

**THE ANNIHILATION GRAPHS OF  
COMMUTATOR  
POSETS AND LATTICES**

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Thesis Presented for the Degree of

**DOCTOR OF PHILOSOPHY**

in the Department of Mathematics and Applied Mathematics

**UNIVERSITY OF CAPE TOWN**

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August 2015

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# Acknowledgements

I would like to express my special gratitude to my supervisor, Professor George Janelidze, for all his support and excellently guiding and advising me through the preparation of this thesis and for his encouragement and inspiration. His contributions and suggestions are gratefully appreciated.

I also acknowledge the Department of Mathematics and Applied Mathematics of the University of Cape Town, specially Professors Hans-Peter A. Künzi and Christopher R. A. Gilmour for their helps during my study at this university.

I express my sincere gratitude to Dr Amir M. Rahimi for the fruitful ideas and his assistance in proofreading of the thesis.

I also wish to express my profound gratitude to my parents for their support and patience for all the times that I have been away from them during the course of this study. I extend my gratitude to my sister and brother, and my inlaws for their support.

# Abstract

We propose a new, widely generalized context for the study of the zero-divisor/annihilating-ideal graphs, where the vertices of graphs are not elements/ideals of a commutative ring, but elements of an abstract ordered set (imitating the lattice of ideals), equipped with a binary operation (imitating products of ideals). The intermediate level of congruences of any algebraic structure admitting a “good” theory of commutators is also considered.

**Keywords:** annihilation graph, annihilating-ideal graph, zero-divisor graph, diameter, girth, clique, connectedness, cut-point, bridge, complete  $r$ -partite graph, poset, lattice, commutator, congruence.

# Introduction

Given a commutative ring  $R$ , one can form a graph whose vertices are (some) elements of  $R$  and edges are pairs  $(x, y)$  with  $xy = 0$  in  $R$ . Or, one can replace elements with ideals of  $R$  and do the same, that is, define edges as pairs  $(A, B)$  with  $AB = \{0\}$ . The study of these so-called zero-divisor graphs and annihilating-ideal graphs were initiated by I. Beck [14] and M. Behboodi and Z. Rakeei [15] respectively, and then continued by various authors.

Since the product of two ideals of a commutative ring is nothing but their commutator in the sense of universal algebra (see Section 3.1 for details), it is natural to:

- replace ideals of a commutative ring with congruences of any algebraic structure that admits a good notion of commutator; "good" might have different meanings (see e.g. R. Freeze and R. McKenzie [22] for different notions of a commutator), and the properties we actually need to hold are listed in Section 3.1;

- replace the property  $AB = \{0\}$  above with the property  $[\alpha, \beta] = 0$ , where  $\alpha$  and  $\beta$  are congruences on a given algebraic structure,  $[\alpha, \beta] = 0$  their commutator, and  $0$  denotes now the equality relation (since it is the smallest congruence)

on that given algebraic structure.

Although this replacement is itself a wide generalization, it immediately suggests a further wide generalization, where congruences on a given algebra are replaced with elements of an abstract lattice, or even just an ordered set, equipped with a binary operation. The condition that operation should be required to satisfy should then imitate the properties of commutators (as in our Section 3.1).

This two-step generalization in the study of *annihilation graphs* (we say "annihilation" instead of "annihilating-") is the author's PhD Thesis' theme, under supervision of Professor G. Janelidze, who suggested it.

Notice that, as suggested by the context considered in [7], we consider a 'relative version' of annihilation, where  $ab = 0$  is replaced with  $ab \leq z$  with a fixed  $z$ .

In this project, our proofs closely follow the ring case. The most surprising fact here is that the binary operation involved is not required to be associative, unlike the ring multiplication; this is important since the commutator operation is almost never associative, except the commutative ring case.

We now review a brief history of the zero-divisor type graphs of commutative rings.

The study of algebraic structures using the properties of graphs has been an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring and semigroup, for instance see [1, 11, 9, 8, 29, 10, 3, 4, 2, 5, 6, 14, 32, 30, 40, 41,

42, 19, 30, 44].

In 1988, Beck [14] introduced the concept of a *zero-divisor graph* of a commutative ring  $R$ , but this work was mostly concerned with colorings of rings. Let  $G$  be a simple graph whose vertices are the elements of  $R$  and two distinct vertices  $x$  and  $y$  are adjacent if and only if  $xy = 0$ . The graph  $G$  is known as the zero-divisor graph of  $R$ . He conjectured that,  $\chi(G)$  (the *chromatic number* of  $G$ ) is same as  $\Omega(G)$  (the *clique number* of  $G$ ). In 1993, Anderson and Naseer [8] gave an example of a commutative local ring  $R$  with 32 elements for which  $\chi(G) = 6 > \Omega(G) = 5$ .

The *zero-divisor graph of a commutative ring*  $R$ , denoted  $\Gamma(R)$ , is an undirected graph whose vertices are the nonzero zero-divisors of  $R$  with two distinct vertices  $x$  and  $y$  joined by an edge if and only if  $xy = 0$ . Thus  $\Gamma(R)$  is the *empty graph* if and only if  $R$  is an *integral domain*.

The above definition first appeared in the work of Anderson and Livingston [11], which contains several fundamental results concerning  $\Gamma(R)$ . This definition, unlike the earlier work of Anderson and Naseer [8] and Beck [14], does not take zero to be a vertex of  $\Gamma(R)$ .

In [39], Redmond introduced the notion of an *ideal-based zero-divisor graph* of a commutative ring and his work continued and developed further by Maimani, Pournaki, and Yassemi in [31] and they called it *zero-divisor graph with respect to an ideal*.

Let  $I$  be a proper ideal of  $R$ . The *zero-divisor graph of  $R$  with respect to  $I$* ,

denoted by  $\Gamma_I(R)$ , is the graph whose vertex set is the set

$$\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$$

with distinct vertices  $x$  and  $y$  adjacent if and only if  $xy \in I$ . Thus if  $I = \{0\}$ , then  $\Gamma_I(R) = \Gamma(R)$ , and  $I$  is a *nonzero prime ideal* of  $R$  if and only if  $\Gamma_I(R) = \emptyset$ .

In both papers [39] and [31], the authors explored the relationship between  $\Gamma_I(R)$  and  $\Gamma(R/I)$ .

The concept of the *annihilating-ideal graph* of a commutative ring, denoted by  $\mathbb{A}G(R)$ , was first introduced by Behboodi and Rakeei in [15] and [16]. Actually,  $\mathbb{A}G(R)$  is the *zero-divisor graph* of the *multiplicative semigroup* of the ideals of  $R$  (see [19]).

Also in [7], Aliniaefard, Behboodi, Rahimi, and the author of this thesis have extended and studied this notion to a more general setting as *the annihilating-ideal graph of a commutative ring  $R$  with respect to an ideal  $I$  of  $R$* , denoted by  $\mathbb{A}G_I(R)$ , by replacing (nonzero) ideals whose product is zero with ideals ( $\not\subseteq I$ ) whose product lies in  $I$ . Clearly,  $I$  is a *nonzero prime ideal* of  $R$  if and only if  $\mathbb{A}G_I(R) = \emptyset$ .

Notice that  $\mathbb{A}G_I(R)$  can be regarded as an *ideal-based zero-divisor graph* of the semiring of the ideals of  $R$  and can be denoted as  $\Gamma_{C_I}(\mathbb{I}(R))$ , where  $\mathbb{I}(R)$  is the semiring of the ideals of  $R$  and  $C_I$  is the set of all ideals of  $R$  that are contained in  $I$ .

As an example of a *graphical representation of rings* different from ‘zero-divisor type graphs’ of a commutative ring, Sharma and Bhatwadekar in [42],

defined a graph  $G$ , on a commutative ring  $R$  with vertices as elements of  $R$ , where two distinct vertices  $a$  and  $b$  are adjacent if and only if  $Ra + Rb = R$ . They showed that  $\chi(G)$  (the chromatic number of  $G$ ) is finite if and only if  $R$  is a finite ring. In this case  $\chi(G) = \omega(G) = t + l$ , where  $\omega(G)$  is the clique number of  $G$ ; and  $t$  and  $l$ , respectively, denote the number of *maximal ideals* and the number of *units* of  $R$  (see Theorem 2.3 in [42]).

Further, in [32], Maimani et al. studied the graph structure defined by Sharma and Bhatwadekar and called it “comaximal graph of a ring”. In their work, they mostly focused on the graph-theoretic and related ring-theoretic properties of the subgraph generated by *nonunit elements* of  $R$ .

Further, Rahimi and the author of this thesis studied the dominating sets of the comaximal and ideal-based zero-divisor graphs of a commutative ring [35]. In their work, besides characterizing the domination number of a comaximal graph, they also compared the domination number of an ideal-based zero-divisor graph  $\Gamma_I(R)$  with the domination number of  $\Gamma(R)$  the zero-divisor graph of a ring, which studied, in detail, by Mojdeh and Rahimi in [36].

Moreover, the concept of a zero-divisor graph of a commutative ring has been generalized to a *k-zero-divisor hypergraph* by Eslahchi and Rahimi in [21]. In their work, they associates a *k-uniform hypergraph* to a commutative ring  $R$ , denoted by  $H_k(R)$ , and besides many results (examples), provide conditions under which  $H_k(R)$  is connected [respectively, bipartite or complete].

In this work, we assume that the reader is familiar with the basic notion and definitions of lattice theory. For the notation and definitions regarding lattice theory, the reader is referred to [17] or any standard text on lattice theory. In

the next chapter, we will write all necessary graph-theoretic definitions that are required in this thesis. Also, for the notation and definitions regarding graph theory, the reader is referred to [18] or any standard text on graph theory.

Note that the results of this thesis are indicated numerically and the propositions that are indicated by upper case letters from A to I, in some remarks, in this thesis are from the other papers.

The thesis consists of the following chapters:

Chapter 1: We begin with presenting the standard definitions of graph theory that will be used in this thesis and then recall some results from [7] on the annihilating-ideal graph of a commutative ring with respect to an ideal.

Chapter 2: We introduce the definition of a commutator poset and define the annihilation graph of a commutator poset  $L$  with respect to an element  $z \in L$ , denoted by  $\mathbb{A}\mathbb{G}_z(L)$  and discuss some basic properties of a commutator poset and  $\mathbb{A}\mathbb{G}_z(L)$ . We show that  $\mathbb{A}\mathbb{G}_z(L)$  is connected with  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$  and if  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle, then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$  (Theorem 2.2.2). We also study the condition(s) under which a vertex [respectively, an edge] is not a *cut-point* [respectively, *bridge*] (Propositions 2.3.2 and 2.4.2, respectively) and by Example 2.3.10 show that  $\mathbb{A}\mathbb{G}_z(L)$  has a cut-point in contrast to the ring case (Proposition 2.3.7). In Theorem 2.5.3 we discuss a relationship between the prime elements of  $L$  and complete bipartiteness of  $\mathbb{A}\mathbb{G}_z(L)$  and provide a different proof (without using associativity of the multiplication) from the ring case for Part (b) of this theorem. We also study a relationship between the prime elements (Definition 2.1.6) of  $L$  and the clique number of  $\mathbb{A}\mathbb{G}_z(L)$  (The-

orem 2.6.8). We show how to construct ideals (Definition 2.5.1) in  $L$  from the *maximal cliques* of  $\mathbb{A}\mathbb{G}_z(L)$  (Theorem 2.6.11) and finally provide some counterexamples (Examples 2.6.12 and 2.6.15) for Theorem 2.6.11.

Chapter 3: We introduce the concept of a commutator lattice and provide some examples for it. Then we define the annihilation graph of a commutator lattice  $L$  with respect to an element  $z \in L$ , denoted by  $\mathbb{A}\mathbb{G}_z(L)$ , and discuss some properties of  $\mathbb{A}\mathbb{G}_z(L)$ . In Theorem 3.2.13 we discuss some properties of a universal vertex of  $\mathbb{A}\mathbb{G}_z(L)$  for a complete commutator lattice  $L$  and by Example 3.2.14 we show that the associativity condition in the last part of Parts (a) and (b) of this theorem is a necessary assumption. We provide a different proof from the ring case for Theorem 3.3.5 that discusses a condition for connectivity of a subgraph of  $\mathbb{A}\mathbb{G}_z(L)$ . We study the condition(s) under which a vertex [respectively, an edge] is not a *cut-point* [respectively, *bridge*] (Propositions 3.4.1 and 3.5.1, respectively). In Theorem 3.6.1, we discuss a relationship between the prime elements of  $L$  and complete bipartiteness of  $\mathbb{A}\mathbb{G}_z(L)$  and by Example 3.6.2 show that the primeness in this theorem is a necessary condition. In Theorem 3.6.4, by a different proof from the ring case, we verify that if  $\mathbb{A}\mathbb{G}_z(L)$  for a complete commutator lattice  $L$  is a complete  $r$ -partite graph with  $r \geq 3$ , then at most one of the parts has more than one vertex. We also show that if  $z$  is an element of a commutator lattice  $L$  such that  $z = \bigwedge_{1 \leq i \leq n} p_i$  and for each  $1 \leq j \leq n$ ,  $z \neq \bigwedge_{1 \leq i \leq n, i \neq j} p_i$ , where  $p_i$  is a prime element of  $L$  for each  $1 \leq i \leq n$ , then  $\omega(\mathbb{A}\mathbb{G}_z(L)) = n$  (Theorem 3.7.5).

# Chapter 1

## Revision of the theory of annihilating ideal-graphs

In this chapter we present the standard definitions of graph theory that will be used in the sequel and recall some results from [7] on the annihilating-ideal graph of a commutative ring with respect to an ideal.

### 1.1 Graphs

By a graph we will mean a pair  $G = (V, E)$ , in which  $V$  is a set and  $E$  a binary irreflexive symmetric relation on  $V$ ; the elements of  $V$  and of  $E$  will be called vertices and edges of  $G$ , respectively.

**Definition 1.1.1.** For a natural number  $n$ , a *path of length  $n$*  in a graph  $G$  is an  $(n + 1)$ -tuple  $(x_0, \dots, x_n)$  of distinct vertices of  $G$ , such that  $(x_{i-1}, x_i)$  is an edge of  $G$ , for each  $i \in \{1, \dots, n\}$ . A path  $(x_0, \dots, x_n)$  is also called a *path from  $x_0$  to  $x_n$* .

- For a graph  $G$ , the *degree* of a vertex  $v$  in  $G$  is the number of edges of  $G$

incident with  $v$ . We denote by  $\delta(G)$  the minimum degree of vertices of  $G$ .

**Definition 1.1.2.** The *distance*  $d(x, y)$  between vertices  $x$  and  $y$  of a graph  $G$  is defined as follows:

- for a (non-zero) natural number  $n$ ,  $d(x, y) = n$ , if  $n$  is the smallest natural number, for which there exists a path of length  $n$  from  $x$  to  $y$ .
- $d(x, y) = 0$ , if  $x = y$ ;
- $d(x, y) = \infty$ , if  $x \neq y$  and there is no path from  $x$  to  $y$ .

Accordingly, the distance between  $x$  and  $y$  is said to be *finite* if either  $x = y$  or there exists a path from  $x$  to  $y$ , and *infinite* otherwise.

**Definition 1.1.3.** A graph  $G$  is said to be *connected*, if for every two distinct vertices  $x$  and  $y$  of  $G$ , there exists a path from  $x$  to  $y$ , or equivalently, the distance  $d(x, y)$  is finite.

**Remark 1.1.4.** Note that, according to the above definition, the empty graph is connected in contradiction with the categorical (and topological) notion of connectedness.

**Definition 1.1.5.** The *diameter*  $\text{diam}(G)$  of a connected graph  $G$  is defined as the largest distance between its vertices. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter

1 if and only if it is complete; that is, each pair of distinct vertices forms an edge.

**Definition 1.1.6.** For a natural  $n \geq 3$ , a *cycle of length  $n$*  in a graph  $G$  is an  $(n + 1)$ -tuple  $(x_0, \dots, x_n)$  of vertices of  $G$  that are distinct except  $x_0 = x_n$ , and such that  $(x_{i-1}, x_i)$  is an edge of  $G$ , for each  $i \in \{1, \dots, n\}$ .

**Definition 1.1.7.** If  $G$  has a cycle, then *girth* of  $G$ , denoted by  $\text{gr}(G)$ , is the smallest number  $n$  such that  $G$  has a cycle of length  $n$ . If  $G$  has no cycle, then  $\text{gr}(G) = \infty$ .

It is well known and easy to show that if  $G$  has a cycle, then

$$\text{gr}(G) \leq 2 \text{diam}(G) + 1;$$

however, in the contexts we shall consider, better inequalities are obtained (see [7, Theorem 3.3] in Section 1.2 and Theorem 2.2.2; [7, Theorem 3.3] is a generalization of a result in [20]).

**Definition 1.1.8.** A vertex  $x$  of a connected graph  $G$  is a *cut-point* of  $G$  if there are vertices  $u$  and  $v$  of  $G$  such that  $x$  is in every path from  $u$  to  $v$  with  $x \neq u$  and  $x \neq v$ . Equivalently, for a *connected graph*  $G$ ,  $x$  is a cut-point of  $G$  if  $G \setminus \{x\}$  (i.e., the graph  $G$  without vertex  $x$ ) is not connected.

**Definition 1.1.9.** An edge  $E$  of a *connected graph*  $G$  is a *bridge* if the graph  $G \setminus \{E\}$  (i.e., the graph  $G$  without edge  $E$ ) is *disconnected*.

**Definition 1.1.10.** An *r-partite graph* is a graph whose vertex set can be partitioned into  $r$  subsets so that no edge has both ends in any one subset.

- A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. The complete graph on  $n$  vertices is denoted by  $K_n$ .

**Definition 1.1.11.** A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset.

- The *complete bipartite graph* (2-partite graph) with parts of size  $m$  and  $n$  is denoted by  $K_{m,n}$ .

- A complete bipartite graph of the form  $K_{1,n}$  is called a *star graph*.

**Definition 1.1.12.** For a graph  $G$ , a complete subgraph of  $G$  is called a *clique*.

- The *clique number* of  $G$ ,  $\omega(G)$ , is the greatest integer  $n \geq 1$  such that  $K_n \subseteq G$ , and  $\omega(G)$  is infinite if  $K_n \subseteq G$  for all  $n \geq 1$ .

- A *maximal clique* is a clique that cannot be extended by including one more adjacent vertex, meaning it is not a subset of a larger clique.

**Definition 1.1.13.** Let  $S$  be a non-empty set of the vertices of a graph  $G$ . The *subgraph induced by  $S$*  is the *maximal subgraph* of  $G$  with vertex set  $S$ , denoted by  $\langle S \rangle$ , that is,  $\langle S \rangle$  contains precisely those edges of  $G$  joining two vertices in  $S$ .

## 1.2 The annihilating-ideal graph of a commutative ring with respect to an ideal

In this section we recall some results from [7] and generalize most of them to commutator posets/commutator lattices (in Chapters 2 and 3 of this thesis, respectively). In this section, we include the ring-theoretic proof of those results not having a commutator poset/commutator lattice version proof in the next chapters of the thesis or having a different proof in Chapter 2 or Chapter 3 of this thesis.

We begin this section by recalling the definitions of an annihilating-ideal graph of a commutative ring  $R$  and the annihilating-ideal graph with respect to an ideal of  $R$ .

