

Price Formation Under Uncertainty

Abstract.

The analysis presented in this thesis is aimed at better understanding the role of expectations in the price formation process. Since general competitive analysis lacks a coherent explanation of how expectations are formulated it is difficult to promote theories that assume agents have no structural knowledge in favour of theories that assume agents have significant structural knowledge, e.g. rational expectations hypothesis versus the theory of rational beliefs. Accordingly, empirical evidence is presented to support analyses of models in which agents are not assumed to have structural knowledge.

Simple general equilibrium models are used to illustrate that modelling risk requires a thorough analysis of investor expectations embedded in asset prices to better understand the information conveyed by observed risk premia. Analysis of the role of diverse expectations in competitive equilibria shows that a prerequisite for the existence of a short-run Walrasian monetary equilibrium is the existence of at least one agent whose expectations are insensitive to current prices.

Ergodic theory shows that any stable dynamical system generates a stationary probability measure based on its underlying generating probability that is unrelated to the data generated by the dynamical system. This result is used to show that the conditions under which diverse beliefs arise are sufficiently general to warrant the study of the impact of diverse expectations on the price formation process.

Enthusiasm for models that allow diverse beliefs is however tempered by a review of Sunspot theory that show that it is not necessary to abandon the rational expectations hypothesis in order for competitive markets to be subject to speculative fluctuations that are driven by expectations.

This analysis is reinforced by a known example that shows that adaptive learning rules can lead rational agents to believe in nonstationary, indeterminate equilibria that are locally stable, such as Sunspot Equilibria. This leads to an important conclusion; diverse beliefs are not temporary phenomena since disequilibrium-learning analysis cannot be relied on to teach investors the economy's equilibrium map.

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Chapter 1 - Introduction

1.1 Introduction.

General competitive analysis considers the simultaneous resolution of the forces acting within an economy on the price formation process, which includes an analysis of the interaction between preferences, technology, expectations formation and the exogenous environment. However, many aspects of the theory still lack clarity. Specifically, the theory lacks a coherent explanation of how expectations are formulated. This implies that general competitive analysis cannot fully explain the impact of expectations on the price formation process, which is the analytical issue considered in this thesis.

The efficient market hypothesis represents an early attempt at explaining the role of expectations on the price-formation process. This hypothesis states that asset-prices instantaneously and fully reflect all available information such that security markets are in continuous stochastic equilibrium. This means that investors cannot, but by chance, outperform the market in a sustained way by deviating from some objective long-term strategic benchmark.

Yet everyday a veritable army of analysts and portfolio managers attempt to add value to their clients' portfolios by taking active positions against their client's benchmarks. This endeavour, if successful, clearly flies in the face of the efficient market hypothesis. This contradiction suggests that an alternative dynamic might be at work that is not fully explained by the efficient market hypothesis.

When forming portfolios, active investors buy and sell securities based on their expectations that reflect their diverse beliefs about the uncertain nature of the world in which they trade. The competitive nature of the act of buying and selling assets and the significant resources dedicated to obtaining an information edge in this activity indicate that investors believe that it is potentially profitable to trade at the ruling price based on their beliefs.

In an important early analysis of this issue Radner (1972 and 1979) shows that investors can hold diverse beliefs about the exogenous states of the economy and still attain the Arrow-Debreu (1954) equilibrium if it is assumed that they have rational expectations. Rational expectations require that all investors have expectations that are self-fulfilled, which means that investors' expectations of the prices that will clear markets under different future states do actually clear them when the state is revealed at the future date. Rational expectations therefore require that investors know the equilibrium map between future exogenous states and spot market prices at future dates. Kurz (1994) equates the information assumed by the Rational expectations assumption to an investor knowing; i) the demand and supply functions of all market participants, ii) how to extract present and future general equilibrium prices and iii) the stochastic law of motion of the economy over time.

Only a moment's thought is needed to realise that the information required for investors to act rationally, such that markets clear in the sense of Radner (1972 and 1979), is onerous. Also, admitting a diversity of beliefs is crucial in any study of markets in which trades occur sequentially in securities rather than in contingent claims. This idea was expressed in the General Theory where Keynes (1936) observed that the distribution of expectations among investors affects aggregate investment and consumption and is therefore central to equilibrium allocation. This observation encapsulates the key questions with which this research agenda is concerned:

- i. What is the state of the empirical debate on whether agents have the structural knowledge implied by the rational expectation hypothesis?
- ii. What is the role of expectations in competitive equilibria under uncertainty?
- iii. What are the conditions under which agents have diverse expectations that are compatible with the data revealed by the dynamics of the economy, whilst having no structural knowledge of the economic system?
- iv. Is it necessary to abandon the rational expectations hypothesis in order for competitive markets to be subject to speculative fluctuations that are driven by expectations?

- v. Are diverse expectations temporary phenomena that will not persist in the limit when investors learn the economy's equilibrium map?

When surveying the state of the empirical debate on whether agents have the structural knowledge implied by the rational expectation hypothesis, a consideration of the efficient market hypothesis is natural. This follows from LeRoy (1989) who observed that the efficient market hypothesis, at its most general level, is the theory of competitive analysis applied to asset markets and that the rational expectations hypothesis is embedded in Fama's (1976) definition of the efficient market hypothesis. Chapter 2 first surveys the theory underpinning the efficient market hypothesis based on work by Fama (1970 & 1976). There exists a large body of research that attempts to discredit the efficient market hypothesis. Given the research agenda, the survey will focus on methods that test whether observed market volatility is greater than that expected by the efficient market hypothesis.

The logic is as follows: If one can observe that market volatility is greater than that expected by the efficient market hypothesis, then one may reasonably conclude that investor expectations play a bigger role in the price formation process than the theory allows for. Such a test constructs a null hypothesis that specifies bounds within which asset prices will vary if the efficient market hypothesis holds. The intuition underlying the construction of these variance bounds is as follows (Shiller 1981b): "Today's price is related to the expected present value of a path of future variables under the efficient market hypothesis. Since present values are long weighted moving averages, it would seem that price data should be very stable and smooth."

This method has been widely used. See, for instance, studies of efficient market models in connection with long-term and short-term bond yields (Shiller - 1979, Singleton - 1980), stock prices and dividends (Shiller 1981b and 1981c, and Mankiw et al 1985), yields on intermediate and short-term bonds (Shiller, 1981a) and foreign exchange rates and money stock differentials (Huang - 1981, Meese and Singleton - 1980). Specifically, we consider whether observed stock prices are more volatile than would be justified by

subsequent dividend changes. The survey is therefore concerned with the research of Shiller (1981b and 1981c) and Mankiw (1985).

Shiller's original method is enhanced to allow the variance bounds to hold even in the presence of a speculative bubble. This method is then applied to monthly US stock data from January 1871 to January 2001 and finds that stock prices are significantly more volatile than predicted by the efficient market hypothesis. A crucial assumption underlying Shiller's method is that a stationary stochastic process generates dividends. A Dicky-Fuller cointegration test of this assumption casts some doubt on its validity. Further tests show that the method suffers from small sample bias and is not stable.

An alternative variance bound test developed by Mankiw et al (1985) is then considered. The main advantages of this method are that it does not suffer from small sample bias and does not assume that a stationary process generates dividends. A cursory analysis finds the results of this test to be highly dependent on the sample used. Specifically, this particular test of market efficiency has relatively little power whilst a speculative bubble is evolving, which indicates that the test is unreliable as it is dependent on the relationship between the starting price and the terminal price prevailing within the sample.

However, an alternative explanation, and more appealing given the research agenda, is that asset price volatility varies over time. This refers to the GARCH phenomenon that observes that asset price volatility can switch between low and high levels of volatility over time. This idea was not explored further in this analysis. The interested reader is referred to a survey of this topic by Bollerslev, Chou and Kroner (1992).

The results of the empirical analysis are not encouraging as the survey concludes that there exists only weak evidence that prices are more unstable than one would expect under the efficient market hypothesis. However, Baumol and Benhabib (1989) use chaos theory to show how a simple and deterministic dynamic mechanism yields a time path so complicated that it will pass most standard tests of randomness. The state of the research

in this direction is still in its infancy but should push the empirical debate on the efficient market hypothesis in a fresh direction.

Finally, remember that the fair game properties of the efficient market hypothesis assumes that: 1) the conditions of market equilibrium can be stated in terms of rational expectations, 2) the available information is fully utilized by the market in forming equilibrium expected returns and thus current prices, and, 3) the empirical models used in chapter 2 restricts the analysis to partial equilibrium infinite-lived, representative agent models.

An analysis of competitive equilibria requires the simultaneous consideration of the whole economy. A partial equilibrium model facilitates the analysis by allowing the researcher to determine the outcome in the market under study in isolation from all other markets. However, this simplification comes at the cost of assuming that the prices of all assets, other than the one under consideration, remain fixed and that there is no income effect in the market under consideration. The latter assumption is not crucial when considering financial markets where the income effect is absent in the short-run, whilst the former assumption is particularly onerous in the context of financial markets where all assets are substitutes such that the cross-effects among markets with changing and interrelated prices cannot be ignored. Also, a side effect of the partial equilibrium interpretation of the efficient market hypothesis employed by Shiller and Mankiw is that cycles cannot be predicted ex-ante on the basis of observed price. Yet Grandmont (1985) provides an example that employs the rational expectations hypothesis that shows that endogenous competitive business cycles can arise in a predictable way when older traders are sufficiently more risk averse than younger traders.

This suggests that, in order to gain deeper insight on how expectations affect the price formation process, one needs to consider the cross-effects among markets with interrelated prices. This necessitates a theoretical approach based on competitive equilibrium analysis in the rest of this thesis.

Chapter 3 takes a step in this direction by constructing, in line with Donaldson and Mehra (1984), a fully endogenous asset-pricing model for an economy with two goods; a non-storable consumption good and an investment good. This model retains the rational expectations hypothesis under conditions of uncertainty through production shocks that are independently distributed.

The focus of the analysis is on the role of relative risk aversion, as measured by the relative curvature of the intertemporal utility functions, in asset pricing. Specifically, the model compares the risk premia of a fully endogenous asset-pricing model with an infinite-lived representative agent (the Lucas asset-pricing model) to the risk premia of a fully endogenous asset-pricing model with overlapping generations (the OLG asset-pricing model).

The comparison allows us to consider the consequences for asset pricing in incomplete markets where the pooling of risks through Arrow-Debreu contingent claims are impossible. The introduction of this market friction plays an important role in the behaviour of the OLG asset-pricing model relative to the frictionless Lucas asset-pricing model. Specifically, the analysis shows that an increase in relative risk aversion is consistent with an increase in the risk premia of the Lucas asset-pricing model, but has an ambiguous effect on the risk premia of the OLG asset-pricing model. This difference arises because of the ambiguity resulting from income and substitution effects in the OLG asset-pricing model, called the intertemporal substitution effect, which is absent in the Lucas asset-pricing model.

Finally, chapter 3 highlights the question of modelling risk by facilitating a deeper understanding of an often-heard premise in markets; “asset prices are volatile because investors have an inconsistent risk appetite”. The difference between the two asset pricing models implies that observed values of risk premia may correspond to vastly different estimates of parameters, which include an analysis of investors’ expectations that are imbedded in the intertemporal substitution effect found in the OLG asset-pricing model, but absent from the intertemporal substitution effect found in the Lucas asset-pricing model. The analysis does not assert that investors do not have an inconsistent risk

appetite, a fact implicitly assumed in competitive equilibrium analysis through the assumption of consistent preferences. It does however indicate a need for a deeper analysis of investor expectations embedded in asset prices.

Chapter 4 explores the relationship between expectations and the intertemporal substitution effect within the framework of a simple exchange economy with money and no external uncertainty. The analysis is based on research by Grandmont (1974 and 1983) on the role of expectations in a short-run Walrasian monetary equilibrium model that relaxes the strict assumption that underlies the rational expectations hypothesis. Specifically, agents are assumed to have diverse beliefs about, i) demand and supply functions, ii) how to extract present and future general equilibrium prices, and iii) the stochastic law of motion of the economy over time.

The analysis illustrates through a series of examples that a strong intertemporal substitution effect is a prerequisite for the existence of a short-run Walrasian monetary equilibrium. The analysis proves, under very general conditions, that this equilibrating mechanism requires that the price expectations of at least one agent are insensitive to current prices (see Grandmont 1983). The presence of an insensitive trader ensures that expectations vary in a manner that is not “wild”, thereby maintaining the upper hemicontinuity of the aggregate excess demand function.

This observation brings the analysis back to the issue of price formation under uncertainty. Specifically, the existence of an ‘insensitive trader’ is questioned; his expectations may become biased by recent experience, requiring him to project a lower/higher than normal equity risk premium (or price) in order to act as a stabilising flywheel and take the opposite trade.

It would be tempting to make bold conclusions about the implications of the above analysis. However, chapter 3 argues that the intertemporal substitution effect is indistinguishable from relative risk aversion given the assumption of additive utility. The implication of this is that it would be difficult to distinguish between market inefficiency and a structural change in asset pricing.

Further, the equilibrium solution to the OLG economy specified under very general assumptions is not necessarily unique. This observation cautions against stating that price instability may be a function of either; a) market inefficiency, i.e. cases when the insensitive trader develops biased expectations, or b) structural changes, i.e. changes in the relative risk aversion of the trader.

Chapter 4 does not attempt to characterise the conditions that permit a diversity of beliefs to arise. Chapter 5 reviews the work of Kurz (1994) and Nielsen (1996) who made important contributions in this area through their development of the theory of rational beliefs. At the heart of the theory is the postulate that agents formulate their beliefs by observing the data generated by a stable dynamic system. Agents are assumed to have no structural knowledge of the economic system, which is a significant weakening of the rational expectations hypothesis. The analytical implication of this is that agents do not know the true probability distribution that generates the dynamic system. However, the frequencies with which events occur in such a system converge in the limit to a stationary probability measure, which agents can learn. Since agents have no structural knowledge they do not necessarily adopt this stationary probability measure as their belief system.

In an important breakthrough the theory of rational beliefs shows how agents can construct beliefs that are allowed to be biased relative to the stationary measure generated by the data in the short-run, but unbiased on average over his investment horizon. Whilst this is a plausible result, it does come with a strong restriction; every agent is required to express a non-zero view on the likelihood of any conceivable combination of events occurring over his investment horizon. A consequence of this method of constructing beliefs is that the range of variation in each trader's expectations is not too large, which rules out significant over- or under reaction by the agent to observed events. Since every agent is "insensitive" in the sense discussed in chapter 4 it follows that an OLG model incorporating rational beliefs will have a strong intertemporal substitution effect, such that the existence of a competitive equilibrium follows in a natural way.

An important insight that follows from the theory of rational beliefs is that agents can express their uncertainty about the true probability distribution that generates the data as uncertainty about the beliefs held by other market participants. This suggests that an agent should compare the distribution of beliefs that currently influence the price formation process with his own. The analysis finds that uncorrelated beliefs have an ambiguous effect on the price formation process; the agent may trade if he perceives that observed beliefs are correlated and biased relative to his own in a significant way.

Chapter 6 explores the impact of correlated beliefs further by considering the effect on the price formation process of extrinsic uncertainty that is perfectly correlated with the rational expectations of agents. As agents' preferences are assumed to be rational with respect to preference, endowments and production sets, price randomness in this extension of the model will be entirely due to the beliefs held by individuals about their environment. Such price randomness may become self-fulfilling if the beliefs are widely held, or correlated, and will not necessarily dissipate, even asymptotically, as individuals accumulate more observations. This phenomenon is also known as sunspot equilibria in the literature. Specifically, chapter 6 reviews the work by Azariadis (1981) and Azariadis and Guesnerie (1986).

A simple two-period OLG model is used to show that stationary sunspot equilibrium exists in the neighbourhood of a monetary stationary equilibrium that is locally dynamically stable. The practical interpretation of this is that market participants do not have to be irrational, defined in the sense of the rational expectations hypothesis, for competitive markets to be subject to speculative fluctuations that are driven by expectations. However, the existence of sunspot equilibria is not sufficient justification to assert that competitive markets with rational participants could be subject to speculative instability.

The use of the rational expectations hypothesis in economic modelling is often justified by using disequilibrium learning analysis to isolate locally unique rational expectations equilibria in which only fundamentals matter. For example, Lucas (1986) shows how a simple adaptive learning rule can be used to isolate the monetary steady state as the

attractor for almost all initial price levels. This is an important result as it is well known that standard versions of OLG models generally have two steady states: 1) an autarkic steady state in which real money balances have zero value, and 2) a monetary steady state in which real money balances are positively valued. All but one of the solution paths discussed above are characterised by an ever increasing price level which converges asymptotically to the autarkic steady state, the exception being the solution path characterised with an initial price level equal to the monetary steady state.

This presents a problem for any research agenda predicated on investors having a diversity of beliefs. Chapter 7 reviews an important counter example proposed by Duffy (1994) of an adaptive learning rule that converges to nonstationary indeterminate equilibria that are locally stable. This result implies that agents will not necessarily learn the information required by the rational expectations hypothesis from disequilibrium adaptive learning rules. A subtext to this result is that agents may well learn to believe in sunspot equilibria, which is a type of indeterminate equilibrium. Finally, without knowledge of how agents form expectations in a particular environment, it is difficult to specify an appropriate learning rule a-priori that rules out a diversity of beliefs.

Chapter 2 - Exploring Aspects of Market Efficiency

2.1 Introduction

The efficient market hypothesis (EMH) states that market prices instantaneously and fully reflect all available information such that security markets should be in continuous stochastic equilibrium. Fama (1970) defines three forms of market efficiency, each with a progressively broader definition of the available information set. Firstly, weak form market efficiency has an information set consisting only of historical prices. Secondly, semi-strong form market efficiency has an information set consisting of all public information. Thirdly, strong form market efficiency has an information set consisting of all public and private information; i.e., a situation in which investors have monopolistic access to information relevant to the price formation process.

The efficient market hypothesis implies that security markets will be in continuous stochastic equilibrium. Fama (1970) represents this concept statistically using the following fair game model:

$$r_{j,t+1} = E[r_{j,t+1} | \Omega_t] + \varepsilon_{j,t+1} \quad (1)$$

where:

$j = 1, 2, 3, \dots, n$;

$r_{j,t+1}$, the one period percent real rate of return for security j during period $t+1$;

Ω_t , the set of information available at time t ;

$\varepsilon_{j,t+1}$, the prediction error on security j in period $t+1$;

The error term $\varepsilon_{j,t+1}$ must be a non-systematic error which means that it is unbiased, independent and efficient. Analysing each of these three properties of the error term we have that:

1. The prediction error will be unbiased if its expected return, conditional on Ω_t , is zero, i.e. $E\{\varepsilon_{j,t+1} | \Omega_t\} = E\{r_{j,t+1} - E\{r_{j,t+1} | \Omega_t\} | \Omega_t\} = 0$; and
2. The prediction error will be independent if it is uncorrelated with the expected return, i.e. $E\{\varepsilon_{j,t+1} E\{r_{j,t+1} | \Omega_t\} | \Omega_t\} = E\{\varepsilon_{j,t+1} | \Omega_t\} E\{r_{j,t+1} | \Omega_t\} | \Omega_t = 0$; and finally
3. The prediction error is efficient if it is both contemporarily and serially uncorrelated, i.e. $E\{\varepsilon_{j,t+1} \varepsilon_{i,t+1} | \Omega_t\} = 0$, $E\{\varepsilon_{j,t+1} \varepsilon_{j,t} | \Omega_t\} = 0$ and $E\{\varepsilon_{j,t+1} \varepsilon_{i,t} | \Omega_t\} = 0$;

These statistical properties must be represented in a price formation process that fully reflects all available information to facilitate testing of the efficient market hypothesis. Such a price formation process specifies that, conditional on some relevant information set, the equilibrium expected return of a security is a function of its risk, where theories differ in their definition of risk. Fama (1970) assumes that the conditions of market equilibrium can be stated in terms of expected returns, i.e.;

$$E_t P_{j,t+1} + E_t C_{j,t} = (1 + E_t r_{j,t+1}) \times P_{j,t} \quad (2)$$

Where;

$j = 1, 2, 3, \dots, n$;

E_t , the mathematical expectation conditional on information at time t , Ω_t ;

$P_{j,t}$, the real price of security j at time t ;

$C_{j,t}$, the real cash flow generated by security j at time t ;

Observe that equations (1) and (2) represent a partial equilibrium price formation process as the prices of all assets other than the one under consideration remain fixed. Also, the investor is assumed to be infinitely lived, and therefore to have an infinite investment horizon. These assumptions will be lapsed in the next chapter when we contrast the implication for risk premia of an investor with an infinite investment horizon vs. an investor with a finite investment horizon within the context of a fully endogenous general equilibrium model of asset pricing.

Any empirical test of equation (2) above is therefore a test of the following multiple hypothesis:

1. The efficient market hypothesis;
2. The assumption that market equilibrium can be stated in terms of expected return, and;
3. The return theory or process that is assumed to underlie $E_t r_{j,t+1}$;

Equation (2) can refer to the price formation process of a single security j , or, an asset class j .

Equation (2) can be rearranged such that:

$$P_{j,t} = \lambda_{j,t+1} \times E_t P_{j,t+1} + \lambda_{j,t+1} \times E_t C_{j,t+1} \quad (3)$$

Where;

$$\lambda_{j,t+1} = \frac{1}{1 + E_t r_{j,t+1}}$$

Equation (3) can be solved by recursive substitution to yield:

$$P_{j,t} = \sum_{k=0}^{\infty} \left(\prod_{n=0}^k \lambda_{j,t+n+1} \right) \times E_t C_{j,t+k} = E_t P_{j,t}^* \quad (4.a)$$

Where;

$$\lim_{k \rightarrow \infty} \left(\prod_{n=0}^k \lambda_{j,t+n+1} \right) \times E_t C_{j,t+k} = 0$$

Equation (4.a) represents a generalised version of the structure common to most models of market efficiency. The assumption that conditions of market equilibrium can be stated in terms of expected returns and that equilibrium expected returns are formed on the basis of the information set Ω_t rule out the possibility of trading systems based only on information in Ω_t that have expected profits or returns in excess of the equilibrium expected profits or returns (Fama 1976). This is the first of two corollaries supported by the efficient market hypothesis. The second corollary states that financial market prices represent rational assessment of fundamental values.

Corollary 1 implies that the efficient market hypothesis can be written as follows:

$$x_{j,t} = P_{j,t} - E_t P_{j,t}^* \quad (5.a)$$

Such that

$$E_t x_{j,t} = 0 \quad (5.b)$$

Or, equivalently;

$$z_{j,t} = r_{j,t} - E_t r_{j,t}^* \quad (6.a)$$

Such that

$$E_t z_{j,t} = 0 \quad (6.b)$$

Thus, $\{x_{j,t}\}$ and $\{z_{j,t}\}$ represent a fair game with respect to information sequence $\{\Omega_t\}$. Equations (5) and (6) above are commonly referred to as the fair game models. It must be noted that the fair game properties of these models are implications of the assumptions that:

1. The conditions of market equilibrium can be stated in terms of expected returns, and;
2. The information set Ω_t is fully utilized by the market in forming equilibrium expected returns and thus current prices;

Two special cases of the fair game model are considered to better illustrate the flexibility of equations (5) and (6) relative to other specifications of the efficient market hypothesis that have been tested. The first case considers the comparison of the performance of a trading rule vs. a simple buy and hold strategy. The second case considers whether successive one period returns or price changes are independently and identically distributed, also commonly known as the random walk model.

Tests of the buy-and-hold strategy assume that $E_t [P_{j,t+1} | \Omega_t] \geq P_{j,t}$, or equivalently, $E_t [r_{j,t+1} | \Omega_t] \geq 0$. This additional constraint is needed to rule out expected profitability under a one-security vs. cash trading system, which can be profitable under the unconstrained fair game model of equations (5) and (6);

Tests of the random walk model are based on the assumption that $f(r_{j,t+1} | \Omega_t) = f(r_{j,t+1})$, i.e. the conditional probability distribution function equals the marginal probability distribution function of an independent random variable. This assumption implies that the sequence of past returns is of no consequence or use in assessing the distributions of future returns.

The above represents a summary of the theory that underpins the efficient market hypothesis. From corollary one and two above it is clear that active management cannot be used as a method of improving the risk adjusted returns of a portfolio of assets. It is therefore worth examining work that criticises the findings of Fama (May 1970, 1976).

Summers (1986), points out that the inability of a body of data to reject a scientific hypothesis does not mean that the tests prove the validity of the hypothesis. This observation is based on statistical theory that states that failure to reject a hypothesis is not equivalent to its acceptance. In fact, the usefulness of a hypothesis depends on its ability to discriminate between it and other plausible formulations.

The above suggests that the validity of corollaries one and two can be questioned by specifying an alternative hypothesis for the price formation process that has a power at least equal to that of the efficient market hypothesis. Summers (1986) suggests the following characterisation of the price formation process:

$$P_{j,t} = E_t P^*_{j,t} + u_t$$

and

$$u_t = \alpha u_{t-1} + v_t$$

where;

- u_t and v_t represent the natural logarithm of random shocks;
- $0 \leq \alpha \leq 1$, i.e. this assumes that deviations persist but do not grow forever;

Summers uses this approach to suggest that certain types of inefficiency in market valuations are not likely to be detected using standard methods. This means that failure to reject the price formation process of the efficient market hypothesis does not lead to the conclusion that market prices represent rational assessments of fundamental valuations

(Summers, 1986). Summers points out that most tests of market efficiency have relatively little power against certain types of market inefficiency.

An alternative approach is to use measures of the variance of $P_{j,t}$ to provide evidence against simple models of market efficiency. It is possible to construct variance bounds for $P_{j,t}$ by assuming that both $P_{j,t}$ and $C_{j,t}$, or some transformation of these, are stationary stochastic processes. This approach can be used to test whether prices show too much variation to be explained in terms of the random arrival of new information about the fundamental determinants of price. Shiller (1981b) shows that given the assumptions outlined above, the conventional regression tests of the efficient market model are no longer suggested, and that volatility tests have more power in certain regions of the parameter space.

This methodology was used to study efficient market models in connection with long-term and short-term bond yields (Shiller - December 1979, Singleton - 1980), stock prices and dividends (Shiller, 1981c), yields on intermediate and short-term bonds (Shiller, 1981a) and foreign exchange rates and money stock differentials (Huang - 1981, Meese and Singleton - 1980).

2.2 A Simplified Empirical Test of the Efficient Market Hypothesis

The variance bound methodology will be used to construct a simple test of the efficient market hypothesis in the context of stock market variability. The derivation of all formulae was obtained from Shiller (1981b and 1981c).

2.2.1 Specifying the Test

By using a constant real discount factor λ , a simplifying assumption that is consistent with that used in simple tests of market efficiency, equation (4.a) reduces to the following:

$$P_{j,t} = \sum_{k=0}^{\infty} \lambda^{k+1} \times E_t C_{j,t+k} \quad (4.b)$$

Equation (4.b) can be restated as a proportion of the long-run dividend growth factor, g :

$$p_{j,t} = \sum_{k=0}^{\infty} \bar{\lambda}^{-k+1} \times E_t c_{j,t+k} \quad (4.c)$$

Where;

$\gamma^{t-T} = (1+g)^{t-T}$, where T is the base year;

$$p_{j,t} = \frac{P_{j,t}}{\gamma^{t-T}};$$

$$c_{j,t} = \frac{C_{j,t}}{\gamma^{t+1-T}};$$

$\bar{\lambda} = \frac{1+g}{1+r} = \frac{1}{1+\bar{r}}$ and $g < r$ such that $\bar{r} > 0$.

By taking unconditional expectations on both sides of (4.c), Shiller (1981c) finds that:

$$E(p_j) = \frac{\bar{\lambda}}{1-\bar{\lambda}} \times E(c_t) \rightarrow \bar{r} = \frac{E(p_j)}{E(c_j)}$$

If $p_{j,t}^*$ is the present value of actual subsequent dividends, then $p_{j,t} = E_t p_{j,t}^*$, where:

$$p_{j,t}^* = \sum_{k=0}^{\infty} \bar{\lambda}^{-k+1} c_{j,t+k}$$

Since this summation extends to infinity, $p_{j,t}^*$ cannot be observed without some error. A long enough price and dividend series may allow us to approximate $p_{j,t}^*$, whilst we can examine the sensitivity of the test results by using alternative terminal values for $p_{j,t}^*$. Shiller (1981b and 1981c) uses the average of the series $\{p_{j,t}^*\}$ as the terminal value for $p_{j,t}^*$ at time T. It is possible that this specification introduces bias into the test since equation (4.a) is analogous to a buy-and-hold strategy; i.e. buy the market for a number of years after which the market is sold at the prevailing price. Therefore, using Shiller's specification of the terminal value for $p_{j,t}^*$ we will have that $p_{j,t} \neq E_t p_{j,t}^*$. If the terminal value for $p_{j,t}^*$ is taken to be the prevailing price at time T (or some observed price after time T) then $p_{j,t} = E_t p_{j,t}^*$. The variance bounds derived using this specification will hold even in the presence of a speculative bubble.

Shiller (1981c) derives the following variance inequalities to test the volatility of $p_{j,t}$ and $\Delta p_{j,t}$ (see the appendix to this chapter for a derivation of these formulae):

$$\sigma(p_{j,t}) \leq \sigma(p^*_{j,t}) \quad (7)$$

$$\sigma(\Delta p_{j,t}) \leq \frac{\sigma(c_{j,t})}{\sqrt{2r}} \quad (8.a)$$

The variance bounds derived by Shiller (1981b) assume that the series $[p_{j,t}, d_{j,t}]$ and $[\Delta p_{j,t}, \Delta d_{j,t}]$ represent a stationary random process. This assumption is part of the efficient market hypothesis and is explicitly modelled in the terminal condition of equation (4.a). This assumption rules out an explosive path for $p_{j,t}$ and $\Delta p_{j,t}$, and suggest tests with greater power that may reject the efficient market hypothesis more decisively than the standard regression test (Shiller, 1981b). This assertion is examined in more detail in the section dealing with the power of the test. The proofs of equations (7) and (8.a) are shown in the mathematical appendix.

In his original test Shiller restricted the solutions to roots of $\sigma[p_{j,t}]$ to real numbers, thereby ignoring complex numbers. This restriction seems arbitrary since the fact that variance matrices are positive semi definite does not imply that the square root of variance matrices is positive semi definite. If we allow complex number solutions for the quadratic then we find that $\sigma[p_{j,t}]$ has the form of a sinusoidal function and hence a cyclical type of time path.

If we reinterpret $r\sigma^2[p_{j,t}] - \rho\sigma[c_{j,t}]\sigma[p_{j,t}] + \frac{1}{2}\sigma^2[\Delta p_{j,t}] = 0$ as the characteristic equation of a second order non-homogenous difference equation of the form:

$$x_{t+2} + ax_{t+1} + bx_t = c_t \quad (b \neq 0) \quad (8.b)$$

Now the general solution of (8.b) is of the form:

$$x_t = Au_t^1 + Bu_t^2 + u_t^* \quad (8.c)$$

Note that this interpretation suggests that the time path of the market evolves according to a difference equation equivalent to (8.b). If the right number of initial conditions is imposed then there is a unique solution for the system. However, the solution to the system changes if one or more of the initial conditions are changed. Under this more general characterisation of Shiller's original test, a test for EMH is equivalent to testing whether small changes in the initial conditions have any effect on the long-run behaviour of the solution, or will the effect die out as $t \rightarrow \infty$.

Equation (8.b) is globally asymptotically stable if the general solution $Au_t^1 + Bu_t^2$ of the associated homogenous equation approaches 0 as $t \rightarrow \infty$, for all values of A and B, as any solution of equation (8.b) approaches the particular solution u_t^* , which is independent of the initial conditions. The criterion for the stability of (8.b) is $|a| < 1 + b$ and $b < 1$. The efficient market hypothesis therefore holds if and only if

$$\left| \frac{\rho\sigma[c_{j,t}]}{r} \right| < 1 + \frac{\sigma^2[\Delta p_{j,t}]}{2r} \text{ and } \frac{\sigma^2[\Delta p_{j,t}]}{2r} < 1 \quad (8.d)$$

Section 2.2.3 below details the results of the original test specified by Shiller in terms of equations (7) and (8.a) based on the updated data set described in section 2.2.2 below. Section 2.2.4 critically examines various aspects of the original test specified by Shiller in terms of equations (7) and (8.a). Section 2.2.5 present the result of the stability test outlined in equation (8.d).

2.2.2 The Data Set

The data set, obtained from Robert J. Shiller (<http://aida.econ.yale.edu/~shiller/>), consists of monthly stock price, dividends, and earnings data and the consumer price index (to allow conversion to real values), all starting January 1871. Monthly dividend and earnings data are computed from the S&P four-quarter tools for the quarter since 1926, with linear interpolation to monthly figures. Dividend and earnings data before 1926 are from Cowles and associates (Common Stock Indexes, 2nd ed. [Bloomington, Ind.: Principia Press, 1939]), interpolated from annual data. Stock price data are monthly

averages of daily closing prices through January 2001. The CPI-U (Consumer Price Index-All Urban Consumers) published by the U.S. Bureau of Labor Statistics begins in 1913; for years before 1913 the CPI Warren and Pearson's price index was spliced in, by multiplying it by the ratio of the indexes in January 1913. For the Plots, the inflation-corrected series is multiplied by a constant so that their value in January 2001 equals their nominal value, i.e., so that all prices are effectively in January 2001 dollars.

2.2.3 The Test Results

The results of applying equations (7) and (8) to the data set are as follows:

Period	130 years
G	1.16%
$p^*_{j,T}$	1208
\bar{r}	4.11%
Monthly \bar{r}	0.34% $\{ 1.0411^{12} - 1 \}$

Equation (7)

$\sigma(p_{j,t})$	207.00
$\sigma(p^*_{j,t})$	183.72

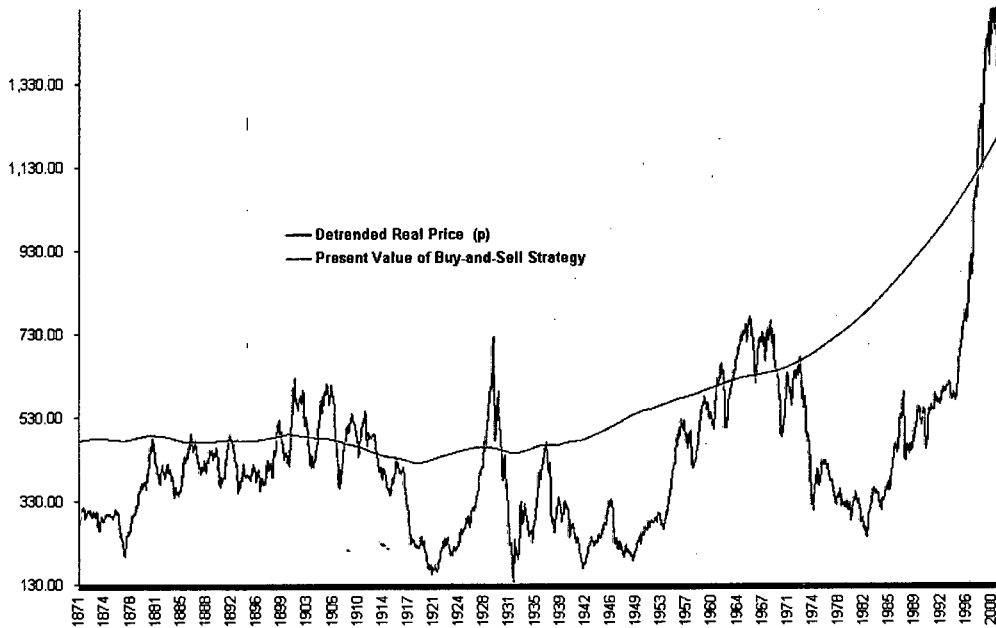
Equation (8)

$\sigma(\Delta p_{j,t})$	61.20
$\sigma(c_{j,t})$	3.26
$\sqrt{2\bar{r}}$	0.29
$\sigma(c_{j,t})/\sqrt{2\bar{r}}$	11.72

From the above it is clear that the data violate equations (7) and (8) as the variance bound implied by equation (7) is violated by a factor of 1.13x and that implied by equation (8) is violated by a factor of 5.22x. The 1.13x ratio by which equation (7) is violated is much lower than that found in Shiller's original test (1981c). This a direct consequence of the alternative specification used for the terminal value of $p^*_{j,t}$ (i.e. $p^*_{j,T} = 1208$). If the average of the series $\{p^*_{j,t}\}$ (i.e. $p^*_{j,T} = 435$) was used the ratio would have been 8.21x .

The relatively low ratio by which the variance bound of equation (7) is violated is concerning given the bias inherent in this particular test statistic (see section 2.3.3 below for a fuller discussion of this issue).

A plot of $p_{j,t}$ relative to $p^*_{j,t}$ using $p^*_{j,T} = 1208$ yields the following:



2.2.4 Critique of the Test

Equation (8) was derived by assuming that the detrended cash flow series $\{c_{j,t}\}$ is stationary. This assumption is necessary as each cash flow is observed only once, and therefore gives no information about the variance of the observed cash flow. To test the variance bounds it is therefore assumed that the cash flow series is stationary around a deterministic trend. This assumption means that the sample variance of the cash flow series will converge to the population variance as the sample becomes large.

In order to test the stationarity assumption dividends were modelled using a simple first order autoregressive process, i.e. $c_{j,t} = \rho \times c_{j,t-1} + e_{j,t}$, using non-overlapping annual data.

The data set contained a 130 annual data points for testing the following hypothesis test:

H0: Dividends follow a nonstationary process, i.e. $\rho = 1$, vs.

H1: Dividends follow a stationary process, i.e. $\rho < 1$.

A Dicky-Fuller Test was used to test H0 v.s. H1 using EViews version 3.1. The test statistic was -0.62 versus a 10% critical value of -1.617 . The simple Dicky-Fuller test only applies to a series following an AR(1) process, and is violated if the series is correlated at higher order lags. This assumes that the error term therefore follows a white noise process, which is tested by examining the correlogram of residuals which reveals no significant spikes. Also, the Q-statistic of the first 9 autocorrelation coefficients is only significant at the 7.6% level, whilst the Q-statistic of the first 10 autocorrelation coefficients is significant at the 3.5% level, i.e.:

Correlogram of Residuals for $c_{j,t} = \rho \times c_{j,t} + e_{j,t}$

Sample: 1872 2000

Included observations: 129

No	AC	PAC	Q-Stat	Prob
1	0.147	0.147	2.8455	0.092
2	-0.172	-0.198	6.7769	0.034
3	-0.119	-0.063	8.6622	0.034
4	-0.13	-0.142	10.931	0.027
5	-0.098	-0.097	12.239	0.032
6	0.008	-0.023	12.247	0.057
7	0.012	-0.052	12.266	0.092
8	-0.139	-0.189	14.974	0.06
9	0.067	0.083	15.598	0.076
10	0.165	0.071	19.445	0.035
11	-0.081	-0.142	20.387	0.04
12	-0.162	-0.135	24.167	0.019

From the above it is clear that the hypothesis test specified above does not suffer from any serious specification problems. This simple hypothesis test therefore does not reject H0 in favour of H1 at the 10% level, which casts some doubt on the validity of the stationarity assumption that was made.

Next, the first order differences of the dividend series were tested for stationarity using the same first order autoregressive process, i.e. $\Delta c_{j,t} = \rho \times \Delta c_{j,t} + e_{j,t}$, again using non-overlapping annual data to test the following hypothesis:

H0: The first order difference of dividends follow a nonstationary process, i.e. $\rho = 1$, vs.

H1: The first order difference of dividends follow a stationary process, i.e. $\rho < 1$.

The test statistic was -9.72 versus a 1% critical value of -2.58 . The Correlogram of Residuals revealed no significant specification problems and is included below for completeness.

Correlogram of Residuals for $\Delta c_{j,t} = \rho \times \Delta c_{j,t} + e_{j,t}$

Sample: 1872 1999

Included observations: 128

No	AC	PAC	Q-Stat	Prob
1	0.147	0.147	2.8267	0.093
2	-0.172	-0.198	6.73	0.035
3	-0.118	-0.063	8.5976	0.035
4	-0.13	-0.142	10.851	0.028
5	-0.098	-0.097	12.151	0.033
6	0.008	-0.024	12.158	0.059
7	0.011	-0.052	12.176	0.095
8	-0.139	-0.189	14.874	0.062
9	0.067	0.083	15.496	0.078
10	0.165	0.071	19.336	0.036
11	-0.081	-0.142	20.267	0.042
12	-0.162	-0.135	24.021	0.02
13	0.095	0.126	25.332	0.021
14	0.091	0.034	26.55	0.022
15	-0.11	-0.159	28.337	0.02

There is significant evidence that the first order difference of dividends is stationary. It therefore seems better to specify the Lagrangean as a function the variance of the first difference in detrended price series and the first difference of the detrended dividend series, i.e.

$$L = \text{var}(\Delta p_{j,t}) + v \left(\text{var}(\Delta c_{j,t}) - \sum_{k=0}^{\infty} f(\sigma_k^2) \right).$$

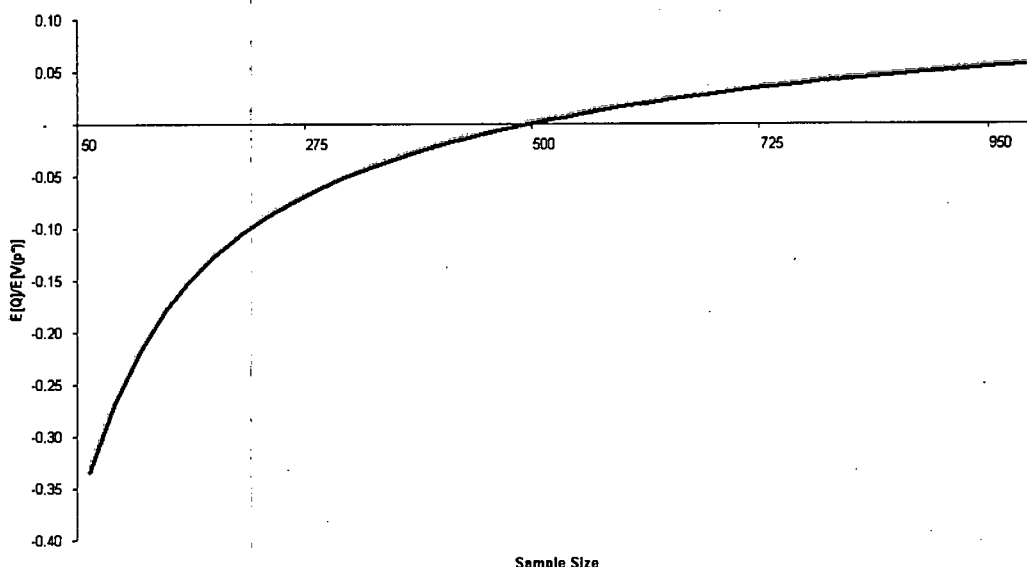
This survey will leave this alternative unexplored, as it will suffice to note that an application of the results from Shiller (1981b) to this data set found that $\text{var}(\Delta p_{j,t})$ did not violate the implied variance bounds derived from the above Lagrangean.

Mankiw et al (1985) show that equation (7) is biased in small samples. This comes from the fact that sample variances are downward-biased estimators of population variances because sample means are used instead of population means. In an independent, identically distributed sample the estimated variance must be corrected by factor of $T/(1-T)$, however, in a serially correlated sample the correction for bias is much larger.

Mankiw et al use the following example to illustrate their point: Market price is proportional to dividends under the assumption that dividends follow a first order autoregressive process. Since p^* is a weighted sum of future dividends, it is a weighted sum of future prices, which tends to smooth p^* relative to p , making the bias in estimating the variance of p^* greater than the bias in estimating the variance of p (see pp 608).

