



50

The Sachs-Wolfe effect

Mark Katz

A Thesis

presented to the University of Cape Town

for the degree of

Master of Science

in

Applied Mathematics

June 11, 1993

The University of Cape Town has been given the right to reproduce this thesis in whole or in part. Copyright is held by the author.

The copyright of this thesis vests in the author. No quotation from it or information derived from it is to be published without full acknowledgement of the source. The thesis is to be used for private study or non-commercial research purposes only.

Published by the University of Cape Town (UCT) in terms of the non-exclusive license granted to UCT by the author.

To my parents.

Acknowledgements

Of all the people who have helped me I would first like to thank George Ellis who introduced the topic of this thesis to me. His vibrant enthusiasm, supportive interest and ready discussion as well as his insights on diverse topics were invaluable.

I would also like to thank Chongming Xu and Bill Stoeger with whom I worked on the material in the first part of this thesis. Their creativity and imagination were inspiring and in particular I would like to thank Chongming Xu for his clear explanations. Through Chongming Xu I met Michael Soffel whose visit of a year ago was most fruitful and rewarding.

Peter Dunsby's broad grasp of perturbative cosmology has proved invaluable in the final write-up and I am indebted to him for his support whether it be finding references or improving my understanding and giving me a better general appreciation of this subject.

The fascinating discussions with Kevin Colville, my office-mate, have been extremely enjoyable. His eclectic mind and original perspective gave a varied working environment as did the presence of two other good friends and colleagues, Gregory Schroeder and Ariel Sumeruk, whose energy and generous sociability were very enlivening.

For my parents' love and support I am deeply indebted and for their generous encouragement I dedicate this thesis to them.

Finally I would like to thank the Foundation for Research and Development for a two year scholarship.

Abstract

This thesis discusses the Sachs-Wolfe effect, which is the variation in the observed temperature of radiation emitted at the last scattering surface which occurs at the place where matter and radiation decouple at about 4000 degrees Kelvin. The work is in two parts, with the first part dealing with extensions made by George Ellis, Chongming Xu, Bill Stoeger and myself to the paper by Mirosław Panek [13] where the gauge invariant formalism of cosmological density perturbations by James Bardeen [1] has been used to find the SW effect in the case of a perturbed Friedman-Lemaître-Robertson-Walker (FLRW) universe with a barotropic equation of state describing the matter in the unperturbed case. In our work we extend the example given by Panek for a flat universe ($K = 0$) filled with dust where the density perturbations are adiabatic, to the case of non-flat universes ($K = -1, 0 + 1$) filled with a mixture of N types of matter where the density perturbations are nonadiabatic. The second part shows the agreement between the formalisms of Sachs and Wolfe's pioneering paper and the recent work of George Ellis and Marco Bruni which presents the study of cosmological perturbations in a gauge invariant and covariant way.

After the overview of the work covered in this thesis, the gauge invariant formulation of Bardeen is discussed where we follow the description by Panek of a universe whose energy content is described by a mixture of N ideal fluids coupled only by gravity. From the Einstein equations we get Bardeen's evolution equation for the gauge invariant energy density perturbation which is now given for the N different matter fluids as it appears in Panek. We then checked Panek's equations where he finds an expression for the placing of the perturbed last scattering surface, after which he derives an equation for the fractional temperature variation and writes it in terms of the perturbation variables. The equation found by SW for their particular choice of $K = 0$, pressure free dust, where the last scattering surface is placed at its unperturbed position, is verified in terms of the Bardeen formalism. Now we extend this simple case to nonadiabatic perturbations in the same scenario and find the SW effect for a mixture of two fluids: dust and radiation, with nonadiabatic perturbations in a not necessarily flat universe. We then generalise to the case of a mixture of baryons and radiation and N types of matter. This section then ends with a calculation of the difference between temperatures taken from two different directions in the sky and is written in terms of the fractional temperature perturbation

defined by Panek.

The second part puts forward the formulation of the gauge problem by Ellis and Bruni (EB), and then writes out their gauge invariant quantities in terms of the SW variables. Their evolution equations are verified in this form, and the shear and vorticity determined as well. Now all of the EB cosmological quantities are listed for the special gauge that SW use and then we explore the relation between the SW metric and that of Bardeen before ending off by verifying that the form for the redshift in the EB approach is in agreement with that given by Panek.

Contents

Acknowledgements	ii
Abstract	iii
Conventions and abbreviations	vi
1 Introduction	1
I	8
2 Bardeen's Formalism	9
2.1 Introduction	9
2.2 Preliminaries	10
2.3 Perturbations of the metric tensor	12
2.4 The general gauge transformation	14
2.5 Gauge choices	15
2.6 The construction of gauge invariant variables and their evolution	18
3 The derivation of Panek's equations	22

3.1	Introduction	22
3.2	The derivation of Panek's equation 29	23
3.3	The derivation of Panek's equation 30	28
3.4	The derivation of Panek's equation 34	32
3.5	The derivation of Panek's equation 38	36
3.6	The derivation of Panek's equation 41	41
3.7	The derivation of Panek's Equation 45	44
3.8	The derivation of Panek's equation 52	48
4	The non-adiabatic perturbation	52
4.1	Introduction	52
4.2	The last scattering surface	53
4.3	Panek's equations extended	54
4.4	A new gauge invariant quantity	56
4.5	Non-adiabatic perturbations for a mixture of radiation and dust	58
4.6	Non-adiabatic perturbations for N types of matter	69
II		80
5	The Ellis-Bruni formalism of the gauge problem	81
5.1	Introduction	81
5.2	Ellis and Bruni	81

5.3	Sachs and Wolfe	85
5.4	Verification of the EB Covariant Propagation equations in the SW variables	87
5.5	Sachs-Wolfe gauge	100
5.6	The redshift written in the EB variables	104
6	Conclusion	110

Conventions and abbreviations

Sign conventions

Signature: $[-, +, +, +]$.
Units: $c = 1$.
Gravitational constant: $\kappa = 8\pi G = 1$.
Latin and Greek indices assume the values 0, 1, 2, 3 and 1, 2, 3 respectively.

Abbreviations

CMBR	Cosmic Microwave Background Radiation.
FLRW	Friedmann-Lemaître-Robertson-Walker.
LSS	Last Scattering Surface.
CDM	Cold Dark Matter.
EB	Ellis and Bruni (1989) [4].
SW	Sachs and Wolfe (1967) [15].

Chapter 1

Introduction

The theory of cosmological density fluctuations has now come fully into its own after an uncertain formative period and shows a number of approaches of varying complexity and application. With the advances made by COBE (Cosmic Background Explorer) in finding anisotropies in the CMBR at a level of $15\mu K$, limits are placed on models of the universe and support is given for the Inflationary and Cold Dark Matter (CDM) scenarios. Fluctuations on the angular scales of 10° to 90° have been found which can be accounted for by no causal processes in standard cosmology and whatever the size of the perturbation, its brightness is the same as predicted by inflation. As well as this, there has not been enough time for the fluctuations detected to form into galaxies if the universe is made solely from baryonic matter, and this suggests perturbations growing in some other kind of matter which interacts only gravitationally with baryons and photons while they are still coupled to each other. It is known that the early universe was filled with an ionised plasma where radiation was absorbed by the free electrons and could not travel far; but several hundred thousand years after the Big Bang when the temperature had dropped to about 4000 K, protons, neutrons and electrons combined to make atoms. Atomic hydrogen, which does not absorb radiation well, became the predominant form of matter and the photons escaping into space are what we see today, highly redshifted, at 2.736 K.

In the relativistic theory of perturbations we come across the notion of 'gauge'. Unfortunately, we cannot use Newtonian theory since primordial density fluctuations are of

superhorizon scale in the early universe, over which no causal contact is possible and a gauge ambiguity consequently arises. For a gauge choice we choose a mapping between the real, perturbed spacetime and the unperturbed background spacetime. From a cosmological quantity defined at a point in the perturbed space, subtract the value that it has at the associated point in the background; then the amplitude of the perturbation in that quantity has been defined for that point. Now change the mapping while keeping the coordinates of the physical space fixed and you change the gauge, and therefore the amplitude of the perturbation.

The notion of gauge invariance developed at the outset of the theory, when a gauge was chosen to simplify the perturbation equations and where confusion in interpreting the physical meaning of the variables gave incorrect predictions from a correct framework. We now discuss the historical background and the physical premises of the theory of density fluctuations.

The Microwave Background Radiation

The isotropy of the Cosmic Microwave Background Radiation (CMBR) is the best reason cosmologists have for using the Friedmann-Lemaître-Robertson-Walker (FLRW) models as a good approximation to the observable universe. The inhomogeneous structures observed today are a result of the growth due to gravitational instability of small primordial density fluctuations [18] (Peebles 1980), where the density perturbations that produced the large scale structures were either present from the beginning and simply determined by some initial condition of the universe, or produced dynamically through the evolution of the universe. To calculate these small inhomogeneities of the CMBR fluctuations, one needs to know the dynamics of decoupling and this is discussed in a well known paper [14]. The study of the propagation and source of the inhomogeneities that form the galaxies and galactic clusters is the study of relativistic perturbation theory. It begins with a small density fluctuation exerting a gravitational attractive force on the surrounding matter; this would result in an exponential growth of density perturbations, but in an expanding universe this contraction is countered by the expansion, with the end result being a power law growth. For fluctuations on scales larger than the horizon at decoupling, it is assumed that decoupling occurs instantaneously. This is because the distance that photons can travel is shorter than the characteristic length of the perturbation, $2\pi/k$, and so cannot influence the CMBR. If the fluctuations necessary came about after the recombination

of hydrogen at about $T \sim 4000K$, then their characteristic scale would be well within the Hubble radius where relativistic effects are no longer important, and the analysis could continue with Newtonian theory. Unfortunately there is no mechanism to allow the localization of energy density on the scales of clusters of galaxies after the recombination time and is mainly due to the slow speed of sound that limits the domain over which the matter energy can be distributed. Perturbations bigger than the Hubble radius at the time of decoupling give rise to galaxies and galactic clusters and we now discuss what occurs when baryons coupled to radiation cross the horizon around the time of recombination [7]. Baryonic perturbations grow under their own gravity when they cross the horizon after recombination; but if on the other hand they cross before recombination, then as the baryons are still coupled to photons by Thomson scattering, there is an effective pressure against the force of gravity. When these fluctuations are smaller than the Jeans mass, which is comparable to the horizon size before recombination, they oscillate; those on the smallest scale, in a process known as Silk damping, dissipate away before recombination as photons diffuse out of the perturbation. After decoupling this pressure disappears; the Jeans mass falls off quickly and the Cold Dark Matter (CDM) scenarios are invoked. These allow the baryonic matter to fall into the potential wells of the CDM perturbations which are unaffected by Silk damping. Now the problem with perturbations on superhorizon scales is that one must necessarily use a general relativistic theory and the presence of gauge freedom now introduces the possibility of spurious gauge modes dominating the physical significance of the perturbations.

The notion of gauge invariance

The association of points in the unperturbed spacetime with those of the perturbed spacetime is the gauge choice. If the points in the background are fixed and a new association is made with the perturbed space time (a new gauge choice is made), then the difference between the values of a cosmological quantity defined in the two different spaces will in general change. An exception is a field defined as a constant scalar. In the background, if a scalar takes the value A , and in the real, perturbed spacetime the value A' , then the difference between any two points one might care to choose from the two spaces simply remains constant and so the choice of gauge is irrelevant. We quote a lemma by Stewart and Walker (1974) [17] which gives the criteria that must be satisfied for a perturbation to be gauge invariant, but first recall that the effect of a gauge transformation induced by

an infinitesimal vector field ξ on a tensorial quantity T in the perturbed universe equals the Lie derivative of the background value T_0 of T along ξ [2] :

$$T' = T + \mathcal{L}_\xi T_0 \Rightarrow \delta T' = \delta T + \mathcal{L}_\xi T_0, \quad \mathcal{L}_\xi T_0 = 0 \Rightarrow \delta T' = \delta T. \quad (1.1)$$

From this follows the Stewart and Walker lemma: perturbations to a background quantity T_0 will be gauge invariant iff T_0 is: (1) a constant scalar, or (2) vanishes, or (3) is a linear combination of products of Kronecker deltas with constant coefficients.

The background before Bardeen

The pioneering work on general relativistic perturbations in Friedman-Lemaître-Robertson-Walker models is that of Lifschitz [9], and is extended by Lifschitz and Khalatnikov [10] with corrections. Their analysis was correct but misinterpreted by authors considering the generation and growth of density perturbations. They used the synchronous gauge to write their equations, which were then too complicated to allow the elimination of unphysical gauge modes. Hawking, in his attempt to eliminate the gauge modes [6], bases his analysis on the gauge dependent density contrast $\delta\mu/\mu$; but his analysis of gravitation waves is correct as the Weyl tensor and its electric and magnetic parts are gauge invariant. His formulation deals with the perturbation of the curvature tensor and avoids any explicit mention of the metric tensor. This was extended by Olson [12] in 1976 free from gauge modes; but in the case of radiation, there is some ambiguity with the gauge as he gives a choice of initial time t_0 which gives a gauge dependence. Meanwhile in 1967 Harrison [5], using a longitudinal gauge free of gauge modes, derived equations for density perturbations. Nariai [11] (1969) derived the perturbation equation in the comoving gauge and free from gauge modes, which was extended by Sakai [16] who investigated the evolution of density perturbations under various gauges.

The papers of Bardeen, Kodama-Sasaki and Panek

Over the past fifteen years new methods for studying linearized gravitational perturbations have been developed, particularly the gauge invariant approach which we follow here. The first fully gauge invariant theory of cosmological perturbations was due to Bardeen [1](1980) and gives a more general analysis of the inhomogeneous version of the equations governing the density perturbation, as well as dealing with vector and tensor perturbations. Based on the Bardeen approach, Kodama and Sasaki [8] (1984) gave an analysis of multicomponent fluids and scalar fields, and in

1986 Miroslav Panek [13], also using the Bardeen formalism, calculated the variation in temperature of the radiation emitted from the last scattering surface across the sky for a multicomponent fluid. He gives a particular example of the pressure free dust case with $K = 0$ and finds that in the adiabatic case he is in agreement with the expression given by Sachs and Wolfe [15] in their pioneering paper of 1967. The key result in the Sachs-wolfe (SW) paper is their calculation of the first-order perturbation in the redshift in a $K = 0$ FLRW background. This is taken to be the temperature anisotropy of the CMWBR as measured by observers (Sachs-Wolfe effect). They ignore any variation in the placement of the last scattering surface as well as any change in temperature on it.

The work in this thesis

The factors that contribute to observed anisotropy in the CMBR are: 1) fluctuations in the temperature and position of the last scattering surface, 2) density fluctuations in the intervening spacetime which perturb the geodesics along which the photons travel, and 3) peculiar velocities of either the observer or emitting atoms at last scattering which give a doppler temperature shift. This is written in terms of the redshift factor $1 + z$,

$$T_{R0} + \delta T_R = \frac{T_{E0} + \delta T_E}{1 + z_{E0}} - \frac{\delta z_E}{(1 + z_{E0})^2}, \quad (1.2)$$

where R and E designate reception and emission respectively, and 0 denotes the value taken in the unperturbed universe.

For the work presented in this thesis we follow the approach of Panek and use his placing of the surface of last scattering. This uses the model of decoupling where the emission of radiation occurs on the hypersurface of the constant density of free electrons, n_b , that couple to photons by Thomson scattering. This density is a function of the local temperature T_S , and the density of baryons E_b , and so for a general perturbation the hypersurface of emission is neither the hypersurface of constant temperature nor that of constant baryon density as is the case with Sachs and Wolfe.

In particular we make extensions to the form that Panek derives for the Sachs-Wolfe effect in the case of a dust filled universe with $K = 0$ and adiabatic perturbations to the energy density. Here he finds himself in agreement with the prediction in the SW paper where they have an unperturbed last scattering surface and no temperature

variation at the LSS. Even though Panek has a constant temperature and baryon density at decoupling, since the density perturbations are adiabatic, he does not have the LSS's position unperturbed, and so there is an apparent variation in temperature across the LSS which he then considers irrelevant by showing it to be dominated by a similar term in the temperature variation arising from the perturbed geodesic.

Here we change the definition that Panek gives for the temperature of reception in the background and give one that seems, to us, more obvious and easier to manipulate in our later work. He defines

$$T_{R0} = \frac{S_{E0}}{S_{R0}} T_E, \quad (1.3)$$

which varies across the sky and uses it to calculate the temperature variation in the CMBR which he defines as

$$\left. \frac{\delta T}{T} \right|_R = \frac{T_R - T_{R0}}{T_{R0}}, \quad (1.4)$$

and which yields a gauge invariant expression that is written in gauge invariant variables. We instead choose the quantity

$$T_{R0} = \frac{S_{E0}}{S_{R0}} T_{E0}, \quad (1.5)$$

which gives a similar expression for the temperature variation calculated by Panek. We find that the difference between the two temperature variations using the different definitions of T_{R0} , is a single gauge invariant variable which implies that the temperature variation here is gauge invariant as well. The new quantity is more easily calculated and we will use it for the estimation of the difference between the temperature received from two different directions. Here our calculations diverge from his, and so we begin by rederiving Panek's equations with our new definition for the background temperature of reception, and after finding a different result for the general expression of the Sachs-Wolfe effect, we continue with the particular examples that he demonstrates and see that this new definition gives the same results as he has for the adiabatic case. We now consider the gauge invariant difference in temperature between two different directions in the sky which is easily determined by experiment; we then express it in terms of the two fractional temperature perturbations defined above, where we find that the second one that we use (equation 5) produces a simpler result.

Following this, a non-adiabatic perturbation in the energy density, $\chi(\tau)$, is defined, which is only dependant on time and which is shown to be gauge invariant. Now

the expressions for the perturbed temperature and baryon density are found to be explicitly dependant on χ as are the definitions of the temperature variation given by Panek in terms of the two different definitions of T_{RO} . The temperature variation is then found for this new case of non-adiabatic perturbations; this is followed by a study of non-adiabatic perturbations in the two component dust-radiation model, with $K = 1, 0$ or -1 . Finally, we extend the previous case to N non-interacting fluids with dust and radiation as two of the components and the constraint that they have barotropic equations of state in the background.

In the second part of this work, we sketch the gauge invariant and covariant formulation of Ellis and Bruni [4] (EB) with their geometrical approach to the theory of density perturbations, express their gauge invariant quantities in terms of the SW variables and verify their evolution equations in this new form. The SW gauge then gives several relations between the EB covariant, gauge invariant variables which can now all be written in terms of their gauge invariant fractional energy density perturbation \mathcal{D}_a . The metrics of Bardeen and SW are now used to determine the gauge used by SW in terms of Bardeen's more familiar variables; we also write Bardeen's quantities in terms of the SW quantities and find that we rederive the agreement between Panek and Sachs and Wolfe. To end off, we verify that the redshift of the last scattering surface given by Panek is in agreement with the same expression written in the EB framework. An expression of the SW effect in the EB formalism has been found by H. Russ, M. Soffel, C. Xu and P.K.S. Dunsby and the paper in which this appears has been accepted by *Phys. Rev. D* and is to be published in 1993.

Part I

Chapter 2

Bardeen's Formalism

2.1 Introduction

Here we develop Bardeen's formulation and give the notation used by Panek in his extension of Bardeen's work to a mixture of N perfect fluids coupled only by gravity. This assumption is somewhat limiting as he demands that the stress-energy be conserved for each of the fluids, and not only the fluid as a whole. In the work of Kodama and Sasaki [8], and Dunsby, Bruni and Ellis [3], the more general case is considered; but for the sake of simplicity, we use the work of Panek. In the next section, the background quantities and equations are outlined and scalar harmonics, which are solutions of the scalar Helmholtz equation, are introduced. The third section gives the perturbations of the metric tensor and the energy-momentum tensor and their interpretation, as well as the definition of the entropy perturbation. In section 4 the general gauge transformation is introduced and the transformations of the scale factor, the metric perturbations and quantities associated with the perturbed stress-energy are given. The following section 5 discusses the choice of gauge and the different choices possible. The last step of constructing suitable gauge invariant variables and their interpretation follows in section 6, with the evolution of the perturbations derived from the Einstein equations and the conservation equations.

2.2 Preliminaries

The background FLRW metric is

$$ds^2 = S^2(\tau)(-d\tau^2 + {}^3g_{\alpha\beta}dx^\alpha dx^\beta), \quad (2.1)$$

with $\alpha, \beta, \dots = 1, 2, 3$ and where ${}^3g_{\alpha\beta}$ is the metric tensor of a three-space of constant spatial curvature \mathcal{K} .

$${}^3g_{\alpha\beta}dx^\alpha dx^\beta = \frac{dr^2}{1 - \mathcal{K}r^2} + r^2 d\Omega_2^2, \quad (2.2)$$

in which $d\Omega_2^2$ is the metric of the 2-dimensional Euclidean sphere. The curvature tensor of this 3-space has Riemann tensor

$${}^3R_{\alpha\beta\gamma\delta} = \mathcal{K}({}^3g_{\alpha\gamma}{}^3g_{\beta\delta} - {}^3g_{\alpha\delta}{}^3g_{\beta\gamma}). \quad (2.3)$$

The scale factor $S(\tau)$ describes the volume expansion of the background as a function of the conformal time τ . A vertical bar $|$ denotes the covariant derivative of a three-tensor (defined only in terms of the spatial coordinates) with respect to ${}^3g_{\alpha\beta}$ and a semicolon is the covariant derivative with respect to the full metric g_{ij} of the physical spacetime.

As the background is FLRW the energy-momentum tensor of the background takes the perfect fluid form

$$T^{(a)}_{ij} = (E_{a0} + P_{a0})U^{(a)}_i U^{(a)}_j + P_{a0}g_{ij}, \quad (2.4)$$

with $i, j, \dots = 0, 1, 2, 3$; $a = 1, \dots, N$, and where $E_{a0}(\tau)$ and $P_{a0}(\tau)$ are the energy density and pressure which depend only on time, and $U_a^i = (S^{-1}, 0, \dots, 0)$. For these quantities the particular fluid being dealt with is indicated by a subscript which in some cases is put in brackets to distinguish it from the spacetime indices. The energy density and pressure are related by the following equation of state for a barotropic perfect fluid in the background

$$P_{a0} = P_{a0}(E_{a0}). \quad (2.5)$$

In the physical universe we have the more general case of

$$P_a = P_a(E_a, s_a), \quad (2.6)$$

where s_a is the entropy density. Now as the energy-momentum tensors of the unperturbed universe are those of N non-interacting perfect fluids at rest with respect to the above coordinates, the only nonzero components are

$$T_{(a)0}^0 = -E_{a0} , \quad T_{(a)\beta}^\alpha = P_{a0} \delta_\beta^\alpha . \quad (2.7)$$

By non-interacting we mean that each fluid component satisfies the background conservation equation with vanishing interaction source term. In the background the time evolution is determined by the field equations

$$(\dot{S}/S) \cdot = -\frac{1}{6} S^2 \sum_{a=1}^N (E_0 + 3P_0) , \quad (2.8)$$

$$(\dot{S}/S)^2 = \frac{1}{3} S^2 \sum_{a=1}^N E_0 - K , \quad (2.9)$$

and the conservation of energy density

$$\frac{\dot{E}_{a0}}{E_{a0} + P_{a0}} = -3 \frac{\dot{S}}{S} . \quad (2.10)$$

where $\dot{S} \equiv dS/d\tau$, and $K = -1, 0, 1$ which correspond to open, flat, and closed universes respectively and the units are chosen so that $c = 8\pi G = 1$.

We will deal here only with perturbations that transform as spatial scalars in the background spacetime and neglect discussion of the vector and tensor perturbations. It is possible to separate the time and the spatial dependence in the perturbation because of the homogeneity and isotropy of the background. The spatial dependence of the perturbation variables is given by solutions of a generalized Helmholtz equation with the scalar harmonics, $Q(x^\mu)$, solutions of the scalar helmholtz equation

$$Q^{|\alpha}{}_{|\alpha} + k^2 Q = 0 , \quad (2.11)$$

where the wave number k sets the spatial scale of the perturbation relative to the comoving coordinates. From the quantity Q , Bardeen defines the vector

$$Q_\alpha = -(1/k) Q_{|\alpha} , \quad (2.12)$$

and the traceless, symmetric, second-rank tensor

$$Q_{\alpha\beta} = k^{-2} Q_{|\alpha\beta} + \frac{1}{3} g_{\alpha\beta} Q . \quad (2.13)$$

2.3 Perturbations of the metric tensor

The conformal factor S^2 is removed from the metric tensor components before defining the perturbations. Bardeen defines

$$g_{00} = -S^2(\tau)[1 + 2A(\tau)Q(x^\mu)], \quad (2.14)$$

$$g_{0\alpha} = -S^2 B(\tau)Q_\alpha(x^\mu) \quad (2.15)$$

$$g_{\alpha\beta} = S^2\{[1 + 2H_L(\tau)Q(x^\mu)]^3 g_{\alpha\beta}(x^\mu) + 2H_T(\tau)Q_{\alpha\beta}(x^\mu)\}, \quad (2.16)$$

and up to first order we have

$$g^{00} = -S^{-2}[1 + 2AQ(x^\mu)], \quad (2.17)$$

$$g^{0\alpha} = -S^{-2}BQ^\alpha(x^\mu), \quad (2.18)$$

$$g^{\alpha\beta} = S^{-2}\{[1 - 2H_L(\tau)Q(x^\mu)]^3 g^{\alpha\beta}(x^\mu) - 2H_T(\tau)Q^{\alpha\beta}(x^\mu)\}. \quad (2.19)$$

The interpretation of these variables is given below.

A is the amplitude of the perturbation in the lapse function. This is the ratio of the proper-time distance and the coordinate-time distance between two neighbouring constant time hypersurfaces.

B is the amplitude of a perturbation in the shift vector which is the rate of deviation of a constant space-coordinate line from a line normal to a constant time hypersurface.

H_L is the perturbation amplitude of a unit spatial volume, and

H_T represents the amplitude of the anisotropic distortion of each constant time hypersurface.

It is understood that each of the quantities A , B , H_L and H_T has a distinct value associated with different wave numbers k , and that an equation presupposing harmonic analysis sums implicitly over all k .

The rest frame is the frame in which the energy flux of the fluid 'a' vanishes and U_a^i is the four velocity of the rest frame of the fluid 'a' relative to the coordinate frame.

The three velocity associated with U_a^i is denoted by

$$U_a^\alpha / U_a^0 = v_a(\tau) Q^\alpha(x^\mu), \quad (2.20)$$

and to first order, $U_{ai} U_a^i = -1$ gives,

$$U_a^0 = S^{-1}[1 - AQ]. \quad (2.21)$$

The components of the perturbed energy-momentum tensor are

$$T_{a0}^0 = -E_{a0}(1 + \delta_a Q), \quad (2.22)$$

$$T_{a0}^\alpha = -(E_{a0} + P_{a0})v_a Q^\alpha, \quad (2.23)$$

$$T_{a\alpha}^0 = (E_{a0} + P_{a0})(v_a - B)Q_\alpha, \quad (2.24)$$

$$T_{a\beta}^\alpha = P_{a0}[(1 + \pi_{La} Q)\delta_\beta^\alpha + \pi_{Ta} Q_\beta^\alpha]. \quad (2.25)$$

where the variables are interpreted as:

δ_a is the amplitude of a density perturbation in the fluid a ,

π_{La} is the amplitude of an isotropic pressure perturbation, and

π_{Ta} is the amplitude of an anisotropic stress perturbation,

From the equation of state the fractional pressure perturbation is given by

$$\delta P = \frac{\partial P}{\partial E} \delta E + \frac{\partial P}{\partial s} \delta s, \quad (2.26)$$

and the entropy perturbation is defined to be the difference between the fractional pressure perturbation and that expected from the background pressure-energy density relation, i.e.

$$\eta_a(\tau) Q = \frac{\partial P}{\partial s} \delta s, \quad (2.27)$$

which is written as

$$\eta_a(\tau) Q = \left(\pi_{La} - \frac{E_{a0}}{P_{a0}} \frac{dP_{a0}}{dE_{a0}} \delta_a \right) Q \quad (2.28)$$

$$= \frac{1}{w_a} (w_a \pi_{La} - C_{Sa}^2 \delta) Q, \quad (2.29)$$

where

$$w_a = P_{a0}/E_{a0}, \quad C_{Sa}^2 = dP_{a0}/dE_{a0}. \quad (2.30)$$

2.4 The general gauge transformation

In scalar perturbations the most general gauge transformation is given by the coordinate transformation

$$\tilde{\tau} = \tau + T(\tau)Q(x^\mu), \quad (2.31)$$

and

$$\tilde{x}^\alpha = x^\alpha + L(\tau)Q^\alpha(x^\mu), \quad (2.32)$$

with T and L arbitrary functions of τ .

We know that the changes in the metric tensor are found from

$$g_{\alpha\beta}(x^c) = \frac{\partial \tilde{x}^k}{\partial x^a} \frac{\partial \tilde{x}^l}{\partial x^b} \tilde{g}_{kl}(\tilde{x}^m), \quad (2.33)$$

and can see that the scale factors in the different coordinate systems are related by the first order Taylor series expansion

$$S(\tilde{\tau}) \simeq S(\tau)[1 + (\dot{S}/S)TQ], \quad (2.34)$$

and the transformation of the three-spaces of constant curvature are

$${}^3g_{\alpha\beta}(\tilde{x}^\mu) \simeq {}^3g_{\alpha\beta}(x^\mu) + LQ^\mu \frac{\partial}{\partial x^\mu} {}^3g_{\alpha\beta}. \quad (2.35)$$

The quantities used in the metric perturbations transform as

$$\tilde{A} = A - \dot{T} - (\dot{S}/S)T, \quad (2.36)$$

$$\tilde{B} = B + \dot{L} + kT, \quad (2.37)$$

$$\tilde{H}_L = H_L - (k/3)L - (\dot{S}/S)T \quad (2.38)$$

and

$$\tilde{H}_T = H_T + kL. \quad (2.39)$$

In the case of the matter perturbations the new three-velocity is found to be

$$\tilde{v}_a Q^\alpha = d\tilde{x}_a^\alpha/d\tilde{\tau} \simeq dx^\alpha/d\tau + \dot{L} Q^\alpha, \quad (2.40)$$

which gives

$$\tilde{v}_a = v_a + \dot{L}, \quad (2.41)$$

and the energy density perturbation changes by

$$\tilde{\delta}_a = \delta_a + 3(1 + \omega_a)(\dot{S}/S)T, \quad (2.42)$$

while the isotropic pressure perturbation becomes

$$\tilde{\pi}_{La} = \pi_{La} - T \dot{P}_{a0} / P_{a0} = \pi_{La} + 3(1 + \omega_a) \frac{C_{Sa}^2 \dot{S}}{\omega_a S} T. \quad (2.43)$$

To conclude we state that the amplitude of the traceless part of the stress tensor π_T is gauge invariant.

2.5 Gauge choices

One now has a choice. In the application of gauge theory one often wants to fix the gauge to interpret the results and compare them with observational data, or simply to set the initial conditions of the perturbation variables. The usual way of doing this is to impose conditions on the form of the metric tensor and/or matter perturbations. For example, when working with the formation of galaxies and galactic clusters, one follows through from the early linear fluctuation stage to the more recent non-linear stage. And here the relativistic linear perturbation theory is well suited to the early stages where the density fluctuations have small amplitudes but their scales are larger than the horizon. In the late stage there are pronounced density perturbations, and the various dissipation processes are now accounted for by non-linear treatments: and so Newtonian theory is more appropriate for the analysis as the scales are now much smaller than the horizon size. It is then best to choose a gauge for the early treatment that will agree with the later Newtonian analysis. There is no gauge in which the evolution equations of perturbations become simpler than the gauge invariant equations and in this work we are interested in the geodesic equations for light rays in perturbed universes, so we do not impose a gauge to simplify the equations; but we will mention a few of the different types of gauges for completeness.

To 'set the gauge', as it were, one must fix the time coordinate and the space coordinates and this requires two relations between the gauge invariant variables. The choice of time slicing of the perturbed spacetime is determined by the gauge condition on one of the gauge dependent variables whose change under the gauge

transformation

$$\tilde{\tau} = \tau + T(\tau)Q(x^\mu) , \quad (2.44)$$

$$\tilde{x}^\alpha = x^\alpha + L(\tau)Q^\alpha(x^\mu) , \quad (2.45)$$

is expressed only in term of $T(\tau)$.

