/index.php>)

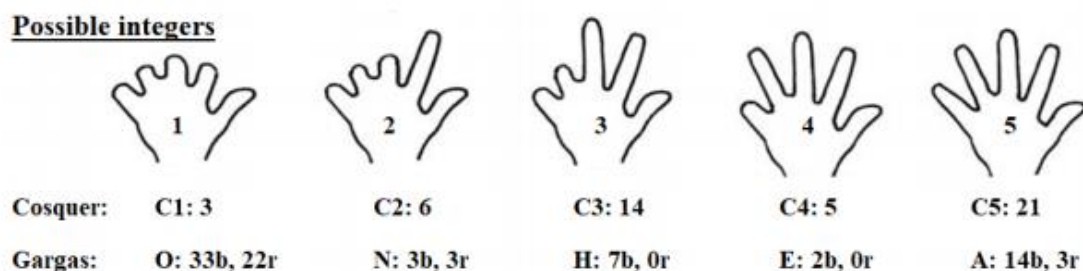


Figure 3.2: Finger-signs depicting the integers 1 through 5, found at Cosquer and Gargas (Source: Overmann, 2014)

Similarly, fingers are spontaneously used early in ontogeny. Studies have shown that new-borns can imitate finger postures and movements relatively quickly (Nagy *et al.*, 2005), and this ability has been linked to the cortical network which is implicated in the acquisition of language.

Fingers are used spontaneously in numerical representations as children start to develop ideas about ordinality, cardinality and one-to-one correspondence. Most children and even adults (Fischer *et al.*, 2014) have been known to use their fingers to count. There is an abundance of research that links the use of fingers to mathematics ability (Fayol, Barrouillet and Marinthe, 1998; Noël, 2005; Penner-Wilger, 2009), but does not consider spatial preference as an expression of SNARC or, as Fischer (2008) hypothesises, as the possible origin of spatial numerical associations (SNAs). Fischer (2008) investigated how finger counting habits contributed to a SNARC effect in adults. He claimed that finger counting habits imposed a stronger SNARC effect than reading direction (Fischer, 2008; Fischer and Brugger, 2011). His study found that adults who started counting on their left hand exhibited more consistent and stronger SNAs than adults who started counting on their right hands. No significant SNARC effect was observed in right-hand starters, which Fischer took as evidence to support his hypothesis that finger counting habits contribute towards a SNARC effect.

DiLuca *et al.* (2006) observed an interesting association between Arabic digits and counting. Participants were asked to identify a single digit by pressing a key using one of their ten fingers. They found that the mapping between Arabic digits and fingers corresponded with the finger counting habits of the participants. The participants' responses conformed to their own finger counting habits, so they would respond faster to small digits on the right with their right hand if they also started counting with their right hand and to larger digits with their left hand. Extending this evidence, Sato and Lalain (2008) tested finger counting direction of French speakers across development (4-5-year-olds, 6-7-year-olds, 10-1-year-old and 24-47-year-olds) and found a stable right hand start across the

different age groups. Dehaene and colleagues (1993) established early on that handedness was not responsible for the SNARC effect. Two subsequent studies support that conclusion.

Brozzoli *et al.*, (2008) tested whether touch stimuli would evoke a number-finger association or number-space association response. Each hand was tested separately under two conditions: palms up and palms down. They reported a SNARC effect. A touch stimulus delivered to the thumb drew a quicker response when it was preceded by number 1 than number 5 in a pronated/palm-down position. When the hand was supinated (palms up), the number 5 drew a quicker response to the thumb. The researchers argued that this number-based tactile priming and manipulation of hand posture supporting the MNL hypothesis since they were able to dissociate the order of the finger on the hand and show how the mapping was related to the position of the fingers in space.

Space-based dominance in number cognition was also confirmed in a separate study by Plasier and Smeets (2011). They asked adult participants to name a number between 1 and 30 and randomly tap one finger on either hand excluding the thumb and little finger. Their hands had to be pronated next to each other, 30 cm apart and crossed but 30 cm wide. The study revealed a tendency to associate tapping a smaller number with fingers using their left hand, demonstrating a spatially congruent movement which represents larger numbers to the right of the smaller numbers.

3.3 Innate or enculturated?

Evidence from neuroscience converges with behavioural studies to corroborate the link between hands/fingers and mental number representations.

The question of whether cultural influences or biological affordances determine the genesis of the MNL has cleared the way for more robust and interesting research paradigms. As we learn more about how numbers are conceptualised in the brain, this polarised view becomes inadequate. The conceptualisation of a MNL is multifaceted, and Fias *et al.* (2011: 145) argues that the “functional overlap between both domains illustrate how environmental, cultural and linguistic experience can shape an innate biological tendency to link different domains of cognition”.

3.3.1 The innate thesis

Various research findings all point to the fact that humans are predisposed to correlate numbers with space before they have considerable experience or understand spatial-quantity relationships, and before we acquire language (DeHe via and Spelke, 2010; Pinel, Piazza, Le Bihan and Dehaene, 2004; Lourenco and Longo, 2010; De Hevia, Izard, Coubert, Spelke and Streri, 2014).

3.3.1.1 Evidence from infants

De Hevia and Spelke's (2010) study provides proof that children as young as 8-months old spontaneously associate collections of differing cardinality (incipient 'numbers') with space. Infants who were exposed to an increasing or decreasing sequence of collections of dots, were able to match collections of dots (by numbers of dots) with length and generalise the relation to new pairings of collections of dots and lengths. Even infants who were not familiar with number-length pairings that varied positively (numbers of dots and lengths related such that fewer dots were consistently associated with shorter lengths while larger collections of dots were associated with longer lengths) paid more attention to positive pairs than when they were exposed to smaller collections of dots associated with longer lengths or larger collections of dots with shorter lengths (inverse pairs).

In a separate study, 7-month-old infants preferred an increasing left-to-right sequence rather than decreasing sequences. DeHevia *et al.* (2014a) take this to mean that spatial orientation impacts on infants' abilities to extract ordinal information from numerical sequences.

More recently, De Hevia *et al.* (2017) reported that new-borns up to 3-day-old infants exhibited a spontaneous preference for ascending numerical sequences exhibited in a left-to-right spatial orientation. Using a looking paradigm, the new-borns were able to associate a small quantity (6 sequential tones) with the left and a large quantity (18) with the right¹¹, suggesting that they are predisposed to relate quantity to different spatial positions prior to any sort of training.

It was also found that 8 to 9-month-old infants orient their attention in response to non-symbolic numerical cues. When cued by a small magnitude (2 dots), the toddlers were faster at detecting targets coming on the left than when cued by a big magnitude (9 dots). Non-symbolic numbers induced spontaneous attentional shifts congruent to the numbers' relevant positions on the line, whereas a size cue (small vs. large) did not (Bulf *et al.* 2016).

Using a new paradigm inspired by Rugani's (2015) experiments with chicks, diGiorgio *et al.* (2019) conducted a series of experiments with hour-old new-borns. New-borns were habituated with a numerosity (12 block squares) displayed side by side. Once habituation was reached, they started two trials. The first had two identical screens with four black squares on them, with the new-borns paying more attention to the screen on the left. When they were exposed to 36 squares presented side-by-

¹¹ Early work by Izard on the ANS suggested that neonates were able to discriminate numerosities with 1:3 ratio.

side, they showed a preference for the screen on the right. The percentage of looking time did not differ during the habituation stage. Two subsequent trials were conducted to control for continuous physical variables and to test the relative nature of the numbers. When they were habituated with the number 4 and presented with 12 during the test trial, they looked longer at the screen to the right. However, when they were habituated with the number 36, the 12 blocks presented during the test trial induced a longer looking time towards the left screen. The data suggest that infants spontaneously associate small numbers to the left and large numbers to the right relevant to the number/numerosity they were habituated with. These results provide evidence that spatial-numerical associations emerge from pre-linguistic biological origins, prior to cultural influences.

3.3.1.2 Evidence from other species

Studies of chicks (*Gallus gallus*) by Rugani and her colleagues have been used to show that the predisposition to map numbers onto space has a biological basis. In one study, 5-day old chicks were trained to choose the third, fourth, or sixth hole in a vertically oriented ten-hole display. (Rugani, Regolin, Vallortigara, 2007). Once the chicks showed mastery, the holes were rotated 90° so that the line was horizontal to the chicks' starting point. In the test condition, the chicks were able to select the correct position and approached the hole from the left, showing a clear leftward bias. Drucker and Brannon (2014) replicated the experiment with rhesus monkeys. They trained the monkeys to choose the fourth item from the bottom in a vertically oriented display. The monkeys used a touch screen to make their selection. After training, the array was rotated 90°, producing a horizontally oriented array. The monkeys behaved similarly to the chicks and preferentially selected the target position from the left suggesting an inherent leftward bias. However, a different study revealed that chicks trained to peck a selected position primarily used ordinal information to identify a target since a leftward bias was not observed (Rugani, Vallortigara, Vallini, Regolin, 2011). The behaviour of both the chicks and monkeys confirms the presence of inherent, biologically endowed number-space mapping in an ordinal rotation task.

More recently, Rugani and her colleagues (2015) found evidence of a non-verbal SNARC-like effect which is similar to the human number line. They trained three-day-old chicks to move around a panel to reach a food reward. At training they were habituated to a target number of elements, for example, 5, 8 or 20. The chicks were then tested with two identical panels on either side of them. Both panels displayed the identical number of elements which differed from the number of elements they were trained with. When the number of elements were less than the target number at training, the chicks spontaneously searched for the food reward on the left side. When the set of elements were larger than the target at training, they searched the right panel. In the series of three experiments, the chicks

consistently associated numerical magnitude to lateralised spatial codes (small numbers were associated to the left and large to the right). Rugani and colleagues have concluded that this could be evidence of a MNL in animals.

However, Fias and Núñez (2015) claim that establishing spatial-numerical associations are not sufficient for constituting mental number lines in preverbal populations. They argue that Rugani *et al.*, (2015) failed to establish a mapping along a line and that the chicks' associations for small/left and large/right could be attributed to the asymmetrical brain caused by the absence of a corpus callosum in the chicks' brain.

3.3.1.3 Cross-cultural evidence

Anthropological studies of indigenous people with a limited vocabulary for numbers provide us with an interesting perspective for an innate MNL. Number-space mappings observed in the Mundurukú illustrate how all humans, even those without a counting algorithm and limited vocabulary for numbers, map numbers onto space. The Mundurukú mapped numerosities on a 1-10 scale in a logarithmic way, as young western children do.

However, Núñez *et al.*, (2012) noticed that a significant number (37%) of Mundurukú participants in a study produced bimodal responses and did not use the full extent of a line to map numerosities. Such behaviour was confirmed when the Yupno¹² clustered the numbers around the endpoints and also failed to distribute them along the line. 1 and 2 mapped to the left endpoint and 3 to the right endpoint. What's more, such a pattern of number placement could be explained by proportional reasoning theory, which states that placements are made closer to the two anchors at the endpoint, while the midway offers the biggest variability in placement (Rouder and Geary, 2014).

Butterworth *et al.*, (2014) found instances of spatial representations among indigenous populations in Australia that did not imply one dimensional oriented lines. The indigenous children who they tested in their study used a pattern strategy and appeared to concatenate an initial array and an added array by using a mapping strategy for items to be counted.

Variances in spatial-numerical associations have also been recorded within numerate populations. Dehaene (2011:70) first documented a reverse SNARC effect in Persian-French bilinguals who started to read and write from right to left in Persian/Farsi and from left to right in French. This led

¹² The Yupno are an indigenous group of Papua New Guinea

Zebian (2005) to investigate the spatial orientation of the MNL in Lebanese participants who were still immersed in their culture. His results confirmed Dehaene's findings that SNARC need not register a left-to-right orientation. Arabic monolinguals displayed a reverse SNARC effect and a weakened SNARC effect for bilinguals. For Shaki, Fischer and Göbel (2012) the leading question was what effect reading direction had on counting. After establishing that the reading direction correlated with the direction of counting in adults, they conducted a test with 3-year-old pre-schoolers and school children to investigate how the systematic acquisition of reading affected counting habits. Their results show that pre-schoolers show a counting bias before they begin to read. If there is a correspondence between counting and reading directions, the bias strengthens. Conversely, the counting bias diminishes in cultures with conflicting reading experiences for words and numbers.

Taken together, one could infer that spatial mapping is culturally mediated through observation and influenced by specific task conditions.

3.4 Neural underpinnings

Evidence for a MNL can also be drawn from neuroimaging studies using fMRI. This technology allows scientists to examine brain areas that are recruited while performing tasks. In this way cortical regions that are active in processing magnitude have been identified. Together with studies of patients with lesions and transcranial magnetic stimulation, a clearer understanding for the cortical connections representing numbers has emerged. The triple code model (Dehaene *et al.*, 2003) identifies three systems for processing number which coexist in the parietal circuits: a core quantity system, a verbal system, and a visual system.

