Steinberg algebras and Leavitt path algebras

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Abstract

Leavitt path algebras are a new and exciting subject in noncommutative ring theory. To each directed graph E, and unital commutative ring R, is associated an R-algebra called the Leavitt path algebra of E with coefficients in R. It was discovered, when the theory of Leavitt path algebras was already quite advanced, that some of the more difficult questions were susceptible to a new approach using topological groupoids. Taking a special kind of groupoid $\mathcal{G}$, one can construct an R-algebra called the Steinberg algebra of $\mathcal{G}$. Many interesting classes of algebras, including Leavitt path algebras, can be obtained from this process. This dissertation is an exposition of the recent advances achieved by the groupoid approach to Leavitt path algebras.

New proofs are presented to show that the boundary path groupoid (which underlies the Steinberg algebra model for Leavitt path algebras) has the necessary topological properties. A new theorem is presented, characterising strongly graded Leavitt path algebras in graphical terms. We show that the main results on the structure theory of Leavitt path algebras, including the simplicity and primitivity theorems, can be recovered using the groupoid approach. We demonstrate how these methods lead to an explicit description of the centre of a Leavitt path algebra.

Contents

In memory of my friend, Steve Bailey

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Summary

The choice of topic stems from a general interest of the author and his supervisor in Leavitt path algebras, a new and beautiful subject in noncommutative algebra. Leavitt path algebras are $\mathbb{Z}$-graded algebras with involution, whose generators and relations are encoded in a directed graph. The construction is somewhat similar to (and indeed motivated by) the construction of graph C^*-algebras. Some of the fundamental examples of Leavitt path algebras are matrix algebras $M_n(R)$, Laurent polynomial algebras $R[x, x^{-1}]$, and the classical Leavitt algebras $L_{n,R}$, introduced by W. G. Leavitt in the 1960s, as the first examples of rings without the invariant basis number property.

To make this more concrete, let $E = (E^0, E^1, r, s)$ be a graph where E^0 is the set of vertices, E^1 is the set of edges, and r and s are the range and source maps respectively. Let R be an arbitrary field, or even a unital commutative ring. The *Leavitt path algebra* $L_R(E)$ is the universal R-algebra generated by pairwise orthogonal idempotents $\{v \mid v \in E^0\}$ and symbols $\{e, e^* \mid e \in E^1\}$, subject to the relations:

(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,

(E2) $e^*s(e) = r(e)e^* = e^*$ for all $e \in E^1$,

(CK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E^1$, and

(CK2) $v = \sum_{s^{-1}(v)} ee^*$ for every vertex $v \in E^0$ where $0 < |s^{-1}(v)| < \infty$.

Steinberg algebras, on the other hand, are even newer and more general than Leavitt path algebras. These are R-algebras of functions defined on a special kind of topological groupoid, called an *ample groupoid*. The idea arose, independently, from two directions. Firstly, Steinberg's paper [77], was motivated by a desire to exploit the connections between groupoids and inverse semigroups, for the purpose of studying inverse semigroup algebras. Secondly, the authors of [28] were motivated by a search for algebraic versions of groupoid C^*-algebras, much like Leavitt path algebras are algebraic versions of graph C^*-algebras. Steinberg algebras can also be viewed as a far-reaching generalisation of group algebras. All Leavitt path algebras are Steinberg algebras, and this new perspective is useful. Many theorems about Leavitt path algebras were subsequently generalised and simplified using the groupoid approach, and some of the more difficult problems were brought within reach for the first time.

Chapter 1: Steinberg algebras

In this chapter, we cover some prerequisites on topological groupoids and define the *Steinberg algebra* $A_R(\mathcal{G})$, where $\mathcal{G}$ is an ample groupoid and R is a unital commutative ring. We proceed to develop the basic theory in a self-contained way, paying attention to what can and cannot be said about non-Hausdorff groupoids. Along the way, we give a very brief treatment of inverse semigroups and their role in the subject. We establish some essential properties of Steinberg algebras, and discuss group-graded rings and group-graded groupoids. In a slightly more original section, we investigate the effects of groupoid-combining operations like products, disjoint unions, and directed unions, and find applications with finite-dimensional Steinberg algebras and the Steinberg algebras of approximately finite groupoids.

Chapter 2: Graphs and groupoids

This is an excursion into topology, in which we define the *path space* of an arbitrary graph E, and prove that it is locally compact, totally disconnected, and Hausdorff. This extends previous results by removing all restrictions on the cardinality of the graph. We then characterise those graphs whose path space is first-countable, second-countable, or σ-compact. A closed subspace of the path space, known as the *boundary path space*, becomes the unit space of the *boundary path groupoid* $\mathcal{G}_E$. We prove it is ample and study its local structure from a topological and an algebraic point of view.

Chapter 3: Leavitt path algebras

In this chapter, we introduce the *Leavitt path algebra* of a graph. It is defined with a universal property, so it becomes important to construct at least one model to establish the existence of the intended object. A key step in this process is the Graded Uniqueness Theorem, which we prove in greater generality than can be found in the literature, making no restrictions on the graph or the commutative ring of coefficients. We use this to prove the existence of a Steinberg algebra model for Leavitt path algebras, and recover some the fundamental properties of Leavitt path algebras. Finally, we prove the Graded and Cuntz-Krieger Uniqueness Theorems for Steinberg algebras.

Chapter 4: Strong grading

Let Γ be a group. A Γ-graded algebra $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is said to be *strongly graded* if $A_\gamma A_\delta = A_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The question "Which graphs have strongly $\mathbb{Z}$-graded Leavitt path algebras" had featured in a few studies (e.g. [49]) but it was not very susceptible to purely ring-theoretic methods: the answer was only known for finite graphs. We were interested in whether a groupoid approach could yield a more complete answer, and indeed it does. Using the Steinberg algebra model, we find necessary and sufficient conditions for strong grading on Leavitt path algebras. We draw analogies from graded rings and modules to other kinds of "graded objects" such as groupoids and sheaves. The research in this chapter was done in collaboration with Lisa Orloff Clark and Roozbeh Hazrat.

Chapter 5: Structure theory

We prove theorems that completely determine, in terms of the underlying graph and ring of coefficients, when a Leavitt path algebra is *prime*, *semiprime*, *primitive*, and *semiprimitive*. These results were all originally proved using non-groupoid methods, but groupoid-based proofs have recently appeared in Steinberg's papers, [79] and [81]. This chapter is an exposition of those proofs, which are achieved by looking directly into the structure theory of Steinberg algebras. We show how graded methods lead to some easy generalisations when the Steinberg algebra is graded by a ordered group.

Chapter 6: Centres and simplicity

In this chapter, we study the *centre* of a Leavitt path algebra. This was one of the more difficult substructures to pin down, and it was the subject of some quite technical papers, [36] and [37]. However, the centre of a Steinberg algebra is very easy to grasp, so it later became a matter of reconciling the features of graphs and groupoids that determine the centres of their respective algebras. This chapter will be an exposition of the recent progress in this area, the main reference being [31]. The chapter ends with the simplicity and graded simplicity theorems for Steinberg and Leavitt path algebras.

Chapter 1

Steinberg algebras

An ample groupoid is a special kind of locally compact topological groupoid. The Steinberg algebra of such a groupoid is a certain R-module of functions defined on it. It becomes an associative R-algebra once it is equipped with a noncommutative operation called the convolution. A standing assumption is that R is a unital commutative ring, and we rarely need to draw attention to it or require it to be anything special. If the groupoid $\mathcal{G}$ is Hausdorff, one can characterise its Steinberg algebra quite succinctly as the convolution algebra of locally constant, compactly supported functions $f : \mathcal{G} \to R$. Steinberg algebras first appeared independently in papers by Steinberg [77] and Clark, Farthing, Sims, and Tomforde [28]. The primary motivation for constructing these algebras was to generalise other classes of algebras, especially group algebras, inverse semigroup algebras, and Leavitt path algebras.

Steinberg algebras provide a unifying theory and a new way of studying these seemingly disparate classes of algebras. Many theorems about Leavitt path algebras and inverse semigroup algebras have since been recovered as specialisations of more general theorems about Steinberg algebras. For instance, various papers [79, 80, 81] have characterised, in terms of the underlying graph or inverse semigroup, when a Leavitt path algebra or inverse semigroup algebra is simple, (semi)prime, (semi)primitive, noetherian, or artinian. The Steinberg algebra model has also been put to use in [31, 77] to describe the centres of these algebras. Other classes of algebras that appear as special cases of Steinberg algebras include partial skew group rings associated to partial dynamical systems [22], and the higher-rank analogues of Leavitt path algebras, known as Kumjian-Pask algebras [33]. Additionally, the theory of Steinberg algebras has succeeded in producing algebras with interesting prescribed properties, including the first examples of simple algebras of arbitrary Gelfand-Kirrilov dimension [65].

The chapter begins, in §1.1 by providing some background on groupoids. In §1.2, we develop some theory of topological groupoids and almost immediately specialise to étale and ample groupoids. In §1.3, we introduce Steinberg algebras, describing them in a few different ways to make the definition more transparent. In §1.4, we investigate some important properties, showing that these algebras are locally unital and enjoy a kind of symmetry that comes from an anti-multiplicative involution. In §1.5, we define products, disjoint unions, and directed unions of groupoids, and use these operations to produce some nontrivial first examples. We add something new to the scene by calculating the Steinberg algebras of *elementary groupoids* and *approximately finite groupoids*: these turn out to be direct limits of direct sums of matrix algebras, with entries in commutative R-algebras. In §1.6, we cover some well-known preliminaries on graded algebras and not so well-known preliminaries on graded topological groupoids.

1.1 Groupoid concepts

The following classical definition of a groupoid is modified from [73]. We have chosen to paint a complete picture; indeed, some parts of the definition can be derived from other parts.

Definition 1.1. A **groupoid** is a system $(\mathcal{G}, \mathcal{G}^{(0)}, \boldsymbol{d}, \boldsymbol{c}, \boldsymbol{m}, \boldsymbol{i})$ such that:

(G1) $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are nonempty sets, called the *underlying set* and *unit space*;

(G2) $\boldsymbol{d}, \boldsymbol{c}$ are maps $\mathcal{G} \to \mathcal{G}^{(0)}$, called *domain* and *codomain*;

(G3) $\boldsymbol{m}$ is a partially defined binary operation on $\mathcal{G}$ called *composition*: specifically, it is a map from the set of *composable pairs*

$$\mathcal{G}^{(2)} = \left\{(g, h) \in \mathcal{G} \times \mathcal{G} \mid \boldsymbol{d}(g) = \boldsymbol{c}(h)\right\}$$

onto $\mathcal{G}$, written as $\boldsymbol{m}(g, h) = gh$, having the properties:

- $\boldsymbol{d}(gh) = \boldsymbol{d}(h)$ and $\boldsymbol{c}(gh) = \boldsymbol{c}(g)$ whenever the composition gh is defined;
- $(gh)k = g(hk)$ whenever either side is defined;

(G4) For every $x \in \mathcal{G}^{(0)}$ there is a unique identity $1_x \in \mathcal{G}$ such that $1_x g = g$ whenever $\boldsymbol{c}(g) = x$, and $h1_x = h$ whenever $\boldsymbol{d}(h) = x$;

(G5) $\boldsymbol{i} : \mathcal{G} \to \mathcal{G}$ is a map called *inversion*, written as $\boldsymbol{i}(g) = g^{-1}$, such that $g^{-1}g = 1_{\boldsymbol{c}(g)}$, $gg^{-1} = 1_{\boldsymbol{d}(g)}$, and $(g^{-1})^{-1} = g$.

If the reader is familiar with categories, the definition can be made more succinct by saying: a *groupoid* is a small category in which every morphism is invertible. Having said this, the elements of $\mathcal{G}$ will usually be called morphisms.

Remark 1.2. We always identify $x \in \mathcal{G}^{(0)}$ with $1_x \in \mathcal{G}$, so $\mathcal{G}^{(0)}$ is considered a subset of $\mathcal{G}$. The elements of $\mathcal{G}^{(0)}$ are called *units*.

Many authors write $\boldsymbol{s}$ (source) and $\boldsymbol{r}$ (range) instead of $\boldsymbol{d}$ and $\boldsymbol{c}$ in the definition of a groupoid. Our notation is chosen to avoid confusion in the context of graphs, where s and r refer to the source and range, respectively, of a directed path.

A *homomorphism* between groupoids $\mathcal{G}$ and $\mathcal{H}$ is a functor $F : \mathcal{G} \to \mathcal{H}$; that is, a map sending units of $\mathcal{G}$ to units of $\mathcal{H}$ and mapping all the morphisms in $\mathcal{G}$ to morphisms in $\mathcal{H}$ in a way that respects the structure. A *subgroupoid* is a subset $\mathcal{S} \subseteq \mathcal{G}$ that is a groupoid with the structure that it inherits from $\mathcal{G}$. We use the notation ${}^x\mathcal{G} = \boldsymbol{c}^{-1}(x)$, $\mathcal{G}^x = \boldsymbol{d}^{-1}(x)$, and ${}^x\mathcal{G}^y = {}^x\mathcal{G} \cap \mathcal{G}^y$. The set ${}^x\mathcal{G}^x$ is a group under composition, called the *isotropy group* based at x, and the set $\mathrm{Iso}(\mathcal{G}) = \bigcup_{x \in \mathcal{G}^{(0)}} {}^x\mathcal{G}^x$ is a subgroupoid called the *isotropy subgroupoid* of $\mathcal{G}$. If $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$ then $\mathcal{G}$ is called *principal*. We say that $\mathcal{G}$ is *transitive* if for every $x, y \in \mathcal{G}^{(0)}$ there is at least one morphism in ${}^x\mathcal{G}^y$. The *conjugacy class* of $x \in \mathrm{Iso}(\mathcal{G})$ is the set

$$\mathrm{Cl}_{\mathcal{G}}(x) = \left\{zxz^{-1} \mid z \in \mathcal{G}^{\boldsymbol{c}(x)}\right\} \subseteq \mathrm{Iso}(\mathcal{G}).$$

The set of conjugacy classes partitions $\mathrm{Iso}(\mathcal{G})$. The conjugacy class of a unit is called an *orbit*. Equivalently, an orbit is the unit space of a maximal transitive subgroupoid. Concretely, the orbit of $x \in \mathcal{G}^{(0)}$ is $\mathrm{Cl}_{\mathcal{G}}(x) = \boldsymbol{c}(\boldsymbol{d}^{-1}(x)) = \boldsymbol{d}(\boldsymbol{c}^{-1}(x))$. The set of orbits partitions $\mathcal{G}^{(0)}$. A subset $U \subseteq \mathcal{G}^{(0)}$ is *invariant* if for all $g \in \mathcal{G}$, $\boldsymbol{d}(g) \in U$ implies $\boldsymbol{c}(g) \in U$, which is to say that U is a union of orbits. If $x, y \in \mathcal{G}^{(0)}$ belong to the same orbit, the isotropy groups ${}^x\mathcal{G}^x$ and ${}^y\mathcal{G}^y$ are isomorphic. In fact, there can be many isomorphisms ${}^x\mathcal{G}^x \to {}^y\mathcal{G}^y$: for every $g \in {}^y\mathcal{G}^x$ there is an "inner" isomorphism ${}^x\mathcal{G}^x \to {}^y\mathcal{G}^y$ given by $x \mapsto gxg^{-1}$. This allows us to speak of the isotropy group of an orbit.

Examples 1.3. Many familiar mathematical objects are essentially groupoids:

(a) Any **group** G with identity ε can be viewed as a groupoid with unit space $\{\varepsilon\}$. Conjugacy classes are conjugacy classes in the usual sense.

(b) If $\{G_i \mid i \in I\}$ is a family of groups with identities $\{\varepsilon_i \mid i \in I\}$, then the disjoint union $\bigsqcup_{i\in I} G_i$ has a groupoid structure with $\boldsymbol{d}(g) = \boldsymbol{c}(g) = \varepsilon_i$ for every $g \in G_i$. The composition, defined only for pairs $(g,h) \in \bigsqcup_{i\in I} G_i \times G_i$, is just the relevant group law. This is known as a **bundle of groups**. The isotropy subgroupoid of any groupoid is a bundle of groups.

(c) Let X be a set with an equivalence relation $\sim$. We define the **groupoid of pairs** $\mathcal{G}_X = \{(x,y) \in X \times X \mid x \sim y\}$ with unit space X, and view (x,y) as a morphism with domain y, codomain x, and inverse $(x,y)^{-1} = (y,x)$. A pair of morphisms $(x,y),(w,z)$ is composable if and only if $y = w$, and composition is defined as $(x,y)(y,z) = (x,z)$. Every principal groupoid is isomorphic to a groupoid of pairs. If $\sim$ is the indiscrete equivalence relation (where $x \sim y$ for all $x, y \in X$) then $\mathcal{G}_X$ is called the *transitive principal groupoid on* X.

(d) Let G be a group with a left action on a set X. There is a groupoid structure on $G \times X$, where the unit space is $\{\varepsilon\} \times X$, or simply just X. We understand that the morphism (g,x) has domain $g^{-1}x$ and codomain x. Composition is defined as $(g,x)(h,g^{-1}x) = (gh,x)$, and inversion as $(g,x)^{-1} = (g^{-1}, g^{-1}x)$. The isotropy group at x is isomorphic to the stabiliser subgroup associated to x. Orbits are orbits in the usual sense, and the groupoid is transitive if and only if the action is transitive. This is called the **transformation groupoid** associated to the action of G on X.

(e) The **fundamental groupoid** of a topological space X is the set of homotopy path classes on X. The unit space of this groupoid is X itself, and the isotropy group at $x \in X$ is the fundamental group $\pi_1(X,x)$. The groupoid is transitive if and only if X is path-connected, and it is principal if and only if every path component is simply connected.

1.2 Topological groupoid concepts

Briefly, here are some of our topological conventions. We use the word *base* to mean a collection of open sets, called *basic open sets*, that generates a topology by taking unions. A neighbourhood base is a filter for the set of neighbourhoods of a point. The word *basis* is reserved for linear algebra. A *compact* topological space is one in which every open cover has a finite subcover, and a *locally compact* topological space is one in which every point has a neighbourhood base of compact sets. If X and Y are topological spaces, a *local homeomorphism* is a map $f : X \to Y$ with the property: every point in X has an open neighbourhood U such that $f|_U$ is a homeomorphism onto an open subset of Y. Every local homeomorphism is open and continuous.

The definition of a topological groupoid is straightforward, but there is some inconsistency in the literature on what it means for a groupoid to be étale or locally compact. While some papers require germane conditions, our definitions are chosen to be classical and minimally restrictive. We are mainly concerned with étale and ample groupoids. Roughly speaking, étale groupoids are topological groupoids whose topology is locally determined by the unit space.

Definitions 1.4. A groupoid $\mathcal{G}$ is

(a) a **topological groupoid** if its underlying set has a topology, and the maps $\boldsymbol{m}$ and $\boldsymbol{i}$ are continuous, with the understanding that $\mathcal{G}^{(2)}$ inherits its topology from $\mathcal{G} \times \mathcal{G}$;

(b) an **étale groupoid** if it is a topological groupoid and $\boldsymbol{d}$ is a local homeomorphism.

Some pleasant consequences follow from these two definitions. In any topological groupoid, $\boldsymbol{i}$ is a homeomorphism because it is a continuous involution, and $\boldsymbol{d}$ and $\boldsymbol{c}$ are both continuous because $\boldsymbol{d}(x) = \boldsymbol{m}(\boldsymbol{i}(x), x)$ and $\boldsymbol{c} = \boldsymbol{d}\boldsymbol{i}$. Additionally, if $\mathcal{G}$ is étale, then $\boldsymbol{d}$, $\boldsymbol{c}$, and $\boldsymbol{m}$ are local homeomorphisms, and $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$ (the openness of $\mathcal{G}^{(0)}$ is proved from first principles in [42, Proposition 3.2]). If $\mathcal{G}$ is a Hausdorff topological groupoid, then $\mathcal{G}^{(0)}$ is closed. Indeed (and this neat proof is from [76]) if $(x_i)_{i\in I}$ is a net in $\mathcal{G}^{(0)}$ with $x_i \to g \in \mathcal{G}$, then $x_i = \boldsymbol{c}(x_i) \to \boldsymbol{c}(g)$ because $\boldsymbol{c}$ is continuous, so $g = \boldsymbol{c}(g) \in \mathcal{G}^{(0)}$ by uniqueness of limits. If $\mathcal{G}$ is any topological groupoid, the maps $\boldsymbol{d} \times \boldsymbol{c} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ and $(\boldsymbol{d}, \boldsymbol{c}) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ are both continuous. If $\mathcal{G}^{(0)}$ is Hausdorff, the diagonal $\Delta = \{(x, x) \mid x \in \mathcal{G}^{(0)}\}$ is closed in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$; consequently, $\mathcal{G}^{(2)} = (\boldsymbol{d} \times \boldsymbol{c})^{-1}(\Delta)$ is closed in $\mathcal{G} \times \mathcal{G}$ and $\mathrm{Iso}(\mathcal{G}) = (\boldsymbol{d}, \boldsymbol{c})^{-1}(\Delta)$ is closed in $\mathcal{G}$.

Let $\mathcal{G}$ be a topological groupoid. If $U \subseteq \mathcal{G}$ is an open set such that $\boldsymbol{c}|_U$ and $\boldsymbol{d}|_U$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$, then U is called an *open bisection*. If $\mathcal{G}$ is étale and $U \subseteq \mathcal{G}$ is open, the restrictions $\boldsymbol{c}|_U$ and $\boldsymbol{d}|_U$ are continuous open maps, so they need only be injective for U to be an open bisection. An equivalent definition of an *étale groupoid* is a topological groupoid that has a base of open bisections. If $\mathcal{G}$ is étale and $\mathcal{G}^{(0)}$ is Hausdorff, then $\mathcal{G}$ is locally Hausdorff, because all the open bisections are homeomorphic to subspaces of $\mathcal{G}^{(0)}$. Another property of étale groupoids is that for any $x \in \mathcal{G}^{(0)}$, the fibres ${}^x\mathcal{G}$ and $\mathcal{G}^x$ are discrete spaces. Consequently, a groupoid with only one unit (i.e., a group) is étale if and only if it has the discrete topology.

Definition 1.5. An **ample groupoid** is a topological groupoid with Hausdorff unit space and a base of compact open bisections.

If $\mathcal{G}$ is an ample groupoid, the notation $B^{\mathrm{co}}(\mathcal{G})$ stands for the set of all nonempty compact open bisections in $\mathcal{G}$, and $\mathcal{B}(\mathcal{G}^{(0)})$ stands for the set of nonempty compact open subsets of $\mathcal{G}^{(0)}$.

Recall that a topological space is said to be *totally disconnected* if the only nonempty connected subsets are singletons, and 0-*dimensional* if every point has a neighbourhood base of clopen (i.e., closed and open) sets. These two notions are equivalent if the space is locally compact and Hausdorff [85, Theorems 29.5 & 29.7]. The following proposition is similar to [43, Proposition 4.1]. It is useful for reconciling slightly different definitions in the literature (e.g., [28]) and for checking when an étale groupoid is ample.

Proposition 1.6. *Let $\mathcal{G}$ be an étale groupoid such that $\mathcal{G}^{(0)}$ is Hausdorff. Then the following are equivalent:*

(1) *$\mathcal{G}$ is an ample groupoid;*

(2) *$\mathcal{G}^{(0)}$ is locally compact and totally disconnected;*

(3) *Every open bisection is locally compact and totally disconnected.*

Proof. (1) $\Rightarrow$ (2) Let $U \subseteq \mathcal{G}^{(0)}$ be open. Since $\mathcal{G}$ is ample and $\mathcal{G}^{(0)}$ is open, for every $x \in U$ there is a compact open bisection B such that $x \in B \subseteq U \subseteq \mathcal{G}^{(0)}$. Moreover, $\mathcal{G}^{(0)}$ is Hausdorff, so B is closed. This shows that $\mathcal{G}^{(0)}$ is locally compact and 0-dimensional (hence totally disconnected).

(2) $\Rightarrow$ (3) Every open bisection is homeomorphic to an open subspace of $\mathcal{G}^{(0)}$, so it is totally disconnected and locally compact.

(3) $\Rightarrow$ (1) Let U be open in $\mathcal{G}$, and $x \in U$. Since $\mathcal{G}$ is étale, it has a base of open bisections, so there is an open bisection B with $x \in B \subseteq U$. Moreover, B is Hausdorff, locally compact, and totally disconnected, so x has a compact neighbourhood $W \subseteq B$ and a clopen neighbourhood $V \subseteq W$. Since B is Hausdorff and V is closed in W, it follows that V is compact. Moreover, V is an open bisection because B is an open bisection. So, V is a compact open bisection. This shows that $\mathcal{G}$ has a base of compact open bisections, so $\mathcal{G}$ is ample. □

Remark 1.7. If $\mathcal{G}$ is a topological groupoid and $\mathcal{E}$ is a subgroupoid of $\mathcal{G}$, then $\mathcal{E}$ is automatically a topological groupoid with the topology it inherits from $\mathcal{G}$. If $\mathcal{G}$ is étale, then so is $\mathcal{E}$. However, if $\mathcal{G}$ is ample, then it is not guaranteed that $\mathcal{E}$ is ample. Indeed, by Proposition 1.6 (2), a subgroupoid $\mathcal{E}$ of an ample groupoid $\mathcal{G}$ is ample if and only if $\mathcal{E}^{(0)}$ is locally compact. In particular, $\mathcal{E}$ is ample if $\mathcal{G}$ is ample and $\mathcal{E}^{(0)}$ is either open or closed in $\mathcal{G}^{(0)}$.

The following lemma is similar to [69, Proposition 2.2.4], but with slightly different assumptions.

Lemma 1.8. *Let $\mathcal{G}$ be an étale groupoid where $\mathcal{G}^{(0)}$ is Hausdorff. If $A, B, C \subseteq \mathcal{G}$ are compact open bisections, then*

(1) *$A^{-1} = \{a^{-1} \mid a \in A\}$ and $AB = \{ab \mid (a, b) \in (A \times B) \cap \mathcal{G}^{(2)}\}$ are compact open bisections.*

(2) *If $\mathcal{G}$ is Hausdorff, then $A \cap B$ is a compact open bisection.*

Proof. (1) Firstly, $A^{-1} = \boldsymbol{i}(A)$ is compact and open because $\boldsymbol{i}$ is a homeomorphism. Clearly, A^{-1} is an open bisection. Secondly, note that AB might be empty, in which case it is trivially a compact open bisection. Otherwise, $(A \times B) \cap \mathcal{G}^{(2)}$ is compact because $\mathcal{G}^{(2)}$ is closed in $\mathcal{G} \times \mathcal{G}$, and $AB = \boldsymbol{m}\big((A \times B) \cap \mathcal{G}^{(2)}\big)$ is compact because $\boldsymbol{m}$ is continuous. Since $\boldsymbol{m}$ is a local homeomorphism, it is an open map, and $AB = \boldsymbol{m}\big((A \times B) \cap \mathcal{G}^{(2)}\big)$ is open. To prove that it is a bisection, suppose (a, b) is a composable pair in $A \times B$ and $\boldsymbol{d}(ab) = x$. Since A and B are bisections, b is the unique element in B having $\boldsymbol{d}(b) = x$, and a is the unique element of A having $\boldsymbol{d}(a) = \boldsymbol{c}(b)$. So, $\boldsymbol{d}|_{AB}$ is injective. Similarly, $\boldsymbol{c}|_{AB}$ is injective.

(2) It is trivial that $A \cap B$ is an open bisection. The Hausdorff property on $\mathcal{G}$ implies A and B are closed, so $A \cap B$ is closed, hence compact. □

Lemma 1.8 remains true if the words "compact" or "open", or both, are removed throughout the statement. Using Lemma 1.8 (2) with mathematical induction shows that when an ample groupoid is Hausdorff, its set of compact open bisections is closed under finite intersections. The converse to this statement is also true: an ample groupoid is Hausdorff if the set of compact open bisections is closed under finite intersections (see [77, Proposition 3.7]).

The main takeaway from Lemma 1.8 (1) is that the compact open bisections in an ample groupoid are important for two reasons: they generate the topology, and they can be multiplied and inverted in a way that is consistent with an algebraic structure called an inverse semigroup. An *inverse semigroup* is a semigroup S such that every $s \in S$ has a unique *inverse* $s^* \in S$ with the property $ss^*s = s$ and $s^*ss^* = s^*$.

Example 1.9. If X is a set, a *partial symmetry* of X is a bijection $s : \mathrm{dom}(s) \to \mathrm{cod}(s)$ where $\mathrm{dom}(s)$ and $\mathrm{cod}(s)$ are (possibly empty) subsets of X. Two partial symmetries s and t are composed in the way that binary relations are composed, so that $st : \mathrm{dom}(st) \to \mathrm{cod}(st)$ is the map $st(x) = s(t(x))$ for all $x \in X$ such that $s(t(x))$ makes sense. It is *not* necessary to have $\mathrm{dom}(s) = \mathrm{cod}(t)$ in order to compose s and t. The semigroup $\mathcal{I}_X$ of partial symmetries on X is called the **symmetric inverse**

semigroup on X. The Wagner-Preston Theorem is an analogue of Cayley's Theorem for groups: every inverse semigroup S has an embedding into $\mathcal{I}_S$.

The following result is an adaptation of [69, Proposition 2.2.3].

Proposition 1.10. *If $\mathcal{G}$ is an ample groupoid, $B^{\mathrm{co}}(\mathcal{G})$ is an inverse semigroup with the inversion and composition rules displayed in Lemma 1.8 (1).*

Proof. Lemma 1.8 (1) proves that $B^{\mathrm{co}}(\mathcal{G})$ is a semigroup and that $A \in B^{\mathrm{co}}(\mathcal{G})$ implies $A^{-1} \in B^{\mathrm{co}}(\mathcal{G})$. If $A \in B^{\mathrm{co}}(\mathcal{G})$ then $AA^{-1} = \boldsymbol{c}(A)$ because all composable pairs in $A \times A^{-1}$ are of the form (a, a^{-1}) for some $a \in A$. Therefore $AA^{-1}A = \boldsymbol{c}(A)A = A$ and $A^{-1}AA^{-1} = A^{-1}\boldsymbol{c}(A) = A^{-1}\boldsymbol{d}(A^{-1}) = A^{-1}$. To show that the inverses are unique, suppose $B \in B^{\mathrm{co}}(\mathcal{G})$ satisfies $ABA = A$ and $BAB = B$. Then for all $a \in A$ there exists $b \in B$ such that $aba = a$. But then $b = a^{-1}aa^{-1} = a^{-1}$. This shows $A^{-1} \subseteq B$. Similarly, $BAB = B$ implies $B^{-1} \subseteq A$ and consequently $B \subseteq A^{-1}$. Therefore $B = A^{-1}$. $\square$

This result has shown how to associate an inverse semigroup to an ample groupoid. The connections between ample groupoids and inverse semigroups run much deeper than this. There are at least two ways to associate an ample groupoid $\mathcal{G}$ to an inverse semigroup S. The first is the discrete *underlying groupoid*, where the unit space of $\mathcal{G}$ is the set of idempotents in S, and $\boldsymbol{d}(s) = s^*s$ while $\boldsymbol{c}(s) = ss^*$, for every $s \in S$. Composition is the same operation as the multiplication in S, just restricted to composable pairs. The second way is more complicated, and it is called the *universal groupoid* of S. It only diverges from the underlying groupoid when S is large (i.e., fails to have some finiteness conditions). The universal groupoid of S is quite powerful (as shown in [77]) because its Steinberg algebra turns out to be the inverse semigroup algebra of S. This takes us beyond our scope and, after all, we still need to define Steinberg algebras.

1.3 Introducing Steinberg algebras

The purpose of this section is to define and characterise the Steinberg algebra of an ample groupoid over a unital commutative ring R. Throughout this section, assume $\mathcal{G}$ is an ample groupoid. In order to make sense of continuity for R-valued functions, assume R has the discrete topology. Recall that *support* of a function $f : X \to R$ is defined as the set $\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}$. When X has a topology, we say that f is *compactly supported* if $\operatorname{supp} f$ is compact. If every point $x \in X$ has an open neighbourhood N such that $f|_N$ is constant, then f is called *locally constant*. It is easy to prove that $f : X \to R$ is locally constant if and only if it is continuous. We use the following notation for the *characteristic function* of a subset U of $\mathcal{G}$:

$$\mathbf{1}_U : \mathcal{G} \to R; \qquad \mathbf{1}_U(g) = \begin{cases} 1 & \text{if } g \in U \\ 0 & \text{if } g \notin U \end{cases}$$

Let $R^{\mathcal{G}}$ be the set of all functions $f : \mathcal{G} \to R$. Canonically, $R^{\mathcal{G}}$ has the structure of an R-module with operations defined pointwise.

Definition 1.11 (The Steinberg algebra)**.** Let $A_R(\mathcal{G})$ be the R-submodule of $R^{\mathcal{G}}$ generated by the set:

$$\{\mathbf{1}_U \mid U \text{ is a Hausdorff compact open subset of } \mathcal{G}\}.$$

The *convolution* of $f, g \in A_R(\mathcal{G})$ is defined as

$$f * g(x) = \sum_{\substack{y \in \mathcal{G} \\ \boldsymbol{d}(y) = \boldsymbol{d}(x)}} f(xy^{-1})g(y) = \sum_{\substack{(z,y) \in \mathcal{G}^{(2)} \\ zy = x}} f(z)g(y) \qquad \text{for all } x \in \mathcal{G}. \tag{1.1}$$

The R-module $A_R(\mathcal{G})$, with the convolution, is called the **Steinberg algebra** of $\mathcal{G}$ over R.

Example 1.12. If Γ is a discrete group, then $A_R(\Gamma) \cong R\Gamma$, the usual **group algebra** of Γ with coefficients in R.

We have yet to justify the definition of the convolution in (1.1). The two sums in the formula are equal, by substituting $z = xy^{-1}$. But it should not be taken for granted that the sum is finite, that $*$ is associative, or even that $A_R(\mathcal{G})$ is closed under $*$. These facts will be proved later. First, we prove the following result (inspired by [77]) that leads to some alternative descriptions of $A_R(\mathcal{G})$ as an R-module.

Proposition 1.13. *Let $\mathcal{B}$ be a base for $\mathcal{G}$ consisting of Hausdorff compact open sets, with the property:*

$$\left\{ \bigcap_{i=1}^{n} B_i \mid B_i \in \mathcal{B}, \ \bigcup_{i=1}^{n} B_i \text{ is Hausdorff} \right\} \subseteq \mathcal{B} \cup \{\emptyset\}.$$

Then $A_R(\mathcal{G}) = \operatorname{span}_R\{\mathbf{1}_B \mid B \in \mathcal{B}\}$.

Proof. Let $A = \operatorname{span}_R\{\mathbf{1}_B \mid B \in \mathcal{B}\}$. From the definition of $A_R(\mathcal{G})$, we have $A \subseteq A_R(\mathcal{G})$. To prove the other containment, suppose U is a Hausdorff compact open subset of $\mathcal{G}$. It is sufficient to prove that $\mathbf{1}_U$ is an R-linear combination of finitely many $\mathbf{1}_{B_i}$, where each $B_i \in \mathcal{B}$. Since $\mathcal{B}$ is a base for the topology on $\mathcal{G}$, we can write U as a union of sets in $\mathcal{B}$, and use the compactness of U to reduce it to a finite union $U = B_1 \cup \cdots \cup B_n$, where $B_1, \ldots, B_n \in \mathcal{B}$. By the principle of inclusion-exclusion:

$$\mathbf{1}_U = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = k}} \mathbf{1}_{\cap_{i \in I} B_i}.$$

The main assumption ensures that the sets $\cap_{i \in I} B_i$ on the right hand side are either empty or in $\mathcal{B}$. Therefore $A_R(\mathcal{G}) \subseteq A$. □

Corollary 1.14. *If $\mathcal{G}$ is Hausdorff and $\mathcal{B}$ is a base of compact open sets that is closed under finite intersections, then $A_R(\mathcal{G}) = \operatorname{span}_R\{\mathbf{1}_B \mid B \in \mathcal{B}\}$.*

We remarked after Lemma 1.8 that if $\mathcal{G}$ is non-Hausdorff, $B^{\mathrm{co}}(\mathcal{G})$ is *not* closed under finite intersections. Strange things can happen in non-Hausdorff spaces and the problem lies in the fact that compact sets are not always closed, and the intersection of two compact sets is not always compact. However, $B^{\mathrm{co}}(\mathcal{G})$ does satisfy the hypothesis of Proposition 1.13.

Corollary 1.15. *[77, Proposition 4.3] The Steinberg algebra is generated as an R-module by characteristic functions of compact open bisections. That is,*

$$A_R(\mathcal{G}) = \operatorname{span}_R\{\mathbf{1}_B \mid B \in B^{\mathrm{co}}(\mathcal{G})\}.$$

Proof. If $B_1, \ldots, B_n \in B^{\mathrm{co}}(\mathcal{G})$, and $U = \cup_i B_i$ is Hausdorff, then each B_i is closed in U because U is compact, so $\cap_i B_i$ is closed in U. And, B_1 is a compact set containing the closed set $\cap_i B_i$, so $\cap_i B_i$ is compact. Clearly $\cap_i B_i$ is an open bisection, so $\cap_i B_i \in B^{\mathrm{co}}(\mathcal{G})$. □

Remark 1.16. If $\mathcal{G}$ is an ample groupoid and $\mathcal{E}$ is an open subgroupoid, then $\mathcal{E}$ is also ample (see Remark 1.7). Let $\iota : \mathcal{E} \hookrightarrow \mathcal{G}$ be the inclusion map. There is a canonical monomorphism $m : A_R(\mathcal{E}) \hookrightarrow A_R(\mathcal{G})$, linearly extended from $\mathbf{1}_U \mapsto \mathbf{1}_{\iota(U)}$ for every Hausdorff compact open set $U \subseteq \mathcal{E}$. If $\mathcal{E}$ is closed, m has a left inverse $e : A_R(\mathcal{G}) \twoheadrightarrow A_R(\mathcal{E})$, linearly extended from $\mathbf{1}_U \mapsto \mathbf{1}_{U\cap\mathcal{E}}$ for every Hausdorff compact open set $U \subseteq \mathcal{G}$.

We still owe a proof that the convolution, from equation (1.1), is well-defined and gives an R-algebra structure to $A_R(\mathcal{G})$. The next two results are similar to [77, Propositions 4.5 & 4.6].

Lemma 1.17. *Let $A, B, C \in B^{\mathrm{co}}(\mathcal{G})$ and $r, s \in R$. Then:*

(1) $\mathbf{1}_{A^{-1}}(x) = \mathbf{1}_A(x^{-1})$ *for all* $x \in \mathcal{G}$*;*

(2) $\mathbf{1}_A * \mathbf{1}_B = \mathbf{1}_{AB}$*;*

Proof. (1) We have that $x \in A^{-1}$ if and only if $x^{-1} \in A$.

(2) Let $x \in \mathcal{G}$. By definition:

$$\mathbf{1}_A * \mathbf{1}_B(x) = \sum_{\substack{y\in\mathcal{G}\\ \boldsymbol{d}(y)=\boldsymbol{d}(x)}} \mathbf{1}_A(xy^{-1})\mathbf{1}_B(y) = \sum_{\substack{y\in B\\ \boldsymbol{d}(y)=\boldsymbol{d}(x)}} \mathbf{1}_A(xy^{-1}) \tag{1.2}$$

Assume x is of the form $x = ab$ where $a \in A$ and $b \in B$. Since B is a bisection, b is the only element of B having $\boldsymbol{d}(b) = \boldsymbol{d}(x)$, and it follows that

$$\mathbf{1}_A * \mathbf{1}_B(x) = \mathbf{1}_A(xb^{-1}) = \mathbf{1}_A(a) = 1.$$

On the other hand, assume $x \notin AB$. If there is $y \in B$ such that $\boldsymbol{d}(y) = \boldsymbol{d}(x)$, then $xy^{-1} \notin A$, for if it were, then $xy^{-1}y = x$ would be in AB. Therefore (1.2) yields $\mathbf{1}_A * \mathbf{1}_B(x) = 0$. □

Lemma 1.17 (2) implies that characteristic functions of compact open subsets of the unit space can be multiplied pointwise. That is, if $V, W \in \mathcal{B}(\mathcal{G}^{(0)})$ then $VW = V \cap W = WV$ and $\mathbf{1}_V * \mathbf{1}_W = \mathbf{1}_{V\cap W} = \mathbf{1}_W * \mathbf{1}_V$. As $\mathcal{G}^{(0)}$ is open in any ample groupoid $\mathcal{G}$, by Remark 1.16, there is a commutative subalgebra $A_R(\mathcal{G}^{(0)}) \hookrightarrow A_R(\mathcal{G})$.

The definition of an *R-algebra* is an R-module A, equipped with a binary operation written as $(x, y) \mapsto xy$. The binary operation should be R-linear in the first and second arguments (that is, bilinear), and it should be associative. There does not need to be a multiplicative identity. Every R-algebra is a ring, so many of our general definitions and theorems will be written for rings. It is tedious to prove that $*$ is associative from its definition in (1.1), so a proof was omitted in [77].

Proposition 1.18. *The R-module $A_R(\mathcal{G})$, equipped with the convolution, is an R-algebra.*

Proof. We need to show that the image of $* : A_R(\mathcal{G}) \times A_R(\mathcal{G}) \to R^{\mathcal{G}}$ is contained in $A_R(\mathcal{G})$, and that $*$ is associative and bilinear. Bilinearity can be proved quite easily from formula (1.1). Recall from Corollary 1.15 that the elements of $A_R(\mathcal{G})$ are R-linear combinations of characteristic functions of compact open bisections. If $f = \sum_i a_i \mathbf{1}_{A_i}$, $g = \sum_j b_j \mathbf{1}_{B_j}$, and $h = \sum_k c_k \mathbf{1}_{C_k}$, where the sums are finite, and $A_i, B_j, C_k \in B^{\mathrm{co}}(\mathcal{G})$ while $a_i, b_j, c_k \in R$ for all i, j, k, then

$$(f * g) * h = \sum_i \sum_j \sum_k a_i b_j c_k \mathbf{1}_{(A_iB_j)C_k} = \sum_i \sum_j \sum_k a_i b_j c_k \mathbf{1}_{A_i(B_jC_k)} = f * (g * h),$$

using Lemma 1.17 (2) and the bilinearity of $*$. This proves $*$ is associative. Evidently, $f * g = \sum_{i,j} a_i b_j \mathbf{1}_{A_iB_j} \in A_R(\mathcal{G})$, so $A_R(\mathcal{G})$ is closed under $*$. □

It is often useful to think of $*$ simply as the extension of the rule $\mathbf{1}_A * \mathbf{1}_B = \mathbf{1}_{AB}$ for all pairs $A, B \in B^{\mathrm{co}}(\mathcal{G})$, rather than the more complicated-looking expression (1.1) that we first defined it with. Moreover, one can infer from it that $A_R(\mathcal{G})$ is a homomorphic image of the semigroup algebra of $B^{\mathrm{co}}(\mathcal{G})$ with coefficients in R.

Proposition 1.19. *If $\mathcal{G}$ is Hausdorff and ample,*

$$A_R(\mathcal{G}) = \{f : \mathcal{G} \to R \mid f \textit{ is locally constant, compactly supported}\}. \tag{1.3}$$

Moreover, if $\mathcal{B}$ is a base for $\mathcal{G}$ consisting of compact open sets, such that $\mathcal{B}$ is closed under finite intersections and relative complements, then every nonzero $f \in A_R(\mathcal{G})$ is of the form $f = \sum_{i=1}^m r_i \mathbf{1}_{B_i}$, where $r_1, \dots, r_n \in R \setminus \{0\}$ and $B_1, \dots, B_n \in \mathcal{B}$ are mutually disjoint.

Proof. Let A be the set of locally constant, compactly supported R-valued functions on $\mathcal{G}$. Let $\mathcal{B}$ be a base of compact open sets for $\mathcal{G}$, such that $\mathcal{B}$ is closed under finite intersections and relative complements. (A worthy candidate for $\mathcal{B}$ is $B^{\mathrm{co}}(\mathcal{G})$.) If $0 \neq f \in A_R(\mathcal{G})$ then according to Corollary 1.14, $f = \sum_{i=1}^n s_i \mathbf{1}_{D_i}$ for some basic open sets $D_i \in \mathcal{B}$ and nonzero scalars $s_i \in R$. We aim to rewrite it as a linear combination of characteristic functions of *disjoint* open sets. If $s \in \operatorname{im} f \setminus \{0\}$, then we have the expression:

$$f^{-1}(s) = \bigcup_{\substack{I \subseteq \{1,\dots,n\} \\ s = \sum_{i \in I} s_i}} B_I, \qquad \text{where} \qquad B_I = \bigcap_{\substack{i \in I \\ j \notin I}} D_i \setminus D_j. \tag{1.4}$$

By assumption, each nonempty B_I in the expression is an element of $\mathcal{B}$; in particular, each B_I is compact and open. Finite unions preserve openness and compactness, so $f^{-1}(s)$ is open and compact for every nonzero $s \in \operatorname{im} f$. It follows that $f^{-1}(0) = \mathcal{G} \setminus \left(\bigcup_{s \in \operatorname{im} f \setminus \{0\}} f^{-1}(s)\right)$ is open. Therefore f is locally constant. As f is a linear combination of n characteristic functions, it is clear that $|\operatorname{im} f \setminus \{0\}| \leq 2^n$. Being a finite union of compact sets, $\operatorname{supp} f = \bigcup_{s \in \operatorname{im} f \setminus \{0\}} f^{-1}(s)$ is compact. Thus $f \in A$, and this shows $A_R(\mathcal{G}) \subseteq A$. To prove the other containment, $A \subseteq A_R(\mathcal{G})$, suppose $f \in A$. As f is continuous and $\operatorname{supp} f$ is compact, $f(\operatorname{supp} f) = \operatorname{im} f \setminus \{0\}$ is compact in R, so it must be finite. Let $\operatorname{im} f \setminus \{0\} = \{r_1, \dots, r_n\}$. Then each set $U_i = f^{-1}(r_i)$ is clopen because f is continuous, and compact because $U_i \subseteq \operatorname{supp} f$. Hence $f = \sum_{i=1}^n r_i \mathbf{1}_{U_i} \in A_R(\mathcal{G})$, and this shows $A \subseteq A_R(\mathcal{G})$.

To prove the "moreover" part, we look again at (1.4). If $I, J \subseteq \{1, \dots, n\}$ and $I \neq J$ then $B_I \cap B_J = \emptyset$. Therefore, $f \in A_R(\mathcal{G})$ can be written as an R-linear combination of characteristic functions of disjoint basic open sets in $\mathcal{B}$:

$$f = \sum_{s \in \operatorname{im} f \setminus \{0\}} s \mathbf{1}_{f^{-1}(s)} = \sum_{s \in \operatorname{im} f \setminus \{0\}} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ s = \sum_{i \in I} s_i}} s \mathbf{1}_{B_I}. \qquad \square$$

1.4 Properties of Steinberg algebras

It is useful to know when $A_R(\mathcal{G})$ is unital or has some property that is nearly as good. The answer is quite easy, and we show it below. We use the definition that a ring (or R-algebra) A is *locally unital* if there is a set of commuting idempotents $E \subseteq A$, called *local units*, with the property: for every finite subset $\{a_1, \dots, a_n\} \subseteq A$, there is a local unit $e \in E$ with $ea_i = a_i = a_i e$ for every $1 \leq i \leq n$. Equivalently, A is the direct limit of unital subrings: $A = \varinjlim_{e \in E} eAe$. The directed system is facilitated by the partial order, $e \leq e'$ if $ee' = e = e'e$, and the connecting homomorphisms (which need not be unit-preserving) are the inclusions $eAe \hookrightarrow e'Ae'$ for $e \leq e'$.

In many respects, working with locally unital rings is like working with unital rings. Every locally unital ring A is idempotent (i.e., $A^2 = A$) and if $I \subseteq A$ is an ideal, then $AI = I = IA$. If A is an R-algebra with local units, then the ring ideals of A are always R-algebra ideals (which, by definition, should be R-submodules of A). These facts are not true in general for arbitrary non-unital rings. Locally unital rings and algebras are always *homologically unital*, in the sense of [64, Definition 1.4.6], which essentially means that they have well-behaved homology. The classical Morita Theorems, with slight adjustments, are valid for rings with local units (see [15]).

Proposition 1.20. *[77, Proposition 4.11] , [26, Lemma 2.6]. Let $\mathcal{G}$ be an ample groupoid. Then $A_R(\mathcal{G})$ is locally unital. Moreover, $A_R(\mathcal{G})$ is unital if and only if $\mathcal{G}^{(0)}$ is compact.*

Proof. We prove the "moreover" part first. If $\mathcal{G}^{(0)}$ is compact, then it is a compact open bisection, and $\mathbf{1}_{\mathcal{G}^{(0)}} \in A_R(\mathcal{G})$. Following Lemma 1.17 (2), $\mathbf{1}_{\mathcal{G}^{(0)}} * \mathbf{1}_U = \mathbf{1}_{\mathcal{G}^{(0)}U} = \mathbf{1}_U$ and, similarly, $\mathbf{1}_U * \mathbf{1}_{\mathcal{G}^{(0)}} = \mathbf{1}_U$, for every $U \in B^{\text{co}}(\mathcal{G})$. Since $\{\mathbf{1}_U \mid U \in B^{\text{co}}(\mathcal{G})\}$ spans $A_R(\mathcal{G})$, it follows by linearity that $\mathbf{1}_{\mathcal{G}^{(0)}} * f = f = f * \mathbf{1}_{\mathcal{G}^{(0)}}$ for every $f \in A_R(\mathcal{G})$. This proves that $\mathbf{1}_{\mathcal{G}^{(0)}}$ is the multiplicative identity in $A_R(\mathcal{G})$.

Conversely, suppose $A_R(\mathcal{G})$ has a multiplicative identity called ξ. The first step is to show that $\xi = \mathbf{1}_{\mathcal{G}^{(0)}}$. Let $x \in \mathcal{G}$ and let $V \subseteq \mathcal{G}^{(0)}$ be a compact open set containing $\boldsymbol{d}(x)$. Then V must be Hausdorff because $\mathcal{G}^{(0)}$ is, so $\mathbf{1}_V \in A_R(\mathcal{G})$. If $x \notin \mathcal{G}^{(0)}$, then

$$0 = \mathbf{1}_V(x) = \xi * \mathbf{1}_V(x) = \sum_{y \in \mathcal{G}^{\boldsymbol{d}(x)}} \xi(xy^{-1})\mathbf{1}_V(y) = \sum_{y \in V \cap \mathcal{G}^{\boldsymbol{d}(x)}} \xi(xy^{-1}) = \xi(x)$$

because $V \cap \mathcal{G}^{\boldsymbol{d}(x)} = \{\boldsymbol{d}(x)\}$. Similarly, if $x \in \mathcal{G}^{(0)}$ then $x = \boldsymbol{d}(x) \in V$ and

$$1 = \mathbf{1}_V(x) = \xi * \mathbf{1}_V(x) = \xi(x).$$

This shows that $\xi = \mathbf{1}_{\mathcal{G}^{(0)}}$. The second step is to show that $\mathbf{1}_{\mathcal{G}^{(0)}} \in A_R(\mathcal{G})$ implies $\mathcal{G}^{(0)}$ is compact. By the definition of $A_R(\mathcal{G})$, there exist scalars $r_1, \dots, r_n \in R \backslash \{0\}$ and compact open sets $U_1, \dots, U_n \subseteq \mathcal{G}$ such that $\mathbf{1}_{\mathcal{G}^{(0)}} = r_1 \mathbf{1}_{U_1} + \dots + r_n \mathbf{1}_{U_n}$. Then $\mathcal{G}^{(0)} \subseteq U_1 \cup \dots \cup U_n$ and consequently $\mathcal{G}^{(0)} = \boldsymbol{d}(U_1) \cup \dots \cup \boldsymbol{d}(U_n)$. Each of the sets $\boldsymbol{d}(U_1), \dots, \boldsymbol{d}(U_n)$ is compact (because $\boldsymbol{d}$ is continuous) and the union is finite, so $\mathcal{G}^{(0)}$ is compact.