The annihilating-ideal graph  $\mathbb{A}\mathbb{G}(R)$  of a commutative ring  $R$  (with 1), introduced by M. Behboodi and Z. Rakeei [15], is defined as follows:

- the vertices of  $\mathbb{A}\mathbb{G}(R)$  are all non-zero ideals of  $R$  with non-zero annihilators;
- a pair  $(A, B)$  of distinct vertices of  $\mathbb{A}\mathbb{G}(R)$  is an edge of  $\mathbb{A}\mathbb{G}(R)$  if and only if  $AB = \{0\}$ .

On the other hand, one can fix an ideal  $I$  in  $R$  and consider the graph  $\mathbb{A}\mathbb{G}_I(R)$ , called *the annihilating-ideal graph of  $R$  with respect to the ideal  $I$*  in [7], in which:

- the vertices of  $\mathbb{A}\mathbb{G}_I(R)$  are all ideals  $A$  of  $R$  not contained in  $I$  and having an ideal  $A'$  not contained in  $I$  with  $AA'$  contained in  $I$ ;

- a pair  $(A, B)$  of distinct vertices of  $\mathbb{A}\mathbb{G}_I(R)$  is an edge of  $\mathbb{A}\mathbb{G}_I(R)$  if and only if  $AB \subseteq I$ .

According to this definition,  $\mathbb{A}\mathbb{G}(R) = \mathbb{A}\mathbb{G}_{\{0\}}(R)$ .

Let us mention some notation and definitions that are used in [7].

Let  $I$  be a proper ideal of a commutative ring  $R$  with nonzero identity. Let  $C_I(R)$  be the set of all ideals of  $R$  that are contained in  $I$ . We also simply write  $C_I$  for  $C_I(R)$  whenever there is no confusion in the context. Note that  $\mathbb{I}(R)$  the set of all ideals of  $R$  is a semiring with zero ( $= (0)$ ) and identity ( $= R$ ) under the multiplication and addition of ideals of  $R$ . It is clear that  $C_I(R)$  is an ideal in the semiring  $\mathbb{I}(R)$ . The radical of an ideal  $\mathcal{I}$  of  $\mathbb{I}(R)$  is the set of all ideals  $A$  of  $R$  such that  $A^n \in \mathcal{I}$  for some positive integer  $n$ . Clearly, the radical of  $C_I(R)$  is the set of all ideals  $A$  of  $R$  such that  $A^n \subseteq I$  for some positive integer  $n$ . Also, for any ideal  $X$  of  $R$ , we write  $(I : X)$  to denote the set of all ideals  $A$  of  $R$  such that  $AX \subseteq I$ .

We now write some of the results from [7] as follows:

**[7, Lemma 2.10]** *Let  $I$  be an ideal of a commutative ring  $R$ . Then*

- $\sqrt{I} = I$  if and only if  $\sqrt{C_I(R)} = C_I(R)$ , where  $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some integer } n \geq 1\}$ .
- $P$  is a prime ideal of  $R$  if and only if  $C_P(R)$  is a prime ideal in the semiring  $\mathbb{I}(R)$ .

(c)  $I = \bigcap_{1 \leq i \leq n} P_i$  if and only if  $C_I = \bigcap_{1 \leq i \leq n} C_{P_i}$ .

*Proof.* The proof follows directly from the definition. □

**[7, Proposition 2.2]** *Let  $A$ ,  $B$ , and  $C$  be vertices of the graph  $\mathbb{A}\mathbb{G}_I(R)$ .*

*Then*

(a) *Suppose  $A \cap B$  is not contained in  $I$ . Then  $(A \cap B)C$  is contained in  $I$  whenever  $AC$  or  $BC$  is a subset of  $I$ .*

(b) *If  $A \cap B$  is contained in  $I$ , then  $(A, B)$  is an edge in  $\mathbb{A}\mathbb{G}_I(R)$  since  $AB \subseteq A \cap B$ .*

(c)  *$(A + B)C$  is contained in  $I$  whenever  $AC$  and  $BC$  are subsets of  $I$ .*

*Proof.* For a commutator lattice version proof, see Proposition 3.2.8. □

**[7, Corollary 2.3]** *Let  $I$  be an ideal of a ring  $R$  such that the radical of  $I$  is equal to  $I$ . If  $\Gamma_I(R)$  the zero-divisor graph of  $R$  with respect to  $I$  contains two edges  $(a, c)$  and  $(b, c)$ , then the ideals  $(a, b)$  and  $(c)$  are adjacent vertices in  $\mathbb{A}\mathbb{G}_I(R)$ .*

*Proof.* See Part (c) of the above proposition. Notice that the assumption  $\sqrt{I} = I$  is necessary in order to prevent, for example,  $(a, b) = (c)$  which implies  $c \in I$ , a contradiction. Of course, by relaxing the condition  $\sqrt{I} = I$ , the edges in  $\Gamma_I(R)$  need not be preserved as edges in  $\mathbb{A}\mathbb{G}_I(R)$  since  $(a, b)$  ( $a \neq b, a, b \in \Gamma_I(R)$ ) does not guarantee  $(a) \neq (b)$ . □

**Remark** Note that in the above corollary,  $1 \notin (a, b)$  since  $c \notin I$ .

**[7, Proposition 2.5]** *Let  $I$  be a nonzero proper ideal of  $R$ . Then*

- (a)  $\mathbb{A}\mathbb{G}_I(R)$  is the empty graph if and only if  $I$  is a prime ideal of  $R$ .
- (b) For any ideal  $I$  of  $R$ ,  $\mathbb{A}\mathbb{G}_I(R)$  contains a copy of  $\mathbb{A}\mathbb{G}(R/I)$  as a subgraph.
- (c)  $\mathbb{A}\mathbb{G}_I(R)$  is the empty graph if and only if  $\mathbb{A}\mathbb{G}(R/I)$  is the empty graph.

*Proof.* We leave the proof of Part (a) to the reader. Part (c) follows directly from Part (a) and the fact that  $\mathbb{A}\mathbb{G}(R)$  is the empty graph if and only if  $R$  is a domain [15]. Part (b) is immediate since there is a one-to-one correspondence between the ideals of  $R/I$  and the ideals of  $R$  containing  $I$ .  $\square$

[7, **Theorem 2.6**] (cf. See Theorem 3.2.13 for a complete commutator lattice version.)

Let  $I$  be a nonzero ideal of  $R$  and  $A \in \mathbb{A}\mathbb{G}_I(R)$ .

- (a) If  $A$  is adjacent to every vertex of  $\mathbb{A}\mathbb{G}_I(R)$  with  $A \cap I \neq I$ , then  $(I : A)$  is a maximal element of the set  $\{(I : X) \mid X \in \mathbb{I}(R) \setminus C_I\}$ . Moreover,  $(I : A)$  is a prime ideal in  $\mathbb{I}(R)$ .
- (b) If  $A$  is adjacent to every vertex of  $\mathbb{A}\mathbb{G}_I(R)$  with  $A^2 \subseteq I$ , then  $(I : A)$  is a maximal element of the set  $\{(I : X) \mid X \in \mathbb{I}(R) \setminus C_I\}$ . Moreover,  $(I : A)$  is a prime ideal in  $\mathbb{I}(R)$ .

*Proof.* (a) Let  $V = V(\mathbb{A}\mathbb{G}_I(R))$ . Choose an ideal  $X \subseteq I$  with  $X$  not contained in  $A \cap I$ . It is easy to see that  $A \neq A + X$ . Also for every  $B$  in  $V$  (different from  $A$ ), by hypothesis, it is clear that  $B(A + X) \subseteq I$  which implies  $A + X \in \mathbb{A}\mathbb{G}_I(R)$  (note that  $A + X \neq R$  since  $B \not\subseteq I$ ). Otherwise, if there is no such  $B$  ( $\neq A$ ) in  $V$ , then  $\mathbb{A}\mathbb{G}_I(R)$  must have only one vertex  $A$  and hence  $A^2 \subseteq I$ . Hence  $A(A + X) \subseteq I$  implies  $A + X \in V$  which is a contradiction (note that  $A \neq A + X$ ).

Consequently,  $\mathbb{A}\mathbb{G}_I(R)$  must have more than one vertex and hence  $A + X \in V$ . Thus by hypothesis,  $A(A + X) = A^2 + AX \subseteq I$  implies  $A^2 \subseteq I$ . Therefore,  $V \cup C_I = (I : A)$ , and so for any  $X \in \mathbb{I}(R) \setminus C_I$ ,  $(I : X)$  is contained in  $V \cup C_I = (I : A)$ . Thus the first assertion holds.

Now, we prove that  $(I : A)$  is a prime ideal of  $\mathbb{I}(R)$ . Let  $XY \in (I : A)$  and  $Y \notin (I : A)$ . Therefore,  $XYA \subseteq I$ . Clearly,  $YA \not\subseteq I$  since  $Y \notin V \cup C_I = (I : A)$ . Hence  $YA \in \mathbb{I}(R) \setminus C_I$ . We know that  $(I : A) \subseteq (I : YA)$ , and by maximality of  $(I : A)$ ,  $(I : A) = (I : YA)$ . Hence,  $X \in (I : A)$ .

(b) Clearly,  $A^2 \subseteq I$  and  $A$  adjacent to every vertex of  $V$  implies  $V \cup C_I = (I : A)$ . Therefore, for any  $X \in \mathbb{I}(R) \setminus C_I$ ,  $(I : X)$  is contained in  $V \cup C_I = (I : A)$ . Thus the first assertion holds and the rest of the proof is similar to Part (a).  $\square$

**[7, Theorem 2.7]** *Let  $I$  be an ideal of  $R$  and let  $S$  be a maximal clique in  $\mathbb{A}\mathbb{G}_I(R)$  such that  $X^2 \subseteq I$  for all  $X \in S$ . Then  $S \cup C_I$  is an ideal of  $\mathbb{I}(R)$ .*

*Proof.* For a commutator poset version proof, see Theorem 2.6.11.  $\square$

For the definition of  $\langle \mathcal{S} \rangle$  (in the following theorem), see Definition 1.1.13.

**[7, Theorem 3.1]** *Let  $I$  be an ideal of  $R$  and consider  $\mathcal{S} = \sqrt{C_I(R)} \setminus C_I(R)$ . If  $\mathcal{S}$  is a nonempty set, then  $\langle \mathcal{S} \rangle$  is connected.*

*Proof.* For a different proof for a commutator lattice, see Theorem 3.3.5.

Let  $X, Y \in \mathcal{S}$ . If  $XY \in C_I$ , then the result is clear. Suppose that  $XY \notin C_I$ , where for some positive integers  $m$  and  $n$ ,  $X^n, Y^m \in C_I$  and  $X^{n-1}, Y^{m-1} \notin C_I$ . Hence, the path

$$(X, X^{n-1}, XY, Y^{m-1}, Y)$$

is a path of length less than or equal to four from  $X$  to  $Y$ . □

**[7, Corollary 3.2]** *Suppose  $N = \sqrt{(0)} \setminus (0) \neq \emptyset$ . Then  $\langle N \rangle$  is a connected subgraph of  $\mathbb{A}\mathbb{G}(R)$ .*

**[7, Theorem 3.3]** *Let  $I$  be an ideal of a ring  $R$ . Then  $\mathbb{A}\mathbb{G}_I(R)$  is connected with  $\text{diam}(\mathbb{A}\mathbb{G}_I(R)) \leq 3$ . Furthermore, if  $\mathbb{A}\mathbb{G}_I(R)$  contains a cycle, then  $\text{gr}(\mathbb{A}\mathbb{G}_I(R)) \leq 4$ .*

*Proof.* For a commutator poset version proof, see Theorem 2.2.2. □

For the definition of the cut-point of a graph, see Definition 1.1.8.

**[7, Proposition 3.5]** *Let  $I$  be a nonzero ideal of  $R$  and  $X$  a vertex in  $\mathbb{A}\mathbb{G}_I(R)$ .*

- (a) *If  $X$  does not contain  $I$ , then  $X$  is not a cut-point of  $\mathbb{A}\mathbb{G}_I(R)$ .*
- (b) *If  $X$  is not a principal ideal of  $R$ , then  $X$  is not a cut-point in  $\mathbb{A}\mathbb{G}_I(R)$ .*
- (c) *Suppose  $X = (x)$  is a principal ideal containing  $I$  and  $\sqrt{I} = I$ . If  $(x) \neq (x)^2$ , then  $X$  is not a cut-point in  $\mathbb{A}\mathbb{G}_I(R)$ .*

*Proof.* (a): For the proof of this part for a commutator lattice, see Proposition 3.4.1.

(b): Let  $x \in X \setminus I$ .

*Case 1:* Suppose  $(U, X, W)$  is a path of shortest length from  $U$  to  $W$ . For the case  $(x) = U$  [or  $(x) = W$ ], then  $(U, W)$  is a path of length 1. Otherwise,  $(U, (x), W)$  is a path of length 2 in  $\mathbb{A}\mathbb{G}_I(R)$  with  $(x) \neq X$  which is a contradiction.  $U = (x)$  or  $W = (x)$ .

*Case 2:* Suppose (without loss of generality)  $(U, X, Y, W)$  is a path of shortest length from  $U$  to  $W$  in  $\mathbb{A}\mathbb{G}_I(R)$ . Clearly,  $(U, (x), Y, W)$  is a path different from  $(U, X, Y, W)$  in  $\mathbb{A}\mathbb{G}_I(R)$  which is a contradiction.

(c): Let  $X = (x)$ . Clearly,  $(x)^2 \subseteq (x)$  and  $(x)^2 \not\subseteq I$  since  $\sqrt{I} = I$  by hypothesis. Arguments like those in Parts (a) and (b) show that  $X$  is not a cut-point in  $\mathbb{A}\mathbb{G}_I(R)$ .  $\square$

For the definition of the bridge of a graph, see Definition 1.1.9.

[7, **Proposition 3.6**] *Let  $I$  be a nonzero ideal of a ring  $R$  and  $A \neq B$  two distinct vertices of the graph  $\mathbb{A}\mathbb{G}_I(R)$  with  $(A, B)$  an edge in  $\mathbb{A}\mathbb{G}_I(R)$ .*

- (a) *Suppose  $A \not\subseteq B$  and  $B \not\subseteq A$  with  $A \cap B \not\subseteq I$ . Then  $(A, B)$  is not a bridge in  $\mathbb{A}\mathbb{G}_I(R)$ .*
- (b) *Assume  $I \not\subseteq A$  and  $I \not\subseteq B$ . Then  $(A, B)$  is not a bridge in  $\mathbb{A}\mathbb{G}_I(R)$ .*
- (c) *Assume  $A^2 \subseteq I$  and  $B^2 \subseteq I$  with  $A \not\subseteq B$  and  $B \not\subseteq A$ . Then  $(A, B)$  is not a bridge in  $\mathbb{A}\mathbb{G}_I(R)$ .*
- (d) *Suppose that neither  $A$  nor  $B$  is a principal ideal of  $R$  with  $A \not\subseteq B$  and  $B \not\subseteq A$ . Then  $(A, B)$  is not a bridge in  $\mathbb{A}\mathbb{G}_I(R)$ .*
- (e) *Assume each of  $A = (x)$  and  $B = (y)$  is a principal ideal of  $R$  containing  $I$  with  $A \not\subseteq B$  and  $B \not\subseteq A$ . Let  $\sqrt{I} = I$  and  $(x)^2 \neq (x)$  and  $(y)^2 \neq (y)$ . Then  $(A, B)$  is not a bridge in  $\mathbb{A}\mathbb{G}_I(R)$ .*

*Proof.* In order to show that  $(A, B)$  is not a bridge in  $\mathbb{A}\mathbb{G}_I(R)$ , it suffices to find another path ( $\neq (A, B)$ ) from  $A$  to  $B$  in  $\mathbb{A}\mathbb{G}_I(R)$ .

(a) Clearly,  $(A, (A \cap B), B)$  is the desired path.

(b) *Case 1:* Suppose  $A \subseteq B$  or  $B \subseteq A$ . In this case,  $(A, (A + I), B)$  or  $(A, (B + I), B)$  is a path from  $A$  to  $B$ . Note that  $A \subseteq B$  [respectively,  $B \subseteq A$ ] implies  $A^2 \subseteq I$  [respectively,  $B^2 \subseteq I$ ].

*Case 2:* Suppose  $A \not\subseteq B$  and  $B \not\subseteq A$ . In this case,  $(A, (B + I), (A + I), B)$  is the desired path.

(c) Clearly,  $(A, (A + B), B)$  is a path from  $A$  to  $B$ .

(d) Let  $x \in (A \setminus I)$  and  $y \in (B \setminus I)$ . Clearly,  $(A, (y), (x), B)$  is a path different from  $(A, B)$ . Note that if  $(x) = (y)$ ,  $(A, (x), B)$  is also a path different from  $(A, B)$ .

(e) Clearly,  $(A, (y)^2, (x)^2, B)$  is a path different from  $(A, B)$ . Note that  $(x)^2, (y)^2 \not\subseteq I$  since  $\sqrt{I} = I$  by hypothesis. Also, for the case  $(x)^2 = (y)^2$ , we have  $(A, (x)^2, B)$  which is a path different from  $(A, B)$ .  $\square$

**[7, Theorem 4.1]** *Let  $I$  be a nonzero ideal of a ring  $R$ . Then*

(a) *Let  $P_1$  and  $P_2$  be two prime ideals of the ring  $R$  such that  $(0) \neq I = P_1 \cap P_2$  (or equivalently,  $C_I = C_{P_1} \cap C_{P_2} \neq 0$ ), then  $\mathbb{A}\mathbb{G}_I(R)$  is a complete bipartite graph.*

(b) *Let  $I$  be a nonzero ideal of the ring  $R$  such that  $I = \sqrt{I}$  (or equivalently,*

$C_I = \sqrt{C_I}$ ). If  $\mathbb{A}\mathbb{G}_I(R)$  is a complete bipartite graph, then there exist prime ideals  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of  $\mathbb{I}(R)$  such that  $C_I = \mathcal{P}_1 \cap \mathcal{P}_2$ .

*Proof.* For the proof of Part(a) and a different proof of Part (b) for a commutator poset, see Theorem 2.5.3.

(b): Suppose that the parts of  $\mathbb{A}\mathbb{G}_I(R)$  are  $V_1$  and  $V_2$ . Set  $\mathcal{P}_1 = V_1 \cup C_I$  and  $\mathcal{P}_2 = V_2 \cup C_I$ . It is clear that  $C_I = \mathcal{P}_1 \cap \mathcal{P}_2$ . We now prove that  $\mathcal{P}_1$  is an ideal of the semiring  $\mathbb{I}(R)$ . To show this, let  $A, B \in \mathcal{P}_1$ .

*Case 1:* If  $A, B \in C_I$ , then  $A + B \in C_I$  and so  $A + B \in \mathcal{P}_1$ .

*Case 2:* If  $A, B \in V_1$ , then there is  $C \in V_2$  such that  $CA \subseteq I$  and  $CB \subseteq I$ . So,  $C(A + B) \subseteq I$ . If  $A + B \subseteq I$ , then  $A + B \in \mathcal{P}_1$ . Otherwise,  $A + B \in V_1$ , which implies  $A + B \in \mathcal{P}_1$ .

*Case 3:* If  $A \in V_1$  and  $B \subseteq I$ , then  $A + B \notin C_I$ , so there is  $C \in V_2$  such that  $C(A + B) \in C_I$ . This implies that  $A + B \in V_1$ , and so  $A + B \in \mathcal{P}_1$ .