The core quantity system is located bilaterally in the horizontal segment of the intraparietal sulcus (hIPS). The hIPS codes the abstract meaning of numbers. It is active in mental arithmetic and number comparison, and it is required for the semantic representation of numerical quantities. Greater neuronal activity has been observed when processing numbers than compared to other categories of objects not pertaining to number.

Next is the verbal system, which is situated in the angular gyrus. The angular gyrus can be found in the posterior inferior regions of the parietal cortex (see Figure 3.3). This area is associated with visuospatial attention. The left angular gyrus is part of a language processing region and purported to have a function in mediating the spatial representation of numbers (Göbel *et al.* 2001, Dehaene, 2003). It activates more for verbal processing and activities that involve rote verbal recollection of mathematical knowledge. The system supports multiplication and other rote memory-based

operations and comparison of number. A distance effect has also been observed, strongly suggesting the presence of a spatial representation of numbers along a MNL (Göbel *et al.*, 2001).

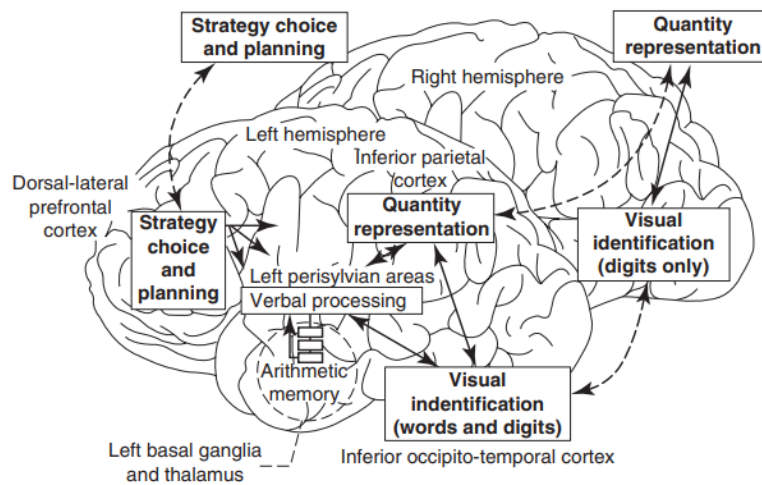


Figure 3.3: A diagram of the brain indicating the various cortical areas implicated in mathematical cognition (Source: Dehaene, 2011).

The visual system (see Figure 3.3) supports the analogue magnitude code and is in the inferior parietal regions of both hemispheres, called the posterior superior parietal lobule (PSPL). The PSPL is active during number comparisons, approximations, counting, subtracting two-digit numbers and has been found to be more active when subjects perform two operations instead of one. The PSPL is not specific to the number domain and is activated during eye movement, attention movement, grasping and pointing. This adds to the growing body of evidence that numerical magnitudes are represented by brain circuitry that also serves spatial representations (Dehaene *et al.*, 2003; Göbel, Walsh and Rushworth, 2001).

3.5 Development of the mental number line

There is now a growing body of evidence to suggest that infants spontaneously associate small magnitudes to the left and large magnitudes to the right, relative to their size (de Hevia *et al.*, 2017; de Giorgio *et al.*, 2018). They relate numerical magnitudes to spatial length (de Hevia and Spelke, 2010), map numerical magnitude to space in a specific direction (de Hevia and Spelke, 2010) and are sensitive to congruent pairings across modalities with numerical cues (tones and line lengths) (de Hevia *et al.*, 2017).

As they progress to the toddler phase, children can explore and engage more actively with the environment and their culture (McCrink *et al.*, 2014). This window of exploration and learning affords researchers an opportunity to observe and document how culture and socially transmitted

information either strengthen or override innate spatial predispositions that we see displayed in infants' behaviours.

More specific to this chapter is the directional nature of spatial associations which, according to de Hevia and colleagues (2017), have not yet been firmly established during infancy and is therefore easily malleable. A study conducted by McCrink, Caldera and Shaki (2018) demonstrated how caregivers spontaneously used gestures and spatial structure to engage their toddler's attention as they told stories and structured scenes and images that were consistent with the dominant reading and writing direction of that culture, prompting the researchers to conclude that cultural bias modulates the direction of the MNL and are present before children start to read and write.

To investigate spatial bias in toddlers, McCrink *et al.*, (2017) adapted a spatial mapping task that was initially used with chicks and monkeys (Rugani *et al.*, 2015; Drucker and Brannon, 2014) to suit toddlers aged 2-to 3-years-old. Toddlers were shown the location of an object in a horizontal array and had to find the object after the line was rearranged as a vertical array. In non-humans the transposition was asymmetrical, and chicks were able to locate the hidden object. However, there was no asymmetrical (left to right) bias observed in children who did not receive numerical or ordinal information on the vertical line. The children who received a numerical and alphabetical labelling condition successfully transposed the line and located the hidden object in the vertical array. Analyses of the results prompted McCrink and her colleagues to infer that hearing the number words activated the associated non-symbolic magnitude.

Patro and Haman (2012), presented 2-4-year-olds with two dot arrays simultaneously to test whether they would select the array that contained more items. The children in the study reacted faster to smaller arrays presented on the left and larger ones presented on the right.

There is limited research investigating spatial-numerical associations in pre-schoolers, mainly because parity judgement tasks designed for adults are developmentally not suited for young children. The challenge has been to find tasks in which number magnitude is not relevant. In a study by Opfer and colleagues (2010, 2011), pre-schoolers were shown a sampling set and matching set of boxes. An object was hidden in one of the boxes and the pre-schoolers were asked to locate the hidden objects. Pre-schoolers were able to locate the hidden object more accurately and quickly when the boxes were labelled in ascending order. Similarly, they report that 73% of pre-schoolers spontaneously started to count from left and applied this bias to the way they added and subtracted in a quantification test. Their results indicate that pre-schoolers could perform a spatial reasoning test

using number labels if the labels matched the cultural left-to-right orientation. Opfer *et al.* (2010) also examined whether the spatial direction of the number line could emerge from the action of tagging items in a left to right orientation while reciting the number sequence. They found that pre-schoolers' ability to exhibit good knowledge of cardinality did not correlate positively with spatial-numeric associations.

Finally, van Gallen and Reitsma (2008) adopted a magnitude judgement task and asked 7-8-year-olds to compare the numeral which they saw with the number five. A SNARC effect was recorded since children responded faster to larger numbers when using the right key and to smaller numbers when using the left key. A distance effect was also observed since response times were slower for numbers further from 5. A SNARC effect was not registered when they performed a detection task in which number magnitude was irrelevant. They interpret this as evidence that up to about 8 years old, children register a SNARC effect only when the task explicitly involves number magnitude. In their investigation, both the 9-year-olds and the adults showed a SNARC effect in the detection task, proving that number magnitude information is immediately engaged even when the number is simply inferred and unrelated to the task (Fischer, 2001).

Taken together, these studies provide support for a MNL as an internal representation of number where numerical value is associated with a spatial location. The next question to think about is whether the directionality of the MNL has any functional significance.

3.6 Concluding remarks

3.6.1 Implications for teaching and learning

The relevance of research findings such as the MNL has implications for teaching in the foundation phase of schooling. It challenges some of the prevalent beliefs we hold about the numerical development of children and what they know. Many of our ideas about how children learn, when to teach certain concepts, and the best ways to engage young children have been influenced by education recontextualisings of the work of the developmental psychologists Lev Vygotsky and Jean Piaget (Woolfolk, 2007). Vygotsky's socio-historical theory emphasises the impact that society has in shaping cognitive development and how cultural tools and social interaction provide children with opportunities to construct meaning. This idea, dovetailing neatly with Piaget's ideas of how children reconstruct and assimilate knowledge, place heavy emphasis on how external forces influence learning.

From the cognitive and neuroscientific research cited here, it is evident that number sense is an innate capacity and the groundwork for mathematical development begins much earlier than was previously thought. Together, they provide educators with a much more robust set of ideas for understanding and appreciating the computational interactions that occur daily in classrooms all over the world.

3.6.2 Curricula

Understanding how concepts are framed by *a priori* mental structures and how these strengthen or inhibit mathematical ideas can contribute to improved teaching and learning in the classroom.

The MNL is an excellent example of how the internal representation of numbers (arranged in a left to right continuum in Western cultures) is externalised as a number line. Three core aspects of number that young child must understand is 1-1-correspondence, cardinality and ordinality. The number line is a convenient resource to map collections to numbers since it represents the cardinal and ordinal features of numbers. Sasanguie and Vos (2018) have shown that cardinal information in a digit comparison task is predictive of arithmetic performance in first graders, whereas ordinal information becomes increasingly important to predict performance in second graders. The number line can therefore be used to assist learners to move from relying on cardinal information in first grade to ordinal information in second grade which becomes a better predictor of arithmetic ability. Understanding how certain tasks are predictive of mathematical abilities can help teachers design tasks to identify the children who may be at risk and once identified, receive additional support.

In the next chapter I will focus on the spatial scaling of the number line and how it relates to the arithmetical development of the child.

Chapter 4

Descriptive accounts of the mental number line

4.1 Introduction

A key feature distinguishing human cognition from that of other animals is our ability to combine and organise distinct core systems in a flexible and productive manner to create new systems of knowledge and to restructure old systems by forming new concepts (Spelke, 2003). One such example is the way in which our mature representation of number is rooted in our core systems of spatial representation and numerical representation to yield the MNL.

In the previous chapter I presented a synthesis of the literature for the spatial orientation of the MNL that relates number magnitude to space. Spatial associations are malleable and largely influenced by the writing and reading conventions of a society (Shaki *et al*, 2009). In mostly western societies, numbers are represented internally in ascending order where small numbers are commonly associated to the left and larger numbers are associated to the right. Non-western societies tend to orient right to left (Shaki *et al*, 2009). An additional feature of the number line, which is also the focus of this chapter, is the non-directional scaling properties of the number line and the extent to which these properties can inform us of numerical development.

4.2 The number line

The Arabic numerical system allows us to represent numbers accurately and precisely, which in turn enables us to surpass the limitations placed on us by core representations of numerosity. But core concepts are resilient, and Dehaene has argued that the influence of the ANS is preserved alongside our ability to express numbers symbolically (2011: 55,61-3). This means that the human brain continues to constitute meaning in a “fuzzy” and imprecise way even though we can express numerosity exactly. The presence of the distance and size effects are often used as evidence for this line of argument because no bias or time delays have been observed when we retrieve numbers digitally. Our perceptions of non-symbolic numerosities and symbolic values are, therefore, conditioned by the ANS and obeys Weber’s law. Although the natural numbers form a discrete set, we can relate it to continuous-analogous quantity, visualised as a number line (Dehaene, 2011; Gallistel, 2011).

In 1685, John Wallis described formally how numbers could be conceived as points on a number line. Dehaene (2011:76) points out that this way of thinking about number provides us with a visual model that corresponds to our intuition about numbers. Núñez (2011) counters this point by arguing that if the number line is indeed an intuitive representation, there would be numerous references to it by ancient mathematically sophisticated civilizations like the Mesopotamians, Egyptians, and the Chinese, but that there are none. However, this line of thinking is very questionable since the neural properties of an organism need not be and are generally not consciously recognised by an organism. In fact, very little of thought is directly accessible to human consciousness (Chomsky, 2002:97).

Nevertheless, number line representations play a key role in developing numerical understanding of children's expanding knowledge of numerical magnitudes and is an important index and predictor of number acuity in children (Siegler, 2016; Siegler, Thompson and Schneider, 2011). Siegler (2016) proposed a theory detailing the dynamic role of the number line in developing numerical understanding since all rational numbers can be represented on the number line. Young children first learn to connect small symbolic natural numbers to their non-symbolic referents and later, as their understanding of the number system expands, they learn to map larger numbers to the right and interstitially to include rational numbers (fractions and decimals). The number line thus provides an important representation of number in the core and noncore domains. It is, therefore, crucial that educators understand the dynamic role that number lines play in facilitating the growth of numerical understanding in young children.

4.3 Representation of number on the number line

Numerical estimation is a central part of the placement of number markers on a number line since it requires being able to map a point on a number line to a number and a number to a spatial position on a number line, moving fluently from an inexact quantitative representation to an exact one. Estimation tasks reveal how our approximate number sense and culturally derived number sense interact in development. By comparing the precise position of the target number on the number line with the estimated position, researchers can construct models to describe how numbers are instantiated in the brain and show how cognitive skills and strategies influence the numerical scaling of number representations.

Researchers have proposed that how we distribute numbers along the number line reflects the development of numerical cognition from early childhood to adolescence (Siegler, 2016). This has the potential to reveal the types of strategies children use when they make estimations (Ashcroft and Moore, 2012; Barth and Paladino, 2011), while others have argued that number line representations

ought to be viewed as a direct reflection of an individual's internal spatial representation of numerical magnitude. The latter view is supported by Zorzi *et al.* (2002) from their studies of neglect patients; by Siegler and Opfer (2003), who argued that numerical quantities can be represented in a variety of ways; by fMRI studies that indicate greater activity in the bilateral parietal lobes for approximation activities (Dehaene *et al.*, 1999; Vogel, Grabner, Schneider, Siegler, Ansari, 2013); and by studies that show a developmental shift from logarithmic to linear patterns in number line tasks (Booth and Sigler, 2006; Siegler and Opfer, 2003). This view is often referred to as the “internal representational view”, which considers how the distance and size effects influence the way we represent numbers imprecisely along the continuum on the number line.