This point can be illustrated empirically by assuming that dividends follow an AR(1) process, i.e. $c_{j,t} = \rho \times c_{j,t-1} + e_{j,t}$ where $E(e_{j,t}) = 0$ and $\text{Var}(e_{j,t}) = \sigma_e^2$, and by defining the test statistic Q , where $Q = \text{Var}(p_{j,t}^*) - \text{Var}(p_{j,t})$, and assuming for simplicity that $c_{j,-1} = 0$. These simplifying assumptions do not detract from the general thrust of the argument presented, but are used to keep the math tractable when calculating the ratio $\frac{E(Q)}{E[\text{Var}(p_{j,t}^*)]}$

for different sample sizes and for different coefficients of the AR(1) process. The Dicky-Fuller test conducted above reported $\rho = 0.994$, which yields the following graph:



This graph clearly illustrates the small sample size bias inherent in equation (7). In fact, it is clear that extremely large sample sizes are needed before the bias inherent in equation (7) become acceptable.

From the above it is therefore clear that the original variance bounds derived by Shiller suffer from bias and use a stationarity assumption that is not supported by the data that was used.

2.2.5 The Stability Test

Stability requires that $\left| \frac{\rho\sigma[c_{j,t}]}{r} \right| < 1 + \frac{\sigma^2[\Delta p_{j,t}]}{2r}$ and $\frac{\sigma^2[\Delta p_{j,t}]}{2r} < 1$. Now, using the data set described above we calculate that $\rho = 0.3252$, $\sigma[c_{j,t}] = 3.36$, $\sigma[\Delta p_{j,t}] = 61.20$ and $r = 4.157\%$. Substituting the values into equation (8.d) we have that $\left| \frac{\rho\sigma[c_{j,t}]}{r} \right| = 26.29$, $\frac{\sigma^2[\Delta p_{j,t}]}{2r} = 45048$. Using these derived quantities it is clear that the solution (8.c) to

equation (8.b) is asymptotically unstable. Thus we have that the efficient market hypothesis cannot be rejected using this methodology.

2.3 A Simplified Unbiased Test of the Efficient Market Hypothesis

Mankiw et al (1985) specify a volatility test that is unbiased in small samples and not dependent on a stationarity assumption. Let $p_{j,t}^n$ be some naïve forecast of the stock price at time t, i.e.:

$$P_{j,t}^n = \sum_{k=0}^{\infty} \gamma^{k+1} F_t C_{j,t+k} \quad (9)$$

where $F_t C_{j,t+k}$ is the naïve forecast of $C_{j,t+k}$ at time t. This naïve forecast does not have to be a rational one. However, it is important that the rational agents at time t have access to this naïve forecast.

Mankiw et al (July 1985) derive the following variance bounds which should hold under market efficiency (see the appendix to this chapter for a derivation of these formulae):

$$E(P_{j,t}^* - P_{j,t}^n)^2 \geq E(P_{j,t}^* - P_{j,t})^2 \quad (10)$$

$$E(P_{j,t}^* - P_{j,t}^n)^2 \geq E(P_{j,t} - P_{j,t}^n)^2 \quad (11)$$

The bias inherent in equation (7) arises from the use of the sample means in the calculation of the test statistics. However, equations (10) and (11) are unbiased because they do not use sample means. This is verified in the mathematical appendix.

In order to apply the test statistics derived in equation (10) and (11), Mankiw et al define the naïve forecast of dividends to be the dividends observed in the previous year, i.e.:

$$F_t C_{j,t+k} = C_{j,t-1} \text{ which gives } P_{j,t}^n = \frac{\gamma}{1-\gamma} C_{j,t-1}.$$

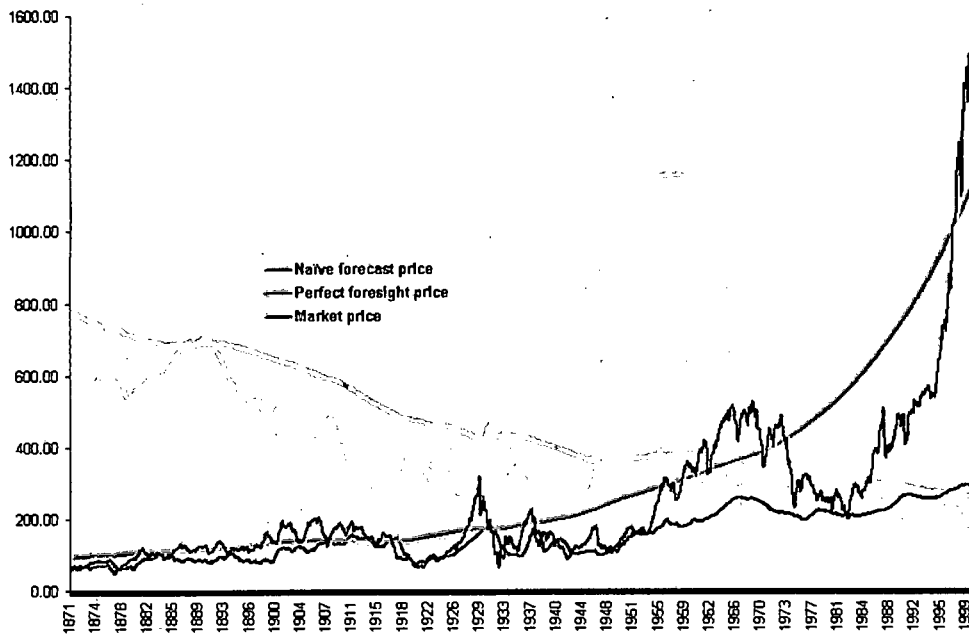
This specification allows the calculation of the test statistics derived in equations (10) and (11) for various different discount rates, i.e.:

r%	$S_1 = E(P_{j,t}^* - P_{j,t}^n)^2$	$S_2 = E(P_{j,t}^* - P_{j,t})^2$	$S_3 = E(P_{j,t} - P_{j,t}^n)^2$	$R_1 = \frac{S_1}{S_2}$	$R_2 = \frac{S_1}{S_3}$
4%	71.85	54.01	26.97	1.33	2.66
5%	64.08	28.34	34.68	2.26	1.85
6%	57.78	17.27	42.49	3.35	1.36
7%	52.58	12.87	49.39	4.09	1.06
8%	48.23	11.73	55.27	4.11	0.87
9%	44.54	12.26	60.27	3.63	0.74
10%	41.38	13.66	64.54	3.03	0.64

Calculations of the test statistics for reasonable discount rates give mixed results. Specifically:

1. The test statistic implied by equation (10) is violated for none of the discount rates, which implies that the market price (i.e. P) is a better forecast of the perfect foresight price (i.e. P*) than the naïve forecast price (i.e. Pⁿ), and
2. The test statistic implied by equation (11) is violated at only the higher discount rates, i.e. volatility of the market price around the naïve forecast price is greater than the volatility of the perfect foresight price around the naïve price at high discount rates. So the test statistic implied by equation (11) is only violated when less weight is placed on the terminal price used in calculating the perfect foresight price P*.

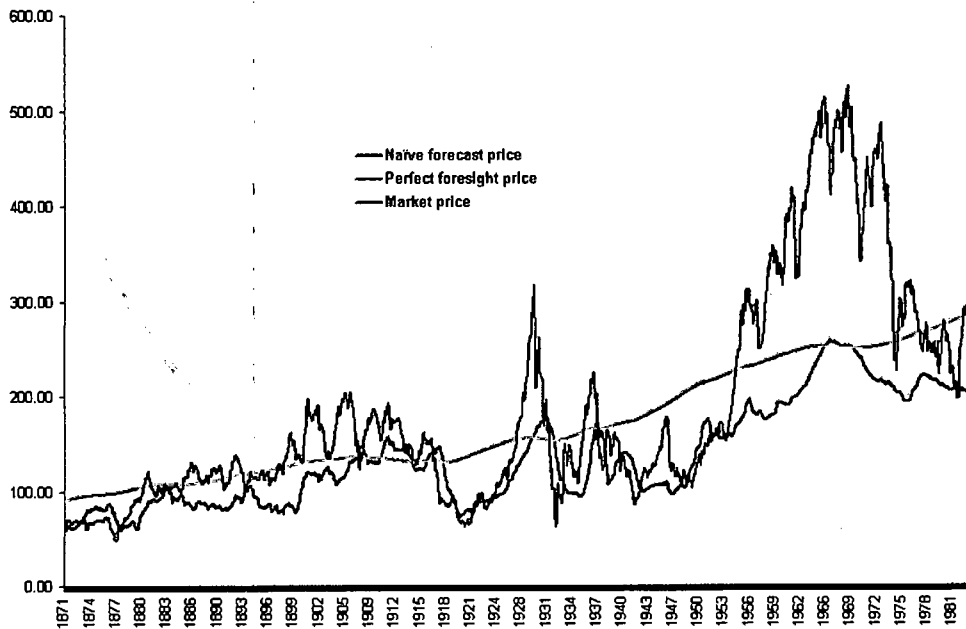
A plot of the data for r = 6% reveals the following:



Close examination of the data reveals that the test results are extremely sensitive to the sample used. If the test was repeated based on data restricted to the period 1871 to 1983, i.e. after the bull market experienced in the 60's and 70's, then both test statistics are uniformly violated, i.e.:

r%	$S_1 = E(P_{j,t}^* - P_{j,t}^n)^2$	$S_2 = E(P_{j,t}^* - P_{j,t})^2$	$S_3 = E(P_{j,t} - P_{j,t}^n)^2$	$R_1 = \frac{S_1}{S_2}$	$R_2 = \frac{S_1}{S_3}$
4%	3.37	9.54	3.38	0.35	1.00
5%	2.21	6.17	4.51	0.36	0.49
6%	1.84	6.10	7.38	0.30	0.25
7%	1.71	7.35	10.47	0.23	0.16
8%	1.66	9.12	13.35	0.18	0.12
9%	1.63	11.02	15.93	0.15	0.10
10%	1.61	12.91	18.21	0.12	0.09

A plot of the data for r = 6% reveals the following:



This suggests that, although the test statistics derived in equation (10) and (11) are unbiased, the resulting test is not sufficiently powerful, as results are significantly influenced by the relationship between terminal price and the initial price. Therefore, this particular test of market efficiency has relatively little power whilst a speculative bubble is evolving, as the evolution of the perfect foresight price series is dependent on the terminal price at which the market is sold at, which may be relatively high in the midst of a speculative bubble (this is observed when applying the test to data spanning 1871 to 2000), or may be low after a speculative bubble has unwound (this is observed when applying the test to data spanning 1871 to 1983).

An alternative explanation for the above phenomena is that asset price volatility varies over time. This refers to the GARCH phenomenon, which observes that asset price volatility can switch between low and high levels of volatility over time. This idea will not be explored in this analysis. The interested reader is referred to the excellent survey of this topic by Bollerslev, Chou and Kroner (1992).

2.4 Comparing Implied Variance Bound Tests and Regression Tests

Equation (4.c) suggests a fairly simple regression test to see whether the price formation process can be forecast. Specifically, let $p_{j,t+1} + c_{j,t} = B_1 + B_2 p_{j,t} + B_3 c_{j,t} + e_{j,t}$ where $e_{j,t}$ is assumed to have a normal distribution with mean zero and variance σ^2 . If the efficient market hypothesis held then we would have $B_1 = B_3 = 0$ and $B_2 = 1 + r > 1$ under the null hypothesis.

The structure of equation (4.c) relates $p_{j,t}$ to an infinite weighted moving average of the expected future cash flows from asset j, partly within the sample and partly without, with successive observations that are not independent. In order to keep the mathematics tractable this discussion will assume that the data consists of T independent normal observations of the vector $z_t = [p_{t+1}, p_t, c_t]$. The likelihood function is then:

$$l(U, \Omega | Z) = (2\pi)^{-3T/2} |\Omega|^{-T/2} \exp\left(-\frac{1}{2} \sum_{t=1}^T (z_t - u) \Omega^{-1} (z_t - u)'\right)$$

Where;

Z is the Tx3 matrix of observations;

U is the 1x3 vector of means;

Ω is the 3x3 covariance matrix;

Let w_t be a linear transformation of z_t such that $w_t = [p_{t+1} + c_t, p_t, c_t]$. Both z_t and w_t are normally distributed. Now, let $w_t = [w_t^1, w_t^2]$ where $w_t^1 = [p_{t+1} + c_t]$ and $w_t^2 = [p_t, c_t]$ and let w , w^1 and w^2 denote the equivalent matrices of observations of order Tx3, Tx1 and Tx2 respectively. Define $Y = w^1$ and $X = [L, w^2]$, where L is a column vector of 1's, V is the two element vector of means of p and c and Φ is the 2x2 covariance matrix for the vector w_t^2 . The likelihood function of w_t can then be written as a product of the likelihood function of w_t^1 , i.e. as the likelihood function for the regression of Y on $X\beta$, and of w_t^2 , i.e. as the likelihood function of a bivariate normal vector, i.e.:

$$\begin{aligned}
l(\sigma^2, \beta, \Phi, V | w) &= l(\sigma^2, \beta, | w^1, w^2) \times l(\Phi, V | w^2) \\
&= (2\pi\sigma^2)^{-T/2} \exp\left(-\frac{(Y - X\beta)'(Y - X\beta)}{2\sigma^2}\right) \\
&\quad \times (2\pi)^{-T} |\Phi|^{-T/2} \exp\left(-\frac{1}{2} \sum_{i=1}^T (w_i^2 - V) \Phi^{-1}(w_i^2 - V)\right)
\end{aligned} \tag{12}$$

The generalized likelihood-ratio test statistic can be written as follows:

$$\lambda = \lambda(x_1, x_2, \dots, x_n) = \frac{\sup_{\theta \in \Theta_0} L(\theta; x_1, x_2, \dots, x_n)}{\sup_{\theta \in \Theta} L(\theta; x_1, x_2, \dots, x_n)} \tag{13}$$

Where the numerator is the maximised likelihood function constrained by the null hypothesis and the denominator is the unconstrained maximum likelihood function. The stationarity assumption that underlies equation (4.c) implies that $E(p_t) = E(p_{t+1})$ and $Var(p_t) = Var(p_{t+1})$. If these restrictions are ignored then it can be seen that test statistic implied by equation (13) will be a transformation of the ordinary regression test statistic, i.e. a transformation of the first factor in equation (11).

If the restrictions implied by the stationarity assumption are not ignored then one can write these restrictions in terms of the parameters of the likelihood function of w_t , i.e. the mean restriction becomes:

$$\beta_1 = (1 - \beta_2)v_p - \beta_3v_d \tag{14}$$

and the variance restriction becomes:

$$\sigma^2 = (1 - \beta_1^2)\sigma^2(p) + 2\beta_2(1 - \beta_3)\sigma(p, c) - (1 - \beta_3)^2\sigma^2(c) \tag{15}$$

Now, $\sigma^2(c) = 0 \rightarrow \sigma^2 = (1 - \beta_1^2)\sigma^2(p)$. Since $B_2 = 1 + r > 1$ under the null hypothesis, then for the null hypothesis to hold we must have $\sigma^2(p) = 0$. The null hypothesis will be rejected with probability 1 if $\sigma^2(p) > 0$. This means that the power of the test tends to

one as $\sigma^2(c)/\sigma^2(p) \rightarrow 0$, which implies that $-2 \log \lambda \rightarrow \infty$, while, whether or not p is explosive in the sample $-2 \log \lambda$ for the regression test will be finite.

The likelihood ratio test defined above therefore has power of one in a region of the parameter space where the regression test does not. It follows that in this region a volatility test with test statistic equal to $\sigma(\Delta p)/\sigma(c)$ will also have power equal to one, which is obtained by using the invariance property of the maximum likelihood function.

2.5 Concluding Remarks

In this chapter a classical challenge by Robert Shiller to the validity of the efficient market hypothesis was reviewed. Although the results of the test showed that markets are more volatile than one would expect under the efficient market hypothesis, it was found that:

1. we cannot accept the assumption that dividends are stationary,
2. that the test results are significantly biased for small sample sizes, and
3. that the test results are not stable.

The original volatility test of Shiller's was extended to remove the small sample bias inherent in it and dependence on the assumption of stationarity of dividends. The results of the extended test was however found to be dependent on the sample used, indicating that this particular test of market efficiency is unreliable as it depends on the relation between the starting price and the terminal price prevailing within the sample.

These results are not encouraging, as they do not present a robust framework within which the validity of the efficient market hypothesis can be challenged. However, it must be remembered that the fair game properties of equations (5) and (6) are implications of the assumptions that: 1) the conditions of market equilibrium can be stated in terms of rational expectations, 2) the information set Ω_t is fully utilized by the market in forming equilibrium expected returns and thus current prices, and; 3) the models employed in this

chapter restrict the analysis to partial equilibrium infinite-lived, representative agent models. These assumptions will be reviewed in more detail in the sequel.

Chapter 3 explores asset pricing in the context of an Overlapping Generations model (OLG model), which establishes the existence of equilibrium as a function of the interaction between preferences, technology, expectations formation and the exogenous stochastic environment.

2.6 Mathematical Appendix

2.6.1 Derivation of equation (7)

⇒ Equation (4.c) implies that $p_{j,t}^* - p_{j,t} \equiv u_{j,t}$ is a forecast error that is uncorrelated with

$$\Rightarrow \sigma^2[p_{j,t}^*] = \sigma^2[p_{j,t}] + \sigma^2[u_{j,t}];$$

⇒ Now, since $\sigma^2[u_{j,t}] \geq 0$ (i.e. variance matrices are positive semi definite) we have that $\sigma^2[p_{j,t}^*] \geq \sigma^2[p_{j,t}]$;

2.6.2 Derivation of equation (8.a)

Equation (4.c) implies that $\Delta p_{j,t+1} + c_{j,t} - r p_{j,t} \equiv u_{j,t}$ is a forecast error that is uncorrelated with $p_{j,t}$. Where $\Delta_t = E_t - E_{t-1}$ is the innovation operator.

$$\Rightarrow \text{Thus, } \sigma[\Delta_{t+1} p_{j,t+1}, p_{j,t}] = 0 \rightarrow \sigma[\Delta p_{j,t+1} + c_{j,t} - r p_{j,t}] = 0$$

$$\Rightarrow \text{Given that } \Delta p_{j,t+1} = p_{j,t+1} - p_{j,t} \text{ we have that } \sigma[p_{j,t+1} + c_{j,t} - (1+r)p_{j,t}] = 0$$

$$\Rightarrow \text{Expanding this covariance yields } \sigma[p_{j,t+1}, p_{j,t}] + \sigma[c_{j,t}, p_{j,t}] - (1+r)\sigma^2[p_{j,t}] = 0$$

⇒ By combining $\sigma^2[p_{j,t+1} - p_{j,t}] = \sigma^2[\Delta p_{j,t+1}]$ with the stationarity assumption which implies that $\sigma^2[p_{j,t+k}] = \sigma^2[p_{j,t}]$, we get that $\sigma[p_{j,t+1}, p_{j,t}] = \sigma^2[p_{j,t}] - \frac{1}{2}\sigma^2[\Delta p_{j,t}]$

⇒ Substituting the equation derived in (4) into equation (3) and simplifying yields the following quadratic equation: $r\sigma^2[p_{j,t}] - \rho\sigma[c_{j,t}]\sigma[p_{j,t}] + \frac{1}{2}\sigma^2[\Delta p_{j,t}] = 0$, where ρ is the correlation coefficient between $c_{j,t}$ and $p_{j,t}$.

⇒ Solving for $\sigma[p_{j,t}]$ in the quadratic yields:

$$\sigma[p_{j,t}] = \frac{\rho\sigma[c_{j,t}] \pm \sqrt{\rho^2\sigma^2[c_{j,t}] - 2r\sigma^2[\Delta p_{j,t}]}}{2r}$$

⇒ If a real number of $\sigma[p_{j,t}]$ is to satisfy the model restriction then the expression inside the discriminant must be non-zero, i.e. $\rho^2\sigma^2[c_{j,t}] - 2r\sigma^2[\Delta p_{j,t}] \geq 0$

$$\Rightarrow \text{Thus, } \sigma^2[\Delta p_{j,t}] \leq \frac{\rho^2\sigma^2[c_{j,t}]}{2r} \rightarrow \sigma^2[\Delta p_{j,t}] \leq \frac{\sigma^2[c_{j,t}]}{2r} \rightarrow \sigma[\Delta p_{j,t}] \leq \frac{\sigma[c_{j,t}]}{\sqrt{2r}} \text{ since } \rho^2 \leq 1$$

2.6.3 Derivation of equations (10) and (11)

- ⇒ Define the identity $P_{j,t}^* - P_{j,t}^n = (P_{j,t}^* - P_{j,t}) + (P_{j,t} - P_{j,t}^n)$
- ⇒ Observe that $P_{j,t}^* - P_{j,t} = v_{j,t}$ and thus uncorrelated with information available at time t.
- ⇒ Thus $E_t[(P_{j,t}^* - P_{j,t})(P_{j,t} - P_{j,t}^n)] = 0$ as $P_{j,t}$ and $P_{j,t}^n$ are known at time t
- ⇒ This implies that $E_t(P_{j,t}^* - P_{j,t}^n)^2 = E_t(P_{j,t}^* - P_{j,t})^2 + E_t(P_{j,t} - P_{j,t}^n)^2$
- ⇒ Such that $E_t(P_{j,t}^* - P_{j,t}^n)^2 \geq E_t(P_{j,t}^* - P_{j,t})^2$ and $E_t(P_{j,t}^* - P_{j,t}^n)^2 \geq E_t(P_{j,t} - P_{j,t}^n)^2$
- ⇒ The law of iterated projections means that one can replace expectations conditional on information at time t with expectations conditional on information prior to the beginning of the sample period. Let E denote the expectation conditional on the initial conditions, then:
- ⇒ $E(P_{j,t}^* - P_{j,t}^n)^2 = E(P_{j,t}^* - P_{j,t})^2 + E(P_{j,t} - P_{j,t}^n)^2$
- ⇒ $E(P_{j,t}^* - P_{j,t}^n)^2 \geq E(P_{j,t}^* - P_{j,t})^2$
- ⇒ $E(P_{j,t}^* - P_{j,t}^n)^2 \geq E(P_{j,t} - P_{j,t}^n)^2$

2.6.4 Lack of Bias Proof for Equations (10) and (11)

- ⇒ To prove that the sample counterparts of these statistics are unbiased define S_1 and S_2 :
- $$S_1 = \frac{1}{T} \sum_{t=1}^T (P_{j,t}^* - P_{j,t}^n)^2 - \frac{1}{T} \sum_{t=1}^T (P_{j,t}^* - P_{j,t})^2 \text{ and } S_2 = \frac{1}{T} \sum_{t=1}^T (P_{j,t}^* - P_{j,t}^n)^2 - \frac{1}{T} \sum_{t=1}^T (P_{j,t} - P_{j,t}^n)^2$$
- ⇒ Now, taking expectations conditional on the initial conditions on each side gives:
- $$\begin{aligned} E(S_1) &= \frac{1}{T} \sum_{t=1}^T E[(P_{j,t} + v_{j,t} - P_{j,t}^n)^2] - \frac{1}{T} \sum_{t=1}^T E[(P_{j,t} + v_{j,t} - P_{j,t})^2] \\ &= \frac{1}{T} \sum_{t=1}^T \{E[v_{j,t}^2] + E[(P_{j,t} - P_{j,t}^n)^2]\} - \frac{1}{T} \sum_{t=1}^T E[v_{j,t}^2] \\ &= \frac{1}{T} \sum_{t=1}^T E[(P_{j,t} - P_{j,t}^n)^2] \geq 0 \end{aligned}$$
- $$\begin{aligned} E(S_2) &= \frac{1}{T} \sum_{t=1}^T \{E[v_{j,t}^2] + E[(P_{j,t} - P_{j,t}^n)^2]\} - \frac{1}{T} \sum_{t=1}^T E[(P_{j,t} - P_{j,t}^n)^2] \\ &= \frac{1}{T} \sum_{t=1}^T E[v_{j,t}^2] \geq 0 \end{aligned}$$

2.2.1 Derivation of equation (14)

The regression test implied by equation (4.c) is $p_{j,t+1} + c_{j,t} = B_1 + B_2 p_{j,t} + B_3 c_{j,t} + e_{j,t}$.

$$\Rightarrow E(p_{j,t+1} + c_{j,t}) = E(B_1 + B_2 p_{j,t} + B_3 c_{j,t} + e_{j,t})$$

$$\Rightarrow E(p_{j,t})(1 - \beta_2) = \beta_1 + E(c_{j,t})(1 - B_3) \text{ given that } E(p_{j,t}) = E(p_{j,t+1}) \text{ and } E(e_{j,t}) = 0$$

$$\Rightarrow \beta_1 = E(p_{j,t})(1 - \beta_2) - E(c_{j,t})(1 - B_3)$$

2.6.5 Derivation of equation (15)

The regression test implied by equation (4.c) is $p_{j,t+1} + c_{j,t} = B_1 + B_2 p_{j,t} + B_3 c_{j,t} + e_{j,t}$.

$$\Rightarrow p_{j,t+1} = B_1 + B_2 p_{j,t} + (1 - B_3)c_{j,t} + e_{j,t}$$

$$\Rightarrow \text{Var}(p_{j,t+1}) = \text{Var}(B_1 + B_2 p_{j,t} + (1 - B_3)c_{j,t} + e_{j,t})$$

$$\Rightarrow \sigma^2(p_{j,t}) = \beta_2^2 \sigma^2(p_{j,t}) + (1 - \beta_3)^2 \sigma^2(c_{j,t}) + \sigma^2 - 2\beta_2(1 - \beta_3)\sigma(p_{j,t}, c_{j,t}) \text{ since}$$

$$\sigma^2(p_{j,t}) = \sigma^2(p_{j,t+1}), \sigma^2 = \sigma^2(e_{j,t}), \sigma^2(p_{j,t}, e_{j,t}) = 0 \text{ and } \sigma^2(c_{j,t}, e_{j,t}) = 0$$

$$\Rightarrow \sigma^2 = (1 - \beta_2^2)\sigma^2(p) + 2\beta_2(1 - \beta_3)\sigma(p, c) - (1 - \beta_3)^2 \sigma^2(c)$$

Chapter 3 - Model Uncertainty and Asset Pricing

3.1 Introduction

In Chapter 2 we found that there is weak evidence of excessive market volatility. However, the significance of these findings was diluted by the modelling problems highlighted in chapter 2. In this chapter we shall review several of the assumptions that underlie the preceding analysis, i.e.:

- (a.1) conditions of market equilibrium can be stated in terms of rational expectations;
- (a.2) the information set Ω_t is fully utilized by the market in forming equilibrium expected returns and thus current prices, and;
- (a.3) the analysis is based on a partial equilibrium infinite-lived, representative agent model.

We compare the risk premia of a fully endogenous asset-pricing model with an infinite-lived representative agent (the Lucas asset-pricing model) to the risk premia of a fully endogenous asset-pricing model with overlapping generations (the OLG asset-pricing model). A fully endogenous asset-pricing model allows us to place the focus on rational expectations in context. We shall see that asset pricing is a function of the interaction between preferences, technology, expectations formation and the exogenous stochastic environment. In this chapter we shall focus on the role of relative risk aversion, as measured by the relative curvature of the intertemporal utility functions, in asset pricing. We find that an increase in relative risk aversion is consistent with an increase in the risk premia of the Lucas asset-pricing model, but has an ambiguous effect on the risk premia of the OLG asset-pricing model. The behaviour of the Lucas asset-pricing model is inconsistent with the results of chapter 2, whilst the behaviour of the OLG asset-pricing model is consistent with the results of chapter 2.

We use an overlapping generations model (OLG) as markets are incomplete such that pooling risks through Arrow-Debreu contingent claims are impossible. This follows from the fact that a consumer cannot sell contingent claims on the future dividend stream of the asset to which he/she has a claim, since potential buyers are not present in the market

5.2.3 Formation of Beliefs

From the analysis presented above it is clear that the object of an agent's uncertainty is $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ itself. Let $\mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ denote the space of all probabilities on $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$. The agent must then select a subjective probability $q \in \mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ for his decision-making. We would then express the agent's expectations by using a Lebesgue integral:

$$Q = \int_{\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)} \mu q(d\mu) \quad (6)$$

Equation (6) provides us with a representation result for rational beliefs. Since $P(\Omega, \mathfrak{S})$ is endowed with the topology of weak convergence it is a complete, separable metric space (see Gray 2001 p52-53). Let $F(P(\Omega, \mathfrak{S}))$ be the space of continuous and bounded real valued functions $f: P(\Omega, \mathfrak{S}) \rightarrow R$. Since $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon) \subset P(\Omega, \mathfrak{S})$ then for each $q \in \mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ the integral $q(f) = \int_{\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)} f(x)q(d\mu)$ is a well-defined linear function on $F(P(\Omega, \mathfrak{S}))$. We say that q represents $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ if $q(f) = f(Q)$ for all $f \in F(P(\Omega, \mathfrak{S}))$. This allows us to define the collection of real valued functions $f^S = \{f: P(\Omega, \mathfrak{S}) \rightarrow R \mid \mu(S) = f^S(\mu), \mu \in P(\Omega, \mathfrak{S}), S \in \mathfrak{S}\}$. (7)

Theorem 5. For any $q \in \mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ let $Q(S) = \int_{\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)} f^S(\mu)q(d\mu) \forall S \in \mathfrak{S}$ (8)

then $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$. Conversely, if $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ then there exists $q \in \mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ such that q represents Q and the support of q is the set of extreme points in $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$.

Proof. See the Mathematical Appendix for details.

The reader may question why we have followed this line of analysis; why not let an agent select an appropriate belief system from $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ in a direct way? The result derived here has a subtle but elegant interpretation: Theorem 5 allows agents to express their uncertainty about Π , the true probability distribution, as uncertainty about the

Definition 6. (Weakly Uniform Stability) Let ε be a convergence function. Then $\varphi^*(\varepsilon) = \{Q \in P_s(\Omega, \mathfrak{S}) : Q(B(\{C\}, N, \nu, \Omega)) \geq 1 - \varepsilon(N, D, \nu), \forall N, D, \nu, \forall C \in \hat{\mathfrak{S}}^D\}$ is said to be a set of weakly ε -uniformly stable probability measures.

The set of weakly ε -uniformly stable probability measures allows the set of realizations to vary relative to cylinders of a particular dimension, but requires that an agent pick a probability belief Q such that; i) the relative frequencies of all cylinders $C \in \hat{\mathfrak{S}}^D$ converge at a uniform rate over a finite investment horizon N to the associated stationary measure m_Q of the events $\mathcal{A} \subset \mathfrak{S}$, and ii) Q is bounded below by a suitable convergence function. By restricting agents' beliefs to rational beliefs (as per definition 4) that are uniformly stable relative to a plausible convergence function (see definition 5) we exclude implausible beliefs that are unlikely to be seen in reality.

This allows us to state the main results of Nielsen (1996).

Proposition 1. The set $\varphi^*(\varepsilon)$ is compact in the topology of weak convergence.

This means that any weakly convergent sequence $\{Q_n\}_{n=1}^{\infty}$ that converges to Q , i.e. $\lim_{n \rightarrow \infty} Q_n = Q$ where that $Q_n \in \varphi^*(\varepsilon) \forall n$, we have that $Q \in \varphi^*(\varepsilon)$.

Proposition 2. Consider the weakly convergent sequence $Q_n \rightarrow Q$ where $Q_n \in \varphi^*(\varepsilon) \forall n$ and m_{Q_n} is the stationary measure associated to Q_n (see theorem 3). Then $m_{Q_n} \rightarrow m_Q$.

Proposition 2 shows that the uniformity condition introduced here also imposes uniformity on the rate of convergence of the average beliefs. These place a restriction on the distribution of beliefs in the economy, which may restrict the ability of the theory to explain market volatility.

From theorem 4 it is clear that further restrictions on the beliefs of agents are required in order to construct a set of admissible beliefs that are closed in the topology of weak convergence. We will proceed as in Nielsen (1996) and restrict agents to beliefs that are compatible with the data (definition 3) and that are Weakly Uniform Stable (definition 6).

The theory of rational beliefs does not predict which beliefs an agent will hold, or how the agent selects a particular belief. It is therefore not unreasonable to expect that agents consider criteria over and above that implied by definition 4 when selecting a rational belief. In order to clarify exposition we consider an agent who makes investment decisions based on the frequencies of a collection of cylinders denoted as $\mathcal{A} \subset \mathfrak{S}$ with which he is concerned. This collection may be the field of cylinders $\mathcal{A} = \hat{\mathfrak{S}}$ or, for an agent with an investment horizon of N years, the collection of cylinders with dimension less or equal to N , i.e. $\mathcal{A} = \{\mathfrak{S}^D : D=1,2,\dots,N\}$ where $\hat{\mathfrak{S}}^D$ denotes the field of all D -dimensional cylinders in \mathfrak{S} . This agent may then restrict his consideration to probability measures in $\mathfrak{R}(\Pi)$ for which the speed of convergence of the data is uniformly high on the specific class of cylinders with which he is concerned. The remainder of this section is a non-technical summary of the main results from Nielsen (1996) section 4. The aim is to clarify and apply the ideas presented by Nielsen to the original results derived by Kurz (1994), rather than reproduce the technical proofs of his propositions.

Definition 5. (Convergence Function) A function $\varepsilon : \mathbb{N}^3 \rightarrow \mathbb{R}_{++}$ is a convergence function if for any D and ν $\lim_{N \rightarrow \infty} \varepsilon(N, D, \nu) = 0$.

Let $P_s(\Omega, \mathfrak{S})$ denote the set of all stable probability measures on (Ω, \mathfrak{S}) . Further, denote the set of realisations $x \in \Omega$ for which the frequencies of $\mathcal{A} \subset \mathfrak{S}$ converge at a uniform rate to the measure of the events $\mathcal{A} \subset \mathfrak{S}$ under the associated stationary measure by

$$B(\mathcal{A}, N, \nu, \Omega) = \left\{ x \in \Omega : \sup_{A \in \mathcal{A}} \left| \frac{1}{K} \sum_{j=0}^{K-1} 1_A(T^j x) - \frac{1}{L} \sum_{j=0}^{L-1} 1_A(T^j x) \right| \leq \frac{1}{\nu} \text{ for } K, L > N \right\},$$

where $\nu \in \mathbb{N}$ measures the speed of convergence. This leads us to a definition of weak uniform stability. Note that $B(\mathcal{A}, N, \nu, \Omega)$ is increasing in N .

$P(\Omega, \mathfrak{S})$ the set of all probabilities on (Ω, \mathfrak{S}) . This allows us to precisely define the implications of having a probability belief Q that is compatible with the data represented by m .

Definition 3. A probability belief $Q \in P(\Omega, \mathfrak{S})$ is compatible with the known data if the system $(\Omega, \mathfrak{S}, Q, T)$ is stable and generates $m(\cdot|x)$, i.e. $m_Q(S) = m(S) \forall S \in \mathfrak{S}$.

Therefore, the belief is not compatible with the data if $m_Q \neq m$, which leads us to define the set of rational beliefs as $\mathfrak{R}(\Pi) = \{Q \in P(\Omega, \mathfrak{S}) : Q \text{ is compatible with the data}\}$ (5)

5.2.2 The Topological Properties of the set of Rational Beliefs

In order to completely specify suitable axioms that underpin the theory of rational beliefs we need to investigate the topological properties of $\mathfrak{R}(\Pi)$. These properties will determine whether we need to impose further restrictions on the beliefs of agents, other than that Q is compatible with the data. Specifically, we wish to establish if the set of rational beliefs relative to any stable measure is closed in the topology of weak convergence, defined as:

Definition 4. The sequence of probability measures $\{Q_n\}_{n=1}^{\infty}$ converges to Q , i.e. $\lim_{n \rightarrow \infty} Q_n = Q$, such that $Q_n \in \mathfrak{R}(\Pi) \forall n$ and there exists no Q such that $Q \notin \mathfrak{R}(\Pi)$.

This is a natural consideration when we wish to establish the existence of equilibrium in general settings as the topology of weak convergence for a sequence space allows one to concentrate on the first members of the space since the set of continuous, bounded functions which are measurable relative to the set of cylinders determines convergence (see Nielsen 1996).

Theorem 4. $\mathfrak{R}(\Pi)$ is not closed in the topology of weak convergence.

Proof. See the Mathematical Appendix for details.

Note that $\hat{m}_\Pi(S) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S)$ has nothing to do with the data, it is simply an analytical expression derived from $(\Omega, \mathfrak{S}, \Pi, T)$. This property says that in the limit the arithmetic mean of iterates of Π with respect to T converges to a set function $\hat{m}_\Pi(\cdot)$. From Lemma 6.2.2 of Gray (2001) we know that $\hat{m}_\Pi(\cdot)$ is a probability function. Since (a.5) assumes that the dynamic system $(\Omega, \mathfrak{S}, \Pi, T)$ is stable, we cannot employ standard ergodic theorems which are based on strong stability. This leads us to theorem 2.

Theorem 2. $(\Omega, \mathfrak{S}, \Pi, T)$ is stable if and only if it is WAMS.

Proof. See the Mathematical Appendix for details.

Theorem 2 allows us to link the finitely additive measure $\hat{m}_\Pi(\cdot)$ and the family of finitely additive measures $m(\cdot|x)$ deduced from the data.

Theorem 3. The probability measure $\hat{m}_\Pi(\cdot)$ on $\hat{\mathfrak{S}}$ can be extended uniquely to a probability measure m_Π on (Ω, \mathfrak{S}) which is stationary with respect to T with the properties that:

- i. $m_\Pi(S|\zeta)(x) = m(S|x)$ m_Π a.e. $\forall S \in \mathfrak{S}$ where $\zeta = \{S \in \mathfrak{S} : T^{-1}S = S\}$, i.e. the collection of all invariant sets, and
- ii. $m(S) = m_\Pi(S) \forall S \in \mathfrak{S}$ where $m(S) = \int_{\Omega} m(S|x) m_\Pi(dx)$

Property i of theorem 3 shows that the limiting sample average of the indicator function of an event equals the descriptive definition of the conditional probability of the event given the σ -field of invariant events. One interpretation of this result is that if we want to calculate the probability of an event S based on the outcome of all invariant measurements, then the answer is equal to the relative frequency of the event in question. Property ii of theorem 3 states that every stable dynamic system $(\Omega, \mathfrak{S}, \Pi, T)$ generates a unique stationary probability m_Π , which is calculated analytically from Π . Denote by

We can now make the following observations:

- i. Given (a.5), in trying to learn Π , agents learn $m(\cdot|x)$, which is a stationary probability. In general we have that $m(\cdot|x) \neq \Pi$ as the true dynamical system $(\Omega, \mathfrak{S}, \Pi, T)$ may not be stationary such that Π may not be learnt,
- ii. Agents know that $m(\cdot|x)$ may not be Π , but $m(\cdot|x)$ is the only measure they can learn and agree upon given the available data, and
- iii. $m(\cdot|x)$ summarises the entire collection of asymptotic restrictions imposed by the true system, i.e. by $(\Omega, \mathfrak{S}, \Pi, T)$, on the empirical distribution of all the observed variables.

We now write $m(S|x) = 0$ for $x \notin C$ and $S \in \mathfrak{S}$, and $m(\Omega|x) = 1 \forall x$. If all agents knew that the system was stationary they would adopt $m(\cdot|x)$ as their rational belief. Assumption (a.5) makes no specific statement about the stationarity or non-stationarity of the system. Therefore, even if the system were stationary, agents would not know this and may still not adopt $m(\cdot|x)$ as their rational belief.

We can now strengthen our early definition of the theory of rational beliefs: The theory of rational beliefs characterises the set of all beliefs that are compatible with the available data by considering what analytical properties the dynamical system $(\Omega, \mathfrak{S}, Q, T)$ must have to generate $m(\cdot|x)$ as a stationary measure. To accomplish this task requires we introduce the concept of a weak asymptotically mean stationary (WAMS) dynamical system in definition 2.

Definition 2. A dynamic system $(\Omega, \mathfrak{S}, \Pi, T)$ is WAMS if for all $S \in \hat{\mathfrak{S}}$ the limit

$$\dot{m}_{\Pi}(S) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S) \text{ exist.} \quad (4)$$

It is strong asymptotically mean stationary (SAMS) if the limit exists for all $S \in \mathfrak{S}$.

adaptation of mathematical techniques that were developed for the study of strongly stable systems.

The set of all $m(S|x)$ defined above describes the normal patterns of the dynamics. Assumption (a.6) assumes that all agents are endowed with this knowledge. This is a simplifying assumption that keeps the mathematics tractable. In practice agents will approximate $m(S|x)$ resulting in a diversity of opinions about the accuracy of the approximations, which will necessarily increase the diversity of beliefs. Further, although all agents are assumed to have the same information about the frequency with which measurable events occur in the limit, they may hold very different views about the interpretation of this information. The latter point will be made more precise later.

The assumption of stability restricts agents in the economy to considering measurable events that are not necessarily simple. Note, however, that the implicit assumption regarding the data required to derive the empirical distribution of the frequency with which events occurred in the past becomes more onerous as the dimension of S increases.

Denoted by $\hat{\mathfrak{S}}$ the field of all cylinders in \mathfrak{S} . Observe that (a.5) assumes that the probability measure $m(S|x)$ exists \prod a.e. $\forall S \in \hat{\mathfrak{S}}$, it does not specify for which $x \in \Omega$ $m(S|x)$ exists \prod a.e. $\forall S \in \hat{\mathfrak{S}}$. Further, the probability measure $m(S|x)$ exists only on the field $\hat{\mathfrak{S}}$. We consider whether one can extend this to a unique probability measure on the σ -field generated by $\hat{\mathfrak{S}}$, denoted $\sigma(\hat{\mathfrak{S}})$. This is an important consideration as $\sigma(\hat{\mathfrak{S}})$ contains all the limiting sequences of cylinders. These results are summarised in Theorem 1.

Theorem 1. There exists a set $C \in \mathfrak{S}$ with $\prod(C)=1$ and for each $x \in C$ a probability measure $m(\cdot|x)$ on (Ω, \mathfrak{S}) which is a unique extension of $m(S|x)$. The dynamical systems $(\Omega, \mathfrak{S}, m(\cdot|x), T)$ are stationary for all $x \in C$.

Proof. See the Mathematical Appendix for details.

We interpret (2) as the relative frequency at which the dynamical system visits the set S , given that it started at x . Observe that agents can learn something about the true Π if $m^n(S|x)$ converges so that with sufficient data $\lim_{n \rightarrow \infty} m^n(S|x)$ can be computed. We make this notion precise in definition 1.

Definition 1. The dynamic system $(\Omega, \mathfrak{S}, \Pi, T)$ is stable if for all finite-dimensional sets, or cylinders, $S \in \mathfrak{S}$ the limit of $m^n(S|x)$ exists Π a.e. The dynamic system is said to be strongly stable if the limit of $m^n(S|x)$ exists Π a.e. for all $S \in \mathfrak{S}$. We denote the limit of $m^n(S|x)$ by $\dot{m}(S|x) = \lim_{n \rightarrow \infty} m^n(S|x)$ Π a.e. (3)

The distinction between stability and strong stability needs clarification; a finite number of observations is needed to verify if $x \in S$ if S is a cylinder, whereas an infinite number of observations is needed to verify if $x \in S$ if S is an infinite dimensional set. We proceed by making the following assumptions:

- (a.4) T is not invertible.
- (a.5) The dynamical system $(\Omega, \mathfrak{S}, \Pi, T)$ is stable.
- (a.6) The limits in (3) are known to all agents.

Assumption (a.4) implies that any particular future evolution x' of the economy is not associated with a unique past $T^{-1}x'$, such that a future x' may arise from many possible pasts. This assumption keeps the dynamics necessarily general.