To show that $A_R(\mathcal{G})$ is locally unital for all ample groupoids $\mathcal{G}$, suppose $F = \{f_1, \dots, f_m\}$ is a finite subset of $A_R(\mathcal{G})$. Since $A_R(\mathcal{G})$ is spanned by $\{\mathbf{1}_U \mid U \in B^{\text{co}}(\mathcal{G})\}$, there exist finite subsets $\{U_1, \dots, U_n\} \subseteq B^{\text{co}}(\mathcal{G})$ and $\{r_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\} \subseteq R$ such that $f_j = r_{1,j}\mathbf{1}_{U_1} + \dots + r_{n,j}\mathbf{1}_{U_n}$ for all $1 \leq j \leq m$. Let $X = \boldsymbol{d}(U_1) \cup \dots \cup \boldsymbol{d}(U_n) \cup \boldsymbol{c}(U_1) \cup \dots \cup \boldsymbol{c}(U_n)$. Then $X \subseteq G^{(0)}$ is compact and open because it is a finite union of compact open sets, and X is Hausdorff because it is a subset of $\mathcal{G}^{(0)}$, so $\mathbf{1}_X \in A_R(\mathcal{G})$. Clearly, $XU_i = U_iX = X \cap U_i = U_i$, so $\mathbf{1}_X * \mathbf{1}_{U_i} = \mathbf{1}_{U_i} * \mathbf{1}_X = \mathbf{1}_{U_i}$, for all $1 \leq i \leq n$. By linearity, $\mathbf{1}_X * f_j = f_j * \mathbf{1}_X = f_j$ for all $1 \leq j \leq m$. The conclusion is that $E = \{\mathbf{1}_X \mid X \in \mathcal{B}(\mathcal{G}^{(0)})\}$ is a set of local units for $A_R(\mathcal{G})$. □

The *characteristic* of a ring A, written $\operatorname{char} A$, is defined as the least positive integer n such that $n \cdot a = 0$ for all $a \in A$, or 0 if no such n exists. If A has a set of local units E, the characteristic of A can be defined as the least n such that $n \cdot e = 0$ for all $e \in E$, or 0 if no such n exists.

Proposition 1.21. *For any ample groupoid $\mathcal{G}$, $\operatorname{char} A_R(\mathcal{G}) = \operatorname{char} R$.*

Proof. If n is a positive integer, $n \cdot \mathbf{1}_U = 0$ for all $U \in \mathcal{B}(\mathcal{G}^{(0)})$ if and only if $n \cdot 1 = 0$. $\square$

Given a topological groupoid $(\mathcal{G}, \mathcal{G}^{(0)}, \boldsymbol{d}, \boldsymbol{c}, \boldsymbol{m}, \boldsymbol{i})$, the *opposite groupoid* is:

$$\mathcal{G}^{\mathrm{op}} = (\mathcal{G}, \mathcal{G}^{(0)}, \boldsymbol{d}^{\mathrm{op}}, \boldsymbol{c}^{\mathrm{op}}, \boldsymbol{m}^{\mathrm{op}}, \boldsymbol{i})$$

where $\boldsymbol{d}^{\mathrm{op}} = \boldsymbol{c}$, $\boldsymbol{c}^{\mathrm{op}} = \boldsymbol{d}$, and $\boldsymbol{m}^{\mathrm{op}}(x, y) = \boldsymbol{m}(y, x)$ for any x, y with $\boldsymbol{c}(x) = \boldsymbol{d}(y)$. We assume it has the same topology as $\mathcal{G}$. We call the opposite groupoid $\mathcal{G}^{\mathrm{op}}$ to distinguish it from $\mathcal{G}$, even though they have the same underlying sets. Naturally, the inversion map $\boldsymbol{i} : \mathcal{G} \to \mathcal{G}^{\mathrm{op}}$ is an isomorphism and a homeomorphism. If A is a ring, an *involution* on A is an additive, anti-multiplicative map $\tau : A \to A$ such that $\tau^2 = \mathrm{id}_A$. If A has an involution, it is called an *involutive ring.* Every involutive ring is self-opposite (i.e., A is isomorphic to its opposite ring A^{op}). If $\mathcal{G}$ is an ample groupoid, the canonical groupoid involution $\boldsymbol{i}$ transfers to $A_R(\mathcal{G})$ in the sense that $f \mapsto f \circ \boldsymbol{i}$ is a canonical involution on $A_R(\mathcal{G})$. More generally, if $^- : R \to R$, written as $r \mapsto \overline{r}$, is an involution on R, then $f \mapsto \overline{f \circ \boldsymbol{i}}$ is an involution on $A_R(\mathcal{G})$. To summarise:

Proposition 1.22. *Let $\mathcal{G}$ be an ample groupoid. There are canonical isomorphisms $\mathcal{G} \cong \mathcal{G}^{\mathrm{op}}$ and $A_R(\mathcal{G}) \cong A_R(\mathcal{G}^{\mathrm{op}}) \cong A_R(\mathcal{G})^{\mathrm{op}}$. Moreover, to each involution $^- : R \to R$ is associated a canonical involution on $A_R(\mathcal{G})$, namely $f \mapsto \overline{f \circ \boldsymbol{i}}$ for all $f \in A_R(\mathcal{G})$.*

This kind of symmetry simplifies and smooths a lot of would-be subtleties. It implies, for example, that the category of left $A_R(\mathcal{G})$-modules is isomorphic to the category of right $A_R(\mathcal{G})$-modules, and the lattice of left ideals in $A_R(\mathcal{G})$ is isomorphic to the lattice of right ideals. Many important notions, like left- and right-primitivity, are equivalent for self-opposite algebras.

1.5 First examples

One or two of the results in this section will be useful later on, but mostly they are just interesting in their own right. Presumably, most of this content is already known, but we do not adhere closely to any references.

Given two groupoids $(\mathcal{G}_1, \boldsymbol{d}_1, \boldsymbol{c}_1, \boldsymbol{m}_1, \boldsymbol{i}_1)$ and $(\mathcal{G}_2, \boldsymbol{d}_2, \boldsymbol{c}_2, \boldsymbol{m}_2, \boldsymbol{i}_2)$, their *disjoint union* $\mathcal{G}_1 \sqcup \mathcal{G}_2$ has the structure of a groupoid with unit space $\mathcal{G}_1^{(0)} \sqcup \mathcal{G}_2^{(0)}$, set of composable pairs $\mathcal{G}_1^{(2)} \sqcup \mathcal{G}_2^{(2)}$, and the following structure maps: for all $x_1, y_1 \in \mathcal{G}_1$ and $x_2, y_2 \in \mathcal{G}_2$,

$$\boldsymbol{d}(x_i) = \boldsymbol{d}_i(x_i), \qquad \boldsymbol{c}(x_i) = \boldsymbol{c}_i(x_i), \qquad \boldsymbol{i}(x_i) = \boldsymbol{i}_i(x_i), \qquad \boldsymbol{m}(x_i, y_i) = \boldsymbol{m}_i(x_i, y_i).$$

The *product* $\mathcal{G}_1 \times \mathcal{G}_2$ also has the structure of a groupoid with unit space $\mathcal{G}_1^{(0)} \times \mathcal{G}_2^{(0)}$, and the following structure maps: for all $x_1, y_1 \in \mathcal{G}_1$ and $x_2, y_2 \in \mathcal{G}_2$,

$$\begin{aligned} \boldsymbol{d}(x_1, x_2) &= (\boldsymbol{d}_1(x_1), \boldsymbol{d}_2(x_2)), & \boldsymbol{c}(x_1, x_2) &= (\boldsymbol{c}_1(x_1), \boldsymbol{c}_2(x_2)), \\ \boldsymbol{i}(x_1, x_2) &= (\boldsymbol{i}_1(x_1), \boldsymbol{i}_2(x_2)), & \boldsymbol{m}((x_1, x_2), (y_1, y_2)) &= (\boldsymbol{m}_1(x_1, y_1), \boldsymbol{m}_2(x_2, y_2)). \end{aligned}$$

These constructions work just as well for the disjoint union or product of arbitrarily many (even infinitely many) groupoids. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are topological groupoids, then $\mathcal{G}_1 \sqcup \mathcal{G}_2$ (with the coproduct topology) and $\mathcal{G}_1 \times \mathcal{G}_2$ (with the product topology) are again topological groupoids. The properties of being étale or ample are preserved by arbitrary disjoint unions and finite products.

Proposition 1.23. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be ample groupoids. The Steinberg algebra of $\mathcal{G}_1 \sqcup \mathcal{G}_2$ is a direct sum of two ideals: $A_R(\mathcal{G}_1 \sqcup \mathcal{G}_2) \cong A_R(\mathcal{G}_1) \oplus A_R(\mathcal{G}_2)$.*

Proof. Let $I_1 = \{f_1 \in A_R(\mathcal{G}_1 \sqcup \mathcal{G}_2) \mid \operatorname{supp} f_1 \subseteq \mathcal{G}_1\}$ and $I_2 = \{f_2 \in A_R(\mathcal{G}_1 \sqcup \mathcal{G}_2) \mid \operatorname{supp} f_2 \subseteq \mathcal{G}_2\}$. Recall from Remark 1.16 that $I_1 \cong A_R(\mathcal{G}_1)$ and $I_2 \cong A_R(\mathcal{G}_2)$. Every $f \in A_R(\mathcal{G}_1 \sqcup \mathcal{G}_2)$ decomposes as $f = f_1 + f_2$ where $f_i \in I_i$ are defined as:

$$f_i(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{G}_i \\ 0 & \text{if } x \notin \mathcal{G}_i \end{cases}$$

for $i = 1, 2$. We claim I_1 and I_2 are orthogonal ideals (that is, $I_1 * I_2 = 0$). For all $f_1 \in I_1$, $f_2 \in I_2$, and $x \in \mathcal{G}_1 \sqcup \mathcal{G}_2$, $f_1 * f_2(x) = \sum_{ab=x} f_1(a) f_2(b)$. So, $\operatorname{supp}(f_1 * f_2) \subseteq \operatorname{supp}(f_1)\operatorname{supp}(f_2) \subseteq \mathcal{G}_1\mathcal{G}_2 = \emptyset$. This implies I_1 and I_2 are ideals, and $A_R(\mathcal{G}_1 \sqcup \mathcal{G}_2) = I_1 \oplus I_2 \cong A_R(\mathcal{G}_1) \oplus A_R(\mathcal{G}_2)$. □

By mathematical induction, the Steinberg algebra of a finite disjoint union of ample groupoids is isomorphic to the direct sum of their respective Steinberg algebras.

Like in [2, Notation 2.6.3], we have reasons to consider matrix rings of a slightly more general nature than usual.

Definition 1.24 (Matrix rings)**.** Let A be a ring (not necessarily commutative or unital). If n is a positive integer, we write $M_n(A)$ for the ring of $n \times n$ matrices with entries in A. If Λ is a set (not necessarily finite) we define $M_\Lambda(A)$ to be the ring of square matrices, with rows and columns indexed by Λ, having entries in A and only finitely many nonzero entries.

Note that $M_\Lambda(A)$ is the direct limit of the finite-sized matrix rings associated to finite subsets of Λ. Also, $M_\Lambda(A)$ is unital if and only if A is unital and Λ is finite. The notation $[a_{ij}]$ stands for the matrix in $M_n(A)$, or $M_\Lambda(A)$, with a_{ij} in its (i,j)-entry. Let $\mathcal{N} = \{1, \dots, n\}^2$ be the transitive principal groupoid on n elements, with the discrete topology, as seen in Example 1.3 (c).

Proposition 1.25. *If $\mathcal{G}$ is a Hausdorff ample groupoid, then $A_R(\mathcal{N} \times \mathcal{G}) \cong M_n(A_R(\mathcal{G}))$.*

Proof. Define the map $F : A_R(\mathcal{N} \times \mathcal{G}) \to M_n(A_R(\mathcal{G}))$:

$$F(f) = [f_{ij}], \qquad \text{where } f_{ij}(x) = f(i,j,x) \text{ for all } f \in A_R(\mathcal{N} \times \mathcal{G}), (i,j) \in \mathcal{N}, \text{ and } x \in \mathcal{G}.$$

If $f \in A_R(\mathcal{N} \times \mathcal{G})$, then f is compactly supported and locally constant, so its restriction to a clopen subset, namely $\{(i,j)\} \times \mathcal{G}$, where $(i,j) \in \mathcal{N}$, is also compactly supported and locally constant. Therefore $f_{i,j} \in A_R(\mathcal{G})$ for all $(i,j) \in \mathcal{N}$. Clearly, F is bijective. Now, let $f, g \in A_R(\mathcal{N} \times \mathcal{G})$. For all $(i,j) \in \mathcal{N}$ and $x \in \mathcal{G}$, the convolution formula yields

$$\begin{aligned}(f * g)_{ij}(x) = f * g(i,j,x) &= \sum_{\substack{(k,\ell,y) \in \mathcal{N} \times \mathcal{G} \\ (\ell, \boldsymbol{d}(y)) = (j, \boldsymbol{d}(x))}} f\big[(i,j,x)(k,\ell,y)^{-1}\big] g(k,\ell,y) \\ &= \sum_{1 \le k \le N} \sum_{\substack{y \in \mathcal{G} \\ \boldsymbol{d}(y) = \boldsymbol{d}(x)}} f(i,k,xy^{-1}) g(k,j,y) = \sum_{1 \le k \le n} f_{ik} * g_{kj}(x)\end{aligned}$$

This shows $F(f * g) = F(f)F(g)$, so F is an isomorphism. □

Remark 1.26. As a specialisation of Proposition 1.25, we obtain $A_R(\mathcal{N}) \cong M_n(R)$. It is well-known that when A is an R-algebra, $M_n(A) \cong M_n(R) \otimes_R A$ (see [23, Example 4.22]). Thus, $A_R(\mathcal{N} \times \mathcal{G}) \cong A_R(\mathcal{N}) \otimes_R A_R(\mathcal{G})$. It is also well-known (see [23, Example 4.20]) that if G and H are groups, then $R(G \times H) \cong RG \otimes_R RH$ or, in our notation, $A_R(G \times H) \cong A_R(G) \otimes_R A_R(H)$. One can show using the standard techniques that when $\mathcal{G}_1$ and $\mathcal{G}_2$ are arbitrary ample groupoids, there is a surjective homomorphism $A_R(\mathcal{G}_1) \otimes_R A_R(\mathcal{G}_2) \to A_R(\mathcal{G}_1 \times \mathcal{G}_2)$. An interesting question is: under what circumstances is it an isomorphism?

Suppose $\mathcal{G}$ is a topological groupoid and $\{\mathcal{G}_i\}_{i\in I}$ is a family of open subgroupoids indexed by a directed set $(I, \leq)$, such that $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$ and $\mathcal{G}_i \subseteq \mathcal{G}_j$ whenever $i \leq j$ in I. If this happens, we say that $\mathcal{G}$ is the *directed union* of the subgroupoids $\{\mathcal{G}_i\}_{i\in I}$.

Proposition 1.27. *If a Hausdorff ample groupoid $\mathcal{G}$ is the directed union of a family of open subgroupoids $\{\mathcal{G}_i\}_{i\in I}$, then $A_R(\mathcal{G})$ is the direct limit of subalgebras $\{A_R(\mathcal{G}_i)\}_{i\in I}$.*

Proof. For all $i \leq j$ in I, let $\varphi_{ij} : A_R(\mathcal{G}_i) \hookrightarrow A_R(\mathcal{G}_j)$ and $m_i : A_R(\mathcal{G}_i) \hookrightarrow A_R(\mathcal{G})$ be the canonical embeddings (see Remark 1.16, which applies here because $\mathcal{G}_i$ is an open subgroupoid of $\mathcal{G}$). We claim that for every $f \in A_R(\mathcal{G})$, there exists $j \in I$ such that $f \in m_j(A_R(\mathcal{G}_j))$. If $f \in A_R(\mathcal{G})$ then $\operatorname{supp} f$ is compact and open. Thus, there is a finite subcover of $\{\mathcal{G}_i\}_{i\in I}$ that covers $\operatorname{supp} f$. If $\operatorname{supp} f \subseteq \mathcal{G}_{i_1} \cup \cdots \cup \mathcal{G}_{i_n}$, then there exists $j \in I$ with $i_1, \ldots, i_n \leq j$, using the fact that $(I, \leq)$ is directed. Thus, $\operatorname{supp} f \subseteq \mathcal{G}_j$, and $f|_{\mathcal{G}_j}$ is compactly supported and locally constant, whereby $f|_{\mathcal{G}_j} \in A_R(\mathcal{G}_j)$. Finally, this shows $f = m_j(f|_{\mathcal{G}_j}) \in m_j(A_R(\mathcal{G}_j))$.

Now assume B is an R-algebra and $\{\beta_i\}_{i\in I}$ is a family of R-homomorphisms $\beta_i : A_R(\mathcal{G}_i) \to B$, such that $\beta_i = \beta_j \varphi_{ij}$ for all $i \leq j$. Then, since every $\varphi_{ij} : A_R(\mathcal{G}_i) \to A_R(\mathcal{G}_j)$ is injective, β_j is an extension of β_i whenever $i \leq j$. Since $A_R(\mathcal{G}) = \bigcup_{i\in I} m_i(A_R(\mathcal{G}_i))$, it follows that there is a unique homomorphism $\beta : A_R(\mathcal{G}) \to B$ such that $\beta_i = \beta m_i$ for all $i \in I$. As such, $A_R(\mathcal{G})$ has the universal property for the directed system $\{A_R(\mathcal{G}_i)\}_{i\in I}$, so we can conclude it is the direct limit of that system. □

We can now extend Propositions 1.23 and 1.25 to allow infinite index sets. This could have been proved directly, mentioning that the functions in $A_R(\mathcal{G})$ have compact supports, but it is nice to demonstrate direct limits.

Proposition 1.28. *Let $\mathcal{G}$ be a Hausdorff ample groupoid, and let Λ be an infinite set.*

(1) *If $\mathcal{D} = \Lambda^2$ is the transitive principal groupoid on Λ, equipped with the discrete topology, then $A_R(\mathcal{D} \times \mathcal{G}) \cong M_\Lambda(A_R(\mathcal{G}))$.*

(2) *If $\mathcal{G} = \bigsqcup_{\lambda\in\Lambda} \mathcal{G}_\lambda$ is the disjoint union of an infinite family of clopen subgroupoids $\{\mathcal{G}_\lambda\}_{\lambda\in\Lambda}$, then $A_R(\mathcal{G}) \cong \bigoplus_{\lambda\in\Lambda} A_R(\mathcal{G}_\lambda)$.*

Proof. (1) Note that $\mathcal{D}\times\mathcal{G}$ is the directed union of the subgroupoids $\mathcal{D}_F \times \mathcal{G}$, as F ranges over all the finite subsets of Λ ordered by inclusion, where $\mathcal{D}_F = \{(d_1, d_2) \in \mathcal{D} \mid d_1, d_2 \in F\}$. By Propositions 1.25 and 1.27, $A_R(\mathcal{D} \times \mathcal{G})$ is the direct limit of matrix algebras $A_R(\mathcal{D}_F \times \mathcal{G}) \cong M_F(A_R(\mathcal{G}))$, which is isomorphic to $M_\Lambda(A_R(\mathcal{G}))$.

(2) Note that $\mathcal{G}$ is the directed union of the subgroupoids $\mathcal{G}_F = \bigsqcup_{\lambda\in F} \mathcal{G}_\lambda$, as F ranges over finite subsets of Λ ordered by inclusion. By Propositions 1.23 and 1.27, $A_R(\mathcal{G})$ is the direct limit of the subalgebras $A_R(\mathcal{G}_F) \cong \bigoplus_{\lambda\in F} A_R(\mathcal{G}_\lambda)$, which is isomorphic to $\bigoplus_{\lambda\in\Lambda} A_R(\mathcal{G}_\lambda)$. □

Here we describe a class of principal groupoids, called *approximately finite* groupoids, that was defined by Renault in his influential monograph [73].

Example 1.29. Let X be a locally compact, totally disconnected Hausdorff space. Consider it as a groupoid with unit space X and no morphisms outside the unit space. Then $A_R(X)$ is the commutative R-algebra of locally constant, compactly supported functions $f : X \to R$, with pointwise addition and multiplication. We adopt the notation $A_R(X) = C_R(X)$ and drop the $*$, because this serves as a reminder that $C_R(X)$ is commutative. An ample groupoid is called **elementary** if it is of the form $(\mathcal{N}_1 \times X_1) \sqcup \cdots \sqcup (\mathcal{N}_t \times X_t)$, where $\mathcal{N}_1, \ldots \mathcal{N}_t$ are discrete, finite, transitive principal groupoids on $n_1, \ldots, n_t$ elements, respectively, and $X_1, \ldots, X_n$ are locally compact, totally disconnected, Hausdorff topological spaces. Using the results of this section:

$$A_R \left(\bigsqcup_{i=1}^{n} (\mathcal{N}_i \times X_i) \right) \cong \bigoplus_{i=1}^{t} M_{n_i}(C_R(X_i)). \tag{1.5}$$

A groupoid is called **approximately finite** if it is the directed union of an increasing sequence of elementary groupoids. The Steinberg algebra of an approximately finite groupoid is a direct limit of matricial algebras, each resembling (1.5).

Definition 1.30. A ring A is called *von Neumann regular* if for every $x \in A$ there exists $y \in A$ such that $x = xyx$.

If $y \in A$ satisfies $x = xyx$ then y is called a *von Neumann inverse* of x. At the time of writing, it is an open problem to determine when a Steinberg algebra is von Neumann regular. Partial progress is achieved in [12]. The next theorem is new, to our knowledge, although the techniques are standard. We use the following fact, from [46, Proposition 3.6]: if R is a commutative von Neumann regular ring, then for every $r \in R$ there exists a unique element $s \in R$ such that $r = r^2 s$ and $s = s^2 r$.

Theorem 1.31. *If $\mathcal{F}$ is an approximately finite groupoid and R is a von Neumann regular unital commutative ring, then $A_R(\mathcal{F})$ is von Neumann regular.*

Proof. Let X be a locally compact, totally disconnected, Hausdorff topological space, and suppose R is von Neumann regular. To verify that $C_R(X)$ is von Neumann regular, take $f \in C_R(X)$ and for every $x \in X$ define $g(x)$ to be the unique element of R such that $f(x) = f(x)^2 g(x)$ and $g(x) = g(x)^2 f(x)$. Note that $g \in C_R(X)$ and $fgf = f$. Now, $C_R(X)$ being regular implies $M_n(C_R(X))$ is regular (this could be argued carefully with Morita equivalence, but one finds in [59, Theorem 24] a clever direct proof by induction). A direct sum of regular rings is regular, so any ring of the form (1.5) is regular, provided R is regular. A direct limit of regular rings is regular: each element in the direct limit must belong to a regular subring, and the von Neumann inverse can be chosen from that same subring. Therefore $A_R(\mathcal{F})$ is von Neumann regular. □

Note that we did not use the assumption that $\mathcal{F}$ is a countable directed union of elementary groupoids; any directed union will do.

This next result is a "baby version" of [80, Proposition 3.1], with an original proof. In preparation for it, we briefly remark that every transitive groupoid $\mathcal{G}$ is isomorphic to the product of a transitive principal groupoid and a group. To construct such an isomorphism, fix a unit $b \in \mathcal{G}^{(0)}$. Let $\Gamma = {}^b\mathcal{G}^b$ be the isotropy group based at b, and let $\mathcal{P} = [\mathcal{G}^{(0)}]^2$ be the transitive principal groupoid on $\mathcal{G}^{(0)}$. Fix a morphism $h_y \in {}^b\mathcal{G}^y$ for every $y \in \mathcal{G}^{(0)}$, and define

$$\begin{aligned} &F : \mathcal{G} \to \mathcal{P} \times \Gamma, && F(g) = \Big(\big(\boldsymbol{c}(g), \boldsymbol{d}(g) \big), h_{\boldsymbol{c}(g)} g h_{\boldsymbol{d}(g)}^{-1} \Big) && \text{for all } g \in \mathcal{G}; \\ &F^{-1} : \mathcal{P} \times \Gamma \to \mathcal{G}, && F^{-1}\big((x, y), \gamma \big) = h_x^{-1} \gamma h_y && \text{for all } x, y \in \mathcal{G}^{(0)}, \gamma \in \Gamma. \end{aligned}$$

This is an isomorphism of groupoids. Note, however, that if $\mathcal{G}$ is a topological groupoid, it might be impossible to find a *topological* isomorphism between $\mathcal{G}$ and $\mathcal{P} \times \Gamma$.

Proposition 1.32. *Let K be a field and $\mathcal{G}$ an ample groupoid. Then $A_K(\mathcal{G})$ is finite-dimensional if and only if $\mathcal{G}$ is finite and has the discrete topology. If $\mathcal{O}_1, \ldots, \mathcal{O}_t$ are the orbits of $\mathcal{G}$, and $\Gamma_1, \ldots, \Gamma_t$ are the corresponding isotropy groups, then $A_K(\mathcal{G}) \cong \bigoplus_{i=1}^{t} M_{\mathcal{O}_i}(R\Gamma_i)$.*

Proof. First of all, if $\mathcal{G}$ is discrete, then $\dim_K A_K(\mathcal{G}) = |\mathcal{G}|$, because $\{\mathbf{1}_{\{g\}} \mid g \in \mathcal{G}\}$ is a basis for $A_K(\mathcal{G})$, by Corollary 1.14. Thus, $A_K(\mathcal{G})$ is finite-dimensional if $\mathcal{G}$ is finite and discrete. Conversely, suppose $A_K(\mathcal{G})$ is finite-dimensional, and let $\{f_1, \dots, f_n\}$ be a basis. The image of each f_i is finite, so $|\operatorname{im} f_1 \cup \cdots \cup \operatorname{im} f_n|$ is bounded by some $M < \infty$. If $|\mathcal{G}^{(0)}| > M^n$ then, by the pigeonhole principle, there exists $u \neq v$ in $\mathcal{G}^{(0)}$ such that $f_i(u) = f_i(v)$ for all $1 \leq i \leq n$, and thus $f(u) = f(v)$ for all $f \in A_K(\mathcal{G})$. But $\mathcal{G}^{(0)}$ is Hausdorff, locally compact, and totally disconnected, so there is a compact open subset $U \subseteq \mathcal{G}^{(0)}$ with $u \in U$ and $v \notin U$. Since $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$, it follows that U is a compact open bisection in $\mathcal{G}$, so $\mathbf{1}_U \in A_K(\mathcal{G})$. We arrive at a contradiction, because $\mathbf{1}_U(u) \neq \mathbf{1}_U(v)$. Therefore $|\mathcal{G}^{(0)}| \leq M^n < \infty$. A finite Hausdorff space is discrete, so $\mathcal{G}^{(0)}$ is discrete. As $\mathcal{G}$ is étale, it must also be discrete. Thus $\dim_K A_K(\mathcal{G}) = n = |\mathcal{G}|$. Given that $\mathcal{G}$ is finite and discrete, it is isomorphic to a disjoint union of transitive groupoids (one for each orbit), each of which is isomorphic to the product of a transitive principal groupoid (with as many elements as the corresponding orbit), and a finite group (the isotropy group of that orbit). The expression giving the structure of $A_K(\mathcal{G})$ follows from Propositions 1.23 and 1.25. $\square$

1.6 Graded groupoids and graded Steinberg algebras

Just as the Steinberg algebra of a groupoid inherits an involution from the groupoid, so it can inherit a graded structure. Many well-studied examples of Steinberg algebras receive a canonical group-grading that comes from a grading on the groupoid itself. We first introduce the concepts and terminology of graded algebras.

A standing assumption is that Γ is a group with identity ε. A ring A is called a Γ-*graded ring* if it decomposes as a direct sum of additive subgroups $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ such that $A_\gamma A_\delta \subseteq A_{\gamma\delta}$ for every $\gamma, \delta \in \Gamma$. The meaning of $A_\gamma A_\delta$ is the additive subgroup generated by all products ab where $a \in A_\gamma, b \in A_\delta$. The additive group A_γ is called the γ-*component* of A. The elements of $\bigcup_{\gamma \in \Gamma} A_\gamma$ in a graded ring A are called *homogeneous elements*. The nonzero elements of A_γ are called γ-*homogeneous*, and we write $\deg(a) = \gamma$ for $a \in A_\gamma \setminus \{0\}$. When it is clear from context that a ring A is graded by the group Γ, we simply say that A is a *graded ring*. If A is an R-algebra, then A is called a *graded algebra* if it is a graded ring and each A_γ is an R-submodule.

An ideal $I \subseteq A$ is a *graded ideal* if $I \subseteq \sum_{\gamma \in \Gamma} I \cap A_\gamma$. Graded left ideals, graded right ideals, graded subrings, and graded subalgebras are defined in a similar manner. If H is a set of homogeneous elements in A, the ideal generated by H is a graded ideal. Likewise, the left and right ideals generated by H are graded. A *graded homomorphism* of Γ-graded rings is a homomorphism $f : A \to B$ such that $f(A_\gamma) \subseteq B_\gamma$ for every $\gamma \in \Gamma$. A Γ-graded ring A is called *strongly* Γ-*graded*, or just *strongly graded*, if $A_\gamma A_\delta = A_{\gamma\delta}$ for all γ, δ in Γ. Finally, we say that a Γ-graded ring A has *homogeneous local units* if A is locally unital, and the set of local units can be chosen to be a subset of A_ε.

Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}$ is called Γ-*graded* if it can be partitioned by clopen subsets $\mathcal{G} = \bigsqcup_{\gamma \in \Gamma} \mathcal{G}_\gamma$, such that $\mathcal{G}_\gamma \mathcal{G}_\delta \subseteq \mathcal{G}_{\gamma\delta}$ for every $\gamma, \delta \in \Gamma$. If a continuous homomorphism $\kappa : \mathcal{G} \to \Gamma$ defines the grading by $\mathcal{G}_\gamma = \kappa^{-1}(\gamma)$ then we call it a *degree map*. (In the literature, κ is often called a 1-*cocycle*.) In analogy with graded rings, we say that the graded groupoid $\mathcal{G}$ is *strongly graded* if $\mathcal{G}_\gamma \mathcal{G}_\delta = \mathcal{G}_{\gamma\delta}$ for every $\gamma, \delta \in \mathcal{G}$. Strongly graded groupoids appeared in [14, Definition 5.3.7] and [16] but otherwise it is a fairly new concept.

We use the notation $\mathcal{G}_\gamma^x = \mathcal{G}^x \cap \mathcal{G}_\gamma$ and ${}^x\mathcal{G}_\gamma = {}^x\mathcal{G} \cap \mathcal{G}_\gamma$ for $x \in \mathcal{G}^{(0)}$ and $\gamma \in \Gamma$. We say a subset $X \subseteq \mathcal{G}$ is γ-homogeneous if $X \subseteq \mathcal{G}_\gamma$. Obviously, the unit space is ε-homogeneous and if X is γ-homogeneous then X^{-1} is γ^{-1}-homogeneous. Moreover, ${\mathcal{G}_\gamma}^{-1} = \mathcal{G}_{\gamma^{-1}}$ for all $\gamma \in \Gamma$. For a Γ-graded ample groupoid, we write $B^{\mathrm{co}}_\gamma(\mathcal{G})$ for the set of all γ-homogeneous compact open bisections of $\mathcal{G}$. The set of all homogeneous compact open bisections is:

$$B^{\mathrm{co}}_*(\mathcal{G}) = \bigcup_{\gamma\in\Gamma} B^{\mathrm{co}}_\gamma(\mathcal{G}) \subseteq B^{\mathrm{co}}(\mathcal{G}).$$

In Proposition 1.10, we proved that $B^{\mathrm{co}}(\mathcal{G})$ is an inverse semigroup, and it is readily apparent that $B^{\mathrm{co}}_*(\mathcal{G})$ is an inverse subsemigroup of $B^{\mathrm{co}}(\mathcal{G})$. In addition, $B^{\mathrm{co}}_*(\mathcal{G})$ is a base of compact open bisections for $\mathcal{G}$. Indeed, since $B^{\mathrm{co}}(\mathcal{G})$ is a base for $\mathcal{G}$, it suffices to show that every $B \in B^{\mathrm{co}}(\mathcal{G})$ is a union of sets in $B^{\mathrm{co}}_*(\mathcal{G})$. This is almost trivial, for if $B \in B^{\mathrm{co}}(\mathcal{G})$ then $B = \bigcup_{\gamma\in\Gamma} B \cap \mathcal{G}_\gamma$ and $B \cap \mathcal{G}_\gamma \in B^{\mathrm{co}}_\gamma(\mathcal{G})$. The next two results are from [34, Lemma 3.1].

Proposition 1.33. *If $\mathcal{G} = \bigsqcup_{\gamma\in\Gamma} \mathcal{G}_\gamma$ is a Γ-graded ample groupoid, then $A_R(\mathcal{G}) = \bigoplus_{\gamma\in\Gamma} A_R(\mathcal{G})_\gamma$ is a Γ-graded algebra with homogeneous local units, where:*

$$A_R(\mathcal{G})_\gamma = \{f \in A_R(\mathcal{G}) \mid \operatorname{supp} f \subseteq \mathcal{G}_\gamma\} \qquad\qquad \text{for all } \gamma \in \Gamma.$$

Proof. From Proposition 1.13, it follows that

$$A_R(\mathcal{G}) = \operatorname{span}_R\{\mathbf{1}_B \mid B \in B^{\mathrm{co}}_*(\mathcal{G})\} = \sum_{\gamma\in\Gamma} \operatorname{span}_R\{\mathbf{1}_B \mid B \in B^{\mathrm{co}}_\gamma(\mathcal{G})\} = \sum_{\gamma\in\Gamma} A_R(\mathcal{G})_\gamma.$$

It is clear that $A_R(\mathcal{G})_\gamma \cap \big(\sum_{\delta\neq\gamma} A_R(\mathcal{G})_\delta\big) = \{0\}$ for all $\gamma \in \Gamma$, so we have $A_R(\mathcal{G}) = \bigoplus_{\gamma\in\Gamma} A_R(\mathcal{G})_\gamma$. Now for all $f \in A_R(\mathcal{G})_\gamma$ and $g \in A_R(\mathcal{G})_\delta$, we have $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f)\operatorname{supp}(g) \subseteq \mathcal{G}_\gamma\mathcal{G}_\delta \subseteq \mathcal{G}_{\gamma\delta}$, and thus $f * g \in A_R(\mathcal{G})_{\gamma\delta}$. Therefore $A_R(\mathcal{G})_\gamma * A_R(\mathcal{G})_\delta \subseteq A_R(\mathcal{G})_{\gamma\delta}$. It follows from Proposition 1.20, and the fact that $\mathcal{G}^{(0)} \subseteq \mathcal{G}_\varepsilon$, that $A_R(\mathcal{G})$ has homogeneous local units. □

Lemma 1.34. *If $\mathcal{G}$ is a Γ-graded Hausdorff ample groupoid, every $f \in A_R(\mathcal{G})$ can be expressed as a finite sum $f = \sum_{i=1}^n r_i\mathbf{1}_{B_i}$ where $r_1, \dots, r_n \in R$ and $B_1, \dots, B_n \in B^{\mathrm{co}}_*(\mathcal{G})$ are mutually disjoint.*

Proof. Since $\mathcal{G}$ is Hausdorff, every homogeneous compact open bisection is closed, so $B^{\mathrm{co}}_*(\mathcal{G})$ is closed under finite intersections and relative complements. The statement now follows from Proposition 1.19. □

Example 1.35. Recall, from Example 1.3 (d), the definition of the transformation groupoid $G \times X$, associated to a group G and a G-set X. Now assume that X is a locally compact, totally disconnected, Hausdorff topological space, and for each $g \in G$ the map $\rho_g : X \to X$, $\rho_g(x) = g \cdot x$, is a homeomorphism. If we assign the discrete topology to G and the product topology to $G \times X$, then $G \times X$ is an ample groupoid. It is easy to verify that this is a G-graded groupoid with homogeneous components $(G \times X)_g = \{g\} \times X$ for all $g \in G$. The Steinberg algebra of $G \times X$ turns out (see [22]) to be the **skew group ring** $C_R(X) \star G$, associated to a certain action of G on $C_R(X)$, canonically induced by the action of G on X.

One can generalise this example quite profitably, by replacing the group action with something more general called a *partial group action* (see [44, Definition 2.1]). In doing so, one obtains a class of algebras so general that it includes all Leavitt path algebras (see [45, Theorem 3.3]) and other interesting things, like the *partial group algebras* that were studied in [41] and [51].

Chapter 2

Graphs and groupoids

In this chapter, we define and study the *boundary path groupoid* of a graph. This groupoid was introduced in its earliest form, for row-finite graphs without sinks, by Kumjian, Pask, Raeburn, and Renault in [61]. It bears a resemblance to a groupoid studied a few years earlier by Deaconu in [40]. The construction was later generalised in a number of different directions, taking a route through inverse semigroup theory [70], and going as far as topological higher-rank graphs (e.g. [60, 75, 86]). Perhaps as an artefact of its history, many fundamental properties of the boundary path groupoid were absorbed into folklore. Some proofs were never written, and others were written at a higher level of generality, and not all in one place, making them difficult to relate back to our present needs.

For instance, when this project began, we could not find a proof that the boundary path groupoid is an ample groupoid, even though this fact was used in all the early papers, [30, 31, 34], that pioneered the use of groupoid methods for Leavitt path algebras. The groupoid approach to Leavitt path algebras is particularly well-suited, compared to traditional, purely algebraic methods, for dealing with graphs of large cardinalities. It was important to make sure that the theorems used to justify these methods could be proved without assuming graphs are countable.

The purpose of §2.1 is to define directed graphs and introduce some terminology. In §2.2, we introduce a topological space called the *path space* of a graph. The path space of a graph is the set of all finite and infinite paths, with a topology described explicitly by a base of open sets. It was previously studied in [84] for its applications to graph C^*-algebras. Generalising [84, Theorem 2.1], we prove in Theorem 2.4 that for graphs of any cardinality, the path space is locally compact and Hausdorff. We also determine which graphs have a second-countable, first-countable, or σ-compact path space. In §2.3, we use the path space (or more precisely, a closed subspace called the boundary path space) to define the *boundary path groupoid* associated to a graph. This groupoid is an intermediate step towards proving that all Leavitt path algebras are Steinberg algebras, and it becomes an important tool for the analysis of Leavitt path algebras.

2.1 Graph concepts I

In this section, we introduce the necessary terminology and conventions pertaining to graphs. We always use the word graph to mean a *directed* graph, defined as follows.

Definition 2.1. A **graph** is a system $E = (E^0, E^1, r, s)$, where E^0 is a set whose elements are called *vertices*, E^1 is a set whose elements are called *edges*, $r : E^1 \to E^0$ is a map that associates a *range* to every edge, and $s : E^1 \to E^0$ is a map that associates a *source* to every edge.

A *countable graph* is one where E^0 and E^1 are countable sets. A *row-finite* (resp., *row-countable*) graph is one in which $s^{-1}(v)$ is finite (resp., countable) for every $v \in E^0$. If e is an edge with $s(e) = v$ and $r(e) = w$ then we say that v *emits* e and w *receives* e. A *sink* is a vertex that emits no edges and an *infinite emitter* is a vertex that emits infinitely many edges. If $v \in E^0$ is either a sink or an infinite emitter (that is, $s^{-1}(v)$ is either empty or infinite) then it is called *singular*, and if v is not singular then it is called *regular*. A vertex that neither receives nor emits any edges is called an *isolated vertex*.

A *finite path* is a finite sequence of edges $\alpha = \alpha_1\alpha_2\ldots\alpha_n$ such that $r(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \ldots, n-1$. The *length* of the path α is $|\alpha| = n$. Reusing notation and terminology, we shall say that $s(\alpha) = s(\alpha_1)$ is the *source* of the path, and $r(\alpha) = r(\alpha_n)$ is the *range* of the path. By convention, vertices $v \in E^0$ are regarded as finite paths of zero length, with $r(v) = s(v) = v$. If $v, w \in E^0$, we write $v \geq w$ if there exists a finite path α with $s(\alpha) = v$ and $r(\alpha) = w$. If a finite path α of positive length satisfies $r(\alpha) = s(\alpha) = v$, then α is called a *closed path* based at v. A closed path α with the property that none of the vertices $s(\alpha_1), \ldots, s(\alpha_{|\alpha|})$ are repeated is called a *cycle*, and a graph that has no cycles is called *acyclic*. An *exit* for a finite path α is an edge $f \in E^1$ with $s(f) = s(\alpha_i)$ for some $1 \leq i \leq |\alpha|$, but $f \neq \alpha_i$.

An *infinite path* is, predictably, an infinite sequence of edges $p = p_1p_2p_3\ldots$ such that $r(p_i) = s(p_{i+1})$ for $i = 1, 2, \ldots$. Again, $s(p) = s(p_1)$ is called the source of the infinite path p. We let $|p| = \infty$ if p is an infinite path. We use the notation $E^\star$ for the set of finite paths, and E^∞ for the set of infinite paths.

Paths can be concatenated if their range and source agree. If $\alpha, \beta \in E^\star$ have positive length and $r(\alpha) = s(\beta)$, then $\alpha\beta = \alpha_1\ldots\alpha_{|\alpha|}\beta_1\ldots\beta_{|\beta|} \in E^\star$. If $p \in E^\infty$ has $r(\alpha) = s(p)$, then $\alpha p = \alpha_1\ldots\alpha_{|\alpha|}p_1p_2\ldots \in E^\infty$. If $v \in E^0$ and $x \in E^\star \cup E^\infty$ has $s(x) = v$, then $vx = x$ by convention. Likewise, if $\alpha \in E^\star$ has $r(\alpha) = v$ then $\alpha v = \alpha$. If $\alpha \in E^\star$, $x \in E^\star \cup E^\infty$, and $x = \alpha x'$ for some $x' \in E^\star \cup E^\infty$, then we say that α is an *initial subpath* of x.

Let $E^0_{\text{sing}} = \{v \in E^0 \mid v \text{ is singular}\}$ and $E^0_{\text{reg}} = \{v \in E^0 \mid v \text{ is regular}\}$. Using the terminology of [84], we define the set of *boundary paths* as

$$\partial E = E^\infty \cup \left\{\alpha \in E^\star \mid r(\alpha) \in E^0_{\text{sing}}\right\}.$$

We employ the following notation from now on:

$$vE^1 = \{e \in E^1 \mid s(e) = v\}; \qquad vE^\star = \{\alpha \in E^\star \mid s(\alpha) = v\};$$
$$vE^\infty = \{p \in E^\infty \mid s(p) = v\}; \qquad v\partial E = \{x \in \partial E \mid s(x) = v\};$$
$$E^\star \times_r E^\star = \left\{(\alpha, \beta) \in E^\star \times E^\star \mid r(\alpha) = r(\beta)\right\}.$$

2.2 The path space of a graph I

We now begin the process of modelling Leavitt path algebras as Steinberg algebras. The first step in this process is "topologising" what is known as the path space of a graph. Throughout this section and the next one, assume $E = (E^0, E^1, r, s)$ is an arbitrary graph. The *path space* of E is the space $E^\star \cup E^\infty$ of all finite and infinite paths. The *boundary path space* is the subspace $\partial E \subseteq E^\star \cup E^\infty$.

For a finite path $\alpha \in E^\star$ we define the *cylinder* set

$$C(\alpha) = \{\alpha x \mid x \in E^\star \cup E^\infty, r(\alpha) = s(x)\}.$$

It is easy to see that the intersection of two cylinders is either empty or a cylinder. Indeed, if $x \in C(\alpha) \cap C(\beta)$ then $x = \alpha y = \beta z$ for some $y, z \in E^\star \cup E^\infty$. If $|\alpha| \leq |\beta|$ then α is an initial subpath of β, implying $C(\beta) \subseteq C(\alpha)$. In symbols:

$$C(\alpha) \cap C(\beta) = \begin{cases} C(\beta) & \text{if } \alpha \text{ is an initial subpath of } \beta \\ C(\alpha) & \text{if } \beta \text{ is an initial subpath of } \alpha \\ \emptyset & \text{otherwise.} \end{cases}$$

This is all we need to conclude that the collection of cylinder sets is a base for a topology on $E^\star \cup E^\infty$. As the authors of [61] have stated, the subspace $E^\infty \subseteq E^\star \cup E^\infty$ with the cylinder set topology is homeomorphic (in the canonical way) to a subspace of $\prod_{n=1}^\infty E^1$, where E^1 is discrete and the product has the product topology. In particular, the cylinder sets generate a Hausdorff topology on E^∞, and if E is row-finite, that topology is locally compact. However, the cylinder set topology is not Hausdorff (or even T_1) on the whole set $E^\star \cup E^\infty$, because a finite path cannot be separated from a proper initial subpath. In order to have enough open sets in hand for a Hausdorff topology, we define a base of open sets called *generalised cylinder* sets:

$$C(\alpha, F) = C(\alpha) \setminus \bigcup_{e \in F} C(\alpha e); \qquad \alpha \in E^\star,\ F \subseteq r(\alpha)E^1 \text{ is finite.} \tag{2.1}$$

We shall write $F \subseteq_{\text{finite}} vE^1$ to mean that F is a finite subset of vE^1. The next lemma (a generalisation of [61, Lemma 2.1]) shows that the collection of generalised cylinders is closed under intersections, so it is a base for a topology on $E^\star \cup E^\infty$. With the generalised cylinder set topology on $E^\star \cup E^\infty$, every finite path is an isolated point unless its range is an infinite emitter.

Lemma 2.2. *If $\alpha, \beta \in E^\star$, $|\alpha| \leq |\beta|$, $F \subseteq_{\text{finite}} r(\alpha)E^1$, and $H \subseteq_{\text{finite}} r(\beta)E^1$, then*

$$C(\alpha, F) \cap C(\beta, H) = \begin{cases} C(\beta, F \cup H) & \text{if } \beta = \alpha \\ C(\beta, H) & \text{if } \exists\ \delta \in E^\star,\ \beta = \alpha\delta,\ |\delta| \geq 1, \text{ and } \delta_1 \notin F \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. By definition of $C(\alpha, F)$ and $C(\beta, H)$, we have

$$C(\alpha, F) \cap C(\beta, H) = C(\alpha) \cap C(\beta) \setminus \left(\bigcup_{e \in F} C(\alpha e) \cup \bigcup_{e \in H} C(\beta e) \right). \tag{2.2}$$

If $\beta = \alpha$, the right hand side of (2.2) is $C(\beta, F \cup H)$. If $\beta = \alpha\delta$ ($|\delta| \geq 1$) and $\delta_1 \notin F$ then $C(\beta) \cap C(\alpha) = C(\beta)$ does not meet $\bigcup_{e \in F} C(\alpha e)$, so the right hand side of (2.2) is $C(\beta, H)$. If $\beta = \alpha\delta$ and $\delta_1 \in F$, then $C(\beta) \cap C(\alpha) = C(\beta) = C(\alpha\delta_1 \ldots \delta_{|\delta|}) \subseteq C(\alpha\delta_1) \subseteq \bigcup_{e \in F} C(\alpha e)$, so the right hand side of (2.2) is empty. If α is not an initial subpath of β then $C(\alpha) \cap C(\beta) = \emptyset$. □

To apply Steinberg's theory from Chapter 1, it is critical that the induced topology on the boundary path space $\partial E \subseteq E^\star \cup E^\infty$ is locally compact and Hausdorff. We proceed by proving that the topology on the path space $E^\star \cup E^\infty$, generated by the base in (2.1), is locally compact and Hausdorff, and that ∂E is closed. As it were, this base is well-chosen: the basic open sets themselves are compact in the Hausdorff topology that they generate.

Recall that a topological space is called *second-countable* if it has a countable base, and *first-countable* if every point has a countable neighbourhood base. We were unable to find a reference in the literature for the following theorem.

Theorem 2.3. *The generalised cylinder sets (2.1) generate a Hausdorff topology on the path space $E^\star \cup E^\infty$. The path space is:*

(1) *second-countable if and only if E is a countable graph;*

(2) *first-countable if and only if E is a row-countable graph.*

Proof. Let $x, y \in E^\star \cup E^\infty$ and $x \neq y$. There are two cases to consider, to demonstrate the Hausdorff property. If $s(x) \neq s(y)$ then $C(s(x))$ and $C(s(y))$ are disjoint open sets separating x and y. If there exists some $t \leq \min\{|x|, |y|\}$ such that $x_t \neq y_t$, then $C(x_1 \dots x_t)$ and $C(y_1 \dots y_t)$ are disjoint open sets separating x and y. Otherwise, one of the paths is an initial subpath of the other. Without loss of generality, assume y is finite and $x = yx'$ for some $x' \in E^\star \cup E^\infty$ with $|x'| \geq 1$. Then $C(y, \{x'_1\})$ and $C(x)$ are disjoint open sets containing y and x respectively.

(1) If E is a countable graph (i.e., $E^0 \cup E^1$ is countable) then $E^\star$ is countable. The base of open sets (2.1) is countable too, because there are only countably many pairs (α, F) where $\alpha \in E^\star$ and $F \subseteq_{\text{finite}} r(\alpha)E^1$. This proves the topology is second-countable. Conversely, if one of E^0 or E^1 is uncountable, then one of $\{C(v) \mid v \in E^0\}$ or $\{C(e) \mid e \in E^1\}$ is an uncountable set of pairwise disjoint open sets, so $E^\star \cup E^\infty$ is not second-countable.

(2) Notice that the following sets are neighbourhood bases at $\alpha \in E^\star$ and $p \in E^\infty$ respectively:

$$\mathcal{N}_\alpha = \big\{C(\alpha, F) \mid F \subseteq_{\text{finite}} r(\alpha)E^1\big\}, \qquad \mathcal{N}_p = \big\{C(p_1 \dots p_m) \mid m \geq 1\big\}.$$

Regardless of the graph, $\mathcal{N}_p$ is countable for every $p \in E^\infty$. If a finite path $\alpha \in E^\star$ has the property that $r(\alpha)E^1$ is countable, then $\mathcal{N}_\alpha$ is countable, because there are only countably many finite subsets F of $r(\alpha)E^1$. So, for every row-countable graph E, the path space $E^\star \cup E^\infty$ is first-countable. Conversely, suppose there exists $v \in E^0$ such that vE^1 is uncountable. Towards a contradiction, assume v has a countable neighbourhood base $\mathcal{B}_v = \{B_1, B_2, \dots, \}$. By replacing B_n, for all $n \geq 1$, with a set of the form $C(v, F_n) \subseteq B_n$, where $F_n \subseteq_{\text{finite}} vE^1$, we have a countable neighbourhood base for v of the form $\mathcal{C}_v = \{C(v, F_1), C(v, F_2), \dots\}$. Since $\bigcup_{n=1}^\infty F_n$ is countable, one can choose $e \in vE^1 \setminus \bigcup_{n=1}^\infty F_n$. Then every neighbourhood of v contains e, which is absurd, because the space is Hausdorff. Therefore $E^\star \cup E^\infty$ is first-countable if and only if E is row-countable. $\square$

Next, we generalise a theorem of Webster [84, Theorem 2.1]. We remove the assumption that E is a countable graph, which was used in that earlier proof. The main idea is to embed $C(v)$ into the space of sequences whose terms belong to the one-point compactification of E^1. We thank Aidan Sims for suggesting this approach. Before proceeding, recall that if $(Y, \mathcal{T})$ is a locally compact, non-compact Hausdorff space, the *one-point compactification* of Y is the space $Y \cup \{\infty\}$, where ∞ is a formal symbol, with the topology:

$$\mathcal{T} \cup \big\{G \cup \{\infty\} \mid G \subseteq Y,\ Y \setminus G \text{ is compact}\big\}.$$

Theorem 2.4. *The collection (2.1) of generalised cylinder sets is a base of compact open sets for a locally compact Hausdorff topology on $E^\star \cup E^\infty$.*

Proof. The Hausdorff part has already been done in Theorem 2.3. We proceed by proving that the basic open sets are compact. Consider E^1 with the discrete topology. If E^1 is finite, let $E^1 \cup \{\infty\}$

be the discrete space with one extra point called ∞. Otherwise, let $E^1 \cup \{\infty\}$ be the one-point compactification of E^1. The product $X = \prod_{n=1}^{\infty}(E^1 \cup \{\infty\})$ is compact by Tychnoff's Theorem, and Hausdorff because products preserve the Hausdorff property. For $v \in E^0$, let X_v be the subspace of X consisting of sequences $(x_n)_1^\infty$ such that:

- $x_1 \in vE^1 \cup \{\infty\}$;
- if $x_n = \infty$ then $x_{n+1} = \infty$;
- if $x_n, x_{n+1} \in E^1$ then $r(x_n) = s(x_{n+1})$.