The opposing view posits that number line representations measure a broad range of mathematical competence, like reasoning about spatial proportions (Barth and Paladino, 2011); knowledge of place-value (Moeller *et al.*, 2009; Thompson and Opfer, 2010); children's fluency with the number range (Ebersbach, Luwel, Frick, Onghena and Verschaffel, 2008); mensuration skills (measurement, calculation and computing) (Cohen and Sarnecka, 2014); language and proficient use of number words (Praet and Desoete, 2014); and analogical reasoning (Sullivan and Barner, 2014), thereby placing heavier emphasis on how experience and training mediates the distribution patterns of number.

Despite these two seemingly opposing views, researchers agree that the number line task is a robust tool to measure a child's competency with numbers and that it can inform teaching practice since extensive evidence suggests a positive correlation with mathematical tasks like counting, arithmetic and school achievement (Ashcroft and Moore, 2012; Petitto, 1990; Schneider, Merz, Stricker, DeSmedt, Torbyns, Verschaffel and Luwel, 2018).

4.4 The number line estimation tasks

This task is a cross-domain task requiring participants to translate continuous and discrete representations (integers from the number domain (number symbols) or non-numerical (e.g., dots)), as length in a spatial domain; that is, a position on the number line. Besides varying the format of stimuli (dots, Arabic numerals, number words) other variations have also been introduced. The traditional version of the estimation task requires participants to estimate and mark the point of numbers on a horizontal line where the endpoints have been marked: 0 at the left endpoint and the larger number (e.g., 10, 20, 100, 1000 or 10 000, etc.), at the right. Participants are then required to plot values that fall within the number range (Ashcroft and Moore, 2012; Siegler and Opfer, 2003). Alternatively, participants must infer the number from the position of a point on the number line

(position-to-number) flanked by a small number to the left and a large number to the right (Berteletti *et al.*, 2012; Petitto, 1990). In both versions of the task, the line is constrained by numbers at either end. Cohen, Blanc-Goldhammer and Quinlan (2018) and Cohen and Quinlan (2018) have argued that this feature truncates the distribution of larger numbers, thereby forcing a logarithmic distribution of values along the continuum.

A modified version is the unbounded number line task (Booth and Siegler, 2006; Cohen and Sarnecka, 2014; Cohen and Bland-Goldhammer, 2011, Cohen, Blanc-Goldhammer and Quinlan, 2018; Link *et al.*, 2014). Participants are given a line denoting the value of one unit, and must use that information to mark the position of a target number on an extended number line. They are typically required to repeat the single unit as many times as indicated by the target. Most participants, across development, use a recursive additive strategy to complete this task and consequently, a linear pattern with scalar variability emerges.¹³

While both tasks require participants to scale numbers to line lengths, researchers have cautioned that we need to consider the different strategies required to complete both tasks successfully when analysing the data before making judgements about underlying numerical representations. Traditionally, the data from the distribution of numbers on the bounded number line tasks completed by children and adults have been interpreted as a developmental shift (Siegler, Thompson and Opfer, 2009) from a logarithmic to a linear pattern, but this hypothesis hinges on the notion that the number line tasks directly reflect psychological representations of number and quantity.

More recently, the distribution patterns from unbounded number line tasks have been used as evidence to suggest that there is no developmental shift (Cohen and Blanc-Goldhammer, 2011; Cohen and Sarnecka, 2014; Cohen, Blanc-Goldhammer and Quinlan, 2018; Cohen and Quinlan, 2018). They take this to mean that the unbounded number line task is a more valid assessment of underlying representations since, (a) it is a less demanding mathematical task, and (b) adults and children used similar strategies to complete the unbounded number line task since both display linear pattern with scalar variability.

¹³ Scalar variability is a signature of the ANS. The ANS is noisy and imprecise and we can therefore assume that the mappings from one system to another will contain errors. The variability can be described as a logarithmic (proportional) model or as linear.

4.4.1 The distribution of numbers on the number line

Three main theories compete for the best way to describe the internal representation of number: (a) the linear account, (b) the logarithmic account and (c) a linear pattern with scalar variance account. The number line estimation task has been used to settle the differences between the different accounts.

a. The linear account

Case and Okamoto (1996: 18) reported that the 4-year children in their study used a global quantity conceptual structure which allowed them to judge collections as “a little” and “a lot”. Siegler and colleagues (Siegler and Robinson, 1982; Laski and Siegler, 2007) also found that pre-schoolers initially categorised numbers as big and small before they constructed more sophisticated groupings such as ‘very small’, ‘small’, ‘medium’, ‘big’ and ‘very big’. Núñez (2011) reached a similar conclusion when he re-analysed the data set of the numerical mappings of adult Mundurukú participants used by Dehaene *et al.*, (2008). Núñez noticed that a significant number of participants placed their dots at the polar ends of the continuum, in a bimodal placement and did not use the full range of the line like the 4-year-olds in Case and Okamoto’s study who referred to “a little” and “a lot”.

A linear model argues that distances between adjacent numbers are invariant regardless of their value. It appears that children become more adept at linear representations as their knowledge of the base-ten system improves. Case and Okamoto concluded that at approximately 6 years old, children merge two conceptual systems (the global quantity system and an object counting system) mediated by the MNL and map numbers 1 to 5 linearly on a number line so that the adjacent numbers are equally spaced. Furthermore, Lipton and Spelke (2005) demonstrated that basic counting knowledge was sufficient to solicit a linear function. In their study, five-year-olds, who were classified as skilled counters, mapped numbers that were within their counting range in a linear pattern whereas unskilled 5-year-old counters’ estimates were unrelated. In addition, the young participants (mean age: 5-years) in Muldoon, Simms, Towse, Perra and Menzies’ (2013) study also produced linear estimates within the 0-10 range. They also report that children who enumerated 20 items were more likely to produce linear estimations on the 0-20 scale than children who failed to enumerate 20 items.

b. The logarithmic account

The prominent view is that humans initially perceive symbolic numbers on a logarithmic scale in which the distance between adjacent numbers are not equal. The perceived distances between adjacent numbers decrease as magnitude increases (Dehaene, 2011:65). Gallistel argued that this way of reasoning was flawed because a logarithmic representation would not be able to account for

addition and subtraction but corresponded instead to multiplication and division of quantities. He also argued that a logarithmic model would not be able to deal with zero because “logarithm goes to infinity as magnitude goes to zero” (Gallistel, 2011:7). For these reasons, Gallistel proposed a scalar variance model, much like Gibbon and Church’s (1981) accumulator model.

c. The linear with scalar variance account

In this model, adjacent numbers are evenly spaced with scalar variability. This means that the representations become imprecise and increase in variability as magnitude increases linearly with the numbers presented. The variability of the estimations is a function of the ratio of the pairs of numbers (Gelman and Gallistel, 2000). Gallistel endorsed an auto-scaling model in non-human animals and humans that could combine continuous and discrete distributions multiplicatively, arguing that:

[m]easuring instruments nonetheless convey information about quantities over a large dynamic range by adjusting their sensitivity to the magnitude of the input signal: the stronger the signal becomes, the more they lower their sensitivity. Automatic scaling adjustment makes minimally representable difference proportional to signal strength, but it does not implement a logarithmic mapping from input to output. At any scale factor, the mapping from input to output is scalar, which means that the full range of arithmetic operations may be validly carried out on this representation of the input. (Gallistel, 2011:8).

Empirical support for Gallistel’s proposition comes from Huntley-Fenner (2001), who found that 5-7-year olds’ estimation patterns revealed a scalar variance signature similar to that of adults when they relied on their analogue system of number to make estimations. Additionally, Cohen and Blanc-Goldhammer (2011) observed that estimations of adult participants on the unbounded number line task resembled a linear function that showed scalar variance. Cohen and Sarnecka (2014) tested young children on the unbounded and bounded number line and found that the unbounded number line yielded a linear pattern with scalar variance. Link *et al.*, (2014) reached a similar conclusion in their comparative study of unbounded and bounded number line estimation tasks. They discovered that differences in error variability increased linearly with scalar variance in the unbounded number line task.

These three accounts do not necessarily compete for descriptive adequacy: the logarithmic account describes how younger children exaggerate the distance between smaller values compared to larger values on a bounded number line and how we revert to logarithmic representations of magnitude when we are faced with unfamiliar number ranges or have time constraints placed upon us. Second, the linear model emerges once numerate children and adults become familiar with the number range,

making their estimations more accurate and precise and, finally, linear estimates with scalar variance describes our perception of magnitude in unbounded number line conditions.

When we consider the various accounts within each context they contribute towards a more nuanced and robust understanding of numerical magnitude representations in various contexts.

4.4.2 Developmental changes in performance in number line estimation tasks

Arguments surrounding which model accurately describes and predicts performance on the bounded number line have also emerged. Three prominent theoretical accounts attempt to describe and explain the developmental change in children's responses: a) the logarithmic-to-linear shift; b) the two linear segmented model; and c) the proportion judgement account. Each account uses a specific statistical model to analyse the mean placement of the distribution of values on the number line and their relative differences.

This debate is of particular interest to educators because it not only attempts to characterise magnitude representations but shifts the conversation from merely accounting for developmental differences to focusing on strategies that children might use to improve their estimations.

a) The logarithmic-to-linear shift

The logarithmic-to-linear position states that although very young children initially generate logarithmic patterns on a bounded number line, the patterns become more linear with age and as children acquire arithmetic skills. To investigate the developmental shift in magnitude representation, Siegler and Booth (2004) presented second, fourth, sixth graders, and adults with a battery of number line estimation tasks. Participants were tested on the 0-100 and 0-1000 number line and tasked with completing number to position (NP) and position-to-number (PN) versions of the estimation tasks. Their responses were timed to prevent the use of counting or repeated division strategies. Analyses of the results for the 0-1000 range indicated logarithmically compressed number lines for second graders while fourth graders' estimation patterns fit both a linear and logarithmic pattern and sixth graders and adults generated linear estimation patterns. In all age groups, a linear pattern was generated on the 0-100 range, indicating that number lines do become more linear with experience and age. The results also demonstrated that while second graders performed linearly within a familiar number range, they had trouble applying linear representations on an unfamiliar and larger number interval (0-1000).

The logarithmic to linear shift has been observed at different ages and for varied number intervals. Berteletti *et al.*, (2010) investigated number line estimation tasks with 3,5-year-olds to 6,5-year-olds. They found that pre-schoolers who could recite the counting sequence 1-10 on their fingers and who could order dots and Arabic digits could map numbers linearly and accurately on the 1-10 interval. When these pre-schoolers were presented with an unfamiliar 0-100 number range, their mapping yielded a logarithmic pattern.

In a second experiment, 4-year-olds to 6-year-olds were tested on a 1-20 interval and a similar pattern emerged. The youngest participants positioned the numbers within the 1-20 range to correspond to a logarithmic mapping whilst the middle and older participants acquired greater precision along the number line and produced linear representations. In addition, second graders who mapped numbers less than 100 linearly on a 0-100 line represented the same digits in a logarithmic pattern on the 0-1000 context (Siegler and Opfer, 2003), demonstrating that context influenced the way individuals represented numbers.

The simultaneous presence of logarithmic and linear mental representations can be extended to adults. Western adults generate linear mappings for integers and small sets that could be enumerated but generate logarithmic mappings for larger collections under conditions that did not encourage counting (Dehaene *et al.*, 2008). In addition, Mundurukú participants who received limited formal education, mapped Portuguese numbers linearly while still mapping Mundurukú number words logarithmically. The Mundurukú number system does not represent numbers exactly and can therefore not lend itself to a linear mapping since all consecutive whole numbers are not separated by the same linear interval. However, western adults performed similarly to their Mundurukú peers when they distributed fictitious and unfamiliar large numbers (e.g., forty-three ‘umptillion’ and twenty-five) in a logarithmic pattern (Rips, 2012).

All of this can be taken as evidence that logarithmic and linear representations coexist and that individuals have access to multiple representations depending on numerical context (Siegler and Opfer, 2003, Siegler, 2016, Booth and Siegler, 2006).

b) Two-linear segmented model

The transformation from a logarithmic to linear hypothesis proposed by Siegler and colleagues (Booth and Siegler, 2006; Laski and Siegler, 2007; 2006; Opfer and Siegler, 2007; Siegler and Booth, 2004; Thompson and Opfer, 2008) has been questioned by Ebersbach, Luwel, Onghena and Verschaffel (2008); Moeller, Pixner, Kaufmann and Nuerk (2009) and Muldoon *et al.*, (2012).