Assumption (a.5) makes no specific statement about the stationarity or non-stationarity of the system. It is the minimal condition needed to establish common learning of meaningful results about probabilities that also allows for the possibility of a stable non-stationary dynamic system, which is an appropriate model for an economy with structural change within which econometric analysis can be carried out. This approach reflects the absence of a conclusive theoretical reason compelling agents to believe in a stationary economic environment. From a technical point of view this assumption requires the

processes that show that agents can learn the causal structure of the economy on which they formulate their beliefs, and hence Π . However, in chapter 5 we show that the convergence of a particular learning rule to a particular unique solution does not imply that learning behaviour will generally lead agents to the same solution. We shall therefore avoid making the assumption of structural knowledge by assuming that:

(a.3) Π is not known by any agent.

An agent who observes the data takes the true probability space $(\Omega, \mathfrak{S}, \Pi)$ as fixed, but does not know the true Π , and therefore will try to learn as much as possible about Π using past data. This knowledge will be reflected in the agent's belief system; define $T^{-n}S = \{x : T^n x \in S \subset \Omega\}$, the set S located n periods into the future such that $Q(T^{-n}S)$ represents the probability that event S occurs n periods later, where Q is the probabilistic expression of an agent's belief system. The agent therefore adopts the theory that the probability space is $(\Omega, \mathfrak{S}, Q)$. The theory of rational beliefs aims to characterise the set of all beliefs that is compatible with the available data.

The objective outlined above is achieved in a natural way by studying the ergodic properties of the system $(\Omega, \mathfrak{S}, \Pi, T)$, which allows us to better understand the normal or average patterns of the dynamic system. The ergodic property of a dynamic system is defined as the existence of the limiting sample average of some function f , where $f : \Omega \rightarrow R$. Specifically, $\frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = f_n$ almost everywhere (a.e.) when $n \rightarrow \infty$. In this analysis we will confine f to the class of all indicator functions of events.

We define:

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \quad (1)$$

$$m^n(S|x) = \frac{1}{n} \sum_{i=0}^{n-1} 1_S(T^i x) \quad (2)$$

5.2 The Theory of Rational Beliefs

5.2.1 Derivation of the Set of Rational Beliefs.

Consider an economy in which an agent can observe a finite number, k , of variables at each date t , $t = 0, 1, 2, \dots$ denoted by $x_t = (x_{t1}, \dots, x_{tk}) \in X \subseteq R^k$. Each variable in the state space represents information such as GNP, stock prices, earnings per share and climate conditions. We make the following assumption on the distribution of information in this economy:

- (a.1) Information is *not* asymmetrically distributed, and all agents know $x_t, \forall t$, and
- (a.2) X is bounded.

Assumption (a.1) means that heterogeneity of beliefs do not result from a diversity in agents' private information, and is therefore consistent with the rational expectations hypothesis (REH) in which prices make public all private information.

Assumption (a.2) is a technical assumption that is not made in Kurz (1994a), which allows the set of rational beliefs to be bounded. This is not an onerous assumption as the information observed in an economy is by definition bounded.

We hypothesize that the economic environment is represented by a dynamical system $(\Omega, \mathfrak{S}, \Pi, T)$ defined on the non-negative integers $t \geq 0$, where:

- i. $\Omega = X^\infty \subseteq (R^k)^\infty$;
- ii. $\mathfrak{S} = \sigma\left(\bigcup_{t=0}^{\infty} \mathfrak{S}'\right)$ where $\mathfrak{S}' = \sigma(x_0, x_1, \dots, x_t), t \geq 0$;
- iii. Π is a probability on measurable sets of infinite sequences in X^∞
- iv. $x^{t+1} = Tx^t, x^t = (x_t, x_{t+1}, \dots)$ where T is a shift transformation and x^t identifies the random sequence from the perspective of time t ;

A central assumption of the REH is that all agents know the causal structure of the economy. This assumption is usually supported by examples of dynamic learning

The analysis of the Kurz's theory of rational beliefs is concluded by characterising an agent's rational belief as a weighted average of the stationary measure generated by the agent's belief system and the agent's beliefs in structural change.

The OLG model introduced in chapter 4 is extended to include both exogenous and endogenous uncertainty expressed through agents' rational beliefs of expected future endowments and prices. It is then shown that a diversity of beliefs does impact the price formation process, which is followed by a discussion of the conditions under which the impact of a change in expectations on equilibrium prices is unambiguous.

Finally, the analysis concludes with a critique of the rational belief model reviewed here.

they can, however, learn the limiting stationary probability measure generated by the system.

Because agents are assumed to have no structural knowledge of the dynamical system that generates the economic data, they do not necessarily adopt this stationary probability measure as their belief system. The analysis shows, however, that any stable dynamical system, which has for instance an agent's belief system as its generating probability, generates a stationary probability measure based on its underlying generating probability that is unrelated to the data generated by the dynamical system. A belief system is then defined as rational if the stationary measure generated by the belief system is equal to the stationary measure generated by the true dynamical system's true probability measure.

In an important application of the theory of rational beliefs Kurz (1994) shows that agents can express their uncertainty about the true probability distribution that generates the data as uncertainty about the beliefs held by other market participants. This result has profound implications for agents trading in the market. Rather than only try and learn the true stationary probability measure generated by the economy, they may elect to learn the distribution of beliefs that determines market pricing at a point in time.

The preceding result is however premised on the set of rational beliefs being closed in the topology of weak convergence. Nielsen (1996) shows that the set of rational beliefs is however not closed in the topology of weak convergence, defined as the limit points of all converging sequences of rational beliefs are all elements of the set of rational beliefs. The development of the theory of rational beliefs then follows Nielsen (1996) by restricting the set of admissible rational beliefs such that; i) the relative frequencies with which events are observed to occur converge uniformly over the agent's finite investment horizon to the stationary measure generated by his belief system, and ii) the probability of any cylinder, i.e. a finite dimensional set, occurring is non-zero. Nielsen (1996) shows that rational beliefs restricted in this way are closed in the topology of weak convergence.

Chapter 5 - Rational Beliefs and Endogenous Uncertainty

5.1 Introduction

Chapter 4 explored the relationship between expectations and the intertemporal substitution effect within the framework of a simple exchange economy with money and no external uncertainty. The model employed relaxed the strict assumptions that underlie the rational expectations hypothesis. Specifically, agents are assumed to have diverse beliefs that result in expectations that are not necessarily self-fulfilling. The analysis did not attempt to characterise the conditions that permit this diversity of beliefs to arise.

The key ideas with which this part of the analysis is concerned is best explained with the aid of a simple example. Define the present value of future dividends payable by a stock

as $p_t = \sum_{k=0}^{\infty} \lambda^{-k+1} c_{t+k}$, where λ is constant discount factor and c_t is the dividends

expected at time t . Agents who observe the data need to evaluate the risky prospect of p_t . However, the agents do not know the true probability distribution of the random sequence c_t . They do, however, have at their disposal a large amount of recorded past data. The agents then learn all they can from this data in order to form a conditional probability belief about the future sequence of random variables c_{t+k} $k > 0$. The analysis aims to establish criteria for determining whether an agent's probability belief is rational in the sense that it is consistent with the observed data. This will allow us to characterise the structure of all rational beliefs, which will enable a better understanding of the causes of diversity among agents' beliefs.

Kurz (1994) made an important contribution in characterising the conditions that permit a diversity of beliefs to arise through his development of the theory of rational beliefs. Kurz postulates that agents formulate their beliefs by observing the history of data generated by the economy. The analysis shows, through an elegant application of ergodic theory to a stable dynamic system, how the frequencies with which events occur converge in the limit to a stationary probability measure. The basic argument is that agents cannot learn the true probability distribution that generates the dynamic system;

⇒ Note that there is no real change in the constraints faced by the agent in equation (c".1) compared to (c'.1) when price expectations are unit elastic with respect to current prices if $(x_a^1, \dots, x_a^{n_a})$ and $(\lambda m_a^1, \dots, \lambda m_a^{n_a})$ are solutions to (c'.1), i.e.:

$$\lambda p_1 x_a^1 + \lambda m_a^1 = \lambda p_1 \zeta_a^1 + \lambda \tilde{m}_a \rightarrow p_1 x_a^1 + m_a^1 = p_1 \zeta_a^1 + \tilde{m}_a, \text{ and}$$

$$\lambda \tilde{p}_{2a}(p_1) x_a^1 = \lambda \tilde{p}_{2a}(p_1) \zeta_a^1 + \lambda m_a^{t-1} \rightarrow \tilde{p}_{2a}(p_1) x_a^1 = \tilde{p}_{2a}(p_1) \zeta_a^1 + m_a^{t-1}$$

⇒ From this we deduce that $z_a(\lambda p_1; \lambda \tilde{m}_a) = x_a^1 - \zeta_a^1 = z_a(p_1; \tilde{m}_a)$ and therefore has homogeneity of degree 0 and that $\eta_a(\lambda p_1; \lambda \tilde{m}_a) = \lambda \eta_a(p_1; \tilde{m}_a)$ and therefore has homogeneity of degree 1;

4.7.5 Derivation of Equations (3.4.a) and (3.4.b)

⇒ (3.4.a) follows from adding the two inequalities, i.e. $p_1 c_1 + m_1^a = p_1 e_1^a + \tilde{m}^a$ and

$$\tilde{p}_{2a}(p_1) c_2 + m_2^a = \tilde{p}_{2a}(p_1) e_1^a + m_1^a, \text{ and noting that } m_2 \geq 0.$$

⇒ (3.4.b) follows from the fact that $m_1 \geq 0$, this inequality represents the economy's liquidity constraint;

$\rightarrow u_a^1(y_{ia}^1, p_i^1) \geq u_a^1(y_{ia}^1(\lambda), p_i^1) \xrightarrow{\lambda} u_a^1(y_{ia}^1, p_i^1) \geq u_a^1(y_{ia}^1, p_i^1)$.
 \rightarrow Now choose $y_{ia}^1 = (x_{ia}^1, m_{ia}^1)$ such that $m_{ia}^1 > {}^0m_{ia}^1$.
 $\rightarrow u_a^1(y_{ia}^1, p_i^1) < u_a^1(y_{ia}^1, p_i^1)$, which is a contradiction.
 \Rightarrow This proves that $\{^j\pi_{ia}\} \rightarrow \infty$ when either $p_i^1 \rightarrow \partial\Delta^{l-1}$ or $p_i^1 \rightarrow R_+^l \setminus \Delta^{l-1}$ such that π_{ia} is upper hemicontinuous.

4.7.3 Proof of Theorem 1.

Assumptions a.1 to a.5 are the intertemporal equivalent of a'.1 to a'.5 for an exchange economy with money.

Lemma 1 proves that the period 1 utility function is continuous, non-satiated and semi-strict quasi-concave.

Assumption a.7 ensures that that aggregate excess demand function of some trader with a strictly positive initial endowment $\zeta_{ia}^1 > 0$ is upper hemicontinuous, which is proved through lemma 2.

Note that proposition 1 is easily extended to the case of an exchange economy with money by incorporating money into conditions c.1, c.2 and c.3. Therefore, conditions c.1 to c.3 are the intertemporal equivalent of c'.1 to c'.3, for the case of an exchange economy with money.

Assumption a.6 prevents an agent from borrowing money, which allows us to have a relatively weak assumption a.7 (see Grandmont (1983) for a fuller discussion of this issue).

4.7.4 Proof of Lemma 3.

\Rightarrow Consider a change of $(p_1; \tilde{m}_a)$ to $(\lambda p_1; \lambda \tilde{m}_a)$ and $\tilde{p}_{2a}(p_1)$ to $\tilde{p}_{2a}(\lambda p_1)$ and call the new problem (c''.1);

4.7.2 Proof of Lemma 2.

\Rightarrow We prove the result by contradiction and assume that the sequence $\{^j \pi_{ia}\} \rightarrow ^0 \pi_{ia} < \infty$ when either $p_t^1 \rightarrow \partial \Delta^{l-1}$ or $p_t^1 \rightarrow R_+^l \setminus \Delta^{l-1}$.

\Rightarrow The continuity property of π_i implies that there must be some convergent sequence $\{^j y_{ia}^1\} \rightarrow ^0 y_{ia}^1$ such that $\{^j \pi_i\} \rightarrow ^0 \pi_i < \infty$ and $\|{}^0 y_{ia}^1\| < \infty$.

\Rightarrow Also, from the compactness of p_{ia}^2 we have that $p_{ia}^2(^j p_t^1, \cdot) \rightarrow p_{ia}^2(^0 p_t^1, \cdot)$.

\Rightarrow Use lemma 1 to show that for any $y_{ia} \in Y_{ia}$ i) $u_a^1(y_{ia}^1, ^j p_t^1) \rightarrow u_a^1(y_{ia}^1, ^0 p_t^1)$ and ii) $u_a^1(^j y_{ia}^1, ^j p_t^1) \rightarrow u_a^1(^0 y_{ia}^1, ^0 p_t^1)$.

i. Consider the case when ${}^0 p_t^1 \in \partial \Delta^{l-1}$:

\rightarrow ${}^0 p_t^1 \cdot x_{ia}^1 + m_{ia}^1 \leq {}^0 p_t^1 \zeta_{ia}^1 + \tilde{m}_{ia} > 0$ since $\zeta_{ia}^1 > 0$.

\rightarrow Thus $u_a^1(^0 y_{ia}^1, ^0 p_t^1) \geq u_a^1(y_{ia}^1, ^0 p_t^1) \forall y_{ia}^1 \in Y_{ia}^1$ such that ${}^0 p_t^1 x_{ia}^1 + m_{ia}^1 \leq {}^0 p_t^1 \zeta_{ia}^1 + \tilde{m}_{ia}$.

\rightarrow But we must have some good h for which ${}^0 p_{th}^1 = 0$;

\rightarrow This leads to a contradiction as the consumer can increase his demand for good h indefinitely without violating Walras' law, i.e. $\lim_{\gamma \rightarrow \infty} {}^0 p_t^1 \cdot ({}^0 x_{ia}^1 + \delta^h \gamma) = 0$.

\rightarrow We must therefore have some $y_{ia}^1 \in Y_{ia}^1$ such that $u_a^1(^0 y_{ia}^1, ^0 p_t^1) < u_a^1(y_{ia}^1, ^0 p_t^1)$.

ii. Consider the case when $p_t^1 \rightarrow R_+^l \setminus \Delta^{l-1}$:

\rightarrow Let $y_{ia}^1 = ({}^0 x_{ia}^1, m_{ia}^1)$ for arbitrary m_{ia}^1 .

\rightarrow Define ${}^j v_t^1 = \frac{{}^j p_t^1}{\|{}^j p_t^1\|}$. Since $\|{}^j v_t^1\| = 1$ we have that ${}^j v_t^1 \rightarrow ^0 v_t^1 > 0$.

\rightarrow Since ${}^0 p_t^1 \cdot x_{ia}^1 + m_{ia}^1 \leq {}^0 p_t^1 \zeta_{ia}^1 + \tilde{m}_{ia}$ we must have ${}^0 v_t^1 \cdot x_{ia}^1 \leq {}^0 v_t^1 \zeta_{ia}^1$.

\rightarrow Now, since ${}^0 v_t^1 \zeta_{ia}^1 > 0$ there exists $\hat{x}_{ia}^1 \in X_{ia}^1$ such that ${}^0 v_t^1 \cdot \hat{x}_{ia}^1 < {}^0 v_t^1 \zeta_{ia}^1$.

\rightarrow Define $x_{ia}^1(\lambda) = \lambda \hat{x}_{ia}^1 + (1-\lambda) x_{ia}^1$ and note that $y_{ia}^1(\lambda) = (x_{ia}^1(\lambda), m_{ia}^1) \in Y_{ia}^1$.

\rightarrow Also, since ${}^0 v_t^1 \cdot x_{ia}^1(\lambda) < {}^0 v_t^1 \zeta_{ia}^1$ and ${}^0 m_{ia}^1 \geq m_{ia}^1$ then ${}^0 p_t^1 \cdot x_{ia}^1 + m_{ia}^1 \leq {}^0 p_t^1 \zeta_{ia}^1 + \tilde{m}_{ia}$.

\rightarrow ${}^j p_t^1 \cdot x_{ia}^1 + m_{ia}^1 < {}^j p_t^1 \zeta_{ia}^1 + \tilde{m}_{ia}$ for j large enough.

\rightarrow $u_a^1(^j y_{ia}^1, ^j p_t^1) \geq u_a^1(y_{ia}^1(\lambda), ^j p_t^1) \xrightarrow{j} u_a^1(^0 y_{ia}^1, ^0 p_t^1) \geq u_a^1(y_{ia}^1(\lambda), ^0 p_t^1)$.

4.7 Mathematical Appendix

4.7.1 Proof of Lemma 1.

i. u_a^1 is continuous on $Y_{ia}^1 \times P_t^1$.

\Rightarrow Pick $({}^0y_{ia}^1, {}^0p_t^1) \in Y_{ia}^1 \times P_t^1$ and consider the sequence $\{{}^Jy_{ia}^1, {}^Jp_t^1\} \rightarrow ({}^0y_{ia}^1, {}^0p_t^1)$, which follows from the compactness of $Y_{ia}^1 \times P_t^1$.

\Rightarrow From (a.3.1) it follows that $u_a({}^Jy_{ia}^1, {}^Jp_t^1)$ is continuous on $R_+ \times P_t^1$ such that $u_a({}^Jy_{ia}^1, {}^Jp_t^1) \rightarrow u_a({}^0y_{ia}^1, {}^0p_t^1)$.

\Rightarrow From assumption (a'''.7) $p_{ia}^2({}^Jp_t^1) \rightarrow p_{ia}^2({}^0p_t^1)$.

\Rightarrow From theorem a.3 in Grandmont (1972) we have $u_a^1({}^Jy_{ia}^1, {}^Jp_t^1) \rightarrow u_a^1({}^0y_{ia}^1, {}^0p_t^1)$.

ii. Non-satiation of u_a^1 follows immediately from (a.3.2).

iii. u_a^1 is semi-strict quasi-concave.

\Rightarrow Since u_a^1 is non-satiated we can select y_{ia}^{n1} and y_{ia}^{m1} such that $p_t^{n1} = p_t^{m1} = p_t^1$ and $u_a^1(\hat{y}_{ia}^{n1}, p_t^1) > u_a^1(\hat{y}_{ia}^{m1}, p_t^1)$.

\Rightarrow Let $\hat{y}_{ia}^{m1} = \varepsilon \hat{y}_{ia}^{n1} + (1 - \varepsilon) \hat{y}_{ia}^{m1}$ for $0 < \varepsilon < 1$.

\Rightarrow Let \hat{x}_{ia}^{n2} and \hat{x}_{ia}^{m2} be possible second-period consumption vectors that maximise $u_a(y_{ia})$ with respect to $y_{ia}^{n1} \times p_t^1$ and $y_{ia}^{m1} \times p_t^1$ subject to $\tilde{p}_{ia}^2 x_{ia}^2 \leq \tilde{p}_{ia}^2 \zeta_{ia}^2 + m_{ia}^1$.

\Rightarrow This yields $u_a^1(\hat{y}_{ia}^{n1}, p_t^1) = \max u_a(\hat{y}_{ia}^{n1}, \hat{x}_{ia}^{n2})$ and $u_a^1(\hat{y}_{ia}^{m1}, p_t^1) = \max u_a(\hat{y}_{ia}^{m1}, \hat{x}_{ia}^{m2})$.

\Rightarrow Let $\hat{x}_{ia}^{m2} = \varepsilon \hat{x}_{ia}^{n2} + (1 - \varepsilon) \hat{x}_{ia}^{m2}$ for $0 < \varepsilon < 1$.

\Rightarrow Since u_a is semi-strict quasi-concave we have $u_a(\hat{y}_{ia}^{m1}, \hat{x}_{ia}^{m2}) \geq u_a(\hat{y}_{ia}^{n1}, \hat{x}_{ia}^{m2})$.

\Rightarrow But we have $\tilde{p}_{ia}^2 x_{ia}^2 \leq \tilde{p}_{ia}^2 \zeta_{ia}^2 + m_{ia}^1$, which combined with the definition of a maximum gives $u_a^1(\hat{y}_{ia}^{m1}, p_t^1) \geq u_a(\hat{y}_{ia}^{m1}, \hat{x}_{ia}^{m2}) > u_a(\hat{y}_{ia}^{n1}, \hat{x}_{ia}^{m2}) = u_a^1(\hat{y}_{ia}^{n1}, p_t^1)$.

or b) structural changes, i.e. changes in the relative risk aversion of the trader. The argument is as follows;

Let $S_t(\cdot)$ be some dynamic function that is used by agents to choose between multiple equilibria at time t . At the next trading opportunity the agents select an equilibrium price which is sufficiently far from that at time t such that the market price “appears to snap” unexpectedly. Back testing may indicate that markets are more volatile than we would otherwise expect, but the presence of the kind of behaviour hypothesised here may render the result of the statistical test ethereal.

4.6 Summary Remarks and Conclusion

In section 4.3 we show that the existence of short-run Walrasian equilibrium is dependent on the existence of a sufficiently strong intertemporal substitution effect to offset or enhance the real balance effect.

Section 4.5 shows that the existence of a sufficiently strong intertemporal substitution effect requires the existence of a trader whose expectations are relatively insensitive to variations of current prices. In the context of chapter 3 this means that the prospective fair value equity risk premium demanded by this trader is insensitive to both the observed level and variation in the current equity risk premium, such that the trader can act as a flywheel that stabilises the price formation process.

We can then consider the issue of price stability in the presence of uncertainty by questioning the existence of this 'insensitive trader'; his expectations may become biased by recent experience, requiring him to project a lower/higher than normal equity risk premium (or price) in order to act as a stabilising flywheel and take the opposite trade.

In chapter 3 we argued that the intertemporal substitution effect is indistinguishable from relative risk aversion given the assumption of additive utility. It is therefore difficult to judge whether unexpected volatility in market prices (or equity risk premium) is due to the absence of the insensitive trader or a change in the relative risk aversion of the trader. The practical implication of this is that it would be difficult to distinguish between market inefficiency and a structural change in asset pricing.

Given that the family of additive utility functions are a subset of the more general class of utility functions, one may be able to make this statement in a more general context.

Furthermore, theorem 1 states simply that an equilibrium solution exists for the OLG economy under very general assumptions. It does not state that this solution is unique. We need to caution against stating that price instability may be a function of either; a) market inefficiency, i.e. cases when the insensitive trader develops biased expectations,

Assume that the trader's expectations are biased downward at the endowment point $(e_1; e_2)$ such that $\frac{p_2}{p_1} \leq \delta \frac{u'_2(e_2)}{u'_1(e_1)} \forall p_1$. Since u'_1 and u'_2 are decreasing functions we have that $c_1 < e_1$ for all values of p_1 . Now, if all traders expectations are biased downwards in this way an aggregate excess supply will exist on the good market at all values of the current price p_1 , such that no short-run Walrasian equilibrium can exist in which money has positive value.

Again, the phenomenon may occur in the particular case where price expectations are unit elastic with respect to the current price. The real balance effect will then be the sole regulating mechanism of the economy, but will be too weak to bring the market into equilibrium.

The results illustrated in examples 1 and 2 and the existence of an equilibrium solution for OLG economy are formalised in theorem 1 above.

The proof of theorem 1 makes the importance of lemma 2 apparent. In order to use the Arrow and Debreu (1954) existence theorem 2 to prove the existence of equilibrium for the OLG economy specified in section 4.3 we need to have at least one agent with bounded expectations. This result does not imply that bounded expectations of such an agent will in any way influence the expectations of other agents or determine the market clearing price \hat{q} . This result does imply that the bounded expectations of such an agent ensures that $\pi > 0$ as $p_1 \rightarrow \partial\Delta^{-1}$.

Assume that the trader's expectations are biased upward at the endowment point $(e_1; e_2)$ such that $\frac{p_2}{p_1} \geq \delta \frac{u'_2(e_2)}{u'_1(e_1)} \forall p_1$. Since u'_1 and u'_2 are decreasing functions we have that $c_1 > e_1$ for all values of p_1 . Now, if all traders expectations are biased upwards in this way an aggregate excess demand will exist on the good market at all values of the current price p_1 , such that no short-run Walrasian equilibrium can exist in which money has positive value.

This phenomenon may occur in the particular case where price expectations are unit elastic with respect to the current price, such that the ratio p_2/p_1 is independent of p_1 . The real balance effect will then be the sole regulating mechanism of the economy, but will be too weak to bring the market into equilibrium. Also note that this conclusion can be valid for a small expectational inflationary bias, since the ratio p_2/p_1 need not be very large.

Example 2: Persistent Excess Supply under Deflationary Expectations

This example is the compliment to that described above and will not be illustrated for the sake of brevity.

Note that that $c_1 < e_1$ if and only if the slope of the normal to the intertemporal budget line $\alpha\beta$ (i.e. p_2/p_1) is less than the marginal rate of substitution (i.e. u'_2/u'_1) at point α , i.e. $p_1/p_2 < u'_2/u'_1$ at point α . If we assume that the utility function has the form $u_1(c_1) + \delta u_2(c_2)$, where u_1 and u_2 is strictly concave and differentiable, and δ is the period discount factor such that $0 < \delta < 1$, then we have:

$$c_1 - e_1 < 0 \text{ if and only if } \frac{p_2}{p_1} < \delta \frac{u'_2(e_2 + \tilde{m}_a/p_2)}{u'_1(e_1)} \quad (3.7)$$

Again, we use this result to construct plausible examples where there is a persistent disequilibrium in the goods market for all values of the current price.

We will illustrate our point more precisely by considering a relatively simple case where $l=1$, which yields figure 3.

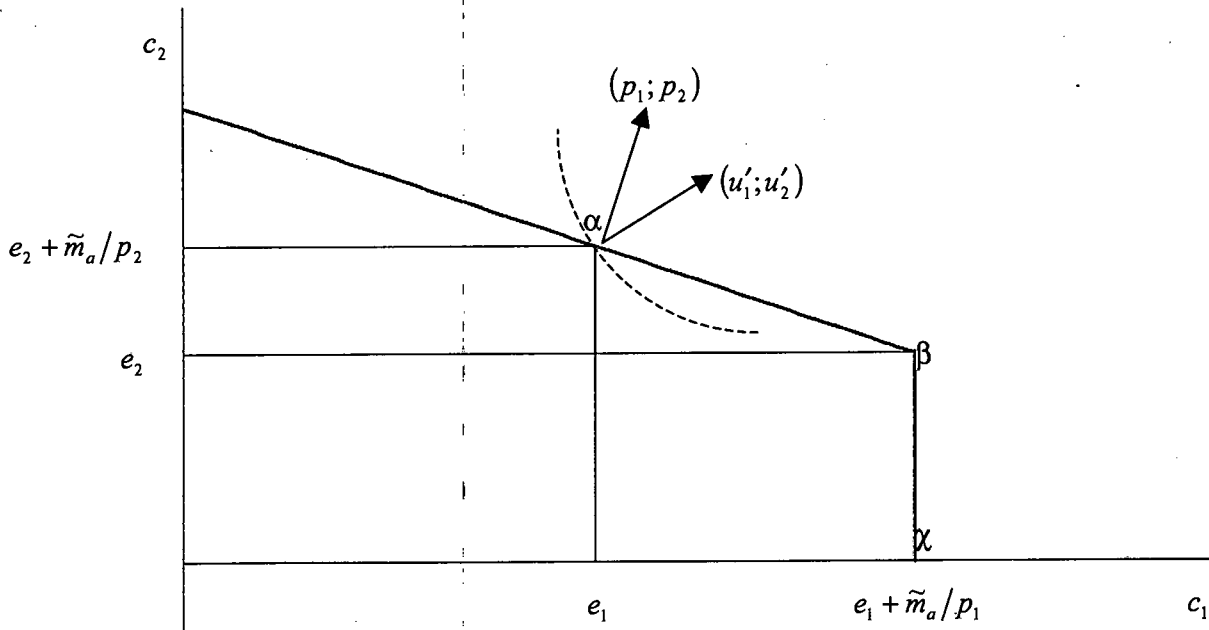


Figure 3

Example 1: Persistent Excess Demand under Inflationary Expectations

From figure 3 it is clear that $c_1 > e_1$ if and only if the slope of the normal to the intertemporal budget line $\alpha\beta$ (i.e. p_2/p_1) exceeds the marginal rate of substitution (i.e. u'_2/u'_1) at point α , i.e. $p_1/p_2 > u'_2/u'_1$ at point α . If we assume that the utility function has the form $u_1(c_1) + \delta u_2(c_2)$, where u_1 and u_2 is strictly concave and differentiable, and δ is the period discount factor such that $0 < \delta < 1$, then we have:

$$c_1 - e_1 > 0 \text{ if and only if } \frac{p_2}{p_1} > \delta \frac{u'_2(e_2 + \tilde{m}_a/p_2)}{u'_1(e_1)} \quad (3.6)$$

We shall use this result to construct plausible examples where there is a persistent disequilibrium in the goods market for all values of the current price.

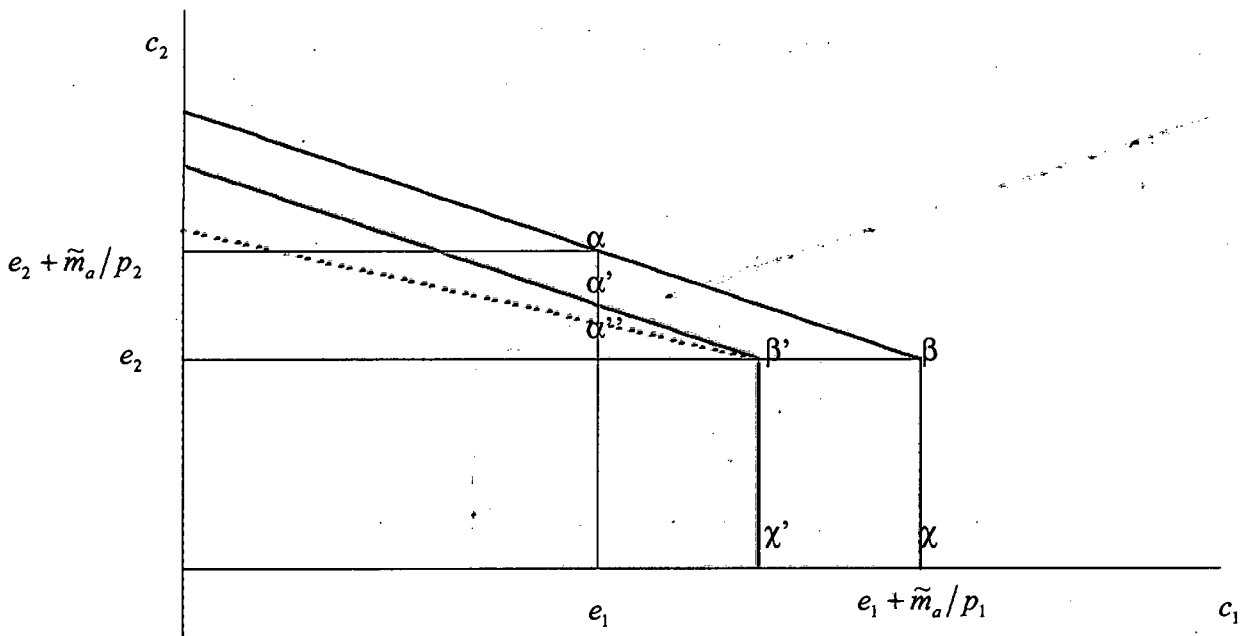


Figure 2

4.5 The Role of Intertemporal Substitution in Short-Run Walrasian Monetary Equilibrium

A short-run Walrasian monetary equilibrium in period 1 exists when the current price p_1 clears both the aggregate demand for goods and the aggregate demand for money, i.e. $\pi = 0$.

At this stage one may reasonably question the relevance of the real balance effect versus the intertemporal substitution effect to the equilibrating process. Grandmont (1983) points out that neoclassical macroeconomists tend to focus on the real balance effect and eliminate the intertemporal substitution effect by assuming that price expectations are unit elastic. We shall therefore consider two examples to illustrate that the real balance effect may be too weak, and that it must be reinforced by a strong intertemporal substitution effect if one wishes to equilibrate the market. We shall conclude this section by proving that this essentially requires that the price expectations of at least one agent be substantially insensitive to current prices.

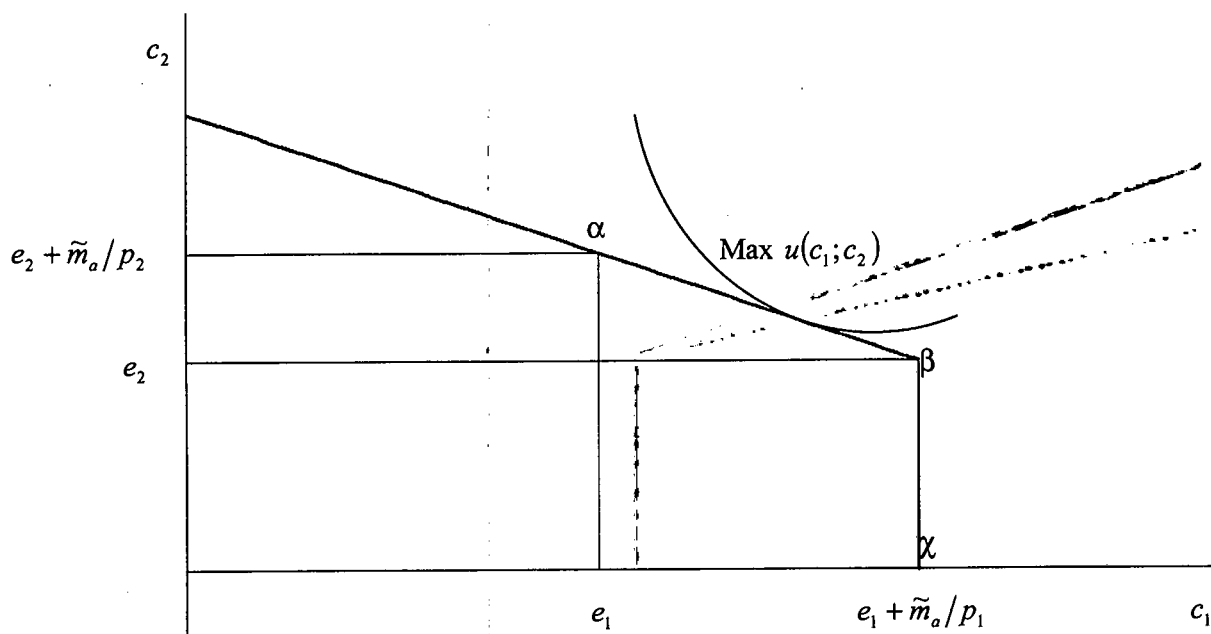


Figure 1

We can use figure 1 to illustrate the absence of money illusion. Consider a change of \tilde{m}_a, p_1, p_2 to $\lambda\tilde{m}_a, \lambda p_1, \lambda p_2$, which leaves unaltered the coordinates of points α and β .

Figure 2 decomposes the impact of an increase in p_1 between the real balance effect and the intertemporal substitution effect:

- If price expectations are unit elastic, an increase in p_1 causes lines $\alpha\beta$ and $\beta\chi$ to move towards the left and become $\alpha'\beta'$ and $\beta'\chi'$, the slopes of the lines being unchanged. This generates the real balance effect.
- If price expectations are not unit elastic, there is an additional rotation of the intertemporal budget line $\alpha'\beta'$ around the point β' , i.e. downward if elasticity exceeds 1 and upward if elasticity is less than 1. This generates the intertemporal substitution effect.

Equation (3.3.b) represents the variation in excess demand that results from the change of expected prices from $\tilde{p}_{2a}(p_1)$ to their true values $\tilde{p}_{2a}(\lambda p_1)$, with the current price being kept at the level λp_1 . We shall call (3.3.b) the intertemporal substitution effect as it measures the consequence of the modification of expected prices relative to the current price. Note that $\Delta z_a^2 = 0$ when price expectations have unit elasticity. However, in general we have that the elasticity of price expectations relative to current prices differ from unity. We consider two cases by way of explanation:

- **Elasticity of price expectations > 1.** This implies that expected prices $\tilde{p}_{2a}(\lambda p_1) > \tilde{p}_{2a}(p_1)$, such that the intertemporal substitution effect will likely favour current consumption and thus counteract the real balance effect.
- **Elasticity of price expectations < 1.** This implies that expected prices $\tilde{p}_{2a}(\lambda p_1) < \tilde{p}_{2a}(p_1)$, such that the intertemporal substitution effect will likely favour increased saving and thus reinforce the real balance effect.

In order to illustrate the properties of short-run demand functions we consider the simple case where there is only one real good, i.e. $l = 1$. Trader a 's budget constraints therefore become $p_1 x_a^1 + m_a^1 = p_1 \zeta_a^1 + \tilde{m}_a$ and $p_2 x_a^2 = p_2 \zeta_a^2 + m_a^1$, where p_2 represents expected prices.

These budget constraints can be written as (see mathematical appendix for derivation):

$$p_1 x_a^1 + p_2 x_a^2 \leq p_1 \zeta_a^1 + p_2 \zeta_a^2 + \tilde{m}_a \quad (3.4.a)$$

$$p_1 x_a^1 \leq p_1 \zeta_a^1 + \tilde{m}_a \quad (3.4.b)$$

Trader a 's opportunity set described by the constraints (3.4.a) and (3.4.b), as well as the result of the utility maximization, is shown in figure 1 below. Note that notation has been simplified as follows: $c_i \equiv x_a^i \forall i$ and $e_i \equiv \zeta_a^i \forall i$

This formulation allows us to make precise the impact of a change in current price on a trader's excess demand z_a . We shall consider the case where there is only one good in the economy, i.e. $l = 1$. This allows us to ignore complications that arise from the possible modification of relative current prices and/or relative expected prices. Consider a variation of current prices from p_1 to λp_1 , which induces the following change on the agent's excess demand for the good:

$$\Delta z_a = z_a(\lambda p_1; \tilde{m}_a) - z_a(p_1; \tilde{m}_a) = \Delta z_a^1 - \Delta z_a^2 \quad (3.2)$$

where;

$$\Delta z_a^1 = z_a(p_1; \tilde{m}_a/\lambda) - z_a(p_1; \tilde{m}_a) \quad (3.3.a)$$

$$\Delta z_a^2 = z_a(\lambda p_1; \tilde{m}_a) - z_a(p_1; \tilde{m}_a/\lambda) \quad (3.3.b)$$

Equation (3.3.a) represents the variation in excess demand that would occur if current prices change from p_1 to λp_1 and if the trader's expected prices move proportionally from \tilde{p}_{2a} to $\tilde{\lambda p}_{2a}$, which is equivalent to assuming that price expectations are unit elastic with respect to current prices, we call this the real balance effect. The solution to equation (c.1) displays straightforward homogeneity properties under this latter assumption that are summarised in lemma 3 (see appendix for proof).

Lemma 3. The functions $z_a(p_1; \tilde{m}_a)$ and $\eta_a(p_1; \tilde{m}_a)$ are homogenous of degree 0 and 1, respectively, with respect to $(p_1; \tilde{m}_a)$, if expected prices are unit elastic with respect to current prices, i.e. $\tilde{p}_{2a} = \tilde{\lambda p}_{2a}$ for every p_1 and $\lambda t = 2, \dots, n_a$.

The property summarised in lemma 3 is also called "the absence of money illusion" property. The absence of money illusion property gives rise to the real balance effect, meaning: multiplying current and expected prices by λ , \tilde{m}_a being fixed, has the same effect on excess demand of the current good as dividing the initial money balance \tilde{m}_a by λ , current and expected prices being kept at their initial level. Therefore, when the good is not inferior an increase of p_1 generates, through the real balance effect, a decrease in demand for the current good.

Lemma 2 gives us a stronger assumption (a.7):

(a.7) There exist at least one trader with either $\zeta_{ia}^1 > 0$ or $\tilde{m}_{ia} > 0$ for which the mapping $p_{ia}^2 : p_i^1 \rightarrow \wp(P_i^2, \beta(P_i^2))$ is continuous and compact.

Let $q_i = (p_i^1; 1)$ and define condition c.3, where $\dot{q}_i = (\dot{p}_i^1; 1)$ and $\dot{\pi}_i$ denotes the equilibrium price and aggregate excess demand respectively, i.e.:

(c.3) $\dot{\pi}_i \leq 0, \dot{q}_i \dot{\pi}_i = 0$.

The maximization problem stated in (c.1) can then be written as an abstract economy denoted by $E_i^{OLG} = [Y_{i1}^1, \dots, Y_{id}^1, A_{i1}(\bar{y}_{i1}^1), \dots, A_{id}(\bar{y}_{id}^1), P_i^1]$, where:

- i. $\bar{y}_{ij}^1 \in \{Y_{i1}^1 \times \dots \times Y_{ij-1}^1 \times Y_{ij+1}^1 \times \dots \times Y_{id}^1 \times P_i^1\}$ represents the strategies chosen by all agents other than j , and
- ii. $A_{ij}(\bar{y}_{ij}^1) = \{y_{ij}^1 | y_{ij}^1 \in Y_{ij}^1, p_i^1 x_{ia}^1 + m_{ia}^1 \leq p_i^1 \zeta_{ia}^1 + \tilde{m}_{ia}\}$ is the domain from which agent j can select his strategies given the strategies chosen by other agents.

The notation defined above allows us to define a competitive equilibrium for this OLG economy as the set of vectors $(\dot{y}_{i1}^1, \dots, \dot{y}_{id}^1; \dot{p}_i^1)$ that satisfies conditions c.1, c.2 and c.3, which is summarised in theorem 1.

Theorem 1. A short-run Walrasian monetary equilibrium exists if assumptions a.1 to a.7 hold.

4.4 The Intertemporal Substitution Effect

The solution to equation (c.1) will yield agent a 's excess demand for goods, i.e. z_a , and demand for money η_a , which are expressed by the agent on the market in response to p_1 . The functional forms of these solutions depend explicitly on the initial money stock \tilde{m}_a and current prices p_1 , and implicitly on Ω_1 as well as on current and future endowments of goods.

17.B.2 for a proof of these results. It follows that z_a is continuous everywhere except at boundary point of the unit simplex.

For the OLG model denote agent a 's excess demand for goods and money by the vector $\pi_{ia} = (z_{ia}; \eta_{ia})$, where $z_{ia} = \hat{x}_{ia}^1 - \zeta_{ia}^1$ and $\eta_{ia} = m_{ia}^1 - \tilde{m}_{ia}$. Aggregate excess demand for the economy is denoted $\pi_i = (z_i; \eta_i)$. The compactness, convexity and continuity of π_{ia} when $p_i^1 \in \Delta^{l-1}$ follow from the compactness and convexity of $y_{ia}^i \in Y_{ia}^i$ $i=1,2$ and the properties of the von Neumann-Morgenstern utility function summarised in lemma 1. To prove that property v hold in this OLG model, and hence that π_{ia} is upper hemicontinuous, observe that we would expect:

- i. As $p_i^1 \rightarrow \partial\Delta^{l-1}$ s.t. $p_n \rightarrow 0$ that $z_i^n \rightarrow \infty$, i.e. the excess demand for the n^{th} good increases indefinitely as its price approaches zero due to lemma 1 part iii.
- ii. As $p_i^1 \rightarrow R_+^l \setminus \Delta^{l-1}$ that $\eta_i \rightarrow \infty$, i.e. the excess demand for money increases indefinitely when the price level tends to infinity due to lemma 1 part iii.

Lemma 2 follows from Grandmont (1974), which states a necessary condition for the above boundary conditions to hold.

Lemma 2. If the set of functions $\{p_{ia}^2(p_i^1) | p_i^1 \in \Delta^{l-1}\}$ is compact in the topology of weak convergence and $\zeta_{ia}^1 > 0$ then $\pi_i \rightarrow \infty$ if either $p_i^1 \rightarrow \partial\Delta^{l-1}$ or $p_i^1 \rightarrow R_+^l \setminus \Delta^{l-1}$.

Proof. See Mathematical Appendix for details

In the proof of existence of temporary equilibria found in Arrow and Hahn (1971) the existence of the period 1 utility function is assumed. Lemma 2 shows that this assumption is equivalent to assuming that at least one agent with a positive endowment of all goods have bounded expectations. The requirement that expectations be bounded means that the range of variation in this trader's expectations must not be too large when p_i^1 varies in Δ^{l-1} . Furthermore, the requirement $\zeta_{ia}^1 > 0$ ensures that the agent can trade when $p_i^1 \in \partial\Delta^{l-1}$. Finally, observe that the upper hemicontinuity of π_{ia} follows from lemma 2.