For example, if one of the two following equations:

$$\tilde{A} = A - \dot{T} - (\dot{S}/S)T , \quad (2.46)$$

and

$$\tilde{v} - \tilde{B} = (v - B) - kT(\tau) , \quad (2.47)$$

vanish, then a time slicing has been specified.

Now that one has a time slicing the next step is to eliminate the spatial coordinate freedom by requiring that a quantity, whose gauge transformation involves L , vanishes and we have only to look to B , v , H_L and H_T for simple examples.

Some typical gauge specifications:

1. Proper-time slicing: $A = 0$.

Here the proper time distance along the normal vector between two neighbouring hypersurfaces coincides with the coordinate time distance between these hypersurfaces. This condition does not completely specify the time slicing and leaves a gauge freedom parameterized by an arbitrary constant and a gauge mode now appears in the density variation δ_a .

1a. Synchronous gauge: $A = B = 0$

This is the most commonly used of the proper-time slicing gauges. The space coordinates are specified by choosing the lines on which the space coordinates are constant, orthogonal to the constant time hypersurfaces. Here it is found that $B = 0$ leaves a residual gauge freedom, but the synchronous gauge was used quite a lot in the early literature and caused problems e.g. Lifschitz [9], and Lifschitz and Khalatnikov [10].

1b. Comoving proper-time gauge: $A = v = 0$

This restricts the residual gauge freedom to

$$L = \beta \tag{2.48}$$

where β is an arbitrary constant.

2. Velocity-orthogonal slicing: $v = B$

$v - B$ represents the deviation of the matter velocity from the vector normal to the constant time hypersurfaces and in this gauge the matter 4-velocity is orthogonal to the constant time hypersurfaces. This gauge completely eliminates the gauge freedom associated with the time slicing.

2a. Comoving time-orthogonal gauge: $v = B = 0$

The gauge freedom here is also expressed by equation (48) in the comoving proper-time gauge.

2b. Velocity-orthogonal isotropic gauge: $v = B, H_T = 0$

There is no residual gauge freedom in this gauge.

3. Newtonian slicing: $(1/k) \dot{H}_T - B = 0$

This eliminates the gauge freedom in T .

3a. Longitudinal gauge: $B = \dot{H}_T = 0$

Here the residual gauge freedom is expressed by the equation in the comoving proper time gauge.

3b. Comoving Newtonian gauge: $B = (1/k) \dot{H}_T$, $v = 0$

The gauge freedom is the same as in the Longitudinal gauge

4. Uniform Hubble slicing: $-A + (\dot{S}/S)^{-1} \dot{H}_L + (1/3)(\dot{S}/S)^{-1} k B = 0$

For this slicing the perturbation in the volume expansion rate of the constant time hypersurfaces vanishes.

2.6 The construction of gauge invariant variables and their evolution

We continue with the foundations of the gauge invariant theory for scalar perturbations and list the gauge invariant quantities constructed by Bardeen from the above variables representing perturbations in the metric and energy-momentum tensor. For a full discussion of the gauge invariant quantities see Bruni, Dunsby and Ellis [2]. Bardeen now constructs the quantities

$$\Phi_A \equiv A + \frac{1}{k} \dot{B} + \frac{1}{k} \frac{\dot{S}}{S} B - \frac{1}{k^2} \left(\ddot{H}_T + \frac{\dot{S}}{S} \dot{H}_T \right) \quad (2.49)$$

$$\Phi_H \equiv H_L + \frac{1}{3} H_T + \frac{1}{k} \frac{\dot{S}}{S} B - \frac{1}{k} \frac{\dot{S}}{S} \dot{H}_T \quad (2.50)$$

$$v_{Sa} \equiv v_a - \frac{1}{k} \dot{H}_T, \quad (2.51)$$

and for the gauge invariant energy density perturbation he has either

$$\epsilon_{ma} \equiv \delta_a + 3(1 + \omega_a) \frac{1}{k} \frac{\dot{S}}{S} (v_a - B), \quad (2.52)$$

or

$$\epsilon_{ga} \equiv \delta_a - 3(1 + \omega_a) \frac{1}{k} \frac{\dot{S}}{S} \left(B - \frac{1}{k} \dot{H}_T \right). \quad (2.53)$$

The gauge invariant matter ‘velocity’ v_{Sa} can be interpreted in terms of the shear of the matter velocity field since for scalar perturbations in the first order the only

non-zero components of the shear tensor are

$$\sigma^{(a)}{}_{\alpha\beta} = S \left(\dot{H}_{T_a} - kv_a \right) Q_{\alpha\beta}. \quad (2.54)$$

It can then be shown that the time dependence of the rate of shear associated with the perturbation is the velocity amplitude v_{S_a} , divided by the proper reduced wavelength S/k .

Equations (52) and (53) are chosen by Bardeen as two obvious possibilities for the gauge invariant measures of the density perturbation. He obtains them by combining the energy density perturbation δ_a with other gauge-dependant quantities and finds that ϵ_{ma} , is a natural choice from the point of view of the matter. The condition that the matter world lines be orthogonal to the τ is constant spacelike hypersurface is expressed by $v_a = B$ and so ϵ_{ma} will reduce to δ_a for this gauge. Bardeen finds that $\epsilon_{g\alpha}$ measures the energy density perturbation relative to the hypersurface whose normal unit vectors have zero shear. This is due to the fact that B is the three-velocity amplitude of the world lines normal to the $\tau = \text{constant}$ hypersurface and that for zero shear we have $\dot{H}_{T_a} = kv_a$. In Bardeen's analysis he focuses on ϵ_{ma} as it acts as the source of the gauge invariant potential in the Einstein equations, and the equations governing the dynamics of the matter are more physically transparent.

Bardeen uses the zero shear hypersurface to give physical meaning to Φ_H and Φ_A . In a gauge where each constant- τ hypersurface has normals with zero shear i.e. $B - \frac{1}{k} \dot{H}_T = 0$, Bardeen finds that the above quantities become

$$\Phi_A = A \quad (2.55)$$

$$\Phi_H = H_L + \frac{1}{3} H_T \quad (2.56)$$

and so Φ_A is now the lapse function (the amplitude of the spatial dependence of the proper time intervals along the normals between two of the neighbouring zero-shear hypersurfaces that have been invoked). Bardeen writes the intrinsic scalar curvature of a zero shear hypersurface, to first order, as

$$\mathcal{R}_{\text{zero shear}} = [6K + 4(k^2 - 3K)\Phi_H Q]/S^2, \quad (2.57)$$

and so in this sense Φ_H is a 'curvature perturbation'. He then continues by saying: "sufficient conditions for the global perturbations of the spacetime geometry to be small are $\Phi_A Q \ll 1$, $\Phi_H Q \ll 1$, but these are not necessary since other hypersurfaces may be less strongly warped by the perturbation".

The entropy perturbation η_a is also gauge invariant. To find

$$\tilde{\eta}_a = \frac{1}{\tilde{\omega}_a}(\tilde{\omega}_a \tilde{\pi}_{La} - \tilde{C}_{sa}^2 \tilde{\delta}_a), \quad (2.58)$$

we substitute equations (42) and (43) and use the fact that $\tilde{\omega}_a = \omega_a$, and $\tilde{C}_{sa}^2 = C_{sa}^2$ to zero order, to obtain

$$\begin{aligned} \tilde{\eta}_a &= \frac{1}{\omega_a}(\omega_a(\pi_{La} + 3(1 + \omega_a)\frac{C_{sa}^2 \dot{S}}{\omega_a} T) - C_{sa}^2(\delta_a + 3(1 + \omega_a)\frac{\dot{S}}{S} T)) \\ &= \frac{1}{\omega_a}(\omega_a \pi_{La} - C_{sa}^2 \delta_a) \\ &= \eta_a. \end{aligned} \quad (2.59)$$

The evolution of the perturbations is derived from the Einstein equations,

$$\frac{2(k^2 - 3K)}{S^2} \Phi_H = \sum_{a=1}^N E_{a0} \epsilon_a \quad (2.60)$$

$$-\frac{k^2}{S^2}(\Phi_A + \Phi_H) = \sum_{a=1}^N P_{a0} \pi_{Ta}, \quad (2.61)$$

and from the conservation equation $T_{aj;i}^i = 0$ Bardeen gets ($a=1, \dots, N$)

$$\begin{aligned} (E_{a0} \epsilon_a S^3) \cdot &+ \frac{3(E_{a0} + P_{a0})S^3}{k} \left\{ \left[\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \right] v_{Sa} + k \dot{\Phi}_H + \frac{1}{3} k^2 v_{Sa} \right\} \\ &- 3S^3 \frac{\dot{S}}{S} \left[(E_{a0} + P_{a0}) \Phi_A - \frac{2}{3} \left[1 - \frac{3K}{k^2} \right] P_{a0} \pi_{Ta} \right] = 0 \end{aligned} \quad (2.62)$$

$$\begin{aligned} \dot{v}_{Sa} + \frac{\dot{S}}{S} v_{Sa} &= k \Phi_A + \frac{k}{1 + \omega_a} (C_{sa}^2 \epsilon_a + \omega_a \eta_a) \\ &- \frac{2}{3} k \left[1 - \frac{3K}{k^2} \right] \frac{\omega_a}{1 + \omega_a} \pi_{Ta}. \end{aligned} \quad (2.63)$$

And so to sum up, after stating the physical properties of the background space-time the perturbations of the metric tensor and the energy-momentum tensor are introduced. A general gauge transformation is then defined and its effect on the

scalar perturbations is given. The gauge invariant quantities were then constructed for these perturbations and given a physical interpretation before turning to the Einstein equations for their evolution equations. The different gauge choices were also listed but this was just mentioned for interest and completeness as the later work deals exclusively with the perturbation of light-like geodesics travelling from the time of decoupling in the early universe to observers today, and for this work the best description is the gauge invariant one which gives the easiest analysis. The next chapter applies Bardeen's theory to the problem of the CMBR as we rederive Panek's equations which give a gauge invariant expression for the observed fluctuations in the temperature of the radiation emitted from the last scattering surface.

Chapter 3

The derivation of Panek's equations

3.1 Introduction

Now the task is to find the variation of the temperature of the Cosmic Microwave Background Radiation (CMBR) across the surface of last scattering. The light emitted from the last scattering surface travels along the perturbed geodesics in the physical spacetime and picks up a first order deviation in the redshift, which appears to observers as a variation in the temperature at reception. This has added to it the variation in the temperature of emission of radiation at the last scattering surface, and together these give what we refer to as the Sachs-Wolfe effect.

In the perturbed spacetime the null-like geodesics are $x^i(\lambda)$, with λ the affine parameter. The null vector tangent to the geodesic is written

$$k^i = \frac{dx^i}{d\lambda}, \quad k^i = (\nu, P^\alpha), \quad k_i k^i = 0, \quad (3.1)$$

where

$$\nu = {}^0\nu(1 - M), \quad P^\alpha = {}^0P^\alpha + {}^1P^\alpha, \quad (3.2)$$

and ${}^0\nu M$ and ${}^1P^\alpha$ are the first-order corrections to the components of the vector k^i . The total derivative with respect to λ is written as

$$\frac{Dk^i}{d\lambda} = k^i{}_{;j} \frac{dx^j}{d\lambda}, \quad (3.3)$$

and the equation of motion for the null vector is then

$$\frac{Dk^i}{d\lambda} = \frac{dk^i}{d\lambda} + \Gamma_{kl}^i k^k k^l = 0. \quad (3.4)$$

For the observed temperature of the CMBR we have

$$\frac{T_R}{T_E} = \frac{1}{1+z} = \frac{(k^i U_{bi})_R}{(k^i U_{bi})_E}, \quad (3.5)$$

where U_{bi} is the four-vector of velocity of the observer at rest with respect to the baryonic fluid. A new parameter $s(\lambda)$ is now introduced where the derivative with respect to s will be denoted by a prime, i.e. $\frac{M}{ds} = M'$. We have

$$\frac{d\lambda}{ds} = \frac{S^2}{S_R^2}, \quad \frac{d}{ds} = \frac{S^2}{S_R^2} \frac{d}{d\lambda} = \frac{S^2}{S_R^2} \left[\nu \frac{\partial}{\partial \tau} + P^\alpha \frac{\partial}{\partial x^\alpha} \right], \quad (3.6)$$

To zero order this gives us

$${}^0\nu = \frac{S_R^2}{S^2}, \quad {}^0\nu^2 = {}^0P^\alpha {}^0P_\alpha. \quad (3.7)$$

where ${}^0P^\alpha$ is defined in terms of the spatial unit vector R^α in the direction of observation: ${}^0P^\alpha = -{}^0\nu R^\alpha$. The normalization ${}^0\nu = 1$ is used to find solutions for the lightlike geodesics to zero order, and we have

$$\tau = \tau_E + s, \quad (3.8)$$

$$x^\alpha = R^\alpha (\tau_R - \tau_E - s), \quad (3.9)$$

$$\frac{dR^\alpha}{ds} = {}^3\Gamma_{\beta\gamma}^\alpha R^\beta R^\gamma, \quad (3.10)$$

where

$$\frac{d}{ds} = \frac{\partial}{\partial \tau} - R^\alpha \frac{\partial}{\partial x^\alpha}, \quad R^\alpha = -\frac{dx^\alpha}{ds}. \quad (3.11)$$

In particular, for the emission event: $s = 0$, $\tau = \tau_E$, $x_E^\alpha = R_E^\alpha (\tau_R - \tau_E)$, and for the reception event, $s = \tau_R - \tau_E$, $\tau = \tau_R$, $x_R^\alpha = 0$. We now turn to Panek's equations.

3.2 The derivation of Panek's equation 29

We now find M' as we will need it to calculate the observed temperature of the CMBR. Panek's equation 29 is

$$M' = \dot{A} Q + 2kA Q_\alpha R^\alpha + \frac{k}{3} B \dot{Q} + \dot{H}_L Q + (\dot{H}_T - kB) Q_{\alpha\beta} R^\alpha R^\beta. \quad (3.12)$$

The '0' component of the equation of motion gives an expression for M' and so the Christoffel symbols are required but firstly M' and $Dk^0/d\lambda$ are written in a manageable form. We have

$$M' = \frac{dM}{ds} = \frac{S^2}{S_R^2} \frac{dM}{d\lambda}, \quad (3.13)$$

$$\frac{S_R^2}{S^2} = {}^0\nu \quad (3.14)$$

and

$$M' = \frac{1}{{}^0\nu} \frac{dM}{d\lambda}. \quad (3.15)$$

The part of the energy equation used to derive M' is

$$\frac{Dk^0}{d\lambda} = \frac{dk^0}{d\lambda} + \Gamma_{kl}^0 k^k k^l. \quad (3.16)$$

We have $k^0 = \nu$ and find

$$\frac{d\nu}{d\lambda} = \frac{d[(1-M){}^0\nu]}{d\lambda} \quad (3.17)$$

$$= -{}^0\nu \frac{dM}{d\lambda} + \nu \frac{\partial {}^0\nu}{\partial \tau} (1-M) \quad (3.18)$$

$$= -{}^0\nu \frac{dM}{d\lambda} + {}^0\nu (1-M)^2 ({}^0\nu (-2\frac{\dot{S}}{S})) \quad (3.19)$$

$$= -{}^0\nu \frac{dM}{d\lambda} + {}^0\nu^2 (1-2M) (-2\frac{\dot{S}}{S}) \quad (3.20)$$

$$= -{}^0\nu \frac{dM}{d\lambda} + {}^0\nu^2 (-2\frac{\dot{S}}{S} + 4\frac{\dot{S}}{S}M), \quad (3.21)$$

which gives

$$M' = -2\frac{\dot{S}}{S} + 4M\frac{\dot{S}}{S} + \frac{1}{{}^0\nu^2} \Gamma_{kl}^0 k^k k^l. \quad (3.22)$$

The Christoffel symbols are now calculated from the metric written below.

Metric:

$$g_{00} = -S^2(1+2AQ) \quad g^{00} = -S^{-2}(1-2AQ) \quad (3.23)$$

$$g_{0\alpha} = -S^2 BQ_\alpha \quad g^{0\alpha} = -S^{-2}(BQ^\alpha) \quad (3.24)$$

$$g_{\alpha\beta} = S^2[(1 + 2H_L Q)^3 g_{\alpha\beta} + 2H_T Q_{\alpha\beta}] , \quad (3.25)$$

$$g^{\alpha\beta} = S^{-2}[(1 - 2H_L Q)^3 g^{\alpha\beta} - 2H_T Q^{\alpha\beta}] , \quad (3.26)$$

and from these one finds the Christoffel symbols to be

$$\Gamma_{00}^0 = \frac{\dot{S}}{S} + \dot{A} Q , \quad (3.27)$$

$$\Gamma_{\alpha 0}^0 = \Gamma_{0\alpha}^0 = -\frac{\dot{S}}{S} B Q_\alpha - k A Q_\alpha , \quad (3.28)$$

$$\begin{aligned} \Gamma_{\alpha\beta}^0 &= -B Q_{\alpha\beta} + \frac{1}{3} k B^3 g_{\alpha\beta} Q + \frac{\dot{S}}{S} [(1 + 2H_L Q)^3 g_{\alpha\beta} + 2H_T Q_{\alpha\beta}] \\ &\quad - 2\frac{\dot{S}}{S} A Q^3 g_{\alpha\beta} + \dot{H}_L Q^3 g_{\alpha\beta} + \dot{H}_T Q_{\alpha\beta} . \end{aligned} \quad (3.29)$$

The equation for null-geodesics, $k_i k^i = 0$ gives a further constraint on M and together with the previous results will give the final form of M' . We write the form for k_i and then derive $k_i k^i$.

$$k_i = g_{ij} k^j : \quad (3.30)$$

$$k_0 = g_{00} k^0 + g_{0\alpha} k^\alpha \quad (3.31)$$

$$k_\alpha = g_{\alpha 0} k^0 + g_{\alpha\beta} k^\beta \quad (3.32)$$

$$k^i k_i = k^0 k_0 + k^\alpha k_\alpha \quad (3.33)$$

$$k^i k_i = k^0 g_{00} k^0 + k^0 k^\alpha g_{0\alpha} + k^\alpha g_{\alpha 0} k^0 + k^\alpha g_{\alpha\beta} k^\beta , \quad (3.34)$$

which gives, after substituting the metric and the null vector tangent to the geodesics,

$$\begin{aligned} k^i k_i &= -S^2(1 + 2AQ)({}^0\nu(1 - M))^2 + (-S^2 BQ_\alpha)({}^0P^\alpha + {}^1P^\alpha){}^0\nu(1 - M) \\ &\quad + (-S^2 BQ_\alpha){}^0\nu(1 - M)({}^0P^\alpha + {}^1P^\alpha) \\ &\quad + S^2[(1 + 2H_L Q)^3 g_{\alpha\beta} + 2H_T Q_{\alpha\beta}]({}^0P^\beta + {}^1P^\beta)({}^0P^\alpha + {}^1P^\alpha) , \end{aligned}$$

$$\begin{aligned}
&= -S^2 {}^0\nu^2(1 - 2M + 2AQ) - 2(S^2 {}^0\nu(-{}^0\nu R^\alpha)(1 - M)BQ_\alpha) \\
&\quad + S^2[2H_L Q {}^0\nu^2 + 2H_T Q_{\alpha\beta} {}^0\nu^2 R^\alpha R^\beta] \\
&\quad + S^2 {}^0\nu^2 + S^2 {}^3g_{\alpha\beta}({}^0P^\alpha {}^1P^\beta + {}^1P^\alpha {}^0P^\beta), \tag{3.35}
\end{aligned}$$

and leaves

$$\begin{aligned}
k^i k_i &= 2S^2 {}^0\nu^2 M - 2S^2 A Q {}^0\nu^2 + 2S^2 B Q_\alpha R^\alpha {}^0\nu^2 \\
&\quad + S^2 {}^3g_{\alpha\beta}({}^0P^\alpha {}^1P^\beta + {}^1P^\alpha {}^0P^\beta) \\
&\quad + S^2[(2H_L Q) {}^3g_{\alpha\beta} + 2H_T Q_{\alpha\beta}]R^\alpha R^\beta {}^0\nu^2. \tag{3.36}
\end{aligned}$$

Returning to the equation of motion, we simplify the terms with the Christoffel symbols:

$$\Gamma_{ki}^0 k^k k^l = \Gamma_{00}^0 k^0 k^0 + 2\Gamma_{\alpha 0}^0 k^\alpha k^0 + \Gamma_{\alpha\beta}^0 k^\alpha k^\beta \tag{3.37}$$

$$\begin{aligned}
\Gamma_{00}^0 k^0 k^0 &= \left(\frac{\dot{S}}{S} + \dot{A} Q\right)({}^0\nu)^2(1 - M)^2 \\
&= \left(\frac{\dot{S}}{S} + \dot{A} Q\right) {}^0\nu^2(1 - 2M) \\
&= \frac{\dot{S}}{S} {}^0\nu^2 - 2M \frac{\dot{S}}{S} {}^0\nu^2 + \dot{A} Q {}^0\nu^2 \tag{3.38}
\end{aligned}$$

$$\begin{aligned}
2\Gamma_{\alpha 0}^0 k^\alpha k^0 &= 2\left(-\frac{\dot{S}}{S} B Q_\alpha - A Q_\alpha k\right)({}^0P^\alpha + {}^1P^\alpha) {}^0\nu \\
&= {}^0\nu^2\left(2\frac{\dot{S}}{S} B Q_\alpha R^\alpha + 2A Q_\alpha k R^\alpha\right) \tag{3.39}
\end{aligned}$$

and,

$$\begin{aligned}
\Gamma_{\alpha\beta}^0 k^\alpha k^\beta &= \Gamma_{\alpha\beta}^0 ({}^0P^\alpha + {}^1P^\alpha) ({}^0P^\alpha + {}^1P^\beta) \\
&= \frac{\dot{S}}{S} {}^3g_{\alpha\beta} {}^0P^\alpha {}^0P^\beta + \frac{\dot{S}}{S} {}^3g_{\alpha\beta} ({}^0P^\alpha {}^1P^\beta + {}^1P^\alpha {}^0P^\beta) \\
&\quad + \frac{\dot{S}}{S} 2H_L Q {}^3g_{\alpha\beta} {}^0P^\alpha {}^0P^\beta + \frac{\dot{S}}{S} 2H_T Q_{\alpha\beta} {}^0P^\alpha {}^0P^\beta \\
&\quad - Bk Q_{\alpha\beta} {}^0P^\alpha {}^0P^\beta + \frac{k}{3} BQ {}^0\nu^2 - 2\frac{\dot{S}}{S} AQ {}^0\nu^2 \\
&\quad + \dot{H}_L Q {}^0\nu^2 + \dot{H}_T Q_{\alpha\beta} {}^0P^\alpha {}^0P^\beta, \tag{3.40}
\end{aligned}$$

which added together become,

$$\begin{aligned}
\Gamma_{kl}^0 k^k k^l &= {}^0\nu^2 \left(2\frac{\dot{S}}{S} BQ_\alpha R^\alpha + 2AQ_\alpha k R^\alpha + \frac{\dot{S}}{S} - 2M\frac{\dot{S}}{S} + \dot{A}Q + \frac{\dot{S}}{S} \right. \\
&\quad + \frac{\dot{S}}{S} 2H_L Q + \frac{\dot{S}}{S} 2H_T Q_{\alpha\beta} R^\alpha R^\beta - kBQ_{\alpha\beta} R^\alpha R^\beta + \frac{k}{3} BQ \\
&\quad \left. - 2\frac{\dot{S}}{S} AQ + \dot{H}_L Q + \dot{H}_T Q_{\alpha\beta} R^\alpha R^\beta \right) \\
&\quad + \frac{\dot{S}}{S} {}^3g_{\alpha\beta} ({}^0P^\alpha {}^1P^\beta + {}^1P^\alpha {}^0P^\beta). \tag{3.41}
\end{aligned}$$

A final form for M' can now be written down:

$$\begin{aligned}
M' &= \underline{2M\frac{\dot{S}}{S} + \dot{A}Q} + \underline{2\frac{\dot{S}}{S} BQ_\alpha R^\alpha} \\
&\quad + 2AQ_\alpha k R^\alpha + \underline{\frac{\dot{S}}{S} 2(-R_\beta) {}^1P^\beta} \\
&\quad + \underline{\frac{\dot{S}}{S} 2H_L Q} + \underline{\frac{\dot{S}}{S} 2H_T Q_{\alpha\beta} R^\alpha R^\beta} + \frac{k}{3} BQ \\
&\quad - kBQ_{\alpha\beta} R^\alpha R^\beta - \underline{2\frac{\dot{S}}{S} AQ} + \dot{H}_L Q + \dot{H}_T Q_{\alpha\beta} R^\alpha R^\beta, \tag{3.42}
\end{aligned}$$

where the underlined terms are equal to

$$\frac{\dot{S}}{S^3 a_{\nu}^2} (k^i k_i) \quad (3.43)$$

which vanish, and so we get Panek's equation (29):

$$\begin{aligned} M' = & \dot{A} Q + 2A Q_{\alpha} k R^{\alpha} - B k Q_{\alpha\beta} R^{\alpha} R^{\beta} \\ & + \frac{k}{3} B Q + \dot{H}_L Q + \dot{H}_T Q_{\alpha\beta} R^{\alpha} R^{\beta} . \end{aligned} \quad (3.44)$$

Equation 30 of Panek gives an expression for the emission temperature divided by the temperature at reception, T_E/T_R . This will eventually be used to calculate an expression for the variation in the temperature of the black-body radiation emitted at the time of decoupling.

3.3 The derivation of Panek's equation 30

We begin with Panek's equation 25

$$\frac{T_E}{T_R} = \frac{1}{1+z} = \frac{(k^i U_{bi})_E}{(k^i U_{bi})_R} , \quad (3.45)$$

where z is the redshift of the point of emission relative to the point of reception and U_{bi} is the four velocity of the observer at rest with respect to the baryonic fluid ignoring local gravity effects. To find the redshift we first find the component of the 4-velocity in the k^i direction.

$$k^i U_{bi} = k^i g_{ij} U_b^j = k^0 g_{0j} U_b^j + k^{\alpha} g_{\alpha j} U_b^j . \quad (3.46)$$

The perturbation of the four-velocity of the fluid a , is

$$U_a^0 = \frac{1}{S} (1 - A Q) , \quad (3.47)$$

$$U_a^{\alpha} = \frac{1}{S} v_a Q^{\alpha} , \quad (3.48)$$

which yields

$$\begin{aligned} k^0 g_{0j} U_b^j &= {}^0\nu(1-M)(-S^2(1+2AQ))\frac{1}{S}(1-AQ) \\ &\quad + {}^0\nu(1-M)(-S^2 BQ_\alpha)\frac{1}{S}v_b Q^\alpha. \end{aligned} \quad (3.49)$$

The second term is discarded as it is second order, and so

$$k^0 g_{0j} U_b^j = -{}^0\nu S(1+2AQ - AQ - M) \quad (3.50)$$

$$\begin{aligned} k^\alpha g_{\alpha j} U_b^j &= P^\alpha(-S^2 BQ_\alpha)\frac{1}{S}(1-AQ) \\ &\quad + P^\alpha S^2[(1+2H_L Q)^3 g_{\alpha\beta} + 2H_T Q_{\alpha\beta}]\frac{1}{S}v_b Q^\beta \end{aligned} \quad (3.51)$$

$$= -SQ_\alpha P^\alpha + S^3 g_{\alpha\beta} v_b Q^\beta P^\alpha, \quad (3.52)$$

which gives

$$k^i U_{bi} = {}^0\nu S(-1 - AQ + M) - \frac{S^2}{S} BQ_\alpha P^\alpha + \frac{S^2}{S} {}^3 g_{\alpha\beta} Q^\beta v_b P^\alpha \quad (3.53)$$

$$k^i U_{bi} = {}^0\nu S(M - 1 - AQ + BQ_\alpha R^\alpha - v_b Q_\alpha R^\alpha). \quad (3.54)$$

The previous equations are written in integral form where

$$\frac{d}{ds} = \frac{\partial}{\partial\tau} - R^\alpha \frac{\partial}{\partial x^\alpha} \quad (3.55)$$

is used to find a form for the redshift using M' . We will use the form already obtained for M' , differentiate all the terms in equation (54) and then write out $k^i U_{bi}$ in terms of their integrals. They are written as

$$\frac{dAQ}{ds} = \dot{A} Q - Q_{|\alpha} R^\alpha A, \quad (3.56)$$

$$\frac{d(BQ_\alpha R^\alpha)}{ds} = \dot{B} Q_\alpha R^\alpha - BQ_{\alpha,\beta} R^\alpha R^\beta + BQ_\alpha \frac{dR^\alpha}{ds}, \quad (3.57)$$

and

$$\frac{d(v_b Q_\alpha R^\alpha)}{ds} = \dot{v}_b Q_\alpha R^\alpha - v_b Q_{\alpha,\beta} R^\alpha R^\beta + v_b Q_\alpha \frac{dR^\alpha}{ds}. \quad (3.58)$$

Now substitute

$$Q_\alpha = -\frac{1}{k} Q_{|\alpha}, \quad (3.59)$$

and

$$\begin{aligned} Q_{\alpha|\beta} &= \left(-\frac{1}{k} Q_{|\alpha} \right)_{|\beta} = -\frac{1}{k} Q_{|\alpha\beta} \\ &= -k Q_{\alpha\beta} + \frac{k}{3} {}^3g_{\alpha\beta} Q, \end{aligned} \quad (3.60)$$

to obtain

$$\frac{dAQ}{ds} = \dot{A} Q + k Q_{\alpha} R^{\alpha} A, \quad (3.61)$$

and

$$\begin{aligned} \frac{d(BQ_{\alpha}R^{\alpha})}{ds} &= \dot{B} Q_{\alpha}R^{\alpha} + BQ_{\alpha} \frac{dR^{\alpha}}{ds} - B(Q_{\alpha|\beta} + \Gamma_{\alpha\beta}^{\sigma} Q_{\sigma}) R^{\alpha} R^{\beta} \\ &= \dot{B} Q_{\alpha}R^{\alpha} - BQ_{\alpha|\beta} R^{\alpha} R^{\beta} \\ &= \dot{B} Q_{\alpha}R^{\alpha} - B(-kQ_{\alpha\beta} + \frac{k}{3} {}^3g_{\alpha\beta}) R^{\alpha} R^{\beta} \\ &= \dot{B} Q_{\alpha}R^{\alpha} + BkQ_{\alpha\beta} R^{\alpha} R^{\beta} - \frac{k}{3} v_b Q. \end{aligned} \quad (3.62)$$

Similarly

$$\frac{d(v_b Q_{\alpha}R^{\alpha})}{ds} = \dot{v}_b Q_{\alpha}R^{\alpha} + v_b k Q_{\alpha\beta} R^{\alpha} R^{\beta} - \frac{k}{3} v_b Q. \quad (3.63)$$

These forms are substituted into equation (54) to get

$$\begin{aligned} k^i U_{bi} &= -{}^0\nu S \left(1 - \int \left\{ \dot{A} Q + 2kAQ_{\alpha}R^{\alpha} + \frac{k}{3} BQ + \dot{H}_L Q + (\dot{H}_T - kB)Q_{\alpha\beta}R^{\alpha}R^{\beta} \right\} ds \right. \\ &\quad + \int \left\{ \dot{A} Q + kAQ_{\alpha}R^{\alpha} \right\} ds - \int \left\{ \dot{B} Q_{\alpha}R^{\alpha} + kBQ_{\alpha\beta}R^{\alpha}R^{\beta} - \frac{k}{3} BQ \right\} ds \\ &\quad \left. + \int \left\{ \dot{v}_b Q_{\alpha}R^{\alpha} + kv_b Q_{\alpha\beta}R^{\alpha}R^{\beta} - \frac{k}{3} v_b Q \right\} ds \right) \\ &= -{}^0\nu S \left(1 - \int \left[kAQ_{\alpha}R^{\alpha} + \dot{H}_L Q + (\dot{H}_T - kv_b)Q_{\alpha\beta}R^{\alpha}R^{\beta} \right. \right. \\ &\quad \left. \left. + \frac{k}{3} v_b Q + (\dot{B} - \dot{v}_b)Q_{\alpha}R^{\alpha} \right] ds \right) \end{aligned}$$

$$\begin{aligned}
&= -{}^0\nu_S \left(1 - \int \left[\left[\dot{H}_L + \frac{k}{3}v_b \right] Q - (\dot{v}_b - kA - \dot{B}) Q_{\alpha} R^{\alpha} \right. \right. \\
&\quad \left. \left. - (kv_b - \dot{H}_T) Q_{\alpha\beta} R^{\alpha} R^{\beta} \right] ds \right), \tag{3.64}
\end{aligned}$$

where the integral part is now written $\int[\dots]ds$. Returning to the redshift, we have

$$\frac{(k^i U_{bi})_E}{(k^i U_{bi})_R} = \frac{{}^0\nu_E S_E}{{}^0\nu_R S_R} \left(1 - \int_E [\dots] ds \right) \left(1 + \int_R [\dots] ds \right), \tag{3.65}$$

and substitute

$${}^0\nu_R = 1; \quad {}^0\nu_E = \frac{S_R^2}{S_E^2}, \tag{3.66}$$

to get the final form

$$\frac{(k^i U_{bi})_E}{(k^i U_{bi})_R} = \frac{S_R}{S_E} \left(1 + \int_R [\dots] ds - \int_E [\dots] ds \right). \tag{3.67}$$

Hence

$$\frac{T_E}{T_R} = \frac{S_R}{S_E} \left(1 + \int_E^R [\dots] ds \right) \tag{3.68}$$

$$\begin{aligned}
&= \frac{S_R}{S_E} \left(1 + \int_E^R \left\{ \left[\dot{H}_L + \frac{k}{3}v_b \right] Q - (\dot{v}_b - kA - \dot{B}) Q_{\alpha} R^{\alpha} \right. \right. \\
&\quad \left. \left. - (kv_b - \dot{H}_T) Q_{\alpha\beta} R^{\alpha} R^{\beta} \right\} ds \right) \tag{3.69}
\end{aligned}$$

where the integral is along the zero-order lightlike geodesic.