We claim that each X_v is closed in X. Let $(y_n)_1^\infty \in X \setminus X_v$. The aim is to show that there is an open set in X containing $(y_n)_1^\infty$ and disjoint from X_v. Write $p_n : X \to E^1 \cup \{\infty\}$ to mean the natural projection map onto the n-th term. If $y_1 \notin vE^1 \cup \{\infty\}$ then $p_1^{-1}(y_1)$ is open, contains $(y_n)_1^\infty$, and does not meet X_v. If there is some n such that $y_n = \infty$ and $y_{n+1} \neq \infty$ then $p_n^{-1}(\infty) \cap p_{n+1}^{-1}(y_{n+1})$ is open, contains $(y_n)_1^\infty$, and does not meet X_v. If there is some n such that $y_n, y_{n+1} \in E^1$ but $r(y_n) \neq s(y_{n+1})$ then $p_n^{-1}(y_n) \cap p_{n+1}^{-1}(y_{n+1})$ is open, contains $(y_n)_1^\infty$, and does not meet X_v. Failing this, $(y_n)_1^\infty \in X_v$, which we assumed is false. Therefore X_v is closed in X, which implies it is compact.

Next, we consider the topological disjoint union $\bigsqcup_{v\in E^0} X_v = \bigcup_{v\in E^0}(\{v\} \times X_v)$ and prove that it is homeomorphic to $E^\star \cup E^\infty$. For a path $q \in E^\star \cup E^\infty$, let $\Psi(q) = (s(q), (x_n)_1^\infty)$ where $x_n = q_n$ if $n \leq |q|$ and $x_n = \infty$ if $n > |q|$. Clearly, this defines a bijection

$$\Psi : E^\star \cup E^\infty \longrightarrow \bigsqcup_{v\in E^0} X_v.$$

For $v \in E^0$, we have $\Psi(C(v)) = \{v\} \times X_v$, and for $\alpha \in E^\star$ with $|\alpha| \geq 1$, we have

$$\Psi(C(\alpha)) = \{s(\alpha)\} \times \left[p_1^{-1}(\alpha_1) \cap \cdots \cap p_{|\alpha|}^{-1}(\alpha_{|\alpha|})\right], \tag{2.3}$$

both of which are open in $\bigsqcup_{v\in E^0} X_v$. When $F \subseteq r(\alpha)E^1$, equation (2.3) yields

$$\bigcup_{e\in F} \Psi(C(\alpha e)) = \{s(\alpha)\} \times \left[p_1^{-1}(\alpha_1) \cap \cdots \cap p_{|\alpha|}^{-1}(\alpha_{|\alpha|}) \cap p_{|\alpha|+1}^{-1}(F)\right]. \tag{2.4}$$

If $\alpha \in E^\star$ and $F \subseteq_{\text{finite}} r(\alpha)E^\star$, equation (2.4) yields

$$\begin{aligned}\Psi(C(\alpha, F)) &= \Psi(C(\alpha)) \setminus \bigcup_{e\in F} \Psi(C(\alpha e)) \\ &= \{s(\alpha)\} \times \left[p_1^{-1}(\alpha_1) \cap \cdots \cap p_{|\alpha|}^{-1}(\alpha_{|\alpha|}) \cap p_{|\alpha|+1}^{-1}\big((E^1 \cup \{\infty\}) \setminus F\big)\right].\end{aligned}$$

Being finite, F is compact, so $(E^1 \cup \{\infty\}) \setminus F$ is open in $E^1 \cup \{\infty\}$ with the topology of one-point compactification. Therefore $\Psi(C(\alpha, F))$ is open, so Ψ is an open map because it maps each basic open set in $E^\star \cup E^\infty$ to an open set. Consequently, Ψ^{-1} is continuous.

Next, write $\iota_v : \{v\} \times X_v \to \bigsqcup_{v\in E^0} X_v$ to mean the natural inclusion map. Observe that $\Psi^{-1} \circ \iota_v : \{v\} \times X_v \to E^\star \cup E^\infty$ is a continuous map from a compact space to a Hausdorff space (see Theorem 2.3), so it is a closed map. From this it follows that $\Psi^{-1} = \bigsqcup_{v\in E^0} \Psi^{-1} \circ \iota_v$ is a closed map, because it is a sum of closed maps, and a sum of closed maps is closed. This shows Ψ is continuous. Therefore Ψ is a homeomorphism. Immediately, it follows that $C(v) = \Psi^{-1}(\{v\} \times X_v)$ is compact because X_v is compact, for every $v \in E^0$.

To show that $C(\alpha)$ is compact for all $\alpha \in E^\star$, we proceed by induction on the length of α. If $e \in E^1$, then $C(s(e)) \setminus C(e) = C(s(e), \{e\})$ is a basic open set, so $C(e)$ is closed in $C(s(e))$, hence compact. Assume $C(\alpha)$ is compact for any $\alpha \in E^\star$ with $|\alpha| = n$. If $\mu \in E^\star$ has $|\mu| = n+1$ then let $\mu' = \mu_1\mu_2 \ldots \mu_n$. We have that $C(\mu') \setminus C(\mu) = C(\mu', \{\mu_{n+1}\})$ is a basic open set, so $C(\mu)$ is closed in $C(\mu')$, hence compact. By induction, $C(\alpha)$ is compact for arbitrary $\alpha \in E^\star$. Finally, if $F \subseteq_{\text{finite}} r(\alpha)E^1$ then $C(\alpha) \setminus\ C(\alpha, F) = \bigcup_{e\in F} C(\alpha e)$ is open, so $C(\alpha, F)$ is compact. □

We can now add another countability property to the list from Theorem 2.3. A topological space is called *σ-compact* if it is a countable union of compact subsets.

Proposition 2.5. *The path space $E^\star \cup E^\infty$ is σ-compact if and only if E^0 is countable.*

Proof. If E^0 is countable then the path space is σ-compact, because $E^\star \cup E^\infty = \bigcup_{v\in E^0} C(v)$ and $C(v)$ is compact for every $v \in E^0$, by Theorem 2.4. For the converse, suppose $E^\star \cup E^\infty$ is σ-compact. Then there is a sequence of compact subsets $(K_n)_1^\infty$ such that $E^\star \cup E^\infty = \bigcup_{n=1}^\infty K_n$. Each K_n is compact, so it can be covered by a finite subcover of $\{C(v) \mid v \in E^0\}$, implying that there is a countable set $S \subseteq E^0$ such that $E^\star \cup E^\infty = \bigcup_{v\in S} C(v)$. But this implies $S = E^0$ because $C(v)$ and $C(w)$ are disjoint unless $v = w$. □

We now prove an easy fact that forms a bridge to the next section, where we shall construct a groupoid with unit space $\partial E = E^\infty \cup \{\alpha \in E^\star \mid r(\alpha) \in E^0_{\text{sing}}\}$.

Proposition 2.6. *The boundary path space ∂E is both closed and open in $E^\star \cup E^\infty$.*

Proof. The complement of ∂E consists of isolated points. Indeed, if $\mu \in (E^\star \cup E^\infty) \setminus \partial E$, then $r(\mu)$ is a regular vertex, and $C(\mu, r(\mu)E^1) = \{\mu\}$ is open. □

An immediate consequence of Theorem 2.4 and Proposition 2.6 is that ∂E is a locally compact Hausdorff space with the base of compact open sets:

$$Z(\alpha, F) = C(\alpha, F) \cap \partial E; \qquad\qquad \alpha \in E^\star,\ F \subseteq_{\text{finite}} r(\alpha)E^1.$$

For $\alpha \in E^\star$, we define $Z(\alpha) = Z(\alpha, \emptyset)$, which is the same as $Z(\alpha) = C(\alpha) \cap \partial E$. As it were, the sets $Z(\alpha, F)$ are very rarely empty. In particular, $Z(\alpha) \neq \emptyset$ for all $\alpha \in E^\star$; in other words, every finite path can be extended to a boundary path.

Lemma 2.7. *Let $\alpha \in E^\star$ and let $F \subseteq_{\text{finite}} r(\alpha)E^1$. Then $Z(\alpha, F) = \emptyset$ if and only if $r(\alpha)$ is a regular vertex and $F = r(\alpha)E^1$.*

Proof. (⇒) Assume $Z(\alpha, F) = \emptyset$. If $r(\alpha)$ were a singular vertex then it would imply $\alpha \in Z(\alpha, F)$. Therefore $r(\alpha)$ is regular, so $r(\alpha)E^1 \neq \emptyset$. Towards a contradiction, assume F is a proper subset of $r(\alpha)E^1$. Then there exists some $x_1 \in r(\alpha)E^1 \setminus F$. Assume that we have a path $x_1x_2 \ldots x_n \in r(\alpha)E^\star$. If $r(x_n)$ is a sink, let $x = x_1 \ldots x_n$. Otherwise, let $x_{n+1} \in r(x_n)E^1$. Inductively, this constructs $x \in r(\alpha)\partial E$ such that $\alpha x \in Z(\alpha, F)$. Since this is a contradiction, it proves $F = r(\alpha)E^1$.

(⇐) If $r(\alpha)$ is regular, then $Z(\alpha) = \bigcup_{e\in r(\alpha)E^1} Z(\alpha e)$, so $Z(\alpha, r(\alpha)E^1) = \emptyset$. □

Theorem 2.8. *The boundary path space ∂E is:*

(1) *second-countable if and only if E is a countable graph,*

(2) *first-countable if and only if E is a row-countable graph, and*

(3) *σ-compact if and only if E^0 is countable.*

Proof. Together with Lemma 2.7, the proof is almost identical to the relevant parts of Theorem 2.3 and Proposition 2.5. □

2.3 The boundary path groupoid

In this section, we define the boundary path groupoid of a graph, and investigate some of its algebraic and topological properties. Throughout, let $E = (E^0, E^1, r, s)$ be an arbitrary graph.

Define the *one-sided shift map* $\sigma : \partial E \setminus E^0 \to \partial E$ as follows:

$$\sigma(x) = \begin{cases} r(x) & \text{if } x \in E^\star \cap \partial E \text{ and } |x| = 1 \\ x_2 \dots x_{|x|} & \text{if } x \in E^\star \cap \partial E \text{ and } |x| \geq 2 \\ x_2 x_3 \dots & \text{if } x \in E^\infty \end{cases}$$

The n-fold composition σ^n is defined on paths of length $\geq n$ and we understand that $\sigma^0 : \partial E \to \partial E$ is the identity map.

Definition 2.9. Let k be an integer and let $x, y \in \partial E$. We say that x and y are **tail equivalent with lag** k, written $x \sim_k y$, if there exists some $n \geq \max\{0, k\}$ such that

$$\sigma^n(x) = \sigma^{n-k}(y).$$

If an integer k exists such that $x \sim_k y$, we say that x and y are *tail equivalent*, and write $x \sim y$.

An equivalent definition is that $x \sim_k y$ if there exists $(\alpha, \beta) \in E^\star \times_r E^\star$ and $z \in r(\alpha)\partial E$, such that $|\alpha| - |\beta| = k$ and $x = \alpha z$, $y = \beta z$. Something that is potentially counter-intuitive about these relations is that the lag is not necessarily unique: it is possible to have $x \sim_k y$ and $x \sim_\ell y$ even when $k \neq \ell$. It is straightforward to prove from the definition that for all $x, y, z \in \partial E$:

$$\begin{gathered} x \sim_0 x, \\ x \sim_k y \implies y \sim_{-k} x, \\ x \sim_k y \text{ and } y \sim_\ell z \implies x \sim_{k+\ell} z, \\ x \sim_k y \implies x, y \in E^\star \text{ or } x, y \in E^\infty. \end{gathered}$$

This shows that $\sim$ is an equivalence relation on ∂E that respects the partition between finite and infinite paths.

Definition 2.10. The **boundary path groupoid** of a graph E is

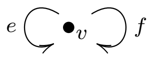
$$\begin{aligned} \mathcal{G}_E &= \big\{(x, k, y) \mid x, y \in \partial E,\ x \sim_k y\big\} \\ &= \big\{(\alpha x, |\alpha| - |\beta|, \beta x) \mid (\alpha, \beta) \in E^\star \times_r E^\star, x \in r(\alpha)\partial E\big\} \end{aligned}$$

where a morphism $(x, k, y) \in \mathcal{G}_E$ has domain y and codomain x. The composition of morphisms and their inverses are defined by the formulae:

$$(x, k, y)(y, l, z) = (x, k + l, z), \qquad\qquad (x, k, y)^{-1} = (y, -k, x).$$

The unit space is $\mathcal{G}_E^{(0)} = \{(x, 0, x) \mid x \in \partial E\}$, which we silently identify with ∂E (see Remark 1.2). The orbits in ∂E are tail equivalence classes.

Example 2.11. Consider this graph, called the *rose with two petals*:

$R_2 \quad =$ 

A standard diagonal argument proves that ∂R_2 is an uncountable set. In fact, it can be deduced from the proof of Theorem 2.4 that ∂R_2 is homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$. There are uncountably many orbits in ∂R_2, but the topology on ∂R_2 is second-countable and σ-compact.

A boundary path $p \in \partial E$ is called *eventually periodic* if it is of the form $p = \mu\epsilon\epsilon \ldots \in E^\infty$ where $\mu, \epsilon \in E^\star$ and ϵ is a closed path of positive length. The following result is [80, Proposition 4.2] except there appears to be a clash between our definitions of cycles and closed paths. We also prove it slightly more formally.

Proposition 2.12. *If E is a graph and $p \in \partial E$, then the isotropy group at p is:*

(1) *infinite cyclic if p is eventually periodic;*

(2) *trivial if p is not eventually periodic.*

Proof. (1) Assume $p = \mu\epsilon\epsilon \ldots \in E^\infty$ where $\mu, \epsilon \in E^\star$, $r(\mu) = s(\epsilon) = r(\epsilon)$, and assume ϵ is minimal in the sense that it has no initial subpath δ such that $\epsilon = \delta^n$ for some $n > 1$. Let $(p, k, p) \in {}^p\mathcal{G}_E{}^p$ and suppose $k \geq 0$. Then $p \sim_k p$ implies that for all sufficiently large $n \geq 0$, we have $\sigma^{|\mu|+n|\epsilon|+k}(p) = \sigma^{|\mu|+n|\epsilon|}(p)$. This yields:

$$\sigma^{|\mu|+n|\epsilon|+k}(p) = \sigma^k(\epsilon\epsilon \ldots) = \sigma^{|\mu|+n|\epsilon|}(p) = \epsilon\epsilon \ldots.$$

Let $m = k \mod |\epsilon|$. Then $0 \leq m < |\epsilon|$ and

$$\sigma^k(\epsilon\epsilon \ldots) = \sigma^m(\epsilon\epsilon \ldots) = \epsilon_{m+1} \ldots \epsilon_{|\epsilon|}\epsilon\epsilon \ldots = \epsilon_1 \ldots \epsilon_m \epsilon\epsilon \ldots.$$

Since ϵ is minimal, this implies $m = 0$, so $k \mid |\epsilon|$. On the other hand, if $k < 0$ then $(p, -k, p) = (p, k, p)^{-1} \in {}^p\mathcal{G}_E{}^p$ and the same argument establishes $k \mid |\epsilon|$. The conclusion is that ${}^p\mathcal{G}_E{}^p$ is the infinite cyclic group generated by $(p, |\epsilon|, p)$.

(2) Let $(p, k, p) \in {}^p\mathcal{G}_E{}^p$. Then $p \sim_k p$ implies $p = \alpha x = \beta x$ for some $(\alpha, \beta) \in E^\star \times_r E^\star$ and $x \in r(\alpha)\partial E$, with $|\alpha| - |\beta| = k$. If p is finite, this implies $\alpha = \beta$, so $k = 0$. That is, the isotropy group at p is trivial. On the other hand, suppose p is infinite and not eventually periodic. If $|\alpha| < |\beta|$, then $\beta = \alpha\beta'$ for some $\beta' \in E^\star$. But then $p = \alpha x = \beta x = \alpha\beta' x$, so $x = \beta' x = \beta'\beta' x = \beta'\beta'\beta' \ldots$, and this proves p is eventually periodic, a contradiction. Similarly, assuming $|\beta| < |\alpha|$ reaches the same contradiction. Therefore, $|\alpha| = |\beta|$ and $k = 0$, implying that the isotropy group at p is trivial. $\square$

The next step is to define a topology on $\mathcal{G}_E$. Let $(\alpha, \beta) \in E^\star \times_r E^\star$, and let $F \subseteq_{\text{finite}} r(\alpha)E^1$. Define the sets:

$$\mathcal{Z}(\alpha, \beta) = \big\{(\alpha x, |\alpha| - |\beta|, \beta x) \mid x \in r(\alpha)\partial E\big\}; \qquad \mathcal{Z}(\alpha, \beta, F) = \mathcal{Z}(\alpha, \beta) \setminus \bigcup_{e \in F} \mathcal{Z}(\alpha e, \beta e).$$

Obviously, $\mathcal{Z}(\alpha, \beta) = \mathcal{Z}(\alpha, \beta, \emptyset)$. Next we present a pair of technical lemmas (generalising [61, Lemma 2.5]) which prove that the collection of sets of the form $\mathcal{Z}(\alpha, \beta, F)$ is closed under pairwise intersections, so it can serve as a base for a topology on $\mathcal{G}_E$.

Lemma 2.13. *Let* $(\alpha, \beta), (\gamma, \delta) \in E^\star \times_r E^\star$. *Then*

$$\mathcal{Z}(\alpha,\beta) \cap \mathcal{Z}(\gamma,\delta) = \begin{cases} \mathcal{Z}(\alpha,\beta) & \text{if } \exists\ \kappa \in E^\star,\ \alpha = \gamma\kappa,\ \beta = \delta\kappa \\ \mathcal{Z}(\gamma,\delta) & \text{if } \exists\ \kappa \in E^\star,\ \gamma = \alpha\kappa,\ \delta = \beta\kappa \\ \emptyset & \text{otherwise} \end{cases}$$

Proof. We prove that when the intersection of the two sets is nonempty, then it must be one of the first two cases in the piecewise expression. To this end, let $(\alpha x, |\alpha| - |\beta|, \beta x) = (\gamma x', |\gamma| - |\delta|, \delta x') \in \mathcal{Z}(\alpha,\beta) \cap \mathcal{Z}(\gamma,\delta)$, where $x \in r(\alpha)\partial E$ and $x' \in r(\gamma)\partial E$. Assume $|\gamma| \leq |\alpha|$, which implies $|\delta| \leq |\beta|$; if not, rearrange. Since $\alpha x = \gamma x'$, it must be that $\alpha = \gamma\kappa$ where κ is the initial subpath of x' of length $|\alpha| - |\gamma|$. Similarly, $\beta = \delta\kappa$. So we are in the first case (or the second case, if a rearrangement took place). In the first two cases in the piecewise expression, it is clear from the definitions what the intersection of $\mathcal{Z}(\alpha,\beta)$ and $\mathcal{Z}(\gamma,\delta)$ must be. □

Lemma 2.14. *Suppose* $(\alpha, \beta), (\gamma, \delta) \in E^\star \times_r E^\star$, $F \subseteq_{\text{finite}} r(\alpha)E^1$, *and* $H \subseteq_{\text{finite}} r(\gamma)E^1$. *Then*

$$\mathcal{Z}(\alpha,\beta,F) \cap \mathcal{Z}(\gamma,\delta,H) = \begin{cases} \mathcal{Z}(\alpha,\beta,F\cup H) & \text{if } \alpha=\gamma, \beta=\delta \\ \mathcal{Z}(\alpha,\beta,F) & \text{if } \exists\ \kappa \in E^\star,\ \alpha = \gamma\kappa,\ \beta = \delta\kappa,\ |\kappa| \geq 1,\ \kappa_1 \notin H \\ \mathcal{Z}(\gamma,\delta,H) & \text{if } \exists\ \kappa \in E^\star,\ \gamma = \alpha\kappa,\ \delta = \beta\kappa,\ |\kappa| \geq 1,\ \kappa_1 \notin F \\ \emptyset & \text{otherwise} \end{cases}$$

Proof. We make a calculation and then proceed by cases.

$$\begin{aligned} &\mathcal{Z}(\alpha,\beta,F) \cap \mathcal{Z}(\gamma,\delta,H) \\ &\qquad = \left[\mathcal{Z}(\alpha,\beta) \setminus \bigcup_{e\in F} \mathcal{Z}(\alpha e,\beta e)\right] \bigcap \left[\mathcal{Z}(\gamma,\delta) \setminus \bigcup_{e\in H} \mathcal{Z}(\gamma e,\delta e)\right] \\ &\qquad = [\mathcal{Z}(\alpha,\beta) \cap \mathcal{Z}(\gamma,\delta)] \setminus \left[\bigcup_{e\in F} \mathcal{Z}(\alpha e,\beta e) \cup \bigcup_{e\in H} \mathcal{Z}(\gamma e,\delta e)\right] \end{aligned} \tag{2.5}$$

Case 1: If $\alpha = \gamma$ and $\beta = \delta$, equation (2.5) yields $\mathcal{Z}(\alpha,\beta,F) \cap \mathcal{Z}(\gamma,\delta,H) = \mathcal{Z}(\alpha,\beta,F\cup H)$.

Case 2: If there exists $\kappa \in E^\star \setminus E^0$ such that $\alpha = \gamma\kappa$ and $\beta = \delta\kappa$ then after applying Lemma 2.13, the right hand side of (2.5) becomes

$$\mathcal{Z}(\alpha,\beta) \setminus \left[\bigcup_{e\in F} \mathcal{Z}(\alpha e,\beta e) \cup \bigcup_{e\in H} \mathcal{Z}(\gamma e,\delta e)\right].$$

Moreover, $\mathcal{Z}(\alpha,\beta) \cap \mathcal{Z}(\gamma e,\delta e) = \emptyset$ for all $e \in H$, provided $e \neq \kappa_1$. If $e = \kappa_1$ then $\mathcal{Z}(\alpha,\beta) \cap \mathcal{Z}(\gamma e,\delta e) = \mathcal{Z}(\alpha,\beta)$. Therefore (2.5) becomes $\mathcal{Z}(\alpha,\beta,F)$ if $\kappa_1 \notin H$ and $\emptyset$ if $\kappa_1 \in H$.

Case 3: If there exists $\kappa \in E^\star \setminus E^0$ such that $\gamma = \alpha\kappa$ and $\delta = \beta\kappa$ then the situation is symmetric to the second case.

Case 4: Otherwise, $\mathcal{Z}(\alpha,\beta) \cap \mathcal{Z}(\gamma,\delta) = \emptyset$, by Lemma 2.13. □

From now on, we assume $\mathcal{G}_E$ has the topology generated by all the sets:

$$\mathcal{Z}(\alpha,\beta,F); \qquad (\alpha,\beta) \in E^\star \times_r E^\star,\ F \subseteq_{\text{finite}} r(\alpha)E^1. \tag{2.6}$$

Some of our references give a different base for the topology on $\mathcal{G}_E$, although it always contains the sets $\mathcal{Z}(\alpha, \beta, F)$. There are advantages to working with a base that is not too large, which is why we have chosen to focus on this one.

Let E be a graph and consider $\mathbb{Z}$ with the discrete topology. The map

$$\theta : \mathcal{G}_E \to \mathbb{Z}, \qquad (x, k, y) \mapsto k,$$

is a continuous groupoid homomorphism. In fact, it is a degree map giving $\mathcal{G}_E$ the structure of a $\mathbb{Z}$-graded groupoid. Some parts of this lemma are reminiscent of [61, Proposition 2.6].

Lemma 2.15. *Let E be a graph.*

(1) *The topology on $\mathcal{G}_E$ is Hausdorff.*

(2) *$\boldsymbol{d} : \mathcal{G}_E \to \partial E$ is a local homeomorphism.*

(3) *If $(\alpha, \beta) \in E^\star \times_r E^\star$ and $F \subseteq_{\text{finite}} r(\alpha)E^1$, then $\mathcal{Z}(\alpha, \beta, F)$ is compact.*

Proof. (1) Take $(x, k, y) \neq (w, \ell, z)$ in $\mathcal{G}_E$. If $k \neq \ell$ then $\theta^{-1}(k)$ and $\theta^{-1}(\ell)$ are disjoint open sets separating the two points. Otherwise, either $x \neq w$ or $y \neq z$. If $w \neq x$ then either: w and x must differ on some initial segment, or one must be an initial subpath of the other. Using Lemma 2.14, it is not difficult to separate the two points by disjoint open sets. If $y \neq z$, the same reasoning applies.

(2) For $(\alpha, \beta) \in E^\star \times_r E^\star$, define

$$h_{\alpha,\beta} : Z(\beta) \to \mathcal{Z}(\alpha, \beta), \qquad \beta x \mapsto (\alpha x, |\alpha| - |\beta|, \beta x).$$

Clearly, $h_{\alpha,\beta}$ is a bijection. By Lemma 2.14, the basic open sets contained in $\mathcal{Z}(\alpha, \beta)$ are all of the form $\mathcal{Z}(\alpha\kappa, \beta\kappa, F')$ where $\kappa \in r(\alpha)E^\star$ and $F' \subseteq_{\text{finite}} r(\kappa)E^1$. Clearly

$$h_{\alpha,\beta}^{-1}\big(\mathcal{Z}(\alpha\kappa, \beta\kappa, F')\big) = Z(\beta\kappa, F')$$

is open in $Z(\beta)$, so $h_{\alpha,\beta}$ is continuous. A continuous map from a compact space to a Hausdorff space is a closed map, so $h_{\alpha,\beta}$ is a closed map. Therefore $h_{\alpha,\beta}$ is a homeomorphism. This proves that $\boldsymbol{d}|_{\mathcal{Z}(\alpha,\beta)}$ is a homeomorphism onto its image (because $\boldsymbol{d}|_{\mathcal{Z}(\alpha,\beta)}^{-1} = h_{\alpha,\beta}$).

(3) According to item (2), $\boldsymbol{d}$ restricts to a homeomorphism $\mathcal{Z}(\alpha, \beta, F) \approx Z(\beta, F)$, and $Z(\beta, F)$ is compact by Theorem 2.4. □

Since $\mathcal{Z}(\alpha, \beta, F) \approx Z(\beta, F)$, Lemma 2.7 implies that $\mathcal{Z}(\alpha, \beta, F) = \emptyset$ if and only if $r(\alpha)$ is a regular vertex and $F = r(\alpha)E^1$.

Remark 2.16. The groupoid $\mathcal{G}_E$ admits continuous maps

$$\boldsymbol{c} : (x, k, y) \mapsto x, \qquad \theta : (x, k, y) \mapsto k, \qquad \boldsymbol{d} : (x, k, y) \mapsto y,$$

so it is tempting to think that the topology on $\mathcal{G}_E$ coincides with the relative topology that it gets from being a subset of $\partial E \times \mathbb{Z} \times \partial E$. However, this is not the case: the topology on $\mathcal{G}_E$ is much finer than the induced topology from the product space $\partial E \times \mathbb{Z} \times \partial E$.

The main theorem that follows is not new, and it has been in use for some time. Indeed, it is implied by [75, Lemma 2.1], although not in a trivial way (see also [70, Theorem 3.5] and [86, Theorem 3.16]). However, due to our Theorem 2.4, this is the first self-contained proof that we know of that applies to ordinary directed graphs, and does not require the graph to be countable.

Theorem 2.17. *Let E be a graph. The groupoid $\mathcal{G}_E$ is a Hausdorff ample groupoid with the base of compact open bisections given in (2.6).*

Proof. The most technical part that remains is showing that the composition map $\boldsymbol{m}$ is continuous. If $x, z \in E^\star \cap \partial E$ are tail equivalent finite paths, then $(x, |x| - |z|, z)$ has a neighbourhood base of open sets, $\mathcal{N}_{(x,|x|-|z|,z)} = \{\mathcal{Z}(x, z, F) \mid F \subseteq_{\text{finite}} r(x)E^1\}$. If $x, z \in E^\infty$ are tail equivalent infinite paths, with lag t, then there exists $N \geq 0$ such that $\sigma^{N+t}(x) = \sigma^N(z)$. Consequently (x, t, z) has a neighbourhood base of open sets, $\mathcal{N}_{(x,t,z)} = \{\mathcal{Z}(x_1 \dots x_{n+t}, z_1 \dots z_n) \mid n > N\}$.

Now suppose U is an open set in $\mathcal{G}_E$ containing a product of two morphisms $(x, k + \ell, z) = (x, k, y)(y, \ell, z)$. It must be that x, y, z are all finite paths or they are all infinite paths. If x, y, z are finite paths, then they must have $r(x) = r(y) = r(z)$ and U must contain some $\mathcal{Z}(x, z, F) \in \mathcal{N}_{(x,|x|-|z|,z)}$. Then $\big((x, k, y), (y, \ell, z)\big)$ is contained in the open set $\big(\mathcal{Z}(x, y, F) \times \mathcal{Z}(y, z, F)\big) \cap \mathcal{G}_E^{(2)}$ which is mapped bijectively by $\boldsymbol{m}$ into $Z(x, z, F) \subseteq U$. Otherwise x, y, z are all infinite paths, and there must exist n large enough that $\sigma^{n+k+\ell}(x) = \sigma^{n+\ell}(y) = \sigma^n(z)$. Making n even larger if necessary, we can assume U contains some $\mathcal{Z}(x_1 \dots x_{n+k+\ell}, z_1 \dots z_n) \in \mathcal{N}_{(x,k+\ell,z)}$. Define:

$$x' = x_1 \dots x_{n+k+\ell}, \qquad y' = y_1 \dots y_{n+\ell}, \qquad z' = z_1 \dots z_n.$$

Then $\big((x, k, y), (y, \ell, z)\big)$ is contained in the open set $\big(\mathcal{Z}(x', y') \times \mathcal{Z}(y', z')\big) \cap \mathcal{G}_E^{(2)}$, which is mapped bijectively by $\boldsymbol{m}$ into $Z(x', z') \subseteq U$. Since $(x, k + \ell, z) = (x, k, y)(y, \ell, z)$ was an arbitrary product in U, this shows that $\boldsymbol{m}^{-1}(U)$ is open in $\mathcal{G}_E^{(2)}$, so $\boldsymbol{m}$ is continuous. It is much easier to show that the inversion map $\boldsymbol{i}$ is continuous, because $\boldsymbol{i}$ puts $Z(\alpha, \beta, F)$ in bijection with $Z(\beta, \alpha, F)$. We have proved $\mathcal{G}_E$ is a topological groupoid. In Lemma 2.15 (2), it is shown that $\boldsymbol{d}$ is a local homeomorphism. Therefore, $\mathcal{G}_E$ is an étale groupoid. The remaining facts from Lemma 2.15 establish that $\mathcal{G}_E$ is a Hausdorff ample groupoid and that the base described in (2.6) consists of compact open bisections. $\square$

Chapter 3

Leavitt path algebras

We now take a different direction: the theory of Leavitt path algebras. Leavitt path algebras, introduced in [3] and [17], are a relatively new development in noncommutative ring theory, although not quite as new as Steinberg algebras. To each graph E is associated an R-algebra, $L_R(E)$, called the Leavitt path algebra of E. The role of the graph may seem unclear at the outset, because all it does is serve as a kind of notational device for the generators and relations that define $L_R(E)$. Surprisingly, it turns out that many of the ring-theoretic properties of $L_R(E)$ are controlled by graphical properties of E. Here is an example of how this phenomenon plays out:

Theorem 3.1 (Abrams & Rangaswamy)**.** *[9, Theorem 1] Let K be a field and let E be a graph. Then $L_K(E)$ is von Neumann regular if and only if E is acyclic.*

The historical background to Leavitt path algebras begins in the late 1950s, with W. G. Leavitt's investigations into the invariant basis number (IBN) property of rings. A nontrivial ring A is said to have the *IBN property* if the only time that the free left A-modules A^n and A^m are isomorphic is when $m = n$. Most of the familiar rings that one encounters have the IBN property, including all commutative rings, one-sided noetherian rings, and one-sided artinian rings [35, Theorem 4.7]. There are, however, some non-IBN rings A such that $A^n \cong A^m$ for *all* positive integer pairs (n, m), and others such that $A^n \cong A^m$ for *some* but not all positive integer pairs (n, m).

With elementary linear algebra, one can find an intrinsic characterisation of the IBN property. A unital ring A satisfies $A \cong A^n$, if and only if there are column vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ with entries in A, such that $\boldsymbol{y}\boldsymbol{x}^t = I_n$ and $\boldsymbol{x}^t\boldsymbol{y} = 1$. The Leavitt algebra $L_{n,K}$ is constructed to be the universal K-algebra generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, satisfying those identities. Leavitt proved that $L_{n,K}$ is the universal K-algebra satisfying:

$$A \cong A^n, \text{ but}$$
$$A \ncong A^m \text{ when } m < n.$$

In other words, $L_{n,K}$ fails to have the IBN property in a very special way.

Leavitt path algebras are a natural generalisation of these classical Leavitt algebras. Since 2005, when they were introduced, there has been an abundance of research on the subject. One of the primary goals is to characterise ring-theoretic properties of $L_R(E)$ by reference to graphical properties of E. As a result, we have a rich supply of algebras with "interesting and extreme properties," as the authors write in [11]. This is useful for generating counterexamples to reasonable-sounding conjectures, e.g. [6, 52], or for supporting other long-standing conjectures by showing they hold within this varied class, e.g. [12, 20].

Additionally, it is well-known by now, and proved in [10, §3], that $L_{\mathbb{C}}(E)$ embeds as a dense $*$-subalgebra of the graph C^*-algebra $C^*(E)$. One expects *a priori* that these two different structures would have little to do with one another, but in fact there are remarkable and often inexplicable similarities. Many results about arbitrary Leavitt path algebras resemble those from graph C^*-algebras. Abrams gives an interesting account of this phenomenon in [1, Appendix 1]. Another branch of the literature concerns itself with finding invariants that determine Leavitt path algebras up to isomorphism, or Morita equivalence. This enterprise is known as the classification question for Leavitt path algebras. It has led to some interesting developments in graded K-theory [48] and has motivated the study of substructures of Leavitt path algebras, like the socle [18] and invariant ideals [57].

Historically, the theory of Leavitt path algebras was developed for the case when R is a field, and E is a row-finite countable graph. The current state of the art allows us to study $L_R(E)$ when R is any unital commutative ring and E is any countable graph [4, 83] or, alternatively, when R is a field and E is an arbitrary graph [2, 47]. The proofs of some key results, including the fact that the relations on $L_R(E)$ do not collapse the algebra to zero ([47, Lemma 1.5] and [83, Proposition 3.4]) and the Graded Uniqueness Theorem ([47, Proposition 3.6] and [83, Theorem 5.3]), have not yet been recorded for the case where simultaneously E is uncountable and R is not a field. One of our goals is to fix this and complete the picture.

In §3.1, we define the Leavitt path algebra of a graph. We define it in terms of its universal property, and then describe how it can be realised as the quotient of a path algebra. Path algebras are, in some sense, the definitive examples of $\mathbb{Z}$-graded algebras, and the $\mathbb{Z}$-grading survives in their Leavitt path algebra quotients. In §3.2, we prove the Graded Uniqueness Theorem for Leavitt path algebras. In §3.3, we lay the cornerstone of this dissertation by proving: the Leavitt path algebra of a graph is isomorphic to the Steinberg algebra of its boundary path groupoid. Through this lens, we rederive some fundamentals of Leavitt path algebras, and classify finite-dimensional Leavitt path algebras. In §3.4, we prove the Graded and Cuntz-Krieger Uniqueness Theorems for Steinberg algebras and specialise to prove the Cuntz-Krieger Uniqueness Theorem for Leavitt path algebras.

3.1 Introducing Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be a graph. We introduce the set of formal symbols $(E^1)^* = \{e^* \mid e \in E^1\}$ and call the elements of $(E^1)^*$ *ghost edges*. For clarity, we will sometimes refer to the elements of E^1 as *real edges*. If $\alpha = \alpha_1 \dots \alpha_{|\alpha|} \in E^\star$ is a finite path of positive length, we define α^* to be the sequence $\alpha^*_{|\alpha|} \dots \alpha^*_1$, and call it a *ghost path*. We also define $v^* = v$ for every $v \in E^0$.

Definition 3.2. [83] Let E be a graph and let A be a ring. Assume $\{v, e, e^* \mid v \in E^0, e \in E^1\}$ is a subset of A; in other words, there is a function $E^0 \sqcup E^1 \sqcup (E^1)^* \to A$ whose image inherits the notation of its domain. Then $\{v, e, e^* \mid v \in E^0, e \in E^1\} \subset A$ is called a **Leavitt E-family** if the following conditions are satisfied:

(V) $v^2 = v$ and $vw = 0$ for all $v, w \in E^0$, $v \neq w$;

(E1) $s(e)e = er(e) = e$ for all $e \in E^1$;

(E2) $e^*s(e) = r(e)e^* = e^*$ for all $e \in E^1$;

(CK1) $e^*e = r(e)$ and $e^*f = 0$ for all $e, f \in E^1$, $e \neq f$;

(CK2) $v = \sum_{e \in vE^1} ee^*$ for all $v \in E^0_{\text{reg}}$.

The interpretation of (V) is that $\{v \in A \mid v \in E^0\}$ is a set of pairwise orthogonal idempotents. The relations (CK1) and (CK2) are called the Cuntz-Krieger relations, and they originate from operator theory. The relevant interpretation, at least in that setting, is that vertices are represented by projections, and edges are represented by partial isometries with mutually orthogonal ranges.

In any algebra A containing a Leavitt E-family $\{v, e, e^* \mid v \in E^0, e \in E^1\}$, one can consider paths $\mu = \mu_1 \dots \mu_{|\mu|}$ and ghost paths $\mu^* = \mu^*_{|\mu|} \dots \mu^*_1$ as elements of A in the obvious way: products of their constituent real edges and ghost edges respectively. The following lemma is straightforward to prove using the relations (E1), (E2), and (CK1). It is so fundamental that we will usually use the result without referring to it.

Lemma 3.3. *If A is an R-algebra generated by a Leavitt E-family $\{v, e, e^* \mid v \in E^0, e \in E^1\}$, the elements of A obey the rule:*

$$(r\mu\nu^*)(r'\gamma\lambda^*) = \begin{cases} (rr')\mu\kappa^*\lambda^* & \text{if } \gamma \text{ is an initial subpath of } \nu, \text{ with } \nu = \gamma\kappa \\ (rr')\mu\kappa\lambda^* & \text{if } \nu \text{ is an initial subpath of } \gamma, \text{ with } \gamma = \nu\kappa \\ 0 & \text{otherwise} \end{cases}$$

for all $r, r' \in R$ and all $\mu, \nu, \gamma, \lambda \in E^\star$, with $r(\mu) = r(\nu)$ and $r(\gamma) = r(\lambda)$.

Corollary 3.4. *Every R-algebra generated by a Leavitt E-family is generated, as an abelian group, by the set $\{r\alpha\beta^* \mid r \in R, (\alpha, \beta) \in E^\star \times_r E^\star\}$.*

Proof. By Lemma 3.3, every word in the generators $\{v, e, e^* \mid v \in E^0, e \in E^1\}$ reduces to an expression of the form $\alpha\beta^*$ where $\alpha, \beta \in E^\star$. Moreover, $\alpha\beta^* = 0$ unless $r(\alpha) = r(\beta)$, by (V), (E1), and (E2). □

Let B be an R-algebra generated by a Leavitt E-family $\{v, e, e^* \mid v \in E^0, e \in E^1\}$. We say that B is *universal* (for Leavitt E-families) if every R-algebra A containing a Leavitt E-family $\{a_v, b_e, c_{e^*} \mid v \in E^0, e \in E^1\}$ admits a unique R-algebra homomorphism $\pi : B \to A$ such that $\pi(v) = a_v$, $\pi(e) = b_e$, and $\pi(e^*) = c_{e^*}$ for every $v \in E^0$ and $e \in E^1$. The universal property determines B up to isomorphism.

Definition 3.5. Let E be a graph. The **Leavitt path algebra** of E with coefficients in R, denoted by $L_R(E)$, is the universal R-algebra generated by a Leavitt E-family.

Technically, $L_R(E)$ is an isomorphism class in the category of R-algebras. If B is a specific R-algebra having the universal property for Leavitt E-families, then B is a *model* of $L_R(E)$. However, it is customary and natural to refer to $L_R(E)$ as if it were a specific model with the standard generators $\{v, e, e^* \mid v \in E^0, e \in E^1\}$. Every element $x \in L_R(E)$, so to speak, is a finite sum of the form $x = \sum_i r_i\alpha_i\beta_i^*$ where $r_i \in R$ and $(\alpha, \beta) \in E^\star \times_r E^\star$ for all i. Such an expression for x is not necessarily unique, owing to the (CK2) relation. If we have reason to consider a different model of $L_R(E)$, say another R-algebra B, then we would write $L_R(E) \cong B$.

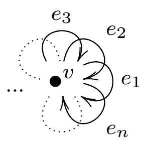
Examples 3.6. [2, §1.3] Sometimes $L_R(E)$ can be recognised as a more familiar algebra. Four fundamental examples of Leavitt path algebras are:

(a) The *finite line graph* with n vertices is the graph pictured below:

$$M_n \quad = \quad \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

It turns out that $L_R(M_n) \cong M_n(R)$, the **matrix algebra** of $n \times n$ matrices over R.

(b) The *rose with n petals* is the graph pictured below (see also Example 2.11):

$R_n \quad =$ 

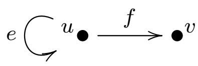
The Leavitt path algebra $L_R(R_n)$ is isomorphic to the **Leavitt algebra** $L_{n,R}$, discovered by W. G. Leavitt in [62]. It is from this example that the Leavitt path algebras get their name.

(c) The *rose with 1 petal*,

$R_1 \quad =$ 

gives rise to the algebra of **Laurent polynomials** $R[x, x^{-1}]$.

(d) The *Toeplitz graph*,

$T \quad =$ 

gives rise to the Toeplitz R-algebra, which has the presentation $R\langle x, y \mid xy = 1\rangle$.

As an alternative to Definition 3.5, it is popular to define the Leavitt path algebra of a graph as a certain quotient of a path algebra. The path algebra of a graph (also called the quiver algebra of a quiver) is an older concept, familiar to a wider audience of algebraists and representation theorists. We have defined $L_R(E)$ by its universal property, so we look towards path algebras to provide a model of $L_R(E)$, thereby proving that $L_R(E)$ exists.

Let $E = (E^0, E^1, r, s)$ be a graph. The *path algebra* of E with coefficients in R is the free R-algebra generated by $E^0 \sqcup E^1$, modulo the ideal generated by the relations (V) and (E1). The *extended graph* of E is defined as $\widehat{E} = (E^0, E^1 \sqcup (E^1)^*, r', s')$, where r' and s' are extensions of r and s, respectively:

$$\begin{aligned} r'(e) &= r(e) \text{ for all } e \in E^1; & \qquad r'(e^*) &= s(e) \text{ for all } e^* \in (E^1)^* \\ s'(e) &= s(e) \text{ for all } e \in E^1; & \qquad s'(e^*) &= r(e) \text{ for all } e^* \in (E^1)^*. \end{aligned}$$

In other words, $\widehat{E}$ is formed from E by adding a new edge e^* for each edge e, such that e^* has the opposite direction to e. The path algebra $R\widehat{E}$ can be characterised as the free R-algebra generated by $E^0 \sqcup E^1 \sqcup (E^1)^*$, subject to the relations (V), (E1), and (E2). Let $\mathcal{A}$ be the quotient of $R\widehat{E}$ by the ideal generated by the relations (CK1) and (CK2). By virtue of its construction, $\mathcal{A}$ has the universal property for Leavitt E-families, and consequently $\mathcal{A} \cong L_R(E)$. The path algebra model is useful for proving the following fact.

Proposition 3.7. *The Leavitt path algebra $L_R(E) = \bigoplus_{n \in \mathbb{Z}} L_R(E)_n$ is a $\mathbb{Z}$-graded algebra, where the homogeneous components are:*

$$L_R(E)_n = \operatorname{span}_R \{\mu\nu^* \mid (\mu, \nu) \in E^\star \times_r E^\star, |\mu| - |\nu| = n\}.$$

Proof. Naturally, the free R-algebra $R\langle E^0 \cup E^1 \cup (E^1)^*\rangle$ is $\mathbb{Z}$-graded by setting $\deg(v) = 0$ for all $v \in E^0$, and $\deg(e) = 1$, $\deg(e^*) = -1$ for all $e \in E^1$. Extending the degree map (in the only possible way) yields $\deg(a_1 \dots a_n) = \sum_{i=1}^n \deg(a_n)$ for any word $a_1 \dots a_n \in R\langle E^0 \cup E^1 \cup (E^1)^*\rangle$. The relations (V), (E1), and (E2) are all homogeneous with respect to the grading on $R\langle E^0 \cup E^1 \cup (E^1)^*\rangle$, so they generate a graded ideal, and the quotient $R\widehat{E}$ is $\mathbb{Z}$-graded. Similarly, (CK1) and (CK2) are homogeneous with respect to the grading on $R\widehat{E}$, so they generate a graded ideal, and the quotient $L_R(E)$ is $\mathbb{Z}$-graded. The word $\mu\nu^*$ has degree $|\mu| - |\nu|$ in $R\langle E^0 \cup E^1 \cup (E^1)^*\rangle$, which gives the expression for the homogeneous components of $L_R(E)$. □

3.2 Uniqueness theorems for Leavitt path algebras

Research on graph algebras has made extensive use of two main kinds of uniqueness theorems: the Cuntz-Krieger uniqueness theorems, and the graded uniqueness theorems. (In the analytic setting, graded uniqueness theorems are replaced by gauge invariant uniqueness theorems.) These theorems give sufficient conditions for a homomorphism to be injective, so they are very useful for establishing isomorphisms between a graph algebra and another algebra that comes from somewhere else. They are also very useful for studying structural properties like primeness and simplicity. Appropriate versions of these theorems have been proved not just for Leavitt path algebras but also (and we refer to [32, 33, 72, 75]) for graph C^*-algebras, as well as Cohn path algebras, higher-rank graph algebras, and even algebras of topological higher-rank graphs.

This section provides a brief account of the uniqueness theorems for Leavitt path algebras. For the Graded Uniqueness Theorem, we adhere to Tomforde's proof from [83].

Lemma 3.8. *[83, Lemma 5.1] Let I be a graded ideal of $L_R(E)$, where E is a graph. Then I is generated as an ideal by its 0-component $I_0 = I \cap L_R(E)_0$.*

Proof. Since I is a graded ideal, $I = \sum_{k\in\mathbb{Z}} I_k$, where $I_k = I \cap L_R(E)_k$. Let $k > 0$ and $x \in I_k$. By Corollary 3.4, we can write $x = \sum_{i=1}^{n} \alpha_i x_i$ where each $x_i \in L_R(E)_0$, and each $\alpha_i \in E^\star$ is distinct with $|\alpha_i| = k$. Then for $1 \le j \le n$, we have $x_j = \alpha_j^* \left(\sum_{i=1}^{n} \alpha_i x_i\right) = \alpha_j^* x \in I_0$. So, I_k is spanned by elements of the form $\alpha_j x_j$ where $\alpha_j \in L_R(E)_k$ and $x_j \in I_0$. That is, $I_k = L_R(E)_k I_0$. Similarly, if $y \in I_{-k}$ then we can write $y = \sum_{i=1}^{m} y_i \beta_i^*$ where each $y_i \in L_R(E)_0$, and each $\beta_i \in E^\star$ is distinct with $|\beta_i| = k$. Then for $1 \le j \le n$, we have $y_j = \left(\sum_{i=1}^{n} y_i \beta_i^*\right) \beta_j = y\beta_j \in I_0$. Therefore I_{-k}, is spanned by elements of the form $y_j\beta_j$ where $\beta_j \in L_R(E)_{-k}$ and $y_j \in I_0$. That is, $I_{-k} = I_0 L_R(E)_{-k}$. Since $I = \sum_{n\in\mathbb{Z}} I_k$, this shows I is the ideal generated by I_0. □

The next lemma is a slight variation of the Reduction Theorem [2, Theorem 2.2.11]. The lemma needs the assumption that $rv \in L_R(E)$ is nonzero for every $r \in R \setminus \{0\}$ and $v \in E^0$. In fact, this is always true, but we shall only prove it later.

Lemma 3.9. *[83, Lemma 5.2]. Let E be an arbitrary graph. Assume $rv \in L_R(E)$ is nonzero for every $r \in R \setminus \{0\}$ and $v \in E^0$. If $x \in L_R(E)_0$ is nonzero, then there exists $(\alpha, \beta) \in E^\star \times_r E^\star$ and $s \in R \setminus \{0\}$, such that $\alpha^* x \beta = sr(\alpha)$.*

Proof. The set $\mathcal{M}_n = \operatorname{span}_R\{\alpha\beta^* \mid 1 \le |\alpha| = |\beta| \le n\}$ is an R-submodule of $L_R(E)_0$, and indeed $L_R(E)_0 = \bigcup_{n=0}^{\infty} \mathcal{M}_n$. The strategy is to prove inductively that for all $n \ge 0$ the claim holds: for all $0 \ne x \in \mathcal{M}_n$ there exists $(\alpha, \beta) \in E^\star \times_r E^\star$ and $s \in R \setminus \{0\}$ such that $\alpha^* x \beta = sr(\alpha)$. The base case is $n = 0$. If $x \in \mathcal{M}_0$ then x is a linear combination of vertices. Say $x = \sum_i r_i v_i$ with the v_i being distinct vertices and the $r_i \in R \setminus \{0\}$. Then $v_1 x v_1 = r_1 v_1$ proves the claim. Now assume the claim holds for $n - 1$. Let $0 \ne x \in \mathcal{M}_n$. We can write

$$x = \sum_{i=1}^{p} r_i \alpha_i \beta_i^* + \sum_{j=1}^{q} s_j v_j \tag{3.1}$$

where for all $1 \leq i \leq p$ and all $1 \leq j \leq q$: $r_i, s_j \in R \backslash \{0\}$, $(\alpha_i, \beta_i) \in E^\star \times_r E^\star$ with $1 \leq |\alpha_i| = |\beta_i| \leq n$, and $v_j \in E^0$. Further assume that all the (α_i, β_i) are distinct and all the v_j are distinct. In the first case, if v_j is a sink for some $1 \leq j \leq q$, then $v_j x v_j = s_j v_j$ proves the claim. In the second case, if v_j is an infinite emitter for some $1 \leq j \leq q$, then there is an edge $e \in v_j E^1 \setminus \{(\alpha_1)_1, \ldots, (\alpha_p)_1\}$ and $e^* x e = s_j r(e)$ proves the claim. Otherwise, in the third case, every v_j is a regular vertex. Applying (CK2), it is possible to expand $v_j = \sum_{e \in v_j E^1} ee^*$ for all $1 \leq j \leq q$. Then (3.1) can be rewritten as

$$x = \sum_{i=1}^{p} t_i e_i \mu_i \nu_i^* f_i^* \tag{3.2}$$

where $t_i \in R \setminus \{0\}$, $e_i, f_i \in E^1$, and $(e_i \mu_i, f_i \nu_i) \in E^\star \times_r E^\star$ for all $i \leq 1 \leq p$. It is safe to assume that

$$\sum_{\substack{1 \leq j \leq p \\ e_j = e_1, f_j = f_1}} t_j e_j \mu_j \nu_j^* f_j^* = e_1 \left(\sum_{\substack{1 \leq j \leq p \\ e_j = e_1, f_j = f_1}} t_j \mu_j \nu_j^* \right) f_1^* \neq 0,$$

otherwise it could just be removed from the sum in (3.2). Then, define

$$x' = \sum_{\substack{1 \leq j \leq p \\ e_j = e_1, f_j = f_1}} t_j \mu_j \nu_j^*,$$

noting that $0 \neq x' \in \mathcal{M}_{n-1}$. By the inductive assumption there exists $(\alpha, \beta) \in E^\star \times_r E^\star$ and $s \in R \setminus \{0\}$ such that $\alpha^* x' \beta = sr(\alpha)$. Clearly $x' = e_1^* x f_1$, so $\alpha^* x' \beta = \alpha^* e_1^* x f_1 \beta = sr(\alpha)$. By assumption, $sr(\alpha) \neq 0$; this implies $e_1 \alpha$ and $f_1 \beta$ are legitimate paths with the same range. The claim is now proved for n, and by mathematical induction it holds for all $n \geq 0$. □

Combining these lemmas proves the Graded Uniqueness Theorem for Leavitt path algebras. This generalises both [83, Theorem 5.3] and [47, Theorem 3.2] by removing any restrictions on the cardinality of E, and by not requiring R to be a field. However, we emphasise that this is essentially Tomforde's proof with the insight that countability is not required.