A convenient approach to solving the agent's maximization problem involves converting it to a two-stage problem. For any $y_{ia}^1 \in Y_{ia}^1$ given some $p_i^2 \in P_i^2$ maximize $u_a(\hat{y}_{ia}(y_{ia}^1, p_i^2))$ with respect to \hat{x}_{ia}^2 subject to $p_i^2 \hat{x}_{ia}^2 \leq p_i^2 \zeta_{ia}^2 + m_{ia}^1$. The period 1 conditional utility function is then $u_a^1(y_{ia}^1 | p_i^2) = \max u_a(\hat{y}_{ia}(y_{ia}^1, p_i^2))$ subject to $p_i^2 \hat{x}_{ia}^2 \leq p_i^2 \zeta_{ia}^2 + m_{ia}^1$. The conditionality in the period-1 utility function is removed by defining the von Neumann-Morgenstern utility, the properties of which is summarised in lemma 1:

$$u_a^1(y_{ia}^1, p_i^1) = \int_{P_i^2} u_a^1(y_{ia}^1 | p_i^2) dp_{ia}^2(p_i^1, p_i^2)$$

Lemma 1. u_a^1 , for $a=1, \dots, d$, has the following properties:

1. u_a^1 is continuous on $Y_{ia}^1 \times P_i^1$.
2. Given $p_i^1 \in \Delta^{l-1}$, u_a^1 is non-satiated.
3. Given $p_i^1 \in \Delta^{l-1}$, u_a^1 is semi-strict quasi-concave.

Proof. See Mathematical Appendix for details

Using this approach, and combining (b.1) and (b.2), the maximisation problem becomes:

(c.1) Maximise $u_a^1(y_{ia}^1, p_i^1)$ subject to:

- i. $x_{ia} \geq 0$;
- ii. $m_{ia}^1 \geq 0$
- iii. $p_i^1 x_{ia}^1 + m_{ia}^1 \leq p_i^1 \zeta_{ia}^1 + \tilde{m}_{ia}$

Given the Arrow and Debreu (1954) framework used here, this maximisation exists for all $p_i^1 \in \Delta^{l-1}$ where the aggregate excess demand function is continuous. Consider the aggregate excess demand function of agent a in the pure exchange economy considered in section 4.2 above, i.e. $z_a : R^l \rightarrow R$. It can be shown that for any regular trader with $p \in \Delta^{l-1}$ that z_a is; i) compact, ii) convex, iii) upper hemicontinuous, iv) $p \cdot z_a = 0 \forall p$ and v) if $p \rightarrow \partial \Delta^{l-1}$ then $z_a \rightarrow \infty$. Note that $\partial \Delta^{l-1}$ and Δ^{l-1} represents the boundary and interior of the unit simplex respectively. See for example Mas-Colell (1995) proposition

(a.6) Let $\tilde{m}_a \geq 0$ such that the total money stock $M = \sum_a \tilde{m}_a > 0$.

For each $p_t^1 \in \Delta^{l-1}$ the trader's expectation will take the form of a probability measure defined on P_t^2 . This idea is represented by the map $p_{ta}^2 : p_t^1 \rightarrow \wp(P_t^2, \beta(P_t^2))$ where:

- i. $\beta(P_t^2)$ is the Borel σ -algebra of P_t^2 ,
- ii. $\wp(P_t^2, \beta(P_t^2))$ is the set of probability measures on the measure space $(P_t^2, \beta(P_t^2))$.

The definition of the agent's expectation function is completed by letting $p_{ta}^2(p_t^1, p_t^2)$ denote the probability assigned by agent a of $p_t^2 \in P_t^2$ occurring in period $t+1$ given p_t^1 in the current period. This leads to the following assumption:

(a'.7) The mapping $p_{ta}^2 : p_t^1 \rightarrow \wp(P_t^2, \beta(P_t^2))$ is continuous.

Consider the maximization problem of agents with $n_a = 2$. Let a denote such an agent and note that he must choose his current consumption of goods $x_a^1 \geq 0$, current money holdings $m_a^1 \geq 0$, and to plan future consumptions $x_a^2 \geq 0$ and money holdings $m_a^2 \geq 0$ subject to the budget constraints:

$$(b.1) \quad p_t^1 x_a^1 + m_a^1 \leq p_t^1 \zeta_a^1 + \tilde{m}_a, \text{ and}$$

$$(b.2) \quad \tilde{p}_{ta}^2 x_a^2 + m_a^2 \leq \tilde{p}_{ta}^2 \zeta_a^2 + m_a^1$$

Set $m_a^2 = 0 \forall a \text{ \& } t$. This is justified, as there is no bequeath in this OLG model.

For any $y_a^1 \in Y_a^1$ given some $p_t^2 \in P_t^2$ define the set of feasible consumption vectors for agent a 's as:

$$\hat{Y}_{ta}(y_a^1, p_t^2) = \{(x_a^1, x_a^2) \mid x_a^2 \in X_a^2, p_t^2 x_a^2 \leq p_t^2 \zeta_a^2 + m_a^1\}.$$

It follows from (a.1) that Y_a^1 is a closed convex subset of R^{l+1} which is bounded and therefore compact and convex.

Since there is no production in this intertemporal pure exchange economy the analogue to assumptions (a.1) to (a.6) is:

(a.1) X_{ta} is a closed convex subset of $R^{1 \times 2}$ which is bounded from below, i.e.
 $\xi_a \leq x_{ta} \forall x_{ta} \in X_{ta}, \forall a$.

(a.2) For some $x_{ta}^i \in X_{ta}^i, x_{ta}^i \leq \zeta_{ta}^i, \forall a \& i$.

At time t an agent makes a decisions about the triple $x_{ta}^1, m_{ta}^1, x_{ta}^2$. Define agents a 's period-1 action set as $y_{ta}^1 \in Y_{ta}^1 = X_{ta}^1 \times R_+$ where $y_{ta}^1 = (x_{ta}^1, m_{ta}^1) i=1,2$ and let $\bar{Y}_{ta} = Y_{ta}^1 \times X_{ta}^2$.

(a.3) u_a , for $a=1, \dots, d$, has the following properties:

1. u_a is a continuous C^1 function on $y_{ta} \in Y_{ta} \forall a$.
2. For every $y_{ta} \in Y_{ta}$, there is an $y'_{ta} \in X_{ta}$ such that $u_a(y'_{ta}) > u_a(y_{ta})$.
3. If $u_a(y_{ta}) > u_a(y'_{ta})$ and $0 < \varepsilon < 1$, then $u_a(\varepsilon y_{ta} + (1-\varepsilon)y'_{ta}) \geq u_a(y'_{ta})$.

Assumption (a.3.2) extends the static equivalent by ensuring that non-satiation exists between time periods as well as within the same period for agents with life expectancy greater than one year. Without this assumption agents would have no demand for money in period one and the period two optimisation problem would be equivalent to that of a pure exchange economy without money.

(a.4) Let $X_t = \sum_a X_{ta}$ and $\zeta_t = \sum_a \zeta_{ta}$ then there exists $x_t \in X_t$ such that $x_t < \zeta_t, \forall t$.

(a.5) Let δ^h be the positive unit vector of the h^{th} axis in R^l . Then, for any $\lambda > 0$, $x_{ta}^i + \lambda \delta^h$ represents an increase of λ in the amount of the h^{th} commodity over x_{ta}^i , with all other commodities remaining unchanged in consumption. Let H be the set of commodities such that for every $a \& i=1,2, x_{ta}^i \in Y_{ta}^i$ and $h \in H$, then there exists $\lambda > 0$ such that $x_{ta}^i + \lambda \delta^h \in X_{ta}^i$ and $u_a(x_{ta}^1 + \lambda \delta^h, x_{ta}^2 + \lambda \delta^h) > u_a(x_{ta}^1, x_{ta}^2), \forall a \& i$.

- v. The agent's initial money stock is denoted by \tilde{m}_{ia} , which is the result of past saving and consumption decisions. Denote the money stock at the end of period i that results from agent a 's savings and consumption decisions by m_{ia}^i .

From iii above it is clear that we have made the simplifying assumption that the agent's real income is fixed in each period in the form of an exogenously given endowment of consumption goods. These endowments cannot be stored, and must be traded and consumed within the period during which they are available. Paper money is the only store of value, and its stock is constant over time. The trader therefore comes to the market at each date with his endowment of goods and non-negative cash balances carried over from the past. The short-run competitive equilibrium of the markets at that date will determine the Walrasian money prices of goods $p_t^1 = (p_{t1}^1, \dots, p_{tn}^1) \in \Delta^{l-1}$, the consumers' net trades in the good markets and the non-negative money balances they will hold until the next period. Formally, redefine condition (c'.2) to represent the current period price system in the OLG model, i.e.:

$$(c.2) P_t^1 = \left\{ p_t^1 \mid p_t^1 \in R^l, p_t^1 \geq 0, \sum_{h=1}^l p_{th}^1 = 1 \right\}.$$

In order to specify the traders short-run behaviour it is necessary to specify how future price expectations are formed. We assume that the agent's expectations are a function of his information on past history and on the current state of the economy, which in period t is represented by the set Ω_t . Furthermore, we assume that the only information an agent has on the current state of the economy is described by the current price system p_t^1 .

Agent a 's price expectations for period 2 is expressed as a continuous function of current price, i.e. $\tilde{p}_{ia}^2 = \tilde{p}_{ia}^2(p_t^1)$ where $P_t^2 = \{p_t^2 \in R^{l-1} \mid p_t^2 \geq 0\}$ if $n_a = 2$. This definition does not make the influence of Ω_t explicit as past history is fixed in the short-run and cannot be altered by current events. Expected prices are therefore independent of the agent's own actions, which is warranted in a large competitive economy where every agent can only have a negligible influence on market prices by varying his own decisions.

of interest from the perspective of time t , e.g. x_{ta}^2 indicates the second period consumption plan of agent a at time t .

Consider an infinite horizon overlapping generations economy without bequest consisting of a successive number of generations at discrete points in time, i.e. $t = 0, 1, \dots, \infty$. There are l consumption goods available in each period.

The analysis is concerned with the short-run properties of the model in a given period t . We therefore only have to know the characteristics of every agent a , $a = 1, \dots, d_t$, living in the period under examination, they are:

- i. The number $n_a = [1, 2]$ of remaining periods for which the agent is going to live, including the current one. In each period t there are d_t^1 new agents born and d_{t-1}^1 old agents such that $d_t = d_t^1 + d_{t-1}^1$. The population is assumed to be constant over time such that the number of old equals the number of newborn, i.e. $d = d_t \forall t$.
- ii. The set of all consumption vectors available to agent a in period $i = 1, 2$ is denoted X_{ta}^i . Let the set of possible 2-period consumption vectors for agent a be denoted $X_{ta} = X_{ta}^1 \times X_{ta}^2$. Let the set of possible consumption allocations for the economy be denoted by $X_t = X_t^1 \times X_t^2$ where $X_t^i = \times_a X_{ta}^i$ $i = 1, 2$.
- iii. The agent's preferences, represented by the utility function u_a which depends upon current and future consumption x_{ta}^i , $i = 1, 2$, where x_{ta}^i is a vector with l nonnegative components.
- iv. The agent's endowment of consumption goods, ζ_{ta}^i , in every remaining period of his or her life, $i = 1, 2$. Let the set of possible 2-period endowment vectors for household a be denoted $\zeta_{ta} = \zeta_{ta}^1 \times \zeta_{ta}^2$ and let the set of possible endowment allocations for the economy be denoted by $\zeta_t = \times_a \zeta_{ta}$.

the simpler version in this analysis too ease exposition. The results of the analysis are however not influenced in any way by this simplification.

The basis of the proof found in Arrow and Debreu (1954) is to show that an economic model satisfying assumptions (a'1) to (a'5) is equivalent to the abstract economy $E^{A\&D} = [X_1, \dots, X_d, u_1, \dots, u_d, A_1(\bar{x}_1), \dots, A_d(\bar{x}_d), P]$ where:

- i. $X_i \forall i$ is a compact and convex space of possible strategies,
- ii. u_i is continuous on X_i and quasiconcave in x_i for every \bar{x}_i ,
- iii. $\bar{x}_i \in \{X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_{i-1} \times P\}$ represents the strategies chosen by all agents other than i , and
- iv. $A_i(\bar{x}_i) = \{x_i | x_i \in X_i, p \cdot x_i \leq p \cdot \zeta_i\}$ is a continuous function whose graph is a closed set, and for every \bar{x}_i , the set $A_i(\bar{x}_i)$ is convex and non-empty. $A_i(\bar{x}_i)$ is the domain from which agent i can select his strategies given the strategies chosen by other agents.

In order to prove existence of equilibrium for this abstract economy Arrow and Debreu (1954) use a generalization of Nash's theorem on the existence of equilibrium points for games (see Nash (1951)). The generalised result will not be illustrated here. The interested reader is referred to Arrow and Debreu (1954) pp 273 to 274 for a precise definition and discussion.

4.3 Model Specification

We consider the short-run properties of an infinite horizon overlapping generation model in which agents only live for two periods. This specification allows one to apply general equilibrium analysis to a sequence of markets, where at each date each economic unit makes decisions according to his expectations of the future. This allows one to explicitly study the impact of each agent's expectations on the competitive equilibrium.

Before developing the model a brief elaboration on the notation adopted is required. The current time period is indicated by subscript t whilst the superscript i indicates the period

(c'.1) Maximise $u_a(x_a)$ with respect to x_a over the set $x_a | x_a \in X_a, p \cdot x_a \leq p \cdot \zeta_a$

The price system of the economy is represented by the set:

$$(c'.2) P = \left\{ p \mid p \in R^l, p \geq 0, \sum_{h=1}^l p_h = 1 \right\}.$$

Note that the price vector p has been normalised such that the sum of its coordinates equal 1. The set P is therefore equivalent to the unit simplex Δ^{l-1} . This does not affect the generality of the result, as all relations are homogenous of degree 1 in p . Also, c'.2 implies that P is compact and convex.

The market for any commodity is in equilibrium when total supply for that commodity equals the total demand; however, allowance must be made for the possibility that at a zero price, supply will exceed demand. Condition c.3 deals with this issue:

$$(c'.3) z^* \leq 0, p^* \cdot z^* = 0,$$

where p^* and z^* denotes equilibrium price and aggregate excess demand respectively.

Condition c'.3 above captures the dynamics of the law of supply and demand in a static function, i.e. prices rise if demand exceeds supply, and fall if supply exceeds demand. Observe that equilibrium is incompatible with excess demand, since prices would simply rise, which gives rise to the first part of condition c'.3. If we have excess supply of commodity h , $\forall h \in \{1, \dots, l\}$ then observe that p^*_h cannot fall below zero, such that either $z^*_h = 0$ or $z^*_h < 0$ and $p^*_h = 0$ in order for the second part of condition c'.3 to hold.

A set of vector $(x_1^*, \dots, x_d^*, p^*)$ is a competitive equilibrium if it satisfies conditions c'.1, c'.2 and c'.3.

Proposition 1. A competitive equilibrium exists for any economic system satisfying assumptions a'.1 to a'.5.

The economy described above is the pure exchange equivalent to the more general case that includes production considered by Arrow and Debreu (1954). I have chosen to use

(a'.2) The agent's preferences are described by the utility function u_a for $a=1, \dots, d$, which has the following properties:

1. u_a is a continuous function on X_a ;
2. For every $x_a \in X_a$, there is an $x'_a \in X_a$ such that $u_a(x'_a) > u_a(x_a)$;
3. If $u_a(x_a) > u_a(x'_a)$ and $0 < t < 1$, then $u_a(tx_a + (1-t)x'_a) \geq u_a(x'_a)$;

a'.2.2 assumes that there is no point of saturation, i.e. there is no consumption vector that the individual would prefer to all others. a'.2.3 assumes that the indifference surfaces are convex in the sense that the set $\{x_a \mid x_a \in X_a \cap u_a(x_a) \geq \alpha\} \forall \alpha \in R$ is a convex set.

(a'.3) Let the agent's endowment of consumption goods be denoted by $\zeta_a \in R^l$ for $a=1, \dots, d$, then for some $x_a \in X_a$, $x_a \leq \zeta_a$. This assumption states that every individual could consume out of his initial stock in some feasible way up to the limit of his endowment.

(a'.4) Let $X = \sum_a X_a$ and $\zeta = \sum_a \zeta_a$ then there exists $x \in X$ such that $x < \zeta$. This assumption states that it is possible to arrange the economic system so that an excess supply of all commodities can be achieved by choosing an appropriate consumption vector.

(a'.5) Let δ^h be the positive unit vector of the h^{th} axis in R^l . Then, for any $\lambda > 0$, $x_a + \lambda\delta^h$ represents an increase of λ in the amount of the h^{th} commodity over x_a , with all other commodities remaining unchanged in consumption. Let H be the set of commodities such that for every $a, x_a \in X_a$ and $h \in H$, then there exists $\lambda > 0$ such that $x_a + \lambda\delta^h \in X_a$ and $u_a(x_a + \lambda\delta^h) > u_a(x_a)$. The set H is not empty. H represents the set of all commodities that are always desired by every consumer.

The basic economic motivation of agents in the economy is that of maximising utility among all consumption vectors that satisfy their budget constraint. This principal is captured in the following condition:

in the real world. Specifically, the rational expectations hypothesis ignores a priori the short-run rigidities in expectations that may result from traders' learning processes in the real world.

Finally, all mathematical proofs have been relegated to the mathematical appendix to facilitate reading.

4.2 The Arrow and Debreu Existence Theorem

The core idea behind the proof of the existence theorem (i.e. theorem 1) for the model specified in section 4.3 is the observation that at least one trader has bounded expectations. This observation is proved by examining the behaviour of the aggregate excess demand function in the context of the Arrow and Debreu existence theorem 2 (1954). Specifically, the behaviour of excess demand functions are examined when prices approach the boundary of the unit simplex that represents the set of admissible prices.

Theorem 2 of Arrow and Debreu (1954) is restated as proposition 1 below, along with the assumptions and conditions that underlie their existence theorem; the interested reader is referred to the classic paper by Arrow and Debreu (1954) for a proof of the result. This approach may appear overly pedantic, but allows the technical reader to appreciate the evolution of assumptions needed to support the general existence theorem for the static economy in theorem 2 of Arrow and Debreu (1954) to the assumptions needed to support the general existence theorem for the OLG economy specified in section 4.3.

A brief note on notation is required before proceeding. If x & $y \in R^n$ then $x \geq y$ means $x_n \geq y_n \forall n$ and $x_n > y_n$ for some n , and $x > y$ means $x_n > y_n \forall n$.

Consider an economy that consists of $a = 1, \dots, d$ agents that can trade in l consumption goods for which the following assumptions hold:

(a'.1) X_a denotes the set of all consumption vectors available to agent a , $a = 1, \dots, d$, where X_a is a closed convex subset of R^l which is bounded from below, i.e.

$$\xi_a \leq x_a \forall x_a \in X_a;$$

Chapter 4 - Short-run Equilibrium Under Uncertainty

4.1 Introduction

Chapter 3 showed that greater risk aversion does not necessarily imply a larger risk premium in an overlapping generations model. The reason for this phenomenon is related to the relative curvature of the utility function that gives rise to the income and substitution effects that are possible in the OLG model. This result means that changes in relative risk aversion do not necessarily explain the excess market volatility observed by some authors (see chapter 2) if an OLG model is used to price assets.

This chapter investigates the intertemporal substitution effect further within the framework of an exchange economy. This analysis is conducted by establishing, under very general conditions, the existence of a short-run equilibrium in an infinite horizon overlapping generations economy in which agents have diverse expectations. This is achieved by first outlining the Arrow and Debreu existence theorem in section 4.2. Section 4.3 then analyses the conditions needed for this theorem to apply to an exchange economy in which agents have diverse expectations.

Although the existence theorem proved in section 4.3 is very general, its implications are illustrated through a series of simple examples. Section 4.4 illustrates the intertemporal substitution effect for a simple one good economy. Section 4.5 illustrates that a strong intertemporal substitution effect is a prerequisite for the existence of a short-run Walrasian monetary equilibrium in a simple one good economy. It shows that this equilibrating mechanism requires that the price expectations of at least one agent are insensitive to current prices.

The model and results used in this chapter can be found in chapter 1 of Grandmont (1983) and in Grandmont (1974). The results shown here differ from Grandmont's original work to the extent that a more general existence theorem is developed that does not exclude the boundary points of the simplex of prices that characterise the economy. The basic thrust of the argument is the same; the results obtained here can be used to question the ability of the rational expectations hypothesis to describe investor behaviour

where:

$$\bar{R}_{m,t}^{LUC}(k_i; \lambda_t) = \int_E R_{m,t}^{LUC}(k_i; \lambda_t; \lambda_{t+1}) dF(\lambda_{t+1})$$

⇒ Rearranging and simplifying yields:

$$\pi^{LUC}(k_i; \lambda_t) = [1 + R_{rf}^{LUC}(k_i; \lambda_t)] \times \text{cov}[-s_t^{LUC}; R_{m,t}^{LUC}(k_i; \lambda_t; \lambda_{t+1}) | (k_i; \lambda_t)] \quad (7.a)$$

where:

$$\pi^{LUC}(k_i; \lambda_t) = \bar{R}_{m,t}^{LUC}(k_i; \lambda_t) - R_{rf}^{LUC}(k_i; \lambda_t)$$

3.7.5 Derivation of Equation (7.b)

⇒ Note that $1 + R_{m,t}^{OLG}(k_i; \lambda_t; \lambda_{t+1}) = \frac{p_z(k_{t+1}; \lambda_{t+1})}{p_l(k_{t+1}; \lambda_{t+1})}$

⇒ Rewriting equation (4) yields:

$$1 = \int_E s_t^{OLG} (1 + R(k_i; \lambda_t; \lambda_{t+1})) dF(\lambda_{t+1})$$

⇒ Expanding equation (6) yields:

$$1 = \left[\int_E s_t^{OLG} dF(\lambda_{t+1}) \right] \times \left[\int_E (1 + R_{m,t}^{OLG}(k_i; \lambda_t; \lambda_{t+1})) dF(\lambda_{t+1}) \right] + \text{cov}[s_t^{OLG}; R_{m,t}^{OLG}(k_i; \lambda_t; \lambda_{t+1}) | (k_i; \lambda_t)]$$

⇒ Substituting in equation (5.b) yields:

$$\frac{1 + \bar{R}_{m,t}^{OLG}(k_i; \lambda_t)}{1 + R_{rf}^{OLG}(k_i; \lambda_t)} = 1 - \text{cov}[s_t^{OLG}; R_{m,t}^{OLG}(k_i; \lambda_t; \lambda_{t+1}) | (k_i; \lambda_t)]$$

where:

$$\bar{R}_{m,t}^{OLG}(k_i; \lambda_t) = \int_E R_{m,t}^{OLG}(k_i; \lambda_t; \lambda_{t+1}) dF(\lambda_{t+1})$$

⇒ Rearranging and simplifying yields:

$$\pi^{OLG}(k_i; \lambda_t) = [1 + R_{rf}^{OLG}(k_i; \lambda_t)] \times \text{cov}[-s_t^{OLG}; R_{m,t}^{OLG}(k_i; \lambda_t; \lambda_{t+1}) | (k_i; \lambda_t)] \quad (7.b)$$

where:

$$\pi^{OLG}(k_i; \lambda_t) = \bar{R}_{m,t}^{OLG}(k_i; \lambda_t) - R_{rf}^{OLG}(k_i; \lambda_t)$$

Case 2: $\hat{i}_0 \rightarrow \hat{w}_0$

Note the $\hat{\cdot}$ notation is dropped in this section for notational expediency.

i. $\lim_{i \rightarrow \infty} -u(w-i) = -\infty$ by assumption 4;

ii. Since $0 < k_0 < \infty \rightarrow 0 < w_0 < \infty$ we have $\beta \int_E v'(i_0, \lambda_1 q'(i_0)) \times [\lambda_1 q'(i_0)] \times dF(\lambda_1) > 0$;

iii. Since $0 < k_0 < \infty \rightarrow 0 < w_0 < \infty$ we have $-\infty < q''(w_0) < 0$ by assumption 2, thus we have $-\infty < \beta \int_E v'(i_0, \lambda_1 q'(i_0)) \times [i_0 \lambda_1 q''(i_0)] \times dF(\lambda_1) < 0$ since $\lim_{i \rightarrow w} v'(iq'(i)) > 0$ by assumption 4;

By combining i to iii above we see that the L.H.S of $\frac{\partial V}{\partial i_0} = 0$ becomes negative as $i_0 \rightarrow w_0$;

\Rightarrow By the continuity property of all the partial derivatives we see that the solution to (P.b.3), denote it $\hat{i}_0(\lambda_0)$, satisfies $0 < \hat{i}_0(\lambda_0) < \hat{w}_0$;

\Rightarrow When \hat{i}_0 is determined, we have $\hat{k}_1 = \hat{i}_0$. Hence the next periods wage $\hat{w}_1 = \lambda_1 q(\hat{k}_1) - \lambda_1 \hat{k}_1 q'(\hat{k}_1)$ is given after observing the production shock λ_1 . and the same process will yields $\hat{k}_2 = \hat{i}_1$;

\Rightarrow This way we construct step by step the competitive equilibrium;

3.7.4 Derivation of Equation (7.a)

\Rightarrow Note that $1 + R_{m,t}^{LUC}(k_t; \lambda_t; \lambda_{t+1}) = \frac{p_z(k_{t+1}; \lambda_{t+1})}{p_i(k_{t+1}; \lambda_{t+1})}$

\Rightarrow Expanding equation (6) yields:

$$1 = \left[\int_E s_t^{LUC} dF(\lambda_{t+1}) \right] \times \left[\int_E (1 + R_{m,t}^{LUC}(k_t; \lambda_t; \lambda_{t+1})) dF(\lambda_{t+1}) \right] + \text{cov} \left[s_t^{LUC}; R_{m,t}^{LUC}(k_t; \lambda_t; \lambda_{t+1}) \mid (k_t; \lambda_t) \right]$$

\Rightarrow Substituting in equation (5.a) yields:

$$\frac{1 + \bar{R}_{m,t}^{LUC}(k_t; \lambda_t)}{1 + R_{r,t}^{LUC}(k_t; \lambda_t)} = 1 - \text{cov} \left[s_t^{LUC}; R_{m,t}^{LUC}(k_t; \lambda_t; \lambda_{t+1}) \mid (k_t; \lambda_t) \right]$$

⇒ We will solve the problem for the young generation at time $t = 0$ for given

$$0 < \hat{k}_0 < \infty;$$

⇒ The first order condition for (P.b.3) is:

$$\frac{\partial V}{\partial i_0} = 0 \rightarrow -u'(\hat{w}_0 - \hat{i}_0) + \beta \int_E v'(\hat{i}_0 \lambda_1 q'(\hat{i}_0)) \times [\lambda_1 q'(\hat{i}_0) + \hat{i}_0 \lambda_1 q''(\hat{i}_0)] \times dF(\lambda_1) = 0$$

⇒ Before proceeding we need to show that $0 < \hat{k}_0 < \infty \rightarrow 0 < \hat{w}_0 < \infty$

→ We use the fact that $\hat{w}_0 = \hat{c}_0^1 + \hat{i}_0 = \lambda_0 q(\hat{k}_0) - \lambda_0 \hat{k}_0 q'(\hat{k}_0)$;

→ Consider the ratio $\frac{q(\hat{k}_0)}{q'(\hat{k}_0)}$, from assumption 2 we have that; i) $\lim_{k_0 \rightarrow 0^+} \frac{q(k_0)}{q'(k_0)} = 0$,

$$\text{and; ii) } \lim_{k_0 \rightarrow \infty} \frac{q(k_0)}{q'(k_0)} = \infty$$

→ Thus we have that $0 < \hat{k}_0 < \infty \rightarrow 0 < \hat{w}_0 < \infty$;

⇒ We now consider the behaviour of the l.h.s. of $\frac{\partial V}{\partial i_0} = 0$ when $\hat{i}_0 \rightarrow 0$ and $\hat{i}_0 \rightarrow \hat{w}_0$

Case 1: $\hat{i}_0 \rightarrow 0$

Note the \wedge notation is dropped in this section for notational expediency.

i. If $\lim_{i \rightarrow 0} iq'(i) = 0$ then $\lim_{i \rightarrow 0} v'(iq'(i)) = \infty$ by assumption 4;

ii. If $\lim_{i \rightarrow 0} iq'(i) \neq 0$ then $\lim_{i \rightarrow 0} v'(iq'(i)) > 0$ by assumption 4, but note that

$$\lim_{i \rightarrow 0} v'(iq'(i)) \times q'(i) = \infty \text{ by assumption 2;}$$

iii. Since $0 < k_0 < \infty \rightarrow 0 < w_0 < \infty$ we have $-\infty < -u'(w_0) < 0$ by assumption 4;

iv. Note that $\lim_{i \rightarrow 0} iq''(i) \neq -\infty$ for $q(\cdot)$ a smooth continuous function, we therefore have

$$\text{that } -\infty < \lim_{i \rightarrow 0} iq''(i) < 0;$$

By combining i to iv above we see that the L.H.S of $\frac{\partial V}{\partial i_0} = 0$ becomes positive as

$$\hat{i}_0 \rightarrow 0;$$

$$\Rightarrow \frac{\partial L}{\partial l_t} = 0 \rightarrow \vartheta_1 p_l(k_t; \lambda_t) = \vartheta_2;$$

$$\Rightarrow \frac{\partial L}{\partial z_t} = 0 \rightarrow \vartheta_1 p_z(k_t; \lambda_t) = \vartheta_3;$$

3.7.3 Proof of Theorem 1

\Rightarrow From section 3.3.1 we have that in equilibrium: i) firm profits will be zero, ii) individuals will supply one unit of labour inelastically and iii) individuals will supply k_t units of capital inelastically such that

$$\hat{c}_t^1 + \hat{c}_t^2 + \hat{i}_t = q(\hat{k}_t) \lambda_t, \text{ where } \hat{\cdot} \text{ denotes equilibrium quantities;}$$

\Rightarrow Note that $\hat{c}_{t+1}^2 = p_z(\hat{k}_{t+1}; \lambda_{t+1}) \hat{z}_{t+1} = \lambda_{t+1} q'(\hat{i}_t) \hat{k}_t$, which follows from (f.2) and from individuals supplying $\hat{k}_{t+1} = \hat{i}_t$ units of capital inelastically in equilibrium;

\Rightarrow Define $\hat{w}_t = \hat{c}_t^1 + \hat{i}_t = \lambda_t q(\hat{k}_t) - \lambda_t \hat{k}_t q'(\hat{k}_t)$ as the equilibrium wage paid to the young at time t ;

\Rightarrow We can use the above to rewrite the equilibrium problem in the following format:

$$V(\hat{k}_t; \lambda_t) = \max \{ u(\hat{c}_t^1) + \beta E_t [v(\hat{c}_{t+1}^2)] \} \text{ subject to:}$$

- i. $\hat{w}_t = \hat{c}_t^1 + \hat{i}_t$
- ii. $\hat{c}_{t+1}^2 = \lambda_{t+1} q'(\hat{i}_t) \hat{k}_t$
- iii. $\hat{c}_t^1 \geq 0$
- iv. $\hat{c}_t^2 \geq 0$

\Rightarrow By substituting constraint i and ii into the maximization problem and noting that $0 \leq \hat{i}_t \leq \hat{w}_t$ as a result of iii, we get the following simplification:

$$V(\hat{k}_t; \lambda_t) = \max \{ u(\hat{w}_t - \hat{i}_t) + \beta E_t [v(\lambda_{t+1} \hat{i}_t q'(\hat{i}_t))] \} \quad (\text{P.b.3})$$

subject to $0 \leq \hat{i}_t \leq \hat{w}_t$ for $t = 0, 1, 2, \dots$

\Rightarrow We have reduced the problem to finding the sequence $\{\hat{i}_t\}_{t=0}^{\infty}$ that maximises the above problem. Thus, the existence of a competitive equilibrium follows from a sequence of generational optimisation problems.

3.7 Mathematical Appendix

3.7.1 Derivation of the First Order Conditions for Problem (P.a)

⇒ Form the following Lagrangian:

$$L = \{p_c(k_i; \lambda_i)c_i + p_i(k_i; \lambda_i)i_i - p_z(k_i; \lambda_i)z_i - p_l(k_i; \lambda_i)l_i\} + \vartheta\{c_i + i_i - l_i, q(z_i/l_i)\lambda_i\}$$

⇒ Now solve $\partial L = 0$ for each element of y_i ;

$$\Rightarrow \frac{\partial L}{\partial c_i} = 0 \rightarrow p_c(k_i; \lambda_i) = \vartheta;$$

$$\Rightarrow \frac{\partial L}{\partial i_i} = 0 \rightarrow p_i(k_i; \lambda_i) = \vartheta;$$

$$\Rightarrow \frac{\partial L}{\partial z_i} = 0 \rightarrow p_z(k_i; \lambda_i) = \vartheta\lambda_i q'(z_i/l_i);$$

$$\Rightarrow \frac{\partial L}{\partial l_i} = 0 \rightarrow p_l(k_i; \lambda_i) = \vartheta\lambda_i [q(z_i/l_i) - (z_i/l_i)q'(z_i/l_i)]$$

⇒ Solving for ϑ yields:

i. $p_c(k_i; \lambda_i) = p_i(k_i; \lambda_i)$

ii. $p_z(k_i; \lambda_i) = \lambda_i p_c(k_i; \lambda_i) q'(z_i/l_i)$

iii. $p_l(k_i; \lambda_i) = \lambda_i p_c(k_i; \lambda_i) [q(z_i/l_i) - (z_i/l_i)q'(z_i/l_i)]$

3.7.2 Derivation of the First Order Conditions for Problem (P.b)

⇒ Form the following Lagrangian:

$$L = u(c_i^1) + \beta \int_E v(p_z(k_{t+1}; \lambda_{t+1})z_{t+1}) dF(\lambda_{t+1}) \\ + \vartheta_1(p_z(k_i; \lambda_i)z_i + p_l(k_i; \lambda_i)l_i - p_c(k_i; \lambda_i)c_i - p_i(k_i; \lambda_i)i_i) \\ + \vartheta_2(1 - l_i) + \vartheta_3(k_i - z_i)$$

⇒ Now solve $\partial L = 0$ for each element of y_i ;

$$\Rightarrow \frac{\partial L}{\partial c_i} = 0 \rightarrow u'(c_i^1) = \vartheta_1 p_c(k_i; \lambda_i);$$

$$\Rightarrow \frac{\partial L}{\partial i_i} = 0 \rightarrow \beta \int_E v(p_z(k_{t+1}; \lambda_{t+1})z_{t+1}) p_z(k_{t+1}; \lambda_{t+1}) dF(\lambda_{t+1}) = \vartheta_1 p_i(k_i; \lambda_i), \text{ since}$$

$$z_{t+1} \leq k_{t+1} = i_i, \text{ and } z_{t+1} = k_{t+1} = i_i \text{ in equilibrium;}$$

learning process that aims to show how agents learn what they know when formulating their rational expectations. We explore this issue in chapter 7 where we show that these results have to be interpreted with care, as the convergence of a *particular* learning process to a particular unique solution does not guarantee that learning behaviour will generally lead agents to the same solution.

Finally, scrutiny of the OLG asset-pricing model used in this chapter shows that the intertemporal substitution effect is also influenced by an investor's expectations. The role of expectations in the intertemporal substitution effect, and hence the price formation process, is explored further in chapter 4.

3.6 Summary Remarks and Conclusion

In this chapter we have shown that greater risk aversion does not necessarily imply a larger risk premia in an overlapping generations model. We showed that the reason for this phenomenon is related to the intertemporal marginal rate of substitution and relative curvature of the utility function that gives rise to the income and substitution effects which are possible in the OLG model, but ruled out in the infinite-lived Lucas asset-pricing model by its construction. This result means that changes in relative risk aversion do not necessarily explain the excess market volatility observed by some authors (see chapter 2) if an OLG model is used to price assets. It is acknowledged in chapter 2 that the issue of excessive market volatility has not been conclusively dealt with for reasons discussed in the analysis.

Both the Lucas and the OLG asset-pricing models used in this chapter determine prices on the basis of consumer preference, initial endowments and production sets. These factors (i.e. preference, endowments and production sets) represent the intrinsic structure of the economy. Price uncertainty within this framework is a reflection of randomness in the underlying intrinsic structure itself, and is not affected by events unrelated to economic fundamentals.

In chapter 6 we extend the basic general equilibrium model introduced in this chapter to one with random prices and no intrinsic uncertainty. Price randomness in this extension of the model will be entirely due to the views and beliefs held by individuals about their environment. These views and beliefs refer to a variety of events or psychological factors that are unrelated to the intrinsic structure of the economy, yet, which influence the forecasts and actions of economic decision-makers.

The analysis of this chapter is based on the rational expectations hypothesis. This hypothesis is based on the premise that agents know a great deal about their environment. Agents are assumed to have knowledge about demand and supply functions, how to extract present and future general equilibrium prices and about the stochastic law of motion of the economy over time. In practice it is hard to conceive how agents come to possess all this knowledge. Recent literature has focused on formulating a dynamic

and these two effects cannot be separated. We therefore examine how $\pi^{LUC}(k_t; \lambda_t)$ and $\pi^{OLG}(k_t; \lambda_t)$ behaves when the level of relative risk aversion changes by considering the effect of a change in λ_{t+1} on (7.a) and (7.b).

Both $R_{m,t}^{LUC}$ and $R_{m,t}^{OLG}$ are increasing functions in λ_{t+1} since (f.1) and (f.2) must hold in equilibrium.

In section 3.5.2 we established that s_t^{LUC} is a decreasing function in λ_{t+1} . We therefore have that $\pi^{LUC}(k_t; \lambda_t)$ increases when the level of relative risk aversion increases.

In section 3.4 we established that s_t^{OLG} can be either an increasing or a decreasing function in λ_{t+1} . It is therefore possible for the risk premia to become small or even negative in an OLG model when the level of relative risk aversion exceeds one and increases. One possible explanation for this is that young agents can substitute between random consumption tomorrow to non-random consumption today by varying savings and hence the proportion of fixed output that they can consume. This means that young consumers with a high second period relative risk aversion (greater than one) may require a lower rate of return to be induced to purchase the security.

The difference between the two asset pricing models implies that observed values of risk premia may correspond to vastly different estimates of parameters, which include an analysis of investors' expectations that are imbedded in the intertemporal substitution effect found in the OLG model, but absent from the intertemporal substitution effect found in the Lucas model. A consequence of the analysis is a deeper understanding of the often heard premise that asset prices are volatile because investors have an inconsistent risk appetite, an observation that is rooted in the behavioural finance literature. The analysis does not assert that investors do not have an inconsistent risk appetite, a fact implicitly assumed in competitive equilibrium analysis. It does, however, indicate a need for a deeper analysis of investor expectations embedded in asset prices.

Equilibrium at each point in time in a deterministic, infinitely lived, representative agent model requires that $c_t = x_t$, where x_t is the current, exogenously determined, per capita real output. Thus, $u'(c_t)$ is constant at time t . Therefore, s_t^{LUC} fluctuates in response to changes in $u'(c_{t+1})$ only, such that $\partial s_t^{LUC} / \partial x_{t+1} = \beta u''(x_{t+1}) / u'(x_t)$. Since $u(\cdot)$ is concave we have that $\partial s_t^{LUC} / \partial x_{t+1} < 0 \forall \lambda_{t+1}$. This holds whether or not there is uncertainty in next period's output.

The aggregate conditional risk premium $\pi^{LUC}(k_t; \lambda_t)$ is (see Mathematical Appendix for derivation):

$$\pi^{LUC}(k_t; \lambda_t) = [1 + R_{rf}^{LUC}(k_t; \lambda_t)] \times \text{cov}[-s_t^{LUC}; R_{m,t}^{LUC}(k_t; \lambda_t; \lambda_{t+1}) | (k_t; \lambda_t)] \quad (7.a)$$

where:

- i. $\pi^{LUC}(k_t; \lambda_t) = \bar{R}_{m,t}^{LUC}(k_t; \lambda_t) - R_{rf}^{LUC}(k_t; \lambda_t)$
- ii. $\bar{R}_{m,t}^{LUC}(k_t; \lambda_t) = \int_E R_{m,t}^{LUC}(k_t; \lambda_t; \lambda_{t+1}) dF(\lambda_{t+1})$

3.5.3 Risk Premia in a OLG Model

The aggregate conditional risk premium $\pi^{OLG}(k_t; \lambda_t)$ is (see Mathematical Appendix for derivation).

$$\pi^{OLG}(k_t; \lambda_t) = [1 + R_{rf}^{OLG}(k_t; \lambda_t)] \times \text{cov}[-s_t^{OLG}; R_{m,t}^{OLG}(k_t; \lambda_t; \lambda_{t+1}) | (k_t; \lambda_t)] \quad (7.b)$$

where:

- i. $\pi^{OLG}(k_t; \lambda_t) = \bar{R}_{m,t}^{OLG}(k_t; \lambda_t) - R_{rf}^{OLG}(k_t; \lambda_t)$
- ii. $\bar{R}_{m,t}^{OLG}(k_t; \lambda_t) = \int_E R_{m,t}^{OLG}(k_t; \lambda_t; \lambda_{t+1}) dF(\lambda_{t+1})$

3.5.4 A Comparison of the Two Models

We wish to establish how $\pi^{LUC}(k_t; \lambda_t)$ and $\pi^{OLG}(k_t; \lambda_t)$ behaves when the level of relative risk aversion changes. The relative curvature of the utility functions measures both relative risk aversion and intertemporal substitution when preferences are additive

- i. β is the period discount factor such that $0 < \beta < 1$;
- ii. s_t^{LUC} is the marginal rate of intertemporal substitution in the Lucas type model;

Define $R_{rf}^{LUC}(k_t; \lambda_t)$ as the period risk free return corresponding to the Lucas type model:

$$1 + R_{rf}^{LUC}(k_t; \lambda_t) = \frac{p_c(k_{t+1}; \lambda_{t+1})}{p_{rf}^{LUC}(k_t; \lambda_t)} = \left[\beta \int_E \frac{u'(c_{t+1})}{u'(c_t)} dF(\lambda_{t+1}) \right]^{-1} = \left[\int_E s_t^{LUC} dF(\lambda_{t+1}) \right]^{-1} \quad (5.a)$$

This result follows by using an argument analogous to (f.1), which is derived under the Lucas type asset-pricing regime (see Donaldson and Mehra (1984)).

We can find $R_{rf}^{OLG}(k_t; \lambda_t)$, the risk free return in the OLG model, by using an argument equivalent to that used to derive (5.a) above:

$$1 + R_{rf}^{OLG}(k_t; \lambda_t) = \frac{p_c(k_{t+1}; \lambda_{t+1})}{p_{rf}^{OLG}(k_t; \lambda_t)} = \left[\beta \int_E \frac{v'(c_{t+1}^2)}{u'(c_{t+1}^1)} dF(\lambda_{t+1}) \right]^{-1} = \left[\int_E s_t^{OLG} dF(\lambda_{t+1}) \right]^{-1} \quad (5.b)$$

3.5.2 Risk Premia in a Lucas Type Model

We consider a simple version of the Lucas asset-pricing model with one asset and independently identically distributed output shocks (see Donaldson and Mehra (1984)).