Ebersbach *et al.* (2008) proposed that a two-linear segmented model offered a better explanation for developmental change. They argued that two separate linear representations with different slopes could provide an alternative interpretation of the estimation patterns observed in young children. Central to their argument is the familiarity of number thesis which states that children generate linear estimations for numbers that they are familiar with. But what entails “familiarity” with number is variable. They tested young children’s counting skills and observed a strong correlation between counting range and the breakpoint¹⁴, prompting them to conclude that estimations beyond the familiarity range resulted in a linear function with a smaller slope moving toward zero. However, correlations between counting and familiarity have not been reliable. Second graders who counted reliably to 200 did not represent numbers linearly (Ebersbach *et al.*, 2008), even when they were presenting with the numerals and their corresponding dots to reduce unfamiliarity with the number, (Ebersbach *et al.*, 2008). Muldoon, Simms, Towse, Perra and Menzies (2013) also argued for a bilinear model, but analysis of their data identified a fixed mean breakpoint at 15. This was observed even though 95% of the five-year-olds tested could count beyond 15.

For Moeller, Pixner, Kaufmann and Nuerk (2009), the breaking point is fixed at the intersection of one- and two-digit numbers. They argue that one- and two-digit numbers are processed separately, because knowledge of the place-value system is fundamental to improving number line estimations. After testing 7-year-olds on the number-to-position task, on a 0-10 and 0-100 range, they proposed an alternate two-linear segmented theory to the multi-representational theory. A linear function best fit the 0-10 range, but Moeller *et al.* (2009) demonstrated how a separate linear representation for units (1-9) and tens (10-99) could account for the variance after the breaking point, which happened to be 10.

Rather than argue for multiple representations within an individual, Ebersbach *et al.* (2008) and Moeller *et al.* (2009) propose that the shape of the number line represents children’s numerical abilities within that number range. Extending on the work of Ebersbach (2008) and Moeller (2009), Xu, Chen, Pan and Li (2013) tested the numerical estimations of Chinese pre-schoolers in the following cohorts: 3-4-year-old, 5-year-olds, and 6-year-olds. All the pre-schoolers were tested on a 0-10 number line using symbols, dots, and objects. Older pre-schoolers were given additional tasks in the 0-100 and 0-1000 ranges. The 6-year olds’ estimation patterns in both number ranges was better represented by two-linear model than when contrasted with a one- and two-cycle model. The team

¹⁴ The breakpoint is the position at which discriminability of a numerical magnitude changes. Numbers within a familiar range are more linearly distributed, whereas numbers beyond the familiarity range displays a different distribution pattern. (Ebersbach *et al.*, 2008).

speculated that among other reasons, the transparent base-ten language system that the Chinese use, may have aided their performance on the number line task. Xu *et al.*, (2013) used the point that separate single and double digits for the 0-100 as a breakpoint and two-and three-digit as the 0-1000 breakpoint, again raising the argument that knowledge of place-value is important in number line estimations.

However, when Young and Opfer (2011) re-analysed data from previous studies, they found little support for the two-linear model and since Weber's law sufficiently predicts the size effects within single- and double-digit comparison. In addition, Laski and Yu (2014) examined estimates of numbers on the 0-100 and 0-1000 scale and found that kindergartners who successfully arranged numbers linearly on the 0-100 interval did not extend their knowledge of base-ten for numbers below 100 on the 0-1000 interval.

c. The proportional judgement account

The proportionate judgement account moves away from assuming mental representation of quantities toward predicting possible cognitive techniques that participants might use to solve estimation problems. This model strongly argues that it is necessary to consider the proportional nature of the estimation task. This means that the target value relative to the size of the whole must be considered when an estimation is made and argues that participants strategically segment the line at reference points to guide their estimations.

Furthermore, the model demonstrates that numbers are not placed directly on a line, but that they are placed in reference to the anchor points. The estimated magnitude and the actual magnitude are calculated as a power function. Estimates are usually more accurate around the midpoint, creating an s-shaped cycle, called the one-cycle power model (CPM). With the addition of reference points, a two-cycle model is created, which improves estimation accuracy. (Barth and Paladino, 2011; Slusser, Santiago and Barth, 2013; Zax, Slusser and Barth, 2019). Rouder and Geary (2014) have argued that the logarithmic pattern evident in young children's estimates result from using zero as a single anchor point and ignoring the physical length of the line or the endpoint. Therefore, the incorporation of additional reference points to break up the line into segments can explain the decline in estimation errors on the number line. Researchers have shown that third graders are more likely to use midpoint strategy representations than younger children (Dackermann, Fischer, Huber, Nuerk and Moeller, 2016; Feldman, Schmuely, Dotan, Tzelgov and Berger, 2019; Peeter, Sekeris, Verschaffel and Luwel, 2017) and that labelled benchmarks positively influence their estimations (Peeters, Sekeris, Verschaffel, Luwel, 2017).

In addition to the strategic use of reference points, Cohen and Sarnecka (2014) have argued that children's arithmetic skills to scale numbers to a line should also be considered. They have demonstrated that successful estimates on the number line require a subtraction strategy where participants estimate the physical length of the line before subtracting the target number from the upper bound value. They therefore modified the cyclic power model and developed a subtraction bias cyclic model (SBCM) to distinguish between perceptual and subtraction bias (overestimation of small values) in young children's estimates.

To accommodate the unbounded number line, Cohen and Blanc-Goldhammer (2011) developed a scalloped power model (SPM)¹⁵, to analyse and describe the repeated addition strategy that was required to perform the unbounded number line estimation task.

4.4.3 Comparing the three developmental accounts

Research teams have debated the source of the developmental changes and several comparative studies have contrasted the various models to try to determine which model would consistently give the best account of numerical development in children.

The most prominent model argues that children access different mental representations of number dependent on numerical context (log-to-linear shift). Competing arguments suggest that psychophysical models of proportion estimation or improved understanding of the Arabic number system can account for developmental change evident across development.

The evidence supporting one model is not conclusive. The variant tasks elicit partly different strategies which influences the patterns of estimation. Perhaps the various strategies that children use when completing estimation tasks will be of more interest to educators.

4.5 Computational strategies on the number line

It is becoming clear that number line estimation tasks do not reflect numerical representation exclusively, but that they offer crucial insights into young children's developing mathematical knowledge and skills. Siegler and Opfer (2003) commented that the logarithmic-to-linear shift is not hostile to the idea that estimation strategies contribute towards the increased linear estimations found in adults and older children.

¹⁵ The scalloped power model (SPM) describes the scalloped pattern that emerges when participants use repeated addition to complete the unbounded number line task i.e. to estimate number like 20 by estimating a line length of 5 (termed the working window) and repeat the length 4 times.

What follows is a brief explanation of the variety of mathematical skills that contribute to improved estimation techniques.

4.5.1 Benchmark-based strategies

More accurate and improved linear estimation patterns have been linked to children's mathematics knowledge and skills. The use of benchmark-based techniques can be credited in part for these advances.

Barth and Paladino (2011) and Slusser, Santiago and Barth (2013) have demonstrated how three cyclic power models could account for the variations of estimations that children demonstrate when they encounter familiar and unfamiliar numerical contexts (various number intervals) without positing developmental change. In an unfamiliar situation, a simple power model explains the pattern of estimates. They found that 5-year-olds' estimates on an unfamiliar number range (0-100 interval) best fit a simple power model which also resembled a logarithmic model. The 5-year-olds used the origin as a reference point but ignored the endpoint, resulting in estimates that did not have a one-to-one relationship with the estimated physical length of the line. As children become more adept with the number range, they start to consider the full range (both endpoints of the line) and a one-cycle power model is used to explain the estimation pattern. This was clearly demonstrated by the 7-year-olds in the task whose estimation patterns were best described by a one cycle power model. The estimation of the 8-10-year olds in this study suggest that additional reference points may have been used to generate accurate estimations. Based on these findings, Slusser, Santiago and Barth (2013) have argued that children's estimates are successively best described by a power model. As their familiarity with the number range improves and they begin to reason about numbers proportionally, they can use either a one-cycle or a two-cycle model. (Barth and Paladino, 2011, Cohen and Blanc-Goldhammer, 2011; Rouder and Geary, 2014; Slusser, Santiago and Barth, 2013).

Ashcroft and Moore (2012) have suggested that the combined effect of increased arithmetic knowledge and underlying number representations positively influenced number line estimation. They noticed that second graders' latencies and errors decreased at the endpoints whereas errors increased at the midpoint. In contrast, third graders demonstrated skilled use of the midpoint strategy resulting in less errors more accurate estimations. This supports Petitto's (1990) argument that children used a midpoint strategy as their counting proficiency and arithmetic skills improved.

Peeters, Degrande, Ebersbach, Verschaffel and Luwel (2015) compared the performance of second graders that were given benchmarks using a 0-200 scale. Second graders were allocated to one of three conditions: (a) a bounded line with the beginning and endpoints labelled; (b) a midpoint condition where a benchmark was given; or (c) a benchmark provided at each quarter (a quartile) condition. The children were also interviewed about the estimation strategies they used, and their responses were coded as perceptual strategies (seeing, guessing, knowing), standard benchmark-based strategies if they partitioned the line into quarters and non-standard benchmark-based strategy if they used another point as a reference. From their analyses, the researchers observed that the use of a midpoint strategy led to more accurate estimates around the point but providing external reference points in the Grade 2 context did not yield more accurate estimations. From their data and participants' verbal reports they inferred that familiarity with the number range played a role in whether second graders were able to make use of such information to inform their estimates. Furthermore, Luwel, Peeters and Verschaffel (2019) demonstrated that the presence of benchmarks did not automatically yield more accurate estimations. On a 1-1000 scale, third and fifth graders were randomly allocated to one of three conditions: control, a midpoint or a quartile condition which differed in the number of benchmarks that was provided on the number line. The fifth graders used the quartile-based strategies more repeatedly to produce more linear estimates compared to the third graders who did not show any improvement. Both age groups, regardless of the condition, demonstrated spontaneous use of the midpoint strategy. Interestingly, Peeters, Sekeris, Verschaffel, and Luwel (2017) demonstrated that third graders' midpoint and quartile benchmark-based strategies improved when the benchmarks on a 0-1000 scale were labelled to corresponded with the numerical values whereas the unlabelled condition had no effect on their strategy use. Sixth graders' number line estimations showed a significant improvement in both labelled and unlabelled conditions when compared to the control condition.

Using finger tracking technology, Feldman, Shmueli, Dotan, Tzelgov and Berger (2019) added to the volume of research that investigates the mathematical strategies. They found that first graders used only the starting point, whereas second graders used the start and endpoints to inform their estimates. Third graders were able to internalize the midpoint and apply that information to distribute the target numbers more accurately along the continuum.

All these studies suggest that improvements in number line estimation could be attributed by the increased use of benchmarks as reference points rather than changes in numerical representation alone.

4.5.2 Analogical reasoning

Several studies have suggested that the number line paradigm measures skills and knowledge unrelated to mathematics. Sullivan and Barner (2014) extended the study of proportional judgement reasoning to investigate the role of analogical reasoning¹⁶ during a number line estimation task.

They demonstrated that 5 years old children would use their previous estimates as a reference point if they had access to it. The results show that most of the children in all age groups (5-, 6- and 7-year-olds) produced highly ordinal responses. Children attended to the numerical ordering of the target number in relation to the reference number to guide their estimates. Across domain numerical estimation required participants to attend to the relative distance between a number being estimated and a numerical reference point other than only ordinality. This skill developed gradually between 5 and 7 years of age.

4.5.3 Children's spatial skills

Gunderson *et al.*, (2012) explored the link between children's spatial ability and estimation skills in relation to mathematics. The team noted that 5-year-olds' spatial skills (mental rotation and transformation) was a good predictor of linear estimates (numbers are equidistant from each other) at age six. Subsequently, they tested whether this knowledge was limited to procedural knowledge or extended to measure conceptual mathematics development. They used a symbolic approximate calculation task on 8-year-olds to test their hypothesis. They concluded that spatial skills, mediated by the 0-100 number line estimation task, played a critical role in approximate calculations, which is an essential indicator of developing number sense.

While LeFevre *et al.*, (2012) and Gunderson *et al.*, (2012) agree that spatial ability can inform how children perform on the number line, Gunderson *et al.* do not support the conjecture that number line knowledge mediates spatial ability with respect to mathematical learning. Rather, they propose that there is a reciprocal relationship between number line knowledge and mathematical performance. The number line problem becomes a result of number knowledge rather than a predictor of it as children's conceptual mastery of the Arabic number system and skilful application of spatial knowledge allows them to create more precise and linear estimations.

¹⁶ Analogical reasoning requires one to find a common relation between two entities in separate domains. In number line estimation tasks, analogical reasoning requires constructing relations between target numbers and numbers to be estimated, and then to extend the relation by making a numerical estimate on a physical number line.

4.5.4 Mensuration skills

The mensuration skills account offered by Cohen and Blanc-Goldhammer (2011) and Cohen and Sarnecka (2012) is another element impacting numerical representation on the number line (2014). According to this account, a child's measurement and arithmetic skills are crucial throughout the estimation task. They suggest that the bounded number line condition requires a subtraction strategy that is too advanced for younger children, resulting in the logarithmic pattern of estimates. Furthermore, they found that young children and adults performed similarly on the bounded number line task since it required an additive process that both groups were competent at.