Theorem 3.10 (Graded Uniqueness Theorem for Leavitt path algebras)**.**

Let E be a graph, and R a unital commutative ring. If A is a $\mathbb{Z}$-graded ring and $\pi : L_R(E) \to A$ is a graded homomorphism with the property that $\pi(rv) \neq 0$ for every $v \in E^0$ and every $r \in R \setminus \{0\}$, then π is injective.

Proof. The first observation is that $rv \neq 0$ (because $\pi(rv) \neq 0$) for every $v \in E^0$ and $r \in R \backslash \{0\}$. The second observation is that $\ker \pi$ is a graded ideal, because π is a graded homomorphism. Suppose $x \in (\ker \pi)_0 = \ker \pi \cap L_R(E)_0$. If $x \neq 0$, then by Lemma 3.9, there exists $(\alpha, \beta) \in E^\star \times_r E^\star$ and $s \in R \setminus \{0\}$, such that $\alpha^* x \beta = sr(\alpha)$. Then $\pi(sr(\alpha)) = \pi(\alpha^*)\pi(x)\pi(\beta) = 0$, which is a contradiction. Therefore $x = 0$, so $(\ker \pi)_0 = 0$. Lemma 3.8 proves that $\ker \pi$ is generated as an ideal by $(\ker \pi)_0 = 0$; consequently, $\ker \pi = 0$, so π is injective. □

Corollary 3.11. *For every nonzero graded ideal I of $L_R(E)$, there exists $r \in R \setminus \{0\}$ and $v \in E^0$ such that $rv \in I$.*

In fact, all of the uniqueness theorems have a corollary of this sort. We will not always write it so explicitly. The Cuntz-Krieger Uniqueness Theorem is similar in spirit to the Graded Uniqueness Theorem. We do not require the homomorphism to be graded, this time, but pay the price of an extra condition on the graph.

Definition 3.12. A graph E satisfies **Condition (L)** if every cycle has an exit.

Note that E satisfies Condition (L) if and only if every closed path has an exit; this is fairly intuitive and it is proved in [3, Lemma 2.5]. Combining [83, Theorem 6.5] and [47, Theorem 3.6] (see also [2, Theorem 2.2.16]) produces a version of the Cuntz-Krieger Uniqueness Theorem for Leavitt path algebras.

Theorem 3.13. *Let E be a graph satisfying Condition (L) and let R be a unital commutative ring, such that either E is countable or R is a field. If A is a ring and $\psi : L_R(E) \to A$ is a homomorphism with the property that $\psi(rv) \neq 0$ for every $v \in E^0$ and every $r \in R \setminus \{0\}$, then ψ is injective.*

This theorem can be proved for a field $R = K$, using the Reduction Theorem [2, Theorem 2.2.11]. However, we shall prove it later using groupoid methods instead. In doing so, we remove the awkward restrictions on E and R.

3.3 The Steinberg algebra model

Here, we prove the existence of a Steinberg algebra model for Leavitt path algebras, and use it to prove some fundamental facts.

Theorem 3.14. *[34] Let E be a graph and R a unital commutative ring. Then $L_R(E)$ and $A_R(\mathcal{G}_E)$ are isomorphic as $\mathbb{Z}$-graded R-algebras.*

Proof. For $v \in E^0$ and $e \in E^1$, define

$$a_v = \mathbf{1}_{Z(v)}, \qquad b_e = \mathbf{1}_{\mathcal{Z}(e,r(e))}, \qquad b_{e^*} = \mathbf{1}_{\mathcal{Z}(r(e),e)}.$$

We can routinely validate that $\{a_v, b_e, b_e^* \mid v \in E^0, e \in E^1\}$ is a Leavitt E-family. For all $e, f \in E^1$, $v, w \in E^0$, and $u \in E^0_{\text{reg}}$:

$$a_v a_w = \mathbf{1}_{Z(v)}\mathbf{1}_{Z(w)} = \mathbf{1}_{Z(v)\cap Z(w)} = \delta_{v,w}\mathbf{1}_{Z(v)}, \tag{V}$$
$$a_{s(e)} b_e a_{r(e)} = \mathbf{1}_{Z(s(e))\mathcal{Z}(e,r(e))Z(r(e))} = \mathbf{1}_{\mathcal{Z}(e,r(e))} = b_e, \tag{E1}$$
$$a_{r(e)} b_{e^*} a_{s(e)} = \mathbf{1}_{Z(r(e))\mathcal{Z}(r(e),e)Z(s(e))} = \mathbf{1}_{\mathcal{Z}(r(e),e)} = b_{e^*}, \tag{E2}$$
$$b_{e^*} b_f = \mathbf{1}_{\mathcal{Z}(r(e),e)\mathcal{Z}(f,r(f))} = \delta_{e,f}\mathbf{1}_{Z(r(e))} = \delta_{e,f} a_{r(e)}, \tag{CK1}$$
$$\mathbf{1}_{Z(u)} = \mathbf{1}_{\bigsqcup_{e\in uE^1} Z(e)} = \sum_{e\in uE^1} \mathbf{1}_{Z(e)} = \sum_{e\in uE^1} b_e b_{e^*}. \tag{CK2}$$

By the universal property of Leavitt path algebras, there is a unique homomorphism of R-algebras $\pi :\ L_R(E) \to A_R(\mathcal{G}_E)$ such that

$$\pi(v) = a_v, \qquad \pi(e) = b_e, \qquad \pi(e^*) = b_{e^*},$$

for all $v \in E^0$ and $e \in E^1$. Evidently π is a graded homomorphism. The Graded Uniqueness Theorem implies π is injective. For a path $\mu \in E^\star$, if we define $b_\mu = b_{\mu_1} \dots b_{\mu_{|\mu|}}$ and $b_{\mu^*} = b_{\mu^*_{|\mu|}} \dots b_{\mu^*_1}$ then it turns out that $b_\mu = \mathbf{1}_{\mathcal{Z}(\mu,r(\mu))}$ and $b_{\mu^*} = \mathbf{1}_{\mathcal{Z}(r(\mu),\mu)}$. Moreover, if $\nu \in E^\star$ is another path with $r(\mu) = r(\nu)$, then $b_\mu b_\nu^* = \mathbf{1}_{\mathcal{Z}(\mu,\nu)}$. If $F \subseteq_{\text{finite}} r(\mu)E^1$, this yields

$$\mathbf{1}_{\mathcal{Z}(\mu,\nu,F)} = \mathbf{1}_{\mathcal{Z}(\mu,\nu)} - \sum_{e\in F} \mathbf{1}_{\mathcal{Z}(\mu e,\nu e)} = b_\mu b_{\nu^*} - \sum_{e\in F} b_{\mu e} b_{e^*\nu^*} = \pi\Big(\mu\nu^* - \sum_{e\in F} \mu e e^* \nu^*\Big). \tag{3.3}$$

Therefore, $\mathbf{1}_{\mathcal{Z}(\mu,\nu,F)}$ is in the image of π. Corollary 1.14 implies that $A_R(\mathcal{G})$ is generated by functions of the form (3.3). We conclude that π is surjective. Therefore, π is an isomorphism. □

In the following, we generalise [83, Propositions 3.4 & 4.9] and [47, Lemmas 1.5 & 1.6] by removing restrictions on the graph and the base ring.

Corollary 3.15. *Let E be a graph and R a unital commutative ring. Then*

(1) *$L_R(E)$ has homogeneous local units, and it has a unit if and only if E^0 is finite;*

(2) *The set $\{\mu, \mu^* \in L_R(E) \mid \mu \in E^\star\}$ is R-linearly independent in $L_R(E)$;*

(3) *For every $v \in E^0$ and $r \in R \setminus \{0\}$, $rv \neq 0$.*

(4) *If $r \mapsto \overline{r}$ is an involution on R, then there exists a unique involution $L_R(E) \to L_R(E)$ such that $r\mu\nu^* \mapsto \overline{r}\nu\mu^*$ for every $r \in R$ and $(\mu, \nu) \in E^\star \times_r E^\star$.*

Proof. (1) From Proposition 1.20, $L_R(E)$ has homogeneous local units, and it has a unit if and only if ∂E is compact. Since $\partial E = \bigsqcup_{v \in E^0} Z(v)$, and each $Z(v)$ is compact and open, it is clear that ∂E is compact if and only if E^0 is finite.

(2) Since $L_R(E) = \bigoplus_{n \in \mathbb{Z}} L_R(E)_n$, it suffices to show that $\{\mu \mid \mu \in E^\star, |\mu| = n\}$ and $\{\mu^* \mid \mu \in E^\star, |\mu| = n\}$ are linearly independent in $L_R(E)$, for every $n \in \mathbb{Z}$. Equivalently, $\{\mathbf{1}_{\mathcal{Z}(\mu, r(\mu))} \mid \mu \in E^\star, |\mu| = n\}$ and $\{\mathbf{1}_{\mathcal{Z}(r(\mu), \mu)} \mid \mu \in E^\star, |\mu| = n\}$ are linearly independent in $A_R(\mathcal{G}_E)$, for every $n \in \mathbb{Z}$. This is clearly true, since $\mathcal{Z}(\mu, r(\mu)), \mathcal{Z}(\nu, r(\nu)) \neq \emptyset$ and $\mathcal{Z}(\mu, r(\mu)) \cap \mathcal{Z}(\nu, r(\nu)) = \emptyset$ for every $\mu, \nu \in E^\star$ such that $\mu \neq \nu$ and $|\mu| = |\nu|$.

(3) This follows directly from (2), or just the fact that $Z(v) \neq \emptyset$ for all $v \in E^0$.

(4) The existence follows from Proposition 1.22. The uniqueness follows from the universal property of $L_R(E)$. □

Item (3) in Corollary 3.15 is entirely disarmed by the Steinberg algebra model. It was noticed in the early years of Leavitt path algebras that a nontrivial proof was needed for Corollary 3.15 (3). The first proofs were written, separately, by Goodearl and Tomforde and they involved a representation of $L_R(E)$ on a free R-module of infinite rank $\aleph \geq \operatorname{card}(E^0 \sqcup E^1)$. Here is another result from the early years of Leavitt path algebras.

Proposition 3.16. *[5, Proposition 3.5] If E is a graph and K a field, then $L_K(E)$ is finite-dimensional if and only if E is acyclic and $E^0 \cup E^1$ is finite. In this case, if $v_1, \dots, v_t$ are the sinks and $n(v_i) = |\{\alpha \in E^\star \mid r(\alpha) = v_i\}|$, then*

$$L_K(E) \cong \bigoplus_{i=1}^{t} M_{n(v_i)}(K).$$

Proof. From Proposition 1.32 we have that $L_K(E)$ is finite-dimensional if and only if $\mathcal{G}_E$ is finite and discrete. If E had a cycle c, then the isotropy group based at $ccc\ldots \in \partial E$ would be infinite. If either E^0 or E^1 were infinite, then ∂E would be infinite, because $\partial E = \bigsqcup_{v \in E^0} Z(v) = E^0_{\text{sing}} \sqcup \big(\bigsqcup_{e \in E^1} Z(e)\big)$. Thus, $\mathcal{G}_E$ is finite only if E is acyclic and $E^0 \cup E^1$ is finite. Conversely, if E is acyclic and $E^0 \cup E^1$ is finite, then there are no infinite paths and no infinite emitters, and no paths with a repeated edge, so $\mathcal{G}_E$ is finite and discrete. To prove the final sentence, note that there are t orbits of sizes $n(v_1), \dots, n(v_t)$, all with trivial isotropy groups. The structure of $L_K(E)$ is now apparent from Proposition 1.32. □

3.4 Uniqueness theorems for Steinberg algebras

Steinberg algebras also support a Cuntz-Krieger Uniqueness Theorem and a Graded Uniqueness Theorem. These were first investigated in [24] and later improved in [25] and [79]. One can think of the Cuntz-Krieger Uniqueness Theorems as saying that a certain property of a graph, namely Condition (L), or a certain property of an ample groupoid, namely *effectiveness*, forces a homomorphism to be injective – provided it does not annihilate any scalar multiples of a local unit. This is interesting as a first example of how a Leavitt path algebra theorem translates into the more general setting of Steinberg algebras.

Briefly, this is the order of events in this section. First, we prove the Graded Uniqueness Theorem for Steinberg algebras of graded ample groupoids. Any groupoid can be graded by the trivial group, and this simple trick obtains the Cuntz-Krieger Uniqueness Theorem for Steinberg algebras. We then use the Cuntz-Krieger Uniqueness Theorem for Steinberg algebras to prove the Cuntz-Krieger Uniqueness Theorem for Leavitt path algebras.

Definitions 3.17. An étale groupoid is

(1) **effective** if $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$, where $^\circ$ denotes the interior in $\mathcal{G}$;

(2) **topologically principal** if $\{x \in \mathcal{G}^{(0)} \mid {}^x\mathcal{G}^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$.

Recall that a groupoid is called principal if the isotropy group at every unit is trivial. Being topologically principal amounts to having a dense set of units with trivial isotropy groups. Obviously, principal implies topologically principal. Effective does not imply topologically principal, with counterexamples in [24, Examples 6.3 and 6.4], and topologically principal does not imply effective, with counterexamples in [27, §5.1]. For a deeper understanding of effective groupoids, the upcoming lemma is essential. We state and prove the lemma for more general groupoids than just ample groupoids, mainly because there was an error in its original proof and we managed to correct it.

First, some topological comments are needed. Sets with compact closure are called *precompact*. A locally compact, Hausdorff étale groupoid $\mathcal{G}$ need not have a base of compact open bisections, but it does have a base of precompact open bisections [24]. Indeed, $\mathcal{G}$ has a base of open bisections. Since it is locally compact and Hausdorff, $\mathcal{G}$ has a base of open bisections, each of which is contained in a (necessarily closed) compact set, and thus has compact closure.

Lemma 3.18. *[24, Lemma 3.1] Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. Then the following are equivalent:*

(1) $\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$ *has empty interior in* $\mathcal{G}$*;*

(2) $\mathcal{G}$ *is effective;*

(3) *Every nonempty open bisection $B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ contains a morphism $g \notin \mathrm{Iso}(\mathcal{G})$;*

(4) *For every compact set $K \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ and every nonempty open $U \subseteq \mathcal{G}^{(0)}$, there exists an open subset $V \subseteq U$ such that $VKV = \emptyset$.*

Proof. (1) $\Rightarrow$ (2) Since $\mathcal{G}$ is étale and Hausdorff, $\mathcal{G}^{(0)}$ is clopen in $\mathcal{G}$, so $\mathcal{G}^{(0)} \subseteq \mathrm{Iso}(\mathcal{G})^\circ$. Now assume $(\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)})^\circ = \emptyset$. If $S \subseteq \mathrm{Iso}(\mathcal{G})$ is open, then S is a disjoint union of two open sets: $S \cap \mathcal{G}^{(0)}$ and $S \cap (\mathcal{G} \setminus \mathcal{G}^{(0)})$. But $S \cap (\mathcal{G} \setminus \mathcal{G}^{(0)}) \subseteq \left(\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}\right)^\circ = \emptyset$, so $S \subseteq \mathcal{G}^{(0)}$. This shows $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$, which means $\mathcal{G}$ is effective.

(2) $\Rightarrow$ (3) Suppose $\mathcal{G}$ is effective. If $B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ is an open bisection, then $B \subseteq \mathrm{Iso}(\mathcal{G})$ implies $B \subseteq \mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$ and therefore $B = \emptyset$.

(3) $\Rightarrow$ (1) If there are no nonempty open bisections contained in $\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$, then there are no nonempty open subsets of $\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$, and therefore $\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$ has empty interior.

(3) $\Rightarrow$ (4) We begin by proving a claim: if $B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ is an open bisection and $U \subseteq \mathcal{G}^{(0)}$ is open and nonempty, then there exists a nonempty open subset $V \subseteq U$ such that $VBV = \emptyset$. If $UBU = \emptyset$, then set $U = V$ and we are done. Otherwise, $UBU \subseteq B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ is a nonempty open bisection. Applying (3), there exists some $g \in UBU$ with $\boldsymbol{d}(g) \neq \boldsymbol{c}(g)$. Naturally, $\boldsymbol{d}(g), \boldsymbol{c}(g) \in U$. By the Hausdorff property, there exist disjoint open sets $W, W' \subseteq U$ with $\boldsymbol{c}(g) \in W$ and $\boldsymbol{d}(g) \in W'$. Set $V = W \cap \boldsymbol{c}(BW')$. Then $\boldsymbol{c}(g) \in V$, so V is nonempty, and

$$VB = \big(W \cap \boldsymbol{c}(BW')\big)B = WB \cap \boldsymbol{c}(BW')B = WB \cap BW'.$$

The last equality uses the fact that B is a bisection, so $\boldsymbol{c}(BW')B = BW'$. Therefore,

$$VBV = (WB \cap BW')V \subseteq (BW')V \subseteq (BW')W = \emptyset,$$

because $W'W = W' \cap W = \emptyset$. This proves the claim.

Now, let $K \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ be a compact set, and let $U \subseteq \mathcal{G}^{(0)}$ be open and nonempty. We set out to construct a nonempty open subset $V \subseteq U$ such that $VKV = \emptyset$. The set K, being compact, can be covered by finitely many open bisections: $K \subseteq B_1 \cup \cdots \cup B_n$. The claim in the previous paragraph proves the existence of a nonempty open set $V_1 \subseteq U$, such that $V_1 B_1 V_1 = \emptyset$. Similarly, there is a nonempty open $V_2 \subseteq V_1$ such that $V_2 B_2 V_2 = \emptyset$. Inductively, this produces a chain of open sets $\emptyset \neq V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq U$ such that $V_i B_i V_i = \emptyset$ for $1 \leq i \leq n$. Setting $V = V_n$, we have

$$VKV \subseteq V(B_1 \cup \cdots \cup B_n)V \subseteq V_1 B_1 V_1 \cup \cdots \cup V_n B_n V_n = \emptyset.$$

(4) $\Rightarrow$ (3) Suppose (3) does not hold, so there is a nonempty open bisection $B_0 \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ with $B_0 \subseteq \mathrm{Iso}(\mathcal{G})$. By shrinking it if necessary, we can assume B_0 is precompact. Let $K_0 = \overline{B_0}$, the closure of B_0. As $\mathrm{Iso}(\mathcal{G})$ is closed in $\mathcal{G}$, we have that $K_0 \subseteq \mathrm{Iso}(\mathcal{G})$. Let $U_0 = \boldsymbol{c}(B_0)$ and take any $\emptyset \neq V \subseteq U_0$. Since $K_0 \subseteq \mathrm{Iso}(\mathcal{G})$, it follows that $VK_0 = K_0 V \neq \emptyset$, so $VK_0V \neq \emptyset$. Therefore (4) does not hold, because there is no $V \subseteq U_0$ such that $VK_0V = \emptyset$. $\square$

Remark 3.19. The original proof of the "(3) $\Rightarrow$ (4)" part of [24, Lemma 3.1], does not appear to be correct. In particular, the set V defined in the proof may be empty. For a concrete example, let $\mathcal{G}$ be the groupoid of the rose with two petals (see Example 2.11). Adopting the notation of [24, Lemma 3.1], let $K = B \cup B'$ where $B = C_B = \mathcal{Z}(ee, e)$ and $B' = C_{B'} = \mathcal{Z}(e, f)$, and let $U = Z(ee)$. Let $\gamma = (eefff\ldots, 1, efff\ldots)$, so $\mathcal{F} = \{B\}$. Setting $V_B = Z(eef)$, we have $\gamma \in V_B B = \mathcal{Z}(eef, ef)$, $V_B \subseteq \boldsymbol{c}(UB)$, and $\boldsymbol{d}(V_B B) \cap V_B = Z(ef) \cap Z(eef) = \emptyset$. But $V = U \cap (V_B \setminus \boldsymbol{c}(B')) = Z(ee) \cap (Z(eef) \setminus Z(e)) = \emptyset$. Fortunately, this problem is resolved by defining V inductively, as we have done in the proof of Lemma 3.18.

Lemma 3.20. *[74, Proposition 3.6 (i)] If a Hausdorff étale groupoid $\mathcal{G}$ is topologically principal, then it is effective.*

Proof. Suppose $\mathcal{G}$ is topologically principal: the set $D = \{x \in \mathcal{G}^{(0)} \mid {}^x\mathcal{G}^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$. If $U \subseteq \mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$ is an open bisection (i.e., open in $\mathcal{G}$) then $\boldsymbol{d}(U)$ is an open subset of $\mathcal{G}^{(0)} \setminus D$, but D is dense in $\mathcal{G}^{(0)}$, so $\boldsymbol{d}(U) = \emptyset$, which implies $U = \emptyset$. This proves $\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$ has empty interior, which implies $\mathcal{G}$ is effective (noting that the proof of (1) $\Rightarrow$ (2) in Lemma 3.18 only requires $\mathcal{G}$ to be Hausdorff and étale). $\square$

The following result is an analogue of [2, Corollary 2.2.13], and it is just an alternative way of presenting some content from [25] and [79].

Proposition 3.21. *Let $\mathcal{G}$ be a Γ-graded Hausdorff ample groupoid such that $\mathcal{G}_\varepsilon$ is effective. Given a nonzero homogeneous element $h \in A_R(\mathcal{G})_\gamma$, there exists $C \in B^{\text{co}}_{\gamma^{-1}}(\mathcal{G})$, nonempty $V \in \mathcal{B}(\mathcal{G}^{(0)})$, and nonzero $r \in R$ such that $\mathbf{1}_C * h * \mathbf{1}_V = r\mathbf{1}_V$.*

Proof. *Step 1* [25, Lemma 3.1]: We show that there exists $B \in B^{\text{co}}_{\gamma^{-1}}(\mathcal{G})$ such that the function $f = \mathbf{1}_B * h$ is ε-homogeneous and its support has nonempty intersection with $\mathcal{G}^{(0)}$. Applying Lemma 1.34, we can write $h = \sum_{i=1}^n r_i \mathbf{1}_{D_i}$, where $r_1, \dots, r_n \in R \setminus \{0\}$ and $D_1, \dots, D_n \in B^{\text{co}}_*(\mathcal{G})$ are mutually disjoint. Since the D_i are disjoint and the r_i are nonzero, we can assume each $D_i \subseteq \mathcal{G}_\gamma$. Let $B = D_1^{-1}$ and define $f = \mathbf{1}_B * h$. Then

$$f = \mathbf{1}_B * h = \sum_{i=1}^n r_i \mathbf{1}_B * \mathbf{1}_{D_i} = \sum_{i=1}^n r_i \mathbf{1}_{BD_i} = r_1 \mathbf{1}_{BB^{-1}} + \sum_{i=2}^n r_i \mathbf{1}_{BD_i} \in A_R(\mathcal{G})_\varepsilon.$$

Note that $BD_1, \dots, BD_n \in B^{\text{co}}_\varepsilon(\mathcal{G})$ are mutually disjoint. Indeed, if $x \in B$ and $y \in D_i$ are composable, then $xy \in BD_j$ implies $y = x^{-1}xy \in B^{-1}BD_j = \boldsymbol{d}(B)D_j \subseteq D_j$. But $y \in D_i \cap D_j$ implies $i = j$ because $D_1, \dots, D_n$ are disjoint. To show that $(\operatorname{supp} f) \cap \mathcal{G}^{(0)} \neq \emptyset$, let $x \in B$. Then $xx^{-1} \in BD_i$ if and only if $i = 1$. Consequently, $f(xx^{-1}) = r_1 \neq 0$, so $xx^{-1} \in (\operatorname{supp} f) \cap \mathcal{G}^{(0)}$.

Step 2 [25, 79]: We show that there exists $V \in \mathcal{B}(\mathcal{G}^{(0)})$ such that $\mathbf{1}_V * f * \mathbf{1}_V = r_1 \mathbf{1}_V$, where f is from Step 1. The set $K = (\operatorname{supp} f) \setminus BB^{-1} = BD_2 \cup \dots \cup BD_n$ is a compact subset of $\mathcal{G}_\varepsilon \setminus \mathcal{G}^{(0)}$. Since $\mathcal{G}_\varepsilon$ is effective, Lemma 3.18 (4) proves that a nonempty open set $V \subseteq BB^{-1} = \boldsymbol{c}(B)$ exists such that $VKV = \emptyset$. By shrinking if necessary, we can assume V is compact. This yields

$$\mathbf{1}_V * f * \mathbf{1}_V = r_1 \mathbf{1}_{V(BB^{-1})V} + \sum_{i=2}^n r_i \mathbf{1}_{V(BD_i)V} = r_1 \mathbf{1}_V.$$

For completion: set $C = VB$ and $r = r_1$. Then $C \in B^{\text{co}}_{\gamma^{-1}}(\mathcal{G})$, $V \in \mathcal{B}(\mathcal{G}^{(0)})$ is nonempty, $r \in R$ is nonzero, and $\mathbf{1}_C * h * \mathbf{1}_V = \mathbf{1}_V * \mathbf{1}_B * h * \mathbf{1}_V = \mathbf{1}_V * f * \mathbf{1}_V = r\mathbf{1}_V$. □

We are now in a position to prove the Graded Uniqueness Theorem for Steinberg algebras.

Theorem 3.22 (Graded Uniqueness Theorem for Steinberg algebras)**.** *[25, Theorem 3.4]*

Let $\mathcal{G}$ be a Γ-graded Hausdorff ample groupoid such that $\mathcal{G}_\varepsilon$ is effective. If A is a Γ-graded ring and $\phi : A_R(\mathcal{G}) \to A$ is a graded homomorphism with the property that $\phi(r\mathbf{1}_V) \neq 0$ for every nonempty $V \in \mathcal{B}(\mathcal{G}^{(0)})$ and every $r \in R \setminus \{0\}$, then ϕ is injective.

Proof. The kernel of ϕ is a graded ideal. Let $h \in (\ker \phi)_\gamma$. If $h \neq 0$ then, according to Proposition 3.21, there exists a compact open bisection $C \subseteq \mathcal{G}_{\gamma^{-1}}$ and a nonempty compact open set $V \subseteq \mathcal{G}^{(0)}$ such that $\mathbf{1}_C * h * \mathbf{1}_V = r\mathbf{1}_V$ for some $r \neq 0$. Then $\phi(r\mathbf{1}_V) = \phi(\mathbf{1}_C)\phi(h)\phi(\mathbf{1}_V) = 0$, which contradicts the assumption about ϕ. Therefore $h = 0$, so $(\ker \phi)_\gamma = 0$. Since this is true for every $\gamma \in \Gamma$, $\ker \phi = \bigoplus_{\gamma \in \Gamma} (\ker \phi)_\gamma = 0$. □

Remark 3.23. If $\mathcal{G} = \mathcal{G}_E$ is the groupoid of a graph E, then

$$\mathcal{G}_0 = \bigcup \{\mathcal{Z}(\alpha, \beta) \mid (\alpha, \beta) \in E^\star \times_r E^\star, |\alpha| = |\beta|\}$$

so $\operatorname{Iso}(\mathcal{G}_0) = \operatorname{Iso}(\mathcal{G}_0)^\circ = \mathcal{G}^{(0)}$, which shows that $\mathcal{G}$ satisfies the hypotheses of Theorem 3.22. The Graded Uniqueness Theorem for Steinberg algebras is a generalisation of the Graded Uniqueness

Theorem for Leavitt path algebras, notwithstanding the fact that the latter theorem is usually called upon to prove that all Leavitt path algebras are Steinberg algebras.

Any groupoid can be graded by the trivial group $\{\varepsilon\}$. With this observation, we immediately obtain the Cuntz-Krieger Uniqueness Theorem for Steinberg algebras [25, Theorem 3.2] .

Corollary 3.24 (Cuntz-Krieger Uniqueness Theorem for Steinberg algebras)**.**

Let $\mathcal{G}$ be an effective Hausdorff ample groupoid. If A is a ring and $\phi : A_R(\mathcal{G}) \to A$ is a homomorphism with the property that $\phi(r\mathbf{1}_V) \neq 0$ for every nonempty $V \in \mathcal{B}(\mathcal{G}^{(0)})$ and every $r \in R \setminus \{0\}$, then ϕ is injective.

We now show how Condition (L) translates to the groupoid setting.

Proposition 3.25. *If E is a graph, then $\mathcal{G}_E$ is effective if and only if $\mathcal{G}_E$ is topologically principal, if and only if E satisfies Condition (L).*

Proof. [81] Assume that E satisfies Condition (L), so that every closed path has an exit. Then every basic open set in ∂E contains a path that is not eventually periodic. Such paths have trivial isotropy groups in $\mathcal{G}$, by Proposition 2.12, so $\mathcal{G}^{(0)}$ has a dense subset with trivial isotropy. This implies $\mathcal{G}$ is topologically principal, hence effective, by Lemma 3.20. On the other hand, if E does not satisfy Condition (L), then there exists a cycle c without an exit, and $\mathcal{G}_E$ is not effective because there is an open set: $\mathcal{Z}(cc, |c|, c) = \{(ccc\dots, |c|, ccc\dots)\} \subseteq \mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$. □

Having proved the Cuntz-Krieger Uniqueness Theorem for Steinberg algebras, we can prove the Cuntz-Krieger Uniqueness Theorem for Leavitt path algebras (see Theorem 3.13), once and for all, in its full generality.

Theorem 3.26 (Cuntz-Krieger Uniqueness Theorem for Leavitt path algebras)**.**

Let E be a graph satisfying Condition (L) and let R be a unital commutative ring. If A is a ring and $\psi : L_R(E) \to A$ is a homomorphism with the property that $\psi(rv) \neq 0$ for every $v \in E^0$ and every $r \in R \setminus \{0\}$, then ψ is injective.

Proof. First of all, suppose $r \in R \setminus \{0\}$, $\mu \in E^\star$, and F is a finite proper subset of $r(\mu)E^1$. Let $x = r\mu\mu^* - r\sum_{e\in F} \mu ee^*\mu^*$. Then $0 \neq x \in L_R(E)_0$, so Lemma 3.9 yields $(\alpha, \beta) \in E^\star \times_r E^\star$, $v \in E^0$, and $s \in R \setminus \{0\}$ such that $\alpha^* x\beta = sv$. This implies that $\psi(\alpha^*)\psi(x)\psi(\beta) = \psi(sv) \neq 0$, so $\psi(x) \neq 0$.

By Proposition 3.25, the groupoid $\mathcal{G}_E$ is effective. Let $\phi : A_R(\mathcal{G}_E) \to A$ be the map $\phi = \psi \circ \pi^{-1}$, where $\pi : L_R(E) \to A_R(\mathcal{G}_E)$ is the isomorphism from Theorem 3.14. Suppose $V \subseteq \partial E$ is compact and open, and $r \in R\setminus\{0\}$. We can find $\mu \in E^\star$ and $F \subseteq_{\text{finite}} r(\mu)E^1$ such that $Z(\mu, F)$ is a nonempty open subset of V. Then $Z(\mu, F)V = Z(\mu, F) \cap V = Z(\mu, F)$, so $r\mathbf{1}_{Z(\mu,F)} = \mathbf{1}_{Z(\mu,F)} * r\mathbf{1}_V$. Noting that $\pi^{-1}\left(r\mathbf{1}_{Z(\mu,F)}\right) = r\mu\mu^* - r\sum_{e\in F} \mu ee^*\mu^*$, the first paragraph proves that $0 \neq \psi\circ\pi^{-1}(r\mathbf{1}_{Z(\mu,F)}) = \phi\left(r\mathbf{1}_{Z(\mu,F)}\right) = \phi\left(\mathbf{1}_{Z(\mu,F)}\right)\phi(r\mathbf{1}_V)$; consequently $\phi(r\mathbf{1}_V) \neq 0$. Applying Corollary 3.24, the map ϕ is injective. Conclude that $\psi = \phi \circ \pi$ is injective. □

Chapter 4

Strong grading

Graded rings possess a high degree of structure, or rigidity, that often compensates for otherwise exotic behaviour. It is not an exaggeration to say that the $\mathbb{Z}$-graded structure of Leavitt path algebras is what makes it possible to study them so successfully. For instance, in the very first paper on Leavitt path algebras, Abrams and Aranda Pino applied a decomposition into homogeneous components to establish a criterion for simplicity (see [3, Theorem 3.11]). The same authors, in [4], used graded ideals to prove that every Leavitt path algebra over a field is semiprime and semiprimitive.

Recall that a Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is called *strongly graded* if $A_\gamma A_\delta = A_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. From our point of view, strong grading is the very best kind of grading. Dade's Theorem, first proved in [39], asserts that A is strongly graded if and only if the category of graded A-modules is naturally equivalent to the category of A_ε-modules (where ε is the identity in Γ). In other words, the ε-component "speaks" for the whole ring and gives information about its other components. If A is strongly graded, there is a correspondence between the graded one-sided ideals of A and the one-sided ideals of A_ε (see [50, Remark 1.5.6]). It follows that A is graded left noetherian (satisfies the ascending chain condition on graded left ideals) if and only if A_ε is left noetherian. In a similar vein, a strongly graded ring is graded von Neumann regular (every homogeneous element has a von Neumann inverse) if and only if its ε-component is von Neumann regular.

Studying graded rings in terms of their modules is an example of an "extrinsic" approach, since modules are external structures on which a ring acts. Dade's Theorem means that we have two equivalent ways of defining strong grading: one intrinsic and the other extrinsic. The same can be said about many concepts in ring theory. For example, a ring A is von Neumann regular if and only if every A-module is flat [46, Corollary 1.13]. Other examples are the IBN property and primitivity, which are discussed from an intrinsic and extrinsic point of view on pages 28 and 55. It is a common theme in mathematics that the theory of an algebraic structure benefits from an extrinsic approach. For example, studying G-sets, on which a group G acts, often illuminates the properties of G. In this chapter, we pursue the idea that a *sheaf* is the appropriate kind of structure on which an étale groupoid should act. In other words, sheaves are to groupoids what modules are to rings.

In [78], Steinberg showed that, for an ample groupoid $\mathcal{G}$, the category of unital right $A_R(\mathcal{G})$-modules is equivalent to the category of $\mathcal{G}$-sheaves of R-modules. This makes precise exactly how we can think of the relationship between modules and sheaves in this context. It turns out that one can characterise strongly graded groupoids both intrinsically and extrinsically, in much the same way as rings. In this chapter, we develop a theory of strongly graded groupoids and graded sheaves, in analogy with the classical theory of strongly graded rings and graded modules. However, there is just a taste of this here, and more results are included elsewhere in [29].

Strongly graded Leavitt path algebras are especially important, because it was shown in [17, Theorem 5.3] that the 0-component of a Leavitt path algebra is ultramatricial (meaning, it is a direct limit of subalgebras, each of which is a direct sum of matrix algebras). Many of the good properties of those ultramatricial 0-components, like von Neumann regularity, are then passed to the other components via strong grading. In another application, Hazrat [50, §3.9.3] applied strong grading to calculate the graded Grothendieck groups of some Leavitt path algebras. Strongly graded Leavitt path algebras have even found an application in noncommutative algebraic geometry [71].

At the workshop, "Recent Trends in Non-Commutative Algebra" held in Pune, in June 2017, Roozbeh Hazrat mentioned in his talk the problem of determining when a Leavitt path algebra is strongly graded. The answer was known for row-finite graphs with finitely many vertices: it is necessary and sufficient that every vertex connects to a cycle (see [49, Theorem 3.15]). That answer seemed like it had something to do with the groupoids underlying the Leavitt path algebras. I managed to prove that the Steinberg algebra of an ample groupoid is strongly graded if and only if the groupoid itself is strongly graded. This led quite easily to a complete characterisation of strongly graded Leavitt path algebras, for graphs of any size.

It is worth explaining briefly why it is much easier to work with gradings on groupoids, compared to gradings on rings. Let $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ be a graded ring and let $\mathcal{G} = \bigsqcup_{\gamma\in\Gamma} \mathcal{G}_\gamma$ be a graded topological groupoid. If $\gamma, \delta \in \Gamma$, then $A_\gamma A_\delta$ contains not only those elements of the form $a_\gamma a_\delta$, where $a_\gamma \in A_\gamma$ and $a_\delta \in A_\delta$, but also all finite sums of such elements. In the groupoid $\mathcal{G}$, however, sums do not exist and $\mathcal{G}_\gamma\mathcal{G}_\delta$ is just the set of all $g_\gamma g_\delta$ where $g_\gamma \in \mathcal{G}_\gamma$ and $g_\delta \in \mathcal{G}_\delta$. Another way in which groupoids tend to be easier, is that every subgroupoid of a graded topological groupoid is a graded topological groupoid. On the other hand, not every subring of a graded ring is graded.

As it was mentioned, some of the results in this chapter are repeated from a preprint [29], written in collaboration with Lisa Orloff Clark and Roozbeh Hazrat. I have chosen to focus here on the parts that I was involved with the most. However, their valuable input was present throughout. The text is almost completely rewritten to fit nicely with the rest of the dissertation. In the preprint, we proved an analogue of Dade's Theorem for $\mathcal{G}$-sheaves of R-modules. Although I was involved in the proof, I claim no credit for the idea. We left it as a remark that this could also be done with the category of graded $\mathcal{G}$-sheaves of sets, which is somewhat easier. In this chapter, I prove the version of Dade's Theorem for $\mathcal{G}$-sheaves of sets, but omit the version for $\mathcal{G}$-sheaves of R-modules.

In §4.1, we define graded modules and revise some results about strongly graded rings, including Dade's Theorem. These results, while they are well-known, needed to be modified slightly for rings with local units instead of rings with a unit. (Thanks to Roozbeh Hazrat for his tip that these modifications can be made quite easily.) In §4.2, we characterise strongly graded groupoids, and prove that the Steinberg algebra $A_R(\mathcal{G})$ is strongly graded if and only if $\mathcal{G}$ is strongly graded. In §4.3, we apply this theorem to find graph-theoretic conditions that characterise strongly graded Leavitt path algebras. We consider not only the canonical grading by $\mathbb{Z}$ but also non-canonical gradings by $\mathbb{Z}/n\mathbb{Z}$. In §4.4, we discuss some preliminaries on sheaf theory. In §4.5, we define the category of graded $\mathcal{G}$-sheaves of sets and prove that an étale groupoid $\mathcal{G}$ is strongly graded if and only if the category of graded $\mathcal{G}$-sheaves of sets is naturally equivalent to the category of $\mathcal{G}_\varepsilon$-sheaves of sets. My own involvement in this proof (and the corresponding version for $\mathcal{G}$-sheaves of R-modules) was mostly in the details, rather than the main conceptual innovations.

4.1 Graded modules and Dade's Theorem

We have been working with modules the entire time: we defined an R-algebra as an R-module with a bilinear product. Now we need some more details, because we will work with graded modules over noncommutative, non-unital rings, and these preliminaries are not quite as well-known.

Let A be a ring. Let M be an abelian group, and let $\operatorname{End} M$ be its ring of endomorphisms. We say that M is a *right A-module* if there is a homomorphism $\rho : A^{\mathrm{op}} \to \operatorname{End} M$. We write it as $\rho(a)(m) = ma$, where $a \in A$ and $m \in M$. The homomorphism ρ is called an *anti-representation* of A in M. We say that M is a *unital* A-module if $MA = M$. If A has a set of local units $E \subseteq A$, then M is a unital A-module if and only if for every $m \in M$ there exists $e \in E$ such that $me = m$. In particular, if A has a 1 then M is a unital A-module if and only if $m1 = m$ for all $m \in M$. From here on, we assume that all modules are unital.

Now let B be another ring. We say that M is a *left B-module* if there is a homomorphism $\lambda : B \to \operatorname{End} M$. We write it as $\lambda(b)(m) = bm$ for $b \in B$ and $m \in M$. The homomorphism λ is called a *representation* of A in M. We say that M is a (B, A)-bimodule if $\lambda(b) \circ \rho(a) = \rho(a) \circ \lambda(b)$ for all $a \in A$ and $b \in B$. If R is a commutative ring, every left R-module can be regarded as a right R-module and vice versa, because R and R^{op} are identical.

Now assume R is a unital commutative ring and A is a locally unital R-algebra. A right A-module M automatically has the structure of an R-module. Specifically, for $m \in M$ there exists a local unit $e \in E$ such that $me = m$, and then we define $mr = m(er)$ for all $r \in R$. This definition is unambiguous, because if $e' \in E$ is another local unit such that $me' = m$, then $m(er) = me'(er) = m(e'er) = m(ee'r) = me(e'r) = m(e'r)$. Moreover, $(ma)r = m(ar)$, for all $m \in M$, $a \in A$, and $r \in R$, so the R-module structure on M is compatible with its A-module structure; that is, $\rho(a)$ is an R-module homomorphism for every $a \in A$.

Remark 4.1. In this chapter, we adopt the convention of using right modules (rather than left modules) because we will be comparing them with sheaves, and it is customary for groupoids to act on sheaves from the right.

Now assume that A is a Γ-graded R-algebra and M is a right A-module. We say that M is a *graded right A-module* if there is a decomposition $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ where each M_γ is an additive subgroup of M, called the *γ-homogeneous component*, and $M_\gamma A_\delta \subseteq M_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. If A has homogeneous local units (that is, a set of local units $E \subseteq A_\varepsilon$) then each M_γ is an A_ε-module. A *graded homomorphism* between Γ-graded A-modules is an A-module homomorphism $f : M \to N$ such that $f(M_\gamma) \subseteq N_\gamma$ for every $\gamma \in \Gamma$. The kernel and image of f are again graded A-modules. We denote by **Mod**-A the category of right A-modules and by **GrMod**-A the category of graded right A-modules with graded homomorphisms. Graded left A-modules are defined dually. The category of left A-modules will be denoted by A-**Mod** and the category of graded left A-modules will be denoted A-**GrMod**.

For a graded A-module M, and $\alpha \in \Gamma$, we define the *α-shifted* graded right A-module $M(\alpha)$ as

$$M(\alpha) = \bigoplus_{\gamma\in\Gamma} M(\alpha)_\gamma, \tag{4.1}$$

where $M(\alpha)_\gamma = M_{\alpha\gamma}$. That is, as an ungraded module, $M(\alpha)$ is a copy of M, but the grading is shifted by α. For $\alpha \in \Gamma$, the *shift functor*

$$\mathcal{T}_\alpha : \mathbf{GrMod}\text{-}A \to \mathbf{GrMod}\text{-}A, \qquad M \mapsto M(\alpha),$$

is an isomorphism of categories, satisfying $\mathcal{T}_\alpha\mathcal{T}_\beta = \mathcal{T}_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$, and $\mathcal{T}_\varepsilon = \mathrm{id}_{\textbf{Gr-}A}$.

The next lemma gives some equivalent conditions for strong grading. It is just a generalisation of [66, Lemma A.I.3.2] for graded rings with homogeneous local units.

Lemma 4.2. *Let A be a Γ-graded ring with homogeneous local units. The following are equivalent:*

(1) *A is strongly graded;*

(2) *$A_\gamma A_{\gamma^{-1}} = A_\varepsilon$ for every $\gamma \in \Gamma$;*

(3) *For every $\gamma \in \Gamma$, the set of homogeneous local units can be chosen from $A_\gamma A_{\gamma^{-1}}$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial from the definitions. For (3) $\Rightarrow$ (1), let $\gamma, \delta \in \Gamma$. If $E \subseteq A_\gamma A_{\gamma^{-1}}$ is the set of local units, then $A_{\gamma\delta} = A_{\gamma\delta}E \subseteq A_{\gamma\delta}(A_{\delta^{-1}}A_\delta) \subseteq A_\gamma A_\delta$. ☐

If A is a Γ-graded ring and Ω is a normal subgroup of Γ, then this gives rise to two more graded rings: $A_\Omega = \bigoplus_{\omega\in\Omega} A_\omega$ is an Ω-graded ring (with the *subgroup grading*), and

$$A = \bigoplus_{[\gamma]\in\Gamma/\Omega} A_{[\gamma]} \tag{4.2}$$

is a Γ/Ω-graded ring (with the *quotient grading*), where $A_{[\gamma]} = \bigoplus_{\omega\in\Omega} A_{\omega\gamma}$.

Next, we discuss Dade's Theorem for graded modules over rings with homogeneous local units. Consider the *restriction functor*

$$\begin{aligned} \mathcal{R} : \textbf{GrMod-}A &\longrightarrow \textbf{Mod-}A_\varepsilon \\ M &\longmapsto M_\varepsilon \\ \psi &\longmapsto \psi|_{M_\varepsilon} \end{aligned} \tag{4.3}$$

and the *induction functor*

$$\begin{aligned} \mathcal{I} : \textbf{Mod-}A_\varepsilon &\longrightarrow \textbf{GrMod-}A \\ N &\longmapsto N \otimes_{A_\varepsilon} A \\ \phi &\longmapsto \phi \otimes \mathrm{id}_A \,. \end{aligned} \tag{4.4}$$

The grading on $N \otimes_{A_\varepsilon} A$ is defined by setting $(N \otimes_{A_\varepsilon} A)_\gamma = N \otimes_{A_\varepsilon} A_\gamma$. One can easily check that $\mathcal{RI}$ is equivalent to $\mathrm{id}_{\textbf{Mod-}A_\varepsilon}$ with the natural isomorphism $\eta : \mathcal{RI} \to \mathrm{id}_{\textbf{Mod-}A_\varepsilon}$ given by

$$\begin{aligned} \eta_N : \mathcal{RI}(N) = \mathcal{R}(N \otimes_{A_\varepsilon} A) = N \otimes_{A_\varepsilon} A_\varepsilon &\longrightarrow N, \\ \eta_N : n \otimes a &\longmapsto na. \end{aligned} \tag{4.5}$$

The inverse of η_N is $n \mapsto n \otimes e$ for some local unit $e \in A_\varepsilon$ such that $ne = n$. One can check that this does not depend on the choice of local unit. On the other hand, there is a natural transformation $\theta : \mathcal{IR} \to \mathrm{id}_{\textbf{GrMod-}A}$ given by

$$\begin{aligned} \theta_M : \mathcal{IR}(M) = \mathcal{I}(M_\varepsilon) = M_\varepsilon \otimes_{A_\varepsilon} A &\longrightarrow M, \\ \theta_M : m \otimes a &\longmapsto ma. \end{aligned} \tag{4.6}$$

The theorem below is Dade's Theorem in the setting of graded rings with homogeneous local units. This proof is adapted from [50, §1.5] where it is proved for unital rings.

Theorem 4.3. *Let A be a Γ-graded ring with homogeneous local units. Then A is strongly graded if and only if the functors $\mathcal{R}$ and $\mathcal{I}$* (see (4.3), (4.4)) *are mutually inverse equivalences of categories.*

Proof. ($\Rightarrow$) We already showed $\mathcal{RI} \cong \mathrm{id}_{\mathbf{Mod}\text{-}A_\varepsilon}$. It is clear that θ_M, from (4.6), is a graded homomorphism. We assume that A is strongly graded and prove that θ_M is an isomorphism. Since there is a set of local units contained in $A_{\gamma^{-1}}A_\gamma$, by Lemma 4.2, we have for all $\gamma \in \Gamma$,

$$M_\gamma = M_\gamma A_{\gamma^{-1}} A_\gamma \subseteq M_\varepsilon A_\gamma \subseteq M_\gamma, \tag{4.7}$$

and thus $\theta_M(M_\varepsilon \otimes_{A_\varepsilon} A_\gamma) = M_\varepsilon A_\gamma = M_\gamma$. This proves that θ_M is surjective. Restricting θ_M to the ε-component yields an isomorphism $M_\varepsilon \otimes_{A_\varepsilon} A_\varepsilon \to M_\varepsilon$ whose inverse is $m \mapsto m \otimes e$ for a local unit e. Let $K = \ker \theta_M$, which is a graded submodule of $M_\varepsilon \otimes_{A_\varepsilon} A$. The component K_ε is the kernel of θ_M restricted to $M_\varepsilon \otimes_{A_\varepsilon} A_\varepsilon$, so $K_\varepsilon = 0$. Performing a calculation like (4.7) yields $K_\gamma \subseteq K_\varepsilon A_\gamma = 0$. Conclude that $K = 0$ and therefore θ_M is an isomorphism. Thus, $\mathcal{IR} \cong \mathrm{id}_{\mathbf{GrMod}\text{-}A}$.

($\Leftarrow$) Conversely, suppose that $\mathcal{R}$ and $\mathcal{I}$ are mutually inverse under the natural transformations (4.5) and (4.6). Fix $\alpha, \beta \in \Gamma$. View A as a graded A-module, and consider the α-shifted module $A(\alpha)$. Applying $\mathcal{IR}$ to $A(\alpha)$ yields

$$\mathcal{IR}(A(\alpha)) = A(\alpha)_\varepsilon \otimes_{A_\varepsilon} A = A_\alpha \otimes_{A_\varepsilon} A.$$

By assumption, $\theta_{A(\alpha)} : A_\alpha \otimes_{A_\varepsilon} A \to A(\alpha)$ is an isomorphism. Consequently,

$$A_\alpha A_\beta = \theta_{A(\alpha)}(A_\alpha \otimes_{A_\varepsilon} A_\beta) = A(\alpha)_\beta = A_{\alpha\beta}$$

and this proves A is strongly graded. $\square$

Note that it is possible to have an equivalence between the categories $\mathbf{Mod}\text{-}A_\varepsilon$ and $\mathbf{GrMod}\text{-}A$, without A being strongly graded (see [67, Example 3.2.4]). It would have to be a different equivalence than (4.3) and (4.4).

4.2 Strongly graded groupoids and Steinberg algebras

We recall that a topological groupoid $\mathcal{G}$ is called Γ-*graded* if $\mathcal{G} = \bigsqcup_{\gamma\in\Gamma} \mathcal{G}_\gamma$, where each $\mathcal{G}_\gamma$ is a clopen subset such that $\mathcal{G}_\gamma\mathcal{G}_\delta \subseteq \mathcal{G}_{\gamma\delta}$ for every $\gamma, \delta \in \Gamma$. Recall that a graded groupoid $\mathcal{G}$ is *strongly graded* if $\mathcal{G}_\gamma\mathcal{G}_\delta = \mathcal{G}_{\gamma\delta}$ for every $\gamma, \delta \in \mathcal{G}$. We begin by proving a groupoid version of Lemma 4.2. It is the first of several instances where graded groupoids and graded algebras display remarkably similar results.

Lemma 4.4. *Let $\mathcal{G}$ be a Γ-graded groupoid. The following are equivalent.*

(1) *$\mathcal{G}$ is strongly graded;*

(2) *$\mathcal{G}_\gamma\mathcal{G}_{\gamma^{-1}} = \mathcal{G}_\varepsilon$, for all $\gamma \in \Gamma$;*

(3) *$\boldsymbol{d}(\mathcal{G}_\gamma) = \mathcal{G}^{(0)}$, for all $\gamma \in \Gamma$;*

(4) *$\boldsymbol{c}(\mathcal{G}_\gamma) = \mathcal{G}^{(0)}$, for all $\gamma \in \Gamma$.*

Proof. (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) For any $\gamma \in \Gamma$, $\mathcal{G}^{(0)} = \boldsymbol{d}(\mathcal{G}_\varepsilon) = \boldsymbol{d}(\mathcal{G}_\gamma\mathcal{G}_{\gamma^{-1}}) \subseteq \boldsymbol{d}(\mathcal{G}_{\gamma^{-1}}) \subseteq \mathcal{G}^{(0)}$.

(3) $\Rightarrow$ (4) For any $\gamma \in \Gamma$, $\mathcal{G}^{(0)} = \boldsymbol{d}(\mathcal{G}_{\gamma^{-1}}) = \boldsymbol{ci}(\mathcal{G}_{\gamma^{-1}}) = \boldsymbol{c}(\mathcal{G}_{\gamma})$.

(4) $\Rightarrow$ (1) For any $x \in \mathcal{G}_{\gamma\delta}$ choose $y \in \mathcal{G}_{\delta^{-1}}$ with $\boldsymbol{c}(y) = \boldsymbol{d}(x)$. Then $x = xyy^{-1} \in \mathcal{G}_{\gamma}\mathcal{G}_{\delta}$. □

If $\mathcal{G}$ is a Γ-graded groupoid and $\Omega \lhd \Gamma$, then this gives rise to two more graded groupoids: the subgroupoid $\mathcal{G}_{\Omega} = \bigsqcup_{\omega\in\Omega} \mathcal{G}_{\omega}$ is an Ω-graded groupoid (with the *subgroup grading*), and

$$\mathcal{G} = \bigsqcup_{[\gamma]\in\Gamma/\Omega} \mathcal{G}_{[\gamma]}, \tag{4.8}$$

is a Γ/Ω-graded groupoid (with the *quotient grading*), where $\mathcal{G}_{[\gamma]} = \bigsqcup_{\omega\in\Omega} \mathcal{G}_{\omega\gamma}$.