With the null vector tangent to the geodesic, its equation of motion and the matter four-velocity, it is seen that Panek has derived an equation for the ratio of temperatures at emission and reception in the perturbed universe. He now continues by modelling a simple approximation for the details of recombination from which the perturbation to the volume expansion S is obtained. This makes it possible to rewrite his equation (30) in gauge invariant quantities and from that find a gauge invariant form for the temperature fluctuations in the general case. In the next section Panek's equation (34) is rederived which gives the perturbed positioning of the last scattering surface in terms of the energy-density perturbations of the baryons and photons.

3.4 The derivation of Panek's equation 34

In Panek's model of decoupling the emission of radiation occurs on the hypersurface of constant density of free electrons that couple to photons by Thompson scattering. This is a function of the local temperature and density of baryons and so the hypersurface of emission is neither the hypersurface of constant baryon density nor that of constant temperature. The time of emission at a point occurs at the time $\tau_{E0} + \Delta\tau$ where the subscript E0 signifies the moment of emission in the zeroth order and $\Delta\tau$ is a function of the perturbations.

We will now show that

$$\frac{\dot{S}}{S}\Delta\tau = \frac{1}{3+D}\delta_b Q + \frac{D}{4(3+D)}\delta_\gamma Q, \quad (3.70)$$

where

$$D = \left[\frac{f(dg/dT)T}{(df/dE_b)gE_b} \right]_E. \quad (3.71)$$

The density of free electrons at emission is denoted by n_{eE} , where

$$n_{eE} = \text{const} = n_e(\tau_{E0} + \Delta\tau) = n_{e0}(\tau_{E0} + \Delta\tau)(1 + \delta_e Q), \quad (3.72)$$

from which we obtain

$$\Delta\tau = -\frac{n_{e0}}{\dot{n}_{e0}}\delta_e Q, \quad (3.73)$$

where δ_e is a perturbation of electron density.

In general

$$n_e = f(E_b)g(T). \quad (3.74)$$

The perturbations of the different quantities are:

$$n_e = n_{e0}(1 + \delta_e Q) \quad \text{electron density} \quad (3.75)$$

$$E_b = E_{b0}(1 + \delta_b Q) \quad \text{baryons} \quad (3.76)$$

$$E_\gamma = E_{\gamma 0}(1 + \delta_\gamma Q) \quad \text{radiation} \quad (3.77)$$

$$T = T_0(1 + \delta_T Q) \quad \text{temperature,} \quad (3.78)$$

where, with the use of the Stefan-Boltzman law,

$$E_\gamma = \sigma T^4, \quad (3.79)$$

we find, to first order

$$\begin{aligned} E_{\gamma 0}(1 + \delta_\gamma Q) &= \sigma T_0^4(1 + \delta_T Q)^4 \\ &= \sigma T_0^4(1 + 4\delta_T Q), \end{aligned} \quad (3.80)$$

and therefore

$$\delta_T Q = \frac{1}{4}\delta_\gamma Q. \quad (3.81)$$

The perturbation in the electron density is written as

$$\begin{aligned} 1) \\ \delta_e Q &= \frac{n_e - n_{e0}}{n_{e0}} \\ &= \frac{f(E_{b0})(dg/dT)|_{T_0} T_0 \delta_T Q + g(T_0)(df/dE_b)|_{E_{b0}} E_{b0} \delta_b Q}{f(E_{b0})g(T_0)} \\ &= \left(\frac{f(E_{b0})(dg/dT)|_{T_0} T_0}{4g(T_0)(df/dE_b)|_{E_{b0}} E_{b0}} \delta_\gamma Q + \delta_b Q \right) \left(\frac{(df/dE_b)|_{E_{b0}} E_{b0}}{f(E_{b0})} \right) \\ &= \frac{((D/4)\delta_\gamma Q + \delta_b Q)(df/dE_b)|_{E_{b0}} E_{b0}}{(f(E_{b0}))}, \end{aligned} \quad (3.82)$$

where, in the last equality, we have made use of equation (71) above. Equation (75), the definition for the electron density, yields

$$2) \\ -\frac{\dot{n}_{e0}}{n_{e0}} = -\frac{f(E_{b0})g(T_0)}{g(T_0)(df/dE_b)(dE_b/d\tau)|_{E_{b0}} + f(dg/dT)(dT/d\tau)|_{T_0}}, \quad (3.83)$$

but, from the condition $P_{b0} \ll E_{b0}$, we have

$$\frac{\dot{E}_{b0}}{E_{b0}} = -3\frac{\dot{S}}{S}, \quad (3.84)$$

and

$$\frac{\dot{T}}{T} = -\frac{\dot{S}}{S}, \quad (3.85)$$

which gives

$$-\frac{\dot{n}_{e0}}{\dot{n}_{e0}} = \frac{f(E_{b0})g(T_0)}{g(T_0)(df/dE_{b0})3\frac{\dot{S}}{S}E_{b0} + f(dg/dT)\frac{\dot{S}}{S}T_0} \quad (3.86)$$

Multiply result (1) with result (2) to obtain

$$\Delta\tau = \frac{\frac{D}{4}\delta_\gamma Q + \delta_b Q}{3\frac{\dot{S}}{S} + D\frac{\dot{S}}{S}}, \quad (3.87)$$

and therefore

$$\frac{\dot{S}}{S}\Delta\tau = \frac{D}{4(3+D)}\delta_\gamma Q + \frac{1}{3+D}\delta_b Q, \quad (3.88)$$

which gives the required result.

In the particular case of adiabatic perturbations, we have $\delta_b/3 = \delta_\gamma/4$, which reduces the above equation to

$$\frac{\dot{S}}{S}\Delta\tau = \frac{1}{3}\delta_b Q. \quad (3.89)$$

A discussion of the distribution of temperature T_S and baryon density E_S on the last scattering surface follows which makes interesting observations about the difference between the adiabatic and non-adiabatic cases. We will also understand how the choice of last scattering surface in the Sachs and Wolfe paper, who do not state explicitly that they use an adiabatic perturbation, gives an equation for $\delta T/T|_R$ that is the same as the one that Panek gives for the adiabatic case.

The temperature perturbation at the last scattering surface is

$$T_S = T_{S0}(\tau_{E0} + \Delta\tau)(1 + \delta_T Q). \quad (3.90)$$

To first order we obtain

$$T_S = T_{S0}(\tau_{E0}) \left(1 + \frac{\dot{T}}{T} \Big|_{\tau_{E0}} \Delta\tau + \delta_T Q \right), \quad (3.91)$$

and substitute $\delta_T = \frac{1}{4}\delta_\gamma$ and $\frac{\dot{T}}{T} \Big|_{\tau_{E0}} = -\frac{\dot{S}}{S} \Big|_{\tau_{E0}}$, to find

$$T_S = T_{S0}(\tau_{E0}) \left(1 - \frac{\dot{S}}{S}\Delta\tau + \frac{1}{4}\delta_\gamma Q \right). \quad (3.92)$$

If we substitute equation (89) into (92), we get

$$T_S = T_{S0}(\tau_{E0}) . \quad (3.93)$$

So for the adiabatic perturbation we find that the temperature on the last scattering surface is constant and takes the value at the last scattering surface in the unperturbed Robertson-Walker model. If we consider this from the physical point of view we find that the temperature perturbation (the third term of (92)) is compensated for by the positioning of the last scattering surface (the second term of (92)). For the baryon density E_{Sb} at the last scattering surface, the same argument holds if the baryon pressure is neglected ($E_b S^3 = \text{constant}$ or $\frac{\dot{E}_b}{E_b} = -3\frac{\dot{S}}{S}$), and then we find

$$E_{Sb} = E_{Sb0}(\tau_{E0}) . \quad (3.94)$$

So not only the perturbed temperature, but also the perturbed baryon density is constant at the last scattering surface in the adiabatic case. We will talk later about the Sachs-Wolfe paper in relation to the Ellis-Bruni formalism but just to sum up we see that Sachs-Wolfe have not considered the real last scattering surface but instead a surface on which the temperature and baryon density are unperturbed as well as its position. As a result of the compensation effect they obtain the result for the adiabatic case of Panek where he considers the extra temperature perturbation term that he gets from the placing of the LSS to be dominated by terms coming from the temperature fluctuations arising from the perturbed geodesic.

We are now ready to discuss the CMBR pattern $T_R(\theta, \phi)$. It is now possible to evaluate $[\frac{\delta T}{T}]_R$ and here we make our break with Panek by defining the quantity T_{R0} in a different way. He gives the following definition

$$T_{R0} = \frac{S_{E0} T_E}{S_{R0}} , \quad (3.95)$$

which varies over the surface of last scattering and is essentially a choice of gauge between the background and real space times. For the background temperature at reception we instead define the zero order quantity

$$T_{R0} = \frac{S_{E0} T_{E0}}{S_{R0}} , \quad (3.96)$$

where for the adiabatic case, $T_E = T_{E0}$, and then this definition is in agreement with Panek but in general the two are different.

Panek defines a temperature variation

$$\left[\frac{\delta T}{T} \right]_R = \frac{T_R - T_{R0}}{T_{R0}}, \quad (3.97)$$

which we use as well.

3.5 The derivation of Panek's equation 38

With the use of equations (68) and (97) we obtain

$$\left. \frac{\delta T}{T} \right|_R = \frac{T_R - T_{R0}}{T_{R0}} = \frac{(S_E)/(S_R)T_E(1 - \int_E^R [\dots] ds) - (S_{E0})/(S_{R0})T_{E0}}{(S_{E0})/(S_{R0})T_{E0}}, \quad (3.98)$$

and the temperature at last scattering can be written in the form

$$T_E = T_{E0}(\tau_{E0} + \Delta\tau)(1 + \delta_T Q) = T_{E0}(1 + \delta_T Q)\left(1 + \frac{\dot{T}}{T} \Delta\tau\right), \quad (3.99)$$

which is analogous to equation (72). This leads to the new relation

$$\begin{aligned} \left. \frac{\delta T}{T} \right|_R &= \frac{(1 + \left. \frac{\dot{S}}{S} \right|_E \Delta\tau)(T_{E0})(1 + \delta_T Q)\left(1 + \frac{\dot{T}}{T} \Delta\tau\right)(1 - \int_E^R [\dots] ds)}{T_{E0}} - 1 \\ &= 1 + \left. \frac{\dot{S}}{S} \right|_E \Delta\tau + \left. \frac{\dot{T}}{T} \right|_E \Delta\tau + \delta_T Q|_E - \int_E^R [\dots] ds - 1, \end{aligned} \quad (3.100)$$

which for the radiation gives

$$\frac{\dot{T}}{T} = -\frac{\dot{S}}{S}, \quad \text{to zeroth order,} \quad (3.101)$$

and so

$$\left. \frac{\delta T}{T} \right|_R = \delta_T Q|_E - \int_E^R [\dots] ds. \quad (3.102)$$

Since $\delta_T = \frac{1}{4}\delta_\gamma$ from Planck (equilibrium spectrum), we have

$$\left. \frac{\delta T}{T} \right|_R = \frac{1}{4}\delta_\gamma Q \Big|_E - \int_E^R [\dots] ds, \quad (3.103)$$

and rewrite the first term with the use of equation (51) and (52) in chapter 2 with $\omega_\gamma = 1/3$ to find

$$\begin{aligned}\delta_T Q|_E &= \frac{1}{4} \delta_\gamma Q \Big|_E \\ &= \frac{1}{4} \epsilon_\gamma Q|_E - \frac{1}{k} \frac{\dot{S}}{S} (v_{S\gamma} - v_{Sb}) Q|_E - \frac{1}{k} \frac{\dot{S}}{S} (v_b - B) Q|_E, \quad (3.104)\end{aligned}$$

where ϵ_{ma} has been replaced by ϵ_a . The last term in the above expression cancels with the same term in the integral which can be written

$$\begin{aligned}\int_E^R [\dots] ds &= \left[\left(\Phi_H - \frac{1}{k} \frac{\dot{S}}{S} v_{Sb} \right) Q \right]_E^R + \frac{Q}{k} \frac{\dot{S}}{S} (v_b - B) \Big|_E^R \\ &\quad + \int_E^R \left\{ \left[\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right] Q_{|\alpha} R^\alpha \right. \\ &\quad \left. - \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right\} ds. \quad (3.105)\end{aligned}$$

With the use of the fact that $w_b = 0$ and $w_\gamma = \frac{1}{3}$, we obtain

$$\begin{aligned}\left(\frac{\delta T}{T} \right)_R &= - \frac{Q}{k} \frac{\dot{S}}{S} (v_b - B) \Big|_R + \frac{1}{4} \epsilon_\gamma Q|_E - \frac{1}{k} \frac{\dot{S}}{S} (v_{S\gamma} - v_{Sb}) Q|_E \\ &\quad - \left[\left(\Phi_H - \frac{1}{k} \frac{\dot{S}}{S} v_{Sb} \right) Q \right]_E^R - \int_E^R \left\{ \left[\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right] Q_{|\alpha} R^\alpha \right. \\ &\quad \left. - \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right\} ds. \quad (3.106)\end{aligned}$$

The first term on the right-hand side of the previous equation is independent of the direction of observation and so is undetectable. Terms like this will be dropped as they have no physical significance.

We now have a new version of Panek's equation (38) which uses a different definition for T_{R0} . The difference between equation (106) and Panek's equation (38) is

$$\frac{3}{3+D} \left[\frac{\epsilon_b}{3} - \frac{\epsilon_\gamma}{4} + \frac{1}{k} \frac{\dot{S}}{S} (v_{S\gamma} - v_{Sb}) \right] = \frac{3}{3+D} \left(\frac{1}{3} \delta_b - \frac{1}{4} \delta_\gamma \right) = \frac{3}{3+D} \chi, \quad (3.107)$$

which is evaluated at the last scattering surface, and so in the important case of the adiabatic perturbation, where we have

$$\delta_\gamma = \frac{4}{3}\delta_b, \quad (3.108)$$

the above equation is zero which shows that equation (106) is the same as Panek's (38) in the adiabatic case. To determine Panek's equation for the temperature variation from the above, we simply add $\frac{3}{3+D}\chi$.

Let us show this in detail. We have

$$-\frac{1}{k}\frac{\dot{S}}{S}(v_{S\gamma} - v_{Sb})Q|_E = \left(\frac{1}{3}\epsilon_b - \frac{1}{4}\epsilon_\gamma\right)Q|_E, \quad (3.109)$$

which leaves us with

$$\begin{aligned} \frac{\delta T}{T}\Big|_R &= \frac{1}{3}(\epsilon_b Q)_E - \left[(\Phi_H - \frac{1}{k}\frac{\dot{S}}{S}v_{Sb})Q \right]_E^R - \int_E^R \left\{ \left[\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right] Q_{|\alpha} R^\alpha \right. \\ &\quad \left. - \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right\} ds. \end{aligned} \quad (3.110)$$

As it is with Panek.

We now derive the terms of equation (106).

The first term is written

$$\begin{aligned} \delta_T Q|_E &= \frac{1}{4}\delta_\gamma Q|_E \\ &= \frac{1}{4}(\epsilon_\gamma Q - \frac{4}{k}\frac{\dot{S}}{S}(v_\gamma - B)Q)\Big|_E \\ &= \frac{1}{4}\epsilon_\gamma Q|_E - \frac{1}{k}\frac{\dot{S}}{S}(v_{S\gamma} - v_{Sb})Q|_E - \frac{1}{k}\frac{\dot{S}}{S}(v_b - B)Q|_E. \end{aligned} \quad (3.111)$$

The integral part,

$$\begin{aligned} &\int_E^R \left[(\dot{H}_L + \frac{k}{3}v_b)Q - (\dot{v}_b - kA - \dot{B})Q_{\alpha} R^\alpha \right. \\ &\quad \left. - (kv_b - \dot{H}_T)Q_{\alpha\beta} R^\alpha R^\beta \right] ds, \end{aligned} \quad (3.112)$$

takes somewhat more manipulation and we begin by substituting

$$Q_\alpha = -\frac{1}{k}Q_{|\alpha}, \quad (3.113)$$

and

$$Q_{\alpha\beta} = \frac{1}{k^2}Q_{|\alpha\beta} + \frac{1}{3}{}^3g_{\alpha\beta}Q, \quad (3.114)$$

to obtain

$$\begin{aligned} \int_E^R [\dots] ds &= \int_E^R \left[(\dot{H}_L + \frac{k}{3}v_b)Q + \frac{1}{k}(\dot{v}_b - kA - \dot{B})Q_{|\alpha}R^\alpha \right. \\ &\quad \left. - (kv_b - \dot{H}_T)\left(\frac{1}{k^2}Q_{|\alpha\beta} + \frac{1}{3}{}^3g_{\alpha\beta}Q\right)R^\alpha R^\beta \right] ds. \end{aligned} \quad (3.115)$$

From the relation

$$g_{\alpha\beta}R^\alpha R^\beta = 1, \quad (3.116)$$

we find that the previous equation becomes

$$\begin{aligned} \int_E^R [\dots] ds &= \\ &= \int_E^R \left[\underbrace{\left(\dot{H}_L + \frac{k}{3}v_b - \frac{k}{3}v_b + \frac{1}{3}\dot{H}_T \right) Q}_{[1]} + \underbrace{\frac{1}{k}(\dot{v}_b - kA - \dot{B})Q_{|\alpha}R^\alpha}_{[2]} \right. \\ &\quad \left. - \underbrace{\frac{1}{k^2}(kv_b - \dot{H}_T)Q_{|\alpha\beta}R^\alpha R^\beta}_{[3]} \right] ds. \end{aligned} \quad (3.117)$$

$$= ([1] + [2] + [3]). \quad (3.118)$$

These terms are now analysed separately.

$$\begin{aligned} [1] &= \int_E^R \left(\dot{H}_L + \frac{1}{3}\dot{H}_T \right) Q ds \\ &= \underbrace{\left[Q \left(H_L + \frac{1}{3}H_T \right) \right]_E^R}_{[1]_1} + \underbrace{\int_E^R \left(H_L + \frac{1}{3}H_T \right) Q_{|\alpha}R^\alpha ds}_{[1]_2}. \end{aligned} \quad (3.119)$$

A substitution from the gauge invariant variables, Φ_H , ϵ_b and v_b ,

$$H_L + \frac{1}{3}H_T = \Phi_H - \frac{1}{k} \frac{\dot{S}}{S} B + \frac{1}{k^2} \frac{\dot{S}}{S} \dot{H}_T \quad (3.120)$$

$$\epsilon_b = \delta_b + \frac{3}{k} \frac{\dot{S}}{S} (v_b - B) \quad (3.121)$$

and

$$v_b = v_{sb} + \frac{1}{k} \dot{H}_T, \quad (3.122)$$

gives:

$$\epsilon_b = \delta_b + \frac{3}{k} \frac{\dot{S}}{S} (v_{sb} - B + \frac{\dot{H}_T}{k}), \quad (3.123)$$

and

$$H_L + \frac{1}{3}H_T = \Phi_H + \frac{1}{k} \frac{\dot{S}}{S} (v_b - B) - \frac{\dot{S}}{kS} v_{sb}, \quad (3.124)$$

which yields

$$[1]_1 = \left[Q \left(\Phi_H - \frac{\dot{S}}{kS} v_{sb} \right) \right]_E^R - \frac{Q}{k} \frac{\dot{S}}{S} (v_b - B)|_E + \frac{Q}{k} \frac{\dot{S}}{S} (v_b - B)|_R. \quad (3.125)$$

If we now make the addition

$$[2] + \int_E^R \underbrace{(H_L + \frac{1}{3}H_T) R^\alpha Q_{|\alpha}}_{[1]_2} ds = \int_E^R (H_L + \frac{1}{3}H_T + \frac{1}{k} \dot{v}_b - A - \frac{1}{k} \dot{B}) Q_{|\alpha} R^\alpha ds, \quad (3.126)$$

where we have

$$H_L + \frac{1}{3}H_T + \frac{1}{k} \dot{v}_b - A - \frac{1}{k} \dot{B} = \Phi_H - \Phi_A + \frac{1}{k} \dot{v}_{sb}, \quad (3.127)$$

we get

$$[2] + \int_E^R (H_L + \frac{1}{3}H_T) R^\alpha Q_{|\alpha} ds = \int_E^R (\Phi_H - \Phi_A + \frac{1}{k} \dot{v}_{sb}) R^\alpha Q_{|\alpha} ds. \quad (3.128)$$

The last term gives

$$\frac{1}{k} v_b - \frac{\dot{H}_T}{k^2} = \frac{1}{k} v_{sb}, \quad (3.129)$$

which implies that

$$[3] = \int_E^R \left(-\frac{1}{k} v_{Sb} Q_{|\alpha\beta} R^\alpha R^\beta \right) ds \quad (3.130)$$

and so we get the final form for the integral

$$\begin{aligned} \{[1] + [2] + [3]\} = & \\ & \left[\left(\Phi_H - \frac{1}{k} \frac{\dot{S}}{S} v_{Sb} \right) Q \right]_E^R + \frac{Q}{k} \frac{\dot{S}}{S} (v_b - B) \Big|_E^R \\ & + \int_E^R \left\{ \left[\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right] Q_{|\alpha} R^\alpha - \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right\} ds . \end{aligned} \quad (3.131)$$

Panek now continues with the case of the adiabatic perturbation, $\delta_\gamma = \frac{4}{3}\delta_b$, where the initial fluctuations of the temperature at the last scattering surface become $\frac{1}{3}\epsilon_b Q$ as we have seen in the derivation of equation 38. In the case of scales larger than the baryon Jeans mass we have $C_{Sb}^2 = 0$ since for $\lambda \geq (ct)_E$ the mass in a sphere of diameter $\sim \lambda$ is much more than the Jeans mass.

3.6 The derivation of Panek's equation 41

For an adiabatic perturbation, Panek's equation (41) takes the form

$$\left[\frac{\delta T}{T} \right]_R = \left(\frac{1}{3} \epsilon_b Q \right)_E + \int_E^R \left[\frac{1}{3} (\dot{\epsilon}_b + k v_{Sb}) Q + \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right] ds , \quad (3.132)$$

where the equations of motion for the baryonic perturbation in this particular scenario are

$$\dot{\epsilon}_b + 3 \left[\left[\frac{\dot{S}}{S} \right]^2 - \left[\frac{\dot{S}}{S} \right] \cdot \right] \frac{v_{Sb}}{k} + 3 \dot{\Phi}_H - 3 \frac{\dot{S}}{S} \Phi_A + k v_{Sb} = 0 , \quad (3.133)$$

and

$$\dot{v}_{Sb} + \frac{\dot{S}}{S} v_{Sb} = k \Phi_A . \quad (3.134)$$

The two previous equations are used to calculate Panek's equation (41). We first

define

$$\begin{aligned}
 I &= - \underbrace{\left[\left(\Phi_H - \frac{1}{k} \frac{\dot{S}}{S} v_{Sb} \right) Q \right]_E^R}_{(1)} - \underbrace{\int_E^R \left[\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right] Q_{|\alpha} R^\alpha ds}_{(2)} \\
 &+ \int_E^R \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta ds
 \end{aligned} \tag{3.135}$$

where

$$\begin{aligned}
 (1) &= - \left[\left(\Phi_H - \frac{1}{k} \frac{\dot{S}}{S} v_{Sb} \right) Q \right]_E^R \\
 &= - \int_E^R \left\{ \left(\dot{\Phi}_H - \frac{1}{k} \left[\frac{\dot{S}}{S} \right]' v_{Sb} - \frac{1}{k} \frac{\dot{S}}{S} \dot{v}_{Sb} \right) Q \right. \\
 &\quad \left. - \left(\Phi_H - \frac{1}{k} \frac{\dot{S}}{S} v_{Sb} \right) Q_{|\alpha} R^\alpha \right\} ds,
 \end{aligned} \tag{3.136}$$

and we have used

$$\frac{d}{ds} = \frac{\partial}{\partial \tau} - R^\alpha \frac{\partial}{\partial x^\alpha}. \tag{3.137}$$

The addition yields

$$\begin{aligned}
 (1) + (2) &= - \int_E^R \left(\dot{\Phi}_H - \frac{1}{k} \left[\frac{\dot{S}}{S} \right]' v_{Sb} - \frac{1}{k} \frac{\dot{S}}{S} \dot{v}_{Sb} \right) Q + \int_E^R \left(\Phi_H - \frac{1}{k} \frac{\dot{S}}{S} v_{Sb} \right) Q_{|\alpha} R^\alpha ds \\
 &\quad - \int_E^R \left(\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right) Q_{|\alpha} R^\alpha ds
 \end{aligned}$$

$$\begin{aligned}
&= \int_E^R \left[\underbrace{\left(-\dot{\Phi}_H + \frac{1}{k} \left[\frac{\dot{S}}{S} \right] v_{S_a} + \frac{1}{k} \frac{\dot{S}}{S} \dot{v}_{S_b} \right)}_{(B)} Q \right. \\
&\quad \left. + \underbrace{\left(-\frac{\dot{v}_{S_b}}{k} - \frac{1}{k} \frac{\dot{S}}{S} v_{S_a} + \Phi_A \right)}_{(A)} Q_{|\alpha} R^\alpha \right] ds, \tag{3.138}
\end{aligned}$$

where from the second equation of motion for the baryonic perturbation, equation (134), we have $\dot{v}_{S_b} + \frac{\dot{S}}{S} v_{S_a} - k\Phi_A = 0$ which implies that $(A) = 0$. For the other term we obtain

$$(B) = \int_E^R \left(-\dot{\Phi}_H + \frac{1}{k} \left[\frac{\dot{S}}{S} \right] v_{S_a} + \frac{1}{k} \frac{\dot{S}}{S} \dot{v}_{S_b} \right) Q ds, \tag{3.139}$$

where (134) has been used again, to find

$$\begin{aligned}
(B) &= \int_E^R \underbrace{\left(-\dot{\Phi}_H + \frac{1}{k} \left[\frac{\dot{S}}{S} \right] v_{S_a} + \frac{\dot{S}}{S} \Phi_A - \frac{1}{k} \left[\frac{\dot{S}}{S} \right]^2 v_{S_a} \right)}_{\frac{1}{3}(\dot{\epsilon}_b + kv_{S_a})} Q ds \\
&= \int_E^R \frac{1}{3} (\dot{\epsilon}_b + kv_{S_a}) Q ds. \tag{3.140}
\end{aligned}$$

This gives

$$I = \frac{1}{3} \int_E^R (\dot{\epsilon}_b + kv_{S_a}) Q ds + \int_E^R \frac{v_{S_a}}{k} Q_{|\alpha\beta} R^\alpha R^\beta ds, \tag{3.141}$$

and so we obtain

$$\left[\frac{\delta T}{T} \right]_R = \left(\frac{1}{3} \epsilon_b Q \right)_E + \int_E^R \left[\frac{1}{3} (\dot{\epsilon}_b + kv_{S_b}) Q ds + \frac{v_{S_b}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right] ds, \tag{3.142}$$

which is what was required.

3.7 The derivation of Panek's Equation 45

We now look at some specific models in which the equations of motion will be simpler. Here we make the assumption that the anisotropic stress and entropy perturbations in any fluid component in the Universe were effective only at the very early stages of evolution. They are omitted and the equations of motion for the N types of fluid become Panek's equations (44a), (44b) and (44c) which are derived from equations 2.62, 2.63 and 2.60 and listed respectively:

$$(E_{a0}\epsilon_a S^3) \cdot + 3S^3 \frac{\dot{S}}{S} (E_{a0} + P_{a0}) \Phi_H + \frac{3(E_{a0} + P_{a0})S^3}{k} \left\{ \left[\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \cdot \right] v_{Sa} + k \dot{\Phi}_H + \frac{1}{3} k^2 v_{Sa} \right\} = 0 \quad (3.143)$$

$$\dot{v}_{Sa} + \frac{\dot{S}}{S} v_{Sa} = -\Phi_H + \frac{k C_{Sa}^2}{1 + \omega_a} \epsilon_a \quad (3.144)$$

$$\Phi_H = \frac{S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \epsilon_a \quad (3.145)$$

We simplify again by considering models dominated by nonrelativistic components with $P_{a0} = 0$ since decoupling, in other words high density universes with $\Omega > 0.1$ [e.g., Cold Dark Matter (CDM) models], and radiation and relativistic neutrinos omitted in the equations of motion.

Panek's equation (44a) - equation (143) above - now becomes

$$\underbrace{(E_{a0}\epsilon_a S^3) \cdot}_{[1]} + \frac{3(E_{a0} + P_{a0})S^3}{k} \left\{ \underbrace{\left[\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \cdot \right]}_{[2]} v_{Sa} + \underbrace{k \dot{\Phi}_H}_{[3]} + \underbrace{\frac{1}{3} k^2 v_{Sa}}_{[4]} \right\} - \underbrace{3S^3 \frac{\dot{S}}{S} (E_{a0} + P_{a0}) \Phi_A}_{[5]} = 0 \quad (3.146)$$

Now since $P_{a0} = 0$ we find from the energy equation that $E_{a0}S^3$ is constant,

$$\frac{\dot{E}_{a0}}{E_{a0} + P_{a0}} = -3\frac{\dot{S}}{S}, \quad (3.147)$$

as with dust. We now look at the terms in equation (146) in combinations.

$$[1] = (E_{a0}S^3 \epsilon_a) \dot{=} \dot{\epsilon}_a E_{a0}S^3. \quad (3.148)$$

The addition of the third and fifth terms gives

$$\begin{aligned} [3] + [5] &= 3E_{a0}S^3 \dot{\Phi}_H + 3S^2 \dot{S} E_{a0} \Phi_H \\ &= \underbrace{3E_{a0}S^3 \frac{2\dot{S}S}{2(k^2 - 3K)} \sum_{c=1}^N E_{c0} \epsilon_c}_{(1)} + \underbrace{3E_{a0}S^3 \frac{S^2 (\sum_{c=1}^N \dot{E}_{c0} \epsilon_c)}{2(k^2 - 3K)}}_{(2)} \\ &\quad + \underbrace{3E_{a0}S^3 \frac{S^2}{2(k^2 - 3K)} \left(\sum_{c=1}^N E_{c0} \dot{\epsilon}_c \right)}_{(3)} \\ &\quad + \underbrace{3S^2 \dot{S} E_{a0} \frac{S^2}{2(k^2 - 3K)} \sum_{c=1}^N E_{c0} \epsilon_c}_{(4)}. \end{aligned} \quad (3.149)$$

Panek defines the constants $E_{a0}S^3 = V_a$, and we find that

$$(3) = E_{a0}S^3 \left(\frac{3}{2(k^2 - 3K)S} \sum_{\substack{c=1 \\ c \neq a}}^N V_c \dot{\epsilon}_c + \frac{3}{2(k^2 - 3K)S} V_a \dot{\epsilon}_a \right). \quad (3.150)$$

The other three terms are going to cancel as is shown below:

$$(1) + (4) = 3E_{a0}S^3 \left(\frac{3\dot{S}S}{2(k^2 - 3K)} \sum_{c=1}^N E_{c0} \epsilon_c \right) \quad (3.151)$$

$$(2) = 3E_{a0}S^3 \left(\frac{S^2}{2(k^2 - 3K)} \sum_{c=1}^N \dot{E}_{c0} \epsilon_c \right). \quad (3.152)$$

From the energy equation

$$\frac{\dot{E}_{a0}}{E_{a0} + P_{a0}} = -3\frac{\dot{S}}{S}, \quad (3.153)$$

with $P_{a0} = 0$, we find

$$\dot{E}_{a0} S^3 + 3 \dot{S} S^2 E_{a0} = 0, \quad (3.154)$$

and so

$$\dot{E}_{a0} S^2 + 3 \dot{S} S E_{a0} = 0, \quad (3.155)$$

which gives

$$(1) + (2) + (4) = 0. \quad (3.156)$$

Hence

$$[3] + [5] = E_{a0} S^3 \left(\frac{3}{2(k^2 - 3K)S} V_a \dot{\epsilon}_a + \frac{3}{2(k^2 - 3K)S} \sum_{\substack{c=1 \\ c \neq a}}^N V_c \dot{\epsilon}_c \right). \quad (3.157)$$

The second and fourth terms become

$$\begin{aligned} [2] + [4] &= \frac{3E_{a0}S^3}{k} \left\{ \left(\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \right) v_{Sa} + \frac{1}{3} k^2 v_{Sa} \right\} \\ &= \frac{E_{a0}S^3 k^2}{k} v_{Sa} \left\{ 1 + \frac{3}{k^2} \left[\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \right] \right\}, \end{aligned} \quad (3.158)$$

and from the substitutions

$$\left(\frac{\dot{S}}{S} \right)^2 = \frac{1}{3} S^2 \sum_{a=1}^N E_{a0} - K, \quad (3.159)$$

and

$$\left(\frac{\dot{S}}{S} \right) = -\frac{1}{6} S^2 \sum_{a=1}^N E_{a0}, \quad (3.160)$$

we find

$$[2] + [4] = E_{a0} S^3 k v_{Sa} \left(1 - \frac{3K}{k^2} + \frac{3}{2k^2 S} \sum_{c=1}^N V_c \right). \quad (3.161)$$

The final result is

$$\begin{aligned}
& [1] + [2] + [3] + [4] + [5] = \\
& \dot{\epsilon}_a \left[1 + \frac{3V_a}{2(k^2 - 3K)S} \right] + kv_{S_a} \left[1 - \frac{3K}{k^2} + \frac{3}{2k^2S} \sum_{c=1}^N V_c \right] \\
& + \frac{3}{2(k^2 - 3K)S} \sum_{\substack{c=1 \\ c \neq a}}^N V_c \dot{\epsilon}_c = 0, \tag{3.162}
\end{aligned}$$

which is Panek's equation (45a).

