Pedagogic evaluation, computational performance and orientations to mathematics: a study of the constitution of Grade 10 mathematics in two secondary schools

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Declaration

This work has not been previously submitted in whole, or in part, for the award of any degree. It is my own work. Each significant contribution to, and quotation in, this dissertation from the work, or works, of other people has been attributed, and has been cited and referenced.

Signature: Signed by candidate Date: 19 February 2018

Abstract

This study takes as its starting point Bernstein's proposition that evaluation is central to pedagogy. Specifically, along with many researchers who draw on his work, Bernstein claims that explicit evaluative criteria are critical to the academic success of learners from working-class families and low economic status communities. The research problem stems from a hypothesis, derived from the literature, that social class differences in learner performances in school mathematics suggest differences in the functioning of pedagogic evaluation, and therefore differences in what is constituted as mathematics, and how, in pedagogic situations differentiated by social class (e.g. Dowling). The contention of this study is that insufficient fine-grained analyses have been undertaken to surface the computational specificity of what it is that constitutes evaluative criteria in mathematics education studies of pedagogy. The study examines the functioning of pedagogic evaluation in what comes to be constituted as mathematics by teachers and their learners, and in the specialisations of mathematical thought in pedagogic situations.

The study set out to investigate the functioning of pedagogic evaluation in two schools differentiated with respect to the social class membership of learners. Two Grade 10 teachers and their learners in each school served as research participants. Methodological resources for describing the functioning of pedagogic evaluation in terms of the computational activity of teachers and learners derive from the work of Davis, which draws on a *computational theory of mind* (e.g., Chomsky; Gallistel & King; Spelke). Bernstein's theory of the *pedagogic device*, with its focus on who gets what knowledge and how, serves as a general descriptive frame structuring the study.

The analysis reveals the following: (1) the commonly used descriptions of evaluative criteria as explicit/implicit are analytically blunt and consequently mask the complexity of criteria operative in pedagogic contexts; (2) differences as well as strong similarities in the functioning of evaluation and, therefore, differences and similarities in what is constituted as mathematics are evident in pedagogic situations differentiated with respect to social class; (3) an orientation to mathematics that constitutes mathematics as computations on the typographical elements of mathematical expressions is common to pedagogic situations involving learners from both upper-middle-class/elite families and working-class families; and (4) greater variation and inter-connectedness in computational resources is realised in pedagogic situations involving learners from upper-middle-class/elite families than in those involving learners from working-class families, where computational resources are relatively restricted and weakly connected. The differences between the two types of situations appear to be enabling of greater flexibility in mathematical thought and action for upper-middle-class/elite learners, on the one hand, and restricting for working-class learners, on the other.

The contribution of the thesis is four-fold. The study: (1) provides a methodology for exploring the complexity of pedagogic evaluation by describing the computations performed by learners and teachers in

mathematical terms, thus contributing to Bernstein's account of pedagogic discourse as it applies to the teaching and learning of mathematics; (2) contributes to our understanding of the structuring effect of evaluation on learners' mathematical thought; (3) contributes to the methodological resources developed by Davis for describing the constitution of mathematics in pedagogic situations; and (4) extends analyses of the constitution of mathematics in pedagogic situations to those populated by learners from upper middle-class/elite families in the South African context, albeit in a limited way.

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Papers generated in the course of the production of the thesis

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Contents

Abbreviations

C2005 Curriculum 2005

CAPS Curriculum and Assessment Policy Statement

CED Cape Education Department

CDM Character distributing matrix

DET Department of Education and Training

ETP Empirico-theoretical propositions

FET Further Education and Training

HOD House of Delegates

HOR House of Representatives

ICM Integrated Causal Model

ID Instructional discourse

NCS National Curriculum Statement

NSC National Senior Certificate

OBE Outcomes-based Education

ORF Official recontextualising field

PRF Pedagogic recontextualising field

RD Regulative discourse

RH Research hypothesis

SMP School Mathematics Projects

SSSM Standard Social Science Model

TIMSS Third International Mathematics and Science Survey

TP Theoretical propositions

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Reading conventions

1. I use the term "Mathematics" for knowledge generated in the field of production and the term "school mathematics" as the knowledge used by teachers and learners in pedagogic contexts.

2. Following the convention used in semiotics, a pair of forward slashes is used to distinguish between the signifier as opposed to the signified, I use a pair of forward slashes to indicate that the mathematical expression(s) included in the slashes are treated as character strings (sequences of alphanumeric characters) as opposed to numbers or mathematical operations e.g. $/-1/$ indicates character string where -1 indicates the number.

Chapter 1
Framing the study

1.1 Introduction

This study explores the Bernsteinian proposition that "the key to pedagogic practice is continuous evaluation" (Bernstein, 1990). The proposition can be understood in two different ways. The first is that pedagogy is fundamentally evaluative because evaluation is central to marking out criteria for the recognition and realisation of legitimate knowledge statements. Secondly, evaluation reveals the "inside" of pedagogy by illuminating who gets what knowledge. More specifically, my interest lies in what evaluation reveals about the production of Grade 10 school mathematics by teachers and learners in two schools differentiated with respect to the social class membership of their learner populations and the implications for the specialisation of learners' mathematical thought. Following Davis (2013a, 2013b), my investigation of the functioning of evaluation in pedagogic situations adopts an approach informed by a computational theory of mind (see Chomsky, 2006, 2007; Fodor, 1998, 2010; Gallistel & King, 2010; Pinker, 1995, 1997, 2007), which posits that thought is computational. Within a computational theory of mind, computations are necessary for generating and processing representations of the world constructed from sensory data (Gallistel & King, 2010, p. x). The methodological implications of the computational nature of thought are elaborated briefly in section 1.5 and in more detail in Chapter 3.

My interest in this study was sparked by research initiated by Davis (2010a, 2010b, 2010c), employing a methodology informed by a computational theory of mind (computational approach) to study what was realised as mathematics in schools populated by learners from working-class[1] families (see Arendse, 2011, 2013; Basbozkurt, 2010a, 2010b; Chitsike, 2011a, 2011b; Davis, 2010a, 2010b, 2010c, 2011a, 2011b, 2011c, 2012, 2013a, 2013b, 2014; Davis & Gripper, 2012a, 2012b; Gripper, 2011a, 2011b; Jaffer, 2009, 2010a, 2010b, 2011a, 2011b, 2012; Jaffer & Davis, 2012). A common finding across the aforementioned studies was that mathematics content emerging in schools populated by learners from working-class families differed from the content typically associated with topic names from the point of view of the field of mathematics. I was curious about whether a computational approach would reveal differences in the mathematics realised in pedagogic contexts differentiated with respect to social class. I was also interested in whether or not there would be differences in how learners thought about mathematics, and what those differences might be, given that schools populated by learners from middle-class families in South Africa continue to produce better outcomes in national and international mathematics assessments than schools

[1] In South Africa unemployment is a feature of structural inequality, thus the term "working-class" used in this study at times refers to learners who come from homes where parents or care-givers are unemployed and without any regular source of income. The social class categories "working-class" and "middle-class" used in the study are discussed in Chapter 4.

populated by learners from working-class families (Bloch, 2009; Fleisch, 2008; Reddy, van der Berg, Lebani, & Berkowitz, 2006; Spaull, 2013; van der Berg, 2007).

Spaull's (2013, p. 5) analysis of the 2011 TIMSS mathematics data on South African schools illustrates that Grade 9 learners in schools populated by learners from working-class families are lagging at least three years behind that of their Grade 9 counterparts in schools populated by learners from middle-class/elite families. Similar differences in performance are evident in the South African national matriculation examination results. For example, 29% of learners in quintile one[2] schools who wrote National Senior Certificate (NSC) Mathematics scored 40% or more compared to 64% of learners in quintile five schools who wrote NSC Mathematics in 2013 (Equal Education, 2015). This difference in achievement in school mathematics takes a particular form given the history of formal education in South Africa. In a context where social class and 'race'[3] are intertwined, national and international assessments continue to reflect both a social class and 'racial' achievement gap in school mathematics (Bloch, 2009; Fleisch, 2008; Reddy et al., 2006; van der Berg, 2007). Such disparity in achievement is strongly correlated with differences in the social class membership of learners and potentially points to differences in pedagogy, and hence to differences in the functioning of evaluation. A possible implication is that what is constituted as mathematics as well as how it is constituted might well vary along social class lines. Whether such differences in what is realised as mathematics exist is a matter for empirical investigation.

In the next section, Bernstein's notion of evaluation is discussed since evaluation constitutes an integral component of the study. In particular, I focus on Bernstein's notions of recognition and realisation rules as a component of evaluation. Then I briefly outline the methodological orientation of the study and locate the study by describing the schooling context and the curriculum context. Finally, I present an overview of the thesis.

1.2 Bernstein's notion of evaluation

Pedagogy entails a relationship between two or more notional pedagogic subjects, the teacher and the learner, with the reproduction of knowledge being the knot that ties the two together, referred to as a *didactic relation* by Chevallard (1989, p.4). Teachers and learners relate to knowledge in different ways. The teacher reproduces knowledge in order to communicate what s/he wants learners to produce and how they should produce what counts as legitimate knowledge in a pedagogic situation. The learner always asks herself what does the teacher expect of her and how she should achieve what the teacher wants from her. The relationship between the teacher and learner is therefore essentially evaluative in that what they produce is structured by evaluation. As such, evaluation is central to the pedagogic reproduction of knowledge.

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[2] Schools are ranked by education departments in terms of poverty levels, with quintile 1 being most poor and quintile 5 least poor. The assignment of poverty indices to schools has been found to be problematic (Hall & Giese, 2009; Kanjee, 2009; Wildeman, 2008). Schools serving poor learners may be placed in a higher quintile because the school is located in an area where the surrounding community is wealthier than the learner population of the school (Hall & Giese, 2009).
[3] I use the term 'race' in quotation marks because 'race' has little biological validity (Yudell, Roberts, DeSalle, & Tishkoff, 2016).

For Bernstein, the centrality of evaluation is emphasised in his discussion of the *pedagogic device*[4], which serves as an analytic and descriptive resource for describing the transformation of knowledge into pedagogic communication (Bernstein, 2000, pp. 40-42).

> We can see that key to pedagogic practice is continuous evaluation. […] This is what the device is about. Evaluation condenses the meaning of the whole device. We are now in a position where we can derive the whole purpose of the device. The purpose of the device is to provide a symbolic ruler of consciousness (Bernstein, 2000, p. 50).

The pedagogic device is Bernstein's attempt to relate social structure to individual consciousness. The pedagogic device entails three hierarchically-related 'rules' – the distributive, recontextualising and evaluative rules (Bernstein, 1996, p. 43), which together structure the production and reproduction of knowledge. The evaluative rule is derived from the recontextualising rule, which is in turn derived from the distributive rule. The distributive rule "mark[s] and distribute[s] who may transmit what to whom" (Bernstein 1990: 158). The distributive rule, which regulates the distribution of "forms of knowledge, forms of consciousness and forms of practice to social groups" (ibid., p. 42), plays a key role in the reproduction of the social division of labour by distributing access to social goods, contributing to the reproduction of patterns of social relations. The recontextualising rule, which governs the selection of knowledge from the field of production and other discourses such as theories of learning and teaching, for the formation of pedagogic discourse (e.g., school mathematics), creates specialised pedagogic subjects (Bernstein, 1996, p. 46)[5]. Bernstein defines pedagogic discourse as an instructional discourse (knowledge and skills) embedded in a regulative discourse (moral discourse), where the latter is dominant (Bernstein, 1990).

> We shall define pedagogic discourse as the rule which embeds a discourse of competence (skills of various kinds) into a discourse of social order in such a way that the latter always dominates the former. We shall call the discourse transmitting specialized competences and their relation to each other *instructional* discourse, and the discourse creating specialized order, relation, and identity *regulative* discourse (Bernstein, 1990, p. 183; italics in orginal).

However, it is the evaluative rule which is key in the pedagogic reproduction of knowledge. Bernstein argues that pedagogic practice is characterised by the ever-present evaluative activity where evaluation distinguishes legitimate from non-legitimate knowledge statements for learners (ibid., p. 50). Note that legitimate knowledge is not necessarily knowledge that is accepted as correct in general. Legitimate knowledge is that sanctioned in a specific pedagogic situation. For example, when a teacher, accepts 2 as the only solution to the equation $x^2 = 4$ where $x \in \mathbb{R}$, then incorrect knowledge is accepted as legitimate. It is also possible that correct mathematical knowledge could be considered non-legitimate in a pedagogic situation. The term evaluation does not refer only to assessment of learners' knowledge claims, but includes

[4] Bernstein's concept of the pedagogic device is loosely modelled on Chomsky's Language Acquisition Device (LAD) (Bernstein, 1996, p. 40). However, while Bernstein describes Chomsky's LAD as social, Chomsky argues that the LAD is biological.

[5] Differences between Bernstein's and my use of the term *recontextualise* are discussed in Chapter 3.

all forms of pedagogic communication such as teacher and learner talk or written productions, textbooks and other curriculum resources, tests and examinations (Davis, 2005b).

Communicative texts produced by teachers and learners make explicit knowledge accepted as legitimate in a pedagogic situation and entail criteria marking out legitimate knowledge statements from non-legitimate statements. The centrality of evaluation is underscored by Bernstein when he declares that "evaluation condenses the meaning of the device" (ibid, p. 50). Bernstein's claim implies that both the recontextualising rule as well as the distributive rule are entailed in the evaluative rule. Researchers, however, sometimes construct their accounts of pedagogic situations in the reverse order by sometimes examining how evaluation functions in pedagogic situations to reveal the content(s) of the recontextualising rules from which the distributive rules are derived. Evaluation is the point at which the content of the instructional discourse (knowledge and skills) and the regulative discourse (moral discourse) are made visible.

In Bernstein's terms, this study investigates the functioning of evaluation with respect to the instructional discourse. Specifically, my interest lies in the recontextualising of mathematics, the distribution of different of forms of knowledge to learners and the specialisation of learners' consciousness by exploring the functioning of evaluation in pedagogic contexts.

One of the reported difficulties with Bernstein's theory of pedagogic discourse is its limited adequacy to describe the content realised in pedagogic contexts. Hoadley (2005, p. 60) argues that Bernstein's theory is concerned with pedagogic discourse as a relay rather than with what is relayed and so augments Bernstein's theory with Dowling's (1998) notions of *domains of practice* to analyse Grade 3 Numeracy and Literacy content. Hugo, Bertram, Green, and Naidoo (2008, p. 33) describe how the application of Bernstein's theory of pedagogic discourse, specifically his concepts of *classification* and *framing*, produced the same analytic description of lessons "even though we intuitively felt that the lessons were of noticeably differing qualities". They turn to Bloom's Revised Taxonomy to supplement Bernstein's theory in their study of the teaching of Grade 10 History. Davis (2005b) points to an operational insufficiency with respect to Bernstein's notion of evaluation.

> While the proposition asserting that the whole of the device is condensed in the evaluative rule does tell us that evaluative judgement must be structured by the contents of the recontextualising rule and the distributive rule, the former in the service of the latter, it reveals little of how such structuring actually works; that is something left in each case as a matter for empirical investigation (Davis, 2005b, pp. 81-82).

Davis (2005b) develops a theoretical account of the inner workings of evaluation by recruiting Hegel's theory of judgement through his investigation of a problem-centred pedagogy. His focus, however, emphasises the regulative discourse, that is the moral ordering of pedagogic discourse. The present study is concerned with the functioning of evaluation in relation to the instructional discourse. In the ensuing discussion, I reflect on another aspect of Bernstein's notion of evaluation, namely recognition and realisation rules.

Recognition and *realisation* rules in Bernstein's theory of pedagogic discourse are components of evaluation located at the level of pedagogic practice intended to regulate the production of privileged texts (Bernstein, 1990, p. 24).

> Recognition rules create the means of distinguishing between and so *recognizing* the speciality that constitutes a context, and realization rules regulate the creation and production of specialized relationships internal to that context (Bernstein, 1990, p. 12, italics in original).
>
> The recognition rules regulate what goes with what: what meanings may be put together, what referential relations are privileged/privileging (ibid., p.24).
>
> Now although realization rules establish what counts as a legitimate text, these rules presuppose and are limited by recognition rules … (ibid., p.30).

For Bernstein, the transmission and acquisition of recognition and realisation rules are required to ensure that learners produce what is expected of them (ibid., p. 58). When learners produce knowledge statements that are considered illegitimate in pedagogic contexts, teachers easily recognise that learners have misread the rules. But when learners produce legitimate knowledge statements, they may or may not be using the recognition and realisation rules used by the teacher. So, as Davis (2013a, p. 35) points out, Bernstein's notion that recognition and realisation rules are 'transmitted' from teacher to learner is problematic. Davis (ibid) argues that Bernstein's treatment of recognition and realisation rules is problematic for two reasons: one located in the nature of the knowledge domain, mathematics and the other in the nature of language and communication. I discuss this argument below briefly and return in Chapter 3 with a more detailed explanation.

Since thought is computational, we can describe recognition and realisation rules used in the production of mathematics in terms of the computations performed by individuals. According to Davis (2013a), computations used in school mathematics largely comprise compositions of operations with operations being functional in nature. As is the case with functions, where it is possible to replace a function rule with a "different, equivalent rule, or an equivalent system of rules" to produce the same output from a given input (Davis, 2013a, p. 35), we can substitute operations with different operations to produce the same output from a given input. For example, in solving the equation $2x + 1 = 5$ both a teacher and a learner produced the following solution.

Line 1: $2x + 1 = 5$

Line 2: $2x = 5 - 1$

Line 3: $2x = 4$

Line 4: $x = 2$

The teacher's explanation of Line 2 of the solution was that she used additive inverses, i.e., "add -1 on both sides of the equation". The learner's explanation of the second line of the solution was "take 1 over to the other side and change the sign". In one sense, the teacher and the learner produce the same outcome but use very different operations, so indicating the use of different recognition and realisation rules. It follows, then, that recognition and realisation rules used by pedagogic agents need not be identical, even when apparently producing the same outcome (Davis, 2013a). So, Bernstein's claim that recognition and realisation rules are 'transmitted' by the teacher and 'acquired' by the learner does not seem to hold as necessary because substitution of computations by replacing operations is always mathematically possible (ibid). This point is elaborated in Chapter 3.

A further complication, noted by Davis (2013a), with respect to the potential for recognition and realisation rules to differ between teacher and learners and amongst learners, relates to the nature of pedagogic situations as communicative contexts. Since communication in pedagogic situations transpires through the medium of ordinary language, the recognition and realisation rules are communicated through ordinary language. One of the characteristics of ordinary language is the lack of a reference function (Chomsky, 2007; Strawson, 1950). Strawson argues that, language does not refer, it is people who use language to refer.

> "Mentioning", or "referring", is not something an expression does; it is something that some one can use an expression to do. Mentioning, or referring to, something is a characteristic of a use of an expression, just as "being about" something, and truth-or-falsity, are characteristics of a use of a sentence (Strawson, 1950, p. 326).

The problem with linguistic reference is at the core of Chomsky's concern with semantics. Reference is a cognitive act and not a denotative relation between language and the world. Pedagogic agents, even when producing the same outcome, may use the same signifiers to refer in completely different ways. It is only when a learner makes an error that a teacher may become aware that the learner uses recognition and realisation rules which differ from theirs. The lack of a reference function in language means that individuals must always check whether or not they are, in fact, referring to the same thing. In pedagogic situations, as illustrated in the example cited above, it is possible for learners and teachers to produce the same outcome expressively using different recognition and realisation rules.

In Bernstein's theory of pedagogic discourse, recognition and realisation rules are related to the concepts of *classification* and *framing* in that "classificatory principles establish recognition rules, and we shall see that framing principles establish realization rules" (Bernstein, 1990, p.30).

1.3 Classification and framing

Bernstein's notions of classification and framing, used to describe pedagogic modalities and recognition and realisation rules, relates the micro-level of the classroom to the macro-level of society. Classification refers to

> relations between categories, these relations being given by their degree of insulation from each other. Thus strong insulation created categories, clearly bounded, with a space for the development of a

> specialised identity, whereas the weaker the insulation, the less specialised the category (Bernstein, 2000, p. 99).

The concept of classification can be used to describe boundaries between agents, spaces and discourses. In this discussion, I am particularly focusing on the notion of classification used by scholars employing Bernstein's notion of classification to describe the distinction between mathematics and 'everyday' knowledge.

Mathematics education research employing Bernstein's notion of classification claims that working-class learners are predisposed to misrecognising the classificatory principle of the schooling context and that middle-class learners are predisposed to recognising the specialised nature of school knowledge. They argue that working-class learners fail to recognise the specificity of school mathematics and tend to use knowledge of 'everyday' or non-mathematical contexts in school mathematics contexts whereas middle-class learners are less likely to draw inappropriately on their 'everyday' knowledge in response to school mathematics tasks (Cooper, 1998; Davis, 2005a; Hoadley, 2005; Jorgensen, Gates, & Roper, 2014; Lerman, 2009; Lubienski, 2000, 2004; Straehler-Pohl, 2010; Zevenbergen & Lerman, 2001). Further, Bernsteinian studies claim that working-class learners are denied access to specialised mathematical knowledge because the content presented to learners blurs the boundary between mathematics and the 'everyday' (Atweh, Bleicher, & Cooper, 1998; Gellert & Straehler-Pohl, 2011; Hoadley, 2005, 2007). So, research literature recruiting Bernstein's theory often employs a distinction between academic knowledge and 'everyday' knowledge to describe the recognition and realisation rules employed in the production of mathematics in pedagogic situations.

Furthermore, a substantial body of literature, employing Bernstein's concepts, argues that the inclusion of the 'everyday' is implicated in the differential achievement in schooling between learners from middle-class families and learners from working-class families (see Cooper & Dunne, 1998, 2000; Davis, 1995b; Dowling, 1993, 1995, 1998; Ensor, 1997; Gellert & Jablonka, 2009; Gellert & Straehler-Pohl, 2011; Hoadley, 2005, 2007; Jablonka, 2009; Le Roux, 2014; Lerman, 2009, 2014; Lerman & Tsatsaroni, 1998; Lerman & Zevenbergen, 2004; Lubienski, 2000, 2004; Morgan, Evans, & Tsatsaroni, 2002; Muller & Taylor, 1995, 2000; O'Halloran, 1996, 2004; Straehler-Pohl, 2010; Taylor, 1999, 2000; Zevenbergen, 2000, 2001b; Zevenbergen & Lerman, 2001). Lerman (2014) draws attention to the use of the academic-everyday distinction in sociological research:

> Key sociological concepts of those researching who succeeds and who fails in school mathematics, and why, include the nature of knowledge discourses, the distinction between the everyday and the "esoteric", and its effect on learners (Dowling 1998; Cooper and Dunne 2000) (Lerman, 2014, p. 556).

The use of the distinction is also employed by research drawing on Systemic Functional Linguistics to explain the social class achievement gap in mathematics (Atweh et al., 1998; O'Halloran, 1996; Veel, 2006; Zevenbergen, 2000, 2001a, 2001b). However, references to 'everyday' objects do not necessarily mean that 'everyday' knowledge is recruited. Since mathematics is agnostic about its objects of reference (Whitehead, 1911), when teachers and learners refer to ‘everyday’ objects, such as balls or cakes, for example, it does not necessarily mean that they are using their everyday knowledge of balls and cakes. The so-called everyday objects referred to by teachers and learners are, in fact, objects involved in the computations performed by teachers and learners (from now on referred to as *computational objects*). This point is elaborated in Chapters 2 and 3.

Classification operates in tandem with another Bernsteinian concept, *framing*, which refers to who has control over pedagogic discourse.

> Framing referred to the locus of control over selection, sequencing, pacing and criteria of knowledge to be acquired. Thus with strong framing (+F) control lies with the teacher, whereas with weak framing (-F) control lies apparently with the student (Bernstein, 2000, p. 99).

Here a key proposition emerging from studies employing Bernstein’s work locates the differential achievement between learners from middle-class families and learners from working-class families in the strength of framing over the evaluative criteria. Research studies recruiting Bernstein claim that pedagogies that are successful in making available specialised knowledge to learners exhibit strong framing over evaluative criteria - criteria that mark for learners legitimate from non-legitimate knowledge statements (see Hoadley, 2005, 2007; Morais, 2002; Morais & Neves, 2001; Morais, Neves, & Pires, 2004; Rose, 2004; Straehler-Pohl, 2010). According to these scholars, strong framing of evaluative criteria is indexed by explicit evaluative criteria whereas weak framing over the evaluative criteria is an indicator of implicit evaluative criteria.

Framing, according to Bernstein, is used to refer to the extent of control over the criteria but does not focus on the nature of the content. It is quite possible to have strong framing over the evaluative criteria yet have mathematically inconsistent criteria. For example, Jaffer (2010b) shows that a teacher’s procedure for converting recurring decimals to common fraction form is very explicit. So, using Bernstein, the teacher’s procedure should be coded as strongly framed. However, her procedure contained mathematical errors but produced the correct outcome. See also Hugo, Bertram, Green, and Naidoo (2008) for a discussion regarding the problem with framing with respect to their analysis of history lessons. In Chapter 2, I elaborate on the problems with the use of framing as a methodological resource with respect to mathematics.

A key concern of my study is to open up the difficulties the Bernsteinian approach gives rise to in terms of what ‘explicit/implicit’ means specifically and more generally with analysing how evaluation, particularly with respect to mathematics, works in pedagogic contexts differentiated by social class. The present study does not seek, in the first instance, to provide explanations for the social class difference in mathematics performance, although it might do so. Rather, the aim is to explore the functioning of evaluation, specifically the recognition and realisation rules employed with respect to the content of the instructional discourse, in pedagogic contexts differentiated with respect to social class. In doing so, the study recruits and develops more mathematically-attuned methodological resources for examining the functioning of evaluation in relation to the instructional discourse in pedagogic contexts differentiated with respect to social class. As such, the study aims to contribute to Bernstein's theory of pedagogic discourse by exploring the complexity of recognition and realisation rules employed by teachers and learners in their productions of mathematics.

Recognition and realisation rules in Bernstein's theory are inextricably connected to the differential specialisation of consciousness on the basis of social class membership. Next, I focus attention on the how specialisation of consciousness is dealt with by Bernstein and by scholars employing Bernstein's theory.

1.4 Specialisation of consciousness

In Bernstein's theory of pedagogic discourse, differential specialisation of consciousness along social class lines becomes visible through his notion of *code,* which he refers to as the principles regulating meaning (Bernstein, 1990, p. 101). Bernstein's (1975, 1990) code theory[6], posits that an individual's *orientation to meaning* is a function of their social class membership. Orientation to meaning refers to "the selection and organization of meaning, of what is seen as relevant and taken as the focus of attention in any situation and the way in which these meanings are organized in practical discourse" (Holland, 1981, p. 1)[7]. Bernstein distinguishes between an *elaborated code* and a *restricted code,* where an *elaborated code* refers to the recognition and realisation of context-independent meanings and a *restricted code* refers to the recognition and realisation of context-dependent meanings (Bernstein, 1990, p. 96)[8]. Orientations to meaning are initially formed through early socialisation of children in the home and in peer groups which varies according to social class (e.g., Heath, 1982, 1983; Painter, 1999; Hasan, 2002). According to Bernstein, middle-class children are more likely than working-class children to form an elaborated code, and thus have an advantage at school (Bernstein, 1975). Further, he argues that schooling either amplifies or disrupts the restricted orientation to meaning working-class children form prior to schooling (Bernstein, 1975).

Holland's (1980) experiment, which involved children sorting pictures of food items into groups and declaring their sorting criteria, served as a key study in validating Bernstein's proposition that working-class children display a restricted orientation to meaning, and that middle-class children have access to both restricted and elaborated orientations to meaning. It is important to note that the Holland experiment suggested a strong correlation between social class and semantic orientation. However, her study is interpreted as establishing a causal relation between social class and semantic orientation (e.g. Cooper & Dunne, 2000; Hoadley, 2005; Hoadley & Ensor, 2009; Lerman, 2014; Lubienski, 2000, 2004; Skerrit, 2017). The use of Bernstein's notion of orientation to meaning in research does not distinguish between group-level correlations and the cognitive features of individuals. The fact that a particular semantic orientation is strongly correlated with social class in one study, does not mean that all working-class learners think in the same way (see also Jaffer & Davis, 2012). Furthermore, Holland's (1981) study contributed significantly to the proposition that working-class learners are more likely than middle-class learners to draw on 'everyday'

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[6] Here I present an overview of Bernstein's code theory. See Appendix 1 for an extended discussion on the evolution of Bernstein's code theory.

[7] Holland's (1981) study was designed by her supervisors, Adlam and Bernstein (Bernstein, 1990, p. 4).

[8] In Bernstein's early work, codes referred to differences in linguistic forms used by the working-class and middle-class. The notions of *restricted* and *elaborated* codes have changed over the course of Bernstein's research (see Appendix 1).

resources in contexts where specialised knowledge is required. In Chapter 2, the notion of orientation to meaning used in mathematics education research is explored in more detail.

In addition to describing orientation to meaning in terms of the academic-everyday distinction, Bernsteinian and neo-Bernsteinian mathematics education literature (e.g., Adler, Pournara, & Graven, 2000; Dowling, 1998; Swanson, 2002) and other mathematics education research (eg., Anyon, 1980, 1981; Baroody, Feil, & Johnson, 2007; Boaler, 2000; Brodie, 2004; Hiebert & Lefevre, 1986) recruit the conceptual-procedural distinction/opposition[9] to describe types of understanding and the form of mathematics realised in pedagogic situations. Conceptual knowledge is aligned with "sense-making" and "meaning" whereas procedural knowledge is cast as "senseless" or "meaningless". The conceptual-procedural distinction has some validity since it is pointing to something different in mathematical processing but it is not productive because "senselessness" or "meaninglessness" is an impossibility given that individuals always assign meaning to signifiers (see Strawson, 1950). Rather, the issue is that there are different types of sense-making. So, procedural understanding is just a different form of sense-making or meaning-making. I argue that sense or meaning can be derived from recognition and realisation rules described in terms of the computations performed by pedagogic agents. I return to a more considered and detailed discussion of the conceptual-procedural opposition employed in mathematics education research in Chapter 2.

The use of the academic-everyday distinction and the conceptual-procedural opposition does describe the cognitive resources used by teachers and learners when doing mathematics but are doing so inadequately with respect to mathematics. Specific cases are discussed in Chapter 2. I tentatively argue that the cognitive science approach to cognition and knowledge, specifically the distinction between *core domain* knowledge and *non-core domain* knowledge, potentially illuminates the use of the academic-everyday distinction and the conceptual-procedural opposition employed in mathematics education research in that it offers more mathematically-attuned descriptions of orientations to knowledge.

I now discuss the distinction between *core domain* knowledge and *non-core domain* knowledge which leads to a brief overview of the methodological orientation of the study. The general methodological orientation, detailed in Chapter 3, refers to the selection of theoretical antecedents that inform the research design, collection of information and production and analysis of data.

1.5 Locating the study methodologically

Core domain knowledge refers to genetically endowed knowledge systems (such as language and number) and *non-core domain* knowledge (such as school mathematics) is knowledge that must be acquired through explicit learning and/or teaching (see Butterworth, 1999; Dehaene, 1997; Gelman, 2009a; Spelke, 2000).

> I call domains that benefit from biological underpinnings *core* domains (Gelman and Williams 1998), in a way that is similar to Spelke (2000). Domains of organized knowledge that are acquired later are called

[9]Procedural knowledge and conceptual knowledge are distinct forms of knowledge. However, a number of schloars treat Procedural knowledge and conceptual knowledge as an opposition.

> *noncore* domains. Thus, I reserve the phrase *core domain* for domains that have an innate origin and noncore domain for those that require the acquisition of both the structure and related content. (Gelman, 2009a, pp. 248-249, italics in the orginal)

The distinction between core and non-core domain knowledge is situated more broadly in an approach that views human cognition as a biological system (e.g., Chomsky, 2005). The discussion of core and non-core domain knowledge as methodologically productive leads me to spell out the methodological orientation of the study.

Tooby and Cosmides (1992, p. 23) claim that the Standard Social Science Model (SSSM) "mischaracterises" phenomena because of the failure to "causally locate their objects of study in the larger network of scientific knowledge". They argue further that the SSSM adopts an insular approach to research by establishing artificial boundaries between bodies of knowledge. They call for an approach to research, referred to as an Integrated Causal Model (ICM), which constructs descriptions and analyses by harnessing the causal connections between related components of a phenomenon, often requiring resources from a broad spectrum of disciplines, thus enabling connections between the social sciences with other sciences. The current study adopts an ICM approach to research by drawing on theoretical resources from a range of disciplines including cognitive science, semiotics, philosophy, and mathematics.

One of the central propositions informing an ICM is that "the human mind consists of a set of evolved information-processing mechanisms instantiated in the human nervous system" (Tooby & Cosmides, 1992, p. 23). From an ICM perspective, culture is considered as a product of the human mind situated in individuals who live communally as opposed to viewing culture exclusively as that which shapes the human mind from the outside. Thus, Tooby and Cosmides (1992) espouse an internalist account of the human mind and of culture i.e., the mind structures experience. This position is shared by Chomsky (2005), who argues that the study of human cognition as a biological system involves the interplay of three factors.[10] The first factor concerns genetically endowed systems (core domain knowledge) required by all humans to function in the world. The second factor refers to the structuring of experience through genetic endowment on the basis of contextual data. Experience, which is structured by genetic endowment, explains within species variation with respect to language acquisition (Chomsky, 2005). For example, all humans have innate language capacity, but depending on the context in which a child is raised, the specific language eventually spoken by the child could be, say, Chinese or Swahili. The third factor includes general properties that impact on our world such as biological and physical laws, and principles of data processing, including computational efficiency.

Adopting an ICM approach in the current study means that the universality of genetic endowment is taken as given. All learners participating in this study, irrespective of their social class membership, are endowed with

[10] Chomsky adopts a Kantian approach to knowledge which is a synthesis of rationalist and empiricist accounts of knowledge. "Kant's version of nativism, with abstract organizing frameworks but not actual knowledge built in to the mind, is the version that is most viable today, and can be found, for example, in Chomskyan linguistics, evolutionary psychology, and the approach to cognitive development called domain specificity" (Pinker, 2007, p.106).

the same core domain knowledge/systems in the sense that all humans, barring pathological cases, are biologically endowed with the same set of core knowledge structures. So, variation in performance in school mathematics (non-core domain knowledge) must be a consequence of the second factor. In other words, learners' mathematical experiences are structured by genetically endowed core domain knowledge which constitutes contextual computational data (what is presented as mathematics). Contextual data in different contexts vary, thereby producing variation in the constitution of non-core domain computational resources. Social class is marked out as an important background contextual variable in this study. The functioning of evaluation in pedagogic situations, another contextual variable, forms the focus of investigation in this study.

The concern with how the inside structures the outside has been extended to a growing research interest in early number acquisition as a sub-field within cognitive science. A number of interesting books in the field, such as Brian Butterworth's *Mathematical Brain* (1999), Rafael Núñez and George Lakoff's *Where Mathematics Comes From* (2000), Stanislas Dehaene's *The Number sense* (1997) and the *Handbook of Mathematical Cognition* (Campbell, 2005) have been disseminated widely in this area (see Butterworth, 1999, 2005, 2010; Dehaene, 1997; Gallistel & Gelman, 2005; Gallistel & King, 2010; Gelman, 2009a; Spelke & Kinzler, 2007). A common proposition, agreed upon in the aforementioned literature and other cognitive science literature (see Chomsky, 2006, 2007; Fodor, 1998, 2010; Gallistel & King, 2010; Pinker, 1995, 1997, 2007), despite differences amongst them, is that thought is computational. Gallistel and King (2010, p. viii) argue that "brains are powerful organs of computation", where computations are regarded as compositions of functions. The argument is that computations are central to thought and to the communication of thought. Although computations are functional in nature, non-functional computations are recognised and routinely employed in pedagogic situations. Precisely how this is possible is discussed in Chapter 3.

Bernstein's proposition regarding the centrality of evaluation taken together with the proposition that the computational nature of thought is a universal human characteristic have been taken up methodologically by Davis in his study of what is constituted as mathematics in pedagogic situations (see for example Davis, 2010c; Davis, 2013a). In particular, Davis is primarily concerned with how mathematical experience is structured by genetic endowment and with the structuring of computational activity (i.e. the computations and domains of objects operated over) in pedagogic situations. The present study employs and adapts Davis' methodology, the details of which are elaborated in Chapters 3 and 5. Furthermore, the methodological propositions that thought is computational, that pedagogy is fundamentally evaluative and that language does not have a reference function serve to frame the review of literature discussed in Chapter 2.

1.6 Locating the study contextually

As stated earlier, there is an expectation that differences in school mathematics performance along social class lines potentially point to differences in what is constituted as mathematics and therefore differences in the functioning of evaluation in pedagogic contexts differentiated by social class. A discussion of the South

African schooling context is intended to illuminate the social class composition of the learner populations of schools.

Next I consider the current curriculum context and the recent history of curriculum reform in South Africa, following the transition to democracy in 1994. Although this study does not focus on the implementation of the curriculum per se, the curriculum is treated as a resource that teachers recruit and which informs the production of textbooks that teachers and learners use. As such, the purpose of discussing the curriculum context is to describe the curriculum milieu that teachers and learners involved in the study find themselves in.

1.6.1 The South African schooling context

Post-Apartheid South Africa inherited a 'racially'[11] segregated and highly unequal education system. Transformation and redressing past inequities within the education system have been the focus of many post-Apartheid government policies, widely documented in the literature (e.g., Jansen & Taylor, 2003; Kanjee, 2009; Veriava, 2010; Wildeman, 2008). Under Apartheid, schooling mirrored the 'racially' segregated South African society. The education system comprised 19 separate government departments with each department administering schools that served different 'racial' groups, for example, 'Africans' by the Department of Education and Training (DET), 'Coloureds' by the House of Representatives (HOR), 'Indians' by the House of Delegates (HOD) and 'Whites' by the Cape Education Department (CED). Schools for different 'race' groups were located in residential areas demarcated for different 'races' according to the Group Areas Act. In other words, schools for 'White' children were to be found in residential areas demarcated for 'Whites', 'Coloured' learners attended schools in 'Coloured' residential areas, etcetera. Former department designation of schools remains in use in government reports and academic literature e.g. ex-DET or ex-CED. Since schools were 'racially' segregated during Apartheid, many schools represented cases of what (Dowling & Brown, 1996, 2009) refer to as "class condensation", that is, schools exhibiting hybridisation in terms of the social class membership of learners.

A relaxation of government policy in the early 1990s allowed 'White' schools, which later became known as 'Model C schools', to admit learners from 'racial' groups other than 'White'. Deregulation of 'race' as an admission requirement in all schools followed shortly after the demise of Apartheid. Subsequently, post-Apartheid South Africa has witnessed substantial transformation in the 'racial' demographics of school populations. 'White', 'Coloured' and 'Indian' schools have changed with respect to their 'racial' composition, but the learner populations of ex-DET schools have to a large extent remained exclusively 'African' (Sujee, 2004 as cited in Chisholm & Sujee, 2007).

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[11] 'Racial' categories were imposed on South African citizens during Apartheid and are not necessarily accepted by individuals categorised as such. I use the Apartheid categories because this historical legacy has shaped and continues to shape schooling in South Africa. Since 1994, census classifications have distinguished between 'black African', 'White', 'Indian/Asian' and 'Coloured'. I use the terms 'African', 'Coloured', 'Indian' and 'White' all with the first letter capitalised.

The negotiated settlement that led to democracy in South Africa was accompanied by an acceptance of private contributions to schooling to maintain the standards that middle-class parents found acceptable (Jansen & Taylor, 2003). The South African Schools Act allows public schools to charge school fees, the amount determined by individual school governing bodies. Thus, the current public education system in South Africa comprises public schools entirely reliant on state funding, currently referred to as *No-fee* schools, and public schools that are resourced through state funding as well as parental contributions mainly in the form of school fees. Since school fees are dependent on parents' capacities to pay, fee-paying schools constitute a broad spectrum from low-fee paying schools (mainly ex-DET, ex-HOR and ex-HOD schools) to high fee-paying schools (mainly ex-Model C schools). An independent schools sector comprising privately funded schools runs parallel to the public education system in South Africa and includes a growing number of low-fee-paying private schools catering for poorer families (Hofmeyr, McCarthy, Oliphant, Schirmer, & Bernstein, 2013; Schirmer, Johnston, & Bernstein, 2010). Public schools are distinguished between Section 20 and Section 21, with Section 20 public schools receiving greater state funding than Section 21 public schools, which are then allowed to charge higher school fees[12]. The policy enabling public schools to charge school fees has the effect of blurring the distinction between public and independent schools. Enrolment patterns, shaped along social class lines, are determined largely by school fees.

The South African schooling system can, thus, broadly be described as a two-tier system. One tier, a privileged and well-resourced schooling sector comprising public schools charging high school fees (Section 21) and independent high-fee-paying schools, serves a minority of children from upper-middle class/elite families. The other tier, an under-resourced and disadvantaged school sector comprising No-fee schools, low-fee-paying public schools (Section 20) and independent No-fee or low-fee schools, serve the majority of South African children - working-class and lower-working class children who are predominantly 'African' and 'Coloured'. Schooling has thus become stratified with respect to social class and remains to a large extent stratified in terms of 'race'. The social class stratification of schooling in South Africa is reflected in the research design of the current study.

The empirical sites for this study include two independent schools, a high-fee paying school and a No-fee independent school as instances of empirical sites that differ with respect to social class membership of their learner populations, with the high-fee paying school catering for learners from upper-middle-class/elite families and the No-fee school populated by learners from working-class families. The justification and selection of schools are discussed in more detail in Chapter 4. The study is exploratory and small-scale, thus placing limits on the potential to generalise to other pedagogic contexts.

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[12] Public schools receive state funding on a sliding scale. Section 21 public schools (includes ex-Model C schools) receive less state funding and are permitted to charge higher school fees. Such schools have greater financial and managerial autonomy to determine the composition of the school population by defining the school's feeder areas, fees and admission policies. Section 20 public schools (mainly ex-DET, ex-HOR and ex-HOD schools) receive more state funding and charge lower fees. Section 20 schools are more strictly regulated financially.

1.6.2 The South African curriculum context

At the time when information[13] for this study was collected, secondary schools were experiencing a third wave of curriculum reform since South Africa's transition to democracy in 1994. The first post-Apartheid curriculum, Curriculum 2005 (C2005) implemented in Grade 1 in 1997, was widely criticised for its outcomes-based education (OBE) philosophy (e.g., Ensor, 1997; Jansen, 1998). Following a Ministerial review (Chisholm et al., 2000), the much-maligned C2005 was replaced with a more streamlined curriculum, the *National Curriculum Statement* (NCS), implemented for the first time at the Further Education and Training (FET) level in 2006 in Grade 10, with the first NCS matriculation examination conducted in 2008. In contrast to C2005, the NCS provided clearer specifications of contents at each grade level, removed the controversial theme-based curriculum organisation and returned to 'subjects' in place of 'learning areas'.

In 2009, a Ministerial Task team reviewed the NCS, following criticisms of various aspects of the implementation of the curriculum and in response to learner underperformance on national and international assessments. The Task team recommended that "the plethora of policies, guidelines and interpretations of policies and guidelines at all levels of the education system" should be consolidated into a single national policy document (Department of Education, 2009, p. 7). The review resulted in the latest iteration of the South African national curriculum, the *Curriculum and Assessment Policy Statements* (CAPS). The Further Education and Training (FET) band (Grades 10-12) CAPS was implemented for the first time in Grade 10 in 2014, when information was collected for this study. CAPS removed the last vestiges of OBE, such as integration of subjects, that were still present in the NCS, foregrounding disciplinary knowledge and significantly increasing the strength of external framing over the selection and sequencing of curriculum topics, with the inclusion of term planners and week-by-week pace-setters.

1.7 Outlining the study

The study is concerned with what evaluation reveals about the constitution of the content of school mathematics in four pedagogic contexts that differ with respect to the social class membership of their learner populations and the implications for the specialisation of learners' mathematical thought. Two independent secondary schools differentiated with respect to the social class membership of their learner populations were selected as the empirical sites. At each school, two mathematics teachers and their Grade 10 mathematics class of learners constituted the research participants of the study.

The focal research question framing the study is now formally stated as follows:

How does pedagogic evaluation function in the instructional discourse of four Grade 10 pedagogic contexts in schools that differ with respect to the social class membership of their learner populations and what are

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[13] I distinguish between information and data, where data is regarded as "information that has been read in terms of an explicitly available theoretical framework" (Brown & Dowling, 1998, p. 42). This is not to deny that the collection of information contained in the archive already constitutes a selection informed by the methodological orientation of the study.

the implications of pedagogic evaluation for learners' computational performances and orientations to mathematics?

In order to address the research question, I observed and video-recorded a sequence of three consecutive Grade 10 lessons taught by each teacher, that is, a total of 12 mathematics lessons. In addition, mathematics test scripts of six learners in each of the four Grade 10 mathematics classes were collected and interviews based on the solutions to selected test items were conducted with the selected learners. An extended discussion of the information archive is presented in Chapter 4 and the procedures for producing data from the information archive are discussed in Chapter 5.

The first focus of attention is the mathematics constituted in the instructional discourse in pedagogic contexts in schools differentiated with respect to learners' social class membership. The observed lessons entail the second factor (in terms of Chomsky's three factors) and provide an opportunity to ascertain the mathematics learners are exposed to, bearing in mind that learners' experiences of mathematics in the pedagogic context are structured by genetically endowed computational nature of thought (core domain knowledge common to all learners). The observed lessons provide empirical instances of the functioning of evaluation at the level of the instructional discourse in each pedagogic context.

The approach entails a fine-grained analysis of teachers' and learners' computational activity i.e., their recognition and realisation rules in terms of the computations employed in the observed lessons. The analysis of teachers' and learners' computational activity (operations and domains operated over) provides a detailed picture of the content realised in association with topic names in the observed lessons. The sub-questions addressing the mathematics constituted in the observed lessons are as follows:

- What does the computational activity reveal about the content realised with respect to Grade 10 mathematics topics in the instructional discourse in pedagogic contexts differentiated with respect to learners' social class membership?
- How is the realisation of content in the instructional discourse regulated in these pedagogic contexts?

The description of the content realised in association with the topic names in the instructional discourse is then re-analysed to produce a description of the computational performance and orientation(s) to mathematics of the learner implied by the computational activity (the *model learner*). Note that the model learner is not an ideal learner but the learner presupposed by the computational activity in the observed lessons. The notion of the model learner used in this study is elaborated in Chapter 3. The sub-questions directing this aspect of the study are as follows:

- What does the computational activity elaborated in the instructional discourse imply about the computational performance of the model learner constructed in pedagogic contexts differentiated with respect to learners' social class membership?
- What orientations to mathematics are implied by the computational activity present in the instructional

discourse in pedagogic contexts that differ with respect to the social class membership of learners?

The second interest of the study involves ascertaining the specialisation of learners' mathematical thought. In order to address this aspect of the study, a fine-grained analysis of learners' computational activity when doing mathematics independently of the teacher, in the context of a test and in a video-recorded interview, was conducted. The sub-questions concerned with the mathematics constituted by learners and the specialisation of mathematical thought displayed in a test and in a video-recorded interview are as follows:

- What does the learner's computational activity imply about their computational performance in pedagogic contexts that differ with respect to their social class membership?
- What does the learner's computational activity imply about their orientation to mathematics in pedagogic contexts that differ with respect to their social class membership?

The computational performances and orientations to mathematics of the model learner implied by the computational activity in the observed lessons are then compared with the computational performances and orientations to mathematics of actual learners displayed in the test and the interview in order to gauge the implications of pedagogic evaluation for the specialisation of learners' mathematical thought.

The study forms part of an enduring concern with differential performance in school mathematics along social class lines and with continuing efforts to understand the teaching and learning of mathematics in South African classrooms. Broadly, the study contributes to the mathematics education field concerned with equity in schooling. In particular, the study recruits and develops more mathematically-attuned methodological resources for examining the functioning of evaluation in relation to the instructional discourse in pedagogic contexts differentiated with respect to social class. As such, the study aims to contribute to Bernstein's theory of pedagogic discourse by exploring the complexity of recognition and realisation rules employed by teachers and learners in their productions of mathematics.

The study is thus a systematic investigation of the functioning of evaluation at the level of the instructional discourse. Specifically, the complexity of the recognition and realisation rules, described computationally, is explored in pedagogic contexts distinguished in terms of learners' social class membership. This comparative exploratory study is unique in terms of the research focus, empirical settings and the methodological approach adopted.

1.8 Overview of the thesis

Chapter 2 critically discusses mathematics education literature which recruits Bernstein's theory of pedagogic discourse as well as mathematics education literature concerned with differential mathematics performance with respect to social class. In particular, the discussion centres around the deployment of the academic-everyday distinction, the conceptual-procedural opposition/distinction and explicit-implicit evaluative criteria distinction in the literature to explain social class differential achievement in mathematics.

The chapter highlights the methodological insufficiencies of the academic-everyday distinction, the conceptual-procedural opposition and the distinction between implicit and explicit evaluative criteria with respect to the teaching and learning of school mathematics. In so doing, a space for more mathematically-attuned methodological resources for addressing the research problem is established.

Chapters 3, 4, and 5 constitute the methodology of the study. The general methodology, discussed in Chapter 3, derives from an engagement with antecedent literature pertinent to the theoretical concerns of the study. The purpose of the chapter is to generate a series of theoretical propositions that serve as the basis for the procedures for the production and analysis of data, which is the focus of Chapter 5. Chapter 4 outlines the research design, including the selection of schools, teachers and learners that constitute the cases of the study and describes the construction of the information archive, which includes the information sources, instruments used for collection of information and modes of collecting information and considers issues pertaining to reliability, validity and generalisability of the study.

Chapter 5 delineates procedures for the production and analysis of data from the information archive. Two stages of data production and analysis enable a discussion of the research question. The first stage entails generating a description of the functioning of evaluation in the instructional discourse and the implications for the computational performance and orientation to mathematics of the model learner in each pedagogic context. The second stage focuses on the recognition and realisation rules employed by learners when doing mathematics independently of the teacher in the context of a test and a clinical interview in order to ascertain their computational performances and orientations to mathematics.

Chapters 6, 7, 8 and 9 present the production and analysis of data. Chapter 6 presents the analysis of the observed lessons in terms of the content realised in association with the announced topics in each pedagogic context and considers how the realised content in the instructional discourse is shaped through the forms of regulation of mathematical activity in each pedagogic context. Chapter 7 presents the analysis of the computational performance of the model learner and the orientation to mathematics implied by the realised content in each pedagogic situation. This chapter describes five pedagogic modalities evident across the four pedagogic contexts and identifies the dominant pedagogic modality in each pedagogic context. Chapter 8 continues the focus on the implied model learner's computational performance and orientation to mathematics as evidenced in the evaluative activity entailed in the setting and marking of a test administered and marked by the teacher in each pedagogic context.

Chapter 9 presents the analysis of the computational activity of learners when doing mathematics independently of the teacher. I present the analysis of learners' solutions to a mathematics test set by their teacher and interviews with selected learners in each pedagogic context on their solutions to selected test problems in order to reveal the specialisation of mathematical thought; i.e., (1) the computational performances of selected learners as opposed to the model learner presupposed by the pedagogic text; and 2) the orientations to mathematics of selected learners. Furthermore, the chapter examines how the dominant pedagogic modality of the pedagogic context has structuring effects on the computational performances of learners and their orientations to mathematics.

In the concluding chapter, Chapter 10, I present a summary of the thesis, discuss the main findings in relation to the research hypotheses established in Chapter 3, reflect on the findings in relation to the literature reviewed in Chapter 2 and identify the potential and limitations of the study.

Chapter 2

Locating the study in the field of mathematics education – a review of the literature

2.1 Introduction

In this chapter I locate the research interest of the thesis in relation to relevant literature in the field of mathematics education and I present an argument for the methodological orientation of the study. The current study is concerned with the functioning of evaluation at the level of the instructional discourse, specifically the complexity of the recognition and realisation rules which are described computationally. The study focuses on what evaluation reveals about the constitution of the content of school mathematics in four Grade 10 pedagogic contexts in two schools that differ with respect to the social class membership of their learner populations and the implications for the specialisation of learners' mathematical thought. Describing *the constitution of mathematics* in pedagogic situations entails describing what comes to be realised as mathematics in a particular pedagogic situation by examining the computational resources employed by teachers and learners rather than on *a priori* notions of what mathematics is (Davis, 2011a, 2013a).

It should be noted that much of the literature in the field of mathematics education does not explicitly refer to the *constitution of mathematics* as such but is concerned with the nature of mathematics produced in pedagogic situations and with orientations to mathematics as displayed by teachers and learners. The literature is expansive, extending across a number of epistemological positions in the field of mathematics education such as anthropological studies of the teaching and learning of school mathematics, psychological, sociological and political accounts of the constitution of mathematics in pedagogic contexts, socio-cultural research in mathematics education and research concerned with linguistic aspects of school mathematics teaching and learning.

Given that the current study centres on investigating Bernstein's proposition that evaluation is central to pedagogic practice, the literature review primarily focuses on literature employing Bernstein's theory but considers other antecedent literature as well. I start by examining comparative studies on mathematics and social class since the present study is designed as a comparative study in two schools differentiated by learners' social class membership. The purpose, here, is to locate the present study in relation to other comparative studies concerned with mathematics in contexts distinguished in terms of learners' social class membership and to mark out the methodological distinctiveness of the current study in relation to the aforementioned studies. I show that the procedural-conceptual opposition and/or the academic-everyday distinction are employed in this literature to describe differences in the constitution of mathematics along social class lines. Next, I consider the procedural-conceptual opposition employed by Bernsteinian and neo-

Bernsteinian scholars and other literature across the epistemological positions described above. This is followed by a discussion of the academic-everyday distinction, particularly focusing on Bernsteinian and neo-Bernsteinian literature.

The intention of this chapter is not to provide an exhaustive review of the literature but to exemplify the oppositions/distinctions used to describe the constitution of mathematics in pedagogic contexts. I shall argue that the use of both sets of oppositions/distinctions tends to produce a misreading, overlooking or distortion of the mathematics emerging in pedagogic situations. In so doing, a gap in the current literature is identified. I argue for a methodology that focuses on the mathematics emerging in pedagogic situations in order to ensure descriptive and explanatory adequacy.

2.2 School mathematics and social class

The difference in mathematics achievement between middle-class and working-class learners, as discussed in Chapter 1, is not a uniquely South African phenomenon. Similar social class-aligned differences in achievement in school mathematics have been reported in United States of America (e.g., Anderson & Tate, 2008; Diversity in Mathematics Education Center for Learning and Teaching, 2007; Gutiérrez, 2008; Lubienski, 2000, 2004). Likewise a growing achievement gap between working-class and middle-class learners is evident in Britain (Cooper & Dunne, 1998, 2000; Reay, 2006) and Europe (see Jurdak, 2014). The performance gap between learners from middle-class families and learners from working-class families in South Africa has to a great extent been documented in large scale quantitative studies (e.g., Reddy, van der Berg, Lebani, & Berkowitz, 2006; Spaull, 2013; van der Berg, 2007). Some smaller qualitative studies are concerned with the nature of the relation between mathematics performance and social class in an attempt to understand the underlying factors impacting on learner performance in mathematics. Most research in mathematics education in South Africa tends to focus on learners from working-class families given that those learners constitute the majority of learners under-performing in mathematics (e.g., Carnoy et al., 2011; Schollar, 2008) as opposed to learners from middle-class families, as part of efforts aimed at closing the social class performance gap (also see Graven, 2014).

My review of the mathematics education literature yielded two South African studies comparing learners from middle-class families with learners from working-class families (Hoadley, 2005, 2007; Jaffer & Davis, 2012) and two studies focusing on learners from working-class families, particularly “Black” learners in schools previously intended for “White” students only (Feza-Piyose, 2011; Swanson, 2002, 2006). Hoadley (2005, 2007) is the only South African study comparing pedagogic practices and learners’ performance on mathematical tasks in schools differentiated in terms of learners’ social class membership. The research design of my study is similar to that of Hoadley (2005, 2007) but they differ methodologically. Her study is concerned with Grade 3 Literacy and Numeracy pedagogic practices and Grade 3 learners’ performance on mathematics tasks in two schools populated by learners from middle-class families and two schools populated by learners from working-class families whereas my study is concerned with the constitution of Grade 10 mathematics and the specialisation of learners’ mathematical thought in two schools that differ with respect to learners' social class membership. Hoadley (2005, 2007) and the current study are concerned with the specialisation of learners' consciousness. However, the current study employed clinical interviews in order to ascertain the recognition and realisation rules employed by learners when doing mathematics. A comparative study focusing on the constitution of secondary school mathematics and the specialisation of learner's mathematical thought in schools differentiated with respect to learners' social class membership in South Africa is thus unique in its empirical focus.

Given the dearth of South African comparative studies of school mathematics in pedagogic contexts differentiated with respect to learners' social class membership, my discussion below centres on local and international comparative studies concerned with mathematics in pedagogic contexts differentiated with respect to learners' social class membership. Despite diversity with respect to methodology, there is convergence in these studies with respect to the differential distribution of knowledge along social class lines. I argue, however, that these studies lack the methodological resources for adequately describing mathematics constituted in pedagogic contexts. A common thread running across the studies to be discussed is the description of mathematics constituted in pedagogic contexts. The literature partitions the constitution of mathematics in pedagogic contexts differentiated in terms of social class in terms of the academic-everyday distinction and procedural-conceptual opposition, and is silent on similarities in the constitution of mathematics in those pedagogic contexts. Mathematics constituted in schools populated by learners from working-class families is often described in this literature as procedural and/or weakly bounded from the 'everyday'. In contrast, mathematics constituted in schools populated by learners from middle-class families is commonly described in this literature as conceptual as opposed to procedural and/or as strongly bounded from the 'everyday'. Given the reported social class differences in the constitution of mathematics, middle-class learners according to the literature are more likely to be successfully inducted into school mathematics than their working-class counterparts. A discussion of particular comparative studies in pedagogic contexts differentiated with respect to learners' social class membership follows below.

In their study on the co-constitution of mathematics and learners, Atweh & Cooper (1995) conducted observations and interviews in two Year 9 classes in two all-girls schools, one populated by learners from low socioeconomic backgrounds and the other by learners from high socioeconomic backgrounds. Located within an interpretive analytic tradition and using a feminist postmodern theoretical framework, Atweh & Cooper (1995) found differences in the constitution of mathematics in the two social class contexts and differences in how learners were constructed in relation to mathematics. In the school populated by learners from low socioeconomic backgrounds, Mathematics was constituted as "a collection of techniques, rather than as strategies, performed on routine tasks" and as "bits and pieces" (ibid, p. 301-302), suggesting an approach to mathematics that is procedural and disconnected. According to Atweh & Cooper (1995), the teacher in this context failed to generalise or to focus on the underlying mathematical principles. In contrast, mathematics in the school populated by learners from high socioeconomic backgrounds was constructed as "a collection of strategies for solving tasks and a system of generalisations and justifications" (ibid, p. 303). According to Atweh & Cooper (1995), the teacher was concerned with teaching for "understanding" and with teaching a form of mathematics that prepared her learners for university mathematics as opposed to useful mathematics for 'everyday' requirements as was the case in the low socioeconomic context.

O'Halloran (1996, 2004), drawing on Systemic Functional linguistics and Bernstein's sociolinguistic thesis, shows similar differences in the constitution of mathematics in schools with contrasting social class membership of their learner populations although the theoretical and methodological orientation of study differs from that used by Atweh & Cooper (1995). O'Halloran (1996, 2004) claims that mathematics presented in a school populated by Year 10 boys from middle-class/elite families corresponds strongly with the field of mathematics and develops context-independent meanings and an elaborated coding orientation. In contrast, mathematics in a Year 10 class in a school populated by learners from working-class families corresponds more closely to 'everyday' meanings and is constructed as a set of rules as opposed to a field of knowledge based on mathematics axioms and definitions. Mathematics lessons exhibit context-dependent meanings and cultivate a restricted orientation to meaning. Mathematics in the middle-class context is intended to induct learners into the discourse of mathematics whereas learners in the working-class context are marginalised with respect to mathematics as a discourse.