Lemma 4.5. *Let $\mathcal{G}$ be a Γ-graded groupoid and Ω a normal subgroup of Γ. Then $\mathcal{G}$ is strongly Γ-graded if and only if $\mathcal{G}$ is strongly Γ/Ω-graded and $\mathcal{G}_{\Omega}$ is strongly Ω-graded.*

Proof. ($\Rightarrow$) If $\mathcal{G}$ is strongly Γ-graded then Lemma 4.4 (2) implies that $\mathcal{G}$ is strongly Γ/Ω-graded and $\mathcal{G}_{\Omega}$ is strongly Ω-graded.

($\Leftarrow$) By Lemma 4.4 (3), $\boldsymbol{d}(\mathcal{G}_{\omega}) = \mathcal{G}^{(0)}$ for any $\omega \in \Omega$, since $\mathcal{G}_{\Omega}$ is strongly Ω-graded. Suppose $\gamma \in \Gamma \setminus \Omega$. Since $\mathcal{G}$ is strongly Γ/Ω-graded, again by Lemma 4.4 (3) we have $\boldsymbol{d}(\mathcal{G}_{[\gamma]}) = \mathcal{G}^{(0)}$. Then for any $u \in \mathcal{G}^{(0)}$, there exists $\omega \in \Omega$ and $g \in \mathcal{G}_{\omega\gamma}$ such that $\boldsymbol{d}(g) = u$. Now $\boldsymbol{c}(g) \in \mathcal{G}^{(0)} = \boldsymbol{d}(\mathcal{G}_{\omega^{-1}})$ so there exists $h \in \mathcal{G}_{\omega^{-1}}$ such that $\boldsymbol{d}(h) = \boldsymbol{c}(g)$. Then $hg \in \mathcal{G}_{\omega^{-1}}\mathcal{G}_{\omega\gamma} \subseteq \mathcal{G}_{\gamma}$ with $\boldsymbol{d}(hg) = u$. Thus $u \in \boldsymbol{d}(\mathcal{G}_{\gamma})$ and so $\boldsymbol{d}(\mathcal{G}_{\gamma}) = \mathcal{G}^{(0)}$. By Lemma 4.4, it follows that $\mathcal{G}$ is Γ-strongly graded. □

Let $\mathcal{G}$ be a Γ-graded groupoid. Recall from Proposition 1.20 and Proposition 1.33 that $A_R(\mathcal{G})$ is a Γ-graded algebra with homogeneous local units. The quotient grading and subgroup grading are preserved by the construction of Steinberg algebras. Specifically, suppose $\Omega \lhd \Gamma$. A Γ/Ω-graded structure on $A_R(\mathcal{G})$ can be obtained in the following two equivalent ways: either by viewing $\mathcal{G}$ as a Γ/Ω-graded groupoid, as in (4.8), and transferring that grading to $A_R(\mathcal{G})$, or by giving $A_R(\mathcal{G})$ the usual Γ-graded structure and taking the quotient grading of $A_R(\mathcal{G})$, as in (4.2). Similarly, $A_R(\mathcal{G}_{\Omega})$ is graded isomorphic to $A_R(\mathcal{G})_{\Omega}$.

We now prove a new theorem that relates the strong grading property for groupoids with the strong grading property for Steinberg algebras.

Theorem 4.6. *Let $\mathcal{G}$ be a Γ-graded ample groupoid. Then $\mathcal{G}$ is strongly graded if and only if $A_R(\mathcal{G})$ is strongly graded.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{G}$ is strongly graded. Fix $\gamma, \delta \in \Gamma$, and suppose $U \in B^{\mathrm{co}}_{\gamma\delta}(\mathcal{G})$. We claim that it is possible to write $\mathbf{1}_U = \sum_j r_j(f_j * g_j)$, where $r_j \in R$, $f_j \in A_R(\mathcal{G})_{\gamma}$, and $g_j \in A_R(\mathcal{G})_{\delta}$. Take some $y \in U$. Since $\mathcal{G}$ is strongly Γ-graded, there exist morphisms $p \in \mathcal{G}_{\gamma}$ and $q \in \mathcal{G}_{\delta}$ such that $y = pq$. From the continuity of groupoid multiplication, there are compact open bisections $V_y \in B^{\mathrm{co}}_{\gamma}(\mathcal{G})$, containing p, and $W_y \in B^{\mathrm{co}}_{\delta}(\mathcal{G})$, containing q, such that $y = pq \in V_yW_y \subseteq U$. Therefore, $U = \bigcup_{y\in U} V_yW_y$ and it can reduce to a finite union $U = \bigcup_{i=1}^{N} V_iW_i$ because U is compact. The principle of inclusion-exclusion yields

$$\mathbf{1}_U = \sum_{j=1}^{N} (-1)^{j-1} \sum_{\substack{I\subseteq\{1,\dots,N\}\\ |I|=j}} \mathbf{1}_{\cap_{i\in I} V_iW_i}.$$

Examining the terms,

$$\begin{aligned}\mathbf{1}_{\cap_{i\in I}V_iW_i} = \mathbf{1}_{V_{i_1}W_{i_1}\cap\cdots\cap V_{i_j}W_{i_j}} &= \mathbf{1}_{(V_{i_1}W_{i_1}\cap\cdots\cap V_{i_j}W_{i_j})W_{i_1}^{-1}W_{i_1}}\\ &= \mathbf{1}_{(V_{i_1}W_{i_1}\cap\cdots\cap V_{i_j}W_{i_j})W_{i_1}^{-1}} * \mathbf{1}_{W_{i_1}} \in A_R(\mathcal{G})_\gamma * A_R(\mathcal{G})_\delta.\end{aligned}$$

The above calculation uses the fact that W_{i_1} is a bisection, which implies $W_{i_1}^{-1}W_{i_1} = \boldsymbol{d}(W_{i_1})$. Therefore, $\mathbf{1}_U \in A_R(\mathcal{G})_\gamma * A_R(\mathcal{G})_\delta$ as claimed. Since the functions $\{\mathbf{1}_U \mid U \in B^{\mathrm{co}}_{\gamma\delta}(\mathcal{G})\}$ span $A_R(\mathcal{G})_{\gamma\delta}$, it follows that $A_R(\mathcal{G})_{\gamma\delta} \subseteq A_R(\mathcal{G})_\gamma * A_R(\mathcal{G})_\delta$ and therefore $A_R(\mathcal{G})$ is strongly graded.

($\Leftarrow$) Suppose that $\mathcal{G}$ is not strongly graded. Then there exists a pair $\gamma, \delta \in \Gamma$ and some $g \in \mathcal{G}_{\gamma\delta}$ such that $g \notin \mathcal{G}_\gamma\mathcal{G}_\delta$. Let $W \in B^{\mathrm{co}}_{\gamma\delta}(\mathcal{G})$ be a neighbourhood of g, so that $\mathbf{1}_W \in A_R(\mathcal{G})_{\gamma\delta}$. It is straightforward to check that

$$A_R(\mathcal{G})_\gamma * A_R(\mathcal{G})_\delta = \mathrm{span}_R\{\mathbf{1}_U * \mathbf{1}_V \mid U \in B^{\mathrm{co}}_\gamma(\mathcal{G}),\ V \in B^{\mathrm{co}}_\delta(\mathcal{G})\}.$$

If it were true that $\mathbf{1}_W \in A_R(\mathcal{G})_\gamma * A_R(\mathcal{G})_\delta$ then it would be possible to write

$$\mathbf{1}_W = \sum_{j=1}^n r_j(\mathbf{1}_{U_j} * \mathbf{1}_{V_j}) = \sum_{j=1}^n r_j \mathbf{1}_{U_jV_j},$$

where each $U_j \subseteq \mathcal{G}_\gamma$ and $V_j \subseteq \mathcal{G}_\delta$. In particular, this would require for at least one j that $\mathbf{1}_{U_jV_j}(g) \neq 0$, which would require $g \in U_jV_j \subseteq \mathcal{G}_\gamma\mathcal{G}_\delta$. This is a contradiction. Therefore, $A_R(\mathcal{G})$ is not strongly Γ-graded. □

4.3 Strongly graded Leavitt path algebras

Our approach to classifying strongly graded Steinberg algebras can be applied to Leavitt path algebras as a special case.

Definition 4.7. A graph E satisfies **Condition (Y)** if for every $k \in \mathbb{N}$ and every infinite path p, there exists an initial subpath α of p and a finite path β such that $r(\beta) = r(\alpha)$ and $|\beta| - |\alpha| = k$.

For example, if E is a graph such that every infinite path contains vertex that is the base of a cycle, then E satisfies Condition (Y). This includes the rose graphs. On the other hand, it is fun and easy to think of infinite acyclic graphs that satisfy Condition (Y).

Theorem 4.8. *Let E be a graph, and R a unital commutative ring. Then $L_R(E)$ is strongly $\mathbb{Z}$-graded if and only if E is row-finite, has no sinks, and satisfies Condition (Y).*

Proof. We prove the statement for $A_R(\mathcal{G}_E)$; by Theorem 3.14 this is equivalent to proving it for $L_R(E)$.

($\Rightarrow$) Firstly, suppose E has a singular vertex. Then there is a finite path $\mu \in \partial E$. The morphism $(\mu, 0, \mu) \in \mathcal{G}_E^{(0)}$ cannot be factored in the form $(\mu, |\mu|+1, x)(x, -(|\mu|+1), \mu)$, where $x \in \partial E$, so $(\mu, 0, \mu) \notin (\mathcal{G}_E)_{|\mu|+1}(\mathcal{G}_E)_{-(|\mu|+1)}$. Therefore, $\mathcal{G}_E$ is not strongly $\mathbb{Z}$-graded, so neither is $A_R(\mathcal{G}_E)$, according to Theorem 4.6. Secondly, suppose E has no singular vertices, but fails to satisfy Condition (Y). This means there is some $k \in \mathbb{N}$, and some infinite path $p \in E^\infty$, such that for every initial subpath α of p, there does not exist a finite path $\beta \in E^\star$ having $r(\beta) = r(\alpha)$ and $|\beta| - |\alpha| = k$. Therefore, the morphism $(p, 0, p) \in \mathcal{G}_E^{(0)}$ does not admit a factoring of the form

$(p, 0, p) = (\alpha p', -k, \beta p')(\beta p', k, \alpha p')$. This implies $(p, 0, p) \notin (\mathcal{G}_E)_{-k}(\mathcal{G}_E)_k$, so $\mathcal{G}_E$ is not strongly graded, and consequently $A_R(\mathcal{G}_E)$ is not strongly graded.

($\Leftarrow$) Suppose E is row-finite, has no sinks, and satisfies Condition (Y). Let $p \in \partial E$ be arbitrary. There are no singular vertices in E, so p is an infinite path. For $n \geq 0$, we have $(p, n, \sigma^n(p)) \in (\mathcal{G}_E)_n$. For $n < 0$, Condition (Y) implies that there exists some initial subpath α of p, and a finite path $\beta \in E^\star$ with $r(\beta) = r(\alpha)$ and $|\beta| - |\alpha| = -n$. Then $(p, n, \beta\sigma^{|\alpha|}(p)) = (\alpha\sigma^{|\alpha|}(p), n, \beta\sigma^{|\alpha|}(p)) \in (\mathcal{G}_E)_n$. Therefore, $p \in \boldsymbol{c}((\mathcal{G}_E)_n)$ for every $n \in \mathbb{Z}$. By Lemma 4.4 (4), $\mathcal{G}_E$ is strongly graded. By Theorem 4.6, $A_R(\mathcal{G}_E)$ is strongly $\mathbb{Z}$-graded. □

It is also possible to equip $L_R(E)$ with a non-canonical graded structure. One way to do this is to take the quotient grading by a subgroup $n\mathbb{Z} \lhd \mathbb{Z}$.

Proposition 4.9. *Let E be a graph. Then $L_R(E)$ is strongly $\mathbb{Z}/n\mathbb{Z}$-graded if and only if every singular vertex receives a path of length $n-1$.*

Proof. ($\Leftarrow$) Let $\mathcal{G} = \mathcal{G}_E$ and take $x \in \partial E$. If x is infinite, or if $|x| \geq n-1$, then for $0 \leq k \leq n-1$, we have $(x, k, \sigma^k(x)) \in \mathcal{G}_{[k]}$ so $x \in \boldsymbol{c}(\mathcal{G}_{[k]})$ for any $[k] \in \mathbb{Z}/n\mathbb{Z}$. Otherwise $0 \leq |x| < n-1$ and $r(x)$ is a singular vertex. By assumption, there exists $\mu \in E^\star$ of length $n-1$, such that $r(\mu) = r(x)$. For all $0 \leq k \leq n-1$ we have

$$\begin{aligned}(x, 0, x) &= \big(x, |x|, r(x)\big)\big(r(x), -k, \sigma^{n-1-k}(\mu)\big)\big(\sigma^{n-1-k}(\mu), k, r(x)\big)\big(r(x), -|x|, x\big)\\ &= \big(x, |x| - k, \sigma^{n-1-k}(\mu)\big)\big(\sigma^{n-1-k}(\mu), k - |x|, x\big),\end{aligned}$$

so $x \in \boldsymbol{c}(\mathcal{G}_{[|x|-k]})$. Therefore $x \in \boldsymbol{c}(\mathcal{G}_{[k]})$ for every $[k] \in \mathbb{Z}/n\mathbb{Z}$, so $\mathcal{G}_E$ is strongly $\mathbb{Z}/n\mathbb{Z}$-graded, by Lemma 4.4 (4). Conclude that $A_R(\mathcal{G}_E) \cong L_R(E)$ is strongly $\mathbb{Z}/n\mathbb{Z}$-graded.

($\Rightarrow$) If $v \in E^0_{\text{sing}}$ does not receive a path of length $n-1$, then v receives no path of length $\geq n-1$, and consequently $v \notin \boldsymbol{d}(\mathcal{G}_{[n-1]})$. By Lemma 4.4 (3), $\mathcal{G}$ is not strongly $\mathbb{Z}/n\mathbb{Z}$-graded, so $L_R(E)$ is not strongly $\mathbb{Z}/n\mathbb{Z}$-graded. □

It is worth noting that there are not many good “recipes” for creating strongly graded algebras, besides the obvious ones like group algebras. One way to do it is by forming *crossed products* (see [50, §1.1.4]). We now can easily identify a large collection of strongly graded algebras, many of which are not crossed products (by [50, Theorem 1.6.16]).

Corollary 4.10. *Let E be a row-finite graph.*

(1) *$L_R(E)$ is a strongly $\mathbb{Z}/2\mathbb{Z}$-graded ring if E contains no isolated vertex.*

(2) *$L_R(E)$ is a strongly $\mathbb{Z}/n\mathbb{Z}$-graded ring if E has no sink.*

If E^0 is finite:

(3) *$L_R(E)$ is a strongly $\mathbb{Z}$-graded ring if and only if E has no sink.*

Part (3) of the corollary recovers [49, Theorem 3.15], one of the main theorems of that paper. The proof of (3) rests on the fact that in a row-finite graph with finitely many vertices and no sinks, every vertex connects to a cycle. In this case, every boundary path is infinite and has tail equivalent paths of arbitrary lag.

Higher-rank graphs of rank k, or *k-graphs* for short, were originally defined in [60] as a generalisation of directed graphs. A higher-rank graph is a category $\Lambda = (\Lambda^0, \Lambda^1, r, s)$ equipped with a *degree functor* $d : \Lambda^1 \to \mathbb{N}^k$ satisfying a certain factorisation property. This generalises the situation in an ordinary graph, where the set of paths has a length function taking values in $\mathbb{N}$. The degree map describes the "shape" of a path. Kumjian-Pask algebras of k-graphs are $\mathbb{Z}^k$-graded algebras defined with Cuntz-Krieger-like relations (see [33, Definition 3.1 & Theorem 3.7]). Within this framework, directed graphs are 1-graphs, and the Kumjian-Pask algebra of a 1-graph is just a Leavitt path algebra. A more general version of Theorem 4.8 can be attained using the techniques in this chapter, together with some technical results on the boundary path space of a k-graph. A discussion of the more general theorem takes us too far from the topic at hand; for this we refer to [29, §4.2].

4.4 Sheaf concepts

Our main reference on sheaves is Tennison's book [82], and we shall summarise some preliminaries here. Given a diagram of topological spaces,

$$\begin{tikzcd} A \arrow[dr, "\alpha"] & & B \arrow[dl, "\beta"'] \\ & C & \end{tikzcd}$$

the set $A \times_C B = \big\{(a,b) \in A \times B \mid \alpha(a) = \beta(b)\big\}$ equipped with the relative product topology is called the *fibre product* of A and B with respect to α and β.

Definition 4.11. Let X be a topological space. A **sheaf space** over X is a pair (E, p) where E is a topological space and $p : E \to X$ is a local homeomorphism.

If (E, p) is a sheaf space, the fibre $p^{-1}(x)$ is denoted E_x, and called the *stalk* of E at x. If $S \subseteq E$ is an open subset such that $p|_S$ is a homeomorphism, we say S is an *open p-section*. This terminology emphasises that open p-sections play a similar role to *open bisections* in the context of étale groupoids (open bisections are simultaneously open $\boldsymbol{c}$-sections and $\boldsymbol{d}$-sections). A *section* (unadorned with a prefix) of p over an open set $U \subseteq X$ is a map $s : U \to E$ such that $p \circ s = \mathrm{id}_U$. A section over X is called a *global* section.

The collection of sheaf spaces over X forms a category. The appropriate definition of a *morphism of sheaf spaces* (E, p) and (F, q) is a continuous map $\phi : E \to F$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

In practice, a sheaf space (E, p) is often referred to by E when there is no need to draw attention to p.

Definition 4.12. A sheaf space E over X is called a **sheaf of R-modules** over X if each stalk E_x is an R-module such that

(1) the zero section $Z : X \to E$ sending $x \in X$ to the zero of E_x is continuous;

(2) addition $+ : E \times_{\mathcal{G}^{(0)}} E \to E$ is continuous;

(3) scalar multiplication $\cdot : R \times E \to E$ is continuous, where R has the discrete topology.

A *morphism of sheaves of R-modules* is a morphism of sheaf spaces that restricts to R-module homomorphisms on the stalks.

No introduction to sheaves is complete if it neglects to mention *presheaves*. The definition of a presheaf looks very different to a sheaf space as it is defined in Definition 4.11. However, with minor caveats, they are the same concept. In fact, most texts on sheaf theory begin with the definition of a presheaf and then define a sheaf as a special kind of presheaf.

Let X be a topological space, and let $(\mathcal{O}_X, \leq)$ be the partially ordered set of open subsets of X, where $U \leq V$ if $V \subseteq U$. Note that this reverses the inclusion relation. One can now regard $\mathcal{O}_X$ as a category in the usual way, with morphisms defined as pairs (U, V) where $U \leq V$. Let $\mathfrak{C}$ be either the category of sets or the category of R-modules.

Definition 4.13. A functor $F : \mathcal{O}_X \to \mathfrak{C}$ is called a **presheaf** over X. More precisely, F is a presheaf of sets or a presheaf of R-modules, according to the category $\mathfrak{C}$.

In other words, a presheaf consists of a set $F(U)$ for each open set U, and a *restriction morphism* $\rho^U_V : F(U) \to F(V)$ for each $V \subseteq U$, such that $\rho^U_U = \mathrm{id}_{F(U)}$ and $\rho^U_V \rho^V_W = \rho^U_W$ whenever $W \subseteq V \subseteq U$. For a presheaf of R-modules, the definition entails that every $F(U)$ is an R-module and every restriction morphism is an R-homomorphism. The elements of $F(U)$ are (again) called *sections*.

Definition 4.14. A **morphism of presheaves** F and G over X is a natural transformation $\nu : F \to G$.

In other words, it comprises a set of maps $\nu_U : F(U) \to G(U)$, $U \in \mathcal{O}_X$, such that for each $V \subseteq U$ the following diagram commutes:

$$\begin{tikzcd} F(U) \arrow[r, "\nu_U"] \arrow[d, "\rho^U_V"'] & F(U) \arrow[d, "\rho^U_V"] \\ F(V) \arrow[r, "\nu_V"] & G(V) \end{tikzcd}$$

Definition 4.15. A presheaf F over X is called a **sheaf** over X if the following two conditions hold for each open set $U \subseteq X$ and open cover $\{U_i\}_{i \in U}$ of U:

(1) If $s, t \in F(U)$ are such that $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$ for every $i \in I$, then $s = t$;

(2) If $\{s_i \mid i \in I\}$ is a family of sections $s_i \in F(U_i)$, and $\rho^{U_i}_{U_i \cap U_j}(s_i) = \rho^{U_j}_{U_i \cap U_j}(s_j)$ for all $i, j \in I$, then there exists $s \in F(U)$ that simultaneously extends all the s_i (that is, $\rho^U_{U_i}(s) = s_i$ for all $i \in I$).

Given a presheaf F, one can build a sheaf space $\mathcal{S}F$ by defining each stalk to be the direct limit in the category $\mathfrak{C}$,

$$\mathcal{S}F_x = \varinjlim_{x \in U} F(U) \tag{4.9}$$

where U ranges over the open sets containing x. The image of $s \in F(U)$ in the direct limit (4.9) is called its *germ* at x, and it is denoted by $[s]_x$. The sheaf space $\mathcal{S}F$ is defined as the (set-theoretic) disjoint union

$$\mathcal{S}F = \bigsqcup_{x \in X} \mathcal{S}F_x$$

equipped with the topology generated by the base of open sets

$$V(s, U) = \big\{[s]_x \in \mathcal{S}F \mid x \in U\big\}; \qquad U \in \mathcal{O}_X, s \in F(U).$$

In general, this is not a Hausdorff topology. The local homeomorphism $\mathcal{S}F \to X$ is the canonical map sending $\mathcal{S}F_x$ onto x.

If F started out as a presheaf of R-modules, then the stalks $\mathcal{S}F_x$ from (4.9) are R-modules; the structure is determined by $r[s]_x = [rs]_x$ and $[s]_x + [t]_x = [s+t]_x$ for sections $s, t \in F(U)$ and $r \in R$. It can be verified with ease that $\mathcal{S}F$ is a sheaf of R-modules in the sense of Definition 4.12.

In the other direction, one can transform a sheaf space into a presheaf that meets the conditions for being a sheaf (Definition 4.15). Given a sheaf space (E, p), define a presheaf $\mathcal{T}E$ by letting $\mathcal{T}E(U)$ be the set of continuous sections over U; that is, continuous maps $s : U \to E$ such that $p \circ s = \mathrm{id}_U$. The restriction morphisms ρ_V^U are, literally, restrictions of sections to smaller open sets. In the event that E is a sheaf of R-modules, $\mathcal{T}E(U)$ is an R-module under pointwise addition and scalar multiplication by elements of R, so $\mathcal{T}E$ is a presheaf of R-modules.

If (and only if) the presheaf F is a sheaf (as in Definition 4.15) then $\mathcal{T}\mathcal{S}F$ is isomorphic to F in the category of presheaves. However, if E is a sheaf space over X then it is always true that $\mathcal{S}\mathcal{T}E$ is isomorphic to E in the category of sheaf spaces. Moreover, one can extend $\mathcal{S}$ and $\mathcal{T}$ to functors by defining what they do to morphisms. The conclusion of this discussion is:

Theorem 4.16. *The functors $\mathcal{T}$ and $\mathcal{S}$ are mutually inverse equivalences between the category of sheaf spaces over X (Definition 4.11) and the category of sheaves over X (Definition 4.15).*

The theorem applies also to sheaves of R-modules.

Corollary 4.17. *The functors $\mathcal{T}$ and $\mathcal{S}$ are mutually inverse equivalences between the category of sheaves of R-modules over X (Definition 4.12) and the category of presheaves of R-modules that are sheaves over X (Definition 4.15).*

We refer to [82, §2.3 & 2.4] for the details.

4.5 Dade's Theorem for $\mathcal{G}$-sheaves of sets

In this section we introduce (graded) $\mathcal{G}$-sheaves over (graded) groupoids $\mathcal{G}$. These are sheaf spaces over $\mathcal{G}^{(0)}$ equipped with a right action of $\mathcal{G}$. Graded $\mathcal{G}$-sheaves have a Γ-indexed decomposition that is compatible with the action of $\mathcal{G}$. Our goal here is to prove Theorem 4.23, a version of Dade's Theorem for sheaves over a groupoid. In doing so, we provide an extrinsic characterisation of strongly graded groupoids.

Definition 4.18. Let $\mathcal{G}$ be an étale groupoid. A **(right) $\mathcal{G}$-sheaf** consists of a sheaf space (E, p) over $\mathcal{G}^{(0)}$ together with a continuous action $a : E \times_{\mathcal{G}^{(0)}} \mathcal{G} \to E$ (where the fibre product is with respect to p and $\boldsymbol{c}$) denoted $a(e, g) = eg$, satisfying the following axioms:

(1) $ep(e) = e$, for all $e \in E$;

(2) $p(eg) = \boldsymbol{d}(g)$, whenever $p(e) = \boldsymbol{c}(g)$;

(3) $(eg)h = e(gh)$, whenever $p(e) = \boldsymbol{c}(g)$ and $\boldsymbol{d}(g) = \boldsymbol{c}(h)$.

If E and F are $\mathcal{G}$-sheaves, a morphism of sheaf spaces $\phi : E \to F$ is called *$\mathcal{G}$-equivariant* if $\phi(eg) = \phi(e)g$ for all $(e, g) \in E \times_{\mathcal{G}^{(0)}} \mathcal{G}$. A *morphism of $\mathcal{G}$-sheaves* is a $\mathcal{G}$-equivariant morphism of sheaf spaces. The category of all $\mathcal{G}$-sheaves, $\mathscr{B}\mathcal{G}$, is called the *classifying topos* of $\mathcal{G}$.

Lemma 4.19. *Let (E, p) be a sheaf space over $\mathcal{G}^{(0)}$, where $\mathcal{G}$ is an étale groupoid.*

(1) *The map* $d_E : E \times_{\mathcal{G}^{(0)}} \mathcal{G} \to \mathcal{G}^{(0)}$ *defined by* $d_E(e, g) = \boldsymbol{d}(g)$ *is a local homeomorphism.*

(2) *A map* $a : E \times_{\mathcal{G}^{(0)}} \mathcal{G} \to E$ *is continuous if and only if it is open, if and only if it is a local homeomorphism.*

(3) *If* E *is a* $\mathcal{G}$*-sheaf and* $\sim$ *is an equivalence relation on* E *such that* $e \sim e'$ *implies* $p(e) = p(e')$ *and* $eg \sim e'g$ *for all* $g \in \boldsymbol{c}^{-1}(p(e))$*, then the quotient space* $E/\sim$ *is a* $\mathcal{G}$*-sheaf.*

Proof. (1) Let $\pi_2 : E \times_{\mathcal{G}^{(0)}} \mathcal{G} \to \mathcal{G}$ be the natural projection onto the second component. Since π_2 and $\boldsymbol{d}$ are open and continuous, so is $d_E = \boldsymbol{d}\pi_2$. Given $(e, g) \in E \times_{\mathcal{G}^{(0)}} \mathcal{G}$, let $U \subseteq E$ be an open p-section containing e, and $W \subseteq \mathcal{G}$ an open bisection containing g. Then d_E restricted to $U \times_{\mathcal{G}^{(0)}} W$ is a homeomorphism onto its image.

(2) Since $d_E = pa$, it is straightforward topology, as in [82, Ch. 2, Lemma 3.5 (c)], to show that a is continuous if and only if it is open, if and only if it is a local homeomorphism.

(3) In the commutative diagram below, the quotient map q is a continuous open surjection so the induced map $\overline{p}$ is continuous and open. Since E has a base of open p-sections, and $e \sim e'$ implies $p(e) = p(e')$, it follows that $E/\sim$ has a base of open sets on which $\overline{p}$ is injective. Hence, $E/\sim$ is a sheaf space over $\mathcal{G}^{(0)}$.

$$\begin{tikzcd} E \arrow[d, "q"'] \arrow[dr, "p"] & \\ E/\sim \arrow[r, "\overline{p}"'] & \mathcal{G}^{(0)} \end{tikzcd}$$

The continuous action $E \times_{\mathcal{G}^{(0)}} \mathcal{G} \to E$ induces an action $(E/\sim) \times_{\mathcal{G}^{(0)}} \mathcal{G} \to (E/\sim)$, defined as $q(e)g = q(eg)$ for all $e \in E$ and $g \in \mathcal{G}$. Every open set in $E/\sim$ is of the form $q(V)$ for some V open in E. Let $W \subseteq \mathcal{G}$ be open. Then $q(V)W = q(VW)$ is open because q and the action of $\mathcal{G}$ on E are open maps. By (2), the induced action is continuous. $\square$

There are two pertinent examples of $\mathcal{G}$-sheaves. The first example is that given an étale groupoid $(\mathcal{G}, \mathcal{G}^{(0)}, \boldsymbol{d}, \boldsymbol{c}, \boldsymbol{m}, \boldsymbol{i})$, the pair $(\mathcal{G}, \boldsymbol{d})$ is a sheaf space over $\mathcal{G}^{(0)}$. With the action $\boldsymbol{m}$, it becomes is a $\mathcal{G}$-sheaf. Lemma 4.19 (2) implies that $\boldsymbol{m}$ is a local homeomorphism, which we stated long ago in §1.2 but avoided proving. For the second example, let E be any sheaf space over $\mathcal{G}^{(0)}$. Lemma 4.19 (1) confirms that

$$(E \times_{\mathcal{G}^{(0)}} \mathcal{G}, d_E) \tag{4.10}$$

is also a sheaf space over $\mathcal{G}^{(0)}$. Define the action

$$a : (E \times_{\mathcal{G}^{(0)}} \mathcal{G}) \times_{\mathcal{G}^{(0)}} \mathcal{G} \to E \times_{\mathcal{G}^{(0)}} \mathcal{G}, \qquad a\big((e, g), h\big) = (e, g)h = (e, gh).$$

An easy way to show that this action is continuous is to observe that if $A \subseteq E$ and $B, C \subseteq \mathcal{G}$ are open sets, then $a((A \times_{\mathcal{G}^{(0)}} B) \times_{\mathcal{G}^{(0)}} C) = A \times_{\mathcal{G}^{(0)}} \boldsymbol{m}(B \times_{\mathcal{G}^{(0)}} C)$ is open, because m is an open map. So a is an open map, hence continuous, by Lemma 4.19 (2). It follows immediately that $(E \times_{\mathcal{G}^{(0)}} \mathcal{G}, d_E)$ is a $\mathcal{G}$-sheaf. Moreover, if F is another sheaf space over $\mathcal{G}^{(0)}$ and $\phi : E \to F$ is a morphism of sheaf spaces, then

$$\phi \times \mathrm{id} : E \times_{\mathcal{G}^{(0)}} \mathcal{G} \to F \times_{\mathcal{G}^{(0)}} \mathcal{G}, \qquad (e, g) \mapsto (\phi(e), g) \tag{4.11}$$

is a morphism of $\mathcal{G}$-sheaves. For the remainder of the section, let $\mathcal{G}$ be a Γ-graded étale groupoid with degree map $\kappa : \mathcal{G} \to \Gamma$.

Definition 4.20. A $\mathcal{G}$-sheaf E is called a **graded $\mathcal{G}$-sheaf of sets** if there is a continuous map $\kappa : E \to \Gamma$ (named again κ, and called the *degree map*) such that $\kappa(eg) = \kappa(e)\kappa(g)$ whenever $p(e) = \boldsymbol{c}(g)$.

We write $E = \bigsqcup_{\gamma\in\Gamma} E_\gamma$, where $E_\gamma = \kappa^{-1}(\gamma)$, and we call E_γ the *γ-component*. Note that the condition $\kappa(eg) = \kappa(e)\kappa(g)$ is equivalent to $E_\gamma\mathcal{G}_\delta \subseteq E_{\gamma\delta}$. A morphism of $\mathcal{G}$-sheaves $\phi : E \to F$ is a *graded morphism* if $\phi(E_\gamma) \subseteq F_\gamma$ for any $\gamma \in \Gamma$. The category of all graded $\mathcal{G}$-sheaves of sets with graded morphisms is denoted by $\mathscr{B}^{\mathrm{gr}}\mathcal{G}$. There is a natural forgetful functor $U : \mathscr{B}^{\mathrm{gr}}\mathcal{G} \to \mathscr{B}\mathcal{G}$.

For a graded $\mathcal{G}$-sheaf of sets E, and $\alpha \in \Gamma$, we define the *α-shifted graded $\mathcal{G}$-sheaf of sets* (compare with (4.1))

$$E(\alpha) = \bigsqcup_{\gamma\in\Gamma} E(\alpha)_\gamma, \tag{4.12}$$

where $E(\alpha)_\gamma = E_{\alpha\gamma}$. As an ungraded sheaf, $E(\alpha)$ is the same as E but the grading is shifted by α.

Example 4.21. Let E be a $\mathcal{G}_\varepsilon$-sheaf. Since $\mathcal{G}_\varepsilon$ is an open subgroupoid of $\mathcal{G}$ containing $\mathcal{G}^{(0)}$, clearly E is a sheaf space over $\mathcal{G}^{(0)}$ and so $E \times_{\mathcal{G}^{(0)}} \mathcal{G}$, from (4.10), is a $\mathcal{G}$-sheaf. Defining $\kappa : E \times_{\mathcal{G}^{(0)}} \mathcal{G} \to \Gamma$, $\kappa(e, g) = \kappa(g)$, gives $E \times_{\mathcal{G}^{(0)}} \mathcal{G}$ the structure of a graded $\mathcal{G}$-sheaf of sets. If F is another $\mathcal{G}_\varepsilon$-sheaf and $\phi : E \to F$ is a morphism of $\mathcal{G}_\varepsilon$-sheaves, then $\phi \times \mathrm{id}$ (see (4.11)) is a graded morphism.

Example 4.22. Take a $\mathcal{G}_\varepsilon$-sheaf E and its associated $\mathcal{G}$-sheaf $E \times_{\mathcal{G}^{(0)}} \mathcal{G}$ from Example 4.21, and define on it an equivalence relation:

$$(eh, g) \sim (e, hg) \qquad \text{for all } e \in E, h \in \mathcal{G}_\varepsilon, g \in \mathcal{G} \text{ such that } p(e) = \boldsymbol{c}(h) \text{ and } \boldsymbol{d}(h) = \boldsymbol{c}(g).$$

Let $E\widehat{\otimes}_{\mathcal{G}_\varepsilon}\mathcal{G}$ be the quotient space of $E \times_{\mathcal{G}^{(0)}} \mathcal{G}$ by $\sim$, and denote the equivalence class of (e, g) by $e\widehat{\otimes}g$. By Lemma 4.19 (3), $E\widehat{\otimes}_{\mathcal{G}_\varepsilon}\mathcal{G}$ is a $\mathcal{G}$-sheaf with the action $(e\widehat{\otimes}g)g' = e\widehat{\otimes}(gg')$ for all $(e, g) \in E \times_{\mathcal{G}^{(0)}} \mathcal{G}$ and $g' \in {}^{\boldsymbol{c}(g)}\mathcal{G}$. Moreover, it is a graded $\mathcal{G}$-sheaf with degree map $\kappa(e\widehat{\otimes}g) = \kappa(g)$.

We now take steps towards proving a version of Dade's Theorem for graded $\mathcal{G}$-sheaves of sets. It turns out that the construction from Example 4.22 describes an *induction functor*

$$\begin{aligned} \mathcal{I} : \mathscr{B}\mathcal{G}_\varepsilon &\longrightarrow \mathscr{B}^{\mathrm{gr}}\mathcal{G} \\ E &\longmapsto E\widehat{\otimes}_{\mathcal{G}_\varepsilon}\mathcal{G}. \end{aligned} \tag{4.13}$$

If $\phi : E \to F$ is a morphism of $\mathcal{G}_\varepsilon$-sheaves, then define

$$\begin{aligned} \mathcal{I}(\phi) = \phi\widehat{\otimes}\,\mathrm{id} : E\widehat{\otimes}_{\mathcal{G}_\varepsilon}\mathcal{G} &\longrightarrow F\widehat{\otimes}_{\mathcal{G}_\varepsilon}\mathcal{G} \\ e\widehat{\otimes}g &\longmapsto \phi(e)\widehat{\otimes}g. \end{aligned}$$

Indeed, $\phi\widehat{\otimes}\,\mathrm{id}$ is a graded morphism of $\mathcal{G}$-sheaves. Clearly, it is graded and $\mathcal{G}$-equivariant, and it is continuous because the following diagram commutes:

$$\begin{tikzcd} E \times_{\mathcal{G}^{(0)}} \mathcal{G} \arrow[r, "\phi\times\mathrm{id}"] \arrow[d] & F \times_{\mathcal{G}^{(0)}} \mathcal{G} \arrow[d] \\ E\widehat{\otimes}_{\mathcal{G}_\varepsilon}\mathcal{G} \arrow[r, "\phi\widehat{\otimes}\,\mathrm{id}"'] & F\widehat{\otimes}_{\mathcal{G}_\varepsilon}\mathcal{G} \end{tikzcd}$$

Consider the *restriction functor*

$$\begin{aligned} \mathcal{R} : \mathscr{B}^{\mathrm{gr}}\mathcal{G} &\longrightarrow \mathscr{B}\mathcal{G}_\varepsilon \\ (D, p) &\longmapsto (D_\varepsilon, p_\varepsilon) \end{aligned} \tag{4.14}$$

Here, $p_\varepsilon = p|_{D_\varepsilon}$ and the action of $\mathcal{G}_\varepsilon$ on $(D_\varepsilon, p_\varepsilon)$ is just a restriction of the original action. Given a morphism $\psi : D \to F$ in $\mathscr{B}^{\mathrm{gr}}\mathcal{G}$, define

$$\mathcal{R}(\psi) = \psi_\varepsilon = \psi|_{D_\varepsilon} : D_\varepsilon \longrightarrow F_\varepsilon.$$

There is a natural isomorphism $\eta : \mathcal{RI} \to \mathrm{id}_{\mathscr{B}\mathcal{G}_\varepsilon}$,

$$\begin{aligned} \eta_E : \mathcal{RI}(E) = \mathcal{R}(E \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}) = E \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}_\varepsilon &\longrightarrow E \\ \eta_E : e \widehat{\otimes} g &\longmapsto eg. \end{aligned} \tag{4.15}$$

Its inverse is $e \mapsto e \widehat{\otimes} p(e)$. On the other hand, there is a natural transformation $\theta : \mathcal{IR} \to \mathrm{id}_{\mathscr{B}^{\mathrm{gr}}\mathcal{G}}$

$$\begin{aligned} \theta_D : \mathcal{IR}(D) = \mathcal{I}(D_\varepsilon) = D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G} &\longrightarrow D \\ \theta_D : d \widehat{\otimes} g &\longmapsto dg \end{aligned} \tag{4.16}$$

Now we have an analogue of Theorem 4.3, for groupoids and sheaves instead of modules and rings.

Theorem 4.23. *Let $\mathcal{G}$ be a Γ-graded étale groupoid. Then $\mathcal{G}$ is strongly graded if and only if the functors $\mathcal{I}$ and $\mathcal{R}$ (see (4.13), (4.14)) are mutually inverse equivalences of categories.*

Proof. ($\Rightarrow$) Assume that $\mathcal{G}$ is strongly graded. For all $\gamma \in \Gamma$,

$$D_\gamma = D_\gamma \mathcal{G}^{(0)} \subseteq D_\gamma \mathcal{G}_\varepsilon = D_\gamma \mathcal{G}_{\gamma^{-1}} \mathcal{G}_\gamma \subseteq D_\varepsilon \mathcal{G}_\gamma \subseteq D_\gamma.$$

This proves that $\theta_D(D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}_\gamma) = D_\varepsilon \mathcal{G}_\gamma = D_\gamma$, from which it follows that θ_D is surjective because it is a graded morphism. To prove that θ_D is injective, it suffices to prove that the restriction of θ_D to the set

$$D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}_\gamma^x = \left\{ d \widehat{\otimes} g \in D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G} \mid g \in \mathcal{G}_\gamma^x \right\},$$

is injective for all $\gamma \in \Gamma$ and $x \in \mathcal{G}^{(0)}$. Let $(\theta_D|_x)_\gamma : D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}_\gamma^x \to D_x \cap D_\gamma$ be the restriction of θ_D. Lemma 4.4 yields $\boldsymbol{d}(\mathcal{G}_\gamma) = \mathcal{G}^{(0)}$, so we can pick $h \in \mathcal{G}_\gamma$ such that $h^{-1}h = x$. Then, define $(\psi|_x)_\gamma : D_x \cap D_\gamma \to D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}_\gamma^x$ as $d \mapsto dh^{-1} \widehat{\otimes} h$. If $d \in D_x \cap D_\gamma$ then

$$(\theta_D|_x)_\gamma (\psi|_x)_\gamma (d) = (\theta_D|_x)_\gamma (dh^{-1} \widehat{\otimes} h) = dh^{-1}h = d.$$

On the other hand, if $d \widehat{\otimes} g \in D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}_\gamma^x$ then

$$(\psi|_x)_\gamma (\theta_D|_x)_\gamma (d \widehat{\otimes} g) = (\psi|_x)_\gamma (dg) = dgh^{-1} \widehat{\otimes} h = d \widehat{\otimes} gh^{-1}h = d \widehat{\otimes} g.$$

This has shown that the maps $(\psi|_x)_\gamma$ and $(\theta_D|_x)_\gamma$ are inverse to each other, implying that θ_D is injective. Thus $\mathcal{IR}(D) = D_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G} \cong D$ and all the morphisms involved are natural. The conclusion is that $\mathcal{IR}$ is equivalent to the identity functor on $\mathscr{B}^{\mathrm{gr}}\mathcal{G}$.

($\Leftarrow$) Suppose that $\mathcal{I}$ and $\mathcal{R}$ are mutually inverse equivalences (under (4.15) and (4.16)). Fix $\alpha, \beta \in \Gamma$. View $\mathcal{G}$ as a graded $\mathcal{G}$-sheaf, and consider (from 4.12) the α-shifted sheaf $\mathcal{G}(\alpha)$. Applying $\mathcal{IR}$ to $\mathcal{G}(\alpha)$ yields

$$\mathcal{IR}(\mathcal{G}(\alpha)) = \mathcal{G}(\alpha)_\varepsilon \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G} = \mathcal{G}_\alpha \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}.$$

The assumption implies $\theta_{\mathcal{G}(\alpha)} : \mathcal{IR}(\mathcal{G}(\alpha)) = \mathcal{G}_\alpha \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G} \longrightarrow \mathcal{G}(\alpha)$ is an isomorphism. Since $\theta_{\mathcal{G}(\alpha)}$ is a graded isomorphism, it is surjective onto the β-component. Therefore $\mathcal{G}$ is strongly graded because $\mathcal{G}_\alpha \mathcal{G}_\beta = \theta_{\mathcal{G}(\alpha)}(\mathcal{G}_\alpha \widehat{\otimes}_{\mathcal{G}_\varepsilon} \mathcal{G}_\beta) = \mathcal{G}(\alpha)_\beta = \mathcal{G}_{\alpha\beta}$, for all $\alpha, \beta \in \gamma$. $\square$

Chapter 5

Structure theory

The notions of primitivity and semiprimitivity are very much at the heart of the classical structure theory of rings. As Jacobson writes in [55], "The structure theory that we shall develop is based on two articles of faith: irreducible representations are the best kind of representations and the best behaved rings are those that have enough irreducible representations to distinguish between elements of the ring." The kinds of rings with "enough irreducible representations to distinguish between elements" are called *semiprimitive*. A ring with a single irreducible representation that distinguishes between elements (that is, a faithful irreducible representation) is called *primitive*. One of the main goals of this chapter is to investigate the primitivity of Steinberg algebras, and pursue a characterisation of primitive Leavitt path algebras. Along the way, by necessity as well as curiosity, we study *prime* and *semiprime* Steinberg and Leavitt path algebras, making use of graded methods to do so.

Let V be a vector space over a division ring D, and let $\operatorname{End}_D V$ be the ring of linear transformations on V. A subset $T \subseteq \operatorname{End}_D V$ is called *dense* if for every finite-dimensional subspace $U \subseteq V$, and every $f \in \operatorname{End}_D V$, there exists $t \in T$ such that $t|_U = f|_U$. The famous Jacobson Density Theorem says that a ring A is primitive if and only if it is isomorphic to a dense subring of $\operatorname{End}_D V$, for some vector space V over a division ring D. This is one reason why primitive rings are so compelling and sought after. By now, we are no strangers to mixing topology and algebra. It turns out that a dense subset of $\operatorname{End}_D V$ really is *dense* with respect to a certain topology on $\operatorname{End}_D V$. It is a slightly mysterious discovery, therefore, that certain dense subsets of groupoids play an important role in the primitivity results on Steinberg algebras.

Over any field, zero divisors exist in the matrix algebras of size $n \geq 2$, the group algebras of finite groups, and all the noncommutative Leavitt path algebras. It is unrealistic, therefore, to expect many noncommutative rings to be domains. In light of this, semiprimeness plays a role in noncommutative ring theory not unlike that of integral domains in commutative ring theory. Matrix algebras and Leavitt path algebras over a field are always semiprime. By Maschke's Theorem, group algebras over a field are semiprime except when the characteristic of the field interferes with the order of the group. We show that a Steinberg algebra over a field (more generally, a reduced ring) is semiprime, provided that the groupoid is effective, so there is not too much isotropy. In some cases where a grading is present, it is only necessary for the ε-component to be effective.

In §5.1, we define the central concepts in the structure theory of rings, discuss how they are related, and provide some characterisations. In §5.2, we develop methods for rings graded by ordered groups. In these cases, it is much easier to check for primeness, semiprimeness, and semiprimitivity.

In §5.3, we prove primeness and semiprimeness criteria for the R-algebras associated to graphs and some ample groupoids. In a technical section, §5.4, we construct irreducible representations of Steinberg algebras from irreducible representations of the isotropy group at a point. In §5.5, we state Steinberg's Equivalence Theorem: $\mathbf{Mod}\text{-}A_R(\mathcal{G})$ is equivalent to the category of $\mathcal{G}$-sheaves of R-modules. In §5.6, we relate the primitivity of $A_K(\mathcal{G})$ to the existence of dense orbits in $\mathcal{G}^{(0)}$. In §5.7, we write the groupoid-theoretic proof of the Abrams-Bell-Rangaswamy Primitivity Theorem for Leavitt path algebras.

5.1 Concepts in the structure theory of rings

In contrast with Chapter 4, the default meaning of a module in this chapter is a *left* module. Most of the time, we will work with left modules, except in one or two places. Since our focus is on involutive rings, it makes no material difference whether we use left or right modules. Recall that all modules are assumed to be unital. An A-module M is called *simple* if $M \neq 0$ and M has no A-submodules besides 0 and M. A module is called *cyclic* if it is generated by a single element. A representation $\rho : A \to \operatorname{End} M$ is *irreducible* if M is a simple A-module, and the representation is *faithful* if ρ is injective. Sometimes we simply say that M is a faithful A-module if the associated representation is faithful.

Definitions 5.1. A ring A is called:

(1) **simple** if $A^2 \neq 0$ and the only two-sided ideals are 0 and A;

(2) **primitive** if it has a faithful irreducible representation;

(3) **semiprimitive** if for every $x \in A \setminus \{0\}$, there exists an irreducible representation ρ such that $\rho(x) \neq 0$.

(4) **prime** if for all $x, y \in A$, $xAy = 0$ implies $x = 0$ or $y = 0$;

(5) **semiprime** if for all $x \in A$, $xAx = 0$ implies $x = 0$.

Primitivity and semiprimitivity can also be defined intrinsically (i.e. without referring to modules). A ring A is primitive if and only if it has a maximal left ideal I such that $A^2 \not\subseteq I$ and I contains no nonzero proper ideals (this is just [55, Proposition 4.2] modified for nonunital rings, using [23, Lemma 3.46]). Obviously, semiprimeness is a necessary condition for von Neumann regularity. It is not quite as obvious, but semiprimitivity is also necessary for von Neumann regularity (see [46, Corollary 1.2(c)])

These are concepts that really come to life in noncommutative ring theory. Nevertheless, it is interesting to know what they can say about commutative rings. A commutative ring R is primitive if and only if it is a field, and it is semiprimitive if and only if it is a subdirect product of fields (see [23, Lemmas 5.7 & 5.72]). Moreover, R is prime if and only if it is an integral domain, and R is semiprime if and only if it is a reduced ring (i.e., it has no nonzero elements whose square is zero). The previous sentence generalises:

Lemma 5.2. *Let A be an R-algebra with local units in which R is represented faithfully. If A is prime (resp., semiprime) then R is an integral domain (resp., reduced ring).*

Proof. If R is not an integral domain then there is a pair of zero divisors $s, t \in R \setminus \{0\}$. Since A is faithful as an R-module, there exist $x, y \in A$ such that $sx \neq 0$ and $ty \neq 0$. Moreover, there is a local unit e such that $s(ex) = sy$ and $t(ex) = tx$, which implies $se, te \neq 0$. But $(se)A(te) = st(eAe) = 0$, so A is not prime. If R is not reduced, then there exists $r \in R \setminus \{0\}$ with $r^2 = 0$, and there exists a local unit e such that $re \neq 0$ (for the same reasons as before). But $(re)A(re) = r^2(eAe) = 0$, so A is not semiprime. □

The properties (2)-(5) in Definition 5.1 have the following interactions.

Lemma 5.3. *[23, Lemma 5.4] For a ring A, the following dependencies hold:*

$$\begin{tikzcd} \textit{primitive} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textit{prime} \arrow[d, Rightarrow] \\ \textit{semiprimitive} \arrow[r, Rightarrow] & \textit{semiprime} \end{tikzcd}$$

Proof. It is straightforward from the definitions that primitive implies semiprimitive, and prime implies semiprime. Now suppose A is primitive, and choose a faithful simple module M. If $x, y \in A$ and $x, y \neq 0$, then there exist $m, m' \in M$ such that $xm, ym' \neq 0$. Since M is simple, $Aym' = M$, so there exists $a \in A$ such that $aym' = m$. Then $xaym' = xm \neq 0$, so $xAy \neq 0$. Therefore A is prime. Next, suppose A is semiprimitive and let $x \in A$ be nonzero. Choosing an irreducible representation in a module M such that $xM \neq 0$, we have that $A(xM) = M$ because M is simple. It follows that $(xAx)M = xM \neq 0$ so $xAx \neq 0$. Therefore A is semiprime. □

We now move towards characterising primeness, semiprimeness, and semiprimitivity. By default, an ideal is two-sided; a one-sided ideal will be called a left or right ideal as appropriate.

Lemma 5.4. *[23, Lemmas 2.17 & 2.21]*

(1) *A ring A is prime if and only if for every pair of ideals I and J,*

$$IJ = 0 \textit{ implies } I = 0 \textit{ or } J = 0.$$

(2) *Likewise, A is semiprime if and only if for every ideal I,*

$$I^2 = 0 \textit{ implies } I = 0.$$

Proof. (1) Suppose A has the property that $IJ = 0$ implies $I = 0$ or $J = 0$ for all ideals I and J. Let $x, y \in A$ and assume $xAy = 0$. Then AxA and AyA are ideals whose product is zero, so either $AxA = 0$ or $AyA = 0$. Assume $AxA = 0$. Given that $AxA = 0$, both xA and Ax are ideals satisfying $A(xA) = (Ax)A = 0$, so $xA = Ax = 0$. (If A has local units, then we conclude at this point that $x = 0$.) Even without local units, $xA = Ax = 0$ implies $\mathbb{Z} \cdot x$ (the additive subgroup generated by x) is an ideal satisfying $(\mathbb{Z} \cdot x)A = 0$, so $\mathbb{Z} \cdot x = 0$, whereby $x = 0$. Therefore A is prime. For the converse, suppose A is prime and I and J are ideals. Then $IJ = 0$ implies $IAJ = 0$ (because $AJ \subseteq J$). If $I \neq 0$, say, then there is a nonzero $x \in I$ such that $xAJ = 0$, which implies $J = 0$, because A is prime.

(2) Suppose A has the property that $I^2 = 0$ implies $I = 0$ for all ideals I. Suppose $xAx = 0$. Then AxA is an ideal with zero square, and consequently $AxA = 0$. Then xA and Ax are ideals with zero square, so $xA = Ax = 0$. Then $\mathbb{Z} \cdot x$ is an ideal with zero square, so $\mathbb{Z} \cdot x = 0$, and $x = 0$. Therefore A is semiprime. For the converse, if A is semiprime and $I^2 = 0$ for an ideal I, then $IAI = 0$, and consequently $I = 0$ because A is semiprime. □

Let A be a ring with a representation $\rho : A \to \operatorname{End} M$. The kernel of ρ is an ideal called the *annihilator* of M. If $a \in \ker \rho$ then we say that a *annihilates* M. An ideal is called *primitive* if it is the annihilator of a simple module.