The first order conditions are:

$$1 = \beta \int_E \frac{u'(c_{t+1})}{u'(c_t)} \times \frac{p_z(k_{t+1}; \lambda_{t+1})}{p_i(k_{t+1}; \lambda_{t+1})} dF(\lambda_{t+1}) = \int_E s_t^{LUC} \times (1 + R_{m,t}^{LUC}(k_t; \lambda_t; \lambda_{t+1})) \times dF(\lambda_{t+1}) \quad (6)$$

Brock and Mirman (1971) show that a one good neoclassical model incorporating uncertainty in the production technology behaves analogously to the deterministic formulation of the model. Specifically, they show that optimal saving and consumption policies exist that are a function of existing capital stock only. The remainder of this subsection will therefore analyse the properties of s_t^{LUC} in the context of a deterministic model, i.e. a model with no uncertainty in the production technology.

3.5 Risk Premia

In this section we show that the risk premia generated by an overlapping generations (OLG) model are fundamentally different from the risk premia determined within an infinitely lived representative consumer model. This difference arises because of the ambiguity resulting from income and substitution effects in an OLG model that is absent in an infinitely-lived, representative agent model such as the asset-pricing model of Lucas.

Since we shall be dealing with a single asset in each case, the return on the asset is also the return on the market, which we denote $R_{m,t}^{LUC}(\cdot)$ and $R_{m,t}^{OLG}(\cdot)$ respectively.

3.5.1 The Risk Free Asset

We introduce a financial instrument market where a riskless asset is traded, this asset being in zero net supply. Since we consider a single consumer economy and the net demand for this asset must be zero in equilibrium (it being in zero net supply), its existence does not affect the equilibrium as long as price is obtained in the usual way from first order conditions of the single consumer. Since the Lucas model and the OLG model are fundamentally different we need to consider the pricing of the risk free asset for each asset-pricing regime. Let $dp_{rf}^{LUC}(k_t; \lambda_t)$ be the Lucas price at time t of a security which pays one unit of consumption next period when the state of nature is between λ_{t+1} and $\lambda_{t+1} + d\lambda_{t+1}$, then in equilibrium:

$$dp_{rf}^{LUC}(k_t; \lambda_t) = \beta u'(c_{t+1}(k_{t+1}; \lambda_{t+1})) \times \text{prob}[\lambda \in (\lambda_{t+1}; \lambda_{t+1} + d\lambda_{t+1}) | \lambda_t]$$

The price of the risk free security that pays a unit of consumption in every state next period is therefore:

$$p_{rf}^{LUC}(k_t; \lambda_t) = \beta \int_E \frac{u'(c_{t+1})}{u'(c_t)} dF(\lambda_{t+1}) = \int_E s_t^{LUC} dF(\lambda_{t+1})$$

where:

the utility functions measures both risk aversion and intertemporal substitution when preferences are additive and that the two effects can not be separated.

From equation 4 we write the marginal rate of intertemporal substitution in equilibrium as: $s_t^{OLG}(\lambda_t; \lambda_{t+1}) = \beta v'(\hat{c}_{t+1}^2) / u'(\hat{c}_t^1)$, where;

- i. $\hat{k}_{t+1} = \hat{i}_t$
- ii. $c_t^1 = \hat{w}(\hat{k}_t; \lambda_t) - \hat{i}_t$
- iii. $\hat{c}_{t+1}^2 = \lambda_{t+1} q'(\hat{k}_{t+1}) \hat{k}_{t+1}$

Condition ii and iii above follow from the proof of theorem 1. By substituting conditions i to iii above into the equation representing the marginal rate of substitution we get:

$$s_t^{OLG}(\lambda_{t-1}; \lambda_t) = \beta \frac{v'(\lambda_{t+1} \hat{i}_t q'(\hat{i}_t))}{u'(\hat{w} - \hat{i}_t)}$$

From the above we have that $\frac{\partial s_t^{OLG}}{\partial \lambda_{t+1}} = \beta \frac{1}{u'(c_t^1)} \left\{ v''(c_{t+1}^2) \hat{i}_t q'(i_t) - \frac{v'(c_{t+1}^2)}{u'(c_t^1)} u''(c_t^1) \frac{i}{\lambda_{t+1}} (1 - q'(i)) \right\}$.

To understand the behaviour of this partial derivative consider the case $\frac{\partial s_t^{OLG}}{\partial \lambda_{t+1}} = 0$, i.e.:

$$\frac{1}{q'(i_t)} - 1 = \frac{u'(c_t^1)}{v'(c_{t+1}^2)} \times \frac{v''(c_{t+1}^2)}{u''(c_t^1)} \times \lambda_{t+1} = \alpha_{t+1} \rightarrow q'(i_t) = \delta_{t+1}$$

From this we see that if:

- i. $q'(i_t) < \delta_{t+1}$ then $\partial s_t^{OLG} / \partial \lambda_{t+1} > 0$. The income effect of an increase in λ_{t+1} dominates the substitution effect and corresponds to a relative risk aversion coefficient greater than one, i.e. $v'(c_{t+1}^2) / u'(c_t^1) > 1$. This means the young are willing to save more by delaying consumption at time t , thereby accept more risk in terms of uncertain future consumption at time $t+1$.
- ii. $q'(i_t) > \delta_{t+1}$ then $\partial s_t^{OLG} / \partial \lambda_{t+1} < 0$. The income effect of an increase in λ_{t+1} is dominated by the substitution effect and corresponds to a relative risk aversion coefficient less than one, i.e. $v'(c_{t+1}^2) / u'(c_t^1) < 1$. This means the young are willing to save less and consume more at time t , thereby accept less risk in terms of uncertain future consumption at time $t+1$.

$$\vartheta_1 p_z(k_t; \lambda_t) - \vartheta_3 = 0; \quad (c.4)$$

Since ϑ_1 , $p_l(k_t; \lambda_t)$ and $p_z(k_t; \lambda_t)$ are all positive, we have $\vartheta_2 > 0$ and $\vartheta_3 > 0$. Hence $l_t = 1$ and $z_t = k_t$; individuals supply one unit of labour and k_t units of capital inelastically.

Again note that we can set $p_c(k_t; \lambda_t) = 1$ without loss of generality:

$$u'(c_t^1) p_l(k_t; \lambda_t) = \beta \int_E v'(c_{t+1}^2) p_z(k_{t+1}; \lambda_{t+1}) dF(\lambda_{t+1}) \quad (4)$$

Equation (4) is the fundamental equation for the pricing of assets. It equates the loss of utility associated with carrying one additional unit of capital to the discounted expected utility of the resulting additional consumption next period.

3.3.3 The Existence Theorem

Theorem 1. Assuming assumptions 1 to 4 hold, there exists a competitive equilibrium solution to the problem specified in section 3.2.3 for each initial $\hat{k}_0 > 0$.

The proof of this theorem can be found in section 3.5.3 of the mathematical appendix. The proof of the existence theorem for a competitive equilibrium for an infinite-lived agent analogue to the OLG model studied above follows along the same lines as that of theorem 1.

3.4 The Marginal Rate of Intertemporal Substitution

To understand the behaviour of the risk premium implied by this model relative to the risk premium of a Lucas asset-pricing model we need to examine the interrelationship between: i) the effect on saving of a change in the perceived riskiness of the environment, and ii) the substitution of current non-stochastic consumption for future stochastic consumption. We examine this relationship by considering the effect of a change in λ_{t+1} on the marginal rate of intertemporal substitution and note that the relative curvature of

$$p_z(k_t; \lambda_t) = \lambda_t q'(k_t) \quad (\text{f.2})$$

$$p_l(k_t; \lambda_t) = \lambda_t [q(k_t) - (k_t)q'(k_t)] \quad (\text{f.3})$$

$$c_t = \lambda_t q(k_t) - i_t \quad (\text{f.4})$$

$$d_t = \lambda_t k_t q'(k_t) - i_t \quad (\text{f.5})$$

3.3.2 The Decision Problem of the Representative Individual

Section 3.2.3 expressed the individual's maximization problem as follows:

$$V(k_0; \lambda_0) = \max \{u(c_t^1) + \beta E_t [v(c_{t+1}^2)]\} \text{ subject to:} \quad (\text{P.b.1})$$

- i. $x_t \in Y(\lambda_t) \cap X(k_t)$
- ii. $k_{t+1} = i_t$ for all t and k_0 and λ_0 given.

Note that the individual's consumption in old age is a function of the amount of capital purchased by the firm in period $t+1$ and the price paid by the firm for the capital, i.e.

$c_{t+1}^2 = p_z(k_{t+1}; \lambda_{t+1})z_{t+1}$. The functional form of the individual's optimisation problem can therefore be written as:

$$V(k_t; \lambda_t) = \max \left\{ u(c_t^1) + \beta \int_E v(p_z(k_{t+1}; \lambda_{t+1})z_{t+1}) dF(\lambda_{t+1}) \right\} \text{ subject to:} \quad (\text{P.b.2})$$

- i. $p_c(k_t; \lambda_t)c_t + p_l(k_t; \lambda_t)i_t = p_z(k_t; \lambda_t)z_t + p_l(k_t; \lambda_t)l_t$;
- ii. $z_t \leq k_t$;
- iii. $k_{t+1} = i_t$;
- iv. $l_t \leq 1$;
- v. $k_{t+1} = g(k_t; \lambda_t)$;

The first order conditions for a maximum are (see the Mathematical Appendix for proof):

$$u'(c_t^1) = p_c(k_t; \lambda_t)\vartheta_1; \quad (\text{c.1})$$

$$u'(c_t^1) \frac{p_l(k_t; \lambda_t)}{p_c(k_t; \lambda_t)} = \beta \int_E v'(c_{t+1}^2) p_z(k_{t+1}; \lambda_{t+1}) dF(\lambda_{t+1}); \quad (\text{c.2})$$

$$\vartheta_1 p_l(k_t; \lambda_t) - \vartheta_2 = 0; \quad (\text{c.3})$$

3.3 Establishing the Existence of Equilibria

We shall first examine the decision problem faced by the firm (section 3.3.1) and the individual (section 3.3.2) separately before establishing the existence of a competitive equilibrium (section 3.3.3).

3.3.1 The Decision Problem of the Representative Firm

The firm faces the static problem of maximizing profits each period, i.e.:

$$\max\{p(k_t; \lambda_t)y_t\} = \max\{p_c(k_t; \lambda_t)c_t + p_i(k_t; \lambda_t)i_t - p_z(k_t; \lambda_t)z_t - p_l(k_t; \lambda_t)l_t\} \quad (\text{P.a})$$

$$\text{s.t. } c_t + i_t \leq l_t q(z_t/l_t) \lambda_t$$

The first order conditions for a maximum are (see the Mathematical Appendix for proof):

- i. $p_c(k_t; \lambda_t) = p_i(k_t; \lambda_t)$
- ii. $p_z(k_t; \lambda_t) = \lambda_t p_c(k_t; \lambda_t) q'(z_t/l_t)$
- iii. $p_l(k_t; \lambda_t) = \lambda_t p_c(k_t; \lambda_t) [q(z_t/l_t) - (z_t/l_t) q'(z_t/l_t)]$

Since firms produce using a constant returns to scale technology we have that $\max\{p(k_t; \lambda_t)y_t\} = 0$, which yields $p_c(k_t; \lambda_t)c_t + p_i(k_t; \lambda_t)i_t - p_z(k_t; \lambda_t)z_t - p_l(k_t; \lambda_t)l_t = 0$.

This equation can be restated as follows:

$$\Rightarrow p_c(k_t; \lambda_t)c_t = p_l(k_t; \lambda_t)l_t + \{p_z(k_t; \lambda_t)z_t - p_i(k_t; \lambda_t)i_t\};$$

\Rightarrow i.e. aggregate consumption is equal to wage plus dividend, where the value of the dividend equals $d_t = p_z(k_t; \lambda_t)z_t - p_i(k_t; \lambda_t)i_t$;

\Rightarrow Substituting constraints i, ii and iii into this equation yields:

$$\Rightarrow p_c(k_t; \lambda_t)c_t = \lambda_t p_c(k_t; \lambda_t) [q(z_t/l_t) - (z_t/l_t) q'(z_t/l_t)] + \lambda_t p_c(k_t; \lambda_t) q'(z_t/l_t) - p_c(k_t; \lambda_t) i_t$$

Note that we can set $p_c(k_t; \lambda_t) = 1$ without loss of generality. From section 3.3.2 below we know that in equilibrium one unit of labour and k_t units of capital are supplied inelastically, i.e. $l_t = 1$ and $z_t = k_t$. These facts can be used to restate the first order conditions as:

$$p_c(k_t; \lambda_t) = p_i(k_t; \lambda_t) = 1 \quad (\text{f.1})$$

3.2.3 Defining Equilibrium

Since the structure of the economy is invariant, economic agents solve a similar problem each period. All relevant information for decision-making in period t can be characterised by the double $(k_t; \lambda)$.

Firms produce to maximize profits in each period given market prices for capital $p_k(k_t; \lambda)$, labour $p_l(k_t; \lambda)$, consumption $p_c(k_t; \lambda)$ and investment goods $p_i(k_t; \lambda)$. Consumers maximize their expected utility of consumption over feasible plans subject to their budget constraint. Expectations are assumed to be rational: prices and price distributions on which economic agents base their consumption/investment/production decisions are exactly the same as those that result as a consequence of their decisions through market clearing.

Define the vector of current prices as $p(k_t; \lambda) = \{p_k(k_t; \lambda), p_l(k_t; \lambda), p_c(k_t; \lambda), p_i(k_t; \lambda)\}$. Knowledge of $p(k_t; \lambda)$, the law of motion of capital stock $k_{t+1} = g(k_t; \lambda)$ and the conditional shock distribution function $F(\lambda_t)$ is sufficient for forming predictive distributions of future prices and selecting optimal current actions.

From the above we can summarise the problem faced by the individual as;

$V(k_0; \lambda_0) = \max \{u(c_t^1) + \beta E_t [v(c_{t+1}^2)]\}$ subject to:

- i. $x_t \in Y(\lambda_t) \cap X(k_t)$
- ii. $k_{t+1} = i_t$ for all t and k_0 and λ_0 given.

3.2.2 Preferences

The young at time t maximize a two period time additive utility function

$$\mu(c_t^1; c_t^2) = u(c_t^1) + \beta E_t[v(c_{t+1}^2)] \quad (2)$$

by deciding how much to consume and how much to save from their current period wages. Notation is defined as follows:

- i. E_t denotes the expectations operator conditional on information at time t , denoted Ω_t ;
- ii. β is the period discount factor such that $0 < \beta < 1$;

Note that capital purchased by firms, z_t , is constrained by capital held by the old, k_t . Thus the young's period t consumption possibility set is defined by:

$$X(k_t) = \{(c_t^1; -l_t; -z_t; i_t) \in R^4 : c_t^1 \geq 0, 0 \leq l_t \leq 1, z_t \leq k_t, i_t \geq 0\} \quad (3)$$

In each period the young selects a commodity vector $x = (c; -l; -z; i)$ from $X(k_t)$ such that the two period utility function defined above is maximized. Note that the amount of capital held by the old, k_{t+1} , at the beginning of the next sale period equals the amount of new investment/saving made during youth, i_t , such that $k_{t+1} = i_t$ specifies how capital holdings depend upon the prior period's decision. Furthermore, consumption in old age, c_{t+1}^2 , depends entirely on the amount of capital demanded by the firm, z_{t+1} , and the price obtained on the sale of capital to the firm.

Assumption 4. The following conditions on $u(\cdot)$ and $v(\cdot)$ hold: i) $u'(c) > 0$ and $v'(c) > 0$, ii) $u''(c) < 0$ and $v''(c) < 0$, iii) u and v three times continuously, iv) $\lim_{c \rightarrow 0} u'(c) = \infty$ and $\lim_{c \rightarrow 0} v'(c) = \infty$ and v) $\lim_{c \rightarrow \infty} u'(c) = 0$ and $\lim_{c \rightarrow \infty} v'(c) = 0$.

where $c_t = c_t^1 + c_t^2$ represents the aggregate consumption at time t such that c_t^1 and c_t^2 denotes consumption in youth and old age respectively.

Assumption 2. The function $q(\cdot)$ is homogenous across all firms, increasing (i.e. $q'(\cdot) > 0$), strictly concave (i.e. $q''(\cdot) < 0$), and twice continuous differentiable and bounded with $q(0) = 0$. We also have that: i) $\lim_{z \rightarrow 0} q'(z) = \infty$, and ii) $\lim_{z \rightarrow \infty} q'(z) = 0$. To clarify notation note that k_t denotes capital held by the old and available for sale to firms in period t , while z_t denotes the capital firms actually choose to purchase in period t . In equilibrium we must have that $k_t = z_t$.

Assumption 3. The production shock λ is independently and identically distributed each period over a finite positive range $E = [\underline{\lambda}; \bar{\lambda}]$ where $\underline{\lambda} > 0$ and $\bar{\lambda} < \infty$. The cumulative distribution function of $F(\cdot)$ of λ is stationary and satisfies: i) $F(\lambda) = 0 \quad \lambda \leq \underline{\lambda}$, and ii) $F(\lambda) = 1 \quad \lambda \geq \bar{\lambda}$.

The firm has already observed the production shock λ at the time the production decision is made.

The firm's static problem is therefore solved by choosing a commodity vector $y = (c; -l; -z; i)$ from $Y(\lambda)$ to maximise period profits. Since firms produce using a constant returns to scale technology they have zero profits and therefore zero value in equilibrium. It is however possible to relate the firm's value to those observed in the capital markets by defining the value of the capital purchased at the beginning of the period by the firm to be its pre-dividend value, denoted $VF_{pre-div}^t$, and the value of the capital sold after production to be the ex-dividend value of the firm, denoted VF_{ex-div}^t . The period dividend is therefore denoted $VF_{pre-div}^t - VF_{ex-div}^t$.

Since firms liquidate at the end of each period there is no capital accumulation such that period t production will not affect future production possibilities.

The economy is assumed to produce only two goods: 1) a non-storable consumption good, and 2) an investment (capital) good. At the beginning of each period t firms observe the shock to productivity λ_t , and purchase capital and labour from individuals at competitively determined rates. Both capital and labour are used to produce the two output goods. The firm is assumed to liquidate at the end of each period, and therefore faces a static problem – maximise period profits.

For every $t \geq 0$ there exists a new fixed-size population of identical individuals who live for two periods and die at the end of $t + 1$. Each generation can only produce in their youth (i.e. in period t) whilst consuming in both their youth and old age (i.e. in period t and $t+1$). The young do not own any capital. The young use their wages to buy the consumption good c_t and the investment good i_t at the end of the first period whilst the investment good is used as capital k_{t+1} available for sale to firms in their old age. The old use the proceeds from the sale of the investment good to buy the consumption good c_{t+1} .

An old generation is assumed to exist at time $t = 0$ that is endowed with $0 < k_0^* < \infty$ units of capital.

3.2.1 Technology

The model uses a simplifying assumption for the production technology that can easily be relaxed:

Assumption 1. Firms are assumed to produce under stochastic constant returns to scale.

This assumption is usually not satisfied when a factor such as land is owned rather than rented by the firm and therefore not included in the commodity vector. In general, a factor can be added to the technology vector such that the resulting technology set displays constant returns to scale. See Mas-Colell (1995) p134 proposition 5.b.2.

The firm's production possibility set $Y(\lambda)$ is defined by:

$$Y(\lambda) = \{(c_t, -l_t, -z_t, i_t) \in R^4 : c_t, l_t, z_t, i_t \geq 0; c_t + i_t \leq l_t q(z_t/l_t) \lambda_t\} \quad (1)$$

at the time of the trade (the unborn future and expired past generations). We shall see that this market friction plays an important role in the behaviour of the OLG asset pricing model relative to the frictionless Lucas asset-pricing model.

Another advantage of using an OLG model is that it allows one to better model intertemporal production and consumption for agents with a finite life expectancy. This is an important benefit as infinitely lived agents can take advantage of any price discrepancy by simply waiting long enough. By introducing death into the model one restricts the extent to which this strategy applies, i.e. death limits the scope for arbitrage.

We avoid the issue of asset bubbles by constructing a model economy in which $L_{t+1} = L_t \forall t$. An asset bubble occurs when the price of an asset is not equal to its market fundamental. Tirole (1985) shows that a bubble cannot occur on the stock market price within an OLG framework if $\lim_{t \rightarrow \infty} L_{t+1}/L_t \leq 1$. We impose this constraint on our analysis as we are concerned with relatively short-term volatility, rather than asset-mispricing that result from the demographics of the model economy.

3.2 Model Specification

The economy introduced in this section has a single nonstorable consumption good and no capital accumulation. An agent can provide for future consumption only through the purchase of an asset today. The model presented here is the overlapping generations analogue of the Lucas asset-pricing model studied by Donaldson and Mehra (1984). The model by Donaldson and Mehra (1984) forms the basis of the Lucas asset-pricing model studied here. Donaldson and Mehra (1984) provide an existence theorem for a very general class of infinite-lived agent economies with correlated stochastic production shocks. In this chapter we study a simple case where production shocks are independently and identically distributed.

Consider an infinite horizon competitive economy consisting of a successive number of generations at discrete points in time, i.e. $t = 0, 1, \dots, \infty$.

beliefs held by other market participants. This result has profound implications for agents trading in the market. Rather than only try and learn m , they may elect to learn what the distribution of beliefs is that determines market pricing at a point in time.

5.2.4 The Axioms of Rationality

For notational convenience we define the set $B_S^+ = \{Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\mathcal{E}) : S \in \mathfrak{S}, Q(S) > 0\}$.

We are now in a position to specify the axioms that a rational choice of q must satisfy:

Axiom 1: (Compatibility with the data) An agent forms belief q with $\mathfrak{R}(\Pi) \cap \varphi^*(\mathcal{E})$ as its support.

Axiom 2: (Continuity with respect to the data) If $S \in \mathfrak{S}$ and $m(S) > 0$ then $q(B_S^+) > 0$.

Axiom 1 requires that an agent form a belief q that places probability 1 on $\mathfrak{R}(\Pi) \cap \varphi^*(\mathcal{E})$. Axiom 2 further restricts the allowable q 's in $\mathfrak{R}(\Pi) \cap \varphi^*(\mathcal{E})$ to those with positive probability when $S \in \mathfrak{S}$ and $m(S) > 0$. Axiom 2 ensures that if a finite dimensional event S is observed infinitely often, thereby generating positive relative frequency, then one cannot be certain that S cannot occur from today's perspective. This prevents the degeneracy of beliefs to a single probability mass point when m gives the point a non-zero probability.

Whilst axiom 1 states that agents can hold diverse beliefs that are consistent with the data, axiom 2 ensures that beliefs are uncertain by excluding beliefs that can be represented as a single probability mass point. By combining axioms 1 and 2 we find that the theory of Rational Beliefs (RB) ensures that agents can hold diverse beliefs and that uncertainty exists over the beliefs held by agents, i.e. agents do not know the demand and supply functions of all other agents.

We can conclude that since 1) agents do not know the true probability law Π , 2) the dynamic system $(\Omega, \mathfrak{S}, \Pi, T)$ is stable and not stationary and 3) agents do not know the demand and supply functions of all other agents it follows that RB theory does not generally lead to a Rational Expectations Equilibrium (REE).

As a consequence of axiom's 1 and 2 we find that endogenous uncertainty, defined as the price volatility caused by the distribution of agents' beliefs, is non-negligible. The existence of endogenous uncertainty is confirmed in section 5.3.2 through an application of the implicit function theorem. We therefore conclude that the price-formation process of an RB economy is necessarily more volatile than in an RE economy. This follows from the fact that in an RB economy uncertainty results from both endogenous and exogenous phenomena, with endogenous uncertainty ensured to exist by axiom 2, whilst in an RE economy uncertainty results only from exogenous phenomena.

5.2.5 The Main Theorem

Theorem 6. Given a dynamical system $(\Omega, \mathfrak{S}, \Pi, T)$, let an agent form a rational belief q that satisfies axioms 1 and 2. Then $q \in \mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ and there exists a probability $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$, which is the expectation under q in the sense of equation (8). Further, there exists probabilities Q^m and Q° on (Ω, \mathfrak{S}) and a constant $0 < \lambda_Q \leq 1$ such that:

- i. Q has a unique representation $Q = \lambda_Q Q^m + (1 - \lambda_Q) Q^\circ$ where Q^m and m are equivalent while Q° and m are singular;
- ii. $(\Omega, \mathfrak{S}, Q^m, T)$ and $(\Omega, \mathfrak{S}, Q^\circ, T)$ are stable with stationary measures \bar{Q}^m and \bar{Q}° that have the property that; 1) \bar{Q}^m and m are equivalent, 2) $\bar{Q}^\circ \ll m$, i.e. \bar{Q}° is absolutely continuous with respect to m , and 3) \bar{Q}^m and \bar{Q}° are weakly uniform stable probability measures;

- iii. There exists regular versions of the conditional probabilities Q_t, m_t and Q_t° and densities $\psi_m = \frac{dm}{dQ}$ $\psi_o = \frac{dQ^\circ}{dQ}$ that satisfy m a.e. for $S \in \mathfrak{S}'$

$$Q_t(S|I_t) = \lambda_Q m_t(S|I_t) \psi_m(I_t) + (1 - \lambda_Q) Q_t^\circ(S|I_t) \psi_o(I_t); \quad (9)$$

Note that $Q_t(S|I_t)$ represents the probability of event S occurring at time t conditional on the history I_t up to time t

Conversely, for any Q , Q^m and Q^o satisfying conditions i to iii there exists a rational belief $q \in \mathcal{P}(\mathcal{R}(\Pi) \cap \varphi^*(\varepsilon))$ that represents Q .

Proof. See the Mathematical Appendix for details.

We use the conditional probability representation of RB, i.e. equation (9), in theorem 6 to show both why and how agents using the same data end-up with different RB. From theorem 1 we see that in trying to learn Π , agents learn m , which is an empirically derived stationary probability measure. From theorem 6 we have that Q^o is a weakly uniform stable probability measures and not stationary, which is compatible with the stationary measure m . The singular measure Q^o may therefore represent agents' beliefs in structural changes. An agent's level of confidence in m , given the agent's belief in the possibility of a structural change measured by Q^o , is measured by the subjective parameter λ_Q . This means that the less confident an agent is in the nature of the world measured by stationary probability m , the more weight is put on the agent's belief in structural change which is represented by Q^o .

The practical implication of theorem 6 is that we can select beliefs that are biased in the short-run relative to the stationary probability generated by the data. However, the rationality requirement implies that these biases average out over the agent's investment horizon. This idea reflects the fact that the timing of events matters in terms of the probabilities that are attached to the events in non-stationary systems.

5.2.6 A Note on Ergodic Dynamical Systems

For the sake of generality the analysis has assumed that the true dynamic system $(\Omega, \mathcal{S}, \Pi, T)$ is not necessarily ergodic. However, in the case where the dynamical system is ergodic we find that the limits of the relative frequencies are independent of x . This result is summarised in Corollary 1

Corollary 1. If $(\Omega, \mathcal{S}, \Pi, T)$ is ergodic then for all $S \in \mathcal{S}$ we have $m(S|x) = m(S) = m_\Pi(S)$ Π and m_Π a.e..

Proof. See the Mathematical Appendix for details.

If $(\Omega, \mathfrak{S}, \Pi, T)$ is ergodic then it follows from definition 2 that the dynamic system $(\Omega, \mathfrak{S}, \Pi, T)$ is SAMS. We then apply the Ergodic Decomposition Theorem 7.4.1 of Gray (2001) and observe that limiting sample averages will behave as if they were produced by a stationary and ergodic dynamic system. Thus, $(\Omega, \mathfrak{S}, m(\cdot|x), T)$ is stationary and ergodic m_Π a.e. if $(\Omega, \mathfrak{S}, \Pi, T)$ is ergodic.

Central to the above analysis is the implicit assumption that agents can observe only one realization of the dynamical system. The Ergodic Decomposition Theorem used above then states that even if $(\Omega, \mathfrak{S}, \Pi, T)$ is not ergodic, the single realization x and all its iterates $T^n x$ will belong to one ergodic component of Ω which is an invariant set. This means that we might as well restrict the space Ω to the ergodic component that contains the single observed realization x , denoted F . The observed dynamical system is then $(F, \mathfrak{S}_F, \Pi_F, T)$, where \mathfrak{S}_F restricts the space \mathfrak{S} to F and Π_F restricts Π to \mathfrak{S}_F . We then have that the observed dynamical system $(F, \mathfrak{S}_F, \Pi_F, T)$ is ergodic.

From the above we see that if the observed data is a single realisation of the dynamical system then we can assume that the system is ergodic such that $m(S|x) = m(S)$ and $m(S) = m_\Pi(S) \quad \forall S \in \mathfrak{S}$.

5.3 Endogenous Uncertainty in a Short-run Walrasian Monetary Equilibrium Model

5.3.1 Model Specification and Existence.

In chapter 4 section 4.3 we specified a short-run Walrasian monetary equilibrium model in which equilibrium is a function of an agent's 1) money stock, 2) endowment of consumption goods and 3) future price expectations. In chapter 3 we did not address the issue of how price expectations are formulated as the analysis of this model was primarily concerned with the Intertemporal Substitution effect. We now extend our analysis to proving the existence of endogenous uncertainty. The remainder of this chapter will use a modified version of the model specified in chapter 4 sections 4.3. We use the same notation for convenience.

The model is extended to the more general case where agents trade in current spot markets based on their beliefs of future prices and endowments of consumption goods. This means that agents form expectations about:

- i. The endowment of consumption goods that will be received in the next period, i.e. $\{\zeta_{ia}^2\}_{a=1,\dots,d}$, which is a function of current and past endowments, and
- ii. The future price of consumption goods, i.e. $\{\tilde{p}_{ia}^2(p_t^1)\}_{a=1,\dots,d}$, which is a function of current and past market clearing prices.

Let the vector of exogenous variables be denoted by $e_t \in E \subset R^{ld}$, where $e_t = (\zeta_{d1}^1, \dots, \zeta_{d1}^1)$ and denote the vector of endogenous variables by $p_t^1 \in \Delta^{l-1}$. Observe that we make the simplifying assumption that each agent observes the endowment of every agent in the economy. This is a benign assumption; an agent can glean no additional information from knowing the current and past endowments of other agents, which is reflected in the market-clearing price.

Each agent holds a rational belief $q_a \in \mathcal{P}(\mathcal{R}(\Pi) \cap \varphi^*(\varepsilon))$, which is expressed as a conditional probability denoted Q_a by using theorem 6 part iii.

The explicit statement of an agent's belief system allows us to write an agent's period two expectations as i) $\tilde{\zeta}_{ia}^2 : Q_{ia} \rightarrow \zeta \subset R^l$ and ii) $\tilde{p}_i^2 : Q_{ia} \rightarrow \Delta^{l-1}$, where ζ represents the space of possible endowments. This is more general than the assumption that future price expectations only depend on current prices adopted in chapter 3. It is necessary to demonstrate that the existence theorem formulated in chapter 3 still applies to the extension of the OLG model used here.

Let $W = E \times Y \subset R^k$ and define the joint dynamical system $(W^\infty, \beta(W^\infty), \Pi, T)$. In the relatively simple OLG model used here we have that:

- i. Each agent's past consumption and savings decisions are represented in aggregate by his time t stock of money, i.e. $\{m_{ia}^1\}_{a=1}^d$, and
- ii. Each agent's belief system $\{Q_{ia}\}_{a=1}^d$ influences current prices p_t through his expectation of future prices and endowments, since uncertainty with respect to current prices is resolved by trading in the current spot market.

This observation implies that the current realisation of this dynamical system, denoted by w_t , is a sufficient statistic for describing equilibrium in this system. We are now in a position to specify the von Neumann-Morgenstern utility of agent a , which expresses his beliefs of the future:

$$u_a^1(y_{ia}^1, w_t) = \int_{w_{t+1}} u_a^1(y_{ia}^1 | w_{t+1}) dQ_{ia}(w_t, w_{t+1}).$$

Notation is clarified as follows:

- i. $u_a^1(y_{ia}^1 | w_{t+1}) = \max u_a(\hat{y}_{ia}(y_{ia}^1, w_{t+1}))$ subject to $p_t^2 \hat{x}_{ia}^2 \leq p_t^2 \zeta_{ia}^2 + m_{ia}^1$, which represents the period-1 utility conditional on the realisation w_{t+1} in the next period,
- ii. $Q_{ia}(w_t, w_{t+1})$ is the probability of realising w_{t+1} given the agent observes w_t in the current period, and
- iii. $u_a^1(y_{ia}^1, w_t)$ is the expected period-1 utility if an agent observes w_t in the current period.

Now, from theorem 5 show that for any $q_a \in \mathcal{P}(\mathcal{R}(\Pi) \cap \varphi^*(\varepsilon))$ there exists $Q_a \in \mathcal{R}(\Pi) \cap \varphi^*(\varepsilon)$, which is compact in the topology of weak convergence by proposition 1. Since $Q_{ia} \subset Q_a$, it follows that Q_{ia} is compact in the topology of weak convergence. This means that the introduction of beliefs formulated along the lines developed in section 5.2 allows us to drop assumption (a.7). This then also shows one of the biggest flaws in the development of the Kurz framework of rational beliefs; each agent holds beliefs that give rise to bounded expectations, which means that the range of variation in each trader's expectations cannot be too large when p_t^1 varies in Δ^{l-1} .

The formulation above specified the following conditional maximisation problem:

(c.1) Maximise $u_a^1(y_{ia}^1, w_t)$ subject to:

- i. $x_{ia} \geq 0$;
- ii. $m_{ia}^1 \geq 0$;
- iii. $p_t^1 x_{ia}^1 + m_{ia}^1 \leq p_t^1 \zeta_{ia}^1 + \tilde{m}_{ia}$

(c.2) $P_t^1 = \left\{ p_t^1 \mid p_t^1 \in R^l, p_t^1 \geq 0, \sum_{h=1}^l p_{th}^1 = 1 \right\}$.

(c.3) $\dot{\pi}_t \leq 0, \dot{q}_t \dot{\pi}_t = 0$.

The maximization problem stated in (c.1) can then be written as an abstract economy denoted by $E_t^{OLG} = [Y_{i1}^1, \dots, Y_{id}^1, A_{i1}(\bar{y}_{i1}^1), \dots, A_{id}(\bar{y}_{id}^1), P_t^1]$, where:

- i. $\bar{y}_j^1 \in \{Y_{j1}^1 \times \dots \times Y_{j-1}^1 \times Y_{j+1}^1 \times \dots \times Y_{jd}^1 \times P_t^1\}$ represents the strategies chosen by all agents other than j , and
- ii. $A_{ij}(\bar{y}_j^1) = \{y_{ij}^1 \mid y_{ij}^1 \in Y_{ij}^1, p_t^1 x_{ia}^1 + m_{ia}^1 \leq p_t^1 \zeta_{ia}^1 + \tilde{m}_{ia}\}$ is the domain from which agent j can select his strategies given the strategies chosen by other agents.

Existence then follows in the usual way by applying the Arrow and Debreu (1954) existence proof to this abstract economy.

5.3.2 Existence of Endogenous Uncertainty.

The existence of a short-run Walrasian monetary equilibrium implies that we have $\dot{\pi} = 0$, where $\dot{\pi} = (\dot{z}, \dot{\eta}) = (0, 0) = 0$ with $\dot{z} = \sum_a \dot{z}_a$ and $\dot{\eta} = \sum_a \dot{\eta}_a$ representing aggregate excess demand for goods and money respectively in equilibrium. We therefore have the following system of equations in equilibrium $\left\{ \dot{z}_a(p_i^1; \vartheta_i) \right\}_{a=1}^d$,

$$\left\{ \dot{z}_a(p_i^1; \vartheta_i) \right\}_{a=1}^d, \quad (10)$$

where $\vartheta_i = (\vartheta_{1i}, \dots, \vartheta_{hi})$ denotes the set of parameters for the system and $\vartheta_{ia} = \{\tilde{m}_a^i, \zeta_a^i, Q_{ia}\}_{a=1}^d$ denotes the set of parameters of agent a at time t . The system represented by (10) is therefore a system with d equations, $l-1$ endogenous price variables and $3 \times d$ parameters.

In order to show that changes in agent's beliefs generate endogenous uncertainty we use Sard's theorem (see Debreu 1970) combined with the Implicit Function Theorem. The Implicit Function Theorem used here is from Mas-Colell p941.

Let $U \subset R^a$ such that U is open and let $F: U \rightarrow R^b$, where F is a continuously differentiable function. Sard's theorem reads as follows:

Proposition 3. (Sard's Theorem) If all the partial derivatives of F to the c^{th} order included, where $c > \max(0, a - b)$, exist and are continuous, then the set elements of R^b for which the Jacobian of F vanishes has Lebesgue measure zero in R^b .

Note that in the context of the model used here we have that $a = l - 1$ since $p_i^1 \in \Delta^{l-1}$, F is equivalent to the aggregate excess demand for goods denoted z and $b = 1$. We now introduce assumption (a.7):

(a.7) The partial derivatives of z to the c^{th} order included, where $c > \max(0, l - 1)$, exist and are continuous.

This is a simplifying assumption made to get round the technicalities surrounding the conditions under which z will be sufficiently differentiable, which is not of interest in the present analysis.

The combination of assumption (a.7) and proposition 3 allow us to invoke the implicit function theorem, which states that:

i. There exist $l-1$ continuously differentiable implicitly defined functions such that

$$v_n(\vartheta) = p_n \quad \forall n = 1, \dots, l-1, \text{ where } p_n \text{ denotes the } n^{\text{th}} \text{ element of } p_t, \text{ and}$$

ii. The first order effects of the parameters on the equilibrium point p_t are given by

$$D_\vartheta v(\vartheta) = -[D_{p_t} z(p_t^1, \vartheta)]^{-1} \times D_\vartheta z(p_t^1, \vartheta), \text{ where } v(\vartheta) = (v_1(\vartheta), \dots, v_{l-1}(\vartheta)).$$

This shows that endogenous uncertainty does indeed exist and influences current price p_t^1 . At this point it is clear that one cannot say much more about the implications of endogenous uncertainty for the price formation process unless very specific assumptions are made about the period-1 utility function $u_a^1(\cdot)$ or z_a .

5.3.3 Correlated Price Expectations.

Of interest is the impact of a change in period-2 price expectations on the period-1 price p_t^1 . The analysis of the intertemporal substitution effect in chapter 4 shows that this impact is ambiguous because the income and substitution effects will be working against each other (see also Grandmont (1985)). This ambiguity is resolved by considering the concavity of an agent's utility function. Consider the Arrow-Pratt coefficient of relative risk aversion (Mas-Colell p 194) that measures the degree of concavity of an agent's utility function:

$$r_R(s_{ta}^1) = -s_{ta}^1 \frac{u_a^{\prime\prime 1}(s_{ta}^1, w_t)}{u_a^{\prime 1}(s_{ta}^1, w_t)}, \text{ where}$$

- i. s_{ta}^1 represents the period-1 savings of agent a, and
- ii. $u_a^{\prime 1}$ and $u_a^{\prime\prime 1}$ are the first and second order derivatives of the period-1 utility function.

Arrow (1970) proves that the boundedness of the utility function is maintained if $r_R(s_{ta}^1) = 1$ when $s_{ta}^1 \rightarrow \infty$ or $s_{ta}^1 \rightarrow 0$. This means that the coefficient of relative risk aversion should broadly be around 1, with $r_R(s_{ta}^1) < 1$ for low wealth and $r_R(s_{ta}^1) > 1$ for

high wealth. Grandmont (1985) proves that an agent's period-1 excess demand is a decreasing function of his period-2 price expectations if $r_R(s_{ia}^1) \leq 1$, i.e. the substitution effect dominates. This is equivalent to the "own-price-effect" dominating the "diagonal-price-effect", i.e.:

$$\left| \frac{\partial x_i}{\partial p_i} \right| > \left| \sum_{j \neq i} \frac{\partial x_i}{\partial p_j} \right|, \text{ where}$$

- i. $\frac{\partial x_i}{\partial p_i}$ is the effect of a change in the price of good i on the demand for good i , and
- ii. $\frac{\partial x_i}{\partial p_j}$ is the effect of a change in the price of good j on the demand for good i .

Again, this is nothing other than the law of demand, i.e. $\Delta p \cdot \Delta x < 0$ where $p = (p_1, \dots, p_n)$ and $x = (x_1, \dots, x_n)$ (Mas-Colell p32).

In short; specific assumptions about the concavity of agents' utility functions are needed in order to show how changes in period-2 price expectations affect period-1 prices. However, note that in the short-run there is no income effect in financial markets such that the impact of a change in period-2 price expectations on period-1 prices can be analysed in a direct way.

The idea of correlated expectations is illustrated through two examples set within the framework of the simple exchange economy defined above.

Example 1.

Consider a generation for which period-2 price expectations instantaneously change as follows, all else being equal:

1. Expectations are initially pair wise uncorrelated such that ${}^1\tilde{p}_i^2 \neq {}^1\tilde{p}_j^2 \forall i \neq j$.
2. Agents then decide that period-2 prices will be higher than their initial expectation such that ${}^2\tilde{p}_i^2 >> {}^1\tilde{p}_i^2 \forall i$.

Observe that an equilibrium price exists for each set of expectations, denoted ${}^1\dot{p}_i^1$ and ${}^2\dot{p}_i^1$ respectively. It follows that ${}^1\dot{p}_i^1 << {}^2\dot{p}_i^1$ if $r_R(s_{ia}^1) \leq 1 \forall a$.

Example 2.

Consider a generation for which period-2 price expectations instantaneously change as follows, all else being equal:

3. Expectations are initially pair wise uncorrelated such that ${}^1\tilde{p}_i^2 \neq {}^1\tilde{p}_j^2 \forall i \neq j$.
4. Agents then decide that period-2 prices will be different from their initial expectation such that ${}^2\tilde{p}_i^2 - {}^1\tilde{p}_i^2 > 0 \forall i$.

Let the equilibrium price associated with each set of expectations be ${}^1\dot{p}_i^1$ and ${}^2\dot{p}_i^1$ respectively. In this case nothing can be said about the impact of the change in expectations because they do not change in a correlated way either up or down, even if $r_R(s_{ia}^1) \leq 1 \forall a$.

5.4 Summary Remarks and Consolation

The research of Kurz (1994) reviewed here contains several important achievements. The first is technical. By showing that stability is a necessary condition for a dynamic system to have the WAMS property, Kurz is able to prove the following set of results:

- i. the frequencies with which events occur in a stable dynamic system converge in the limit to a stationary probability measure,
- ii. a stable dynamic system has the WAMS property such that the arithmetic means of a sequence of probability measures converges in the limit, and most importantly,
- iii. the stationary probability measure derived from the data is equivalent to the stationary probability measure derived from the WAMS property, which is unrelated to the data.

An important achievement is the insight that agents can express their uncertainty about the true probability distribution that generates the data as uncertainty about the beliefs held by other market participants. This is a powerful concept: an agent can compare the distribution of beliefs that currently influence the price formation process with his own. He will trade if the observed distribution of beliefs is correlated in a way that is significantly biased relative to his beliefs. This means that he will only act when he perceives that observed beliefs are correlated and biased relative to his own in an extreme way, as uncorrelated beliefs have an ambiguous effect on the price formation process.

The theory of rational beliefs ensures that the range of variation in each trader's expectations is not too large when w_t varies in W , as each trader selects his rational belief from a compact set. Whilst this deals with the technicalities associated with proving the existence of the short-run Walrasian equilibrium in an elegant way, it does rule out the possibility of biased expectations impacting the current price formation process in a significant way. However, axiom's 1 and 2 ensure that agents can hold diverse beliefs and that uncertainty exits over the beliefs held by agents. A consequence of this is that endogenous uncertainty is non-negligible, which leads to the conclusion that the price-formation process of a rational belief economy is necessarily more volatile than in a rational expectations economy.

Finally, the compactness of the set of rational beliefs follows from two restrictions, specifically:

- i. Agents are required to assign a non-zero probability to all combinations of events that may conceivably occur over his investment horizon,
- ii. Such that the biases inherent in an agent's belief system average out over his investment horizon.

It is plausible that a rational agent will learn from the mistakes implied by biased beliefs in such a way that he will compensate for them. However, requiring him to express a view, which is unbiased on average, on all combinations of events that may conceivably occur over his investment horizon is a tall order. Presented in this way one may conclude that the requirements of the theory of rational beliefs is as onerous as that of the rational expectations framework.