Now let  $S \in \mathbb{I}(R)$  and  $A \in \mathcal{P}_1$ .

*Case 1:* If  $A \subseteq I$ , then  $SA \in C_I$  and so  $SA \in \mathcal{P}_1$ .

*Case 2:* If  $A \in V_1$ , then there exists  $C \in V_2$  such that  $CA \in C_I$ . So,  $C(SA) \in C_I$ . If  $SA \in C_I$ , then  $SA \in \mathcal{P}_1$  and if  $SA \notin C_I$ , then  $SA \in V_1$  which implies  $SA \in \mathcal{P}_1$ . Therefore,  $\mathcal{P}_1$  is an ideal of  $\mathbb{I}(R)$ .

We now prove  $\mathcal{P}_1$  is prime.

Let  $AB \in \mathcal{P}_1$  and  $A, B \notin \mathcal{P}_1$ . Since  $\mathcal{P}_1 = V_1 \cup C_I$ ,  $AB \in C_I$  or  $AB \in V_1$ , so there exists  $C \in V_2$  such that  $C(AB) \in C_I$ . Thus,  $A(CB) \in C_I$ .

If  $CB \in C_I$ , then by the definition of  $\mathbb{A}\mathbb{G}_I(R)$  we have  $B \in V_1$ , that is a con-

tradiction. Hence,  $CB \notin C_I$ , so  $CB \in V_1$ . Therefore,  $C^2B \in C_I$ .

Since  $I = \sqrt{I}$  (equivalently,  $\sqrt{C_I} = C_I$ ),  $C^2 \notin C_I$ . Hence,  $C^2 \in V_2$ . So  $B \in V_1$  which is a contradiction. Therefore,  $\mathcal{P}_1$  is a prime ideal of  $\mathbb{I}(R)$ .  $\square$

A subtractive ideal (=k-ideal)  $I$  of a semiring  $S$  is an ideal such that if  $a, a + b \in I$ , then  $b \in I$  (so 0 is a  $k$ -ideal of  $S$ ). An ideal  $I$  is said to be strongly  $k$ -ideal if and only if  $a + b \in I$  implies that  $a \in I$  and  $b \in I$ . Clearly, every strongly  $k$ -ideal of a semiring  $S$  is a  $k$ -ideal of  $S$ . For example, the set  $2N$  of all nonnegative even integers is a subtractive ideal of the semiring of all nonnegative integers. It is not a strongly  $k$ -ideal since  $3 + 5 \in 2N$  while neither 3 nor 5 belong to  $2N$ . Note that in [23], Golan uses the term “subtractive ideal”, but in the literature of semirings, authors use equivalently the term “ $k$ -ideal”.

**[7, Lemma 4.2]** *Let  $A$  and  $B$  be two  $k$ -ideals of a semiring  $S$ . Then for any ideal  $I \subseteq A \cup B$  of  $S$ , either  $I \subseteq A$  or  $I \subseteq B$ .*

**[7, Lemma 4.3]** *Let  $I$  and  $X$  be two ideals of a commutative ring  $R$ . Then  $(I : X) = \{A \in \mathbb{I}(R) | AX \subseteq I\}$  is a strongly  $k$ -ideal of  $\mathbb{I}(R)$ .*

*Proof.* Let  $A$  and  $B$  be two ideals of  $R$ . Suppose  $A + B$  is in  $(I : X)$ . Clearly,  $A \subseteq A + B$  and  $B \subseteq A + B$ . Hence,  $AX \subseteq (A + B)X \subseteq I$  implies  $A \in (I : X)$  by definition. Similarly, by the same argument,  $B$  is in  $(I : X)$ .  $\square$

**[7, Theorem 4.4]** *Let  $I$  be a nonzero proper ideal of  $R$ . If  $\mathbb{A}\mathbb{G}_I(R)$  is a complete  $r$ -partite graph with  $r \geq 3$ , then at most one of the parts has more than one vertex. If  $V_i$  and  $V_j$  ( $i \neq j$ ) are two parts such that  $V_i = \{A\}$  and*

$V_j = \{B\}$ , then either  $A^2 \subseteq I$  or  $B^2 \subseteq I$ . Furthermore, if  $B \not\subseteq A$ , then  $A^2 \subseteq I$ .

*Proof.* For the proof of the second part and a different proof for the first part for a commutator lattice, see Theorems 3.6.5 and 3.6.5 respectively.

The proof is essentially the same as the proof of Theorem 3.2 of [31]. Assume that  $V_1, \dots, V_r$  are parts of  $\mathbb{A}\mathbb{G}_I(R)$ . Let  $V_t$  and  $V_s$  have more than one element.

Choose  $X \in V_t$  and  $Y \in V_s$ . Let  $V_l$  be a part of  $\mathbb{A}\mathbb{G}_I(R)$  such that  $V_l \neq V_t$  and  $V_l \neq V_s$ . Let  $Z \in V_l$ .

Since  $\mathbb{A}\mathbb{G}_I(R)$  is a complete  $r$ -partite graph,  $(I : X) = (\bigcup_{1 \leq i \leq r, i \neq t} V_i) \cup \{I\}$ ,  $(I : Y) = (\bigcup_{1 \leq i \leq r, i \neq s} V_i) \cup \{I\}$ , and  $(I : Z) = (\bigcup_{1 \leq i \leq r, i \neq l} V_i) \cup \{I\}$ .

Therefore,  $(I : Z) \subseteq (I : X) \cup (I : Y)$ , so by Lemmas 4.2 and 4.3 (above), we have  $(I : Z) \subseteq (I : X)$  or  $(I : Z) \subseteq (I : Y)$ . Let  $(I : Z) \subseteq (I : X)$  and choose  $X' \in V_t$  such that  $X' \neq X$ . Then we have  $X' \in (I : Z) \setminus (I : X)$ . This is a contradiction and completes the first part of the proof.  $\square$

**[7, Proposition 5.1]** Let  $I$  be an ideal of a ring  $R$ .

- (a) If  $\mathbb{A}\mathbb{G}(R/I)$  contains a cycle, then  $\text{gr}(\mathbb{A}\mathbb{G}_I(R)) \leq 4$ .
- (b) If  $\mathbb{A}\mathbb{G}_I(R)$  has no cycles, then girth of  $\mathbb{A}\mathbb{G}(R/I)$  is infinite.

*Proof.* From [15, Theorem 2.1],  $\mathbb{A}\mathbb{G}(R)$  is connected with diameter less than or equal to 3 for any ring  $R$ , and  $\text{gr}(\mathbb{A}\mathbb{G}(R)) \leq 4$  whenever  $\mathbb{A}\mathbb{G}(R)$  contains a cycle. Hence  $\text{gr}(\mathbb{A}\mathbb{G}_I(R)) \leq \text{gr}(\mathbb{A}\mathbb{G}(R/I))$  since  $\mathbb{A}\mathbb{G}_I(R)$  contains a copy of  $\mathbb{A}\mathbb{G}(R/I)$ .  $\square$

[7, **Lemma 5.3**] Suppose  $I$ ,  $A_1$ , and  $A_2$  are ideals of a ring  $R$ . Let  $(I : A_1)$  and  $(I : A_2)$  be two distinct prime ideals of the semiring  $\mathbb{I}(R)$ . Then  $A_1A_2 \subseteq I$ .

*Proof.* Without loss of generality, assume  $B \in (I : A_1) \setminus (I : A_2)$ . Thus,  $BA_1 \subseteq I$  implies  $BA_1 \in (I : A_2)$ . Hence,  $A_1 \in (I : A_2)$ .  $\square$

[7, **Proposition 5.4**] Let  $I$  be a proper ideal of a ring  $R$ . If  $\mathbb{I}(R)$  contains at least three distinct prime ideals of the form  $(I : A)$ ,  $(I : B)$ , and  $(I : C)$  for some (proper) ideals  $A$ ,  $B$ , and  $C$  of  $R$ , then  $\text{gr}(\mathbb{A}\mathbb{G}_I(R)) = 3$ .

*Proof.* The proof follows directly from the above lemma.  $\square$

[7, **Theorem 5.5**] Let  $I$  be a nonzero proper ideal of a ring  $R$ , and let  $P_1$  and  $P_2$  be two prime ideals of  $R$  such that  $I = P_1 \cap P_2$  (or equivalently,  $C_{P_1}$  and  $C_{P_2}$  are prime ideals of the semiring  $\mathbb{I}(R)$  and  $C_I = C_{P_1} \cap C_{P_2}$ ). If  $|C_{P_1} \setminus C_{P_2}| \geq 2$  and  $|C_{P_2} \setminus C_{P_1}| \geq 2$ , then  $\text{gr}(\mathbb{A}\mathbb{G}_I(R)) = 4$ .

*Proof.* Since  $|C_{P_i} \setminus C_{P_j}| \geq 2$  for  $i \neq j$  and  $1 \leq i, j \leq 2$ , then Theorem 2.5.3 implies that  $\text{gr}(\mathbb{A}\mathbb{G}_I(R)) = 4$ .  $\square$

In the above theorem, suppose  $|C_{P_1} \setminus C_{P_2}| = 1$  and  $|C_{P_2} \setminus C_{P_1}| \geq 2$ . In this case,  $\mathbb{A}\mathbb{G}_I(R)$  is a star graph. But by Proposition 3.4.1 [Parts (a) or (b)], this cannot happen if the ideal in  $C_{P_1} \setminus C_{P_2}$  is not containing  $I$  or is not a principal ideal of  $R$  since the center of a star graph is a cut-point. Similarly, the same argument is also valid for the case  $|C_{P_2} \setminus C_{P_1}| = 1$  and  $|C_{P_1} \setminus C_{P_2}| \geq 2$ .

[7, **Theorem 5.6**] Let  $I$  be an ideal of  $R$  such that  $I = \bigcap_{1 \leq i \leq n} P_i$  and for each  $1 \leq j \leq n$ ,  $I \neq \bigcap_{1 \leq i \leq n, i \neq j} P_i$ , where  $P_i$ 's are prime ideals of  $R$ . Then  $\omega(\mathbb{A}\mathbb{G}_I(R)) = n$ .

*Proof.* For a commutator poset version proof, see Theorem 2.6.8. □

[7, Corollary 5.7] *Let  $I = \bigcap_{1 \leq i \leq n} P_i \neq 0$  and  $J = \bigcap_{1 \leq j \leq m} Q_j \neq 0$  such that for each  $1 \leq k \leq n$ ,  $I \neq \bigcap_{1 \leq i \leq n, i \neq k} P_i$  and for each  $1 \leq k \leq m$ ,  $J \neq \bigcap_{1 \leq i \leq m, j \neq k} Q_j$ , where  $P_i$ 's and  $Q_j$ 's are prime ideals of  $R$ . Then  $m = n$  when  $\mathbb{A}\mathbb{G}_I(R) \cong \mathbb{A}\mathbb{G}_J(R)$ .*

# Chapter 2

## Commutator posets and their annihilation graphs

In this chapter we introduce the definition of a commutator poset and define the annihilation graph of a commutator poset  $L$  with respect to an element  $z \in L$ , denoted by  $\mathbb{A}\mathbb{G}_z(L)$  and discuss some basic properties of a commutator poset and  $\mathbb{A}\mathbb{G}_z(L)$ . We show that  $\mathbb{A}\mathbb{G}_z(L)$  is connected with  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$  and if  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle, then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$  (Theorem 2.2.2). We also study the condition(s) under which a vertex [respectively, an edge] is not a *cut-point* [respectively, *bridge*] (Propositions 2.3.2 and 2.4.2, respectively) and by Example 2.3.10 show that  $\mathbb{A}\mathbb{G}_z(L)$  has a cut-point in contrast to the ring case (Proposition 2.3.7). In Theorem 2.5.3, we discuss a relationship between the prime elements of  $L$  and complete bipartiteness of  $\mathbb{A}\mathbb{G}_z(L)$  and provide a different proof (without using associativity of the multiplication) from the ring case for Part (b) of this theorem. We also study a relationship between the prime elements (Definition 2.1.6) of  $L$  and the clique number of  $\mathbb{A}\mathbb{G}_z(L)$  (Theorem 2.6.8). We show how to construct ideals (Definition 2.5.1) in  $L$  from *maximal cliques* of  $\mathbb{A}\mathbb{G}_z(L)$  (Theorem 2.6.11) and finally provide some counterexamples (Examples 2.6.12 and 2.6.15) for Theorem 2.6.11.

## 2.1 The annihilation graph of a commutator poset

As motivated by Introduction, we introduce the following definition.

**Definition 2.1.1.** A *commutator poset* is a bounded poset (= partially ordered set)  $L$  with least element 0 and greatest element 1 equipped with a binary operation  $[-, -]$ , also written as  $[x, y] = xy$ , and satisfying

$$\mathbf{P1} \quad xy \leq x,$$

$$\mathbf{P2} \quad xy = yx,$$

$$\mathbf{P3} \quad x \leq y \Rightarrow xz \leq yz,$$

for all  $x, y, z$  in  $L$ .

For some examples of *commutator posets*, see Example 3.2.2.

The definition of the annihilating-ideal graph,  $\mathbb{AG}_I(R)$ , of a ring  $R$  with respect to the ideal  $I$  (see Section 1.2) immediately extends to the context of a commutator poset as follows:

**Definition 2.1.2.** For an element  $z$  in a commutator poset  $L$  we define *the annihilation graph of  $L$  with respect to  $z$* , denoted by  $\mathbb{AG}_z(L)$ , in which:

- the vertices of  $\mathbb{A}\mathbb{G}_z(L)$  are all elements  $x$  of  $L$  not less than or equal to  $z$  and having an element  $y$  in  $L$  not less than or equal to  $z$  with  $xy \leq z$ ;
- a pair  $(x, y)$  of distinct vertices of  $\mathbb{A}\mathbb{G}_z(L)$  is an edge of  $\mathbb{A}\mathbb{G}_z(L)$  if and only if  $xy \leq z$ .

We shall also write  $\mathbb{A}\mathbb{G}(L) = \mathbb{A}\mathbb{G}_0(L)$ , and call this graph the annihilation graph of  $L$ .

In particular we have

$$\mathbb{A}\mathbb{G}_I(R) = \mathbb{A}\mathbb{G}_I(L) \text{ and } AG(R) = AG(L),$$

where  $\mathbb{A}\mathbb{G}_I(R)$  and  $\mathbb{A}\mathbb{G}(R)$  are as in Section 1.2, while  $L$  is the commutator poset of ideals of  $R$  with the commutator operation as in Example 3.1.1(b).

**Definition 2.1.3.** An element  $x$  in a commutator poset  $L$  is said to be a *zero-divisor* if there exists a nonzero element  $y \in L$  such that  $xy = 0$ .

In the following simple example, we show that  $\mathbb{A}\mathbb{G}_z(L)$  is non-empty (Part (a)) and define a commutator poset  $L$  whose annihilation graph is complete bipartite (Part (b)).

**Example 2.1.4.** Let  $L = L_1 \times L_2$  be the direct product of two commutator posets  $L_1$  and  $L_2$  with at least two elements 0 and 1. Clearly,  $L$  is a commutator poset by defining its multiplicative operation and its order  $\leq$  componentwise.

- (a) Let  $a_1 \neq 1$  and  $a_2 \neq 1$  be two elements of  $L_1$  and  $L_2$ , respectively. Suppose  $a = (a_1, 1)$ ,  $b = (1, a_2)$ , and  $z = (a_1, a_2)$ . Clearly  $ab \leq z$ , but neither  $a$  nor  $b$  is less than or equal to  $z$ .
- (b) If each of  $L_1$  and  $L_2$  contains no nonzero zero-divisors, then  $\mathbb{A}\mathbb{G}(L)$  is a complete bipartite graph with parts  $\{(a, 0) | 0 \neq a \in L_1\}$  and  $\{(0, b) | 0 \neq b \in L_2\}$ .

We now construct two isomorphic graphs in a commutator poset  $L$  with respect to two different elements of  $L$ .

**Example 2.1.5.** Let  $X = \{1, 2, 3\}$  and  $L = (P(X), \cap, \subseteq)$  be a commutator poset, where  $P(X)$  is the power set of  $X$ . Let  $z = \{1\}$  and  $w = \{2\}$  be two elements of  $L$ . It is not difficult to show that each of  $\mathbb{A}\mathbb{G}_z(L)$  and  $\mathbb{A}\mathbb{G}_w(L)$  is a square and so  $\mathbb{A}\mathbb{G}_z(L) \cong \mathbb{A}\mathbb{G}_w(L)$ .

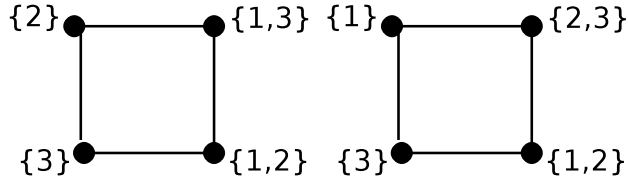


Figure 2.1:  $\mathbb{A}\mathbb{G}_z(L) \cong \mathbb{A}\mathbb{G}_w(L)$

**Definition 2.1.6.** An element  $z \neq 1$  in a commutator poset  $L$  is said to be *prime* when  $xy \leq z$  implies either  $x \leq z$  or  $y \leq z$  for all  $x, y \in L$ .

We now consider a condition under which  $\mathbb{A}\mathbb{G}_z(L)$  is the *empty graph*.

**Proposition 2.1.7.** (cf.[7, Proposition 2.5(a)] and [39, Proposition 2.2(b)]) Let  $z \neq 1$  be an element of a commutator poset  $L$ . Then  $\mathbb{A}\mathbb{G}_z(L)$  is the empty graph if and only if  $z$  is a prime element of  $L$ .

*Proof.* Suppose that  $z$  is a prime element of  $L$ . Then  $xy \leq z$  implies  $x \leq z$  or  $y \leq z$ . Hence the vertex set of  $\mathbb{A}\mathbb{G}_z(L)$  is empty by definition.

Conversely, suppose that  $\mathbb{A}\mathbb{G}_z(L) = \emptyset$ . Therefore, if  $x \in L$  such that  $x \not\leq z$  and  $xy \leq z$  for some  $y \in L$ , we must have  $y \leq z$  (otherwise,  $x$  is a vertex of  $\mathbb{A}\mathbb{G}_z(L)$ ). Hence  $z$  is a prime element of  $L$ .  $\square$

**Remark 2.1.8.** In the next proposition, we provide a necessary and sufficient condition for the multiplicative operation of a commutator poset to be associative. Notice that (by using induction) in order to show that a binary operation of an algebraic structure  $A$  is associative, it suffices to prove  $a(bc) = (ab)c$  for all elements  $a, b, c \in A$ . For example,  $a(bcd)$  can be written as  $a(ed) = (ae)d$ , where  $e = bc$ .

**Proposition 2.1.9.** *For all  $a, b$ , and  $c$  in a commutator poset  $L$ , the following are equivalent:*

- (a) *The multiplicative operation in  $L$  is associative.*
- (b)  $a(bc) \leq (ab)c$ .
- (c)  $(ab)c \leq a(bc)$ .

*Proof.* Clearly, (a) implies both (b) and (c). Now we prove (b) implies (a):  $a(bc) \leq (ab)c = c(ab) = c(ba) \leq (cb)a = a(bc)$ . Thus  $a(bc) = (ab)c$  by the antisymmetric property of  $\leq$ . By the similar argument (c) implies (a) and by the above remark the proof is complete.  $\square$

## 2.2 Connectivity of $\mathbb{A}\mathbb{G}_z(L)$

For the definitions of the connectedness, diameter and girth of a graph, see Definitions 1.1.3, 1.1.5 and 1.1.7 respectively.