Definition 5.5. Let A be a ring. The **Jacobson radical** $\boldsymbol{J}(A)$ is defined as the intersection of all the primitive ideals of A. If A has no primitive ideals (that is, no irreducible representations) then by definition $\boldsymbol{J}(A) = A$.

Every unital ring except the zero ring has an irreducible representation, so the second sentence in the definition only exists to deal with (very pathological) nonunital rings. Clearly, $\boldsymbol{J}(A)$ is an ideal because it is an intersection of ideals. From the definitions, it is clear that A is semiprimitive if and only if $\boldsymbol{J}(A) = 0$. Indeed, if $0 \neq a \in \boldsymbol{J}(A)$ then a annihilates all simple A-modules, so A fails to be semiprimitive. Conversely, if A is not semiprimitive, then there exists some $a \in A \setminus \{0\}$ having no irreducible representation that separates a from 0, so $a \in \boldsymbol{J}(A)$. The next lemma is fairly easy to prove, following the ideas from [23, Lemma 5.53].

Lemma 5.6. *If A is a ring, $\boldsymbol{J}(A)$ contains no nonzero idempotents.*

Proof. The statement is equivalent to the claim: for every nonzero idempotent $e \in A$, there is a simple A-module M such that $eM \neq 0$. To this end, let $e \in A$ be a nonzero idempotent, and consider the left ideal $L = \{ae - a \mid a \in A\}$. Firstly, notice that $e \notin L$, for if it were, then there would exist $a \in A$ such that $ae - a = e$, and so $ae - ae = e = 0$. Secondly, if $L \subseteq L'$ for a proper left ideal L', then we claim that $e \notin L'$. For if $e \in L'$, and $b \in A$, then $be \in L'$, and so $be - (be - b) = b \in L'$. This would imply $L' = A$. By Zorn's lemma, there exists a maximal proper left ideal $\mathcal{L}$ with $L \subseteq \mathcal{L}$. We established that $e \notin \mathcal{L}$. We claim that $M = A/\mathcal{L}$ is a simple A-module. Indeed, if N is a submodule of M, then $K = \{a \in A \mid a + \mathcal{L} \in N\}$ is a left ideal of A that contains $\mathcal{L}$, so $K = \mathcal{L}$ or $K = A$. But $K = \mathcal{L}$ implies $N = 0$, while $K = A$ implies $N = M$. Therefore M has no submodules besides 0 and M. For M to be simple, we also need that $AM = A(A/\mathcal{L}) \neq 0$. This is equivalent to $A^2 \not\subseteq \mathcal{L}$, which is obvious because $e = e^2 \in A^2 \setminus \mathcal{L}$. Thereore M is simple. Moreover, $e + \mathcal{L} \neq 0$, so $eM \neq 0$. □

Simple unital rings are always primitive. More generally:

Corollary 5.7. *If A is a simple ring containing a nonzero idempotent, then A is primitive.*

Proof. By Lemma 5.6, A has an irreducible representation ρ, because $\boldsymbol{J}(A) \neq A$. Since A is simple, $\ker \rho = 0$, so ρ is faithful, and thus A is primitive. □

A *corner* of a ring A, associated to a nonzero idempotent $e \in A$, is the unital ring eAe. We close this section with a multi-part lemma that will allow us to "cut some corners" later on in the chapter. Like many of the statements in this section, it is difficult to find a reference for these facts in the setting of nonunital rings.

Lemma 5.8. *Let A be a ring and $e \in A$ a nonzero idempotent.*

(1) *There exists a simple A-module M such that $(eAe)M \neq 0$.*

(2) *If M is a simple A-module such that $(eAe)M \neq 0$ then eM is a simple eAe-module.*

(3) *If A is a ring and $e \in A$ is a nonzero idempotent, then $\boldsymbol{J}(eAe) \subseteq \boldsymbol{J}(A)$;*

(4) *If A is semiprimitive then eAe is semiprimitive;*

(5) *If A is primitive then eAe is primitive.*

Proof. (1) If $(eAe)M = 0$ for every simple A-module M, then $eM = 0$ for every simple A-module M, which would imply that $e \in \boldsymbol{J}(A)$, and this is impossible by Lemma 5.6.

(2) Suppose N is an eAe-submodule of eM. Then $eN = N$, and AN is an A-submodule of M, because $AN \supseteq A(AN) \supseteq A(eN) = AN$. Since M is simple, either $AN = 0$ or $AN = M$. If $AN = 0$ then $N = (eAe)N = 0$. Whereas, if $AN = M$ then $N = (eAe)N = e(AN) = eM$. This, together with the fact that $(eAe)M \neq 0$, proves that eM is a simple eAe-module.

(3) Let $a \in \boldsymbol{J}(eAe)$. Then a annihilates every simple eAe-module. Part (2) shows there is a dichotomy for simple A-modules M: either $(eAe)M = 0$ or eM is a simple eAe-module. Either way, $aM = 0$ for every simple A-module M, and thus $a \in J(A)$.

(4) If A is semiprimitive then $\boldsymbol{J}(eAe) \subseteq \boldsymbol{J}(A) = 0$ and thus eAe is semiprimitive.

(5) If A is primitive then A has a faithful representation ρ in a simple module M. Since ρ is faithful, $aM \neq 0$ for every nonzero $a \in eAe$. If $a \in eAe$ annihilates eM then $a = ae$ annihilates M, which is a contradiction. So $a(eM) \neq 0$. This shows that the representation $\rho' : eAe \to \operatorname{End} eM$, defined by $\rho'(a) = \rho(a)|_{eM}$, is faithful. Moreover (2), proves that eM is simple, and thus eAe is primitive. ☐

5.2 Graded methods

In [79], Steinberg develops some sufficient conditions for a Steinberg algebra to be semiprimitive. For effective Hausdorff ample groupoids, it is shown that $A_R(\mathcal{G})$ is semiprimitive if R is semiprimitive (which applies of course to the case when $R = K$ is a field). When the groupoid is non-effective, Steinberg's results essentially reduce it to the problem of whether there is a dense subset of units whose isotropy groups have semiprimitive group algebras. However, the problem of determining the semiprimitivity of a group algebra is still very much unsolved (see [23, Problem 5.69]). Except, it is known by Amitsur's theorem [13] that a group algebra KG is semiprimitive if K is a transcendental field extension of $\mathbb{Q}$. It is also known that $K[x, x^{-1}]$, the group algebra of $\mathbb{Z}$, is semiprimitive for any field K. Boundary path groupoids have the feature that all of their isotropy groups are either trivial or infinite cyclic, and this leads to a fact that has been known for some time:

Proposition 5.9. *[2, Proposition 2.3.2] or [79, Corollary 4.5].*

The Leavitt path algebra $L_K(E)$ is semiprimitive for every graph E and field K.

This seems to be a rare occasion in which the theory of semiprimitive Steinberg algebras (as it currently stands) yields such a *definitive* answer about the semiprimitivity of a class of algebras. The groupoid models for inverse semigroup algebras, or partial skew group rings, generally involve more complicated isotropy groups, so it reduces to a problem not much less difficult than the original one. So, rather than developing this theory, we prefer to use a graded technique to prove Proposition 5.9.

We have powerful tools at our disposal for studying graded rings. One of these is the following theorem, first proved for $\mathbb{Z}$-graded groups in an unpublished (but widely cited) note by Bergman [21], and generalised by a number of different individuals.

Theorem 5.10 (Jespers & Puczyłowski)**.** *[56, Corollary 3.5]*

If A is a G-graded ring, where G is a free group or a finitely generated torsion-free nilpotent group, then $\boldsymbol{J}(A)$ is a graded ideal.

In fact, the theorem is more general than this: free groups and finitely generated torsion-free nilpotent groups are examples of groups that are *residually p-finite* for two distinct primes p, and the theorem holds when the grading group is one of these. More particularly, the theorem applies to $\mathbb{Z}$-graded rings and $\mathbb{Z}^k$-graded rings. The idea for the proof in the next paragraph probably originates from [4].

Proof of Theorem 5.9. Let E be a graph and K a field. By Theorem 5.10, $\boldsymbol{J}(L_K(E))$ is a graded ideal. If $\boldsymbol{J}(L_K(E))$ were nonzero, then it would contain a vertex, by Corollary 3.11. But this is impossible, by Lemma 5.6. Therefore $L_K(E)$ is semiprimitive. $\square$

The Graded Uniqueness Theorem for Steinberg algebras (Theorem 3.22) implies every graded ideal of $A_K(\mathcal{G})$, for a field K, contains an idempotent of the form $\mathbf{1}_V$, where $V \in \mathcal{B}(\mathcal{G}^{(0)})$. This leads to a generalisation of Proposition 5.9, and the proof is identical.

Proposition 5.11. *Let $\mathcal{G}$ be a Γ-graded Hausdorff ample groupoid such that $\mathcal{G}_\varepsilon$ is effective. If K is a field and Γ is a free group or a finitely generated torsion-free nilpotent group, the $A_K(\mathcal{G})$ is semiprimitive.*

Graded techniques are also powerful for studying the primeness or semiprimeness of a G-graded algebra, at least when G is an orderable group.

Definition 5.12. An **ordered group** is a group G whose elements are linearly ordered by a relation $<$, such that for all $a, b, c \in G$, $a < b$ implies $ac < bc$ and $ca < cb$.

Given an ordered group G, we define the *positive cone* $G^+ = \{g \in G \mid \varepsilon < g\}$ and the *negative cone* $G^- = \{g \in G \mid g < \varepsilon\}$. If $g, h \in G^+$, then $gh \in G^+$ and $g^{-1} \in G^-$. In fact, given a group G with a partition $G = G^- \sqcup \{\varepsilon\} \sqcup G^+$, meeting some reasonable conditions [68, Ch. 13, Lemma 1.3], one can define an order, $x < y \iff yx^{-1} \in G^+$, with respect to which G is an ordered group. An *orderable group* is a group that can be given the structure of an ordered group.

Examples 5.13. Besides the obvious examples like $\mathbb{Z}$, there is an abundance of orderable groups. They are usually "large" in some sense of the word (e.g. torsion-free).

(1) Every orderable group is either trivial or torsion-free. Indeed, if G is ordered and $g \in G^+$ has finite order n, then $g^{n-1} \in G^+$ but $g^{-1} \in G^-$, which is a contradiction because $g^{-1} = g^{n-1}$.

(2) [68, Lemma 1.6] Every torsion-free nilpotent group is orderable. Consequently, $\mathbb{Z}^k$ is orderable for any $k \in \mathbb{N}$.

(3) Every subgroup of an orderable group is orderable.

(4) A direct product of ordered groups is orderable. Indeed, if α is a cardinal and $\{G_\lambda \mid \lambda \in \alpha\}$ is a family of ordered groups, then $\prod_{\lambda\in\alpha} G_\lambda$ is an ordered group with the lexicographic order: $(g_\lambda)_{\lambda\in\alpha} < (h_\lambda)_{\lambda\in\alpha}$ if and only if there exists $\kappa \in \alpha$ such that $g_\kappa < h_\kappa$ and $g_\lambda = h_\lambda$ for all $\lambda < \kappa$.

(5) [68, Theorem 2.7] A free product of ordered groups is orderable. Consequently, every free group is orderable.

It is curious that the kinds of groups needed in Theorem 5.10, which simplified the calculation of the Jacobson radical, are included in the class of orderable groups. In a similar spirit, grading by an orderable group leads to easier criteria for primeness and semiprimeness. One could say that these are the best kinds of groups with which to grade a ring. The next result is implied by [66, Ch. A., Proposition II.1.4].

Lemma 5.14. *Let $A = \bigoplus_{g\in G} A_g$ be a G-graded ring, where G is an orderable group. Then the following are equivalent:*

(1) *A is prime;*

(2) *For all homogeneous elements $x, y \in A$, $xAy = 0$ implies $x = 0$ or $y = 0$;*

(3) *For all graded ideals I and J, $IJ = 0$ implies $I = 0$ or $J = 0$.*

Proof. Clearly, (1) implies (2) and (3). To show (2) implies (1), let $x, y \in A$ be such that $xAy = 0$. Assuming $x \neq 0$, we can write $x = x_{g_1} + \cdots + x_{g_n}$ where $g_1 < \cdots < g_n$ and $x_{g_i} \in A_{g_i}$. We can assume that the x_{g_i} are nonzero. Similarly, write $y = y_{h_1} + \cdots + y_{h_m}$ where $h_1 < \cdots < h_m$ and $y_{h_j} \in A_{h_j}$. Then $xAy = 0$ implies $x_{g_n} A y_{h_m} = 0$, so $y_{h_m} = 0$. Repeating the argument yields $y_{h_1} = \cdots = y_{h_m} = 0$. Therefore A is prime. To show (3) implies (2), we use the fact that a principal (one- or two-sided) ideal generated by a homogeneous element is a graded ideal. If $x, y \in A$ are homogeneous and $xAy = 0$, then the proof of Lemma 5.4 (1) works (because all the ideals involved are graded) to show that either $x = 0$ or $y = 0$. □

There is a corresponding result for semiprimeness.

Lemma 5.15. *Let $A = \bigoplus_{g\in G} A_g$ be a G-graded ring, where G is an orderable group. Then the following are equivalent:*

(1) *A is semiprime;*

(2) *For all homogeneous elements $x \in A$, $xAx = 0$ implies $x = 0$;*

(3) *For all graded ideals I, $I^2 = 0$ implies $I = 0$.*

Proof. Trivially, (1) implies (2) and (3). To show that (2) implies (1), let x be such that $xAx = 0$ and write it as an ordered decomposition $x = x_{g_1} + \cdots + x_{g_n}$, where $x_{g_i} \in A_{g_i}$, as in the previous lemma. This implies $x_{g_n} A x_{g_n} = 0$, so $x_{g_n} = 0$, and so forth. Therefore $x = 0$, proving that A is semiprime. To show (3) implies (2), the argument from Lemma 5.4 (2) applies because all the ideals involved are graded. □

Remark 5.16. If a graded ring has the property in Lemma 5.14 (2), or in Lemma 5.15 (2), then it is called *graded prime*, or *graded semiprime*, respectively. What we have proved is that a ring graded by an orderable group is (semi)prime if and only if it is graded (semi)prime.

5.3 Prime and semiprime Steinberg and Leavitt path algebras

We shall study results about prime and semiprime Steinberg algebras, recently proved in [81], and generalise them slightly for Steinberg algebras graded by an ordered group.

Theorem 5.17. *Let $\mathcal{G}$ be a Hausdorff ample groupoid, graded by an orderable group Γ such that $\mathcal{G}_\varepsilon$ is effective. Then $A_R(\mathcal{G})$ is semiprime if and only if R is reduced.*

Proof. Assume R is a reduced ring, and let I be a nonzero graded ideal of $A_R(\mathcal{G})$. By the Graded Uniqueness Theorem for Steinberg algebras (Theorem 3.22), I contains an element of the form $r\mathbf{1}_V$

where $r \in R \setminus \{0\}$ and $V \subseteq \mathcal{G}^{(0)}$ is compact and open. Then $(r\mathbf{1}_V)^2 = r^2\mathbf{1}_V \neq 0$, because R is reduced, so $I^2 \neq 0$. By Lemma 5.15, $A_R(\mathcal{G})$ is semiprime. The converse comes from Lemma 5.2. □

Although the theorem is stated for Γ-graded groupoids whose ε-component is effective, it also applies to any effective groupoid (which we can think of as an $\{\varepsilon\}$-graded groupoid). This yields [81, Theorem 4.8]: If $\mathcal{G}$ is an effective Hausdorff ample groupoid, then $A_R(\mathcal{G})$ is semiprime if and only if R is reduced.

Corollary 5.18. *[81, Corollary 5.4]. Let E be a graph. The Leavitt path algebra $L_R(E)$ is semiprime if and only if R is reduced.*

Proof. Leavitt path algebras arise as Steinberg algebras of $\mathbb{Z}$-graded groupoids whose 0-component is effective. □

The following definition is equivalent to the one given in [81].

Definition 5.19. An étale groupoid $\mathcal{G}$ is **topologically transitive** if for every pair of nonempty open sets $U, V \subseteq \mathcal{G}^{(0)}$ there exists a nonempty open set $W \subseteq \mathcal{G}$ such that $\boldsymbol{c}(W) \subseteq U$ and $\boldsymbol{d}(W) \subseteq V$.

Equivalently, given any pair of nonempty open sets $U, V \subseteq \mathcal{G}^{(0)}$, there exists an element $g \in \mathcal{G}$ with $\boldsymbol{c}(g) \in U$ and $\boldsymbol{d}(g) \in V$. Then, since $\emptyset \neq \boldsymbol{c}^{-1}(U) \cap \boldsymbol{d}^{-1}(V)$ is open, there exists a nonempty open set $W \subseteq \boldsymbol{c}^{-1}(U) \cap \boldsymbol{d}^{-1}(V)$. With this observation, it is clear that every transitive groupoid is topologically transitive. Like many topological concepts, we can also reason with bases. Suppose $\mathcal{B}$ is a base for the topology on $\mathcal{G}$ and $\mathcal{B}^0$ is a base for the topology on $\mathcal{G}^{(0)}$. Then $\mathcal{G}$ is topologically transitive if and only if for every $U, V \in \mathcal{B}^0$ there exists $W \in \mathcal{B}$ such that $\boldsymbol{c}(W) \subseteq U$ and $\boldsymbol{d}(W) \subseteq V$. Topological transitivity of groupoids is related to the primeness of their Steinberg algebras.

Proposition 5.20. *[81, Proposition 4.3] Let $\mathcal{G}$ be an ample groupoid. If $A_R(\mathcal{G})$ is prime then $\mathcal{G}$ is topologically transitive and R is an integral domain.*

Proof. If $\mathcal{G}$ is not topologically transitive, then there exists a pair of nonempty compact open sets $U, V \subseteq \mathcal{G}^{(0)}$ such that $\boldsymbol{c}^{-1}(U) \cap \boldsymbol{d}^{-1}(V) = \emptyset$. Then for all $W \in B^{\mathrm{co}}(\mathcal{G})$,

$$\mathbf{1}_U * \mathbf{1}_W * \mathbf{1}_V = \mathbf{1}_{UWV} = \mathbf{1}_\emptyset = 0.$$

Since $A_R(\mathcal{G})$ is spanned by characteristic functions of compact open bisections, this implies $\mathbf{1}_U * A_R(\mathcal{G}) * \mathbf{1}_V = 0$, and therefore $A_R(\mathcal{G})$ is not prime. If R is not an integral domain, then $A_R(\mathcal{G})$ is not prime, by Lemma 5.2. □

The converse holds if $\mathcal{G}$ is Hausdorff and effective. More generally:

Theorem 5.21. *Let $\mathcal{G}$ be a Hausdorff ample groupoid, graded by an orderable group such that $\mathcal{G}_\varepsilon$ is effective. Then $A_R(\mathcal{G})$ is prime if and only if $\mathcal{G}$ is topologically transitive and R is an integral domain.*

Proof. Assume $\mathcal{G}$ is topologically transitive and R is an integral domain. Let I, J be nonzero graded ideals of $A_R(\mathcal{G})$. By the Graded Uniqueness Theorem for Steinberg algebras (Theorem 3.22), $a\mathbf{1}_U \in I$ and $b\mathbf{1}_V \in J$, for some $a, b \in R \setminus \{0\}$ and nonempty $U, V \in \mathcal{B}(\mathcal{G}^{(0)})$. Since $\mathcal{G}$ is topologically transitive, there exists $W \in B^{\mathrm{co}}(\mathcal{G})$ such that $UWV \neq \emptyset$. Then $0 \neq a\mathbf{1}_U * \mathbf{1}_W = a\mathbf{1}_{UW} \in I$ and $(a\mathbf{1}_{UW}) * (b\mathbf{1}_V) = ab\mathbf{1}_{UWV} \neq 0$. Therefore $IJ \neq 0$, and by Lemma 5.14, $A_R(\mathcal{G})$ is prime. The converse comes from Proposition 5.20. □

As a consequence (by assuming the grading group is trivial), we have:

Corollary 5.22. *[81, Theorem 4.5] Let $\mathcal{G}$ be an effective Hausdorff ample groupoid. Then $A_R(\mathcal{G})$ is prime if and only if $\mathcal{G}$ is topologically transitive and R is an integral domain.*

Recall that for vertices v and w in a graph E, $v \geq w$ means there exists a finite path α such that $s(\alpha) = v$ and $r(\alpha) = w$.

Definition 5.23. A graph E is **downward directed** if for every $u, v \in E^0$ there exists some $w \in E^0$ such that $v \geq w$ and $u \geq w$.

On the surface, downward directedness already looks a bit like topological transitivity.

Proposition 5.24. *[81, Proposition 5.1] A graph E is downward directed if and only if $\mathcal{G}_E$ is topologically transitive.*

Proof. ($\Rightarrow$) Suppose E is downward directed, and let

$$U = Z(\alpha, F) \neq \emptyset, \qquad\qquad V = Z(\beta, H) \neq \emptyset,$$

where $\alpha, \beta \in E^\star$, $F \subseteq_{\text{finite}} r(\alpha)E^1$, and $H \subseteq_{\text{finite}} r(\beta)E^1$. If $r(\alpha)$ and $r(\beta)$ are both sinks then $r(\alpha) = r(\beta)$, by downward directedness, and $\boldsymbol{c}(\mathcal{Z}(\alpha,\beta)) = \{\alpha\} = U$ while $\boldsymbol{d}(\mathcal{Z}(\alpha,\beta)) = \{\beta\} = V$. If $r(\alpha)$ is a sink but $r(\beta)$ is not, then there exists $e \in r(\beta)E^1 \setminus H$, because $V \neq 0$. Then, by downward directedness, there exists some $\mu \in r(e)E^\star$ with $r(\mu) = r(\alpha)$. Then $\boldsymbol{c}(\mathcal{Z}(\alpha, \beta e\mu)) = \{\alpha\} = U$ while $\boldsymbol{d}(\mathcal{Z}(\alpha, \beta e\mu)) = \{\beta e\mu\} \subseteq V$. If $r(\beta)$ is a sink but $r(\alpha)$ is not, the same kind of argument applies. Otherwise, neither $r(\alpha)$ nor $r(\beta)$ is a sink. Since $U, V \neq \emptyset$, there exists $e \in r(\alpha)E^1 \setminus F$ and $f \in r(\beta)E^1 \setminus H$. By downward directedness, there exists $\gamma \in r(e)E^\star$ and $\lambda \in r(f)E^\star$ such that $r(\gamma) = r(\lambda)$. Then $\boldsymbol{c}(\mathcal{Z}(\alpha e\gamma, \beta f\lambda)) = Z(\alpha e\gamma) \subseteq U$ while $\boldsymbol{d}(\mathcal{Z}(\alpha e\gamma, \beta f\lambda)) = Z(\beta f\lambda) \subseteq V$. Therefore $\mathcal{G}_E$ is topologically transitive.

($\Leftarrow$) Suppose $\mathcal{G}_E$ is topologically transitive and let $u, v \in E^0$. Let $U = Z(u)$ and $V = Z(v)$. Then there exists some nonempty $W = \mathcal{Z}(\alpha, \beta, F)$, where $(\alpha, \beta) \in E^\star \times_r E^\star$, such that $\boldsymbol{c}(W) = Z(\alpha, F) \subseteq U$ and $\boldsymbol{d}(W) = Z(\beta, F) \subseteq V$. Then $s(\alpha) = u$ and $s(\beta) = v$, so $u \geq r(\alpha)$ and $v \geq r(\beta) = r(\alpha)$. Therefore E is downward directed. ∎

In the following, we recover a proof of [6, Theorem 2.4].

Theorem 5.25. *Let E be a graph. Then $L_R(E)$ is prime if and only if E is downward directed and R is an integral domain.*

Proof. The groupoid model for $L_R(E)$ involves a $\mathbb{Z}$-graded Hausdorff ample groupoid whose 0-component is effective. The conclusion follows from Theorem 5.21 and Proposition 5.24. ∎

5.4 Irreducible representations of Steinberg algebras

In this section, we describe two collections of functors:

$$\operatorname{Ind}_x : R^x\mathcal{G}^x\text{-}\mathbf{Mod} \to A_R(\mathcal{G})\text{-}\mathbf{Mod};$$
$$\operatorname{Res}_x : A_R(\mathcal{G})\text{-}\mathbf{Mod} \to R^x\mathcal{G}^x\text{-}\mathbf{Mod}.$$

The functor Ind_x gives us a method to produce a simple $A_R(\mathcal{G})$-module, given any simple $R^x\mathcal{G}^x$-module. We assume some notation throughout §5.4, some of which is already standard and is repeated here for convenience.

- $\mathcal{G}$ is an ample groupoid and R is a unital commutative ring;
- $x \in \mathcal{G}^{(0)}$ and ${}^x\mathcal{G}^x$ is the isotropy group based at x;
- $R^x\mathcal{G}^x$ is the group algebra of ${}^x\mathcal{G}^x$ with coefficients in R;
- $\mathscr{N}_x = \{U \in \mathcal{B}(\mathcal{G}^{(0)}) \mid x \in U\}$ where $\mathcal{B}(\mathcal{G}^{(0)}) = \{\text{compact open subsets of } \mathcal{G}^{(0)}\}$.
- A-**Mod** stands for the category of unital left A-modules;
- J_x is the free R-module with basis $\mathcal{G}^x = \boldsymbol{d}^{-1}(x)$.
- $\mathcal{O}_x \subseteq \mathcal{G}^{(0)}$ is the orbit of x.
- $T = \{t_y \mid y \in \mathcal{O}_x\}$ is a fixed transversal of $\mathcal{G}^x/{}^x\mathcal{G}^x$ and $t_x = x$.

The meaning of $\mathcal{G}^x/{}^x\mathcal{G}^x$ is the set of orbits of $\mathcal{G}^x$ with respect to the canonical right action of ${}^x\mathcal{G}^x$ on $\mathcal{G}^x$. A *transversal* of $\mathcal{G}^x/{}^x\mathcal{G}^x$ is a set having a single representative of each orbit. Explicitly, we can fix the set $T = \{t_y \mid y \in \mathcal{O}_x\}$ by choosing a single morphism $t_y \in {}^y\mathcal{G}^x$ for each $y \in \mathcal{O}_x$. The choices of t_y may be arbitrary, except we require $t_x = x$.

Observe that J_x has the structure of a right $R^x\mathcal{G}^x$-module. Its module structure is defined by the anti-representation $\lambda : (R^x\mathcal{G}^x)^{\mathrm{op}} \to \operatorname{End} J_x$,

$$\lambda(z)(t) = tz \qquad \text{for all } z \in {}^x\mathcal{G}^x, t \in \mathcal{G}^x. \tag{5.1}$$

Since both J_x and $R^x\mathcal{G}^x$ are free R-modules, this information defines λ uniquely.

Proposition 5.26. *[77, Proposition 7.7] J_x is a free right $R^x\mathcal{G}^x$-module with basis T.*

Proof. Clearly, T is $R^x\mathcal{G}^x$-linearly independent. Indeed, if

$$\sum_{y\in\mathcal{O}_x} t_y\Big(\sum_{i=1}^n r_y^i z_y^i\Big) = \sum_{y\in\mathcal{O}_x}\sum_{i=1}^n r_y^i(t_y z_y^i) = 0$$

where every $z_y^i \in {}^x\mathcal{G}^x$ is distinct and every $r_y^i \in R$, then $r_y^i = 0$ for all i and all y, because J_x is freely generated by $\mathcal{G}^x$. If $s_y \in {}^y\mathcal{G}^x$ then $s_y = t_y(t_y^{-1}s_y) \in t_y R^x\mathcal{G}^x$, so J_x is generated by T. $\square$

Proposition 5.27. *[77, Proposition 7.8] $A_R(\mathcal{G})$ has a representation*

$$\rho : A_R(\mathcal{G}) \to \operatorname{End} J_x, \qquad \rho(f)(t) = \sum_{y\in\mathcal{G}^x} f(yt^{-1})y \qquad \textit{for all } f \in A_R(\mathcal{G}) \textit{ and } t \in \mathcal{G}^x. \tag{5.2}$$

As such, J_x is an $(A_R(\mathcal{G}), R^x\mathcal{G}^x)$-bimodule.

Proof. Let $U \in B^{\mathrm{co}}(\mathcal{G})$ and $t \in \mathcal{G}^x$. A straightforward computation yields:

$$\rho(\mathbf{1}_U)(t) = \mathbf{1}_U t = \begin{cases} U\cdot t & \text{if } \boldsymbol{c}(t) \in \boldsymbol{d}(U) \\ 0 & \text{otherwise.}\end{cases} \tag{5.3}$$

(Here, $U \cdot t$ stands for the unique element ut where $u \in U$ and $\boldsymbol{d}(u) = \boldsymbol{c}(t)$.) This can be used to verify that $\rho(\mathbf{1}_{UV}) = \rho(\mathbf{1}_U * \mathbf{1}_V) = \rho(\mathbf{1}_U)\rho(\mathbf{1}_V)$ for all $U, V \in B^{\mathrm{co}}(\mathcal{G})$. It is clear from the definition (5.2) that ρ is R-linear. Since $A_R(\mathcal{G})$ is generated by characteristic functions of compact open bisections, this shows that ρ is a homomorphism. Moreover, if $t \in \mathcal{G}^x$ and $U \in B^{\mathrm{co}}(\mathcal{G})$ contains $\boldsymbol{c}(t)$, then $Ut = t$, which shows J_x is a unital $A_R(\mathcal{G})$-module. To show that J_x is a bimodule, it suffices to show that ρ and λ commute, where λ is from (5.1). Given $f \in A_R(\mathcal{G})$, $z \in {}^x\mathcal{G}^x$, and $t \in \mathcal{G}^x$, one has

$$\rho(f)\big(\lambda(z)(t)\big) = \rho(f)(tz) = \sum_{y\in\mathcal{G}^x} f(yz^{-1}t^{-1})y,$$
$$\lambda(z)\big(\rho(f)(t)\big) = \sum_{s\in\mathcal{G}^x} f(st^{-1})sz = \sum_{y\in\mathcal{G}^x} f(yz^{-1}t^{-1})y,$$

where the final equality is achieved by substituting $s = yz^{-1}$. Therefore ρ and λ commute, and J_x is a bimodule. □

Proposition 5.28. *[77, Proposition 7.10] Let $a, b, t_1, \dots, t_n \in \mathcal{G}^x$ be such that $\boldsymbol{c}(a)$ differs from $\boldsymbol{c}(t_1), \dots, \boldsymbol{c}(t_n)$. Then there exists $U \in B^{\mathrm{co}}(\mathcal{G})$ such that $\mathbf{1}_U a = b$ and $\mathbf{1}_U t_i = 0$ for $i = 1, \dots, n$.*

Proof. Let U' be a compact open bisection containing ba^{-1}. Since $\mathcal{G}^{(0)}$ is Hausdorff, there exists a compact open neighbourhood $U \subseteq U'$ of ba^{-1} such that $\boldsymbol{d}(U)$ does not contain $\boldsymbol{c}(t_i)$ for any $i = 1, \dots, n$. Applying (5.3) yields $\mathbf{1}_U a = b$ and $\mathbf{1}_U t_i = 0$ for $i = 1, \dots, n$. □

Corollary 5.29. *[77, Corollary 7.11] J_x is a cyclic $A_R(\mathcal{G})$-module generated by x.*

Proof. Let $w = \sum_{i=1}^n r_i t_i \in J_x$, where each $t_1, \dots, t_n \in \mathcal{G}^x$ are distinct, and $r_1, \dots, r_n \in R$. By the previous lemma, there exists for each $1 \le i \le n$, some $U_i \in B^{\mathrm{co}}(\mathcal{G})$ such that $\mathbf{1}_{U_i} x = t_i$. Then $w = fx$ where $f = \sum_i r_i \mathbf{1}_{U_i}$. □

Definition 5.30. [77, Definition 7.9 & 7.12] Let M be a left $R^x\mathcal{G}^x$-module. Define the **induced** $A_R(\mathcal{G})$-module:

$$\mathrm{Ind}_x(M) = J_x \otimes_{R^x\mathcal{G}^x} M.$$

Let N be a left $A_R(\mathcal{G})$-module. Define the **restricted** $R^x\mathcal{G}^x$-module:

$$\mathrm{Res}_x(N) = \bigcap_{U\in\mathcal{N}_x} \mathbf{1}_U N$$

One defines the action of $R^x\mathcal{G}^x$ on $\mathrm{Res}_x(N)$ by defining, for $g \in {}^x\mathcal{G}^x$ and $n \in \mathrm{Res}_x(N)$:

$$gn = \mathbf{1}_U n \tag{5.4}$$

where $U \in B^{\mathrm{co}}(\mathcal{G})$ is a neighbourhood of g. Steinberg in [77] showed that this is a well-defined unital module action of $R^x\mathcal{G}^x$ on $\mathrm{Res}_x(N)$.

Although we only need some of these facts, it turns out that: Ind_x is an exact functor that preserves simplicity and semisimplicity, Res_x is left exact, Ind_x is the left adjoint of Res_x, and $\mathrm{Res}_x \mathrm{Ind}_x$ is naturally isomorphic to the identity functor on $R^x\mathcal{G}^x$-**Mod**. (The reference is [77, §7.1].)

Proposition 5.31. *[77, Proposition 7.16] The functor $\mathrm{Res}_x \mathrm{Ind}_x$ is naturally isomorphic to the identity functor on $R^x\mathcal{G}^x$-**Mod**.*

Proof. Let M be an $R^x\mathcal{G}^x$-module. According to Proposition 5.26, J_x is a free $R^x\mathcal{G}^x$-module with basis T, so

$$\mathrm{Ind}_x(M) = J_x \otimes_{R^x\mathcal{G}^x} M = \bigoplus_{t\in T} t \otimes M.$$

We claim that

$$\operatorname{Res}_x \operatorname{Ind}_x(M) = \operatorname{Res}_x(J_x \otimes_{R^x\mathcal{G}^x} M) = \operatorname{Res}_x\left(\bigoplus_{t\in T} t \otimes M\right) = x \otimes M. \tag{5.5}$$

Indeed, if $U \in \mathscr{N}_x$ then $\mathbf{1}_U(x \otimes m) = x \otimes m$ for all $m \in M$, since $\mathbf{1}_U x = x$ by equation (5.3). Whereas, if $t_1 \otimes m_1 + \cdots + t_n \otimes m_n \in \operatorname{Ind}_x(M)$ where $t_i \in T \setminus \{x\}$ and $m_i \in M$, then there exists a neighbourhood $U \in \mathscr{N}_x$ such that $\boldsymbol{c}(t_1), \ldots, \boldsymbol{c}(t_n) \notin U$. Then for all $1 \leq i \leq n$, $\mathbf{1}_U(t_i \otimes m_i) = 0$ since $\mathbf{1}_U t_i = 0$ by equation (5.3). This proves the claim (5.5).

Finally, let $\theta_M : x \otimes M \to M$ be the map sending $x \otimes m \mapsto m$. This is indeed a homomorphism of $R^x\mathcal{G}^x$-modules, since for all $g \in {}^x\mathcal{G}^x$ and $m \in M$, the action of g is performed as follows, with some choice of $U \in B^{\mathrm{co}}(\mathcal{G})$ containing g:

$$g(x \otimes m) \overset{(5.4)}{=} \mathbf{1}_U(x \otimes m) = \mathbf{1}_U x \otimes m \overset{(5.3)}{=} g \otimes m = x \otimes gm.$$

So, θ is a natural isomorphism $\operatorname{Res}_x \operatorname{Ind}_x \to \mathrm{id}_{R^x\mathcal{G}^x\text{-}\mathbf{Mod}}$. □

Proposition 5.32. *[77, Proposition 7.19] If M is a simple $R^x\mathcal{G}^x$-module, then $\operatorname{Ind}_x(M)$ is a simple $A_R(\mathcal{G})$-module.*

Proof. Suppose B is a nonzero $A_R(\mathcal{G})$-submodule of $\operatorname{Ind}_x(M)$. Then $\operatorname{Res}_x(B) \subseteq B$ (as sets) and $\operatorname{Res}_x(B)$ is an $R^x\mathcal{G}^x$-submodule of $\operatorname{Res}_x \operatorname{Ind}_x(M) = x \otimes M \cong M$ (this isomorphism is proved in Proposition 5.31). We claim that $\operatorname{Res}_x(B) \neq 0$. Fix some nonzero $b \in B \subseteq \operatorname{Ind}_x(M) = J_x \otimes_{R^x\mathcal{G}^x} M$, and write $b = t_1 \otimes m_1 + \cdots + t_n \otimes m_n$ for $m_i \in M$ and $t_i \in T$, where T is the transversal serving as an $R^x\mathcal{G}^x$-basis for J_x (see Proposition 5.26). We can assume that $m_1 \neq 0$, and then Proposition 5.28 proves that some $U \in B^{\mathrm{co}}(\mathcal{G})$ exists for which $\mathbf{1}_U t_1 = x$ while $\mathbf{1}_U t_2 = \cdots = \mathbf{1}_U t_n = 0$. Thus, $\mathbf{1}_U b = x \otimes m_1 \neq 0$ because $x \otimes m_1$ corresponds to $m_1 \neq 0$ in the isomorphism $x \otimes M \cong M$. Moreover, $\mathbf{1}_U b = x \otimes m_1 \in \operatorname{Res}_x(B)$ because $x \otimes m_1$ is invariant under the action of $\{\mathbf{1}_V \mid V \in \mathscr{N}_x\}$. This proves $\operatorname{Res}_x(B) \neq 0$. Therefore $\operatorname{Res}_x(B) = \operatorname{Res}_x \operatorname{Ind}_x(M)$ because $\operatorname{Res}_x \operatorname{Ind}_x(M) \cong M$ and M is simple. Since $J_x = A_R(\mathcal{G}) \cdot x$, by Corollary 5.29, we have that $\operatorname{Ind}_x(M) = A_R(\mathcal{G}) \cdot (x \otimes M)$, and it follows that:

$$\operatorname{Ind}_x(M) = A_R(\mathcal{G}) \cdot (x \otimes M) = A_R(\mathcal{G}) \cdot \operatorname{Res}_x \operatorname{Ind}_x(M) = A_R(\mathcal{G}) \cdot \operatorname{Res}_x(B) \subseteq A_R(\mathcal{G}) \cdot B \subseteq B.$$

The conclusion is that $B = \operatorname{Ind}_x(M)$, so $\operatorname{Ind}_x(M)$ is simple. □

Example 5.33. [79] Let K be a field, let $\mathcal{G}$ be an ample groupoid, and let $x \in \mathcal{G}^{(0)}$. We can consider K as a $K^x\mathcal{G}^x$-module, with the "trivial action" given by $(kz)k' = kk'$ for all $k, k' \in K$ and $z \in {}^x\mathcal{G}^x$. In fact, we get this by taking the quotient of $K^x\mathcal{G}^x$ by its augmentation ideal. Clearly, K is a simple $K^x\mathcal{G}^x$-module, so applying the functor Ind_x yields a simple $A_K(\mathcal{G})$-module,

$$\operatorname{Ind}_x(K) = J_x \otimes_{K^x\mathcal{G}^x} K \cong J_x / K^x\mathcal{G}^x$$

But J_x is a free $K^x\mathcal{G}^x$-module with basis T, so as vector spaces $J_x/K^x\mathcal{G}^x \cong K\mathcal{O}_x$ where $K\mathcal{O}_x$ is the K-vector space freely generated by $\mathcal{O}_x$. The action of $A_K(\mathcal{G})$ on $K\mathcal{O}_x$, induced by this isomorphism, is

$$f \cdot u = \sum_{g \in \mathcal{G}^u} f(g)\boldsymbol{c}(g) \qquad\qquad \text{for all } f \in A_R(\mathcal{G}) \text{ and } u \in \mathcal{O}_x.$$

In case $B \in B^{\mathrm{co}}(\mathcal{G})$, this translates to

$$\mathbf{1}_B \cdot u = \begin{cases} B \cdot u & \text{if } u \in \boldsymbol{d}(B) \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

where $B \cdot u$ stands for the unique element in $\boldsymbol{c}(Bu)$.

5.5 Steinberg's Theorem on the equivalence of categories

Let G be a group. It is a triviality (if you know about representations of groups) that the category of complex representations of G is equivalent to the category of $\mathbb{C}G$-modules. In this section, we present a vast generalisation. Steinberg's Equivalence Theorem [78, Theorem 3.5] proves that there is an equivalence between the category of $\mathcal{G}$-sheaves of R-modules and the category of right $A_R(\mathcal{G})$-modules. The theorem will not be proved here, but we hope to convey an understanding of the statement.

In this section, $\mathcal{G}$ is an arbitrary ample groupoid and R an arbitrary unital commutative ring. First, recall from Definition 4.18 that a $\mathcal{G}$*-sheaf* is a topological space E equipped with a local homeomorphism $p : E \to \mathcal{G}^{(0)}$ and a continuous action of $\mathcal{G}$. A *sheaf of R-modules* is defined in Definition 4.12: it is a sheaf whose stalks are all R-modules, satisfying some continuity conditions. The next definition does not appear in Chapter 4.

Definition 5.34. [78] A $\mathcal{G}$-sheaf (E, p) is called a **$\mathcal{G}$-sheaf of R-modules** if it is a sheaf of R-modules over $\mathcal{G}^{(0)}$ and for each $g \in \mathcal{G}$, the map $r_g : E_{\boldsymbol{c}(g)} \to E_{\boldsymbol{d}(g)}$, given by $r_g(e) = eg$, is a homomorphism of R-modules.

The category of $\mathcal{G}$-sheaves of R-modules is denoted $\mathbf{Sh}\text{-}_R\mathcal{G}$, and the morphisms in this category are $\mathcal{G}$-equivariant morphisms of sheaves of R-modules (see Definition 4.12).

Steinberg defined two functors

$$\Lambda : \mathbf{Sh}\text{-}_R\mathcal{G} \to \mathbf{Mod}\text{-}A_R(\mathcal{G})$$
$$\Omega : \mathbf{Mod}\text{-}A_R(\mathcal{G}) \to \mathbf{Sh}\text{-}_R\mathcal{G}.$$

The first of these functors is defined in the following way: If (E, p) is a $\mathcal{G}$-sheaf of R-modules, then $\Lambda(E, p)$ is the set of continuous, compactly supported global sections $s : \mathcal{G}^{(0)} \to E$ of p. Pointwise addition makes $\Lambda(E, p)$ an abelian group, and there is a right $A_R(\mathcal{G})$-module structure:

$$\big(sf\big)(x) = \sum_{g \in \mathcal{G}^x} f(g) s\big(\boldsymbol{c}(g)\big) g \tag{5.7}$$

for all $f \in A_R(\mathcal{G})$ and $s \in \Lambda(E, p)$. The sum is finite because f is compactly supported and $\mathcal{G}^x$ is discrete. In case $f = \mathbf{1}_U$ where $U \in B^{\mathrm{co}}(\mathcal{G})$, the formula (5.7) simplifies to

$$\big(s\mathbf{1}_U\big)(x) = \begin{cases} s(\boldsymbol{c}(g))g & \text{if } g \in U \cap \mathcal{G}^x \\ 0 & \text{if } x \notin \boldsymbol{d}(U). \end{cases} \tag{5.8}$$

Note that there can be at most one element $g \in U \cap \mathcal{G}^x$. If $U \subseteq \mathcal{G}^{(0)}$ then (5.8) simplifies to $\big(s\mathbf{1}_U\big)(x) = \mathbf{1}_U(x)s(x)$. Note, $\mathbf{1}_U(x)s(x)$ equals $s(x)$ or the 0 of E_x. It is not too difficult to check that $\Lambda(E, p)$ is a unital right $A_R(\mathcal{G})$-module. Given a morphism of $\mathcal{G}$-sheaves of R-modules $\phi : (E, p) \to (F, q)$, one defines its image under the functor Λ:

$$\Lambda(\phi) : \Lambda(E, p) \to \Lambda(F, q), \qquad\qquad \Lambda(\phi)(s) = \phi \circ s.$$

The second of Steinberg's two functors is defined in the following way: given a unital right $A_R(\mathcal{G})$-module M, let $\Omega(M) = (\widetilde{M}, p_M)$ where

$$\widetilde{M} = \bigsqcup_{x \in \mathcal{G}^{(0)}} \widetilde{M}_x.$$

and $p_M : \widetilde{M} \to \mathcal{G}^{(0)}$ maps $\widetilde{M}_x \mapsto x$. The definition of $\widetilde{M}_x$ is the limit

$$\widetilde{M}_x = \varinjlim_{U \in \mathscr{N}_x} M\mathbf{1}_U$$

of a directed system of R-modules $\{M\mathbf{1}_U\}_{U\in\mathscr{N}_x}$, with index set $\mathscr{N}_x = \{U \in \mathcal{B}(\mathcal{G}^{(0)}) \mid x \in U\}$, ordered by $U \leq V$ if $V \subseteq U$. Note for all $U, V \in \mathcal{B}(\mathcal{G}^{(0)})$, $V \subseteq U$ implies $\mathbf{1}_V * \mathbf{1}_U = \mathbf{1}_V$, so $M\mathbf{1}_V \subseteq M\mathbf{1}_U$. The connecting homomorphisms in the system $\{M\mathbf{1}_U\}_{U\in\mathscr{N}_x}$ are the surjections $\rho_V^U : M\mathbf{1}_U \to M\mathbf{1}_V$, $m \mapsto m\mathbf{1}_V$, for all $V \subseteq U$ in $\mathscr{N}_x$.

Assume $m \in M$ and $x \in \mathcal{G}^{(0)}$. Since M is a unital right $A_R(\mathcal{G})$-module, and $A_R(\mathcal{G})$ has local units, for each $m \in M$ there exists some $U \in \mathcal{B}(\mathcal{G}^{(0)})$ with $m = m\mathbf{1}_U$. If $x \notin U$ then U can be enlarged to include x, and remain compact and open, which shows $m \in M\mathbf{1}_U$ for some $U \in \mathscr{N}_x$. Let $[m]_x$ be the image of m in the direct limit $\widetilde{M}_x$. The map $m \mapsto [m]_x$ defines an R-module homomorphism $M \to \widetilde{M}_x$. One can easily show that the collection of sets

$$Z(m, U) = \big\{[m]_x \mid x \in U\big\}; \qquad m \in M, \quad U \in \mathcal{B}(\mathcal{G}^{(0)}).$$

is a base of open sets that generates a topology on $\widetilde{M}$, with respect to which p_M is a local homeomorphism, and therefore $(\widetilde{M}, p_M)$ is a sheaf of R-modules over $\mathcal{G}^{(0)}$.

Remark 5.35. This process is very similar to the sheaf functor $\mathcal{S}$ defined on page 49, which builds a sheaf space $\mathcal{S}F$ from a presheaf F. The major difference is that the functor

$$\widetilde{M} : \mathcal{B}(\mathcal{G}^{(0)}) \to R\text{-}\mathbf{Mod}, \qquad U \mapsto \widetilde{M}(U) = M\mathbf{1}_U,$$

(and indeed it is a functor) is not defined on all the open subsets U of $\mathcal{G}^{(0)}$.

To conclude that $(\widetilde{M}, p_M)$ is a $\mathcal{G}$-sheaf of R-modules, the only piece of information still needed is an action of $\mathcal{G}$. For $g \in \mathcal{G}$ we define the right action $r_g : M_{\boldsymbol{c}(g)} \to M_{\boldsymbol{d}(g)}$,

$$r_g([m]_{\boldsymbol{c}(g)}) = [m]_{\boldsymbol{c}(g)} \cdot g = [m\mathbf{1}_U]_{\boldsymbol{d}(g)},$$

where $U \in B^{\mathrm{co}}(\mathcal{G})$ is any compact open bisection containing g. One can check that this action is unambiguously defined (does not depend on the choice of U), that r_g is a homomorphism of R-modules, and that $(\widetilde{M}, p_M)$ satisfies Definition 4.18.

Furthermore, if $f : M \to N$ is a homomorphism of $A_R(\mathcal{G})$-modules, there is a natural transformation $\nu : \widetilde{M} \to \widetilde{N}$, $\nu_U : m \mapsto f(m)$, along the lines of Definition 4.14. This induces the morphism of $\mathcal{G}$-sheaves of R-modules:

$$\Omega(f) = \widetilde{f} : \widetilde{M} \to \widetilde{N} \qquad \widetilde{f}([m]_x) = [f(m)]_x \qquad \text{for all } [m]_x \in \widetilde{M}_x.$$

It is shown in [78, pp. 423-426] that these functors Λ and Ω are well defined.

Theorem 5.36 (Steinberg's Equivalence Theorem)**.** *[78, Theorem 3.5]*

Let $\mathcal{G}$ be an ample groupoid and R a unital commutative ring. The functors

$$\Lambda : \mathbf{Sh}\text{-}_R\mathcal{G} \to \mathbf{Mod}\text{-}A_R(\mathcal{G})$$
$$\Omega : \mathbf{Mod}\text{-}A_R(\mathcal{G}) \to \mathbf{Sh}\text{-}_R\mathcal{G}$$

are mutually inverse equivalences of categories.

5.6 Primitive Steinberg algebras

Given an ample groupoid $\mathcal{G}$ and a subset $U \subseteq \mathcal{G}^{(0)}$, define $\mathcal{G}|_U = U\mathcal{G}U = \boldsymbol{c}^{-1}(U) \cap \boldsymbol{d}^{-1}(U)$, the *full subgroupoid* with unit space U. If U is open (resp., closed) then $\mathcal{G}|_U$ is an open (resp., closed) ample subgroupoid of $\mathcal{G}$.

Lemma 5.37. *Let $U \in \mathcal{B}(\mathcal{G}^{(0)})$. Then $A_R(\mathcal{G}|_U) \cong \mathbf{1}_U * A_R(\mathcal{G}) * \mathbf{1}_U$.*

Proof. Of course, $A_R(\mathcal{G}|_U)$ is isomorphic to a unital subalgebra $A \subseteq A_R(\mathcal{G})$ (see Remark 1.16) and the unit element of A is $\mathbf{1}_U$. This shows that

$$A = \mathbf{1}_U * A * \mathbf{1}_U \subseteq \mathbf{1}_U * A_R(\mathcal{G}) * \mathbf{1}_U.$$

Whereas, if $f \in A_R(\mathcal{G})$ then $\operatorname{supp}(\mathbf{1}_U * f * \mathbf{1}_U) \subseteq (\operatorname{supp} \mathbf{1}_U)(\operatorname{supp} f)(\operatorname{supp} \mathbf{1}_U) \subseteq U\mathcal{G}U = \mathcal{G}|_U$. Therefore $\mathbf{1}_U * f * \mathbf{1}_U \in A$, proving that $\mathbf{1}_U * A_R(\mathcal{G}) * \mathbf{1}_U \subseteq A$. □

Proposition 5.38. *[79, Proposition 4.7] Let $\mathcal{G}$ be an ample groupoid and let $U \in \mathcal{B}(\mathcal{G}^{(0)})$. If $A_R(\mathcal{G})$ is primitive (resp., semiprimitive) then $A_R(\mathcal{G}|_U)$ is primitive (resp., semiprimitive).*

Proof. It follows directly from Lemma 5.37 and Lemma 5.8 (4,5). □

Before embarking on the next proposition, we point out a change in left-right conventions. In §5.1, primitivity was defined in terms of left modules, and in general this is not the same as primitivity in terms of right modules. However, Steinberg algebras are self-opposite, so the two notions are equivalent. That is, $A_R(\mathcal{G})$ has a faithful simple left module if and only if it has a faithful simple right module.