To find Panek's 45(b) we begin with his 44(b)

$$\dot{v}_{S_a} + \frac{\dot{S}}{S} v_{S_a} = -k\Phi_H + \frac{kC_{S_a}^2}{1 + \omega_a} \epsilon_a, \tag{3.163}$$

where

$$\omega_a = \frac{P_{a0}}{E_{a0}} = 0 \tag{3.164}$$

and

$$C_{S_a} = \frac{dP_{a0}}{dE_{a0}} = 0. \tag{3.165}$$

This yields

$$\dot{v}_{S_a} + \frac{\dot{S}}{S} v_{S_a} = -k\Phi_H = -k \frac{S^2}{2(k^2 - 3K)} \sum_{c=1}^N E_{c0} \epsilon_c, \tag{3.166}$$

with the use of the Einstein equation

$$\frac{2(k^2 - 3K)}{S^2} \Phi_H = \sum_{a=1}^N E_{a0} \epsilon_a. \tag{3.167}$$

So, finally, we have

$$\dot{v}_{S_a} + \frac{\dot{S}}{S} v_{S_a} = -\frac{1}{2k(1 - \frac{3K}{k^2})S} \sum_{c=1}^N V_c \epsilon_c, \tag{45b} \tag{3.168}$$

which is Panek's equation (45b).

3.8 The derivation of Panek's equation 52

For the derivation of his equation (52), Panek makes many simplifications and introduces several new concepts. The first is that of the present density parameters Ω_{Ra} , which are closely related to the constants V_a

$$\Omega_{Ra} = \frac{E_{a0R}}{3H_R^2} = \frac{V_a}{\sum_{c=1}^N V_c - 3KS_R}. \quad (3.169)$$

In the case of $K = 0$, suggested by many inflationary scenarios, $\Omega_a = \text{constant} = \Omega_{Ra}$ and the sum of all the V_c 's is denoted by V :

$$\Omega_{Ra} \sum_{c=1}^N V_c = V_a \quad (3.170)$$

which implies that

$$V_a = \Omega_a V. \quad (3.171)$$

Now Panek's 45(a) (equation 162) becomes

$$\dot{\epsilon}_a \left[1 + \frac{3\Omega_a V}{2k^2 S} \right] + kv_{Sa} \left[1 + \frac{3V}{2k^2 S} \right] + \frac{3V}{2k^2 S} \sum_{\substack{c=1 \\ c \neq a}}^N \Omega_c \dot{\epsilon}_c = 0, \quad (47a) \quad (3.172)$$

which is Panek's 47(a), and his 45(b) becomes

$$\dot{v}_{Sa} + \frac{\dot{S}}{S} v_{Sa} = -\frac{V}{2kS} \sum_{c=1}^N \Omega_c \epsilon_c. \quad (3.173)$$

Greatly simplified equations of motion are obtained from (45a) and (45b) if $N = 1$ or, for $c = 1, \dots, N$ all the $\dot{\epsilon}_c$'s are equal, and so for baryons we have

$$\dot{\epsilon}_b + \left[1 - \frac{3K}{k^2} \right] kv_{Sb} = 0, \quad (48a) \quad (3.174)$$

which is Panek's equation (48a). This is the case in most of the dark matter scenarios - on scales larger than the Jeans mass, the baryons sink in the potential wells of the dark component and in a few expansion times all the ϵ_c 's become equal. The second equation of motion now yields Panek's equation (48b)

$$\dot{v}_{Sb} + \frac{\dot{S}}{S} v_{Sb} = -\frac{V}{2k[1 - 3K/k^2]S} \epsilon_b, \quad (48b) \quad (3.175)$$

and we continue by writing equation (41) under the assumption of equal ϵ_c 's.

Recall equation 41

$$\frac{\delta T}{T} \Big|_R = \frac{1}{3}(\epsilon_b Q)_E + \int_E^R \left[\frac{1}{3}(\dot{\epsilon}_b + k v_{Sb})Q + \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right] ds . \quad (3.176)$$

Now substitute the form for v_{Sb} from 48(a) (equation 174)

$$\frac{1}{3}(k v_{Sb} + \dot{\epsilon}_b) = \frac{\dot{\epsilon}_b}{3} \left(-\frac{1}{1 - \frac{3K}{k^2}} + 1 \right) = -\frac{\dot{\epsilon}_b K/k^2}{1 - \frac{3K}{k^2}} = -\frac{\dot{\epsilon}_b K}{k^2 - 3K} , \quad (3.177)$$

and find

$$\frac{v_{Sb}}{k} = -\frac{\dot{\epsilon}_b}{\left(1 - \frac{3K}{k^2}\right) k^2} = -\frac{\dot{\epsilon}_b}{k^2 - 3K} , \quad (3.178)$$

which gives Panek's equation 49

$$\frac{\delta T}{T} \Big|_R = \frac{1}{3}(\epsilon_b Q)_E - \frac{1}{k^2 - 3K} \int_E^R \dot{\epsilon}_b (KQ + Q_{|\alpha\beta} R^\alpha R^\beta) ds . \quad (3.179)$$

To rederive SW's result we now take $K = 0$, and equation 49 becomes

$$\frac{\delta T}{T} \Big|_R = \frac{1}{3}(\epsilon_b Q)_E - \frac{1}{k^2} \int_E^R \dot{\epsilon}_b (Q_{|\alpha\beta} R^\alpha R^\beta) ds . \quad (3.180)$$

When $K = 0$, Q can be considered to be a plane wave with wave vector k^α :

$$Q = \exp(ik_\alpha x^\alpha) , \quad (3.181)$$

and we use the relation

$$Q_{|\alpha\beta} = Q_{,\alpha\beta} - {}^3\Gamma_{\alpha\beta}^\gamma Q_{,\gamma} , \quad (3.182)$$

to find

$$\begin{aligned} \frac{\delta T}{T} \Big|_R &= \frac{1}{3}(\epsilon_b Q)_E + \frac{1}{k^2} \int_E^R \dot{\epsilon}_b (k_\alpha R^\alpha)^2 Q ds \\ &\quad - \frac{1}{k^2} \int_E^R \dot{\epsilon}_b (-{}^3\Gamma_{\alpha\beta}^\gamma Q_{,\gamma} R^\alpha R^\beta) ds . \end{aligned} \quad (3.183)$$

Which is different from Panek's equation 50 but it appears to be a printing error as his following equations are obviously derived using this form.

We can now substitute a form for the density perturbation which, for the case of flat spacetime, has the solution

$$\epsilon_b = A\tau^2 + B\tau^{-3}, \quad (3.184)$$

where Panek now chooses the growing mode $\epsilon_b = \epsilon_{bE}(\tau/\tau_E)^2$ for the rest of the analysis, and the integral in (180) can be integrated by parts. From Panek's equation (49) (equation 179) we have

$$-\frac{1}{k^2} \int_E^R \dot{\epsilon}_b (Q_{|\alpha\beta} R^\alpha R^\beta) ds = -\frac{2\epsilon_{bE}}{k^2 \tau_E^2} \int_E^R \tau \left(-\frac{d}{ds} (R^\alpha Q_{,\alpha}) \right) ds, \quad (3.185)$$

since

$$R^\alpha R^\beta Q_{|\alpha\beta} = R^\alpha R^\beta Q_{,\alpha\beta} - \frac{dR^\gamma}{ds} Q_{,\gamma}, \quad (3.186)$$

where we have made use of the relation

$${}^3\Gamma_{\alpha\beta}^\gamma R^\alpha R^\beta = \frac{dR^\gamma}{ds}, \quad (3.187)$$

which implies

$$R^\alpha R^\beta Q_{|\alpha\beta} = R^\alpha R^\beta Q_{,\alpha\beta} - \frac{d(R^\gamma Q_{,\gamma})}{ds} + R^\gamma Q_{,\gamma\alpha} \frac{dx^\alpha}{ds}, \quad (3.188)$$

and since

$$\frac{dx^\alpha}{ds} = -R^\alpha, \quad (3.189)$$

we have

$$R^\alpha R^\beta Q_{|\alpha\beta} = -\frac{d(R^\gamma Q_{,\gamma})}{ds}. \quad (3.190)$$

This gives the final form of the integral,

$$\begin{aligned} & -\frac{1}{k^2} \int_E^R \dot{\epsilon}_b (Q_{|\alpha\beta} R^\alpha R^\beta) ds \\ &= \frac{2\epsilon_{bE}}{k^2 \tau_E^2} \int_E^R \tau \frac{d}{ds} (R^\gamma Q_{,\gamma}) ds \end{aligned} \quad (3.191)$$

$$= \frac{2\epsilon_{bE}}{k^2 \tau_E^2} \left[(\tau R^\gamma Q_{,\gamma}) \Big|_E^R - \int_E^R R^\gamma Q_{,\gamma} \underbrace{\frac{d\tau}{ds}}_1 ds \right] \quad (3.192)$$

$$= \frac{2\epsilon_{bE}}{k^2\tau_E^2} \left[(\tau R^\gamma Q_{,\gamma})|_E^R + \int_E^R \underbrace{Q_{,\gamma} \frac{dx^\gamma}{ds}}_{dQ} ds \right] \quad (3.193)$$

$$= \frac{2\epsilon_{bE}}{k^2\tau_E^2} [\tau_R(R^\gamma Q_{,\gamma})_R - \tau_E(R^\gamma Q_{,\gamma})_E + Q_{(R)} - Q_{(E)}] , \quad (3.194)$$

and we write

$$\frac{\delta T}{T} \Big|_R = \frac{1}{3} \epsilon_b Q|_E + \frac{2\epsilon_{bE}}{k^2\tau_E^2} [\tau_R(R^\gamma Q_{,\gamma})_R - \tau_E(R^\gamma Q_{,\gamma})_E + Q_{(R)} - Q_{(E)}] \quad (3.195)$$

which is Panek's equation 52. Panek now argues that the comoving coordinate distance to the horizon at τ_E is $k\tau_E$ and so the criterion for the comoving scale to be larger than the horizon at τ_E is $k\tau_E \ll 1$ implying that the first term is much smaller than the integral term which gives the relation first given by Sachs and Wolfe in their pioneering paper i.e.

$$\frac{\delta T}{T} \Big|_R = \frac{2\epsilon_{bE}}{k^2\tau_E^2} [\tau_R(R^\gamma Q_{,\gamma})_R - \tau_E(R^\gamma Q_{,\gamma})_E + Q_{(R)} - Q_{(E)}] , \quad (3.196)$$

where the term $\sim Q_R$ has been discarded since it is independent of the direction of observation.

The SW form is written as

$$\frac{\delta T_R}{T_R} = \frac{1}{10} [\eta_R(B_{,\mu}e^\mu)_R - \eta_E(B_{,\mu}e^\mu)_E + B_R - B_E] \quad (3.197)$$

where η replaces τ , e^μ replaces R^α , B takes the place of Q and SW say that their interpretations are only valid when the redshifts are considered due to perturbations of the relatively increasing kind. We will show this in greater detail in the second part which deals with the SW paper more closely.

Chapter 4

The non-adiabatic perturbation

4.1 Introduction

For a non-adiabatic perturbation we define

$$\chi(\tau) = \frac{\delta s}{s} = \frac{1}{3}\delta_b - \frac{1}{4}\delta_\gamma. \quad (4.1)$$

As has been stated before, a single ideal fluid with barotropic equation of state has no entropy perturbation, but in the universe where there are at least two fluid components, e.g. one baryons and the other radiation, the entropy perturbations may be important. The non-adiabatic perturbation discussed here is not general but dependant only on time and not on space; when we speak of the non-adiabatic case we are referring to this. We now derive the above equation. The entropy per baryon is proportional to T^3/n_b , where n_b is the number density of the baryons, T is the temperature, and $E_\gamma \propto T^4$. The entropy perturbation is

$$\begin{aligned} \frac{\delta s}{s} &= \frac{(3T^2/n_b)\delta T - (T^3/n_b^2)\delta n_b}{T^3/n_b} \\ &= \frac{3\delta T}{T} - \frac{\delta n_b}{n_b}. \end{aligned} \quad (4.2)$$

Substitute $E_\gamma \propto T^4$ and $E_b \propto n_b$, to obtain

$$\frac{\delta s}{s} = \frac{3}{4} \frac{\delta E_\gamma}{E_\gamma} - \frac{E_b}{E_b}. \quad (4.3)$$

which is generally the case for the times we are interested in. Entropy perturbations are found where the different matter components are distributed nonuniformly in space but with a uniform total energy density. For instance, the inhomogeneous distribution of baryons in a background of radiation, where initially the energy density surplus in baryons is made up by a corresponding deficit in the radiation energy. These fluctuations are often called isocurvature perturbations and because of causality constraints, the formation of adiabatic perturbations on scales larger than the Hubble radius is impossible. Before continuing any further we show that the definition of χ is gauge invariant. From Panek's equation (11c) we have

$$\epsilon_b = \delta_b + \frac{3}{k} \frac{\dot{S}}{S} (v_b - B) \quad (4.4)$$

$$\epsilon_\gamma = \delta_\gamma + \frac{4}{k} \frac{\dot{S}}{S} (v_\gamma - B), \quad (4.5)$$

and therefore

$$\begin{aligned} \chi &= \frac{1}{3} \delta_b - \frac{1}{4} \delta_\gamma \\ &= \frac{1}{3} \epsilon_b - \frac{1}{4} \epsilon_\gamma - \frac{1}{k} \frac{\dot{S}}{S} (v_b - v_\gamma) \\ &= \left(\frac{1}{3} \epsilon_b - \frac{1}{4} \epsilon_\gamma \right) - \frac{1}{k} \frac{\dot{S}}{S} (v_{Sb} - v_{S\gamma}). \end{aligned} \quad (4.6)$$

And since ϵ_a and v_{Sa} are gauge invariant, χ is as well.

4.2 The last scattering surface

As for the derivation of equation 34 of Panek we discuss the temperature and baryon density on the last scattering surface. From Panek's equation (34) we have

$$\begin{aligned} \frac{\dot{S}}{S} \Delta\tau &= \frac{1}{3+D} \delta_b Q + \frac{D}{4(3+D)} \delta_\gamma Q \\ &= \frac{\delta_b Q}{3} - \frac{\chi D Q}{3+D} \Big|_E. \end{aligned} \quad (4.7)$$

On the perturbed last scattering surface the perturbed temperature T_S and baryon density E_b are not constant and we now discuss this in detail.

From the last chapter we have

$$T_S = T_{S0}(\tau_{E0}) \left(1 + \frac{\dot{T}}{T} \Big|_{\tau_{E0}} \Delta\tau + \delta_T Q \right), \quad (4.8)$$

and since

$$\delta_T = \frac{1}{4} \delta_\gamma \quad \text{and} \quad \frac{\dot{T}}{T} \Big|_{\tau_{E0}} = -\frac{\dot{S}}{S} \Big|_{\tau_{E0}}, \quad (4.9)$$

we get

$$\begin{aligned} T_S &= T_{S0}(\tau_{E0}) \left(1 - \frac{\dot{S}}{S} \Delta\tau + \frac{1}{4} \delta_\gamma Q \right) \\ &= T_{S0}(\tau_{E0}) \left(1 - \frac{1}{3} \delta_b Q + \frac{\chi D Q}{3 + D} + \frac{1}{4} \delta_\gamma Q \right) \\ &= T_{S0}(\tau_{E0}) \left(1 + \frac{\chi D Q}{3 + D} - \chi Q \right), \end{aligned} \quad (4.10)$$

which leaves us with the final form

$$T_S = T_{S0}(\tau_{E0}) \left(1 - \frac{3\chi Q}{3 + D} \right) \Big|_E, \quad (4.11)$$

and similarly

$$E_b = E_{b0}(\tau_{E0}) \left(1 + \frac{3\chi D Q}{3 + D} \right) \Big|_E, \quad (4.12)$$

which was obtained from

$$E_b = E_{b0}(\tau_{E0} + \Delta\tau) (1 + \delta_b Q) = E_{b0}(\tau_{E0}) \left(1 + \frac{\dot{E}_b}{E_b} \Delta\tau + \delta_b Q \right), \quad (4.13)$$

where $\dot{E}_b / E_b = -3 \dot{S} / S$.

So we have expressions explicitly dependant on the non-adiabatic perturbation variable χ for the perturbations of the temperature and baryon density at the last scattering surface.

4.3 Panek's equations extended

The temperature variation, $\frac{\delta T_R}{T}$, in the non-adiabatic case will now be calculated.

The general formula is

$$\begin{aligned} \frac{\delta T}{T} \Big|_R &= - \frac{Q \dot{S}}{k S} (v_b - B) \Big|_R \\ &+ \frac{1}{4} \epsilon_\gamma Q \Big|_E - \frac{1}{k S} \dot{S} (v_{S\gamma} - v_{Sb}) Q \Big|_E - \left[\left(\Phi_H - \frac{1}{k S} \dot{S} v_{Sb} \right) Q \right]_E^R \\ &- \int_E^R \left\{ \left[\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right] Q_{|\alpha} R^\alpha - \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right\} ds . \end{aligned} \quad (4.14)$$

Substitute equation (6) evaluated at E ,

$$\frac{1}{4} \epsilon_\gamma Q \Big|_E - \frac{1}{k S} \dot{S} (v_{S\gamma} - v_{Sb}) Q \Big|_E = \frac{1}{3} \epsilon_b Q \Big|_E - \chi Q \Big|_E , \quad (4.15)$$

into the above to obtain

$$\begin{aligned} \frac{\delta T}{T} \Big|_R &= \frac{1}{3} \epsilon_b Q \Big|_E - \chi Q \Big|_E - \left[\left(\Phi_H - \frac{1}{k S} \dot{S} v_{Sb} \right) Q \right]_E^R \\ &- \int_E^R \left\{ \left[\frac{\dot{v}_{Sb}}{k} + \Phi_H - \Phi_A \right] Q_{|\alpha} R^\alpha - \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right\} ds , \end{aligned} \quad (4.16)$$

where terms evaluated at R in equation (14) are independent of the direction of observation and have been attributed to T_{R0} . The difference between equation (16) and Panek's equation (38) is, as was seen earlier,

$$\frac{3}{3+D} \left[\frac{\epsilon_b}{3} - \frac{\epsilon_\gamma}{4} + \frac{1}{k S} \dot{S} (v_{S\gamma} - v_{Sb}) \right] = \frac{3}{3+D} \left(\frac{1}{3} \delta_b - \frac{1}{4} \delta_\gamma \right) , \quad (4.17)$$

and so in the case of adiabatic perturbations equation (17) is zero and formula (16) is the same as Panek's equation (38) in the adiabatic case i.e.

$$\delta_\gamma = \frac{4}{3} \delta_b . \quad (4.18)$$

We have also seen in the introduction to this chapter that χ is gauge invariant and this shows that the new version of $\frac{\delta T}{T} \Big|_R$ is gauge invariant.

In the extension to Panek's equation (41) we have the same analysis as before and find that

$$\begin{aligned} \frac{\delta T}{T} \Big|_R &= \frac{1}{3} \epsilon_b Q \Big|_E - \chi Q \Big|_E \\ &+ \int_E^R \left[\frac{1}{3} (\dot{\epsilon}_b + k v_{Sb}) Q + \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right] ds, \end{aligned} \quad (4.19)$$

our new equation for Panek's equation (41).

We now continue as before and find

$$\frac{\delta T}{T} \Big|_R = \frac{1}{3} \epsilon_b Q \Big|_E - \chi Q \Big|_E - \frac{1}{k^2 - 3K} \int_E^R \dot{\epsilon}_b (KQ + Q_{|\alpha\beta} R^\alpha R^\beta) ds \quad (4.20)$$

which is the modified equation (49) of Panek.

For the case of $K = 0$,

$$Q = \exp(ik_\alpha x^\alpha), \quad (4.21)$$

and taking the growing mode again for the density perturbation:

$$\epsilon_b = \epsilon_{bE} \left(\frac{\tau}{\tau_E} \right)^2, \quad (4.22)$$

we find

$$\begin{aligned} \frac{\delta T}{T} \Big|_R &= \\ &\frac{1}{3} \epsilon_b Q \Big|_E - \chi Q \Big|_E + \frac{2\epsilon_{bE}}{k^2 \tau_E^2} [\tau_R (R^\alpha Q_{,\alpha})_R - \tau_E (R^\alpha Q_{,\alpha})_E + Q_R - Q_E], \end{aligned} \quad (4.23)$$

which is the nonadiabatic version of Panek's equation (52). Sachs and Wolfe are concerned about their assumption of intrinsic uniformity of the temperature emitted from the last scattering surface and say that any variation in its temperature might easily dominate the effects they analyse. The last equation shows this effect in terms of a non-adiabatic perturbation to the energy density as well as a first order correction to the LSS.

4.4 A new gauge invariant quantity

We now discuss the difference the difference in the observed temperatures between two different directions, but first see that the temperature at reception can be ex-

pressed as follows:

$$T_R = T_{R0} \left(\frac{\delta T}{T} \Big|_R + 1 \right), \quad (4.24)$$

and so we define the difference between temperatures measured in the two directions A and B as

$$\begin{aligned} \Delta_{AB} T_R &= T_{RA} - T_{RB} = T_{R0} \left(\frac{\delta T}{T} \Big|_{RA} - \frac{\delta T}{T} \Big|_{RB} \right) \\ &= T_{R0} \left(\frac{1}{3} \epsilon_b Q|_{EA} - \chi Q|_{EA} + \int_{EA}^R [\dots] ds \right) \\ &\quad - T_{R0} \left(\frac{1}{3} \epsilon_b Q|_{EB} - \chi Q|_{EB} + \int_{EB}^R [\dots] ds \right). \end{aligned} \quad (4.25)$$

We can also express this in Panek's formalism where his quantities have a superscript P. From earlier work we have

$$\frac{\delta T}{T} \Big|_R^{(P)} - \frac{\delta T}{T} \Big|_R = \frac{3}{3+D} \chi \Big|_E, \quad (4.26)$$

and so we write

$$\begin{aligned} \Delta_{AB} T_R &= T_{R0} \left(\frac{\delta T}{T} \Big|_{RA} - \frac{\delta T}{T} \Big|_{RB} \right) \\ &= T_{R0} \left(\frac{\delta T}{T} \Big|_{RA}^P - \frac{\delta T}{T} \Big|_{RB}^P - \frac{3}{3+D} (\chi_{EA} - \chi_{EB}) \right). \end{aligned} \quad (4.27)$$

The gauge invariant quantity $\Delta_{AB} T_R$ is measured in observations, and to determine it one must be careful in the choice of T_{R0} which gives the form one uses for $\frac{\delta T}{T} \Big|_R$.

4.5 Non-adiabatic perturbations for a mixture of radiation and dust

We now discuss a non-adiabatic perturbation,

$$\frac{1}{3}\delta_b - \frac{1}{4}\delta_\gamma = \chi(\tau), \quad (4.30)$$

with $K = 1, 0, \text{ or } -1$, and a mixture of two fluids: one dust and the other radiation with $E_\gamma = \frac{1}{3}P_\gamma$. At late times, when the temperature T is low compared with the baryon mass, the pressure of baryons is negligible and the total pressure of the fluid is given by the radiation. From our modified formula of Panek's (41), we have

$$\begin{aligned} \left. \frac{\delta T}{T} \right|_R &= \frac{1}{3}(\epsilon_b Q)_E - (\chi Q)|_E \\ &+ \int_E^R \left[\frac{1}{3}(\dot{\epsilon}_b + k v_{Sb})Q + \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right] ds. \end{aligned} \quad (4.31)$$

We begin with some basic preparation.

For the Baryon fluid the energy equation with $P_{b0} = 0$ (which is always valid after recombination) is

$$\frac{\dot{E}_{b0}}{E_{b0}} = -3 \frac{\dot{S}}{S}, \quad (4.32)$$

which gives the constant

$$S^3 E_{b0} = V_b, \quad (4.33)$$

and the entropy perturbation

$$\eta_b = \pi_{Lb} - \frac{C_{Sb}^2}{\omega_b} \delta_b, \quad (4.34)$$

is taken to be zero, as

$$C_{Sb}^2 = \frac{dP_{b0}}{dE_{b0}} = 0, \quad (4.35)$$

for scales larger than Jeans mass - $P_b = 0$, and

$$\pi_{Lb} = 0, \quad (4.36)$$

i.e. the isotropic pressure perturbation is zero for the same reason as above - $P_b = 0$. So $\eta_b = 0$.

For the Photon fluid we have

$$P_{\gamma 0} = \frac{1}{3}E_{\gamma 0}, \quad (4.37)$$

which gives the constant

$$S^4 E_{\gamma 0} = V_{\gamma}. \quad (4.38)$$

The ratio of these two constants for the baryons and radiation is given by

$$C_{\rho} = V_b/V_{\gamma}, \quad (4.39)$$

which gives a relation between the unperturbed energy densities of the two fluids:

$$E_{b0} = C_{\rho} S E_{\gamma 0} = 3C_{\rho} S P_{\gamma 0}. \quad (4.40)$$

For the radiation the entropy perturbation is

$$\eta_{\gamma} = \pi_{L\gamma} - \frac{C_{S\gamma}^2}{\omega_{\gamma}} \delta_{\gamma}, \quad (4.41)$$

where

$$C_{S\gamma}^2 = w_{\gamma} = \frac{1}{3}, \quad (4.42)$$

and so we obtain

$$\eta_{\gamma} = \pi_{L\gamma} - \delta_{\gamma}. \quad (4.43)$$

In the previous section we wrote the entropy perturbation as

$$\frac{\delta s}{s} = \frac{3}{4} \frac{\delta E_{\gamma}}{E_{\gamma}} - \frac{\delta E_b}{E_b}, \quad (4.44)$$

and for our present case the total energy density is

$$\delta E = \delta E_{\gamma} + \delta E_b. \quad (4.45)$$

These two relations can be used to express δE_b in terms of δE and δE_{γ} . The total pressure is written

$$P = P_{\gamma} = \frac{1}{3}E_{\gamma} \quad (4.46)$$

and so,

$$\delta P = \frac{1}{3}\delta E_{\gamma}. \quad (4.47)$$

To begin with we have

$$\begin{aligned} \frac{\delta s}{s} &= \frac{3}{4} \frac{\delta E_{\gamma}}{E_{\gamma}} - \frac{\delta E - \delta E_{\gamma}}{E_b} \\ &= \frac{(\frac{3}{4}E_b + E_{\gamma})\delta E_{\gamma}}{E_b E_{\gamma}} - \frac{\delta E}{E_b}, \end{aligned} \quad (4.48)$$

which gives an expression for δE_γ

$$\delta E_\gamma = \left(\frac{\delta s}{s} + \frac{\delta E}{E_b} \right) \frac{E_b E_\gamma}{\frac{3}{4} E_b + E_\gamma}. \quad (4.49)$$

Substitute this into the pressure perturbation to obtain

$$\delta P = \frac{1}{3} \frac{E_\gamma}{(\frac{3}{4} E_b + E_\gamma)} \delta E + \frac{1}{3} \frac{E_\gamma}{(\frac{3}{4} E_b + E_\gamma)} E_b \frac{\delta s}{s}. \quad (4.50)$$

In chapter 2 we wrote an expression for the pressure perturbation when it is a function of the energy density and entropy.

$$\delta P = \left. \frac{\partial P}{\partial E} \right|_s \delta E + \left. \frac{\partial P}{\partial s} \right|_E \delta s \equiv C_s^2 \delta E + \eta \quad (4.51)$$

From the two previous equations we see that

$$C_s^2 = \frac{1}{3} \frac{1}{(\frac{3}{4} E_b / E_\gamma + 1)}, \quad (4.52)$$

and so, for the early universe, this model for the mixture of dust and radiation describes a smooth transition from the period of radiation domination ($E_\gamma \gg E_b$) with $C_s^2 = \frac{1}{3}$ to the matter dominated epoch ($E_b \gg E_\gamma$) with $C_s^2 = 0$.