Like O'Halloran (1996, 2000), Atweh, Bleicher, and Cooper (1998) recruit Halliday's Systemic Functional linguistics in their comparative study on the constitution of mathematics in a Year 9 class in a school populated by boys from middle-class/elite families and a Year 9 class in a school populated by girls from working-class families. Atweh et al. (1998) conclude that mathematics in the middle-class school tends to focus on formal definitions and the use of mathematical language in order to display the formal logic of mathematics. The teacher is concerned with developing learners' 'understanding' of mathematics in order to prepare them for university mathematics. Mathematics in the school populated by learners from working-class families tends to focus on the use of 'everyday' terms to describe mathematics rather than formal mathematical language. So, the boundary between mathematics and 'everyday' speech is less discernible in the working-class context than in the middle-class context. Furthermore, the teacher is "less concerned with developing their meanings (mathematical notions) than with developing their intuitive and algorithmic usage" (ibid, p.71).

In contrast to the studies discussed above, Anyon (1980, 1981) is primarily interested in the reproduction of social class in four school types: working-class, middle-class, affluent professional-class and executive elite-class. Her study shows differences in what is constituted as Grade 2 and 5 mathematics in schools that differ with respect to the social class membership of learners. Mathematics in the school populated by learners in working-class families is described as "often restricted to the procedures or steps to be followed in order to add, subtract, multiply, or divide" (Anyon, 1981, p.7), where the purposes of procedures are not explained and are "seemingly unconnected to thought processes or decision making" of learners (Anyon, 1981, p.8). Mathematics in the middle-class/elite schools is described as conceptual in order to develop learners' 'understanding' of mathematics.

Similarly, Bernsteinian and neo-Bernsteinian scholars, Hoadley (2005, 2007), Cooper & Dunne (1998, 2000) and Dowling (1998), describe the differences in constitution of mathematics along social class lines in terms of the academic-everyday distinction and/or the conceptual-procedural opposition. An in-depth discussion of those studies follows later in this chapter.

Studies comparing mathematics presented to learners in diverse social class contexts and comparing learners' experiences of mathematics in such contexts come to similar conclusions as outlined by the studies discussed above. Swanson (2002, 2006), concerned with the construction of "deficit" and "disadvantage" in an upper-middle class/elite school for boys, focuses on how "Black Scholarship" learners are constructed by their teachers in relation to mathematics. She argues that "Black Scholarship" learners, placed within the "lower set", were denied access to the regulating principles of "upper stream" mathematics despite the Academic Support Programme which was intended to assist "Black Scholarship" learners to transition into "Mainstream" mathematics. Furthermore, in the Academic Support Programme, "Black Scholarship" learners were "granted access to procedural practices and mere rules rather than the regulating principles of school mathematics" (Swanson, 2006, p.211).

Lubienski (2004), drawing on Bernstein's code theory, investigated how learners from diverse social class backgrounds experienced a mathematics curriculum which employed an invisible problem-based pedagogy in which evaluative criteria were not made explicit. Open-ended discussions and contextualised mathematics problems served as the basis of the mathematics curriculum. Lubienski found social class differences in the responses of her learners to the pedagogy. Learners from middle-class families were more confident and successful in decoding the invisible pedagogy than learners from working-class families who had difficulty in reading the evaluative criteria implicit in open-ended discussions. Furthermore, Lubienski (2004) claims that learners from middle-class families were more adept at distilling the mathematical principles from contextualised problems whereas learners from working-class families tended to focus on the 'everyday' contexts such as pizzas or popcorn.

The studies discussed above vary with respect to theoretical and methodological orientations but recruit the procedural-conceptual opposition and/or the academic-everyday distinction. Some studies such as O'Halloran (1996, 2004) have stronger internal languages of description (theories) than other studies. All the studies lack specific methodological resources for describing the constitution of mathematics in pedagogic contexts. O'Halloran (1996, 2004) and Atweh et al. (1998) draw on theoretical and methodological resources that focus attention on the linguistic features of pedagogic practice rather than the mathematical. None of the studies cited above employ a methodology that focuses on the computational activity of teachers and learners. So methodologically, the current study as a comparative study focusing on mathematics in pedagogic contexts that differ with respect to learners' social class membership is unique with respect its methodological orientation.

What follows is a detailed examination of the procedural-conceptual opposition and the academic-everyday distinction as resources for describing the constitution of mathematics in pedagogic situations. As discussed in Chapter 1, the methodological propositions that thought is computational, that pedagogy is fundamentally evaluative and that language does not have a reference function, serve to frame the review of literature.

2.3 The conceptual-procedural opposition/distinction

The distinction between *conceptual* knowledge/understanding and *procedural* knowledge/understanding has been and continues to be extensively employed to describe the constitution of mathematics in pedagogic situations in terms of the knowledge produced by teachers and learners and their orientations to mathematics (see for example, Anyon, 1980, 1981; Boaler, 2000; Brodie, 2004, 2010; Dowling, 1998; Hoadley, 2005, 2007; Kieran, 2013; Ma, 1999; Sfard, 1991; Steinbring, 1989; Tall, 2008)[14].

Hiebert & Lefevre's (1986) notions of conceptual and procedural knowledge/understanding and to a lesser extent Skemp's (1976) concepts of *relational* and *instrumental* knowledge/understanding[15] serve as key resources recruited by scholars to describe the constitution of mathematics in pedagogic contexts. Hiebert & Lefevre (1986, p. 6) describe conceptual knowledge as "knowledge that is rich in relationships" that forms "a connected web of knowledge" and procedural knowledge as the "formal language, or symbol representation system of mathematics" and "the algorithms, or rules, for completing mathematical tasks". Their descriptions of conceptual and procedural knowledge imply that connections between knowledge only occur with respect to conceptual knowledge and that procedural knowledge is devoid of such connections. However, connections between knowledge occurs naturally and routinely as a feature of the way in which the human mind functions bearing in mind that the connections between bits of information may differ across individuals. Star (2005, p. 407) points out that the literature defines conceptual knowledge in terms of the quality of an individual's knowledge rather than as knowledge of concepts or principles.

Hiebert & Lefvre align conceptual knowledge with 'meaningfulness' or 'sense-making' whereas procedural knowledge is deemed as 'meaningless' (Hiebert & Lefevre, 1986, p. 8). Surely, 'meaningless' knowledge is contradictory because knowledge entails meanings that individuals assign to information. Recall that words do not refer, it is people who use words to refer (Strawson, 1950). So, individuals assign meaning and use terms in their individual ways. At times the meanings assigned by teachers and learners to mathematical notions correspond with the meaning generally accepted by mathematics adepts, but sometimes they do not.

Skemp's (1976) notions of relational and instrumental understanding/knowledge are correlates of Hiebert & Lefevre's (1986) conceptual knowledge and procedural knowledge respectively. Skemp describes the former as "rules with reasons" and the latter as "rules without reasons" (Skemp, 1976, p. 2). For example, the rule, "change sides, change signs" often used in solving equations, is regarded by Skemp (1976) as instrumental knowledge or a "rule without a reason". But, a "rule without a reason" is an impossibility because individuals can always provide reasons for their actions or behaviour. A learner who responds that she "changes sides, changes signs" when solving equations because this is what she was taught, has a reason and

[14] Dowling distinguishes between principled and procedural discourse where principled discourse "exhibits connective complexity" and procedural discourse "tends to impoverish complexity, minimizing rather than maximizing connections and exchanging instructions for definitions" (Dowling, 1998, p. 146).
[15] The terms *relational* and *instrumental* understanding were in fact coined by Stieg Mellin-Olsen (Skemp, 1976, p. 2).

is acting rationally even though her reason would not be used or accepted as a valid mathematical reason by mathematics adepts. Conceptual and procedural knowledge are, therefore, set up as dichotomous by Hiebert and Lefvre (1986), as well as by Skemp (1976).

Conceptual knowledge is considered as the valued knowledge associated with "meaning", "sense-making" and "connections", and procedural knowledge is deemed as a lesser knowledge form, often described as "meaningless" and as rote learning, and as the cause of learner failure in mathematics more generally (e.g., Gray & Tall, 1994; Kieran, 2013) and of working-class learners in particular (Anyon, 1980, 1981; Atweh, Bleicher, and Cooper, 1998): Atweh & Cooper, 1995; Hoadley, 2005, 2007; Swanson, 2002, 2006).

Star (2005, 2007) levelled a series of criticisms against the widespread deployment of the procedural-conceptual opposition in mathematics education. He argued against setting procedural knowledge in opposition to conceptual knowledge and proposed that both forms of knowledge are required. He argued that procedural knowledge is viewed as a lesser knowledge form because it is regarded as less complex than conceptual knowledge. He maintains that both procedural and conceptual knowledge can be considered on two levels of complexity, viz., superficial or deep knowledge.

Other scholars also argue that the dichotomy between conceptual knowledge and procedural knowledge is a false dichotomy (see Kieran, 2013; Star, 2005; Vergnaud, 1998; Wu, 1999). Wu (ibid) shows how the standard algorithms (e.g. addition or multiplication algorithms) embed conceptual knowledge. Kieran (ibid), drawing on the work of Jean-Baptiste Lagrange and the cognitive scientist Merlin Donald, argues that procedures are conceptual both at the times of their elaboration by individuals and when they have become automatised and that procedural and conceptual knowledge interact constantly and iteratively in the development and doing of mathematics.

Despite Kieran's (2013, p. 169) critique of the conceptual-procedural dichotomy deployed in mathematics education, she attributes the failure of learners in mathematics to "the teaching of algebra as a set of concept free, manipulative procedures", suggesting that it is possible for procedures to be concept-free. Vergnaud (1998) argues that procedures cannot be divorced from concepts because concepts are central to cognition, a view supported by many cognitive scientists and philosophers, who have long held the view that human thought is conceptual (e.g., Chomsky, 2006; Pinker, 1997). So, the idea of concept-free knowledge is illogical. Star (2005) argues that deep procedural knowledge is distinct from, but related to, conceptual knowledge. Conceptual knowledge and procedural knowledge are, therefore, not in opposition to each other.

Both forms of knowledge are required to do mathematics. It is impossible for an individual to do mathematics without using any concepts. That the concepts employed may be located outside of mathematics from the perspective of mathematicians is always possible in pedagogic situations. The proposition that procedural knowledge is concept-free, unprincipled, 'meaningless' and 'senseless' and that conceptual knowledge is principled, 'meaningful' and entails 'sense-making', is only valid if one is producing a prescription of knowledge that *ought* to be present in pedagogic situations. Thus, the conceptual-procedural opposition as deployed in the mathematics education literature functions as a moral distinction.

Furthermore, the use of the procedural-conceptual opposition by many researchers does not read the empirical for concepts that fall outside of the field of Mathematics education and as such produces a misreading of the empirical.

The significance of the above discussion regarding the procedural-conceptual opposition/distinction to the current study relates to the methodology of the study. An important component of the current study entails examining the procedures as well as the concepts employed by teachers and learners. However, the study does not employ the opposition between procedural and conceptual knowledge and does not associate "meaninglessness", "rules without reason" or the lack of "sense-making" with the use of procedures. A fundamental proposition underpinning the study is that thought is conceptual. Thus, the study aims to ascertain the 'sense' or 'meaning of pedagogic agents by examining their recognition and realisation rules described in terms of the computations employed. In Chapter 10 I tentatively propose a link between the procedural-conceptual opposition and the distinction between core domain knowledge and non-core domain knowledge.

2.4 The academic-everyday distinction and explicit-implicit evaluative criteria

With regard to the academic-everyday distinction, definite camps amongst mathematics education scholars are discernible, with some arguing for the inclusion of the 'everyday' as a means of facilitating the induction of learners into school mathematics (e.g., D'Ambrosio, 1985; Gerdes, 1985; Gutstein, 2003, 2006, 2016; Julie, 2004)[16] and others arguing that the very presence of the 'everyday' is what hampers learners' access into school mathematics (e.g. Hoadley, 2005, 2007; Cooper & Dunne, 1998, 2000; Le Roux, 2014).

Vociferous arguments against the incorporation of the 'everyday' into school mathematics emanates from scholars employing Bernstein's theory of pedagogic discourse and his theory of codes, and Dowling's (1993, 1998) notions of domains of practice. They argue that the inclusion of the 'everyday' is implicated in the differential achievement in schooling between learners from middle-class families and learners from working-class families (see Cooper & Dunne, 1998, 2000; Davis, 1995b; Ensor, 1997; Gellert & Jablonka, 2009; Gellert & Straehler-Pohl, 2011; Hoadley, 2005, 2007; Jablonka, 2009; Lerman, 2009, 2014; Lerman & Tsatsaroni, 1998; Lerman & Zevenbergen, 2004; Lubienski, 2000, 2004; Morgan, Evans, & Tsatsaroni, 2002; Muller & Taylor, 1995, 2000; O'Halloran, 1996, 2004; Straehler-Pohl, 2010; Taylor, 1999, 2000; Zevenbergen, 2000, 2001b; Zevenbergen & Lerman, 2001). Furthermore, Bernsteinian literature claims that working-class learners are predisposed to an orientation to meaning which weakens the classification between mathematics and the 'everyday' and that pedagogies in working-class contexts fail to interrupt learners' orientation to meaning that they bring to school because knowledge produced in such contexts tends to blur the boundary between mathematics and the 'everyday' (e.g., Cooper & Dunne, 1998, 2000; Hoadley, 2005, 2007; Zevenbergen, 2000, 2001b; Zevenbergen & Lerman, 2001).

[16] See Appendix 2 for a discussion on the arguments by scholars advocating the inclusion of the 'everyday' into mathematics.

What follows is an analysis of studies using Bernstein's notion of classification and framing. The studies rely on the academic-everyday distinction as an explanatory proposition for the differential achievement in mathematics along social class lines. The intention, as stated earlier, is not to produce an exhaustive reading of the field but to discuss how the theoretical and methodological resources are employed in the literature. Firstly, I examine the work of Cooper and colleagues in order to examine their employment of Bernstein's notion of orientation to meaning and recognition and realisation rules. This is followed by a discussion of Hoadley (2005, 2007) in order to illustrate her use of Bernstein's notions of classification and framing in her study of pedagogy in schools populated by learners from middle-class families and schools populated by learners from working-class families. Finally, I attend to Dowling's (1998) domains of practice that are often employed by researchers in conjunction with Bernstein's theory to argue that the inclusion of the 'everyday' into school mathematics denies learners access into school mathematics.

2.4.1 The deployment of the notion of orientation to meaning[17]

The work of Cooper and his colleagues focuses on examining the relationship between learners' performances on mathematics test items and their social class membership and gender. This work is recruited by a number of scholars as empirical support for the claim that the distinction between the 'everyday' and the academic is implicated in the success and failure of learners in mathematics along social class lines (Gellert 2008; Hoadley, 2005, 2007; Lubienski, 2000, 2004; Straehler-Pohl, 2010; Zevenbergen, 2001a).

One of the findings of Cooper and his colleagues is that working-class learners performed less well than middle-class learners on 'realistic' test items (Cooper & Dunne, 2000, p. 199). They base their explanation for the social class disparity in performance on Bernstein's proposition that states that an individual's semantic orientation is a function of their social class membership. *Realistic* items, glossed by Cooper and Dunne as items which "contains either persons or non-mathematical objects from everyday settings" (Cooper & Dunne, 2000, p. 84) are contrasted with so-called *esoteric* mathematics items[18]. A test item that involves finding the price of a box of popcorn given two bits of information: (1) a coke and a box of popcorn cost 90p and (2) two cokes and a box of popcorn costs £1.45 is classified as a *realistic* item. In Bernstein's terms, the boundary between mathematics and the 'everyday' is weak. The test item, "n stands for a number. $n + 7 = 13$. Find the value of $n + 10$" is an example of an *esoteric* mathematics item. In Bernstein's terms, classification between mathematics and the 'everyday' is strong (Cooper & Dunne, 2000, pp. 84-85). Cooper and Dunne (2000, p. 3) are concerned with how learners cope with the "boundary problem". From Dowling's point of view, the coke-and-popcorn item is located in the public domain and the other in the esoteric domain (to be discussed later).

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[17] Sections 2.4.1 and 2.4.2 were published in Jaffer (2011b).

[18] Here Cooper and Dunne (2000) appear to draw on Dowling's (1998) notions of esoteric and public domain discussed in Section 2.3.3 but are simultaneously using Bernstein's notion of classification that refers to the boundary between categories. The esoteric domain represents the domain of practice of an activity, for example school mathematics, which is most strongly classified with respect to other activities. The public domain is the domain of practice that has the form of a non-specialised practice but is nevertheless a domain of school mathematics.

Since evaluation reveals criteria for the recognition and realisation of mathematical objects or procedures in pedagogic contexts, an examination of the evaluation implied by the coke-and-popcorn item reveals the computational objects required for solving the test item. The problem is to determine the price of popcorn and the price of coke. The solution to this problem can be solved through (1) setting up and solving two equations simultaneously; or, (2) using explicit syllogistic reasoning based on the cost of one coke and one box of popcorn and two cokes and one box of popcorn.

Solution 1: If the cost of coke is C and the cost of popcorn is P, then we have two equations $C + P = 90$ and $2C + P = 145$. Solving the two equations simultaneously, produces the solution $C = 55$ and $P = 35$. So a coke costs 55p and popcorn 35p.

Solution 2: If a coke and box of popcorn cost 90p,

and a coke and a coke and a box of popcorn cost £1.45 (145p),

then a coke must cost 55p.

So a box of popcorn must cost 35p.

The solutions show that the item requires, at times, the recognition of the computational objects *coke*$\wedge$*popcorn* and *coke*$\wedge$*coke*$\wedge$*popcorn* as single objects rather than as two and three distinct objects, respectively. Values are assigned to *coke*$\wedge$*popcorn* and *coke*$\wedge$*coke*$\wedge$*popcorn*, and it is the value of the computational object *popcorn* that has to be calculated.

Mathematics is agnostic about its objects (see Whitehead, 1911, p. 9), so it does not matter that the computational objects are coke and popcorn. The coke-and-popcorn item is categorised as realistic by Cooper and Dunne (2000) because it contains references to 'everyday' objects. Similarly, the second item is referred to as an esoteric mathematics item because it does not contain any references to the 'everyday'. Cooper & Dunne's (2000) use of the notion of classification ('boundary') with respect to mathematics and the 'everyday' is based on *reference* to 'everyday' objects rather than *evaluation*.

A response of a working-class learner to the coke-and-popcorn item was as follows:

> I said to myself in a sweetshop a can of coke is normally 40p so I thought of a number and the number was 50p so I add 40p and 50p and it equalled 90p. (Cooper & Dunne, 2000, p. 41)

The learner's response to the realistic test item is typical of working-class learners according to Cooper and Dunne. Their explanation for the learner's response is as follows: the learner recruits her everyday knowledge of shopping as a resource to solve the problem and does not recognise the system of simultaneous equations implied by the problem. That is, the learner responds in a 'realistic' rather than 'esoteric' manner to the item (Cooper & Dunne, 2000, p. 199). In Bernstein's terms, the learner fails to recognise the specificity of the context, blurs the boundary between school mathematics and the 'everyday' context of shopping, thereby weakening the classification with respect to mathematics. In other words the learner does not posses the realisation rule to produce the legitimate text. With respect to Dowling, the learner is operating in the public domain of practice as opposed to the esoteric domain.

However, Cooper and Harries (2005) report that it is working-class learners who were least likely to recruit 'everyday' considerations when dealing with "calculational word problems" such as the lift item. The particular item referred to a notice, situated in an office block lift, which stated that: "The lift can carry up to 8 people" [19]. The task was to calculate how many times the lift "must go up" to transport 76 people during morning rush hour. If we were to consider this problem realistically, we would require queuing theory to solve the problem. However, the expected solution to the problem from the examiners is 10 lift trips. The problem, therefore, requires an inductive leap from the learner in that the learner has to assume that the lift is always full when possible and that everyone uses the lift.

Cooper and Harries (2005) found that many working-class learners correctly calculated 76 divided by 8 to produce the answer 9,5 but they ignored the fact that the problem referred to *lift trips* and so failed to see that the answer should be 10 trips. Here, according to Cooper & Harries, there is an expectation that learners would weaken the boundary between mathematics and the 'everyday', but it is the working-class learners who create strong boundaries between mathematics and the 'everyday'. On the one hand, according to Cooper and colleagues, the failure of working-class learners is located in their recruitment of the 'everyday' (weakening classification with respect to mathematics) and, on the other hand, their failure is located in their suspension of 'everyday' considerations (strengthening classification with respect to mathematics). So, what is it to be?

Cooper and Harries (2005) recognise the anomaly in their findings. They argue that for the short "calculational word problems" (like the lift item), in contrast to more extended context problems (like the coke-and-popcorn item), the presence of numbers prompts learners to perform calculations and to ignore the 'everyday'. However, the coke-and-popcorn problem also contains numbers. It is curious that the classificatory boundary seems to shift depending on the type of problem presented to learners. This is even more curious given that Cooper and colleagues base their arguments on the purported predisposition of working-class children to use 'everyday' knowledge inappropriately when solving mathematics test items (Cooper & Dunne, 1998, p. 125).

> This relative failure (and it is relative, not absolute) to recognise the strongly classified nature of school mathematics in the face of surface appearances which suggest everyday knowledge may be an aspect of sociocultural predispositions discussed by Bourdieu and Bernstein (Cooper & Dunne, 1998, p. 140).

This paradox in Cooper and colleagues' findings renders their deployment of the concept classification with respect to mathematics and 'everyday' knowledge inconsistent because if they base their arguments on the predisposition of working-class learners, then this predisposition should be consistent and should explain their data.

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[19] The original lift problem question involved 269 people using a lift which can carry up to 14 people where the expected solution is 20 (Cooper & Dunne, 2000, pp. 35-37).

It seems that the working-class learners in Cooper & Dunne's study recognise *coke*, *popcorn* and *money* as computational objects which are immediately recognisable in the coke-and-popcorn item but fail to recognise the more complex computational objects *coke*$\wedge$*popcorn* and *coke*$\wedge$*coke*$\wedge$*popcorn*, which are central to solving the problem and thus realise an illegitimate responses to the item. The syllogistic reasoning of the working-class learner described above is as follows: the learner needs the value associated with *coke* so she can find the value associated with *popcorn*. Since the value associated with *coke* is not provided in the test item, the learner derives the value associated with *coke* from experience and is then able to find the value associated with *popcorn* through some computation.

In the lift trip item, the *total number of people*, *maximum number of people in the lift at one time* and a *lift trip* are the computational objects required to be recognised. The *total number of people* and the *maximum lift capacity* are easily recognisable as computational objects from the question but the *lift trip* is a less obvious computational object. It seems that the working-class learners recognise the quantities provided and recognise that these have to be operated with. Some of these learners choose the correct operation, division, but neglect to convert the fractional answer to a whole number because they fail to recognise that the number of *lift trips* is a natural number. Learners have to recognise that the answer (9,5) to the division problem $76 \div 8$, in fact means that nine trips have a full load of eight people and one trip has half full load. So, there are 10 lift trips if we accept the inductive assumptions spelt out earlier.

In both test items, working-class learners recognise that they must find suitable computational objects with which to calculate and to relate these computational objects appropriately. However, it seems that the learners' failure to produce correct solutions to the problems can be traced to their inability to select the appropriate computational objects from the range of computational objects available in the test items rather than to decisions about whether or not to recruit 'everyday' knowledge. As stated earlier, if one took into account 'everyday' knowledge of lifts, the solution to the problem becomes much more complex than simply dividing 76 by 8. In other words, the learners fail to grasp the classificatory principle encoded in the test items.

I agree with the conclusion reached by Cooper and his colleagues that the working-class learners in their study fail to recognise the classificatory principles of the test items. However, their analysis of learners' responses to realistic mathematics items in terms of reference to the 'everyday' is problematic because reference to the 'everyday' generates inconsistencies in their analysis, as illustrated above. Furthermore, the deployment of Bernstein's proposition regarding orientation to meaning which entails the concept of classification with respect to mathematics and the 'everyday' produces a misreading of the empirical and generates inadequate descriptions of mathematics constituted by the learners in their study.

Orientation to meaning may still be an issue, but not in the way in which the construct is described in the work of Cooper and colleagues. Their work displays descriptive inadequacy with respect to mathematics and raises questions about the explanatory adequacy of their deployment of Bernstein's proposition on the social class basis of an individual's orientation to meaning.

Through an analysis of the work of Cooper and his colleagues, I show how the methodological resources for describing the recognition and realisation rules employed by learners and their orientation to meaning, although purporting to describe mathematical objects, generate sociological objects as the primary objects and consequently produce inadequate accounts of the constitution of mathematics in pedagogic situations. The above analysis demonstrates that the academic-everyday distinction which is tied to learners' orientations to meaning is problematic as a methodological resource, particularly in relation to mathematics.

2.4.2 The use of Bernstein's notions of classification and framing

As discussed above, studies recruiting Bernstein's theory (e.g., Adler, Pournara, & Graven, 2000; Cooper & Dunne, 2000; Hoadley, 2005, 2007; Lerman, 2009; Reeves, 2005; Pausigere, 2015, 2016; Reeves & Muller, 2005; Straehler-Pohl, 2010, 2015; Straehler-Pohl & Gellert, 2013) utilise the concepts of *classification* and *framing* as methodological resources to describe the relationship between school mathematics and 'everyday' knowledge in pedagogic contexts. Strong classification in such studies refers to strong boundaries between school mathematics and the 'everyday' and weak classification refers to greater integration of school mathematics and the 'everyday'. They claim that classification between mathematics and the 'everyday' is strong in middle-class contexts whereas classification between mathematics and the 'everyday' in working-class contexts tends to be weak. In part, the argument goes, it is this difference in the constitution of mathematics in the pedagogic situations of schooling that contributes to social class differences in performance in mathematics. Below I consider the study conducted by Hoadley (2005, 2007) in order to exemplify how the academic-everyday distinction and the explicit-implicit distinction, operationalized through Bernstein's notions of classification and framing, function as methodological resources.

Hoadley (2005, 2007) uses Bernstein's code theory including his concepts of *classification* and *framing,* together with Dowling's (1998) notions of *domains of practice* and *distributing strategies,* to describe variations in pedagogic modalities of Grade 3 classrooms that differ with respect to the social class membership of learners. She claims that, in middle-class schools, mathematics distributed to learners is specialised in that it is strongly bounded from 'everyday' knowledge and that the evaluative criteria (that which marks out legitimate knowledge from non-legitimate knowledge statements for learners) are explicit and strongly controlled by the teacher. In contrast, she claims that mathematics in working-class schools is weakly bounded from 'everyday' knowledge, and evaluative criteria are implicit and, on many occasions, absent. Hoadley claims that her study shows how pedagogy in different social class settings serves to reproduce social class and, in particular, how pedagogy in the working-class contexts fails to interrupt working-class learners' orientation to meaning acquired in the home. Hoadley's (2007) claim regarding the differential distribution of knowledge along social class lines is cited and reproduced by a number of scholars who recruit the academic-everyday distinction in studies that similarly explore the reproduction of social class (e.g., Barrett, 2017; Graven, 2014; Hoadley, 2016; Knipping, Reid, & Straehler-Pohl, 2015; Pausigere, 2015; Straehler-Pohl, 2015; Valero et al., 2014; Venkat, 2013). Furthermore, Hoadley's (2007) finding with regard to social class difference in terms of explicit/implicit evaluative criteria is likewise recruited by scholars (e.g., Aploon-Zokufa, 2013; Barrett, 2017). Aploon-Zokufa (2013) and Barrett (2017) confirm the proposition that explicit evaluative criteria are strongly correlated with better performance in their studies located in schools, populated by learners from working-class families, that are performing better than other working-class schools.

What follows is an in-depth analysis of the academic-everyday distinction and the explicit-implicit distinction operationalised through Bernstein's notions of classification and framing in Hoadley (2005, 2007). A description of a Grade 3 mathematics lesson used in her analysis illustrates her argument (Hoadley, 2007, pp. 687-689).

The teacher started the numeracy lesson by reading a word problem from a textbook. An extract of a transcript of the lesson illustrates how the teacher engaged the learners in repeating a sentence she read from the textbook in a chorus-like fashion:

Teacher:	Listen, on page 63, how a tree lives and grows. It says that …what does it say people. How a tree lives and grows. What does it say?
Learners:	How a tree lives and grows.
Teacher:	What does it say?
Learners:	How a tree lives and grows.
Teacher:	What does it say?
Learners:	How a tree lives and grows. (Extract from Hoadley, 2007, p. 687).

The chorusing was followed by the teacher translating the sentence into isiXhosa[20] and the learners repeating the sentence in isiXhosa followed by chanting the sentence in English again. The teacher and the learners dealt with the rest of the word problem—"Pulani has about 289 trees on her farm. Write the number of trees to the nearest hundred"—in the same way. After reading the word problem, the teacher drew two trees on the board and talked about trees being shaped differently. The teacher, then, wrote the symbol '79' on the board for learners to round off to the nearest hundred (Hoadley, 2007, pp. 687-689).

When the learners could not solve the problem, the teacher stated a rule that the learners chanted:

Teacher:	I say to you if the number is over 50 then it's a 100, if it's over 100 then it's 200, if it's over 300 …
Learners:	[chant] Then it's 400, if it's over 400 then it's 500.

(Extract from Hoadley, 2007, p. 688)

The teacher recorded six numbers on the board that the learners read from the textbook. She asked them to "write the numbers to the nearest hundred". When a learner rounded up 114 to 200, the teacher asked the

[20] IsiXhosa is one of the 11 official South African languages.

researcher whether the learner was correct. After the researcher responded that the learner was incorrect, the teacher provided a revised rule for rounding numbers to the nearest hundred.

> Teacher: You haven't started writing. I want the nearest hundred. Write. I said to you if it is over 50 the nearest hundred, it goes to hundred. If it is below 50 it doesn't go to a 100. If it is above 150 something it goes to 200. If it is below 150 something then it doesn't go. Do you understand? Same as if it's above 200 and something. If it is 250 something and above it goes to 300, if it doesn't go above 250 something, then it doesn't go to 300. It remains 200, ne? Do you understand? We are going to explain it again tomorrow.
>
> (Extract from Hoadley, 2007, p. 688)

According to Hoadley (2007), by the end of the lesson the learners copied the word-problem and the numbers from the board but did not successfully complete the problems and the teacher did not return to the exercise later.

Hoadley (2007, p. 688) reports that "the semantic resources for the lesson lay in everyday knowledge or the theme, and the object of the lesson was the theme and not the mathematical knowledge". She coded the lesson as very weakly classified with respect to mathematics because she saw the boundary between school knowledge and 'everyday' knowledge as being very weak and framing with respect to the evaluative criteria as weak because the criteria were implicit and unclear.

If we examine the teacher's lesson more closely, we notice that the teacher focused on three different aspects in the lesson: 1) reading English and translating from English to isiXhosa; 2) shapes of trees; and 3) 'rounding' numbers. The lesson can therefore be partitioned into three segments, each with its own focus. In the first segment, the teacher read what appeared to be the title of a section "How a tree lives and grows" from the textbook, asking learners to read the sentence repeatedly. It is not clear from Hoadley's description whether the learners read the sentences or simply repeated what the teacher said. The teacher then translated the sentence into isiXhosa and asked learners to repeat what she said. In this segment the teacher appeared to be doing 'literacy' before doing mathematics. In the second segment, the teacher discussed shapes of trees with learners. This segment was very short with the teacher stating that trees have the shape of umbrellas or circles. When learners were asked to provide other shapes, they merely repeated what the teacher had said. We should bear in mind that this lesson took place at a time when C2005 was in place. It seems highly likely that the teacher was attempting to 'integrate' learning areas given that integration across learning areas was one of the key principles of that curriculum. The first and second segments of the lessons took about 24 minutes of the 35-minute lesson. In the third segment, the teacher dealt with rounding numbers to the nearest hundred.

The lesson as a whole was fragmented with the link between the segments being the word problem involving trees and less than a third of the time was spent on rounding numbers to the nearest hundred. That trees seemed to be the organising principle of the lesson is true, but that 'trees' served as the 'semantic resources' for the lesson is questionable. What does it mean to use the 'everyday' knowledge of trees in a school mathematics lesson? Everyday knowledge of trees, for example that a tree has leaves or provides shade, is not used in the lesson. When the teacher eventually dealt with rounding numbers, she no longer referred to trees.

In fact the teacher changed the problem, strengthening the boundary between mathematics and the everyday in terms of Hoadley's formulation of classification. The original question referred to the computational object a *collection of trees*, with which was associated an approximate value, 289. So the idea, in 'everyday' terms, seems to be: if Pulani has about 289 trees then we can say that she has about 300 trees. Here, however, the solitary 79 of the teacher no longer referred to an amount of everyday objects. By focusing on the computational requirement of the problem the teacher attempted to fashion a procedure for rounding and she ejected the "context".

Let's examine the teacher's first rule for rounding numbers to the nearest hundred: "I say to you if the number is over 50 then it's a 100, if it's over 100 then it's 200, if it's over 300 …". Her rule (where n is the number to be rounded to the nearest 100 and m the nearest 100 to which n should be rounded to) can be written as follows:

If $n > 50$, then $m = 100$

If $n > 100$, then $m = 200$

If $n > 200$, then $m = 300$

If $n > 300$, then $m = 400$; and so on ….

We observe that the teacher had an explicit procedure for rounding-up numbers to the nearest hundred and it was a procedure that the learners grasped inductively. The teacher however reconfigured the procedure after a learner rounded-up the number 114 to 200, a correct response according to the teacher's original procedure.

Teacher: I said to you if it is over 50 the nearest hundred, it goes to hundred. If it is below 50 it doesn't go to a 100.

Teacher: If it is above 150 something it goes to 200. If it is below 150 something then it doesn't go. Do you understand? Same as if it's above 200 and something. If it is 250 something and above it goes to 300, if it doesn't go above 250 something, then it doesn't go to 300. It remains 200, ne? (Extract from Hoadley, 2007, p. 688)

The teacher's revised procedure can be written as:

If $n > 50$, then $m = 100$

If n > 150, then $m = 200$

If $n > 250$, then $m = 300$

Here the teacher introduced a procedure for rounding-up and it was only when she said "if it doesn't go above 250 something, then it doesn't go to 300. It remains 200, ne?" that you realise that she also had a procedure for rounding-down numbers to the nearest hundred. It is highly unlikely that the learners picked up on this criterion. There is no doubt that Hoadley is correct in her overall conclusion that the teacher created confusion about rounding numbers to the nearest hundred, that the topic was not satisfactorily dealt with in the lesson and that about two thirds of the lesson was spent on non-mathematical activity of reading and translating a couple of sentences from a textbook. But contrary to Hoadley's analysis, the section of the lesson on rounding numbers to the nearest hundred focuses entirely on mathematics with no reference to 'everyday' objects or contexts. As such, the boundary between mathematics and the 'everyday', in Hoadley's terms, appears to be strongly classified.

Does it mean that when the teacher is reading the word problem that contains the word "trees" that she is focusing on "trees" as 'everyday' knowledge or when she is describing the shapes of trees that the topic is 'trees'? So what is meant by 'everyday knowledge' and how is the use of 'everyday knowledge' recognised empirically? Hoadley (2007, p. 682) uses Bernstein's distinction between educational knowledge and 'everyday' knowledge where Bernstein defines educational knowledge as "uncommonsense knowledge […] knowledge freed from the particular, the local (Bernstein 1971: 215)". It appears that for, Hoadley, it is *references* to extra-mathematical objects rather than *knowledge* of the computational objects (in this case trees) that serve as indices of so-called context-dependent meanings and so of 'everyday' knowledge. 'Everyday' knowledge appears to be a catchall term for extra-mathematical referents or even extra-topic referents.

Hoadley coded the evaluative criteria evident in the lesson extract involving rounding numbers to the nearest hundred as unclear and implicit. However, the teacher's first rule for rounding-up numbers to the nearest hundred is clear and explicit; it is a rule that the learners grasped. When the teacher reformulated the rule, the rule was explicit, although mathematically imprecise, since the teacher said "if it is above 150 something" probably meaning if the number is greater than 150.

How are we to make sense of Hoadley's analysis of the Grade 3 lesson? Her analysis only makes sense if we read 'explicit evaluative criteria' as criteria that resonate with the mathematics that *ought* to be present, in other words, mathematically correct criteria and not the evaluative criteria *actually* present in the pedagogic context. Hoadley's (2007) use of the notion of implicit criteria is based on an idea of what ought to be present mathematically in the pedagogic situation. Davis & Johnson make a similar comment about the ESSA group[21]'s deployment of the notion of explicit evaluative criteria.

> When ESSA researchers speak of the necessity to make explicit evaluative criteria they have in mind only very specific criteria, which may or may not be circulating in any empirically given pedagogic context. For them the criteria to be made explicit are criteria that enable the legitimate reproduction of science in a principled fashion. That is, of course, highly desirable, but such criteria need not be present in a given empirical pedagogic context and are, therefore, externally defined by ESSA as necessary. In other words,

[21] ESSA (Sociological Studies of the Classroom) group is a research project based in Portugal and led by Anna Morais. ESSA recruits Berins theory of pedagogic discourse to study the primary and secondary science teaching and science curriculum.

> for ESSA productive teaching and learning *should* explicitly exhibit the operation of particular evaluative criteria—a prescription for the realisation of good pedagogic practice (Davis & Johnson, 2007, p. 130, italics in original).

Hoadley's deployment of Bernstein's concept of classification with *reference* to 'everyday' objects as the index of classification and the focus on the clarity and explicitness of the evaluative criteria masks the content emerging as a consequence of the recognition and realisation rules operating in the pedagogic situation. Hoadley's deployment of Bernstein's concept of classification and framing resonates with the deployment of classification and framing in other Bernstein-related research (see Adler et al., 2000; Hoadley, 2005; Lerman, 2009; Morais, 2002; Muller & Taylor, 1995, 2000; Reeves, 2005; Reeves & Muller, 2005; Rose, 2004; Straehler-Pohl, 2010; Taylor, 1999, 2000).

Although Hoadley's analyses may still hold sociologically in relation to the analytic categories she establishes, the methodological resources deployed produce a misreading of the empirical at the level of mathematics. Her account of the functioning of evaluation and so the constitution of mathematics in the lesson extract discussed above is mathematically inadequate. The descriptive inadequacy of the account of the constitution of mathematics raises questions about her explanation of the differences in performance between working-class and middle-class learners as residing in the mathematics distributed to them. Her conclusions may be valid but the descriptive inadequacy of classification and framing as methodological resources, described in terms of the academic-'everyday' distinction and explicit/implicit evaluative criteria respectively, produces an inaccurate account of mathematics constituted in the pedagogic situations. Furthermore, her analysis generates an inaccurate account of the functioning of evaluation in the pedagogic situation, and so consequently jeopardises the explanatory adequacy of classification and framing with respect to evaluation.

My criticisms of the mathematical inadequacy of classification with respect to mathematics and the 'everyday' are not directed against Bernstein's concept of classification per se but at the way that the concept has been deployed in Bernstein-related research. Bernstein's concept of implicit/explicit criteria, however, is generally problematic at the level of the theory and consequently in its deployment. What does it mean for criteria to be implicit? How are implicit criteria recognised empirically? Bernstein's sociolinguistic thesis provides a clue to the notions of implicit/explicit. For Bernstein implicit/explicit language is aligned with particularistic/universalistic meanings and context-dependent/context-independent meanings.

> It is possible to distinguish between forms of speech where the referents are not in the text, but in the *context*. In the latter case, unless the listener has access to critical features of the context in which the speech is imbedded, the meanings are not clear. Dr Hasan calls such context imbedded speech exophoric. When speech is exophoric, the meanings are highly context dependent (Bernstein, 1971, p. 14, italics in original).
>
> On the other hand, context-independent meanings did not rely to anywhere near the same extent upon shared, unspoken understandings of critical features of the context, for context-independent meanings are

> linguistically explicit. From this point of view, context-independent meanings are available to all. Thus context-independent meanings are *universalistic* (Bernstein, 1971, p. 14, italics in original).

Firstly, meanings are always context-dependent. If the context (such as a bit of linguistic text) obscures particular features necessary to understand the context (and hence, the meaning) appropriate meanings cannot be arrived at. It might be more appropriate to refer to context-independent expressions than to context-independent meanings. Secondly, the alignment of universality with context-independence is problematic since it is possible to form expressions, the meaning of which can clearly be ascertained outside of the context of initial enunciation, but which are not semantically universal.

Bernstein's notions of implicit/explicit criteria seem to refer to the nature of expressions used. However, "shared, unspoken understandings" does not necessarily mean that criteria are implicit only that the criteria are context-dependent. In other words, if "shared, unspoken understandings" are present in a pedagogic context, the criteria should be available to those in the pedagogic context. In Bernstein-related research, as is evident in Hoadley's (2007) study, the deployment of the notions of implicit/explicit criteria rests on an expected idea of what knowledge ought to be present in the pedagogic situation. This moral imperative, as we have seen with Hoadley (2007), renders inaccurate accounts of mathematics constituted in pedagogic contexts because the methodological apparatus is searching for what *ought* to be present rather than what *is* present.

In the next section I consider Dowling's (1998) categories of *domains of practice* which are often used by researchers (e.g., Hoadley, 2007; Sethole, 2007; Straehler-Pohl & Gellert, 2013) to overcome what they perceive as the inadequacy of Bernstein's theory to describe and analyse the subject matter.

2.4.3 Dowling's domains of practice

Dowling's (1998) analysis of a School Mathematics Project (SMP) textbook series developed from his engagement with the work of Basil Bernstein. Dowling's concept of domains of practice is embedded in his language of description, Social Activity Theory, developed to analyse the SMP textbook series. In his Social Activity Theory, an activity is an analytic space for describing the empirical and (re)produces the social division of labour through regulating what subjects may say, do or mean. At the structural level, an activity specializes practices and constitutes positions within practices. At the textual level, an activity distributes messages through strategies over a range of positions (voices). Although his language of description is extensive, only his concept of domains of practice is discussed here.

Dowling's theory has been used by a number of mathematics education researchers (e.g., Adler et al., 2000; Brantlinger, 2011; Christiansen, 2007; Coombe & Davis, 1995; Cooper & Dunne, 2000; Davis, 1995a, 1995c; Ensor, 1997; Galant, 1999a; Gellert & Jablonka, 2009; Hoadley, 2005, 2007; Lerman, 2014; Parker, 2006). Central to Dowling's domains of practice is his notion of classification, which differs from Bernstein's notion of classification (Dowling, 1998, p. 117). Dowling redefines classification as the degree of specialisation rather than the strength of boundaries between categories, thus ejecting the notion of boundary. Furthermore, he employs the Saussurean distinction between *expression* (signifier) and *content* (signified). The cross product of weak and strong classification of expression with weak or strong content produces four domains of practice, as shown in Figure 2.1. Dowling (2009) changed the term *domains practice* to *domains of action* and the notion of *classification* is replaced with the notion of *institutionalisation,* both of which are discussed later.

The *esoteric domain* represents the domain of practice of an activity, for example school mathematics, which is most strongly classified with respect to other activities. The *public domain* (content weakly classified and expression weakly classified) is the domain of practice that has the form of a non-specialised practice but is nevertheless a domain of practice of an activity. Furthermore, with respect to the public domain, Dowling notes that the principles of an activity such as school mathematics "cannot be adequately expressed within this domain, because there can be no certainty of the prioritizing of specialised denotations and connotations" (Dowling, 1998, p. 136). He simultaneously maintains that the public domain is a portal for novices to enter the practice. The two notions of the public domain, one as a portal for novices, and the other, as the domain in which the principles cannot adequately be expressed, are contradictory since we cannot make assumptions about the mathematics content used to introduce learners to mathematics. What is actually used in any given situation requires empirical investigation.

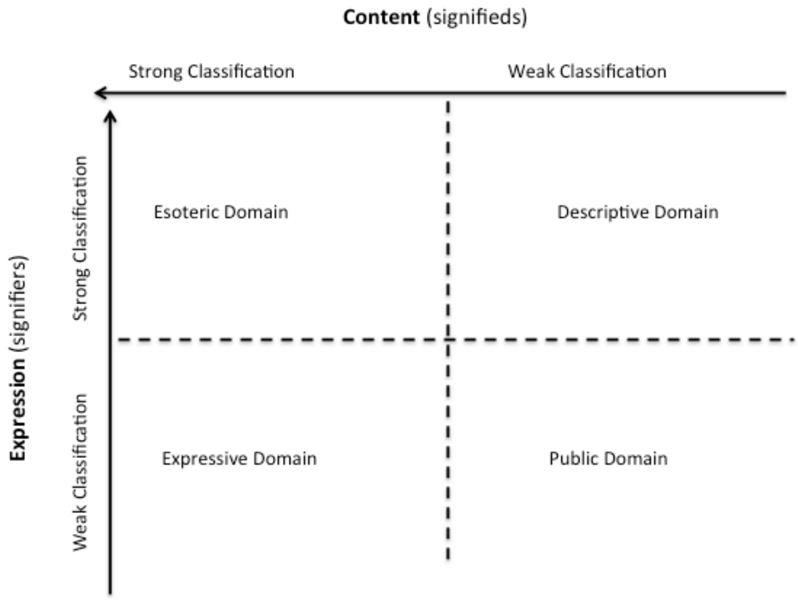

Figure 2.1. Dowling's domains of practice (Dowling, 1998, p. 135)

Dowling (2009, pp. 234-239), drawing on the work of his PhD students, provides various reconfigurations of the public domain generating a fractal structure of the domains of action (previously referred to as domains of practice). One configuration is shown in Figure 2.2, where *I* stands for institutionalisation.

He argues that the fractal structure enables analysis at different levels, akin to Bernstein's pedagogic device. The fractal structure appears to be an attempt to deal with the concern that the portal cannot be restricted to public domain text (i.e., content and expression as weakly classified with respect to mathematics) since novices can be inducted into school mathematics through a range of tasks that differ in terms of their domain location. However, placing the esoteric domain or any other domains into the public domain is curious because the public domain is defined as weakly classified with respect to mathematics in terms of content and expression. An alternative explanation for carving up the public domain into the four domains of action (practice) is to consider the public domain as referring to extra-mathematical contents. So, the esoteric domain within the public domain refers to the esoteric domain of a non-mathematical activity. However, this explanation is also problematic because, according to (Dowling, 2009, p. 99), non-mathematical practices recontextualised to constitute the public domain have to conform to the principles of specialised mathematical practice, i.e., to the esoteric domain of school mathematics.

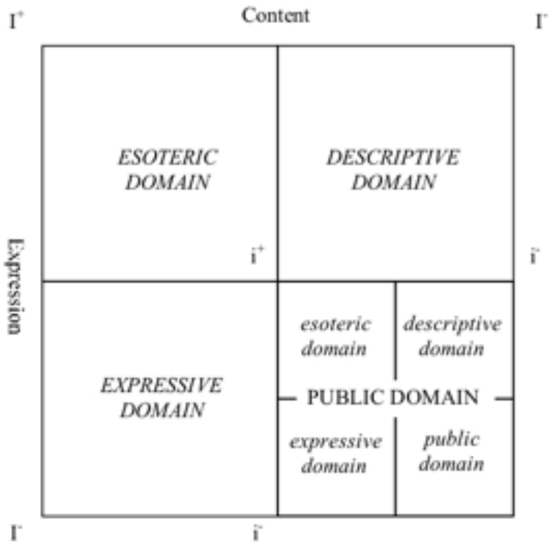

Figure 2.2. Dowling's (2009, p.235) domains of action

Dowling uses the extracts in Figures 2.3, 2.4, 2.5 and 2.6 from the SMP textbook series to exemplify his domains of practice. He categorises the extracts in the following way: Extract A (Figure 2.3) is strongly classified in terms of content and expression and is therefore located in the *esoteric domain*.

EXTRACT A

Solve (a) $18x+92=137$ (b) $0.7x+3.2=4.88$
(c) $2.9x-3.5=19.7$ (d) $0.4x-4.6x-4.6=-2.0$

(*SMP 11–16 Book Y1*, p. 66)

Figure 2.3. Extract from SMP textbook (Dowling, 1998, p. 133)

The text in Extract B (Figure 2.4), depicted as an original shopping list in the textbook, indexes shopping rather than mathematics and is weakly classified with respect to content and expression. Dowling (1998) locates Extract B in the *public domain*.

EXTRACT B

A4 What is the bill for each of these shopping lists?
Work out each bill in your head.

(a) 1 kg potatoes
1 grapefruit

(b) 2 oranges
1 cauli

(c) 2 kg spuds

(d) 1kg bananas
100 gmushrooms
1kg bananas

(e) 1 grapefruit
1 orange

(*SMP 11–16 Book G1*, p. 14; original shopping lists drawn as fragments of paper)

Figure 2.4. Extract from SMP textbook (Dowling, 1998, p. 133)

Extract C (Figure 2.5) represents a non-mathematical activity because it indexes an everyday practice of running a business but the task utilises algebraic symbols. At the level of content the task is weakly classified because it refers to loaves of bread but at the level of expression the task represents strongly classified mathematical knowledge. Extract C is located in the *descriptive domain.*

EXTRACT C

A café orders p white loaves and q brown loaves every day for r days.
What does each of these expressions tell you?

(a) $p+q$ (b) pr (c) qr (d) $(p+q)r$

(*SMP 11–16 Book Y1*, p. 32)

Figure 2.5. Extract from SMP textbook (Dowling, 1998, p. 134)

Extract D (Figure 2.6) is strongly classified with respect to content because the signifiers are mathematical symbols. However because the 'machine' (a non-mathematical expression) is used, the task is weakly classified with respect to expression. Extract D is located in the *expressive domain*. Pedagogic metaphors such as a "fraction is a piece of cake" are included in the expressive domain. Thus, the academic-everyday distinction is central to Dowling's domains of practice.

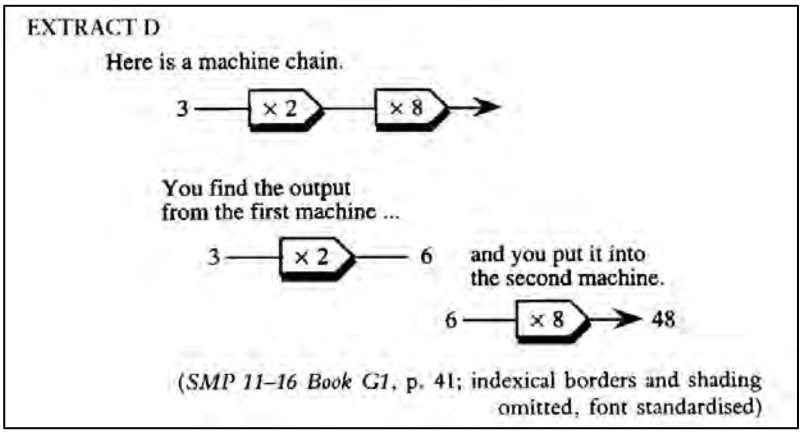

Figure 2.6. Extract from SMP textbook (Dowling, 1998, p. 134)

As alluded to earlier, in his more recent work, *Sociology as Method*, Dowling (2009) substitutes *classification* with *institutionalisation* and *domains of practice* with *domains of action* (see Figure 2.7).

Expression (signifiers)	Content (signifieds)	
	I^{+}	I^{-}
I^{+}	*esoteric domain*	*descriptive domain*
I^{-}	*expressive domain*	*public domain*

$I^{+/-}$ represents strong/weak institutionalisation

Figure 2.7. Domains of action (Dowling, 2009, p. 206)

Dowling (2009) defines institutionalisation as "a regularity of practice emergent on autopoietic action" (Dowling, 2009, pp. 272-273). What is weakly or strongly institutionalised is determined by autopoietic action. Autopoiesis implies that Dowling would need to consider whatever emerges as mathematics in a pedagogic situation as strongly institutionalised according to his definition of institutionalisation. For example, since the CAPS (FET) stipulate the use of "real life contexts" (Department of Basic Education, 2011, p. 8), these documents should be located in the esoteric domain by Dowling. However, according to his schema shown in Figure 2.7, Dowling would be forced to place the curriculum texts use of "real life contexts" into the public domain because both expression and content are weakly institutionalised. Dowling's notion of autopoiesis, although it purports to describe what emerges empirically, is overshadowed by his earlier use of classification that appeals to mathematics in the field of its production when deciding on the strength of classification of expression and content of school mathematics texts.

Dowling's domains of practice represent a fetishising of certain signifiers because the more the signifiers look like mathematics found in the field of production, the more likely for the text to be categorised as esoteric domain or descriptive domain. Classification/institutionalisation with respect to content is dependent on the presence or absence of *reference* to 'everyday' objects. In other words, the presence or absence of references to 'everyday' objects constitute the criteria for identifying whether the content (signified/meaning) is weakly or strongly classified with respect to mathematics in Dowling's schema. Methodologically, using reference to everyday objects as the criterion for recognising content is problematic because mathematics entails operations performed on objects irrespective of whether the objects referred to are dogs, numbers or sets. Dowling, therefore, conflates *referring* with *content*.

Now, looking back at Cooper and Dunne (1998, 2000) and Hoadley (2005, 2007), we observe that their use of classification with respect to mathematics and the 'everyday' is more aligned to Dowling's notion of classification than Bernstein's. So their work also suffers from the reference problem discussed earlier. The discussion of the work of Cooper and colleagues, Hoadley (2005, 2007) and Dowling (1998, 2009) serves to exemplify the methodological inadequacies with the academic-everyday distinction with respect to mathematics.

2.5 Summary

This chapter outlined some empirical antecedents to this study in order to firstly, locate and frame the study and secondly, to argue for the methodological orientation of the study. The review of comparative studies on mathematics in pedagogic contexts differentiated with respect to learners' social class membership illustrated that the current study is the only South African comparative study focusing on secondary school mathematics in pedagogic contexts that differ with respect to the social class membership of learners. Furthermore, the review of the aforementioned literature confirmed that the methodological orientation of study, with its focus on the computational activity of teachers and learners, is unique with respect to studies focusing on mathematics in pedagogic contexts that differ with respect to the social class membership of learners.

This chapter focused centrally on the academic-everyday distinction and the procedural-conceptual opposition/distinction employed by mathematics education researchers generally and in particular by Bernsteinian and neo-Bernsteinian scholars, who in addition sometimes employ the distinction between explicit/implicit evaluative criteria particularly in relation to differential achievement in school mathematics along social class lines. The main purpose of the chapter was to critically discuss the key methodological resources entailed in the aforementioned oppositions/distinctions in order to open up the methodological space that the thesis seeks to generate and populate.

The chapter demonstrated that the aforementioned oppositions/distinctions are methodologically inadequate for describing the functioning of evaluation, and so the recognition and realisation rules employed, with regards to mathematics in pedagogic contexts in that they misread or ignore mathematics realised in pedagogic contexts. I illustrated that the methodological resources tend to generate imprecise and inaccurate accounts of mathematics constituted in pedagogic contexts because analyses of mathematics in pedagogic situations are based on *a priori* notions of what mathematics is rather than what emerges as mathematics in pedagogic situations.

In my discussion, I drew attention to the descriptive inadequacies at the level of mathematics of the methodological resources deployed across the literature reviewed. I argued that while the general disciplinary propositions generated within these accounts may be valid, the mathematical descriptions produced were mostly imprecise and often misleading. I argued that the descriptive inadequacy of the methodological resources with respect to mathematics raises questions about their explanatory adequacy.

As discussed briefly in Chapter 1, the study recruits and further develops Davis' (2013a, 2013b) computational approach to analysing mathematics emerging in pedagogic situations. I argue that a computational approach is an attempt at providing greater descriptive adequacy in relation to mathematics in pedagogic situations. In the next chapter, I engage with antecedent literature pertinent to the theoretical concerns of the computational approach adopted in the current study with the aim of producing a set of theoretical propositions that are intended as the basis for the procedures for the production and analysis of data discussed in Chapter 5.

Chapter 3

General methodology

3.1 Introduction

This chapter sets out the general methodology of the study in order to frame its theoretical and methodological orientation. The general methodology derives from an engagement with antecedent literature pertinent to the theoretical concerns of the study. The purpose of the chapter is to generate a series of theoretical propositions that serve as the basis for the procedures for the production and analysis of data, which is the focus of Chapter 5.

The research problem identified in Chapter 1 comprises two aspects. The first focuses on how evaluation functions at the level of the instructional discourse. The second concerns the specialisation of learners' mathematical thought as evidenced in their mathematical work performed independently of the teacher. Here, learners' recognition and realisation rules displayed in a test and in an interview context are described computationally. The learners' computational activity is then read in terms of the content realised, their computational performances and orientations to mathematics.

The study is concerned with what evaluation reveals about the constitution of the content of school mathematics in two schools that differ with respect to the social class membership of their learner populations and how evaluation structures learners' computational performance and orientation to mathematics. The key theoretical resources recruited derive from: (1) Bernstein's theory of the pedagogic device (discussed in Chapter 1); (2) Eco's semiotic concepts, *topic* and *isotopy*; (3) his notion of the *Model Reader* and *closed* and *open texts*, which are adapted for analysing mathematics pedagogic texts; and (4) Lotman's concepts for describing culture, *grammar/text-orientation* and *content/expression-orientation* as adapted by Davis (2011b) for analysing mathematics pedagogic texts. In addition, the methodology for describing the constitution of mathematics is based on that developed by Davis (see Davis, 2011a, 2013a).

3.2 Davis' computational approach

As discussed in Chapter 1, a central proposition underpinning the methodology employed by Davis (2013a, 2013b) to investigate the constitution of mathematics, derives from a computational theory of mind, which posits that thought is computational in nature (Chomsky, 2006, 2007; Gallistel & King, 2010). Another key proposition foundational to Davis' methodology is located in Bernstein's (1996, 2000) proposition that pedagogy is necessarily evaluative.

Davis (2011a, 2013a) argues that the research is obliged to accept whatever emerges as mathematics in a pedagogic situation, whether or not what materialises might be considered questionable as mathematics from the point of view of a mathematics adept. In other words, decisions by the researcher regarding what is mathematics are not based on *a priori* notions of what the mathematics of the situation is, but entail describing what is realised as the computational apparatus in a particular pedagogic situation. Categories for the production and analysis of data thus emerge from an initial computational analysis rather than exist prior to it.

Following Davis, *pedagogic situation* is a term used in this study that attempts to capture Badiou's (2005) sense of *situation*[22] and refers to the computational 'stuff' that teachers and learners do and say as they go about doing mathematics. Specifically, the objects employed in their computations represent the elements of the pedagogic situation and the operations or operation-like manipulations performed over the collections of objects constitutes the computational logic of the pedagogic situation. The term *pedagogic situation* is used to refer to a particular lesson or part of a lesson and *pedagogic contexts* are taken as made up of pedagogic situations.

Davis (2010b, p. 102; 2013a, p. 35) claims that in order to ascertain what is constituted as mathematics in a pedagogic situation, we need to describe what it is that teachers and learners say and do as they go about doing mathematics, which is read off their oral, written, or gestural acts. 'Doing mathematics' involves computations which comprise compositions of operations/manipulations on objects, such as numbers or symbols representing numbers and other mathematical objects. Such operations and objects may not necessarily be those that are accepted as mathematical in the field of production or recognized by researchers as mathematics. This is in line with Badiou's (2005) indifference to the nature of the objects that belong to a situation – an idea derived from set theory. As such, the methodology does not only privilege objects and operations that are recognised as mathematics by researchers and adepts but considers all objects and operations or operation-like manipulations that emerge in a pedagogic situation. The objects involved in computations provide insight into the domains and codomains associated with the operations since operations and their objects are compossible. In other words, particular operations require particular types of objects. Alternate operations or manipulations suggest domains of objects different from those we conventionally associate with particular signifiers (Davis, 2010b). For example, the general domain and codomain of multiplication in school mathematics is the set of real numbers. In other words, the domain for multiplication comprises the cross product of the set of real numbers which is mapped to the codomain, the set of real numbers. However, an operation-like manipulation such as "change sides, change signs" cannot operate over numbers because numbers cannot be spatially displaced. So, the domain of the operation-like manipulation is in fact a set of symbols or characters which are amenable to spatial displacement.