Cohen and Sarnecka (2014) have argued that children's improved estimates could be attributed to having acquired the relevant mathematical skills needed to scale a number to a line. Furthermore, they found that performance on the unbounded number line was similar for adults and children because the task could be completed by using simple addition to scale the number to the line. These findings led them to conclude that improvements reflected the changes in children's mensuration skills and not integer representations and, secondly, that the unbounded number line task was better suited to assess numerical representation.

4.5.5 The influence of language

Humans can construct natural number concepts and quantify large sets (numerosities/collections) exactly, in part because of the combinatorial properties of natural language (Spelke, 2003). Children are encouraged to learn the counting sequence of numbers from an early age, but it is not until much later that they grasp the cardinal principle for all of the numbers in their count list. (Lipton and Spelke, 2005). Several studies have sought to document the effects of number word structure on number processing and development.

The linguistic structure of counting systems varies across languages in apparently idiosyncratic ways: (a) transparency between the verbal number code (spoken words) and the Arabic notation. (a) Asian languages like Chinese and Japanese have a transparent counting structure (e.g., 56 is spoken as “five-ten-six”) (Laski and Yu, 2014; Poncin, Van Rinsveld and Schiltz, 2020); (b) order of tens and units connection between number words and numerical codes (e.g., 56 is spoken as six and fifty) (Imbo, Van Den Bulcke and Fias, 2014) and for the “teen” numbers in English (e.g., 15); (c) the base of the number system. The French counting system follows the vigesimal notational system (base-20) for numbers greater than 60 (e.g., 72 would be sixty-twelve) (Van Rinsveld and Schiltz, 2016).

Laski and Yu (2014) observed that Chinese kindergartners produced linear number line estimations in the 0-100 range, much earlier than their American peers who only reached this level in Grade 2. The authors attributed part of the children's success to the consistent rules for combining numbers and the similarity between the number-word systems which makes it easier for children to acquire basic numerical concepts.

Meanwhile, researchers observed that languages that invert the tens and units imposed additional cognitive challenges when transcoding from one form of notation to another. Imbo *et al.*, (2014) examined second graders' competence for number transcoding (translating the number from verbal to digit) and concluded that the Belgian Dutch-speakers who inverted two-digit numbers made significantly more transcoding errors than their French speaking peers who did not invert two-digit numbers (but used the vigesimal notational system). They also observed that the Dutch-speaking children who were more skilled at transcoding numbers achieved higher mathematics scores than the less skilled transcoders.

The inversion property also seemed to add a level of difficulty when comparing two-digit numbers. Pixner, Moeller, Hermanova, Nuerk and Kaufman (2011) compared how 6- and 7-year-old children performed across three countries: Austria, the Czech Republic and Italy. The Italian speaking children (with a non-inverted number system) performed better than their Czech and Austrian peers in the magnitude comparison tasks, lending additional support to the theory that a transparent number word system supports place-value integration in number processing tasks. The Czech language uses both a non-inverted number system in formal situations and an inverted system in informal situations. The authors argue that on the one hand this might slow down place-value integration, but it might also create an awareness of the place-value principle.

Praet and Desoete (2014) reported that although language could account for the variances in kindergarten, it did not affect the development of estimation accuracy. Laski and Yu (2014) reached a similar conclusion. As you may recall, the Chinese kindergartners in their study produced more linear estimations in the 0-100 number range, but it was not until the second grade that many children produced linear estimates on 0-1000 number line, similar to American children's performance. The impacts of language, according to Moeller *et al.*, (2009), are relatively small and unlikely.¹⁷

4.5.6 Counting skills

¹⁷ Mostert (2019) examines how the linguistic features of African languages impacts teaching and learning of number names.

Muldoon, Simms, Towse, Perra and Menzies (2013) conducted a longitudinal study with 5-year-olds to determine how counting might influence number line estimation performance. They asked children to count to their highest number (reciting task) and to enumerate sets to assess counting and found significant correlations between counting, estimation, and mathematics ability. Schneider *et al.*, (2018) revealed in a meta-analysis that correlations between number line estimations were a strong predictor of mathematical ability in longitudinal investigations.

Whether the number line reflects an organised internal knowledge structure or a product of a broader range of mathematical acuity, the usefulness of the number line estimation task as a tool to inform mathematical growth and development is invaluable.

4.6 Approximate calculation

At this point one should consider how spontaneous operations on collections of dots interact with formal operations that are being taught at school. The operational momentum (OM) effect defines the observed bias to overestimate non-symbolic addition (e.g., dots) and to underestimate non-symbolic subtraction on the number line. This bias was observed in adults (McCrink, Dehaene and Dehaene-Lambertz, 2007), 9-month-old-infants (McCrink and Wynn, 2009), and 8–9-year-olds, but in 6-year-old participants, the OM effect was reversed (Pinheiro-Chagas, Didino, Haase, Wood and Knops, 2018). Most researchers agree that there is sufficient evidence to link this phenomenon to the approximate magnitude system. The outcomes of approximate addition and subtraction operations show a consistent bias that conforms with Weber's law (McCrink *et al.*, 2007). Three mechanisms underlying the OM bias have been proposed: the attentional shift/momentum effect, the compression effect, and the heuristic account. Central to all accounts is the visuospatial nature of the MNL. During approximate estimations, visuospatial attention shifts in the same direction as the operation: to the right for addition and left for subtraction. The compression effect is said to be the consequence of a linear operation performed on a logarithmic continuum. Addition on a logarithmic number line would shift towards larger numbers while subtraction would yield a reverse effect. Finally, the heuristic account argues that for approximate calculations performed with imprecise values and under time constraints, we generate a rule that adding yields more than the initial amount and subtracting leaves us with less, so we will reject smaller numbers in adding situations and reject larger numbers for subtraction.

The comparison task draws on key principles of natural number: the unit, succession and exact equality (Spelke, 2017). The ANS not only supports comparison of aggregates, but allows us to form representations of approximate quantities, perform approximate calculation expressed in non-

symbolic notation and to understand logical principles that underlie arithmetic (Barth, LaMont, Lipton and Spelke, 2005; Barth, LaMont, Lipton, Dehaene, Kanwisher and Spelke, 2006; Gilmore, McCarthy and Spelke, 2007, 2010; Gilmore and Spelke, 2008 Wynn, 1992). Barth *et al.*, (2005), found that five-year old pre-schoolers were able to add two successive arrays of dots and compare their sum with a third set of dots or auditory sequence of tones accurately.

Secondly, Gilmore, McCarthy and Spelke (2007) conducted a series of experiments to study how the ANS and symbolic and non-symbolic numerical abilities relate to each other prior to formal education (Gilmore, McCarthy and Spelke, 2010). They found that 5- to 6-year-old pre-schoolers could solve approximate addition and subtraction problems independent of their knowledge of exact number. Furthermore, their performance on symbolic approximate calculations is subject to the signature limits of the ANS; that is, a ratio effect on accuracy, performance on comparison and addition tasks and similar and subtraction is less accurate than addition.

Thirdly, recent studies detail how approximate calculations are subject to the OM effect found across development in adults, children, and infants (Dunn, Bernstein, de Hevia, Macchi Cassia, Bulf and McCrink, 2018; McCrink, Dehaene and Dehaene-Lambertz, 2007; Pinheiro-Chagas, Didino, Haase, Wood and Knops, 2018). The OM effect is seen as a by-product of how we represent number along a logarithmically compressed continuum, spatially organised with numbers arranged in ascending order. This interaction (between the spatial mapping and number systems) imposes a systematic bias when we perform approximate calculations on non-symbolic collections. McCrink, Dehaene and Dehaene-Lambertz (2007) observed this behaviour in adults who displayed attentional shifts along the MNL in the direction of the operation, causing the participants to systematically overestimate their answers to addition problems and underestimate their answers to subtraction problems. Similar studies using a ‘looking paradigm’ conducted on 9-month-old infants reveal an OM effect too. Infants also demonstrated a sensitivity to addition and subtraction transformations. When they saw visual sequences of addition or subtraction transformations using non-symbolic numerosities they looked significantly longer at outcomes that violated their expectations (such as $14 - 4 = 5$) than correct outcomes, or ones that exceeded the correct amount but moved in the direction of the operation. However, a reverse OM effect was observed in studies with 6-7-year-old children. That is, subtraction led to larger overestimations than addition problems (Knops, Zitzmann and McCrink, 2013).

Together, these studies provide evidence that ANS representations possibly serve as a foundation for basic understanding of exact arithmetic. Young children can perform approximate calculations on symbolic quantities before they start formal schooling and highlights the supporting role that the ANS

plays in numerical development. A natural question for educators is how this intuitive knowledge connects with the curriculum and how it can be harnessed to teach mathematical concepts.

A significant amount of time is set aside in the primary years to teach calculation of exact numbers and number operations carry a weighting of 65%, 60% and 58% of notional time in Grade 1 to Grade 3 respectively (DBE, 2011:10). If children can already estimate large-number addition and subtraction results, we need to figure out how to better use this capacity to aid in the teaching of exact quantities.

4.7 Conclusion: Implications for educational practice

The number line estimation task has great value in educational contexts. The task requires few resources, it is easy to administer, and can be modified in a variety of ways to test for a wide range of mathematical skills. The task also draws from a wide range of mathematical skills such as knowledge of place-value, proportional reasoning, analogical reasoning, and spatial skills to successfully complete the task.

The task has the potential to be used as a diagnostic tool since a strong association between mathematical competence and number line estimation tasks has been extensively researched. It would be beneficial for teachers to understand this relationship and to understand how altering the task could be used to test children's use of different mathematical skills. Children start out by relying on an inherent, but inexact magnitude estimation system (Gallistel & Gelman, 1992) that obeys the natural logarithm of number. This means that with larger numbers, representations are compressed, resulting in a perceived distance between 8 and 9 being smaller than the perceived distance between 2 and 3. Learning of mathematics, in contrast, mediates the distributive patterns of number on the number line, ensuring that the distances between adjacent numbers are the same regardless of the size of the numbers.

Atypical or poor performance on number line tasks could assist early identification of low achieving mathematics students, which could mean that intervention can occur much earlier.

Chapter 5

Representation and the pedagogical use of the number line

Teachers of young children are known for accumulating a multitude of household items such as bottle tops and buttons to use as resources for inducting learners into basic arithmetic. They are certainly not the first to realise that material objects can aid in the development of natural number notions. There is evidence to suggest that artefacts for representing quantity date back as long as 77 000 years ago when Middle Stone-Age hunter-gathers may have counted using stringed beads (in addition to using them as social signalling and personal adornments); notched tallies and hand stencils painted on caves to represent finger counting from the Upper Palaeolithic Age; and clay tokens dated to Neolithic farmers for keeping an account of animals, measures of grain and so on (Overmann, 2016). Clay tokens, notched tallies on bones and knotted strings have been replaced with the abacus, unifix cubes, the hundreds square and plastic bead strings in the modern classroom, but they are still used for the same purpose: to connect numbers to the world, creating a mapping from entities in the represented system (the world) to symbols in the representing system (the brain).

Mathematical models have their limitations because they depend on human intention and on the child's interpretation of how a physical situation can be turned into a mathematical object. Some mathematical models have been criticised because of the limited potential for developing good number sense since they focus on helping children arrive at the correct answers, but do not preserve the intuitive idea of magnitude (of number) and they do not encourage relational thinking about numbers (Beishuizen, Fosnot and Dolk, 2001: 90,95).

What, then, are the essential features of a good mathematical model? Krause (1969) explains that a mathematical model of a physical action involves setting up a correspondence between the appropriate physical objects in the situation and mathematical objects of interest, and then defining one or more operations over the mathematical objects that correspond to the operations over the physical objects. Mathematical models are representations that preserve the essential features of the mathematical idea whilst moving from one form to another. To illustrate the idea briefly, we might consider the number line as a representation. There is a consistent relation between numbers and the various lengths that can be marked off on a line segment. The lengths are not the numbers, but they can be used to illustrate the ordinal relationship of numbers and highlight the spatial arrangement of larger numbers to the right and smaller numbers to the left, for example. Left and right spatial displacements along the number line, which entail the marking out of different successive lengths

and considering some final length as a solution to a problem, are recognised and exploited as analogues of arithmetic operations defined over whichever types of numbers are of concern (e.g., natural numbers, integers, rational numbers or real numbers).

The mathematical specificities of representations are examined in a subsequent section of this chapter. Before doing so, I discuss some of the aspects of a cognitive scientific account of representation.

5.1 Cognitive science and representation

Cognitive science provides us with a computational theory of how the mind is innately designed to learn. Gallistel and King (2010:55) argue that the brain is to be viewed as a powerful computational organ and learning is to be understood as a computational process employing inputs to and outputs from efficient algorithms that update the probabilities that we assign to represent states of the world, arguing that the human brain is a “collection of representing systems capable of assigning probabilities to possible messages”.