**Remark 2.2.1.** Let  $R$  be a commutative ring with identity and  $L$  the commutator poset of ideals of  $R$ . In [7, Theorem 3.3], there is a discussion about connectivity, diameter and girth of  $\mathbb{A}\mathbb{G}_z(L)$  and we write it here as follows.

**Proposition A** Let  $z$  be an element of  $L$ . Then  $\mathbb{A}\mathbb{G}_z(L)$  is connected with  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$ . Furthermore, if  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle, then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$ .

In the following theorem, for a commutator poset  $L$ , we obtain the same results for  $\mathbb{A}\mathbb{G}_z(L)$  similar to the case of rings.

**Theorem 2.2.2.** (cf. [7, Theorem 3.3]) Let  $z \neq 1$  be an element of a commutator poset  $L$ . Then  $\mathbb{A}\mathbb{G}_z(L)$  is connected with  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$ . Furthermore, if  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle, then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$ .

*Proof.* Note that for all elements  $a, b, c \in L$ ,  $(ab)c \leq bc$  since  $ab \leq b$  and we use this fact in our proof implicitly. Let  $x$  and  $y$  be distinct vertices of  $\mathbb{A}\mathbb{G}_z(L)$ . To show  $\mathbb{A}\mathbb{G}_z(L)$  is connected with  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$ , we consider five cases:

*Case 1:*  $xy \leq z$ . Then  $(x, y)$  is a path in  $\mathbb{A}\mathbb{G}_z(L)$ .

*Case 2:*  $xy \not\leq z$ ,  $x^2 \leq z$ , and  $y^2 \leq z$ . Then  $(x, xy, y)$  is a path.

*Case 3:*  $xy \not\leq z$ ,  $x^2 \leq z$ , and  $y^2 \not\leq z$ . Clearly there exists an element  $b$  in  $\mathbb{A}\mathbb{G}_z(L)$  such that  $by \leq z$ . If  $bx \leq z$ , then  $(x, b, y)$  is a path. If  $bx \not\leq z$ , then  $(x, bx, y)$  is a path.

*Case 4:*  $xy \not\leq z$ ,  $y^2 \leq z$ , and  $x^2 \not\leq z$ . Then we obtain a path as in the above case.

*Case 5:*  $xy \not\leq z$ ,  $x^2 \not\leq z$ , and  $y^2 \not\leq z$ . Then there exist  $a$  and  $b$  in  $L$  such that  $a, b \not\leq z$  and  $a, b \notin \{x, y\}$  with  $ax \leq z$  and  $by \leq z$ .

If  $a = b$ , then  $(x, a, y)$  is a path.

If  $a \neq b$  and  $ab \leq z$ , then  $(x, a, b, y)$  is a path.

If  $a \neq b$  and  $ab \not\leq z$ , then  $(x, ab, y)$  is a path. Thus,  $\mathbb{A}\mathbb{G}_z(L)$  is connected and  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$ .

Now to show  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$ , suppose that  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle, and let  $C = (a_1, \dots, a_n, a_1)$  be a cycle with the least length. If  $n \leq 4$ , we are done. Otherwise, we have  $a_1a_4 \not\leq z$ . We need only consider 3 cases:

*Case 1:*  $a_1a_4 = a_1$ . Then  $a_1 = a_1a_4 \leq a_4$  implies  $a_1a_3 \leq a_4a_3 \leq z$  and

$$(a_1, a_2, a_3, a_1)$$

is a cycle, a contradiction.

The case  $a_1a_4 = a_4$  is similar to the above case, which implies that  $a_2, a_3$ , and  $a_4$  form a cycle of length three.

*Case 2:*  $a_1a_4 = a_2$ . Then  $a_2 = a_1a_4 \leq a_1$  implies  $a_2a_n \leq a_1a_n \leq z$  and so

$$(a_2, \dots, a_n, a_2)$$

is a cycle with length  $n - 1$ , a contradiction.

The case  $a_1a_4 = a_3$  is similar.

*Case 3:*  $a_1a_4 \neq a_1, a_2, a_3, a_4$ . Then  $a_2(a_1a_4) \leq a_2a_1 \leq z$ ,  $(a_1a_4)a_3 \leq a_4a_3 \leq z$ , and

$$(a_2, (a_1a_4), a_3, a_2)$$

is a cycle, a contradiction. Thus  $n \leq 4$ , i.e.,  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$ .  $\square$

**Remark 2.2.3.** There are simple counterexamples showing that  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$  is not always true, even if there are cycles of length  $\geq 5$ . For instance, if  $R$  is an integral domain that is not a field, then obviously,  $\text{gr}(\mathbb{A}\mathbb{G}(R \times R)) = 4$  (where  $\mathbb{A}\mathbb{G}(\dots)$  is as in [15]; see Section 1.2). In this well-known case all cycles in  $\mathbb{A}\mathbb{G}(R \times R)$  have even number of elements, and for every  $n$ -tuple  $(A_1, \dots, A_n)$  of distinct ideals in  $R$ , the sequence

$$(A_0 \times \{0\}, \{0\} \times A_0, A_1 \times \{0\}, \{0\} \times A_1, \dots, A_{n-1} \times \{0\}, \{0\} \times A_{n-1}, A_n \times \{0\}, \{0\} \times A_n),$$

where  $n \geq 2$  and  $A_0 = A_n$ , is a cycle of length  $2n$  in  $\mathbb{A}\mathbb{G}(R \times R)$ . See also Example 2.1.4(b).

## 2.3 On cut-points of $\mathbb{A}\mathbb{G}_z(L)$

In this section, we study a condition under which a vertex is not a *cut-point* (Proposition 2.3.2) and introduce a new class of ideals (multiplication ideals) in a commutative ring with identity, in order to provide a counterexample (Example 2.3.10) for Proposition 2.3.7 (that shows if a vertex is not a multiplication ideal, then it is not a cut point) and by Example 2.3.10 show that  $\mathbb{A}\mathbb{G}_z(L)$  has a cut-point in contrast to the ring case (Proposition 2.3.7).

For the definition of the cut-point of a graph, see Definition 1.1.8.

**Remark 2.3.1.** Let  $R$  be a commutative ring with identity and  $L$  the commutator poset of ideals of  $R$ . In [7, Proposition 3.5], there is a discussion, under which  $\mathbb{A}\mathbb{G}_z(L)$  ( $0 \neq z \in L$ ) has no cut-points and we write it here as follows.

**Proposition B** Let  $z$  be a nonzero element of  $L$  and  $x$  a vertex in  $\mathbb{A}\mathbb{G}_z(L)$ .

- (a) If  $z \not\leq x$ , then  $x$  is not a cut-point of  $\mathbb{A}\mathbb{G}_z(L)$ .
- (b) If  $x \in L$  is not a principal ideal of  $R$ , then  $x$  is not a cut-point in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (c) Suppose  $x \in L$  is a principal ideal of  $R$  with  $z \leq x$  and  $\sqrt{z} = z$ , where  $\sqrt{z} = \{a \in L \mid a^n \leq z \text{ for some positive integer } n\}$  (see [7, Lemma 2.1]). If  $x \neq x^2$ , then  $x$  is not a cut-point in  $\mathbb{A}\mathbb{G}_z(L)$ .

The following proposition provides a condition under which  $\mathbb{A}\mathbb{G}_z(L)$  has no cut-points, where  $L$  is a commutator poset.

**Proposition 2.3.2.** *Let  $z \neq 1$  be a nonzero element of a commutator poset  $L$  and  $x$  a vertex in  $\mathbb{A}\mathbb{G}_z(L)$ . If  $S = \{a \in L \mid a \leq x \text{ and } a \not\leq z\}$  has more than one element, then  $x$  is not a cut-point in  $\mathbb{A}\mathbb{G}_z(L)$ .*

*Proof.* Assume the vertex  $x$  of  $\mathbb{A}\mathbb{G}_z(L)$  is a cut-point. Then there exist vertices  $u$  and  $v$  ( $\neq x$ ) in  $\mathbb{A}\mathbb{G}_z(L)$  such that  $x$  lies on every path from  $u$  to  $v$ . Clearly, by Theorem 2.2.2, the shortest path from  $u$  to  $v$  is of length 2 or 3.

Consider the following two cases.

*Case 1:* Suppose  $(u, x, v)$  is a path of shortest length from  $u$  to  $v$  and  $a \in S$ . For the case  $a = u$  [or  $a = v$ ], then  $(u, v)$  is a path of length 1 since  $a \leq x$  implies  $av \leq xv \leq z$  [or  $ua \leq ux \leq z$ ]. Otherwise,

$$(u, a, v)$$

is a path of length 2 in  $\mathbb{A}\mathbb{G}_z(L)$  with  $a \neq x$  which is a contradiction.

*Case 2:* Suppose (without loss of generality)  $(u, x, y, v)$  is a path of shortest length from  $u$  to  $v$  in  $\mathbb{A}\mathbb{G}_z(L)$ . Clearly,

$$(u, a, y, v)$$

is a path different from  $(u, x, y, v)$  in  $\mathbb{A}\mathbb{G}_z(L)$  which is a contradiction.  $\square$

In the following example, we provide a commutator poset whose annihilation graph has a cut-point.

**Example 2.3.3.** Consider the totally-ordered set  $L = \{0, a, 1\}$  with  $0 < a < 1$ . Clearly,  $L$  is a commutator poset by assuming  $xy = yx, 0x = 0, 1x = x, a^2 = a$  for all  $x, y \in L$ . Let  $C = L \times L$  be the commutator poset of the direct product of commutator posets. Let  $z = (0, a) \in C$ .

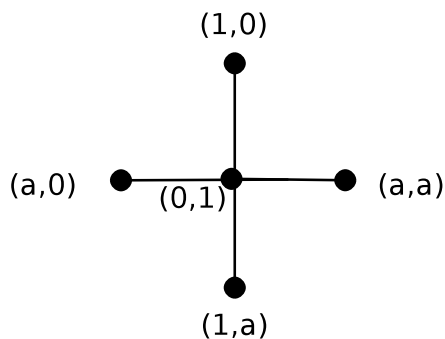


Figure 2.2:  $\mathbb{A}\mathbb{G}_{(0,a)}(C)$

Now it can easily be seen that  $\mathbb{A}\mathbb{G}_z(C)$  is a star graph on 5 vertices with center  $(0, 1)$  and vertex set

$$V = \{(0, 1), (1, 0), (a, a), (a, 0), (1, a)\}.$$

Clearly, the center of  $\mathbb{A}\mathbb{G}_z(C)$  is a cut-point.

We now introduce a new class of ideals in a commutative ring with identity, which will be used later in order to provide a counterexample (Example 2.3.10) for Proposition 2.3.7.

**Definition 2.3.4.** An ideal  $X$  in a commutative ring  $R$  with identity is said to be a *multiplication ideal of  $R$* , if for any ideal  $Y$  of  $R$  with  $Y \subseteq X, Y = AX$  for some ideal  $A$  in  $R$ .

**Theorem 2.3.5.** *Every principal ideal in a commutative ring  $R$  with identity is a multiplication ideal of  $R$ .*

*Proof.* Let  $R$  be a commutative ring with 1,  $x$  an element in  $R$  and  $Y$  an ideal of  $R$  with  $Y$  being a subset of  $Rx$ . Let  $I$  be the ideal of  $R$  consisting of all elements  $r \in R$  with  $rx \in Y$ . Then  $Y = IRx$ . Indeed, since  $Y$  is an ideal containing  $Ix$ , we know that  $Y$  contains  $IRx = RIx$ . Conversely, for  $y \in Y$  we can write  $y = rx$  since  $Y$  is a subset of  $Rx$ ; moreover such  $r$  must be in  $I$ , by the definition of  $I$ . Therefore  $y = r1x$  tells us that  $y$  is in  $IRx$ .  $\square$

In the following example we show that the converse of the above theorem need not be true in general.

**Example 2.3.6.** Let  $X$  be an ideal of a Boolean ring  $R$ . Then for every ideal  $Y$  of  $R$  with  $Y \subseteq X$ ,  $Y = Y \cap X = YX$  since the product of ideals in a Boolean ring is the same as their intersection. Therefore  $X$  is a multiplication ideal in  $R$ . On the other hand, every infinite Boolean ring has non-principal ideals. That means a multiplication ideal in a ring  $R$  need not be a principal ideal.

We now extend Part(b) of [7, Proposition 3.5] (see Section 1.2) for a non-multiplication ideal.

**Proposition 2.3.7.** *Let  $I$  be a nonzero ideal of commutative ring  $R$  with identity and  $X$  a vertex in  $\mathbb{A}\mathbb{G}_I(R)$ . If  $X$  is not a multiplication ideal of  $R$ , then  $X$  is not a cut-point in  $\mathbb{A}\mathbb{G}_I(R)$ .*

*Proof.* Clearly by Theorem 2.3.5, every non-multiplication ideal is not a principal ideal and so not a cut point by the proof of Part(b) of [7, Proposition 3.5] (see Section 1.2).  $\square$

**Definition 2.3.8.** An element  $x$  in a commutator poset  $L$  is said to be a *multiplication element* of  $L$ , if for any element  $y \in L$  with  $y \leq x$ ,  $y = rx$  for some  $r \in L$ .

**Remark 2.3.9.** From the above definition, it is clear that a multiplication ideal of a ring  $R$  is a multiplication element in the commutator poset of ideals of  $R$ .

Now, in contrast to Proposition 2.3.7, we construct a commutator poset  $C$  in which a non-multiplication element is a cut-point of  $\mathbb{A}\mathbb{G}_z(C)$ .

**Example 2.3.10.** Let  $L = \{0, a, b, 1\}$  be a totally ordered set with  $0 < a < b < 1$ . Clearly  $L$  is a commutator poset by assuming  $xy = yx, 0x = 0, 1x = x$  for all  $x, y \in L$  and  $bb = aa = ab = 0$ . Let  $C = L \times L$  be the commutator poset of the direct product of commutator posets  $L$ . Let  $z = (0, a)$ . Then the graph  $\mathbb{A}\mathbb{G}_z(C)$  has a cut-point  $(0, b)$ .

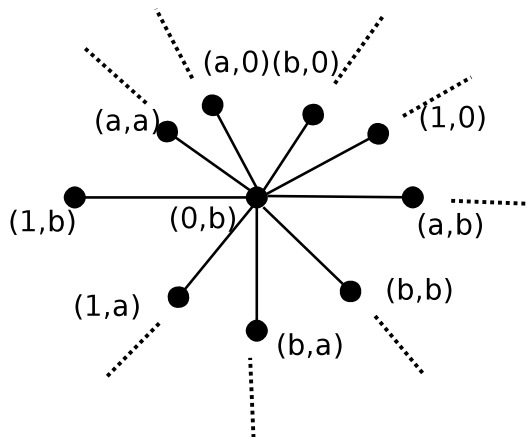


Figure 2.3:  $(0, b)$  is a cut-point in  $\mathbb{A}\mathbb{G}_{(0,a)}(C)$

Clearly  $(0, a) \leq (0, b)$  but  $(0, a)$  is not a multiple of  $(0, b) \in C$ . Thus  $(0, b)$  is not a multiplication element in  $C$ , but it is a cut-point of  $\mathbb{A}\mathbb{G}_z(C)$  which provides a counterexample for Proposition 2.3.7.

## 2.4 On bridges of $\mathbb{A}\mathbb{G}_z(L)$

For the definition of the bridge of a graph, see Definition 1.1.9.

**Remark 2.4.1.** Let  $R$  be a commutative ring with identity and  $L$  the commutator poset of ideals of  $R$ . In [7, Proposition 3.6], there is a discussion, under which  $\mathbb{A}\mathbb{G}_z(L)$  ( $0 \neq z \in L$ ) has no bridges and we write it here as follows.

**Proposition C** Let  $z$  be a nonzero element of  $L$  and  $a \neq b$  two distinct vertices of the graph  $\mathbb{A}\mathbb{G}_z(L)$  with  $(a, b)$  an edge in  $\mathbb{A}\mathbb{G}_z(L)$ .

- (a) Suppose  $a \not\leq b$  and  $b \not\leq a$  and there exists an element  $x \in L$  such that  $x \leq a$  and  $x \leq b$  with  $x \not\leq z$ . Then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (b) Assume  $z \not\leq a$  and  $z \not\leq b$ . Then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (c) Assume  $a^2 \leq z$  and  $b^2 \leq z$  with  $a \not\leq b$  and  $b \not\leq a$ . Then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (d) Suppose that neither  $a$  nor  $b$  is a principal ideal of  $R$  with  $a \not\leq b$  and  $b \not\leq a$ . Then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (e) Assume each of  $a$  and  $b$  is a principal ideal of  $R$  with  $z \leq a$ ,  $z \leq b$ ,  $a \not\leq b$ , and  $b \not\leq a$ . Let  $\sqrt{z} = z$  and  $a^2 \neq a$  and  $b^2 \neq b$ . Then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .

The following proposition provides some conditions under which  $\mathbb{A}\mathbb{G}_z(L)$  has no bridges for a commutator poset  $L$ .

**Proposition 2.4.2.** *Let  $z \neq 1$  be a nonzero element of a commutator poset  $L$  and  $a \neq b$  two distinct vertices of the graph  $\mathbb{A}\mathbb{G}_z(L)$  with  $(a, b)$  an edge in  $\mathbb{A}\mathbb{G}_z(L)$ . Let  $S = \{x \in L \mid x \leq a, x \leq b \text{ and } x \not\leq z\}$ .*

- (a) *If  $S$  has more than one element, then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .*
- (b) *Assume  $a \not\leq b$  and  $b \not\leq a$ . If  $|S| = 1$ , then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .*

*Proof.* In order to show that  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ , it suffices to find another path ( $\neq (a, b)$ ) from  $a$  to  $b$  in  $\mathbb{A}\mathbb{G}_z(L)$ .

(a) Since  $a \neq b$ , then by hypothesis there exists  $x \in S$  such that  $x \neq a$  and  $x \neq b$ . Clearly,  $x \leq a$  and  $x \leq b$  implies that

$$(a, x, b)$$

to be the desired path.

(b) Clearly by hypothesis, there exists  $x \in S$ , such that  $a \neq x$  and  $b \neq x$ .  
Now

$$(a, x, b)$$

is a path different from  $(a, b)$  since  $x \leq b$ ,  $x \leq a$ , and  $ab \leq z$ . □

In the following example, we provide a commutator poset whose annihilation graph has four bridges.

**Example 2.4.3.** Let  $\mathbb{A}\mathbb{G}_z(C)$  be the star graph on 5 vertices as defined in Example 2.3.3. Clearly,  $\mathbb{A}\mathbb{G}_z(C)$  has 4 bridges (i.e., each edge is a bridge).

## 2.5 $\mathbb{A}\mathbb{G}_z(L)$ as a complete $r$ -partite graph

For the definition of a complete  $r$ -partite graph, see Definition 1.1.11.

As we proved in Proposition 2.1.7,  $z$  is a prime element of  $L$  if and only if  $\mathbb{A}\mathbb{G}_z(L) = \emptyset$ . In the following, for the prime elements  $p_1$  and  $p_2$  of  $L$ , we show that the condition  $x \leq z$  if and only if  $x \leq p_1$  and  $x \leq p_2$  for any  $x \in L$ , implies that  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph. In the next section we study the girth and clique number of  $\mathbb{A}\mathbb{G}_z(L)$  under the condition  $x \leq z$  if and only if  $x \leq p_i$  for any  $x \in L$  and  $1 \leq i \leq n$ , where  $p_i$ 's are prime elements of  $L$ .