From the Einstein equation

$$\frac{-k^2}{S^2} (\Phi_A + \Phi_H) = \sum_{a=1}^2 P_{a0} \pi_{T_a} \quad (4.53)$$

assuming that the anisotropic perturbation $\pi_{T_a} = 0$ (as we are dealing with late times) we obtain $\Phi_A = -\Phi_H$. So from the equation of motion it is found that

$$\Phi_H = -\Phi_A = \frac{S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \epsilon_a, \quad (4.54)$$

and the propagation equation for the velocity

$$\dot{v}_{S_a} + \frac{\dot{S}}{S} v_{S_a} = -k \Phi_H + \frac{k}{1 + \omega_a} (C_{S_a}^2 \epsilon_a + \omega_a \eta_a), \quad (4.55)$$

in the particular case of the baryons is

$$\dot{v}_{S_b} + \frac{\dot{S}}{S} v_{S_b} = -k \Phi_H, \quad (4.56)$$

and for the photons is

$$\dot{v}_{S\gamma} + \frac{\dot{S}}{S} v_{S\gamma} = -k\Phi_H + \frac{\dot{S}}{S}(v_\gamma - B) + \frac{k}{4}\pi L_\gamma. \quad (4.57)$$

The propagation of the energy density perturbation together with the above equations for the velocity will allow us to find an expression for the temperature variation of the CMBR. Its evolution is given in the general case and then evaluated separately for the particular cases of baryons and radiation. From equation (3.143) we have

$$\begin{aligned} (E_{a0}\epsilon_a S^3)' + \frac{3(E_{a0} + P_{a0})S^3}{k} \left\{ \left[\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right)' \right] v_{Sa} \right. \\ \left. + k \dot{\Phi}_H + \frac{1}{3}k^2 v_{Sa} \right\} - 3S^3 \frac{\dot{S}}{S} (E_{a0} + P_{a0}) \Phi_A = 0. \end{aligned} \quad (4.58)$$

This equation is now written for the case of baryons i.e. $a = b$

The first term is

$$(\epsilon_b E_{b0} S^3)' = V_b \dot{\epsilon}_b, \quad (4.59)$$

and adding the third and fifth terms gives

$$\begin{aligned} \frac{3(E_{b0} + P_{b0})S^3}{k} k \dot{\Phi}_H + 3S^3 \frac{\dot{S}}{S} (E_{b0} + P_{b0}) \Phi_H \\ = 3V_b \left\{ \frac{1}{2(k^2 - 3K)S} E_{\gamma 0} S^3 \dot{\epsilon}_\gamma + \frac{V_b \dot{\epsilon}_b}{2(k^2 - 3K)S} \right. \\ \left. + \frac{3S^2 \dot{S}}{2(k^2 - 3K)S} E_{\gamma 0} \epsilon_\gamma + \frac{S^3}{2(k^2 - 3K)S} \dot{E}_{\gamma 0} \epsilon_\gamma \right\}, \end{aligned} \quad (4.60)$$

where the last two terms for the baryons may be neglected as,

$$3S^2 \dot{S} E_{b0} + S^3 \dot{E}_{b0} = 0 \quad (\text{from } (S^3 E_{b0})' = 0). \quad (4.61)$$

Adding the second and fourth terms yields

$$\begin{aligned} \frac{3(E_{b0} + P_{b0})S^3}{k} \left\{ \left[\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right)' \right] v_{Sb} + \frac{1}{3}k^2 v_{Sb} \right\} \\ = \frac{3V_b}{k} \frac{1}{3}k^2 v_{Sb} \left[1 + \frac{3}{k^2} \left(\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right)' \right) \right]. \end{aligned} \quad (4.62)$$

We first determine $(\dot{S}/S)^2 - (\dot{S}/S) \cdot$ from the Friedmann equations.

$$\left(\frac{\dot{S}}{S}\right) \cdot = -\frac{1}{6}S^2 \sum_{a=1}^2 (E_{a0} + 3P_{a0}) \quad (4.63)$$

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{1}{3}S^2 \sum_{a=1}^2 E_{a0} - K \quad (4.64)$$

$$\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right) \cdot = \frac{1}{6}S^2 \sum_{a=1}^2 (3E_{a0} + 3P_{a0}) - K. \quad (4.65)$$

Substituting this into equation (62), we get

$$V_b k v_{sb} \left\{ 1 - \frac{3K}{k^2} + \frac{1}{2k^2 S} \sum_{a=1}^2 (3E_{a0} S^3 + 3P_{a0} S^3) \right\}. \quad (4.66)$$

The general equation for the evolution of the energy density (58), in the case of baryons, becomes

$$\begin{aligned} & V_b \left\{ \dot{\epsilon}_b \left(1 + \frac{3V_b}{2(k^2 - 3K)S} \right) \right. \\ & + k v_{sb} \left(1 - \frac{3K}{k^2} + \frac{\sum_{c=1}^2 (3E_{c0} S^3 + 3P_{c0} S^3)}{2k^2 S} \right) \\ & \left. + \frac{3}{2(k^2 - 3K)S} \left[E_{\gamma 0} S^3 \dot{\epsilon}_\gamma + (3S^2 \dot{S} E_{\gamma 0} + S^3 \dot{E}_{\gamma 0}) \epsilon_\gamma \right] \right\} = 0 \quad (4.67) \end{aligned}$$

The last term in the above expression can be reduced; we take

$$(3S^2 \dot{S} E_{\gamma 0} + S^3 \dot{E}_{\gamma 0}) \epsilon_\gamma = \left(\frac{V_\gamma}{S}\right) \cdot \dot{\epsilon}_\gamma, \quad (4.68)$$

and we find

$$\frac{3}{2(k^2 - 3K)S} \left[E_{\gamma 0} S^3 \dot{\epsilon}_\gamma + (3S^2 \dot{S} E_{\gamma 0} + S^3 \dot{E}_{\gamma 0}) \epsilon_\gamma \right] = \frac{3V_\gamma}{2(k^2 - 3K)S} \left(\frac{\epsilon_\gamma}{S}\right) \cdot. \quad (4.69)$$

We continue by evaluating the propagation equation in the case of radiation, $a = \gamma$. The first term is

$$(E_{\gamma 0} \epsilon_\gamma S^3) \cdot = \left(\frac{\epsilon_\gamma}{S}\right) \cdot E_{\gamma 0} S^4$$

$$= \frac{\dot{\epsilon}_\gamma V_\gamma}{S} - \dot{S} S^2 E_{\gamma 0} \epsilon_\gamma, \quad (4.70)$$

and the third and fifth terms are added to obtain

$$\begin{aligned} & 3(E_{\gamma 0} + P_{\gamma 0})S^2(\Phi_H S) = \\ & 4E_{\gamma 0}S^2 \left\{ \frac{3S^2 \dot{S}}{2(k^2 - 3K)} \sum_{a=1}^2 E_{a0} \epsilon_a + \frac{S^3}{2(k^2 - 3K)} \sum_{a=1}^2 \dot{E}_{a0} \epsilon_a \right. \\ & \left. + \frac{S^3}{2(k^2 - 3K)} \sum_{a=1}^2 E_{a0} \dot{\epsilon}_a \right\}. \end{aligned} \quad (4.71)$$

Adding the second and fourth we find

$$\begin{aligned} & \frac{3(E_{\gamma 0} + P_{\gamma 0})S^3}{k} \left\{ \left[\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \right] v_{S\gamma} + \frac{1}{3} k^2 v_{S\gamma} \right\} \\ & = \frac{4}{3S} V_\gamma k v_{S\gamma} \left\{ 1 - \frac{3K}{k^2} + \frac{S^2}{2k^2} \sum_{a=1}^2 (3E_{a0} + 3P_{a0}) \right\}. \end{aligned} \quad (4.72)$$

The equation for the photon fluid energy density perturbation has become

$$\begin{aligned} & V_\gamma \left\{ \frac{\dot{\epsilon}_\gamma}{S} - \frac{\dot{S}}{S} \epsilon_\gamma + \frac{2}{(k^2 - 3K)} \left[3 \dot{S} \sum_{a=1}^2 \epsilon_a E_{a0} + S \sum_{a=1}^2 \dot{E}_{a0} \epsilon_a + S \sum_{a=1}^2 E_{a0} \dot{\epsilon}_a \right] \right. \\ & \left. + \frac{4k}{3S} v_{S\gamma} \left[1 - \frac{3K}{k^2} + \frac{S^2}{2k^2} \sum_{a=1}^2 (3E_{a0} + 3P_{a0}) \right] \right\} = 0 \end{aligned} \quad (4.73)$$

To complete these derivations the last term in the previous expression is expanded

$$\begin{aligned} \frac{S^2}{2k^2} \sum_{a=1}^2 (3E_{a0} + 3P_{a0}) & = \frac{3S^2}{2k^2} \left(\frac{V_b}{S^3} + \frac{4V_\gamma}{3S^4} \right) \\ & = \frac{3V_b}{2k^2 S} \left(1 + \frac{4}{3SC_\rho} \right) \\ & = \frac{2V_\gamma}{k^2 S^2} \left(1 + \frac{3SC_\rho}{4} \right). \end{aligned} \quad (4.74)$$

We now look at the two equations for the propagation of the energy density for baryon and photon together.

$$V_b \left\{ \dot{\epsilon}_b \left[1 + \frac{3V_b}{2(k^2 - 3K)S} \right] + kv_{sb} \left[1 - \frac{3K}{k^2} + \frac{3V_b}{2k^2S} \left(1 + \frac{4}{3C_\rho S} \right) \right] + 3 \frac{V_\gamma}{2(k^2 - 3K)} \left(\frac{\epsilon_\gamma}{S} \right)' \right\} = 0 \quad (4.75)$$

$$V_\gamma \left\{ \left(\frac{\epsilon_\gamma}{S} \right)' + \frac{2}{k^2 - 3K} \left[-\dot{S} E_{\gamma 0} \epsilon_\gamma + S^{-2} V_b \dot{\epsilon}_b + \frac{V_\gamma}{S^3} \dot{\epsilon}_\gamma \right] + \frac{4k}{3S} v_{s\gamma} \left(1 - \frac{3K}{k^2} + \frac{2V_\gamma}{3S^2} \left(1 + \frac{3SC_\rho}{4} \right) \right) \right\} = 0, \quad (4.76)$$

which simplifies to

$$\dot{\epsilon}_b \left[1 + \frac{3V_b}{2(k^2 - 3K)S} \right] + kv_{sb} \left[1 - \frac{3K}{k^2} + \frac{3V_b}{2k^2S} \left(1 + \frac{4}{3C_\rho S} \right) \right] + 3 \frac{V_b}{2(k^2 - 3K)SC_\rho} \left(\frac{\epsilon_\gamma}{S} \right)' = 0 \quad (4.77)$$

$$\left(\frac{\epsilon_\gamma}{S} \right)' + \frac{2V_\gamma}{k^2 - 3K} \left[S^{-2} \left(\frac{\epsilon_\gamma}{S} \right)' + S^{-2} C_\rho \dot{\epsilon}_b \right] + \frac{4k}{3S} v_{s\gamma} \left[1 - \frac{3K}{k^2} + \frac{2V_\gamma}{3S^2} \left(1 + \frac{3SC_\rho}{4} \right) \right] = 0. \quad (4.78)$$

The next step in finding the temperature variation considers the propagation equation for the gauge invariant velocity. Repeating equations 56 and 57

$$\dot{v}_{sb} + \frac{\dot{S}}{S} v_{sb} = -k\Phi_H \quad (4.79)$$

$$\dot{v}_{s\gamma} + \frac{\dot{S}}{S} v_{s\gamma} = -k\Phi_H + \frac{\dot{S}}{S} (v_\gamma - B) + \frac{k}{4} \pi_{L\gamma}, \quad (4.80)$$

we find their difference can be expressed as

$$[(v_{s\gamma} - v_{sb})S]' = \dot{S} (v_\gamma - B) + \frac{Sk}{4} \pi_{L\gamma}, \quad (4.81)$$

which gives

$$\frac{1}{S} [(v_{s\gamma} - v_{sb})S]' = \frac{k}{4} \left(\pi_{L\gamma} + \frac{4\dot{S}}{kS} (v_\gamma - B) \right). \quad (4.82)$$

This is linked with the non-adiabatic perturbation

$$\frac{1}{4}\epsilon_\gamma - \frac{1}{3}\epsilon_b = -\chi(\tau) + \frac{1}{k} \frac{\dot{S}}{S} (v_{S\gamma} - v_{Sb}), \quad (4.83)$$

so that $(v_{S\gamma} - v_{Sb})$ can be substituted from (82) and (83) to find an expression for ϵ_γ .

$$\frac{1}{S} \frac{d}{d\tau} \left[\frac{1}{\dot{S}/S^2} \left(\frac{1}{3}\epsilon_b - \frac{1}{4}\epsilon_\gamma - \chi(\tau) \right) \right] = - \left(\frac{1}{4}\pi_{L\gamma} - \frac{1}{4}\delta_\gamma + \frac{1}{4}\epsilon_\gamma \right) \quad (4.84)$$

$$\frac{1}{3}\epsilon_b - \frac{1}{4}\epsilon_\gamma = -\frac{\dot{S}}{4S^2} \int (\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) S d\tau + \chi(\tau) \quad (4.85)$$

$$\epsilon_\gamma = \frac{4}{3}\epsilon_b + \frac{\dot{S}}{S} \int (\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) S d\tau - 4\chi(\tau). \quad (4.86)$$

We now find an expression for $\left(\frac{\epsilon_\gamma}{S}\right)^\cdot$ which can be substituted into equation (77) to give an expression for kv_{Sb} .

$$\begin{aligned} \left(\frac{\epsilon_\gamma}{S}\right)^\cdot &= \left(\left(\frac{\epsilon_\gamma}{S}\right) \left(\frac{S^3}{\dot{S}}\right) \right)^\cdot \\ &= \left(\frac{\epsilon_\gamma}{S}\right)^\cdot \frac{S^3}{\dot{S}} + \left(\frac{\epsilon_\gamma}{S}\right) \left(-\frac{(\dot{S}/S^3)^\cdot}{(\dot{S}/S^3)^2} \right). \end{aligned} \quad (4.87)$$

Changing the subject of the formula to $\left(\frac{\epsilon_\gamma}{S}\right)^\cdot$, we get

$$\left(\frac{\epsilon_\gamma}{S}\right)^\cdot = \frac{\dot{S}}{S^3} \left\{ \left(\frac{\epsilon_\gamma}{\dot{S}}\right)^\cdot - \left(\frac{\epsilon_\gamma}{S}\right) \left(-\frac{(\dot{S}/S^3)^\cdot}{(\dot{S}/S^3)^2} \right) \right\}, \quad (4.88)$$

and the second term is put in a more recognizable form

$$\begin{aligned} &- \left(\frac{\epsilon_\gamma}{S}\right) \left(-\frac{(\dot{S}/S^3)^\cdot}{(\dot{S}/S^3)^2} \right) \\ &= \frac{\epsilon_\gamma}{S} \left(\frac{(\dot{S}/S)^\cdot S^{-2} - 2S^{-3} \dot{S} (\dot{S}/S)}{(\dot{S}/S)^2 S^{-4}} \right) \\ &= \frac{\epsilon_\gamma}{S} \left(\frac{(\dot{S}/S)^\cdot}{(\dot{S}/S)^2 S^2} - \frac{2(\dot{S}/S)^2 S^{-2}}{(\dot{S}/S)^2 S^{-4}} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\epsilon_\gamma}{S} \left(\frac{(\dot{S}/S) \cdot S^2}{(\dot{S}/S)^2} - 2S^2 \right) \\
&= \epsilon_\gamma S \left(\frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} - 2 \right), \tag{4.89}
\end{aligned}$$

which gives

$$\left(\frac{\epsilon_\gamma}{S} \right) \cdot = \frac{\dot{S}}{S^3} \left[\left(\frac{\epsilon_\gamma}{\dot{S}/S^2} \right) \cdot - \epsilon_\gamma S \left[2 - \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right] \right]. \tag{4.90}$$

The first term in the previous equation is evaluated using equation 84, which gives

$$\left(\frac{\epsilon_\gamma}{\dot{S}/S^2} \right) \cdot = S(\pi_{L\gamma} - \delta_\gamma + \epsilon_\gamma) + \left(\frac{\frac{4}{3}\epsilon_b - 4\chi(\tau)}{\dot{S}/S^2} \right) \cdot. \tag{4.91}$$

Substitution of (91) into (90) yields

$$\left(\frac{\epsilon_\gamma}{S} \right) \cdot = \frac{\dot{S}}{S^3} \left[\left(\frac{\frac{4}{3}\epsilon_b - 4\chi(\tau)}{\dot{S}/S^2} \right) \cdot + S(\pi_{L\gamma} - \delta_\gamma + \epsilon_\gamma) - S\epsilon_\gamma \left[2 - \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right] \right], \tag{4.92}$$

where the $S\epsilon_\gamma$ in the second term cancels with the same in the third term and so the previous equation becomes

$$\left(\frac{\epsilon_\gamma}{S} \right) \cdot = \frac{\dot{S}}{S^3} \left\{ \left(\frac{\frac{4}{3}\epsilon_b - 4\chi(\tau)}{\dot{S}/S^2} \right) \cdot + S(\pi_{L\gamma} - \delta_\gamma) - \epsilon_\gamma S \left[1 - \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right] \right\}. \tag{4.93}$$

For the last term a substitution is made from (86) and we have

$$v_{Sb} - v_{S\gamma} = \frac{k}{4S} \int S(\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) d\tau, \tag{4.94}$$

and if the substitutions

$$-\delta_\gamma = 4\chi(\tau) - \frac{4}{3}\delta_b \tag{4.95}$$

$$\epsilon_\gamma = \frac{4}{3}\epsilon_b - 4\chi(\tau) - \frac{4\dot{S}}{kS}(v_{Sb} - v_{S\gamma}), \tag{4.96}$$

are made, $(\epsilon_\gamma/S) \cdot$ becomes

$$\begin{aligned}
 \left(\frac{\epsilon_\gamma}{S}\right) \cdot &= \frac{\dot{S}}{S^3} \left[\left(\frac{4}{3} \dot{\epsilon}_b - 4 \dot{\chi}(\tau) \right) \frac{S^2}{\dot{S}} + \left(\frac{4}{3} \epsilon_b - 4\chi(\tau) \right) \underbrace{\left(S - S \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right)}_{(S^2/\dot{S}) \cdot} \right. \\
 &+ S \left(\pi_{L\gamma} + 4\chi(\tau) - \frac{4}{3} \delta_b \right) - S \left(\frac{4}{3} \epsilon_b - 4\chi(\tau) \right) \left(1 - \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right) \\
 &\left. - S \frac{4 \dot{S}}{k S 4 S} \int S(\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) d\tau \left(1 - \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right) \right] \quad (4.97)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\dot{S}}{S^3} \left[\left(\frac{4}{3} \dot{\epsilon}_b - 4 \dot{\chi}(\tau) \right) \frac{S^2}{\dot{S}} + \left(4\chi(\tau) - \frac{4}{3} \delta_b + \pi_{L\gamma} \right) S \right. \\
 &\left. - \frac{\dot{S}}{S} \int S(\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) d\tau \left(1 - \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right) \right] \quad (4.98)
 \end{aligned}$$

It is now possible to calculate v_{Sb} and we do so by substituting (92) into (77).

$$\begin{aligned}
 kv_{Sb} &= \frac{-1}{1 - \frac{3K}{k^2}} \left\{ \dot{\epsilon}_b \left(1 - \frac{3V_b}{2(k^2 - 3K)S} \left(1 + \frac{4}{3C_\rho S} \right) \right. \right. \\
 &+ \left. \left. \frac{3V_b}{2(k^2 - 3K)S} \right) + \frac{3V_b}{2(k^2 - 3K)SC_\rho} \left[\left(\frac{4}{3} \dot{\epsilon}_b - 4 \dot{\chi}(\tau) \right) \frac{1}{S} \right. \right. \\
 &+ \left. \left. \frac{\dot{S}}{S^3} \left[S(\pi_{L\gamma} - \frac{4}{3} \delta_b + 4\chi(\tau)) \right. \right. \right. \\
 &\left. \left. \left. - \int (\epsilon_\gamma - \delta_\gamma + \frac{3}{4} \pi_{L\gamma}) S d\tau \left(\frac{\dot{S}}{S} - \frac{(\dot{S}/S) \cdot}{(\dot{S}/S)^2} \right) \right] \right] \right\} \quad (4.99)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{1 - \frac{3K}{k^2}} \left\{ \dot{\epsilon}_b - \frac{V_b}{(k^2 - 3K)S^2 C_\rho} (6 \dot{\chi}(\tau)) \right. \\
&\quad + \frac{V_b}{2(k^2 - 3K)S^3 C_\rho} \left(\dot{S} \left(\pi_{L\gamma} - \frac{4}{3} \delta_b + 4\chi(\tau) \right) \right. \\
&\quad \left. \left. - \int S(\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) d\tau \left(\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \cdot \right) \right) \right\}. \quad (4.100)
\end{aligned}$$

We now evaluate the last expression required for determining the Sachs-Wolfe effect in this scenario.

$$\begin{aligned}
&\frac{1}{3}(\dot{\epsilon}_b + kv_{sb}) = \\
&\quad \frac{-1}{3(1 - \frac{3K}{k^2})} \left\{ \frac{K}{k^2} \dot{\epsilon}_b - \frac{V_b(6 \dot{\chi}(\tau))}{(k^2 - 3K)S^2 C_\rho} \right. \\
&\quad + \frac{V_b}{2(k^2 - 3K)S^3 C_\rho} \left(\dot{S} \left(\pi_{L\gamma} - \frac{4}{3} \delta_b + 4\chi(\tau) \right) \right. \\
&\quad \left. \left. - \int S(\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) d\tau \left[\frac{V_b}{2S} \left(1 + \frac{4}{3SC_\rho} \right) - K \right] \right) \right\} \quad (4.101)
\end{aligned}$$

The observed temperature variation across the last scattering surface for a mixture of dust and radiation with $K = 1, 0, -1$, and non-adiabatic perturbations to the gauge invariant energy density is

$$\begin{aligned}
\left[\frac{\delta T}{T} \right]_R &= \left(\frac{1}{3} \epsilon_b Q \right)_E - (\chi(\tau) Q)_E - \frac{1}{(k^2 - 3K)} \int_E^R \dot{\epsilon}_b (KQ + Q_{|\alpha\beta} R^\alpha R^\beta) ds \\
&\quad + \frac{1}{(k^2 - 3K)C_\rho} \int_E^R \left(\frac{6V_b \dot{\chi}(\tau)}{(k^2 - 3K)S^2} - \frac{V_b}{2(k^2 - 3K)S^3} \left[\dot{S} \left(\pi_{L\gamma} - \frac{4}{3} \delta_b + 4\chi(\tau) \right) \right. \right. \\
&\quad \left. \left. - \int S(\epsilon_\gamma - \delta_\gamma + \pi_{L\gamma}) d\tau \left[\frac{V_b}{2S} \left(1 + \frac{4}{3SC_\rho} \right) - K \right] \right] \right) \\
&\quad \times \left(\frac{k^2}{3} Q + Q_{|\alpha\beta} R^\alpha R^\beta \right) ds \quad (4.102)
\end{aligned}$$

This considerably more complex case reduces to the single component fluid perturbation expression at the end of the last section. To see this we neglect the radiation fluid which is the same as putting $C_\rho = V_b/V_\gamma$ to infinity.

In this chapter we have determined the quantity $\frac{\delta T}{T}|_R$ for several scenarios each more general than the one before and it is assumed that they will be used in determining the quantities $\Delta_{AB}T_R$ in each case as this is easily observed.

4.6 Non-adiabatic perturbations for N types of matter

We now extend the previous work to non-adiabatic perturbations of a mixture of N fluids with $K = 1, 0, -1$ and ω_a not constant.

The energy equation gives us

$$\frac{\dot{E}_{a0}}{E_{a0} + P_{a0}} = -3\frac{\dot{S}}{S} = -\theta, \quad (4.103)$$

and for the pressure we substitute from equation 2.30

$$\omega_a = \frac{P_{a0}}{E_{a0}}. \quad (4.104)$$

We use the above relations to rewrite the propagation equation for the perturbation in the energy density where again we have neglected the anisotropic stress perturbation. The first term becomes

$$\begin{aligned} (E_{a0}\epsilon_a S^3)^\cdot &= \dot{E}_{a0} \epsilon_a S^3 + E_{a0} \dot{\epsilon}_a S^3 + \theta E_{a0} \epsilon_a S^3 \\ &= -\theta E_{a0}(1 + w_a)\epsilon_a S^3 + \theta E_{a0}\epsilon_a S^3 + E_{a0} \dot{\epsilon}_a S^3 \\ &= -\theta w_a E_{a0}\epsilon_a S^3 + E_{a0} \dot{\epsilon}_a S^3. \end{aligned} \quad (4.105)$$

The third and fifth terms, when added, give

$$\begin{aligned} &\frac{3(E_{a0} + P_{a0})S^3}{k} k \dot{\Phi}_H + 3S^3 \frac{\dot{S}}{S} (E_{a0} + P_{a0})\Phi_H \\ &= 3(E_{a0} + E_{a0}w_a)S^3 \dot{\Phi}_H + \theta S^3 (E_{a0} + E_{a0}w_a)\Phi_H \\ &= E_{a0}(1 + w_a)S^3 (3 \dot{\Phi}_H + \theta \Phi_H). \end{aligned} \quad (4.106)$$

As we have seen before

$$\Phi_H = \frac{S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \epsilon_a, \quad (4.107)$$

and so the time derivative becomes

$$\begin{aligned} \dot{\Phi}_H &= \frac{2 \dot{S} S}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \epsilon_a + \frac{S^2}{2(k^2 - 3K)} \sum_{a=1}^N \dot{E}_{a0} \epsilon_a \\ &\quad + \frac{S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \dot{\epsilon}_a \\ &= \frac{\dot{S}}{S} \frac{S^2}{k^2 - 3K} \sum_{a=1}^N E_{a0} \epsilon_a + \frac{S^2}{2(k^2 - 3K)} \sum_{a=1}^N (-\theta E_{a0} (1 + w_a)) \epsilon_a \\ &\quad + \frac{S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \dot{\epsilon}_a. \end{aligned} \quad (4.108)$$

From the above we obtain

$$\begin{aligned} (3 \dot{\Phi}_H + \theta \Phi_H) &= \\ &\quad \frac{\theta S^2}{k^2 - 3K} \sum_{a=1}^N E_{a0} \epsilon_a + \frac{\theta S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \epsilon_a \\ &\quad - \frac{3S^2}{2(k^2 - 3K)} \sum_{a=1}^N \theta E_{a0} (1 + w_a) \epsilon_a + \frac{3S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} \dot{\epsilon}_a \\ &= - \frac{3S^2 \theta \sum_{a=1}^N E_{a0} w_a \epsilon_a}{2(k^2 - 3K)} + \frac{3S^2 \sum_{a=1}^N E_{a0} \dot{\epsilon}_a}{2(k^2 - 3K)} \\ &= \frac{3S^2}{2(k^2 - 3K)} \sum_{a=1}^N E_{a0} (\dot{\epsilon}_a - \theta w_a \epsilon_a), \end{aligned} \quad (4.109)$$

and this eventually gives

$$S^3 E_{a0} (1 + w_a) \frac{3S^2}{2(k^2 - 3K)} \sum_{c=1}^N E_{c0} (\dot{\epsilon}_c - \theta w_c E_{c0}). \quad (4.110)$$

Now, for the second and fourth terms we obtain from equation 4.62

$$\frac{3(E_{a0} + w_a E_{a0}) S^3}{k} \left\{ \left[\left(\dot{S} / S \right)^2 - \left(\dot{S} / S \right) \right] v_{S_a} + \frac{1}{3} k^2 v_{S_a} \right\}, \quad (4.111)$$

we find the new expressions for the Friedmann equations in terms of the more general variables.

$$\left(\frac{\dot{S}}{S}\right) = -\frac{1}{6}S^2 \sum_{a=1}^N (E_{a0} + 3E_{a0}w_a) \quad (4.112)$$

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{1}{3}S^2 \sum_{a=1}^N E_{a0} - K \quad (4.113)$$

$$\begin{aligned} \left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right) &= \frac{2}{6}S^2 \sum_{a=1}^N E_{a0} - K + \frac{1}{6}S^2 \sum_{a=1}^N (E_{a0} + 3E_{a0}w_a) \\ &= \frac{1}{2}S^2 \sum_{a=1}^N E_{a0} + \frac{1}{2}S^2 \sum_{a=1}^N E_{a0}w_a - K \\ &= \frac{S^2}{2} \sum_{a=1}^N E_{a0}(1 + w_a) - K \end{aligned} \quad (4.114)$$

So, returning to the second and fourth term addition, we find

$$\begin{aligned} &\frac{3E_{a0}(1 + w_a)S^3}{k} v_{S_a} \left(\frac{S^2}{2} \sum_{c=1}^N E_{c0}(1 + w_c) - K + \frac{1}{3}k^2 \right) \\ &= E_{a0}(1 + w_a)S^3 v_{S_a} k \left(1 - \frac{3K}{k^2} + \frac{3S^2 \sum_{c=1}^N E_{c0}(1 + w_c)}{2k^2} \right) \end{aligned} \quad (4.115)$$

All the terms added together become

$$\begin{aligned} &-\theta w_a E_{a0} \epsilon_a S^3 + E_{a0} S^3 \dot{\epsilon}_a + \frac{3S^3 E_{a0}(1 + w_a)}{2(k^2 - 3K)} S^2 \sum_{c=1}^N E_{c0} (\dot{\epsilon}_c - \theta \omega_c E_{c0}) \\ &+ E_{a0}(1 + w_a) S^3 v_{S_a} k \left(1 - \frac{3K}{k^2} + \frac{3S^2}{2k^2} \sum_{c=1}^N E_{c0}(1 + w_c) \right) \end{aligned}$$

$$\begin{aligned}
&= E_{a0}S^3 \left\{ \dot{\epsilon}_a \left(1 + 3(1 + w_a) \frac{E_{a0}S^2}{2(k^2 - 3K)} \right) + \frac{3(1 + w_a)S^2}{2(k^2 - 3K)} \sum_{\substack{c=1 \\ c \neq a}}^N E_{c0} \dot{\epsilon}_c \right. \\
&\quad + kv_{Sa}(1 + w_a) \left(1 - \frac{3K}{k^2} + \frac{3S^2}{2k^2} \sum_{c=1}^N E_{c0}(1 + \omega_c) \right) \\
&\quad \left. - \left(\frac{3(1 + w_a)S^2 \sum_{c=1}^N E_{c0}}{2(k^2 - 3K)} \theta \omega_c E_{c0} + \theta w_a \epsilon_a \right) \right\} = 0. \tag{4.116}
\end{aligned}$$

For the mixture of N arbitrary matter fluids, one of them radiation, we continue in a way that makes the previous result for baryons and radiation more general. We obtain

$$\begin{aligned}
&\dot{\epsilon}_a \left(1 + \frac{3(1 + w_a)}{2(k^2 - 3K)} E_{a0}S^2 \right) + \frac{3(1 + w_a)S^2}{2(k^2 - 3K)} \sum_{\substack{c \neq a \\ c \neq \gamma}}^N \dot{\epsilon}_c E_{c0} \\
&\quad - \theta w_a \epsilon_a + kv_{Sa}(1 + w_a) \left(1 - \frac{3K}{k^2} + \frac{3S^2}{2k^2} \sum_{c=1}^N E_{c0}(1 + \omega_c) \right) \\
&\quad - \frac{3(1 + w_a)S^2}{2(k^2 - 3K)} \sum_{c \neq \gamma}^N E_{c0} \theta \omega_c E_{c0} + \frac{3(1 + w_a)}{2(k^2 - 3K)} S^2 E_{\gamma 0} \dot{\epsilon}_\gamma \\
&\quad - \frac{(1 + w_a)}{2(k^2 - 3K)} E_{\gamma 0} S^2 3\theta \omega_\gamma \epsilon_\gamma = 0, \tag{4.117}
\end{aligned}$$

and manipulate the last two terms just a little

$$\begin{aligned}
&\frac{3(1 + w_a)S^2}{2(k^2 - 3K)} E_{\gamma 0} \dot{\epsilon}_\gamma - \frac{(1 + w_a)}{2(k^2 - 3K)} E_{\gamma 0} S^2 3\theta \omega_\gamma \epsilon_\gamma \\
&= \frac{3(1 + w_a)E_{\gamma 0}S^2}{2(k^2 - 3K)} (\dot{\epsilon}_\gamma - \theta \omega_\gamma \epsilon_\gamma) \\
&= \frac{3(1 + w_a)E_{\gamma 0}S^2}{2(k^2 - 3K)} \left(\dot{\epsilon}_\gamma - \frac{\dot{S}}{S} \epsilon_\gamma \right) \\
&= \frac{3(1 + w_a)E_{\gamma 0}}{2(k^2 - 3K)} S^3 \left(\frac{\epsilon_\gamma}{S} \right).
\end{aligned}$$

$$= \frac{3(1+w_a)V_\gamma}{2(k^2-3K)S} \left(\frac{\epsilon_\gamma}{S} \right) , \quad (4.118)$$

where we have used $\omega_\gamma = 1/3$ for the radiation fluid and $V_\gamma = E_{\gamma 0}/S^4$.