[22] For Badiou, any collection of things ("what is there") such as a group of school children, a staff meeting, prayer service or political rally represents a multiplicity which is counted as one in some or other way (Ling, 2010) and it is this collection of 'stuff' which he refers to as a *situation* (Badiou, 2005) or a *world* (Badiou, 2009, p. 45). Multiplicities are always collected by an operation referred to by Badiou as the *count-for-one* which structures or organises a situation. The count determines *what* is presented as belonging to the situation and *how* it is presented. Badiou argues that there is always an excess beyond what is recognised as the elements in the count-for-one. Hence, the multiple is more than the one and a situation cannot be distinguished from the count (Ling, 2010).

Davis (2013a, p. 35) maintains that "compositions of operations are, of course, always regulated by higher level propositions and decision-making". So, *computational activity* includes the composition of operations on objects as well as the propositions and definitions or descriptions of mathematical terms which govern such operations. It is through an analysis of the computational activity of teachers and learners that we gain insight into the recognition and realisation rules entailed in the production of mathematical statements and their transformations. A description of the computational activity of teachers and their learners illuminates the functioning of evaluation in a pedagogic situation. The above is a broad outline of Davis' general methodology, the details follow in Chapter 5.

In order to better understand the way in which evaluation functions in pedagogic situations we need to consider pedagogic situations as communicative contexts (Bernstein, 1990).

3.3 Evaluation and communication in pedagogic situations

Pedagogy is a communicative context in which natural language serves as the general medium through which knowledge is communicated between teacher and learners and amongst learners. As such, communication in pedagogic situations is inextricably bound up with evaluative activity because recognition and realisation rules are communicated using verbal, scriptural or gestural semiotic resources. Following Davis (2013a), the general methodological orientation adopted here in relation to language and communication is framed by Chomsky's account of language in which language is conceived of as a biological organ. Thought and language are internal processes, while a very small component of this language is used for communication, which is the externalisation of thought through speech, writing, gesture or some other semiotic media (McGilvray, 2011, pp. 13-14).

Recall from Chapter 1 that language does not have a reference function (Chomsky, 2007; Strawson, 1950). Consequently, since reference is tied up with the *use* of language which has been traditionally located in pragmatics, Chomsky proposes that reference should be disassociated from semantics to be placed in language use (pragmatics) instead and goes further to claim that possibly language only has syntax and pragmatics (Bilgrami & Rovane, 2007, p. 190). Chomsky argues for the elimination of an externalist referential semantics by placing meaning in syntax (McGilvray, 2011, p. 228). The displacement of semantics into syntax has methodological implications, briefly addressed below but discussed in more detail in Chapter 5.

The computational activity of teachers and learners entail three inter-related levels - expression, syntax and semantics (Davis, 2013a, p. 35). The level of expression refers to the expressive or lexical elements entailed in communicating mathematical thinking through speech, writing, gesture or any other semiotic medium. This is the level which is directly observable from what teachers and learners say and do. The syntax, which comprises the operations and associated domains and codomains, is central in relating the level of expression to the level of semantics, which is not directly observable but inferred from an account of the syntax. So methodologically, I make decisions regarding what teachers and learners refer to by examining their computational activity.

Given the problems with reference, evaluative acts performed by both the teacher and the learner are required in order for teachers and learners to check whether they more or less agree with each other. Often this agreement is assumed if both parties produce the same outcome for a given mathematical problem (Davis, 2013a). However, it is always possible to use entirely different domains and codomains and operations or operation-like manipulations, yet achieve the same outcome expressively. For example, a teacher or learner who factorises the quadratic expression $x^2 - 6x + 5$ to produce $(x - 5)(x - 1)$ as the factors may generate a procedure[23] that treats numbers as symbols and so therefore comprise operation-like manipulations that are not recognised as mathematics by adepts but produces the correct outcome expressively (See Chapter 5 for a fuller discussion of this example). So, it is possible to achieve the same outcome expressively across agents using different recognition and realisation rules.

Furthermore, Chomsky maintains that reference is highly contextual because a word can be used to refer to a wide range of things and is dependent on the intention of the speaker (Bilgrami & Rovane, 2007). The contextual nature of language use can be related to Saussure's notion of the arbitrariness of the sign[24].

> The link between signal [signifier] and signification [signified] is arbitrary. Since we are treating a sign as the combination in which a signal [signifier] is associated with a signification [signified], we can express this more simply as: *the linguistic sign is arbitrary*.
>
> There is no internal connexion, for example, between the idea 'sister' and the French sequence of sounds *s-ö-r* which acts as its signal [signifier]; the same idea might as well be represented by any other sequence of sounds. This is demonstrated by differences between languages, and even by the existence of different languages. The signification [signified] 'ox' has as its signal [signifier] *b-ö-f* on one side of the frontier, but o-k-s (*Ochs*) on the other side." (Saussure de, 1983, pp. 67-68, italics in orginal).

Saussure's proposition regarding the arbitrary nature of the sign implies that different signifiers can have the same meaning. For example, the signified 'ox' has several variants in different languages. The arbitrary nature of the sign, however, does not imply that the signifier-signified coupling is random and unmotivated. However, according to Saussure there is no necessity for a particular signifier-signified coupling.

> the arbitrary nature of the sign is really what protects language from any attempt to modify it. Even if people were more conscious of language than they are, they will still not know how to discuss it. The reason is simply that any subject in order to be discussed must have a reasonable basis. […] …but language is a system of arbitrary signs and lacks the necessary basis, the solid ground for discussion. There is no reason for preferring *soeur* to sister, *Ochs* to *boeuf*, etc. (Saussure, 1983, p. 73, italics in original).

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[23] The procedure entails calculating the factors of the last term, using the factors that add or subtract to give the numeral associated with the middle term and then establishing whether positive or negative signs should occupy the factorisation brackets.

[24] I am aware that the Saussurean signifier-signified couplet is in tension with Chomsky's internalist account of language since for Saussure language is considered as sound with meaning attached to it, that is, the signifier precedes signification which differs from Chomsky's internalist account where meaning precedes sound/signifier. Saussure's position is associationist.

> Because the sign is arbitrary, it follows no law other than that of tradition, and because it is based on tradition, it is arbitrary (Saussure, 1983, p. 74).

Saussure's proposition can also be considered in reverse, that is, for any given signifier (words, expression, mathematical statement), a number of different signifieds are possible. For example the word 'groom' means a person who takes care of horses or it could mean a man in the process of being married. In pedagogic situations, we have cases where the computations of teachers and learners related to the familiar signifiers of mathematics differ from those we conventionally associate with those signifiers (see factorisation example discussed earlier). Although teachers and learners may ascribe alternate signifieds for given signifiers, signifier-signified couplings are bounded by the connection of school mathematics to Mathematics[25]. In other words, the outcome of computations must satisfy Mathematics. For example, $-2+5$ must always equal 3 even though -2 is sometimes treated as though it is a composed of minus sign and natural number 2 rather than as an integer.

As discussed earlier, doing mathematics necessarily entails some form of regulation of the activity given that individuals use recognition and realisation rules that they believe could achieve their desired goals. So, individuals necessarily evaluate their own activity as they go about doing mathematics. This brings me to consider the regulative aspects of pedagogic evaluation.

3.4 The regulative dimension of pedagogic evaluation

Recall from Chapter 1 that for Bernstein evaluation regulates the production of legitimate texts through the transmission and acquisition of recognition and realisation rules. Recall, too, that Davis (2013a) argues that that different recognition and realisation rules may be operative even in cases where the same outcome is produced at the level of expression given that it is entirely possible to produce the same mathematical outcome expressively using different operations and domains of objects.

How, then, do we recover Bernstein's proposition regarding the centrality of evaluation given the aforementioned problems with his recognition and realisation rules (see Section 1.2)? This issue is addressed in two ways. The first recognises that holding onto the notion that evaluation resides solely in the hands of the teacher results in observationally inadequate research. Thus, the ambit of evaluation is extended beyond the teacher to include learners as necessarily evaluating and so regulating their own activity when faced with instructions to solve mathematics problems. The second entails expanding the methodological resources to deal with the regulation of mathematical activity. Here, I employ Davis' (2010a, 2011b) notion of *ground*.

3.4.1 Davis' categories of ground

By recruiting Hegel and Peirce, Davis (2010a) posits that thought is always subject to some form of internal regulation. From Hegel, Davis (2010a) argues that our experiences are always mediated and it is *ground* which plays that mediating role through regulating thought and language. In other words, whatever we

[25] I use the term Mathematics for knowledge generated in the field of production and school mathematics as the knowledge used by teachers and learners in pedagogic contexts.

encounter already entails some or other recognition and thus has some semantic intentionality. Furthermore, Davis (2010a) reminds us that Peirce's notion of the sign (representamen, interpretant and object) entails a fourth element, *ground,* which for Peirce is a function of the act of referring.

> A sign, or *representamen*, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. It stands for that object, not in all respects, but in reference to a sort of idea, which I have sometimes called the *ground* of the representamen (Peirce, 1931, p. 135, italics in original).

Drawing together Hegel's notion of ground and the role of ground in Peirce's notion of the sign, (Davis, 2010a) fashioned a notion of *ground* to describe the regulation of mathematical thinking.

> Ground, then, entails an ontological decision about what the objects upon which to operate are, and that decisions about the nature of the object appear to be informed by decisions about which operations to perform, so that operations might even be thought of as ontologically prior to the objects upon which they operate (Davis, 2010a, p. 379).

For Davis, the regulation of mathematical activity of teacher's and learners can be discerned from their computational activity. More precisely, the nature of the domains of the operations performed by teachers and learners are indicative of the form of regulation operative in the computational activity. So, although ground circumscribes the nature of the object referred to, methodologically the domains of objects along with the operations are used to infer the type of ground. Chapter 5 details the methodological procedures for recognising and realising categories of ground. Features of Davis' (2010a) four categories of ground - iconic, empirical, propositional and algorithmic[26] - are shown in Figure 3.1. Examples of each category of ground follows.

Ground	Central grounding resource	Objects of central concern
Iconic	Comparisons centred on iconic features, including similarities and differences of expressions	Graphical and/or symbolic expressions treated as images
Empirical	Empirical testing of expressions	Graphical and/or symbolic expressions treated as in some way "measurable"
Propositional	Knowledge of the mathematical objects and relations between such objects referenced by mathematical statements	Mathematical objects indexed by the axioms, definitions and propositions that are signified by expressions
Algorithmic	Meta-rules governing an algorithm, regulating the selection and sequencing of operations on mathematical signifiers	Operations commonly used within a particular algorithm as well as their sequencing

Figure 3.1. Categories of ground (Davis, 2011b, p. 311)

[26] Algorithmic ground was previously referred to as syntactic ground and procedural ground (Davis, 2010a). Propositional ground was referred to as fundamental ground. See Davis & Johnson (2008).

His categories of ground index the domains of objects that serve as the arguments to operations. Each category indicates a particular domain of objects, as outlined in Figure 3.1. *Iconic ground* underpins the production of knowledge statements when the objects involved in an operation are graphical or symbolic expressions themselves. In such cases, mathematical activity is regulated by the iconic similarity of expressions. For example, when a teacher describes a parabola as "smiley-faced" rather than as a parabola which has a minimum, iconic ground is said to be regulating mathematical activity because the attention is on imagistic similarity, focusing on what the graph looks like. Similarly, when a teacher insists that learners set out a solution to match hers exactly or when learners slavishly copy the form of the solution to a particular problem from the teacher, iconic ground is said to be regulating mathematical activity because the focus is on what the solution looks like (see Davis & Johnson, 2007; Jaffer, 2010b, 2011a; Johnson & Davis, 2010).

Empirical ground regulates mathematical activity when teachers or learners employ some testing or trial-and-error methods. *Algorithmic ground* refers to instances where the selection and sequencing of operations regulate the mathematical activity of teachers and learners. For example, a teacher's insistence that the equation $(x-3)^2 = 16$ has to be transformed into 'standard form' before solving the equation is an example of algorithmic ground since there is no procedural necessity for converting the equation to standard form before solving it. *Propositional ground* is evident when teachers and learners explicitly attend to Mathematics axioms, definitions and propositions underlying a particular topic[27].

Drawing on Badiou (2006), Davis (2011b, p. 310) claims that the categories of ground serve to highlight that decisions regarding the primary computational objects have an over-determining effect on what comes to be constituted as mathematics in a pedagogic situation, bearing in mind that such decisions are always regulated by pedagogic evaluation. Categories of ground do not in any way indicate a developmental sequence nor are they hierarchical. Furthermore, more than one category of ground may be operative simultaneously (Davis, 2010a).

Figure 3.2 shows Davis' (2011b) organisation of the list of categories of ground into a Greimassian semiotic rectangle which illustrates the inter-relatedness of the categories as a logical system. The utility of the Greimassian organisation of the categories of ground is that it yields six binary oppositions shown in Figure 3.2: algorithmic-iconic, iconic-empirical, propositional-empirical, algorithmic-propositional, propositional-iconic, and algorithmic-empirical (ibid., p. 314). See Davis (2011b, pp. 313-315) for his argument regarding the arrangement of the Greimassian semiotic rectangle and for a description of the binaries.

The iconic-algorithmic is evident when the iconic serves as the primary regulative resource in the selection and sequencing of computational objects and resources. Jaffer (2012b) details an example of iconic-algorithmic ground regulating the computational activity of a teacher whose procedure for "changing"

[27] Dowling (2013, p. 318) refers to theorem, procedure, template and operational matrix as strategies of the esoteric domain (specialised mathematics). Although these strategies appear similar to Davis' categories of ground - particularly theorem and propositional ground, procedure and algorithmic ground and template and iconic ground - the two sets of concepts differ methodologically in that Dowling does not focus on computational activity.

expressions with negative exponents to expressions with positive exponents entails a series of computations focused on generating and distributing symbols. Iconic-algorithmic ground was found to be most dominant in pedagogic situations populated by working-class learners (see Davis & Johnson, 2007; Jaffer, 2010b, 2011a; Johnson & Davis, 2010).

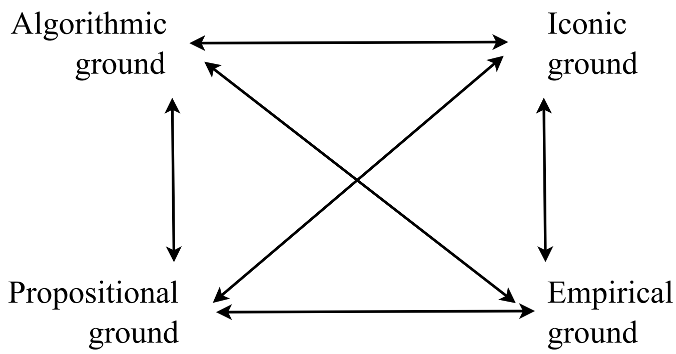

Figure 3.2. A Greimassian organisation of the categories of ground (Davis, 2011b, p. 315)

Ground is tied up with the way in which *necessity*[28] functions in pedagogic situations. Debates regarding necessity have occupied philosophy since Aristotle (see Bunnin & Yu (2004) for different positions regarding the notion of necessity).

3.4.2 Necessity in pedagogic situations and the notion of ground

In pedagogic situations, necessity refers to how knowledge statements are established as necessary 'truths'. In other words, what is known, or taken to be the case, in a pedagogic situation can be read off the computational activity of teachers and their learners. However, necessity also derives from other sources beyond the computational activity of teachers and their learners. Unlike Mathematics, which is a strictly denotative context, where necessity is an irrefutable deductive outcome of the coordination of a series of fundamental axioms, propositions and theorems, necessity in socio-cultural contexts can take different forms. Particularly in a curriculum context which backgrounds the axiomatic nature of Mathematics, mathematical necessity is replaced with other forms of necessity to justify propositions and procedures for solving mathematical problems and to regulate mathematical activity. For example, examination-marking criteria, which award partial marks for incorrect answers if a full solution is provided and zero marks for an incorrect answer if a full solution is not provided, are often used by teachers to regulate learners into producing full solutions to a mathematical problem rather than just providing answers. Such secondary regulative resources operative in pedagogic situations used to authorise mathematical statements, are referred to by Davis, Parker, and Adler (2005) and Davis, Adler, and Parker (2007) as authorising ground (see also Chitsike, 2011b). I first discuss categories of ground in relation to the notion of necessity and then discuss secondary regulative resources.

[28] "Necessity is ascribed to a state that must occur or is always the same, irrespective of changing circumstances or of our interventions. Necessity is distinguished from contingency or possibility, which is ascribed to a state that may or may not occur and that varies with circumstances." Bunnin & Yu (2004, p. 462).

Propositional ground is operative when teachers and learners explicitly draw on Mathematics axioms, definitions and propositions underlying a particular topic as regulative resources. So when propositional ground is used to regulate mathematical activity in a pedagogic situation, necessity is situated internal to the field of Mathematics. The other forms of ground locate necessity with the teacher rather than in the field of Mathematics because the supports for the computational activity derive from criteria that are generated by the teacher in the absence of Mathematics axioms, definitions and propositions. This is not to say that empirical, iconic and algorithmic resources are necessarily situated outside of Mathematics. However, it is when there is a reliance on these forms of ground without the support of propositional ground that necessity is said to be located outside of the field of Mathematics.

Davis (2011b, p. 314) argues that the propositional becomes encoded into formal rules for generating a desired mathematical outcome thus rendering the algorithmic as the pinnacle of propositional thought and as the basis of the formal production of mathematical necessity. This use of the algorithmic is supported by the propositional and differs from the form of the algorithmic that operates without the support of Mathematical axioms, definitions and propositions as is the case when a teacher, for example, demands that the equation $(x-3)^2 = 16$ has to be transformed into the form $x^2 - 6x - 7 = 0$ before solving it. In this case, the algorithmic has a strong regulative effect on the selection and sequencing of operations and at times appears to be motivated by the need to transform expressions into ones which serve as triggers of particular procedures[29] or are attempts at generating 'easier' expressions for learners to work with so as to ensure the reproduction of mathematics despite their lack of knowledge of fundamental mathematical notions. The latter use of the algorithmic situates necessity in the teacher's criteria and thus external to the field of mathematics.

The empirical, which takes the form of testing expressions or measuring, is strongly indicative of inductive forms of reasoning. Inferences based on induction draw generalisations with respect to a class or set based on a few observed singular instances (Hegel, 1969; Peirce, 1931, 1992). Hegel describes induction as "the syllogism of *experience* - of the subjective taking together of the individuals into the genus and of the conjoining of the genus with a universal determinateness because this latter is found in all the individuals" (ibid. pp. 612-613, italics in original). He argues that since singulars are distinct from the universal used to characterise them, there can be no necessity on which to base conclusions. A fuller discussion of inductive reasoning follows in Chapter 7. When the empirical is used in conjunction with the propositional, as is the case with mathematical induction, then necessity is located in the field of Mathematics. Mathematical induction is a form of deductive reasoning but includes an inductive move as part of its proof (Brown, 2008). When empirical ground is employed without the propositional in play, then necessity is located outside the field of mathematics and instead is situated with the teacher's criteria.

[29] Davis (1984, p.35) introduced the notion of a *visually moderated sequence*, which "can be thought of as a visual cue *V1* which elicits a procedure *P*, whose execution produces a new visual cue *V2*, which elicits a procedure *P2*,... and so on".

The role of the iconic in mathematics is not as straightforward as that of the algorithmic and empirical. In the work of Peirce (1931), icons are defined as signs referring to objects by virtue of similarity. Thus, the pertinent feature of an icon is its perceived resemblance or similarity to an object (Peirce, 1931, 2.276, 3.362). So, when Peirce likens mathematical expressions to icons, we have to bear in mind that he distinguishes between three different kinds of icons, namely images, metaphors and diagrams. For him mathematical expressions are diagrams rather than images, which have a "sensuous resemblance between it and its object".

> Every picture (however conventional its method) is essentially a representation of that kind. So is every diagram, even although there be no sensuous resemblance between it and its object, but only an analogy between the relations of the parts of each. Particularly deserving of notice are icons in which the likeness is aided by conventional rules. Thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols (ibid., 2.279).

Peirce argues further that diagrams such as algebraic expressions or geometric figures have "the capacity to reveal unexpected truths" because they possess "a distinguishing property" which is "that by the direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction" (ibid., 2.279) Thus, treating mathematical expressions as though they are diagrams rather than images strongly suggests the presence of the propositional because unknown properties of the icon can only be determined by subjecting it to some deductive thought (ibid., 66; 3.363). The presence, therefore, of the propositional renders the iconic intelligible as is the case with "picture-proofs" (see Brown, 2008; Stjernfelt, 2000) or "a system of equations written under one another so that their relations can be seen at a glance" (Peirce & Moore, 2010, p. 40). Peirce even goes as far to claim that abstract rules are only "understandable" when rendered in sensible form.

> The letters of applied algebra are usually tokens, but the x, y, z, etc., of a general formula, such as $(x+y)z = xz + yz$ are blanks to be filled up with tokens, they are indices of tokens. Such a formula might, it is true, be replaced by an abstractly stated rule (say that multiplication is distributive); but no application could be made of such an abstract statement without translating it into a sensible image. (ibid., 3.363)

In contrast to the aforementioned use of the iconic, when mathematical expressions are treated as though they are images and subjected to spatial transformations in pedagogic situations e.g. "change sides, change signs" used in solving equations (see Davis & Johnson, 2007; Jaffer, 2010b, 2011a; Johnson & Davis, 2010; Moschkovich, 1996; Radford & Puig, 2007; Rivera, 2010; Staats & Batteen, 2009, 2010; Wittmann, Flood, & Black, 2013), the iconic operates without the support of mathematical axioms, definitions and propositions, and as such situates necessity external to Mathematics.

In summary, the binaries (propositional-empirical, algorithmic-propositional, propositional-iconic) locate necessity internal to the field of Mathematics whereas the other three binaries (algorithmic-iconic, iconic-empirical, algorithmic-empirical[30]) situate necessity external to the field of Mathematics.

Before discussing other secondary regulative resources, I examine iconic regulation in more detail given that spatial displacement of algebraic symbols in the regulation of the computational activity of teachers and learners appears to be prevalent in pedagogic situations (Jaffer, 2012; Johnson & Davis, 2010; Staats & Batteen, 2009, 2010; Wittmann et al., 2013).

3.4.3 Character distribution matrices

Johnson & Davis (2010) provide an extended discussion of spatial displacement of symbols as central to the procedure for factorising quadratic trinomials. They illustrate how the iconic features of a solution procedure regulates the mathematics. They exemplify their notion of a regulatory mechanism which they refer to as a *character distribution matrix*, described as follows:

> We call such a type of regulatory text a *character distribution matrix* and define it as a resource for the regulation of the mathematical activity demanding the use of very particular spatial distributions of symbols in the organisation and presentation of transformations from one mathematical expression to another as a solution is generated according to a procedure (Johnson & Davis, 2010, p. 135, italics in original).

The notion of a character distribution matrix in terms of a spatial template resonates with others in the field of mathematics education. For example Wittmann et al. (2013, p. 178), working within an embodied cognition framework, refer to a *landscape* with "defined regions and physical structure" in which parts of mathematical expressions are moved by learners as though they are physical objects. Radford & Puig (2007, p. 157) describe learners' treatment of an equation as a "situated spatial object with two sides". However, Johnson & Davis (2010) differ with respect to the aforementioned studies in that their focus is on how a character distribution matrix regulates the production and spatial distribution of symbols, that is, its structuring effect on the computational activity of pedagogic agents.

Johnson & Davis (2010) stress that the use of a character distribution matrix differs from the secondary role that notation plays in the presentation and communication of mathematical ideas. Mathematics comprises syntactical rules applied to lexical elements (symbols, e.g., numerals or letters) that enable the production of different combinations of symbols for expressing mathematical ideas. Notation is closely tied to the syntax of mathematics since it serves as the means for registering syntax. Current notational conventions[31] employ spatial positioning of symbols as a central resource for the display of mathematical expressions. For example,

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[30] The algorithmic-empirical is sometimes used in the field of Mathematics in the absence of other available resources and often gives rise to the propositional.

[31] Generally three historical periods are identified. During the rhetorical period mathematical ideas were expressed only in words without symbols, later symbols were introduced as abbreviations that accompanied words during the syncopated period. Syncopated texts were eventually replaced mainly by symbols during the symbolic period, still in use in modern mathematics texts (see O'Halloran (1996) and Radford &Puig (2007) for details of this history).

$\frac{a}{b}$ where $a, b \in \mathbb{Z}, b \neq 0$ is a notational form that exploits the positioning of numerals to express rational numbers as shown in Figure 3.3, where the top position (a) is reserved for the dividend (numerator) and the bottom position (b) the divisor (denominator) of a fraction. So $\frac{2}{3}$ has a different value to $\frac{3}{2}$.

$$\frac{\boxed{a}}{\boxed{b}}$$

Figure 3.3. Rational number template

The Arabic base 10 number system is another example which uses positioning together with the ten digits 0 to 9 to display numbers as shown in the template (Figure 3.4), where the first position from the right represents ones, the second represents tens and the third represents hundreds. A specific example, e.g., 354 is a symbolic representation of the number three hundred and fifty-four, where 3 represents 3 hundreds, 5 represents 5 tens and 4 represents 4 ones by virtue of the position of the digits. Figures 3.3 and 3.4 are referred to by Dowling (2010) as *templates* and the particular instantiations e.g. 351 or $\frac{2}{3}$ as *graphs*.

H	T	U

Figure 3.4. Thee digit number template

Learners of school mathematics are expected to learn the notational conventions associated with the display of mathematical thought alongside the syntax (composition of mathematical operations) and the associated mathematics contents. The expressive elements play a secondary role when they function merely as semiotic resources for expressing mathematical ideas. A number of studies, however, show that teachers and learners at times treat the manipulations of symbols and the display of mathematical symbols as the content (e.g. Arendse, 2013; Chitsike, 2011b; Jaffer, 2012; Johnson & Davis, 2010; Wittmann et al., 2013). In such cases the display template functions as a character distribution matrix which structures the production of the appropriate mathematical expressions without the presence of the associated fundamental mathematics ideas.

3.4.4 Secondary regulative resources

As discussed above, teachers use other criteria beyond computational resources to regulate mathematical activity. Criteria that function as secondary regulative resources are used by teachers to justify or authorise mathematical statements or actions. They include (1) pragmatic; (2) bureaucratic' (3) technological; and (4) personal forms of regulative resources.

Pragmatic resources include the use of examination criteria or promotion of particular solution methods because they are perceived to be easier for the learner. Often, 'easier' methods are motivated to avoid learner errors so that learners can reproduce mathematics despite their lack of the apposite fundamental mathematical knowledge, for example, "changing the signs" of the equation $-x^2 - x + 6 = 0$ to produce the equation $x^2 + x - 6 = 0$ which is considered easier to solve.

Regulative resources may have a legal or bureaucratic basis when the teacher appeals to the curriculum to establish necessity. For example, a teacher explains to her Grade 10 learners that the formula for the axis of symmetry of a parabola is $x = \frac{-b}{2a}$ without establishing how the formula is derived. She justifies her explanation by appealing to the curriculum which has allocated the topic as part of the Grade 11 curriculum. Learners are expected to accept that the formula for the axis of symmetry is $x = \frac{-b}{2a}$ on the basis that this explanation will be covered in Grade 11 and is excluded from their curriculum. A teacher may authorise mathematical claims by appealing to curriculum technologies such as textbooks or computer software programmes. For example, learners are obliged to accept that a parabola looks the way it does because a computer programme generates the graph from a given equation. Necessity is thus located in the software.

The teacher may also act as the locus of authority when the s/he merely asserts "this is so because I say so". Here necessity rests on the personal authority of the teacher. It is the case that the teacher holds the symbolic mandate conferred through symbolic investiture[32] (Bourdieu, 1991; Davis et al., 2005; Santner, 2001, p. 47) which authorises her to speak and act on behalf of a social institution, in this case the field of Mathematics education. But this authority is inscribed in a "symbolic space" or a legitimating field, external to the individual (Davis et al., 2005; Santner, 2001).

Next, I examine the nature of school mathematics and its relation to Mathematics as constituted in the field of production.

3.5 Mathematics and its relation to school mathematics

Recall that for Bernstein (1996, 2000) school knowledge derives, in part, from knowledge produced in the field of production. So school mathematics is recontextualised from Mathematics to school mathematics curricula, texts for teaching (e.g. textbooks or worksheets) and pedagogy. Chevallard (1989) in his *Theory of Didactic Transposition* (TDT), later referred to as *Anthropological Theory of Didactics* (ATD), distinguishes between (1) 'original' or 'scholarly' mathematical knowledge as it is produced by mathematicians; (2) knowledge to be taught as prescribed in school curricula; (3) knowledge as it is actually used by teachers in their classrooms; and (4) knowledge as it is learnt by learners (Bosch, Chevallard, and Gascón, 2006, p. 4). The focus of my project is on the teaching and learning of school mathematics but the methodology, following Davis (2011a, 2013a), derives partially from Mathematics. Thus, Mathematics itself is used as a methodological resource.

Similarly to Davis (2011a, 2013a), a number of scholars such as the French school led by Chevallard's *Anthropological Theory of Didactics* (ATD) and Brousseau's *Theory of Didactical Situations* (TDS), *Stoffdidaktik* (subject matter didactics, content-oriented analysis) methodologies employed by German scholars and the *Realistic Mathematics Education* research tradition initiated by Hans Freudenthal use

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[32] "By symbolic investiture I mean, more generally, those social acts, often involving a ritualized transferal of a title and mandate, whereby an individual is endowed with a new social status and role within a shared symbolic universe. It is how one comes into being as - comes to enjoy the predicate/value of - husband, professor, judge, psychoanalyst, and so forth." (Santner, 2001, p. 47).

Mathematics as a reference point in their analyses of pedagogic texts (see Bergsten, 2014 for a summary of this literature)[33]. A key difference between Davis (2011a, 2013a) and the aforementioned research traditions is that his methodology is premised on a computational theory of mind. The details of the methodology employed in the current study, which recruits Davis (2011a, 2013a), will be discussed in Chapter 5. Here, I briefly sketch the essential features of Mathematics and its relation to school mathematics.

In a lecture given in 1919, Hilbert stated that Mathematics is "a conceptual system possessing internal necessity that can only be so and by no means otherwise" (as quoted in Corry, 2007, p. 23). However, Mac Lane (1986, p. 409) argues that Mathematics is not a single system but "a tightly connected network of formal rules, concepts, and systems" and it derives internal consistency from its definitions, axioms and proofs (p.142). It is this network which Davis (2010a) refers to as the *Mathematics encyclopaedia*.

The term *encyclopaedia* typically connotes an alphabetical ordering of knowledge in a single text, even if the text comprises a collection of books. However, alphabetical order merely serves as a mechanism to facilitate the location of specific facts, a feature made redundant by current electronic search facilities. There is no consensus as to what the ordering principle of the Mathematics encyclopaedia ought to be. The only principle adhered to by all Mathematics encyclopaedic texts is the necessity for internal consistency and coherence. While for some texts the ordering principle may be alphabetical, others are organised on the basis of conceptual categories. See for example the Encyclopaedia on General Typology (Hart, Nagata, & Vaughan, 2004) where the text is ordered according to categories derived from "section 54 of the 2000 Mathematics Subject Classification as used by Mathematical Reviews and Zentralblatt MATH" (ibid., p. vii).

The idea of a Mathematics encyclopaedia used here resonates with Badiou's (2005) notion of an encyclopaedia as an authoritative systematised collection of agreed-upon facts or propositions.

> Knowledge is realized as an encyclopaedia. An encyclopaedia must be understood here as a summation of judgements under a common determinant. […] The encylopaedia (sic) contains a classification of parts of the situation which group together terms having this or that explicit property. One can 'designate' each of these parts by the property in question and thereby determine it within the language. It is this designation which is called a determinant of the encyclopaedia (Badiou, 2005, pp. 328-329).

Facts or propositions that do not currently exist but that could be logically implied from existing axioms and propositions are included in an encyclopaedia even if not vetted by any mathematician. Badiou (2005) argues that the fidelity of the encyclopaedia "has nothing to do with ethics or the moral virtues and everything to do with domain-specific requirements such as consistency, rigour, demonstrative force, logical explicitness and so forth" (Norris, 2009, p. 84). The Mathematics encyclopaedia is not located in a single text but comprises

[33] See Appendix 3 for a brief review of the French School's ATD and TDS.

all the agreed upon Mathematical axioms, definitions and propositions that are stated in numerous primary[34] as well as secondary texts.

The foundational elements (definitions, axioms, propositions) that make up the Mathematics encyclopaedia form the basis of school mathematics even though teachers and learners may only be implicitly aware of and use the axiomatic structure of Mathematics. Whatever form recontextualisation takes, school mathematics (curricula, text books, teacher/learner mathematical productions) is bound to conform in some or other way to the axiomatic structure of Mathematics. For example, 1 + 1 must always equal 2 when working in base 10, irrespective whether such a statement emerges in the context of the field of production or in the field of reproduction.

School mathematics curricula, textbooks and pedagogy entail a collection of specific mathematical topics selected from the Mathematics encyclopaedia. Such selections are controlled by government agencies, referred to by Bernstein (1990) as the Official Recontextualising Field (ORF). The Pedagogic Recontextualising Field (PRF), like textbook publishers and agencies involved in school mathematics education, may be recruited by the ORF to sanction official curricula. Selections of topics differ across national school mathematics curricula and differ with respect to grades within a particular country. In the current South African FET school mathematics curriculum (CAPS), the main topics include: 1) functions; 2) number patterns, sequences, series; 3) finance, growth and decay; 4) algebra; 5) differential calculus; 6) probability; 7) Euclidean geometry and measurement; 8) analytical geometry; 9) trigonometry; and 10) statistics (Department of Basic Education, 2011). A cursory examination of the 2014 final NSC Grade 12 Mathematics question papers (Department of Basic Education, 2014a, 2014b) indicates that the set of real numbers serves as the domain over which operations are performed. Although the axiomatic structure of the field of the reals is not explicitly taught and examined in school mathematics, the calculations performed are bound by the definitions, axioms and operations of the field of the reals. Teachers and learners, irrespective of whether they explicitly refer to the field axioms, have to account for them in some way in their computational activity.

Before discussing recontextualisation from the Mathematics encyclopaedia to school mathematics, I examine aspects of the Mathematics encyclopaedia. First, I consider definitions of mathematical terms, then I discuss mathematical propositions and finally I examine operations and associated domains and codomains.

3.5.1 Definitions of mathematical terms

Recall that the foundational elements that make up the Mathematics encyclopaedia are axioms, definitions, and propositions that constitute axiomatic systems. We have to start with some undefined or primitive terms stated at the outset of a deductive theory in order to avoid infinite regress.

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[34] It should be noted that new knowledge in the field of Mathematics routinely circulates on list serves prior to publication in journals.

> One central element in the exposition will be explicit *definitions* to explain our use of various words and symbols. It is a requirement of such a definition that it should be formally eliminable, so that every occurrence of the word defined could in principle be replaced by the phrase that defines it without affecting the correctness of the proof. But this process of elimination must stop eventually: at the beginning of our exposition there must be mathematical words or symbols which we do not define in terms of others but merely take as given: they are called *primitives* (Potter, 2004, p. 6; italics in orginal).

Definitions are developed from these primitive terms and serve to explain the meaning of new mathematical terms introduced into the axiomatic deductive system[35].

> The role of definitions is to clarify the meaning of terms of which the propositions are composed, that of demonstration, the gaining of acceptance for those propositions. […] they have, a purely logical function, namely, to interrelate all the terms and all the propositions into a systematic whole (Blanché, 1962, p. 12).

Definitions are, therefore, the building blocks of propositions and represent abbreviations of mathematical terms or notions in concise terms, thus establishing a logical connection between a new term and a set of established definitions and propositions (Blanché, 1962; Potter, 2004, p. 6).

A Mathematical definition is considered to be a *definite description,* which is a formal mechanism for introducing an abbreviation for the phrase that defines it (Potter, 2004) (see also Kripke's (1980) discussion of definite descriptions). A definite description outlines the cluster of predicates that describes a particular mathematical object. For example, the definition of a prime number as a positive integer having exactly one positive divisor other than one constitutes a definite description. Any change to the definition of a mathematical object produces a term with a set of predicates that refers to a different object or a term with an empty extension and in the process nullifies the definition as a definite description.

In the context of pedagogic situations, particularly those regulated by school curricula such as CAPS, which downplay formal definitions, we can expect that when teachers 'define' mathematical terms, that such 'definitions' often cannot function as definite descriptions (see Arendse, 2011, 2013). This point is elaborated in Chapter 5.

3.5.2 Propositions

Comparable to establishing definitions in an axiomatic deductive system, propositions function in the same manner. Starting with a small number of *primitive terms* and a few basic propositions called *axioms*, other propositions are established through formal proof (Blanché, 1962; Potter, 2004).

> This ensures that every proposition is linked to certain other propositions from which it is deduced as a consequence, so that a close network is progressively established in which all propositions are directly or indirectly related. The result is a system of which no part can be transposed or altered without affecting the whole (Blanché, 1962, p. 1).

[35] This is the case in the presentation of new mathematical terms but not in the development of new terms.

Mathematical propositions are statements which are either true or false, but cannot be both true and false. To avoid infinite regress, we have to start with a few basic propositions which are referred to as axioms (Blanché, 1962; Potter, 2004). Subsequent propositions are derived from the basic axioms via rigorous proof and can then be used as the basis for deriving other propositions or as facts in computations. For example, once we have proved the proposition "if lines are parallel, then the alternate angles formed by a transversal are equal", we can then use this proposition to prove that the sum of the interior angles of a triangle is one 180 degrees and we can use both propositions to compute unknown angles in a given geometric problem.

Mathematical propositions may take the form *P* is *Q*. The statement "two is an even number" is an example of such a proposition. However, many propositions in Mathematics are of the form "if *P*, then *Q*" or *P* implies *Q,* which are causal statements or implications, where *P* refers to the *premise* and *Q* the *conclusion* (Houston, 2009, pp. 63-64).[36] For example a proposition in the form of an implication is the following: "If a > 0 then the parabola $y = ax^2 + bx + c$ has a minimum value." This proposition may be used by someone when sketching the graph of a parabola given its equation. By examining the value of a for a given parabola $y = ax^2 + bx + c$, the individual could deduce whether the parabola has a minimum or a maximum value.

Propositions describe the existential features of mathematical objects and function to regulate mathematical activity of individuals since they serve as rationales or justifications for mathematical computations. Propositions thus authorise the computational activity of individuals. In pedagogic situations, we expect that teachers and learners may replace mathematical propositions with alternate propositions for two reasons. Firstly, as discussed in Chapter 1 and 2, contrary to the widely held position in mathematics education specifically and in education more generally thought is considered to be conceptual or propositional (Chomsky, 2006, 2007; Fodor, 1998, 2010; Pinker, 2007). Thus, even "procedural" mathematical thinking considered in opposition to conceptual or propositional thought in the literature (see for example Hiebert & Lefevre, 1986) recruits propositions but these propositions may differ from those found in the Mathematics encyclopaedia. Secondly, we can expect teachers and learners to substitute mathematical propositions particularly since schooling excludes from explicit study the axiomatic features of Mathematics. However, teachers and their learners cannot escape the field of reals, so substitution of mathematical propositions with alternate propositions is a means of compensating for the absence of the explicit use of axiomatic features of Mathematics. For example, the proposition "If a > 0 then the parabola $y = ax^2 + bx + c$ has a minimum value" is often replaced with the proposition "If a is positive, then the parabola has a smiley face". The substitution of mathematical propositions with alternate propositions is discussed further in Chapter 5.

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[36] Davis (1984, p39, italics in the original) refers to propositions "If P is the case then do Q" as *condition-action pairs,* where *P* is considered the condition and "do *Q*" as the action. In some knowledge representation structures (KRS) *every item is of this same form*, a knowledge-action pair. Such a KRS is called a production system."

3.5.3 Operations, domains and codomains

Recall from Chapter 1 that the functioning of evaluation at the level of the instructional discourse is ascertained by examining the recognition and realisation rules described in terms of the computations performed by teachers and learners and that an analysis of the computational activity tells us what is constituted as mathematics in pedagogic contexts and how such constitution comes about. Recall further, that the objects computed with represent the elements of a pedagogic situation and the operations or operation-like manipulations performed over the objects constitute the logic of the pedagogic situation. This leads me to define what an operation is.

An operation is a function consisting of a domain (a set of arguments of the operation) and the co-domain (a set of outputs or values of the operation).

> An operation, *, is defined in general terms as a function of the form $*: D_1 \times D_2 \times \ldots \times D_k \rightarrow C$, where the sets D_j are the domains of the operation, the set C is the codomain of the operation; the fixed non-negative integer k, which indicates the number of arguments, is the arity of the operation (Davis, 2011a, p. 99).

In addition, the functional nature of mathematical operations enables the identification of what Davis (2013a) refers to as operation-like manipulations that are ubiquitous in pedagogic situations of schooling, and which are not familiar operations in the Mathematic encyclopaedia. It is not always the case that the manipulations used by teachers and learners are operations in a mathematical sense. It is quite common that "operations" used in pedagogic situations are not necessarily functions. A function is defined as follows.

> A *function* f on the set D to the set C is a set $S \subset D\ X\ C$ of ordered pairs which to each $x \in D$ contains exactly one ordered pair $< x, y >$ with first component x. The second component of the pair is the value of the function f at the argument x, written $f(x)$.We call C the *domain* and D the *codomain* of the function f (adapted from Mac Lane, 1986, p. 129, italics in orginal).

That operations are functional in nature is central to the stability enjoyed by Mathematics since functions generally produce stable outputs from given inputs[37]. The definition of an operation provided above covers all mathematical operations including the basic arithmetic operations (addition, multiplication, division and subtraction) which are binary, and are usually defined as functions of the form $*:D \times D \rightarrow D$ and unary operations such as squaring described as $*:D \rightarrow D$. However, because my methodology requires me to be attuned to identifying the operation-like manipulations used by teachers and learners, the more general definition of an operation is required (Davis, 2013a, p. 36).

Using Lawvere & Schanuel (1997), Davis (2011a) highlights a unique feature of mathematics, that is, it is always possible to substitute a particular rule of a function. The rule describes the process by means of which each element of the domain of a function is uniquely associated with an element of its codomain. Lawvere &

[37] The definition of an operation as function does not restrict the project to the study of functions nor is school mathematics treated as though it is abstract algebra. Rather the proposition that thought is computational requires a description of the computations and operations employed by pedagogic agents.

Schanuel (1997) contend that a particular rule for a function is itself not unique with respect to a function. For example, the rule, *f*, 'add 1 to the input value and then square', and the rule, *g*, 'square the input value, double the input value, add the two results and then add 1', produce the same output values for given input values.

> What the equation $(x + 1)^2 = x^2 + 2x + 1$ says is precisely that $f = g$, not that the two rules are the same rule (which they obviously are not; in particular, one of them takes more steps than the other) (Lawvere & Schanuel, 1997, pp. 22-23).

In other words, *f* and *g* produce the same output (the value) from a given input (the argument of the function). Thus *f* and *g* are equivalent at the level of value but not at the level of expression and operations.

Davis (ibid.) argues that the substitution of rules, which is an essential feature internal to Mathematics itself, is prevalent in school mathematics. However, in contrast to Mathematics, the substitution of operations indicated by mathematical statements common in the pedagogic situations of schooling often involves auxiliary operations or operation-like manipulations such as 'change sides, change signs' used in solving equations (see Arendse, 2013; Basbozkurt, 2010a, 2010b; Chitsike, 2011a, 2011b; Davis, 2010a, 2010b, 2010c, 2011a, 2011b, 2011c, 2012, 2013a, 2013b, 2014; Jaffer, 2009, 2010a, 2010b, 2011a, 2012). Although not described in the same terms as Davis, examples of operation-like manipulations can be found in mathematics education literature (see for example Lima & Tall, 2008; Ma, 1999; Sfard, 2007; Staats & Batteen, 2009; Wittmann et al., 2013).

As discussed previously, teachers and learners are bound by Mathematics, even if only implicitly, because school mathematics owes its fidelity to Mathematics. However, the substitution of definitions, propositions and operations with auxiliary 'definitions', propositions and operations or operation-like manipulations allows for the production of mathematical statements that conform to the Mathematics encyclopaedia at the level of expression but not necessarily at the level of value. In other words, it is possible for teachers and learners to produce signifiers that correspond to what is expected from the point of view of the Mathematics encyclopaedia but with very different signifieds. For example, when solving the equation $2x + 1 = 5$, the expression $2x = 5 - 1$ can be obtained by (1) using an operation-like manipulation, "taking 1 over and changing the sign" or (2) using the additive inverse -1 on both sides of the equation. In both cases, the expression $2x = 5 - 1$ conforms with the Mathematics encyclopaedia. However, with respect to (1) an existential shift from a number to character strings has occurred, so we have difference at the level of value despite equivalence at the level of expression whereas in (2) we have preservation at the level of value.

This discussion leads directly to recontextualisation from the field of Mathematics to school mathematics.

3.6 Recontextualisation with respect to school mathematics

Recall that recontextualisation represents one of the moments in the transformation of knowledge into pedagogic communication in Bernstein's account of the pedagogic device. According to Bernstein (1996, 2000), the recontextualising rule of the pedagogic device governs the creation of pedagogic discourse by selecting knowledge produced in the field of production together with other discourses such as accounts of how that knowledge ought to be taught and when it should be taught. Pedagogic discourse, according to Bernstein, consists of an instructional discourse (content and skills) and a regulative discourse (social or moral order). Bernstein defines pedagogic discourse as a

> principle for delocating a discourse, for relocating it, for refocusing it, according to its own principle. […]
>
> As the discourse moves from its original site to its new positioning as pedagogic discourse, a transformation takes place. The transformation takes place because every time a discourse moves from one position to another, there is a space in which ideology can play. No discourse ever moves without ideology at play. As this discourse moves, it is ideologically transformed; it is not the same discourse any longer. I will suggest that as this discourse moves, it is transformed from an actual discourse, from an unmediated discourse to an imaginary discourse. As pedagogic discourse appropriates various discourses, unmediated discourses are transformed into mediated, virtual or imaginary discourses. From this point of view, pedagogic discourse selectively creates *imaginary subjects* (Bernstein, 1996, p. 47, italics in original).

Bernstein, thus, argues that pedagogic discourse differs from the discourse from which it derives, i.e., the discourse in the field of production. For example, the school subject woodwork differs from carpentry and school mathematics from Mathematics. Bernstein's argument with respect to the differences in discourses hinges on the "delocation" and "relocation" of practices from the field of production to the field of reproduction. In other words, he maintains, for example, that the practices of the mathematician differs from the practices of teachers and learners. However, his argument conceals the complexity entailed in recontextualisation.

Bernstein is correct when he claims that carpentry and woodwork are different discourses because the practices of the carpenter, which typically take place in a carpenter's workshop, differ from the practices of teachers and learners in woodwork classrooms. So carpentry and woodwork differ with respect to their practices and contexts. However, the causal relations between knowledge propositions used in carpentry and woodwork remain stable irrespective of the practice or the context. For example, what constitutes a dovetail joint is the same whether produced by a carpenter, woodwork learner or DIY enthusiast. A carpenter would most likely use a dovetail joint as part of the construction of a cabinet whereas in a woodwork classroom learners may be expected to produce dovetail joints until they perfect the joint without necessarily producing joints as part of making a cabinet. Similarly, the propositions entailed in the concept of ratio whether used by a mathematician, school mathematics teacher/learner or by someone making a best-buy purchase in a supermarket are the same. The purpose of using the ratio concept may differ across contexts but the knowledge propositions entailed in the concept of ratio remain stable across contexts.

Dowling (1998, 2009), who shares Bernstein's sociological concerns, is interested in the relation between school mathematics and everyday practices. Dowling (1998, 2009) refashioned Bernstein's notion of recontextualisation as follows:

> I shall use the term *gaze* to refer to a mechanism which delocates and relocates, that is, which recontextualizes ideological expression and content. The result of such recontextualizing is to subordinate the recontextualized ideology to the regulating principles of the recontextualizing ideology (Dowling, 1998, p. 121, italics in original).

For Dowling, school mathematics recontextualise everyday practices and in the process subordinates everyday practice to the principles of mathematics. In his later work, Dowling (2010, p. 5) describes the *gaze* of mathematics as a *fetching* activity where resources from other practices are recruited and recontextualised as school mathematics and then *pushed* back into mathematics education. For example, a school mathematics textbook best-buy task (Dowling, 1998, pp. 248-249) represents the recontextualisation of the practice of making a best-buy purchase into a school mathematics task. Dowling argues that a transformation has taken place in that the context of making a best-buy purchase has been delocated and relocated from a domestic practice into the practice of school mathematics, and in the process distorts the practice of making a best-buy purchase. However, the mathematics content, in this case the notion of ratio, is preserved although there is transformation at the level of practice.

Furthermore, recontextualisation is always possible irrespective of whether a practice is "relocated" or "delocated". Recall that it is always possible to achieve the same outcome by substituting function rules or operations with alternate operations or operation-like manipulations (Davis, 2011a, Davis 2013a). Therefore, within any pedagogic situation, we can expect learners to recontextualise mathematics contents in their attempts to reproduce mathematics. Secondly, we know from cognitive science that structure preservation is key to learning or thought (Gallistel & King, 2010; Gelman, 2009). Gallistel & King (2010, p. 55) posit that cognitive representation entails establishing a relation between the representing system (brain) and represented systems (aspects of the world). The structure-preserving mapping between the representing and represented systems are morphisms.

A morphism is a function which links two structures, f: $(A,\circ)\longmapsto(f(A),\square)$ such that $f(a_1)\square f(a_2) = f(a_1\circ a_2)$ $\forall a_1,a_2 \in A$. When the function f is many-to-one, the morphism is referred to as a homomorphism that is a structure-preserving map between two algebraic structures (see Baker, Bruckheimer, & Flegg, 1971; Krause, 1969). Therefore, recontextualisation is always possible since individuals create isomorphisms or homomorphisms as part of thought in general.

The salient point to be distilled from the above discussion is that transformation at the level of practice is always involved when discourses are delocated or relocated (recontextualised) but this does not necessarily entail transformation at the level of knowledge. When knowledge statements made in a pedagogic situation correspond with knowledge statements in the Mathematics encyclopaedia, then it is understood that there has been no content substitution of knowledge propositions. However, when teachers or their learners replace propositions and operations with auxiliary propositions and auxiliary operations (operation-like manipulations) then it is understood that substitution of knowledge has taken place. Thus, recontextualisation of mathematics to school mathematics occurs with or without content substitution.

The notion of recontextualisation is important in my study as a key methodological procedure involves comparing what teachers and learners do computationally with the Mathematics encyclopaedia. The use of the Mathematics encyclopaedia as a methodological resource is also employed in Chevallard's (1989, 1992) Anthropology of theory of Didactic Transposition (ATD) where the initial task of the researcher is to describe the relevant "scholarly mathematics'' as the referent knowledge which "knowledge to be taught", the "knowledge taught" and the "knowledge learnt" is compared to. The aim of ATD is to examine the "ecology" of the didactic situation for the conditions and constraints of a particular didactic relation that shapes the knowledge that emerges in the didactic situation (Chevallard, 1989, p8). In my study, the focus is on examining the recognition and realisation rules entailed in the computational activity of teachers and learners in order to determine the substitution of definite descriptions, encyclopaedic propositions and encyclopaedic operations with auxiliary descriptions, propositions and operations.

The substitution of definite descriptions, encyclopaedic propositions and encyclopaedic operations has consequences for what comes to be constituted as the content of a mathematics topic. It is to this discussion that I now turn.

3.7 Realisation of content

School mathematics curricula, as is the case with CAPS, often list topics (e.g. parabola, linear equations) and may even elaborate the expected content associated with topics that teachers and learners are expected to teach and learn. In the mathematics education literature, teachers and learners are assumed to realise the expected content when their solutions to mathematics tasks match the demands of the tasks. But is the relation between topic and content as straightforward and unproblematic as that portrayed in the literature?

Let us consider the topic linear equations. Teachers and learners are considered to realise the expected content associated with the topic linear equations when they solve equations like $5x + 2 = 3x + 8$, irrespective of the methods used. One teacher may solve the equation by 'moving' all the terms that include x to the left-hand side of the equation and all the constant terms to the right-hand side of the equation. Another teacher may solve the equation by using additive inverses. Both teachers arrive at the correct answer and so we assume that the content realised by both teachers is linear equations in the particular pedagogic situation. Such an assumption is based on what we expect *ought* to be realised as the content rather than what *is* realised as the content associated with a topic name. Secondly, the assumption does not take into account that language does not have a reference function. So, a one-to-one onto mapping between the solution of the equation and the content realised does not exist. In other words, just because the solution to the linear equation are equivalent at the level of expression, does not necessarily imply that the content associated with the expression is the same. The content realised is dependent on the operations and domains employed in the solution of the equation. It is assumed that what is realised as the content associated with an announced topic remains stable across pedagogic contexts. The idea that the content associated with a topic does not necessarily remain stable across contexts has been shown in other work (see Chitsike, 2011b; Davis, 2011a; Jaffer, 2012).

I draw on Eco's (1984b) theory of textual coherence as a resource to explore the idea that the content associated with a topic name is constructed in a pedagogic situation. The notion of *topic* is conceptualised in semiotics as referring to the subject of a text in relation to its levels of coherence. The notion of *isotopy,* first introduced by Greimas (1970, as cited in Eco, 1984b) and later adapted by Eco, is associated with the level of coherence of a text (Eco, 1984b). In reference to literary texts, Eco describes topic as

> an abductive schema that helps the reader to decide which semantic properties have to be actualised whereas isotopies are the actual textual verification of that tentative hypothesis (Eco, 1979 as cited in Eco, 1984b, p. 189).

Eco regards topic as a pragmatic resource deployed by the reader to focus the interpretation of the meaning of a text and isotopy as "a level of possible semantic actualisation of the text" (Eco, 1984b, p. 189). The glossing of the notion of isotopy as 'repetition' in a text, due to Greimas, was found to be problematic by Eco (1984), prompting him to redefine the idea.

> Indeed, isotopy almost always refers to constancy in going in a direction that a text exhibits when submitted to rules of interpretative coherence […] What should be clear in any case is that the identification of the topic is cooperative (pragmatic) movement guiding the reader to individuate the isotopies as a semantic feature of a text (Eco, 1984b, p. 201).

Topic imposes a rule of interpretative coherence and isotopy refers to the semantic result of that coherent interpretation (Eco, 1984b, p. 193). Further, Eco recognises the "the actualized isotopy as the 'objective' content of the expression" (Eco, 1984b, p. 193).

Eco's (1984) notions of topic and isotopy are semiotic resources for describing and analysing literary texts. This study focuses on mathematics texts produced by teachers and learners in pedagogic situations of schooling. The texts referred to by Eco include conversations, comic books, films, poems and so forth, that may be used as pedagogic resources but clearly differ from pedagogic texts. Pedagogic texts include sequences of verbal, written, visual or gestural significations such as teacher and learner speech, worked examples or notes written on a chalkboard or a textbook. Bernstein (2000, p. 18) asserts that a text, by which he means pedagogic text, is "anything which attracts evaluation". Pedagogic texts are, therefore, understood as necessarily evaluative in that they distinguish legitimate from non-legitimate knowledge statements for learners and reveal criteria for the recognition and realisation of mathematical objects and procedures in pedagogic contexts. Such evaluations are, of course, not identical across pedagogic situations, and so the criteria that are posited could, in principle, be rather different across contexts.

According to Eco (1984b), the reader posits the topic of a literary text in generating a theory about its semantic content and isotopies verify or disrupt the reader's construction of the topic. In the latter case, the topic is altered to conform to the isotopies. In the reading of literary texts the reader usually has a store of cultural information that can be selected from to actualise a topic. With pedagogic texts, however, the learner is routinely confronted with content that is unfamiliar, and has to rely on the evaluative criteria operative in the pedagogic situation to constitute the topic. It is also the case that the learner cannot but draw on their reservoir of mathematics in their attempts to constitute the topic. Isotopy, which, in the case of mathematics, the learner reads off from the computational activity deployed in the pedagogic situation, confirms (or disrupts) the learner's construction of the content of the topic.

The pedagogic organisation of a content area and an elaboration of the content reveals the content associated with the announced topic. An examination of the elaboration of the content in the pedagogic situation reveals the isotopy constituted by the teacher's computational activity and the isotopy confirms (or disrupts) the content of the topic for the learner. The computational activity in a pedagogic situation is described by considering the operations that emerge in that situation along with the collections of objects over which the operations are performed. In addition, the criteria regulating the selection and sequencing of the operations are also described (Davis, 2011b).

Topic, for Eco (1984b), is always constructed by the reader or listener from isotopic markers or clues. In contrast to Eco (1984b), who does not refer to content, I distinguish between *announced topic* and *realised content*. The *announced topic* refers to a name used by teachers, learners and textbooks to indicate a particular selection of mathematics contents, and the *realised content* is the content that becomes associated with the topic name as the topic unfolds in the pedagogic situation and it is the content actually taught and learnt. The realised content is constructed from the isotopic markers or computational clues produced in the pedagogic situation.

The realised content associated with a topic may conform to or differ from the content associated with the topic from the point of view of the Mathematics encyclopaedia. So, a central feature of my methodology for analysing what comes to be constituted as mathematics in pedagogic situations entails comparing the realised content with the Mathematics encyclopaedia in order to ascertain points of commonality and points of difference.

3.8 Computational performance

Having discussed the theoretical resources that enable us to link the functioning of evaluation at the level of the instructional discourse to the content realised in pedagogic situations, I now focus attention on the learner implied by the computational activity.

Instructional discourse contains within itself a model of the acquirer (Bernstein, 1990, p. 163). I recruit Eco's notion of the *Model Reader* and his concepts of *open* and *closed* texts to fashion methodological resources for describing the relation between the computational activity emergent in the observed lessons and the computational performance of the learner implied by the computational activity as well as the computational performance of actual learners when doing mathematics independently of the teacher in the context of a test and an interview based on the test.

For Eco (1984a), texts always constitute a model of a possible reader, the *Model Reader.* Eco (1984a) distinguishes between the Model Reader as a theoretical or textual category and the empirical reader, an actual reader of a text. The Model Reader is assumed to be capable of interpreting a text in a manner comparable to the author's construction of the text.

> To organize a text, its author has to rely upon a series of codes that assign given contents to the expressions he uses. To make his text communicative, the author has to assume that the ensemble of codes he relies upon is the same as that shared by his possible reader. The author has thus to foresee a model of the possible reader (hereafter Model Reader) supposedly able to deal interpretatively with the expressions in the same way as the author deals generatively with them (Eco, 1984a, p. 7).

Eco (ibid.) argues that a text not only presupposes a model of competence external to the text but also constructs this competence. Dowling (1998), who recruits Eco's notion of a model reader in his analysis of a textbook series, emphasises that the model reader does not exist in the text itself but in the reading of the text. Drawing on Barthes (1972), he bases his argument on a distinction between *text-as-text* and *text-as-work*, with the former referring to the product of an analysis of the text and the latter the generation of text by its author.

As discussed earlier, the texts referred to by Eco are literary texts. However, Eco's notion of a Model Reader has been applied to pedagogic texts (see Chitsike, 2011b; Dowling, 1998; Ernest, 2008; Herbel-Eisenmann & Wagner, 2005; Jaffer, 2011a; Sierpinska, 1997; Wagner, 2012). In contrast to Dowling, who is concerned with the reproduction of ideology through pedagogic texts, and Herbel-Eisenmann & Wagner (2005) and Wagner (2012), who focus on the way in which pedagogic texts position learners, I am concerned with the reproduction of mathematics and the mathematical performance of the model learner implied by pedagogic texts. In this respect, my use of Eco's notion of the Model Reader overlaps with that of Sierpinska (1997) and Chitsike (2011b) who are both interested in the way that the text both presumes and creates the competence of its Model Reader. Like Dowling, I am interested in whether pedagogic texts differentially distribute different forms of mathematical knowledge to their Model Readers. In other words, my concern is the extent to which pedagogic texts construct the computational performance of learners along social class lines.