**Definition 2.5.1.** A subset  $I$  of a commutator poset  $L$  is said to be an *ideal* of  $L$  when  $rL \subseteq I$  for each  $r \in I$ . An ideal  $P$  of a commutator poset  $L$  is *prime* when  $P \neq L$  and  $xy \in P$  implies either  $x \in P$  or  $y \in P$  for all elements  $x$  and  $y$  in  $L$ .

**Remark 2.5.2.** Let  $R$  be a commutative ring with identity and  $L$  the commutator poset of ideals of  $R$ . In [7, Theorem 4.1(a)], there is a discussion on the complete bipartiteness of  $\mathbb{A}\mathbb{G}_z(L)$  ( $0 \neq z \in L$ ) and we write it here as follows.

**Proposition D** Let  $z \neq 1$  be a nonzero element of  $L$ .

- (a) Let  $p_1$  and  $p_2$  be two prime elements of  $L$  such that  $x \leq z$  if and only if  $x \leq p_1$  and  $x \leq p_2$  for any  $x \in L$ , then  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph.

- (b) Let  $z = \sqrt{z}$ . If  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph, then there exist prime ideals  $P_1$  and  $P_2$  of  $L$  such that  $C_z = P_1 \cap P_2$  where  $C_z$  is the set of all elements  $x \in L$  with  $x \leq z$ .

In the following theorem, we discuss some conditions under which  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph and vice versa (see also the above remark).

**Theorem 2.5.3.** (cf. [7, Theorem 4.1(a)], see also [31, Theorem 3.1]) Let  $z \neq 1$  be a nonzero element of a commutator poset  $L$ .

- (a) Let  $p_1$  and  $p_2$  be two prime elements of  $L$  such that  $z = \inf\{p_1, p_2\}$ , then  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph.
- (b) Suppose  $C_z = \{x \in L \mid x \leq z\}$  and for any  $x \in L$ ,  $x^2 \leq z$  if and only if  $x \leq z$  (i.e.,  $x \in C_z$ ). If  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph with parts  $V_1$  and  $V_2$ , then  $V_1 \cup C_z$  and  $V_2 \cup C_z$  are prime ideals of  $L$  with  $C_z = (V_1 \cup C_z) \cap (V_2 \cup C_z)$ .

*Proof.* (a): Let  $a, b \in L$  and  $a, b \not\leq z$  with  $ab \leq z$ . Then  $ab \leq p_1$  and  $ab \leq p_2$ .

Since  $p_1$  and  $p_2$  are prime elements of  $L$ , we have  $a \leq p_1$  or  $b \leq p_1$  and  $a \leq p_2$  or  $b \leq p_2$ .

Therefore, suppose  $a \leq p_1$  with  $a \not\leq p_2$  (if  $a \leq p_1$  and  $a \leq p_2$ , then  $a \leq z$ , a contradiction) and  $b \leq p_2$  with  $b \not\leq p_1$ . Thus,  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph with parts  $\{x \in L \mid x \leq p_1 \text{ and } x \not\leq p_2\}$  and  $\{x \in L \mid x \leq p_2 \text{ and } x \not\leq p_1\}$  since  $a$  and  $b$  are chosen arbitrarily.

(b): Set  $S = V_1 \cup C_z$  and  $T = V_2 \cup C_z$ . It is clear that  $C_z = S \cap T$  (note that  $V_1 \cap V_2 = \emptyset$ ). We first prove that  $S$  is an ideal of  $L$ . To show this, let  $s \in L$  and  $a \in S$  and consider two cases:

*Case 1:* If  $a \in C_z$ , then  $sa \in C_z$  and so  $sa \in S$ .

*Case 2:* If  $a \in V_1$ , then there exists  $c \in V_2$  such that  $ca \leq z$ . So  $c(sa) \in C_z$  since  $sa \leq a$ . If  $sa \in C_z$ , then  $sa \in S$  and if  $sa \notin C_z$ , then  $sa \in V_1$  since  $c(sa) \leq z$ , which implies  $sa \in S$ . Therefore,  $S$  is an ideal of  $L$ .

We now prove  $S$  is prime.

Let  $S = V_1 \cup C_z$  and  $ab \in S$ . Suppose to the contrary that  $a, b \notin S$ . Clearly  $ab \notin C_z$  since  $ab \in C_z$  implies either  $a \in V_1$  or  $b \in V_1$ , yielding a contradiction. Thus  $ab \in V_1$ .

We now consider two cases: (i)  $a \neq b$  or (ii)  $a = b$  and just discuss Case (i) and leave the other part to the reader.

Since  $ab \in V_1$ , there exists  $c \in V_2$  such that  $c(ab) \in C_z$ .

We claim that  $a(cb) \in C_z$ . Otherwise, suppose that  $a(cb) \notin C_z$ . Then  $c(a(cb)) \leq c(ab) \in C_z$  and then  $a(cb)$  is a vertex in  $V_1$  since  $c \in V_2$ . Note that  $c \neq a(cb)$ , since  $c$  is a vertex and hence  $c^2 \not\leq z$  by hypothesis. On the other hand,  $a(cb) \leq cb \leq c$ , implies that  $(ab)(a(cb)) \leq (ab)c \in C_z$ , which is a contradiction since two vertices in one part do not have any edge. If  $ab = a(cb)$ , then  $(ab)^2 \leq z$  and then  $ab \leq z$  which is a contradiction by hypothesis. Consequently,

$a(cb) \in C_z$ .

It is clear that either  $(c^2)b \leq z$  or  $(c^2)b \not\leq z$ .

Suppose  $(c^2)b \not\leq z$ . Clearly  $a((c^2)b) \leq a(cb) \leq z$  and then  $(c^2)b \in V_1$  and  $a \in V_2$  since  $a \notin V_1$ . Now  $(c^2)b \in V_1$  and  $ab \in V_1$ , but their product is in  $C_z$  which is a contradiction. Note that if  $(c^2)b = ab$ , we get a contradiction by hypothesis.

Now, let  $(c^2)b \leq z$ . Since  $b \notin V_1$  and  $c^2 \not\leq z$ , then  $c^2 \in V_1$  and  $b \in V_2$ . But if  $c^2 \in V_1$ , then  $ab(c^2) \leq (ab)c \leq z$  which is a contradiction (two vertices in one part can not have an edge). Also  $c^2 \neq ab$  since the equality implies that  $(ab)^2 \leq z$ , yielding a contradiction by hypothesis. Therefore,  $S$  is a prime ideal.  $\square$

We conclude this section by characterizing all finite *regular graphs* which can be realized as the annihilation graph of a special class of *decomposable commutator posets*.

- A poset  $L$  is said to be *indecomposable* if  $L$  can not be represented as  $L_1 \times L_2$ , where  $L_1$  and  $L_2$  are two posets, otherwise  $L$  is said to be *decomposable*.

- A graph  $G$  is *regular* if the degrees of all vertices of  $G$  are the same.

- The *annihilator* of any element  $x$  in a commutator poset  $L$ , denoted by  $\text{Ann}(x)$ , is the set of all  $r \in L$  such that  $rx = 0$ .

**Theorem 2.5.4.** (cf. [4, Theorem 7]) Let  $L$  be a finite commutator poset such that  $\mathbb{A}\mathbb{G}(L)$  is a regular graph. If  $L = L_1 \times L_2$  is a finite decomposable poset

with  $1_{L_1}x = x$  for all  $x \in L_1$  and  $1_{L_2}x = x$  for all  $x \in L_2$ , then  $\mathbb{A}\mathbb{G}(L)$  is a complete bipartite graph.

*Proof.* Assume that  $\mathbb{A}\mathbb{G}(L)$  is a regular graph of degree  $r$ . Suppose that  $L = L_1 \times L_2$ , is a decomposable poset. Since the degree of  $(1, 0)$  is  $|L_2| - 1$  and the degree of  $(0, 1)$  is  $|L_1| - 1$ , we have  $|L_1| = |L_2| = r + 1$ . We show that  $L_1$  is a poset with no nontrivial zero-divisors.

If not, then there exist two non-zero elements  $a$  and  $b$  in  $L_1$  such that  $ab = 0$ . But  $(\{0\} \times L_2) \cup \{(b, 1)\} \subseteq \text{Ann}((a, 0))$  and this implies that  $r = \deg((a, 0)) \geq r + 1$ , a contradiction.

Similarly,  $L_2$  must be a poset with no nontrivial zero-divisors. So in this case,  $\mathbb{A}\mathbb{G}(L) \cong K_{r,r}$ . □

## 2.6 Girth and clique number of $\mathbb{A}\mathbb{G}_z(L)$

For the definitions of the clique number and maximal clique see Definition 1.1.12.

**Remark 2.6.1.** In Theorem 2.2.2, for a commutator poset, we directly showed that  $\mathbb{A}\mathbb{G}_z(L)$  is connected with diameter less than or equal to 3 and  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$  when  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle.

**Definition 2.6.2.** For any elements  $x$  and  $z$  of a commutator poset  $L$ , we define  $l_z(x) = \sup\{a \in L | ax \leq z\}$  such that  $x \sup\{a \in L | ax \leq z\} = \sup\{ax \in L | ax \leq z\}$ . Clearly  $y \leq l_z(x) \Leftrightarrow yx \leq z$  for each  $y \in L$ .

The following lemma is similar to Lemma 2.1 in [3] that states for any two distinct *associated prime ideals*  $\text{Ann}(x_1)$  and  $\text{Ann}(x_2)$  of a ring  $R$ , then we have  $x_1x_2 = 0$ .

We use the following lemma in Proposition 2.6.4, to show that  $\mathbb{A}\mathbb{G}_z(L)$  has girth equal to 3.

**Lemma 2.6.3.** *Suppose  $z$ ,  $a_1$ , and  $a_2$  are elements of a commutator poset  $L$  with the largest elements  $l_z(a_1)$  and  $l_z(a_2)$  (as defined in Definition 2.6.2). Let  $l_z(a_1)$  and  $l_z(a_2)$  be two distinct prime elements of  $L$ . Then  $a_1a_2 \leq z$ .*

*Proof.* Without loss of generality, assume  $b \leq l_z(a_1)$  and  $b \not\leq l_z(a_2)$ . Thus,  $ba_1 \leq z$  implies  $(ba_1)a_2 \leq za_2 \leq z$ . Thus  $ba_1 \leq l_z(a_2)$ , which implies  $a_1 \leq l_z(a_2)$  since  $l_z(a_2)$  is prime.  $\square$

In the following proposition, we state a condition that proves  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 3$ .

**Proposition 2.6.4.** (cf. [7, Proposition 5.4]) Let  $z$  be an element of a commutator poset  $L$ . If  $L$  contains at least three distinct prime elements of the form  $l_z(a)$ ,  $l_z(b)$ , and  $l_z(c)$  (as defined in Definition 2.6.2) for some elements  $a$ ,  $b$ , and  $c$  of  $L$ , then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 3$ .

*Proof.* The proof follows directly from the above lemma.  $\square$

In the next theorem, we show that for a commutator poset  $L$ ,  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph when its girth is 4.

**Theorem 2.6.5.** (cf. [12, Theorem 2.2]) Let  $z \neq 1$  be an element of a commutator poset  $L$ .

(a) If  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 4$  and  $a^2 \not\leq z$  for all  $a \in L$  with  $a \not\leq z$ , then  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph.

(b) If  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph, then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 4$  or  $\infty$ .

*Proof.* First, we show that  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) = 2$ .

If  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) = 0$  or  $1$ , then  $\mathbb{A}\mathbb{G}_z(L)$  is a complete graph and so  $\text{gr}(\mathbb{A}\mathbb{G}_z(L))$  is either  $\infty$  or  $3$ , yielding a contradiction.

If  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) = 3$ , then there exist  $a_1, a_2, a_3, a_4 \in \mathbb{A}\mathbb{G}_z(L)$  such that  $(a_1, a_2, a_3, a_4)$  is a path,  $a_1a_3 \not\leq z$ ,  $a_2a_4 \not\leq z$  and  $a_1a_4 \not\leq z$ .

If  $a_1a_4 = a_2$ , then  $(a_2)^2 \leq z$  since  $a_1a_4 \leq a_1$  implies  $a_2(a_1a_4) \leq a_2a_1 \leq z$ , yielding a contradiction by hypothesis.

Similarly  $a_1a_4 \neq a_3$ . Thus  $(a_2, a_3, a_1a_4, a_2)$  is a cycle and so  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 3$ , yielding a contradiction. Therefore,  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) = 2$ .

We now show that  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph.

Since  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 4$ , there exist  $a, b, c, d \in \mathbb{A}\mathbb{G}_z(L)$  such that

$$(a, b, c, d, a).$$

We show that  $\mathbb{A}\mathbb{G}_z(L) \cong K_{|V_1|,|V_2|}$ , where  $V_1 = \{t \in L | t \not\leq z \text{ and } ta \leq z\}$  and  $V_2 = \{s \in L | s \not\leq z \text{ and } sa \leq z\}$ .

Let  $t, t_1 \in V_1$  and  $s, s_1 \in V_2$ .

Assume that  $ts \not\leq z$ . Since  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) = 2$ , there exists  $x \in L$  with  $x \not\leq z$  such that

$$(a, x, s).$$

Now by using  $ts \leq s$  and  $ts \leq t$ , if  $ts = x$  or  $ts = a$ , then  $(ts)^2 \leq z$ , yielding a contradiction by hypothesis.

Therefore,  $(a, ts, x, a)$  is a cycle, yielding a contradiction since  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 4$ .

Thus  $ts \leq z$ .

If  $tt_1 \leq z$ , then  $(a, t, t_1, a)$  is a cycle, yielding a contradiction. So,  $tt_1 \not\leq z$ .

Similarly,  $ss_1 \not\leq z$ .

Also  $V_1 \cap V_2 = \emptyset$ . Therefore,  $\mathbb{A}\mathbb{G}_z(L) \cong K_{|V_1|,|V_2|}$  and so  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph. The proof of the other part is clear.  $\square$

In the next theorem, we show that the clique number of an annihilation graph  $\mathbb{A}\mathbb{G}(L)$  is  $n \geq 2$ , where  $L$  is the product of  $n$  commutator posets with no non-trivial zero-divisors.

**Theorem 2.6.6.** (cf. [9, Theorem 3.7]) Let  $L_1, \dots, L_n$  be  $n \geq 2$  commutator posets with no zero-divisors and let  $L = L_1 \times \dots \times L_n$ . Then  $\omega(\mathbb{A}\mathbb{G}(L)) = n$ .

*Proof.* Clearly using the set consisting of  $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1),$   
 $\omega(\mathbb{A}\mathbb{G}(L)) \geq n.$

The case  $n = 2$  is clear; so let  $X = \{x_1, \dots, x_m\}$  be a complete subgraph of  $\mathbb{A}\mathbb{G}(L)$  with  $n \geq 3$  and each  $x_i = (x_{i1}, \dots, x_{in}).$

We may assume that  $x_{11} \neq 0,$  and thus  $x_{21} = \dots = x_{m1} = 0.$

Hence we may consider  $X \setminus \{x_1\}$  as a complete subgraph of  $\mathbb{A}\mathbb{G}(L_2 \times \dots \times L_n).$   
 By induction, we have  $m - 1 \leq n - 1,$  and thus  $m \leq n.$   $\square$

**Remark 2.6.7.** Let  $R$  be a commutative ring with identity and  $L$  the commutator poset of ideals of  $R.$  In [7, Theorem 5.6], it is shown that the clique number of  $\mathbb{A}\mathbb{G}_z(L) (0 \neq z \in L)$  is  $n \geq 2$  under some condition of prime elements of  $L$  and we write it here as follows.

**Proposition E** Let  $z$  be an element of  $L$  and  $p_1, p_2, \dots, p_n$  prime elements of  $L.$  Suppose  $x \leq z$  if and only if  $x \leq p_i$  for any  $x \in L$  and  $1 \leq i \leq n,$  and for each  $1 \leq j \leq n,$  there exists  $y \leq p_i$  for all  $1 \leq i \neq j \leq n$  such that  $y \not\leq z.$  Then  $\omega(\mathbb{A}\mathbb{G}_z(L)) = n.$

In the following theorem, we study a relationship between prime elements of a commutator poset  $L$  and the clique number of  $\mathbb{A}\mathbb{G}_z(L)$  (see also the above remark).

**Theorem 2.6.8.** (cf. [7, Theorem 5.6], see also Theorem 4.2 of [31]) *Let  $z \neq 1$  be a nonzero element of a commutator poset  $L$  and  $p_1, p_2, \dots, p_n$  prime elements*

of  $L$ . Suppose  $z = \inf\{p_1, \dots, p_n\}$  for  $1 \leq i \leq n$ , and for each  $1 \leq j \leq n$ , there exists  $y \leq p_i$  for all  $1 \leq i \neq j \leq n$  such that  $y \not\leq z$ . Then  $\omega(\mathbb{A}\mathbb{G}_z(L)) = n$ .

*Proof.* Now, consider  $x_j \leq p_i$  for all  $i \neq j$  with  $x_j \not\leq p_j$ .

It is easy to see that  $X = \{x_1, \dots, x_n\}$  is a clique in  $\mathbb{A}\mathbb{G}_z(L)$ . Hence,  $\omega(\mathbb{A}\mathbb{G}_z(L)) \geq n$ .

To show  $\omega(\mathbb{A}\mathbb{G}_z(L)) \leq n$ , we proceed by induction on  $n$ .

For  $n = 2$ , by Theorem 2.5.3(a),  $\mathbb{A}\mathbb{G}_z(L)$  is a bipartite graph and hence  $\omega(\mathbb{A}\mathbb{G}_z(L)) = 2$ .

Suppose  $n > 2$  and the result is true for any integer less than  $n$ .

Let  $\{x_1, \dots, x_m\}$  be a clique in  $\mathbb{A}\mathbb{G}_z(L)$ . Hence,  $x_1 x_j \leq p_i$  ( $1 \leq i \leq n$ ) for any  $2 \leq j \leq m$ .

Without loss of generality, suppose that  $x_1 \not\leq p_1$ . Therefore,  $x_2, \dots, x_m \leq p_1$ , so  $x_2, \dots, x_m \not\leq p_i$  for some ( $2 \leq i \leq n$ ).

Let  $w$  be an element of  $L$  such that for any  $x \in L$ ,  $x \leq w$  if and only if  $x \leq p_i$  for any  $x \in L$  and  $2 \leq i \leq n$ . Hence,  $\{x_2, \dots, x_m\}$  is a clique in  $\mathbb{A}\mathbb{G}_w(L)$ . Therefore,  $m - 1 \leq n - 1$ , and the proof is complete.  $\square$

**Corollary 2.6.9.** *Let  $z \neq 1$  and  $w \neq 1$  be two nonzero elements of a commutator poset  $L$ . Let  $x \leq z$  if and only if  $x \leq p_i$  for any  $x \in L$  and  $1 \leq i \leq n$*

such that for each  $1 \leq k \leq n$ , there exists  $y \leq p_i$  for all  $1 \leq i \neq k \leq n$ , with  $y \not\leq z$ , where  $p_i$ 's are prime elements of  $L$ . Let  $x \leq w$  if and only if  $x \leq q_j$  for any  $x \in L$  and  $1 \leq j \leq m$  such that for each  $1 \leq k \leq m$ , there exists  $y \leq q_j$  for all  $1 \leq j \neq k \leq m$  such that  $y \not\leq w$ , where  $q_j$ 's are prime elements of  $L$ . Then  $m = n$  when  $\mathbb{A}\mathbb{G}_z(L) \cong \mathbb{A}\mathbb{G}_w(L)$ .