More specifically, the human mind is taken to be endowed with a small number of core systems for representing objects, actions, numbers, places and social partners (Spelke, 2003). Such systems provide the foundation for all human cognitive achievement, much of which is shared with other animals. This means that the brain must be structured to encode messages from the world, or the sensory inputs that we receive would be meaningless. It also means that much of the potential data that might be available in the environment cannot be accessed by the innate human computational systems and are never recognised (Chomsky, 2016: 27ff.). This is so because the sensorimotor interfaces through which data can enter the computational-intentional system are specialised to recognise only certain types of data. To convince yourself of the validity of this claim, consider that different types of animals recognise only certain types of data available in their environments. For example, bees recognise light of only a certain band of wavelengths, different from the band of wavelengths accessible to humans; dogs recognise a wide range of odours entirely inaccessible to humans; bats and other organisms recognise bands of sound frequencies inaccessible to humans; and so forth, all of which ought to be taken into account as we consider the sensorimotor systems that different species are biologically endowed with.

It is essential to recognise that core domain systems are activated in learning in general and, therefore, in the learning of elementary mathematics, and that such systems are the cognitive basis upon which noncore domain systems, like mathematics, are grown. Core domain systems can be characterised by a distinct set of signature limits that operate on specific inputs and generate restricted outputs. Such

systems direct attention to salient features of the world that support a critical but small set of inferences. That we hold innate assumptions of the world is illustrated when, for example, pre-verbal infants show surprise when events violate their expectations (Pinker, 1998; 2008). Core systems are independent, encapsulated and universal. They exist prior to learning and function throughout life. Non-core knowledge systems, like formal mathematics, build on the core domain systems.

The core number system shows distinctive signature limits; number-related representations are abstract and apply over visual and auditory sensory modalities. Recall that humans and other animals are endowed with a system for precise representation of small numbers of objects, with a size limit of three in human children and four in adults, referred to as the Object Tracking System (OTS) (Feigenson and Carey, 2003; Feigenson, Dehaene and Spelke, 2004). Also recall the existence of a second system for apprehending large sets with a ratio limited precision which is available to humans and other animals, referred to as the Approximate Number System (ANS) (Hyde and Spelke, 2011, Xu and Spelke, 2000; Dehaene, 2011). Our nascent mathematical abilities allow us to recognise the quantity of small sets, understand comparative relations like greater than and less than, order numbers and perform basic arithmetic operations like addition and subtraction approximately. But each system is encapsulated and develops separately, so cannot yield the system of natural numbers. Spelke (2003, 2017: 155-158) argues that the acquisition of natural language affords humans the capacity to productively combine separate domains (the ANS and OTS) to generate and make sense of the logic of the natural number system. Mastery of noun phrases like, “a cup” and “the dog”, along with language expressions like, “my cup”, “your cup” and “this cup”, map to representations of small distinct arrays and to representations of numerical magnitude. The use of expressions like two cats and three dogs help children to overcome the constraints placed on them by ANS and OTS to construct natural number concepts (Spelke, 2017;157-8). Language makes representation easier and assists in making connections between representing and represented systems explicit. Chomsky has made a related argument for the innateness of natural numbers from the fundamental linguistic operation, referred to as *merge*, applied to a single lexical object (Chomsky, 2008; Watanabe, 2017).

5.2 The mathematics of representation

Krause (1969) has argued that the concept of representation is fundamental to all branches of mathematics because it allows us to connect a newly encountered system to a known system in ways that allow us to construct new knowledge systems. Numbers are abstract concepts that can be used to model aspects of things in the world. Actions on objects in the world can be exploited to teach basic arithmetic concepts like addition and subtraction as long as the action is analogous to the

mathematical construct it is meant to describe. To make these ideas more mathematically precise a number of mathematical resources have to be introduced.

First, the notion of *set* is fundamental to the construction of representations. Cantor's 1915 definition of a set is helpful in this regard: "By [a set] [...] we are to understand any collection into a whole [...] of definite and separate objects of our intuition or our thought. These objects are called the 'elements'" (Cantor 1952, 85). The fundamental idea for the composition of sets is that of *membership*. Members of a set are referred to as *elements* and sets are defined by their elements.

Second, the idea of a mathematical *function* is necessary. A function is an association of each of the elements of an initial set, referred to as the *domain* of the function, with one and only one of the elements of a second set, referred to as the *codomain* of the function. The elements of the codomain selected by the function constitute its *range*. Where the elements of the codomain are associated with only one element of the domain, the function is described as *into* (or *1-to-1* or *injective*). Where one or more elements of the codomain are associated with more than one element of the domain, the function is referred to as *many-to-one*. Where the range of a function is identical to the codomain, the function is referred to as *onto* (or *surjective*). Where a function is both 1-to-1 and onto (that is, both injective and surjective), it is referred to as a one-to-one correspondence (or, *bijective*). An association between sets that is *one-to-many*, it is *not* a function.

Third, an *operation* is a function defined over a set of objects, and takes in one or more objects as *inputs*, referred to as its *argument(s)*, and generates an *output*, referred to as its *value*. The number of arguments required by an operation is referred to as its *arity*. The standard basic operations used in arithmetic usually have an arity of two. Further, the arguments and values of such operations are of the same type. For example, if the arguments are sets, then the values of the operation are also sets. Similarly, if the arguments are numbers, then the values of the operation are numbers. However, there are processes in which the arguments and values of processes are of different types. For example, much of what happens in elementary school mathematics requires teachers and their learners to use the cardinalities of finite sets in their computations. In such instances, the arguments are usually finite discrete sets, and the values are natural numbers. There are also instances where the arguments are numbers, but the values are sets, or other types of mathematical objects, like line segments. In order to deal with such requirements, I shall use the terms *map* and *mapping* to refer to all processes that take in some argument(s) and generate some value. In other words, the terms *map* and *mapping* will be used to refer to the familiar basic operations as well as to functions where the arguments and values

are of different types. Where one is referring only to a specific basic operation, the term *operation* will be used.

Fourth, the idea of *structure* is required. A structure is defined as a set with an operation or mapping defined over a set. For example, the natural numbers, which will be indicated by the symbol $\mathbb{N}$, along with addition, is an example of a structure. That structure has a specific series of properties like, for example, commutativity, associativity and closure. When zero is included in $\mathbb{N}$, the structure also has an identity element, 0. For the sake of convenience, zero will be taken as included in $\mathbb{N}$. A structure will be indicated by symbols of the form $(S, *)$, where S is some set and $*$ is some operation or mapping. Addition defined over the natural numbers will, therefore, be indicated by $(\mathbb{N}, +)$. Operations like subtraction and division defined over $\mathbb{N}$ require restrictions on the domain to prevent the loss of closure. For example, in the case of subtraction over $\mathbb{N}$, the first argument will always have to be greater than or equal to the second, otherwise there is no available value for the subtraction over $\mathbb{N}$. In the case of division over $\mathbb{N}$, the first argument has to be multiple of the second—again, if such is not the case, no value exists in $\mathbb{N}$ for such a computation.

Fifth, representations require the articulation of at least two structures. The structures are related by mappings that associate the objects of one structure with that of the other, as well as the mappings internal to one structure with those of the other. For example, consider the common use of the combination of sets associated with addition over the natural numbers. A typical Grade 1 problem is shown in Figure 5.1.

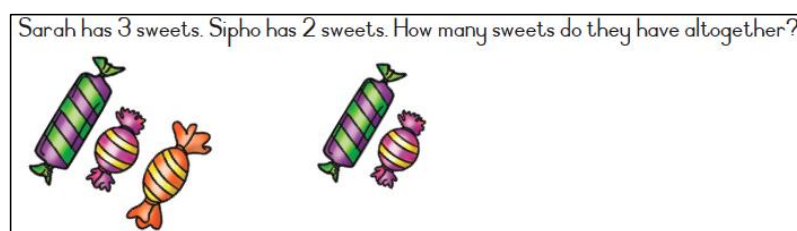


Figure 5.1: An implied mapping from $(\text{FINSET}, \cup)$ to $(\mathbb{N}, +)$ (Source: DBE, 2019: 94)

Here we have two structures that are to be related, one of which concerns disjoint union defined over finite sets, $(\text{FINSET}, \cup)$, while the other concerns addition defined over $\mathbb{N}$, $(\mathbb{N}, +)$. The learner is required to combine the two sets and then count the elements of the resultant set to arrive at a value for the computation $3 + 2$. The mappings that associate $(\text{FINSET}, \cup)$ with $(\mathbb{N}, +)$ are COUNT, which associates a set with its cardinality, and another which associates disjoint union with addition. The computation involving the sets in Figure 5.1 are to be represented by the computations involving the

addition of natural numbers. The mapping from $(\text{FINSET}, \cup)$ to $(\mathbb{N}, +)$ is shown diagrammatically in Figure 5.2.

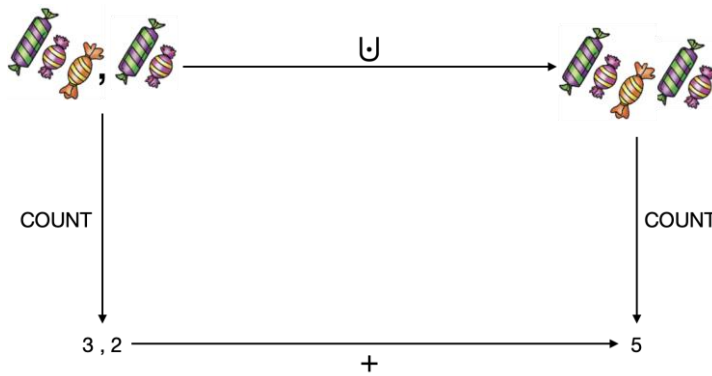


Figure 5.2: COUNT maps $(\text{FINSET}, \cup)$ to $(\mathbb{N}, +)$.

We write $\cup: \text{FINSET} \times \text{FINSET} \rightarrow \text{FINSET}$ and $+: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ to indicate the general mappings associated with the structures $(\text{FINSET}, \cup)$ and $(\mathbb{N}, +)$, respectively, and we write $\text{COUNT}: (\text{FINSET}, \cup) \rightarrow (\mathbb{N}, +)$ to indicate the general mapping from $(\text{FINSET}, \cup)$ to $(\mathbb{N}, +)$. The structure $(\text{FINSET}, \cup)$ is referred to as the *represented system*, while $(\mathbb{N}, +)$ is referred to as the *representing system*. The relation between the two systems is referred to as a *representation*.

We can now detail the important general criteria for representations as spelt out by Gallistel and King (2009: 55; italics in the original); namely, that representations must be *causal*, *structure preserving* and *behaviourally efficacious*:

1. The mapping from entities in the represented system to their symbols in the representing system is *causal* (as, for example, when light reflected off an object in the world acts on sensory receptors in an eye causing neural signals that eventuate in a percept of the object [...]).
2. The mapping is *structure preserving*: The mapping from entities in the represented system to their symbols is such that functions defined on the represented entities are mirrored by functions of the same mathematical form between their corresponding symbols. Structure-preserving mappings are called homomorphisms.
3. Sequences of symbolic operations (procedures) in the representing systems are (ideally) *behaviourally efficacious*: they control and direct appropriate computational behaviour within, or with respect to, the represented system.

In Figure 5.2 we see that the mapping COUNT relates different sets to their sizes, expressed as cardinal values. The symbols for a given set in FINSET are associated with a symbol for number in $\mathbb{N}$ in a consistent manner. That is, all sets in FINSET that are bijectively related constitute a class of sets, each element of which is associated with the same, specific element of $\mathbb{N}$. Sets belonging to different classes are associated with different elements of $\mathbb{N}$. This serves to satisfy the causal criterion.

Further, since $\text{COUNT}:(\text{FINSET}, \cup) \rightarrow (\mathbb{N}, +)$, disjoint union defined over FINSET works in a manner analogous to addition over $\mathbb{N}$. This satisfies the structure preservation criterion.

Finally, given that addition understood in a general sense resonates so strongly with both disjoint union over FINSET and addition over $\mathbb{N}$, addition over $\mathbb{N}$ is naturally efficacious with respect to disjoint union. In fact, the extensive pedagogical use of operations defined over FINSET as a ground for introducing learners to elementary arithmetic operations is a marker of the efficacious effects of operations defined over $\mathbb{N}$ on operations over FINSET. This satisfies the behaviourally efficacious criterion.

Similar remarks can be made for the construction of mappings from addition over $\mathbb{N}$ to disjoint union over FINSET. However, in such cases the mapping that relates the structures $(\text{FINSET}, \cup)$ and $(\mathbb{N}, +)$ go from $(\mathbb{N}, +)$ to $(\text{FINSET}, \cup)$ and we need a mapping that we might refer to as $\text{SET}:(\mathbb{N}, +) \rightarrow (\text{FINSET}, \cup)$, as is shown in Figure 5.3.