Eco's (1984a) Model Reader of literary texts is comparable to the model learner presupposed by pedagogic texts. Similar to the way in which the semiotic resources of a literary text presupposes and constructs the competence of its Model Reader, so, too, does the computational activity circulating in a pedagogic context presuppose and construct a particular mathematical or computational performance. As such, the model learner constitutes an analytic category that describes the computational performance that the computational activity assumes and structures. The methodological apparatus developed in Chapter 5 provide the terms to describe categories of the model learner.

According to Eco (1984a), different types of text construct different Model Readers. He distinguishes between two types of texts – texts that can be described as *open* and those that can be described as *closed*. He glosses *closed* and *open* texts as follows:

> Those texts that obsessively aim at arousing a precise response on the part of more or less precise empirical readers (be they children, soap opera addicts, doctors, law-abiding citizens, swingers, Presbyterians, farmers, middle-class women, scuba divers, effete snobs, or any other imaginable sociopsychological category) are in fact open to any possible 'aberrant' decoding. A text so immoderately 'open' to every possible interpretation will be called a *closed* one (Eco, 1984a, p. 8, italics in original).

> [Open texts] work at their peak revolutions per minute only when each interpretation is reechoed by the others, and vice versa. [...] You cannot use the text as you want, but only as the text wants you to use it. An open text, however 'open' it be, cannot afford whatever interpretation. An open text outlines a 'closed' project of its Model Reader as a component of its structural strategy (Eco, 1984a, p. 9).

A *closed* text attempts to elicit a very particular reading and is consequently subject to divergent, 'aberrant' readings. In contrast, *open* texts are structured so that all the elements work together to produce a reading that converges on a particular reading. An *open* text assumes the competence of its Model Reader while simultaneously constructing the Model Reader's competence.

> An 'open' text cannot be described as a communicative strategy if the role of its addressee (the reader, in the case of verbal texts) has not been envisaged at the moment of its generation *qua* texts (Eco, 1984a, p. 3, italics in original).

In addition to assuming and constructing the competence of its Model Reader, an *open* text, as discussed earlier, constructs a 'closed' project by making available combinatorial resources that enable a convergence of interpretations. *Closed* texts, on the other hand, are structured to steer readers along a set path through the text.

> They seem to be structured according to an inflexible project. Unfortunately, the only one not to have been 'inflexibly' planned is the reader. These texts are potentially speaking to everyone (Eco, 1984a, p. 3).

So, by attempting to fix a particular interpretation, *closed* texts have the effect of producing divergent 'aberrant' interpretations unanticipated by their author.

At the level of intention, all pedagogic texts might be considered *open* because they are structured by assumptions about the competence of their model learners, since a teacher always has to decide on the extent of the prior knowledge of the learner. Secondly, pedagogic texts might be considered *open* because such texts set up "closed projects" for their model learners. In other words, pedagogic texts are intended to produce convergence with respect to interpretation of content. However, at the level of the actual computational activity that emerges, a pedagogic text may be constructed as either *open* or *closed* with respect to the particular topic referenced by the pedagogic text.

A number of studies in the field of mathematics education recruit Eco's notions of open and closed texts (see Chitsike, 2011b; Davis, 2011b; Otte, 1983; Sierpinska, 1997; Weinberg, Wiesner, Benesh, & Boester, 2012; Weiss, 2011) to analyse pedagogic texts. However, their use of Eco's concepts differs from my recruitment of his concepts. Similarly to (Chitsike, 2011b), I use Eco's concepts as methodological resources for describing how the computational activity circulating in the pedagogic situation both presupposes and structures the computational performance of learners. In other words, my adaptation of Eco's concepts as methodological resources for reading mathematics pedagogic texts entails ascertaining what is assumed about learners' computational performance and how their computational performance is structured by the computational activity that emerges in the pedagogic situation.

In my study, an *open pedagogic text* is one in which Mathematical axioms, definitions and propositions are explicitly recruited as computational resources. An open pedagogic text provides learners with computational resources that enable the realisation of content that converges at the level of content with the topic from the point of view of the Mathematics encyclopaedia but may differ at the level of expression. A *closed pedagogic text*, on the other hand, does not explicitly draw on Mathematical axioms, definitions and propositions but recruits auxiliary 'definitions' and propositions as computational resources. A closed pedagogic text provides learners with computational resources that enables the realisation of content that diverges from the topic from the point of view of the Mathematics encyclopaedia with respect to content even if expressively convergent. Closed pedagogic texts attempt to obtain expressively precise responses from learners and thus have the potential to produce unanticipated divergent 'aberrant' interpretations.

The notions of open and closed pedagogic texts are used to describe the computational performance of the model learner presupposed by the computational activity emerging in the observed lessons and to describe the computational performance exhibited by actual learners when doing mathematics independently of the teacher in the context of a test and an interview based on the test.

Just as the computational activity structures the computational performance of learners, so too does it reveal and shape their orientation to mathematics in a pedagogic situation.

3.9 Orientations to mathematics

Lotman's distinction between *text-oriented* cultures and *grammar-oriented* cultures can be usefully appropriated for describing orientations to mathematics (see Davis, 2011b; Dowling, 1998; Jaffer, 2010a). He describes grammar-oriented and text-oriented societies as follows:

> Cultures can be governed by a *system* of rules or by a *repertoire* of texts imposing models of behaviour. In the former category, texts are generated by combinations of discrete units and are judged correct or incorrect according to their conformity to the combinational rules. In the latter category, society directly generates texts, which constitute macro units from which rules can eventually be inferred, but which initially and most importantly propose models to be followed and imitated.
>
> A grammar-oriented culture depends on 'Handbooks', while a text-oriented culture depends on 'The Book'. A handbook is a code which permits further messages and texts, whereas a book is a text, generated by an as-yet-unknown rule which, once analyzed and reduced to a handbooklike form, can suggest new ways of producing further texts (Eco in Lotman, 1990, p. xi, italics in orginal).

Lotman's categories, while useful as a heuristic for thinking about pedagogy, require adaptation for analysis of pedagogic situations. Pedagogic modalities in which learners are encouraged to reproduce texts, through repetition and rehearsal, that precisely conform with texts considered as legitimate in the pedagogic context, are suggestive of Lotman's text-oriented cultures. His concept of grammar-oriented cultures, on the other hand, is comparable with pedagogic modalities which encourage "syntactical symbol manipulation and propositional descriptions of relations between mathematical objects" (Davis, 2011b, p. 315).

Davis (2011b) argues that Lotman's distinction between text-oriented and grammar-oriented cultures in some ways resonates with mathematical activity as described in the mathematics education literature. However, the association of grammar-oriented generated texts with combinatorial rules suggests an absence of combinatorial rules in text-oriented cultures. Davis (ibid) argues further that this opposition cannot apply in pedagogic situations because, following Chomsky (2009) and Fodor (2010), combinatorial rules are essential components of thought and language and so of mathematics, and so ought to be present in text-oriented pedagogic situations. Davis (2011b, pp. 316-317) adopts Lotman's distinction between *expression-* and *content-orientation* as a more appropriate distinction for describing aspects of the constitution of mathematics, given that Lotman aligns text-orientation with expression-orientation and grammar-orientation with content-orientation respectively (see Eco, 1976, p. 138). So, for Davis (2011b, pp. 316-317), both expression-oriented and content-oriented activities are combinatorial. Lotman distinguishes between content-oriented and expression-oriented societies as follows:

> Lotman suggests that text-oriented societies are at the same time expression-oriented ones, while grammar-oriented societies are content-oriented. The reason for such a definition becomes clear when one considers the fact that a culture which has evolved a highly differentiated content-system has also provided expression-units corresponding to its content-units, and may therefore establish a so-called 'grammatical' system — this simply being a highly articulated code. On the contrary a culture which has not yet differentiated its content-units expresses (through macroscopic expressive grouping: the texts) a sort of *content-nebula* (Eco, 1976, p. 138, italics in orginal).

Davis (2011b) defines expression-orientation as one which focuses primarily on the expressive elements required and entails a system of combinatorial rules that operates directly on the expressive elements to generate texts. In other words, the domains and codomains comprise alphanumeric strings or graphical images and the operations recruited are auxiliary operation-like manipulations. The use of the operation-like manipulation "change sides, change signs" in the solving of linear equations is an example of expression-orientation in school mathematics because the domains and codomains are character strings and the operation-like manipulation is one that is not found in the Mathematics encyclopaedia. Similarly, the use of character distribution matrices, which involve the generation and distribution of symbols as the realised content, is also described as expression-oriented. I include the repetition and rehearsal of texts that precisely imitate texts considered as legitimate in the pedagogic situation as expression-oriented. In other words, teaching and learning modalities that resonate with Lotman's text-orientation are considered as expression-oriented because it is the case that learners are expected to use the texts as though they were images.

With content-orientation, Davis (2011b) argues that the expressive elements are secondary, functioning merely as resources for communicating mathematics. The system of combinatorial rules, in contrast to expressive resources, comprise domains and codomains and operations drawn from the field of the reals. For example, a teacher who explicitly recruits Viete's theorems in the solving of quadratic equations can be said to be content-oriented because they are explicitly recruiting computational resources derived from the field of the reals.

In this chapter, I considered the general theoretical resources that I will deploy to both produce data and generate analyses of the information collected for the study, which aims to answer the research question:

How does pedagogic evaluation function in the instructional discourse of four Grade 10 pedagogic contexts in schools that differ with respect to the social class membership of their learner populations and what are the implications of pedagogic evaluation for learners' computational performances and orientations to mathematics?

3.10 Summary: Propositions underpinning the study

From an engagement with antecedent literature, I derive a number of propositions that will be used as the basis for the production and analysis of data. The propositions are of two types: theoretical propositions (TP) and empirico-theoretical propositions (ETP) (Davis, 2005b; Dowling, 1993). The theoretical propositions derive directly from theoretical resources discussed in this chapter and relate to three aspects pertinent to my study, namely, (1) the nature of pedagogy (2) the nature of language and thought and (3) the nature of Mathematics and its relationship to school mathematics. The empirico-theoretical propositions emerge from research studies conducted in contexts similar to the empirical contexts of the study.

The theoretical propositions as well as the empirico-theoretical propositions form the basis for developing research hypotheses (RH) which serve to guide the production and analysis of data.

3.10.1 Theoretical propositions

Since the current study is concerned with the functioning of evaluation in pedagogic situations that differ with respect to the social class membership of learners, Bernstein's account of the pedagogic device, with its focus on who gets what knowledge and how, serves as the general frame structuring the analysis of the study.

TP1: *Pedagogy is necessarily evaluative* since it entails the attempt to 'transmit' evaluative criteria, which are the rules for regulating pedagogic activity and which reveal what is considered as legitimate knowledge in particular pedagogic contexts (Bernstein, 1996, 2000).

TP2: The *distributive and recontextualising rules are condensed in the evaluative rule* (Bernstein, 1996, 2000). An examination of how evaluation functions in pedagogic situations is necessary in order to reveal the recontextualising and distributive principles.

TP3: *Evaluation structures what comes to be constituted as mathematics.* Mathematical activity entails what people say, do or mean and is regulated by the recognition and realisation rules that they use to generate their mathematical productions (Davis, 2011a). Recognition and realisation rules are described in terms of the computational activity of teacher and learners. *Computational activity includes the composition of operations or operation-like manipulations over domains as well as the propositions and definitions or descriptions of mathematical terms which govern such operations*.

TP4: *Recognition and realisation rules may differ across pedagogic agents even when producing the same outcome* expressively (Davis, 2013a).

TP5: Pedagogy acts as *a symbolic ruler of consciousness* (Bernstein, 2000). According to Bernstein, the ruler of consciousness can be interpreted as either having power over consciousness or as a measure of the realisation of specialised consciousness, loosely equivalent to what is referred to as mathematical thinking.

TP6: *Pedagogic discourse specialises time, text and space* (Bernstein, 1996, 2000).

Pedagogy is essentially a communicative context in which recognition and realisation rules are employed by pedagogic agents when doing mathematics. As such communication is tied up with evaluation. Pedagogic agents communicate in ordinary language to achieve the transmission and acquisition of school mathematics. However, given the problem of reference entailed in communication, teachers and learners have to rely on evaluation to sort out the meanings implied by the signifiers they use. And even then, there is no certainty that they share the same meanings even though they may produce the same apparent outcome.

TP7: *Thought and language are internal.* Communication is considered as the externalisation of thought (Chomsky, 2006, 2007).

TP8: *Thought is computational* (Chomsky, 2006, 2007; Fodor, 1998, 2010; Gallistel & King, 2010; Pinker, 1997, 2007).

TP9: *Language does not have a reference relation*. In other words, language does not refer, it is people who use language to refer (Strawson, 1950). Language does not have stable word-object relations where the object is external to the mind (Chomsky, 2007, p. 21). The linguistic sign is arbitrary - there is no necessity for a particular signifier-signified coupling. There is no fixed connection between a signified and a signifier or for any given signifier (words, expression, mathematical statement), a number of different signifieds are possible (Saussure, 1983).

TP10: Semantics (meaning) is not associated with reference. *Semantics is considered to be entailed in syntax* which is internal to the mind (Bilgrami & Rovane, 2007; Chomsky, 2006; McGilvray, 2011). Hence the focus on the computational activity of teachers and learners.

TP11: *Cognitive representation entails establishing a relation between the representing system (brain) and represented systems (aspects of the world) through structure-preserving mappings referred to as morphisms* (Gallistel & King, 2010).

Adopting an Integrated Causal Model (ICM) approach to research entails approaching the research from multiple theoretical perspectives. In this study, theoretical resources include semiotics, Mathematics and cognitive science.

> **TP12:** This study takes as given that *humans are genetically endowed with core domain* knowledge which remains in use throughout their lives and enables them to acquire school and advanced mathematics (*non-core domain* knowledge), the focus of schooling and further studies (see for example Butterworth, 1999, 2005; Butterworth, 2010; Dehaene, 1997; Gallistel & Gelman, 2005; Gallistel & King, 2010; Gelman, 2009; Spelke & Kinzler, 2007). Non-core domain mathematical knowledge has to be explicitly taught and/or learnt and is essentially what constitutes school mathematics.

> **TP13:** The first factor (genetic endowment) and third factor (biological and physical laws) are beyond human control. *This means that within species differences emerge from the second factor, which refers to the structuring of experience through genetic endowment on the basis of contextual data* (Chomsky, 2005).

Pedagogic discourse (school mathematics) instantiated in curricula, textbooks and pedagogy is recontextualised from knowledge in the field of production (Mathematics) and other discourses (Bernstein, 2000, pp. 31-35). The notion of recontextualisation employed in the study differs from both Bernstein (1996, 2000) and Dowling (1998). In alignment with Bernstein and Dowling, recontextualisation of practices always occurs when discourses are "relocated or delocated". However, recontextualisation of mathematics to school mathematics may or may not involve content substitutions. When knowledge statements in a pedagogic situation converge at the level of content with knowledge statements in the field of production, then content substitution has not taken place even though there may be differences at the level of expression. When knowledge statements in a pedagogic situation diverge at the level of content with knowledge statements in the field of production, then content substitution has taken place even though there may be convergence at the level of expression.

> **TP14:** Decisions regarding what is constituted as mathematics in pedagogic situations are not based on *a priori* notions of what mathematics is, but instead entail deploying resources for describing what materialises as mathematics in pedagogic situations (Davis, 2010a).

> **TP15:** *Recontextualisation of mathematics to school mathematics may or may not involve content substitutions* (Davis, 2013a; Jaffer, 2012). The content associated with announced topics may be convergent or divergent with the content associated with the topic from the point of view of the Mathematics encyclopaedia (Davis & Ensor, 2018).

> **TP16:** The essential properties of mathematics cannot change as we move from the field of production (Mathematics) to the fields of recontextualisation or reproduction (school mathematics) because *school mathematics derives its internal consistency from Mathematics* (Davis, 2010a). School mathematics is

underpinned by the field of the reals and teachers and learners, irrespective of whether they explicitly use the field axioms, have to account for them in some way.

TP17: The operations that populate mathematics are functions. This feature of operations being functions is essential for providing mathematics with stability since functions have unique outputs for given inputs. However, operation-like manipulations that are not necessarily functions are present in pedagogic situations (e.g., Davis, 2013a; Staats & Batteen, 2009, 2010).

TP18: It is always possible to substitute a particular rule of a function with another rule, providing the same output for a given input (Lawvere & Schanuel, 1997). Thus the same expression can be associated with very different content, making it possible for the a topic name to be associated with very different contents (Chitsike, 2011b; Davis, 2013a, 2013b; Jaffer, 2012). In other words, *content substitution is always possible.*

TP19: *Pedagogy implies a model learner* due to the way in which the evaluation both presupposes and structures the mathematical performance of the learner, which is similar to the way in which a text presupposes and structures the performance of its reader (Chitsike, 2011b; Jaffer, 2011a).

TP20: The orientations to the productions of mathematics constituted in pedagogic situations can be either content-oriented or expression-oriented (Davis, 2011b).

3.10.2 Empirico-theoretical propositions

ETP1: Procedures, elaborated mainly through worked examples for particular classes of mathematical problems, serve as the primary vehicle for the transmission and acquisition of school mathematics in schools populated by learners from working-class families (Anyon, 1980, 1981; Chitsike, 2011b; Davis & Johnson, 2007; Jaffer, 2010b).

ETP2: Teachers and learners in schools populated by learners from working-class families employ a number of operations and operation-like manipulations that are not usually recognised in the Mathematics encyclopaedia but produce the desired outcome (Arendse, 2013; Basbozkurt, 2010a, 2010b; Chitsike, 2011b; Davis, 2010a, 2010b, 2011a; Jaffer, 2009, 2010a, 2010b, 2011a). The content realised in pedagogic contexts populated by learners from working-class families does not correspond with content indexed by the announced topics (Chitsike, 2011b, Jaffer, 2011a, Davis, 2010a, 2010b, 2011a, 2012).

ETP3: The dominant resources recruited as supports for computational activity of teachers and learners in pedagogic situations populated by learners from working-class backgrounds are algorithmic and iconic in nature (Chitsike, 2011b; Davis & Johnson, 2007, 2008; Jaffer, 2009, 2010a, 2010b, 2011a). The interest of this study is to explore whether this extends to schools populated by middle-class learners.

ETP4: Schools populated by learners from working-class backgrounds are not required to engage with mathematics that extends much beyond basic arithmetic i.e. operations (addition, subtraction, multiplication and division) over natural numbers and fractions (Arendse, 2013; Chitsike, 2011b; Davis, 2010a, 2010b, 2010c, 2011a, 2011b, 2011c, 2012, 2013a, 2013b, 2014; Jaffer, 2001; Johnson & Davis, 2010; Schollar, 2008). The interest of this study is to explore whether this is a general feature of school mathematics or whether there are social class differences in the constitution of mathematics.

Although the domain underpinning school mathematics topics is the field of the reals, except for encounters with real numbers such as π and the square root of negative numbers in the context of solving equations, real numbers are rarely explicitly attended to in pedagogic contexts. In fact, it is the set of rationals, particularly the set of positive rationals that, for the most part, serve as the computational domain for operations and operation-like manipulations performed by teachers and learners.

ETP5: Schools populated by learners from working-class backgrounds construct model learner of closed pedagogic texts (Chitsike, 2011b; Jaffer, 2011a). The model learner implied by the computational activity is unable to engage with the propositional ground underpinning the announced topic.

3.10.3 Research hypotheses

The following research hypotheses for my study derives from the theoretical and empirico-theoretical propositions discussed above. I started by establishing the research hypotheses related to pedagogic contexts populated by learners from working-class families based on the empirico-theoretical propositions describing pedagogic contexts. Given the difference in mathematics performance along social class lines and the strong partitioning of learners and schools differentiated in terms of social class in the literature discussed in Chapter 2, the research hypotheses pertaining to pedagogic contexts populated by learners from middle-class/elite families are set up in opposition to the research hypotheses concerned with pedagogic contexts populated by learners from working-class families.

RH1: The realised content associated with announced topic(s) in the instructional discourse diverges from the Mathematics encyclopaedic content associated with the topic(s) in pedagogic contexts populated by learners from working-class families.

RH2: In pedagogic contexts populated by learners from upper-middle-class/elite families, the recognition and realisation rules employed realise content associated with announced topic(s) in the instructional discourse that converges with the Mathematics encyclopaedic content associated with the topic(s)

RH3: Pedagogic contexts populated by learners from working-class families construct model learners of closed pedagogic texts that are expression-oriented.

RH4: Pedagogic contexts populated by learners from upper-middle-class/elite families construct model learners of open pedagogic texts that are content-oriented.

RH5: The computational performance of learners from working-class families exhibits closed pedagogic texts and their orientation to mathematics is expression-oriented.

RH6: The computational performance of learners from upper-middle-class/elite families exhibits open pedagogic texts and their orientation to mathematics is content-oriented.

The interest in this study is to examine the structuring of computational activity in pedagogic situations that differ with respect to the social class membership of learners.

Having established the propositions underpinning the study, the next chapter focuses on procedures for the production and analysis of data.

Chapter 4

Research design and construction of the information archive

4.1 Introduction

Chapter 3 set out the general methodology of the study. In this chapter, I describe issues pertaining to research design and construction of the information archive. This chapter bridges Chapter 3 and Chapter 5, which deal with procedures for the production and analysis of data from the information archive. So taken together, Chapters 3, 4 and 5 constitute the methodology of the study.

The first part of this chapter describes the design of the study and the selection of schools, teachers and learners that constitute the cases of the study. The second part of this chapter deals with the construction of the information archive, that is the collection of information sources, instruments used for collection of information and modes of collecting information. The main information collected included the video-recording of three consecutive lessons taught by each selected teacher and the clinical interviews with selected learners in each of the selected Grade 10 classes. Secondary sources of information relating to contextual and biographical information with respect to the schools, teachers and learners were also collected.

Finally, the chapter considers issues pertaining to reliability, validity and generalisability of the study.

4.2 Designing the study

This study, which is concerned with the functioning of evaluation at the level of the instructional discourse and the relation between the constitution of mathematics and the production and reproduction of learners' computational performance and orientations to mathematics, requires in-depth, close analysis of pedagogic situations to establish the order or logic of the pedagogic situation. To this end, the study has been designed as a case study (Yin, 1984, 2009). Although Yin (1984, 2009) refers to *case study* as a method of research which allows for in-depth study of a small number of "cases" where the focus is on understanding a complex phenomenon, I do not treat *case study* as a specific research method. Instead I view a "case" simply as a means of describing the selection procedure of a study (Brown & Dowling, 1998, p. 151). A range of research methods could be employed within case study research and is dependent on the specific methodological orientation and methodological resources recruited. The general methodological approach and methodological resources adopted in this study were discussed in Chapter 3 and the specific procedures for generating and analysing data follows in Chapter 5.

Nevertheless, a number of features identified as particular to case studies are pertinent to my study. Firstly, in-depth close analysis, a central feature of case studies, is achieved through the use of an analytic method that pays close attention to the computational activity of teachers and learners and requires detailed analyses of what teachers and learners say and do. Secondly, given the close attention to detail, the scope of the study is small, focusing on two independent schools. At each school two teachers and their Grade 10 mathematics class comprise the research participants of the study. In total, the study involves four teachers each teaching one Grade 10 mathematics class. The mathematical work of four teachers together with that of their learners represent the four cases under consideration in this study. The study is therefore conceived of as a multiple comparative case study, with the cases purposively selected in order to explore potentially contrasting cases of the constitution of mathematics. Schools were selected on the basis of the social class membership of their learner population – a school with learners drawn from upper-middle-class/elite families and a school populated by learners from working-class backgrounds. The selection of schools on the basis of the social class membership of their learner population was guided by the assumption that because social class continues to be aligned with differential mathematics achievement in the research literature, such differences potentially point to differences in the functioning of evaluation and so to differences in the constitution of mathematics.

Thirdly, the complexity of the phenomenon under investigation required systematic collection of information from multiple sources in order to establish the functioning of pedagogic evaluation and what comes to be constituted as mathematics in each case. The multiple sources contained in the information archive of the study include video-recorded lessons (observations), interviews with principals and teachers, clinical interviews with selected learners, questionnaires completed by the principal and by learners, curriculum resources such as textbooks, worksheets, tests etc. and learner test scripts. A discussion of the selection of the cases follows.

4.3 Selecting the cases

As discussed above, the selection of empirical sites on the basis of the social class membership of a school's learner population was guided by the assumption that because social class continues to be aligned with differential mathematics achievement, such differences potentially point to contrasts in the constitution of mathematics. Social class, therefore, functions as a background variable in the study.

4.3.1 Selection of schools

In Chapter 1, I argued that the right of public schools to charge school fees blurs the boundary to some extent between independent schools and public schools. The principal criterion for selecting schools as empirical sites of the study is the social class membership of a school's learner population. The poverty index[38] (quintile 1 to quintile 5) assigned to public schools by education departments seems at first glance to be a

[38] Poverty quintiles serve as the basis for determining the amount of recurrent, non-personnel expenditure allocated to schools.

means of selecting schools on the basis of the social class membership of a school's learner population. However, poverty quintiles are an unstable measure of the social class membership of learners (Hall & Giese, 2009; Kanjee, 2009; Wildeman, 2008) and do not apply to independent schools. School fees serve as a more reliable index since school fees are an indicator of parents' ability and willingness to pay for their children's education (Soudien, 2004). The use of school fees as a proxy for social class has been used in other studies (see Luckay, 2010; Reeves, 2005).

Two schools (and hence the cases) were purposively selected so that social class membership of the school's learner population contrasted with each other. Following Luckay (2010, p. 61),'No-fee' schools and schools charging school fees less than R3,300 per annum were categorised as schools populated by learners from working-class families and schools charging school fees in excess of R25,000 per annum were considered to serve learners from upper-middle-class/elite families[39]. I selected a 'No-fee' independent school and an independent fee-paying school. The 'No-fee' independent school was chosen because the number of learners per class was small and comparable to the numbers of learners in the independent school.

I chose to focus on FET level mathematics because teachers teaching mathematics at this level are most likely specialist mathematics teachers who have some university level training in mathematics. Secondly, mathematics is an elective subject at the FET level unlike the GET (General Education and Training) level, where mathematics is compulsory, so learners doing mathematics at the FET level have elected to do so. It would have been ideal to focus on Grade 12 since it is the last year of schooling. However, teachers and schools are often reluctant to allow their Grade 12 learners to participate in a research study given their concern with the final National Senior Certificate examination. The choice therefore was between Grade 11 and Grade 10. I chose Grade 10 since it is the first year of the FET level and the first year of the implementation of FET CAPS at the time of collecting information for the study. Secondly, Grade 10 was safer given that if I was unable to complete the collection of information while learners were in Grade 10 I would have greater access to learners in their Grade 11 year than if I chose Grade 11 and was forced to collect information in the next year when learners would be in Grade 12.

4.3.2 School profiles

The profiles of schools constructed below are based on a school questionnaire administered to principals (Appendix 4.1) as well as interviews with the principal (Appendix 4.2) and with the selected Grade 10 mathematics teachers (Appendix 4.3). The school populated by learners from upper-middle-class/elite backgrounds is referred to as Prestige College and the school populated by learners from working-class and lower working-class backgrounds is referred to as Evergreen High.

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[39] Luckay (2010), using a box-and-whisker plot of school fee data, categorised low Socio-economic Status (SES) as those charging school fees of less than R300 per month, between R300 and R2250 per month as medium SES and more than R2250 per month as high SES. I adjusted the fees by 10% to accommodate for inflation and considered annual fees instead of monthly fees. Note that schools collect fees over 10 months if parents choose to pay monthly.

Prestige College

Prestige College is an all-boys independent school, established 163 years ago, situated in an affluent, previously "White" suburb of Cape Town. Although there are some learners on scholarships at the school, most learners paid school fees which were set at R85 000 per annum in 2012, at the time of the collection of information of the study.

The school recruited its learners mainly from upper-middle-class and elite families, with parents in mainly high-status professional jobs and businesses. Most of the learners lived in the more affluent Cape Town suburbs but some were from outside of Cape Town and used the school's boarding facilities. The total learner population at the school was 750 learners with an average of 24 learners per class, although class size varied depending on subject choice. There were six Grade 10 classes comprising about 150 learners in total. The learner body was 'racially' mixed but predominantly "White" with 79% of the learners being "White", 10% "Coloured", 7% "African" and 2,5% "Indian". Learners were admitted on the basis of academic performance and had to be able to communicate in English. Learners from the primary school associated with the school were accepted automatically and family and other connections were advantageous in the learner-selection process.

The school had a large staff component of 124 members which included 12 in secretarial positions and 40 in support positions such as gardeners, cleaners and security guards. The senior leadership of school consisted of the principal, three deputy principals and 15 heads of department. In addition, the school employed 55 teachers over and above the 15 heads of department who also had teaching duties. Most of the staff were "White" (65%) occupying senior leadership positions or teaching positions. Although some "African", "Coloured" and "Indian" staff occupied teaching positions, the majority were in support and administrative posts.

The school was extremely well-resourced with extensive, well-utilised facilities, including a library, a computer laboratory, and several science laboratories. The grounds were spacious with well maintained gardens and trees and ample sporting facilities. The school buildings were well-maintained. Access to the school was tightly controlled by security. The classrooms were well equipped with computers, data projectors, electronic whiteboards and document cameras. There were more than sufficient desks for learners and ample space to move around the classrooms.

The school offered mathematics as well as mathematical literacy. At Grade 10 level, there were six mathematics classes and one mathematics literacy class. So although learners could choose to do either mathematics or mathematics literacy, very few opted to do the latter. Learners were streamed into ability sets based on their Grade 9 performance in mathematics. Sets 1 and 2 were also offered the Additional Mathematics (Admaths) curriculum. The mathematics department consisted of 10 teachers. Each teacher was allocated four classes to teach, spread across grades and sets, with the exception of the deputy principal who taught three classes. The department met once a week to discuss organisational issues as well as issues pertaining to teaching and learning of mathematics. Planning for the entire year was done by the head of the mathematics department in conjunction with the mathematics team. The team agreed on what topics would be examined and dates for common tests after each topic was completed as well as submission dates for tutorials set on each topic were set at the beginning of the year. The year planner set out common goals but provided flexibility for each teacher to achieve the goals in the manner they thought fit. The year planner was guided by CAPS but did not necessarily adhere to the CAPS sequence of topics nor the grade-specific topics. For example, although CAPS specifies the parabola $y = ax^2 + q$ for Grade 10 and $y = ax^2 + bx + q$ for Grade 11, the mathematics department opted to do both $y = ax^2 + q$ and $y = ax^2 + bx + q$ in Grade 10. Teachers produced and shared worksheets amongst themselves. However, each teacher was given the freedom to use whatever they thought worked best for them and their learners.

Grade 10 mathematics was allocated nine periods of 45 minutes over a two-week cycle, giving a total pedagogic time of 3,4 hours per week which is less than the 4,5 hours stipulated by the Department of Basic Education (2011, p. 7). The school was not regulated by the education department because it is an independent school. The mathematics department offered mathematics support to learners every afternoon (except on Thursdays) after school for about an hour and half. These support sessions were open to all learners from Grades 8 to 12 and were voluntary. Learners come to the sessions to seek help with a particular mathematics problems or topics. One of the mathematics teachers would be available to help learners. In addition to this support, teachers were available to help learners during breaks.

Evergreen High

Evergreen High is a co-ed independent 'No-fee' school established 12 years ago, offering Grades 9 to 12. The school is situated in a former "White" suburb of Cape Town and draws its learner population from former "African" townships, Langa, Gugulethu, Crossroads, Delft and Khayelitsha. The school was funded by corporate funders mainly, some private funding and a very small portion of state funding. Learners contribute to social development activities[40] and pay for bus fare but do not pay school fees.

The total learner population at the school was 330 learners with an average of 22 learners per class. There were four Grade 10 classes comprising about 170 learners in total. The learner body was exclusively "African" and learners were bussed to and from school. Grade 8 learners at schools in the designated township areas were targeted for enrolment. Learners were selected based on scores obtained on English and Mathematics tests given to applicants. In addition, applicants were interviewed before being selected.

The school had a staff complement of 52 which included two in secretarial positions. Many teachers were from the African continent and USA. The majority of the staff was "African" (46%), 27% "White" and 19% "Coloured". As reported in the interview conducted with the school leadership group, leadership of the school was not located in a single person. Instead, the school used a collective leadership - operations leader (finance, administration etc.), instructional leader (curriculum and classroom practice, relationship leader

[40] Social development activities involve students and staff working with community organisations after school at scheduled times during the year.

(with families) and life orientation leader (with learners)). An individual held leadership in one aspect but was involved in all aspects of leadership. The collective met twice per week to discuss the running of the school. Leaders were allocated half of the normal teaching load.

The school was much less resourced than Prestige College. The classrooms lacked computers, data projectors and electronic whiteboards. Although there were sufficient desks for learners and ample space to move around the classroom, the furniture looked worn and some chairs were broken. There were no science laboratories, even though its mission emphasised science and mathematics. The school had a computer laboratory and a library, although the library lacked sufficient resources for teaching. The grounds were far less spacious than that of Prestige College and lacked gardens and sporting facilities. Access to the school grounds was not controlled but access to the buildings was controlled by security.

The school only offered Mathematics and not Mathematical literacy at Grade 10 level. There were four Grade 10 mathematics classes. Learners were not streamed into ability groupings as was the case with Prestige College. The mathematics department consisted of six teachers servicing 16 classes, four classes per grade. Teachers were allocated three mathematics classes on average for the year. The department met every week to discuss organisational issues as well as issues pertaining to teaching and learning of mathematics. Common tests were written at the end of each term. The school followed CAPS both in terms of content as well as sequencing. The Grade 10 teachers, however, decided not to include statistics and geometry riders because of a shortage of time to complete the curriculum in the year that information was collected for the study.

The school day extended from 8.15 to 5.15 pm. Double time was given for Mathematics, Science, English and Life Orientation. Grade 10 Mathematics was allocated 11 periods of 35 minutes per week, giving a total pedagogic time of 6,4 hours per week which is more than the 4,5 hours stipulated by the DBE Department of Basic Education (2011, p. 7). The school conducted Saturday classes. This was independent time for learners to do school work supervised by a teacher. When a mathematics teacher was on duty for Saturday classes, they often used one session for mathematics. In addition, the mathematics department offered mathematics support (a maths clinic) to learners every day (except on Thursdays) during lunch time. The mathematics teachers took turns to run the maths clinics. However, the teachers claimed that learners did not make sufficient use of this facility except when a class test or an examination was approaching.

Table 4.1 summarises the key distinguishing features of the two schools. The schools are comparable in the sense that the class sizes are relatively similar, there are however disparities in available resources: specialised teaching rooms, technology in classrooms and sporting facilities available at the school. The presence of audio-visual equipment in Prestige college classrooms impacted positively on the presentation of mathematics lessons compared to Evergreen High. Prestige College teachers were able to incorporate mathematics computer software into lessons and display solutions to homework problems, which helped them to cover more work in a lesson.

Table 4.1. Characteristics of the two schools

School	# learners	# teaching staff	school fees	average class size	sufficient desks	specialist rooms	classroom resourcing
Prestige College Grades 8 -12	750	70	R85 000	24	yes	10 science laboratories, computer laboratory,	Computer, Smartboard, document camera
Evergreen High Grades 9 -12	330	50	0	22	yes	Computer laboratory but no science laboratories	No audio-visual equipment in classrooms

4.3.3 Selection and profile of mathematics teachers

At each school, I had had an initial meeting with the head of the Mathematics department to explain the study. At Evergreen High, there were only two teachers (Maya and Jono) teaching mathematics at Grade 10 level, so there was no need to select teachers. Each teacher was assigned two Grade 10 mathematics classes. I met with the teachers to explain the study and their involvement as well as the involvement of their learners. Since the Grade 10 learners were not streamed according to performance, the teachers selected the Grade 10 class to observe. At Prestige College, two teachers agreed to participate in the study following a meeting with the head of the mathematics department. Each had one Grade 10 mathematics class assigned to them. So there was no need to select the class to observe. Sara taught a Set-2 Grade 10 mathematics class and Jada taught a Set-3 Grade 10 mathematics class. Table 4.2 summarises the details of the four teachers participating in the study and is followed by profiles of each teacher.

Table 4.2. Details of teachers participating in the studies

Teacher	School	Gender	Race	Position	Qualification in mathematics	Teaching experience	Experience of teaching G 10 mathematics
Sara	Prestige College	female	“White”	HOD	Mathematics major	30 years	30 years
Jada	Prestige College	female	“White”	Educator	Mathematics major	31 years	31 years
Maya	Evergreen High	female	“White”	Educator	Mathematics in an education qualification	12 years	First year in Grade 10
Jono	Evergreen High	male	“African”	Educator	Honours in Statistics	10 years (unqualified)	2 years

Sara

Sara had 30 years of experience in teaching Grade 8-12 mathematics. Her teacher training was preceded by an undergraduate degree majoring in Mathematics and Psychology at the University of Cape Town where she also completed a post-graduate teaching qualification. She had been teaching at Prestige College for 11 years and had been in the position as the head of Mathematics for six years.

Jada

Jada had 31 years of experience in teaching Grade 8-12 mathematics, although she had a 10-year break from formal teaching when she tutored learners outside of formal schooling. She had been teaching at Prestige College for four years. Jada completed an undergraduate degree at the University of Stellenbosch, majoring in Mathematics and Computer science, followed by a postgraduate teaching qualification. Jada also held a Masters in Education.

Maya

Maya was trained as a middle-school teacher in the United States where she completed an undergraduate degree in mathematics and science education which certified her to teach children aged 10-14 years. She was also certified as a master teacher in the USA and completed a Masters in Education (Curriculum and Instruction). She taught mathematics for 12 years with 10 years of teaching Grade 5 in the USA. She was appointed at Evergreen High in June 2010 when she taught Grade 9 for the first time and had only started teaching Grade 10 shortly before the observed lessons due to re-allocation of mathematics classes when a teacher left the school.

Jono

Jono had 10 years experience in teaching mathematics – two years in a South African school teaching Grade 10 and five years in the Democratic Republic of Congo (DRC) where he taught Grades 11 and 12 as an unqualified teacher. For three years he was outside the formal schooling system operating as a tutor for high-school learners and university learners. He completed an Honours degree in Statistics in the DRC which he described as Applied Mathematics and a Masters in Information Systems at the University of Cape Town. He did not have a teaching qualification.

4.3.4 Profile of the Grade 10 learners

A questionnaire to capture biographical information of the Grade 10 learners and to confirm their social class membership was administered to all learners in the classes observed (see Appendix 4.4). The questionnaire was conducted in September 2012, shortly after observing the lessons. Learners completed the questionnaire in a classroom setting where each question was read to them by the researcher. Questions of clarification from learners were dealt with by the researcher as they arose.

The ages of learners in Sara and Jada's class ranged between 15 and 16 years when the questionnaire was conducted whereas Maya and Jono's learners ranged from 14 to 17 years of age. Sara and Jada's classes consisted only of boys given that the school was an all-boys school, whereas Maya's and Jono's classes were mixed gender classes, where most of the learners were girls in both classes.

The majority of the learners in Sara's class, with the exception of one learner who came from a neighbouring country and another learner who lived in a former "Coloured" area, lived in areas formerly designated as "Whites"-only residential areas during the Apartheid era and which today are considered as the more affluent residential areas of Cape Town. Similarly, the majority of learners in Jada's class, with the exception of one learner who lived in a former "African" township and another learner who lived in a former "Indian" area, lived in areas formerly demarcated for "Whites" only. The learner population in both Sara's and Jada's classes was "racially" heterogeneous but mainly "White". All the learners in Sara and Jada's classes lived in brick buildings, with access to electricity, running tap water, hot water and water-flushed toilets. Furthermore, all the learners had access to internet at home and all of them owned a mobile phone.

All the learners in Maya's and Jono's classes lived in former "African" townships and the learners were "racially" homogenous. isiXhosa was the first language of all the learners with English being their second language and the language of learning and teaching at the school. All the learners, with the exception of one, had access to electricity, two learners in Jono's class and four in Maya's class did not have access to running tap water inside the home, only four learners in Jono's class and five learners in Maya's class had access to hot water and running water in their homes, five learners in both classes did not have access to a water-flushed toilet in their homes. A quarter of the learners in Maya's class and a third of the learners in Jono's class did not own a mobile phone. Half of the learners in Maya's class and about a third of the learners in Jono's class had access to internet at home. The number of learners with internet access seems high but internet access was most likely via mobile phones. The stark difference in the residential areas and living conditions of the learners in the two schools are indicators that the social class membership of the learners at the two schools are on opposite ends of the social class spectrum. The economic differences in living conditions are mirrored in the reported educational levels of the learners' caregivers.

All the learners in Sara and Jada's class considered either their mother or father as their primary caregiver whereas seven of Maya's learners and 11 of Jono's learners indicated that either their grandmother or another person such as an aunt was their primary caregiver. All the caregivers of learners in Sara's and Jada's classes completed high school according to the learners whereas only 40% of learners in Maya's class and half the learners in Jono's class indicated that their caregivers completed high school. Furthermore, 88% of learners in Sara's class and 84% of learners in Jada's class indicated that their caregivers had a higher education qualification (see Table 4.3). In comparison, 10% of the learners in Maya's class and 21% of the learners in Jono's class indicated that their caregivers had any higher education qualifications. In fact, about more than half of these learners indicated that their caregivers did not have any post-school education (see

Table 4.3). Thus the learners in Sara's and Jada's classes had access to greater *cultural capital* (Bourdieu, 1989)[41] than the learners in Maya's and Jono's classes.

The living conditions of the learners and the educational levels of the caregivers indicate difference in the social class membership of the learners in the two schools but this data is not sufficiently nuanced to determine specific social class locations.

Table 4.3. Post-high school education levels of caregivers

Post-high school education	Sara (n = 16)	Jada (n = 19)	Maya (n = 20)	Jono (n =24)
University	13	14	0	4
College	1	2	1	1
Technikon	0	0	1	0
On-the-job training	0	2	1	1
Other post-school institutions	0	0	2	1
No post high school education	1	1	13	13
I don't know	1	0	2	3

Social class theories fall mainly into two broad schools of thought, namely, those in the Marxist and neo-Marxist tradition and those following a Weberian framework. For the Marxists and neo-Marxists, an individual's relationship to the means of production is of central concern (Marx, 1859), whereas for Weber, class membership is regulated by an individual's life chances, the opportunities available to them to improve their lives, which in a capitalist society are primarily determined by the market (Weber, 1978). Since social class serves as a second factor issue and not as an explanatory category in this study, I have not adopted any particular theory of social class. Wright (1977) claims that class typologies can be used empirically within a Marxist or Weberian framework. Furthermore, Seekings and Nattrass (2005) argue that considerable convergence with respect to the categories is now evident in Marxist and Weberian frameworks. Seekings and Nattrass (2005) modified the most widely used neo-Marxist class typology constructed by Wright (1977) and the neo-Weberian class schema developed by Goldthorpe (2000, p.223) to capture the social class structure unique to South African society, with the caveat that "alternative approaches generate somewhat different class categories" (Seekings & Nattrass, 2005, p. 237). They argue that "given the overwhelming dependence of South African households on wages as a source of income … occupations must be the starting point for analyses of class in South Africa" (Seekings & Nattrass, 2005, p. 241).

Their social class schema (Table 4.4) is based on the occupations of individuals comprising 10 social class categories. Each category attempts to capture similar occupations in terms of "economic security, career prospects, and autonomy" (Seekings & Nattrass, 2005, p. 247).

[41] Bourdieu (1989, p. 17) refers to *capital* as the multi-faceted set of resources and powers possessed by an individual, acquired primarily through the family and entrenched through schooling. For Bourdieu, capital is central in locating individuals within the social division of labour.

From the questionnaire administered to learners, Table 4.5 was used to map the occupations of learners' caregivers as reported by learners to social class categories. Categories 1, 2 and 3 were mapped to the upper-middle-class/elite, categories 4 and 5 to the middle-class and categories 6, 7 and 8 to the working-class. Category 9 includes those temporarily without employment but who are seeking a job and those who have given up on finding a job, the latter forming part of the structurally unemployed underclass. Similarly, category 10 is a hybrid category comprising individuals who have similar economic conditions to that of the marginal working-class but also includes economically well-off individuals who have opted not to work to look after the home and children.

Table 4.4. Social class locations in terms of occupations (Seekings & Nattrass, 2005, pp. 247-252)

Social class location	Code
1. High income entrepreneurs (do not work and employ others)	WE1
2. Upper class managers and professionals	UC
3. Entrepreneurs (work and employ others)	WE2
4. Semi-professional class: teachers and nurses	SPC
5. Intermediate class: routine white-collar, skilled, and supervisory workers	IC
6. "Petty traders" (work in their own business and do not employ others)	WE3
7. Core working class: semi-skilled and unskilled workers (except farm and domestic workers)	CWC
8. Marginal working class: farm and domestic workers	MWC
9. Unemployed (the jobless seeking employment or those disenchanted with seeking employment)	UE
10. Other (not employed but some form income such as old-age pensions, home executives etc.)	O

The result of mapping the occupations of the primary caregivers as reported by the learners to the social class locations shown in Table 4.4 is presented in Table 4.5.

Table 4.5. Social class locations of learners' primary caregivers[42]

	Prestige College		Evergreen High	
Social class location	**Sara (n = 16)**	**Jada (n =19)**	**Maya (n=20[43])**	**Jono (n-24)**
WE1	1			
UC	7	10		
WE2		2		
SPC	3	2	2	5
IC			1	
WE3			2	2
CWC				
MWC			11	7
UE	2	1	3	7
O	3	4	1	3

[42] One student in Maya's class and one student in Jono's class did not participate in the survey because they were absent on the day this information was collected.

[43] One student in Maya's class and one student in Jono's class did not participate in the survey because they were absent on the day this information was collected.

Table 4.5 reveals the social class differences between the learner populations of the two schools. At Prestige College 57% of the primary caregivers of the learners were located in the upper-middle-class/elite category, 14% in the middle-class category and none were located in the working-class category. At Evergreen High, none of the primary caregivers were located in the upper-middle-class/elite category, 18% in the middle-class category employed in teaching, nursing or policing professions and 50% of primary caregivers were located in the working-class category. Nearly a quarter (23%) of primary caregivers were unemployed at the time of the survey but were seeking employment and 9% were pensioners. It is highly likely that 23% seeking employment formed part of an underclass who are structurally unemployed and the pensioners were living off state pensions. This means that in essence 82% of primary caregivers at Evergreen High are in fact from working-class families.

The primary caregivers of learners at Prestige College (Sara's and Maya's classes) were predominantly classified as UC with a few categorised as WE1 (57%). These primary caregivers were employed as professionals such as academics, doctors/surgeons/medical specialists (one was a cartographer) and as general managers or CEOs of companies. About 30% were not working, three of the primary caregivers were seeking employment and the rest were university graduates who were full-time caregivers, presumably being financially supported by some other means. A small number of primary caregivers (20%) were classified as SPC, employed as teachers or librarians.

At Evergreen High 18% of primary caregivers were classified as middle-class, employed in teaching, nursing or policing professions, classified as SPC and the primary caregiver of one learner was employed as a receptionist classified as IC. Half of the primary caregivers were classified as MWC and WE3, employed as cleaners, gardeners, domestic workers, security guards, shop assistants and kitchen staff and four learners' primary caregivers were self-employed running informal businesses such as take-away food, bed and breakfast, taxi business etc. The rest, as indicated above, were unemployed or pensioners on state pensions.

The stark differences in the stated occupations and educational levels of the primary caregivers of the learners in the two schools reflect differential access to economic and cultural capital and is indicative of the social class membership of the learners at the two schools.

4.4 Information archive

Dowling (1993, p. 87) distinguishes between *information* and *data*, where data are produced from information which is read in terms of a theoretical framework or in terms of an analytic structure of some kind. In other words data are produced from information gathered empirically through the process of analysis. As such data are principled and consciously derived as opposed to information, which although informed by theory has not yet been read in terms of a theoretical framework. This is not to deny that the collection of information contained in the archive already represents a selection informed by the methodological orientation of the study. The information required to answer the research problem is outlined in Table 4.6.

Table 4.6. Information archive

Information source	Research participants	Information required	Instrument	Collection mode
Lesson observations (Instructional discourse)	Teachers Learners	recognition and realisation rules (computational activity)	None	Video-recording
Curriculum resources	Teachers	Constitution of mathematics	None	Researcher collects
Teacher interviews	Teachers	Biographical information Teaching & learning of mathematics	Semi-structured interview	Audio-recording
Learners' test papers	Learners	recognition and realisation rules (computational activity)	None	Researcher collects
Clinical interviews based on tests	Learners	recognition and realisation rules (computational activity)	Unstructured interview	Video-recording
Learner questionnaire	Learners	Home background of learners' caregivers occupations and education levels	Questionnaire	Participants' recording
School questionnaire	Principal	Information about schools, teachers & learners	Questionnaire	Participants' recording
Principal interview	Principals	Information about schools, teachers & learners	Semi-structured interview	Audio- recording

4.4.1 Lesson observations

The study is concerned with the functioning of evaluation at the level of the instructional discourse. Specifically I am interested in the recognition and realisation rules used by teachers and learners when doing mathematics. Recognition and realisation rules are described in terms of the computational activity of teachers and learners (TP8).

Mathematics lessons are empirical instances of instructional discourse. Since pedagogy is fundamentally evaluative (TP1), what is constituted as mathematics is rendered visible through recognition and realisation rules used in pedagogic situations (TP3). Given that pedagogy is necessarily extended over time, it was therefore important to observe how the teaching of a particular topic or sub-topic was dealt with over more than one lesson. Three consecutive lessons taught by each of the four teachers to their Grade 10 class were observed and video-recorded. Video records are helpful to capture pedagogic communication between teacher and learners. Two video-recorders were used, one focused on the teacher to record all of his/her communication with individual learners and the whole class. The second video camera was used as a roving camera focusing on learners - particularly on what they recorded in their notebooks or learner-learner discussions when working on classroom tasks. I was interested in capturing the speech of teachers and learners as well as other semiotic resources such as gestures, written productions such as texts on a whiteboard or chalkboard or displayed via an overhead projector as well as texts produced by learners in their notebooks.

The video-records produced by the camera following the teacher were transcribed and speech was translated from isiXhosa to English where necessary. Where helpful, stills of written text produced by the teacher or learners were captured to augment the speech captured in transcripts. The video-records from the second camera were used to check what learners produced in their notebooks and to capture speech that was not audible or indistinct on the video records captured on the first camera.

In addition to the video-recording of the lessons, I collected teaching and learning resources such as worksheets and textbooks used during the lesson.

4.4.2 Teacher interviews

A semi-structured interview (Appendix 4.3) was conducted with each of the four teachers. The interview was multi-focused. First, the interview captured biographical information about the teacher to establish the teacher's qualifications in mathematics and experience of teaching mathematics, particularly at Grade 10 level. Second, the interview focused on the teaching and learning of mathematics at the school with respect to planning, assessment, the use of curriculum resources such as textbooks and worksheets, the relationship with CAPS, learner support outside of mathematic lessons, and the functioning of the mathematics department at the school. Third, the interview explored the observed lessons in an attempt to locate the lessons in the curriculum plan. So, teachers were asked what they had covered prior to the observed lessons and what they planned to do after the observed lessons. Questions relating to aspects of the observed lessons that were not immediately obvious were covered with each teacher. Each interview was audio-recorded. The information collected using the interviews with teachers was used to construct the teacher profiles in Section 4.3.3.

4.4.3 Clinical interviews

Pedagogy aims at the reproduction of knowledge on the part of the learner. The extent of a learner's acquisition of knowledge is established through the demonstration of the appropriate use of recognition and realisation rules, which reveal the criteria used by learners when doing mathematics (TP3). The key methodological issue here relates to the proposition discussed in Chapter 3 that the recognition and realisation rules used by learners may not accord with those used by the teacher even when what they produce expressively is in agreement (TP4).

In each pedagogic context, the teacher set, administered and marked a test on the topic(s) covered during the observed lessons. Test scripts reveal very little about the recognition and realisation rules that learners use to solve mathematical problems. Because of the arbitrary nature of the signifier-signified relation, the expressions produced by learners could refer to a variety of different signifieds (content) (TP9). Thus, clinical interviews were conducted with selected learners in order to ascertain their computational activity which provides the basis for establishing the nature of the computational performance and the orientation to mathematics displayed by these learners. I interviewed six learners (two top performers, two mid performers and two bottom performers) in each pedagogic situation with the exception of Jono's class, where only four of six learners selected to be interviewed turned up for interviews.

From the learners' test scripts, I selected a test question or questions, that corresponded with the topics covered during the observed lessons, as the focus of the interviews with learners. In video-recorded individual clinical interviews, I presented each learner with a copy of their marked test script as well as the test question paper, and I asked them to explain the methods they used to produce their solutions. I explained that it did not matter whether their solutions were correct or incorrect because I was interested in their reasoning. I also explained that they could provide an alternate solution to the one presented in their test script. During the interview, I probed each learner's reasoning of the computations employed, asking for clarifications where necessary. When the learner presented incorrect mathematical reasoning, I would first attempt to elicit from the learner their reasoning, and then directed the learner towards more mathematically appropriate reasoning. If the learner was unable to correct their reasoning, I would point out their error and provide the correct mathematical explanation in order to assist the learner with the particular mathematical problem. Learners were encouraged to write if they needed to. The clinical interview with each learner was video-recorded and transcribed and, where necessary, translations from isiXhosa to English were done. The analyses of the tests and clinical interviews are presented in Chapters 8 and 9.

4.4.4 Summary of the information archive

The information that was collected from the four pedagogic situations is summarised in Table 4.7.

Table 4.7. Summary of the information archive

Information source	Information record
Lesson observations (instructional discourse) (3 lessons per pedagogic context)	12 video-recorded lessons (focusing on the teacher) and 12 lesson transcripts 12 video-recorded lessons (focusing on learners)
Curriculum resources	4 sets of teaching resources (textbooks, worksheets, lesson plans) FET CAPS
Teacher interviews (1 per pedagogic context)	4 audio-recorded interviews
Learner clinical interviews (6 learners per pedagogic situation except in Jono's class where only 4 learners were interviewed)	22 learner tasks 22 video-recorded interviews and interview transcripts
Learner questionnaire (All learners in each of the four classes completed a questionnaire)	4 sets of learner questionnaires (one per pedagogic situation)
School questionnaire	4 questionnaires completed by school principal
Principal interviews	4 audio-recorded interviews

4.4.5 Transcription and translation of videos

The video recordings of the observed lessons (specifically the one focusing on the teacher) and the clinical interviews were transcribed into written text, capturing what teachers and learners said (speech) and what they did (gestures that referred to what was spoken about, and diagrams or written texts produced on the board or on paper). Speech was transcribed verbatim and then annotated to include gestures and written text when it was difficult to discern what teachers and learners were referring to without access to the video-records. The result was a transcript that could be read without the need to refer to the video-record.

I transcribed three of the 12 video-recordings of the observed lessons in order to establish conventions for transcription (see Appendix 4.5). The remaining nine video-recorded lessons and the 22 clinical interviews were transcribed by professional transcribers who followed the same transcript protocol. These initial transcripts only captured the speech of teachers and learners. I then produced a second version of each transcript, checking the transcript by examining the video-records and annotating the transcript by including what teachers and learners referred to while speaking (mathematical expressions or diagrams).

All the lessons and the clinical interviews were conducted in English in all four pedagogic contexts. In Maya's and Jono's observed lessons, learners occasionally spoke in isiXhosa, particularly when conversing with fellow learners and on occasion learners used isiXhosa phrases during the clinical interviews. The fragments of isiXhosa speech were translated into English by an isiXhosa first language speaker who was a Masters research student in Mathematics education.

The translation of information into data is the subject of Chapter 5.

4.5 Research quality criteria

The quality of the research design can be established through the following criteria: reliability, internal validity and external validity (Yin, 2009).

4.5.1 Reliability

"Reliability is the measure of the consistency of the coding process when carried out on different occasions and/or by different researchers" (Brown & Dowling, 1998, p. 26). For Brown & Dowling (1998) reliability can be achieved by making the process of data analysis explicit. In other words, an elaborated description of the data must make visible the operationalisation of the analytic framework and thereby the relationship between theory and data in the process of analysis.

Chapter 5 provides a detailed description of the procedures for producing and analysing data. However, prior to the production and analysis of the data, detailed accurate transcripts were produced for all the lessons video-recorded and all the clinical interviews with learners. The production and analysis of data proceeded by applying the analytic framework to the transcripts as well as video-records to produce a description of the computational activity of teachers and learners.

4.5.2 Internal validity

Maxwell (1996, p. 87) defines validity as "the correctness or credibility of a description, conclusion, explanation, interpretation, or other sort of account". This however does not imply the existence of an objective truth. He distinguishes between descriptive, interpretative and theoretical validity. Descriptive validity entails ensuring the accuracy and completeness of the information (Maxwell, 1996, p. 89). In this study, this was achieved the use of two video cameras during the observed lessons to capture the teacher's speech, gestures and written texts as well as that of learners. In addition, transcripts of video-recorded lessons included capturing speech mainly and gestures and written texts where appropriate. Furthermore, clinical interviews with selected learners were video-recorded in order to establish the recognition and realisation rules used by learners in their productions of mathematics. Transcripts, capturing speech, gestures and written texts produced by learners in the interviews were also produced.

The main threat to validity at the level of interpretation involves the imposition of the researcher's "own framework or meaning, rather than understanding the perspective of the people studied and the meanings they attach to their words and actions" (Maxwell, 1996, p. 90). Interpretative validity was addressed through making explicit the theoretical and empirical propositions underpinning the study (see Chapter 3) and the elaboration of a detailed analytic framework (Chapter 5) that illustrates the transformation of information into data. Furthermore, as explained in Chapter 3, I adopt an internalist account in the production and analysis of data which means searching for what participants refer to rather than imposing *a priori* notions onto the speech and actions of research participants.

Theoretical validity could be compromised "through not paying attention to discrepant data or not considering alternative explanations or understandings of the phenomenon under study" (Maxwell, 1996, p. 90). Theoretical validity is addressed partly by explicit procedures for producing and analysing data, thus establishing explicit connections between information, theory and data as well as through reporting on instances of discrepant data where these occur.

Maxwell (1996, pp. 90-91) identifies two further threats to validity, namely, researcher bias and reactivity. Researcher bias involves the selection of data that fit the researcher's existing theory. The threat of researcher bias was avoided through the use of explicit procedures for producing and analysing data. Furthermore, making available transcripts and detailed analyses of lessons and clinical interviews allows for the production and analysis of data to be scrutinised for any evidence of researcher bias.

Reactivity refers to the influence of the researcher on the setting or individuals studied. Researcher reactivity for observations was minimised by acclimatising participants to the presence of the researcher and video camera. As three consecutive lessons were recorded, this allowed participants to become comfortable with being recorded over time. Reactivity in the clinical interview setting was minimised by assuring learners that the correctness of their responses to questions was not being judged. Instead, it was their reasoning that was of importance. In addition, learners were given the opportunity to change their responses if they wished to do so.

4.5.3 External Validity (Generalisation)

Maxwell (1996, p. 97) distinguishes between internal generalisability and external generalisability. Internal generalisability refers to the generalisability of the conclusion within the research setting and is an important issue for case studies. External generalisability refers to its generalisability beyond the research setting.

Critics either claim that external generalisations cannot be made from case studies or argue that external generalisability is not a crucial issue for qualitative studies (Maxwell, 1996).

In contrast, Yin (2009) contends that external generalisations from case studies are possible. Yin (2009) argues that critics of case-study research tend to conflate statistical generalisability with analytic generalisability. Statistical generalisability is used to generalise from the sample to the broader population whereas analytic generalisability is used in case-study research to generalise from the case study to a broader population. As Yin (2009) points out, a small number of cases cannot generalise from a sample to a broader population. This study does not make use of statistical evidence to infer from one case to a broader population.

Analytic generalisability enables generalisation from the theoretical propositions underpinning the research. So this study, on the basis of the analysis using a robust analytic method which establishes a relationship between the information collected for study and the theoretical propositions, generates the potential for generalisability to other pedagogic situations This mode of analysis is consistent with Yin's analytic generalisability in which validity is established through conceptual coherence and logical reasoning.

4.6 Ethical considerations

Ethical considerations including anonymity, confidentiality and voluntary participation were discussed with participants in the study.