Proposition 5.39. *[79, Proposition 4.9] Let $\mathcal{G}$ be an ample groupoid and let K be a field. If $A_K(\mathcal{G})$ is primitive then there is a dense orbit in $\mathcal{G}^{(0)}$.*

Proof. Let M be a faithful simple right $A_K(\mathcal{G})$-module. By Steinberg's Equivalence Theorem (Theorem 5.36), there exists a $\mathcal{G}$-sheaf of K-modules (E, p) such that $M \cong \Lambda(E, p)$ as right $A_K(\mathcal{G})$-modules. If every stalk E_x ($x \in \mathcal{G}^{(0)}$) were zero, then $\Lambda(E, p)$ would be the zero module, which contradicts that it is simple (or indeed faithful). With this in mind, fix some $x \in \mathcal{G}^{(0)}$ such that the stalk E_x is nonzero. Let $\mathcal{O}_x \subseteq \mathcal{G}^{(0)}$ be the orbit of x and let $X = \boldsymbol{d}^{-1}(\mathcal{O}_x) \subseteq \mathcal{G}$. Consider the ideal

$$I = \{f \in A_K(\mathcal{G}) \mid \operatorname{supp} f \subseteq \mathcal{G} \setminus X\}.$$

Towards an eventual contradiction, suppose $I \neq 0$. Then, since $\Lambda(E, p)$ is a faithful right $A_K(\mathcal{G})$-module, there exists $t \in \Lambda(E, p)$ such that $tI \neq 0$. Moreover, tI is an $A_K(\mathcal{G})$-submodule of $\Lambda(E, p)$, so $tI = \Lambda(E, p)$ because $\Lambda(E, p) \cong M$ is simple. For every $s \in \Lambda(E, p)$, therefore, there exists $f \in I$ such that $tf = s$. By definition of the action of $A_K(\mathcal{G})$ on $\Lambda(E, p)$, given in Equation (5.7),

$$s(x) = \big(tf\big)(x) = \sum_{g \in \mathcal{G}^x} f(g)t\big(\boldsymbol{c}(g)\big)g.$$

This implies $s(x) = 0$, because $f \in I$ and $\mathcal{G}^x \subseteq X$, so $f(g) = 0$ for every $g \in \mathcal{G}^x$. But if $s(x) = 0$ for every $s \in \Lambda(E, p)$ then $E_x = 0$, which is a contradiction. We conclude that $I = 0$. Since $I = 0$, the interior of $\mathcal{G} \setminus X$ is empty, so X is dense in $\mathcal{G}$, and thus $\mathcal{O}_x = \boldsymbol{d}(X)$ is dense in $\mathcal{G}^{(0)}$. □

A converse comes at the price of assuming that $\mathcal{G}$ is effective and Hausdorff.

Theorem 5.40. *[79, Theorem 4.10] Let $\mathcal{G}$ be an effective Hausdorff ample groupoid and K a field. Then $A_K(\mathcal{G})$ is primitive if and only if there is a dense orbit in $\mathcal{G}^{(0)}$.*

Proof. ($\Rightarrow$) This follows from Proposition 5.39.

($\Leftarrow$) Suppose there is a dense orbit $\mathcal{O}_x$ in $\mathcal{G}^{(0)}$. Consider the simple $A_K(\mathcal{G})$-module $M = K\mathcal{O}_x$ from Example 5.33. We claim that M is faithful. Towards a contradiction, suppose the annihilator $\operatorname{ann} M \subseteq A_K(\mathcal{G})$ of M is nonzero. The Cuntz-Krieger Uniqueness Theorem (Corollary 3.24) implies that $\operatorname{ann} M$ (which is a primitive ideal) contains an idempotent $\mathbf{1}_U \neq 0$, for some $U \in \mathcal{B}(\mathcal{G}^{(0)})$. Since $\mathcal{O}_x$ is dense, there exists some $y \in U \cap \mathcal{O}_x$. Applying the rule (5.6) we have $\mathbf{1}_U y = U \cdot y = y$, which contradicts that $\mathbf{1}_U \in \operatorname{ann} M$. Therefore M is a faithful simple module, and the conclusion is that $A_K(\mathcal{G})$ is primitive. □

5.7 Primitive Leavitt path algebras

In this section, we set out to prove a theorem of Abrams, Bell, and Rangaswamy [6, Theorem 5.7] that completely describes the graphical conditions that give rise to primitive Leavitt path algebras. It is a beautiful theorem, not least because it contributes to an old question of Kaplansky [58] concerning the interactions between primitivity, primeness, and von Neumann regularity: "Is a regular prime ring necessarily primitive?"

While writing this dissertation, we set a goal to prove the primitivity theorem for Leavitt path algebras [6, Theorem 5.7]. The results of [79], some of which are included in §5.6, seemed to suggest that this theorem was within reach using a groupoid approach. Indeed, Steinberg writes in [79] that the results therein could be used to recover the primitivity theorem for Leavitt path algebras. After that goal was set, Steinberg uploaded a new paper onto the arXiv [81], in which he does go all the way to proving the primitivity theorem for Leavitt path algebras. This made the task much easier.

Definition 5.41. Let E be a graph. We say that E has the **countable separation property** if there is a countable subset $T \subseteq E^0$ (called a *countable separating set* for E) such that for every $w \in E^0$ there is some $t \in T$ with $w \geq t$.

Clearly, if E^0 is countable, then E^0 is a countable separating set for E.

Lemma 5.42. *If E is a row-countable and downward directed graph, then E has the countable separation property.*

Proof. Choose some $v \in E^0$, and let $T(v) = \{w \in E^0 \mid v \geq w\}$. Let $X_1 = \{v\}$ and define inductively $X_{n+1} = r(s^{-1}(X_n))$. If X_n is countable then $s^{-1}(X_n)$ is countable, because every $v \in X_n$ emits countably many edges, so $X_{n+1} = r(s^{-1}(X_n))$ is countable. Since X_1 is finite, it follows by induction that all the sets X_n are countable, and thus $T(v) = \bigcup_{n=1}^{\infty} X_n$ is countable. Downward directedness implies that for all $w \in E^0$ there exists $u \in T(v)$ such that $w \geq u$. So, $T(v)$ is a countable separating set for E. □

Theorem 5.43 (Primitivity Theorem for Leavitt path algebras)**.** *[6, Theorem 5.7]*

Let E be a graph and K a field. Then $L_K(E)$ is primitive if and only if

(i) *E is downward directed;*

(ii) *E satisfies Condition (L);*

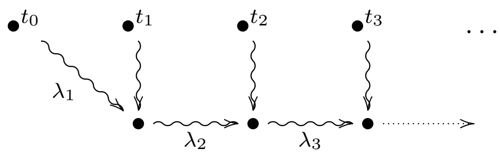
(iii) *E has the countable separation property.*

Proof. Necessity of (i): Suppose $L_K(E)$ is primitive. Then $L_K(E)$ is prime, and Theorem 5.25 implies E is downward directed.

Necessity of (ii): Suppose E does not satisfy Condition (L). Then there is a cycle c without exits, and the infinite path $c^\infty = ccc\ldots$ is an isolated point in ∂E because $Z(c) = \{c^\infty\}$. Let $U = Z(c)$. Then $\mathcal{G}_E|_U = \mathrm{Iso}(c^\infty) \cong \mathbb{Z}$, so $A_K(\mathcal{G}_E|_U) \cong K[x, x^{-1}] \cong \mathbf{1}_U * A_K(\mathcal{G}_E) * \mathbf{1}_U$, by Lemma 5.37. But $K[x, x^{-1}]$ is not primitive because it is commutative but not a field. Since $A_K(\mathcal{G}_E) \cong L_K(E)$ has a corner that is not primitive, $A_K(\mathcal{G}_E) \cong L_K(E)$ is not primitive, by Lemma 5.8 (5).

Necessity of (iii): Assume $L_K(E) \cong A_K(\mathcal{G}_E)$ is primitive. By Proposition 5.39, there exists $x \in \partial E$ such that the orbit $\mathcal{O}_x$ is dense in ∂E. Set $T = \{s(x), r(x_1), r(x_2), \ldots\}$ if x is infinite, and set $T = \{s(x), r(x_1), \ldots, r(x_{|x|})\}$ if x is finite. Let $w \in E^0$. Since $\mathcal{O}_x$ is dense, $Z(w) \cap \mathcal{O}_x \neq \emptyset$, so there exists $y \in Z(w)$ such that y is tail equivalent to x. Then there exists $(\alpha, \beta) \in E^\star \times_r E^\star$ and $z \in r(\alpha)\partial E$ such that $y = \alpha z$ and $x = \beta z$. This shows that $w \geq r(\alpha) = r(\beta) \in T$, so T is a countable separating set for E^0.

Sufficiency of (i), (ii), and (iii): Assume E is downward directed, satisfies Condition (L), and has the countable separation property. Then $\mathcal{G}_E$ is effective, by Proposition 3.25. If we can show there is a dense orbit in ∂E, then Theorem 5.40 will prove $A_K(\mathcal{G}_E)$ is primitive. First, suppose $E^0 = \{v_1, \ldots, v_n\}$ is finite. Using downward directedness, there is some $v_j \in E^0$ such that $v_1, v_2 \geq v_j$, and some $v_k \in E^0$ with $v_j, v_3 \geq v_k$, so $v_1, v_2, v_3 \geq v_k$, and so on. Let $v_\ell \in E^0$ be such that $v_i \geq v_\ell$ for $i = 1, \ldots, n$. Fix $x \in v_\ell \partial E$. Then for every $\alpha \in E^\star$, there exists $\mu \in r(\alpha)E^\star$ such that $r(\mu) = v_\ell$. Then $\alpha\mu x \in Z(\alpha) \cap \mathcal{O}_x$, where $\mathcal{O}_x$ is the orbit of x. If $F \subsetneq r(\alpha)E^1$ is finite, then there exists $e \in r(\alpha)E^1 \setminus F$ and $Z(\alpha, F) \cap \mathcal{O}_x \supseteq Z(\alpha e) \cap \mathcal{O}_x$ is nonempty, as we have just shown. This shows $\mathcal{O}_x$ is dense. Second, suppose E^0 is infinite, so E has a countable separating set $T = \{t_0, t_1, \ldots\}$. Using downward directedness, choose $\lambda_1 \in E^\star$ such that $s(\lambda_1) = t_0$ and $t_1 \geq r(\lambda_1)$. Assume we have $\lambda_1, \lambda_2, \ldots, \lambda_n \in E^\star$ such that $r(\lambda_i) = s(\lambda_{i+1})$ for $i = 1, \ldots, n-1$, and $t_i \geq r(\lambda_i)$ for $i = 1, \ldots, n$. Then, using downward directedness again, choose λ_{n+1} such that $r(\lambda_n) = s(\lambda_{n+1})$ and $t_{n+1} \geq r(\lambda_{n+1})$. Inductively, this defines an infinite path $x = \lambda_1\lambda_2\ldots$, belonging to a subgraph of E, of the form



(where the curly arrows represent paths). If $\alpha \in E^\star$ then by definition of a countable separating set, there exists $t_i \in T$ with $r(\alpha) \geq t_i$. Let $\mu \in r(\alpha)E^\star$ have $r(\mu) = t_i$. Let $\gamma \in t_i E^\star$ have $r(\gamma) = r(\lambda_i)$. Then $\alpha\mu\gamma\lambda_{i+1}\lambda_{i+2}\cdots \in Z(\alpha) \cap \mathcal{O}_x$. If $F \subsetneq r(\alpha)E^1$ is finite, then there exists $e \in r(\alpha)E^1 \setminus F$ and $Z(\alpha, F) \cap \mathcal{O}_x \supseteq Z(\alpha e) \cap \mathcal{O}_x \neq \emptyset$, as just shown. This proves $\mathcal{O}_x$ is dense. Therefore $A_K(\mathcal{G}_E)$ is primitive, by Theorem 5.40. $\square$

Although some relatively advanced technology (sheaves, for instance) was needed to get to this point, the above proof of the primitivity theorem is quite brief, in contrast with the "heavy lifting" needed to achieve it in [6].

Corollary 5.44. *Let E be a graph such that E^0 is countable or E is row-countable. Let K be a field. Then $L_K(E)$ is primitive if and only if E is downward directed and satisfies Condition (L).*

Proof. If E^0 is countable then E obviously has the countable separation property. If E is row-countable, then by Lemma 5.42, the countable separation property is implied by downward directedness. □

A weaker form of the corollary could also be deduced from [81, Corollary 4.6]. It is shown there that if K is a field and $\mathcal{G}$ is an effective Hausdorff ample groupoid, with second-countable unit space, then $A_K(\mathcal{G})$ is prime if and only if it is primitive. Only countable graphs have a second-countable boundary path space (see Theorem 2.8). So the translation of [81, Corollary 4.6] into Leavitt path algebras is that if E is a countable graph that satisfies Condition (L), then $L_K(E)$ is primitive if and only if it is prime.

Examples 5.45. Briefly, we describe the examples of prime, nonprimitive, von Neumann regular rings, that were discovered in [6]. We have taken care throughout the dissertation to make sure that our proofs work for arbitrary-sized graphs. These examples provide some reward for the effort.

(1) Let X be a set, and let $\mathcal{F}(X)$ be the set of nonempty finite subsets of X. The graph $E_{\mathcal{F}(X)}$ is defined as follows:

$$E^0_{\mathcal{F}(X)} = \mathcal{F}(X), \qquad E^1_{\mathcal{F}(X)} = \{e_{A,B} \mid A, B \in \mathcal{F}(X),\ A \subsetneqq B\},$$

where $s(e_{A,B}) = A$ and $r(e_{A,B}) = B$ for every $e_{A,B} \in E^1_{\mathcal{F}(X)}$.

(2) Let $\kappa > 0$ be an ordinal, and define the graph E_κ:

$$E^0_\kappa = \{\alpha \mid \alpha < \kappa\} \qquad E^1_\kappa = \{e_{\alpha,\beta} \mid \alpha < \beta\},$$

where $s(e_{\alpha,\beta}) = \alpha$ and $r(e_{\alpha,\beta}) = \beta$ for every $e_{\alpha,\beta} \in E^1_\kappa$.

The graphs here are obviously acyclic, so their Leavitt path algebras over fields are von Neumann regular, by Theorem 3.1. It is also not hard to become convinced that they are downward-directed, so their Leavitt path algebras are prime. However, $E_{\mathcal{F}(X)}$ is primitive if and only if X is at most countable. Likewise, E_κ is primitive if and only if κ has *countable cofinality* (that is, κ has a countable cofinal subset).

We can infer from Theorem 4.8, Theorem 3.1, and Corollary 5.44, that there are no strongly $\mathbb{Z}$-graded Leavitt path algebras that are prime, nonprimitive, and von Neumann regular.

Chapter 6

Centres and simplicity

If A is a ring, $Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}$ is its *centre*. The main goal for this chapter is to explicitly describe the elements in the centre of a Steinberg algebra and a Leavitt path algebra. The centre of a Leavitt path algebra can be zero, and if it is nonzero then it is isomorphic to a (possibly large) direct sum of components R and $R[x, x^{-1}]$. This is a consequence of the results in [31]. Centres of Leavitt path algebras seem to have captured the interests of quite a few mathematicians working in the area. This might be because it was one of the more difficult substructures to pin down, compared to others like the socle and the Jacobson radical.

There are several reasons to take an interest in the centre of a ring. Let A be a unital R-algebra and let M be an (A, A)-bimodule. The n-th *Hochschild cohomology* module $H^n(A, M)$ is not just an R-module but also a $Z(A)$-module (details in [64]). When $M = A$, the 0-th cohomology is exactly the centre: $H^0(A, A) = Z(A)$. An M-valued *derivation* of A is an R-linear map $d : A \to M$ such that $d(xy) = xd(y) + d(x)y$ for all $x, y \in A$. The set of all such derivations, $\mathrm{Der}_R(A, M)$, is a $Z(A)$-module, and $H^1(A, M)$ is equal to $\mathrm{Der}_R(A, M)$ modulo the inner derivations. *Inner derivations*, by definition, are of the form $\mathrm{ad}(m) : a \mapsto ma - am$, for some $m \in M$. The centre of A is the intersection of the kernels of all inner derivations in $\mathrm{Der}_R(A, A)$. It becomes apparent that any study of the derivations of an algebra is likely to require a good understanding of its centre. Similarly, the automorphisms of A are related to the centre, inasmuch as $Z(A)$ is fixed by every inner automorphism.

The centre of a ring is also an important invariant that can be used for classification. Morita's First Theorem (see [55, p. 167]) entails that Morita equivalent rings have isomorphic centres. Additionally, if a unital Leavitt path algebra $L_K(E)$ over a field K is central (i.e., $Z(L_K(E)) = K1$), and $L_K(F)$ is another Leavitt path algebra, then [57, Proposition 1.2] shows that $L_K(E)$ and $L_K(F)$ are isomorphic as rings if and only if they are isomorphic as K-algebras.

Simplicity is a structural property of rings, so perhaps it belongs in Chapter 5, where it is defined in Definition 5.1 (1). However, there is a rich relationship between centres and simplicity, that seems to justify organising it this way. The centre of a group G is defined as $Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$, and it is a normal subgroup. Thus, every simple group is either commutative or has trivial centre. The centre of a Lie algebra $\mathfrak{g}$ is defined as $Z(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [z, a] = 0 \text{ for all } a \in \mathfrak{g}\}$, and it is a Lie ideal, so the centre of a semisimple Lie algebra is zero. No analogous statement holds for rings (the centre is certainly not an ideal, in general), but in spite of this, many interesting things can be said. For example, if A is a simple unital ring, its centre is a field (see [23, Example 1.21]). We show in this chapter that if $A_K(\mathcal{G})$ is a simple unital Steinberg algebra, its centre is K.

Herstein, in [53] and other papers, derived many interesting facts about nonassociative structures built from simple rings. For instance, one can take a simple ring A of characteristic not 2, and define a Lie ring structure on $[A, A]$, the additive subgroup generated by the commutators in A. If A is simple, then $[A, A]/(Z(A) \cap [A, A])$ is a simple Lie ring [53, Theorem 4]. The centre of a Leavitt path algebra has been used in [7] and [8] to determine when the Lie algebras, arising like this from Leavitt path algebras, are simple.

A second goal for this chapter is to give necessary and sufficient conditions on Hausdorff ample groupoids (resp., graphs) that give rise to simple Steinberg algebras (resp., Leavitt path algebras). Simplicity theorems play a very important role in graph algebras and related classes of algebras. This theme goes right back to the beginning, when Leavitt proved in [63] that the Leavitt algebras $L_{n,K}$ $(n \geq 2)$ are all simple. On a different side of the mathematical universe, Cuntz proved in [38] that the Cuntz algebras $\mathcal{O}_n$ $(n \geq 2)$, analytic cousins of the Leavitt algebras, are simple C^*-algebras. The analytic cousins of Steinberg algebras are groupoid C^*-algebras. Given any ample groupoid $\mathcal{G}$, the Steinberg algebra $A_{\mathbb{C}}(\mathcal{G})$ sits inside the full groupoid C^*-algebra $C^*(\mathcal{G})$, as a dense $*$-subalgebra. If E is a graph, an analogue of $A_R(\mathcal{G}_E) \cong L_R(E)$ is that $C^*(\mathcal{G}_E) \cong C^*(E)$, and this explains much about the relationship between Leavitt path algebras and graph C^*-algebras. An accessible reference on groupoid C^*-algebras is [76], and the definitive text is probably [73].

When Leavitt path algebras were introduced, in the very first paper on the subject, Abrams and Aranda Pino [3] wrote the simplicity theorem for Leavitt path algebras of row-finite graphs. It was extended to Leavitt path algebras of arbitrary graphs, as soon as these were defined in [4]. Once Steinberg algebras appeared on the scene, Brown, Clark, Farthing, and Sims [24] proved a simplicity theorem for Steinberg algebras of Hausdorff ample groupoids over $\mathbb{C}$. That effort led them to unlock a remarkable piece of research in which they derived a simplicity theorem for the C^*-algebras of second-countable, locally compact, Hausdorff étale groupoids. It speaks to the significance of these new ideas, that they were put to use in solving a problem that was open for many decades. The effort has recently been repeated for non-Hausdorff groupoids, in [27], where it is said that "We view Steinberg algebras as a laboratory for finding conditions to characterize C^*-simplicity for groupoid C^*-algebras."

In §6.1, we show that the centre of a Steinberg algebra is the set of class functions, which include the characteristic functions of compact open invariant sets. In §6.2, we define some germane graph-theoretic data types, necessary to calculate the centre of a Leavitt path algebra. In §6.3, we study the lattice of open invariant subsets of the boundary path space, and the sublattice of compact open invariant sets. In §6.4, we study the conjugacy classes of cycles without exits in the boundary path groupoid. Putting this all together in §6.5, we calculate the centre of a Leavitt path algebra, and the centres of prime Steinberg and Leavitt path algebras. In §6.6, we prove that $A_K(\mathcal{G})$, for a field K and Hausdorff groupoid $\mathcal{G}$, is simple if and only if $\mathcal{G}$ is minimal and effective. Since it has not been done before, we show that $A_K(\mathcal{G})$, for a Γ-graded groupoid $\mathcal{G}$, is graded simple if and only if $\mathcal{G}$ is minimal and $\mathcal{G}_\varepsilon$ is effective. This leads easily to the simplicity and graded simplicity theorems for Leavitt path algebras.

6.1 The centre of a Steinberg algebra

We record some general facts before specialising to Steinberg algebras. If A is an R-algebra, then $Z(A)$ is a commutative R-subalgebra. If $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is Γ-graded and Γ is abelian, then $Z(A)$ is a graded subalgebra. To prove this, let $z = \sum_{\gamma \in \Gamma} z_\gamma$ be in the centre of A, where $z_\gamma \in A_\gamma$ for all $\gamma \in \Gamma$.

To show that $z_\gamma \in Z(A)$, it suffices to show that z_γ commutes with the homogeneous elements in A. If $a_\mu \in A_\mu$, then $\sum_{\gamma\in\Gamma} z_\gamma a_\mu = z a_\mu = a_\mu z = \sum_{\gamma\in\Gamma} a_\mu z_\gamma$. Comparing the $\gamma\mu$-homogeneous ($=\mu\gamma$-homogeneous) parts yields $z_\gamma a_\mu = a_\mu z_\gamma$. Therefore $z_\gamma \in Z(A)$, so $Z(A) = \bigoplus_{\gamma\in\Gamma} Z(A)_\gamma$ is a graded subalgebra. If Γ happens to not be abelian, the situation can be saved by replacing Γ with its abelianisation Γ' (and taking the quotient grading) so that $Z(A)$ becomes a Γ'-graded subalgebra of A.

Let $\mathcal{G}$ be an ample groupoid and R a commutative ring with unit. A function $f \in A_R(\mathcal{G})$ is called a *class function* if it vanishes outside the isotropy subgroupoid and it is constant on conjugacy classes. In other words, it should satisfy:

(C1) $f(x) = 0$ for all $x \in \mathcal{G}$ such that $\boldsymbol{d}(x) \neq \boldsymbol{c}(x)$;

(C2) $f(zxz^{-1}) = f(x)$ for all $x, z \in \mathcal{G}$ such that $\boldsymbol{c}(x) = \boldsymbol{d}(x) = \boldsymbol{d}(z)$.

Theorem 6.1. *[77, Theorem 4.13] Let $\mathcal{G}$ be an ample groupoid. The centre of $A_R(\mathcal{G})$ is the set of class functions.*

Proof. First of all, suppose $f \in A_R(\mathcal{G})$ is a class function, and let $g \in A_R(\mathcal{G})$. Then

$$f * g(x) = \sum_{y\in\mathcal{G}^{\boldsymbol{d}(x)}} f(xy^{-1})g(y) = \sum_{y\in\ {}_{\boldsymbol{c}(x)}\mathcal{G}^{\boldsymbol{d}(x)}} f(xy^{-1})g(y). \tag{6.1}$$

The second equality uses the fact (C1) that $f(xy^{-1}) = 0$ if $\boldsymbol{c}(y) \neq \boldsymbol{c}(x)$. Now, for all $y \in \mathcal{G}^{\boldsymbol{d}(x)}$, xy^{-1} and $y^{-1}x$ are conjugates ($xy^{-1} = y(y^{-1}x)y^{-1}$), and (C2) implies $f(xy^{-1}) = f(y^{-1}x)$. After a change of variables, $z = y^{-1}x$, equation (6.1) becomes

$$f * g(x) = \sum_{z\in\ {}_{\boldsymbol{d}(x)}\mathcal{G}^{\boldsymbol{d}(x)}} f(z)g(xz^{-1}) = \sum_{z\in\mathcal{G}^{\boldsymbol{d}(x)}} f(z)g(xz^{-1}) = g * f(x).$$

The second equality, where the number of terms in the sum appears to increase, is justified because $f(z) = 0$ whenever $\boldsymbol{c}(z) \neq \boldsymbol{d}(z)$. Therefore $f \in Z(A_R(\mathcal{G}))$. Secondly, we show that $f \in Z(A_R(\mathcal{G}))$ implies f is a class function. Suppose $x \in \mathcal{G}$ has $\boldsymbol{d}(x) \neq \boldsymbol{c}(x)$. If $U \in \mathcal{B}(\mathcal{G}^{(0)})$,

$$\mathbf{1}_U * f(x) = \sum_{y\in\mathcal{G}^{\boldsymbol{d}(x)}} \mathbf{1}_U(xy^{-1})f(y) = \mathbf{1}_U(\boldsymbol{c}(x))f(x) \tag{6.2}$$

because $xy^{-1} \in U \subseteq \mathcal{G}^{(0)}$ implies $xy^{-1} = \boldsymbol{c}(x)$ and thus $y = x$. Similarly,

$$f * \mathbf{1}_U(x) = \sum_{y\in\mathcal{G}^{\boldsymbol{d}(x)}} f(xy^{-1})\mathbf{1}_U(y) = f(x)\mathbf{1}_U(\boldsymbol{d}(x)) \tag{6.3}$$

because $y \in U \cap \mathcal{G}^{\boldsymbol{d}(x)}$ implies $y = \boldsymbol{d}(x)$. Choosing U so that it contains $\boldsymbol{c}(x)$ but not $\boldsymbol{d}(x)$ means that (6.2) becomes $f(x)$ while (6.3) becomes 0. But the two equations must be equal because $f \in Z(A_R(\mathcal{G}))$, so $f(x) = 0$. That is, f has property (C1). Now suppose $x, z \in \mathcal{G}$ have $\boldsymbol{c}(x) = \boldsymbol{d}(x) = \boldsymbol{d}(z)$. If $V \in B^{\mathrm{co}}(\mathcal{G})$ contains z, then

$$f * \mathbf{1}_V(zx) = \sum_{y\in\mathcal{G}^{\boldsymbol{d}(z)}} f(zxy^{-1})\mathbf{1}_V(y) = f(zxz^{-1}) \tag{6.4}$$

because no morphism $y \in V$ besides z has the same domain as z. Similarly,

$$\mathbf{1}_V * f(zx) = \sum_{y\in\mathcal{G}^{\boldsymbol{d}(z)}} \mathbf{1}_V(zxy^{-1})f(y) = f(x) \tag{6.5}$$

because if $zxy^{-1} \in V$ then $zxy^{-1} = z$ and $y = x$. Since f is in the centre, (6.4) and (6.5) are equal and this proves that f has property (C2). $\square$

We have generalised a well-known fact: the centre of a group algebra is the set of class functions. Since $A_R(\mathcal{G})$ is generated by characteristic functions, it is natural to ask when such a function is in the centre. If $\mathbf{1}_U \in Z(A_R(\mathcal{G}))$, then condition (C1) implies that $U \subseteq \mathrm{Iso}(\mathcal{G})$. Condition (C2) implies that U is a union of conjugacy classes. In particular, if $U \subseteq \mathcal{G}^{(0)}$, then U is invariant. On the other hand, if $U \subseteq \mathcal{G}$ is a union of conjugacy classes and $\mathbf{1}_U \in A_R(\mathcal{G})$, then $\mathbf{1}_U$ satisfies (C1) and (C2), so it is a class function. The discussion has just proved:

Lemma 6.2. *Let $\mathcal{G}$ be an ample groupoid and suppose $\mathbf{1}_U \in A_R(\mathcal{G})$. Then $\mathbf{1}_U \in Z(A_R(\mathcal{G}))$ if and only if $U \subseteq \mathrm{Iso}(\mathcal{G})$ and $\mathrm{Cl}_{\mathcal{G}}(z) \subseteq U$ for every $z \in U$.*

If $\mathcal{G}$ is a Γ-graded ample groupoid, then $\mathcal{G}_\varepsilon$ is a clopen ample subgroupoid of $\mathcal{G}$, and $\mathrm{Iso}(\mathcal{G}) = \bigsqcup_{\gamma\in\Gamma} \mathrm{Iso}(\mathcal{G})_\gamma$ is a closed graded ample subgroupoid of $\mathcal{G}$, where $\mathrm{Iso}(\mathcal{G})_\gamma = \mathrm{Iso}(\mathcal{G}) \cap \mathcal{G}_\gamma$. Note that $\mathrm{Iso}(\mathcal{G})_\varepsilon = \mathrm{Iso}(\mathcal{G}_\varepsilon)$. The situation can be depicted:

$$\begin{tikzcd}
& \mathcal{G} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{G}_\varepsilon \arrow[dr, dash] & & \mathrm{Iso}(\mathcal{G}) \arrow[dl, dash] \\
& \mathrm{Iso}(\mathcal{G}_\varepsilon) \arrow[d, dash] & \\
& \mathcal{G}^{(0)} &
\end{tikzcd}$$

Recall that $A_R(\mathcal{G}) = \bigoplus_{\gamma\in\Gamma} A_R(\mathcal{G})_\gamma$, and if Γ is abelian then the centre of $A_R(\mathcal{G})$ is a graded subalgebra, so it has a decomposition: $Z(A_R(\mathcal{G})) = \bigoplus_{\gamma\in\Gamma} Z(A_R(\mathcal{G}))_\gamma$, where $Z(A_R(\mathcal{G}))_\gamma = Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\gamma$. Having said this, the next three lemmas are most useful when Γ is abelian; if it is not, then knowing about $Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\gamma$ does not tell us much about $Z(A_R(\mathcal{G}))$.

Lemma 6.3. *Suppose $\mathcal{G}$ is an effective Γ-graded Hausdorff ample groupoid. Then $Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\gamma = 0$ for all $\gamma \neq \varepsilon$.*

Proof. Let $f \in Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\gamma$ where $\gamma \neq \varepsilon$. Since f is a continuous class function, $\mathrm{supp}\, f$ is an open subset of $\mathrm{Iso}(\mathcal{G})_\gamma \subseteq \mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$. But the interior of $\mathrm{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}$ is empty, by Lemma 3.18, so $\mathrm{supp}\, f = \emptyset$ and $f = 0$. $\square$

Lemma 6.4. *Let $\mathcal{G}$ be a Γ-graded Hausdorff ample groupoid, with $\mathcal{G}_\varepsilon$ effective. Then $Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\varepsilon$ is spanned by characteristic functions of compact open invariant subsets of $\mathcal{G}^{(0)}$.*

Proof. The characteristic functions of compact open invariant sets are class functions, so they are in $Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\varepsilon$. On the other hand, if $f \in Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\varepsilon$ then it must be a class function supported on an open subset of $\mathrm{Iso}(\mathcal{G}_\varepsilon)$. Then, $\mathrm{supp}\, f \subseteq \mathrm{Iso}(\mathcal{G}_\varepsilon)^\circ = \mathcal{G}^{(0)}$. Since f is a class function, (C2) implies that for all $0 \neq r \in \mathrm{im}\, f$, the set $f^{-1}(r) \subseteq \mathcal{G}^{(0)}$ is invariant. Since f is continuous and compactly supported, $f^{-1}(r)$ is also compact and open. Clearly $f = \sum r\mathbf{1}_{f^{-1}(r)}$, where r ranges over $\mathrm{im}\, f \setminus \{0\}$, and this proves the conclusion. $\square$

Lemma 6.5. *Suppose $\mathcal{G}$ is a Γ-graded Hausdorff ample groupoid, and let $\gamma \neq \varepsilon$. Then $Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\gamma$ is spanned by characteristic functions of compact open subsets of $\operatorname{Iso}(\mathcal{G})_\gamma$ that are unions of conjugacy classes.*

Proof. If $U \subseteq \operatorname{Iso}(\mathcal{G})_\gamma$ is a compact open union of conjugacy classes, then $\mathbf{1}_U$ is a class function, so $\mathbf{1}_U \in Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\gamma$. On the other hand, if $f \in Z(A_R(\mathcal{G})) \cap A_R(\mathcal{G})_\gamma$ then it is a class function supported on $\operatorname{Iso}(\mathcal{G})_\gamma$. For all $0 \neq r \in \operatorname{im} f$, $f^{-1}(r)$ is a compact open subset of $\operatorname{Iso}(\mathcal{G})_\gamma$. Condition (C2) implies that for all such r, the set $f^{-1}(r)$ contains the conjugacy classes of all of its elements. $\square$

6.2 Graph concepts II

We introduce a festival of new terminology and notation for graphs. Let E be a graph. The definition of x^0 for a path $x \in E^\star \cup E^\infty$ is the set of vertices that occur as ranges or sources of the edges that constitute x. That is:

$$\begin{array}{rcll} v^0 & = & \{v\} & \text{if } v \in E^0, \\ \alpha^0 & = & \{s(\alpha_1), r(\alpha_1), \dots, r(\alpha_{|\alpha|})\} & \text{if } \alpha \in E^\star \setminus E^0, \\ p^0 & = & \{s(p_1), r(p_1), r(p_2), \dots\} & \text{if } p \in E^\infty. \end{array}$$

If $\emptyset \neq H \subseteq E^0$, define $F_E(H)$ to be the set of paths of positive length whose final edge has its range in H, and no other edges have their range in H:

$$F_E(H) = \{\alpha \in E^\star \setminus E^0 \mid s(\alpha_1), r(\alpha_1), \dots, r(\alpha_{|\alpha|-1}) \notin H, r(\alpha_{|\alpha|}) \in H\}.$$

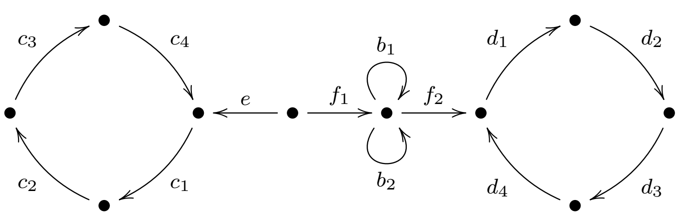
If $c = c_1 \dots c_{|c|}$ is a cycle based at $s(c_1)$, then for $1 \leq i \leq |c|$, define:

$$c^\infty = cc \dots \in E^\infty,$$
$$c^{(i)} = c_i \dots c_{|c|} c_1 \dots c_{i-1} \in E^\star.$$

Each $c^{(i)}$ is a cycle based at $s(c_i)$. Let $[c] = \{c^{(1)}, \dots, c^{(|c|)}\}$. If any $d \in [c]$ has an exit, then every $d \in [c]$ has an exit. Let $\mathscr{C}_{\text{ne}}$ be the set of all cycles without exits. Define $\mathscr{C}^{\text{f}}_{\text{ne}} = \{c \in \mathscr{C}_{\text{ne}} \mid F_E(c^0) \text{ is finite}\}$, the set of all cycles c without exits such that the number of paths sharing exactly one vertex with c is finite. Furthermore, define:

$$[\mathscr{C}_{\text{ne}}] = \big\{[c] \mid c \in \mathscr{C}_{\text{ne}}\big\}, \qquad [\mathscr{C}^{\text{f}}_{\text{ne}}] = \big\{[c] \mid c \in \mathscr{C}^{\text{f}}_{\text{ne}}\big\}.$$

Example 6.6. Let E be the graph:



In this example, $c = c_1c_2c_3c_4$ and $d = d_1d_2d_3d_4$ are cycles without exits, whereas b_1 and b_2 are cycles with exits. So $\mathscr{C}_{\text{ne}} = [c] \cup [d] = \{c, c^{(2)}, c^{(3)}, c^{(4)}, d, d^{(2)}, d^{(3)}, d^{(4)}\}$. The set $F_E(c^0) = \{e\}$ is finite, but $F_E(d^0)$ is infinite. So $\mathscr{C}^{\text{f}}_{\text{ne}} = [c]$ and $[\mathscr{C}^{\text{f}}_{\text{ne}}] = \{[c]\}$.

If $v, w \in E^0$, then $v \geq w$ means there is a path $\alpha \in E^\star$ with source v and range w. A set $H \subseteq E^0$ is called *hereditary* if for all $v \in H$, $v \geq w$ implies $w \in H$. A set $J \subseteq E^0$ is called *saturated* if for every $v \in E^0_{\text{reg}}$, $r(vE^1) \subseteq J$ implies $v \in J$. If $X \subseteq E^0$, then $\overline{X}$ is the smallest hereditary and saturated subset of E^0 containing X, called the *hereditary saturated closure* of X. The closure $\overline{X}$ exists because E^0 is hereditary and saturated, and intersections of hereditary (resp., saturated) subsets are hereditary (resp., saturated). That is,

$$\overline{X} = \bigcap \{Y \mid X \subseteq Y \subseteq E^0,\ Y \text{ is hereditary and saturated}\}.$$

Moreover, there exists an algorithm to construct $\overline{X}$ by systematically adding vertices to X. Define the *tree* of a set of vertices $X \subseteq E^0$ as the hereditary set $T(X) = \bigcup_{v \in X} \{w \in E^0 \mid v \geq w\}$. Define $S(X) = X \cup \{v \in E^0_{\text{reg}} \mid r(vE^1) \subseteq X\}$. The following lemma is straightforward to prove.

Lemma 6.7. *[2, Lemma 2.0.7] Let E be a graph and $X \subseteq E^0$. Define $X_0 = T(X)$ and define inductively $X_{n+1} = S(X_n)$ for all $n \geq 0$. Then $\overline{X} = \bigcup_{n=1}^{\infty} X_n$.*

If H is hereditary and saturated, a *breaking vertex* for H is an infinite emitter $v \notin H$ such that all but finitely many edges in vE^1 have ranges in H, and at least one edge in vE^1 has its range outside H. Formally, the set of breaking vertices of H is

$$B_H = \left\{v \in E^0 \setminus H \mid \ v \text{ is an infinite emitter and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty \right\}.$$

Define the set of pairs:

$$\mathscr{T}_E = \left\{(H, S) \mid H \subseteq E^0 \text{ is hereditary and saturated},\ S \subseteq B_H \right\}.$$

For $(H, S) \in \mathscr{T}_E$, define $F_E(H, S)$ to be the set of paths in $F_E(H)$ whose final edge is not emitted by a breaking vertex in S:

$$\begin{aligned} F_E(H, S) &= \{\alpha \in F_E(H) \mid s(\alpha_{|\alpha|}) \notin S\} \\ &= \{\alpha \in E^\star \setminus E^0 \mid s(\alpha_1), r(\alpha_1), \ldots, r(\alpha_{|\alpha|-1}) \notin H, s(\alpha_{|\alpha|}) \notin S, r(\alpha_{|\alpha|}) \in H\}. \end{aligned}$$

Note that $F_E(H, \emptyset) = F_E(H)$. Let $F'_E(H) = F_E(H, B_H)$.

Definition 6.8. A hereditary and saturated set $H \subseteq E^0$ satisfies **Condition (F)** if the following set is finite:

$$H \sqcup F'_E(H) \sqcup \{\beta \in E^\star \mid r(\beta) \in B_H\}. \tag{6.6}$$

In words, H satisfies Condition (F) if and only if H is finite, there are finitely many paths of positive length that enter H through a non-breaking vertex and share exactly one vertex with H, and there are finitely many paths whose range is a breaking vertex for H. Define

$$\mathscr{T}_E^{\text{c}} = \left\{(H, B_H) \mid H \text{ satisfies Condition (F)}\right\}.$$

Let $\mathscr{T}_E^{\text{cm}}$ be the set of all pairs $(H, B_H) \in \mathscr{T}_E^{\text{c}}$ such that $H \neq \emptyset$ and H has no proper hereditary and saturated subsets satisfying Condition (F).

If $v \in B_H$ is a breaking vertex for H, we partition $vE^1 = F_v \sqcup F_v^{\text{inf}}$, where:

$$F_v = \{e \in vE^1 \mid r(e) \notin H\}, \qquad F_v^{\text{inf}} = \{e \in vE^1 \mid r(e) \in H\}.$$

Note that F_v is finite and F_v^{inf} is infinite, because v is a breaking vertex for H. A vertex v can be a breaking vertex for several different hereditary saturated sets H, and indeed the definition of F_v

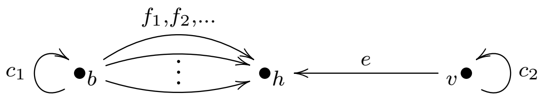
and $F_v^{\inf}$ depends on H. However, we will only use this notation when it is clear which H is meant, so there is no need to further complicate the typography. Given $(H,S) \in \mathscr{T}_E$, we define

$$U_H = \{x \in \partial E \mid x^0 \cap H \neq \emptyset\}, \qquad U_S = \{\alpha \in E^\star \mid r(\alpha) \in S\}, \qquad U_{H,S} = U_H \sqcup U_S.$$

Note that $U_{H,S} \subseteq \partial E$. When we refer to U_H and U_S individually, the meaning will be clear from context, despite the potential ambiguity of the notation. Finally, let $\mathscr{O}_E$ be the set of open invariant subsets of $\mathcal{G}_E^{(0)} = \partial E$, and let $\mathscr{O}_E^{\mathrm{c}} \subseteq \mathscr{O}_E$ be the set of compact open invariant sets. (Recall that a subset U of ∂E is invariant if and only if it contains the tail equivalence classes of all its elements.) For $U \in \mathscr{O}_E$, we define:

$$H_U = \{v \in E^0 \mid Z(v) \subseteq U\}, \qquad S_U = \{v \in E^0 \mid v \in U \text{ is an infinite emitter }, v \notin H_U\}.$$

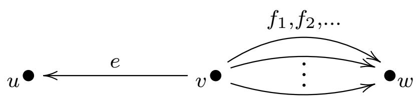
Example 6.9. Let E be the following graph:



The vertex set is $E^0 = \{b, h, v\}$ and the edge set is $E^1 = \{c_1, c_2, e, f_j \mid j \geq 1\}$. The set $H = \{h\}$ is hereditary and saturated, and b is a breaking vertex for H because it emits infinitely many edges f_j whose range is h, and one edge c_1 whose range is not h. Indeed, $B_H = \{b\}$. However, H does not satisfy Condition (F) because the set in (6.6) is infinite. The other data pertaining to H are:

$$\begin{aligned} F_E(H) &= \{e, c_2^n e \mid n \geq 1\} \cup \{f_j, c_1^n f_j \mid n \geq 1, j \geq 1\} & F'_E(H) &= \{e, c_2^n e \mid n \geq 1\} \\ F_b &= \{c_1\} & F_b^{\inf} &= \{f_j \mid j \geq 1\} \\ U_H &= F_E(H) \cup \{h\} & U_{\{b\}} &= \{c_1^n \mid n \geq 1\} \end{aligned}$$

Example 6.10. Let E be the following graph:



The vertex set is $E^0 = \{u, v, w\}$ and the edge set is $E^1 = \{e, f_j \mid j \geq 1\}$. The sets $H_1 = \{w\}$, $H_2 = \{u\}$, and $H_1 \cup H_2 = \{u, w\}$ are hereditary and saturated. Although $r(vE^1) \subseteq H_1$ and $v \notin H_1$, this does not contradict the saturation of H_1 because v is not a regular vertex. Then, $B_{H_1} = \{v\}$, $B_{H_2} = \emptyset$, and $B_{H_1 \cup H_2} = \emptyset$. The vertex v is not a breaking vertex for $H_1 \cup H_2$ because *all* the edges in vE^1 have their ranges in $H_1 \cup H_2$. Clearly, H_1 and H_2 satisfy Condition (F). On the other hand, $H_1 \cup H_2$ does not satisfy Condition (F) because $F'_E(H_1 \cup H_2) = E^1$ is infinite. For completeness, we give:

$$\begin{aligned} \mathscr{T}_E &= \big\{(\emptyset, \emptyset), (H_1, \emptyset), (H_1, v), (H_2, \emptyset), (H_1 \cup H_2, \emptyset), (E^0, \emptyset)\big\}; \\ \mathscr{T}_E^{\mathrm{c}} &= \big\{(\emptyset, \emptyset), (H_1, v), (H_2, \emptyset), (E^0, \emptyset)\big\}; \\ \mathscr{T}_E^{\mathrm{cm}} &= \{(H_1, v), (H_2, \emptyset)\} \end{aligned}$$

The next section discusses how to determine $\mathscr{O}_E$ and $\mathscr{O}_E^{\mathrm{c}}$ using these data.

6.3 The path space of a graph II

In this section and the next, we characterise the (compact) open subsets of the boundary path groupoid that are unions of conjugacy classes. The starting point is studying the conjugacy classes of units, which are called invariant sets. Open invariant sets have a description in terms of hereditary and saturated sets of vertices, and their associated breaking vertices. With this description, it is possible to determine when the open invariant sets are compact, and when they are minimal as compact open invariant sets. These sets play an important role in the 0-component of the centre of a Leavitt path algebra.

Theorem 6.11. *[30, Theorem 3.3] Let E be a graph. Open invariant subsets of $\mathcal{G}_E^{(0)}$ are in one-to-one correspondence with pairs (H, S) where $H \subseteq E^0$ is hereditary and saturated, and $S \subseteq B_H$. The correspondence is given by the mutually inverse bijections:*

$$\begin{aligned} \phi &: \mathscr{T}_E \to \mathscr{O}_E, & \phi &: (H,S) \mapsto U_{H,S}, \\ \rho &: \mathscr{O}_E \to \mathscr{T}_E, & \rho &: U \mapsto (H_U, S_U). \end{aligned} \tag{6.7}$$

Proof. The proof is done in four steps, all of which are quite technical.

Step 1: If $(H,S) \in \mathscr{T}_E$ then $U_{H,S} \in \mathscr{O}_E$.

If $x \in U_H$, then it has an initial subpath x' such that $(x')^0 \cap H \neq \emptyset$, so $x \in Z(x') \subseteq U_H$. This shows U_H is open. If $x \in U_S$, then $Z(x, F_{r(x)})$ consists of x together with all $xy \in \partial E$ ($|y| \geq 1$) such that $y_1 \notin F_{r(x)}$. But then $r(y_1) \in H$, so $xy \in U_H$. Therefore $x \in Z(x, F_{r(x)}) \subseteq U_{H,S}$. This shows $U_{H,S}$ is open. To show that it is invariant, let $(\alpha x, k, \beta x) \in \mathcal{G}_E$ and suppose $\beta x \in U_H$. Then $(\beta x)^0 \cap H \neq \emptyset$ implies $x^0 \cap H \neq \emptyset$, because H is hereditary, which implies $(\alpha x)^0 \cap H \neq \emptyset$. So $\alpha x \in U_H$. On the other hand, if $\beta x \in U_S$, then $r(\beta x) = r(\alpha x)$ so $\alpha x \in U_S$. This shows that $U_{H,S}$ is invariant.

Step 2: If $U \in \mathscr{O}_E$ then $(H_U, S_U) \in \mathscr{T}_E$.

Since U is invariant, it is a union of tail equivalence classes. Let $v \in H_U$. If $w \in E^0$ and $v \geq w$, then there is a path $\alpha \in vE^\star$ with $r(\alpha) = w$. If $x \in Z(w)$ then x is tail equivalent to $\alpha x \in Z(v) \subseteq U$, so $Z(w) \subseteq U$. Hence $w \in H_U$, so H_U is hereditary. Let $v \in E^0_{\text{reg}}$ be such that $r(e) \in H_U$ for every $e \in vE^1$. Then for every such e, $Z(e) \subseteq U$ because each $ex \in Z(e)$ is tail equivalent to $x \in Z(r(e)) \subseteq U$. Now, $Z(v) = \bigcup_{e \in s^{-1}(v)} Z(e) \subseteq U$ shows that $v \in H_U$. Therefore H_U is saturated. Now let $v \in S_U$. Then v is an infinite emitter in U but not in H_U. Since U is open, v has an open neighbourhood $Z(v, F)$ contained in U. Then for every $e \in vE^1 \setminus F$, $Z(e) \subseteq Z(v,F) \subseteq U$. Since U is invariant, this implies $Z(r(e)) \subseteq U$, and therefore $r(e) \in H_U$. This shows that all but finitely many edges in vE^1 have ranges in H_U. If *all* the edges in vE^1 have ranges in H_U, then $Z(r(e)) \subseteq U$ and consequently $Z(e) \subseteq U$, for every $e \in vE^1$. But then $Z(v) = \{v\} \cup \bigcup_{e \in vE^1} Z(e) \subseteq U$ implies $v \in H_U$, a contradiction. Therefore at least one edge in vE^1 has its range outside H_U. This shows v is a breaking vertex for H_U, and therefore $(H_U, S_U) \in \mathscr{T}_E$.

Step 3: $\phi \circ \rho = \mathrm{id}_{\mathscr{O}_E}$.

Let $U \in \mathscr{O}_E$. We must prove that $\phi \circ \rho(U) = \phi((H_U, S_U)) = U_{H_U, S_U} = U$.

Let $x \in U_{H_U,S_U}$. To show that $x \in U$, consider two cases. In the first case, suppose $x \in U_{H_U}$. Then $x^0 \cap H_U \neq \emptyset$, so there exists $v \in x^0$ such that $Z(v) \subseteq U$. Then x is tail equivalent to some $y \in Z(v)$, so $x \in U$. In the second case, suppose $x \in U_{S_U}$. Then $r(x) \in S_U$; in particular, $r(x) \in U$ and so $x \in U$ because x is tail equivalent to $r(x)$. Therefore $U_{H_U,S_U} \subseteq U$.

For the reverse containment, let $y \in U$. Then, since $y \in \partial E$, we divide the possibilities into three cases. In the first case, $y \in E^\star$ and $r(y)$ is a sink. Then $r(y) \in U$ implies $Z(r(y)) = \{r(y)\} \subseteq U$, and thus $r(y) \in H_U$. This implies $y \in U_{H_U}$. In the second case, $y \in E^\star$ and $r(y)$ is an infinite emitter. But then, since $r(y) \in U$, either $r(y) \in H_U$ or $r(y) \in S_U$. If $r(y) \in H_U$, then $y \in U_{H_U}$, while if $r(y) \in S_U$ then $y \in U_{S_U}$. Either way, $y \in U_{H_U,S_U}$. In the third and final case, $y \in E^\infty$ and, since U is open, y must have a neighbourhood $Z(y_1 \dots y_n) \subseteq U$. Then $Z(r(y_n)) \subseteq U$, which implies $r(y_n) \in H_U$, and in turn this implies $y \in U_{H_U}$. The conclusion is that $U = U_{H_U,S_U}$.

Step 4: $\quad \rho \circ \phi = \mathrm{id}_{\mathscr{T}_E}$.

Let $(H, S) \in \mathscr{T}_E$. We must prove $\rho \circ \phi((H, S)) = \rho(U_{H,S}) = (H_{U_{H,S}}, S_{U_{H,S}}) = (H, S)$.

We first claim that $H = H_{U_{H,S}}$. Let $w \in H$. Then by definition $Z(w) \subseteq U_H \subseteq U_{H,S}$, so $w \in H_{U_{H,S}}$ and therefore $H \subseteq H_{U_{H,S}}$. For the reverse containment, let $v \in H_{U_{H,S}}$. This means $Z(v) \subseteq U_{H,S}$. If v is a singular vertex, then $v \in Z(v) \subseteq U_{H,S}$. But $v \notin U_S$ because $v \notin S$, so $v \in U_H$, which implies $v \in H$. So this reduces to the case where v is a regular vertex. From $v\partial E = Z(v) \subseteq U_{H,S}$, we deduce that for every $x \in v\partial E$: either $x \in vE^\star$ and $r(x) \in S$, or $x^0 \cap H \neq \emptyset$. But if $r(x) \in S$ for some $x \in vE^\star$, then $r(x) \notin H$ and this contradicts that H is hereditary. Therefore,

$$x^0 \cap H \neq \emptyset \text{ for every } x \in v\partial E. \tag{6.8}$$

Suppose, for the sake of contradiction, that $v \notin H$. If every $e \in vE^1$ has its range in H, then $v \in H$ because H is saturated. But we assumed this is false, so there exists $e_1 \in vE^1$ with $r(e_1) \notin H$. But then $(e_1 y)^0 \cap H \neq \emptyset$ and hence $y^0 \cap H \neq \emptyset$, for every $y \in r(e_1)\partial E$, by (6.8). Now, $r(e_1)$ is regular because otherwise $r(e_1) \in r(e_1)\partial E$, which would imply $r(e_1) \in H$, which it is not. If every $f \in r(e_1)E^1$ has its range in H, then saturation implies $r(e_1) \in H$, which it is not. So there exists $e_2 \in r(e_1)E^1$ with $r(e_2) \notin H$. Continuing inductively, we can construct $p = e_1 e_2 \ldots \in vE^\infty$ such that $p^0 \cap H = \emptyset$, which contradicts (6.8). Thus, $v \in H$, so $H_{U_{H,S}} \subseteq H$.