**Remark 2.6.10.** Let  $R$  be a commutative ring with identity and  $L$  the commutator poset of ideals of  $R$ . In [7, Theorem 5.6], it is shown how to construct an ideal in  $L$  from a maximal clique of  $\mathbb{A}\mathbb{G}_z(L)$  ( $0 \neq z \in L$ ) and we write it here as follows.

**Proposition F** Let  $z$  be an element of  $L$  and let  $S$  be a maximal clique in  $\mathbb{A}\mathbb{G}_z(L)$  such that  $x^2 \leq z$  for all  $x \in S$ . Then  $S \cup C_z$  is an ideal of  $L$ , where  $C_z = \{x \in L \mid x \leq z\}$ .

We now show how to construct an ideal in a commutator poset  $L$  from a maximal clique of  $\mathbb{A}\mathbb{G}_z(L)$ .

**Theorem 2.6.11.** (cf. [7, Theorem 2.7], see also [31, Theorem 2.5]) Let  $z \neq 1$  be an element of a commutator poset  $L$  and let  $S$  be a maximal clique in  $\mathbb{A}\mathbb{G}_z(L)$  such that  $x^2 \leq z$  for all  $x \in S$ . Then  $S \cup C_z$  is an ideal of  $L$ , where  $C_z = \{x \in L \mid x \leq z\}$ .

*Proof.* Let  $x \in S \cup C_z$  and  $a \in L$ . If  $ax \in C_z \subseteq S \cup C_z$ , we are done. Otherwise, suppose that  $ax \notin C_z$ . Thus  $ax \leq x$  implies  $(ax)y \leq xy \leq z$  for each  $y \in S$ . Hence,  $ax \in S$  by maximality of  $S$ .  $\square$

We now provide two examples to show that for each  $x$  in the maximal clique  $S$  in the above theorem  $x^2 \leq z$  is not a superfluous assumption.

**Example 2.6.12.**

- (a) Consider the totally-ordered set  $L = \{0, a, 1\}$  with  $0 \leq a \leq 1$ . Clearly,  $L$  is a commutator poset by assuming  $xy = yx$  and  $0x = 0$  for all  $x, y \in L$  with  $1 \cdot 1 = 1a = a$  and  $a^2 = 0$ . Let  $C = L \times L$  be the commutator poset of the direct product of commutator posets  $L$ . Let  $z = (0, a) \in C$  and then  $C_z = \{(0, 0), (0, a)\}$ . It is clear that the edge  $((1, a), (0, 1))$  is one of the maximal cliques in  $\mathbb{A}\mathbb{G}_z(C)$ . Let  $S = \{(1, a), (0, 1)\}$ . Clearly  $(1, a)^2 = (a, 0) \not\leq z$  and  $(0, 1)^2 = (0, a) \leq z$ . Now it is easy to see that  $S \cup C_z$  is not an ideal in  $C$  since  $(1, a)(a, a) = (a, 0) \notin S \cup C_z$ .

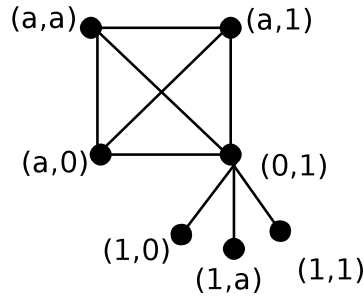


Figure 2.4:  $\mathbb{A}\mathbb{G}_{(0,a)}(C)$

- (b) Consider the commutator poset  $C$  as defined in Example 2.3.3. Let  $z = (a, a) \in C$ . Clearly  $C_z = \{(0, 0), (a, 0), (0, a), (a, a)\}$  and  $\mathbb{A}\mathbb{G}_z(C)$  is the square

$$((1, 0), (0, 1)), ((0, 1), (1, a)), ((1, a), (a, 1)), ((a, 1), (1, 0))$$

in which every edge can be regarded as a maximal clique.

Let  $S = \{(1, a), (a, 1)\}$ . Clearly  $(a, 1)^2 = (a, 1) \not\leq z$  and  $(1, a)^2 = (1, a) \not\leq z$ . Now it is easy to see that  $S \cup C_z$  is not an ideal in  $C$  since  $(0, 1)(a, 1) = (0, 1) \notin S \cup C_z$ .

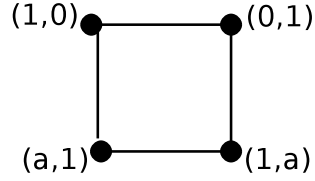


Figure 2.5:  $\mathbb{A}\mathbb{G}_{(a,a)}(C)$

The following proposition provides another counterexample for Theorem 2.6.11.

**Proposition 2.6.13.** *Let  $X$  be a nonempty set and  $P(X)$  the power set of  $X$ . Let  $L = (P(X), \cap, \subseteq)$  be a commutator poset,  $Z \neq X$  an element of  $P(X)$ , and  $\mathcal{Q}$  a clique in  $\mathbb{A}\mathbb{G}_Z(L)$ . Then  $\mathcal{Q}$  is a maximal clique in  $\mathbb{A}\mathbb{G}_Z(L)$  if and only if  $\{A \setminus Z \mid A \in \mathcal{Q}\}$  is a partition of  $X \setminus Z$ .*

*Proof.* For the necessary part, suppose  $\mathcal{Q}$  is a maximal clique in  $\mathbb{A}\mathbb{G}_Z(L)$ . Take any  $x \in X \setminus Z$  and consider the set  $\{x\}$ . If  $\{x\} \in \mathcal{Q}$ , then  $x \in \bigcup_{A \in \mathcal{Q}} A \setminus Z$ . On the other hand, if  $\{x\} \notin \mathcal{Q}$ , by maximality of  $\mathcal{Q}$ , there exists  $A \in \mathcal{Q}$  with  $\{x\} \cap A \not\subseteq Z$ . In this case,  $\{x\} \cap A \neq \emptyset$  and so  $x \in A$ .

Conversely, let  $\{A \setminus Z \mid A \in \mathcal{Q}\}$  be a partition of  $X \setminus Z$ . We now show that  $\mathcal{Q}$  is a maximal clique in  $\mathbb{A}\mathbb{G}_Z(L)$ . Suppose to the contrary that  $\mathcal{Q}$  is not a maximal clique. Thus, there exists  $C \notin \mathcal{Q}$  with  $C \not\subseteq Z$  such that  $C \cap A \subseteq Z$  for each  $A \in \mathcal{Q}$ . Now let  $x \in C \setminus Z$ . Clearly,  $x \in X \setminus Z$  and so  $x \in A \setminus Z$  by hypothesis. Consequently,  $x \in C \cap A \subseteq Z$ , yielding a contradiction since  $x \notin Z$ .  $\square$

**Corollary 2.6.14.** *Let  $Z = \emptyset$  and  $\mathcal{Q}$  be a clique in  $\mathbb{A}\mathbb{G}(L)$  as defined in the above proposition. Then  $\mathcal{Q}$  is a maximal clique in  $\mathbb{A}\mathbb{G}(L)$  if and only if  $\{A \mid A \in \mathcal{Q}\}$  is a partition of  $X$ .*

**Example 2.6.15.** Suppose  $\mathbb{A}\mathbb{G}_Z(L)$  is a noncomplete graph as defined in the above proposition. Clearly, maximality of  $\mathcal{Q}$ , in the above proposition, implies the existence of a  $B$  in  $\mathbb{A}\mathbb{G}_Z(L) \setminus \mathcal{Q}$  such that  $B \cap A \notin \mathcal{Q} \cup C_Z$  for some  $A \in \mathcal{Q}$ . This shows that  $\mathcal{Q} \cup C_Z$  is not an ideal of  $L$ . Therefore, by this example, we conclude that the condition  $x^2 \leq z$  in Theorem 2.6.11 is not a superfluous assumption.

# Chapter 3

## Commutator lattices and their annihilation graphs

In this chapter we introduce the concept of a commutator lattice and provide some examples for it. Then we define the annihilation graph of a commutator lattice  $L$  with respect to an element  $z \in L$ , denoted by  $\mathbb{A}\mathbb{G}_z(L)$ , and discuss some properties of  $\mathbb{A}\mathbb{G}_z(L)$ . In Theorem 3.2.13, we discuss some properties of a universal vertex of  $\mathbb{A}\mathbb{G}_z(L)$  for a complete commutator lattice  $L$  and by Example 3.2.14 we show that the associativity condition in the last part of Parts (a) and (b) of this theorem is a necessary assumption. We provide a different proof from the ring case for Theorem 3.3.5 that discusses a condition for connectivity of a subgraph of  $\mathbb{A}\mathbb{G}_z(L)$ . We study the condition(s) under which a vertex [respectively, an edge] is not a *cut-point* [respectively, *bridge*] (Propositions 3.4.1 and 3.5.1, respectively).

In Theorem 3.6.1, we discuss a relationship between the prime elements of  $L$  and complete bipartiteness of  $\mathbb{A}\mathbb{G}_z(L)$  and by Example 3.6.2 show that the primeness in this theorem is a necessary condition. In Theorem 3.6.4, by a different proof from the ring case, we verify that if  $\mathbb{A}\mathbb{G}_z(L)$  for a complete commutator lattice  $L$  is a complete  $r$ -partite graph with  $r \geq 3$ , then at most one of the parts

has more than one vertex. We also show that if  $z$  is an element of a commutator lattice  $L$  such that  $z = \bigwedge_{1 \leq i \leq n} p_i$  and for each  $1 \leq j \leq n$ ,  $z \neq \bigwedge_{1 \leq i \leq n, i \neq j} p_i$ , where  $p_i$  is a prime element of  $L$  for each  $1 \leq i \leq n$ , then  $\omega(\mathbb{A}\mathbb{G}_z(L)) = n$  (Theorem 3.7.5).

### 3.1 Commutators

The familiar group-theoretic notion of a commutator has been generalized to various contexts of universal algebra and category theory. The universal-algebraic references to commutators usually begin with J. D. H. Smith [43], and then mention various further generalizations of Smith's definition (see e.g. [22] and references therein, although there are many more recent ones). The categorical notions of commutators of subobjects and of internal equivalence relations first appear in S. A. Huq's papers (see [24]), and in M. C. Pedicchio's papers (see [38]), respectively.

As formulated in [25] (based on the approach of [28]), the commutator  $[\alpha, \beta]$  of two congruences  $\alpha$  and  $\beta$  on an algebra  $A$  in a Mal'tsev (=congruence permutable) variety  $\mathbf{C}$  with a Mal'tsev term  $p$  can be defined as the smallest congruence  $\gamma$  on  $A$  such that the map

$$(*) \{(x, y, z) \in A^3 \mid (x, y) \in \alpha \ \& \ (y, z) \in \beta\} \rightarrow A/\gamma$$

sending  $(x, y, z)$  to the  $\gamma$ -class of  $p(x, y, z)$ , is a homomorphism of algebras (that is, a morphism in  $\mathbf{C}$ ).

As also mentioned in [25], this commutator has the following properties:

$$\mathbf{C1} \quad [\alpha, \beta] \leq \alpha \wedge \beta$$

$$\mathbf{C2} \quad [\alpha, \beta] = [\beta, \alpha],$$

$$\mathbf{C3} \quad [\alpha, \beta \vee \gamma] = [\alpha, \beta] \vee [\alpha, \gamma]$$

where  $\wedge$  and  $\vee$  are the meet and the join in the lattice of congruences on a given algebra  $A$ . In fact this properties are well-known to hold also in various more general contexts, in particular for commutators in a congruence modular varieties (see e.g. [22]) and in an exact Mal'tsev category with coequalizers (see [38]).

When the ground variety  $\mathbf{C}$  is semi-abelian in the sense of  $G$ . Janelidze, L. Marki, and W. Tholen [26] (in some more general contexts, earlier studied by A. Ursini; see e.g. in [27] and references on Ursini's papers there), for each algebra  $A$  in  $C$ , there is a lattice isomorphism

$$\text{Con}(A) \approx \text{NSub}(A)$$

between the lattice  $\text{Con}(A)$  of congruences on  $A$  and the lattice  $\text{NSub}(A)$  of normal subalgebras of  $A$ , under which congruences correspond to their '0-classes'. This immediately allows us to define commutators in semi-abelian varieties using  $(*)$  as above, even though the so defined commutator will not necessarily coincide with the Huq commutator (see [33] for the clarification of their relationship). Accordingly, for normal subalgebras  $H$  and  $K$  of  $A$  and the corresponding congruences  $\alpha$  and  $\beta$  on  $A$ , we shall write  $[H, K]_{\text{Smith}}$  for the normal subalgebra of  $A$  corresponding to  $[\alpha, \beta]$ . Note also that, the so-defined  $[H, K]_{\text{Smith}}$  is at the same time a special case of the commutator introduced by A. Ursini in [45], as shown in that paper.

Let us recall the simplest examples:

**Example 3.1.1.**

- (a) If  $\mathbf{C}$  is the variety of groups, then normal subalgebras of  $A$  in  $\mathbf{C}$  are the same as normal subgroups of  $A$ , and, for two normal subgroups  $H$  and  $K$  of  $A$ ,  $[H, K]_{Smith}$  is the ordinary commutator of  $H$  and  $K$ . That is,

$[H, K]_{Smith}$  = the subgroup of  $A$  generated by all  $hkh^{-1}k^{-1}$  with  $h \in H$  and  $k \in K$ .

- (b) If  $\mathbf{C}$  is the variety of commutative rings (here and below rings are not required to have an identity element), then normal subalgebras of  $A$  in  $\mathbf{C}$  are the same as ideals of  $A$ , and, for two ideals  $H$  and  $K$  of  $A$ ,

$$[H, K]_{Smith} = HK,$$

the product of  $H$  and  $K$ .

- (c) If  $\mathbf{C}$  is the variety of rings (not necessarily commutative), then normal subalgebras of  $A$  in  $\mathbf{C}$  are the same as ideals of  $A$ , and, for two ideals  $H$  and  $K$  of  $A$ ,

$$[H, K]_{Smith} = HK + KH,$$

the product of  $H$  and  $K$ .

## 3.2 The annihilation graph of a commutator lattice

As motivated by Introduction and suggested by commutator theory, we introduce the following definition.

**Definition 3.2.1.** A *commutator lattice* is a bounded lattice  $L$  with least element  $0$  and greatest element  $1$  equipped with a binary operation  $[-, -]$ , also written as  $[x, y] = xy$ , and satisfying the conditions similar to **C1-C3** in Section 3.1, that is, satisfying

$$\mathbf{L1} \quad xy \leq x,$$

$$\mathbf{L2} \quad xy = yx$$

$$\mathbf{L3} \quad x(y \vee z) = (xy) \vee (xz)$$

for all  $x, y, z$  in  $L$ .

Our obvious examples of interest of a commutator lattice are:

### Example 3.2.2.

- (a) For an algebra  $A$  in a Mal'tsev variety  $\mathbf{C}$ , the lattice  $\text{Con}(A)$  of congruences on  $A$ , equipped with commutator operation defined as in Section 3.1, is a commutator lattice. The same is obviously true in all more contexts where commutators satisfy properties **C1-C3**, including the context of congruence modular varieties considered in [22].

- (b) For an algebra  $A$  in a semi-abelian variety  $\mathbf{C}$ , the lattice  $\text{NSub}(A)$  of normal subalgebras of  $A$ , equipped with commutator operation  $[-, -]_{Smith}$  defined as in Section 3.1, is a commutator lattice. In particular, this is the case for the varieties considered in Example 3.1.1 with commutators described there.

Let us mention two other obvious examples:

**Example 3.2.3.** An arbitrary lattice  $L$  becomes a commutator lattice if we put either

- (a)  $xy = x \wedge y$  for all  $x, y \in L$ , provided  $L$  is distributive, or
- (b)  $xy = 0$  for all  $x, y \in L$ .

As suggested by commutator theory, we might call these two kinds of commutator lattices *arithmetical* and *abelian*, respectively.

In the next remark, we recall some facts in a lattice and commutator lattice which will be used implicitly in the sequel.

**Remark 3.2.4.** Let  $L$  be a bounded lattice with 0 and 1. Using the fact that  $x \leq y$  if and only if  $x \vee y = y$  or  $x \wedge y = x$ , we know that for each  $x, y, z \in L$ ,

- (a) (i)  $x \leq x \vee y$ , (ii)  $x \wedge y \leq x$ .
- (b) (i)  $x \vee 0 = x$ , (ii)  $x \wedge 1 = x$ , (iii)  $x \vee 1 = 1$ , (iv)  $x \wedge 0 = 0$ .
- (c) (i) If  $x \leq y$ , then  $x \wedge z \leq y \wedge z$ , (ii) if  $x \leq y$ , then  $x \vee z \leq y \vee z$ .

Further, if  $L$  is a commutator lattice, we have:

- (d)  $x0 = 0$  for any  $x \in L$ .
- (e)  $xy \leq x \wedge y$ , since  $xy \leq x$  and  $xy \leq y$  for any  $x, y \in L$ .
- (f) If  $x \leq y$ , then  $xz \leq yz$  for any  $x, y, z \in L$ .
- (g)  $u(x \wedge y) \leq ux \wedge uy$  for any  $u, x, y \in L$ .

The definition of the annihilating-ideal graph  $\mathbb{A}\mathbb{G}_I(R)$  of a ring  $R$  with respect to the ideal  $I$  immediately extends to the context of a commutator lattice as follows:

**Definition 3.2.5.** For an element  $z$  in a commutator lattice  $L$  we define the annihilation graph of  $L$  with respect to  $z$ , denoted by  $\mathbb{A}\mathbb{G}_z(L)$ , in which:

- the vertices of  $\mathbb{A}\mathbb{G}_z(L)$  are all elements  $x$  of  $L$  not less than or equal to  $z$  and having an element  $y$  in  $L$  not less than or equal to  $z$  with  $xy \leq z$ ;
- a pair  $(x, y)$  of distinct vertices of  $\mathbb{A}\mathbb{G}_z(L)$  is an edge of  $\mathbb{A}\mathbb{G}_z(L)$  if and only if  $xy \leq z$ .

We shall also write  $\mathbb{A}\mathbb{G}(L) = \mathbb{A}\mathbb{G}_0(L)$ , and call this graph the annihilation graph of  $L$ .

In particular we have

$$\mathbb{A}\mathbb{G}_I(R) = \mathbb{A}\mathbb{G}_I(L) \text{ and } AG(R) = AG(L),$$

where  $\mathbb{A}\mathbb{G}_I(R)$  and  $\mathbb{A}\mathbb{G}(R)$  are as in Section 1.2, while  $L$  is the commutator lattice of ideals of  $R$  with the commutator operation as in Example 3.1.1(b).

In the following simple example, we show that  $\mathbb{A}\mathbb{G}_z(L)$  is non-empty (Part (a)) and define a commutator lattice  $L$  whose annihilation graph is complete bipartite (Part (b)).