In order to ensure that participants in the study engaged so voluntarily, I made sure that they understood the nature of the study which was described in letters to the principal, teacher and parent. In addition I explained the study in meetings with the principal and teachers as well as to the learners. Thereafter, I sought written informed consent from all participants, giving participants the opportunity not to participate in the study. Written informed consent was also obtained from parents since the children were mostly minors. Voluntary participation was borne out by the refusal of two learners at Evergreen High to participate in the clinical interviews.

Confidentiality and anonymity were achieved through the usual practice of using pseudonyms for all participants as well as for the school in my dissertation. Video-recordings of learners during the clinical interviews deliberately did not focus on faces in order to protect the identity of learners, The same level of anonymity could not be achieved with the videos of the lesson observations. However, the videos have been seen by myself only and snippets of the videos were watched by the translator.

4.7 Summary of the chapter

In this chapter, I outlined the research design of the study and provided a rationale for the selection of schools and teachers. I presented profiles of the schools constructed from interviews conducted with the principals of each school and the school questionnaire completed by each principal. I also presented profiles of the teachers involved in the study constructed from the interviews conducted with each teacher. This was followed by a description of the profiles of the Grade 10 learners participating in the study constructed from the questionnaires completed by learners. In particular, I focused on the occupations and education levels of the primary caregivers of learners in order to verify the social class membership of learners at the two schools. Then, I provided a justification of the information in order to answer the research question framing the study, thus providing a description of the information archive. I also discussed the instruments used to collect the information for the study. I concluded the chapter with a consideration of the validity and reliability of the study and ethical concerns.

The next chapter lays out the procedures for the production and analysis of data from the information collected.

Chapter 5

Analytic framework for the production and analysis of data

5.1 Introduction

I outlined the general methodological orientation of the study in Chapter 3, and considered issues of research design and the construction of the study's information archive in Chapter 4. The central concern of this chapter is the development of procedures for the production and analysis of data from the information archive in order to address the central research question of the study:

How does pedagogic evaluation function in the instructional discourse of four Grade 10 pedagogic contexts in schools that differ with respect to the social class membership of their learner populations and what are the implications of pedagogic evaluation for learners' computational performances and orientations to mathematics?

As stated in Chapter 4, I use Brown & Dowling's (1998, pp. 62-64) distinction between *information* and *data*. Thus, the data production procedures constitute analytic and methodological resources for the transformation of information[44] contained in the archive to data. The first interest of the study focuses on how evaluation functions at the level of the instructional discourse in the observed lessons. The second interest concerns the specialisation of mathematical thought of learners as evidenced in their mathematical work displayed in a test and in an interview context. So, *Stage 1* of the data production and analysis focuses on the observed lessons and *Stage 2* on the clinical interviews conducted with learners on the basis of the tests administered by the teachers.

Stage 1: First, using primarily the video-records of the observed lessons and the accompanying transcripts, the recognition and realisation rules employed by teachers and learners (described computationally) were identified in order to ascertain the content realised in the instructional discourse. The teacher interviews and curriculum resources such as curriculum documents, textbooks, worksheets and assessment tasks serve as supplementary information sources. This constitutes the primary data. Second, the descriptions of the computational activity of teachers and their learners emerging in the observed lessons are then analysed further using theoretical resources discussed in Chapter 3 to produce secondary data, i.e., descriptions of the realised content and regulation of mathematics (Chapter 6) and descriptions of the computational performance and orientation to mathematics of the model learner emerging in the observed lessons (Chapter 7).

[44] The information contained in the archive already constitutes a selection informed by the methodological orientation of the study.

Stage 2 involved using the clinical interviews and test scripts to produce the primary data, i.e., learners' recognition and realisation rules described computationally. The descriptions of the computational activity of learners when doing mathematical work independently of the teacher together with theoretical resources discussed in Chapter 3 were used to generate descriptions of the content realised by learners, and the computational performances and orientations to mathematics of learners when doing mathematical work independently of the teacher (Chapter 9).

5.2 Stage 1: procedures for producing primary data with respect to the instructional discourse

Mathematics lessons entail empirical instances of pedagogic evaluation (TP1) and therefore are a source of information with respect to what is constituted as mathematics (TP3). A description of the computational activity of teachers and their learners reveals how evaluation at the level of the instructional discourse functions in a pedagogic situation (TP3). The focus here is to examine the recognition and realisation rules as evidenced in the computations employed by teachers and learners in order to ascertain the content realised in the observed lessons (TP3 & TP14) and to describe the model learner's computational performance and orientation to mathematics implied by the computational activity evident in the observe lessons (TP4).

It is important to note that the notions of recognition and realisation rules used in this study differ from the way in which the concepts are employed in other studies recruiting Bernstein (1996, 2000). Those studies distinguish between recognition rules and realisation rules (e.g. Hoadley, 2005; Morais, Fontinhas & Neves, 1992) and learners are said to possess realisation rules only if they produce the legitimate text. In this study, I refrain from making statements such as there is an "error in realisation" (Hoadley, 2005, p.8) and I do not distinguish between recognition and realisation rules. Instead, recognition and realisation rules are entailed in each other. My interest is in understanding what is recognised and realised as mathematics by examining the computation employed by pedagogic agents, bearing in mind that it is possible to produce the same mathematical expressions using very different recognition and realisation rules.

As discussed previously, what is constituted as mathematics in pedagogic situations of schooling is an empirical question which requires a methodology that accepts whatever emerges as mathematics in a pedagogic situation, whether or not such might be considered questionable from the point of view of a Mathematics adept (TP13). The following steps were involved in producing data for describing the computational activity of teachers and learners in the observed lessons: step 1 (segmenting lessons); step 2 (classifying mathematics problems); step 3 (describing the computational activity of the teacher and learners); step 4 (comparing computational activity with the Mathematics encyclopaedia, the field of production); and step 5 (describing the treatment of the announced topic in curriculum and textbook, the field of recontextualisation).

5.2.1 Step 1: segmenting lessons

The first step in the production of data entails segmenting lesson transcripts. Recall that evaluation specialises time, therefore time officially allocated for the teaching and learning of school mathematics in each pedagogic context is of concern (TP5). The classification of pedagogic time was done in two parts. Firstly, the use of time spent during the lesson was identified using categories adapted from the TIMSS 1999 Video Study (U.S. Department of Education, 2003). The purpose was to mark out the time spent on mathematics, the core interest of the study, as opposed to time spent on other aspects of classroom life such as discipline or classroom organisation. Pedagogic time, that is time officially allocated for teaching and learning of school mathematics, was categorised in terms of time used for (1) mathematical activity (2) pedagogic organisation and (3) non-pedagogic activity[45].

Mathematical activity (**MA**) refers to activities involving the teacher or learners stating mathematical propositions and definitions, the teacher working through examples to elaborate procedures for solving particular classes of problems or the teacher marking homework tasks, learners applying procedures in classwork tasks or writing a mathematics test. In other words, this time is marked by recognition and realisation criteria used by either the teacher or learners for the production of mathematics.

Pedagogic organisation (**PO**) refers to activities that involve preparing materials or discussing information related to mathematics, but not qualifying as mathematical activity, e.g., distributing worksheets used to solve problems, distributing homework tasks, learners copying notes or problems from the board.

Non-pedagogic activity (**NP**) refers to activities that are not related to mathematics or pedagogy e.g., talking about a social function, disciplining a learner while other learners wait, or listening to school announcements on a public-address system.

Table 5.1. Use of pedagogic time in Maya's lesson S02T03L03[46]

Segment	Start Time	Duration	Transcript lines	Activity	Type
S1	00:00	05 min 13s	1 - 19	Announcements	NP
S2	05:13	42 min 13s	20 - 346	Teacher explaining worked examples and learners working on exercises	MA
S3	47:44	05 min 32s	347-385	Discussion of school programme	NP
S4	53:16	13 min 19s	386 - 436	Teacher explains example learners worked through	MA
S5	1: 06: 35	00 min 53 s	437 - 447	Instructions on homework task	PO

[45] These categories are derived from the TIMSS 1999 Video study categories "mathematical work", "mathematical organisation" and "non-mathematical work".
[46] Each lesson is coded as follows S02 refers to School 2 (Evergreen High), T03 refers to Teacher 3 and L03 refers to lesson 3.

Table 5.1 provides an example of how pedagogic time is coded. The broad categorisation of the use of pedagogic time was followed by an examination of time devoted to mathematical activity. The next step involved partitioning the lesson transcript into what Davis (2011a) refers to as an *evaluative event*. As discussed in Chapter 3, an evaluative event is:

> composed of a sequence of pedagogic activity, starting with a presentation of specific content in some initial form, and concluding with the presentation of the realisation of the content in a (provisionally) final form. Such finality might be only temporary, as in cases where content requires elaboration over several lessons (or even over several grades) (Davis, 2011a, pp. 97-98).

The evaluative event which serves as the unit of analysis for investigating the functioning of evaluation with respect to the instructional discourse is generated through the segmentation of records of lessons (video and transcripts of the speech of teachers and their learners) into segments that are homogenous with respect to the mathematics topic announced by teachers and learners[47]. The process of segmenting lessons into evaluative events involves identifying the starting point and terminal point of an evaluative event. The starting point is marked by an introduction to a particular topic. The terminal point is indicated by the realisation of the content in some final form that may be in a temporary state of finality since the topic may be further elaborated later in the lesson or in subsequent lessons or even in later grades. The elaboration of particular topic or sub-topic takes place in the period between the initial and terminal points of an evaluative event. A new evaluative event commences with the announcement of a new topic or sub-topic (Davis, 2011a). An example of a lesson segmented into evaluative events is shown in Table 5.2.

Table 5.2. Segmentation of Jada's lesson S01T02L01 into evaluative events[48]

EE	Start Time	Duration	Transcript lines	Activity	Type
EE1	00:00	16 min 49s	1 - 80	Writing a test on parabola	CWK[49]
EE2	16:49	08 min 07s	103 - 213	Sketching graph of a parabola	EXP
EE3	24:56	06 min 10s	214 - 264	Reflecting and translating parabolas	CWK
EE4	30:06	07min 10s	265 - 365	Calculating turning point of $y = (x-1)^2 - 4$	EXP
EE5	37:16	03 min 23s	366 - 404	Reflecting $y = x^2 - 5x - 6$ in the axes	EXP
EE6	40:39	02 min 45s	405 - 408	Calculating critical points of $y = x^2 - 7x + 12$	EXP
EE7	43:24	04 min 00s	409 - 440	Calculating the equation of a parabola	EXP

Lesson S01T02L01 consists of seven evaluative events. Each event is homogenous with respect to the mathematics topic announced in some way by either the teacher or the learners. Such announcements may take the form of headings written on the board, explicitly stated definitions or propositions related to particular content or posing of a particular mathematics problem to be solved such as a worked example or some exploration of mathematical content. Learners may introduce a topic for discussion by posing

[47] Similarly, Ensor et al. (2009, p. 142) partitioned pedagogic records into what they call *pedagogic tasks*, "usually signalled by the teacher as she changed focus from one topic to another, or, within the same topic, changed the mode of classroom organisation".

[48] CWK refers to classwork and EXP to exposition, both of which are glossed later.

[49] CWK refers to classwork and EXP to exposition, both of which are glossed later.

questions that divert the lesson to focus on a different topic. In Evaluative Event 2 (EE2), the transcript of which is shown in the lesson Extract 5.1, the teacher announced the topic by stating: “there are four critical points with a parabola. Can anybody tell me what they are?” To illustrate procedures for finding the “critical points” of a parabola, she then used a specific example, $y = 2x^2 - 4x - 6$. Thus the starting point of an evaluative event is marked by the teacher’s question that introduces a new topic for consideration in the lesson.

Extract 5.1. transcript S01T02L01 (lines 103 – 111)

Teacher:	There are four critical points with a parabola. Can anybody tell me what they are?
Learner:	Turning point.
Learner:	The turning point.
Teacher:	Hey Michael. Turning point. What else?
Learners:	*y*-cut.
Teacher:	*y*-cut.
Learners:	*x*-cut.
Learner:	There are two *x*-cuts.
Teacher:	There are two *x*-cuts. So I need two *x*-cuts one *y*-cut one turning point.

The terminal point of an evaluative event is often indicated by a shift to the next topic or sub-topic, which in effect signals the beginning of the next evaluative event. As illustrated in the Extract 5.2, the teacher signalled the end of EE2 by stating that “Okay. Now you have got the graph.” The beginning of the next evaluative event, EE3, is indicated by “I want you to reflect that graph in the *x*-axis”.

Extract 5.2. transcript S01T02L01 (line 214)

Teacher:	Okay. Now you have got the graph [Referring to the graph of $y = 2x^2 - 4x - 6$] I want you to think a little bit. I want you to reflect that graph in the *x*-axis. Quite a big ask that I want from you. I want you to reflect it in the *x*-axis.

The column headed *Activity* in Table 5.2 refers to mathematical activity of the teacher and/or learners as well as the topic or sub-topic announced or inferred from the instructions provided by the teacher or stated in mathematical problems dealt with.

An evaluative event takes place over a particular time segment. In the case of EE2, the time segment runs from 16 minutes 49 seconds to 24 minutes 56 seconds, constituting a single evaluative event which has a duration of 8 minutes 26 seconds and which coheres around a specific topic - sketching graphs of parabola functions defined by the general equation $y = ax^2 + bx + c$.

A lesson may consist of one or more evaluative events. As shown in Table 5.2, evaluative events vary in duration. While evaluative events have temporal extent, the unit is really concerned with the elaboration of content. The notation used to label an evaluative event is EEi, which indicates that it is the ith evaluative event of that particular lesson. So EE2 indicates the second evaluative event of the lesson.

An evaluative event may consist of several sub-events, which are segments of evaluative events. Sub-events focus on separate sub-topics of related mathematical content and together the sub-events constitute a particular topic. Sub-events may also be used to mark instances within an evaluative event when a teacher diverts from the main topic in order to refer to content related to, but not central to the main topic of the evaluative event. Such digressions sometimes occur when a teacher unexpectedly encounters learners' lack of prerequisite knowledge for the topic under consideration, or when the teacher needs to motivate for dealing with a specific topic. An example of the former is evident when the teacher deals with factorisation of a quadratic trinomial because she realises that learners are unable to factorise quadratic trinomials required for finding the x-intercepts of a parabola. An instance of the latter is apparent when a teacher, for example, explains addition of fractions to motivate for the necessity for using multiples. A third way in which sub-events are used is to indicate when there is change in the activity of teacher and/or learners, such as when the teacher completes the exposition of a procedure for solving a particular class of problems and then proceeds to give learners exercises on applications of the procedure.

Table 5.3 shows the segmentation of an evaluative event into sub-events. The second evaluative event (EE2) is segmented into three sub-events, with each sub-event presenting sub-topics of the main topic, viz., sketching graphs of the parabola $y = ax^2 + bx + c$. Together the sub-topics 2.1 to 2.3 constitute the content associated with the announced topic for EE2.

Table 5.3. Segmentation of S01T02L01 evaluative event 2 into sub-events

EE	Start Time	Duration	Transcript lines	Content Activity	Type
EE2	16:49	08 min 07s	103 - 213	Sketching a graph of a parabola	
EE2.1	16:49	01 min 21s	103 -113	Identifying the critical points of a parabola	EXP
EE2.2	18:10	02 min 55s	114 -159	Computing x and y-intercepts of $y = 2x^2 - 4x - 6$	EXP
EE2.3	21:05	03 min 51s	160 -213	Calculating the turning point of $y = 2x^2 - 4x - 6$	EXP

Sub-events are labelled EE$i.j$, indicating that it is the jth sub-event of the ith evaluative event. So EE2.3 refers to the third sub-event of EE2.

Evaluative events and/or sub-events are described in terms of two activity types: exposition (EXP) and classwork (CWK). In cases where an evaluative event is segmented into sub-events, the sub-events are classified in terms of activity types rather than the evaluative event as a whole because an evaluative event may be composed of sub-events of both types of activity.

Exposition (EXP) involves the teacher and learners engaging publicly through speech, writing and/or gesture. Exposition marks out an activity which in most cases is led by the teacher. Learners generally respond to questions directed at them by the teacher or pose questions of clarification to the teacher. Exposition includes the teacher or learners (1) explaining mathematical propositions or stating definitions or descriptions of mathematical terms; (2) providing an overview of a lesson or a summary of the main points of a prior lesson;

(3) working through examples publicly to elaborate procedures for solving particular classes of problems; (4) providing oral answers to classwork or homework exercise questions or written solutions of the exercise problems on the board; and (5) monitoring learners who were instructed by the teacher to reproduce the solutions to homework tasks on the board.

Classwork (CWK) is dominated by private interactions between the teacher and a learner or groups of learners or amongst groups of learners. The teacher may occasionally address the class publicly during classwork activity but this type of activity predominantly involves learners completing mathematics tasks individually, in pairs, or in small groups and the teacher assisting learners when called upon to do so or when s/he observes that a learner or a group of learners requires assistance (U.S. Department of Education, 2003). Classwork generally entails learners working on a task assigned by the teacher. The task may take the form of (1) a mathematics exercise where learners work individually, in pairs or in groups, (2) exercises where they are required to 'discover' mathematical notions or concepts, or, (3) mathematics tests completed by learners working independently of the teacher

In addition to classifying evaluative events in terms of activity type, I also categorise evaluative events in terms of whether they were procedural events (PE) or non-procedural events (NPE).

Procedural evaluative events are defined as events in which mathematical activity involves the elaboration of a procedure by the teacher in terms of a general heuristic or through a worked example to exemplify a general procedure for solving a particular class of mathematical problems. Included here are lesson segments that involve the teacher marking homework exercises or classroom exercises which involve the application of procedures. Furthermore, procedural evaluative events include segments of the lesson where learners are involved in the application of a procedure in classroom exercises and tests or reproducing the solutions to classroom exercises or homework exercises as a form of marking.

A *non-procedural* evaluative event[50] is defined as an event in which mathematical activity does not involve the elaboration of a procedure by the teacher or application of the procedure by learners. In these evaluative events, the teacher and or learners may be engaged in stating definitions or descriptions of mathematical terms, pointing out relationships among ideas in the lesson and previous lessons, or investigating mathematical ideas or concepts through an exploratory task

Having categorised evaluative events and sub-events in terms of activity types and procedural or non-procedural evaluative events, I construct descriptions of computational activity in each evaluative event and sub-event. I start by discussing the mathematics problems dealt with in each pedagogic situation before focusing on the computational activity displayed in each pedagogic situation.

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[50] Adapted from U.S. Department of Education (2003, p. 42).

5.2.2 Step 2: classifying mathematics problems

Previous studies show that typically a large proportion of mathematics lessons is spent on solving mathematics problems (see, for example, U.S. Department of Education (2003)). In my study, all evaluative events classified as procedural were concerned with solving mathematics problems either as worked examples by the teacher or as classwork exercises or tests by learners. Mathematics problems are classified in terms of the number of announced topics referenced by the problem.

A mathematics problem which indexes one announced topic is referred to as a *mono-topic* problem, typically accompanied by problem statements such as "sketch the parabola" or "calculate the equation of the function". *Multi-topic* mathematics problems involve more than one announced topic. Figure 5.1 shows an example of a multi-topic mathematics problem which consists of three mono-topic problems: (1) sketch graphs of a parabola and a line on the same system of axes (2.1); (2) read off the values of x for which $f(x) = g(x)$ (2.2); and (3) state the domain and range of f (2.3).

Question 2

$f: y = x^2 - 2x - 3$ and $g: y = -3 + x$

2.1 Draw neat sketch graphs of f and g on the same system of axes and show the intercepts on both axes as well as the co-ordinates of the turning point of the graph of f. (8)

2.2 Use your graphs to read off the values of x for which $f(x) = g(x)$ (2)

2.3 State the domain and the range of f. (4)

Figure 5.1. Extract from tutorial used by Prestige College teachers

Multi-topic problems encourage *inter-topic connectivity* because they require learners to use previously encountered procedures together with the newly introduced procedures to solve mathematics problems.

5.2.3 Step 3: describing recognition and realisation rules in terms of computational activity

Typically school mathematics is concerned with procedures for solving standard problems as is evident from the CAPS FET assessment guidelines, adapted from TIMSS (1999), which stipulate the following distribution of problem types: *knowledge* (20%), *routine procedures* (35%), *complex procedures* (30%) and *problem solving* (15%) (Department of Basic Education, 2011, p. 53). The assessment guidelines nevertheless provide a strong indication of what is expected from teachers and learners. Likewise, research studies on South Africa characterise mathematics teaching as teaching that predominantly focuses on procedures for solving mathematical problems (Carnoy & Chisholm, 2008; Davis & Johnson, 2007; Taylor & Vinjevold, 1999) and, in many instances, without any explicit attention to mathematical definitions and propositions (Chitsike, 2011b; Davis & Johnson, 2007; Jaffer, 2010b; Mackay, 2010).

I produce descriptions of the recognition and realisation criteria described computationally by closely analysing the execution of procedures for solving standard problems in school mathematics and, where provided, from definitions or descriptions of mathematical terms and propositions used to support the computational activity. My analysis of the recognition and realisation criteria described through the computational activity involves a discussion of the components shown in Figure 5.2: (1) descriptions/definitions of mathematical terms; (2) propositions underpinning procedures; (3) procedures used to solve classes of mathematical problems; and (4) operations and associated domains and codomains (TP3).

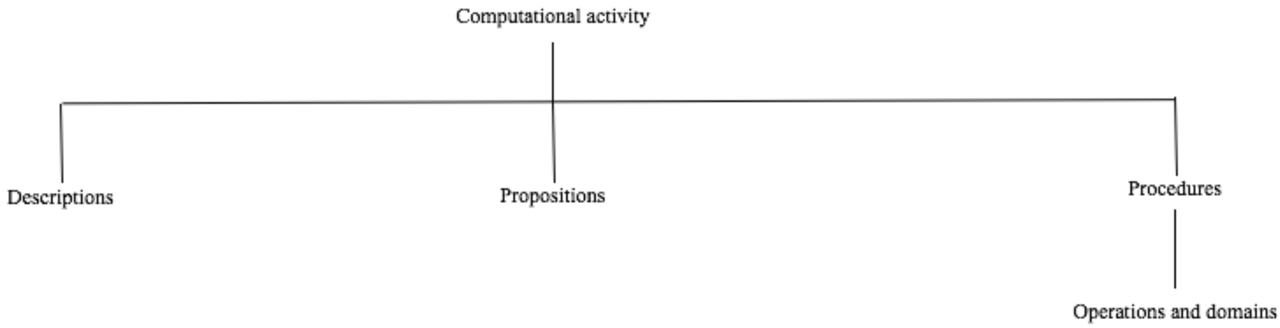


Figure 5.2. Network for the recognition and realisation rules in terms of computational activity

The components of computational activity are treated separately analytically, but in practice may be intertwined since explication of definitions/descriptions and propositions often emerges in the process of the elaboration of procedures, but may well be dealt with outside of the elaboration of procedures. However, procedures represent the macro-level computations and the operations with associated domains and codomains signify the micro-level computations, the latter being entailed in the former.

5.2.3.1 Step 3.1 Descriptions or definitions of mathematical terms

Recall from Chapter 3 that a mathematical definition is a *definite description,* which is a formal mechanism for introducing an abbreviation for the phrase that defines it (Potter, 2004) and which delineates a cluster of predicates that describes a particular mathematical object. However, in pedagogic situations teachers or learners sometimes state they are providing definitions but these do not function as definite descriptions, as shown in Extract 5.3, when Jona attempts to 'define' a function as "the relationship between two variables". I use the term, *auxiliary description*, to refer to teachers' and learners' explanations that do not function as definite descriptions[51].

The process of describing a term is sometimes preceded by the question "what is X?" or "what does X mean?" or "what is the definition of X?" e.g. "what is the definition of a function?". Teachers, at times, elaborate the meaning of a mathematical term by providing a phrase which describes a new mathematical term introduced without asking learners any questions, as indicated when Jono provides a description of the

[51] Frege used the term *elucidations* to refer to informal explanations used by mathematicians to indicate the meanings of terms (Potter, 2004, p. 7). Teachers and learners' auxiliary descriptions of mathematical terms do not necessarily resemble the informal explanations of mathematicians.

term ‘domain’ by stating, “So this is the domain. Any number where the independent variable can take” (transcript S02T04L01: line 140).

Extract 5.3. transcript S02T04L01 (lines 37 – 45)

Teacher:	…Now we know that a function is … who can define a function?
Learner:	It is a relationship between …
Teacher:	So when we have a function it is a relationship between what and what?
Learner:	[inaudible]
Teacher:	[draws diagram of a “function machine” while learners chat amongst themselves]
Learner:	Ja. Between two variables.
Teacher:	All right we have the [Points to the board]…
Learners:	The input.
Teacher:	The input. The input is which variable? A function is the relationship between two variables.

Since auxiliary descriptions are not definite descriptions, the explanations offered by teachers or learners change the objects described. The concept of *natural kinds* is useful to consider in relation to auxiliary descriptions used in pedagogic contexts. Kripke uses the term *natural kinds* to refer to objects such as biological entities, natural substances or natural phenomena that are encountered in everyday life (Kripke, 1980, pp. 135-137). Changing a property of a natural kind does not alter the type of object being described, which contrasts with definitions of mathematical objects, where any change in the predicates changes the type of object referred to, and so its category membership. Using Kripke’s example, the natural kind tiger remains a tiger even though it may only have three legs. However, the predicates of a prime number such as that it is a positive integer with only one positive divisor other than 1 are essential predicates of primeness, Jono’s description of a function as a relation between two variables, treats the mathematical object ‘function’ as a natural kind because the description captures relations which are not necessarily functions. So, the description works as a natural kind description rather than as a definite description. (see also Arendse, 2011, 2013).

Auxiliary descriptions are categorised into two types: (1) *iconic* and (2) *non-iconic. Iconic auxiliary descriptions* are explanations of mathematical terms that treat mathematical objects as though they are physical objects or an image of a physical object. Jono’s description of the mathematical notion of infinity as “a number that we can’t touch. It is not really quantifiable. It is a big big number but you can’t quantify it” (transcript S02T04L01: line 373) is an example of an iconic auxiliary description in that he treats numbers as physical objects.

Non-iconic auxiliary descriptions are explanations of mathematical terms but they are not definite descriptions and they also do not treat mathematical objects as physical objects. Jono’s description of a function as “a relationship between two variables” is an example of a non-iconic auxiliary description.

It is possible for definite descriptions to have iconic features but here the iconic operates together with propositional as opposed to the iconic referred to in relation to iconic auxiliary resources. Recall Peirce's (1931) notion of the iconic discussed in Chapter 3.

Figure 5.3 presents the network for categorising descriptions of mathematical objects provided by teachers and learners.

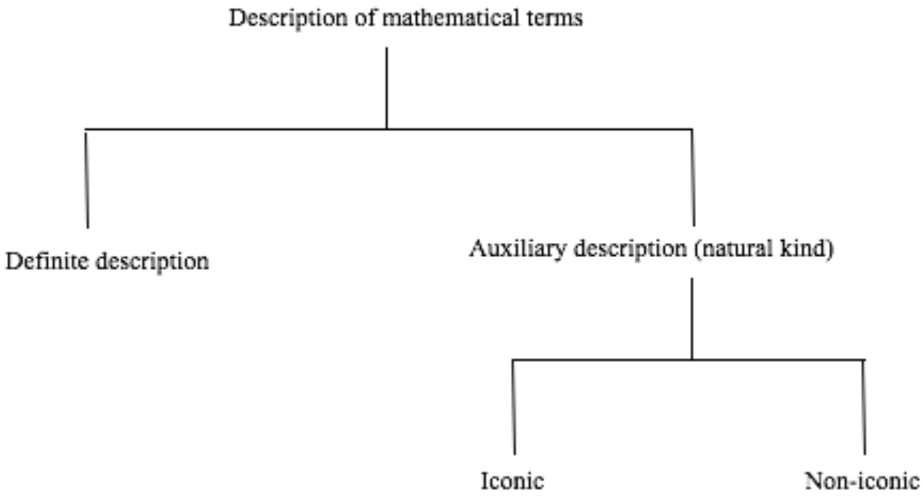


Figure 5.3. Network for categorising descriptions of mathematical object

5.2.3.2 Step 3.2 Propositions

In Chapter 3, I distinguished between propositions found in the Mathematics encyclopaedia, referred to as *encyclopaedic propositions,* from *auxiliary propositions* that stand in place of encyclopaedic propositions. The proposition "the parabola $y = ax^2 + bx + c$ has a smiley face if a is positive" (see Extract 5.4) is an example of an auxiliary proposition used by Mona to describe the shape of a parabola. The latter proposition is used in place of the encyclopaedic proposition: "if $a > 0$ then the parabola $y = ax^2 + bx + c$ has a minimum value"

Extract 5.4. transcript S02T03L01 (line 37)

Teacher:	This is a a parabola because it has the *x* squared. So that is going to end up looking like a big smiley face because this [points at the coefficient of x^2 in $y = 2x^2 - 8$] is positive [Teacher gestures shape with hands] and if this [points at /2/ in $y = 2x^2 - 8$] was negative it would be looking like a big grumpy face [Teacher gestures shape with hands.]

Auxiliary propositions are further categorised into *iconic auxiliary propositions* and *non-iconic auxiliary propositions*. Those propositions that focus on imagistic features are referred to as iconic auxiliary propositions. The proposition used by Mona when discussing the mathematics problem, sketch the function $y = 2x^2 - 8$, is an example of an iconic auxiliary proposition because her proposition treats a parabola as a physical object – a parabola is described in terms of facial expressions, which creates an image of the "shape" of the parabola.

Propositions that fall outside of the Mathematics encyclopaedia but do not draw attention to iconic features are referred to as *non-iconic auxiliary propositions.* For example, the proposition established by Jono in Extract 5.5 when discussing the domain of the function $y = x + 1$ is that the domain of a function is any real number if there is no real number for the function is undefined. The proposition is non-iconic because it does not have any imagistic references.

Encyclopaedic propositions may also have iconic features but the use of the iconic differs from the iconic associated with auxiliary propositions. Whitehead (1911, p. 61) argues that the use of symbolism when stating mathematical propositions e.g. commutative property of addition, $x + y = y + x$, illustrates how "we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play higher faculties of the brain".

Extract 5.5. transcript S02T04L01 (lines 317-324)

Learner:	Sir there's nothing that you can put that makes the equation undefined.
Teacher:	So there is not. There is no number that we can put here [referring to x] to make this [referring to $y = x + 1$] undefined. So our x can take any?
Learner:	Number.
Teacher:	Are we together?
Learner:	Yes.
Teacher:	Are we together? So our x can take any [Writes /$x \in \mathbb{R}$/]
Learner:	Real number.
Teacher:	Any real number.

Figure 5.4 shows the network of categories used to classify propositions used by teachers and learners to support procedures employed.

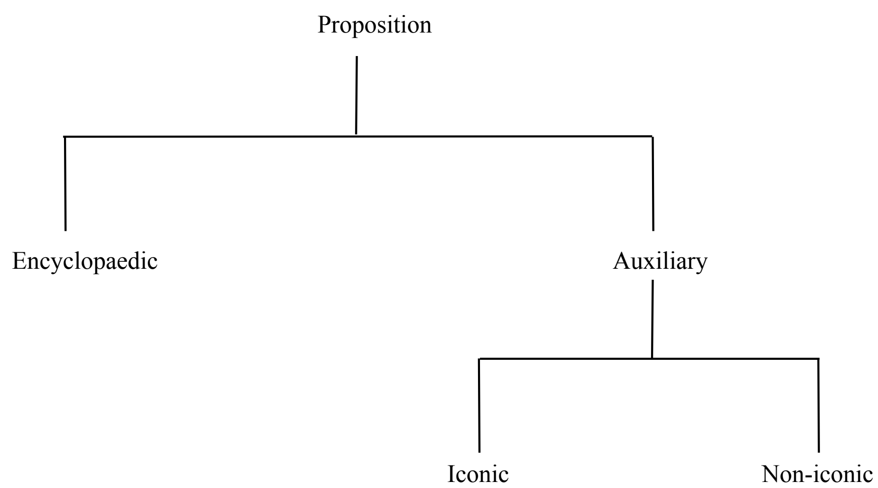


Figure 5.4. Network for categorising propositions used by teachers and learners

5.2.3.3 Step 3.3 Mathematical procedures

A procedure entails a series of computations for the production of solutions to a class of mathematical problems. Typically, a procedure starts with some initial expression (E_o), which is substituted by another expression (E_1), which in turn is substituted by a different expression (E_2) if necessary, and so on, halting at the production of some final expression (E_n). The production of an E_{i+1} from E_i are the 'steps' of a procedure. Each E_i is an expression referencing one or more operations or operation-like manipulations which, together with the propositions appealed to, enables the production of E_{i+1}[52].

$$E_0 \to E_1 \to E_2 \to \cdots \to E_{n-1} \to E_n$$

Of central importance in mathematics is that the production of an E_{i+1} from E_i necessarily produces differences at the level of expression but must preserve identity at the level of value. For example, let us examine a procedure for solving linear equations typically used in South African schools. The solution to the problem, solve for x if $2(x-1)=5$ is shown in Figure 5.5.

E_0: $2(x-1)=5$	
E_1: $2x-2=5$	(Remove the brackets)
E_2: $2x=5+2$	(Take over and change the sign)
E_3: $2x=7$	(Add)
E_4: $x=\frac{7}{2}$	(Take over and divide)
E_5: $x=3{,}5$	(Divide by 2)

Figure 5.5. Procedure for solving a linear equation

The procedure shown in Figure 5.5 illustrates a series of productions of E_{i+1} from E_i, generated by some rule(s) or operations to produce the final expression $x = 3{,}5$ (E_5) from the initial expression $2(x-1)=5$ (E_0). We note that for each E_{i+1} produced from E_i, i.e., from E_0 to E_1 and E_1 to E_2 etc., the expressions differ but the implicit value of the unknown remains the same.

[52] Hiebert and Lefevre (1986, p. 6) have a similar description of a mathematical procedure. They distinguish between procedures (symbol manipulation rules) that use symbols as objects and procedures that use non-symbolic objects such as the concrete, mental images or diagrams as objects. It is not clear how they recognise whether the objects are treated as symbols or numbers by a particular user. It seems as though the nature of the object is dependent on whether an individual has procedural or conceptual knowledge, the latter being evident when an individual is able to link different bits of information and the former when an individual is unable to do so. Hiebert & Lefevre's notion of symbols as input and output objects differs from that of Davis (2011a) for whom the nature of the object is dependent on the type of the operation performed. So, operations or operation-like manipulations such as sundering a sign from a number cannot take a number as its input. The input of such an operation is a symbol such as /-2/ rather than the number -2.

Table 5.4. Procedures used by Sara for elaborating the topic Parabola

Announced topic 1: Parabola		
Problem type	**Procedure**	**# problems**
1. Sketch parabola	**PR1.** Sketching parabola given the form $y = a(x-p)^2 + q$ (turning point method 1)	11 mono-topic
	PR2. Sketching parabola given the form $y = a(x-p)^2 + q$ (turning point method 2)	
	PR3. Sketching parabola given the form $y = ax^2 + bx + c$ (root method)	6 mono-topic
	PR4. Sketching parabola given the form $y = ax^2 + bx + c$ (completing the square method)	4 mono-topic
	PR5. Sketching parabola given the form $y = ax^2 + bx + c$ (formula method)	1 mono-topic
Calculate turning point of parabola	Sub-procedures related to procedures 1 - 5	3 multi-topic
2. Calculate equation of straight line	**PR6**. Calculating equation of straight line	
3. Calculate horizontal length between two points on a graph	**PR7** Calculating horizontal lengths	
4. Calculate vertical length between two points on a graph	**PR8** Calculating vertical lengths	
5. Calculate points of intersection of two graphs	**PR9.** Calculating points of intersection	
6. Calculate graphical inequalities	**PR10.** Calculating graphical inequalities	

A description of the procedures used in Sara's observed lessons is shown in Table 5.4. Procedures are labelled PRi where i refers to the procedure number and are associated with the problem statement. In addition, the number of problems (and problem type) in which the procedure is used is also shown.

Procedures used in pedagogic contexts are often tied to particular expression states. Teachers and learners focus on producing a standard form before employing a particular procedure. For example, $(x-3)^2 = 16$ can be solved by taking square roots on both sides of the equation. However, teachers may insist on first generating $x^2 - 6x - 7 = 0$ which represents the equation in standard form before solving the equation. Procedures in Sara's pedagogic context are not tied to specific problem types which makes it possible for her learners to construct a number of procedures not explicitly elaborated by her, by combining sub-procedures in different ways. As such, she encourages *intra-topic connectivity*. In contrast, when teachers or learners associate particular procedures with particular expression states, this indicates a lack of intra-topic connectivity.

The procedures used in the observed lessons for each pedagogic context are described as shown in Figure 5.5 and Table 5.4 but this description does not capture all the operations and domains and codomains entailed in the procedure.

5.2.3.4 Step 3.4 Operations, domains, codomains

Having discussed the procedures used by teachers and learners for the solutions to particular classes of problems, I now examine the operations or operation-like manipulations (auxiliary operations) involved in the procedures (TP17). The operations or auxiliary operations provide information about the objects operated with. That is, the operations tell us about the domain(s) and codomain(s). Arithmetic operations from the Mathematics encyclopaedia, such as multiplication, involve domains and codomains from the field of the reals. Auxiliary operations such as "take over and change signs" involve domains and codomains that are made up of character strings. I turn to an example in order to illustrate the methodology for generating a description of the operations, domains and codomains used. In Extract 5.6, Sara explains a procedure for calculating the x-intercepts of the parabola $y = -x^2 + 6x - 5$. The procedure outlined by Sara for calculating the x-intercepts of the parabola $y = -x^2 + 6x - 5$ is summarised below:

(1) Make $y = 0$ to produce $-x^2 + 6x - 5 = 0$.

(2) "Make x squared positive" i.e. "change all the signs" or "multiply by minus one" to produce $x^2 - 6x + 5 = 0$

(3) Factorise $x^2 - 6x + 5 = 0$ to produce $(x - 5)(x - 1) = 0$

(4) Read off the solution of $(x - 5)(x - 1) = 0$ to produce x-intercepts $x = 5$ or $x = 1$.

This description of the procedure, however, describes transformations at the level of the expression and provides some syntactical rules but masks what is actually happening at the operational level (syntax) (TP10).

Extract 5.6. transcript S01T01L01 (lines 374 – 380[53])

Teacher:	Now some of you might have taken this expression our standard form expression. Minus *x* squared plus six *x* minus five is nought. How do we solve that?
Learner:	*x* equal five and minus five.
Teacher:	No. I know that's the answer.[54]
Learner:	Make that plus *x* squared. Take everything to the other side.
Teacher:	Right. So we have to make *x* squared positive. Change all the signs. In other words we are multiplying everything by minus one. And now what to do? It's a trinomial.
Learner:	We try to factorise.
Teacher:	Factorise.
Teacher*:	Right [Writes brackets / ( )( )/]
Teacher*:	.*x* *x*. [Writes in brackets / (*x*)(*x*)/]
Teacher*:	Five one. [Writes in brackets /(*x* 1)(*x* 5)/]
Teacher*:	Minus and minus. [Writes in brackets/(*x* - 1)(*x* - 5)/]
Teacher*:	So *x* is one or *x* is five.

The teacher's instruction to "change all the signs" implies splitting the expression $/-x^2 + 6x - 5/$ to perform operations on the signs, suggesting for example that the number must be split into the symbol $/-/$

[53] Line 380 (marked with *) has been split into different lines for ease of reading to capture the board work annotated next to the speech.
[54] The answer produced by the learner is incorrect but the teacher does not correct him - presumably because she is interested in discussing the procedure.

and the numeral $/5/$. But numbers cannot serve as arguments to operations that detach signs from numerals, indicating that an existential shift that converts a number -5 to a character string $/-5/$ must have taken place prior to the 'operation' that splits $/-5/$ into the symbol $/-/$ and the numeral $/5/$. Existential shifts, such as this, change the nature of objects at the level of value, but the changes cannot be detected at the level of expression. In other words, the signifier remains the same but the signified changes (TP9). The particular signifier-signified couple operating in this pedagogic situation differs from the signifier-signified relation found in the Mathematics encyclopaedia. Secondly, the operation used to split the symbol $/-/$ from the numeral is an operation that is not found in the Mathematics encyclopaedia (TP17). This operation-like manipulation, referred to as *sundering* by Davis (2010b, p. 107)[55] is used alongside the operations from the field of the reals and is referred to as an *auxiliary operation*[56].

We also note that the teacher substitutes the learner's operation "take everything to the other side" with another operation "change all the signs", used in conjunction with the operation "multiply by -1" which is a familiar operation in the field of the reals, referred to as an *encyclopaedic operation*. She thus moves seamlessly between the field of the reals and auxiliary domains and operations, at times consciously and at times unconsciously. The teacher's language "change all the signs. In other words we are multiplying everything by minus one" makes both expression and content explicit. The use of the operation, multiplying by minus one, has an expressive effect in that it produces changes at the level of expression ("change all the signs") but the content is convergent with the Mathematics encyclopaedia. In contrast, the learner's language ("take everything to the other side") collapses the content into the expressive elements, that is, the change effected at the level of expression is simultaneously the content realised and is the content that diverges from the Mathematics encyclopaedic content (TP18).

In calculating the x-intercepts, the first move is to "make the x squared positive" as a condition for solving the equation. However, since $\forall x \in \mathbb{R}, x^2 \geq 0$, this is not what the teacher and learners are referring to (TP9). The statement "make the x squared positive" refers to replacing the symbol $/-/$ with the symbol $/+/$ or removing $/-/$. Note that there is no procedural necessity to "make the x squared positive" before solving the equation. It is possible to solve the equation $-x^2 + 6x - 5 = 0$ without "making x squared positive". The purpose of "make the x squared positive" appears to be motivated by the desire to produce an expression that is more easily factorisable for the learner and is predicated on a particular method for factorising quadratic trinomials that requires the coefficient of x^2 to be positive. It is well known that learners find working with positive integers much easier than working with negative integers (e.g., Gallardo, 2002; Gelman, 2015; Linchevski & Williams, 1999).

[55] See Appendix 5 for a full description of the auxiliary operation.
[56] Harel, Fuller, and Rabin (2008, p. 116) refer to operations on symbols as non-referential symbolic reasoning where symbols "possess a life of their own" and no attention is given to the meaning of the symbols. Here "meaning" is specifically that which aligns with the Mathematics encyclopaedia. See also Goldin (1998, p. 151) who refers to imagistic manipulations on symbols as if they are "objects" which can be moved e.g. "move x to the other side of = sign, and put a – sign in front of it".

In order to capture the detail of the computational activity, I record the input object, domain, output object, codomain, encyclopaedic or auxiliary operation entailed in the procedure as displayed in Tables 5.5, 5.6 and 5.7 where $\mathbb{X}$ indicates the domain of character strings and $\mathbb{R}$ the reals; where applicable, $\mathbb{Z}$ and $\mathbb{N}$ are used to represent integers and natural numbers respectively. Table 5.5 shows the computations used by the teacher to "make x squared positive" by "multiplying by minus one" to produce $x^2 - 6x + 5 = 0$.

Table 5.5. Encyclopaedic operations used by teacher to "make the x squared positive"

	Input	Domain	Encyclopaedic or auxiliary operation	Output	Codomain
1	$-x^2, -1$	$\mathbb{R}$	multiplication	x^2	$\mathbb{R}$

Table 5.6 shows computations used by the teacher to "make x squared positive" by "changing all the signs" to produce $x^2 - 6x + 5 = 0$.

Table 5.6. Encyclopaedic and auxiliary operations used by the teacher to "make the x squared positive"

	Input	Domain	Encyclopaedic or auxiliary operation	Output	Codomain
1	$-x^2 + 6x - 5 = 0$	$\mathbb{R}$	string	$/-x^2 + 6x - 5 = 0/$	$\mathbb{X}$
2	$/-x^2 + 6x - 5 = 0/$	$\mathbb{X}$	sundering	$/-x^2/,/+6x/,/-5/,/{=}0/$	$\mathbb{X}$
3	$/-x^2/$	$\mathbb{X}$	sundering	$/-/, /x^2/$	$\mathbb{X}$
4	$/-/$	$\mathbb{X}$	"change sign"	$/+/$	$\mathbb{X}$
5	$/+/, /x^2/$	$\mathbb{X}$	concatenate	$/+x^2/$	$\mathbb{X}$
6	$/+6x/$	$\mathbb{X}$	sundering	$/+/, /6x/$	$\mathbb{X}$
7	$/+/$	$\mathbb{X}$	"change sign"	$/-/$	$\mathbb{X}$
8	$/-/, /6x/$	$\mathbb{X}$	concatenate	$/-6x/$	$\mathbb{X}$
9	$/-5/$	$\mathbb{X}$	number	-5	$\mathbb{Z}$
10	-5	$\mathbb{Z}$	string	$/-5/$	$\mathbb{X}$
11	$/-5/$	$\mathbb{X}$	sundering	$/-/, /5/$	$\mathbb{X}$
12	$/-/$	$\mathbb{X}$	"change sign"	$/+/$	$\mathbb{X}$
13	$/+/, /5/$	$\mathbb{X}$	concatenate	$/+5/$	$\mathbb{X}$
14	$/x^2/, /6x/, /+5/$	$\mathbb{X}$	concatenate	$/x^2 - 6x + 5 = 0/$	$\mathbb{X}$
15	$/x^2 - 6x + 5 = 0/$	$\mathbb{X}$	number	$x^2 - 6x + 5 = 0$	$\mathbb{R}$

The auxiliary operations *string*, *sundering*, *concatenate* and *number* are defined by Davis(201b) (see Appendix 5. Other auxiliary operations are defined as they emerge in the analysis and are included in Appendix 5. For example, the operation-like manipulation "change sign", from now on referred to as ALT, is defined as follows. ALT maps the symbol $/+/$ to the symbol $/-/$ or vice versa. The objects that serve as arguments for ALT cannot be real numbers because one can not change the sign of real numbers. We can, however, perform the operation-like manipulation ALT on numerical and literal symbols. ALT therefore implies that an existential shift that transforms the number -5 to character string /-5/ has to take place. In other words, /-5/ has to be conceived of as a sign and a numeral. Secondly, the operation-like manipulation, sundering, that detaches the sign from the numeral is a prior operation that has to be performed. As noted by Davis (2010b, p. 107) auxiliary operations are not necessarily functions (TP17). For example, sundering is not a function because the output of sundering is not necessarily unique as its output is often dependent on the decision of the agent effecting the sundering.

Let $S = \{+; -\}$ and C be a subset of the set of numerals and letters. $C = \{a, b, \dots x, 0,1,2 \dots\}$. The input elements for ALT are contained in a set obtained from the cross product $S \times C$. These input elements are mapped to elements in $S \times C$. So $S \times C$ constitutes both the domain and codomain of ALT. The application of the operation-like manipulation ALT maps $/+/$ to $/-/$ or vice versa. In other words, ALT: $(+, a) \to (-, a)$ and ALT: $(-, a) \to (+, a)$ where $a \in C$. The output requires an operation-like manipulation, concatenation, which combines the sign and the alphanumeric character to produce $+a$ or $-a$, which represents the required outcome. Other auxiliary operations will be discussed and defined as they emerge in the analysis.

Figure 5.6 illustrates the morphism relating the structure, multiplication by minus one over the reals ($\mathbb{R}, \times(-1)$) with the structure, "changing signs" performed on characters ($\mathbb{X}$, ALT). Both structures produce the same outcome expressively but are associated with different content. Recall from Chapter 3 that structure preservation is central to thought (Gallistel & King, 2010) and it is what enables content and expression to connect in different ways thus allowing for content substitution without necessarily displaying that substitution at the level of expression (TP11). Content substitution is made explicit by the teacher but the content substitution implied by the learner's "take over and change the sign" operation-like manipulation remains implicit (TP18).

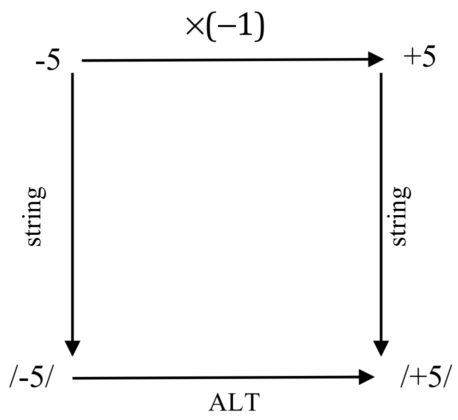


Figure 5.6. A morphism mapping ($\mathbb{R}, \times(-1)$) to ($\mathbb{X}$, ALT)

Similarly, Figure 5.7 represents a morphism relating between the learner's "take over and change the sign" or transposition (TRP)[57] over characters ($\mathbb{X}$, TRP) and the teacher's "multiplying by minus one" over reals $(\mathbb{R},\times(-1))$.

$$\begin{tikzcd}
-5 \arrow[r, "\times(-1)"] \arrow[d, "\text{string}"'] & +5 \arrow[d, "\text{string}"] \\
/\text{-}5/ \arrow[r, "\text{TRP}"'] & /\text{+}5/
\end{tikzcd}$$

Figure 5.7. A morphism mapping $(\mathbb{R},\times -1)$ to ($\mathbb{X}$, TRP)

Table 5.7 shows computations used by the learner to "make x squared positive" by "taking everything to the other side" to produce $x^2 - 6x + 5 = 0$.

Table 5.7. Computations used by learner to "make the x squared positive"

	Input	Domain	Encyclopaedic or auxiliary operation	Output	Codomain
1	$-x^2 + 6x - 5 = 0$	$\mathbb{R}$	string	$/-x^2 + 6x - 5 = 0/$	$\mathbb{X}$
2	$/-x^2 + 6x - 5 = 0/$	$\mathbb{X}$	sundering	$/-x^2/,/+6x/,/-5/$	$\mathbb{X}$
3	$/-x^2/$	$\mathbb{X}$	transposition	$/x^2/$	$\mathbb{X}$
4	$/+6x/$	$\mathbb{X}$	transposition	$/-6x/$	$\mathbb{X}$
5	$/-5/$	$\mathbb{X}$	transposition	$/+5/$	$\mathbb{X}$
6	$/x^2/, /-6x/, /+5/$	$\mathbb{X}$	concatenate	$/x^2 - 6x + 5 = 0/$	$\mathbb{X}$
7	$/x^2 - 6x + 5 = 0/$	$\mathbb{X}$	number	$x^2 - 6x + 5 = 0$	$\mathbb{R}$

Tables 5.5, 5.6 and 5.7 illustrate the use of different sets of operations and/or operation-like manipulations that all start from the same initial expression $-x^2 + 6x - 5 = 0$ and ends with the same final expression $x^2 - 6x + 5 = 0$. Tables 5.5, 5.6 and 5.7 also show the existential shifts, i.e., operations that require a change in the nature of the objects operated on, that take place during the execution of part of the procedure for calculating x-intercepts used by the teacher and her learners. Existential shifts are evident in Tables 5.6 and 5.7. In Table 5.6, we note a shift from the reals ($\mathbb{R}$) to character strings ($\mathbb{X}$) to integers

[57] Al-Khwarizmi introduced the notion of transposition in the context of his method for solving equations, through the use the twin concepts *al-jabr* and *muqabalah*. *Al-jabr*, meaning "restoration", refers to adding equal quantities to both sides of an equation in order to remove a negative quantity from one side of an equation, the effect of which is the transposition of a term to the other side of an equation. *Muqabalah*, meaning "reduction", refers to the 'removal' of a positive term on one side of an equation by subtracting equal quantities on both sides of the equation. Transposition with respect to multiplication and division is also possible (Joseph, 2000, pp. 475-476).

($\mathbb{Z}$) and back to character strings ($\mathbb{X}$). Such existential shifts are indicative of recontextualisation at the level of the domains, codomains and operations involved in the computational activity and so have implications for the constitution of mathematics in the local pedagogic situation (TP2).

A network for categorising operations employed by teachers and learners in pedagogic situations is displayed in Figure 5.8.

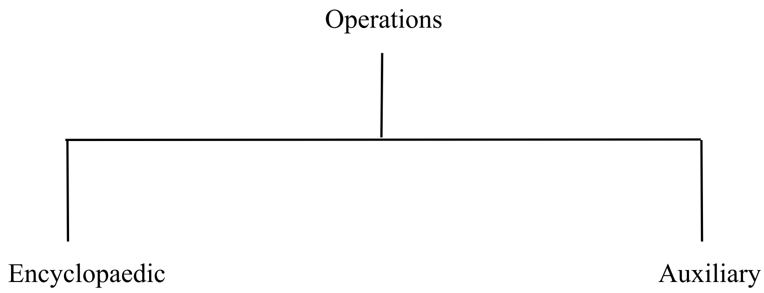
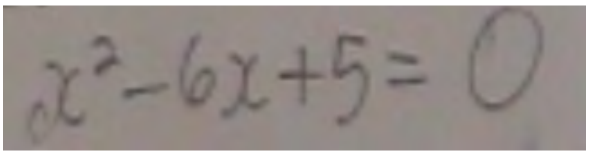
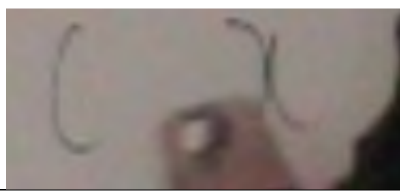


Figure 5.8. Network for categorising operations

The domain and codomain associated with encyclopaedic operations is the set of reals or subsets of the reals such as rationals, integers or natural numbers and domains and codomains associated with auxiliary operations are character strings.

5.2.3.5 Step 3.5 Character distribution matrices

Recall the procedure for calculating the x-intercepts of a parabola $y = -x^2 + 6x - 5$, discussed in 5.2.3.4. Embedded in the procedure is a sub-procedure for factoring a quadratic trinomial (referred to as a "trinomial" by the teacher). Having produced the equation $x^2 - 6x + 5 = 0$, her quadratic trinomial factorisation sub-procedure is captured very briefly: "Factorise. Right. x x . Five one. Minus and minus. So x is one or x is five" (see Line 379 in Extract 5.6), presumably because this procedure is not new to learners. The teacher's procedure for solving the equation $x^2 - 6x + 5 = 0$ in order to calculate the x-intercepts of $y = -x^2 + 6x - 5$ is captured in Extract 5.6 and the accompanying writing of the solution on the board (see Figure 5.9).

Speech	**Writing/gesturing**
T: And now what do? It's a trinomial.	
S: We try to factorise.	
T: Factorise. Right.	 Writes brackets ( )( )

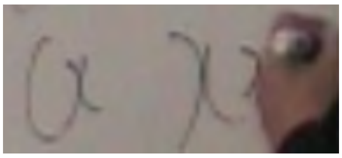
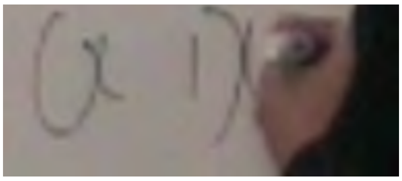
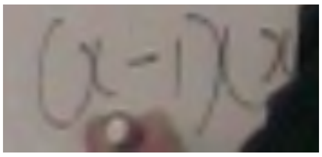
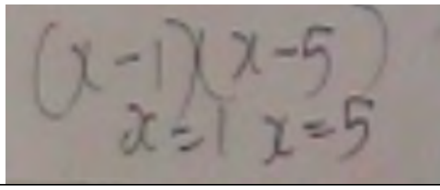
T: x x	Writes in brackets (x)(x)] 
T: Five one	Writes in brackets (x 1)(x 5)] 
T: Minus and minus	Writes in brackets $(x - 1)(x - 5)$ 
T: So x is one or x is five.	Writes /$x = 1$ $x = 5$/] 

Figure 5.9. Solving quadratic equation - transcript S01T01L01 (lines 373-379)

Her procedure for factoring $x^2 - 6x + 5$ as a sub-procedure for solving the equation $x^2 - 6x + 5 = 0$ is as follows:

(1) Write the brackets ( )( ).

(2) Factorise x^2 to produce the factors x and x then distribute the factors to the brackets as follows: $(x \quad)(x \quad)$.

(3) Factorise 5 to produce the factors 5 and 1 then distribute the factors to the brackets as follows: $(x \quad 5)(x \quad 1)$.

(4) Identify and distribute the signs to the brackets to produce the factorised expression $(x - 5)(x - 1)$.

Having produced the equation $(x - 5)(x - 1) = 0$ from $x^2 - 6x + 5 = 0$, the teacher records the solution to the equation and thus the x-intercepts as $x = 5$ and $x = 1$.

The procedure for factoring $x^2 - 6x + 5$, outlined above describes the computations at the level of the expression but masks what is happening at the level of syntax. In order to understand this level of the computational activity, we note that the procedure used by the teacher and her learners relies on a *character distribution matrix* (CDM) (Johnson & Davis, 2010) discussed in Chapter 3. The population of the particular CDM employed in calculating the x-intercepts of $y = -x^2 + 6x - 5$ is shown in Figure 5.10.

The *character distribution matrix* (CDM) used in the teacher's factorisation procedure has the form (□□□)(□□□) as shown in Figure 5.10. Now, it is the case that there are always accepted norms for displaying mathematical expressions, with the display of secondary importance in the production of mathematics. What we observe in the procedure for factorising quadratic trinomials is that the spatial distribution of symbols is prioritised and is used to regulate mathematical activity. The procedure generates symbols to populate the character distribution matrix. Before I discuss the production of symbols generated by the procedure, Figure 5.10 shows the spatial frame marked with the spaces A to F which serve as placeholders for particular symbols.

(	A	E	C	)	(	B	F	D	)

Figure 5.10. "Trinomial" factorisation character distribution matrix

A and B are placeholders for the symbol $/x/$, C and D are places for numerals and E and F are placeholders for symbols $/-/$ or $/+/$. So, there are three sets of characters that are required to populate the spatial frame, that is, a three-stage process of generating symbols to occupy spaces in the spatial frame. Prior to producing the characters the brackets (   )(   ) mark out the CDM to be populated.

Stage 1, *populating spaces A and B*: The first term, x^2 is factorised to produce x and x. The outcome of the operation is placed in positions A and B in the CDM.

Stage 2, populating spaces C and D: It appears that $/+5/$ is split into the sign /+/ and $/5/$, suggesting that the teacher and her learners conceive of integers as whole numbers that have signs attached to them. As discussed above such an operation-like manipulation implies that the arguments are character strings and indicates that an existential shift had to occur in order to perform an operation-like manipulation, sundering. Such existential shifts entail changes in the nature of objects at the level of the content but are not obvious at the level of expression. The operation that factorises 5 as 5×1 again implies that another existential shift has occurred this time from a numeral /5/ to the natural number 5 since the operation factorisation takes numbers as arguments not characters. The numbers 5 and 1 are placed in positions C and D in the CDM.

Stage 3, populating the spaces E and F. The teacher's propositions for generating the symbols /-/ and /-/ in the brackets to occupy spaces E and F are implicit. It may be the case that the teacher uses a computational rule "a minus minus is a plus" which has its roots in the axioms associated with addition and multiplication of integers[58]. It is also possible that the teacher uses a proposition that goes something like "if the last term is plus then the signs in the brackets are the same and if the middle term is minus then the signs in the brackets must both be minus" (see extract from textbook used by the teacher in Section 5.2.4).

[58] See Stewart and Tall (1977, pp. 172-174) for discussion of the operatory properties associated with $(\mathbb{Z}, +)$ and $(\mathbb{Z},\times)$ discussed in Section 5.2.3).

Figure 5.11 summarises the network of categories for producing and analysing primary data, a description of the functioning of evaluation in the instructional discourse. In particular, the network shows analytic resources employed to describe the recognition and realisation rules used by teachers and learners in computational terms.