5.5 Mathematical Appendix

5.5.1 Proof of Theorem 1.

⇒ Define the set $S \in \mathfrak{S}$ as $S = A \times \prod_{t+1}^{\infty} X = \{x \in \Omega \mid (x_0, \dots, x_t) \in X^{t+1}, x_j \in R^K \text{ for } j > t\}$ for

$t \geq 0$;

⇒ Observe that A is a measurable rectangle, i.e. a rectangle with rational endpoints, and that $A \in \mathfrak{S}'$;

⇒ Let \mathfrak{S}' denote the countable collection of all such sets in \mathfrak{S}' and $\mathfrak{S}' = \sigma(\mathfrak{S}')$;

⇒ Define the field $\mathfrak{S} = \left(\bigcup_{t=0}^{\infty} \mathfrak{S}' \right) = \{F_i; i = 1, 2, \dots\}$ as the collection of all subsets $\mathfrak{S}' \subset \Omega$,

and note that the field \mathfrak{S} is a semi-algebra;

⇒ From the construction of \mathfrak{S} we have that $\mathfrak{S} = \sigma(\mathfrak{S})$;

⇒ There must exist $C_i \in \mathfrak{S}$ such that i) $\Pi(C_i) = 1$ and ii) $\dot{m}(F_i|x)$ exists Π a.e. $\forall x \in C_i, F_i \in \mathfrak{S}$;

⇒ Now define $C = \bigcap_{i=1}^{\infty} C_i$. It follows that $\Pi(C) = 1$ since all the sets C_i are necessarily independent such that $\dot{m}(F_i|x)$ exists Π a.e. $\forall x \in C, F_i \in \mathfrak{S}$;

⇒ Let $A_n \in \mathfrak{S} \forall n = 1, 2, \dots$ such that $A_{n+1} \subset A_n$;

⇒ Observe that the A_n are measurable rectangles and that $\dim(A_{n+1}) < \dim(A_n)$ such that i) $\lim_{n \rightarrow \infty} A_n = 0$ and ii) $\bigcap_n A_n = 0$;

⇒ We can then use theorems 0.2 and 0.4 of Walter (1982) pages 4 and 5 to show that there exists a unique extension of $\dot{m}(\cdot|x)$, denoted $m(\cdot|x)$, on $(\Omega; \mathfrak{S})$ where $m(\cdot|x)$ is a countably additive probability measure on the σ -field \mathfrak{S} for each $x \in C$;

⇒ From the definition of stability we have $\lim_{n \rightarrow \infty} m^n(S|x) = m(S|x) \Pi$ a.e. $\forall x \in C, S \in \mathfrak{S}$ such that $\dot{m}(T^{-1}S|x) = m(S|x) \Pi$ a.e. $\forall x \in C, S \in \mathfrak{S}$;

⇒ This property is inherited by $m(\cdot|x)$ such that the dynamic system $(\mathfrak{S}; \mathfrak{S}; m(\cdot|x); T)$ is measure preserving and hence stationary (see Petersen p7);

5.5.2 Proof of Theorem 2

Part 1: Show that a Stable System is WAMS

⇒ Let $S \in \mathfrak{S}$ be a cylinder;

⇒ Define $1_S(T^k x) = \begin{cases} 1 & \text{if } x \in T^{-k}S \\ 0 & \text{if } x \notin T^{-k}S \end{cases}$

⇒ $\frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S) = \frac{1}{n} \sum_{k=0}^{n-1} \int_{\Omega} 1_S(T^k x) \Pi(dx) = \int_{\Omega} \frac{1}{n} \sum_{k=0}^{n-1} 1_S(T^k x) \Pi(dx) = \int_{\Omega} m^n(S|x) \Pi(dx)$;

⇒ Taking limits on both sides yields $\int_{\Omega} \bar{m}(S|x) \Pi(dx) = \bar{m}_{\Pi}(S)$;

⇒ Stability implies WAMS;

Part 2: Show WAMS implies Stability

⇒ Assume that $(\Omega; \mathfrak{S}; \Pi; T)$ is WAMS;

⇒ Note that if a system is stable we have that $\inf m^n(S|x) = \sup m^n(S|x) = m^n(S|x)$ in the limit (see definition 1);

⇒ Define i) $\underline{g}_S(x) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S)$ and ii) $\bar{g}_S(x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S)$;

⇒ Since we have that i) $\underline{g}_S(x) \leq \bar{g}_S(x)$ and ii) $\underline{g}_S(x) \geq 0$ it is sufficient to show that

$$\underline{g}_S(x) \geq \bar{g}_S(x) \Pi \text{ a.e.};$$

⇒ WAMS implies that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S) = \bar{m}_{\Pi}(S)$ exists;

⇒ Sufficient to show that $\int_{\Omega} \bar{g}_S(x) \Pi(dx) \leq \bar{m}_{\Pi}(S) \leq \int_{\Omega} \underline{g}_S(x) \Pi(dx)$;

⇒ Since $\bar{g}_S(x) \leq 1$, for any given $\varepsilon > 0$ there exists n such that $\frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S) \geq \bar{g}_S(x) - \varepsilon$;

⇒ Let $\underline{n}(x)$ be the smallest integer for which this holds;

⇒ Note that $\bar{g}_S(x)$ is an invariant function in T since $\bar{g}_S(T^k x) = \bar{g}_S(x)$;

$$\Rightarrow \frac{1}{\underline{n}(x)} \sum_{k=0}^{\underline{n}(x)-1} 1_S(T^k x) \geq \bar{g}_S(x) - \varepsilon; \quad (1)$$

$$\Rightarrow \sum_{k=0}^{\underline{n}(x)-1} 1_S(T^k x) \geq \sum_{k=0}^{\underline{n}(x)-1} \bar{g}_S(T^k x) - \underline{n}(x)\varepsilon;$$

$$\Rightarrow \sum_{k=0}^{\underline{n}(x)-1} 1_S(T^k x) + \underline{n}(x)\varepsilon \geq \sum_{k=0}^{\underline{n}(x)-1} \bar{g}_S(T^k x);$$

$$\Rightarrow \sum_{k=0}^{\underline{n}(x)-1} \bar{g}_S(T^k x) \leq \sum_{k=0}^{\underline{n}(x)-1} 1_S(T^k x) + \underline{n}(x)\varepsilon; \quad (2)$$

\Rightarrow Since $\underline{n}(x) < \infty$ there exists N such that $\Pi(\{x : \underline{n}(x) > N\}) = \sum_{k=N+1}^{\infty} \Pi(\{x : \underline{n}(x) = k\}) \leq \varepsilon$,

where the inequality follows from the fact that $\lim_{N \rightarrow \infty} \sum_{k=N+1}^{\infty} \Pi(\{x : \underline{n}(x) = k\}) = 0$; (3)

\Rightarrow Observe that although $\underline{n}(x)$ is chosen to be the smallest n for which (1) holds there may be $\{x\}$ such that $\underline{n}(x) \rightarrow \infty$ and the RHS of (2) becomes unbounded. We collect all such bad sets in the set B where $B = \{x : \underline{n}(x) > N\}$;

\Rightarrow By construction of equations (1) and (2) we have that x determines $\underline{n}(x)$, such that if $x \in B^c$ then $T^k x \in B^c$ $k = 1, 2, \dots, \underline{n}(x) - 1$;

\Rightarrow To handle the case where $x \in B$ we define:

$$\text{i) } \tilde{1}_S(x) = \begin{cases} 1_S(x) & \text{if } x \notin B \\ 1 & \text{if } x \in B \end{cases} \quad \text{and} \quad (4.a)$$

$$\text{ii) } \tilde{\underline{n}}(x) = \begin{cases} \underline{n}(x) & \text{if } x \notin B \\ 1 & \text{if } x \in B \end{cases}; \quad (4.b)$$

\Rightarrow From (2) we get $\sum_{k=0}^{\tilde{\underline{n}}(x)-1} \bar{g}_S(T^k x) \leq \sum_{k=0}^{\tilde{\underline{n}}(x)-1} 1_S(T^k x) + \tilde{\underline{n}}(x)\varepsilon$ where $\tilde{\underline{n}}(x) \leq N$; (5)

\Rightarrow Now, if $x \in B$ then $\bar{g}_S(x) \leq 1 + \varepsilon$. Since $\bar{g}_S(x) \leq 1$ we have that (5) holds if $x \in B$;

\Rightarrow Now, if $x \notin B$ then equation (5) is equivalent to (2) since $x \& T^k x \in B^c$ $k = 1, 2, \dots, \underline{n}(x) - 1$;

\Rightarrow Chose L such that $\frac{N}{L} < \varepsilon$;

⇒ Define $n_k(x)$ inductively such that i) $n_0(x)=0$ and ii) $n_k(x)=n_{k-1}(x)+\underline{n}(T^{n_{k-1}(x)}x)$ and let $\bar{k}(x)$ be the largest k such that $n_k(x)\leq L-1$;

$$\Rightarrow \sum_{l=0}^{L-1} \bar{g}_s(T^l x) = \sum_{k=1}^{\bar{k}(x)} \sum_{l=n_{k-1}(x)}^{n_k(x)-1} \bar{g}_s(T^l x) + \sum_{l=n_{\bar{k}(x)}}^{L-1} \bar{g}_s(T^l x); \quad (6)$$

⇒ Applying the bound in (5) to each of the $\bar{k}(x)$ blocks in the inner sum in (6) yields

$$\sum_{l=n_{k-1}(x)}^{n_k(x)-1} \bar{g}_s(T^l x) \leq \sum_{l=n_{k-1}(x)}^{n_k(x)-1} \bar{1}_s(T^l x) + [n_k(x) - n_{k-1}(x)]\mathcal{E}; \quad (7)$$

⇒ Note that $L \in (n_k(x), n_{k+1}(x)]$ since $n_{k+1}(x) = n_k(x) + \underline{n}(T^{n_k(x)}x)$ and that $\underline{n}(x) = [1; N]$ such that $L - n_k(x) \leq n_{k+1}(x) - n_k(x) \leq N - 1$;

$$\Rightarrow \bar{g}_s(x) \leq 1 \Rightarrow \sum_{l=n_k(x)}^{L-1} \bar{g}_s(T^l x) \leq N - 1; \quad (8)$$

⇒ Now, summing the right hand bound of the inner sum in equation (7) yields:

$$\begin{aligned} \sum_{k=1}^{\bar{k}(x)} \sum_{l=n_{k-1}(x)}^{n_k(x)-1} \bar{g}_s(T^l x) &\leq \sum_{k=1}^{\bar{k}(x)} \sum_{l=n_{k-1}(x)}^{n_k(x)-1} \bar{1}_s(T^l x) + \sum_{k=1}^{\bar{k}(x)} [n_k(x) - n_{k-1}(x)]\mathcal{E} \\ &= \sum_{l=1}^{n_{\bar{k}(x)}-1} \bar{1}_s(T^l x) + L\mathcal{E} \\ &\leq \sum_{l=0}^{n_{\bar{k}(x)}-1} \bar{1}_s(T^l x) + L\mathcal{E} \end{aligned} \quad (9)$$

⇒ Combining (8), (9) and (6) we get

$$\sum_{l=0}^{L-1} \bar{g}_s(T^l x) \leq \sum_{l=0}^{n_{\bar{k}(x)}-1} \bar{1}_s(T^l x) + L\mathcal{E} + (N-1) \leq \sum_{l=0}^{L-1} \bar{1}_s(T^l x) + L\mathcal{E} + (N-1); \quad (10)$$

⇒ Take expectation w.r.t. Π on both sides of inequality (10) yields:

$$\sum_{l=0}^{L-1} \int_{\Omega} \bar{g}_s(T^l x) \Pi(dx) \leq \sum_{l=0}^{L-1} \int_{\Omega} \bar{1}_s(T^l x) \Pi(dx) + L\mathcal{E} + (N-1); \quad (11)$$

⇒ By applying (4.a) and (3) to $\int_{\Omega} \bar{1}_s(x) \Pi(dx) = \int_B \bar{1}_s(x) \Pi(dx) + \int_{B^c} \bar{1}_s(x) \Pi(dx)$ we get:

$$\begin{aligned}
\int_{\Omega} \tilde{1}_S(x) \Pi(dx) &= \int_B \tilde{1}_S(x) \Pi(dx) + \int_{B^c} \tilde{1}_S(x) \Pi(dx) = \int_{B^c} 1_S(x) \Pi(dx) + \int_B 1 \Pi(dx) \\
&\leq \int_{\Omega} 1_S(x) \Pi(dx) + \Pi(B) \\
&\leq \int_{\Omega} 1_S(x) \Pi(dx) + \varepsilon \\
&= \Pi(S) + \varepsilon
\end{aligned} \tag{12}$$

$$\Rightarrow \text{Inserting (12) into (13) gives } \sum_{l=0}^{L-1} \int_{\Omega} \bar{g}_s(T^l x) \Pi(dx) \leq \sum_{l=0}^{L-1} \Pi(T^l x) + 2L\varepsilon + (N-1); \tag{13}$$

\Rightarrow Since $\bar{g}_s(x)$ is invariant and $\frac{N}{L} < \varepsilon$, by dividing (13) by L we get:

$$\int_{\Omega} \bar{g}_s(x) \Pi(dx) \leq \frac{1}{L} \sum_{l=0}^{L-1} \Pi(T^l x) + 3\varepsilon$$

\Rightarrow Since S is a cylinder and $(\Omega; \mathfrak{S}; \Pi; T)$ is WAMS by assumption we have that

$$\int_{\Omega} \bar{g}_s(x) \Pi(dx) \leq \dot{m}_{\Pi}(x); \tag{14}$$

\Rightarrow We can show that $\dot{m}_{\Pi}(S) \leq \int_{\Omega} \underline{g}_s(x) \Pi(dx)$ by using a proof similar to that used above;

\Rightarrow Thus we have shown that if $(\Omega; \mathfrak{S}; \Pi; T)$ is WAMS then $(\Omega; \mathfrak{S}; \Pi; T)$ is stable;

5.5.3 Proof of Theorem 3.

Part 1: We first proof part ii of theorem 3.

\Rightarrow Note that the sets $T^{-k}S$ have the following properties: i) $T^{-k}S \subset \Omega$ and ii)

$$T^{-j}S \cap T^{-i}S = \emptyset;$$

$\Rightarrow \dot{m}_{\Pi}$ is an countably additive probability measure on $\hat{\mathfrak{S}}$;

\Rightarrow We use the Caratheodory extension theorem to uniquely extend \dot{m}_{Π} to a measure m_{Π} on $(\Omega; \mathfrak{S})$;

\Rightarrow Define for $S \in \hat{\mathfrak{S}}$ $\dot{m}(S) = \int_{\Omega} \dot{m}(S|x) \Pi(dx)$, which is a finitely additive measure on $\hat{\mathfrak{S}}$;

\Rightarrow Define for $S \in \mathfrak{S}$ $m(S) = \int_{\Omega} m(S|x) \Pi(dx)$ which is a countably additive measure on \mathfrak{S} ;

\Rightarrow From proposition 2 we have that $\dot{m}(S) = \dot{m}_{\Pi}(S) = m_{\Pi}(S) \forall S \in \hat{\mathfrak{S}}$;

⇒ Since the extension m_{Π} is unique we have $m(S) = \int_{\Omega} m(S|x)\Pi(dx) = m_{\Pi}(S) \forall S \in \mathfrak{S}$;

⇒ From the definition of WAMS we have that \dot{m}_{Π} is invariant w.r.t. T, i.e. $\dot{m}_{\Pi}(T^{-k}S) = \dot{m}_{\Pi}(S)$. This property is inherited by m_{Π} such that $(\Omega; \mathfrak{S}; m_{\Pi}; T)$ is stationary;

⇒ Observe that $m(C) = 1$ since:

i. $m(S) = \int_{\Omega} m(S|x)\Pi(dx) \forall S \in \mathfrak{S}$;

ii. $m(S|x) = 0$ for $x \notin C$ and $S \in \mathfrak{S}$;

iii. $m(\Omega|x) = 1 \forall x$, which follows from a) $m(C|x) = 1$ for $x \in C$ and b) ii above;

⇒ From proposition 2 it follows that $m_{\Pi}(C) = 1$ such that $\Pi(C) = m(C) = m_{\Pi}(C) = 1$;

⇒ From part 1 of the proof of proposition 1 we have that $\int_{\Omega} \dot{m}(S|x)\Pi(dx) = \dot{m}_{\Pi}(S)$. We

will now show that this relation holds using m_{Π} instead of Π ;

→ Since m_{Π} is invariant in T we have that $m_{\Pi}(T^{-k}S) = m_{\Pi}(S) \forall S \in \mathfrak{S}$;

→ Since $m(\cdot|x)$ is a unique extension of $\dot{m}(\cdot|x)$ we can rewrite the above relation as

$$\text{follows } \int_{\Omega} m(S|x)m_{\Pi}(dx) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} m_{\Pi}(T^{-k}S) = m_{\Pi}(S);$$

$$\Rightarrow m_{\Pi}(S) = m(S) = \int_{\Omega} m(S|x)m_{\Pi}(dx);$$

Part 2: We now proof part i of theorem 3.

⇒ m_{Π} is strong asymptotically mean stationary since $m_{\Pi}(S)$ is stationary for $S \in \mathfrak{S}$;

⇒ From proposition 2 it follows that $(\Omega; \mathfrak{S}; m_{\Pi}; T)$ is strongly stable;

⇒ $\lim_{n \rightarrow \infty} m^n(S|x)$ exists m_{Π} a.e. $\forall S \in \mathfrak{S}$;

⇒ $\lim_{n \rightarrow \infty} m^n(S|x) = m(S|x)$ exists m_{Π} a.e. $\forall S \in \mathfrak{S}$ since $m(S|x)$ is a unique extension on $(\Omega; \mathfrak{S})$ of $\dot{m}(S|x)$;

⇒ Denote by $m_{\Pi}(S|\zeta)(x)$ the conditional probability of m_{Π} given the σ -field of invariant events;

⇒ By theorem 6.6.1 of Gray (2001) we have that $m_{\Pi}(S|\zeta)(x) = m(S|x) m_{\Pi}$ a.e.
 $\forall S \in \mathfrak{S}$;

5.5.4 Proof of Theorem 4.

⇒ Since Ω is a complete and separable metric space (see Gray 2001 p 52-53 for definition) we have that $P(\Omega, \mathfrak{S})$ is closed in the topology of weak convergence (see definition 4 and Gray 2001 p 6-7);

⇒ We show that this property is not inherited by $\mathfrak{R}(\Pi)$ by showing that there exists Q such that $Q \notin \mathfrak{R}(\Pi)$ and $\lim_{n \rightarrow \infty} Q_n = Q$;

⇒ By lemma 3.2.4 of Gray 2001 we have that $P(\Omega)$ is a complete and separable metric space;

⇒ Let m_{Π} be defined as in theorem 3;

⇒ Let $H(\Omega, \mathfrak{S}, m_{\Pi})$ denote the set of stable measures μ such that $\bar{\mu} = m_{\Pi}$, where $\bar{\mu}$ is the stationary measure associated with μ (see definition 2);

⇒ Let ν be another probability measure on (Ω, \mathfrak{S}) which need not be stable or belong to $H(\Omega, \mathfrak{S}, \pi)$;

⇒ Define for $n \in \mathbb{N}$ the truncated probability measure ν^n on $(X^n, \sigma(X^n))$ such that $\nu^n(A) = \nu(A \times \Omega)$ for $A \in \sigma(X^n)$;

⇒ Define the composite probability measure ν_n on (Ω, \mathfrak{S}) to be the unique probability measure on (Ω, \mathfrak{S}) such that $\nu_n(E) = \nu^n(A)\mu(B)$ for any rectangle $E = A \times B$ such that $A \in \sigma(X^n)$ and $B \in \mathfrak{S}$;

⇒ Now, for any cylinder S and observe that ν_n is stable and that $\bar{\nu}_n = m_{\Pi}$:

$$\rightarrow \frac{1}{J} \sum_{i=0}^{J-1} \nu_n(T^{-i}S) = \frac{1}{J} \left[\sum_{i=0}^{n-1} \nu_n(T^{-i}S) + \sum_{i=n}^{J-1} \nu_n(T^{-i}S) \right];$$

→ Concentrating on the second summation we let $E_i = T^{-i}S = \{x : T^i x \in S\} \forall i \geq n$;

→ Write $E_i = A_n \times B_{i-n}$ and observe that if we let $B_{i-n} = T^{-i+n} = \{x : T^{i-n}x \in S\}$ then there exists a convergence set for $T^{-n}S$, such that $A_n \in \sigma(X^n)$ and $v^n(A_n) = 1$, see proof of theorem 1 and theorem 3;

$$\rightarrow \frac{1}{j} \sum_{i=0}^{j-1} v_n(T^{-i}S) = \frac{1}{m} \left[\sum_{i=0}^{n-1} v_n(T^{-i}S) + \sum_{i=0}^{m-1-n} \mu(T^{-i}S) \right] \xrightarrow{m \rightarrow \infty} \bar{\mu}(S) = \pi(S)$$

⇒ For any cylinder S we have $\lim_{n \rightarrow \infty} V_n(S) = V(S)$;

⇒ By corollary 8.2.2. of Gray (2002) we have that V_n converges to V weakly;

⇒ Now if we pick V which is not stable then we have proved that $H(\Omega, \mathfrak{F}, \pi)$ is not closed in the topology of weak convergence;

5.5.5 Proof of Theorem 5.

Part 1: Let $q \in \mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$:

⇒ First we need to prove that Q is a probability, we proceed as follows:

→ If $S = \Omega$ then $f^{\Omega}(\mu) = \mu(\Omega) = 1$;

→ If $S = \emptyset$ then $f^{\emptyset}(\mu) = \mu(\emptyset) = 0$;

→ If $S = \bigcup_{i=1}^{\infty} S_i$ with $S_i \cap S_j = \emptyset \forall i \neq j$ then $f^S(\mu) = \lim_{n \rightarrow \infty} \mu(S_i) = \lim_{n \rightarrow \infty} f^{S_i}(\mu)$, which exit q a.e.;

→ By the bounded convergence theorem (see Billingsley 1978 Theorem 16.5) we

$$\text{have } Q(S) = \lim_{n \rightarrow \infty} \int_{\mathfrak{R}(\Pi) \cap \varphi(\varepsilon)} f^{S_i}(\mu) q(d\mu) = \sum_{i=1}^{\infty} Q(S_i);$$

→ This proves the Q is a probability;

⇒ We now need to show that $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$. Let S be a cylinder and proceed as follows:

$$\rightarrow \frac{1}{n} \sum_{k=0}^{n-1} Q(T^{-k}S) = \int_{\mathfrak{R}(\Pi) \cap \varphi(\varepsilon)} \left(\frac{1}{n} \sum_{k=0}^{n-1} f^{T^{-k}S}(\mu) \right) q(d\mu);$$

→ But from (7) we have that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f^{T^{-k}S}(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}S) = m(S) \quad q \text{ a.e.};$

→ Applying the bounded convergence theorem again we have that:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(T^{-k}S) &= \int_{\mathfrak{R}(\Pi) \cap \varphi(\varepsilon)} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f^{T^{-k}S}(\mu) \right) q(d\mu) \\
&= \lim_{n \rightarrow \infty} \int_{\mathfrak{R}(\Pi) \cap \varphi(\varepsilon)} \left(\frac{1}{n} \sum_{k=0}^{n-1} f^{T^{-k}S}(\mu) \right) q(d\mu) \\
&= m(S)
\end{aligned}$$

→ We thus have shown that $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$;

Part 2: Let $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$:

⇒ First show that $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ is a convex, compact subset of $P(\Omega, \mathfrak{S})$:

→ Convexity follows from the stability assumption (a.5);

→ Boundedness follows from (a.2):

- If we let \bar{X} be the closure of X , then $P(\bar{\Omega}, \bar{\mathfrak{S}})$ is compact in the topology of weak convergence, where:

i. $\bar{\Omega} = \bar{X}^{\infty}$;

ii. $\bar{x}_t \in \bar{X}$;

iii. $\bar{\mathfrak{S}} = \sigma\left(\bigcup_{t=0}^{\infty} \bar{\mathfrak{S}}^t\right)$ and $\mathfrak{S}^t = \sigma(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_t) \ t \geq 0$;

- $\varphi^*(\varepsilon)$ is therefore a closed subset of compact set, and hence compact;
- Nielsen (1996) shows that (a.2) is necessary for the compactness of $\varphi^*(\varepsilon)$;

→ Compactness follows from the fact that $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ is closed and bounded;

⇒ We now refer to the Choquet integral representation theorem (see Walters (1982) p153) which states that there exists $q \in P(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ such that q represents Q and the support of q is the set of extreme points in $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$;

5.5.6 Proof of Theorem 6.

Part 1: Let q be a rational belief. By axiom 1 $q \in P(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ and by theorem 5 there exists $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ which represents q :

- i. From the Lebesgue Decomposition Theorem (see Gray (2002) theorem 5.6.2 p 107) there exists probabilities Q^m and Q^o on (Ω, \mathfrak{S}) , sets $A \subset \Omega$ and $B = \Omega - A$, and a

constant $0 \leq \lambda_Q \leq 1$ such that $Q = \lambda_Q Q^m + (1 - \lambda_Q) Q^\circ$ where $Q^m \ll m$, $Q^\circ \perp m$, $\lambda_Q = Q(A)$, $m(A) = 1$ and $m(B) = 0$;

\Rightarrow By axiom 2, $m(A) = 1$ implies $Q(A) > 0$ such that $0 < \lambda_Q \leq 1$;

\Rightarrow Now show that Q^m and m are equivalent. Since we already have that $Q^m \ll m$, we need to show that $m \ll Q^m$. Assume $m \ll Q^m$ is false;

\rightarrow Let $S \in \mathfrak{S}$ such that $Q^m(S) = 0$ and $m(S) > 0$;

\rightarrow From axiom 2 it follows that $Q(S) > 0$ such that

$$0 < Q(S) = \lambda_Q Q^m(S) + (1 - \lambda_Q) Q^\circ(S) = (1 - \lambda_Q) Q^\circ(S)$$

$$0 < m(S) = m(S \cap A) + m(S \cap B) = m(S \cap A)$$

\rightarrow Let $\hat{S} = S \cap A$. Now $\hat{S} \subset S$ implies $Q^m(\hat{S}) = 0$ and $\hat{S} \subset A$ implies that $Q^\circ(\hat{S}) = 0$ since $Q^\circ \perp m$;

\rightarrow Thus $Q(\hat{S}) = 0$ which contradicts axiom 2;

ii. From i above we have that $\frac{1}{n} \sum_{k=0}^{n-1} Q(T^{-k}S) = \lambda_Q \frac{1}{n} \sum_{k=0}^{n-1} Q^m(T^{-k}S) + (1 - \lambda_Q) \frac{1}{n} \sum_{k=0}^{n-1} Q^\circ(T^{-k}S)$;

\Rightarrow Since $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ the L.H.S converges to $m(S)$ for all cylinders $S \in \mathfrak{S}$;

\Rightarrow Since $Q^m \ll m$ and m is stationary we apply theorem 4 and theorem 2 of Gray and Kieffer (1980) to show that Q^m is strong asymptotically mean stationary;

\Rightarrow Therefore, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q^m(T^{-k}S) = \bar{Q}^m$ exists $\forall S \in \mathfrak{S}$;

\Rightarrow It follows that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q^\circ(T^{-k}S) = \bar{Q}^\circ$ exists for all cylinders $S \in \mathfrak{S}$;

\Rightarrow Now extend \bar{Q}° to a measure Q° on (Ω, \mathfrak{S}) , and it follows that

$$m(S) = \lambda_Q \bar{Q}^m(S) + (1 - \lambda_Q) \bar{Q}^\circ(S) \quad \forall S \in \mathfrak{S},$$

which shows that $(\Omega, \mathfrak{S}, Q^m, T)$ and $(\Omega, \mathfrak{S}, Q^\circ, T)$ are stable with stationary measures \bar{Q}^m and \bar{Q}° ;

\Rightarrow We now need to show that \bar{Q}^m and m is equivalent. Since we already have that $\bar{Q}^m \ll m$, we need to show that $m \ll \bar{Q}^m$;

- Assume $m \ll \bar{Q}^m$ and select $S \in \mathfrak{S}$ with $\bar{Q}^m(S) = 0$ and $m(S) > 0$;
 - Since Q^m is equivalent to m , $Q^m(S) > 0$;
 - Define $\hat{S} = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} (T^{-k}S) = \limsup_{k \rightarrow \infty} (T^{-k}S)$;
 - Now, from the continuity of probability it follows that

$$\bar{Q}^m(\hat{S}) = \bar{Q}^m\left(\bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} (T^{-k}S)\right) = \lim_{n \rightarrow \infty} \bar{Q}^m\left(\bigcup_{k=n}^{\infty} (T^{-k}S)\right) \leq \bar{Q}^m\left(\bigcup_{k=0}^{\infty} (T^{-k}S)\right) \leq \sum_{k=0}^{\infty} \bar{Q}^m(T^{-k}S) = 0$$
 - Since \hat{S} is an invariant set and Q^m is strong asymptotically mean stationary it follows from lemma 6.3.1 of Gray (2001) that $\bar{Q}^m(\hat{S}) = Q^m(\hat{S}) = 0$;
 - Since Q^m is equivalent to m it follows that $m(\hat{S}) = 0$;
 - Since $m(\hat{S}) > -\infty$ we can apply Fatou's lemma to derive the following

$$m(\hat{S}) = m\left(\limsup_{k \rightarrow \infty} (T^{-k}S)\right) \geq \limsup_{k \rightarrow \infty} m(T^{-k}S) = m(S) > 0;$$
 - This is a contradiction, which proves that $m \ll \bar{Q}^m$;
- ⇒ Since $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ we have that Q is a weakly uniform stable probability measure;
- $\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$ is a convex, compact subset of $P(\Omega, \mathfrak{S})$, see Proof of Theorem 5;
 - It therefore follows that Q^m and Q^o are weakly uniform stable probability measures;
- iii. This part of the proof uses a result from Blackwell and Dubins (1962) which reads: Since m and Q^m are probability measures on (Ω, \mathfrak{S}) and $Q^m \ll m$, then for each t and every regular conditional probability m_t there exists a corresponding conditional probability Q_t^m such that $\forall S \in \mathfrak{S}'$ $m_t(S|I_t) \approx Q_t^m(S|I_t)$ a.e. Q^m and m ;
- ⇒ By definition we have that $Q(S \cap I_t) = \int_{I_t} Q_t(S|I_t) dQ$;
- ⇒ Applying the Lebesgue Decomposition Theorem we have that

$$Q(S \cap I_t) = \lambda_Q Q^m(S \cap I_t) + (1 - \lambda_Q) Q^o(S \cap I_t)$$

⇒ Since $Q^m \ll m$ we can apply the Randon-Nikodym Theorem to $Q^m(S \cap I_t)$ and

$$\text{get } \int_{I_t} Q_t(S|I_t) dQ = \lambda_Q \int_{I_t} Q_t^m(S|I_t) dm + (1 - \lambda_Q) \int_{I_t} Q_t^o(S|I_t) dQ^o;$$

$$\Rightarrow Q_t(S|I_t) dQ = \lambda_Q Q_t^m(S|I_t) dm + (1 - \lambda_Q) Q_t^o(S|I_t) dQ^o$$

$$\Rightarrow Q_t(S|I_t) = \lambda_Q Q_t^m(S|I_t) \psi_m + (1 - \lambda_Q) Q_t^o(S|I_t) \psi_o$$

$$\Rightarrow Q_t(S|I_t) \approx \lambda_Q m_t(S|I_t) \psi_m + (1 - \lambda_Q) Q_t^o(S|I_t) \psi_o \text{ a.e. } Q^m \text{ and } m;$$

Part 2: Let Q, Q^m, Q^o and λ_Q be defined as in i to iii.

⇒ Q is weakly uniform stable, hence $Q \in \mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon)$;

⇒ It follows from theorem 5 that there exists $q \in \mathcal{P}(\mathfrak{R}(\Pi) \cap \varphi^*(\varepsilon))$ such that q represents Q , which proofs axiom 1;

⇒ Now choose $S \in \mathfrak{S}$ such that $m(S) > 0$;

⇒ Since $Q^m \ll m$ and $m \ll Q^m$ it follows that $Q^m(S) > 0$;

⇒ Since $\lambda_Q > 0$ it follows that $Q(S) > 0$;

⇒ $q(B_S) > 0$, which proofs axiom 2;

5.5.7 Proof of Corollary 1.

⇒ We proceed by showing that if $(\Omega, \mathfrak{S}, \Pi, T)$ is ergodic then $m(S|x)$ is independent of x Π a.e. $\forall S \in \mathfrak{S}$;

⇒ Let $S \in \mathfrak{S}$ and define:

$$1) A_C = \{x \in \Omega : m(S|x) = m_\Pi(S)\}$$

$$2) A_C^+ = \{x \in \Omega : m(S|x) > m_\Pi(S)\}$$

$$3) A_C^- = \{x \in \Omega : m(S|x) < m_\Pi(S)\}$$

⇒ Observe that all three sets are invariant in T and since $(\Omega, \mathfrak{S}, \Pi, T)$ is ergodic we have that $\Pi(A_C) = 0$ or $\Pi(A_C) = 1$ (see Gray (2001) p147);

\Rightarrow From theorem 3 we have that $m(S) = \int_{\Omega} m(S|x)\Pi(dx) = m_{\Pi}(S) \quad \forall S \in \mathfrak{S}$, it therefore follows that we must have that $\Pi(A_C) = 1$ if we do not want to contradict the fact that m is a unique extension of \dot{m} ;

\Rightarrow As a consequence of the above we have $\Pi(A_C^+) = \Pi(A_C^-) = 0$;

Chapter 6 - General Equilibrium Under Uncertainty

6.1 Introduction

Chapter 5 discussed how agents could formulate diverse beliefs by observing the data generated by the stochastic process that underlies the economy. It was shown that correlation of agent's beliefs could impact the price formation process thereby causing speculative fluctuations in markets. Chapter 6 explores this issue further by considering the effect of extrinsic uncertainty that is perfectly correlated with the rational expectations of agents on the price formation process.

We begin in section 6.1 by considering the properties of a simplified general equilibrium model within the context of an overlapping generations (OLG) model. Models of rational expectations typically determine prices on the basis of consumer preference, initial endowments and production sets. These factors (i.e. preference, endowments and production sets) represent the intrinsic structure of the economy. Price uncertainty within this framework is a reflection of randomness in the underlying intrinsic structure itself, and is not affected by events unrelated to economic fundamentals.

Section 6.3 sets the scene by extending the basic general equilibrium model introduced in section 6.2 to one with random prices and no intrinsic uncertainty. Price randomness in this extension of the model will be entirely due to the views and beliefs held by individuals about their environment. A sunspot equilibrium occurs when these beliefs and views are widely held by consumers such that price randomness may become self-fulfilling. Under these conditions price randomness will not necessarily dissipate, even asymptotically, as individuals accumulate more observations.

It is worthwhile expanding on the meaning of sunspots. In the context of market equilibrium one can view sunspots as a variety of events or psychological factors that are unrelated to the intrinsic structure of the economy, yet, which influence the forecasts and actions of economic decision-makers. Examples of these psychological factors include the animal spirits referred to by Keynes, fears and Bayesian learning theories.

The notion of periodicity is important in this analysis. In particular; the sequence of prices $\{p_t\}_{t=1}^{\infty}$ is a periodic competitive equilibrium of order k (or k -cycle) if $p_t = p_{t+k}$ for $t = 1, 2, \dots$ and $k \geq 2$ while $p_t \neq p_{t+j} \forall j \in [0; k]$.

Section 6.4 establishes a sufficient condition for the existence of a stationary sunspot equilibrium (SSE), thereby describing a class of economies for which sunspot equilibria exist. This condition is related to the stochastic characteristics of extrinsic uncertainty and affects the shape of the savings function. Section 6.5 describes how the stationary equilibria bifurcate to sunspot equilibria, where bifurcation can approximately be defined as the passage through zero of the derivative of a continuous function. Section 6.6 establishes the connection between SSE and a deterministic periodic equilibrium of order two.

Section 6.7 rounds up this chapter by considering a very perplexing fact; stationary sunspot equilibria may exist in theory, but this does not answer the key question as to why investors initially should believe in them to the extent that they should influence equilibrium prices. This question will be addressed in chapter 7.

The results derived in this chapter are based on previous work in this area by Costas Azariadis (1981, 1986), Roger Guesnerie (1986) and David Cass and Karl Shell (1983). The interested reader will find all proofs of results derived in the main body of the text in the appendix.

6.2 A Simplified Overlapping Generations Model Under Perfect Foresight

Consider an infinite horizon economy consisting of a successive number of generations at discrete points in time, i.e. $t = 0, 1, \dots, \infty$.

For every $t \geq 1$ there exists a new fixed-size population of identical individuals who live for two periods and die at the end of $t + 1$. Each generation can only produce in their youth (i.e. in period t) whilst consuming only in old age (i.e. in period $t+1$). Let $e = (e_1, e_2) \gg 0$ represent each generation's endowment vector, where e_1 represents units of divisible leisure in youth and e_2 units of a single, perishable consumption good in old age.

The initial generation of period $t = 0$ is already old and endowed with e_2 units of the consumption good and one unit of fiat money, an intrinsically worthless paper asset, which is the sole store of value in the economy.

Each member of the young generation may use a constant returns to scale technology to transform $n \in [0; e_1]$ units of his own leisure into $n \leq y$ of the perishable consumption good in excess of e_2 . The entire stock of fiat money is owned by the old generation at period $t \geq 0$, provided it has positive value. All individuals are price takers and possess perfect foresight about future prices.

Individuals have preferences over intertemporal bundles that are represented by a utility function denoted by $u(c_{t+1}; y_t)$, assumed to be monotone, twice continuously differentiable and strictly concave. Note that the utility of an individual born at time t depends upon: 1) the leisure (denoted by e_1) he gives up at time t , or, equivalently upon the amount of goods he offers (denoted by y_t) and 2) his consumption c_{t+1} in period $t+1$.

Two additional assumptions need to be made to complete the description of this simple economy: 1) consumption and leisure are strictly normal goods, and 2) young individuals choose positive savings if confronted with a zero real rate of interest.

Let $D(p_t, p_{t+1})$ represent the aggregate excess demand function at time t (i.e. for the old and young alive at that time). The excess demand of the old at time t equals the purchasing power of the fiat money, i.e. $1/p_t$, whilst the excess demand of the young equals $-s(R)$, where $s(R)$ is the young's saving function, from this it follows that:

$$D(p_t, p_{t+1}) = 1/p_t - s(p_t/p_{t+1}) \quad (1)$$

$$s(R) = \max u(\omega_2 + Ry, y) \text{ where } y \in [0, e_1] \text{ and } R = p_t/p_{t+1} \quad (2)$$

A competitive equilibrium with perfect foresight is defined as:

- 1) a sequence $\{p_t\}_{t=0}^{\infty}$ of non-negative prices that satisfy $D(p_t, p_{t+1}) = 0 \forall t$, or
- 2) a sequence $\{1/m_t\}_{t=0}^{\infty}$ of real money balances that satisfy $D(1/m_t, 1/m_{t+1}) = 0$ where $1/m_t = p_t, \forall t$

To find equilibria with perfect foresight one needs to solve the difference equation $D(p_t, p_{t+1}) = 0$ either: 1) backwards, where the solution has the form $m_t = f(m_{t+1})$, or; 2) forwards, where the solution has the form $m_{t+1} = \phi(m_t)$, where f and ϕ are known maps such that $\phi = f^{-1}$.

To establish sufficient conditions for the existence of SSE, we first need to establish under what conditions a stationary equilibrium under perfect foresight exists.

Proposition 1. If the individual's indifference map satisfies standard boundary assumptions and if consumption and leisure are normal goods, then a backward-looking competitive equilibrium exists and it is unique. Further, if the current price is not too small, a forward-looking competitive equilibrium exists as well, but it is not necessarily unique.

Section 6.3 introduces uncertainty in the form of a two state transition probability matrix to model the occurrence of sunspot activity or the absence thereof through a Markov process. This introduces the notion of periodicity, which is important in establishing sufficient conditions for the existence of a two-state SSE. More formally, the sequence

$\{p_t\}_{t=0}^{\infty}$ is a periodic competitive equilibrium of order k , i.e. a k -cycle, if $p_t = p_{t+k}$ for $t = 0, 1, 2, \dots$ and $k \geq 2$ while $p_t \neq p_{t+j} \forall j \in (0; k)$. This leads us to proposition 2.

Proposition 2. If the monetary stationary equilibrium is locally dynamically stable, then a 2 cycle exists.

Proposition 2 is from Grandmont (1985), the proof of which is left until later when it is restated and proved as corollary 2. Figure 1 a & b illustrates proposition 1 and 2. Panel a represents the sequence (p_1, p_2, p_3, \dots) as a competitive equilibrium. Panel b represents a two cycle with alternating price sequence (p_1, p_2, p_1, \dots) .

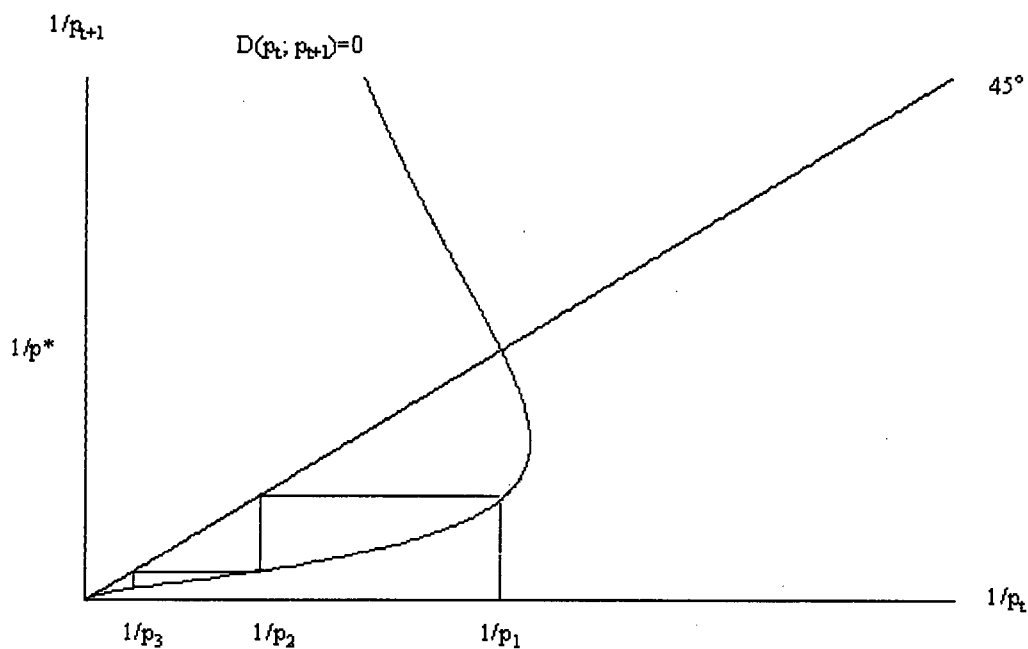


Figure 1.a.