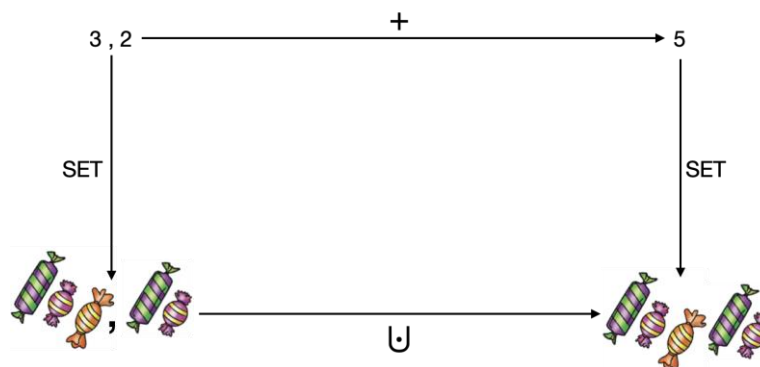


Figure 5.3: SET maps $(\mathbb{N}, +)$ to $(\text{FINSET}, \cup)$.

5.3 Connecting the mathematical and cognitive scientific accounts of representation

Grade R and Grade 1 learners are expected to demonstrate competence in describing and comparing two or more sets according to greater than, less than or equal (DBE, 2011:19). Results from several experiments suggest that infants spontaneously judge relations of *more* and *less* successfully in comparison tasks, subject to developmental limitations. In one study, 10- and 12-month-old infants

successfully chose two crackers over one, and when given the choice of 2 or 3 crackers they consistently chose the larger group. Infants failed when they had to choose larger numbers that exceeded 3, such as 3 vs. 4; 2 vs. 4 and 3 vs. 6 when these collections were concealed. However, when the crackers were presented on a tray, they successfully discriminated 3 vs. 6 crackers even though the numbers exceeded 3 (Feigenson, Carey and Hauser, 2002). When presented with large numbers of objects, infants and adults both focus on the set rather than each individual element (Xu and Spelke, 2000; Spelke, 2017), allowing them to successfully go beyond the set size limit. ANS acuity improves throughout human development and while 6-month-old infants need a ratio of 1:2, 9-month-old infants can differentiate between collections with a ratio of 2:3 (Lipton and Spelke, 2003), while 6-year-olds record success with a ratio limit of 6:7 (Halberda & Feigenson, 2008).

Figure 5.4 shows how a comparison task at school level can be connected to our intuitive number sense concerning numerical order, using a mapping from the comparison of small finite sets to numerical order via COUNT.

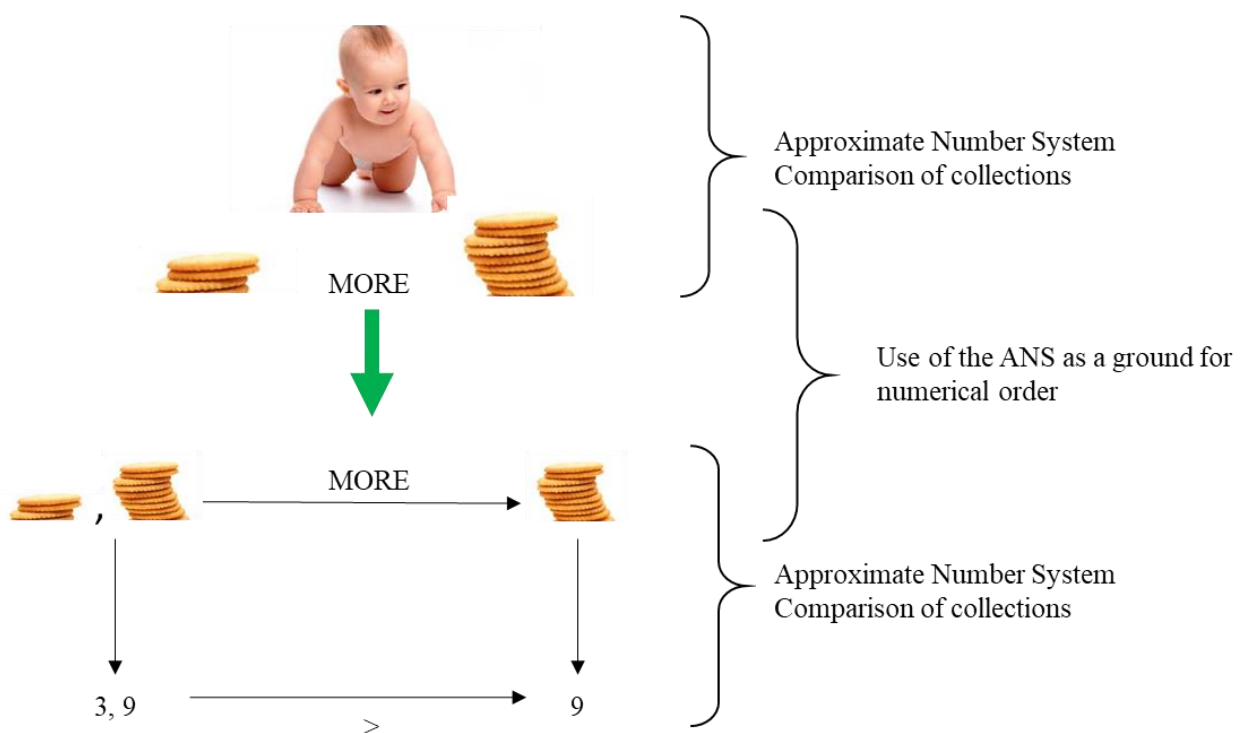


Figure 5.4: Using the ANS as a ground for learning numerical order. How the ANS relates to comparison of numbers in mathematics

The comparison task draws on key principles of natural number: the unit, succession and exact equality (Spelke, 2017).

The ANS not only supports comparison of aggregates, but allows us to form representations of approximate quantities, perform approximate calculation expressed in non-symbolic notation and to understand logical principles that underlie arithmetic (Wynn, 1992; Barth, LaMont, Lipton and Spelke, 2005; Barth, LaMont, Lipton, Dehaene, Kanwisher and Spelke, 2006, Gilmore, McCarthy and Spelke, 2007,2010; Gilmore and Spelke, 2008). Barth *et al.* (2005), found that 5-year-old pre-schoolers' are able to add two successive arrays of dots and compare their sum with a third set of dots or auditory sequence of tones accurately.

Consider the task shown in Figure 5.5.

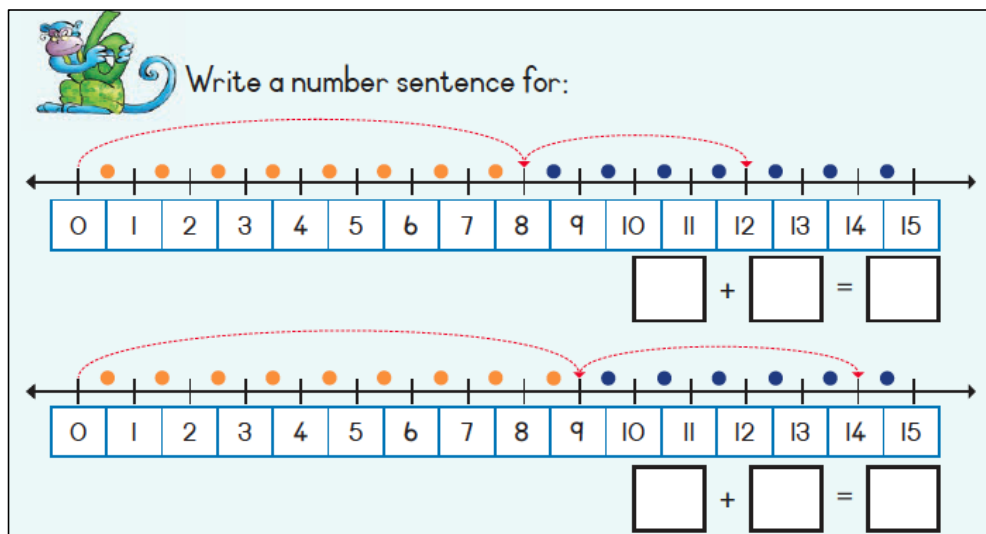


Figure 5.5: Task 72, “Addition - building up and breaking down numbers up to 20”, Problem 6 – Grade 1 (Source: DBE, 2019: 17)

In this problem the learner has to generate a number sentence by using ‘jumps’ on the number line. To solve the problem, learners must understand how the ‘jumps’ relate to the finite sets, moving from a continuous extent indicated as length to discrete sets (dots and unit lengths). Each ‘jump’ on the number line denotes a unit length. The image attempts to make the connection more explicit by placing the yellow and blue dots between the spaces to show that each unit length is equivalent to one dot, making it possible for the learner to construct a mapping from the set of dots to the corresponding unit lengths, thereby suggesting that each unit length is to be treated as an element of a set. Next, they have to construct a mapping from the sets of jumps over finite sets to addition over the natural numbers.

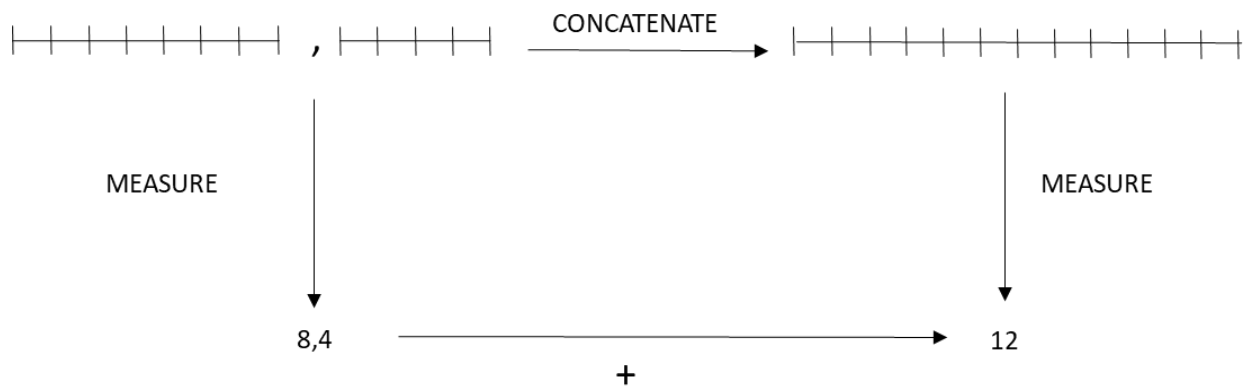


Figure 5.6: The mapping indicates how the markings on the line, maps to the aggregate of the set. The output, 8 and 4 become the arguments for addition.

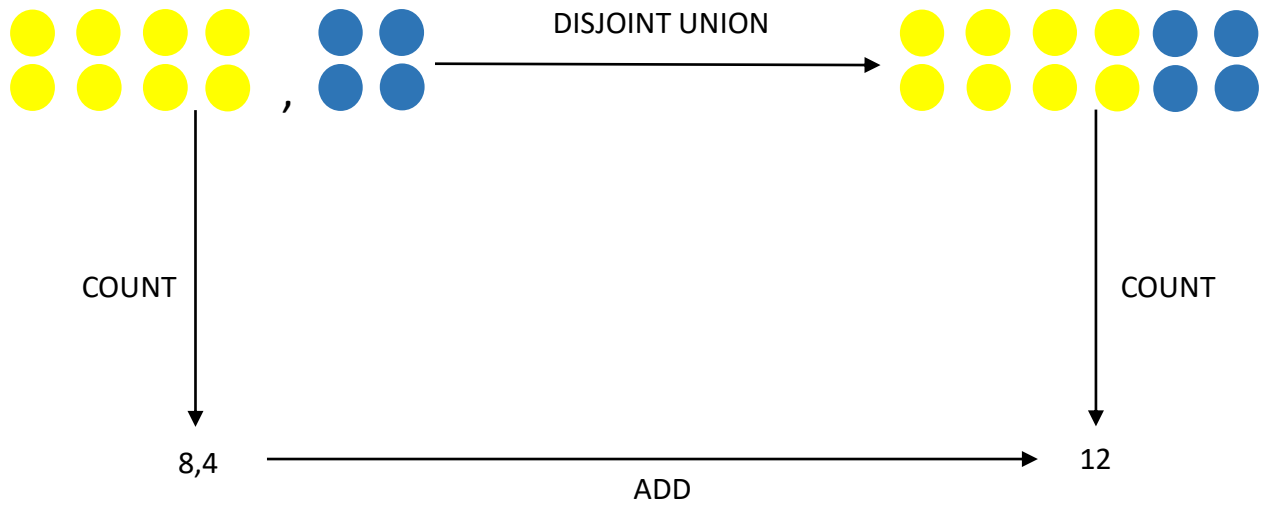


Figure 5. 7: Here the blue dots indicate the disjoint sets. The finite set maps to the set of natural numbers.

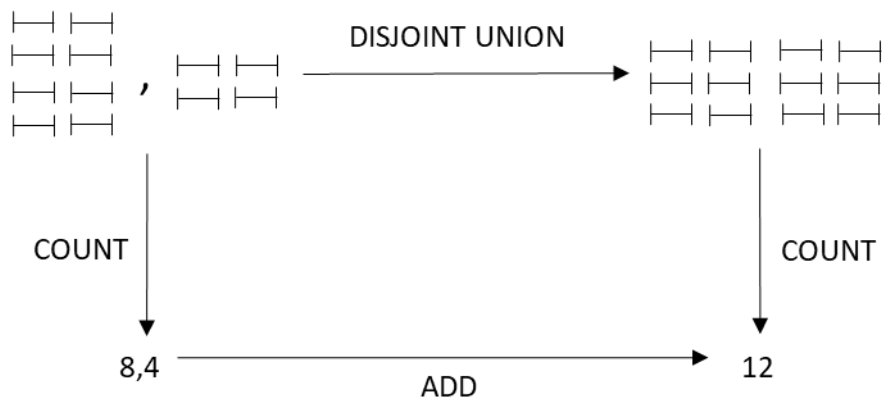


Figure 5. 8: Finally, we have the unit lengths that become the elements of each set. The unit lengths map to the set of natural numbers.