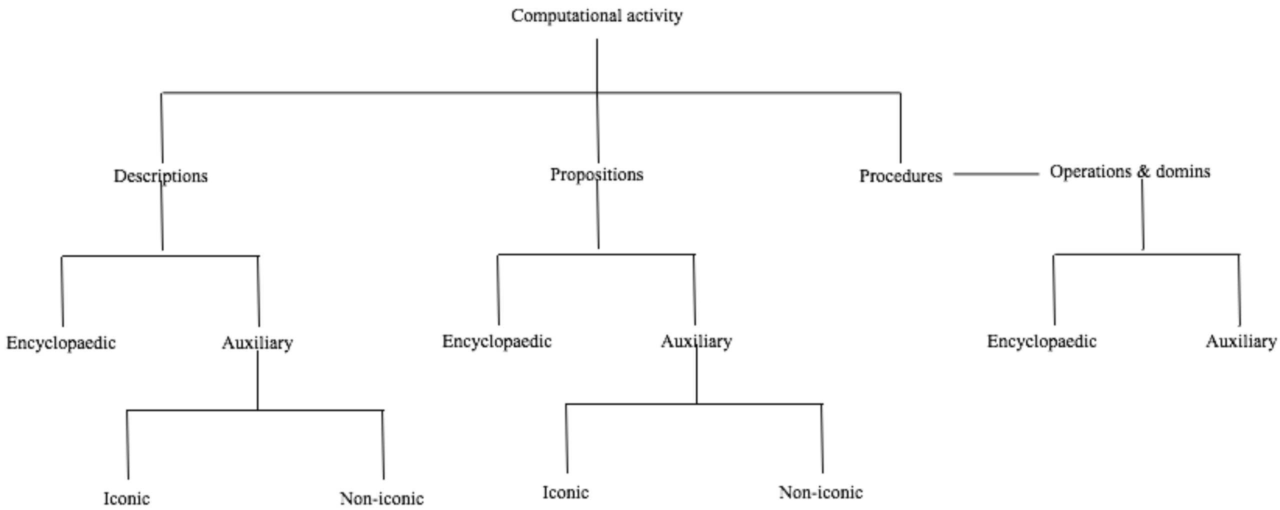


Figure 5.11. Network of categories for producing descriptions of recognition and realisation rules

Now that I have described the recognition and realisation rules in terms of the computational activity of the teacher and her learners with respect to: (1) definitions or descriptions of mathematical terms; (2) propositions used to support computational activity; (3) procedures for solving classes of problems; and (4) the operations/operation-like manipulations performed and the domains and codomains operated over, I turn to related aspects of Mathematics in the field of production in order to examine the teacher's procedure from the perspective of Mathematics.

5.2.4 Step 4: consulting the field of production – the Mathematics encyclopaedia

The point of consulting aspects of content in the field of production is to identify points of commonality and points of difference in order to ascertain what content substitutions, if any, are present in a particular instance of school mathematics compared to formal Mathematics (TP15). As will be discussed later, such a comparison provides insight into the content realised in relation to topics announced by teachers and so the recontextualisation of content (TP2) as well as the performance of the model learner implied by the computational activity (TP5).

As indicated in Chapter 3, the Mathematics encyclopaedia is not contained in any one text. I refer to mathematics texts that deal with the foundations of Mathematics in order to describe aspects of announced topics referred to by teachers and learners (e.g. Bronshtein, Semendyayev, Musiol, & Muehlig, 2007; Mac Lane, 1986; Stewart & Tall, 1977; Usiskin, Perrisini, & Marchisotto, 2003). For example, what does the Mathematics encyclopaedia have to say about quadratic factorisation, which is a sub-procedure used by the teacher when calculating the x-intercepts of a parabola shown in Extract 5.6?

Factorisation of quadratic trinomials is governed by the fundamental theorem of algebra, which states that "every polynomial of degree n, $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, can be factored into the product of exactly n factors where $f(x) = (x - \propto_1)(x - \propto_2) \dots (x - \propto_n)$, where $\propto_1, \propto_2, \propto_3, \dots, \propto_n$ are complex numbers, the roots of the equation $f(x) = 0$" (Courant & Robbins, 1996, p. 101). Quadratic trinomials of the form $ax^2 + bx + c$ where $a, b, c, x \in \mathbb{R}$ can be factored into two binomials with integer coefficients through the use of Viete's theorem can be stated as follows: for any quadratic trinomial $ax^2 + bx + c = (mx + p)(nx + q)$where $a, b, c, m, n, p, q \in \mathbb{Z}, mn = a, pq = c$ and $pn + mq = b$ (Usiskin et al., 2003). Compared to the teacher's procedure, Viete's theorem does not depend on the positions of terms in the expression and does not involve operation-like manipulations listed in Table 5.7. For the case $x^2 + bx + c$, Viete's theorem can be restated as follows: for any quadratic trinomial $x^2 + bx + c = (x + p)(x + q)$ where $b, c, p, q \in \mathbb{Z}, pq = c$ and $p + q = b$. For example, Viete's theorem can be used to factorise $x^2 - 2x - 3$.

$pq = c = -3$. So, $p = 3$ and $q = -1$ or $p = -3$ and $q = 1$. But $p + q = -2$. So, $p = -3$ and $q = 1$. Therefore $x^2 - 2x - 3 = (x - 3)(x + 1)$.

Table 5.8. Operatory properties of $(\mathbb{Z},\times)$ and $(\mathbb{Z},+)$ (Stewart & Tall, 1977, p. 172)

Axioms	Properties
$\forall a, b, c \in \mathbb{Z}, a + (b + c) = (a + b) + c$	Associativity of addition
$\forall a, b \in \mathbb{Z}, a + b = b + a$	Commutativity of addition
$\forall a \in \mathbb{Z}, a + 0 = a = 0 + a$	Additive identity
$\forall a \in \mathbb{Z}, \exists(-a) \in \mathbb{Z}$ such that $a + (-a) = 0$	Additive inverse
$\forall a, b, c \in \mathbb{Z}, a\times(b\times c) = (a\times b)\times c$	Associativity of multiplication
$\forall a, b \in \mathbb{Z}, a\times b = b\times a$	Commutativity of multiplication
$\forall a, b, c \in \mathbb{Z}, a\times(b + c) = (a\times b) + (b\times c)$	Distributivity of multiplication over addition
$\forall a \in \mathbb{Z}, a\times 1 = a = 1\times a$	Multiplicative identity
$\forall a \in \mathbb{Z}, a \neq 0, a \times a^{-1} = 1$	Multiplicative inverse

From the field of production, the operational resources required to use Viete's theorem are the axioms of the field $(\mathbb{R}, +, \times)$, specifically multiplication over integers, $(\mathbb{Z},\times)$ and addition over integers, $(\mathbb{Z}, +)$. The operatory properties of $(\mathbb{Z},\times)$ and $(\mathbb{Z}, +)$ are listed in Table 5.8.

I return to the teacher's propositions, which substitute for the operatory properties listed in Table 5.8. It could well be the case that her propositions are conceived as a shorthand for Viete's theorem, which guarantees the result of factorisation. However, for many learners in her class, the propositions are likely to entail the use of a character distribution matrix (as discussed in Section 5.2.3.5) and a combination of arithmetic over natural numbers and auxiliary operations used to generate the symbols to occupy the character distribution matrix. The character distribution matrix together with the auxiliary calculus enable the production of expressions that are convergent with the Mathematics encyclopaedia in terms of expression but which are divergent with respect to the content (TP15).

5.2.5 Step 5: consulting the field of recontextualisation – the curriculum and textbooks

Above I discussed the teacher's procedure from the perspective of the Mathematics encyclopaedia. However, teachers are subject to state policy with respect to what to teach and how to teach topics specified for each grade of schooling. Government policies on school mathematics are contained in curriculum documents that make grade-specific prescriptions of topics. With the implementation of the Curriculum and Assessment Policy Statements (CAPS) (Department of Basic Education, 2011) in 2012, state control over grade-specific topics and sequencing of topics increased substantially. CAPS not only specifies topics, but also the sequencing and pacing of topics, referred to as 'pace-setters', which outline week-by-week learning programmes for the year. In addition, topic-related curriculum statements are clarified with examples. Thus, there is an attempt on the part of the state to strongly regulate what should be taught, how it should be taught and when topics should be taught.

The specific curriculum document relevant to my study is CAPS Grades 10-12 Mathematics (Department of Basic Education, 2011). Curriculum documents offer a particular realisation of mathematics topics and represent the first level of recontextualisation from the field of production. The second level of recontextualisation is found in school mathematics textbooks and other texts used by teachers and learners. The Department of Basic Education (DBE) regulates school mathematics textbooks, which are evaluated before the textbooks are approved. Textbooks, therefore, have to meet the requirements specified in CAPS and schools select a particular textbook from the approved list as their main texts for teaching. Only four Grade 10 Mathematics textbooks were approved by the DBE for use in schools in 2012. In addition, the open source textbook, *Siyavula,* was distributed to schools by the Western Cape Education Department. Curriculum documents and textbooks, together, provide a window into the recontextualisation of topics from field of production from the point of view of the Official Recontextualising Field (ORF) and the Pedagogic Recontextualising Field (PRF) respectively (Bernstein, 2000).

In relation to solving quadratic equations and quadratic trinomial factorisation, it is interesting to note that CAPS, like the teacher described above, refer to quadratic trinomials simply as "trinomials" (Department of Basic Education, 2011, pp. 13, 21). Likewise, an examination of the textbook series used by the teacher illustrates that quadratic trinomials are referred to as "trinomials" as the teacher does (see Extract 5.6). Although CAPS does not specify particular methods for factorising "trinomials", the textbook does.

WORKED EXAMPLES

1. $12x^2 - 56x + 9$
2. $8x^2 + 13xy - 6y^2$
3. $18x^2 - 9x - 20$

SOLUTIONS

1. $12x^2 - 56x + 9$
$= (2x - 9)(6x - 1)$

	$6x$	-1
$2x$	$12x^2$	$-2x$
-9	$-54x$	

1 2 3 | 1 3 −9
12 6 4 | 9 3 −1

2. $8x^2 + 13xy - 6y^2$
$= (x + 2y)(8x - 3y)$

1 2 | 1 +2 3 6   +16 The bigger result carries the middle sign '+'.
8 4 | 6 −3 2 1   −3 The smaller result gets the '–'.
13

Figure 5.12. "Trinomial" worked example from textbook (Campbell & McPetrie, 2011, p. 19)

The method for factorising "trinomials" and the procedure elaborated by the teacher corresponds with the method outlined by the textbook shown in Figures 5.12 and 5.13.

Trinomials

When you factorise trinomials, remember these rules:

- Always take out the highest common factor first if there is one.
- Make sure that there are no common factors within brackets.
- If the last sign in the trinomial is a '+' sign, both sets of brackets will have the same sign.
- Both sets of brackets will have '+' signs if the trinomial has only '+' signs.
- Both sets of brackets will have '–' signs if the trinomial has a '–' followed by a '+' sign.
- If the last sign in the trinomial is a '–' sign, the sets of brackets will have opposite signs.
- Trinomials are often easy to factorise and you may write down the solution by inspection.

However, the worked example below gives the steps you may need to follow to factorise more challenging trinomials.

Figure 5.13. "Trinomial" procedural rules from textbook (Campbell & McPetrie, 2011, p. 18)

When we examine the treatment of quadratic equations in the textbook (see Campbell and McPetrie (2011, pp. 62-64), we observe that the notion of an equation is secondary to factorisation, which appears as the dominant notion structuring the unit, *Solve quadratic equations by factorisation*. Sub-sections of the unit on quadratic equations have the different forms of factorisation as the headings, suggesting that the unit is over-determined by the notion of factorisation. The zero product property (if $a.b = 0$ then $a = 0$ or $b = 0$) is not made explicit as a fundamental idea underpinning the solution of quadratic equations that have been rewritten as the product of linear factors. Instead, it seems that the primary task for the learners is to recognise what type of factorisation is required. The implicitness of the zero product property as a regulative resource is mirrored by the teacher in her procedure for solving quadratic equations.

5.2.6 Summarising the procedure for producing Stage 1 primary data - the instructional discourse

The following is a summary of the procedure for producing a description of the computational activity of teachers and learners emerging in a particular lesson in order to illuminate the functioning of evaluation in a pedagogic situation:

Step 1: Segment the lesson into evaluative events and classify evaluative events.

Step 2: Classify mathematics problems.

Step 3.1: Describe the computational activity of the teacher and learners in terms of descriptions of mathematical objects used.

Step 3.2: Describe the computational activity of the teacher and learners in terms of the propositions used.

Step 3.3: Describe the computational activity of the teacher and learners in terms of procedures used.

Step 3.4: Describe procedures and other computations in terms of operation(s) and domain(s) and co-domain(s).

Step 3.5: Describe the use of character distribution matrices where applicable

Step 4: Compare aspects of the Mathematics encyclopaedia (field of production) such as definitions, propositions and axioms related to the announced mathematical topic with the realised content.

Step 5: Describe the treatment of the announced topic in the field of recontextualisation - curriculum documents (ORF) and textbooks (PRF).

5.3 Stage 1: procedures for producing secondary data with respect to the instructional discourse

The primary data which entails a description of the computational activity emerging in the observed lessons generated as discussed above is analysed further to produce a second level of data which include (1) the realisation of content and (2) the regulation of computational activity in the observed lessons.

5.3.1 The realisation of content

In Chapter 3, I distinguished between the *announced topic* and the *realised content* drawing on Eco's (1984a) theory of textual coherence[59]. The announced topic refers to a name used by teachers, learners, and

[59] Note, as discussed in Chapter 3, that Eco (1984b) does not refer to content. Instead, topic, for Eco (1984b), is always constructed by the reader or listener from isotopic markers or clues.

textbooks to indicate a particular selection of mathematics contents, and contents are what are realised in relation to a topic name. So the realised content is the content that becomes associated with the topic name as the topic unfolds in the pedagogic situation and it is the content taught and learnt (TP14).

Recall that for Eco (1984a), a topic is constructed through isotopies, which are textual markers or clues used by readers or listeners to constitute the topic. Mathematics learners read the computational activity in their pedagogic situation for isotopic markers that provide computational clues in order to constitute the content of mathematics topics. What this means is that (1) the realised content constituted in relation to a particular topic name may differ across and even within pedagogic contexts and (2) the realised content constituted may differ from the content associated with the topic in the Mathematics encyclopaedia (TP15).

To illustrate the analysis of the realised content, I return to the teacher's procedure for calculating x-intercepts discussed in detail in Section 5.2 (see Extract 5.8). The announced sub-topic is calculating x-intercepts which emerges in relation to sketching the graph of the parabola $y = -x^2 + 6x - 5$. It is interesting that the teacher does not explicitly refer to $-x^2 + 6x - 5 = 0$ as a quadratic equation suggesting that the notion of equation remains implicit. Instead, central to the teacher's procedure is "making x^2 positive" thus generating a particular expression state, $x^2 - 6x + 5 = 0$ which serves as a visual cue or what Davis (1984) refers to as a *visually-moderated-sequence* which prompts the sub-procedure trinomial factorisation that produces the product of two linear binomials. The use of expression states as prompts for recalling procedures is further supported by the teacher's reference to $x^2 - 6x + 5 = 0$ as a "trinomial" and not a quadratic equation. The term *trinomial* in the Mathematics encyclopaedia refers to any three-term expression whereas the term *quadratic trinomial* refers to a general class of expressions of the form $ax^2 + bx + c$ where $a, b, c, \in \mathbb{R}$ and $a \neq 0$.

The teacher's procedure, which relies on the use of a character distribution matrix (CDM), centres on generating elements required to occupy particular positions in the CDM, as discussed in detail in Section 5.2.3.5. Her procedure suggests that the rules of her procedure rather than the mathematical properties underpinning factorisation of quadratic trinomials regulate the mathematical activity. Alongside the familiar structures, such as the magma[60] $(\mathbb{R},\times)$ found in the Mathematics encyclopaedia the teacher's procedure involves an auxiliary calculus that consists of auxiliary operations such as sundering, concatenation, and ALT that take alphanumeric characters as their arguments and values. So, the expressive elements are operated on directly and they constitute the domains and codomains of the auxiliary operations that form part of her procedure.

Furthermore, she implicitly relies on a strict order of the terms in the quadratic trinomial, the 'standard form' – the "first term" must be the x^2 term, the "middle term" is the bx term and the "last term" the c term of the quadratic trinomial, $x^2 + bx + c$. The spatial arrangement of the terms is central to the workings of the procedure since the execution of the transformations that make up the procedure are framed in terms of the

[60] A magma is a mathematical structure which consists of a set with a single binary operation which must be closed.

positions of particular terms in the expression. So, the iconic features of the expression are key to the procedure.

As discussed, the content realised in association with the topic name in this pedagogic situation does not explicitly draw on the Mathematics encyclopaedic propositions and theorems (fundamental theorem of algebra and Viete's theorem) underlying the announced topic. The aforementioned theorems are substituted by a procedure for calculating a parabola's x-intercepts, which includes a sub-procedure, the "trinomial" factorisation procedure. The computational activity, discussed above, focuses centrally on a particular expression state and the spatial arrangement of terms in the "trinomial" in order to generate and distribute characters required to populate a CDM. The isotopies, read off the computational activity, constitute the content associated with the topic, calculating x-intercepts, as whole number arithmetic (factorising whole numbers) and character generation and distribution. So the realised content differs from the Mathematics encyclopaedic content associated with the topic name.

As discussed earlier, the realised content is read off the computational activity which comprises the operations performed and associated domains and codomains together with definitions or descriptions of mathematical terms and/or propositions that support the computational activity of teachers and their learners. I compare the computational activity emerging in the pedagogic situation with the content typically associated with the announced topic in the Mathematics encyclopaedia as a means of gauging the recontextualisation of content[61]. I examine the following computational resources in order to identify points of similarity and differences with respect to computational resources indexed by the particular topic in the Mathematics encyclopaedia in order to identify whether content substitutions have occurred:

(1) *Descriptions of mathematical terms*: Where descriptions of mathematical terms in an evaluative event are offered, are these definite descriptions or do they merely function as explications of mathematical terms, i.e., as auxiliary descriptions?
(2) *Propositions* recruited to support computations: Are the propositions used in an evaluative event propositions found in the Mathematics encyclopaedia or are encyclopaedic propositions replaced with auxiliary propositions or with computational rules?
(3) *Operations* recruited by teachers and learners in an evaluative event: Are the operations solely recruited from the field of the reals or are one or more of the operations auxiliary operations? As noted earlier, auxiliary operations are not necessarily functions which have unique, stable outputs for each input and that maintain identity at the level of value.
(4) *Domains and codomains*: Do the domains and codomains differ from that indexed by the topic in the Mathematics encyclopaedia? If so, are domains and codomains subsets of $\mathbb{R}$ or are they character strings? Recall that operations and domains are compossible. The domains and codomains of auxiliary

[61] This is not to say that the pedagogic agents consciously recontextualise mathematics contents from the field of production. Rather, it is the case that we can view the version of mathematics constituted in the pedagogic context as a recontextualisation when compared with the Mathematics encyclopaedic content.

operation-like manipulations tend to be character strings rather than numbers. So, auxiliary operations simultaneously indicate a shift in domain indexed by the topic in the Mathematics encyclopaedia.

(5) *Computational objects*: Does the computational object match the object indexed by the topic in the Mathematics encyclopaedia? Do the computational objects indexed by the topic in the Mathematics encyclopaedia function explicitly as regulative resources? For example, when solving the quadratic equation $-x^2 + 6x - 5 = 0$ discussed above, the notion of an equation is not explicitly recruited as a regulative resource.

Having examined computational resources discussed above, evaluative events are coded as those where no content substitution has occurred or those where content substitution has taken place. Evaluative events in which content substitution has not occurred represent instances where the realised content corresponds with content associated with the topic from the point of view of the Mathematics encyclopaedia. In other words, there is *content convergence.* An evaluative event is *content convergent* when it contains operations, domains and codomains recruited from the field of the reals, i.e., when all operations used in the event are functions, identity is preserved at the level of value, although it may change at the level of expression and computations may be supported by encyclopaedic propositions and/or definite descriptions. When differences occur with respect to one or more of the computational resources outlined above, then this is taken as an indicator that content substitution has occurred. Evaluative events where content substitution has occurred are marked by the absence of encyclopaedic descriptions or encyclopaedic propositions. Evaluative events in which content substitution has occurred represent instances where the realised content differs from the content associated with the topic from the point of view of the Mathematics encyclopaedia. In other words, we have *content divergence.* (See also Davis and Ensor (2018) where they argue that both content convergence and content divergence can take the form of expression convergence or expression divergence.)

The computational resources described above can be grouped into two broad categories, namely, *encyclopaedic computational resources* (definite descriptions, encyclopaedic propositions and encyclopaedic operations) and auxiliary computational resources (auxiliary descriptions, auxiliary propositions and auxiliary operations). If we arrange the two categories of computational resources in terms of presence or absence, we generate a system of logically inter-related categories - either both categories are present or both are absent or one category is present and the other is absent, therefore generating four inter-related categories (see Figure 5.16). The resultant four inter-related categories describe the realised content in relation to announced topics.

If encyclopaedic computational resources are present and auxiliary computational resources are absent, the realised content is referred to as *canonical*. Extract 5.7 shows a portion of an evaluative event classified as canonical.

Extract 5.7. transcript S01T01L02 (lines 63 – 70)

Teacher:	Okay and then this [pointing at $y = ax^2 + bx + c$] could then be written in the form *x* minus *x* one *x* minus *x* two which is the factorised form [points at $y = a(x - x_1)(x - x_2)$]. So we have seen that the parabola can actually be written in three forms. One two three [points at the board]. What did I call this form [referring to $y = ax^2 + bx + c$]?
Learners:	Standard form.
Teacher:	Standard form. This one I called? [pointing at $y = a(x - p)^2 + q$]
Learner:	[indistinct]
Teacher:	Turning point form
Learners:	Turning point form.
Teacher:	Okay. We said *p q*. If it is in that form *p q* is equal to the turning point. So these are our *x* cuts and this would be B [pointing at *x*-intercept on graph] and this would be C [pointing at the second *x*-intercept] Okay. when you factorise it. That point [referring to A (*y*-intercept on graph)] is the …
Learner:	*y* cut.

In this evaluative event, Sara discusses different forms of a parabola's equation. She recruits two encyclopaedic propositions that support the computational activity in this evaluative event. Firstly, she refers to the proposition, if $y = a(x - p)^2 + q$ then the turning point is $(p; q)$ and, secondly, if $y = a(x - x_1)(x - x_2)$ then x_1 and x_2 are the *x*-intercepts of the parabola. In particular, this evaluative event does not contain any auxiliary descriptions, auxiliary propositions or auxiliary operations. The only operation referred to in this event is multiplication in connection with transforming the turning point form of the equation, $y = a(x - p)^2 + q,$ to the standard form, $y = ax^2 + bx + c.$ Multiplication is an operation located in the Mathematics encyclopaedia which takes real numbers as its domain.

When auxiliary computational resources are present and encyclopaedic computational resources are absent, the realised content is described as *ancillary*. Here the auxiliary resources are often accompanied by the basic arithmetic operations over the set of natural numbers, although at times positive rationals in the form of fractions may be operated over. The teachers procedure for calculating the *x*-intercepts of $y = -x^2 + 6x - 5$ discussed in Section 5.2.3.4 (Extract 5.6 is repeated here as Extract 5.8) is typical of content realised as ancillary which relies on auxiliary descriptions, propositions and operations and is devoid of any definite descriptions and encyclopaedic propositions. Here the realised content differs from the content associated with the topic name from the point of view of the Mathematics encyclopaedia because (1) the encyclopaedic proposition, Viete's theorem, is replaced with the trinomial factorisation procedure which includes computational rules for generating the signs in the brackets; (2) the procedure for calculating the *x*-intercepts include auxiliary operations such as ALT and sundering, which are located external to the Mathematics encyclopaedia and which take character strings as their domains and codomains; (3) a character distribution matrix is employed; and (4) the notion of an equation which is central to calculating the *x*-intercepts of a parabola does not function explicitly as a regulative resource.

Extract 5.8. transcript S01T01L01 (lines 374 – 380[62])

Teacher:	Now some of you might have taken this expression our standard form expression. Minus *x* squared plus six *x* minus five is nought. How do we solve that?
Learner:	*x* equal five and minus five.
Teacher:	No. I know that's the answer.[63]
Learner:	Make that plus *x* squared. Take everything to the other side.
Teacher:	Right. So we have to make *x* squared positive. Change all the signs. In other words we are multiplying everything by minus one. And now what to do? It's a trinomial.
Learner:	We try to factorise.
Teacher:	Factorise.
Teacher*:	Right [Writes brackets / ( )( )/]
Teacher*:	.*x* *x*. [Writes in brackets / (*x*)(*x*)/]
Teacher*:	Five one. [Writes in brackets/(*x* 1)(*x* 5)/]
Teacher*:	Minus and minus. [Writes in brackets/(*x* - 1)(*x* - 5)/]
Teacher*:	So *x* is one or *x* is five.

If both encyclopaedic computational resources and auxiliary computational resources are present, the realised content is described as *symbiotic*. The extract of a lesson shown in Extract 5.9 focuses on determining the turning point of the function $y = -(x-1)^2 + 4$.

The teacher, Sara, refers to a number of auxiliary operations. Firstly, she focuses on the minus sign in front of $(x-1)^2$. In doing so, she is employing the operation *sunder* which splits the sign from the rest of the expression. Sunder is an auxiliary operation which operates on the symbols. Similarly, she focuses on $/-1/$ and $/+4/$ which also entails the use of auxiliary operation, *sundering.* She then employs operations that effect a 'shift' of the graph to the right and upwards. These auxiliary operations take the graphical images as inputs and entail the image 'moving' left or right. These spatial auxiliary operations stand in place of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ entailed in the transformation of functions. Lastly, she employs an auxiliary operation, "opposite sign" which takes the character string $/-1/$ as its input and produces the output $/+1/$. The auxiliary operations discussed above are used together with encyclopaedic propositions. Sara refers to the proposition "this graph [referring to $y = -(x-1)^2 + 4$] has a maximum because a is negative" (S01T01L02: line 102) which is an encyclopaedic proposition which states that a parabola $y = a(x-p)^2 + q$ has a maximum value when $a < 0$. She also refers to the encyclopaedic proposition which states that a parabola $y = a(x-p)^2 + q$ has a turning point $(p;\ q)$. Extract 5.9, therefore , illustrates the coexistence of encyclopaedic and auxiliary computational resources.

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[62] Line 380 (marked with *) has been split into different lines for ease of reading to capture the board work annotated next to the speech.

[63] The answer produced by the learner is incorrect but the teacher does not correct him - presumably because she is interested in discussing the procedure.

Extract 5.9. transcript S01T01L02 (lines 98 – 104)

Teacher:	So here the graph the original graph if it was y equals x squared. What did .. how did the minus transform the graph [referring to $y = -(x-1)^2 + 4$] ?
Learner:	Reflect the curve.
Teacher:	Right. Reflected it [gestures to show across the x-axis]. What did the minus one do? Shifted it plus one [gestures to the right]. It does the opposite sign for that shifting and what does the plus four do? Shifted it …?
Learner:	Up or down.
Teacher:	Right. Another way of thinking about the turning point rather than minus one goes one to the right is to say that this graph [referring to $y = -(x-1)^2 + 4$] has a maximum because a is negative. Okay. And for this [referring to /4/] to be a maximum this [referring to $(x-1)^2$] has to be nought. Right and for that [referring to $(x-1)^2$] to be nought what does x have to be? x has to be?
Learner:	One.
Teacher:	x has to be one. So you can see it is the opposite sign. So the turning point is one four. Right and then we said here that you have to recognise that form [referring to $y = a(x-p)^2 + q$ that the turning point is p q. Okay you all happy with that?

If both encyclopaedic computational resources and auxiliary computational resources are absent, the realised content is described as *elementary*. The immediate question that arises, is it possible for both encyclopaedic and auxiliary computational resources to be absent? Does the absence of both categories of computational resources then not imply a non-existent category of realised content? In order to examine this issue, let us consider an example shown in Figure 5.14.

Exercise 1

1 Copy the following table into your exercise book and complete it for $f(x) = \frac{x^2 - x - 2}{x - 2}$.

x	1	1,5	1,75	1,85	1,95	1,96	1,97	1,98	1,99	1,999	1,9999
$f(x)$			2,75						2,99		

2 What happens to the value of $f(x)$ as the value of x approaches (tends towards) $x = 2$ from $x = 1$?

3 Copy the following table into your exercise book and complete it for $f(x) = \frac{x^2 - x - 2}{x - 2}$.

x	3	2,5	2,25	2,15	2,05	2,04	2,03	2,02	2,01	2,001	2,0001
$f(x)$		3,5								3,001	

4 What happens to the value of $f(x)$ as the value of x approaches (tends towards) $x = 2$ from $x = 3$?

5 Can you now estimate the value of $f(2)$?

Figure 5.14. Extract from Botsane et al. (2013, p. 170)

Here the notion of a limit of a function is explicated through a task that requires learners to complete a table of values by calculating function values for given values of x. Learners are expected to observe from the table of values that "the value of $f(x)$ approaches (tends towards) 3 as x approaches (tends towards) 2" (see Figure 5.15).

> We say that the value of $f(x)$ approaches (tends towards) 3 as the value of x approaches (tends towards) 2. We write this as: $f(x) \to 3$ as $x \to 2$. We use the symbol '$\to$' to indicate 'approaches'.
>
> Another way to write this is:
> $$\lim_{x \to 2} f(x) = 3$$

Figure 5.15. Extract from Botsane et al. (2013, p. 170)

The table of values serves as the principal computational resource for developing the notion of a limit. The notion of a limit of a function is therefore arrived at empirically and inductively. A definite description of a limit of a function is absent. In fact, the epsilon-delta definition of a limit is encoded into the task and is used in structuring the task itself. The task therefore excludes fundamental axioms, definitions and propositions (encyclopaedic resources) and excludes any auxiliary computational resources. The operations that are entailed in completing the table of values are the basic arithmetic operations (addition, multiplication and division) and possibly squaring over rational numbers.

The above discussion is summarised in Figure 5.16 which displays the four inter-related categories which describe the realised content in relation to announced topics.

		Encyclopaedic computational resources	
		present	**absent**
Auxiliary computational resources	**absent**	*canonical*	*elementary*
	present	*symbiotic*	*ancillary*

Figure 5.16. Realised content typology for evaluative events

Because pedagogy has a regulative dimension, the criteria that regulate the selection and sequencing of the operations and collections of objects operated over are always present (Davis, 2011a) and it is the regulation of mathematical activity that shapes the realisation of content in the pedagogic situation. So, another layer of my analysis entails examining the regulation of the computational activity, which I now discuss.

5.3.2 Regulation of computational activity

Recall from Chapter 3 that Davis (2010a, 2011b), drawing on Hegel and Peirce, constructed a notion of ground to describe the regulation of mathematical thought. Features of Davis' (2010) four categories of ground: iconic, empirical, algorithmic and propositional[64], discussed in Chapter 3, are repeated here as Figure 5.17.

Ground	Central grounding resource	Objects of central concern
Iconic	Comparisons centred on iconic features, including similarities and differences of expressions	Graphical and/or symbolic expressions treated as images
Empirical	Empirical testing of expressions	Graphical and/or symbolic expressions treated as in some way "measurable"
Fundamental (propositional)	Knowledge of the mathematical objects and relations between such objects referenced by mathematical statements	Mathematical objects indexed by the axioms, definitions and propositions that are signified by expressions
Algorithmic	Meta-rules governing an algorithm, regulating the selection and sequencing of operations on mathematical signifiers	Operations commonly used within a particular algorithm as well as their sequencing

Figure 5.17. Categories of ground adapted from Davis (2010a, p. 382)

Iconic ground is recognised as regulating mathematical activity when: (1) there is an emphasis on what the solution or part of a solution should look like; (2) graphical or symbolic expressions are operated on directly i.e., the presence of operation-like manipulations ; (3) metaphors are used to refer to mathematical ideas; (4) images are used to refer to mathematical notions; and (5) there is use of a character distribution matrix. The use of the iconic outlined above operates without the support of mathematical axioms, definitions and propositions, and as such situates necessity external to Mathematics.

The teacher's use of a metaphor in Extract 5.10 is an example of iconic ground regulating mathematical activity.

Extract 5.10. transcript S01T02L01 (line 221)

Teacher:	If you take a skipping rope and you skip. When the rope is up and then flips down you are reflecting in the *x*-axis. Okay. So it's a skipping rope effect. So if your graph was smiling and you reflected that correctly. It should be that your *x*-axis is acting as a handle and it is flipping perfectly on that axis. Anyone want to suggest a method?

The teacher uses the metaphor of skipping to explain the reflection of a parabola in the x-axis. "Skipping" substitutes for a function (T_r) which maps every pair of coordinate pairs $(x; y)$ of the quadratic function $f(x) = ax^2 + bx + c$ (where $a, b, c \in \mathbb{R}$) to the coordinate pairs $(x; -y)$. The auxiliary operation "skipping" takes as its argument a skipping rope and produces a skipping rope in a reflected orientation. "Skipping" is an auxiliary operation because the objects that serve as the arguments are not real numbers. The auxiliary operation, *imaging* (IMG), takes a parabola as input and produces an image of a physical object, thus

[64] Algorithmic ground was previously referred to as syntactic ground and procedural ground (Davis, 2010a). Propositional ground was referred to as fundamental ground (See Davis & Johnson, 2008, Davis, 2010a). I have reverted to the term, *fundamental ground*, because I have argued that propositions always underpin computational activity even though the propositions are not necessarily recruited from the Mathematics encyclopaedia.

entailing an existential shift from a mathematical object to a physical object. The auxiliary operation, IMG, is central in structure preservation from the transformation of a parabola to skipping with a skipping rope as shown in Figure 5.18.

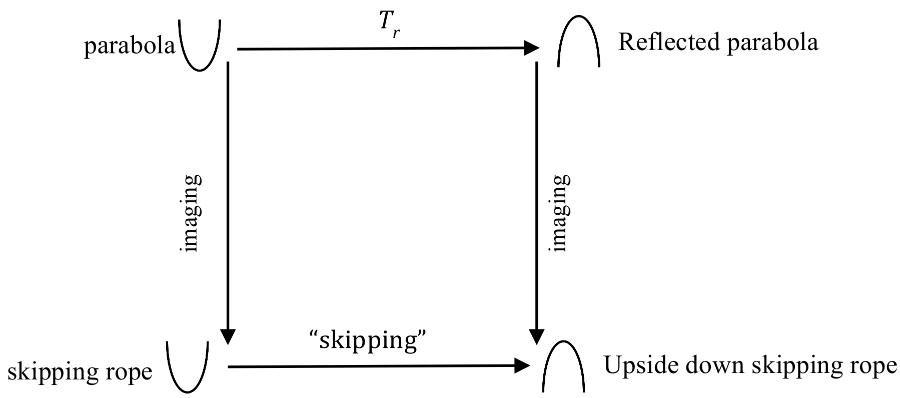


Figure 5.18. Structure preservation map - skipping metaphor

There is of course another metaphor implied by the teacher's explanation because she refers to a "smiling graph" which refers to a parabola described by $y = ax^2 + bx + c$ where $a > 0$. The auxiliary operation, "flipping" takes the graphical image in its initial position as its input and produces the reflected graphical image as its output. "Flipping" is an auxiliary operation because the object that serves as the argument is a facial image and the output is a facial image rather than real numbers. A structure preservation map (Figure 5.19) shows the relation between the physical movement of facial expression images to the operation reflection of the parabola.

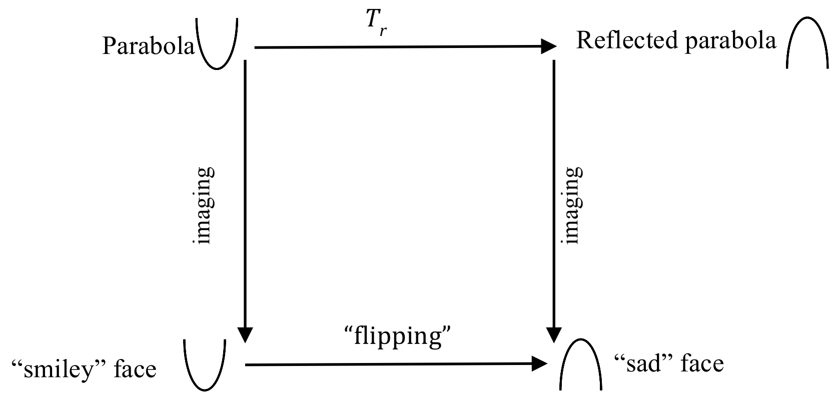


Figure 5.19. Structure preservation map – facial expressions metaphor

We can of course present a composite map of the two structures.

Algorithmic ground refers to instances where the selection and sequencing of operations regulate the mathematical activity of teachers and learners. Consider the same teacher's explanation of solving the equation $(x - 3)^2 - 16 = 0$ shown in Extract 5.11.

Extract 5.11. transcript S01T02L02 (line 5 and 96-98)

Teacher:	Multiply out x minus three all squared to get that [referring to $x^2 - 6x + 9 - 16$] … To get the standard form here all you have to do is to add in the sixteen the minus sixteen on the end. So if I want to get this into standard form you multiply that out and then you add minus sixteen and you get that [referring to $x^2 - 6x - 7 = 0$] and it's not difficult.
After *writing the equation in standard form, the teacher returns much later in the lesson to solve the equation* $x^2 - 6x - 7 = 0$.	
Teacher:	Okay. You take the seven has very few options which is fantastic. What are the options?
Learners:	Seven and one.
Teacher:	Seven and one. It's a no brainer. It's the only option you got. So use seven and one. I need a negative and the opposite sign and I need the bigger result to be negative. If the bigger one is negative the other one is positive. Should be easy.

From Extract 5.11, we note that the teacher's procedure is regulated by the selection and sequencing of operations: (1) write the equation in standard form; (2) factorise the constant term, 7; (3) work out the signs associated with each factor and (4) write down the solution to the equation. It is not procedurally necessary to transform the equation $(x-3)^2 - 16 = 0$ to $x^2 - 6x - 7 = 0$ to solve the equation. We can solve the equation very easily in the following ways: $(x-3+4)(x-3-4) = 0$, so $x = 7$ or -1 or $(x-3)^2 = 16$, so $x - 3 = \pm 4$, so $x = 7$ or -1. The notion of an equation, the mathematical object signified, appears to be absent as a regulative resource in this teacher's computational activity. Instead, it's the factorisation procedure which regulates the teacher's mathematical activity and so the activity of her learners. Here, necessity resides in the teacher's procedure.

Fundamental ground is evident when teachers and learners explicitly attend to the Mathematics encyclopaedic axioms, definitions or propositions underlying a particular topic as illustrated by Jada in Extract 5.12 when discussing a procedure for finding the equation of a parabola if given its graph:

Extract 5.12. transcript S01L02T02 (line 107)

Teacher:	So I can't just take the x-cuts and just say well here is my equation. I have to control the graph through something else. Now if I put any other point in to play. That point for example. It's absolutely impossible to draw a perfect parabola through those three points but that's different from what I have. I can draw millions (parabolas) through the x cuts. That's what I am worrying about. But once if I put a third point in doesn't matter where, there is no other parabola than that one that will go through those three points.

Here, fundamental ground underpins the teacher's procedure for finding the equation of a parabola from a given sketch. She specifically draws learners' attention to a fundamental proposition relating to parabolas which is that there is a family of parabolas with the same x-intercepts, so a third point on the parabola is required in addition to the x-intercepts to calculate the equation unique to the given parabola.

Empirical ground regulates mathematical activity when teachers or learners employ some testing or trial-and-error methods. For example, learners at Evergreen High were given the task of finding the equation of a parabola from a sketch of a graph passing through the x-axis at $(2; 0)$ and $(-2; 0)$ with turning point $(0; 12)$. They were expected to solve the problem without being taught a method by the teacher. A group of learners working on the problem proceeded in the following way. The learners knew that the general equation of the parabola is $y = ax^2 + q$ and worked out that since the graph has a "sad" face the value of a should be negative and since the y -intercept is 12 that q is 12. They then guessed the equation to be $y = -x^2 + 12$, calculated the x-intercepts as a check and discarded the equation when they noticed that the x-intercepts were not 2 and -2. They subsequently tried the equation $y = -2x^2 + 12$ but discarded this equation because the x-intercepts did not match the given graph. Finally, they tried $y = -3x^2 + 12$ which yielded the correct x-intercepts. They subsequently accepted $y = -3x^2 + 12$ as the equation of the given graph. Here we observe learners guessing and checking until they arrived at the correct equation. The form of ground regulating their mathematical activity is therefore empirical. Since empirical ground is operative in the absence of the fundamental mathematics axioms, propositions and definitions, necessity is located outside the field of mathematics and instead is situated with the teacher's criteria.

As pointed out by Davis (2010a), the categories of ground are not hierarchical and it is possible for more than one category of ground to be present simultaneously. Returning to the teacher's procedure for calculating x-intercepts discussed in Section 5.1, we can now discuss the regulation of mathematical activity. The teacher refers to quadratic trinomials as "trinomials", placing the emphasis on the fact that the expression contains three terms. The quadratic nature of the expression, therefore, does not serve as a regulative resource even though the validity of the teacher' solution procedure is dependent on it being a quadratic trinomial. Secondly, that $-x^2 + 6x - 5 = 0$ is an equation remains implicit suggesting that the notion of an equation does not function as regulative resource indicating the absence of fundamental ground.

Instead, it is the expression state $x^2 + 6x - 5 = 0$ which acts as one of the primary regulative resources invoking the trinomial factorisation sub-procedure. The spatial arrangement of the expression, although implicit, with the first term being the x^2 term, the middle term the bx term and the last term the c term, plays a central regulative role. Thus, the ground regulating mathematical activity relies on the iconic features of the expression. Secondly, the teacher's procedure for factorising a quadratic trinomial rests on a character distribution matrix, which she invokes when she generates the brackets on the board as a spatial template while verbally producing the symbols to occupy the character distribution matrix (See Figure 5.10).

Thirdly, the procedure contains operation-like manipulations which take parts of the expression as domains and codomains, yet another indicator of iconic ground in operation.

Algorithmic ground is present given that the teacher specifies that "making x^2 positive" is required to solve the equation, which as discussed earlier is not a procedural necessity but serves to regulate the mathematical activity. Algorithmic ground plays a secondary role in the regulation of mathematical activity. The teacher's procedure is not explicitly regulated by the notion of an equation and does not explicitly draw on the fundamental ground underlying the procedure - the fundamental theorem of algebra and Viete's theorem. So, fundamental ground is absent. We thus observe the iconic-algorithmic binary regulating the teacher's procedure, thus locating necessity external to mathematics.

5.3.3 The implied model learner

Recall from Chapter 3 that the model learner, which is derived from Eco's (1984) Model Reader, constitutes an analytic category that describes the competence of the notional learner implied by the computational activity prevalent in a particular pedagogic situation. Recall too that according to Eco (1984a), different types of text construct different Model Readers. His distinction between *open* and *closed* texts was used in Chapter 3 to construct the notions of *open pedagogic texts* and *closed pedagogic texts.*

An *open pedagogic text* is one in which Mathematical axioms, definitions and propositions are explicitly recruited as computational resources. A *closed pedagogic text*, on the other hand, does not draw explicitly on Mathematical axioms, definitions and propositions but recruits auxiliary descriptions and propositions as computational resources. Closed pedagogic texts attempt to obtain a precise response from learners thus curtailing flexibility to produce different solution procedures.

Returning to the teacher's procedure for calculating the *x*-intercepts of a parabola discussed in Section 5.3.1 and Extract 5.8, we can now examine whether her procedure functions as an open or closed pedagogic text.

As discussed earlier, from the point of view of the Mathematics encyclopaedia the teacher's procedure does not explicitly reference the fundamental mathematical ideas underpinning the procedure such as the fundamental theorem of algebra or Viete's theorem to factorise "trinomials" which suggests that the model learner does not require these fundamental ideas as computational resources. Furthermore, the notion of an equation as well as the zero product property required to solve a quadratic equation that has been factorised into linear factors remains implicit. So, the procedure is closed with respect to the sub-topic, calculating *x*-intercepts of a parabola.

The text is further closed down by the insistence on a particular expression state, "x^2 positive", as a prerequisite for factorising the expression. The motivation for "making x^2 positive" does not have any mathematical necessity, instead the expression state serves as a prompt for the sub-procedure, "trinomial" factorisation. The expression state, thus, acts as a regulative resource with the notion of a quadratic equation remaining implicit. Insisting on a particular expression state strongly suggests attempts to, consciously or unconsciously, prevent errors on the part of learners who it seems are deemed more likely to produce a successful outcome when factorising a "trinomial" with "x^2 positive" than with "x^2 negative". Thus, the procedure attempts to steer learners down a particular path, suggestive of Eco's closed text. The model learner implied by the procedure is constructed as one who is not sufficiently competent to factorise "trinomials" in general. The sub-topic is recontextualised in a manner that ensures that learners produce the required outcome expressively, the factors of the "trinomial" and the solution to the quadratic equation, despite their lack of knowledge of the topic from the point of view of the Mathematics encyclopaedia.

Despite the closed nature of the procedure with respect to the Mathematics encyclopaedia, we have to examine the elaboration of the procedure in relation to its place in the computational activity and its connections to other computational resources. The teacher's procedure for calculating the *x*-intercepts described in detail in this chapter represents one of two procedures that emerged in the pedagogic situation. The equation of the parabola stated in the problem statement was in fact $y = -(x-3)^2 + 4$ and was used to derive the equation $y = -x^2 + 6x - 5$. An alternate method for calculating the *x*-intercepts was based on solving the equation $-(x-3)^2 + 4 = 0$ by using square roots rather than factorisation as discussed above. The teacher's employment of alternate methods encourages procedural flexibility and contributes towards weakening the closed nature of the text. The teacher's procedure for calculating the *x*-intercepts of $y = -x^2 + 6x - 5$ is thus coded as a weak closed pedagogic text.

In summary, an *open pedagogic text* (T_o) explicitly recruits Mathematical axioms, definitions and propositions and so does not entail substitution of content. So, an open pedagogic text is content convergent and is made less open (weakened) by:

- inclusion of the iconic and/or empirical but these resources are not dominant
- privileging particular methods without providing mathematical necessity for the particular method
- specifying a particular selection and sequencing of operations that has no mathematical necessity
- using particular expression states (visually-moderated sequences) as prompts for a particular procedure

A *closed pedagogic text* (T_c) forecloses access to Mathematical axioms, definitions and propositions and is weakened by the following, that is closed pedagogic texts are made less closed by:

- inclusion of the empirical
- procedural flexibility – multiple methods made available irrespective of the form of the expression
- auxiliary propositions accompanied by encyclopaedic correlates
- connections to other topics or sub-topics made explicit.

So, open and closed pedagogic texts are each realised in terms of two modalities, strong or weak (Davis, 2011b). Strong open pedagogic texts (T_o^+) exhibit none of the criteria that weaken an open pedagogic text elaborated above and weak open pedagogic texts (T_o^-) display one more of these criteria. Strong closed texts (T_c^+) are characterised by an absence of any of the criteria that weaken the closedness of a pedagogic text listed above and weak closed texts (T_c^-) are marked by the presence of one more of these criteria.

Table 5.9 summarises the criteria for coding evaluative events as open pedagogic texts (weak or strong) or closed pedagogic texts (weak or strong).

Table 5.9. Coding categories for characterising pedagogic texts as open or closed

Pedagogic text	Strong	Weak
Open	• Mathematical axioms, definitions and propositions explicit • No content substitution • Absence of iconic computational resources • Absence of empirical computational resources	• Mathematical axioms, definitions and propositions explicit • Inclusion of empirical and/or iconic • Fixed selection and sequencing of operations - no mathematical necessity for order of operations • Privileging solution methods • Solution methods dependent on expression states •

Closed	• Mathematical axioms, definitions and propositions implicit/absent • Fixed selection and sequencing of operations - no mathematical necessity for order of operations • Privileging solution methods -dependent on expression states • Strong focus on imagistic features of expressions	• Mathematical axioms, definitions and propositions implicit/absent • Strong focus on the empirical • Procedural flexibility – multiple methods made available irrespective of the form of the expression • Auxiliary propositions accompanied by encyclopaedic correlates • Connections to other topic sub-topics explicit

Having considered the nature of the pedagogic text as either open or closed which illuminates the implied mathematical competencies of the model learner, I now consider what the computational activity reveals about the orientation to mathematics in the pedagogic context.

5.3.4 Orientation to mathematics

Recall from Chapter 3 that Davis (2011b) reconfigured Lotman's, *expression-orientation* and *content-orientation,* to produce notions that are more appropriate for describing orientations to mathematics. Davis (2011b, pp. 316-317) defines expression-orientation as one which focuses primarily on the expressive elements required and entails a system of combinatorial rules that operate directly on the expressive elements to generate texts. In other words, the domains and codomains comprise alphanumeric strings or graphical images and the operations recruited are auxiliary operation-like manipulations. With content-orientation, the expressive elements are secondary, functioning merely as conventions for communicating mathematics and the system of combinatorial rules comprise domains and codomains drawn from the field of the reals and operations are encyclopaedic.

Considering the teacher's procedure for calculating x-intercepts discussed above, we observe that the expressive resources are of primary importance suggesting that the orientation to the privileged text is expression-oriented rather than content-oriented. However, as discussed earlier, the teacher does counter the learners' use of auxiliary operations when she transforms the learner's operation "take everything to the other side" to another operation "change all the signs" which she immediately restates as the operation "multiply by -1". All three operations produce the same outcome. However, as discussed earlier, "take everything to the other side" and "change all the signs" are operation-like manipulations that are auxiliary to the Mathematics encyclopaedia. The teacher's parallel usage of the Mathematics encyclopaedic operation, multiplication by -1, serves to explicitly remind learners of the appropriate Mathematical operation, thus weakening the expression-orientation. The orientation to mathematics as exhibited in this procedure is categorised as weak expression-oriented (O_e^-).

In summary, an orientation to mathematics is described as expression-oriented when expressive elements (domains and codomains comprise of character strings or graphical images and auxiliary operations) are primary and the use of character distribution matrices (CDM) or spatial templates are evident. An expression-orientation is weakened by the following:

- auxiliary operations accompanied by encyclopaedic correlates

- empirical resources such as calculations or the use of computer software because the empirical steers the computational activity towards the numerical, reducing the emphasis on the expressive resources (Davis, 2011b, p. 319)
- Mathematical axioms, definitions and propositions are recruited e.g. “proofs-without-words” where the inclusion of Mathematical axioms, definitions and propositions reduces the sensibility of the iconic, like in so-called “proofs-without-words” or “iconic-proofs”
- Deductive explanations or reasoning
- Connections to related sub-topics explicit

An orientation to mathematics is described as content-oriented when the expressive elements play a secondary role and the main function of the expressive resources lies in communicating mathematical activity. Strong content-orientation (O_C^+) is characterised by the explicit presence of Mathematics axioms, definitions and propositions. The inclusion of empirical resources such as calculations or the use of computer software as part of the computational resources weakens content-orientation (O_C^-) because the empirical opens up the possibility for induction which diminishes the regulative effect of the fundamental Mathematical axioms, definitions and propositions. Likewise, the presence of the iconic weakens content-orientation (O_C^-) because it diminishes the regulative effect of the fundamental Mathematical axioms, definitions and propositions. Table 5.10 summarises the criteria for coding orientations to mathematics as content-oriented (weak or strong) or expression-oriented (weak or strong).

Table 5.10. Coding categories for content-orientations and expression-orientations

Orientation	Strong	Weak
Content	• Mathematical axioms, definitions and propositions explicit • Expressive elements secondary	• Mathematical axioms, definitions and propositions explicit • Expressive elements secondary • Inclusion of empirical and/or iconic • Auxiliary operations accompanied by encyclopaedic correlates • Auxiliary calculus present but not dominant
Expression	• Mathematical axioms, definitions and propositions implicit/absent • Combinatorial rules that operate directly on the expressive elements • Presence of character distribution matrices	• Mathematical axioms, definitions and propositions implicit/absent • Combinatorial rules that operate directly on the expressive elements • Auxiliary operations accompanied by encyclopaedic correlates • Deductive arguments provided • Connections to related sub-topics explicit

5.4 Stage 2: procedures for producing data - specialisation of learners' mathematical thought

With respect to the clinical interviews of learners on the basis of the tests conducted by the teachers, all the procedures entailed in Stage 1 barring step 1 (segmenting lessons) and step 2 (classifying mathematics problems) are used to generate descriptions of the mathematics constituted by the learners. As discussed above, the primary data production process involves generating descriptions of the computational activity of learners. The descriptions of the computational activity serve as the basis for generating secondary data which involves describing (1) the realised content; (2) the nature of the regulation of the mathematical activity of learners; (3) the computational performance of the learners in the interview context; and (4) orientation to mathematics of learners in the interview context.

5.5 Conclusion

This chapter elaborated the procedures for producing and analysing data from the information archive. The main concern was to demonstrate in detail how research information was transformed into data in order to answer the central research question framing the study. The chapter described two stages in the production and analysis of data. Stage 1, primarily using the video records of the observed lessons, is concerned with how evaluation functions at the level of the instructional discourse. Stage 2 focuses on the specialisation of learners' mathematical thought as evidenced in their mathematical work displayed in a test and in an interview context. The descriptions of the computational activity in the instructional discourse and in the tests and clinical interviews are used to read off what is realised as the content with respect to the announced topic and how the realised content is shaped by the regulation of mathematical activity. Figure 5.20 displays the network categories that will be used to generate a description of the realised content in the instructional discourse in each pedagogic context (Chapter 6) and a description of the realised content produced by learners when doing mathematical work independently of the teacher (Chapter 9).

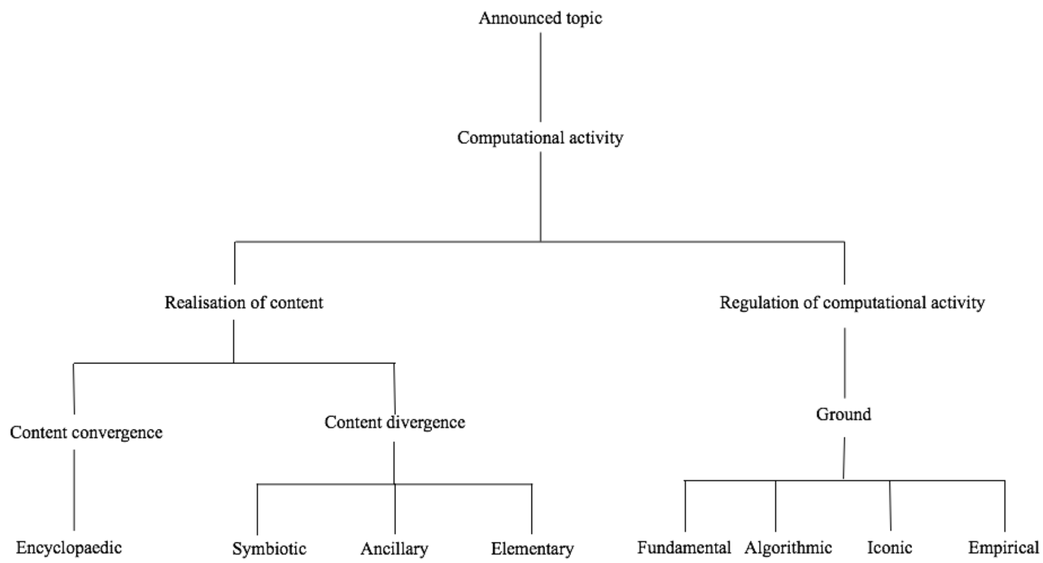


Figure 5.20. Network for describing realisation of content and regulation of mathematical activity

The realisation of content and regulation of mathematical activity are then used to read off the mathematical performance of the model learner and the orientation to mathematics implied by the computational activity evident in the instructional discourse, which is the principal concern of Chapter 7. An analysis of the marking of the test scripts which is the focus of Chapter 8 is used to support the analysis discussed in Chapter 7. Chapter 9 is dedicated to the analysis of mathematics test scripts and clinical interviews in order to reveal specialisation of learner's mathematical thought. Figure 5.21 displays the network of categories to be deployed in Chapters 7, 8 and 9.

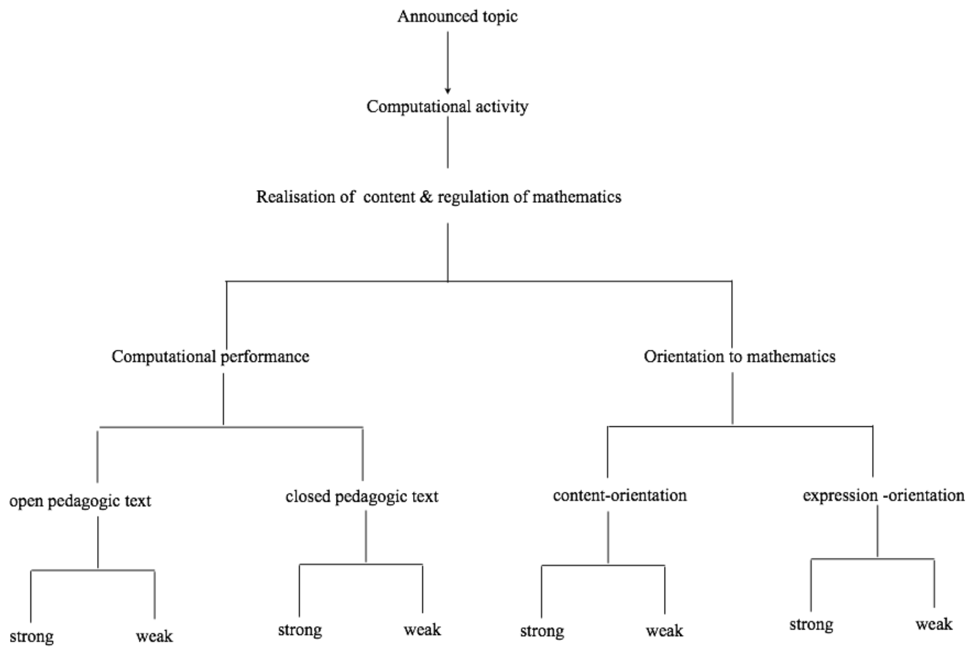


Figure 5.21. Network for describing computational performance and orientation to mathematics

The relation between the methodological resources, the specific research questions discussed in Chapter 1, and the chapters in which sub- research questions are addressed are shown in Table 5.11.

Table 5.11. Summary of methodological framework

Research sub-question	Information source(s)	Methodological resource(s)	Chapter
1. What does the computational activity reveal about the content realised with respect to Grade 10 mathematics topics in the instructional discourse in pedagogic contexts differentiated with respect to learners' social class membership?	Video-recorded lessons Lesson transcripts Curriculum resources Mathematics test	Davis' computational approach (Davis, 2010a, 2011a, 2013a; Davis & Ensor, 2018) *Realised content* (Eco, 1984b)	Chapter 6
2. How is the realisation of content in the instructional discourse regulated in these pedagogic contexts?	Video-recorded lessons Lesson transcripts Curriculum resources	*Ground* (Davis, 2010b, 2011b)	Chapter 6
3. What does the computational activity elaborated in the instructional discourse imply about the computational performance of the model learner constructed in pedagogic contexts differentiated with respect to learners' social class membership? 4. What orientations to mathematics are implied by the computational activity present in the instructional discourse in pedagogic contexts that differ with respect to the social class membership of learners?	Video-recorded lessons Lesson transcripts Curriculum resources Mathematics test	*Open* and *closed texts* and the *Model Reader* (Eco, 1984a) *Open* and *closed pedagogic texts* and the *Model Learner* (Davis, 2011b; Jaffer, 2011a) *Expression orientation* and *content-orientation* (Davis, 2011b; Lotman, 1990)	Chapter 7 and 8

5. What does learners' computational activity imply about their computational performance in pedagogic contexts that differ with respect to their social class membership? 6. What does learners' computational activity imply about their orientation to mathematics in pedagogic contexts that differ with respect to their social class membership?	Mathematics tests scripts Clinical interviews with selected learners	Davis' computational approach Davis, 2010a, 2011a, 2013a; Davis & Ensor, 2018) *Realised content* (Eco, 1984b) *Ground* (Davis, 2010b, 2011b) *Open* and *closed texts* and the *Model Reader* (Eco, 1984a) *Open* and *closed pedagogic texts* and the *Model Learner* (Davis, 2011b; Jaffer, 2011a) *Expression orientation* and *content-orientation* (Davis, 2011b; Lotman, 1990)	Chapter 9

Chapter 6

Realised content in the instructional discourse: recontextualisation and distribution of mathematics

6.1 Introduction

Having situated the study empirically, theoretically and methodologically in the preceding chapters, the following four chapters present the production and analysis of data. In this chapter and the next, I discuss the results of the data production and analysis of the instructional discourse instantiated in the observed lessons using the analytic framework outlined in Chapter 5. The observed lessons comprise Grade 10 mathematics lessons taught by four teachers, two (Sara and Jada) at Prestige College and two (Maya and Jono) at Evergreen High. Prestige College is populated by learners from upper-middle-class/elite families and Evergreen High is populated by learners from working-class families. Each teacher was observed and video-recorded teaching three consecutive lessons each, amounting to 12 lessons in total under consideration in this chapter. As discussed in Chapter 4, each lesson was video-recorded using two cameras, one focusing on the teacher and the other on the learners. The transcripts and video-recordings together with the analytic framework discussed in Chapter 5 were used to generate detailed analyses of each lesson. A transcript of one lesson (see Appendix 6.1) and the detailed analysis of that lesson (see Appendix 6.2) serve to illustrate how the analysis was carried out. Summaries of the analysis of each set of lessons for each pedagogic context can be found in Appendix 6.3 (Sara), Appendix 6.4 (Jada), Appendix 6.5 (Maya) and Appendix 6.6 (Jono).