**Example 3.2.6.** Let  $L = L_1 \times L_2$ , where each of  $L_1$  and  $L_2$  is a commutator lattice with at least two elements 0 and 1. Clearly,  $L$  is a commutator lattice by defining its operations and its order  $\leq$  componentwise.

- (a) Let  $a_1 \neq 1$  and  $a_2 \neq 1$  be two elements of  $L_1$  and  $L_2$ , respectively. Let  $a = (a_1, 1)$ ,  $b = (1, a_2)$ , and  $z = (a_1, a_2)$ . Clearly  $ab \leq z$ , but neither  $a$  nor  $b$  is less than or equal to  $z$ .
- (b) If each of  $L_1$  and  $L_2$  contains no nonzero zero-divisors, Then  $\mathbb{A}\mathbb{G}(L)$  is a complete bipartite graph with parts  $\{(a, 0) | 0 \neq a \in L_1\}$  and  $\{(0, b) | 0 \neq b \in L_2\}$ .

**Remark 3.2.7.** Let  $R$  be a commutative ring with identity and  $L$  the commutator lattice of ideals of  $R$ . In [7, Proposition 2.2], there is a discussion about a relationship between the vertices of  $\mathbb{A}\mathbb{G}_z(L)$  ( $0 \neq z \in L$ ) and we write it here as follows.

**Proposition G** Let  $a$ ,  $b$ , and  $c$  vertices of the graph  $\mathbb{A}\mathbb{G}_z(L)$ .

- (a) Suppose  $a \wedge b$  is not less than or equal to  $z$ . Then  $(a \wedge b)c \leq z$  whenever  $ac \leq z$  or  $bc \leq z$ .

(b) If  $a \wedge b \leq z$ , then  $(a, b)$  is an edge in  $\mathbb{A}\mathbb{G}_z(L)$ .

(c) Suppose that  $ac \leq z$  and  $bc \leq z$ . Then  $(a \vee b)c \leq z$ .

In the following proposition, we state some results which will be used (implicitly) in the sequel for commutator lattices (see also the above remark).

**Proposition 3.2.8.** *Let  $z \neq 1$  be an element in a commutator lattice  $L$  and  $a$ ,  $b$ , and  $c$  vertices of the graph  $\mathbb{A}\mathbb{G}_z(L)$ .*

(a) *If  $a \wedge b \leq z$ , then  $(a, b)$  is an edge in  $\mathbb{A}\mathbb{G}_z(L)$ .*

(b) *Suppose that  $ac \leq z$  and  $bc \leq z$ . Then  $(a \vee b)c \leq z$ .*

*Proof.* Part (b) is immediate since  $(a \vee b)c = ac \vee bc$ ; and Part (a) is clear since  $ab \leq a \wedge b$ . □

**Definition 3.2.9.** A partially ordered set  $(L, \leq)$  is a *complete lattice*, if every subset  $A$  of  $L$  has both a greatest lower bound (meet) and a least upper bound (join) in  $(L, \leq)$ .

**Definition 3.2.10.** A *complete commutator lattice* is a complete lattice  $L$  in which the commutator operation of  $L$  is distributive over arbitrary join in  $L$ . That is,  $x \vee S = \bigvee_{s \in S} xs$  for all subsets  $S \subset L$  and all  $x \in L$ .

We now from the above definition can define:

**Definition 3.2.11.** Let  $x$  and  $z$  be two elements of a complete commutator lattice  $L$ . We define  $l_z(x) = \bigvee\{y \in L \mid xy \leq z\}$ , and it satisfies the equivalence

$$xy \leq z \Leftrightarrow y \leq l_z(x).$$

**Remark 3.2.12.** Let  $R$  be a commutative ring with identity and  $L$  the commutator lattice of ideals of  $R$ . In [7, Theorem 2.6] (see Section 1.2), there is a discussion of some properties of a universal vertex of  $\mathbb{A}\mathbb{G}_z(L)$  ( $0 \neq z \in L$ ) and we write it here as follows.

**Proposition H** Let  $z$  be a nonzero element of  $L$  and  $a \in \mathbb{A}\mathbb{G}_z(L)$ .

- (a) If  $a$  is adjacent to every vertex of  $\mathbb{A}\mathbb{G}_z(L)$  with  $a \wedge z \neq z$ , then  $(z : a)$  (the set of all elements  $x \in L$  such that  $xa \leq z$ ) is a maximal element of the set  $\{(z : x) \mid x \in L \text{ and } x \not\leq z\}$ . Moreover,  $(z : a)$  is a prime ideal in  $L$ .
- (b) If  $a$  is adjacent to every vertex of  $\mathbb{A}\mathbb{G}_z(L)$  with  $a^2 \leq z$ , then  $(z : a)$  is a maximal element of the set  $\{(z : x) \mid x \in L \text{ and } x \not\leq z\}$ . Moreover,  $(z : a)$  is a prime ideal in  $L$ .

In the next theorem, we discuss some properties of a universal vertex of  $\mathbb{A}\mathbb{G}_z(L)$  for a complete commutator lattice (see also the above remark). Note that the commutator lattices arising from commutative rings are examples of complete commutator lattices. For the definition of  $l_z(a)$ , see Definition 3.2.11.

**Theorem 3.2.13.** (cf. [7, Theorem 2.6], see also [31, Theorem 2.4]) Let  $z \neq 1$  be a nonzero element of a complete commutator lattice  $L$  and  $a \in \mathbb{A}\mathbb{G}_z(L)$ .

- (a) If  $a$  is adjacent to every vertex of  $\mathbb{A}\mathbb{G}_z(L)$  with  $a \wedge z \neq z$ , then  $l_z(a)$  is a largest element of the set  $\{l_z(x) \mid x \in L \text{ and } x \not\leq z\}$ . Moreover,  $l_z(a)$  is a prime element of  $L$  provided that multiplication is associative in  $L$ .

(b) If  $a$  is adjacent to every vertex of  $\mathbb{A}\mathbb{G}_z(L)$  with  $a^2 \leq z$ , then  $l_z(a)$  is a largest element of the set  $\{l_z(x) | x \in L \text{ and } x \not\leq z\}$ . Moreover,  $l_z(a)$  is a prime element of  $L$  provided that multiplication is associative in  $L$ .

*Proof.* (a): Let  $V = V(\mathbb{A}\mathbb{G}_z(L))$ . Choose an element  $0 \neq x \leq z$  with  $x \not\leq a$ .

It is easy to see that each of  $a$  and  $x$  is different from  $a \vee x$  by hypothesis. Also for every  $b$  (different from  $a$ ) in  $V$ , by hypothesis, it is clear that  $b(a \vee x) \leq z$  which implies  $a \vee x \in V$ .

Otherwise, if there is no such  $b$  ( $\neq a$ ) in  $V$ , then  $\mathbb{A}\mathbb{G}_z(L)$  must have only one vertex  $a$  and hence  $a^2 \leq z$ .

Hence  $a(a \vee x) \leq z$  implies  $a \vee x \in V$  which is a contradiction (note that  $a \neq a \vee x$ ).

Consequently,  $\mathbb{A}\mathbb{G}_z(L)$  must have more than one vertex and hence  $a \vee x \in V$ .

Thus by hypothesis,  $a(a \vee x) = a^2 \vee ax \leq z$  implies  $a^2 \leq z$  since  $a^2 \leq a^2 \vee ax$ . Therefore  $x \leq l_z(a)$  if and only if  $x \in V$  or  $x \leq z$ , and so for any  $x \in L$  with  $x \not\leq z$ ,  $l_z(x) \leq l_z(a)$ . Note that if  $l_z(x) \not\leq z$ , then  $l_z(x) \in V$  (since  $xl_z(x) \leq z$  and  $x \not\leq z$ ) and  $al_z(x) \leq z$  and therefore  $l_z(x) \leq l_z(a)$ . Otherwise  $l_z(x) \leq z \leq l_z(a)$  (clearly  $az \leq z$  implies  $z \leq l_z(a)$ ). Thus the first assertion holds.

Now, we prove that  $l_z(a)$  is a prime element of  $L$ .

Let  $xy \leq l_z(a)$  and  $y \not\leq l_z(a)$ . Therefore  $(xy)a \leq z$ . Clearly,  $ya \not\leq z$  since

$y \not\leq l_z(a)$ .

We know that  $l_z(a) \leq l_z(ya)$  since  $ya \leq a$  implies  $r(ya) \leq ra \leq z$  for each  $r \leq l_z(a)$ . Thus by maximality of  $l_z(a)$ ,  $l_z(a) = l_z(ya)$ . Hence,  $x \leq l_z(a)$  since  $x(ya) \leq (xy)a \leq z$  by the associativity of the multiplication in  $L$ .

(b): Clearly,  $a^2 \leq z$  and  $a$  adjacent to every vertex of  $V$  imply that  $x \leq l_z(a)$  if and only if  $x \in V$  or  $x \leq z$ . Therefore, for any  $x \in L$  with  $x \not\leq z$ ,  $l_z(x) \leq l_z(a)$ . Thus the first assertion holds and the rest of the proof is similar to Part (a).  $\square$

In the following example, we show that the associativity condition in the last part of Parts (a) and (b) of the above theorem is a necessary assumption.

**Example 3.2.14.** Consider the totally-ordered set  $L = \{0, a, 1\}$  with  $0 \leq a \leq 1$ . Clearly,  $L$  is a lattice and turns to a commutator lattice by assuming  $xy = yx$  and  $0x = 0$  for all  $x, y \in L$  with  $1 \cdot 1 = 1a = a$  and  $a^2 = 0$ . Since  $a = 1(1a) \neq (1 \cdot 1)a = 0$ , the multiplication is nonassociative. Let  $C = L \times L$  be the nonassociative commutator lattice of the direct product of commutator lattices  $L$ .

(a) Let  $z = (0, 1) \in C$ . Clearly, the vertex  $\alpha = (a, a) \in \mathbb{A}\mathbb{G}_z(C)$  is adjacent to all other vertices of  $\mathbb{A}\mathbb{G}_z(C)$  and  $\alpha \wedge z \neq z$ . Let  $x = (1, 0) \not\leq z$  and  $y = (1, a) \not\leq z$  be two elements of  $C$ . Now, it is easy to see that  $(xy)\alpha \leq z$  and then  $xy \leq l_z(\alpha)$  but  $x\alpha \not\leq z$  and  $y\alpha \not\leq z$ , which implies that  $l_z(\alpha)$  is not a prime element of  $C$ . Consequently, the associativity of the commutator lattice is not a superfluous assumption for Theorem 3.2.13 Part (a).

(b) Let  $z, x$ , and  $y \in C$  be defined as in Part (a). Clearly, the vertex  $\beta = (a, 1) \in \mathbb{A}\mathbb{G}_z(C)$  is adjacent to all other vertices of  $\mathbb{A}\mathbb{G}_z(C)$  and  $\beta^2 \leq z$ .

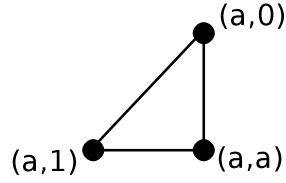


Figure 3.1:  $\mathbb{A}\mathbb{G}_{(0,1)}(C)$

Now, it is easy to see that  $(xy)\beta \leq z$  but  $x\beta \not\leq z$  and  $y\beta \not\leq z$ , which implies that  $l_z(\beta)$  is not a prime element of  $C$ . Consequently, the associativity of the commutator lattice is a necessary condition for Theorem 3.2.13 Part (b).

### 3.3 Connectivity of $\mathbb{A}\mathbb{G}_z(L)$

Recall that in Theorem 2.2.2 we showed that for a commutator poset  $L$ ,  $\mathbb{A}\mathbb{G}_z(L)$  is connected with  $\text{diam}(\mathbb{A}\mathbb{G}_z(L)) \leq 3$  and also if  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle, then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$ .

In the following, we define  $a^n$  for an element  $a$  of a commutator lattice  $L$  and a positive integer  $n$ .

**Definition 3.3.1.** Let  $n$  be a positive integer and  $a$  an element of a commutator lattice  $L$ . We define  $a^n$  by induction as follows:  $a^1 = a$ ,  $a^2 = aa$ ,  $a^3 = aa^2 \vee a^2a$  and assuming that  $a^m$  is already defined for all  $m < n$ , then  $a^n = aa^{n-1} \vee a^2a^{n-2} \vee \dots \vee a^{n-1}a$ .

**Remark 3.3.2.** Since the commutator operation of  $L$  is commutative and  $a \vee a = a$  for any  $a \in L$ , we have  $a^n = aa^{n-1} \vee a^2a^{n-2} \vee \dots \vee a^k a^k$  when  $n = 2k$  and

$$a^n = aa^{n-1} \vee a^2a^{n-2} \vee \dots \vee a^k a^{k+1} \text{ when } n = 2k + 1 \text{ for } k \geq 1.$$

By the above definition of  $a^n$ , we define the *radical* of an element  $z$  of  $L$  as follows:

**Definition 3.3.3.** Let  $L$  be a commutator lattice. The *radical* of an element  $z$  of  $L$ , denoted by  $\sqrt{z}$ , is the set of all elements  $a$  of  $L$  such that  $a^n \leq z$  for some positive integer  $n$ . Clearly,  $x \in \sqrt{z}$  when  $x \leq z$ .

In the following theorem, we discuss a condition for the connectivity of a subgraph of  $\mathbb{A}\mathbb{G}_z(L)$ . For the definition of  $\langle S \rangle$ , see Definition 1.1.13.

**Lemma 3.3.4.** *Let  $x$  be an element of a commutator lattice  $L$ . Then  $x^n \leq x$  for any positive integer  $n$ .*

*Proof.* Proof by induction on  $n$ . Clearly  $x^2 \leq x$ ,  $x^3 = xx^2 \leq x$ ,  $x^4 = xx^3 \vee x^2x^2 \leq x \vee x = x$ ,  $\dots$ . Now suppose for each positive integer  $m \leq n$ ,  $x^m \leq x$ . Therefore, by definition,  $x^{n+1} = xx^n \vee x^2x^{n-1} \vee \dots \vee x^n x$  and so  $x^{n+1} \leq x \vee x \vee \dots \vee x = x$ .  $\square$

**Theorem 3.3.5.** (cf. [7, Theorem 3.1], see also [31, Theorem 2.6]) Let  $z \neq 1$  be an element of a commutator lattice  $L$  and  $S = \{x \in \sqrt{z} \mid x \not\leq z\}$ . Then  $\langle S \rangle$  is connected.

*Proof.* If  $S$  is empty, then  $\langle S \rangle$  is connected by Remark 1.1.4. Otherwise, let  $x, y \in S$ . If  $xy \leq z$ , then the result is clear. Suppose for some positive integers

$m$  and  $n$ ,  $x^n, y^m \leq z$  and  $x^{n-1}, y^{m-1} \not\leq z$ . Clearly,  $xx^{n-1} \leq x^n$  by the definition of  $x^n$ . Now by the above lemma,  $x^{n-1} \leq x$  and so  $(x^{n-1})^2 = x^{n-1}x^{n-1} \leq xx^{n-1} \leq x^n \leq z$ , which implies that  $x^{n-1} \in S$ . Similarly,  $y^{m-1} \in S$ .

We now consider two cases: (i)  $x^{n-1} \wedge y^{m-1} \leq z$  or (ii)  $x^{n-1} \wedge y^{m-1} \not\leq z$ .

Hence, for Case (i), we have the path

$$(x, x^{n-1}, y^{m-1}, y)$$

since  $x^{n-1}y^{m-1} \leq x^{n-1} \wedge y^{m-1} \leq z$ , which is a path of length less than or equal to three from  $x$  to  $y$ .

For Case (ii),  $x^{n-1} \wedge y^{m-1} \in S$  since by the above lemma  $x^{n-1} \leq x$  and so  $(x^{n-1} \wedge y^{m-1})^2 = (x^{n-1} \wedge y^{m-1})(x^{n-1} \wedge y^{m-1}) \leq x^{n-1}x^{n-1} \leq xx^{n-1} \leq x^n \leq z$ .

Now it is clear that

$$(x, x^{n-1} \wedge y^{m-1}, y)$$

is a path of length less than or equal to two since  $x^{n-1} \wedge y^{m-1} \leq x^{n-1}$  and  $x^{n-1} \wedge y^{m-1} \leq y^{m-1}$ .  $\square$

**Corollary 3.3.6.** (cf. [7, Corollary 3.2]) Suppose  $N = \sqrt{0} \setminus \{0\}$  is a non-empty set. Then  $\langle N \rangle$  is a connected subgraph of  $\mathbb{A}\mathbb{G}(L)$ .

**Remark 3.3.7.** If in the above theorem we assume  $L$  is a totally ordered commutator lattice, then it is easy to show that the distance between any two

distinct elements of  $S$  is less than or equal to 2. For instance, suppose  $x \leq y$ , where  $x \neq y$  and  $x, y \in S$ . Then  $d(x, y) \leq 2$  as shown in the path

$$(x, y^{m-1}, y)$$

since  $xy^{m-1} \leq yy^{m-1} \leq z$ .

### 3.4 On cut-points of $\mathbb{A}\mathbb{G}_z(L)$

For the definition of the cut-point of a graph, see Definition 1.1.8.

In the following proposition, we discuss some condition under which  $\mathbb{A}\mathbb{G}_z(L)$  has no cut-points for a commutator lattice  $L$  (see also Remark 2.3.1 and Proposition 2.3.2).

**Proposition 3.4.1.** (cf. [7, Proposition 3.5], see also Section 3 of [39]) Let  $z \neq 1$  be a nonzero element of a commutator lattice  $L$  and  $x$  a vertex in  $\mathbb{A}\mathbb{G}_z(L)$ . If  $z \not\leq x$ , then  $x$  is not a cut-point of  $\mathbb{A}\mathbb{G}_z(L)$ .

*Proof.* Assume the vertex  $x$  of  $\mathbb{A}\mathbb{G}_z(L)$  is a cut-point. Then there exist vertices  $u$  and  $v$  ( $\neq x$ ) in  $\mathbb{A}\mathbb{G}_z(L)$  such that  $x$  lies on every path from  $u$  to  $v$ . Clearly, by Theorem 2.2.2, the shortest path from  $u$  to  $v$  is of length 2 or 3.

To show that  $x$  is not a cut-point of  $\mathbb{A}\mathbb{G}_z(L)$ , we consider two cases:

*Case 1:* Suppose  $(u, x, v)$  is a path of shortest length from  $u$  to  $v$ . If  $x \vee z = u$ , then  $u$  is adjacent to  $v$ . Similarly, if  $x \vee z = v$ , then  $u$  is adjacent to  $v$ .

So suppose  $(x \vee z) \neq u$  and  $(x \vee z) \neq v$ . Clearly,  $x \vee z \neq x$  since  $z \leq x \vee z = x$  implies  $z \leq x$ , which is not true by hypothesis. Also  $x \vee z \leq z$  implies  $x \leq z$  which is impossible since  $x$  is a vertex in  $\mathbb{A}\mathbb{G}_z(L)$ . Hence

$$(u, (x \vee z), v)$$

is a path different from  $(u, x, v)$  in  $\mathbb{A}\mathbb{G}_z(L)$  which is a contradiction.

*Case 2:* Suppose (without loss of generality)  $(u, x, y, v)$  is a path of shortest length from  $u$  to  $v$  in  $\mathbb{A}\mathbb{G}_z(L)$ . Clearly,

$$(u, (x \vee z), y, v)$$

is a path different from  $(u, x, y, v)$  in  $\mathbb{A}\mathbb{G}_z(L)$  which is a contradiction.  $\square$

**Remark 3.4.2.** For an example of a commutator lattice whose annihilation graph has a cut-point, see Example 2.3.3.

### 3.5 On bridges of $\mathbb{A}\mathbb{G}_z(L)$

For the definition of the bridge of a graph, see Definition 1.1.9.