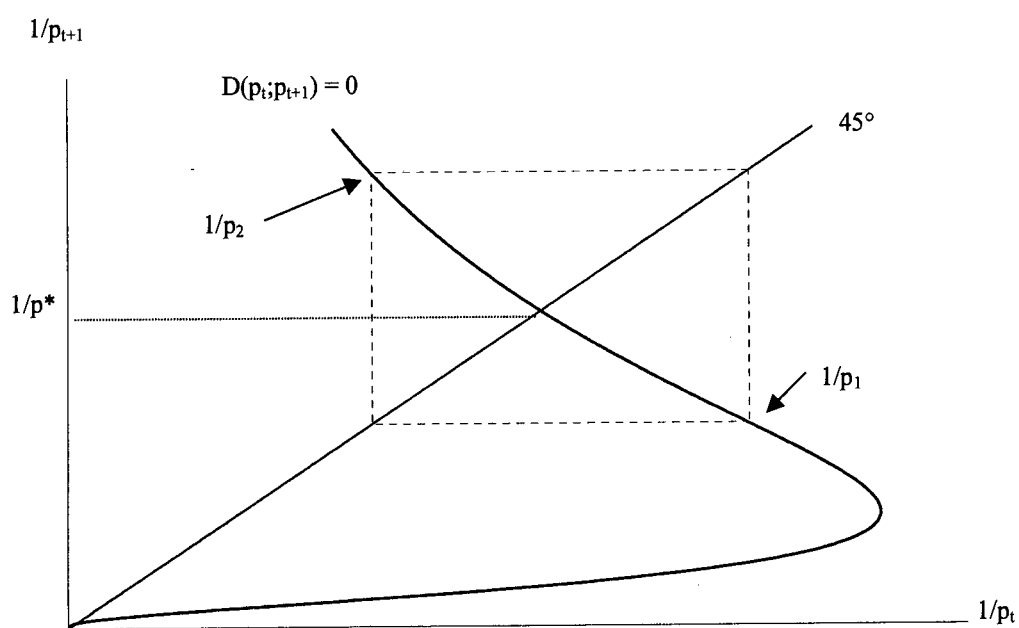


Figure 1.b.

The simplistic structure of the economy described above deserves some comments as it is represented by successive generations of identical consumers that use a constant returns to scale technology to produce a single physical commodity. A more generalised framework will be considered in section 6.7.

6.3 Properties of Stationary Sunspot Equilibria

Sunspot equilibria are rational expectations equilibria that are perfectly correlated with extraneous events (i.e. with events that preclude the preferences, endowments and production sets of individuals in the economy). These equilibria are not necessarily stationary and there may be many different extraneous events that are perfectly correlated with rational expectations equilibria. Let us consider a set of S events, the occurrence of which is governed by a Markov process with the following transition probability matrix;

$$\Pi_S = \begin{pmatrix} \pi_{1,1} & \dots & \pi_{i,1} & \dots & \pi_{S,1} \\ \dots & \dots & \dots & \dots & \dots \\ \pi_{1,i} & \dots & \pi_{i,i} & \dots & \pi_{S,i} \\ \dots & \dots & \dots & \dots & \dots \\ \pi_{1,S} & \dots & \pi_{i,S} & \dots & \pi_{S,S} \end{pmatrix} \quad (3.a)$$

Where $\pi_{i,j}$ is the probability that sunspot activity will be i tomorrow given that it is j today.

We will initially establish the existence of a very limited class of sunspot equilibria, i.e. a two-state stationary sunspot equilibrium (SSE) characterised as follows: either the existence of sunspot activity (state 1), or the absence of sunspot activity (state 2), which will establish how the set of stationary equilibria is enlarged by the sunspot hypothesis. Given (3.a), we can represent a two-event Markov process with the following transition probability matrix;

$$\Pi = \begin{pmatrix} \pi_{1,1} & \pi_{2,1} \\ \pi_{1,2} & \pi_{2,2} \end{pmatrix} \quad (3.b)$$

Furthermore, we confine the analysis to the stationary case because: 1) stable beliefs are likely to be the asymptotic outcome of many well defined learning processes; and 2) understanding stationary sunspot equilibria is a prerequisite to understanding transitory sunspot equilibria.

Since SSE are rational expectations equilibria that are perfectly correlated with extraneous events other than the intrinsic structure of the economy we have that forecast price behaviour is validated by actual price behaviour. This allows us to define the forecast of future prices to be $p_i = \phi(i)$, for $i = (1,2)$, if state i occurs tomorrow.

Let $z(R, \pi)$ be the rational expectations counterpart of the perfect-foresight savings function s , the properties of which are summarised under lemma 1 and 2.

Lemma 1. The function $z(R, \pi)$ has the following properties;

- i. $z(R, \pi) = \max_{y \in [0, e_1]} \{ \pi u(e_2 + y, y) + (1 - \pi) u(e_2 + Ry, y) \}$;
- ii. $z(R, \pi)$ is single-valued and continuous;
- iii. $z(R, 0) = s(R)$ for all R ;
- iv. $z(1, \pi) = s(1)$ for all π ;
- v. $z(R, \pi)$ lies between $s(R)$ and $s(1)$ for all R and π ;
- vi. $z(R, \hat{\pi})$ lies between $z(R, 0)$ and $z(R, \pi)$ if $\hat{\pi} < \pi$;

Lemma 2. Let $\eta(R; \pi)$ be the wage elasticity of saving under stochastic beliefs w.r.t. the real wage R , evaluated at $(R; \pi)$. Then $\eta(R; \pi) = (1 - \pi)\epsilon(1)$ for all π , where $\epsilon(R)$ is the corresponding elasticity of saving under perfect foresight.

Lemma 1 and 2 allows us to present a formal definition for a stationary sunspot equilibrium, i.e.;

Definition 1. A SSE is a vector $(p_1 \ p_2 \ \pi_{11} \ \pi_{22})$ of positive numbers such that;

- i. π_{11} and $\pi_{22} \in (0, 1)$, where π_{11} and π_{22} are the diagonal elements of the matrix Π defined in equation 3.b;
- ii. $p_1 \neq p_2$ (Note that this restriction is necessary to avoid the SSE degenerating to a stationary equilibrium where the sequence $\{p_t\}_{t=0}^{\infty}$ follows a golden rule path), and;
- iii. the excess demand function is zero for each state, i.e.

$$D^1 \equiv 1/p_1 - z\left(p_1/p_2; \pi_{11}\right) = 0 \quad (4.a)$$

$$D^2 \equiv 1/p_2 - z\left(p_2/p_1; \pi_{22}\right) = 0 \quad (4.b)$$

It can be seen that this definition is equivalent to the one given earlier as the prices $p_i = \phi(i)$, for $i = (1,2)$ are self-fulfilling since by equation 4.a. we have that p_1 is the equilibrium price if state 1 occurs in the present period, and by equation 4.b. we have that p_2 is the equilibrium price if state 2 occurs in the present period.

It is worthwhile considering the special case in which transitions along the diagonal of matrix Π is ruled out, i.e. when $\pi_{11} = 0$ and $\pi_{22} = 0$. In this case the equilibrium prices p_1 and p_2 will necessarily succeed each other. This leads us to corollary 1:

Corollary 1. A deterministic two-cycle is the limiting sunspot equilibrium associated with a 2x2 degenerate matrix Π for which $\pi_{11} = 0$ and $\pi_{22} = 0$.

Proof. The proof follows easily by combining property iii of lemma 1 with equations (4.a) and (4.b).

Corollary 1 allows us to establish a weak relationship between sunspots and cycles in the following theorem.

Theorem 1.a. In an economy that admits a periodic equilibrium of order two, there is generally a subset $\nu(\overline{\Pi}) \subset \Pi$, where Π is defined as before, such that a SSE of order two exists w.r.t. every $\Pi \in \nu(\overline{\Pi})$.

Throughout this chapter I shall only consider sunspots of order 2. However, it is worth considering whether one can extend theorem 1.a to include sunspots of a higher order, say a sunspot of order $m > 2$. Before we proceed with an analysis of this problem it is necessary to extend definition 1 to the more general class of sunspot equilibria.

Definition 2. A SSE of order m is a vector of positive numbers $(p; \pi_1; \dots; \pi_m)$ such that the excess demand function $f(p; \pi_1; \dots; \pi_m) = 0$, or for notational efficiency $f(p; \Pi_m) = 0$, where:

- i. $p = (p_1; \dots; p_m)$ where p_j is the future price if state j occurs tomorrow such that $p_i \neq p_j \forall i \neq j$;
- ii. $\pi_j = (\pi_{j1}; \pi_{j2}; \dots; \pi_{jm}) \forall j = 1; \dots; m$ such that $\sum_m \pi_{jm} = 1 \forall j$ where π_{ji} is the probability that sunspot activity will be j tomorrow given that it is i today;
- iii. The excess demand function $f(p; \Pi_m)$ is defined as follows;

$$f(p; \Pi_m) = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = \begin{bmatrix} \frac{1}{p_1} - Z_1 \\ \vdots \\ \frac{1}{p_m} - Z_m \end{bmatrix} \text{ where}$$

$$f_j = \frac{1}{p_j} - Z_j \text{ and } Z_j = \max_{y \in (0; e_1)} \left\{ \sum_m \pi_{jm} u \left(e_2 + \frac{p_j}{p_m} y; y \right) \right\} \forall j = 1, \dots, m$$

Definition 2 allows us to explore a possible extension to theorem 1.a in the form of theorem 1.b. Theorem 1.b relies on two mathematical results summarised in lemmas 3 and 4 respectively.

Lemma 3. Let $D_p f$ be the $\eta = m \times m$ derivative of the excess demand function f w.r.t. p_m . Then $Df(p_m) p_m^T = 0$ where $Df(p_m)$ is the derivative of f evaluated at p_m .

This result is represented graphically in figure 2 below for the simple case where $p_m^* = (x_1^*; x_2^*)$.

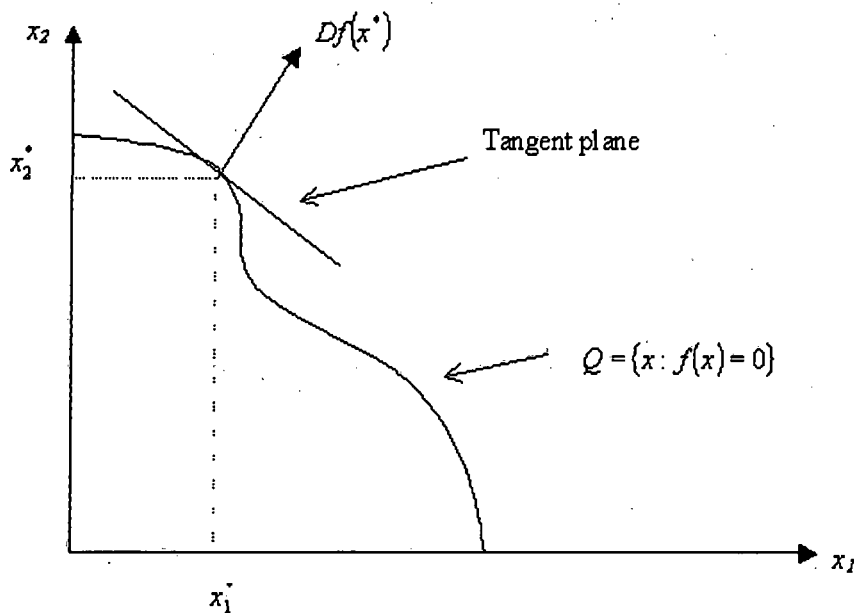


Figure 2

Lemma 4. Let B be a $M \times K$ matrix with a non zero entry in every row. Then by perturbing only the nonzero entries of B , we can find a nearby matrix B' such that if $B'q = 0$ has a solution $q \gg 0$, then $\text{rank } B' = M$.

Lemma 3 is from Hall pp 317 and lemma 4 is from Mas-Colell (1985) pp 339. Theorem 1.b. follows directly.

Theorem 1.b. Assume that the system has (at least) one stationary equilibrium. To every $m-1$ sunspot equilibrium which is not a bifurcation point of the mapping defining $m-1$ sunspot equilibria (an extension to $m > 2$ of $F(\cdot)$ defined in section 6.4), one can associate (an infinity of) m -sunspot equilibrium.

Note that theorem 1.b does not imply that the existence of an SSE of order 2 implies the existence of an SSE of order $m > 2$, as in the limit when m increases to an arbitrary large number one will exhaust the subset of Π_m for which the implicit function theorem holds, i.e. for which $f(p_m; \Pi_m) = 0$.

6.4 Establishing the Existence of SSE

By solving equations 4.a and 4.b simultaneously we can define a single valued function $F(\cdot)$ that can be used to represent the existence of a SSE as defined in definition 1 above. The properties of $F(\cdot)$ is summarised in Lemma 5.

Lemma 5. The function $F(\cdot)$ has the following properties:

- i. $F(w, \pi_{11}, \pi_{22}) = w \times z(w, \pi_{11}) - z(1/w, \pi_{22})$
- ii. A SSE exists if and only if F has a positive root $w \neq 1$ for some $\pi_{11} \in (0,1)$ and $\pi_{22} \in (0,1)$;
- iii. $F(1, \pi_{11}, \pi_{22}) = 0$
- iv. $F \rightarrow \infty$ as $w \rightarrow \infty$
- v. $F(w, \pi_{11}, \pi_{22}) < 0$ for w small enough;
- vi. If w is a root of $F(w, \pi_{11}, \pi_{22})$, then $\hat{w} = 1/w$ is a root of $F(\hat{w}, \pi_{22}, \pi_{11})$;

Theorem 2 uses the results derived so far to establish sufficient conditions for the existence of a two-state SSE.

Theorem 2. If the utility function satisfies regularity assumptions on differentiability, concavity and boundary behaviour, then a sufficient condition for the existence of sunspot equilibrium with respect to a given Markov transition probability matrix Π is

$$\varepsilon(1) < 0 \text{ and } \pi_{11} + \pi_{22} < 2 - \frac{1}{|\varepsilon(1)|}.$$

This result can be used to prove proposition 2, i.e. a 2 cycle exists if the monetary stationary equilibrium is locally dynamically stable.

Corollary 2. If $\varepsilon(1) < -\frac{1}{2}$ then there exists a periodic equilibrium of order 2;

An intuitive explanation of these results is in order. Corollary 2 states that a two-cycle will exist, when for any given wage rate, the income effect of a wage change outweighs

the substitution effect by a sufficient margin (i.e. by more than 2:1). Theorem 2 states that if $\varepsilon(1) < 0$ and we reduce the size of the diagonal elements in Π sufficiently, i.e. only consider a subset of Π , we strengthen the income effect of a wage change relative to the substitution effect and facilitate the existence of a SSE. This subset is shown as the shaded area in figure 3, where the complete set of matrices coincide with the unit square, and the origin represents the degenerate matrix associated with a periodic two-cycle.

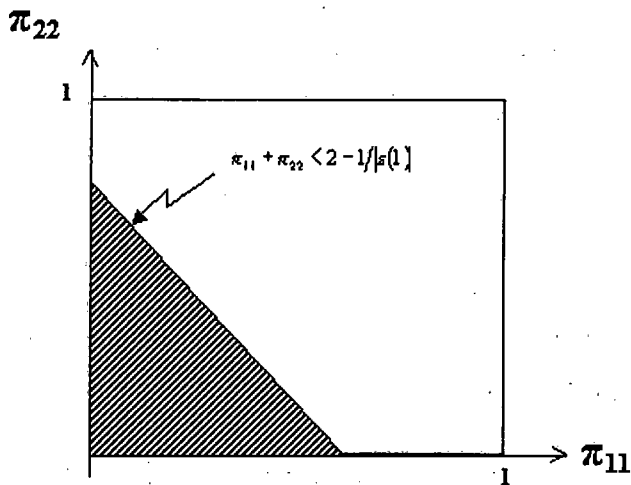


Figure 3.

6.5 The Bifurcation of Stationary Equilibria to SSE

In this section we show how deterministic equilibria bifurcate into sunspot equilibria. To understand this process we assume firstly that $\varepsilon(1) < 0$, and consider a one-dimensional path P on the (π_{11}, π_{22}) plane that crosses transversely the line $\pi_{11} + \pi_{22} < K$ (where $K = 2 - 1/|\varepsilon(1)|$) at some point G . Note that the qualitative features of this graph is the same along a diagonal path P with $\pi_{11} = \pi_{22}$, this is shown in figure 4.a. Area ABC in figure 4.a represent the subset of Π for which SSE exist. From theorem 2 it is clear that as $\pi_{11} + \pi_{22}$ decreases and passes through $K = 2 - 1/|\varepsilon(1)|$, $\partial_w F$ evaluated at $w = 1$ is at first strictly positive, then vanishes and becomes strictly negative thereafter. Figure 4.b represents the single equilibrium solution of function F with respect to w when $\pi_{11} + \pi_{22} \geq K$. There must exist two additional roots arbitrarily close to $w = 1$ when $\pi_{11} + \pi_{22} < K$, this is represented in figure 4.c. From lemma 5 we know that these two additional roots are symmetric around $w = 1$. Thom's classification theorem applies: we have a cusp catastrophe of the pitchfork type since the additional roots are symmetrical and the equilibrium point $w = 1$ is invariant. We therefore have that the graph of $w = p_1/p_2$ as a function of the curvilinear abscissa along P has the shape of a pitchfork bifurcation, with only one equilibrium before point G and three thereafter. The pitchfork bifurcation is illustrated in figure 4.d.

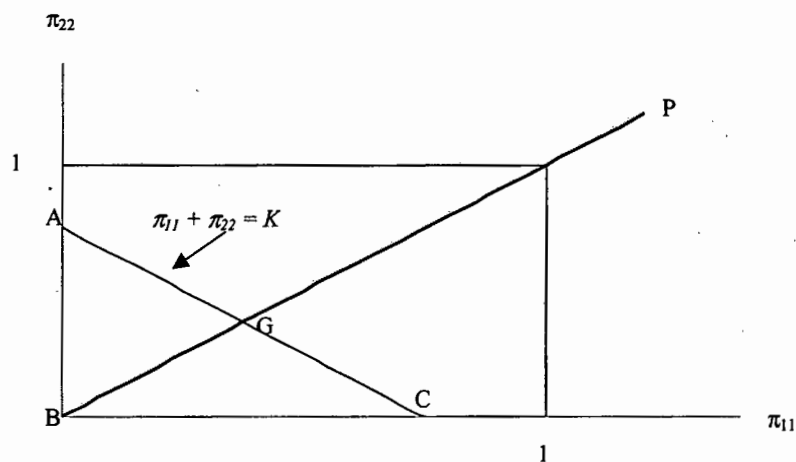


Figure 4.a.

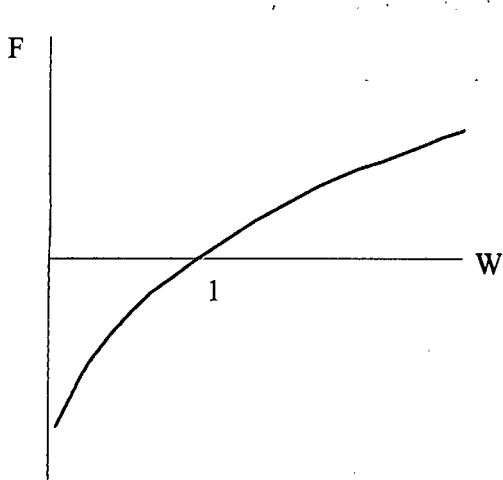


Figure 4.b.

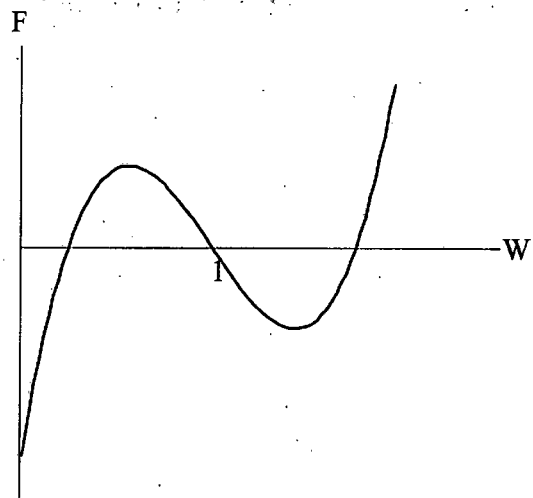


Figure 4.c.

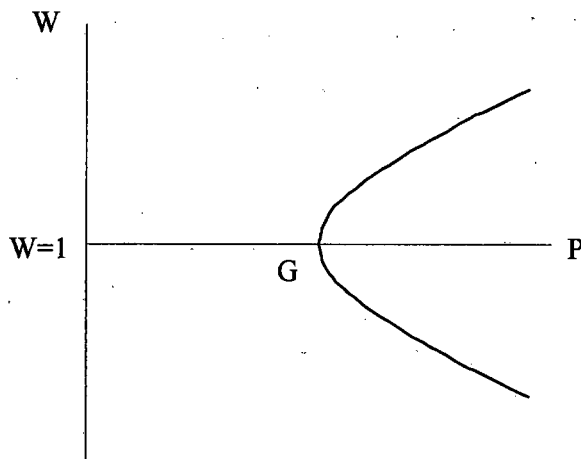


Figure 4.d.

One limitation of theorem 2 is that it does not assert that the existence of SSE depends on $\pi_{11} + \pi_{22}$, i.e. theorem 2 only derives a sufficient condition for the existence of SSE, not a necessary and sufficient condition. Theorem 3 shows that this claim can only be made strictly on the borderline of the subset shaded in Figure 2, and approximately in the neighbourhood of that borderline.

Theorem 3. Degenerate transition probability matrices of the form $\pi_{11} = \pi_{22} = 0$ admit sunspot equilibria in the neighbourhood of the deterministic stationary state if that stationary state possesses borderline dynamic stability.

From corollary 2 we observe that a two cycle requires that, for a given level of wage, the income effect of a wage change should outweigh the substitution effect by a sufficient margin. Theorem 3 takes this one step further by proving that transition probability matrices with infinitesimal diagonal elements do not alter by much the relative strength of substitution and income effects. This means that the saving behaviour remains unchanged and that a SSE exists whenever a two-cycle does.

6.6 The Relationship Between SSE and Cycles

It is however possible to derive a stronger relationship between a two-state SSE and a deterministic periodic equilibrium of order two. This result is derived in theorem 4.

Theorem 4. Given standard assumptions on preferences and strict normal goods, a two-state stationary sunspot equilibrium exists if, and only if, a regular determinist periodic equilibrium of order two exists.

6.7 Summary Comments and Conclusions

The key results derived in this chapter are summarised in figure 2 below. Each panel in figure 5 page 17 graphs the SSE price ratio w against the one dimensional parameter $\pi = \pi_{11} + \pi_{22}$, which represents the transition probability matrix Π . It must be noted that this one-dimensional characterization of the transition probability matrices wastes some information on the set of matrices compactable with stationary sunspot equilibria.

Theorem 1.a, illustrated in panel a, states that the two broken lines exist for any two-cycle. Theorem 1.a was shown to hold for the simplistic economy outlined in this chapter. The use of the standard transversality theorem in the proof should allow the extension of the proof to a more general case; i.e. that of an n -commodity economy in which expectations look one period forward and for cycles of any order k .

Theorem 2, illustrated in panel b, shows that the graph is non-empty above the thick line drawn in panel b. The argument based on the slope of the tangent is a one-dimensional

version of the Poincaré-Hopf theorem. Guesnerie (1985) generalized the sufficient condition derived in theorem 2 to n -dimensional systems.

Panel c demonstrates the bifurcation studied in section 6.5. The generalizations mentioned above will not be pursued here, but can be the topic of future research.

In this chapter we have shown that competitive equilibria may possess sunspot equilibria, otherwise known as rational expectations equilibria, in which purely extrinsic uncertainty affects equilibrium prices and allocations. This means that market participants do not have to be irrational for competitive markets to be subject to speculative fluctuations, which are driven by expectations.

The mere fact that sunspot equilibria exist as a solution to a system of market clearing conditions is not a sufficient justification to assert that competitive markets with rational participants could be subject to speculative instability. Farmer (1991) points out that there are many ways in which agents could form beliefs, each of which may be self-fulfilling, in any rational expectations model with multiple equilibria. In order to understand this issue better one needs to compare the properties of alternative forecasting rules in the face of an unforeseeable switch in the sunspot variable discussed in this chapter. Chapter 7 will explore this issue by examining an example by Duffy (1993) of an adaptive learning rule that converges to a nonstationary sunspot equilibrium that is locally stable.

The methodology used here imposed a particular expectation formation process on the consumer, with no consideration given as to why this should in fact hold in practice. This is not dissimilar to the rational expectations hypothesis used in general equilibrium analysis, since there is no explanation given as to how agents should come to possess the information required by this assumption. It is worth repeating a point made in the conclusion to chapter 4: the rational expectations hypothesis is based on the premise that agents know a great deal about their environment. Agents are assumed to have knowledge about demand and supply functions, how to extract present and future general equilibrium prices and about the stochastic law of motion of the economy over time. In practice it is hard to conceive how agents come to possess all this knowledge. Recent

literature has focused on formulating a dynamic learning process that aims to show how agents learn what they know when formulating their rational expectations. We explore this issue in chapter 7 where we show that these results have to be interpreted with care, as the convergence of a *particular* learning process to a particular unique solution does not guarantee that learning behaviour will generally lead agents to the same solution.

In chapter 4 we have seen that expectations play a role in price formation. The representative consumer assumption can be relaxed in order to develop a more general framework within which to consider the role of expectations in the price formation process. In this setting it is hard to believe that consumers will en-mass believe in the validity of sunspot phenomena without a credible theory on how beliefs are formed. This issue has been examined in some detail in chapter 5 that dealt with the role of rational beliefs.

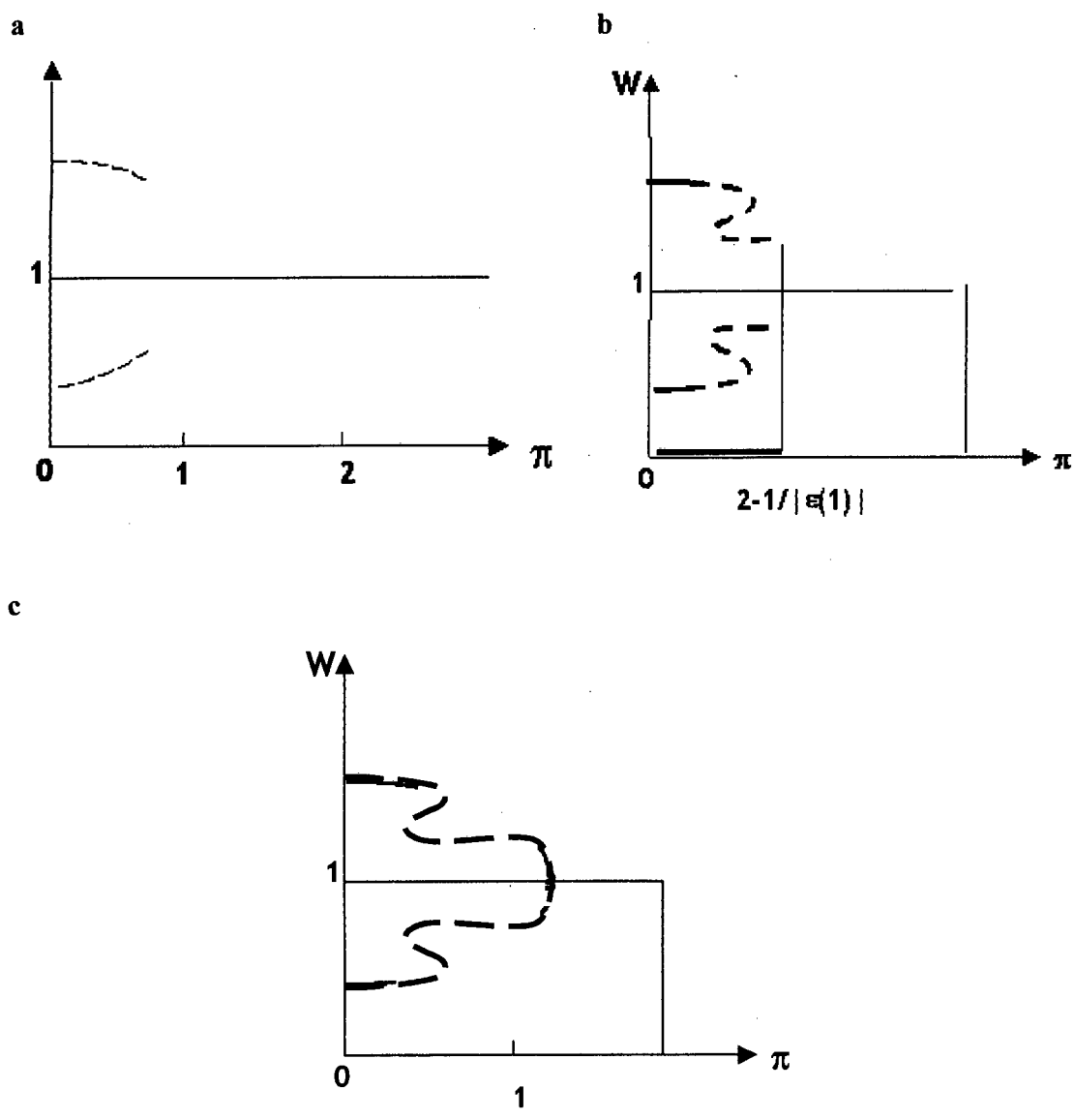


Figure 5

6.8 Mathematical Appendix

6.8.1 Proof of Lemma 1.

⇒ Define π = probability that sunspot activity occurs, i.e. $\pi = (\pi_{11}; \pi_{21})$ given the framework defined above, in which case $p_t = p_{t+1}$ and $R = 1$;

⇒ Now it follows that $(1 - \pi)$ = probability that sunspot activity does not occur, i.e. $\pi = (\pi_{12}; \pi_{22})$ given the framework defined above, in which case $p_t \neq p_{t+1}$ and

$$R = \frac{p_t}{p_{t+1}} ;$$

⇒ From equation (1) it follows that $z(R, \pi) = \max_{y \in [0, e_1]} \{\pi u(e_2 + y, y) + (1 - \pi)u(e_2 + Ry, y)\}$;

⇒ Property ii follows directly from the assumption that $u(c_{t+1}; y_t)$ is monotone and strictly concave;

⇒ Now from property i it follows that $z(R, 0) = \max_{y \in [0, e_1]} u(e_2 + Ry, y) = s(R)$;

⇒ Now from property i it follows that $z(1, \pi) = \max_{y \in [0, e_1]} u(e_2 + y, y) = s(1)$;

⇒ To prove properties v and vi we again note that $u(c_{t+1}; y_t)$ is assumed to be monotone, twice continuously differentiable and strictly concave, where $c_{t+1} = e_e + Ry_t$, where $R > 1$ or $R < 1$;

⇒ Consider the case where $R < 1$, i.e. $p_t < p_{t+1}$:

→ $\frac{\partial u}{\partial y_t} > 0 \rightarrow s(R) < s(1)$, which follows from the strict concavity assumption;

→ $\max_{y \in [0, e_1]} u(e_2 + Ry, y) < \max_{y \in [0, e_1]} u(e_2 + y, y)$. Now, by multiplying by $(1 - \pi)$ on both sides and then adding $\max_{y \in [0, e_1]} \pi u(e_2 + y, y)$ on both sides yields;

→ $\max_{y \in [0, e_1]} \{(1 - \pi)u(e_2 + Ry, y) + \pi u(e_2 + y, y)\} < \max_{y \in [0, e_1]} \{(1 - \pi)u(e_2 + y, y) + \pi u(e_2 + y, y)\}$

→ Now, $s(R) = \max_{y \in [0, e_1]} u(e_2 + Ry, y) = \max_{y \in [0, e_1]} \{\pi u(e_2 + Ry, y) + (1 - \pi)u(e_2 + Ry, y)\}$

→ Since $s(R) < s(1) \rightarrow \max_{y \in [0, e_1]} u(e_2 + Ry, y) < \max_{y \in [0, e_1]} u(e_2 + y, y)$, we have that;

→ $s(R) < \max_{y \in [0, e_1]} \{\pi u(e_2 + Ry, y) + (1 - \pi)u(e_2 + y, y)\}$

$$\rightarrow s(R) < z(R; \pi) < s(1);$$

\Rightarrow Consider the case where $R > 1$, i.e. $p_t > p_{t+1}$:

$$\rightarrow \frac{\partial u}{\partial y_t} > 0 \rightarrow s(R) > s(1), \text{ which follows from the strict concavity assumption;}$$

$$\rightarrow \text{Using a similar procedure we can show that } s(R) > z(R; \pi) > s(1);$$

\Rightarrow Combining the above means that $z(R; \pi)$ lies between $s(R)$ and $s(1)$ for R and π ;

\Rightarrow To prove property vi we note that $z(R, 0) = \max_{y \in [0, e_1]} u(e_2 + Ry, y) = s(R)$, where $R < 1$ or $R > 1$;

\Rightarrow Consider the case where $R < 1 \rightarrow s(R) < s(1)$;

$$\begin{aligned} \rightarrow z(R, 0) &= \max_{y \in [0, e_1]} \{ \hat{\pi} u(e_2 + Ry, y) + (1 - \hat{\pi}) u(e_2 + Ry, y) \} \\ &< \max_{y \in [0, e_1]} \{ \hat{\pi} u(e_2 + Ry, y) + (1 - \hat{\pi}) u(e_2 + y, y) \} \\ &= z(R; \hat{\pi}) \end{aligned}$$

\rightarrow Similarly we have that $z(R; 0) < z(R; \pi)$;

$$\rightarrow \text{We have that } \frac{\partial z}{\partial \pi} = \max_{y \in [0, e_1]} \{ u(e_2 + y, y) - u(e_2 + Ry, y) \}, \text{ such that } \frac{\partial z}{\partial \pi} > 0 \text{ iff } R < 1$$

$$\text{and } \frac{\partial z}{\partial \pi} < 0 \text{ iff } R > 1, \text{ since } u() \text{ is assumed to be strictly concave;}$$

$$\rightarrow z(R; 0) < z(R; \hat{\pi}) < z(R; \pi);$$

\Rightarrow We can use a similar technique to show that $z(R; 0) > z(R; \hat{\pi}) > z(R; \pi)$ if $R > 1$ and $\hat{\pi} < \pi$

6.8.2 Proof of Lemma 2.

⇒ We have that $\varepsilon(R) = \frac{\partial S}{\partial R} \times \frac{R}{s(R)} = \max_{y \in [0; e_1]} \{u'(e_2 + Ry, y) \times y\} \times \frac{R}{s(R)}$

⇒ $\varepsilon(1) = \max_{y \in [0; e_1]} \{u'(e_2 + y, y) \times y\} \times \frac{1}{s(1)}$

⇒ It follows that $\eta(R; \pi) = \max_{y \in [0; e_1]} \{(1 - \pi) \times y \times u'(e_2 + Ry, y)\} \times \frac{R}{z(R; \pi)}$;

⇒ $\eta(1; \pi) = \max_{y \in [0; e_1]} \{(1 - \pi) \times y \times u'(e_2 + y, y)\} \times \frac{1}{z(1; \pi)}$

⇒ But by iv of lemma 1 we have that $z(1, \pi) = s(1)$ for all π ,

⇒ $\eta(1; \pi) = \max_{y \in [0; e_1]} \{(1 - \pi) \times y \times u'(e_2 + y, y)\} \times \frac{1}{s(1)} = (1 - \pi) \times \varepsilon(1)$

6.8.3 Proof of Theorem 1.a.

⇒ We can interpret equations (4.a) and (4.b) as a system of M=2 equations with N=2 unknowns, represented by $p = (p_1, p_2)$, with S=2 parameters represented by $\bar{\pi} = (\pi_{11}, \pi_{22})$, such that for every $\bar{\pi}$ we have a system of equations $f(p; \bar{\pi}) = 0$;

⇒ The Transversality Theorem states that if the Mx(N+S) matrix $Df(p; \bar{\pi})$ has rank M whenever $f(p; \bar{\pi}) = 0$, then for almost every $\bar{\pi}$, the MxN matrix $D_p f(p; \bar{\pi})$ has rank M whenever $f(p; \bar{\pi}) = 0$;

⇒ By corollary 1 it follows that if $\bar{\pi} = (0, 0)$ the 2x4 matrix $Df(p; \bar{\pi})$ has rank 2, as the vector of unknowns, i.e. p, has exactly one solution;

⇒ Applying the transversality theorem stated above it follows that $D_p f(p; \bar{\pi})$ has rank 2 whenever $f(p; \bar{\pi}) = 0$ for almost every $\bar{\pi}$.

⇒ The set of feasible $\bar{\pi}$ can be represented by the subset $\nu(\bar{\Pi}) \subset \Pi$;

6.8.4 Proof of Lemma 3.

- ⇒ We can interpret the gradient of f at p_m as $Df(p_m)$. The reason being: The derivative of f is a linear function in R^n and the gradient is a vector in R^n . Also, linear functions can be represented by vectors, and the representation of the derivative has the same components as the gradient;
- ⇒ Therefore, $Df(p_m)p_m^T$ represents the dot product of p_m with the Jacobian matrix $Df(p_m)$, which will be a maximum when p_m points in the same direction as $Df(p_m)$;
- ⇒ The gradient vector point in the direction in which f is increasing most;
- ⇒ Since p_m is chosen such that $f(p_m)=0$, we define the surface $Q = \{x \in R^n : f(x)=0\}$, then the value of the function $f(p_m)$ is constant along the surface Q ;
- ⇒ Thus, the derivative of $f(p_m)$ in a direction tangent to the surface Q should be zero as the function does not change in that direction;
- ⇒ Formally: Let p_m^* be a point on the surface Q such that $f(p_m^*)=0$;
- ⇒ Note that the linear map $f(p_m^*) + Df(p_m^*)(p_m - p_m^*)$ approximates the nonlinear map $f(p_m)$ near x_m ;
- ⇒ Let Q' be the linear approximation to Q for $|p_m - p_m^*| \leq \varepsilon$:
$$Q' = \{p_m : f(p_m^*) + Df(p_m^*)(p_m - p_m^*) = 0\} = \{p_m : Df(p_m^*)(p_m - p_m^*) = 0\};$$
- ⇒ Thus, $p - p_m^*$ is orthogonal to the gradient of f at p_m^* ;

6.8.5 Proof of Lemma 4.

- ⇒ Let \bar{B} satisfy the hypothesis of the lemma;
- ⇒ Define B to be an open neighbourhood of \bar{B} constituted by admissible matrices (i.e. zero entries of \bar{B} are kept equal to zero);
- ⇒ Define $E : B \times R_+^k \rightarrow R^M$ by $E(B; q) = Bq$;
- ⇒ Assume that $E(B; q) = 0$;

- ⇒ Now, $\partial_{b_j} E(B; q) = q_j e_i \neq 0$ and every row has a nonzero entry we deduce that rank $\partial_B E(B; q) = M$;
- ⇒ Applying the transversality theorem we have that there is B' near \bar{B} such that $E_{B'}(q) = 0$ implies rank $\partial_{B'} E(q) = M$;
- ⇒ But $\partial_q E_{B'}(q) = B'$, and so B' is as we wanted.