This chapter addresses the first two sub-questions outlined in Chapter 1. The first sub-question concerns what comes to be realised as the content associated with the announced topic(s) in the observed lessons in each pedagogic context. The second question focuses on how the realised content comes to be shaped in the way that it does by considering the forms of regulation of mathematical activity in the observed lessons. Recall that evaluation is central to pedagogy. Methodologically, descriptions of the inner workings of evaluation are achieved by examining computational activity of teachers and their learners, that is the recognition and realisation rules used by teachers and students in the production of mathematics in pedagogic situations. What is realised as the content associated with announced topics in the instructional discourse is read off from a description of the computational activity displayed by teachers and their learners. As such the realised content is a product of evaluation. The analysis of the recognition and realisation rules employed in the instructional discourse is used to read off the content realised in relation to announced topics. The realised content emerging in the pedagogic context is compared with the content typically associated with the announced topic in the Mathematics encyclopaedia as a means of gauging the recontextualisation of content in the transformation of knowledge into pedagogic communication and to ascertain the distribution of knowledge along social class lines.

I begin the chapter with an overview of the lessons in order to provide a context for the results of my analysis and discussion of these results, outlined in this chapter and the following three chapters.

6.2 Overview of lessons

6.2.1 Use of pedagogic time in general

As discussed in Chapter 3, specialisation of time is a component of evaluation (Bernstein, 2000). Furthermore, time spent on the teaching and learning of content has been identified as the most important factor implicated in learner success or failure on standard tests and examinations (see for example, Carnoy et al., 2011; Reeves and Muller, 2005; Smith, 1998; Smith, Smith and Bryk, 1998). I provide an overview of the use of pedagogic time allocated for the teaching and learning of mathematics across the three observed lessons in each pedagogic context. Comparing the use of pedagogic time across schools and teachers based solely on information from the three observed lessons may not accurately reflect the amount of time spent on mathematics over longer periods such as the course of a year. However, although the current study cannot make generalised claims about the use of pedagogic time, the data does provide a picture of the specialisation of pedagogic time for the period of the observed lessons.

Figure 6.1 shows the distribution of pedagogic time in terms of three categories of activity: (1) *mathematical activity* (MA) which refers to time used for teaching and learning mathematics content; (2) *pedagogic organisation* (PO) which includes time used for organising a lesson such as grouping learners for a particular task, distributing worksheets used to solve problems, discussing the results of a test or learners copying notes or problems from the board; and (3) *non-pedagogic activity* (NP) such as talking about a social function, disciplining a learner while other learners wait or discussing extra-mural activities. Chapter 5 detailed the criteria for the recognition and coding of these categories.

The scheduled lesson time at Prestige College was 45 minutes with a five-minute break between lessons to accommodate for movement of learners from one room to the next. The average time used for the lessons observed was about 48 minutes for Sara and 47 minutes for Jada. These teachers started their lessons as soon as most of the learners were present in the classroom rather than waiting for the actual start of the lesson. At Evergreen High the scheduled lesson time was 67 minutes. With the exception of Maya's second lesson which started later than the scheduled time because the class was held up by another teacher, most lessons were generally longer than the scheduled time, given the three-minute break between lessons for movement of learners from one room to the next. The use of pedagogic time in the observed lessons in the four pedagogic contexts is shown in Figure 6.1.

All the teachers, with the exception of Maya, spent minimal time on activities that were not related to the teaching and learning of school mathematics, with the teachers at Prestige College not using any time for non-pedagogic activity. It should be noted that beyond the 13,4% of lesson time used by Maya on activities unrelated to the teaching and learning of school mathematics, she used some of the time coded as mathematical time for organisational tasks not related to the lesson. For example, she started each lesson with a task referred to as PDN ("please do now") in the form of a mini-test which learners worked on independently. In most cases, Maya used this time to do organisational tasks unrelated to the lesson. She indicated that she used the PDN to settle learners down into working and to provide opportunities for them to assess whether they understood the content. Her engagement with learners and with mathematical activity during this time was minimal and when she did engage learners it was often to discipline them.

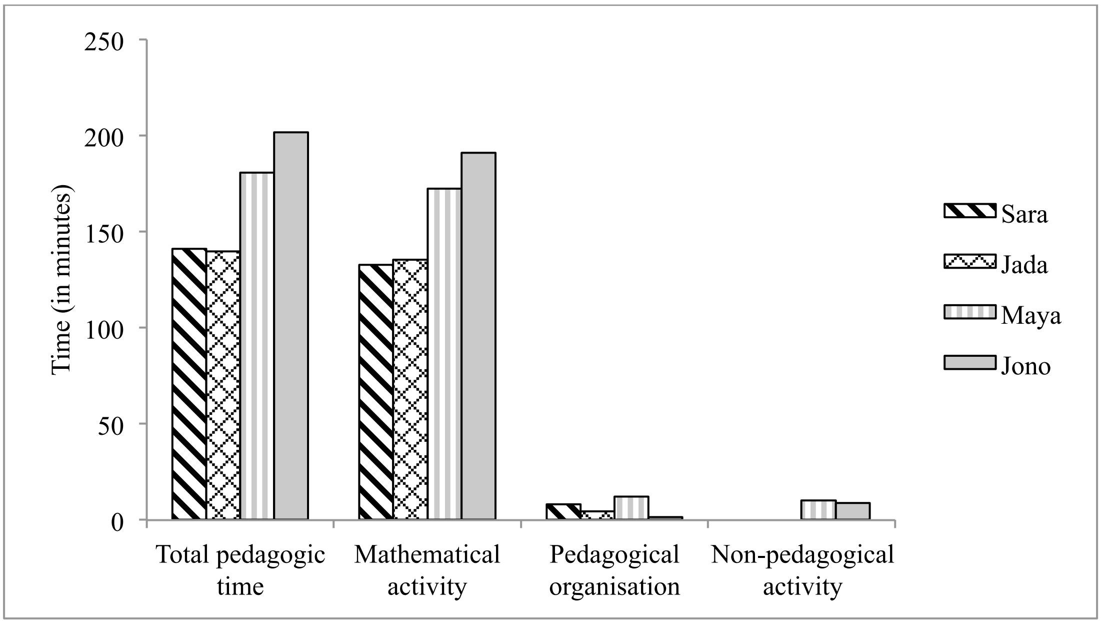


Figure 6.1. Use of pedagogic time during the observed lessons

From Figure 6.1, we note that more time was spent on mathematical activity by teachers at Evergreen High than by teachers at Prestige College, yet Prestige College produced better results than Evergreen High on NSC Mathematics examinations. It should be noted, however, that Prestige College teachers expected learners to do a substantial amount of work outside of class time. Their learners were provided with a set of worksheets which they were expected to work on independently of the teacher. The teachers made solutions of worksheets available and encouraged learners to approach them if they were not coping with the worksheets. In addition, learners were tasked to submit a tutorial (see Appendix 6.7) on the topic in preparation for a test which took place about two weeks after the observed lessons. Both teachers at Prestige College and Jono at Evergreen High assigned homework tasks to learners during the period of observation. Homework in Maya's class, however, was only assigned once during the period of observation when learners did not complete the classwork exercises during the assigned time in the last observed lesson. Thus, although Evergreen High teachers spent more time on mathematical activity than Prestige College during mathematics lessons, learners at Prestige College were expected to spend more independent time on mathematics than their counterparts at Evergreen High. Learners at Evergreen High also attended Saturday morning sessions which focused on mathematics every alternate week.

The pertinent point emerging from the distribution of pedagogic time represented in Figure 6.1 is the amount of time spent by all the teachers on mathematical activity, ranging from 86,6% to 96,9% of pedagogic time, which suggests that the difference, then, between the schools and teachers appears not to be so much about the time spent on school mathematics but perhaps on what comes to be realised as the mathematics content that is taught and learnt, which implies differences in how evaluation functions in pedagogic contexts.

6.2.2 Use of pedagogic time for mathematical activity

Mathematical activity was disaggregated into two activity types, exposition and classwork, each of which are discussed in detail in Chapter 5. The key factor distinguishing mathematical activity types from each other is the nature of the interaction between the teacher and learners, that is whether the interaction is public or private. Exposition involves the teacher and learners engaging publicly through speech, writing or gesture, with the intent that all such public interactions are available to everyone in the classroom. Classwork activity, on the other hand, is dominated by private interactions between the teacher and a learner or groups of learners or amongst groups of learners. The teacher may occasionally address the class publicly during classwork activity but this type of activity predominantly involves learners completing mathematics tasks individually, in pairs or in small groups and the teacher assisting learners when called upon to do so or when s/he observes that a learner or group of learners requires assistance (U.S. Department of Education, 2003).

The proportion of mathematical activity time, during the observed lessons, spent on each type of activity across the four pedagogic contexts is shown in Figure 6.2. We see that both Jada and Jono spent a greater proportion of their mathematical activity time on exposition. Sara and Maya, on the other hand, spent a greater proportion of their mathematical activity time on classwork activities. However, classwork time was dealt with very differently by the two teachers. Sara spent all the classwork time engaging with individual learners or groups of learners about the mathematical task under discussion whereas in a few lesson segments such as the PDN segments Maya tended to focus on organisational tasks with minimal attention given to the mathematical task the learners were working on. She often observed learners' work without commenting or she merely alerted learners to errors without indicating what the error was or assisting learners to correct the error.

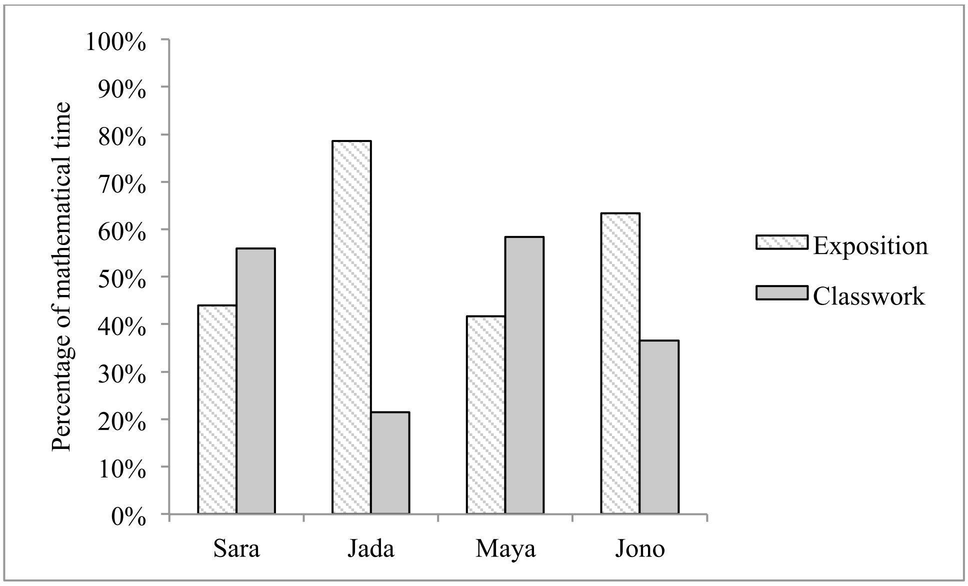


Figure 6.2. Use of mathematical activity time during the observed lessons by activity type

Although almost 80% of mathematical activity time involved exposition in Jada's lessons, these segments were at times interspersed with instructions to learners to investigate mathematical ideas by using the software *Geogebra* and *Autograph*. So learners were engaged in more than just listening to the teacher and copying notes from the board during the exposition segments as learners in Maya and Jono's lessons during exposition segments. Furthermore, as discussed above, Jada's learners were expected to work independently on worksheets and a tutorial outside of pedagogic time. The time spent on activity types, however, does not provide any insight into the computational activity of teachers and their learners.

6.2.3 Announced topics

Announced topics were inferred from what teachers announced as the topic orally or what they wrote on the board as topic headings or from the headings of worksheets provided to learners. With respect to Prestige College, the topic was read off the school's term planner for Grade 10. The four teachers dealt with different mathematics topics in the observed lessons as reflected in Table 6.1. Since the realisation of announced topics is discussed in terms of the computational activity, a comparison of the realised content is made possible despite differences at the level of the announced topics. On the other hand, even if topic names are the same, the content realised in relation to the topic name may differ across pedagogic contexts. This point is elaborated on in Section 6.3.

The announced topics in the observed lessons of the four teachers are all related to the CAPS curriculum topic, functions, which according to the CAPS pace setter is scheduled for teaching in the second term of the school year (Department of Basic Education, 2011, p. 17). All the lessons were observed in the third term. Prestige College teachers were observed before Evergreen High teachers.

Table 6.1. Announced topics in the four pedagogic contexts

School	**Teacher**	**Announced topic(s)**
Prestige College	Sara	1. Parabola
Prestige College	Jada	1. Parabola 2. Exponential functions (not completed in the observed lessons) 3. Hyperbola (not completed in the observed lessons)
Evergreen High	Maya	1. Sketching functions: linear, parabola, exponential function, hyperbola (revision) 2. Calculating equations of functions: linear function (revision) and parabola
Evergreen High	Jono	1. Domain and range of functions: linear function, quadratic function, hyperbolic function and exponential function (trigonometric functions introduced but not completed in the observed lessons)

Sara covered the topic, parabola, focusing on one problem type, sketching the parabola. Jada also dealt with the topic parabola but focused primarily on three types of problems: sketching the parabola, calculating the equation of parabolas and reflecting parabolas in the x-axis and the y-axis. Both Sara and Jada combined the topic parabola with previously encountered topics: (1) straight line function, specifically calculating the equation of a straight line and (2) graph interpretation which included calculating horizontal and vertical lengths, and points of intersection. In addition, Sara included graphical inequalities as part of graph interpretation. Jada started the topics, exponential functions and hyperbola, in the third lesson but she did not complete these topics during the period of observation. From the test administered by Prestige College teachers, it appears that the topics covered during the observed lessons were in fact sub-topics of a much larger topic, functions.

Maya dealt with two topics as indicated in Table 6.1 and Jono focused on one topic, the domain and range of functions. Although he introduced the sub-topic, domain and range of trigonometric functions, during the last observed lesson, the period ended before he was able to complete the sub-topic. Maya was the only teacher who spent most of the first lesson, barring the last six minutes, on revising content dealt with previously. The other teachers focused on new topics during the three observed lessons, revising content when and if required by learners.

6.2.4 Mathematics problems

The mathematics problems used to elaborate the content of the announced topics were either used as worked examples by teachers to illustrate procedures for solving classes of problems or given to learners as classwork or homework exercises to practise the application of procedures. As discussed in Chapter 5 mathematics problems were classified as *mono-topic* or *multi-topic* mathematics problems, where the former indexes one announced topic (e.g. "sketch the parabola") and the latter entails more than one announced topics. A summary of the number of mathematics problems and the nature of these problems is provided in Table 6.2.

Table 6.2. Summary of problems and problem types across the four pedagogic contexts[65]

School	Teacher	# mathematics problems	Mono-topic problems	Multi-topic problems	Time per problem (minutes)
Prestige College	Sara	25	22	3	5
Prestige College	Jada	16	12	3	9
Evergreen High	Maya	12	12	0	14
Evergreen High	Jono	21	21	0	7

From Table 6.2, we note that Sara dealt with the most mathematics problems and Maya the fewest during the observed lessons. The number of problems used by Sara and Jada excludes the mathematics problems contained in the *Tutorial* on *graphs* (Appendix 6.7) which was handed out to learners before the observed lessons and which was submitted by learners after the observed lessons and before writing the *Common Test Functions 2012* (Appendix 6.8) Since the mathematics problems differed across pedagogic contexts, comparisons in terms of time spent per problem is difficult. However, Maya appeared to exhibit the slowest

[65] Four mathematics problems given to Maya's learners to work on in the last session were not completed and were assigned as homework since many learners opted not to work on the task during the lesson.

pacing across the four pedagogic contexts, spending on average 14 minutes per mathematics problem. A notable difference between the upper-middle-class/elite school and the working-class school lies in the nature of the mathematics problems presented to learners.

6.2.4.1 Mathematics problems used by Sara and Jada

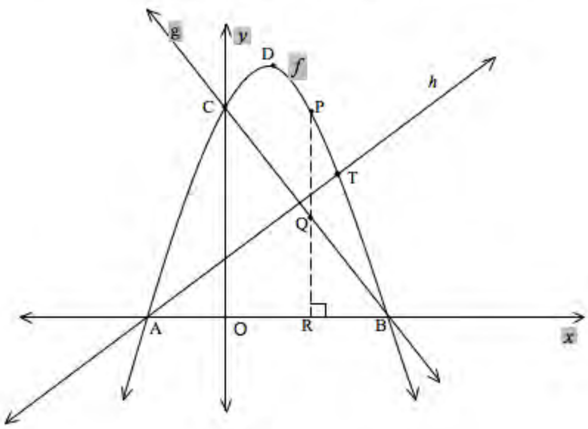
Prestige College teachers provided learners with mono-topic mathematics problems such as “sketch the function” or “reflect the function in the *x*-axis” when introducing new procedures to learners and as initial practice exercises. However, they consolidated the newly introduced procedures by using multi-topic mathematics problems, which required the newly introduced procedures as well as previously encountered procedures (see Figure 6.3), thus connecting the new topic with prior topics. The *Parabola worksheet* (Appendix 6.9) provided to learners in Sara’s and Jada’s lessons contained three such multi-topic mathematics problems that served to prepare learners for the *Tutorial on graphs* as well as the *Common test Functions 2012* and the final examination. The mathematics problems contained in the *Parabola worksheet* are comparable to the problems in the tutorial, test and final examination, all of which comprise multi-topic mathematics problems.

Figure 6.3 provides an example of a multi-topic mathematics problem (labelled Question 1) which consists of five final computational objects: (1) length of AB; (1.1), (2) turning point of parabola (1.2 and 1.3); (3) equation of straight line (1.4); (4) length of PQ (1.5); and (5) coordinates of point T (1.6).

Parabola Worksheet

Question 1

In the accompanying sketch, which is not drawn to scale, *f* and *g* intersect at points B and C.
f and *h* intersect at points A and T. $f(x) = -x^2 + 2x + 8$ and $h(x) = x + 2$.
The parabola intersects the x-axis at points A and B and the *y*-axis at C.



1.1	Determine the length of AB.	(4)
1.2	Write down the equation of the axis of symmetry of *f*.	(1)
1.3	Determine the co-ordinates of D, the turning point of *f*.	(2)
1.4	Determine the equation of g in the form $y = mx + c$.	(3)
1.5	Calculate the length of PQ if OR = 2 units.	(3)
1.6	Determine, by calculation, the co-ordinates of point T.	(6)

Figure 6.3. Question 1 from *Parabola worksheet* used by Sara and Jada

Question 1 of the *Parabola worksheet* (see Figure 6.3) used by Sara and Jada consists of problems related to the topic parabola (1.1–1.3) together with a problem related to the topic linear functions (1.4) and general graph problems, such as calculating vertical lengths (1.5) and points of intersections (1.6). As such, the connections between topics are encoded into the mathematics problems provided to Prestige College learners. In this way, Prestige College teachers achieved *inter-topic connectivity* at the level of the mathematics problems presented to learners.

Furthermore, the *Parabola worksheet* used by Prestige College teachers contained mathematics problems which did not directly name the procedure required to solve the problem. Thus learners were expected to analyse the problem statement in order to decipher the appropriate procedure required. For example, Problem 1.6, which involves calculating the coordinates of point T, is a problem where learners are first required to identify that the point T is a point of intersection of the parabola and line and to then select an appropriate procedure to solve the problem. Similarly, Problem 1.1, which involves calculating the length of AB, entails identifying that A and B are the x-intercepts before calculating the horizontal length. Thus Prestige College learners were provided with opportunities that required analysis of problems in order to identify the required procedure.

6.2.4.2 Mathematics problems used by Maya and Jono

All the mathematics problems worked on in Maya's and Jono's lessons were mono-topic mathematics problems. In Maya's lessons, mathematics problems could be grouped into two clusters of problems: (1) "sketch the graph of the function" and (2) "calculate the equation of the function".

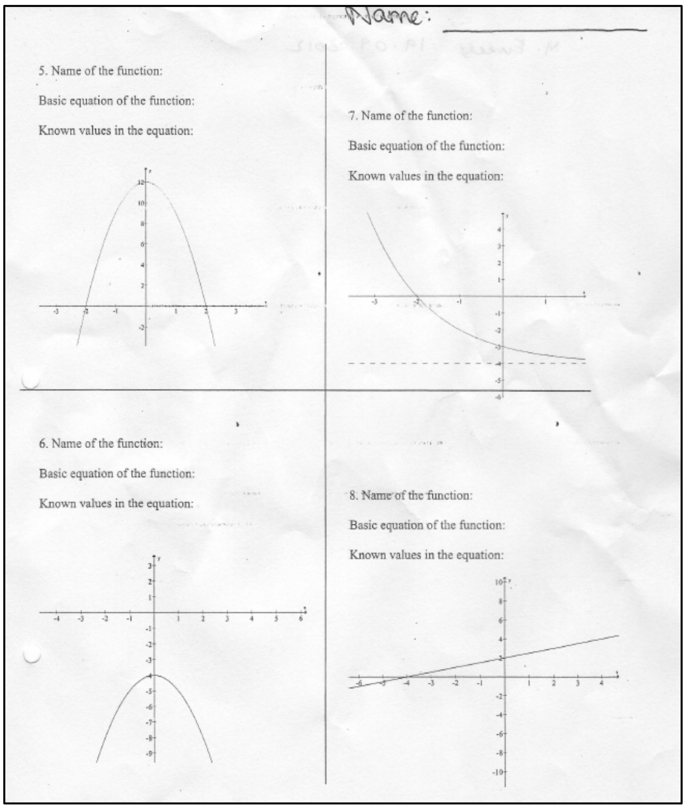


Figure 6.4. Worksheet on finding equations of functions used by Maya

Figure 6.4 shows the worksheet used by Maya in the last observed lesson. Although not stated on the worksheet itself, the instruction by Maya was to calculate the equation of the given function. All the problems contained in the worksheet are mono-topic mathematics problems because they index one announced topic, equations of functions. Note that Problem 6 contains insufficient information to calculate the equation of the function. However, the omission was not picked up by Maya during the observed lesson.

In Jono's lessons, the mathematics problems focused on calculating the domain and range of one of four function types (linear, quadratic, hyperbolic and exponential), expressed in interval and set-builder notation. The worksheet used during the observed lessons contained four sections (see Appendix 6.10) with each section focusing on a different function. Section A of the worksheet used by Jono is shown in Figure 6.5. All the problems contained in the worksheet are mono-topic mathematics problems.

A. Find the **domain** and **range** of following linear functions (write it as **interval** and **set notations**):

1. y = x+1
2. H(x) = 2x-1
3. 3y-2x = 6
4. X= 3y
5. $\frac{x}{2} - \frac{y}{3} = 1$

Figure 6.5. Part A of the worksheet on domain and range used in Jono's lessons

The function type in Jono's worksheet is stated in the problem statement, which means that it was not required of learners to deduce the type of function from the equation provided. Therefore, analysis of the problem statement was absent.

The absence of multi-topic mathematics problems during the observed lessons at Evergreen High indicates that topics were treated in isolation by Evergreen High teachers thus resulting in a lack of *inter-topic connectivity* in the observed lessons. As such, Evergreen High learners were left to make connections between topics independently of the teacher. It could be argued that synthesis of school mathematics topics into a coherent whole was made much harder for the working-class learners than the upper-middle-class/elite learners

Furthermore, all the mathematics problems that Evergreen High learners encountered as classwork or homework exercises and tests were of the type which explicitly announced the procedure required to solve the problem. While Prestige College learners were provided with opportunities that required analysis of problems in order to identify the required procedure, such opportunities for analysis of mathematics problems were absent for Evergreen High learners during the observed lessons.

6.3 Describing recognition and realisation rules in terms of computational activity

Recall that examining the computational activity of teachers and their learners provides insights into the inner workings of evaluation. The recognition and realisation rules used by teachers and learners when doing mathematics in pedagogic situations are central to inferring the content realised with respect to announced topics. In order to construct descriptions of the realised content in relation to the announced topic(s) in each pedagogic context, my analysis focused on examining (1) definitions and/or descriptions of mathematics terms, objects and processes (2) the propositions that underpin the procedures used by teachers and learners; (3) procedures used by teachers and learners to solve classes of mathematics problems; and (4) the domains, codomains and operations recruited by teachers and learners to perform computations involved in the procedures discussed in (3).

In addition, I compare the content realised under a topic name with the content associated with that topic in the Mathematics encyclopaedia in order to establish whether content substitution has occurred and what that substitution entails. As discussed in Chapter 3, content substitution occurs when the realised content associated with an announced topic differs from the content associated with the topic name from the point of view of the Mathematics encyclopaedia.

6.3.1 Mathematical definitions and descriptions

Teaching and learning school mathematics must at times entail the introduction of mathematical terms or explanations of mathematical terms that learners may or may not have not dealt with before. On such occasions, teachers may explain mathematical terms by providing descriptions which may take the form of formal definitions referred to as *definite descriptions* or informal clarification of mathematical terms referred to as *auxiliary descriptions*. The distinction between definite descriptions and auxiliary descriptions is discussed in Chapter 3 and Chapter 5. *Auxiliary descriptions* have been categorised into *iconic auxiliary descriptions* and *non-iconic auxiliary descriptions*. Table 6.3 shows the distribution of description types across the four pedagogic contexts.

As discussed in Chapter 5, we expect that formal definitions may not be a dominant feature of the computational activity of teachers and learners since CAPS downplays formal definitions. In fact, the only references to definitions in CAPS are the definition of a function which CAPS specifies should be dealt with in Grade 12 (Department of Basic Education, 2011, p. 24) and the definition of a logarithm to be addressed in Grade 11 (Department of Basic Education, 2011, p. 41). This hypothesis is confirmed by the data displayed in Table 6.3, which shows that none of the descriptions of mathematical terms offered by teachers are classified as definite descriptions. Thus, the descriptions provided by teachers do not form part of an axiomatic deductive system. Instead, their main purpose appears to be computational. In other words, descriptions of mathematical terms are required to perform one or more computations.

Table 6.3. Distribution of description types[66]

School	Teacher	Definite descriptions	Non-iconic auxiliary descriptions	Iconic auxiliary descriptions	Total auxiliary descriptions
Prestige College	Sara	0	5(4)	0	5
Prestige College	Jada	0	3(3)	1	4
Evergreen High	Maya	0	2(2)	0	2
Evergreen High	Jono	0	10(4)	5	15
Total		**0**	**20**	**6**	**26**

Recall that modifications of the features of a mathematical object shifts the category membership of the object. For example, Jono's description of a function as "a relationship between two variables" (S02T04L01: line 43) describes a relation but does not distinguish relations that are functions. So, his description alters the features of a function thus treating a function as though it is a natural kind rather than as a definite description of a mathematical object. So the data (see Table 6.3) suggests that all the teachers treat mathematical objects as though they are natural kinds since none of the descriptions offered by teachers are definite descriptions.

Of the 20 non-iconic auxiliary descriptions, 13 are in fact formulae of basic functions such as the general formula of a parabola ($y = ax^2 + bx + c$), or descriptions of symbols in formulae such as m represents the gradient and c represents the y-intercept in the general formula $y = mx + c$. The number of formulae used by teachers are indicated in the brackets in the column for non-iconic auxiliary descriptions in Table 6.3. General equations of functions are required either to generate a sketch of a function, calculate the specific equation of a given function or to identify equations as specific function types in order to calculate the domain and range of a function. In Sara's lessons, the only non-iconic auxiliary description that was not a general formula was her description of a function "something that for every *x* there is one corresponding .. *y* value" (S01T01L01: lines 39-41). Her description of a function is context-dependent because it implies that x always represents the argument of a function and y the value of a function. However, x can only represent the argument of a function and y the value of a function if defined as such. Therefore her description of a function does not function as a definite description and stands in place of a definite description such as the one offered by Bronshtein, Semendyayev, Musiol, and Muehlig (2007):

> If *x* and *y* are two variable quantities, and if there is a rule which assigns a unique value of *y* to a given value of *x*, then we call *y* a function of *x* and we use the notation *f*(*x*). The variable *x* is called the independent variable or the argument of the function. The values of *x*, to which the values of *y* are assigned, form the domain *D* of the function $f(x)$. The variable *y* is called the dependent variable, the values of *y* form the range *W* of the function $f(x)$. (Bronshtein et al., 2007, p. 47)

[66] The brackets refer to formulae of functions such as the formula for a parabola is $y = ax^2 + bx + c$.

All the non-iconic auxiliary descriptions used by Jada and Maya were general formulae and four non-iconic auxiliary descriptions used by Jono were formulae.

Iconic auxiliary descriptions refer to descriptions which focus on imagistic features of a text or focus on what the text looks like. For example, the description of an asymptote provided by Jada: “An asymptote is a line that you can’t cross. It’s impossible to cross the line” (S01T02L03: line 270) generates an image of a line as a fence or barrier. Hence, this description was characterised as an iconic auxiliary description. Of the 26 auxiliary descriptions offered by the four teachers, 23,1% are iconic in nature as shown in Table 6.3. Both Sara and Maya did not use iconic auxiliary descriptions.

The key point highlighted by the analysis of descriptions used in the pedagogic contexts is that formal definitions of mathematical objects and processes do not feature as part of the computational activity of teachers and learners in all the pedagogic contexts and are replaced with descriptions of mathematical terms, which function as auxiliary descriptions rather than as definite descriptions. Thus, mathematical objects are treated as though they are natural kind objects rather than as objects which have definite descriptions.

The absence of definite descriptions from the computational activity of teachers and learners is not surprising at all given the diminished emphasis placed by CAPS on formal definitions. This finding, the absence of formal definitions from the computational activity of teachers and learners, supports that of Davis and Johnson (2007), Chitsike (2011b) and Arendse (2013). However, it should be noted that those studies focused only on schools with learner populations drawn from working-class families whereas my study includes a school with an upper-middle-class/elite learner population. This finding suggests that perhaps the absence of formal definitions from the computational activity of teachers and learners is a general feature of South African schooling and as such requires further investigation.

6.3.2 Mathematics propositions

As discussed in Chapter 5, propositions authorise the mathematical activity of individuals since they serve as rationales or justifications for mathematical computations. I distinguish between two types of propositions, *encyclopaedic propositions* and *auxiliary propositions.* Propositions that are found in the Mathematics encyclopaedia are referred to as *encyclopaedic propositions.* For example, the proposition “if the parabola $y = ax^2 + bx + c$ has a minimum value then $a > 0$” is an example of an encyclopaedic proposition.

Encyclopaedic propositions are at times replaced by teachers and learners with other propositions which do the work of encyclopaedic propositions. Such propositions are referred to as *auxiliary propositions.* The proposition “if the parabola $y = ax^2 + bx + c$ has a “smiley face” then a is positive” is an example of an auxiliary proposition used by a teacher and stands in place of the encyclopaedic proposition described above. A structure preservation map which illustrates the substitution of the encyclopaedic proposition with the auxiliary proposition is shown in Figure 6.6. Note that structure preservation is enabled by the auxiliary operation *imaging* which maps the notion of a parabola’s minimum to a “smiley-faced” image and the notion of $a > 0$ to a “plus” sign.

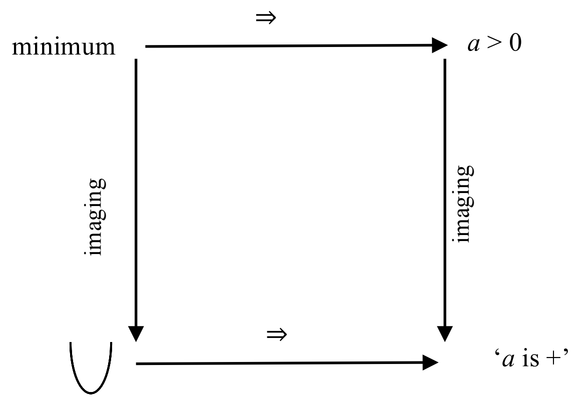


Figure 6.6. A mapping from encyclopaedic proposition to an auxiliary proposition

Auxiliary propositions are further categorised into *iconic auxiliary propositions* and *non-iconic auxiliary propositions*. Auxiliary propositions featured as part of the computational activity observed in all four pedagogic contexts. Those propositions that focus on imagistic features are referred to as iconic auxiliary propositions. The proposition "if the parabola $y = ax^2 + bx + c$ has a "smiley face" then a is positive" focuses on an imagistic feature of the parabola, the shape of the parabola and is therefore classified as an iconic auxiliary proposition.

Propositions that fall outside of the Mathematics encyclopaedia but do not draw attention to iconic features are referred to as *non-iconic propositions.* Jono was the only teacher who used non-iconic propositions. When discussing the notion of the domain of a function, Jono stated that if the variable x is not assigned anything (e.g. loaves of bread), then the domain is the set of real numbers (see S02T04LO1, line 328) and "(if) there is no number that we can put here to make this undefined (then) our x can take any … real number" (S02T04LO1, line 318 and 324).

Table 6.4. Distribution of propositions used by teachers

Teacher	Total	Encyclopaedic	Auxiliary	Auxiliary Iconic	Auxiliary Non- iconic	Auxiliary without encyclopaedic correlate	Auxiliary with encyclopaedic correlate
Sara	17	8	9	9	0	4	5
Jada	15	6	9	9	0	9	0
Maya	9	0	9	9	0	9	0
Jono	6	0	6	4	2	6	0
Total	**47**	**14**	**33**	**31**	**2**	**28**	**5**

Encyclopaedic propositions were used by both Prestige College teachers (Sara and Jada) but were absent from the computational activity of both teachers at Evergreen High (Maya and Jono) (see Table 6.4). Sara is the only teacher who used five of the eight auxiliary propositions interchangeably with their encyclopaedic correlates as reflected in Table 6.4. For example, she used the encyclopaedic proposition "if $a > 0$ then the parabola $y = ax^2 + bx + c$ has a minimum value" (S01T01L02, line 102) interchangeably with the auxiliary propositions "if a is positive then the arms of the parabola goes up" (S01T01L02, line 204) and "if a is positive then the parabola has a smiley face" (S01T01L02, line 375). Evergreen High teachers relied only on auxiliary propositions. For all teachers, the auxiliary propositions were predominantly iconic in nature.

Table 6.5. Computational rules used by teachers

Teacher	Computational rules	Encyclopaedic propositions
Sara Jada	For trinomial factorisation, if the last term is + then the signs in the brackets are the same (x + □)(x + □) or (x - □)(x - □) For trinomial factorisation, if the last term is negative then the signs in the brackets are opposite (x + □)(x - □) or (x - □)(x + □)	Viete's theorem for factorising quadratic trinomials
Jada	For the trinomial $x^2 + bx + c$, if c is negative and b is negative then the "bigger factor" of $\lvert c \rvert$ is assigned to the minus factorisation bracket (the brackets are $(\ -\ \ )(\ +\ \ )$	Viete's theorem for factorising quadratic trinomials
Sara	A minus times a minus is a plus or minus minus is a plus.	Integer multiplication: If $a, b \in \mathbb{Z}$ and $a, b < 0$ then $a \times b > 0$

In addition to the substitution of encyclopaedic propositions with auxiliary propositions, teachers at times substituted mathematical propositions with computational rules or operations. See Table 6.5 for a list of computational rules. For example, Sara used the computational rule "minus minus is a plus" and "a minus times a minus is a plus" which substitutes for the integer multiplication proposition, if $a, b \in \mathbb{Z}$ and $a, b < 0$ then $a \times b > 0$.

Figure 6.7 shows that three auxiliary operations are involved in the structure preservation map which relates multiplication of the integers to "timesing" over characters. Firstly, the number -1 is transformed into a character string $/-1/$ through the auxiliary operation *string*. Then the character string $/-1/$ is split into the characters $/-/$ and $/1/$ through the auxiliary operation *sunder*. Finally, $/-/$ is selected from the list of characters using the auxiliary operation ALT. The structure preservation map illustrates the substitution of encyclopaedic content with auxiliary content.

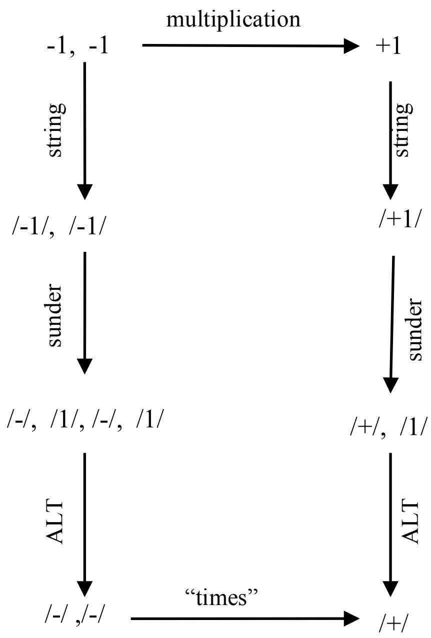


Figure 6.7. A mapping of multiplication over the reals to “timesing” over characters

Both Sara and Jada used a set of computational rules when factorising quadratic trinomials to generate the symbols /+/ or /-/ required to populate the character distribution matrix $(m \square n)(p \square q)$, where $\square$ represents the spaces for the characters /+/ or /-/ and m, n, p, q are whole numbers. These computational rules substitute for Viete’s theorem. Jada spent some time recapping these rules when she re-explained the procedure for factorising “trinomials” as part of her procedure for calculating x-intercepts of a parabola. Sara, on the other hand, factorised a quadratic trinomial (discussed in detail in Chapter 5) without explicitly elaborating these rules. So, it seems as though Sara expected her learners to be familiar and competent in factorising quadratic trinomials whereas Jada retaught the procedure for factorising trinomials several times during the observed lessons. The difference between Sara and Jada with respect to previously encountered procedures may be related to who their learners are. Sara’s learners were placed in Set 2 of Grade 10 whereas Jada’s learners were in Set 3.

In summary, encyclopaedic propositions were substituted with auxiliary propositions, computational rules or auxiliary operations by all four teachers, to a lesser extent by Prestige College teachers than by Evergreen High teachers and to a greater extent by Jada than by Sara in Prestige College since Sara at times used auxiliary propositions together or interchangeably with their encyclopaedic correlates.

6.3.3 Procedures

In this section, I discuss the details of the procedures employed in each pedagogic context. Table 6.6 displays the number of procedures elaborated in each pedagogic context in relation to the number of announced topics, number of procedures and number of problems.

Table 6.6. Procedures in relation to announced topics and problems

Teacher	# Announced topics	# Procedures	# Problems
Sara	1	10	25
Jada	3 (2 incomplete)	11	16
Maya	2 (1 revision)	6	12
Jono	1	8	21

Table 6.6 shows that the number of procedures dealt with by Sara and Jada exceeds the number of procedures elaborated by Maya and Jono during the observed lessons. More details regarding the announced topics, procedures and problem types used in each pedagogic context are displayed in Tables 6.7-6.10.

Table 6.7 illustrates that Sara focused on one announced topic, parabola, which involved five procedures used to solve the mathematical problem, sketch the parabola. Sara connected the parabola topic to the broader topic functions by integrating the parabola with prior topics, the straight-line function and graphical interpretations, indicated by problem types 2-5. For Sara's learners, procedures for problem types 1 and 6 were new procedures whereas the other procedures were encountered in the context of linear functions, dealt with earlier in the term according to Prestige College's Grade 10 term planner. As discussed earlier, Sara established *inter-topic connectivity* by integrating the new topic, parabola, with topics encountered earlier. In total, Sara elaborated 10 procedures for six problem types when solving 25 mathematics problems.

Table 6.7. Sara's topic, problem types and procedures

Announced topic: Parabola		
Problem type	**Procedure**	**# problems**
1. Sketch parabola	**PR1.** Sketching parabola given the form $y = a(x-p)^2 + q$ (turning point method 1)	11 mono-topic
	PR2. Sketching parabola given the form $y = a(x-p)^2 + q$ (turning point method 2)	
	PR3. Sketching parabola given the form $y = ax^2 + bx + c$ (root method)	6 mono-topic
	PR4. Sketching parabola given the form $y = ax^2 + bx + c$ (completing the square method)	4 mono-topic
	PR5. Sketching parabola given the form $y = ax^2 + bx + c$ (formula method)	1 mono-topic
Calculate turning point of parabola	Sub-procedures related to procedures 1 - 5	3 multi-topic
2. Calculate equation of straight line	**PR6**. Calculating equation of straight line	
3. Calculate horizontal length	**PR7** Calculating horizontal lengths	
4. Calculate vertical length	**PR8** Calculating vertical lengths	
5. Calculate points of intersection of two graphs	**PR9.** Calculating points of intersection	
6. Calculate graphical inequalities	**PR10.** Calculating graphical inequalities	

Sketching a parabola is the main problem type associated with the topic parabola and the only type of problem which has more than one procedure associated with it. Although Sara explicitly elaborated five procedures to calculate key points required to sketch a parabola, she enabled learners to construct other combinations of procedures by not restricting procedures to particular forms of the equation and by demonstrating the conversion of parabola equations from standard form to turning point form and vice versa. She explicitly stated that multiple methods could be used to solve the mathematics problem, sketch the parabola (S01T01L02: lines 498-505). Given that Sara constitutes the announced topic parabola as comprising one problem type, sketching the parabola, and that she establishes connections between the procedures for sketching parabola, she establishes *intra-topic connectivity*.

Jada, in contrast to Sara, dealt with three announced topics with only the announced topic, parabola, considered in depth. The other two announced topics were dealt with towards the end of the last observed lesson and hence not completed during the period of observation. Table 6.8 shows the announced topics in Jada's lessons which entailed 11 problem types employing 11 procedures for solving 16 mathematical problems. Sketching parabola is the only problem type for which Jada employed more than one procedure. However, in contrast to Sara, the procedures for sketching parabola elaborated by Jada were dependent on the form of the equation. PR1 is used if the equation is given in the form $y = a(x - p)^2 + q$ and PR2 is used when the equation is in the form $y = ax^2 + bx + c$. PR1 includes two different sub-procedures for calculating the turning point of the parabola, using the x-intercepts or using the formula ($x = \frac{-b}{2a}$). However, Jada's preferred method was to use the x-intercepts with the formula as a "double check method" (S01T02L02: line 69).

Table 6.8 Jada's topics, problem types and procedures

	Announced topic 1: Parabola	
Problem type	**Procedure**	**# problems**
1. Sketch parabola	**PR1:** Sketching parabola from $y = a(x - p)^2 + q$ **PR2:** Sketching parabola $y = ax^2 + bx + c$	3 mono-topic
2. Reflect parabola in x-axis	**PR3:** Reflecting parabola in the x-axis	2 mono-topic
3. Reflect parabola in y-axis	**PR4:** Reflecting parabola in the y-axis	2 mono-topic
4. Calculate equation of parabola	**PR5:** Calculate equation of parabola	1 mono-topic
Calculate turning points	Sub-procedures related to Procedures 1 and 2	2 mono-topic
Calculate key points of parabola	Sub-procedures related to Procedures 1 and 2	3 multi-topic
5. Calculate horizontal length	**PR6** Calculating horizontal lengths	
6. Calculate equation of straight line	**PR7:** Calculating equation of straight line	
7. Calculate vertical length	**PR8**: Calculating vertical lengths	
8. Calculate points of intersection of two graphs	**PR9:** Calculating points of intersection	
	Announced topic 2: Exponential functions	
9. Sketch exponential function	**P10:** Sketching exponential functions	1 mono-topic
10. Reflect exponential graphs in the x-axis	**PR11:** Reflecting exponential graphs in the x-axis	1 mono-topic
	Announced topic 3: Hyperbola	
11. Sketch Hyperbola	Topic started but not completed- no procedure elaborated	1 mono-topic

The procedure using the turning point form involves converting the turning point form equation to standard form for the purposes of calculating the intercepts even though it is possible to calculate the intercepts from the turning point form equation, which suggests regulation of the mathematical activity of the learner since there is no procedural necessity for converting the equation to standard form to calculate the intercepts. The difference in Sara's and Jada's approaches to the broader topic, functions is really one of sequencing. For example, Sara focused on sketching parabola, leaving calculating the equations of parabolas to a later stage. Jada included sketching parabolas together with reflecting parabolas and calculating the equation of parabolas but left converting the standard form equation to turning point form for Grade 11. Both teachers would be expected to cover the same topics by the end of Grade 10.

Maya covered two announced topics: (1) sketching functions; and (2) calculating equations of functions (see Tables 6.9). The announced topic, sketching functions, was taught prior to the observed lessons. In the first observed lessons, learners discussed a test written on the topic and groups were tasked to work on areas of weakness identified by the teacher. One group was assigned a task on sketching the parabola while another group worked on sketching hyperbola. Each group was given four problems to work on but only appeared to complete one. The tasks, which were taken in by the teacher to mark, were not returned to learners during the period of observation. Two groups were provided with a worksheet to explore calculating the equation of a parabola, a topic that they had not been taught yet because they had performed well on the test. The same worksheet was used in the last observed lesson.

Table 6.9. Maya's topics, tasks and procedures

Announced topic 1: Sketching functions		
Problem type	**Procedure**	**# problems**
1. Graph function/ Sketch	**PR1**: Sketching linear functions	1 mono-topic
	PR2: Sketching parabola	1 mono-topic
	PR3: Sketching hyperbola	1 mono-topic
Announced topic 2: Calculating equations of functions		
2. Calculate equation of function	**PR4**: Calculating equation of linear function (dual intercept)	2 mono-topic
	PR5: Calculating equation of parabola with "same" x-intercepts	2 mono-topic
	PR6: Calculating equation of parabola with "different" x-intercepts	3 mono-topic
	PR7: Calculating equation of parabola with no x-intercepts	1 mono-topic
	PR8: Calculating the equation of exponential function	1 mono-topic

During the observed lessons, Maya provided learners with12 problems, four of which were not completed in the last observed lesson and so were assigned as homework. Procedures, PR7 and PR8, were not discussed during the observed lessons but learners were required to do two of the problems involving PR7 and PR8 on the worksheet handed out in the last observed lesson. One of the problems involved a parabola with no x-intercepts. The teacher had not dealt with this type of problem in the observed lessons. Learners appeared to treat this problem as though it required PR6.

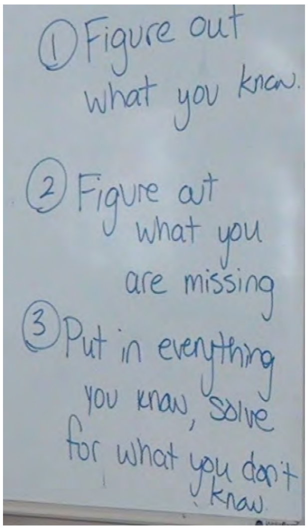


Figure 6.8. Heuristic used by Maya for solving "calculate the equation of a function" type

The problem type, sketching functions, involves three procedures. Each procedure differed depending on the function type. Similarly, the problem type, calculating equation of functions, comprised three procedures. Again the procedures were dependent on the function type as represented in the given sketch. Although three different methods of calculating the gradient of a linear function were mentioned, calculating the gradient by "counting spaces" or the formula were not elaborated. Instead, Maya focused on the dual intercept procedure indicating her preferred method for calculating the equation of a linear function. The announced topic, calculating equations of functions, on the whole is disconnected because procedures were dependent on the function type. It should be noted though that Maya used a general heuristic (shown in Figure 6.8) that served to connect the three procedures used to calculate equations of functions.

The third point of the heuristic which involved "putting in everything you know, solve for what you don't know" was used in a 'trial-and-error' manner. Maya encouraged learners to substitute points into an equation even though the same points were used to construct the equation, thus leading to an identity which did not help them to solve the problem.

We observe that Jono covered one announced topic, domain and range of functions, utilising eight procedures to solve 21 problems (see Table 6.10) . Although he started the problem type, domain and range of trigonometric functions, in the third observed lesson, he did not complete this problem type. Instead, he used the spread of AIDS as an example to explain the notion of a function, presumably in order for learners to consider trigonometric ratios such as sine x as functions. It should be noted, however, that the procedures in Jono's lessons differed from the procedures in the other three teachers' lessons. Here, the procedures served the purpose of arriving inductively at propositions for the domain and range of the four function types rather than as procedures to be applied in solving mathematical problems of the same type. His use of the procedures will be discussed in more detail in Chapter 7.

Table 6.10. Jono's topics, problem types and procedures sub-topics and procedures

Announced topic: Domain and range of functions		
Problem type	**Procedure**	**# problems**
1. Find domain and range of linear functions	**PR1:** Calculating the domain of linear functions	6 mono-topic
	PR2: Calculating the range of linear functions	
2. Find domain and range of quadratic functions	**PR3:** Calculating the domain of quadratic functions	4 mono-topic
	PR4: Calculating the range of quadratic functions	
3. Find domain and range of hyperbolic functions	**PR5**: Calculating the domain of hyperbolic functions	6 mono-topic
	PR6: Calculating the range of hyperbolic functions	
4. Find domain and range of exponential functions	**PR7**: Calculating the domain of exponential functions	5 mono-topic
	PR8: Calculating the range of exponential functions	
5. Domain and range of trigonometric functions	No procedures elaborated – focus on what is a trigonometric function	n/a

Looking across the four pedagogic contexts, the above discussion on the use of procedures is summarised in Table 6.11. We note that teachers at Prestige College (Sara and Jada) introduced learners to more procedures than both teachers at Evergreen High (Maya and Jono) during the period of observation. This difference may be topic related. However, recall from Section 6.2.2 that Evergreen High teachers had more pedagogic time available than Prestige College teachers, thus suggesting slower pacing for Evergreen High teachers than Prestige College teachers.

As discussed, Sara achieved both inter-topic as well as intra-topic connectivity during the observed lessons whereas Jada achieved inter-topic connectivity but not intra-topic connectivity during the observed lessons. For both Evergreen High teachers, there was an absence of inter-topic and intra-topic connectivity during the observed lessons. The limited period of observation means that I have no way of assessing whether Evergreen High teachers established inter-topic connections at a later stage.

Table 6.11. Summary of procedures elaborated by teachers

Teacher	Total number of procedures	Connected procedures	Intra-topic connectivity	Inter-topic connectivity
Sara	10	5	Yes	Yes
Jada	11	0	No	Yes
Maya	6	0	No	No
Jono	8	0	No	No

This discussion of the procedures merely provides the mathematical context of the analysis but does not reveal much about the actual content realised in relation to the announced topics and so therefore very little information with respect to the realisation of content. For this we examine the lower-level computations that make up the procedures by examining the domains, codomains and operations that make up these computations.

6.3.4 Domains, codomains and operations

I distinguished evaluative events in which domains, codomains and operations were recruited solely from the field of the reals from evaluative events in which domains, codomains and operations from both the field of the reals and auxiliary operations or operation-like manipulations are employed. Operations from the field of the reals include for example, addition, multiplication, subtraction, division, squaring and square rooting. Auxiliary operations used by teachers and learners stand in place of operations from the field of the reals or as substitutes for encyclopaedic propositions. Recall that the domains and codomains of auxiliary operations comprise alpha-numeric characters and that auxiliary operations are not operations because they are not necessarily functions which ensure unique outputs for a given input, and which preserve identity at the level of content but not necessarily at the level of expression.

Table 6.12 contains a summary of this analysis. Evaluative events in which computations were not performed were not included in the count since operations and domains are not in use in such evaluative events. Two such evaluative events occurred in Jono's lessons. Both L01EE1 and LO1EE5 involve descriptions of mathematical terms. In L01EE1 he described what a function is without performing any computations and in L01EE5 he described the difference between exponential functions, power functions and quadratic functions but did not perform any computations. There was only one non-computational event in Sara's lessons, L01EE1, where she revised features of basic functions dealt with previously.

Table 6.12 shows that in 68,4% of evaluative events teachers and learners in all four pedagogic contexts recruit operations from both the field of the reals as well as auxiliary operations. This indicates that large parts of the realised content across the four pedagogic contexts differ from the content associated with the topic from the point of view of the Mathematics encyclopaedia. Maya used the reals and auxiliary operations or only auxiliary operations in all computational evaluative events, Jono in 88,9%, Jada in 83,3% and Sara in 64,5%, of computational evaluative events.

Table 6.12. Classifying events in terms of domains, codomains and operations deployed

Teacher	Lesson	# evaluative events	Reals only $(\mathbb{R},+,\times)$	Reals $(\mathbb{R},+,\times)$ & auxiliary operations $(\mathbb{X},*)$	Auxiliary operations only $(\mathbb{X},*)$	Non-computational
Sara	L01	3	0	2	0	1
	L02	5	2	3	0	0
	L03	7	3	4	0	0
Jada	L01	7	1	4	2	0
	L02	6	0	6	0	0
	L03	5	2	3	0	0
Maya	L01	4	0	3	0	1
	L02	4	0	3	0	1
	L03	4	0	3	1	0
Jono	L01	6	0	4	0	2
	L02	3	0	3	0	0
	L03	3	1	1	0	1
Total		**57**	**9**	**39**	**3**	**6**

A list of auxiliary operations used by the four teachers and their learners is presented in Tables 6.13–6.16. It should be noted that the operations "move up/down" and "move left/right" used by Sara and Jada in the context of transformations of the parabola are very similar to their encyclopaedic correlates, translate vertically and translate horizontally respectively. However, the operations used by the teachers take the graph as an input and the "shifted graph" as the output whereas their encyclopaedic correlates are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. The auxiliary operations, "move up/down" and "move left/right" entail physical movements of graphs. In other words, these auxiliary operations imply shifts of graphical images. So, the transformation is viewed as one where the parabola has moved to a different position on the Cartesian plane. The teacher's treatment of transformation of functions is therefore very different from transformations as mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. In the latter case, the transformation generates a different parabola from the initial parabola.

All the teachers made use of the operation-like manipulation, transposition, which substitutes for the right-cancellation theorem: if a, b and c are real numbers, then $a + c = b + c \implies a = b$ or the left-cancellation theorem if a, b and c are real numbers, then $c + a = c + b \implies a = b$. Maya was the only teacher who used the operation "add the same thing" to both sides of an equation, interchangeably with the operation, transposition, which is also a substitute for the right-cancellation or left-cancellation theorem. There are strong suggestions that some auxiliary operations, such as transposition, may serve as abbreviations or short cuts for operations located in the Mathematics encyclopaedia. This remains a hypothesis which requires further investigation since it is beyond the scope of this study.

Table 6.13. Auxiliary operation used in Sara's lessons

Auxiliary operation	**Substitutes for**
Move up/down	Proposition: For the function $y = af(x) + p$, the parameter p shifts the graph of f(x) up or down. If $p > 0$, the graph $f(x)$ shifts up and if p < 0, then the graph of $f(x)$ shifts down. Proposition: For the function $y = a(x - p)^2 + q$, the parameter q shifts the graph $y = x^2$ up or down. If $q > 0$, the graph $y = x^2$ shifts up and if $q < 0$, the graph $y = x^2$ shifts down.
Stretch in/out	Proposition: For the function $y = af(x) + p$, the parameter a moves $f(x)$ in or out. Proposition: For the function $y = a(x - p)^2 + q$, the parameter a results in a vertical stretch of $y = x^2$, that is, "the graph gets wider or narrower".
Move left/right	Proposition: for the function $y = a(x - p)^2 + q$, p effects a horizontal shift of the graph with a shift to the right if $p > 0$ and a shift to the left if $p < 0$.
Take the opposite sign	Proposition: for the parabola $y = a(x - p)^2 + q$, the maximum or minimum value of the parabola is q when $x = p$ i.e. that the turning point of the parabola is $(p; q)$
Change sign (ALT)	Multiply by -1 (used simultaneously) or transposition (used interchangeably)
"drop the minus"	Multiply by -1
Transposition	Right cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $a + c = b + c \implies a = b$ and Left cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $c + a = c + b \implies a = b$

Sara recruited eight different auxiliary operations (see Table 6.13). On one occasion, Sara restated the learner's use of "take over and change signs" when solving an equation to "multiplying by -1", thus immediately inserting the encyclopaedic correlate of the auxiliary operation. A structure preservation map illustrating the relation between multiplication over the reals and the auxiliary operation, transposition which uses characters as its domain and codomain is shown in Figure 6.9. The commutative diagram shows that it is the auxiliary operation *string* which enables structure preservation and so the substitution of encyclopaedic content with auxiliary content.

$$
\begin{tikzcd}
-5 \arrow[r, "\text{Multiplication by -1}"] \arrow[d, "\text{string}"'] & +5 \arrow[d, "\text{string}"] \\
/-5/ \arrow[r, "\text{transposition}"] & /+5/
\end{tikzcd}
$$

Figure 6.9. A mapping of multiplication over the reals to transposition over characters

The auxiliary operation, transposition, is structurally similar to the auxiliary operations “change sign”, “take opposite sign” and “drop the minus”.

Jada recruited five auxiliary operations (see Table 6.14). The auxiliary operations used by Jada correspond with those used by Sara. This may be related to the fact the two teachers dealt with similar topics, even though there were differences across the two teachers.

Table 6.14. Auxiliary operation used in Jada’s lessons

Auxiliary operation	**Substitutes for**
Move left/right	Proposition: for the function $y = a(x-p)^2 + q$, p effects a horizontal shift of the graph $y = x^2$ with a shift to the right if $p > 0$ and a shift to the left if $p < 0$.
Take the opposite sign	Proposition: for the parabola $y = a(x-p)^2 + q$, the maximum or minimum value of the parabola is q when $x = p$ i.e. that the turning point of the parabola is $(p; q)$
Change sign (ALT)	Multiply by -1
“ignore the negative”	Multiply by -1
Transposition	Right cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $a + c = b + c \Rightarrow a = b$ and Left cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $c + a = c + b \Rightarrow a = b$

Maya used three auxiliary operations (see Table 6.15). Two of the auxiliary operations are in fact mappings from an image to an equation of a function. The one maps an image of a line to the equation $y = mx + c$ and the other maps an image to the equation $y = ax^2 + bx + c$.

Table 6.15. Auxiliary operations used in Maya's lessons

Auxiliary operation	Substitutes for
Transposition	right cancellation theorem: if a, b and c are real numbers, then $a + c = b + c \Rightarrow a = b$ (Used interchangeably with add/subtract the same on both sides of the equation)
Line image – linear function mapping	Stating type of function or providing general equation $y = mx + c$
facial image – quadratic function mapping	Stating type of function or providing general equation $y = ax^2 + bx + c$
Auxiliary operation	**Substitutes for**
Transposition	Right cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $a + c = b + c \Rightarrow a = b$ and Left cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $c + a = c + b \Rightarrow a = b$ (Used interchangeably with add/subtract the same on both sides of the equation)
Line image – linear function mapping	Stating type of function or providing general equation $y = mx + c$
facial image – quadratic function mapping	Stating type of function or providing general equation $y = ax^2 + bx + c$

Jono utilised three auxiliary operations (see Table 6.16). The auxiliary operation, spatial order, maps smaller numbers to the left of an imaginary number line and bigger numbers to the right of it.