Finally

$$\begin{aligned} \dot{\epsilon}_a \left(1 + \frac{3(1+w_a)}{2(k^2-3K)} E_{a0} S^2 \right) + \frac{3(1+w_a)S^2}{2(k^2-3K)} \sum_{\substack{c \neq a \\ c \neq \gamma}}^N \dot{\epsilon}_c E_{c0} \\ - \theta \omega_a \epsilon_a + k v_{S_a} (1+w_a) \left(1 - \frac{3K}{k^2} + \frac{3S^2}{2k^2} \sum_{c=1}^N E_{c0} (1+\omega_c) \right) \\ - \frac{3(1+w_a)S^2}{2(k^2-3K)} \sum_{c \neq \gamma}^N E_{c0} \theta \omega_c E_{c0} + \frac{3(1+w_a)V_\gamma}{2(k^2-3K)S} \left(\frac{\epsilon_\gamma}{S} \right) = 0 . \end{aligned} \quad (4.119)$$

We now derive a new form for $\left(\frac{\epsilon_\gamma}{S} \right)$. The entropy perturbation is left in its original form

$$\eta_a = \pi_{L_a} - \frac{C_{S_a}^2}{w_a} \delta_a , \quad (4.120)$$

and the anisotropic stress is neglected in the propagation equation for the velocity

$$\dot{v}_{S_a} + \frac{\dot{S}}{S} v_{S_a} = -k \Phi_H + \frac{k}{(1+w_a)} (C_{S_a}^2 \epsilon_a + w_a \eta_a) . \quad (4.121)$$

From these more general equations we obtain

$$\frac{1}{S} [(v_{S_\gamma} - v_{S_a}) S] \dot{} = \frac{k}{4} (\epsilon_\gamma + \eta_\gamma) - \frac{k}{(1+w_a)} (C_{S_a}^2 \epsilon_a + w_a \eta_a) , \quad (4.122)$$

and in the case of non-adiabatic perturbations we use the same definition and extend it to all fluids

$$\frac{1}{4} \epsilon_\gamma - \frac{1}{3} \epsilon_a = -\chi_a + \frac{1}{k} \frac{\dot{S}}{S} (v_{S_\gamma} - v_{S_a}) . \quad (4.123)$$

And so from the previous two equations we find

$$\frac{1}{S} \frac{d}{d\tau} \left(\left(\frac{1}{4} \epsilon_\gamma - \frac{1}{3} \epsilon_a + \chi_a \right) \frac{S^2}{\dot{S}} \right) = \frac{1}{4} (\epsilon_\gamma + \eta_\gamma) - \frac{1}{(1+w_a)} (C_{S_a}^2 \epsilon_a + w_a \eta_a) , \quad (4.124)$$

which yields

$$\begin{aligned} \frac{1}{4} \epsilon_\gamma - \frac{1}{3} \epsilon_a + \chi_a = \\ \frac{\dot{S}}{S^2} \int S \left(\frac{1}{4} (\epsilon_\gamma + \eta_\gamma) - \frac{1}{(1+w_a)} (C_{S_a}^2 \epsilon_a + w_a \eta_a) \right) d\tau , \end{aligned} \quad (4.125)$$

and from which it follows that

$$\epsilon_\gamma = \frac{4}{3}\epsilon_a - 4\chi_a + 4\frac{\dot{S}}{S^2} \int S \left(\frac{1}{4}(\epsilon_\gamma + \eta_\gamma) - \frac{1}{(1+w_a)}(C_{S_a}^2\epsilon_a + w_a\eta_a) \right) d\tau. \quad (4.126)$$

The next task is to find an expression for $\left(\frac{\epsilon_\gamma}{S}\right)^\cdot$.

$$\left(\frac{\epsilon_\gamma}{S}\right)^\cdot = \dot{S}/S^3 \left[\left(\frac{\epsilon_\gamma}{\dot{S}/S^2}\right) - \epsilon_\gamma S \left[2 - \frac{(\dot{S}/S)^\cdot}{(\dot{S}/S)^2} \right] \right] \quad (4.127)$$

From equation (124) we get

$$\begin{aligned} \left(\frac{\epsilon_\gamma}{\dot{S}/S^2}\right)^\cdot &= \\ S(\epsilon_\gamma + \eta_\gamma) - \frac{4S}{(1+w_a)}(C_{S_a}^2\epsilon_a + w_a\eta_a) + \left(\frac{S^2}{\dot{S}} \left(-4\chi_a + \frac{4}{3}\epsilon_a\right)\right)^\cdot \end{aligned} \quad (4.128)$$

and this is now substituted into (127), and we find a new form for $\left(\frac{\epsilon_\gamma}{S}\right)^\cdot$.

$$\begin{aligned} \left(\frac{\epsilon_\gamma}{S}\right)^\cdot &= \frac{\dot{S}}{S^3} \left[S(\epsilon_\gamma + \eta_\gamma) - \frac{4S}{(1+w_a)}(C_{S_a}^2\epsilon_a + w_a\eta_a) + \left(\frac{4}{3}\dot{\epsilon}_a - 4\dot{\chi}_a\right) \frac{S^2}{\dot{S}} \right. \\ &\quad \left. + S \left(\frac{4}{3}\epsilon_a - 4\chi_a\right) \left(1 - \frac{(\dot{S}/S)^\cdot}{(\dot{S}/S)^2}\right) - \epsilon_\gamma S \left[2 - \frac{(\dot{S}/S)^\cdot}{(\dot{S}/S)^2} \right] \right]. \end{aligned} \quad (4.129)$$

The expression for ϵ_γ from equation (126) gives a final form for $\left(\frac{\epsilon_\gamma}{S}\right)^\cdot$.

$$\begin{aligned}
\left(\frac{\epsilon_\gamma}{S}\right)^\cdot &= \frac{\dot{S}}{S^3} \left[S\eta_\gamma - \frac{4S}{(1+w_a)}(C_{S_a}^2\epsilon_a + w_a\eta_a) + \left(\frac{4}{3}\dot{\epsilon}_a - 4\dot{\chi}_a\right) \frac{S^2}{\dot{S}} \right. \\
&\quad + S \left(\frac{4}{3}\epsilon_a - 4\chi_a\right) \left(1 - \frac{(\dot{S}/S)^\cdot}{(\dot{S}/S)^2}\right) - S \left(\frac{4}{3}\dot{\epsilon}_a - 4\dot{\chi}_a\right) \left(1 - \frac{(\dot{S}/S)^\cdot}{(\dot{S}/S)^2}\right) \\
&\quad \left. - \frac{4\dot{S}}{S} \int S \left(\frac{1}{4}(\epsilon_\gamma + \eta_\gamma) - \frac{1}{(1+w_a)}(C_{S_a}^2\epsilon_a + w_a\eta_a)\right) d\tau \left(1 - \frac{(\dot{S}/S)^\cdot}{(\dot{S}/S)^2}\right) \right] \\
&= -\frac{\dot{S}}{S^2} \frac{4}{(1+w_a)}(C_{S_a}^2\epsilon_a + w_a\eta_a - \frac{(1+w_a)}{4}\eta_\gamma) \\
&\quad + \frac{1}{S} \left(\frac{4}{3}\dot{\epsilon}_a - 4\dot{\chi}_a\right) - \frac{4}{S^2} \int S \left(\frac{1}{4}(\epsilon_\gamma + \eta_\gamma) \right. \\
&\quad \left. - \frac{1}{(1+w_a)}(C_{S_a}^2\epsilon_a + w_a\eta_a)\right) d\tau \left(\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^\cdot \right) \tag{4.130}
\end{aligned}$$

For the evaluation of kv_{S_a} we use equation 119

$$\begin{aligned}
kv_{S_a} &= -(1+w_a)^{-1} \left(1 - \frac{3K}{k^2}\right)^{-1} \left(\dot{\epsilon}_a \left(1 - \frac{3S^2}{2(k^2-3K)} \sum_{\substack{c=1 \\ c \neq a}}^N E_{c0}(1+\omega_c)\right) \right. \\
&\quad + \frac{3(1+w_a)}{2(k^2-3K)} S^2 \sum_{\substack{c \neq a \\ c \neq \gamma}}^N \dot{\epsilon}_c E_{c0} - \theta w_a \epsilon_a - \frac{3(1+w_a)S^2}{2(k^2-3K)} \sum_{c \neq \gamma}^N E_{c0} \theta \omega_c E_{c0} \\
&\quad \left. + \frac{3S^2}{2(k^2-3K)} \theta w_a \epsilon_a \sum_{c=1}^N E_{c0}(1+\omega_c) + \frac{3(1+w_a)}{2(k^2-3K)S} V_\gamma \left(\frac{\epsilon_\gamma}{S}\right)^\cdot \right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{k^2}{(1+w_a)(k^2-3K)} \left(\dot{\epsilon}_a \left(1 - \frac{3S^2}{2(k^2-3K)} \sum_{\substack{c \neq a \\ c \neq \gamma}}^N E_{c0}(1+\omega_c) \right) \right. \\
&\quad + \frac{3(1+w_a)}{2(k^2-3K)} S^2 \sum_{\substack{c \neq a \\ c \neq \gamma}} \dot{\epsilon}_c E_{c0} - \theta w_a \epsilon_a - \frac{3(1+w_a)}{2(k^2-3K)} S^2 \sum_{\substack{c \neq a \\ c \neq \gamma}}^N E_{c0} \theta \omega_c E_{c0} \\
&\quad + \frac{3S^2}{2(k^2-3K)} \theta w_a \epsilon_a \sum_{\substack{c \neq a \\ c \neq \gamma}}^N E_{c0}(1+\omega_c) + \frac{4\theta w_a \epsilon_a}{2(k^2-3K)S^2} V_\gamma \\
&\quad \left. + \frac{3(1+w_a)}{2(k^2-3K)S} V_\gamma \left(\frac{\epsilon_\gamma}{S} \right) \right) \quad (4.131)
\end{aligned}$$

And from this the integral term of the variation in the CMBR temperature can be found. The last step is simply to find

$$\frac{1}{3}(\dot{\epsilon}_a + kv_{Sa}) =$$

$$\begin{aligned}
&\frac{k^2}{(3(1+w_a)(k^2-3K))} \left\{ \dot{\epsilon}_a \left(w_a - \frac{3K}{k^2}((1+w_a)) + \frac{3S^2}{2(k^2-3K)} \sum_{\substack{c=1 \\ c \neq a}}^N E_{c0}(1+\omega_c) \right) \right. \\
&\quad - \frac{3(1+w_a)}{2(k^2-3K)} S^2 \sum_{\substack{c \neq a \\ c \neq \gamma}}^N \dot{\epsilon}_c E_{c0} + \theta w_a \epsilon_a + \frac{3(1+w_a)}{2(k^2-3K)} S^2 \sum_{\substack{c \neq a \\ c \neq \gamma}}^N E_{c0} \theta \omega_c E_{c0} \\
&\quad - \frac{3S^2}{2(k^2-3K)} \theta w_a \epsilon_a \sum_{\substack{c \neq a \\ c \neq \gamma}}^N E_{c0}(1+\omega_c) - \frac{4\theta w_a \epsilon_a}{2(k^2-3K)S^2} V_\gamma \\
&\quad \left. - \frac{3(1+w_a)}{2(k^2-3K)S} V_\gamma \left(\frac{\epsilon_\gamma}{S} \right) \right\} \quad (4.132)
\end{aligned}$$

For the temperature variation in the previous cases we have assumed that the observers are travelling with the baryonic fluid and that the decoupling process is dependant only on the interactions of the baryonic and radiation fluids. Here we have done the same, but now assume that the universe is filled with $N - 2$ other sorts of matter which do not affect the decoupling process and so the Sachs-Wolfe

effect becomes

$$\begin{aligned}
\left. \frac{\delta T}{T} \right|_{Rb} &= \\
& \frac{1}{3} \epsilon_b Q|_E - \chi_b Q|_E + \int_E^R \left[\frac{1}{3} (\dot{\epsilon}_b + k v_{Sb}) Q + \frac{v_{Sb}}{k} Q_{|\alpha\beta} R^\alpha R^\beta \right] ds \quad (4.133) \\
&= \frac{1}{3} \epsilon_b Q|_E - \chi_b Q|_E + \frac{1}{(1+w_b)(k^2-3K)} \int_E^R \left\{ \dot{\epsilon}_b \left(\frac{w_b}{3} (k^2-3K) - K \right) Q \right. \\
& \quad \left. + \frac{k^2}{3} \theta w_b \epsilon_b Q + (\theta w_b \epsilon_b - \dot{\epsilon}_b) Q_{|\alpha\beta} R^\alpha R^\beta \right\} ds \\
&+ \frac{1}{(1+w_b)(k^2-3K)} \int_E^R \left(\frac{3S^2 \dot{\epsilon}_b}{2(k^2-3K)} \sum_{\substack{c \neq b \\ c \neq \gamma}}^N E_{c0} (1 + \omega_c) \right. \\
& \quad \left. + \frac{4 \dot{\epsilon}_b}{2(k^2-3K)S^2} V_\gamma - \frac{3(1+w_b)}{2(k^2-3K)} S^2 \sum_{\substack{c \neq b \\ c \neq \gamma}}^N \dot{\epsilon}_c E_{c0} \right. \\
& \quad \left. + \frac{3(1+w_b)}{2(k^2-3K)} S^2 \sum_{\substack{c \neq b \\ c \neq \gamma}}^N E_{c0} \theta \omega_c E_{c0} - \frac{3S^2}{2(k^2-3K)} \theta w_b \epsilon_b \sum_{\substack{c \neq b \\ c \neq \gamma}}^N E_{c0} (1 + \omega_c) \right. \\
& \quad \left. - \frac{4\theta w_b \epsilon_b}{2(k^2-3K)S^2} V_\gamma - \frac{3(1+w_b)V_\gamma}{2(k^2-3K)S} \left(\frac{\epsilon_\gamma}{S} \right) \right) \left(\frac{k^2}{3} Q + Q_{|\alpha\beta} R^\alpha R^\beta \right) ds. \quad (4.134)
\end{aligned}$$

For a mixture of two fluids - radiation and dust - we show that the above equation agrees with the results from the previous section. Equation 134 is rewritten for this particular case where the summation terms are left out and the above equation reduces to

$$\begin{aligned}
\left. \frac{\delta T}{T} \right|_{Rb} &= \frac{1}{3} \epsilon_b Q|_E - \chi_b Q|_E - \frac{1}{(k^2-3K)} \int_E^R \dot{\epsilon}_b (KQ + Q_{|\alpha\beta} R^\alpha R^\beta) ds \\
&+ \frac{1}{(k^2-3K)} \int_E^R \left(\frac{2 \dot{\epsilon}_b V_\gamma}{(k^2-3K)S^2} - \frac{3V_\gamma}{2(k^2-3K)S} \left(\frac{\epsilon_\gamma}{S} \right) \right) \\
&\times \left(\frac{k^2}{3} Q + Q_{|\alpha\beta} R^\alpha R^\beta \right) ds, \quad (4.135)
\end{aligned}$$

where the expression for $(\epsilon_\gamma/S)^\cdot$ now becomes

$$\left(\frac{\epsilon_\gamma}{S}\right)^\cdot = S\eta_\gamma + \frac{1}{S} \left(\frac{4}{3}\dot{\epsilon}_b - 4\dot{\chi}_b\right) - \frac{4}{S^2} \int S \left(\frac{1}{4}(\epsilon_\gamma + \eta_\gamma)\right) \left(\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^\cdot \right) d\tau. \quad (4.136)$$

Substitute $-\delta_\gamma = 4\chi_\gamma - \frac{4}{3}\delta_b$ into the previous equation to find

$$\begin{aligned} \left(\frac{\epsilon_\gamma}{S}\right)^\cdot &= \frac{\dot{S}}{S^2}(\pi_{L\gamma} + 4\chi_\gamma - \frac{4}{3}\delta_b) + \frac{1}{S} \left(\frac{4}{3}\dot{\epsilon}_b - 4\dot{\chi}_b\right) \\ &\quad - \frac{1}{S^2} \int S(\epsilon_\gamma + \pi_{L\gamma} - \delta_\gamma) \left(\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^\cdot \right) d\tau. \end{aligned} \quad (4.137)$$

The last line of equation 135 becomes

$$\begin{aligned} &\frac{1}{(k^2 - 3K)} \int_E^R \left\{ \frac{6V_\gamma \dot{\chi}_b}{kS^2} - \frac{3V_\gamma}{2(k^2 - 3K)S^3} \left[\dot{S} \left(\pi_{L\gamma} + 4\chi_\gamma - \frac{4}{3}\delta_b \right) \right. \right. \\ &\quad \left. \left. - \int S(\epsilon_\gamma + \pi_{L\gamma} - \delta_\gamma) \left(\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^\cdot \right) d\tau \right] \right\} \\ &\quad \times \left(\frac{k^2}{3}Q + Q_{|\alpha\beta}R^\alpha R^\beta \right) ds, \end{aligned} \quad (4.138)$$

where instead of our general expression for

$$\left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^\cdot = \frac{S^2}{2} \sum_{a=1}^N E_{a0}(1 + w_a) - K, \quad (4.139)$$

we have, in the particular case of radiation and dust,

$$\begin{aligned} \left(\frac{\dot{S}}{S}\right)^2 - \left(\frac{\dot{S}}{S}\right)^\cdot &= \frac{S^2}{2} \left(E_{b0} + \frac{4}{3}E_{\gamma 0} \right) - K \\ &= \frac{V_b}{2S} \left(1 + \frac{4}{3S^2 C_{pb}} \right) - K \end{aligned} \quad (4.140)$$

and so finally we obtain

$$\begin{aligned}
\frac{\delta T}{T} \Big|_{Rb} &= \frac{1}{3} \epsilon_b Q|_E - \chi_b Q|_E - \frac{1}{(k^2 - 3K)} \int_E^R \dot{\epsilon}_b (KQ + Q_{|\alpha\beta} R^\alpha R^\beta) ds \\
&+ \frac{1}{(k^2 - 3K)} \int_E^R \left\{ \frac{6V_\gamma \dot{\chi}_b}{kS^2} - \frac{3V_\gamma}{2(k^2 - 3K)S^3} \left[\dot{S} (\pi_{L\gamma} + 4\chi_\gamma - \frac{4}{3}\delta_b) \right. \right. \\
&\left. \left. - \int S(\epsilon_\gamma + \pi_{L\gamma} - \delta_\gamma) \left(\left(\frac{\dot{S}}{S} \right)^2 - \left(\frac{\dot{S}}{S} \right) \cdot \right) d\tau \right] \right\} \\
&\times \left(\frac{k^2}{3} Q + Q_{|\alpha\beta} R^\alpha R^\beta \right) ds, \tag{4.141}
\end{aligned}$$

which agrees with the temperature variation given in the previous section for a universe composed of a mixture of radiation and dust.

Part II

Chapter 5

The Ellis-Bruni formalism of the gauge problem

5.1 Introduction

In this chapter we write the covariant and gauge invariant quantities of Ellis and Bruni in terms of the variables used in Sachs and Wolfe's early paper. The propagation equations are then verified and the two formalisms are shown to agree for the simple case of a flat, $K = 0$, universe with pressure free dust. To begin with, we give the motivations for the EB approach and then follow their construction of gauge invariant quantities; this is followed by the outline of the SW paper with the assumptions they make and the explicit form of their metric.

5.2 Ellis and Bruni

In their approach, Ellis and Bruni suggest a scheme that avoids the gauge problem of perturbation theory by being fully covariant and gauge invariant. Their paper considers density inhomogeneities in an almost FLRW universe and they perform their analysis on the hypersurfaces orthogonal to the fluid flow lines. The four-velocity vector tangent to these lines is

$$U^a = dx^a/d\tau \tag{5.1}$$

where τ is the proper time along the fluid flow lines. From this is defined the projection tensor into the tangent three-spaces orthogonal to U^a :

$$h_{ab} \equiv g_{ab} + U_a U_b . \quad (5.2)$$

It is only when the fluid vorticity vanishes that a family of three surfaces, everywhere orthogonal to the fluid flow U^a , exists. Quantities that vanish in the FLRW background are taken to be the gauge invariant variables as follows from the Stewart and Walker lemma [17] at the top of pp. 4 of the introduction. Several well known examples in the FLRW universes are the, (1) vorticity, shear and acceleration:

$$\omega_{ab} \equiv h_a^c h_b^d U_{[c;d]} , \quad (5.3)$$

$$\sigma_{ab} \equiv h_a^c h_b^d U_{(c;d)} - \frac{1}{3} U^c{}_{;c} h_{ab} \quad (5.4)$$

and

$$a_a = \dot{U}^a \equiv U^a{}_{;b} U^b , \quad (5.5)$$

which we will consider in more detail here,

(2) The electric and magnetic parts E_{ab} , H_{ab} of the Weyl tensor C_{abcd} and,

(3) the matter tensor components:

$$q_a \quad \text{and} \quad \pi_{ab} . \quad (5.6)$$

The projected covariant derivative operator orthogonal to U^a , ${}^3\nabla_a$, is obtained by totally projecting the 4-dimensional covariant derivative operator. From the momentum equation we see that it determines the fluid acceleration

$$(\mu + p)a_a + {}^3\nabla_a p = 0 \quad (5.7)$$

where the pressure p and the energy density μ are related by a barotropic equation of state

$$p = p(\mu) \quad \Rightarrow \quad p_{[a}\mu_{,c]} = 0 \quad (5.8)$$

To include zero order quantities such as the energy density μ , the pressure p , and the fluid expansion θ the following gauge invariant quantities are constructed from them

$$X_a \equiv \kappa h_a^b \mu_{,b} , \quad Y_a \equiv \kappa h_a^b p_{,b} , \quad Z_a \equiv h_a^b \theta_{,b} . \quad (5.9)$$

Two other gauge invariant quantities that are important in the general theory but for our pressure free case will not be relevant, are the divergence of the acceleration and its spatial gradient

$$A \equiv \dot{U}^c{}_{;c}, \quad A_a \equiv h_a^b A_{,b}. \quad (5.10)$$

We first discuss the spatial projection of the energy density gradient which in this theory describes density inhomogeneities. The quantity defined by

$$X_a \equiv h_a^b \mu_{,b} \quad (5.11)$$

not only vanishes in the background and is covariantly defined, but is observable by virial theorem estimates. From this, the fractional density gradient is defined

$$\mathcal{X}_a \equiv \frac{X_a}{\kappa\mu} = h_a^b \left(\frac{\mu_{,a}}{\mu} \right), \quad (5.12)$$

which allows the comparison of the density gradient with the existing density and is also gauge invariant. This is superseded though, by a quantity that allows one to consider density variations at a fixed comoving scale and we come to the final definition which is of the dimensionless comoving fractional density gradient and is written

$$\mathcal{D}_a \equiv S \mathcal{X}_a, \quad (5.13)$$

or, equivalently

$$\mathcal{D}_a \equiv S \left(\frac{{}^3\nabla_a \mu}{\mu} \right) \quad (5.14)$$

where $S(\tau)$ is the scale factor. \mathcal{D}_a gives the relative growth of energy density perturbations in neighbouring comoving volumes and is closely related to the other vectors Y_a and Z_a .

With the help of the energy- and momentum-conservation equations EB find propagation equations for the acceleration, the spatial gradient of the energy density and the expansion.

$$h_a^c (\dot{U}_c)^\cdot = \dot{U}_a \theta \left(\frac{dp}{d\mu} - \frac{1}{3} \right) + h_a^b \left(\frac{dp}{d\mu} \theta \right)_{,b} - \dot{U}_c (\omega_a^c + \sigma_a^c), \quad (5.15)$$

$$S^{-4} h_a^c (S^4 X_a)^\cdot = -\kappa(\mu + p) Z_c - (\omega_a^c + \sigma_a^c) X_a, \quad (5.16)$$

and

$$S^{-3} h_c^a (S^3 Z_a)^\cdot = \dot{U}_c \mathcal{R} + h_c^a \left(-\frac{1}{2} X_a - 2(\sigma^2)_{,a} + 2(\omega^2)_{,a} + A_a \right) - Z_b (\sigma_c^b + \omega_c^b), \quad (5.17)$$

where

$$\mathcal{R} \equiv -\frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 + A + \kappa\mu + \Lambda. \quad (5.18)$$

The propagation equation for \mathcal{D}_a is

$$h_c^a(\mathcal{D}_a)^\cdot = \frac{p}{\mu}\theta\mathcal{D}_c - \left(1 + \frac{p}{\mu}\right)\mathcal{Z}_a - \mathcal{D}_a(\omega_c^a + \sigma_c^a), \quad (5.19)$$

where $\mathcal{Z}_a \equiv S\mathcal{Z}_a$, or equivalently, $\mathcal{Z}_a \equiv S^{(3)}\nabla_a\theta$.

For the perfect fluid assumption these are exact propagation equations and the next step is to linearize them about an almost FLRW universe. The basic perturbation equations are

$$\begin{aligned} \dot{\mathcal{D}}_{\perp a} &= h_a^c(\mathcal{D}_c)^\cdot = \frac{p}{\mu}\theta\mathcal{D}_a - \left(\frac{p}{\mu} + 1\right)\mathcal{Z}_a \\ &= w\theta\mathcal{D}_a - (1 + w)\mathcal{Z}_a, \end{aligned} \quad (5.20)$$

$$\dot{\mathcal{Z}}_{\perp a} = h_c^a(\mathcal{Z}_a)^\cdot = -\frac{2}{3}\theta\mathcal{Z}_c - \frac{1}{2}\kappa\mu\mathcal{D}_c + S\left(\frac{3k}{S^2}a_a + A_c\right). \quad (5.21)$$

(See below for the definition of k .)

From the 3-curvature scalar in the tangent space

$${}^{(3)}R = 2\left(-\frac{1}{3}\theta^2 + \sigma^2 - \omega^2 + \kappa\mu + \Lambda\right), \quad (5.22)$$

a gauge-invariant and covariant quantity related to \mathcal{D}_a and \mathcal{Z}_a is defined:

$$C_a \equiv S^3 {}^{(3)}\nabla_a({}^{(3)}R) = -\frac{4}{3}\theta S^2\mathcal{Z}_a + 2\kappa\mu S^2\mathcal{D}_a, \quad (5.23)$$

and the evolution equation is written

$$\dot{C}_{\perp a} = \frac{6k}{S^2}\theta^{-1}\left(\frac{1}{2}C_a - \kappa\mu S^2\mathcal{D}_a\right) - \frac{3}{4}\theta S^3\left(\frac{3k}{S^2}a_a + A_a\right), \quad (5.24)$$

where the covariant derivatives (implied by the overdot) may all be taken in the background spacetime, and

$${}^{(3)}R = 6k/S^2, \quad \dot{k} = 0. \quad (5.25)$$

To end off this introduction to the EB variables, we give the equation for the dynamics of the basic variable \mathcal{D}_a which is the analogue of Bardeen's equation (4.9) for his gauge-invariant energy density perturbation ϵ_m .

$$\ddot{\mathcal{D}}_{\perp a} + A(t)\dot{\mathcal{D}}_{\perp a} - B(t)\mathcal{D}_a - C_S^2 {}^{(3)}\nabla_a({}^{(3)}\nabla^b\mathcal{D}_b) = 0 \quad (5.26)$$

where the coefficients

$$\mathcal{A}(t) = \left(\frac{2}{3} - 2w + C_S^2\right)\theta, \quad (5.27)$$

$$\begin{aligned} \mathcal{B}(t) = & \left(\frac{1}{2} + 4w - \frac{3}{2}w^2 - 3C_S^2\right)\kappa\mu + (C_S^2 - w)\frac{12k}{S^2} \\ & + (5w - 3C_S^2)\Lambda, \end{aligned} \quad (5.28)$$

are determined in the background model.

5.3 Sachs and Wolfe

Sachs and Wolfe assume that the curvature constant K is zero and the universe is filled with a non-interacting dust with pressure $p = 0$, otherwise with radiation with $p = \frac{1}{3}\rho$. The units are $c = 8\pi G = 1$ as in the earlier part of this work, and as before the Latin indices run from 0 to 3, Greek indices from 1 to 3, the signature of the metric is taken as $(+1, -1, -1, -1)$, and the Minkowski metric is written as

$$\eta_{ab} = \text{diagonal}(+1, -1, -1, -1) = \eta^{ab}. \quad (5.29)$$

In the unperturbed universe the Einstein field equations for a perfect fluid with energy density ρ , pressure p and average world velocity U^a are

$$G_b^a = -(\rho + p)U^a U_b + p\delta_b^a, \quad (5.30)$$

and for this case the form of the metric for the Friedmann-Tolman models with $K = 0$ is written

$$ds^2 = a^2(\eta)[d\eta^2 - dx^2 - dy^2 - dz^2] = a^2(\eta)\eta_{ab}dx^a dx^b. \quad (5.31)$$

Let H_R be the Hubble parameter now, and then for $p = 0$ the scale factor $a(\eta)$, the density ρ and the cosmological proper time t are

$$p = 0 \quad a(\eta) = \frac{2\eta^2}{H_R}, \quad \rho = \frac{3H_R^2}{\eta^6}, \quad t = \frac{2\eta^3}{3H_R}. \quad (5.32)$$

The perturbed metric is

$$g_{ab} = a^2\bar{g}_{ab} = a^2(\eta_{ab} + \bar{H}_{ab}) \quad (5.33)$$

where $H_{ab} = a^2 \bar{H}_{ab}$ is the first order perturbation to the metric and the coordinates x^μ are comoving coordinates where

$$U^a = \frac{\delta_0^a}{a(\eta)} \iff G_0^\mu = 0, \quad \bar{H}_{00} = 0. \quad (5.34)$$

The allowed gauge transformations are now

$$[x]^a = x^a - \xi^a(x^b), \quad (5.35)$$

where ξ^a is small in the same sense that H_{ab} is. The first order vectors and tensors are raised and lowered with the Minkowski metric η_{ab} , η^{ab} and from the Einstein equations, SW find

$$\bar{H}_{00} = 0 \quad (5.36)$$

$$\bar{H}_{0\alpha} = \bar{H}_{\alpha 0} = -\frac{2\nabla^2 C_\alpha}{\eta^2} \quad (5.37)$$

$$\begin{aligned} \bar{H}_{\alpha\beta} = & \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} D_{\alpha\beta} \right) - 2 \left(\frac{8}{\eta^3} - \frac{\nabla^2}{\eta} \right) (C_{\alpha,\beta} + C_{\beta,\alpha}) \\ & + \frac{A_{,\alpha\beta}}{\eta^3} + \eta_{\alpha\beta} B - \frac{\eta^2}{10} B_{,\alpha\beta}, \end{aligned} \quad (5.38)$$

where $A(x^\alpha)$, $B(x^\alpha)$ and $C_\alpha(x^\alpha)$ are dependant on the space coordinate only, and $D_{\alpha\beta}(x^\alpha, \eta)$ is dependant on space and time coordinates. We have the properties:

$$C^\alpha_{,\alpha} = 0, \quad (5.39)$$

and

$$D_\alpha^\alpha = 0 \quad D_{\alpha\beta}{}^{,\beta} = 0 \quad D_{\alpha\beta} = D_{\beta\alpha}. \quad (5.40)$$

The products of A , B , C_α and $D_{\alpha\beta}$ are second order and so neglected. The projection tensor into the three spaces orthogonal to the fluid flow (the rest space of the fundamental observer) is written as

$$h_{ab} = g_{ab} - U_a U_b \quad (5.41)$$

where the four velocity of the fluid flow U^a satisfies $U^a U_a = 1$. In the comoving frame $U^a = (a^{-1}, 0, 0, 0)$.