Now that $H = H_{U_{H,S}}$, we claim that $S = S_{U_{H,S}}$. Let $v \in S$. Clearly, $v \in U_S \subseteq U_{H,S}$; also, $v \in B_H$, so $v \in E^0_{\mathrm{inf}}$ and $v \notin H = H_{U_{H,S}}$; so by definition $v \in S_{U_{H,S}}$. Thus, $S \subseteq S_{U_{H,S}}$. Now let $w \in S_{U_{H,S}}$. In particular, $w \in U_{H,S}$ and $w \notin H_{U_{H,S}} = H$. More specifically, $w \in U_S$ because $w \in U_H$ would imply $w \in H$. But $w \in U_S$ implies $r(w) = w \in S$. This shows $S_{U_{H,S}} \subseteq S$, and concludes the proof. $\square$

We have shown that every open invariant subset of ∂E is of the form $U_{H,S}$ for some $(H, S) \in \mathscr{T}_E$. These can now be written down explicitly.

Lemma 6.12. *[31, Lemma 2.1] Let E be a graph, and $(H, S) \in \mathscr{T}_E$. Then*

$$U_{H,S} = \left(\bigsqcup_{v \in H} Z(v) \right) \sqcup \left(\bigsqcup_{\alpha \in F_E(H,S)} Z(\alpha) \right) \sqcup \left(\bigsqcup_{\substack{\beta \in E^\star \\ r(\beta) \in S}} Z(\beta, F_{r(\beta)}) \right). \tag{6.9}$$

Proof. Let the three sets appearing in brackets in (6.9) be called A_1, A_2, and A_3. Note that $\alpha \in F_E(H, S)$ implies that $s(\alpha) \notin H$, so A_1 and A_2 are disjoint. Likewise, if $r(\beta) \in S$ then $s(\beta) \notin H$ because H is hereditary, so A_1 and A_3 are disjoint. If $\alpha x \in Z(\alpha)$ for some $\alpha \in F_E(H, S)$, then αx first enters H through a vertex not in S, so $\alpha x \notin Z(\beta, F_{r(\beta)})$ for any $\beta \in E^\star$ with $r(\beta) \in S$. This shows A_2 and A_3 are disjoint.

Next, we show that $A_1 \cup A_2 \cup A_3 \subseteq U_{H,S}$. It is easy to see that $A_1 \cup A_2 \subseteq U_H \subseteq U_{H,S}$. On the other hand, let $\beta \in E^\star$ and $r(\beta) \in S$. Then $\beta \in U_S$ and everything in $Z(\beta, F_{r(\beta)}) \setminus \{\beta\}$ is of the form βx where $x \in r(\beta)\partial E$, $|x| \geq 1$, and $x_1 \notin F_{r(\beta)}$. But then $r(x_1) \in H$, so $\beta x \in U_H$. This shows $A_3 \subseteq U_{H,S}$.

Lastly, we show that $U_{H,S} \subseteq A_1 \cup A_2 \cup A_3$. By the definition of U_S it is clear that $U_S \subseteq A_3$. Let $y \in U_H$, which means $y \in \partial E$ and $y^0 \cap H \neq \emptyset$. Immediately rule out the case where y is a vertex not in H. If $s(y) \in H$ then $y \in Z(s(y)) \subseteq A_1$. What remains is the case where $s(y) \notin H$ and $|y| \geq 1$. Write $y = \beta e x$ where $\beta \in E^\star$, $e \in r(\beta)E^1$, $x \in r(e)\partial E$, and $s(e) \notin H$ while $r(e) \in H$. If $s(e) \notin S$, then $\beta e \in F_E(H, S)$ and $y \in Z(\beta e) \subseteq A_2$. Otherwise, $s(e) = r(\beta) \in S$, and we have $y = \beta e x \in Z(\beta, F_{r(\beta)}) \subseteq A_3$. □

Proposition 6.13. *[31, Proposition 2.3] Let E be a graph, and $(H, S) \in \mathscr{T}_E$. Then $U_{H,S}$ is compact if and only if $S = B_H$ and H satisfies Condition (F).*

Proof. Define
$$I = H \sqcup F_E(H, S) \sqcup \{\beta \in E^\star \mid r(\beta) \in S\}.$$
By Lemma 2.7, each basic open set appearing in (6.9) is nonempty. If $|I| = \infty$ then (6.9) gives an expression for an I-indexed open cover of $U_{H,S}$ with no finite subcover, so $U_{H,S}$ is not compact. On the other hand, if $|I| < \infty$ then (6.9) gives an expression for $U_{H,S}$ as a finite union of compact sets, which implies $U_{H,S}$ is compact. So, $U_{H,S}$ is compact if and only if I is finite. We claim that I is finite if and only if $S = B_H$ and H satisfies Condition (F). If $S \neq B_H$, then there exists a breaking vertex $b \in B_H \setminus S$. Then $e \in F_E(H, S)$ for every $e \in F_b^{\mathrm{inf}}$, of which there are infinitely many, so I is infinite. If $S = B_H$ then by definition, H satisfies Condition (F) if and only if I is finite. □

It is worth pursuing an even better description of the compact open invariant sets. The summary so far is that $\mathscr{O}_E$ is in one-to-one correspondence with $\mathscr{T}_E$ (Theorem 6.11) and $\mathscr{O}_E^{\mathrm{c}}$ is in one-to-one correspondence with $\mathscr{T}_E^{\mathrm{c}}$ (Proposition 6.13).

A *lattice* is a partially ordered set $(L, \leq)$ where any two elements $l, l' \in L$ have a least upper bound $l \vee l' \in L$ and a greatest lower bound $l \wedge l' \in L$. A *sublattice* of L is a subset M that is closed under $\vee$ and $\wedge$. A *homomorphism* between lattices L and L' is a map $f : L \to L'$ that preserves $\vee$ and $\wedge$. Lattice homomorphisms are order-preserving, but not all order-preserving maps between lattices are lattice homomorphisms. A lattice *isomorphism* is a bijective homomorphism of lattices. The *principal downset* of $p \in L$ is the set $p^\downarrow = \{l \in L \mid l \leq p\}$. We chall use [54, Theorem 8.2]: a bijection of lattices $\phi : L \to L'$ is a lattice isomorphism if and only if ϕ and ϕ^{-1} are order-preserving.

A lattice structure $(\mathscr{O}_E, \subseteq)$ is available on $\mathscr{O}_E$, because unions and finite intersections preserve openness and invariance. There is a lattice structure on $\mathscr{T}_E$ too, which comes from the partial order:
$$(H_1, S_1) \leq (H_2, S_2) \iff H_1 \subseteq H_2 \text{ and } S_1 \subseteq S_2 \cup H_2.$$
The formulae for $\vee$ and $\wedge$ in $(\mathscr{T}_E, \leq)$ can be found in [2, Proposition 2.5.6] or [19, Remark 3.4].

Theorem 6.14. *[30, Lemma 3.8] Let E be a graph. The map $\phi : \mathscr{T}_E \to \mathscr{O}_E$ from (6.7) is a lattice isomorphism.*

Proof. If ϕ and $\rho = \phi^{-1}$ are order-preserving, then [54, Theorem 8.2] implies ϕ is an isomorphism. Firstly, let $(H_1, S_1), (H_2, S_2) \in \mathscr{T}_E$ and $(H_1, S_1) \leq (H_2, S_2)$. By definition, $H_1 \subseteq H_2$, so $U_{H_1} \subseteq U_{H_2} \subseteq U_{H_2,S_2}$. Also by definition, $S_1 \subseteq S_2 \cup H_2$, so $U_{S_1} \subseteq U_{H_2} \cup U_{S_2} = U_{H_2,S_2}$. Thus $\phi(H_1, S_1) = U_{H_1,S_1} = U_{H_1} \cup U_{S_1} \subseteq U_{H_2,S_2} = \phi(H_2, S_2)$. Secondly, let $U, V \in \mathscr{O}_E$ and $U \subseteq V$. This implies $H_U \subseteq H_V$. If $v \in S_U$ then by definition $v \in U$ is an infinite emitter and $v \notin H_U$. Immediately, $v \in V$ because $U \subseteq V$. If $v \notin H_U$ then either $v \notin H_V$ (so $v \in S_V$) or $v \in H_V \setminus H_U$ (so $v \in H_V$). Either way, $v \in S_V \cup H_V$. This shows $S_U \subseteq S_V \cup H_V$. Therefore $\rho(U) = (H_U, S_U) \leq (H_V, S_V) = \rho(V)$. Conclude that ϕ is a lattice isomorphisms. □

The set $\mathscr{O}_E^{\mathrm{c}}$ of compact open invariant sets is a sublattice of $\mathscr{O}_E$ because $\cup$ and $\cap$ preserve compactness (because ∂E is Hausdorff). Recall that $\mathscr{T}_E^{\mathrm{c}}$ is the set of all pairs $(H, S) \in \mathscr{T}_E$ such that $S = B_H$ and H satisfies Condition (F). The elements of $\mathscr{T}_E^{\mathrm{c}}$ are precisely those pairs whose associated open invariant sets are compact (Proposition 6.13). As a result, the lattice isomorphism whose existence is proved by Theorem 6.14 restricts to a lattice isomorphism $\mathscr{T}_E^{\mathrm{c}} \cong \mathscr{O}_E^{\mathrm{c}}$.

Lemma 6.15. *Let H_1 and H_2 be hereditary saturated subsets of E^0, both satisfying Condition (F). Then $(H_1, B_{H_1}) \leq (H_2, B_{H_2})$ if and only if $H_1 \subseteq H_2$.*

Proof. By definition, $(H_1, B_{H_1}) \leq (H_2, B_{H_2})$ implies $H_1 \subseteq H_2$. Conversely, suppose $H_1 \subseteq H_2$. Let $v \in B_{H_1}$. Towards a contradiction, suppose $v \notin B_{H_2} \cup H_2$. Since it is a breaking vertex for H_1, v is an infinite emitter and $0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H_1)| < \infty$. Since $E^0 \setminus H_2 \subseteq E^0 \setminus H_1$, it follows that $|s^{-1}(v) \cap r^{-1}(E^0 \setminus H_2)| < \infty$. Therefore $0 = |s^{-1}(v) \cap r^{-1}(E^0 \setminus H_2)|$ because otherwise v would be a breaking vertex for H_2. This implies $s^{-1}(v) \subseteq r^{-1}(H_2)$. Then every $e \in s^{-1}(v)$ has $s(e) = v \notin B_{H_2} \cup H_2$, and $r(e) \in H_2$, so $s^{-1}(v) \subseteq F'_E(H_2)$. Since v is an infinite emitter, $F'_E(H_2)$ is infinite, which contradicts the assumption that H_2 satisfies Condition (F). The conclusion is that $B_{H_1} \subseteq B_{H_2} \cup H_2$, and this shows $(H_1, B_{H_1}) \leq (H_2, B_{H_2})$. □

Surprisingly, the lemma does not hold if Condition (F) is removed from the statement. For a counterexample, look at Example 6.10, where $H_1 \subseteq H_1 \cup H_2$ but $(H_1, B_{H_1}) \not\leq (H_1 \cup H_2, B_{H_1 \cup H_2})$.

Recall that we defined $\mathscr{T}_E^{\mathrm{cm}}$ to be the set of all $(H, B_H) \in \mathscr{T}_E^{\mathrm{c}}$ such that $H \neq \emptyset$ and H has no proper hereditary and saturated subsets satisfying Condition (F). By Lemma 6.15, $\mathscr{T}_E^{\mathrm{cm}}$ is the set of all minimal nontrivial pairs in $\mathscr{T}_E^{\mathrm{c}}$, and it corresponds to the set of all minimal nonempty compact open invariant sets in $\mathscr{O}_E^{\mathrm{c}}$.

The following lemma demonstrates a special feature of boundary path groupoids, which is probably not shared by many other ample groupoids.

Lemma 6.16. *[31, Lemmas 3.5 & 3.6] Let E be a graph.*

(1) *$\mathscr{O}_E^{\mathrm{c}}$ satisfies the descending chain condition (meaning, every chain $U_1 \supseteq U_2 \supseteq \ldots$ is eventually constant);*

(2) *Every nonempty $U \in \mathscr{O}_E^{\mathrm{c}}$ contains a subset $V \in \mathscr{O}_E^{\mathrm{c}}$ that is minimal as a nonempty compact open invariant set;*

(3) *If $U \in \mathscr{O}_E^{\mathrm{c}}$ is nonempty, then $U = \bigsqcup_{V \in \mathcal{M}} V$ where $\mathcal{M}$ is the finite set of all minimal nonempty compact open invariant sets contained in U.*

Proof. (1) If $U_1 \supseteq U_2 \supseteq \ldots$ is a chain of compact open invariant sets, then there exists a chain $(H_1, B_{H_1}) \geq (H_2, B_{H_2}) \geq \ldots$ in $\mathscr{T}_E^{\mathrm{c}}$ such that $U_i = U_{H_i, B_{H_i}}$. Each H_i is finite because it satisfies Condition (F), so $(H_1, B_{H_1}) \geq (H_2, B_{H_2}) \geq \ldots$ is eventually constant. This implies $U_1 \supseteq U_2 \supseteq \ldots$ is eventually constant.

(2) Let $\mathcal{P} = \{V \in \mathscr{O}_E^{\mathrm{c}} \mid \emptyset \neq V \subseteq U\}$. By (1), $\mathcal{P}$ satisfies the descending chain condition. By Zorn's Lemma, $\mathcal{P}$ has minimal elements.

(3) From (2), the set $\mathcal{M}$ is nonempty. Let $U = U_{H, B_H} \in \mathscr{O}_E^{\mathrm{c}}$. Then H is finite, because it satisfies Condition (F), so the principal downset $(H, B_H)^{\downarrow} = \{(X, B_X) \in \mathscr{T}_E^{\mathrm{c}} \mid X \subseteq H\}$ is finite. Therefore

$\mathcal{M}$ is finite. To show that $U = \bigcup_{V\in\mathcal{M}} V$, let $U' = U \setminus \bigcup_{V\in\mathcal{M}} V$. Then U' is compact open and invariant, because $\bigcup_{V\in\mathcal{M}} V$ is both open and closed. But if $U' \neq \emptyset$, then one of its subsets is in $\mathcal{M}$, which is a contradiction. Thus $U' = \emptyset$. To show that the union is disjoint: if $V, V' \in \mathcal{M}$ then $V \cap V'$ is also compact open and invariant, so either $V \cap V' = \emptyset$ or $V = V'$, because V and V' are minimal as nonempty compact open invariant sets. □

6.4 The isotropy subgroupoid

Let E be a graph that has cycles without exits or, in other words, does not satisfy Condition (L). Then $\mathcal{G}_E$ is not effective, by Proposition 3.25. This means there are nonempty compact open sets in $\mathrm{Iso}(\mathcal{G}_E) \setminus \mathcal{G}_E^{(0)}$. In this section, we shall characterise the homogeneous (compact) open subsets of $\mathrm{Iso}(\mathcal{G}_E) \setminus \mathcal{G}_E^{(0)}$ that are unions of conjugacy classes. These will feature in the centre of the Leavitt path algebra, just like compact open invariant sets do. Put differently, this section pursues the same goals as the previous section, working now in the n-components ($n \neq 0$) of the isotropy subgroupoid, instead of the unit space. Recall that the conjugacy class of $x \in \mathrm{Iso}(\mathcal{G})$ is defined as $\mathrm{Cl}_{\mathcal{G}}(x) = \{gxg^{-1} \mid g \in \mathcal{G}^{\mathbf{c}(x)}\}$. The inverse of a conjugacy class is the conjugacy class of the inverse. That is, for $x \in \mathrm{Iso}(\mathcal{G})$:

$$\mathrm{Cl}_{\mathcal{G}}(x)^{-1} = \{(gxg^{-1})^{-1} \mid g \in \mathcal{G}^{\mathbf{c}(x)}\} = \{gx^{-1}g^{-1} \mid g \in \mathcal{G}^{\mathbf{c}(x^{-1})}\} = \mathrm{Cl}_{\mathcal{G}}(x^{-1}).$$

Lemma 6.17. *Let E be a graph and $\mathcal{G} = \mathcal{G}_E$. Let $0 \neq n \in \mathbb{Z}$. Then:*

$$\mathrm{Iso}(\mathcal{G})_n{}^\circ \subseteq \bigsqcup_{\substack{[c]\in[\mathscr{C}_{\mathrm{ne}}]\\ m|c|=n}} \mathrm{Cl}_{\mathcal{G}}\left((c^\infty, m|c|, c^\infty)\right).$$

Proof. First assume $n > 0$. If $(z, n, z) \in \mathrm{Iso}(\mathcal{G})_n^\circ$, then $z \in E^\infty$ and z is eventually periodic, by Proposition 2.12. So (z, n, z) has an open neighbourhood $\mathcal{Z}(\alpha\tau, \alpha) \subseteq \mathrm{Iso}(\mathcal{G})_n$, where $r(\alpha) = s(\tau) = r(\tau)$ and $|\tau| = n$. This implies that for every $x \in r(\alpha)\partial E$, we have $(\alpha\tau x, n, \alpha x) \in \mathrm{Iso}(\mathcal{G})$, so $\alpha\tau x = \alpha x$, then $x = \tau x = \tau\tau x = \tau\tau \dots$. So, $r(\alpha)\partial E = \{\tau^\infty\}$. This implies τ is a closed path without exits, so we can write $\tau = c^m$ for some $c \in \mathscr{C}_{\mathrm{ne}}$ and $m \geq 1$, such that $m|c| = |\tau| = n$. Then,

$$(z, n, z) = (\alpha c^\infty, m|c|, \alpha c^\infty) = (\alpha c^\infty, |\alpha|, c^\infty)(c^\infty, m|c|, c^\infty)(c^\infty, -|\alpha|, \alpha c^\infty),$$

and this shows $(z, n, z) \in \mathrm{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$. On the other hand, if $n < 0$ and $(z, n, z) \in \mathrm{Iso}(\mathcal{G})_n^\circ$, then $(z, n, z)^{-1} = (z, -n, z) \in \mathrm{Iso}(\mathcal{G})_{-n}^\circ$ because inversion is a homeomorphism. Therefore $(z, -n, z) \in \mathrm{Cl}_{\mathcal{G}}((c^\infty, -m|c|, c^\infty))$ for some $c \in \mathscr{C}_{\mathrm{ne}}$ with $-m|c| = -n$. This implies $(z, n, z) \in \mathrm{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$ where $m|c| = n$. Conclude $\mathrm{Iso}(\mathcal{G})_n^\circ \subseteq \bigcup \mathrm{Cl}_{\mathcal{G}}\left((c^\infty, m|c|, c^\infty)\right)$ where c ranges over all cycles without exits whose length divides n. Two conjugacy classes $\mathrm{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$ and $\mathrm{Cl}_{\mathcal{G}}((d^\infty, m|d|, d^\infty))$ are disjoint when $[c], [d] \in \mathscr{C}_{\mathrm{ne}}$ are distinct, because then c^∞ and d^∞ are not tail equivalent, and there is no morphism in $\mathcal{G}$ with domain c^∞ and codomain d^∞. Therefore the union on the right hand side is a disjoint union. □

Lemma 6.18. *Let E be a graph and $\mathcal{G} = \mathcal{G}_E$. Let $c \in \mathscr{C}_{\mathrm{ne}}$ and $m \in \mathbb{Z}$. Then*

$$\mathrm{Cl}_{\mathcal{G}}\left((c^\infty, m|c|, c^\infty)\right) = \bigsqcup_{\substack{\alpha\in F_E(c^0)\cup\{c^0\}\\ d\in[c],\ r(\alpha)=s(d)}} \mathcal{Z}(\alpha d^{|m|}, \alpha)^{\mathrm{sgn}(m)}. \tag{6.10}$$

Our convention is that $\operatorname{sgn}(m) = 1$ if $m \geq 0$ and $\operatorname{sgn}(m) = -1$ if $m < 0$. The right hand side of the equation is written that way to emphasise the openness of the set (and in certain cases, its compactness). However, it is simpler than it appears because $\mathcal{Z}(\alpha d^{|m|}, \alpha)^{\operatorname{sgn}(m)} = \{(\alpha d^\infty, m|c|, \alpha d^\infty)\}$, given that $d \in [c]$ is a cycle without exits.

Proof. Clearly, $\mathrm{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty)) = \{(p, m|c|, p) \mid p \sim c^\infty\}$. Since c is a cycle without exits, every path p that is tail equivalent to c^∞ is of the form $p = \alpha d^\infty$ for some $\alpha \in F_E(c^0) \cup \{c^0\}$ and $d \in [c]$, with $r(\alpha) = s(d)$. Moreover, p is uniquely determined by the initial subpath α. $\square$

Lemma 6.19. *Let E be a graph and $\mathcal{G} = \mathcal{G}_E$. Let $0 \neq n \in \mathbb{Z}$. Then:*

$$\mathrm{Iso}(\mathcal{G})_n{}^\circ = \bigsqcup_{\substack{[c]\in[\mathscr{C}_{\mathrm{ne}}] \\ m|c|=n}} \mathrm{Cl}_{\mathcal{G}}\left((c^\infty, m|c|, c^\infty)\right)$$

Proof. Let C be the right hand side. Clearly, $C \subseteq \mathrm{Iso}(\mathcal{G})_n$. Lemma 6.18 proves C is open, so $C \subseteq \mathrm{Iso}(\mathcal{G})_n^\circ$. Lemma 6.17 proves that $\mathrm{Iso}(\mathcal{G})_n^\circ \subseteq C$. $\square$

The notation first introduced in §6.2 now comes to fruition:

$$[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}] = \big\{[c] \mid c \in \mathscr{C}_{\mathrm{ne}},\ F_E(c^0) \text{ is finite}\big\}.$$

Lemma 6.20. *Let E be a graph and $\mathcal{G} = \mathcal{G}_E$. Let $c \in \mathscr{C}_{\mathrm{ne}}$ and $m \in \mathbb{Z}$. Then $\mathrm{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$ is open, and it is compact if and only if $F_E(c^0)$ is finite.*

Proof. Let $I = F_E(c^0) \cup \{c^0\}$ and $D = \mathrm{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$. If $|I| = \infty$, then (6.10) gives an expression for an I-indexed open cover of D with no finite subcover, so D is not compact. On the other hand, if $|I| < \infty$ then D is a finite union of compact sets, so it is compact. Of course, c^0 is finite, so I is finite if and only if $F_E(c^0)$ is finite. $\square$

Lemma 6.21. *Let $c \in \mathscr{C}_{\mathrm{ne}}$ and let $H = \overline{c^0}$. Then $U_{H,\emptyset}$ is the orbit of c^∞ in ∂E.*

Proof. From the definitions, $U_{H,\emptyset} = \{x \in \partial E \mid x^0 \cap H \neq \emptyset\}$ and the orbit of c^∞ is the set $\{x \in \partial E \mid x \sim c^\infty\}$. Now, if $x \in U_{H,\emptyset}$ then x has an initial subpath $\gamma \in H \cup F_E(H)$. Let $v = r(\gamma)$, and note that $v \in H$. Lemma 6.7 implies that eventually $r(r(r(\ldots r(vE^1)\ldots))) \subseteq c^0$ if there are sufficiently many iterations. The same lemma implies every vertex in H is regular, so x must be an infinite path. Therefore $x \cap c^0 \neq \emptyset$, so $x \sim c^\infty$, because c is a cycle without exits. This shows that $U_{H,\emptyset}$ is contained in the orbit of c^∞. The other containment is obvious. $\square$

As a result, we have [30, Lemma 2.5], but our proof is shorter and avoids a lot of details, as a result of the work so far.

Lemma 6.22. *Let E be a graph and let $c \in \mathscr{C}_{\mathrm{ne}}$ be a cycle without exits. Then $F_E(c^0)$ is finite if and only if $\overline{c^0}$ satisfies Condition (F) and $\overline{c^0}$ has no breaking vertices.*

Proof. Combining Proposition 6.13 and Lemmas 6.20 and 6.21: $F_E(c^0)$ is finite if and only if the orbit of c^∞ (equivalently, the conjugacy class of $(c^\infty, 0, c^\infty)$) is compact, if and only if $U_{\overline{c^0},\emptyset}$ is compact, if and only if $B_{\overline{c^0}} = \emptyset$ and $\overline{c^0}$ satisfies Condition (F). $\square$

6.5 The centre of a Leavitt path algebra

The primary goal for this section is to describe the centre of a Leavitt path algebra as explicitly as possible, by giving a basis for it. This follows [31] quite closely. A secondary goal is to "calculate" the centre by showing that it is isomorphic to a more recognisable commutative ring. Another secondary goal is to specialise and describe the centres of Leavitt path algebras of row-finite graphs, and prime Leavitt path algebras. In doing so, we shall recover some of the main theorems of [36] and [37].

Theorem 6.23. *[31, Theorem 3.8] Let E be a graph and let R be a commutative ring with unit.*

(1) *The 0-component of the centre of $L_R(E)$ is the free R-module:*

$$\bigoplus_{(H,B_H)\in\mathscr{T}_E^{\mathrm{cm}}} R\left(\sum_{v\in H} v + \sum_{\alpha\in F'_E(H)} \alpha\alpha^* + \sum_{\substack{\beta\in E^\star\\ r(\beta)\in B_H}} \left(\beta\beta^* - \sum_{e\in F_{r(\beta)}} \beta ee^*\beta^*\right)\right)$$

(2) *Let $n\neq 0$. The n-component of the centre of $L_R(E)$ is zero unless E has a cycle c without exits such that $F_E(c^0)$ is finite and $|c|$ divides n. When $L_R(E)_n$ is nonzero, it is the free R-module:*

$$\bigoplus_{\substack{[c]\in[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}]\\ m|c|=n}} R\left(\sum_{\substack{\alpha\in F_E(c^0)\cup\{c^0\}\\ d\in[c],\ r(\alpha)=s(d)}} \alpha d^m\alpha^*\right) \qquad \textit{if } n>0;$$

$$\bigoplus_{\substack{[c]\in[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}]\\ -m|c|=n}} R\left(\sum_{\substack{\alpha\in F_E(c^0)\cup\{c^0\}\\ d\in[c],\ r(\alpha)=s(d)}} \alpha (d^*)^m\alpha^*\right) \qquad \textit{if } n<0.$$

Proof. Since the proof will be done using the Steinberg algebra model, it is worth translating the statement of the theorem into that setting before proceeding further. The relevant graded isomorphism $\pi : L_R(E)\to A_R(\mathcal{G}_E)$ is described on page 34. Let $\mathcal{G}=\mathcal{G}_E$. Applying Lemma 6.12 and Lemma 6.18, we see that Theorem 6.23 is equivalent to:

$$Z(A_R(\mathcal{G}))_0 = \bigoplus_{(H,B_H)\in\mathscr{T}_E^{\mathrm{cm}}} R\mathbf{1}_{U_{H,B_H}}, \tag{6.11}$$

$$Z(A_R(\mathcal{G}))_n = \bigoplus_{\substack{[c]\in[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}]\\ m|c|=n}} R\mathbf{1}_{\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty))} \quad \text{for all } n\neq 0. \tag{6.12}$$

(1) The sets U_{H,B_H}, where $(H,B_H)\in\mathscr{T}_E^{\mathrm{cm}}$, are minimal compact open invariant sets, so they are all mutually disjoint. This shows that the sum in (6.11) is direct, so it is a free R-module. Since $\mathcal{G}_0$ is effective, Lemma 6.4 asserts that $Z(A_R(\mathcal{G}))_0$ is spanned by characteristic functions of compact open invariant sets. Immediately, the right hand side of (6.11) is contained in the left hand side. If W is compact open and invariant, Lemma 6.16 asserts that $W=\bigsqcup_{V\in\mathcal{M}} V$, where $\mathcal{M}$ is a set of minimal compact open invariant sets, and thus $\mathbf{1}_W=\sum_{V\in\mathcal{M}}\mathbf{1}_V$. Each $V\in\mathcal{M}$ is of the form U_{H,B_H} for some $(H,B_H)\in\mathscr{T}_E^{\mathrm{cm}}$, and this shows that any such $\mathbf{1}_W$ (and hence all of $Z(A_R(\mathcal{G}))_0$) is contained in the right hand side of (6.11).

(2) Let $n \neq 0$. If E has no cycles without exits, then $\mathcal{G}$ is effective by Proposition 3.25, and $Z(A_R(\mathcal{G}))_n = 0$ by Lemma 6.3. Otherwise, Lemma 6.5 asserts that $Z(A_R(\mathcal{G}))_n$ is spanned by characteristic functions of compact open subsets of $\operatorname{Iso}(\mathcal{G})_n$ that are unions of conjugacy classes. Lemma 6.20 establishes that for all $[c] \in \mathscr{C}_{\text{ne}}^{\text{f}}$ with $m|c| = n$, the conjugacy class of $((c^\infty, m|c|, c^\infty))$ is a compact open subset of $\operatorname{Iso}(G)_n$. Immediately, the right hand side of (6.12) is contained in the left hand side. Assume W is a compact open subset of $\operatorname{Iso}(\mathcal{G})_n$ and W is a union of conjugacy classes. Then $W \subseteq \operatorname{Iso}(\mathcal{G})_n^\circ = \bigsqcup_{\substack{[c]\in[\mathscr{C}_{\text{ne}}]\\ m|c|=n}} \operatorname{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$, by Lemma 6.19. Since W is a union of conjugacy classes, there is a subset $F \subseteq \{[c] \in [\mathscr{C}_{\text{ne}}] \mid m|c| = n\}$ such that $W = \bigsqcup_{[c]\in F} \operatorname{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$. Moreover, each $\operatorname{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$ is open, so F must be finite, otherwise W admits an open cover with no finite subcover. Furthermore, $F \subseteq [\mathscr{C}_{\text{ne}}^{\text{f}}]$ because otherwise W contains a noncompact closed subset, by Lemma 6.20. Therefore $\mathbf{1}_W = \sum_{[c]\in F} \operatorname{Cl}_{\mathcal{G}}((c^\infty, m|c|, c^\infty))$. This shows that any such $\mathbf{1}_W$ (hence all of $Z(A_R(\mathcal{G}))_n$) is contained in the sum of R-modules appearing on the right hand side of (6.12). Clearly, that sum is direct, because all of the conjugacy classes that appear in it are disjoint. □

If E is row-finite, then there are no breaking vertices, so Theorem 6.23 (1) says that $Z(L_R(E))_0$ is the direct sum of components $R\left(\sum_{v\in H} v + \sum_{\alpha\in F_E(H)} \alpha\alpha^*\right)$ as H ranges over the minimal hereditary saturated sets satisfying Condition (F). This can be compared with [37, Theorem 3.18], where these sets H are constructively defined as the hereditary and saturated closures of certain equivalence classes, comprising line points and vertices in extreme cycles or cycles without exits.

Next is a (new) theorem in which we determine the algebraic structure of the centre. For a cardinal number κ, let $\bigoplus_\kappa R$ be the direct sum of R with itself, κ-many times.

Theorem 6.24 (Structure theorem for the centre of a Leavitt path algebra)**.** *Let E be a graph.*

(1) *If E has no cycle c without exits such that $F_E(c^0)$ is finite, then*

$$Z(L_R(E)) \cong \bigoplus_\kappa R$$

where κ is the cardinality of $\mathscr{T}_E^{\text{cm}}$.

(2) *Otherwise,*

$$Z(L_R(E)) \cong \left(\bigoplus_{\kappa-\lambda} R\right) \oplus \left(\bigoplus_\lambda R[x, x^{-1}]\right)$$

where κ is the cardinality of $\mathscr{T}_E^{\text{cm}}$, λ is the cardinality of $[\mathscr{C}_{\text{ne}}^{\text{f}}]$, and $\kappa - \lambda$ is the unique cardinal number such that $\lambda + (\kappa - \lambda) = \kappa$.

Proof. (1) follows from equation (6.11), which expresses $Z(A_R(\mathcal{G}_E)) \cong Z(L_R(E))$ as a free R-module generated by a set of orthogonal idempotents with cardinality κ.

For (2), let $\mathcal{X} = \{(H, B_H) \in \mathscr{T}_E^{\text{cm}} \mid H \cap c^0 = \emptyset \text{ for all } c \in \mathscr{C}_{\text{ne}}^{\text{f}}\}$ and let $\mathcal{Y} = \mathscr{T}_E^{\text{cm}} \setminus \mathcal{X}$. Suppose $(H, B_H) \in \mathcal{Y}$. Then H is minimal as a hereditary and saturated set satisfying Condition (F), and H contains the vertices of a cycle $c \in \mathscr{C}_{\text{ne}}^{\text{f}}$. The closure $\overline{c^0}$ is minimal (as a hereditary saturated set) and by Lemma 6.22, $\overline{c^0}$ satisfies Condition (F), so it follows that $H = \overline{c^0}$ and also that $B_H = \emptyset$. This proves that $\mathcal{Y} = \left\{\left(\overline{c^0}, \emptyset\right) \mid c \in \mathscr{C}_{\text{ne}}^{\text{f}}\right\}$. By Lemma 6.21, for all $c \in \mathscr{C}_{\text{ne}}^{\text{f}}$, $U_{\overline{c^0},\emptyset}$ is the orbit of c^∞, which we identify with $\operatorname{Cl}_{\mathcal{G}}((c^\infty, 0, c^\infty))$. We can adapt equation (6.11) and write $Z(A_R(\mathcal{G}))_0 =$

$\left(\bigoplus_{(H,B_H)\in\mathcal{X}} R\mathbf{1}_{U_{H,B_H}}\right) \oplus \left(\bigoplus_{[c]\in[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}]} R\mathbf{1}_{\mathrm{Cl}_{\mathcal{G}}((c^\infty,0,c^\infty))}\right)$. Combining it with equation (6.12) yields the following expression for $Z(A_R(\mathcal{G}))$ as a free R-module:

$$Z(A_R(\mathcal{G})) = \left(\bigoplus_{(H,B_H)\in\mathcal{X}} R\mathbf{1}_{U_{H,B_H}}\right) \bigoplus \left(\bigoplus_{[c]\in[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}]} \bigoplus_{m\in\mathbb{Z}} R\mathbf{1}_{\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty))}\right) \tag{6.13}$$

The part in the first pair of brackets is an orthogonal direct sum of copies of R indexed by $\mathcal{X}$. Recall from Lemma 6.18 that $\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty))$, where $c\in\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}$, is a discrete compact open bisection: $\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty)) = \{(\alpha d^\infty, m|c|, \alpha d^\infty) \mid \alpha\in F_E(c^0)\cup\{c^0\},\ d\in[c],\ r(\alpha)=s(d)\}$. It follows that $\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty))\,\mathrm{Cl}_{\mathcal{G}}((c^\infty,k|c|,c^\infty)) = \mathrm{Cl}_{\mathcal{G}}((c^\infty,(m+k)|c|,c^\infty))$ for all $k,m\in\mathbb{Z}$, so $\mathbf{1}_{\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty))}\mathbf{1}_{\mathrm{Cl}_{\mathcal{G}}((c^\infty,k|c|,c^\infty))} = \mathbf{1}_{\mathrm{Cl}_{\mathcal{G}}((c^\infty,(m+k)|c|,c^\infty))}$. As such,

$$\bigoplus_{m\in\mathbb{Z}} R\mathbf{1}_{\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty))} \cong R[x,x^{-1}].$$

If $[c],[d]\in[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}]$, $[c]\neq[d]$, and $m,k\in\mathbb{Z}$, then $\mathrm{Cl}_{\mathcal{G}}((c^\infty,m|c|,c^\infty))\,\mathrm{Cl}_{\mathcal{G}}((d^\infty,k|c|,d^\infty)) = \emptyset$. This shows the part in the second pair of brackets in (6.13) is an orthogonal direct sum of copies of $R[x,x^{-1}]$ indexed by $[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}]$. To complete the proof, notice that $\kappa = \mathrm{card}(\mathcal{X}\sqcup\mathcal{Y}) = \mathrm{card}(\mathcal{X}\sqcup[\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}])$ so $\mathrm{card}(\mathcal{X}) = \kappa-\lambda$, where $\lambda = \mathrm{card}([\mathscr{C}^{\mathrm{f}}_{\mathrm{ne}}])$. □

Theorem 6.24 was achieved in [37, Theorem 3.27] for row-finite graphs, where the index sets (i.e., κ and λ) are described more explicitly Our next proposition and its corollary are similar to [79, Proposition 3.7] and [25, Theorem 5.2] although none of them is a complete generalisation of another.

Proposition 6.25. *Let $\mathcal{G}$ be a Γ-graded Hausdorff ample groupoid, such that $\mathcal{G}_\varepsilon$ is effective and $A_R(\mathcal{G})$ is prime. Then $Z(A_R(\mathcal{G}))\cap A_R(\mathcal{G})_\varepsilon \cong R$ if $\mathcal{G}^{(0)}_E$ is compact, and otherwise it is zero.*

Proof. We claim that $\mathcal{G}^{(0)}$ is the only nonempty clopen invariant subset of $\mathcal{G}^{(0)}$. For contradiction, suppose U is a proper clopen invariant subset of $\mathcal{G}^{(0)}$. Then its complement $V=\mathcal{G}^{(0)}\setminus U$ is also clopen and invariant, and $\mathcal{G}|_U$ and $\mathcal{G}|_V$ are ample clopen subgroupoids of $\mathcal{G}$. Moreover, $\mathcal{G}=\mathcal{G}|_U\sqcup\mathcal{G}|_V$, because U and V are invariant. Referring to Proposition 1.23, $A_R(\mathcal{G})$ is a direct sum of two ideals whose product is zero, and this contradicts that $A_R(\mathcal{G})$ is prime.

We know from Lemma 6.4 that $Z(A_R(\mathcal{G}))\cap A_R(\mathcal{G})_\varepsilon$ is spanned by characteristic functions of compact open invariant subsets of $\mathcal{G}^{(0)}$. From the previous paragraph, there is at most one such set. If $\mathcal{G}^{(0)}$ is compact, then it is the only compact open invariant set, so $Z(A_R(\mathcal{G}))\cap A_R(\mathcal{G})_\varepsilon = R\mathbf{1}_{\mathcal{G}^{(0)}}$. Otherwise, there are no compact open invariant sets, so $Z(A_R(\mathcal{G}))\cap A_R(\mathcal{G})_\varepsilon = 0$. □

Corollary 6.26. *Let $\mathcal{G}$ be an effective Hausdorff ample groupoid such that $A_R(\mathcal{G})$ is prime. Then $Z(A_R(\mathcal{G}))\cong R$ or 0, according as $\mathcal{G}^{(0)}$ is or is not compact.*

We now recover [36, Theorem 6] with a very different proof.

Theorem 6.27. *Let E be a graph such that $L_R(E)$ is prime. Then $Z(L_R(E))\neq 0$ if and only if E^0 is finite. In this case,*

(1) *$Z(L_R(E))\cong R$ if and only if E satisfies Condition (L) or E has a cycle c without exits such that $F_E(c^0)$ is finite;*

(2) *$Z(L_R(E))\cong R[x,x^{-1}]$ if and only if E has a cycle c without exits such that $F_E(c^0)$ is infinite.*

Proof. Finiteness of E^0 is equivalent to compactness of ∂E. If E^0 is infinite and $L_R(E)$ is prime, then Proposition 6.25 yields $Z(L_R(E))_0 \cong Z(A_R(\mathcal{G}_E))_0 = 0$. But if $Z(L_R(E))_0 = 0$, then $Z(L_R(E)) = 0$, because $Z(L_R(E))$ is generated as an ring by its 0-component. If E^0 is finite and $L_R(E)$ is prime, then Proposition 6.25 yields $Z(L_R(E))_0 \cong Z(A_R(\mathcal{G}_E))_0 \cong R$. By Theorem 6.24, $Z(L_R(E)) \cong R$ or $R[x, x^{-1}]$, depending on the existence of a cycle c without exits such that $F_E(c^0)$ is finite. □

6.6 The simplicity theorems

Here we shall prove: given a Hausdorff ample groupoid $\mathcal{G}$, its Steinberg algebra $A_R(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is effective and minimal, and R is a field. Effectiveness has been defined already, but minimality has not.

Definition 6.28. An étale groupoid $\mathcal{G}$ is **minimal** if the only open invariant subsets of $\mathcal{G}^{(0)}$ are $\emptyset$ and $\mathcal{G}^{(0)}$.

An étale groupoid $\mathcal{G}$ is minimal if and only if all its orbits are dense in $\mathcal{G}^{(0)}$. Indeed, if every orbit is dense, a nonempty open invariant set must intersect with every orbit and thus contain every orbit. Conversely, if $\mathcal{O}_x$ is an orbit in a minimal groupoid $\mathcal{G}$, then it must be dense: if $U \subseteq \mathcal{G}^{(0)} \setminus \mathcal{O}_x$ is open, then $\boldsymbol{d}(\boldsymbol{c}^{-1}(U))$ is open and invariant and does not contain $\mathcal{O}_x$, so $\boldsymbol{d}(\boldsymbol{c}^{-1}(U)) = \emptyset$, whereby $U = \emptyset$. This also implies: if an étale groupoid $\mathcal{G}$ is minimal and has at least one unit with trivial isotropy, then $\mathcal{G}$ is topologically principal.

Before proceeding, we need to remember the definition of J_x from §5.4: it is the free R-module generated by $\mathcal{G}^x$ ($x \in \mathcal{G}^{(0)}$) and carries the structure of an $(A_R(\mathcal{G}), R^x\mathcal{G}^x)$-bimodule (see Proposition 5.27). By virtue of J_x being unital as a left $A_R(\mathcal{G})$-module, the annihilator of J_x is an ideal properly contained in $A_R(\mathcal{G})$. The proof of the next proposition is adapted from [79] with the intention of using a bit less inverse semigroup theory. The essential ideas are the same.

Proposition 6.29. *[79, Proposition 3.4] Let $\mathcal{G}$ be an ample groupoid. Then $\mathcal{G}$ is minimal if and only if $A_R(\mathcal{G})$ has no proper ideals containing an idempotent $\mathbf{1}_W$ where $W \in \mathcal{B}(\mathcal{G}^{(0)})$.*

Proof. ($\Rightarrow$) Assume $\mathcal{G}$ is minimal, $W \in \mathcal{B}(\mathcal{G}^{(0)})$, and I is an ideal of $A_R(\mathcal{G})$ containing $\mathbf{1}_W$. Let $x \in \mathcal{G}^{(0)}$ be arbitrary. Since $\mathcal{O}_x$ is dense in $\mathcal{G}^{(0)}$, there exists a morphism $y \in \mathcal{O}_x \cap W$. As $y \in \mathcal{O}_x$, there is a morphism $g \in {}^y\mathcal{G}^x$. Let $V \in B^{\text{co}}(\mathcal{G})$ be a neighbourhood of g. Then $g = yg \in WV$ and $x = g^{-1}g \in (WV)^{-1}(WV) = \boldsymbol{d}(WV)$. Moreover, $\mathbf{1}_{\boldsymbol{d}(WV)} = \mathbf{1}_{(WV)^{-1}} * \mathbf{1}_W * \mathbf{1}_V \in I$ because I is an ideal containing $\mathbf{1}_W$. If $A \in \mathcal{B}(\mathcal{G}^{(0)})$ has $x \in A \subseteq \boldsymbol{d}(WV)$ then $\mathbf{1}_A = \mathbf{1}_A * \mathbf{1}_{\boldsymbol{d}(WV)} \in I$, and it follows that x has a neighbourhood base of compact open sets whose characteristic functions are all in I. Thus $\mathcal{G}^{(0)}$ has a base $\mathcal{B}$ of compact open sets whose characteristic functions are in I. Considering $A_R(\mathcal{G}^{(0)})$ as a subalgebra of $A_R(\mathcal{G})$ and applying Corollary 1.14, it follows that $A_R(\mathcal{G}^{(0)}) = \operatorname{span}_R\{\mathbf{1}_B \mid B \in \mathcal{B}\} \subseteq I$. But $A_R(\mathcal{G}^{(0)})$ contains a set of local units for $A_R(\mathcal{G})$, which yields $A_R(\mathcal{G}) = A_R(\mathcal{G}^{(0)}) * A_R(\mathcal{G}) = I$.

($\Leftarrow$) Assume that no proper ideal of $A_R(\mathcal{G})$ contains an idempotent $\mathbf{1}_W$ where $W \in \mathcal{B}(\mathcal{G}^{(0)})$. Fix some $x \in \mathcal{G}^{(0)}$. The annihilator of J_x is a proper ideal of $A_R(\mathcal{G})$, so $\mathbf{1}_W J_x \neq 0$ for every $W \in \mathcal{B}(\mathcal{G}^{(0)})$. Therefore, for every such W, there exists $t \in \mathcal{G}^x$ with $\mathbf{1}_W t \neq 0$, which implies $\boldsymbol{c}(t) \in W$. Hence $\boldsymbol{c}(t) \in W \cap \mathcal{O}_x$, which proves $\mathcal{O}_x$ is dense in $\mathcal{G}^{(0)}$ because $\mathcal{O}_x$ has nonempty intersection with every open subset of $\mathcal{G}^{(0)}$. □

Comparing this result with Corollary 3.24, it becomes clear that minimality is in tension with effectiveness. Let $\mathcal{G}$ be Hausdorff and let K be a field. If $\mathcal{G}$ is effective, *every* nonzero ideal of $A_K(\mathcal{G})$ contains an idempotent $\mathbf{1}_W$ where $W \in \mathcal{B}(\mathcal{G}^{(0)})$. Whereas, if $\mathcal{G}$ is minimal, *no* proper ideal of $A_K(\mathcal{G})$ contains such an idempotent. This is the key that unlocks the simplicity theorem.

Theorem 6.30. *[79, Theorem 3.5] Let $\mathcal{G}$ be an ample groupoid. If $A_K(\mathcal{G})$ is simple, then K is a field and $\mathcal{G}$ is effective and minimal. The converse holds if $\mathcal{G}$ is Hausdorff.*

Proof. If K is not a field then it has a nontrivial ideal I, and $IA_K(\mathcal{G}) \cong A_I(\mathcal{G})$ is a nontrivial ideal of $A_K(\mathcal{G})$; this uses the fact that K is represented faithfully in $A_K(\mathcal{G})$. If $\mathcal{G}$ is not minimal then Proposition 6.29 implies there is a proper ideal of $A_K(\mathcal{G})$ containing a nonzero idempotent, so $A_K(\mathcal{G})$ is not simple. Now suppose $A_K(\mathcal{G})$ is simple. Suppose $U \subseteq \operatorname{Iso}(\mathcal{G})$ is a compact open bisection. Fix some $x \in \mathcal{G}^{(0)}$. Since $U \subseteq \operatorname{Iso}(\mathcal{G})$, we get from Example 5.33 that $\mathbf{1}_U - \mathbf{1}_{\boldsymbol{d}(U)}$ annihilates the $A_K(\mathcal{G})$-module $K\mathcal{O}_x$. But $A_K(\mathcal{G})$ is simple and the annihilator ideal of $K\mathcal{O}_x$ is properly contained in $A_K(\mathcal{G})$, because $K\mathcal{O}_x$ is a unital $A_K(\mathcal{G})$-module, so the annihilator must be zero. Thus $\mathbf{1}_U - \mathbf{1}_{\boldsymbol{d}(U)} = 0$ and $U = \boldsymbol{d}(U) \subseteq \mathcal{G}^{(0)}$. This shows $\operatorname{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$; in other words, $\mathcal{G}$ is effective. In the scenario where K is a field and $\mathcal{G}$ is Hausdorff, minimal, and effective, $A_K(\mathcal{G})$ must be simple. This is because every nontrivial ideal must simultaneously contain some idempotent $\mathbf{1}_W$, $W \in \mathcal{B}(\mathcal{G}^{(0)})$, and also not contain such an idempotent (per the discussion after Proposition 6.29). □

Specialising, we obtain a proof of [4, Theorem 3.1].

Theorem 6.31 (Simplicity Theorem for Leavitt path algebras)**.**

Let E be a graph and let K be a field. The Leavitt path algebra $L_K(E)$ is simple if and only if

(i) *E satisfies Condition (L);*

(ii) *the only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

Proof. As usual, we prove the statement for $A_K(\mathcal{G}_E)$ instead of $L_K(E)$. According to Theorem 6.30, $A_K(\mathcal{G}_E)$ is simple if and only if $\mathcal{G}_E$ is effective and minimal. Recall from Proposition 3.25 that $\mathcal{G}_E$ is effective if and only if E satisfies Condition (L). We claim that $\mathcal{G}_E$ is minimal if and only if E satisfies (ii). Indeed, by Proposition 6.14, there is a lattice isomorphism between $\mathscr{T}_E$ and $\mathscr{O}_E$. The maximal and minimal elements in $\mathscr{T}_E$ are $(E^0, \emptyset)$ and $(\emptyset, \emptyset)$, respectively, and they correspond to the open invariant sets ∂E and $\emptyset$. This proves the claim. □

In keeping with our interest in graded properties of rings, we briefly investigate graded simplicity for Steinberg algebras.

Definition 6.32. A graded ring A is **graded simple** if $A^2 \neq 0$ and there are no graded ideals besides 0 and A.

The following theorem is new, and the proof is very similar to that of Theorem 6.30.

Theorem 6.33. *Let $\mathcal{G}$ be a Γ-graded ample groupoid. If $A_K(\mathcal{G})$ is graded simple, then K is a field, $\mathcal{G}$ is minimal, and $\mathcal{G}_\varepsilon$ is effective. The converse holds if $\mathcal{G}$ is Hausdorff.*

Proof. Assume $A_K(\mathcal{G})$ is graded simple. If K is not a field then it has a nontrivial ideal I, and $IA_K(\mathcal{G}) = \bigoplus_{\gamma \in \Gamma} IA_K(\mathcal{G})_\gamma$ is a nontrivial graded ideal, contradicting that $A_K(\mathcal{G})$ is graded simple. If $\mathcal{G}$ is not minimal, then there is a proper ideal I containing a nonzero idempotent $\mathbf{1}_W$ with $W \in \mathcal{B}(\mathcal{G}^{(0)})$. The ideal generated by $\mathbf{1}_W$ is a graded ideal contained in I, because $W \in \mathcal{B}(\mathcal{G}^{(0)})$ is homogeneous. This contradicts that $A_K(\mathcal{G})$ is graded simple. Suppose $U \subseteq \operatorname{Iso}(\mathcal{G}_\varepsilon)$ is a compact open bisection. As in the proof of Theorem 6.30, fix some $x \in \mathcal{G}^{(0)}$ and notice that the homogeneous element $\mathbf{1}_U - \mathbf{1}_{\boldsymbol{d}(U)} \in A_K(\mathcal{G})_\varepsilon$ annihilates $K\mathcal{O}_x$. The ideal I generated by $\mathbf{1}_U - \mathbf{1}_{\boldsymbol{d}(U)}$ is a graded ideal contained in the (not necessarily graded) annihilator ideal of $K\mathcal{O}_x$. The annihilator of $K\mathcal{O}_x$ is a proper ideal, so I is a proper graded ideal and is therefore zero. Thus $\mathbf{1}_U - \mathbf{1}_{\boldsymbol{d}(U)} = 0$ and $U = \boldsymbol{d}(U)$. This shows $\mathrm{Iso}(\mathcal{G}_\varepsilon)^\circ = \mathcal{G}^{(0)}$; in other words, $\mathcal{G}_\varepsilon$ is effective.

Assume K is a field, $\mathcal{G}$ is Hausdorff and minimal, and $\mathcal{G}_\varepsilon$ is effective. By Proposition 6.29, there are no proper ideals of $A_K(\mathcal{G})$ containing an idempotent $\mathbf{1}_W$ where $W \in \mathcal{B}(\mathcal{G}^{(0)})$. By Theorem 3.22, every nonzero graded ideal of $A_K(\mathcal{G})$ contains such an idempotent. Therefore, there are no graded ideals in $A_K(\mathcal{G})$ besides 0 and $A_K(\mathcal{G})$. □

The 0-component of $A_K(\mathcal{G}_E)$ is effective for any graph E, and $A_K(\mathcal{G}_E)$ is minimal if and only if $\emptyset$ and E^0 are the only hereditary and saturated subsets of E^0. This gives a groupoid-theoretic proof of the Graded Simplicity Theorem for Leavitt path algebras [2, Corollary 2.5.15].

Corollary 6.34. *Let E be a graph and K a field. Then $L_K(E)$ is graded simple if and only if the only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

Index of notation

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