Later, when learners become more adept at using the number line they can dispense with the graded number line and use the jumps on an open number line to represent the values that are needed to solve the problem.

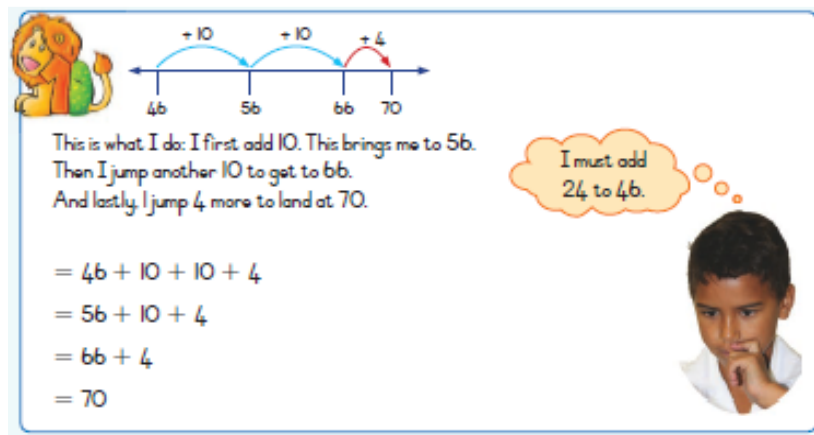


Figure 5.9: Decomposing the second addend and counting on in tens and units.

The next problem is taken from the DBE Mathematics workbook for Grade 3, Book 1 (pp.42-45) under the heading: Add on a number line. Here the learners are shown how the open number line can support mental arithmetic for solving two-digit addition problems. Three variations of decomposing the addends are described in the book. Figure 5.9 is one example that I shall discuss.

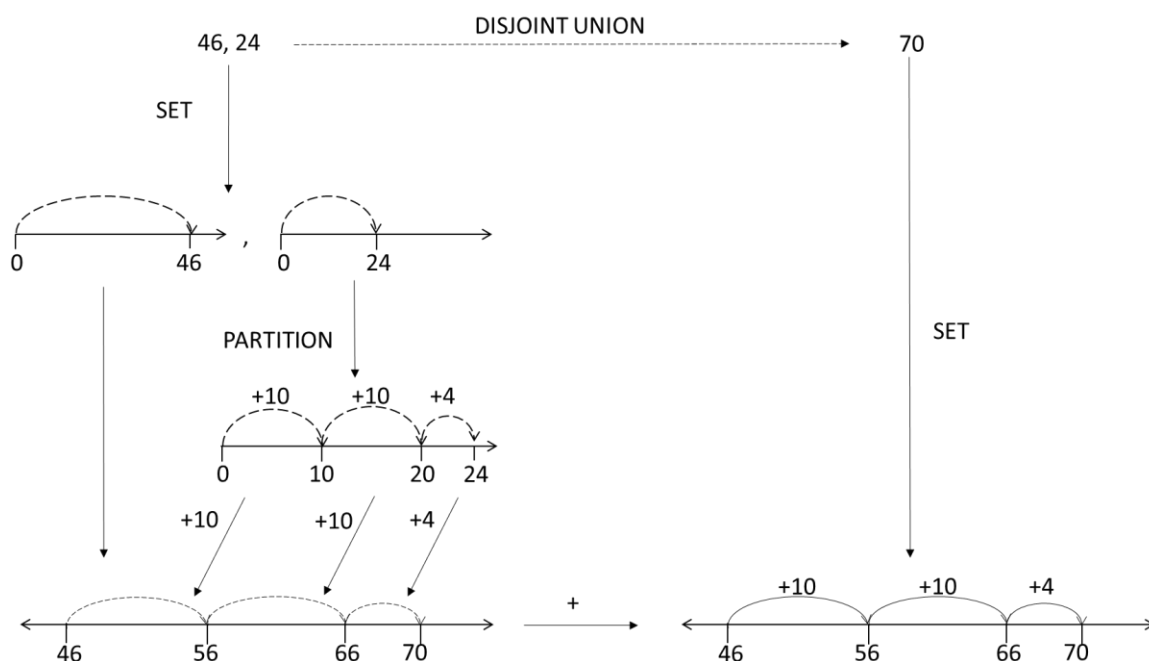


Figure 5.9: Indicates the mapping from addition over natural numbers to the disjoint union

In this example the learner is presented with an initial mapping of addition over natural numbers to disjoint union over finite sets. In this example, the learner applies the commutative property for addition and uses the larger value as the first argument. The second argument of the disjoint union can then be decomposed into various ways. One way to do this is to use canonical partition of natural numbers and in this instance, the learner uses tens and units. The second addend is decomposed into groups of ten. Each group of ten serves as an argument to the disjoint union-until all the groups of ten has been added. Finally, the remaining units are added to solve the problem.

The dotted arrow for $+$: $(46,24) \rightarrow 70$ is used to show that the mapping is implied since the second argument is decomposed into canonical partitions. The dotted jumps indicate the partitions for 24.

5.4 Concluding remarks

The central thesis of this chapter is to show how representation and structure preservation can be used to connect the research in cognitive science with the mathematical content that young children encounter daily. The use of representation to illustrate the homomorphic relationship between core knowledge and noncore, formal mathematics, must be considered because it can inform intervention programmes and help to improve instructional/teaching and learning practices.

One benefit is that it allows us to analyse the types of representation we use to teach mathematical concepts and ideas. By examining how closely it aligns with our natural intuitions, we can see how core knowledge either facilitates or impinges on new concepts and ideas we are trying to construct. The examples I have chosen to illustrate the connection between mathematical and cognitive scientific accounts with specific focus on the use of the number line. The number line is a reflection of an organised internal knowledge structure and there is a transparent connection with the ordinal and cardinal aspects of natural number. This relationship becomes explicit with the use of a representational diagram.

Chapter 6

Conclusion

6.1 Introduction

This study generated a synthesis of some of the research in cognitive science, the purpose of which was to explore why the number line might be an effective pedagogical tool to promote the growth of number sense in the child during the foundation phase of schooling. Research in mathematics education typically considers how the number line is used to facilitate number sense and support basic concepts of exact number, but for my study I approached the number line from a different perspective. I wanted to describe how the number line aligns with our natural intuitions about number and magnitude, making it an effective resource for teaching mathematical concepts in the early years. For that reason, I had to consider the internal workings of the mind and draw on research in cognitive science.

Chapter two compares the two major approaches to the study of the mind, viz., empiricism and rationalism. I proceeded to explain why a rationalist account of the mind is better suited to my inquiry (and to inquiry in general). Rationalism proceeds from an internalist position by looking at neurological structures of the brain and how it processes data. The chapter details how biologically endowed computational systems like the ANS and OTS structure the ways in which we experience our environment and connect with school mathematics. I concluded by arguing that it is important for educators to understand how genetic endowment enables us to think in particular ways and to have a greater awareness of how concepts are conditioned by *a priori* mental structures.

Chapter 3 is a synthesis of the research in cognitive science that connects the spatial orientation of the MNL to magnitude. Spatial associations, while innate, are malleable and influenced by environmental factors. While most Western societies exhibit a left-to-right orientation bias, it is not the only way in which numbers are represented internally. However, an oriented ordering of numbers internally as a MNL (whether left-to-right or not) maps to the orientation of the ordered number line used in schooling and supports three fundamental aspects of number: 1-1-correspondence as used in counting, ordinality and cardinality, making it a valuable teaching resource.

The scaling properties of number representation internally and the extent to which they inform us of numerical properties is the focus of Chapter 4. Number line estimation tasks measure the relative accuracy of numbers positioned on a line. This can either be interpreted to reflect the internal spatial

representation of number or it can be interpreted to reveal the types of strategies children use when making estimations.

Finally, Chapter 5 evaluates the efficacy of the number line as a mental model. In that chapter I connected the research in cognitive science with mathematics by using a mathematical account of representation to illustrate the structure preserving relationship between core domain knowledge and formal mathematics. Structure preserving mappings also illustrate the relationship between the ordinal and cardinal aspects of natural number.

6.2 Limitations of the use of the number line in schooling

It is unsurprising that additive reasoning dominates early mathematics instruction given that the innate human sensitivity to forming aggregates, merging aggregates, and breaking them apart is part of core domain systems. In fact, additive reasoning appears to have an overdetermining effect on both the curriculum and teaching with respect to elementary arithmetic. Consequently, pedagogical uses of the number line tend to emphasise additive relations (addition and subtraction). Further, multiplication and division are largely treated additively, as ‘repeated addition’ and ‘repeated subtraction’, respectively.

Such treatments of multiplication and division, even when aided by the use of the number line, constitute multiplication and division as derived from additive relations and, therefore, not as basic operations. Were multiplication and division explicitly treated as operations constructed around the notion of scaling, they would be constituted as basic operations in their own right and aligned with proportional rather than additive reasoning. Such alignment ought to be enabling of pedagogical uses of the number line in ways that support the growth of proportional reasoning, which is crucial for the growth of mathematically adequate conceptions of rational numbers (fractions).

6.3 Final remarks

Research findings in cognitive science gives us insights into the origins and early development of the knowledge on which children build when they go to school. However, the interdisciplinary nature of cognitive science and rapid expansion means that much of these insights are not accessible to teachers and makes it quite difficult to navigate. There is then an urgent need for researchers in mathematics education to synthesise the findings in cognitive science thereby making this new knowledge available to educators since much of our educational practices and understanding about learning continue to rely heavily on outdated theories.

Although we can learn a lot about the underlying cognitive processes that support learning in noncore domains, this knowledge does not directly translate to how teachers should teach, or, to how children learn in classrooms. What is strikingly absent from the array of research available to me, were pilot studies conducted in schools to test these theories. Research in education and cognitive science would benefit from a more collaborative approach to develop and test hypotheses and explore the best ways to teach children.

Finally, training in Early Childhood Development (ECD) still remains a neglected and underappreciated and under resourced sector of the Department of Basic Education. A general problem with ECD curriculum framework is that it lacks rigorous training with regards to mathematics and psychology. The South African National Curriculum Framework for children from Birth to Four (2015), recognises the importance of how young children use their senses and increased motor skills to explore their environment, but ignores how our minds construct ideas so that we can create the conditions for optimal learning to occur. ECD educators receive little to no training in how biologically innate structures assist learning in non-core areas like mathematics and reading. This is compounded by the misconception that mathematics is simple in the early years because learners work with small numbers, when much of the necessary groundwork for more sophisticated mathematics is laid.

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Addendum

Appendix 1: Weber Fraction (page 24)

Park and Starns (2015) proposed a new approach to measure ANS acuity which not only accounts for accuracy in numerosity comparison tasks, but also includes the speed with which a participant decides. The rate of information accumulated is called the *drift rate*. The drift diffusion model that they propose considers accuracy levels and response times which, they argue, captures ANS acuity more precisely and is, therefore, a better predictor of symbolic arithmetic performance. Alternatively, Sasanguie and her colleagues (2013) consider the Weber fraction and reaction time when they measure ANS acuity. The comparison distance effect (CDE) “computed as a difference between the conditions based on reaction times, (Sasanguie *et al.*, 2013)” reveal the ratio effect observed when reaction times are longer when two numerically close numbers are being compared. This, they argue, is crucial for predicting mathematical performance.

Appendix 2: Number string bisection tasks (pg. 33)

Fischer and colleagues (2003) findings’ support the neuro-imaging studies for partially overlapping regions in the parietal cortex. They report that spatial biases in numerical tasks can be extended to saccadic movement and are therefore not limited to motor responses. They also discovered that the size of the number preceding a target influences the speed with which adult participants detect it. Targets in the left visual area were recognized faster when preceded by a lower digit (1 or 2) than when preceded by a high digit (8 or 9), whereas targets in the right visual field were detected faster when preceded by a high digit than when preceded by a low digit. This behavior has also been observed in infants aged 8 to 9 months (Bulf, de Hevia, Macchi Cassia, 2016). They found evidence that perceiving a numerical cue (array of dots) boosted attentional shifts to the visual region of space that was congruent with the digit’s relative position on the MNL. Infants who saw a numerical cue reacted faster to a target on the right when it was preceded by a large numerosity cue, and towards the left when it was preceded by a small numerosity cue, like adults (Fischer et al, 2003). Non-symbolic numbers also induced attentional shifts akin to the SNARC effect. However, size cues did not induce attentional shifts since there was no difference in the reaction time of infants who were presented with a size cue prior to the target.