5.4 Verification of the EB Covariant Propagation equations in the SW variables

The derivations begin with the comoving fractional density gradient orthogonal to the fluid flow

$$\begin{aligned} \mathcal{D}_b &= S \left(\frac{{}^{(3)}\nabla_b \mu}{\mu} \right) \\ &= S \left(\frac{(h_b^a \nabla_a) \mu}{\mu} \right), \end{aligned} \quad (5.42)$$

where we first find the projection tensor in terms of the perturbed SW metric,

$$h_b^a = g_b^a - U_b U^a = \delta_b^a + H_b^a - \frac{\delta_0^a}{a} \bar{g}_{b0a} \quad (5.43)$$

$$H_0^0 = a^2 \bar{H}_{0c} a^{-2} \eta^{c0} = H_{00} \eta^{00} = 0 \quad (5.44)$$

$$H_0^\alpha = H_{0c} \eta^{c\alpha} = -\delta^{\mu\alpha} H_{\mu 0} = \delta^{\mu\alpha} 2 \nabla^2 C_\mu / \eta^2 \quad (5.45)$$

$$H_\alpha^0 = H_{\alpha c} \eta^{c0} = -2 \nabla^2 C_\alpha / \eta^2 \quad (5.46)$$

$$\begin{aligned} H_\beta^\alpha &= H_{\beta c} \eta^{c\alpha} = -\delta^{\gamma\alpha} H_{\beta\gamma} \\ &= -\delta^{\gamma\alpha} \left(\frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} D_{\beta\gamma} \right) - 2 \left(\frac{8}{\eta^3} - \frac{\nabla^2}{\eta} \right) (C_{\beta,\gamma} + C_{\gamma,\beta}) \right. \\ &\quad \left. + \frac{A_{,\beta\gamma}}{\eta^3} + \eta_{\beta\gamma} B - \frac{\eta^2}{10} B_{,\beta\gamma} \right). \end{aligned} \quad (5.47)$$

Now we substitute for μ .

$$\mu = \frac{3H_R^2}{\eta^6} + \frac{H_R^2}{4} \nabla^2 \left(\frac{6A}{\eta^9} - \frac{3B}{5\eta^0} \right) \quad (5.48)$$

$$\begin{aligned} \mathcal{D}_b &= S \frac{h_b^a \mu_{,a}}{\mu} \\ &= S \frac{\delta_b^a \mu_{,a} + H_b^a \mu_{,a} - \delta_0^a g_{b0} \mu_{,a}}{\mu} \\ &= S \frac{\mu_{,b} - \mu_{,0} (\eta_{b0} + \bar{H}_{b0}) + H_b^a \mu_{,a}}{\mu} \end{aligned} \quad (5.49)$$

For the time coordinate we get

$$\begin{aligned}
\mathcal{D}_0 &= S \frac{\mu_{,0} - \mu_{,0}(\eta_{00}) + H_0^2 \mu_{,a}}{\mu} \\
&= S \frac{H_0^\alpha \mu_{,a}}{\mu} \\
&= 0
\end{aligned} \tag{5.50}$$

to the first order. The space components, as we shall see, are a linear combination of the gradient of the spatial Laplacian of A and B divided by suitable powers of the time coordinate η . We have

$$\mathcal{D}_\alpha = S \frac{\mu_{,\alpha} - \mu_{,0}(\eta_{\alpha 0} + \bar{H}_{\alpha 0}) + H_\alpha^c \mu_{,c}}{\mu}, \tag{5.51}$$

where

$$S \frac{H_\alpha^c \mu_{,c}}{\mu} = S \frac{H_\alpha^0 \mu_{,0}}{\mu} + S \frac{H_\alpha^\beta \mu_{,\beta}}{\mu}, \tag{5.52}$$

with the second term being second order as H_α^β and $\mu_{,\beta}$ are first order. We are left with

$$\begin{aligned}
&S \frac{H_\alpha^c \mu_{,c}}{\mu} = \\
&S (-2\nabla^2 C_\alpha / \eta^2) \left(\frac{3H_R^2(-6)}{\eta^7} + \frac{H_R^2}{0} \nabla^2 \left(\frac{6A(-9)}{\eta^{10}} - \frac{3B(-4)}{5\eta^5} \right) \right) \\
&\div \frac{3H_R^2}{\eta^6} \left(1 + \frac{\nabla^2}{12} \left(\frac{6A}{\eta^3} - \frac{3B}{5}\eta^2 \right) \right) \\
&= S \frac{36H_R^2 \nabla^2 C_\alpha / \eta^9}{3H_R^2 / \eta^6} \left(1 - \frac{\nabla^2}{12} \left(\frac{6A}{\eta^3} - \frac{3B\eta^2}{5} \right) \right) \\
&= S \frac{12\nabla^2 C_\alpha}{\eta^3}
\end{aligned} \tag{5.53}$$

to first order, and so to return to \mathcal{D}_α we find

$$\begin{aligned}
\mathcal{D}_\alpha &= S \frac{(H_R^2/4)\nabla^2}{3H_R^2/\eta^6} \left(\frac{6A_{,\alpha}}{\eta^9} - \frac{3B_{,\alpha}}{5\eta^4} \right) - S \frac{(3H_R^2(-6)/\eta^7)}{3H_R^2/\eta^6} (-2\nabla^2 C_\alpha / \eta^2) \\
&\quad + S \frac{12\nabla^2 C_\alpha}{\eta^3}
\end{aligned}$$

$$\begin{aligned}
&= S \left(\frac{\nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^3} - \frac{3B_{,\alpha}\eta^3}{5} \right) - \frac{12\nabla^2 C_\alpha}{\eta^3} + \frac{12\nabla^2 C_\alpha}{\eta^3} \right) \\
&= S \frac{\nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^3} - \frac{3B_{,\alpha}\eta^2}{5} \right). \tag{5.54}
\end{aligned}$$

Another important EB quantity is the comoving expansion gradient orthogonal to the fluid flow \mathcal{Z}_a . This is defined as

$$\mathcal{Z}_a = S h_a^b \theta_{,b}, \tag{5.55}$$

where the volume expansion θ is defined as

$$\begin{aligned}
\theta &= U_{;a}^a = U_{,a}^a + \Gamma_{ba}^a U^b \\
&= -\frac{a'}{a^2} + \frac{\Gamma_{00}^0}{a} + \frac{\Gamma_{0\alpha}^\alpha}{a}. \tag{5.56}
\end{aligned}$$

The Christoffel symbols are listed as follows.

$$\Gamma_{00}^0 = \frac{a'}{a} \tag{5.57}$$

to zero order, and vanishes to first order and where we have written $a_{,0} = \frac{da}{d\eta}$ as a' . We also have

$$\Gamma_{0\alpha}^\alpha = \frac{3a'}{a} + \frac{\nabla^2}{2} \left(\frac{3A}{\eta^4} + \frac{2\eta B}{10} \right), \tag{5.58}$$

and so we find θ to be

$$\theta = -\frac{a'}{a^2} + \frac{a'/a + 3a'/a}{a} \tag{5.59}$$

to zero order, and in full it is written as

$$\theta = 3\frac{a'}{a^2} + \frac{1}{a} \left(\frac{3}{2\eta^4} \nabla^2 A + \frac{\eta}{10} \nabla^2 B \right). \tag{5.60}$$

Before we continue with the calculation of \mathcal{Z}_a , we discuss the difference between the different scale factors used in the two theories that are being compared here and understand that they can be interchanged so long as one takes care which time coordinate is being used when determining the volume expansion.

The representative length scale along the fluid flow lines in the different formulations of SW and EB, respectively a and S , when taken to be equal, give the same value

for the volume expansion θ . We have

$$S = a \quad (5.61)$$

$$\frac{\dot{S}}{S} = \frac{dS/dt}{S} \quad \text{and,} \quad \frac{a'}{a} = \frac{da}{d\eta} a^{-1} \quad (5.62)$$

$$t = \frac{2\eta^3}{3H_R} \quad \frac{dt}{d\eta} = \frac{6\eta^2}{3H_R} \quad (5.63)$$

$$\frac{d\eta}{dt} = \frac{H_R}{2\eta^2} = \frac{1}{a} \quad (5.64)$$

$$\frac{dS}{dt} = \frac{dS}{d\eta} \frac{d\eta}{dt} = \frac{da}{d\eta} \frac{d\eta}{dt} = \frac{a'}{a}, \quad (5.65)$$

which leaves

$$\frac{\dot{S}}{S} = \frac{a'}{a^2}, \quad (5.66)$$

and so from now on we will differentiate only with the η time coordinate used in the SW paper.

We continue with the evaluation of \mathcal{Z}_a in terms of the SW variables.

$$\begin{aligned} \mathcal{Z}_a &= S h_a^b \theta_{,b} = S (\delta_a^b + H_a^b - U_a U^b) \theta_{,b} \\ &= S \left(\delta_a^b - \frac{\delta_0^b}{a} a \bar{g}_{a0} + H_a^b \right) \theta_{,b} \\ &= S (\delta_a^b + H_a^b - \delta_0^b (\eta_{a0} + \bar{H}_{a0})) \theta_{,b} \\ &= S (\theta_{,a} + H_a^b \theta_{,b} - \theta_{,0} (\eta_{a0} + \bar{H}_{a0})) \end{aligned} \quad (5.67)$$

For the time component of this vector,

$$\begin{aligned} \mathcal{Z}_0 &= S (\theta_{,0} + H_0^b \theta_{,b} - \theta_{,0}) = S H_0^b \theta_{,b} \\ &= S H_0^\alpha \theta_{,\alpha} \\ &= 0, \end{aligned} \quad (5.68)$$

we find that as in the case of \mathcal{D}_a , it vanishes to first order.

The space components turn out to be

$$\begin{aligned}
\mathcal{Z}_\alpha &= S(\theta_{,\alpha} + H_\alpha^b \theta_{,b} - \theta_{,0} \bar{H}_{\alpha 0}) \\
&= S(\theta_{,\alpha} + H_\alpha^0 \theta_{,0} + H_\alpha^\beta \theta_{,\beta} - \theta_{,0} \bar{H}_{\alpha 0}) \\
&= S(-2\nabla^2 C_\alpha / \eta^2 \theta_{,0} - (-2\nabla^2 C_\alpha / \eta^2) \theta_{,0} + \theta_{,\alpha}) \\
&= S \frac{\nabla^2}{a} \left(\frac{A_{,\alpha}(3)}{2\eta^4} + \frac{\eta B_{,\alpha}}{10} \right), \tag{5.69}
\end{aligned}$$

where we have used $H_\alpha^\beta \theta_{,\beta} = 0$ to first order.

We now have expressions for the comoving expansion and fractional density gradients and can verify their propagation equations. The propagation of \mathcal{D}_a is written

$$\begin{aligned}
h_a^c (\mathcal{D}_c)^\cdot &= h_a^c \mathcal{D}_{c;b} U^b = h_a^c \mathcal{D}_{c,b} U^b - h_a^c \Gamma_{cb}^d \mathcal{D}_d U^b \\
&= (\delta_a^c + H_a^c) \frac{\mathcal{D}_{c,0}}{a} - \delta_a^c \Gamma_{c0}^\delta \mathcal{D}_\delta \frac{1}{a} \\
&= -\Gamma_{a0}^\delta \mathcal{D}_\delta \frac{1}{a} + \mathcal{D}_{a,0} \frac{1}{a}, \tag{5.70}
\end{aligned}$$

and so for the particular case of the time coordinate we have

$$h_0^c (\mathcal{D}_c)^\cdot = 0 \tag{5.71}$$

as \mathcal{D}_0 vanishes and the Christoffel symbol Γ_{00}^δ is at least first order. For the space components, $a = \alpha$, we have

$$\mathcal{D}_\alpha = -\Gamma_{\alpha 0}^\delta \mathcal{D}_\delta \frac{1}{a} + \mathcal{D}_{\alpha,0} \frac{1}{a}. \tag{5.72}$$

The Christoffel symbols are

$$\Gamma_{\alpha 0}^\delta = \delta_\alpha^\delta \frac{a'}{a}, \tag{5.73}$$

and this, put into the propagation equation, gives

$$h_\alpha^c (\mathcal{D}_c)^\cdot = -\delta_\alpha^\delta \frac{a'}{a^2} \mathcal{D}_\delta + \mathcal{D}_{\alpha,0} \frac{1}{a}, \tag{5.74}$$

with the substitution of

$$a = \frac{2\eta^2}{H_R} \qquad \frac{a'}{a} = \frac{2}{\eta}, \tag{5.75}$$

to find

$$\begin{aligned}
h_\alpha^c(\mathcal{D}_c) &= \frac{1}{a} \left(\mathcal{D}_{\alpha,0} - \frac{2}{\eta} \mathcal{D}_\alpha \right) \\
&= \frac{1}{a} \left(S_{,0} \frac{\nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^3} - \frac{3B_\alpha}{5} \right) + \frac{S \nabla^2}{12} \left(\frac{6A_{,\alpha}(-3)}{\eta^4} - \frac{3B_{,\alpha} 2\eta}{5} \right) \right. \\
&\quad \left. - \frac{2}{\eta} \left(S \frac{\nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^3} - \frac{3B_{,\alpha} \eta^2}{5} \right) \right) \right) \\
&= \frac{1}{a} \left(S_{,0} \frac{\nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^3} - \frac{3B_\alpha \eta^2}{5} \right) + S \frac{\nabla^2}{12} \left(-\frac{5.6A_{,\alpha}}{\eta^4} \right) \right). \tag{5.76}
\end{aligned}$$

If we now take $S = a$ and $S_{,0} = \frac{2}{\eta}a$ we have

$$\begin{aligned}
h_\alpha^c(\mathcal{D}_c) &= \frac{1}{a} \left(\frac{2a \nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^4} - \frac{3B_{,\alpha} \eta}{5} \right) + a \frac{\nabla^2}{12} \left(-\frac{5.6A_{,\alpha}}{\eta^4} \right) \right) \\
&= \frac{\nabla^2}{12} \left(-\frac{3.6A_{,\alpha}}{\eta^4} - \frac{6B_{,\alpha} \eta}{5} \right) \\
&= -\mathcal{Z}_\alpha, \tag{5.77}
\end{aligned}$$

as required.

The same process for \mathcal{Z}_α now follows.

$$\begin{aligned}
h_\alpha^c(\mathcal{Z}_c) &= (\eta_\alpha^c + H_\alpha^c - \delta_0^c \bar{g}_{\alpha 0}) \mathcal{Z}_{c;d} U^d \\
&= (\eta_\alpha^c + H_\alpha^c - \delta_0^c \bar{g}_{\alpha 0}) (\mathcal{Z}_{c,d} U^d - \Gamma_{cd}^b \mathcal{Z}_b U^d) \tag{5.78}
\end{aligned}$$

For the time component, we have

$$h_0^c(\mathcal{Z}_c) = 0 \tag{5.79}$$

since, if $c = 0$ then $h_0^0 = 0$ and $\mathcal{Z}_0 = 0$, and if $c = \beta$ then h_0^β is first order as is \mathcal{Z}_β .

For the space components the equations yield

$$\begin{aligned}
h_\alpha^c(\mathcal{Z}_c) &= \frac{\mathcal{Z}_{\alpha,0}}{a} - \Gamma_{\alpha 0}^b \frac{\mathcal{Z}_b}{a} = \frac{1}{a} \left(\mathcal{Z}_{\alpha,0} - \Gamma_{\alpha 0}^\beta \mathcal{Z}_\beta \right) \\
&= \frac{1}{a} \left(-\nabla^2 A_{,\alpha} \frac{3.4}{2\eta^5} + \nabla^2 \frac{B_{,\alpha}}{10} - \delta_\alpha^\beta \frac{a'}{a} \mathcal{Z}_\beta \right) \\
&= -\frac{1}{a} \left(\nabla^2 A_{,\alpha} \frac{9}{\eta^5} + \nabla^2 \frac{B_{,\alpha}}{10} \right), \tag{5.80}
\end{aligned}$$

which we would now like to show equal to

$$-\frac{2}{3}\theta\mathcal{Z}_c - \frac{1}{2}\kappa\mu\mathcal{D}_c \quad (5.81)$$

where $\kappa = 1$. We have

$$\begin{aligned} -\frac{2}{3}\theta\mathcal{Z}_c - \frac{1}{2}\kappa\mu\mathcal{D}_c &= -\frac{2}{3}\frac{3a_{,0}}{a^2} \left(\nabla^2 A_{,\alpha} \left(\frac{3}{2\eta^4} \right) + \frac{\eta}{10} \nabla^2 B_{,\alpha} \right) \\ &\quad - \frac{1}{2} \frac{3H_R^2}{\eta^6} \frac{S\nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^3} - \frac{3B_{,\alpha}\eta}{5} \right) \\ &= -\frac{1}{a} \left(\nabla^2 A_{,\alpha} \left(\frac{6}{\eta^5} \right) + \nabla^2 B_{,\alpha} \frac{4}{10} \right) \\ &\quad - \frac{1}{a} \frac{6}{\eta^2} \frac{\nabla^2}{12} \left(\frac{6A_{,\alpha}}{\eta^3} - \frac{3B_{,\alpha}\eta}{5} \right) \\ &= -\frac{1}{a} \left(\nabla^2 \frac{A_{,\alpha}9}{\eta^5} + \frac{\nabla^2 B_{,\alpha}}{10} \right), \end{aligned} \quad (5.82)$$

as required.

The next gauge invariant EB quantity \mathcal{C}_a , is the spatial variation of the three-curvature scalar ${}^{(3)}R$ in the tangent space.

$$\mathcal{C}_a = S^{(3)}\nabla_a({}^{(3)}R) \quad (5.83)$$

$${}^{(3)}R = 2\left(-\frac{1}{3}\theta^2 + \sigma^2 - \omega^2 + \kappa\mu + \Lambda\right) \quad (5.84)$$

To linear order we obtain

$$\mathcal{C}_a = -\frac{4}{3}\theta S^2 \mathcal{Z}_a + 2\kappa\mu S^2 \mathcal{D}_a, \quad (5.85)$$

as we will demonstrate.

$$\begin{aligned} S^3 h_a^b ({}^{(3)}R)_{,b} &= S h_a^b 2\left(-\frac{1}{3}\theta^2 + \sigma^2 - \omega^2 + \kappa\mu + \Lambda\right)_{,b} \\ &= S 2\left(-\frac{2}{3}\theta\theta_{,b} + 2\sigma\sigma_{,b} - 2\omega\omega_{,b} + \kappa\mu_{,b}\right) h_a^b \end{aligned} \quad (5.86)$$

The σ and ω terms are both second order and so

$$C_a = -\frac{4}{3}\theta Z_a S^2 + 2\kappa\mu D_a S^2. \quad (5.87)$$

It is possible to verify the propagation equation for C_a ,

$$\dot{C}_{\perp a} = \frac{6k}{S^2}\theta^{-1}\left(\frac{1}{2}C_a - \kappa\mu S^2 D_a\right) - \frac{3}{4}\theta S^3\left(\frac{3k}{S^2}a_a + A_a\right) \quad (5.88)$$

by simply making use of our knowledge of the quantities D_a and Z_a , their propagation in the case of zero pressure, and the quantities k , a_a and A_a , which are defined as

$${}^{(3)}R = 6k/S^2, \quad \dot{k} = 0 \quad (5.89)$$

$$a_a = \dot{U}_a \quad (5.90)$$

and,

$$A_a = {}^{(3)}\nabla_a(a_{;c}^c). \quad (5.91)$$

We make use of the general relation

$$\kappa(\mu + p)\dot{U}_a + {}^{(3)}\nabla_a p = 0, \quad (5.92)$$

where for pressure free matter we have

$$\dot{U}_a = 0, \quad (5.93)$$

which from the above definitions leaves us with

$$a_a = 0 \quad (5.94)$$

$$A_a = 0 \quad (5.95)$$

and

$$6k/S^2 = 2\left(-\frac{1}{3}\theta^2 + \kappa\mu\right) \quad (5.96)$$

where, in the last equation the cosmological constant Λ is neglected. The propagation equation for C_a is now

$$2\left(-\frac{1}{3}\theta^2 + \kappa\mu\right)\theta^{-1}\left(\frac{1}{2}C_a - \kappa\mu S^2 D_a\right). \quad (5.97)$$

As the propagation equations for \mathcal{Z}_a and \mathcal{D}_a have already been verified we find that \mathcal{C}_a , which is defined in terms of the first two physical quantities, will have its propagation equation satisfied as well when translated into the SW variables. We need to show that the acceleration a_a vanishes to complete this case.

$$\begin{aligned}
a_a &= \dot{U}_a = U_{a;c} U^c \\
&= (U^b g_{ba})_{;c} U^c = \left(\frac{g_{0a}}{a} \right)_{;c} U^c \\
&= \left(\frac{g_{0a}}{a} \right)_{,c} U^c - \Gamma_{a^d c}^d \left(\frac{g_{0d}}{a} \right) U^c \\
&= \left(\frac{g_{0a}}{a} \right)_{,0} a^{-1} - \Gamma_{a0}^d \left(\frac{g_{0d}}{a^2} \right)
\end{aligned} \tag{5.98}$$

For the time component of the acceleration we obtain

$$\begin{aligned}
a_0 &= \frac{a'}{a} - \Gamma_{00}^d \left(\frac{g_{0d}}{a^2} \right) \\
&= \frac{a'}{a} - \Gamma_{00}^0 \left(\frac{g_{00}}{a^2} \right) - \Gamma_{00}^\alpha \left(\frac{g_{0\alpha}}{a^2} \right) \\
&= a^{-1} \left(a' - a' - \Gamma_{00}^\alpha \left(\frac{g_{0\alpha}}{a} \right) \right)
\end{aligned} \tag{5.99}$$

The Christoffel symbol is first order, and so we have

$$a_0 = 0 . \tag{5.100}$$

For the space components the task is slightly more complex but nevertheless straightforward.

$$a_\alpha = - \left(\frac{2\nabla C_\alpha}{\eta^2} \frac{2\eta^2}{HR} \right)_{,0} \frac{1}{a} - \Gamma_{\alpha 0}^\delta \left(\frac{g_{0\delta}}{a^2} \right) - \Gamma_{\alpha 0}^0 \left(\frac{g_{00}}{a^2} \right) , \tag{5.101}$$

where

$$\Gamma_{\alpha 0}^0 = \frac{a'}{a} \left(\frac{2\nabla^2 C_\alpha}{\eta^2} \right) , \tag{5.102}$$

and since

$$\Gamma_{\alpha 0}^\delta = \frac{a'}{a} \delta_\alpha^\delta , \tag{5.103}$$

we have

$$a_\alpha = 0 . \tag{5.104}$$

And so we have shown that

$$a_a = 0 , \tag{5.105}$$

which implies that

$$A_a = 0, \quad (5.106)$$

and the above propagation equation for C_a is satisfied.

The vorticity and shear for the SW formalism are now calculated. The vorticity is defined as

$$\begin{aligned} \omega_{ab} &= \frac{1}{2} h_a^c h_b^d (U_{c;d} - U_{d;c}) \\ &= \frac{1}{2} (\delta_a^c - U^c U_a) (\delta_b^d - U^d U_b) (U_{c;d} - U_{d;c}). \end{aligned} \quad (5.107)$$

An expression for $U_{c;d}$ is obtained first:

$$U_{c;d} = (U^a g_{ac})_{;d} = \left(\frac{g_{0c}}{a} \right)_{,d} - \frac{\Gamma_{cd}^a g_{0a}}{a}, \quad (5.108)$$

and so for the particular case where $c, d \neq 0$ we find a first order expression for the vorticity,

$$\begin{aligned} U_{\alpha;\gamma} - U_{\gamma;\alpha} &= \left(\frac{g_{0\alpha}}{a} \right)_{,\gamma} - \left(\frac{g_{0\gamma}}{a} \right)_{,\alpha} \\ &= -a \frac{2\nabla^2}{\eta^2} (C_{\alpha,\gamma} - C_{\gamma,\alpha}) \\ &= -\frac{4\nabla^2}{H} (C_{\alpha,\gamma} - C_{\gamma,\alpha}). \end{aligned} \quad (5.109)$$

For the $\{0,0\}$ term, i.e. $c, d = 0$, the vorticity vanishes, and since (g_{0c}/a) is dependant only on the time coordinate when $c = 0$, and dependant only on the spatial coordinates when $c \neq 0$, we see that

$$\omega_{\alpha 0} = \omega_{0\alpha} = 0, \quad (5.110)$$

and so we are left with

$$\omega_{ab} = \omega_{\alpha\beta} = -\frac{2\nabla^2}{H_R} (C_{\alpha,\beta} - C_{\beta,\alpha}) \quad (5.111)$$

For the shear we have the equation

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3} \theta h_{ab}, \quad (5.112)$$

where we have already calculated the volume expansion θ and so only need the first term on the right hand side which is

$$\theta_{ab} = \frac{1}{2}(\delta_a^c - U^c U_a)(\delta_b^d - U^d U_b)(U_{c;d} + U_{d;c}), \quad (5.113)$$

We begin by finding

$$\frac{\Gamma_{cd}^a g_{0a}}{a} = a\Gamma_{cd}^0 + a\Gamma_{cd}^\alpha \frac{(-2\nabla^2 C_\alpha)}{\eta^2}. \quad (5.114)$$

and list the relevant Christoffel symbols:

$$\Gamma_{00}^0 = \frac{a'}{a} \quad (5.115)$$

$$\Gamma_{\alpha 0}^0 = \frac{a'}{a} \left(\frac{-2\nabla^2 C_\alpha}{\eta^2} \right) \quad (5.116)$$

$$\Gamma_{\beta 0}^\alpha = \delta_\beta^\alpha \frac{a'}{a} \quad (5.117)$$

$$\Gamma_{00}^\alpha = -\frac{4\nabla^2 C_\beta \eta^{\beta\alpha}}{\eta^3} \quad (5.118)$$

$$\Gamma_{\alpha\beta}^0 = -\frac{a'}{a}\eta_{\alpha\beta} + \frac{\nabla^2}{\eta^2}(C_{\alpha,\beta} + C_{\beta,\alpha}) - \frac{1}{2}\bar{H}'_{\alpha\beta} - \frac{a'}{a}\bar{H}_{\alpha\beta} \quad (5.119)$$

$$\Gamma_{\mu\nu}^\alpha = {}^3\Gamma_{\mu\nu}^\alpha = 0, \quad (5.120)$$

where ${}^3\Gamma_{\mu\nu}^\alpha$ is the Riemann tensor for Euclidean three-space and so $\Gamma_{\mu\nu}^\alpha$ vanishes to zero order. This is all that concerns us as we will not need its first order contribution. This gives the following relations:

$$\frac{\Gamma_{00}^a g_{0a}}{a} = a' \quad (5.121)$$

$$\frac{\Gamma_{0\beta}^a g_{0a}}{a} = -\frac{8\nabla^2 C_\beta}{H_R \eta}. \quad (5.122)$$

We return to θ_{ab} .

$$\begin{aligned} \theta_{ab} = & \frac{1}{2} (2U_{(a;b)} - \delta_a^c U^d U_b (U_{c;d} + U_{d;c}) \\ & - \delta_b^d U^c U_a (U_{c;d} + U_{d;c}) + U^c U_a U^d U_b (U_{c;d} + U_{d;c})) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(2U_{(a;b)} - \frac{g_{0b}}{a^2} (U_{a;0} + U_{0;a}) \right. \\
&\quad \left. - \frac{g_{0a}}{a^2} (U_{0;b} + U_{b;0}) + \frac{g_{a0}g_{b0}}{a^4} (2U_{0;0}) \right). \quad (5.123)
\end{aligned}$$

This gives a vanishing time-time component $\theta_{00} = 0$, and the time-space component is also zero as we now show

$$\begin{aligned}
\theta_{\alpha 0} &= \frac{1}{2} \left((U_{\alpha;0} + U_{0;\alpha}) - (U_{\alpha;0} + U_{0;\alpha}) - \frac{2g_{0\alpha}}{a^2} U_{0;0} + \frac{2g_{\alpha 0}g_{00}}{a^4} U_{0;0} \right) \\
&= 0. \quad (5.124)
\end{aligned}$$

The space-space component is a zero order quantity as the first order terms in its expression vanish.

$$\theta_{\mu\nu} = \frac{1}{2} \left(2U_{(\mu;\nu)} - \frac{g_{0\nu}}{a^2} (U_{\mu;0} + U_{0;\mu}) - \frac{g_{0\mu}}{a^2} (U_{0;\nu} + U_{\nu;0}) + \frac{2g_{\mu 0}g_{\nu 0}}{a^4} U_{0;0} \right) \quad (5.125)$$

The last term is second order and so we find

$$\theta_{\mu\nu} = \frac{1}{2} \left(U_{\mu,\nu} + U_{\nu,\mu} - 2(a\Gamma_{\mu\nu}^0 + \Gamma_{\mu\nu}^\alpha U_\alpha) - \left(-\frac{2}{a}\Gamma_{\mu 0}^0\right)\frac{g_{0\nu}}{a^2} - \left(-\frac{2}{a}\Gamma_{0\nu}^0\right)\frac{g_{0\mu}}{a^2} \right). \quad (5.126)$$

We use the Christoffel symbols above and the fact that $g_{0\alpha}$ is first order to obtain

$$\begin{aligned}
\theta_{\mu\nu} &= \frac{1}{2} \left(-\frac{4\nabla^2}{H_R} (C_{\mu,\nu} + C_{\nu,\mu}) + 2a'\eta_{\mu\nu} + 2a'\bar{H}_{\mu\nu} + a\bar{H}'_{\mu\nu} + \frac{4\nabla^2}{H_R} (C_{\mu,\nu} + C_{\nu,\mu}) \right) \\
&= a'\eta_{\mu\nu} + a'\bar{H}_{\mu\nu} + \frac{1}{2}a\bar{H}'_{\mu\nu}. \quad (5.127)
\end{aligned}$$

The evaluation of the shear tensor follows, but we first calculate the projection tensor in the three-spaces orthogonal to the fluid flow,

$$h_{ab} = g_{ab} - U_a U_b. \quad (5.128)$$

For the particular time-space component of this metric we have

$$\begin{aligned}
h_{\alpha 0} &= g_{\alpha 0} - U_\alpha U_0 \\
&= g_{\alpha 0} - \frac{g_{\alpha 0}}{a} \\
&= 0, \quad (5.129)
\end{aligned}$$

and the time-time component also vanishes

$$\begin{aligned} h_{00} &= g_{00} - U_0 U_0 \\ &= a^2 - a^2. \end{aligned} \quad (5.130)$$

The shear tensor is defined as

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta h_{ab}, \quad (5.131)$$

where

$$\begin{aligned} \frac{1}{3}\theta h_{\mu\nu} &= \frac{1}{3}\theta g_{\mu\nu} = \frac{a^2\theta}{3}(\eta_{\mu\nu} + \bar{H}_{\mu\nu}) \\ &= a'\eta_{\mu\nu} + a'\bar{H}_{\mu\nu} + \frac{a}{3}\left(\frac{3\nabla^2 A}{2\eta^4} + \frac{\eta}{10}\nabla^2 B\right)\eta_{\mu\nu}, \end{aligned} \quad (5.132)$$

and so we obtain

$$\sigma_{\mu\nu} = \frac{\eta^2}{H_R}\bar{H}'_{\mu\nu} - \frac{2\eta^2}{3H_R}\left(\frac{3\nabla^2 A}{2\eta^4} + \frac{\eta}{10}\nabla^2 B\right)\eta_{\mu\nu} \quad (5.133)$$

$$\begin{aligned} &= \frac{1}{H_R}\left\{-\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}D_{\mu\nu}\right) + \eta\frac{\partial^2}{\partial^2\eta}\left(\frac{1}{\eta}D_{\mu\nu}\right) - 2\left(-\frac{24}{\eta^2} + \nabla^2\right)(C_{\mu,\nu} + C_{\nu,\mu})\right. \\ &\quad \left. - \frac{3A_{,\mu\nu}}{\eta^2} - \frac{2\eta^3}{10}B_{,\mu\nu}\right\} - \frac{2\eta^2}{3H_R}\left(\frac{3\nabla^2 A}{2\eta^4} + \frac{\eta\nabla^2 B}{10}\right)\eta_{\mu\nu}. \end{aligned} \quad (5.134)$$

Now that we have found these physical quantities, we apply the gauge that Sachs and Wolfe use which takes the algebraic form

$$A = 0 \quad C_\mu = 0 \quad D_{\mu\nu} = 0. \quad (5.135)$$

This simplifies the equations considerably and we can rewrite the physical quantities of EB in terms of the SW variable B .

$$\mathcal{Z}_\alpha = -\frac{\nabla^2 B_{,\alpha}\eta}{10} = \frac{2\mathcal{D}_\alpha}{a\eta} \quad (5.136)$$

$$\mathcal{D}_\alpha = -\frac{a\nabla^2 B_{,\alpha}\eta^2}{20} \quad (5.137)$$

$$\theta = 3\frac{a'}{a^2} + \frac{1}{a}\left(-\frac{\eta}{10}\nabla^2 B\right) \quad (5.138)$$

$$\sigma_{\mu\nu} = -\frac{2\eta^3}{10H_R} \left(B_{,\mu\nu} + \frac{\nabla^2 B}{3} \eta_{\mu\nu} \right) \quad (5.139)$$

$$\omega_{\mu\nu} = -\frac{2\nabla^2}{H_R} (C_{\mu,\nu} - C_{\nu,\mu}) = 0 \quad (5.140)$$

$$\dot{Z}_{\perp\alpha} = \frac{1}{a} \frac{\nabla^2 B_{,\alpha}}{10} = -\frac{2\mathcal{D}_\alpha}{a^2\eta^2} \quad (5.141)$$

$$\dot{D}_{\perp\alpha} = \frac{\nabla^2 B_{,\alpha}\eta}{10} = -\frac{2\mathcal{D}_\alpha}{a\eta} \quad (5.142)$$

$$C_\alpha = \frac{4\nabla^2 B_{,\alpha}\eta^2}{5H_R} = -\frac{8\mathcal{D}_\alpha}{\eta^2} \quad (5.143)$$

$$\dot{C}_{\perp\alpha} = 0. \quad (5.144)$$

5.5 Sachs-Wolfe gauge

The Sachs-Wolfe gauge is now expressed in terms of Bardeen's variables. To do this the perturbed metrics are equated as well as the perturbations in the energy momentum tensor. For the SW formalism

$$\delta G_0^0 = -\delta\rho \quad \delta G_0^\mu = 0 \quad \delta G_\beta^\mu = \delta_\beta^\mu \delta p = 0, \quad (5.145)$$

and for Bardeen

$$\begin{aligned} T_0^0 &= -E_0(\tau)[1 + \delta Q] & T_0^\alpha &= -(E_0 + P_0)vQ^\alpha & T_\alpha^0 &= (E_0 + P_0)(v - B)Q_\alpha \\ T_\beta^\alpha &= P_0[1 + \pi_L Q]\delta_\beta^\alpha + P_0\pi_T Q_\beta^\alpha, \end{aligned} \quad (5.146)$$

which leads to the conclusion that

$$-\delta\rho = -E_0(\tau)\delta Q \quad v = 0. \quad (5.147)$$

We proceed with the metric perturbations and list them for the two formalisms.