6.8.6 Proof of Theorem 1.b.

- ⇒ Let $p_{m-1} = (p_1; \dots; p_{m-1})$ be the equilibrium prices of a SSE of order $m-1$ with a transition probability matrix Π_{m-1} ;
- ⇒ Let p^* be the equilibrium price of the stationary equilibrium which exists by assumption;
- ⇒ Now assume that event m exist such that $\pi_{mj} = 0 \forall j = 1, \dots, m-1$ and $\pi_{mm} = 1$ such that $\bar{\Pi}_m = \begin{bmatrix} \Pi_{m-1} & 0 \\ 0 & 1 \end{bmatrix}$, where the bar-sign indicates the degenerate case constructed;
- ⇒ By applying $p_m^* = (p_1; \dots; p_{m-1}; p^*)$ and $\bar{\Pi}_m$ to definition 2 we see that $(p_m^*; \bar{\Pi}_m)$ represents a degenerate SSE of order m ;
- ⇒ If $D_p f(p_m; \Pi_m)$ has full rank m , then any arbitrary perturbation of $\bar{\Pi}_m$ will break us away from the degenerate solution such that an SSE of order m exists;
- ⇒ By combining lemma 3 and 4 we have that rank $D_p f(p_m; \Pi_m) = m$ for every Π_m when $f(p_m; \Pi_m) = 0$;
- ⇒ Note that the existence of $D_p f(p_m; \Pi_m)$ is guaranteed as we are using a SSE of order $m-1$ which is not a bifurcation point;
- ⇒ By continuity of f we can perturb $\bar{\Pi}$ such that $f(p_m; \Pi_m) = 0$ to find true SSE of order m ;

6.8.7 Proof of Lemma 5.

⇒ Solving equations 4.a and 4.b simultaneously yields:

$$p_1 \times z(p_1/p_2; \pi_{11}) = p_2 \times z(p_2/p_1; \pi_{22});$$

⇒ If we set $w = p_1/p_2 \rightarrow w \times z(w; \pi_{11}) - z(1/w; \pi_{22}) = 0$;

⇒ This allows us to define the function $F(w, \pi_{11}, \pi_{22}) = w \times z(w, \pi_{11}) - z(1/w, \pi_{22})$;

⇒ From the above simultaneous solution and by definition 1 it is clear that a SSE will exist if and only if $F(w, \pi_{11}, \pi_{22}) = 0$ and $w \neq 1$;

⇒ For property ii we have that $F(1, \pi_{11}, \pi_{22}) = z(1, \pi_{11}) - z(1, \pi_{22})$;

→ By property iv of lemma 1 we have that $z(1, \pi) = s(1)$

$$\rightarrow F(1, \pi_{11}, \pi_{22}) = s(1) - s(1) = 0$$

⇒ To prove property iii note that $\lim_{w \rightarrow \infty} z(1/w; \pi_{22}) = z(0; \pi_{22}) < \infty$;

→ From property v of lemma 1 we have that $z(w; \pi)$ lies between $s(1)$ and $s(w)$, and, specifically we have that $s(1) < z(w; \pi) < s(w)$ if $w > 1$ (see proof above);

$$\rightarrow ws(1) < wz(w; \pi) < ws(w) \text{ if } w > 1$$

→ Now since $ws(1) \rightarrow \infty$ as $w \rightarrow \infty$ we have that $F \rightarrow \infty$ as $w \rightarrow \infty$;

→ Note that Azariades and Guesnerie assumed that $ws(w) \rightarrow \infty$ as $w \rightarrow \infty$ in their original paper. This assumption is however not required as it is sufficient to note that $ws(1) < wz(w; \pi) < ws(w)$ if $w > 1$ and $ws(1) \rightarrow \infty$ as $w \rightarrow \infty$;

⇒ To prove property iv note that $F(w, \pi_{11}, \pi_{22}) = w\{z(w, \pi_{11}) - 1/w z(1/w, \pi_{22})\}$;

→ Note that by property iii above we have that

$$z(w, \pi_{11}) - 1/w z(1/w, \pi_{22}) \rightarrow -\infty \text{ as } w \rightarrow 0;$$

→ $F \rightarrow -\infty$ as $w \rightarrow 0 \rightarrow F < 0$ for w sufficiently small;

⇒ Finally, if w is a root of $F(\cdot)$, then we have that $F(w, \pi_{11}, \pi_{22}) = 0$;

→ From property 1 of Lemma 5 we have that $w = \frac{z(1/w; \pi_{22})}{z(w; \pi_{11})}$ if w is a root;

→ Now assume that \hat{w} is a root of $F(\hat{w}, \pi_{22}, \pi_{11})$ such that $\hat{w} = \frac{z(1/\hat{w}; \pi_{11})}{z(\hat{w}; \pi_{22})}$;

→ But this is equivalent to writing $\frac{1}{\hat{w}} = \frac{z(\hat{w}; \pi_{22})}{z(1/\hat{w}; \pi_{11})} \leftrightarrow w = \frac{z(1/w; \pi_{22})}{z(w; \pi_{11})} \rightarrow \hat{w} = \frac{1}{w}$;

6.8.8 Proof of Theorem 2.

⇒ Given property iii of Lemma 5 we have that F has a root at $w = 1$;

⇒ Given that $F \rightarrow \infty$ as $w \rightarrow \infty$ and $F \rightarrow -\infty$ as $w \rightarrow 0$ (i.e. property iv and v of Lemma 5) one will not be guaranteed of at least one other pair of roots at $w = 1$ if

$\left. \frac{\partial F}{\partial w} \right|_{w=1} > 0$, i.e. if F is an increasing function at $w = 1$;

⇒ However, if we can show that $\left. \frac{\partial F}{\partial w} \right|_{w=1} < 0$, then there must be at least one other pair of root that satisfy the properties of F summarised under Lemma 5;

⇒ $\frac{\partial F}{\partial w} = \frac{\partial}{\partial w} \{wz(w; \pi_{11}) - wz(w; \pi_{22})\} = z(w; \pi_{11}) + wz'(w; \pi_{11}) - z'(w; \pi_{22})$;

⇒ $z'(w; \pi) = \frac{\partial}{\partial w} z(w; \pi) = -\pi y u'(e_2 + wy; y)$;

⇒ If we evaluate the derivative of $z'(w; \pi)$ at $w = 1$, then by lemma 2 we have that

$$z'(1; \pi) = s(1) \times \eta(1; \pi) = s(1)(1 - \pi)\varepsilon(1);$$

⇒ Thus using this we get $\left. \frac{\partial F}{\partial w} \right|_{w=1} = s(1) + s(1) \times \eta(1; \pi_{11}) - s(1) \times \eta(1; \pi_{22})$;

⇒ Simplifying yields $\left. \frac{\partial F}{\partial w} \right|_{w=1} = s(1)\{1 + (1 - \pi_{11})\varepsilon(1) - (1 - \pi_{22})\varepsilon(1)\}$
 $= s(1)\{1 + \{2 - \pi_{11} - \pi_{22}\}\varepsilon(1)\}$

⇒ Now, $\left. \frac{\partial F}{\partial w} \right|_{w=1} < 0$ if $\{2 - \pi_{11} - \pi_{22}\}\varepsilon(1) < -1$ or $\{\pi_{11} + \pi_{22} - 2\} \times (-1) \times \varepsilon(1) < -1$;

⇒ If $\varepsilon(1) > 0$ then $(-1) \times \varepsilon(1) < 0$;

→ But $\{\pi_{11} + \pi_{22} - 2\} < 0$ since π_{11} and $\pi_{22} \in [0; 1]$;

→ $\{\pi_{11} + \pi_{22} - 2\} \times (-1) \times \varepsilon(1) > 0$, and hence we find that the required condition is violated;

⇒ We can use a similar method to show that the required condition is not violated if $\varepsilon(1) < 0$;

⇒ Also note that $\{\pi_{11} + \pi_{22} - 2\} \times (-1) \times \varepsilon(1) < -1 \leftrightarrow \pi_{11} + \pi_{22} < 2 - \frac{1}{|\varepsilon(1)|}$;

6.8.9 Proof of Corollary 2.

⇒ The proof of theorem 2 uses the fact that π_{11} and $\pi_{22} \in [0;1]$;

⇒ Corollary 1 states that a deterministic two cycle if $\pi_{11} = 0$ and $\pi_{22} = 0$;

⇒ Thus when $\pi_{11} = 0$ and $\pi_{22} = 0$ the occurrence of state-1 ensures the occurrence of state-2 next, and vice-versa;

⇔ $0 + 0 < 2 - \frac{1}{|\varepsilon(1)|} \leftrightarrow |\varepsilon(1)| < \frac{1}{2} \leftrightarrow \varepsilon(1) < -\frac{1}{2}$;

⇒ Thus a periodic equilibrium of order two exists when $\varepsilon(1) < -\frac{1}{2}$;

6.8.10 Proof of Theorem 3.

⇒ Equation 1 writes the excess demand function as $D(p_t, p_{t+1}) = 1/p_t - s(p_t/p_{t+1})$;

⇒ Use Taylor's theorem to expand the excess demand function around the equilibrium price p^* ;

→ Note that this is similar to expanding the excess demand function around $R = 1$, where $R = p_t/p_{t+1}$;

→ Now, $D'(R) = -\frac{1}{p_{t+1}R^2} - \max_{y \in [0; \varepsilon_1]} \{u'(e_2 + Ry; y)\}$;

→ Evaluating at $R=1$ yields: $D'(R=1) = -\frac{1}{p^*} - s(1)\varepsilon(1)$;

→ Now, applying Taylor's theorem and assuming derivatives of order two and higher evaluate to approximately zero, i.e. $D''(R=1) \approx 0$, $D'''(R=1) \approx 0$, yields:

$$D(R) = [1/p^* - s(1)] - [1/p^* + s(1)\varepsilon(1)] \times [R - 1] \approx 0;$$

→ By definition we have that $D(R=1) = 0$ for a stationary price equilibrium, which means that $1/p^* = s(1)$;

- $\rightarrow D(R) = 0 - [1/p^* + s(1)\varepsilon(1)] \times [R - 1] = [1 + \varepsilon(1)] - [1 + \varepsilon(1)]R = 0;$
- \rightarrow Now use the property that under stationarity $1 = p^*/p^* \equiv p^*/p_{t+1};$
- $\rightarrow D(R) = [p^*/p_{t+1} + \varepsilon(1)] - [1 + \varepsilon(1)] \times p_t/p_{t+1} = 0;$
- $\rightarrow D(R) = \varepsilon(1)p_{t+1} - [1 + \varepsilon(1)]p_t + p^* + [\varepsilon(1)p^* - \varepsilon(1)p^*] = 0;$
- $\rightarrow D(R) = \varepsilon(1) \times [p_{t+1} - p^*] - [1 + \varepsilon(1)] \times [p_t - p^*] = 0;$
- \Rightarrow We can therefore write the perfect foresight excess demand function as approximately
$$\varepsilon(1) \times [p_{t+1} - p^*] - [1 + \varepsilon(1)] \times [p_t - p^*] = 0;$$
- \Rightarrow By corollary 2 we know that local dynamic stability corresponds to $\varepsilon(1) < -\frac{1}{2};$
- \Rightarrow To generalize this result to the rational expectation case write $x_t = x_1 = p_t - p^*$ and $x_{t+1} = x_2 = p_{t+1} - p^*;$
- \Rightarrow We can make the same linear approximation for equations 4.a and 4.b by noting that $1 - \pi_{11} = \pi_{12}$ and $1 - \pi_{22} = \pi_{21}$ and by using lemma 2, i.e.:
$$(1 - \pi_{11})\varepsilon(1)x_2 - [1 + (1 - \pi_{11})\varepsilon(1)]x_1 = 0 \rightarrow [1 + \pi_{12}\varepsilon(1)]x_1 - \pi_{12}\varepsilon(1)x_2 = 0; \text{ and } (5.a)$$
- $$(1 - \pi_{22})\varepsilon(1)x_1 - [1 + (1 - \pi_{22})\varepsilon(1)]x_2 = 0 \rightarrow \pi_{21}\varepsilon(1)x_1 - [1 + \pi_{21}\varepsilon(1)]x_2 = 0; (5.b)$$
- \Rightarrow The trivial solution to equations 5.a and 5.b are $x_1 = 0$ and $x_2 = 0$, i.e. the stationary equilibrium price $p^*;$
- \Rightarrow A SSE is obtained when $x_1 \neq 0$ and $x_2 \neq 0$ and equations 5.a and 5.b are solved;
$$\Rightarrow \frac{\pi_{12}\varepsilon(1)}{1 + \pi_{12}\varepsilon(1)} = \frac{1 + \pi_{21}\varepsilon(1)}{\pi_{21}\varepsilon(1)};$$
- $\Rightarrow \pi_{12} + \pi_{21} = -\frac{1}{\varepsilon(1)};$
- \Rightarrow This equation was derived assuming local dynamic stability, i.e. we must have $\varepsilon(1) = -\frac{1}{2}$ if this equation is to hold;
- \Rightarrow Now: $\pi_{12} + \pi_{21} = 2 \leftrightarrow \pi_{11} + \pi_{22} = 0;$

6.8.11 Proof of Theorem 4.

⇒ If we assume that a regular deterministic equilibrium of order two exists, then theorem 1.a holds and we have that a two-state SSE exists.

⇒ This then proves the first part of theorem 4, we now need to show that the reciprocal holds, i.e. the existence of a two-state SSE implies the existence of a regular deterministic equilibrium of order two;

⇒ Assume that a two-state SSE exists defined by $(\bar{w}; \pi_{11}; \pi_{22})$, such that:

- $\pi_{11} < 1$ and $\pi_{11} < 1$;
- $F(\bar{w}) = \bar{w}z(\bar{w}; \pi_{11}) - z(\frac{1}{\bar{w}}; \pi_{22}) = 0$;
- $\Omega_1 = [w | s(w) \geq s(1)]$ and $\Omega_2 = [w | s(w) \leq s(1)]$;

⇒ The remainder of the proof will be broke down into five smaller steps, the first four of which will derive results that will be drawn together in step 5;

⇒ **Step 1.** Show that $\psi(w) \equiv z(w; \pi) < s(1)$ for $w < 1$;

- By lemma 1 $\Omega_1 \Rightarrow s(w) \geq z(w; \pi) \geq s(1)$ and $\Omega_2 \Rightarrow s(w) \leq z(w; \pi) \leq s(1)$;
- Consider, $w \in \Omega_2 \Rightarrow s(w) \leq z(w; \pi) \leq s(1) \Rightarrow ws(w) \leq wz(w; \pi) \leq ws(1) < s(1)$;
- Thus $w \in \Omega_2 \Rightarrow \psi(w) \equiv wz(w; \pi) < s(1)$;
- Also consider, $w \in \Omega_1 \Rightarrow s(1) \leq z(w; \pi) \leq s(w) \Rightarrow \psi(w) \equiv wz(w; \pi) \leq ws(w)$;
- By applying standard boundary assumptions on individual behaviour we have that $ws(w) \rightarrow \infty$ as $w \rightarrow \infty$;
- We therefore have that $ws(w)$ is an increasing function such that $ws(w) < s(1)$;
- $w \in \Omega_1 \Rightarrow \psi(w) < s(1)$ for $w < 1$;
- Combining the above gives $\psi(w) \equiv z(w; \pi) < s(1)$ for $w < 1$;

⇒ **Step 2.** No $w > 1$ exists such that $w \in \Omega_1$ and $F(w) = 0$;

- Note that $F(w) = 0 \Rightarrow z(w; \pi_{11}) = \frac{1}{w} z(w; \pi_{22})$;
- $w > 1 \Rightarrow \frac{1}{w} z(w; \pi_{22}) < s(1)$ by step 1 above;
- $z(w; \pi_{22}) < s(1) \Rightarrow s(w) < s(1)$ by lemma 1;
- Note however that this can not happen for any $w \in \Omega_1$ given the definition of Ω_1 ;

- ⇒ **Step 3.** No $w > 1$ exists such that $\gamma_w \in \Omega_2$ and $F(w) = 0$;
- This proof is similar to that of step 2, and is left for the interested reader;
- ⇒ **Step 4.** Show that if $F(\bar{w}) = 0$ for some $\bar{w} > 1$ then $\bar{w}s(\bar{w}) - s\left(\frac{1}{s(\bar{w})}\right) \leq 0$:
- By applying the result of step 2 it follows that $\bar{w} \in \Omega_2$:
- ⇒ $s(\bar{w}) < z(\bar{w}; \pi)$ which follows from lemma 1;
- ⇒ $\bar{w}s(\bar{w}) < \bar{w}z(\bar{w}; \pi)$;
- By applying the result of step 3 it follows that $\gamma_{\bar{w}} \in \Omega_2$:
- ⇒ $z(\gamma_{\bar{w}}; \pi) < s(\gamma_{\bar{w}})$ which follows from lemma 1;
- ⇒ $-z(\gamma_{\bar{w}}; \pi) > -s(\gamma_{\bar{w}})$;
- Using the fact that $F(\bar{w}) = \bar{w}z(\bar{w}; \pi) - z(\gamma_{\bar{w}}; \pi) \Rightarrow 0 > \bar{w}s(\bar{w}) - s(\gamma_{\bar{w}})$;
- ⇒ **Step 5.**
- Property vi of Lemma 5 we have that root are symmetrical;
- We can therefore assume that $\bar{w} > 1$ without loss of generality;
- If $w > 1$ we have that $\bar{w}s(\bar{w}) - s(\gamma_{\bar{w}}) < 0$;
- But $\bar{w}s(\bar{w}) - s(\gamma_{\bar{w}}) \rightarrow \infty$ as $w \rightarrow \infty$;
- Which means that at least one finite root greater than unity exist;

Chapter 7 - Adaptive Learning Behaviour

7.1 Introduction

Standard versions of OLG models with fiat money may be characterized by a continuum of rational expectations solution paths, each of which may be a valid description of the economy, that are indexed by different initial price levels. These OLG models generally have two steady states: 1) an autarkic steady state in which real money balances have zero value, and 2) a monetary steady state in which real money balances are positively valued. All but one of the solution paths discussed above are characterised by an ever increasing price level which converges asymptotically to the autarkic steady state, the exception being the solution path characterised with an initial price level equal to the monetary steady state.

The monetary steady state is therefore a determinate steady state as it has a locally unique solution path, whereas the autarkic steady state is an indeterminate steady state as it does not have a locally unique solution path, i.e. within an arbitrary small neighbourhood of the autarkic steady state there exist many other solution paths that are uniformly close to the autarkic steady state.

Indeterminacy of equilibria means that standard comparative static analysis cannot be performed as one has difficulty isolating the effect of small changes in parameters, as there are many solution paths arbitrarily close to the one being examined. This problem is exaggerated when one considers stationary sunspot equilibrium of order- s , which has s indeterminate equilibria.

Various authors have used disequilibrium learning analysis as a tool to isolate locally unique rational expectations equilibria in which only fundamentals matter. Lucas (1986) shows that the dynamics of a Samuelson-type overlapping generations model are reversed if a simple adaptive learning rule is used in place of rational expectations forecasts. Under rational expectations the indeterminate autarkic steady state is the attractor for almost all initial price levels. In the Lucas model, which uses a simple

adaptive learning rule, the monetary steady state is the attractor for almost all initial price levels.

This chapter will concentrate on an alternative learning process by Duffy (1994) that is expectationally stable and that leads agents to believe in a continuum of solution paths converging to an indeterminate steady state of an OLG model. This model is very useful for two very important and unrelated reasons:

1. It causes one to question the usefulness of disequilibrium learning analysis as a selection criterion for isolating rational expectations equilibria, as it implies that the convergence of a particular learning process to a particular unique solution does not guarantee that disequilibrium learning behaviour will generally lead agents to the same solution.
2. The model presented by Duffy allows agents to adjust quantity first in response to a monetary disturbance, whilst the learning rule allows price to adjust more slowly such that it converges only asymptotically to the new monetary steady state.

The second feature of the model is particularly useful as it allows one to build a model that better reflects the sluggishness observed in the adjustment of the price level to monetary disturbances, e.g. Sims (1989). This has potential benefits in modelling asset price behaviour, as there is much research that indicates that asset prices adjust only slowly to new information that adjusts return expectations.

Woodford (1990) presents a learning model that leads agents to believe in an equilibrium that is driven by purely self-fulfilling beliefs. Although the model presented by Woodford achieves the same results as that of Duffy, as discussed here, it does not make use of a dynamic function that adjusts either price or quantity to arrive at new market clearing conditions in response to new information. Moreover, the model imposes a rational expectations equilibrium market clearing condition, as per Azariadis (1981) and Azariadis & Guesnerie (1986), on a disequilibrium learning process, a methodology that presents conceptual problems for this author.

7.2 Model Specification

The OLG model used in this chapter is based on work by Farmer (1991) and Duffy (1993). Two goods are traded in the economy; 1) fiat money, and 2) a non-storable consumption good. The aggregate production function is derived from the individual firm's production function, i.e. $y_t^i = \max[\psi n_{t-1}^i; \phi n_t^i]$, and is described by the equation:

$$y_t = \psi n_{t-1} + \phi n_t \quad (1)$$

Where,

- y_t is the aggregate production function at time t ;
- n_t is labour input at time t , and;
- $\psi > 0$ and $\phi \geq 0$;

In this model both labour demand and supply are sensitive to intertemporal prices as it takes two periods to produce the consumption good y_t . Agents in this economy can consume during each of the two periods for which they live, but they can only work in the first period. A representative agent born at time t face two budget decisions, one for each period of life:

$$\omega_t (n_t^s - n_t^d) + \phi n_t^d - c_t^1 - m_t \geq 0 \quad (2.a)$$

$$\psi n_t^d + \frac{m_t}{l_{t+1}^e} - c_{t+1}^2 \geq 0 \quad (2.b)$$

Where;

- ω_t denotes the real wage;
- n_t^s denotes labour supply which is restricted to the interval $0 \leq n_t^s \leq \bar{n}$, with \bar{n} equal to the endowment of leisure time;
- n_t^d denotes labour demand;
- c_t^1 and c_{t+1}^2 denote the quantities of the consumption good consumed in each period;
- m_t is the agents demand for real money balances;
- p_t is the price of the consumption good in terms of money; and
- l_{t+1}^e is the expectation at date t of the inflation factor $l_{t+1} \equiv p_{t+1}/p_t$.

An intuitive discussion of these budget constraints is in order. The first budget constraint (equation 2.a) states that net real wage income (i.e. real wage income from supplying labour to the production process, less the cost of labour used in the production process) plus inventories built up from engaging in the production process in period 1 must at least exceed the quantity of the good consumed in period one plus the demand for real money balances in period one. The second budget constraint (equation 2.b) states that the inventories built up by the young in period one from engaging in the production process plus the expected value of the real money balances in period two must at least exceed the quantity of the good consumed in period two.

An agent can store wealth between the two periods by either holding money or by setting up a firm, demanding labour and holding inventories of goods in process. The rates of return from holding money and from holding inventories must be equal to ensure that both means of storing wealth are present in equilibrium. Therefore, since agents earn zero interest from holding money they must earn zero profits from holding inventories. By combining this equilibrium condition with the two budget constraint one easily finds that labour demand, and hence demand for money balances, are maximized when

$$\omega_t = \phi + \psi n_{t+1}^e. \quad (3)$$

The aggregate supply and demand functions that reflect an agents' rational choices are:

$$\rightarrow n_t^s = n(\omega_t) \quad (4)$$

$$\rightarrow c_t^1 = c(\omega_t, n_t^s) \quad (5)$$

$$\rightarrow m_t = \omega_t n_t^s - \omega_t n_t^d - c_t^1 + \phi n_t^d \quad (6)$$

$$\rightarrow c_{t+1}^2 = \frac{m_t}{l_{t+1}^e} + \psi n_t^d \quad (7)$$

where $n(\cdot)$ and $c(\cdot)$ are continuous, increasing, and differentiable functions. It is important to note that these functions are not generated by any explicit utility maximization problem. However, since equations (4) to (7) are all continuous and satisfy Walras' Law, it follows that there exists an economy with at most four agents in every generation which generates these functions as a result of utility maximization behaviour.

In this economy we assume that the government finances a fixed level of real expenditure g through seigniorage and that government expenditure does not yield agents any additional utility and therefore do not affect agents' decisions:

$$g = m_{t+1} - \frac{m_t}{l_{t+1}} \quad (8)$$

Given the information provided above, it is possible to characterise all equilibrium solutions for this economy by the following first difference equation, i.e.:

$$l_{t+1} = \frac{s(\omega_t)}{s(\omega_{t+1}) - g} \quad (9)$$

where $s(\cdot)$ is the net demand for real money balances as a store of wealth.

Without loss of generality we shall consider equilibrium solutions where $g = 0$ for the remainder of this section. In the next section we shall consider possible equilibrium solutions that result from learning behaviour as a result of a one-time macroeconomic disturbance, e.g. government decides to increase the stock of money. Furthermore, using 3 it is possible to write the real wage ω_t in terms of the expected inflation at date t , i.e. ι_t^e . Equation (9) then becomes:

$$l_{t+1} = \frac{s(\iota_{t+1}^e)}{s(\iota_{t+2}^e)} \quad (10)$$

If we assume perfect foresight such that $\iota_t^e = \iota_t$, then the model with $g = 0$ has a single steady state: $\iota = 1$. This steady state will have positively valued fiat money, e.g. $m_t > 0$, if $\phi n(\iota) > c_t^1[\iota \times n(\iota)]$, which is derived by evaluating (6) in equilibrium (i.e. labour demand equals labour supply) with $m_t > 0$. The rest of the paper assumes that this inequality is always satisfied. Thus, inventories built up at time t must always exceed the consumption of the younger generation at time t , in order for fiat money to be positively valued, i.e. the young must be net savers. This is a reasonable assumption as this economy makes no allowance for welfare spending by the government that subsidises the consumption habits of the various generations, since we assume that that government

expenditure does not yield agents with any additional utility and therefore does not affect agents' decisions.

Equation (10) establishes an inflation rule that can be compared with the equilibrium condition found in the OLG model used in chapter 6, i.e. equation (1) establishes the following equilibrium price rule: $1/p_t = s(p_t/p_{t+1})$ with $M = 1$ under perfect foresight, i.e. $p_{t+1}^e = p_{t+1}$. It is easy to generalise this formulae as follows:

$$M/p_t = s(p_t/p_{t+1}^e) \text{ with } M > 0. \quad (11)$$

The equivalence between (10) and (11) is readily established by dividing the (11) evaluated at time $t + 1$ by (11) evaluated at time t . However, these equations have very different information requirements, i.e. in (10) expectations depend on information available at both time t and at time $t+1$ (i.e. a multi-period expectations equilibrium), whereas in (11) expectations depend only on information available at time t (i.e. a single-period expectations equilibrium). It is this difference in informational requirements of the two equilibrium conditions that lead to different results in terms of the convergence of adaptive learning rules to unique solutions of an overlapping generations economy. If equilibria are characterised by (11) and agents form expectations of the price level using a simple adaptive learning rule, then agents will learn to believe in the locally unique monetary equilibrium. However, section 3 shows that if equilibria are characterised by (10) and agents form expectations of the inflation factor using a simple adaptive learning rule, then agents will learn to believe in a continuum of nonunique, nonstationary equilibrium paths.

Since establishing that (10) has a single monetary steady state at $\iota = 1$, we can linearise (10) at this point to obtain the following first difference equation:

$$\iota_{t+1} = 1 + b(\iota_{t+1}^e - \iota_{t+2}^e) \text{ with } b = \frac{s'(1)}{s(1)} \quad (12.a)$$

If the model is closed under perfect foresight, then we can write (2.12.a) as follows:

$$\iota_{t+1} = \frac{1}{b} + \left(1 - \frac{1}{b}\right)\iota_t \quad (12.b)$$

This allow us to see that there are two classes of solutions to (12.b): 1) the zero order solution where $\iota_{t+1} = 1$, and 2) the set of nonstationary first order solutions which take the form of (12.b) with $\iota_{t+1} \neq 1$. Note that there are an infinite number of these first order solutions with each solution being indexed by a different initial value for the inflation factor.

The implicit function theorem allows us to describe the behaviour of dynamic equilibria around the steady state, i.e. $\iota = 1$, by using solutions to a first order difference equation of the form $\iota_{t+1} = f(\iota_t)$. Substituting $\iota_{t+1} = f(\iota_t)$ into (12.a), assuming perfect foresight, we get $\iota_{t+1} = 1 + b[\iota_{t+1} - f(\iota_{t+1})]$. The first derivative of the implicit function f at $\iota = 1$ is then: $f' = 1 - \frac{1}{b}$. When $|f'| < 1$, sequences of inflation expectations that begin close to the steady state will converge back to it. Note that, $|f'| < 1 \rightarrow b > 1$, which means that the economy must be parameterised in such a way that both s' and s are positive at the steady state. By imposing the condition $b > 1$ on stationary monetary equilibria, members of the set of nonstationary equilibria all converge back to the monetary steady state. Therefore, the condition $b > 1$ is sufficient to guarantee that the monetary steady state is indeterminate. This can be seen as follows: Start with ι_0 . Recursive substitution of equation (12.b) into itself yields:

$$\iota_t = \frac{1}{b-1} \sum_{r=1}^t \left(1 - \frac{1}{b}\right)^r + \left(1 - \frac{1}{b}\right)^t \iota_0.$$

Now, $\lim_{t \rightarrow \infty} \iota_t = \frac{b-1}{b-1} = 1$, i.e. the monetary steady state.

It is worth examining the comparative statics of this model at the indeterminate steady state, i.e. at $\iota = 1$. Of interest is the effect of an increase in the level of government spending on steady state value of inflation. We proceed as follows:

⇒ Use equation (3) to write equation (9) in terms of the inflation factor ι and assume

$$\text{perfect foresight, i.e. } \iota_{t+1} = \frac{s(\iota_{t+1})}{s(\iota_{t+2}) - g};$$

⇒ Now evaluate the total derivative of ι w.r.t. g at $g = 0$, i.e.

$$\rightarrow \left. \frac{d\iota}{dg} \right|_{g=0} = \frac{\partial \iota}{\partial \iota} \times \frac{d\iota}{dg} + \frac{\partial \iota}{\partial g} = \frac{\partial \iota}{\partial g};$$

$$\rightarrow \text{At the steady state value of inflation we have } \iota = \frac{s(\iota)}{s(\iota) - g} \text{ or } g = \frac{s(\iota)(\iota - 1)}{\iota};$$

$$\rightarrow \text{Now, } \frac{\partial g}{\partial \iota} = \frac{s'(\iota)(\iota - 1) + s(\iota)}{\iota} - \frac{s(\iota)(\iota - 1)}{\iota^2} = \frac{s'(\iota)(\iota - 1) + s(\iota)}{\iota} = \frac{s(\iota)}{\iota} \text{ since } \iota = 1 \text{ at the monetary steady state;}$$

$$\rightarrow \left. \frac{d\iota}{dg} \right|_{g=0} = \frac{\partial \iota}{\partial g} = \frac{\iota}{s(\iota)} = \frac{1}{s(1)} > 0 \text{ since } \iota = 1 \text{ at the monetary steady state;}$$

Therefore, in this model the single indeterminate steady state is one where fiat money is positively valued, and an increase in government spending leads to an increase in the steady state value of inflation, i.e. the comparative statics at this steady state are not perverse.

7.3 Adaptive Learning

The perfect foresight assumption is a useful simplification that allows us to understand the dynamics of the model. This assumption can be relaxed by assuming that agents are aware of the general form of the model's perfect foresight solutions, but that they do not know the true parameterisation of the model's solutions and must learn these parameter values. Specifically, agents are assumed to form their expectations of future inflation using a simple adaptive learning rule of the form

$$\iota_{t+1}^e = \alpha + \beta \iota_t \tag{13}$$

where α and β are unknown parameter values. It is assumed that all agents have homogenous beliefs and therefore use the same learning rule (13) to learn about inflation.

Substituting the adaptive learning rule (13) into the true law of motion (12.a) gives the actual law of motion of the economy (14),

$$\iota_{t+1} = \frac{1}{1 + \beta b} + \frac{\beta b}{1 + \beta b} \iota_t \tag{14}$$

We can write (14) more compactly using matrix notation:

$$T(\theta) = \begin{bmatrix} 1 \\ \frac{1 + \beta b}{\beta b} \\ \frac{\beta b}{1 + \beta b} \end{bmatrix} \quad (15)$$

An intuitive explanation of (15) is in order. T is an operator that maps the learning rule (13), represented by the parameter vector $\theta^T = (\alpha; \beta)$, into the true law of motion (12.a) to derive the actual law of motion (14). Agents, who begin with an initial set of beliefs for the values of θ , update these beliefs through a learning process that is modelled using the following differential equation;

$$\frac{d\theta}{d\zeta} = T[\theta(\zeta)] - \theta(\zeta) \quad (16)$$

Equation (16) represents a revision process by which the learning rule θ is adjusted towards the actual law of motion at a fixed rate per unit of fictional time ζ .

Given the actual law of motion (15), the learning process (16) can be written as a system of two differential equations:

$$\dot{\alpha} = \frac{1}{1 + \beta b} - \alpha \quad (17.a)$$

$$\dot{\beta} = \frac{\beta b}{1 + \beta b} - \beta \quad (17.b)$$

where the independent variable is fictional time, ζ . It is relatively straightforward to solve this system of equations simultaneously with both $\dot{\alpha}$ and $\dot{\beta}$ equal to zero, which yields the fixed points of this system of equations, i.e.:

$$\bar{\alpha} = 1; \quad \bar{\beta} = 0 \quad (18.a)$$

$$\bar{\alpha} = \frac{1}{b}; \quad \bar{\beta} = \frac{b-1}{b} \quad (18.b)$$

By substituting the first set of fixed point (i.e. 18.a) into the actual law of motion (14) for the economy we see that they correspond to the zero order solution in which price level is constant. Likewise, by substituting the second set of fixed point (i.e. 18.a) into the actual

law of motion (14) for the economy we see that they correspond to the first order solutions in which price level is not constant. We shall refer to the particular first order solution where the initial inflation level equals that observed historically as the predetermined solution.

The zero order solution and the predetermined solution predict very different responses to a macroeconomic disturbance. Consider a one-time permanent increase in the stock of money at an arbitrary moment in time, say time t :

1. If agents learn using the zero order solution then the inflation level will immediately jump to the new equilibrium level in response to this monetary disturbance. The jump in price level would cause holders of money balances to be taxed exactly enough to pay for the increase in government expenditure. This inflation tax would have been unforeseen in the sense that agents who chose to hold money balances did so in the belief that the distribution of price level had point mass at its historically predetermined level. Therefore, the unforeseen shock is inconsistent with a single rational expectations equilibrium that characterises the economy both before and after the time of the shock as it is based on beliefs of the distribution of price level that turn out to be incorrect. See figure 1 below.

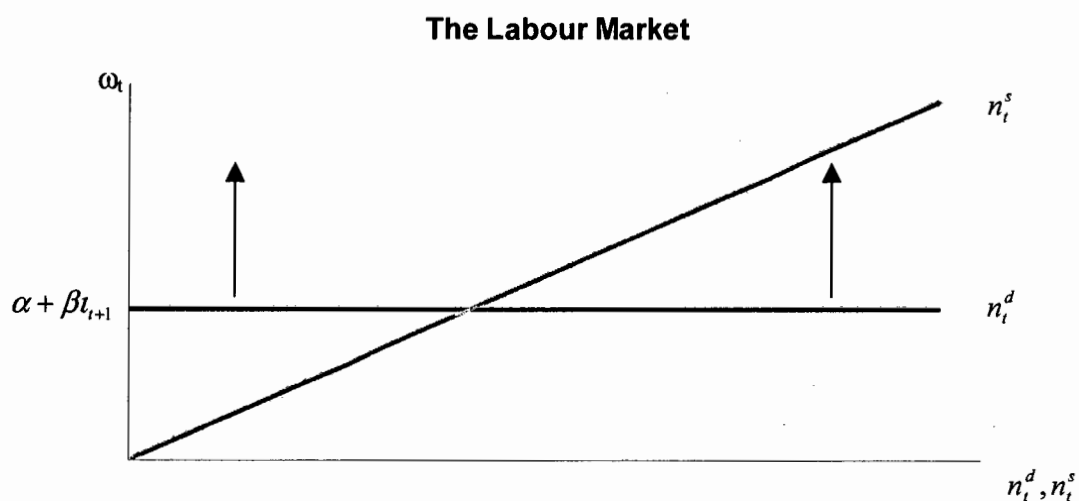


Figure 1

2. If agents learn using the predetermined solution, then the inflation level will adjust rather slowly in response to this monetary disturbance and converge only asymptotically to the new monetary steady state. The market clearing mechanism works as follows:

In period T: Given that all agents use the same learning rule (13), firms will anticipate that p_{T+1}/p_T will be positive, and this expectation implies that, given the level of real wages that existed historically, they should switch from holding money as a store of wealth into holding inventory. Firms will therefore hire more labour and drive up wages to the point where $\omega_T = \alpha + \beta t_{T+1}$.

In period T + 1 and thereafter: The money stock will remain constant at the new higher level but output, employment and real wages contract back towards the stationary state in which $\omega = \alpha + \beta$ (since $b > 1$). Since the demand for money is an increasing function of real wage (see equation (6)), the real demand for money in period T + 1 will be lower than in period T. The price level in time T + 1 must exceed the price level at time T to equate demand and supply. Note that this is the rationally anticipated increase in price level that triggered the expansion in economic activity in period T. See figure 2 below.

Response to a Monetary Shock

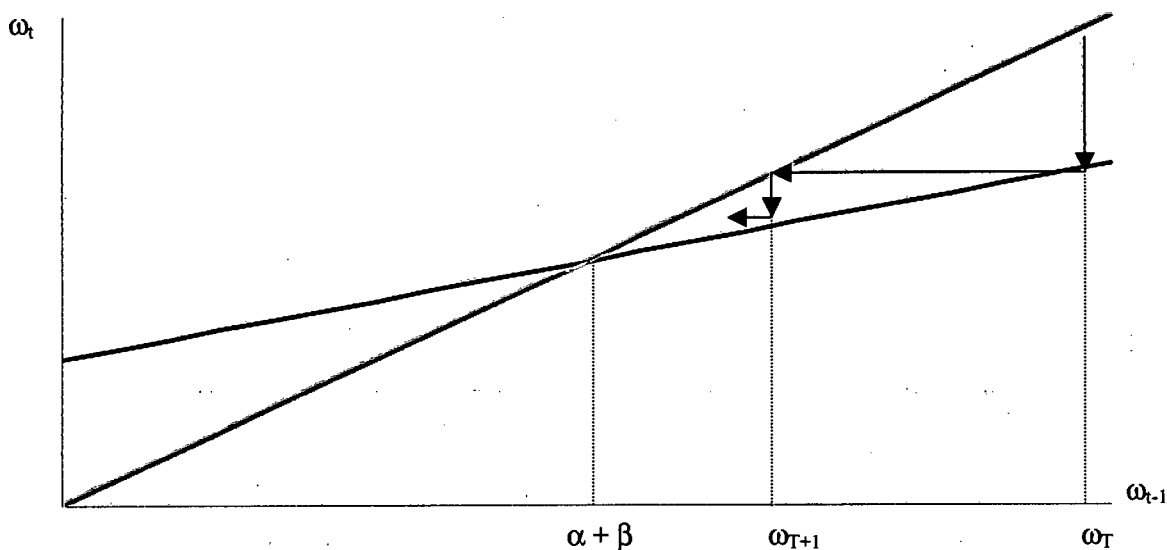


Figure 2

It is worth noting that the sluggish adjustment in the price level that occurs when agents learn using the predetermined solution is consistent with the time series properties of U.S. macroeconomic data (see Sims 1989, in the Amer. J. Agr. Econ. volume 71).

If the learning process governed by (16) is locally stable at either one of the two fixed points, i.e. (18.a) and (18.b), then that fixed point is a good candidate for the equilibrium solution of the model (see Evans (1989)). The stability of these fixed points can be tested by evaluating the Jacobian of (17.a) and (17.b) at each of the fixed points. If the roots of the resulting characteristic equation are all negative, then the particular fixed point is locally stable, or else it is locally unstable.

The Jacobian is then given by:

$$J = \begin{bmatrix} -1 & -b/(1+b\beta)^2 \\ 0 & b/(1+b\beta)^2 - 1 \end{bmatrix} \quad (19)$$

Theorem 1. The zero order solution (18.a) is locally unstable under the learning rule (16).

Theorem 2. The set of first order solutions (18.b) is locally stable under the learning rule (16).

7.4 Summary Remarks and Conclusion

Theorems 1 and 2 demonstrate that adaptive learning rules can lead rational agents to believe in nonstationary, indeterminate equilibria that are locally stable. Therefore, disequilibrium adaptive learning rules cannot be relied upon to isolate only stationary determinate equilibria. I will not dispute the fact that the learning rule used in this model is one of many rules that agents in the economy can adopt. However, without knowledge of how agents form expectations in a particular environment, it is difficult to specify an appropriate learning rule a-priori. In summary, this analysis indicates that the convergence of a particular learning rule to a particular unique solution does not imply that learning behaviour will generally lead agents to the same solution.

The model presented here has profound implications for the rational expectations hypothesis as it casts some doubt on recent literature that uses dynamic learning process to show how agents learn the information required by the rational expectations hypothesis: i.e. it assumes that agents have knowledge about the demand and supply functions they face, how to extract present and future general equilibrium prices and about the stochastic law of motion of the economy over time.

7.5 Mathematical Appendix

7.5.1 Derivation of equation 3

⇒ Equation (2.b) can be written as follows: $l_{t+1}^e \psi n_t^d + m_t - l_{t+1}^e c_{t+1}^2 \geq 0$;

⇒ Add equation (2.a) to (3.2.b): $\omega_t (n_t^s - n_t^d) + \phi n_t^d + l_{t+1}^e \psi n_t^d \geq c_t^1 + l_{t+1}^e c_{t+1}^2$;

⇒ Differentiate w.r.t. n_t^d gives: $\phi + l_{t+1}^e \psi \geq \omega_t$,

⇒ In order to have both methods of storing wealth in equilibrium there must be no arbitrage opportunities between holding real money balances and inventory, such that the differential equation becomes $\phi + l_{t+1}^e \psi = \omega_t$,

7.5.2 Derivation of equation 9

⇒ Equation 8 can be written as follows: $l_{t+1} = \frac{m_t}{m_{t+1} - g}$

⇒ At any time agents can choose between demanding labour and demanding money. Solving equations (4) to (6), and noting that labour demand equals labour supply in equilibrium we get the following:

$$\rightarrow m_t = \omega_t n_t^s - \omega_t n_t^d - c_t^1 + \phi n_t^d$$

$$\rightarrow m_t = \phi n(\omega_t) - c(\omega_t, n(\omega_t)) = s(\omega_t)$$

$$\Rightarrow l_{t+1} = \frac{s(\omega_t)}{s(\omega_{t+1}) - g}$$

⇒ From the assumptions made in equations (4) and (5) we have that $s(\cdot)$ is an increasing function of real wealth.

7.5.3 Derivation of 12

⇒ Use the Taylor expansion rule to expand (10) around $l = 1$, i.e.:

$$\rightarrow l_{t+1} = \frac{s(1)}{s(1)} + \frac{s'(1)}{s(1)} [l_{t+1}^e - 1] - \frac{s'(1)s(1)}{s(1)^2} [l_{t+2}^e - 1]$$

$$\rightarrow l_{t+1} = 1 + \frac{s'(1)}{s(1)} [l_{t+1}^e - l_{t+2}^e]$$

$$\rightarrow l_{t+1} = 1 + b [l_{t+1}^e - l_{t+2}^e]$$

7.5.4 Derivation of Equation 14

⇒ From (12.a) we have $l_{t+1} = 1 + b(l_{t+1}^e - l_{t+2}^e)$;

⇒ Substituting (13) into (12.a) yields $l_{t+1} = 1 + b[(\alpha + \beta l_t) - (\alpha + \beta l_{t+1})]$;

⇒ Collecting terms on the l.h.s. $(1 + \beta b)l_{t+1} = 1 + b\beta l_t$;

$$\Rightarrow l_{t+1} = \frac{1}{1 + \beta b} + \frac{\beta b}{1 + \beta b} l_t$$

7.5.5 Derivation of Equation (19)

The Jacobian is derived by calculating the first derivative of the differential equation system (17) with respect to α and β , i.e.:

$$\Rightarrow \frac{\partial \dot{\alpha}}{\partial \alpha} = -1 \text{ and } \frac{\partial \dot{\alpha}}{\partial \beta} = -\frac{b}{(1 + b\beta)^2};$$

$$\Rightarrow \frac{\partial \dot{\beta}}{\partial \alpha} = 0 \text{ and } \frac{\partial \dot{\beta}}{\partial \beta} = \frac{b}{(1 + b\beta)^2} - 1;$$

7.5.6 Proof of Theorem 1

⇒ At the zero order solution (18.a), the Jacobian matrix becomes

$$J = \begin{bmatrix} -1 & -b \\ 0 & b-1 \end{bmatrix}$$

⇒ The characteristic equation is given by $\det[\lambda I - J] = 0 \rightarrow (\lambda - 1)(\lambda - [b - 1]) = 0$;

⇒ The characteristic equation has one negative root, -1, and one positive root, b-1;

⇒ The fixed point (18.a) is therefore locally unstable as all the roots of the characteristic equation are not negative;

7.5.7 Proof of Theorem 2

⇒ At the zero order solution (18.b), the Jacobian matrix becomes

$$J = \begin{bmatrix} 1 & -1/b \\ 0 & 1/b - 1 \end{bmatrix}$$

⇒ The characteristic equation is given by $\det[\lambda I - J] = 0 \rightarrow (\lambda + 1) \left(\lambda - \left[\frac{1-b}{b} \right] \right) = 0$;

⇒ The characteristic equation has two negative roots, -1 and $(b-1)/b$, since $b > 1$;

⇒ The fixed point (18.b) is therefore locally stable as all the roots of the characteristic equation are negative;

Chapter 8 - Conclusion

The analysis conducted in this thesis was aimed at better understanding the role of expectations in the price formation process. Since general competitive analysis lacks a coherent explanation of how expectations are formulated it is difficult to promote one theory over another on purely theoretical grounds, e.g. rational expectations hypothesis versus the theory of rational beliefs.

Of key interest is the question of whether agents have the structural knowledge assumed by the rational expectations hypothesis, which is embedded in Fama's (1976) definition of the efficient market hypothesis. The research agenda presented some empirical evidence that supports analysis of models in which agents are not assumed to have structural knowledge about the nature of the world in which they invest. However, Samuelson (1965 and 1973) first showed that the efficient market hypothesis is closely related to the principle of a martingale. Indeed, close examination of the formulae derived based on Shiller (1981b and 1981c) and Mankiw's (1985) interpretation of the efficient market hypothesis reveals that the tests merely deal with particular martingales. Given that the definition of the martingale principle is unique, but not its interpretation, one has to be careful when interpreting the empirical results reviewed here.

Relatively simple general equilibrium models were employed to illustrate that modelling risk requires a thorough analysis of investor expectations embedded in asset prices in order to better understand the information conveyed by observed risk premia. Analysis of the role of diverse expectations in competitive equilibria show that a prerequisite for the existence of a short-run Walrasian monetary equilibrium is the existence of at least one agent whose expectations are insensitive to current prices. The presence of such an insensitive trader ensures that expectations vary in a manner that is not "wild", thereby maintaining the upper hemicontinuity of the aggregate excess demand function, such that a competitive equilibrium results. Since this analysis was concerned with the short-run properties of OLG models, it would be constructive to consider what characteristics agents' diverse expectations must have in order for a dynamic system to be locally stable.

Merely assuming that agents have diverse beliefs is unsatisfactory as this assumption avoids an important analytical question: are the conditions under which diverse beliefs arise sufficiently general to warrant studying the impact of diverse expectations on the price formation process? This issue is addressed by showing that any stable dynamical system generates a stationary probability measure based on its underlying generating probability that is unrelated to the data generated by the dynamical system. An agent's belief system is then consistent with the observed data if the stationary measure generated by his belief system is equal to the stationary measure generated by the dynamical system's true probability measure. Agents can then construct beliefs that are biased relative to the stationary measure generated by the data in the short-run, but unbiased on average over his investment horizon, since an agent's belief can be characterised as a weighted average of the stationary measure generated by the data and the agent's beliefs in structural change.

Sunspot theory is used to show that it is not necessary to abandon the rational expectations hypothesis in order for competitive markets to be subject to speculative fluctuations that are driven by expectations. The analysis shows that correlated beliefs can cause competitive markets to be subject to speculative fluctuations that are driven by expectations. This model only considers the effect on the price formation process of extrinsic uncertainty that is perfectly correlated with the rational expectations of agents. It would be worthwhile to consider whether the same results apply to the case where agents hold diverse beliefs that may become perfectly correlated depending on the observed environment in a dynamic system where beliefs are constrained such that the system is locally stable.

Finally, an example by Duffie is used to show that diverse beliefs are not temporary phenomena since disequilibrium learning analysis can not be relied on to teach investors the economy's equilibrium map. This leads to an important conclusion; it is difficult to specify an appropriate learning rule a-priori that rules out a diversity of beliefs without knowledge of how agents form expectations in a particular environment.

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