In the following proposition, we discuss some conditions under which  $\mathbb{A}\mathbb{G}_z(L)$  has no bridges for a commutator lattice  $L$  (see also Remark 2.4.1 and Proposition 2.4.2).

**Proposition 3.5.1.** (cf. [7, Proposition 3.6]) Let  $z \neq 1$  be a nonzero element of a commutator lattice  $L$  and  $a \neq b$  two distinct vertices of the graph  $\mathbb{A}\mathbb{G}_z(L)$  with  $(a, b)$  an edge in  $\mathbb{A}\mathbb{G}_z(L)$ . Let  $S = \{x \in L \mid x \leq a \wedge b \text{ and } x \not\leq z\}$ .

- (a) If  $S$  has more than one element, then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (b) Assume  $a \not\leq b$  and  $b \not\leq a$ . If  $|S| = 1$ , then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (c) Assume  $a \not\leq b$ ,  $b \not\leq a$ , and  $S = \emptyset$ . If  $a^2 \leq z$  and  $b^2 \leq z$ , then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .
- (d) Assume  $a \leq b$  and  $z \not\leq b$  [or  $b \leq a$  and  $z \not\leq a$ ]. Then  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ .

*Proof.* In order to show that  $(a, b)$  is not a bridge in  $\mathbb{A}\mathbb{G}_z(L)$ , it suffices to find another path ( $\neq (a, b)$ ) from  $a$  to  $b$  in  $\mathbb{A}\mathbb{G}_z(L)$ .

(a): Let  $x \in S$  such that  $x \neq a \wedge b$ . Clearly,

$$(a, x, b)$$

is the desired path. Note that  $a \neq x$  and  $b \neq x$  since  $x \leq a \wedge b \leq a$ ,  $x \leq a \wedge b \leq b$ , and  $x \neq a \wedge b$ .

(b): Clearly  $a \neq a \wedge b$  and  $b \neq a \wedge b$  since  $a = a \wedge b \leq b$  and  $b = a \wedge b \leq a$  contradicts the hypothesis. Now

$$(a, a \wedge b, b)$$

is a path different from  $(a, b)$  since  $a \wedge b \leq b$ ,  $a \wedge b \leq a$ , and  $ab \leq z$ .

(c): Clearly,  $(a, a \vee b, b)$  is the desired path from  $a$  to  $b$ .

(d): Let  $a \leq b$  and  $z \not\leq b$ . Clearly  $a^2 \leq ab \leq z$  and  $z \not\leq a$ . Hence

$$(a, a \vee z, b)$$

is the desired path. Note that since  $z \not\leq a$  and  $z \not\leq b$ ,  $z \leq a \vee z$  implies that  $a \vee z$  is different from both  $a$  and  $b$ . The other part of the proof is similar to the previous argument.  $\square$

**Remark 3.5.2.** For an example of a commutator lattice whose annihilation graph containing four bridges, see Example 2.4.3.

### 3.6 $\mathbb{A}\mathbb{G}_z(L)$ as a complete $r$ -partite graph

As we proved in Proposition 2.1.7,  $z$  is a prime element of  $L$  if and only if  $\mathbb{A}\mathbb{G}_z(L) = \emptyset$ . In the following, we show that if  $z = p_1 \wedge p_2$  with  $p_1$  and  $p_2$  prime elements of  $L$ , then  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph. In the next section we study the girth and clique number of  $\mathbb{A}\mathbb{G}_z(L)$  for  $z = p_1 \wedge p_2 \cdots \wedge p_n$  with  $p_i$  a prime element of  $L$  for each  $1 \leq i \leq n$ .

In the following theorem, we discuss some condition under which  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph for a commutator lattice  $L$  (see also Remark 2.5.2).

**Theorem 3.6.1.** (cf. [7, Theorem 4.1(a)], see also [31, Theorem 3.1]) Let  $z \neq 1$  be a nonzero element of a commutator lattice  $L$ . Let  $p_1$  and  $p_2$  be two prime elements of  $L$  such that  $z = p_1 \wedge p_2$ . Then  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph.

*Proof.* The proof is similar to the proof of Theorem 2.5.3 (a). □

In the following examples, we show that the primeness condition in the above theorem is not a superfluous assumption.

**Example 3.6.2.**

- (a) Let  $L = \{0, a, b, 1\}$  be a totally ordered set with  $0 \leq a \leq b \leq 1$ . Then  $L$  is a commutator lattice by assuming  $xy = yx$  and  $x0 = 0$  for all  $x, y \in L$  with  $1 \cdot 1 = 1$  and  $1a = 1b = aa = bb = ab = a$ . Let  $C = L \times L$  be the commutator lattice of the direct product of commutator lattices  $L$ . Let

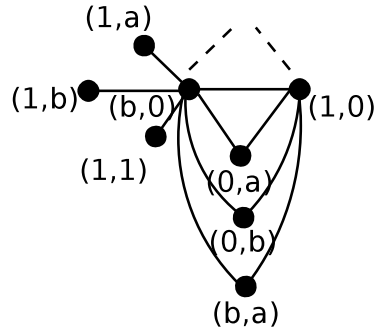


Figure 3.2:  $\mathbb{A}\mathbb{G}_{(a,0)}(C)$

$z = (a, 0) = (1, 0) \wedge (a, a)$ . It can easily be seen that  $(1, 0)$  is a prime element of  $C$  and  $(a, a)$  is not a prime element of  $C$  since  $(b, 0)(0, b) \leq (a, a)$  but  $(b, 0), (0, b) \not\leq (a, a)$ . Now from the above figure,  $\mathbb{A}\mathbb{G}_z(C)$  is not a complete bipartite graph since its girth is 3, which shows that the primeness condition for both elements  $p_1$  and  $p_2$  is necessary in the above theorem.

(b) Let  $\mathbb{A}\mathbb{G}_z(C)$  be defined as in Example 2.6.12(a). It is easy to see that  $C$  is a commutator lattice. We now see  $z = (0, a) = (0, 1) \wedge (a, a)$ . It is clear that  $(0, 1)$  and  $(a, a)$  are not prime elements of  $C$  since  $(a, 1)(a, a) \leq (0, 1)$  and  $(a, 1)(1, a) \leq (a, a)$ , but  $(a, 1), (a, a) \not\leq (0, 1)$  and  $(a, 1), (1, a) \not\leq (a, a)$ . Now from the Figure 2.4 of Example 2.6.12(a),  $\mathbb{A}\mathbb{G}_z(C)$  is not a complete bipartite graph since its girth is 3.

**Remark 3.6.3.** Let  $R$  be a commutative ring with identity and  $L$  the commutator lattice of ideals of  $R$ . In [7, Theorem 4.4], it is shown that if  $\mathbb{A}\mathbb{G}_z(L)(0 \neq z \in L)$  is a complete  $r$ -partite graph ( $r \geq 3$ ), then at most one of its parts has more than one element and we write it here as follows.

**Proposition I** Let  $z$  be a nonzero element of  $L$ . If  $\mathbb{A}\mathbb{G}_z(L)$  is a complete  $r$ -partite graph with  $r \geq 3$ , then at most one of the parts has more than one vertex. If  $V_i$  and  $V_j$  ( $i \neq j$ ) are two parts such that  $V_i = \{a\}$  and  $V_j = \{b\}$ , then either  $a^2 \leq z$  or  $b^2 \leq z$ . Furthermore, if  $b \not\leq a$ , then  $a^2 \leq z$ .

In the following theorems, we characterize some properties of  $\mathbb{A}\mathbb{G}_z(L)$  when  $L$  is a (complete) commutator lattice and  $\mathbb{A}\mathbb{G}_z(L)$  is a complete  $r$ -partite graph with  $r \geq 3$  (see also the above remark).

**Theorem 3.6.4.** (cf. [7, Theorem 4.4], see also [31, Theorem 3.2]) Let  $z \neq 1$  be a nonzero element of a complete commutator lattice  $L$ . If  $\mathbb{A}\mathbb{G}_z(L)$  is a complete  $r$ -partite graph with  $r \geq 3$ , then at most one of the parts has more than one vertex.

*Proof.* Assume that  $V_1, \dots, V_r$  are parts of  $\mathbb{A}\mathbb{G}_z(L)$ . Suppose that each of  $V_t$

and  $V_s$  has more than one element.

Choose  $b \in V_t$  and  $c \in V_s$ . Let  $V_l$  be a part of  $\mathbb{A}\mathbb{G}_z(L)$  such that  $V_l \neq V_t$  and  $V_l \neq V_s$ .

Let  $a \in V_l$ . Since  $\mathbb{A}\mathbb{G}_z(L)$  is a complete  $r$ -partite graph,  $ca \leq z$  and then  $c \leq l_z(a)$  and similarly  $b \leq l_z(a)$  by definition of  $l_z(a)$ . So  $l_z(a) \not\leq z$ . Clearly  $l_z(a)$  is a vertex (and lies in some of the parts) since  $al_z(a) \leq z$  and  $a, l_z(a) \not\leq z$ . Now if  $c \neq c' \in V_s$  and  $l_z(a) \notin V_s$  then  $cc' \leq l_z(a)c' \leq z$  and therefore  $cc' \leq z$  which is a contradiction (two vertices in one part can not have an edge). If  $b \neq b' \in V_t$  and  $l_z(a) \in V_s$  then  $bb' \leq l_z(a)b' \leq z$  and therefore  $bb' \leq z$  which is a contradiction and completes the proof.  $\square$

**Theorem 3.6.5.** (cf. [7, Theorem 4.4], see also [31, Theorem 3.2]) Let  $z \neq 1$  be a nonzero element of a commutator lattice  $L$  and  $\mathbb{A}\mathbb{G}_z(L)$  a complete  $r$ -partite graph with  $r \geq 3$ . If  $V_i$  and  $V_j$  ( $i \neq j$ ) are two parts such that  $V_i = \{a\}$  and  $V_j = \{b\}$ , then either  $a^2 \leq z$  or  $b^2 \leq z$ . Furthermore, if  $b \not\leq a$ , then  $a^2 \leq z$ .

*Proof.* Suppose  $a \in V_i$  and  $b \in V_j$  ( $1 \leq i \neq j \leq r$ ). Hence, there exists  $c \in \mathbb{A}\mathbb{G}_z(L)$  such that  $ac \leq z$  and  $bc \leq z$  with  $c \notin V_i$  and  $c \notin V_j$  since  $r \geq 3$ .

Clearly,  $a \vee b \not\leq z$  and  $(a \vee b)c \leq z$  and then  $a \vee b$  is a vertex. Now if  $a \vee b \notin V_i$ , then  $a^2 \vee ba \leq z$  yielding  $a^2 \leq z$ . Otherwise,  $a \vee b = a$  implies  $b^2 \leq ab \vee b^2 = ab \leq z$ . The last part is immediate since  $b \not\leq a$  implies  $a \vee b \notin V_i$ .  $\square$

Theorem 3.6.5 together with the fact that every complete graph is a complete  $r$ -partite graph gives:

**Corollary 3.6.6.** *Let  $z \neq 1$  be a nonzero element of a commutator lattice  $L$ . If  $\mathbb{A}\mathbb{G}_z(L)$  is a complete graph with more than two vertices, then the square of at most one vertex of  $\mathbb{A}\mathbb{G}_z(L)$  is not less than or equal to  $z$ .*

*Proof.* The proof follows directly from the above theorem. □

### 3.7 Girth and clique number of $\mathbb{A}\mathbb{G}_z(L)$

For the definition of the clique number, see Definition 1.1.12.

**Remark 3.7.1.** In Theorem 2.2.2, for a commutator poset, we directly showed that  $\mathbb{A}\mathbb{G}_z(L)$  is connected with diameter less than or equal to 3 and  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) \leq 4$  when  $\mathbb{A}\mathbb{G}_z(L)$  contains a cycle.

**Remark 3.7.2.** Let  $z = p_1 \wedge p_2$  for two prime elements  $p_1$  and  $p_2$  in a commutator lattice  $L$ . In the next theorem, we use the fact that  $p_1 \not\leq p_2$  and  $p_2 \not\leq p_1$  since  $z$  is not a prime element of  $L$  by the fact that we always assume  $\mathbb{A}\mathbb{G}_z(L)$  is a non-empty graph. Notice that in Proposition 2.1.7 we showed that  $\mathbb{A}\mathbb{G}_z(L)$  is the empty graph if and only if  $z$  is a prime element of  $L$ .

**Theorem 3.7.3.** (cf. [7, Theorem 5.5]) Let  $z \neq 1$  be a nonzero element of a commutator lattice  $L$ , and  $p_1$  and  $p_2$  two prime elements of  $L$  such that  $z = p_1 \wedge p_2$ . Let  $S = \{x \in L \mid x \leq p_1 \text{ and } x \not\leq p_2\}$  and  $T = \{x \in L \mid x \leq p_2 \text{ and } x \not\leq p_1\}$ .

- (a) Suppose there exist some  $a \in S$  and  $b \in T$  such that  $a \neq a \vee z$  and  $b \neq b \vee z$ . Then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 4$ .
- (b) If  $|S| \geq 2$  and  $|T| \geq 2$ , then  $\text{gr}(\mathbb{A}\mathbb{G}_z(L)) = 4$ .

*Proof.* The proof of Part (b) is an immediate consequence of Theorem 3.6.1.

For Part (a), clearly, by the remark preceding the theorem,  $S$  and  $T$  are non-empty sets since we always assume that  $\mathbb{A}\mathbb{G}_z(L)$  is a non-empty graph.

Now by Theorem 3.6.1,  $\mathbb{A}\mathbb{G}_z(L)$  is a complete bipartite graph with parts  $V_1 = S$  and  $V_2 = T$ .

Clearly, for any  $a \neq x \in V_1$  [respectively,  $b \neq y \in V_2$ ],  $ax \not\leq z$  [respectively,  $by \not\leq z$ ] (i.e. there is no edge between  $a$  and  $x$  [respectively,  $b$  and  $y$ ]).

Otherwise, since by hypothesis  $z = p_1 \wedge p_2$ ,  $ax \leq p_2$  [respectively,  $by \leq p_1$ ] implies that  $a \leq p_2$  or  $x \leq p_2$  [respectively,  $b \leq p_1$  or  $y \leq p_1$ ], yielding a contradiction by primeness of  $p_2$  [respectively,  $p_1$ ].

Now in order to complete the proof, it suffices to show that  $V_1$  contains  $a \vee z$  and  $V_2$  contains  $b \vee z$ .

It is obvious that  $a \vee z \leq p_1$  and  $b \vee z \leq p_2$ .

On the other hand, it is clear that  $a \vee z \not\leq p_2$  [respectively,  $b \vee z \not\leq p_1$ ]. Otherwise,  $a \vee z \leq z = p_1 \wedge p_2$  [respectively,  $b \vee z \leq z = p_1 \wedge p_2$ ] implies  $a \leq a \vee z \leq z = p_1 \wedge p_2 \leq p_2$  [respectively,  $b \leq b \vee z \leq z = p_1 \wedge p_2 \leq p_1$ ] implies  $a \leq p_2$  [respectively,  $b \leq p_1$ ], yielding a contradiction. Thus

$$a \vee z \in V_1 \text{ and } b \vee z \in V_2$$

and the proof is complete. □

**Remark 3.7.4.** In the above theorem, suppose  $|S| = 1$  and  $|T| \geq 2$ . In this case,  $\mathbb{A}\mathbb{G}_z(L)$  is a star graph. But by Proposition 3.4.1, this cannot happen if the element in  $S$  is not a cut-point since the center of a star graph is a cut-point. For example, Proposition 3.4.1(a) states that  $x$  is not a cut-point when  $|\{a \in L | a \leq x \text{ and } a \not\leq z\}| \geq 2$ . Similarly, the same argument is also valid for the case  $|T| = 1$  and  $|S| \geq 2$ .

In the following theorem, we show that the clique number of  $\mathbb{A}\mathbb{G}_z(L)$  is  $n \geq 2$  when  $z = \bigwedge_{1 \leq i \leq n} p_i$ , where  $L$  is a commutator lattice and  $p_i$ 's are prime elements of  $L$  (see also Remark 2.6.7).

**Theorem 3.7.5.** (cf. [7, Theorem 5.6]) Let  $z \neq 1$  be a nonzero element of a commutator lattice  $L$  such that  $z = \bigwedge_{1 \leq i \leq n} p_i$  and for each  $1 \leq j \leq n$ ,  $z \neq \bigwedge_{1 \leq i \leq n, i \neq j} p_i$ , where  $p_i$ 's are prime elements of  $L$ . Then  $\omega(\mathbb{A}\mathbb{G}_z(L)) = n$ .

*Proof.* The proof is immediate from the Theorem 2.6.8. □

**Corollary 3.7.6.** Let  $z \neq 1$  and  $w \neq 1$  be two nonzero elements of a commutator lattice  $L$ . Let  $z = \bigwedge_{1 \leq i \leq n} p_i$  and  $w = \bigwedge_{1 \leq j \leq m} q_j$  such that for each  $1 \leq k \leq n$ ,  $z \neq \bigwedge_{1 \leq i \leq n, i \neq k} p_i$  and for each  $1 \leq k \leq m$ ,  $w \neq \bigwedge_{1 \leq i \leq m, i \neq k} q_i$ , where  $p_i$ 's and  $q_j$ 's are prime elements of  $L$ . Then  $m = n$  when  $\mathbb{A}\mathbb{G}_z(L) \cong \mathbb{A}\mathbb{G}_w(L)$ .

# List of symbols

$\langle S \rangle$	the subgraph of a given graph induced by a subset $S$ of its vertices	15
$d(x, y)$	the distance between the vertices $x$ and $y$ of a given graph	13
$diam(G)$	the diameter of a connected graph $G$	13
$gr(G)$	the girth of a graph $G$	14
$K_{m,n}$	the complete bipartite graph with parts of size $m$ and $n$	15
$\omega(G)$	the clique number of $G$	9
$\sqrt{z}$	the radical of an element $z$ of a given commutator lattice	71
$Ann(x)$	the annihilator of any element $x$ in a given commutator poset	47
$AG_z(L)$	the annihilation graph of a commutator poset/commutator lattice $L$ with respect to an element $z$ of $L$	10

$AG(L)$	the annihilation graph of a commutator poset/commutator lattice $L$ with respect to $0 \in L$	31
$l_z(x)$	for a commutator poset $L$ , $l_z(x) = \sup\{a \in L   ax \leq z\}$ such that $x \sup\{a \in L   ax \leq z\} = \sup\{ax \in L   ax \leq z\}$	48
$\mathbb{I}(R)$	the set of all ideals of a commutative ring $R$	8
$C_I(R)$	the set of all ideals of a commutative ring $R$ that are contained in the ideal $I$ of $R$	8
$(I : X)$	the set of all ideals $A$ of a given commutative ring such that $AX \in I$ where $X$ and $I$ are ideals in the given ring	17
$\Gamma_I(R)$	the zero-divisor graph of a commutative ring $R$ with respect to an ideal $I$ of $R$	7
$AG_I(R)$	the annihilating-ideal graph of a commutative ring $R$ with respect to an ideal $I$ of $R$	8

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