The Bardeen metric for the different perturbations - scalar, vector and tensor, are:

Scalar

$$g_{00} = -S^2(1 + 2AQ) \quad (5.148)$$

$$g_{0\alpha} = -S^2 B Q_\alpha \quad (5.149)$$

$$g_{\alpha\beta} = S^2 \{ [1 + 2H_L Q]^3 g_{\alpha\beta} + 2H_T Q_{\alpha\beta} \}. \quad (5.150)$$

Vector

$$g_{0\alpha} = -S^2 B^{(1)} Q^{(1)}_\alpha \quad (5.151)$$

$$g_{\alpha\beta} = S^2 [{}^3 g_{\alpha\beta} + 2H^{(1)}_T Q^{(1)}_{\alpha\beta}]; \quad (5.152)$$

where we have the divergenceless part of the vector field proportional to a vector harmonic $Q^{(1)\alpha}$ which is a solution of the vector Helmholtz equation

$$Q^{(1)\alpha|\beta}{}_\beta + k^2 Q^{(1)\alpha} = 0, \quad (5.153)$$

and the tensor quantity

$$Q^{(1)\alpha\beta} = -\frac{1}{2k} (Q^{(1)\alpha|\beta} + Q^{(1)\beta|\alpha}), \quad (5.154)$$

where $Q^{(1)\alpha}{}_\alpha = 0$ and $Q^{(1)}_{\alpha\beta} = Q^{(1)}_{\beta\alpha}$.

Tensor

$$g_{\alpha\beta} = S^2 [{}^3 g_{\alpha\beta} + 2H^{(2)}_T Q^{(2)}_{\alpha\beta}], \quad (5.155)$$

where

$$Q^{(2)}_{\alpha\beta|\beta} = 0, \quad Q^{(2)}_{\alpha\beta} = Q^{(2)}_{\beta\alpha}, \quad Q^{(2)\alpha}{}_\alpha = 0, \quad Q^{(2)\gamma}{}_{\alpha\beta\gamma} = -k^2 Q^{(2)}_{\alpha\beta}. \quad (5.156)$$

The Sachs-Wolfe metric is written as a combination of all three types of perturbations

$$g_{00} = a^2 \quad (5.157)$$

$$g_{0\alpha} = -a^2 2\nabla^2 C_\alpha / \eta^2 \quad (5.158)$$

$$g_{\alpha\beta} = a^2 \left(\frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} D_{\mu\nu} \right) - 2 \left(\frac{8}{\eta^3} - \frac{\nabla^2}{\eta} \right) (C_{\alpha,\beta} + C_{\beta,\alpha}) + \frac{A_{,\alpha\beta}}{\eta^3} + \eta_{\alpha\beta} B - \frac{\eta^2}{10} B_{,\alpha\beta} \right), \quad (5.159)$$

where

$$C^\mu{}_{,\mu} = 0 = C^\mu{}_{|\mu} \quad (5.160)$$

and

$$D_\alpha^\alpha = 0 \quad D_{\alpha\beta}{}^{,\beta} = 0 \quad D_{\alpha\beta} = D_{\beta\alpha}. \quad (5.161)$$

Implicit in the SW metric is the condition that Bardeen's scalar quantity A vanishes, as we see from the g_{00} component of the perturbed metric.

We suggest the association

$$\frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} D_{\mu\nu} \right) = 2H_T^{(2)} Q^{(2)}{}_{\mu\nu}, \quad (5.162)$$

which correspond to gravitational waves. For the vector perturbations we have

$$-2 \left(\frac{8}{\eta^3} - \frac{\nabla^2}{\eta} \right) (C_{\alpha,\beta} + C_{\beta,\alpha}) = 2H_T^{(1)} Q^{(1)}{}_{\alpha\beta} \quad (5.163)$$

and

$$-S^2 (B Q_\alpha + B^{(1)} Q^{(1)}{}_\alpha) = -a^2 \nabla^2 C_\alpha / \eta^2 \quad (5.164)$$

which, if we use the fact that

$$C^\mu{}_{|\mu} = C^\mu{}_{,\mu} = 0 \quad \text{and,} \quad Q_\alpha^{(1)\alpha} = 0, \quad (5.165)$$

then Bardeen's quantity B must be zero. Finally, the same process for scalar perturbations gives

$$\frac{A_{,\alpha\beta}}{\eta^3} + \eta_{\alpha\beta} B - \frac{\eta^2}{10} B_{,\alpha\beta} = {}^3g_{\alpha\beta} 2H_L Q + 2H_T Q_{\alpha\beta}. \quad (5.166)$$

In the last equation we take the SW quantity A to be zero as they demand in their gauge specification and use

$$Q_{\alpha\beta} = \frac{1}{k^2} Q_{|\alpha\beta} + \frac{1}{3} {}^3g_{\alpha\beta} Q \quad (5.167)$$

to find

$$\begin{aligned}\eta_{\alpha\beta}B - \frac{\eta^2}{10}B_{,\alpha\beta} &= {}^3g_{\alpha\beta}2H_LQ + 2H_TQ_{\alpha\beta} \\ &= {}^3g_{\alpha\beta}(2H_L + \frac{2}{3}H_T)Q + \frac{2H_T}{k^2}Q_{|\alpha\beta},\end{aligned}\quad (5.168)$$

which implies that since we have $\eta_{\alpha\beta} = {}^3g_{\alpha\beta}$ in this particular case, we can say

$$B = (2H_L + \frac{2}{3}H_T)Q, \quad (5.169)$$

and

$$\frac{2H_T}{k^2}Q_{|\alpha\beta} = -\frac{\eta^2}{10}B_{,\alpha\beta} \quad (5.170)$$

and so, as B is a constant in time, we find that $2H_L + \frac{2}{3}H_T$ is a constant which leads to the relations

$$H'_L = -\frac{1}{3}H'_T \quad (5.171)$$

and

$$H_T \propto -\frac{k^2\eta^2}{20} \quad \Longrightarrow \quad H_T = -\frac{Jk^2\eta^2}{20}. \quad (5.172)$$

where J is the constant of proportionality. This yields

$$B = -\frac{J}{5}Q \quad (5.173)$$

where we have used $\Phi_H = -\Phi_A$, obtained from the Einstein equations.

This agrees with the equation that Panek found for the SW effect in the case of pressure free matter in a flat universe, where we take the SW quantity B as proportional to the Bardeen quantity Q :

$$\left.\frac{\delta T}{T}\right|_R = \frac{2\epsilon_{bE}}{k^2\tau_E^2}[\tau_R(R^\alpha Q_{,\alpha})_R - \tau_E(R^\alpha Q_{,\alpha})_E - Q_E], \quad (5.174)$$

since the equivalent expression in the SW paper is

$$\frac{\delta T_R}{T_R} = \frac{1}{10}[(B_{,\mu}e^\mu)_R \eta_R - (B_{,\mu}e^\mu)_E \eta_E + B_R - B_E], \quad (5.175)$$

where the vector e^β represents to zero order the spatial direction of the light signal as seen by the observer moving with the fluid. The zero order tangent ${}_0k^a$ is given by

$${}_0k^a = (e^\beta, -1), \quad (5.176)$$

with $e^\beta e_\beta = -1$. Panek also considers the term B_R to be irrelevant as it is independent of direction and so we obtain the SW result by equating the metrics of the two formalisms. If we continue and as SW do, take $C_\mu = 0$, then we have $B^{(1)} = 0$ from equation 164 above. Finally, for the Bardeen quantities we have $A = B = v = 0$, and this gives: The Synchronous gauge: $A = B = 0$, the Comoving proper time gauge $A = v = 0$ which leaves the gauge freedom $L = \beta$, and the Comoving Time Orthogonal gauge $v = B = 0$ which has the gauge freedom $L = \beta$, where β is an arbitrary constant. The Synchronous gauge is a full gauge specification and so there are no gauge ambiguities left after assuming the associations made between the SW and Bardeen formalisms.

5.6 The redshift written in the EB variables

We continue by showing that Panek's equation (25),

$$\frac{T_R}{T_E} = \frac{1}{1+z} = \frac{(k^i U_{bi})_R}{(k^i U_{bi})_E}, \quad (5.177)$$

can be written in terms of the EB formalism. This equation gives the ratio of the temperature of the radiation at emission to that observed at reception and we show that when we write it in the EB variables and substitute for the quantities of Bardeen, we are fully in agreement with the form given by Panek for the redshift. The appendix lists the Christoffel symbols used here as well as the null vector tangent to the geodesic and the fluid four velocity.

$$\frac{(k^i U_{bi})_E}{(k^i U_{bi})_R} = \frac{\int_R^E \frac{d(k^i U_{bi})}{d\lambda} d\lambda + (k^i U_{bi})_R}{(k^i U_{bi})_R} = 1 + z, \quad (5.178)$$

which gives

$$z = \frac{\int_R^E \frac{d}{d\lambda}(U_{bi} k^i) d\lambda}{(k^i U_{bi})_R} = \frac{\int_R^E U_{(a;b)} k^a k^b \frac{d\lambda}{ds} ds}{(k^i U_{bi})_R}, \quad (5.179)$$

where in the numerator of the second equality the fact that a baryonic fluid is being dealt with has not been emphasised and the summation is now taken over the letters 'a' and 'b' which run from 0 to 3, and so we have

$$U_{(a;b)} k^a k^b = \sigma_{ab} k^a k^b + \frac{1}{3} \theta h_{ab} k^a k^b - \dot{U}_a U_b k^a k^b \quad (5.180)$$

and

$$\frac{d\lambda}{ds} = \frac{S^2}{S_R^2}. \quad (5.181)$$

Each of these terms is now evaluated in turn. From Bardeen's equation (3.12) we have

$$\begin{aligned} \sigma_{ab} k^a k^b &= S(\dot{H}_T^{(0)} - kv_b^{(0)}) Q_{\alpha\beta} k^\alpha k^\beta \\ &= \left(\frac{S_R^2}{S^2}\right)^2 S(\dot{H}_T^{(0)} - kv_b^{(0)}) Q_{\alpha\beta} R^\alpha R^\beta \end{aligned} \quad (5.182)$$

$$\sigma_{ab} k^a k^b \frac{d\lambda}{ds} = \frac{S_R^2}{S^2} S(\dot{H}_T^{(0)} - kv_b^{(0)}) Q_{\alpha\beta} R^\alpha R^\beta. \quad (5.183)$$

For the second term of equation 180 we find

$$\theta = U_{;a}^a \quad \text{and} \quad k^a k^b h_a h_b = (U_a k^a)^2 \quad (5.184)$$

which gives

$$\frac{1}{3} \theta k^a k^b h_{ab} = \frac{1}{3} U_{;a}^a (U_a k^a)^2. \quad (5.185)$$

For the volume expansion we have

$$U_{;a}^a = U_{,a}^a + \Gamma_{ba}^a U^b = \frac{3\dot{S}}{S^2} - 3AQ \frac{\dot{S}}{S^2} + \frac{1}{S} v_a Q_{|a}^\alpha + \frac{3\dot{H}_L}{S} Q, \quad (5.186)$$

and the component of the null vector tangent to the geodesic in the direction of the fluid flow is given by

$$U_b k^b = \frac{S_R^2}{S^2} (-1 + M - AQ + (B - v_b) Q_\alpha R^\alpha), \quad (5.187)$$

which yields

$$\begin{aligned} \frac{1}{3} \theta h_{ab} k^a k^b \frac{d\lambda}{ds} &= \frac{1}{3} U_{;a}^a (U_b k^b)^2 \frac{d\lambda}{ds} \\ &= \frac{S_R^2}{S^2} \left[\frac{\dot{S}}{S} + \dot{H}_L Q + AQ \frac{\dot{S}}{S} + v_a \frac{Q_{|a}^\alpha}{3} - 2M \frac{\dot{S}}{S} \right. \\ &\quad \left. - \frac{\dot{S}}{S} (B - v_b) Q_\alpha R^\alpha \right]. \end{aligned} \quad (5.188)$$

We move to the third and last term where we find

$$-\dot{U}_a U_b k^a k^b = -\dot{U}_a k^a U_b k^b, \quad (5.189)$$

$$\begin{aligned}
\dot{U}_a k^a &= U_{a;b} U^b k^a = (U_{a,b} - \Gamma_{ab}^c U_c) U^b k^a \\
&= \underbrace{U_{0,0} U^0 k^0}_{(1)} + \underbrace{U_{\alpha,0} k^\alpha U^0}_{(2)} + \underbrace{U_{0,\alpha} k^0 U^\alpha}_{(3)} + \underbrace{U_{\alpha,\beta} k^\alpha k^\beta}_{(4)} - \\
&\quad \left[\underbrace{\Gamma_{00}^0 U_0 k^0 U^0}_{(5)} + \underbrace{\Gamma_{00}^\alpha U_\alpha U^0 k^0}_{(6)} + \underbrace{\Gamma_{\alpha 0}^0 U_0 k^\alpha U^0}_{(7)} \right. \\
&\quad + \underbrace{\Gamma_{\alpha 0}^0 U_0 U^\alpha k^0}_{(8)} + \underbrace{\Gamma_{\alpha\beta}^0 U_0 U^\beta k^\alpha}_{(9)} + \underbrace{\Gamma_{\beta 0}^\alpha U_\alpha k^\beta U^0}_{(10)} \\
&\quad \left. + \underbrace{\Gamma_{0\beta}^\alpha U_\alpha k^0 U^\beta}_{(11)} + \underbrace{\Gamma_{\beta\gamma}^\alpha U_\alpha k^\beta U^\gamma}_{(12)} \right]. \tag{5.190}
\end{aligned}$$

If we consider terms up to first order and discard those of higher order, we find

$$U_{0,0} U^0 k^0 = {}^0\nu \left(-\frac{\dot{S}}{S} + \frac{\dot{S}}{S} M - \frac{\dot{S}}{S} A Q - \dot{A} Q + \frac{\dot{S}}{S} A Q \right) \tag{5.191}$$

$$U_{\alpha,0} k^\alpha U^0 = -{}^0\nu \frac{\dot{S}}{S} Q_\alpha R^\alpha (v_\alpha - B) - {}^0\nu Q_\alpha R^\alpha (\dot{v}_\alpha - \dot{B}) \tag{5.192}$$

$$(3) = (4) = 0 \tag{5.193}$$

$$\Gamma_{00}^0 U_0 k^0 U^0 = -{}^0\nu \left(\frac{\dot{S}}{S} - \frac{\dot{S}}{S} M + \dot{A} Q \right) \tag{5.194}$$

$$(6) = 0 \tag{5.195}$$

$$\Gamma_{\alpha 0}^0 U_0 k^\alpha U^0 = -{}^0\nu \left(A k + \frac{\dot{S}}{S} B \right) Q_\alpha R^\alpha \tag{5.196}$$

$$\Gamma_{\alpha\beta}^0 U_0 U^\beta k^\alpha = {}^0\nu v_\alpha \frac{\dot{S}}{S} Q_\alpha R^\alpha \tag{5.197}$$

$$\Gamma_{\beta 0}^\alpha U_\alpha k^\beta U^0 = -{}^0\nu \frac{\dot{S}}{S} (v_\alpha - B) Q_\alpha R^\alpha \tag{5.198}$$

$$(8) = (9) = (10) = 0, \tag{5.199}$$

which implies that

$$\dot{U}_a k^a = {}^0\nu \left[-Q_\alpha R^\alpha (\dot{v}_a - \dot{B}) + AkQ_\alpha R^\alpha - Q_\alpha R^\alpha (v_a - B) \frac{\dot{S}}{S} \right], \quad (5.200)$$

and

$$- \dot{U}_a k^a (U_b k^b) \frac{d\lambda}{ds} = - \frac{S_R^2}{S} \left[(\dot{v}_a - \dot{B}) + (v_a - B) \frac{\dot{S}}{S} - Ak \right] Q_\alpha R^\alpha, \quad (5.201)$$

where we have used $U_b k^b$ to zero order only and the fact that ${}^0\nu S = S_R^2/S$. This is now all put together to find the redshift:

$$\begin{aligned} z &= \int_E^R \{(2) + (3) + (4)\} ds \\ &= \int_E^R \frac{S_R^2}{S} \left\{ (\dot{H}_T - kv_b) Q_{\alpha\beta} R^\alpha R^\beta \right. \\ &\quad + \frac{\dot{S}}{S} [1 + AQ - 2M - (B - v_b)Q_\alpha R^\alpha] + (\dot{B} - \dot{v}_b)Q_\alpha R^\alpha \\ &\quad \left. + AkQ_\alpha R^\alpha + \dot{H}_L Q + \frac{v_a}{3} Qk \right\} ds. \end{aligned} \quad (5.202)$$

We have just found a form for the redshift in the EB framework in terms of the Bardeen variables and will now show that it agrees with the standard Panek version of the redshift, in particular we will now show that

$$z = \frac{1}{(k^i U_{bi})_R} \int_E^R d(k^i U_{bi}), \quad (5.203)$$

by differentiating $k^i U_{bi}$ directly. Beginning with

$$\frac{d(k^i U_{bi})}{d\lambda} \frac{d\lambda}{ds} ds = \frac{d}{d\lambda} [{}^0\nu S (-1 + M - AQ + (B - v_b)Q_\alpha R^\alpha)] \frac{S^2}{S_R^2} ds, \quad (5.204)$$

and

$$\frac{d}{d\lambda} = \nu \frac{\partial}{\partial \tau} + P^\alpha \frac{\partial}{\partial x^\alpha}, \quad (5.205)$$

we obtain

$$\begin{aligned}
& \frac{d(k^i U_{bi})}{d\lambda} \frac{d\lambda}{ds} = \\
& \left\{ {}^0\nu(1-M)(-\dot{S} \ {}^0\nu)(-1+M-AQ+(B-v_b)Q_\alpha R^\alpha) \right. \\
& + {}^0\nu S({}^0\nu M' - {}^0\nu \dot{A} Q - AQ_{,\alpha}(-{}^0\nu R^\alpha) + {}^0\nu(\dot{B} - \dot{v}_b)Q^\alpha R_\alpha) \\
& \left. + (B-v_b)Q_{,\beta}^\alpha(-{}^0\nu R^\beta)R_\alpha + (B-v_b)Q^\alpha R_{\alpha,\beta}(-{}^0\nu R^\beta) \right\} \frac{S^2}{S_R^2} ds \\
= & \left\{ -{}^0\nu^2 \dot{S}(-1+2M-AQ+(B-v_b)Q^\alpha R_\alpha) + \right. \\
& {}^0\nu^2 S \left[\dot{A} Q + 2kAQ_\alpha R^\alpha + \frac{k}{3}BQ + \dot{H}_L Q + (\dot{H}_T - kB)Q_{\alpha\beta} R^\alpha R^\beta - \right. \\
& \left. \dot{A} Q - kAQ_\alpha R^\alpha + (\dot{B} - \dot{v}_b)Q_\alpha R^\alpha - (B-v_b) \left(-\frac{1}{k}Q_{|\alpha\beta} R^\alpha R^\beta \right) \right] \left. \right\} \frac{S^2}{S_R^2} ds \\
= & {}^0\nu S \left\{ \frac{\dot{S}}{S} (1 + AQ - 2M - (B-v_b)Q_\alpha R^\alpha) + kAQ_\alpha R^\alpha + \frac{k}{3}BQ \right. \\
& + \dot{H}_L Q + (\dot{B} - \dot{v}_b)Q_\alpha R^\alpha + (\dot{H}_T - kB)Q_{\alpha\beta} R^\alpha R^\beta \\
& \left. + \frac{1}{k}(B-v_b) \left(k^2 Q_{\alpha\beta} - \frac{k^2}{3} {}^3g_{\alpha\beta} Q \right) R^\alpha R^\beta \right\} ds \\
= & \frac{S_R^2}{S} \left\{ \frac{\dot{S}}{S} [1 + AQ - 2M - (B-v_b)Q_\alpha R^\alpha] + (\dot{H}_T - kv_b)Q_{\alpha\beta} R^\alpha R^\beta \right. \\
& \left. + (\dot{B} - \dot{v}_b)Q_\alpha R^\alpha + kAQ_\alpha R^\alpha + \dot{H}_L Q + \frac{v_a Q}{3} k \right\} ds. \tag{5.206}
\end{aligned}$$

And so we see that equation 204 obtained by the direct method is in agreement with the expression found earlier for the redshift (equation 199) where the cosmological quantities of EB were used. This suggests that it is possible to write an equation for the SW effect in terms of the EB variables and this has been accomplished by Peter Dunsby and Heinz Russ (1993).

APPENDIX.

$$\Gamma_{00}^0 = \frac{\dot{S}}{S} + \dot{A} Q \quad (5.207)$$

$$\Gamma_{0\alpha}^0 = -kA Q_\alpha - \frac{\dot{S}}{S} B Q^\beta {}^3g_{\alpha\beta} \quad (5.208)$$

$$\begin{aligned} \Gamma_{\alpha\beta}^0 &= -kBQ_{\alpha\beta} + \frac{k}{3} B {}^3g_{\alpha\beta} Q + \frac{\dot{S}}{S} [(1 + 2H_L Q) {}^3g_{\alpha\beta}] \\ &\quad - 2\frac{\dot{S}}{S} A Q {}^3g_{\alpha\beta} + \dot{H}_L Q {}^3g_{\alpha\beta} + \dot{H}_T Q_{\alpha\beta} \end{aligned} \quad (5.209)$$

$$\Gamma_{00}^\alpha = \frac{\dot{S}}{S} B Q^\alpha - 2\frac{\dot{S}}{S} {}^3g^{\alpha\beta} B Q_\beta - \dot{B} {}^3g^{\alpha\beta} Q_\beta - kA {}^3g^{\alpha\beta} Q_\beta \quad (5.210)$$

$$\Gamma_{0\beta}^\alpha = \frac{\dot{S}}{S} \delta_\beta^\alpha + \delta_\beta^\alpha \dot{H}_L Q + Q_{\beta\gamma} {}^3g^{\alpha\gamma} \dot{H}_T \quad (5.211)$$

$$\begin{aligned} \Gamma_{\beta\gamma}^\alpha &= {}^3T_{\beta\gamma}^\alpha + \frac{\dot{S}}{S} B Q_{\beta\gamma}^\alpha - kH_L \delta_\beta^\alpha Q_\gamma - kH_L Q_\beta \delta_\gamma^\alpha \\ &\quad - kH_L Q_\delta {}^3g^{\alpha\delta} {}^3g_{\beta\gamma} + H_T {}^3g^{\alpha\delta} [Q_{\beta\delta|\gamma} + Q_{\gamma\delta|\beta} - Q_{\beta\gamma|\delta}] \end{aligned} \quad (5.212)$$

$$\begin{aligned} k_\alpha &= -S_R^2 (R_\alpha + BQ_\alpha + (1 + 2H_L Q) R_\alpha + 2H_T Q_{\alpha\beta} R^\beta \\ &\quad - {}^1P^\alpha / {}^0\nu) \end{aligned} \quad (5.213)$$

$$k_0 = -S_R^2 (1 + 2AQ - M - BQ_\alpha R^\alpha) \quad (5.214)$$

$${}^0\nu = \frac{S_R^2}{S^2} \quad (5.215)$$

$$U_\alpha^0 = \frac{1}{S} (1 - AQ) \quad (5.216)$$

$$U_{\alpha 0} = -S(1 + AQ) \quad (5.217)$$

$$U_\alpha^\alpha = \frac{1}{S} v_\alpha Q^\alpha \quad (5.218)$$

$$U_{\alpha\alpha} = S(v_\alpha Q^\beta {}^3g_{\alpha\beta} - BQ_\alpha) = S Q_\alpha (v_\alpha - B). \quad (5.219)$$

Chapter 6

Conclusion

The first task of this work was to make sure that all the equations in Panek's paper were correct and extend his simple example to more general cases. Here we found that the obscure definition for the background temperature of the reception event, T_{R0} , in his expression for the temperature perturbation as seen by the observer is somewhat arbitrary, and so it is changed for a simpler form which allows one to give a simple expression for the gauge invariant difference between the temperatures measured in two different directions in the sky. We found that we were able to give an expression for this quantity using Panek's definition, $\frac{\delta T}{T}|_R$. It is this quantity, $\Delta_{AB}T_R$, defined in equation (4.25), that would be a comparison to the COBE data which are the differences in temperature taken from different directions in the sky. The new work presented here also includes extensions to the form given for the temperature variation found by Sachs and Wolfe for a flat universe with dust, to the case of dust and radiation mixing with N other types of matter in a non-flat universe with non-adiabatic perturbations to the energy density. This will be used further in the study of low-density ($K = -1$) universes with the goal of finding a relation between $\frac{\delta T}{T}|_R$ and the density parameter Ω . For this we need to have the equation that expresses the temperature anisotropy in the CMBR explicitly in terms of the density parameter Ω , the exact positioning of the last scattering surface, and the energy content of the universe made up of the two baryon and photon fluids as well as two dark matter components.

In the second part of the thesis it was shown that the covariant and gauge invariant equations in the formalism of Ellis and Bruni do indeed agree with Sachs and Wolfe

for the case of a $K = 0$ universe filled with dust. All of the gauge invariant, covariant quantities of the Ellis-Bruni formalism and their propagation equations were found to be satisfied and written in terms of the Sachs-Wolfe variable, B , which is the only metric perturbation left after they completely specify their gauge.

The metrics and energy momentum tensors of Bardeen, and Sachs and Wolfe are equated, and it is found that Panek's agreement with the form found by Sachs and Wolfe for their temperature variation across the sky, is confirmed. The gauge Sachs and Wolfe use is also found in terms of the Bardeen variables and, on the assumption that the two theories are in agreement, the Sachs-Wolfe gauge is completely specified.

An obvious task is now to write the perturbed temperature variation in the Ellis-Bruni formulation. We perform the first step in this process by showing that the perturbed redshift in the form given by Panek is indeed in agreement with that given by EB and so the entire subject can now be written in their more geometric variables which has recently been completed by Heinz Russ, Michel Soffel, Chongming Xu and Peter Dunsby, and accepted for publication in *Phys. Rev. D*, to appear in 1993 under the title: A Covariant and Gauge-Invariant Formulation of the Sach Wolfe Effect.

The work in the first part of this thesis will be appearing in *Phys. Rev. D*, (1993) under the authorship of G.F.R. Ellis, C. Xu, W.R. Stoeger and M.P. Katz.

Bibliography

- [1] BARDEEN JM 1980 Gauge invariant cosmological perturbations *Phys. Rev. D* **22** 1882
- [2] BRUNI M, DUNSBY PKS AND ELLIS GFR 1991 Cosmological perturbations and the meaning of gauge invariant variables *Astrophys. J.* **395** 34
- [3] DUNSBY PKS, BRUNI M AND ELLIS GFR 1992 Covariant perturbations in a multifluid cosmological medium *Astrophys. J.* **395** 54
- [4] ELLIS GFR AND BRUNI M 1989 A better choice of variables for density inhomogeneities *Phys. Rev. D* **40** 1819
- [5] HARRISON ER 1967 Normal modes of vibration of the universe *Rev. Mod. Phys.* **39** 862
- [6] HAWKING SW 1966 Perturbations of an expanding universe *Astrophys. J.* **145** 544
- [7] HOLTZMAN JA 1989 Microwave background anisotropies and large-scale structure in universes with cold dark matter, baryons, radiation, and massive and massless neutrinos *Astrophys. J. Supp.* **71** 1
- [8] KODAMA H AND SASAKI M 1984 Cosmological perturbation theory *Prog. Theor. Phys. Suppl.* **78** 1
- [9] LIFSCHITZ EM 1946 On the gravitational stability of the expanding universe *J. Phys.* **12** 116
- [10] LIFSCHITZ EM AND KHALATNIKOV IM 1963 Investigations in relativistic cosmology *Adv. Phys.* **12** 185

- [11] NARIAI H 1969 The Lagrangian approach to the gravitational instability in an expanding universe *Prog. Theor. Phys.* **41** 686
- [12] OLSON DW 1976 Density perturbations in cosmological models *Phys. Rev. D* **14** 327
- [13] PANEK M, 1986 Large-scale microwave background fluctuations: Gauge invariant formalism. *Phys. Rev. D* **34** 416
- [14] PEEBLES PJE AND YU JT 1970 Primeval adiabatic perturbations in an expanding universe *Ap. J.* **162** 815
- [15] SACHS RK AND WOLFE AM 1967 Perturbations of a cosmological model and variations of the microwave background *Ap. J* **147** 73
- [16] SAKAI K 1969 Density perturbations and preferential coordinate systems in an expanding universe *Prog. Theor. Phys.* **41** 1461
- [17] STEWART JM AND WALKER M 1974 Perturbations of spacetimes in general relativity *Proc. R. Soc. A* **341** 49
- [18] PEEBLES PJE 1980 THE LARGE-SCALE STRUCTURE OF THE UNIVERSE (Princeton: Princeton University Press).

Heydenrych & Van der Nest

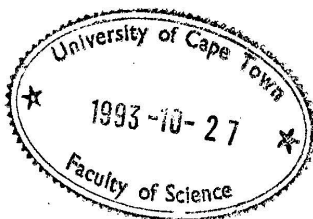
REHABILITATION TECHNOLOGISTS

18 Van der Hoff Avenue ☒ 20813, Noordbrug 2522 ☎ (0148) 297 7455 Fax (0148) 297 7458

VNSLAM001

Scido1

ADMISSIONS SECRETARY
CENTRAL ADMISSIONS OFFICE
UNIVERSITY OF CAPE TOWN
PRIVATE BAG
RONDEBOSCH
7700 CAPE



1993-09-30

Dear Sir

ADDITIONAL INFORMATION FOR PhD APPLICATION

The area of specialization for research will be: Rehabilitation of Gold and Chrysotile tailings.

Name of proposed supervisor: Prof M Fey (Geochemistry)

Please bring the matter under the attention of Prof M Fey or Prof A le Roex should you require any additional information regarding the detail of the study since especially Prof M Fey has all the planned detail of the study at his disposal.

Yours faithfully

LAMBERT J. VAN DER NEST

Heydenrych & Van der Nest (Pty) Ltd

Reg No 93/04578/07

Directors: G V Heydenrych, L J Van der Nest



POTCHEFSTROOMSE UNIVERSITEIT
VIR CHRISTELIKE HOËR ONDERWYS

Telefoon (0148) 992635/992636 Telegramme: PUK Teliks 34-6019
2520 Potchefstroom Republiek van SUID-AFRIKA AKADEMIESE ADMINISTRASIE

VERKLARING

Hiermee word verklaar dat LAMBERTUS JOHANNES VAN DER NEST

STUDENTENOMMER 8605505

Op 18/02/92 aan al die vereistes vir die graad

MAGISTER SCIENTIAE

voldoen het en dat genoemde graad amptelik op 24 APRIL 1992

aan hom / haar toegeken is / is toegeken.

REGISTRATEUR





POTCHEFSTROOMSE UNIVERSITEIT
VIR CHRISTELIKE HOËR ONDERWYS

Telefoon (0148) 992635/992636 Telegramme: PUK Teleks 34-6019
2520 Potchefstroom Republiek van SUID-AFRIKA AKADEMIESE ADMINISTRASIE

VNS LAMOOJ
Sci DO1

VERKLARING

Hiermee word verklaar dat LAMBERTUS JOHANNES VAN DER NEST

Studentenommer 8605505

Op 5/12/88 aan al die vereistes vir die graad

BACCALAUREUS SCIENTIAE

voldoen het en dat genoemde graad amptelik op 18 APRIL 1989

hom toegeken is / ~~saakxxx~~

REGISTRATEUR



POTCHEFSTROOMSE UNIVERSITEIT
VIR CHRISTELIKE HOËR ONDERWYS

Telefoon (0148) 992635/992636 Telegramme: PUK Teleks 34-6019
2520 Potchefstroom Republiek van SUID-AFRIKA AKADEMIESE ADMINISTRASIE

VERKLARING

Hiermee word verklaar dat LAMBERTUS JOHANNES VAN DER NEST

STUDENTENOMMER 8605505

Op 31/01/90 aan al die vereistes vir die graad

HONNS. SCIENTIAE

voldoen het en dat genoemde graad amptelik op 20 MAART 1990

aan hom / ~~saak~~ toegeken is / ~~saakxxx~~

REGISTRATEUR