Table 6.16. Auxiliary operations used in Jono's lessons

Auxiliary operation	Substitutes for
Transposition	Right cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $a + c = b + c \Rightarrow a = b$ and Left cancellation theorem, which tells us that if a, b and c are real numbers, then it must be the case that if $c + a = c + b \Rightarrow a = b$
Cross multiply (top left-hand side (LHS) with bottom right-hand side (RHS) and vice versa)	$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{ad}{bd} = \frac{bc}{bd} \Leftrightarrow ad = bc$
Spatial order (smallest number on LHS, bigger number RHS)	Numerical order

Sara, Maya and Jono used the term "over" as a substitute for division. Jono was the only teacher, though, who restated a learner's use of "over" with the term "division". "Over" seems to be a synonym for division rather than an auxiliary operation since the domain and codomain of "over" appear to be the set of integers. Since auxiliary operations involve character strings as domains and codomains, it appears that "over" is not an auxiliary operation. Further investigation such as follow-up interviews with teachers and learners are required to better establish what is meant when teachers/learners use the term "over" as part of their computational activity.

In summary, the presence of auxiliary structures $(\mathbb{X},*)$ where $\mathbb{X}$ represents the domain of characters and * represents an auxiliary operation, are elements of the computational activity realised in all four pedagogic contexts, which again indicates substitution of encyclopaedic content at the micro-level of the computations employed by teachers and their learners when doing mathematical work.

By focusing on the computational activity in terms of descriptions of mathematical terms, propositions, procedures and the operations and associated domains and codomains, I have described the recognition and realisation rules employed by teachers and learners. I now examine what the description of the computational activity tells us about the realisation of content in each pedagogic context.

6.4 Realised content

As discussed earlier, the announced topic is that which is marked out by the teacher as the topic to be taught and learnt. The realised content is the content that becomes associated with the topic name as the topic unfolds in the pedagogic context and it is the content actually taught and learnt. The realised content emerges as an outcome of the functioning of evaluation in a pedagogic context and is read off the computational activity which comprises the operations performed and associated domains and codomains together with definitions or descriptions of mathematical terms and propositions that support the computational activity of teachers and their learners. My analysis compared the content emerging in the pedagogic context with the content typically associated with the announced topic in the Mathematics encyclopaedia as a means of gauging the recontextualisation of content in the transformation of knowledge into pedagogic communication i.e. establishing whether content substitution has taken place. Instances where substitution of content has not taken place are said to be *content convergent* and instances where substitution of content have taken place are said to be *content divergent* with respect to the Mathematics encyclopaedia.

Substitution of content occurred in all of the evaluative events for both Evergreen teachers (see Figure 6.10). In other words, the content realised in both Maya's and Jono's pedagogic contexts represent content that diverges from the content associated with the announced topic(s) from the point of view of the Mathematics encyclopaedia. With respect to Prestige College teachers, content substitution took place in 80% of Sara's evaluative events and 89% of Jada's evaluative events. So, the realised content for both Sara and Jada represents a hybrid of content that converges with the Mathematics encyclopaedia and content that diverges from the Mathematics encyclopaedia.

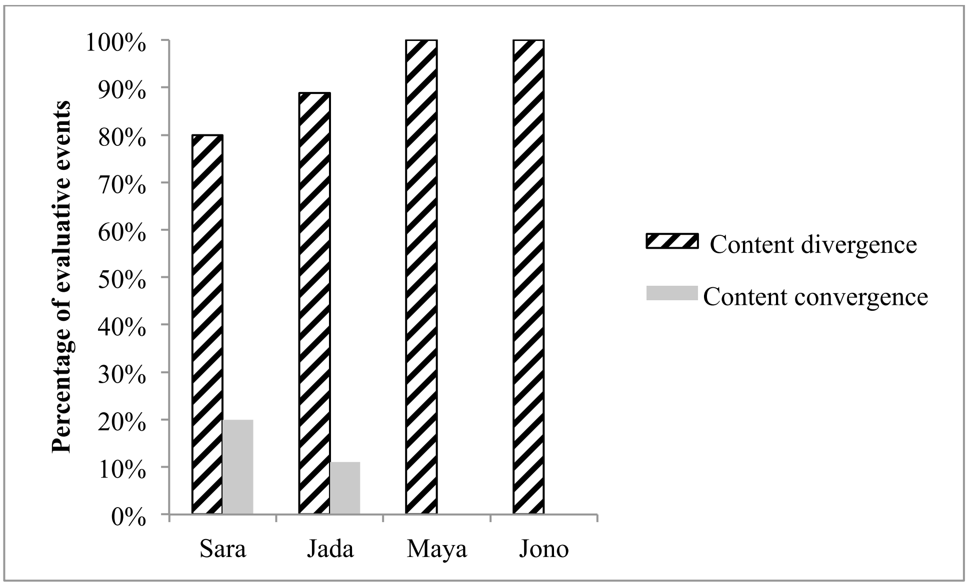


Figure 6.10. Content convergence and content divergence by evaluative event

Recall from Chapter 5 that the realised content is described in terms of the combination of presence/absence of encyclopaedic computational resources and presence/absence of auxiliary computational resources, producing four content types: *canonical*, *ancillary*, *symbiotic* and *elementary* (see Figure 6.11). Encyclopaedic computational resources refer to definite descriptions and/or propositions from the Mathematics encyclopaedia and encyclopaedic operations over the domain of real numbers. Auxiliary computational resources refer to auxiliary descriptions, auxiliary propositions and/or auxiliary operations over the domains of characters.

		Encyclopaedic computational resources	
		present	**absent**
Auxiliary computational resources	**absent**	*Canonical*	*Elementary*
	present	*Symbiotic*	*Ancillary*

Figure 6.11. Realised content types

Table 6.17 shows that the canonical content type is only present in 9% of all of the evaluative events across the four pedagogic contexts and only occurs at Prestige College.

Table 6.17. Categorisation of evaluative events with respect to realised content types

Teacher	Lesson	No. of EEs	Canonical	Elementary	Ancillary	Symbiotic
Sara	L01	3	0	0	0	3
	L02	5	1	0	0	4
	L03	7	2	0	4	1
	Total	**15**	**3**	**0**	**4**	**8**
Jada	L01	7	0	0	6	1
	L02	6	0	0	5	1
	L03	5	2	0	3	0
	Total	**18**	**2**	**0**	**14**	**2**
Maya	L01	4	0	0	4	0
	L02	4	0	0	4	0
	L03	4	0	0	4	0
	Total	**12**	**0**	**0**	**12**	**0**
Jona	L01	6	0	0	6	0
	L02	3	0	0	3	0
	L03	3	0	0	3	0
	Total	**12**	**0**	**0**	**12**	**0**
Total		**57**	**5**	**0**	**42**	**10**

In other words, the inclusion of encyclopaedic content without the presence of any auxiliary computational resources is evident in 20% of Sara's evaluative events and 11% of Jada's evaluative events (see also Figure 6.12), confirming that content convergence is particular to the middle-class pedagogic contexts.

The dominant form of content divergence across the four pedagogic contexts is realised as the ancillary content type, which constitutes 42 (74%) of the 57 evaluative events, indicating that substitution of content is achieved chiefly through replacing encyclopaedic computational resources with auxiliary computational resources (or auxiliary calculus). Recall from Chapter 5 that computational activity using auxiliary operations and domains is often present together with arithmetic (addition, subtraction, multiplication and division) over whole numbers mostly and sometimes rational numbers. Recall too that auxiliary computational resources in the absence of encyclopaedic computational resources realises content described as ancillary.

In Maya's and Jono's pedagogic contexts, populated by learners from working-class families, the content type associated with all the evaluative events is categorised as ancillary. This means that the realised content associated with the announced topics is essentially arithmetic in combination with an auxiliary calculus. This finding confirms previous research which also found that mathematics encountered in schools populated by learners from working-class backgrounds was constituted mainly as counting, arithmetic and as content that involved an alternate calculus on characters (see Arendse, 2013; Chitsike, 2011b; Davis, 2010a, 2010b, 2010c, 2011a, 2011b, 2011c, 2012, 2013a, 2013b, 2014; Jaffer, 2009, 2010a, 2010b, 2011a, 2012; Johnson & Davis, 2010) So, it seems as though learners from working-class families are distributed mathematics that does not extend much beyond primary school arithmetic.

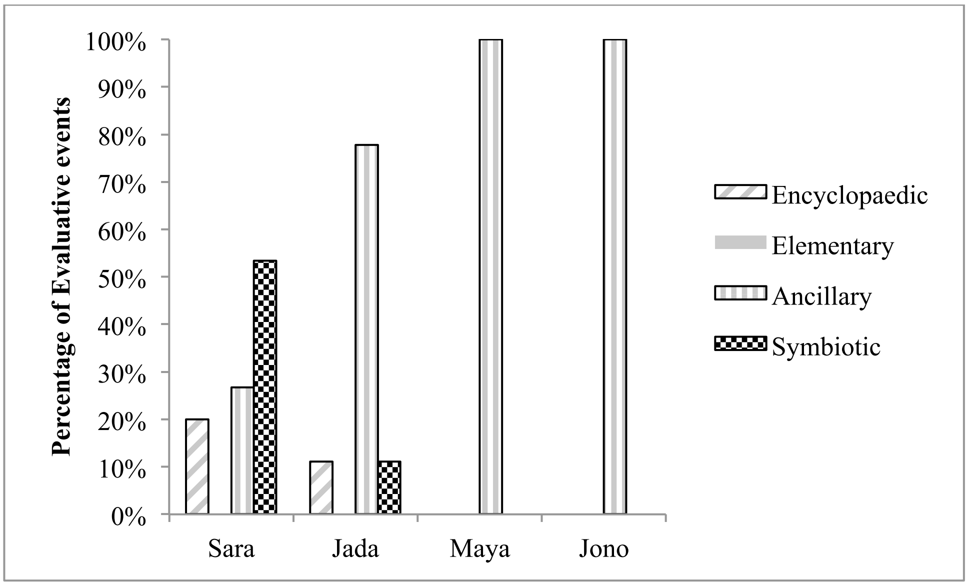


Figure 6.12. Distribution of content types across the four pedagogic contexts

In the pedagogic contexts populated by learners from upper-middle-class/elite families (Prestige College) we observe that the realised content corresponds in part with the realised content in the working-class context, more so in the case of Jada than Sara, in that the ancillary content type is present in 27% of Sara's evaluative events and 78% of Jada's evaluative events. However, the realised content in the upper-middle-class/elite contexts differs from that realised content in the pedagogic contexts populated by learners from working-class families (Evergreen High) in that eight evaluative events (53%) in Sara's lessons and two (11%) in Jada's lessons are classified as representing the symbiotic content type, which includes encyclopaedic computational resources recruited alongside or interchangeably with an auxiliary calculus, to a greater extent in the case of Sara than in the case of Jada. The difference in the realised content emerging in the two middle-class pedagogic contexts could be attributed to the streaming of learners into ability sets since Sara's learners were placed in a higher set (set 2) than Jada's learners (set 3). This hypothesis is left for further investigation since it is beyond the scope of this study.

Analysis of the realised contents shows the absence of fundamental mathematics axioms, definitions and propositions for both Maya and Jono. This raises the question as to what regulates the mathematical activity of teachers and their learners if the primary regulative resources in the most part do not include the fundamental mathematics axioms, definitions and propositions pertinent to the announced topics. It is to this discussion that we now turn.

6.5 Regulation of mathematical activity

When examining the regulation of mathematics, I classified each evaluative event in terms of the *ground* used to regulate the computational activity. Recall from Chapter 3 and Chapter 5 that (Davis, 2010a, 2011b) defines *ground* as "an ontological decision about what the objects upon which to operate are" He further states "that decisions about the nature of the object appear to be informed by decisions about which operations to perform" (Davis, 2010a, p. 379). Identifying the ground regulating mathematical activity entails examining teachers' and learners' statements and actions (oral, written or gestural) for clues regarding the nature of the objects of central concern in their computational activity. I used Davis' (2011b) four categories of ground: fundamental (propositional), algorithmic, empirical and iconic. The criteria for classifying evaluative events in terms of ground types were discussed in detail in Chapter 5. The distribution of the categories of ground across the four pedagogic contexts is shown in Figure 6.13.

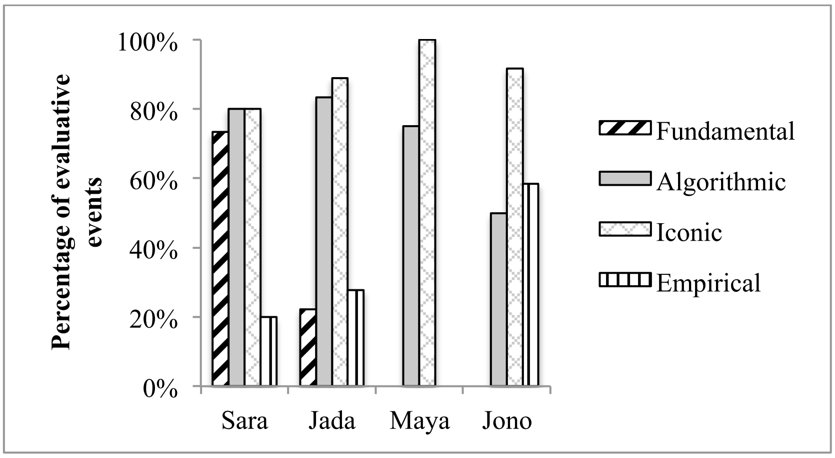


Figure 6.13. Distribution of the categories of ground

Figure 6.13 confirms that fundamental ground only features in the computational activity circulating in pedagogic contexts of Sara and Jada. With respect to Sara, 73% of evaluative events relied on fundamental ground as regulative resources compared to 22% of evaluative events in Jada's pedagogic context, suggesting that the computational activity in Sara's pedagogic context is more aligned with the Mathematics encyclopaedia than the computational activity in Jada's pedagogic context. The absence of fundamental ground as regulative resources in the pedagogic contexts of Maya and Jono suggests that alternate forms of regulative resources are recruited by Maya and Jono and that mathematical necessity is replaced by other forms of necessity.

The use of empirical resources to regulate computational activity was most dominant in Jono's lessons and absent from that of Maya. The reliance on testing or "measuring" symbolic and/or graphical expressions was achieved differently in the three pedagogic contexts. Sara and Jada used computer geometry software, *Autograph* and/or *Geogebra,* as a means to justify and establish propositions and computational rules. This occurred in three evaluative events (20%) with respect to Sara and in five evaluative events (22%) with respect to Jada. Jada resorted to calculations in one evaluative event as a means of justifying the formula for the *x*-coordinate of a parabola's turning point. Jono relied on computing tables of values as a means of establishing the notions of domain and range of functions. Although Maya did not rely on empirical computational resources, at times she encouraged learners to try any points on the graph in order to calculate the equations of graphs. If a chosen point did not work, then learners were encouraged to try other points until they found the point which produces the required outcome. So she sometimes used a "guess-and-check" strategy.

Iconic ground is present in the computational activity of all four teachers, and it is the dominant ground for Jada, Maya and Jono. Similarities and differences of expressions and the treatment of graphical and/or symbolic expressions as images are evident in all the evaluative events in Maya's lessons, 92% of the evaluative events in Jono's lessons, 89% of evaluative events in Jada's lessons and 73% of evaluative events in Sara's lessons. The iconic in all pedagogic contexts took the form of auxiliary operations, iconic auxiliary propositions and iconic auxiliary descriptions.

The computational activity in all four pedagogic contexts is regulated by algorithmic ground, that is by what procedures need to be carried out and the order of operations that make up the procedures. Almost all the evaluative events (83%) in Jada's lessons are regulated by algorithmic ground compared to 80% of evaluative events in the case of Sara and 75% of Maya's evaluative events. Jono's lessons comprised the lowest proportion of evaluative events regulated by algorithmic ground. As discussed earlier and to be elaborated in Chapter 7, the procedures carried out by Jono served the purpose of establishing propositions regarding the domain and range of the four function types rather than as procedures to be practised by learners.

6.6 Summary

The recognition and realisation rules in terms of the computational activity of teachers and their learners, displayed in the instructional discourse, were discussed in terms of four dimensions: (1) descriptions of mathematical objects and processes; (2) propositions underpinning procedures; (3) procedures; and (4) operations and associated domains and codomains entailed in procedures. The description of the computational activity was then used to read off the content realised in relation to the topics announced by the teacher and to examine the regulation of the computational activity of teachers and their learners. The realised content in each pedagogic context I compared with the content typically associated with the announced topic in the Mathematics encyclopaedia as a means of gauging the recontextualisation of content in the transformation of knowledge into pedagogic communication. Where appropriate, similarities and differences between school contexts (i.e. social class contexts) and within school contexts were identified. Key features of the computational activity in the observed lessons are summarised in Table 6.18.

A key finding distilled from Table 6.18 is that substitution of Mathematics encyclopaedic content takes place in all the pedagogic contexts but to differing degrees. One of the main differences between the upper-middle-class context and the working-class context, is the presence of encyclopaedic content in the form of encyclopaedic propositions and operations, domains and codomains. Definite descriptions, however, are absent from all pedagogic contexts, an expected finding given a curriculum context which downplays formal definitions. Sara, in contrast to Jada, at times used encyclopaedic propositions simultaneously or interchangeably with auxiliary propositions and used encyclopaedic operations simultaneously or interchangeably with auxiliary operations. Most notably, the analysis reveals that auxiliary content in the form of auxiliary descriptions, auxiliary propositions and auxiliary operations, domains and codomains feature in the computational activity across all four pedagogic contexts.

Table 6.18. Summary of the computational activity across the four pedagogic contexts

Computational activity	Prestige College (upper middle-class/elite)		Evergreen High (Working-class)	
	Sara	Jada	Maya	Jono
Descriptions	Definite descriptions absent	Definite descriptions absent	Definite descriptions absent	Definite descriptions absent
Propositions	Auxiliary propositions dominant Encyclopaedic propositions present Some auxiliary propositions used interchangeably or simultaneously with encyclopaedic correlates	Auxiliary propositions dominant Encyclopaedic propositions present	Auxiliary propositions dominant Encyclopaedic propositions absent All propositions iconic	Auxiliary propositions dominant Encyclopaedic propositions absent
Procedures	Intra-topic connectivity present	Intra-topic connectivity absent	Intra-topic connectivity absent	Intra-topic connectivity absent
Operations, domains, codomains	field of the reals and auxiliary structures	field of the reals and auxiliary structures	field of the reals and auxiliary structures	field of the reals and auxiliary structures

Table 6.19 summarises the degree of content substitution in each of the four pedagogic contexts, illustrating that content convergence with respect to the Mathematics encyclopaedia is only evident in the upper-middle-class context, to a greater extent in set 2 than set 3.

Table 6.19. Summary: realised content and regulation of mathematical activity in observed lessons

	Prestige College (Upper middle-class/elite)		Evergreen High (Working-class)	
	Sara (Set 2)	**Jada (Set 3)**	**Maya**	**Jono**
Realised content	Content convergence in 20% of EEs Content types: Encyclopaedic (20%), Ancillary (27%), Symbiotic (53%) of EEs	Content converges in 11% of EEs Content types: Encyclopaedic (11%), Ancillary (78%), Symbiotic (11%) of EEs	Content divergence in all EEs Content types: Ancillary content in all EEs	Content divergence in all EEs Content types: Ancillary content in all EEs
Regulation of mathematical activity	Fundamental ground present fundamental, algorithmic and iconic ground equally present	Fundamental ground present algorithmic and iconic ground dominant	Fundamental ground absent iconic ground dominant	Fundamental ground absent iconic ground dominant

Secondly, Table 6.19 summarises the distribution of content types in each pedagogic context, highlighting that the realised content in the working-class context takes the form of ancillary content type only whereas the realised content in the upper-middle-class context comprises a combination of the content types, encyclopaedic, ancillary as well as symbiotic. As such, Table 6.19 displays the recontextualisation of knowledge into pedagogic communication and the distribution of knowledge across social class pedagogic contexts.

The analysis, thus, reveals that the realised content in the working-class context was constituted as arithmetic over rationals combined with an auxiliary calculus. The significant point of difference between the two social class contexts with respect to content types is that although auxiliary content is present in the middle-class context as indexed by the presence of ancillary and symbiotic content types, the auxiliary content is presented alongside or interchangeably with encyclopaedic content in evaluative events characterised as symbiotic, to a greater extent in Sara's pedagogic context than in Jada's pedagogic context.

Fundamental ground featured in the regulation of computational activity in the middle-class contexts but not so in the working-class pedagogic contexts where iconic ground was the dominant form of regulation. In Jada's pedagogic context like that of the working-class pedagogic contexts, iconic ground was also present as the dominant form of computational activity regulation.

Table 6.20. Nature of mathematics problems used in observed lessons

Computational activity	Prestige College (upper middle-class/elite)		Evergreen High (working-class)	
	Sara	Jada	Maya	Jono
Mathematical problems	Mono-topic and multi-topic problems Inter-topic connectivity present	Mono-topic and multi-topic problems Inter-topic connectivity present	Mono-topic problems only Inter-topic connectivity absent	Simple problems only Inter-topic connectivity absent

A further significant difference between the two social class contexts relates to the nature of the mathematics problems given to learners and the connections forged between mathematics topics (see Table 6.20). Both Prestige College teachers, in contrast to Evergreen High teachers, used mono-topic and multi-topic mathematics problems during the observed lessons. As such, the problems presented to Prestige College learners included typical examination-like problems. Moreover, Prestige College teachers encouraged inter-topic connectivity by presenting learners with mathematics problems that required new procedures as well as procedures encountered in topics dealt with previously. The problems encountered by Prestige college learners were therefore more complex than those presented to Evergreen High learners. Synthesis of mathematics topics was made explicit to Prestige College learners during the period of observation. So, synthesis and connections between topics ought to be much easier for Prestige College learners than Evergreen High learners who were left to make connections between topics independently of the teacher.

Prestige College learners were provided with in-class opportunities for analysing mathematics problems in that some of the problems presented to them did not explicitly reference the procedure to be carried out whereas all the mathematics problems presented to Evergreen High learners as classwork or homework exercises were of the type which explicitly referred to the procedure required to solve the mathematics problem. Prestige College learners were thus provided with opportunities to analyse mathematics problems whereas Evergreen High learners were not trained to do so in the observed lessons.

6.7 Concluding remarks

This chapter set out to answer the first two research questions: (1) What does the computational activity reveal about the content realised with respect to Grade 10 mathematics topics in the instructional discourse in pedagogic contexts differentiated with respect to learners' social class membership? And (2) How is the realisation of content in the instructional discourse regulated in these pedagogic contexts?

In answering the questions, the analysis proceeded by illuminating the inner workings of evaluation, particularly the recognition and realisation rules employed by teachers and learners in the observed lessons through generating a description of the computational activity of teachers and their learners. The analysis is summarised in the foregoing discussion. The analysis reveals that evaluation produces similarities and differences with respect to what is realised as the content in relation to announced topics across social class pedagogic contexts and within social class pedagogic contexts and differences and similarities with respect to the form of regulation of computational activity across and within social class pedagogic contexts. As such, the analysis provides insight into the recontextualisation of knowledge into pedagogic discourse and the distribution of knowledge across and within social class pedagogic contexts,

The next chapter focuses on what the implications of the realised content are for the computational performance of the model learner and the orientation to mathematics revealed by the computational activity in each teacher's lessons.

Chapter 7

The model learner implied by evaluation in the instructional discourse

7.1 Introduction

This chapter focuses on a secondary analysis of the recognition and realisation rules described in terms of the computational activity evident in the observed lessons in the four pedagogic contexts discussed in Chapter 6. The interest in this chapter lies in describing what the recognition and realisation rules employed by teachers and learners reveal about the computational performance of the implied model learner and the orientation to mathematics exhibited in the observed lessons. In particular, the chapter focuses on the third and fourth research sub-questions, namely: (3) What does the computational activity elaborated in the instructional discourse imply about the computational performance of the model learner[67] constructed in pedagogic contexts that differ with respect to learners' social class membership? And (4) What orientations to mathematics are implied by the computational activity present in the instructional discourse in pedagogic contexts that differ with respect to the social class membership of learners?

For the analysis of the implied model learner's computational performance, I draw on Eco's (1984a) concept of the *Model Reader* in relation to his notion of *open* and *closed* texts and for the analysis of the orientation to mathematics implied by the realised content I recruit Davis' (2011b) reconfiguration of Lotman's notions of *expression-oriented* and *content-oriented* cultures (Eco, 1976).

7.2 The computational performance of the implied model learner

Recall from Chapter 3 that the model learner presupposed by pedagogic texts is comparable to Eco's (1984a) notion of the Model Reader. Pedagogic texts, which include sequences of oral, written or gestural significations such as teacher and learner speech, worked examples or notes written on a board or a textbook, are fundamentally evaluative in that they reveal criteria for the recognition and realisation of content in pedagogic contexts (Bernstein, 1996). Recall, too, that it is through an analysis of the computational activity of teachers and learners that we gain insight into the recognition and realisation rules entailed in the production of mathematical statements and their transformations and consequently the functioning of evaluation in the pedagogic contexts.

I argued, in Chapter 3, that the computational activity circulating in a pedagogic context presuppose and construct a particular computational performance of its implied model learner comparable to the way in which the semiotic resources of a literary text presupposes and constructs the performance of its Model

[67] The term *model learner* does not refer to the ideal learner but to the learner presupposed by pedagogy. The notion of the model learner used in this study was elaborated in Chapter 3.

Reader. Secondly, different types of texts construct different model learners and so assume and construct different mathematical performances as read off the computational activity circulating in a pedagogic situation.

As discussed, an *open pedagogic text* is one in which Mathematical axioms, definitions and propositions are explicitly recruited as computational resources. The model learner implied by an open pedagogic text is one who is capable of engaging with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia and so is provided with the computational resources that enable the implied model learner to flexibly (re)produce solutions to mathematical problems that converge on the same content associated with the announced topic even if expressively different.

A *closed pedagogic text*, on the other hand, substitutes Mathematical axioms, definitions and propositions with auxiliary descriptions, propositions and/or auxiliary operations as computational resources. The model learner implied by a closed pedagogic text is constructed as one who is not able to engage with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia. So, the computational activity focuses on producing mathematics that is expressively convergent with the announced topic but divergent from the content associated with the announced topic from the point of view of the Mathematics encyclopaedia. Furthermore, a *closed* text, in attempting to elicit a very particular reading, is subject to divergent, 'aberrant' decodings.

Chapter 5 outlined the procedure for distinguishing between open and closed pedagogic texts and for coding the computational activity as strong or weak versions of open or closed pedagogic texts. Each evaluative event in each pedagogic context was coded as one of four modes of textual orientation – strong open pedagogic text (T_O^+), weak open pedagogic text (T_O^-), weak closed pedagogic text (T_C^-), or strong closed pedagogic text (T_C^+). A summary of the coding of evaluative events for each teacher is shown in Table 7.1.

Table 7.1 Summary of evaluative events coded in terms of the nature of the pedagogic texts

Teacher	Lesson	No. of Evaluative events	strongly open pedagogic texts	weakly open pedagogic texts	weakly closed pedagogic texts	strongly closed pedagogic texts
Sara	L01	3	0	3	0	0
	L02	5	1	4	0	0
	L03	7	2	1	3	1
	Total	**15**	**3**	**8**	**3**	**1**
Jada	L01	7	0	1	4	2
	L02	6	0	2	0	4
	L03	5	1	1	1	2
	Total	**18**	**1**	**4**	**5**	**8**
Maya	L01	4	0	0	0	4
	L02	4	0	0	0	4
	L03	4	0	0	0	4
	Total	**12**	**0**	**0**	**0**	**12**
Jono	L01	6	0	0	4	2
	L02	3	0	0	2	1
	L03	3	0	0	2	1
	Total	**12**	**0**	**0**	**8**	**4**

An open pedagogic text is marked by the explicit recruitment of Mathematics axioms, definitions and propositions whereas the absence of Mathematics axioms, definitions and propositions as computational resources indicates a closed pedagogic text. As discussed in Chapter 5, a closed pedagogic text is weakened by (1) a strong focus on the empirical; (2) procedural flexibility – multiple methods made available irrespective of the form of the expression; (3) auxiliary propositions accompanied by encyclopaedic correlates; and (4) connections to other topics or sub-topics made explicit. Figure 7.1 displays the percentage of evaluative events categorised in terms of the four modes of textual orientation discussed above for each of the four pedagogic contexts.

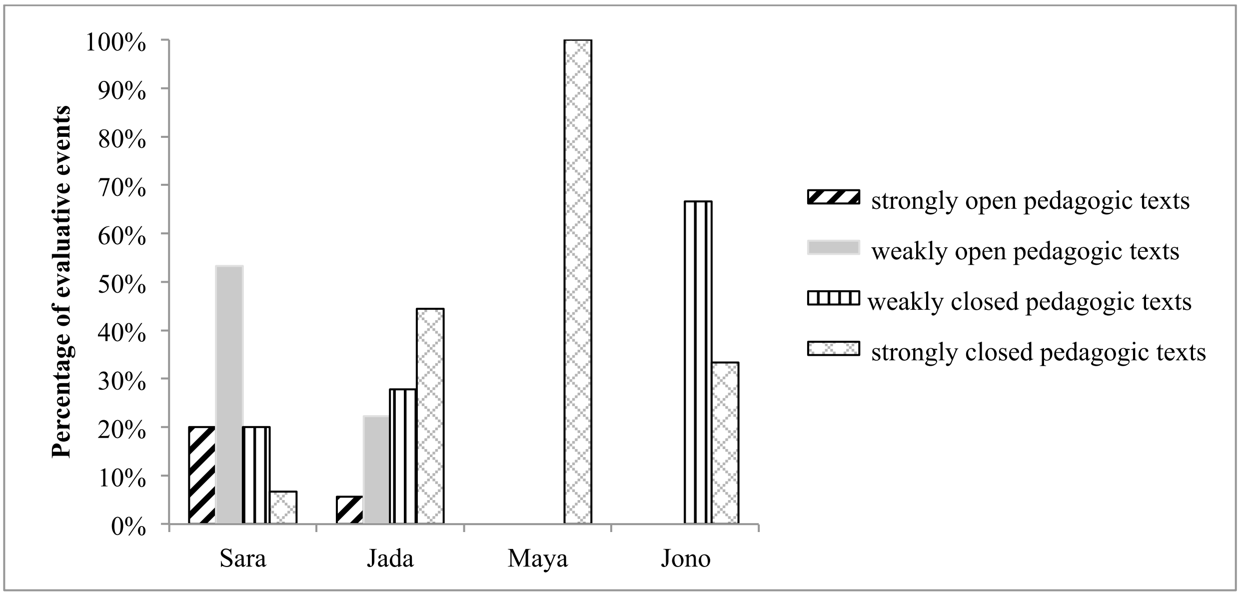


Figure 7.1. Distribution of textual orientations in terms of evaluative events

The most striking result evident from Figure 7.1 is the dominance of closed pedagogic texts across three of the four pedagogic contexts and the absence of open pedagogic texts in the Evergreen High pedagogic contexts. All the evaluative events for both Evergreen High teachers (Jono and Maya) and 72% of evaluative events for Jada (Prestige College) were coded as closed pedagogic texts. So the implied model learner in those three pedagogic contexts is predominantly one who is unable to engage with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia, to a lesser extent in Jada's pedagogic context. Evaluation constitutes content in a way that enables the implied model learner to produce mathematics that corresponds with the announced topic at the level of expression but which diverges from the content associated with the announced topic. Thus, the implied model learner's performance is constructed as that which enables the reproduction of mathematics expressively, often in very precise ways that mimic the teacher's solution procedures and so curtails the implied model learner's flexibility to produce alternate solution procedures. The presupposed model learner of pedagogic texts where content diverges from the content associated with the announced topic from the point of view of the Mathematics encyclopaedia has implications for the construction of actual learners' computational performance. This point is taken up in Chapter 8 where the computational performances of actual learners as displayed in the solutions to test questions and clinical interviews are discussed.

Open pedagogic texts are only evident in Prestige College, with 80% of Sara's evaluative events and 28% of Jada's evaluative events coded as open pedagogic texts. Sara is the only teacher with evaluative events exhibiting strongly open pedagogic texts. In 80% of evaluative events, Sara and her learners recruit Mathematical axioms, definitions or propositions explicitly as computational resources, and in so doing the implied model learner is endowed with flexibility to produce solution procedures that converge on the same content associated with the announced topic from the point of view of the Mathematics encyclopaedia even if expressively different. The model learner is, therefore, constructed as one who can competently combine computational resources in a flexible and novel manner and who is capable of reproducing content that corresponds with content associated with the announced topic from the point of view of the Mathematics encyclopaedia. Procedures encountered when dealing with other topics earlier in the year or in previous grades such as calculating intercepts of graphs or factorising quadratic expressions are referred to in ways that signal that the teacher expects the implied model learner to be already competent. In other words, these procedures are not re-explained in detail. The pedagogic text emerging in the pedagogic context thus presupposes a particular computational performance of the implied model learner and at the same time constructs the model learner's performance as a flexible user of computational resources. The similarities and differences across the four pedagogic contexts discussed above reveal the structuring effect of evaluation on the computational performance of the model learner implied by the computational activity.

7.3 Orientation to mathematics

Recall from Chapter 5 that orientations to mathematics are described via the analytic constructs *expression-orientation* and *content-orientation*, developed by Davis, (2011b) as mathematically-attuned versions of the terms used by Lotman to describe orientations to the reproduction of culture. For Davis (2011b), an orientation to mathematics is described as expression-oriented when expressive elements (domains and codomains that comprise character strings or graphical images and auxiliary operations) are primary. An orientation to mathematics is described as content-oriented when the expressive elements play a secondary role and the main function of the expressive resources lies in communicating mathematical activity. Also discussed in Chapter 5, is that expression-orientation and content-orientation can be realised as either weak or strong. Table 7.2 and Figure 7.2 present the distribution of orientation to mathematics across the four pedagogic contexts.

All the evaluative events for both Evergreen High teachers and 83% of the evaluative events in the case of Jada (Prestige College) are classified as expression-oriented. So, the expressive resources are of primary concern for Maya, Jono and Jada. For Maya, all the evaluative events are coded as strongly expression-oriented indicating an absence of empirical and fundamental computational resources to offset the strong focus on the iconic.

Table 7.2 Distribution of orientations to mathematics across the four pedagogic contexts

Teacher	Lesson	No. of Evaluative events	strong content-orientation	weak content-orientation	weak expression-orientation	strong expression-orientation
Sara	L01	3	0	3	0	0
	L02	5	1	4	0	0
	L03	7	2	2	2	1
	Total	**15**	**3**	**9**	**2**	**1**
Jada	L01	7	0	1	4	2
	L02	6	0	2	0	4
	L03	5	1	1	1	2
	Total	**18**	**1**	**4**	**5**	**8**
Maya	L01	4	0	0	0	4
	L02	4	0	0	0	4
	L03	4	0	0	0	4
	Total	**12**	**0**	**0**	**0**	**12**
Jono	L01	6	0	0	0	6
	L02	3	0	0	0	3
	L03	3	0	0	1	2
	Total	**12**	**0**	**0**	**1**	**11**

In Sara's case, content-orientation as opposed to expression-orientation is dominant since 73% of evaluative events are classified as content-oriented, with 53% of evaluative events being weakly content-oriented and 20% strongly content-oriented. Thus, the predominant model learner implied by the computational activity in Sara's pedagogic context is oriented towards the content associated with the announced topic from the point of view of the Mathematics encyclopaedia. For the remainder of the evaluative events, barring one which focused on calculating the equation of a linear function, the orientation to mathematics is classified as weakly expression-oriented where the auxiliary calculus is tempered by connections within topics and multiple methods for calculating *x*-intercepts and turning point of parabola.

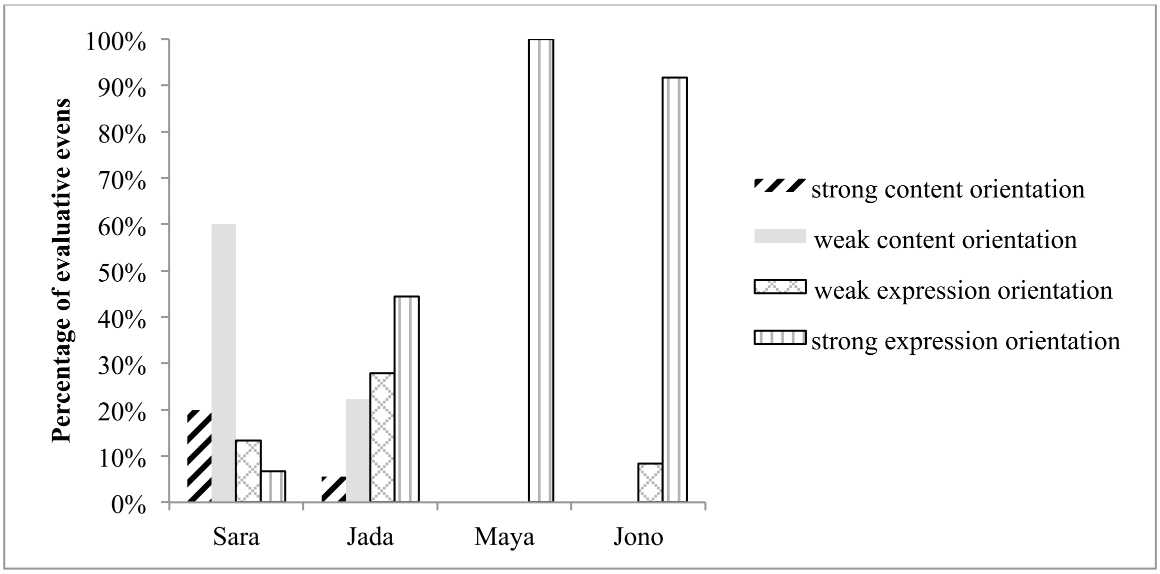


Figure 7.2. Distribution of orientations to mathematics across the pedagogic contexts

As discussed in Davis (2011b, p. 318), combinations of the four modes of textual orientation and the four orientations to mathematics generate 16 pedagogic modalities (see Figure 7.3).

Pedagogic text: open/closed

Orientation to pedagogic text: content/expression	$\times$	T_o^+	T_o^-	T_c^-	T_c^+
	O_c^+	O_c^+/T_o^+	O_c^+/T_o^-	O_c^+/T_c^-	O_c^+/T_c^+
	O_c^-	O_c^-/T_o^+	O_c^-/T_o^-	O_c^-/T_c^-	O_c^-/T_c^+
	O_e^-	O_e^-/T_o^+	$/O_e^-/T_o^-$	O_e^-/T_c^-	O_e^-/T_c^+
	O_e^+	O_e^+/T_o^+	O_e^+/T_o^-	O_e^+/T_c^-	O_e^+/T_c^+

Figure 7.3. Pedagogic modalities - cross product of $O \times T$ (Davis, 2011b: p318)

Although 16 pedagogic modalities are theoretically possible, empirically five pedagogic modalities are evident across the four pedagogic contexts (see Table 7.3):

(1) strong open pedagogic text accompanied by a strong content-orientation (T_o^+/O_c^+);

(2) weak open pedagogic text accompanied by a weak content-orientation (T_o^-/O_c^-);

(3) weak closed pedagogic text accompanied by a weak expression-orientation (T_c^-/O_e^-);

(4) weak closed pedagogic text accompanied by a strong expression-orientation (T_c^-/O_e^+) and

(5) strong closed pedagogic text accompanied by a strong expression-orientation (T_c^+/O_e^+).

Table 7.3. Distribution of pedagogic modalities in terms of evaluative events

	T_o^+/O_c^+	T_o^-/O_c^-	T_c^-/O_e^-	T_c^-/O_e^+	T_c^+/O_e^+
Sara	20%	53%	20%	0%	7%
Jada	6%	22%	28%	0%	44%
Maya	0%	0%	0%	0%	100%
Jono	0%	0%	8%	58%	33%

From Table 7.3, we note that, with the exception of Maya, the other three pedagogic contexts are hybrids exhibiting different combinations of the five pedagogic modalities. Maya's pedagogic context is characterised by T_c^+/O_e^+ which is evident in all the evaluative events across the three observed lessons. Jono's pedagogic context comprises a combination of the pedagogic modalities $(T_c^+/O_e^+), (T_c^-/O_e^-)$ and (T_c^-/O_e^+), with the latter being more prevalent. This pedagogic context can be described as one that realises a closed pedagogic text that is strongly expression-oriented, since only one event is coded as weakly expression-oriented. Jada's pedagogic context is a hybrid tending towards a closed pedagogic text and expression-orientation since the dominant pedagogic modality is T_c^+/O_e^+. Sara's pedagogic context is also a hybrid but one that leans towards an open pedagogic text that is content-oriented since the dominant pedagogic modality realised is T_O^-/O_c^- and only one event is coded as T_c^+/O_e^+.

The above discussion provides an overview of the dominant pedagogic modalities in each pedagogic context. Below I discuss each of five pedagogic modalities in order to illustrate the dominant pedagogic modality in each of the four pedagogic contexts and to describe what this implies about the computational performance of the model learner and the orientation to mathematics and so how evaluation functions in each pedagogic context.

7.4 Describing the five pedagogic modalities

7.4.1 Strong closed text and strong expression-orientation

Almost 46% of evaluative events across the four pedagogic contexts are coded as strongly closed pedagogic texts, with almost half of those located in Maya's lessons, nine in Jada's, four in Jono's and one in Sara's. To illustrate an example of a strongly closed pedagogic text, I refer to the first evaluative event in Maya's second lesson (L02EE1), in which she explicates the solution for calculating the equation of the function depicted in Figure 7.4.

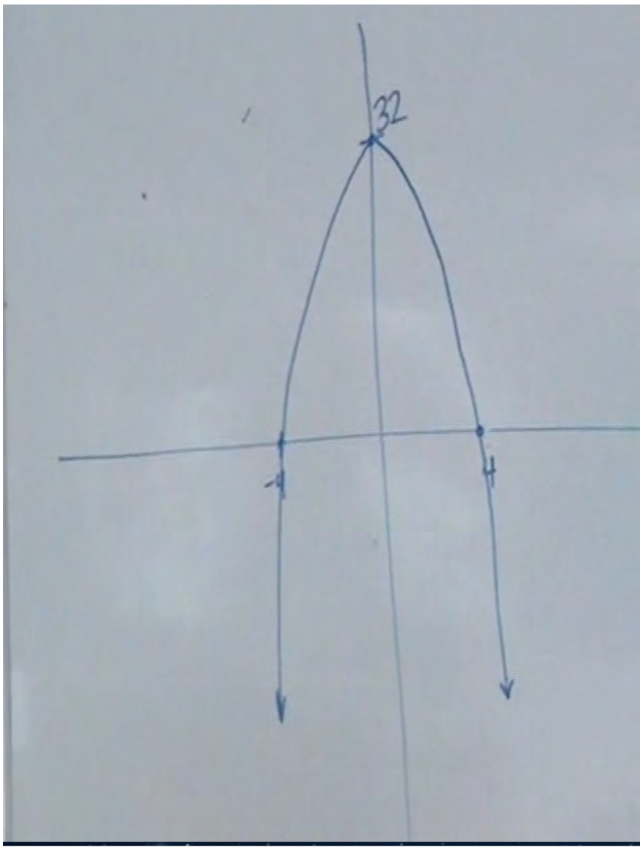


Figure 7.4. Mathematic problem provided in S02T03L02 EE2

Extract 7.1 illustrates the starting point of her procedure for calculating the equation of the function shown in Figure 7.4, which was accompanied with the verbal instruction "find the equation of the function".

Extract 7.1. S02T03L02 transcript: lines 34-54

Teacher:	Okay. This is four and negative four. We were looking at this kind of equation. Today what I am going to do is I am going to show you a way that you can solve a kind of equation that doesn't have the same numbers here. We will get to that. Let's start with this. Now. Now what?
Learners call out different responses, "nothing", "equation", "formula".	
Teacher:	Okay. Okay. Formula first. Okay. Alright. Uhmm Sisanda what kind of function is this?
Learner1:	[indistinct]
Sisanda:	Parabola.
Teacher:	How do you know?
Other learners shout out responses but the teacher insists that they give Sisanda an opportunity to respond	
Sisanda:	Because it has a sad face Miss.
Teacher:	Okay it has a sad face. It is sad.
Learners:	[Laugh]
Teacher:	Okay. In this case it looks like this [Draws a parabola on board] which lets us know that this [refers to the parabola] is what?
Learner:	Negative.
Teacher:	Okay good. This is negative. So this is what we know so far. We know that this is a parabola and we know that when we go to do our check step that our our first term in the equation is going to be negative because it is giving us a sad face. Got it. Next. Uhmm Redi what is the formula for a parabola?
Learner:	*y* is equal to *a x* squared plus *q*.
Teacher:	Okay. There are some parabolas that look like this *y* is equal to *a x* squared plus *b x* plus *q* [draws graph on board] and we're going to get to that in a minute.

We observe Maya and her learners using the imagistic features of the text as resources to identify the function type as a parabola and the associated formula as $y = ax^2 + q$ since Figure 7.4 is not explicitly defined as a parabola either as part of a written instruction or marked as being parabola through an oral statement. Firstly, the shape of the graph seems to be used as an indicator of the type of function represented. Maya and her learners identify the graph as a parabola because the graph has a "sad face", thus invoking an auxiliary proposition "if the graph has this $\frown$ shape then it is a sad-faced 'parabola'" which stands in place of the encyclopaedic proposition, "if the equation of the function is $y = ax^2 + bx + c$, then the graph of function is a parabola". The substitution of the encyclopaedic proposition with the auxiliary proposition is shown diagrammatically in Figure 7.5.

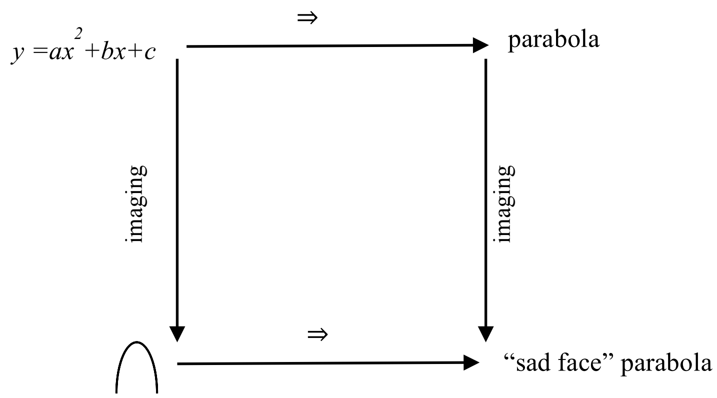


Figure 7.5. Structure preservation map – facial expression proposition

They then recruit a second auxiliary proposition which states that for a ‘sad-faced’ parabola, the first term (i.e. the x^2 term) is negative. The auxiliary proposition substitutes for the encyclopaedic proposition, if a parabola $y = ax^2 + q$ has a maximum value then $a < 0$. The substitution is illustrated in Figure 7.6.

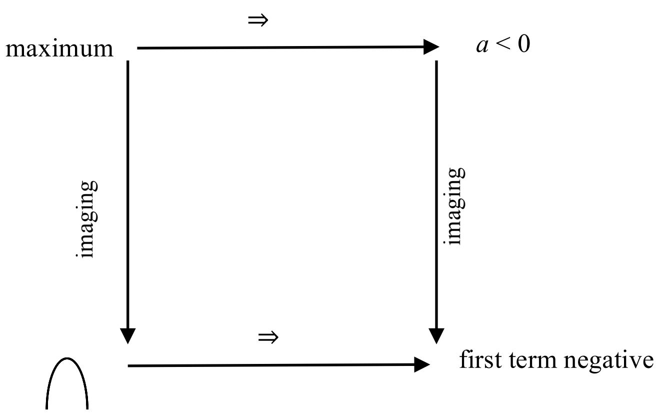


Figure 7.6. Structure preservation map – maximum of a parabola

However, it is not mathematically necessary for the x^2 term to be first. Given that addition is commutative, we could write the general equation of the parabola as $y = q + ax^2$ although conventionally we write $y = ax^2 + q$. So, the teacher’s fixed ordering of terms in the parabola’s equation suggests that she treats the parabola’s equation as a character distribution matrix.

The formula associated with the graph is identified as $y = ax^2 + q$ because it is a parabola which has the “same numbers” as x-intercepts and this function is distinguished from parabolas with x-intercepts that are “not the same” as indicated by the teacher saying: “Okay. This is four and negative four [referring to the x-intercepts]. […] Today what I am going to do is I am going to show you a way that you can solve a kind of equation that doesn’t have the same numbers here [referring to the x-intercepts]” (S02T03L02: line 33). Maya views the x-intercepts -4 and 4 as the “same number”. Surely, she does not mean that –4 equals 4. Her statement, “the same number” refers to x-intercepts that are equidistant from the y-axis.

Furthermore, she indicates that a different method is required for finding an equation of a graph that doesn't have the "same" x-intercepts, which is dealt with later in the same lesson. Maya's iconic mapping of parabolas with the "same" x-intercepts to the general formula $y = ax^2 + q$ substitutes for identifying $y = ax^2 + q$ as representing a class of parabolas with turning points $(0; q)$. In addition, parabolas of the form $y = ax^2 + q$ are not treated as a subset of parabolas defined by $y = ax^2 + bx + q$ where $b = 0$. Here, we observe another auxiliary proposition, "if the x-intercepts of a parabola are the "same", then the equation of the parabola is $y = ax^2 + q$", which substitutes for the encyclopaedic proposition, "if the symmetry-axis of a parabola is the y-axis, then the equation of the parabola is $y = ax^2 + q$".

Having identified that $q = 32$ because the y-intercept of the function is 32, Maya and her learners produce the equation $y = ax^2 + 32$ into which the point (4;0) is substituted to produce the equation $0 = \mathrm{a}(4)^2 + 32$. We pick up the discussion at a point where the task now focuses on computing the missing value a. A learner's solution to this task is shown in Extract 7.2.

Extract 7.2 S02T03L02 transcript: lines 98-104

Learner1:	Then take thirty two to the other side.
Learner2:	[Laughs]
Teacher:	Take what?
Learner1:	Thirty two to the other side. [background mutter indistinct]Then it makes negative thirty two.
Teacher:	We need to do the order of operations. We need to do BODMAS.
Learner3:	[indistinct] four squared is sixteen.
Learner2:	Okay Miss. Four times four equals to sixteen.

The learner decides to "take 32 over" first, but the teacher insists that he must apply the BODMAS rule[68], which means squaring 4 first. However, that is not mathematically necessary since any order of operations is possible due to the associativity of multiplication and addition over the reals. The teacher's insistence on applying BODMAS is indicative of strict selection and sequencing of operations which is an attempt at shepherding learners in a particular direction. The emphasis on the order of operations reveals a difference from the content associated with solutions of equations and is suggestive of computational rigidity. It is interesting that the learner does not challenge the teacher. He merely continues the computation in the order that she sets out. The learner's reaction to the teacher's instruction indicates that necessity lies in the authority of the teacher, situating necessity external to Mathematics.

In this evaluative event, evaluation produces a closed pedagogic text because the underlying fundamental Mathematics axioms, definitions and propositions are not drawn on explicitly as computational resources.

[68] BODMAS, an acronym for brackets, of, division, multiplication, addition and subtraction is used as a pedagogic strategy in many South African classrooms to teach order of operations in arithmetic computations.

Instead, iconic auxiliary propositions stand in place of encyclopaedic propositions and shape the selection and sequencing of computational resources for calculating the equation of a parabola. The pedagogic text is therefore closed with respect to the announced topic and the selection and sequencing of computational resources are curtailed with respect to the Mathematics encyclopaedia. The model learner implied by the pedagogic text is constructed as one who is not competent to engage with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia and one who is unable to fashion alternative procedures for solving a mathematics problem by re-using and re-mixing known procedures. Recall that a closed pedagogic text potentially produces "aberrant readings", which will be discussed in Chapter 9.

Furthermore, this closed pedagogic text is simultaneously one which treats the expressive elements of the pedagogic text as the primary computational resources by attending primarily to the imagistic features of the expression. The orientation to mathematics realised through this strongly closed pedagogic text is characterised as expression-oriented as opposed to content-oriented, and is calibrated as strongly expression-oriented given the absence of empirical and/or fundamental computational resources that weaken an expression-orientation to mathematics.

The realised content associated with the announced topic focuses on basic arithmetic on whole numbers combined with an auxiliary calculus which involves auxiliary operations (such as "take over and change the sign") for which domain and codomain is a set of symbols. The realised content diverges from the content associated with topic from the point of view of the Mathematics encyclopaedia but converges at the level of expression. The recognition and realisation rules, described in terms of computational activity, and so functioning of evaluation discussed in this evaluative event is emblematic of all the evaluative events in Maya's pedagogic context and is found in half of Jada's evaluative events, a third of Jono's evaluative events and in one of Sara's evaluative events.

7.4.2 Weak closed pedagogic text, strong expression-orientation and inductive reasoning

Seven evaluative events across the four pedagogic contexts are coded as weak closed pedagogic texts but with a strong expression-orientation. All of these events are located in Jono's lessons. I discuss the weak closed pedagogic text with accompanying strong expression-orientation in two parts. The first considers the role of the empirical in weakening a closed pedagogic text and the second discussed in Section 7.4.3 focuses on the use of character distribution matrices in contributing to a strong expression-orientation.

Below, I discuss an evaluative event (SO2T04L01 EE2) where Jono uses the function $y = 8x + 6$ to develop the notion of the domain and range of a linear function. For the function $y = 8x + 6$, y represents the cost of buying loaves of bread, x represents the number of loaves of bread purchased, with the cost of a loaf of brown bread being R8 each and the cost of the taxi fare being R6 (see Figure 7.7).

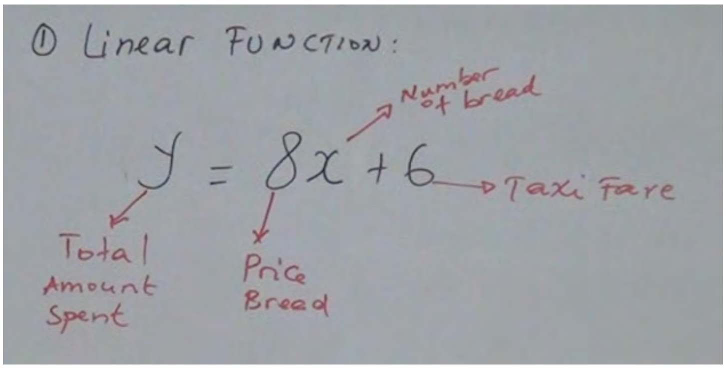


Figure 7.7. Teacher's representation of the bread-buying problem

He sets up a table of values to show the relation between the number of loaves purchased and total amount spent by calculating the cost of buying 0 loaves, then 1 loaf, 2 loaves etc. (see Figure 7.8), inserting dots in the row for x values to indicate that x continues indefinitely.

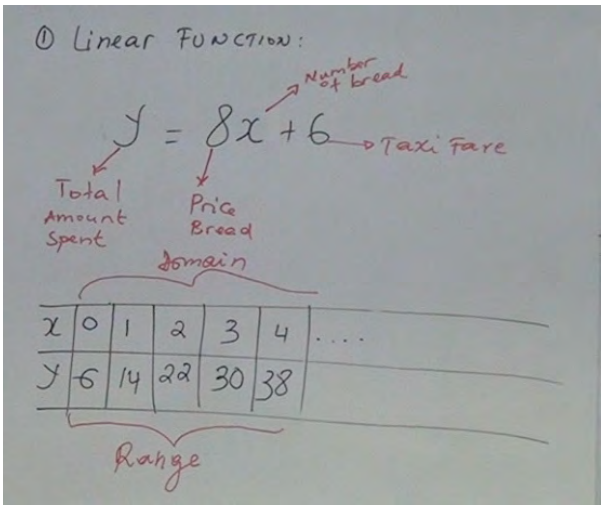


Figure 7.8. The table of values generated for the bread-buying problem

He labels the set of numbers allocated to x in the table of values as the domain and the set of values allocated to y as the range (see Figure 7.8). The set of numbers $\{0; 1; 2; 3; 4; \ldots\}$is identified with the domain as discussed in Extract 7.3. From the table of values, he concludes that the domain can be represented in what he refers to as set notation in the following way: $\{x: x \in \mathbb{N}_0; x \geq 0\}$ and the range as $\{y: y \in \mathbb{N}_0; y \geq 6\}$ where $\mathbb{N}_0$ represents whole numbers according to Jono and $\mathbb{N}$ represents natural numbers. It is not immediately clear why he needs to state that $x \geq 0$ since its inclusion is redundant given that whole numbers ($\mathbb{N}_0$) as defined by him are $\{0; 1; 2; 3; 4; \ldots\}$. The reason becomes clearer later when he rewrites the domain in interval notation as $[0; \infty)$. So, the inclusion of $x \geq 0$ serves as a source of data for the symbols required for the interval notation. His attention to symbol generation to populate a character distribution matrix is discussed in Section 7.4.3 where strong expression-orientation is discussed.

Figure 7.8 and Extract 7.3. show that the table of values serves as a central regulative resource in establishing the domain and range of the function.

Extract 7.3 S02T04L01 transcript: lines 136 -141

Teacher:	So here [referring to table of values see Figure 7.8] we have only positive numbers. So this set of independent variable represents the domain. .. Here we have the domain. And when we look at the dependent variable we have the range.
Learners:	[some background chatter]
Teacher:	What this means? We see we can buy from .. a bread or if there is no bread we spend our money for taxi but we end up not buying bread so we have zero bread here. But we can buy as much as we can if we have money. Isn't it?
Learner:	Yes.
Teacher:	So this is the domain. Any number where the independent variable can take. You can go to shop and buy two bread. I can go and buy three. So this is the domain of our?
Learner:	Independent variable.

The descriptions of domain and range used by the teacher do not meet the requirements of definite descriptions and stand in place of the definite descriptions of domain and range found in the Mathematics encyclopaedia, where the domain is defined up front.

> If x and y are two variable quantities, and if there is a rule which assigns a unique value of y to a given value of x, and we use the notation $y = f(x)$. The variable x is called the *independent* variable or the *argument* of the function y. The values of x, to which a value of y is assigned, form the domain D of the function $f(x)$. The variable y is called the *dependent* variable, the values of y form the *range W* of the function $f(x)$. (Bronshtein, Semendyayev, Musiol, & Muehlig, 2007, p. 47, italics in original)

In the case of the bread-buying problem discussed above, Jono implicitly defines the domain as whole numbers because only whole loaves of bread can be bought, not fractional loaves of bread when he states that "you can go to shop and buy two bread. I can go and buy three" (S02L01T04: line 140). He, however, does not make these assumptions explicit to learners. Instead, the definite description of domain which requires that one defines the domain of a function upfront is replaced with empirically establishing a table of values, from which the domain is read off as though the domain needs to be calculated. The emphasis on the empirical in this evaluative event constitutes domain and range of a function as something sensible which can be discovered inductively by learners. The presence of the empirical in the absence of the fundamental Mathematics axioms, definitions and propositions is indicative of inductive reasoning, which pervades the three observed lessons and locates necessity external to Mathematics.

Inductive reasoning was achieved as discussed above by using calculation as a means of establishing the notions of domain and range of a function. Another example of inductive reasoning prevalent in this pedagogic context was achieved through the development of a general proposition regarding the domain and range of all linear functions. And this was repeated for all four functions dealt with.

Having established that the domain of the function $y = x + 1$ is $\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$, learners were expected to work through the rest of Section A of the worksheet (see Figure 7.9).

A. Find the **domain** and **range** of following linear functions (write it as **interval** and **set notations**):

1. y = x+1
2. H(x) = 2x-1
3. 3y-2x = 6
4. X= 3y
5. $\frac{x}{2} - \frac{y}{3} = 1$

Figure 7.9. Part A of the worksheet on domain and range

After answering the five problems shown in Figure 7.8, a learner remarks that he obtained the same answer for all five problems, to which the teacher responds as follows "Ja. I want that thank you very much. I want everyone to come to that conclusion" (S02T04L01: line 577), indicating that the intention of the task is to arrive at a general proposition regarding all linear functions which is established in Extract 7.4.

Extract 7.4 S02T04L01 transcript: lines 747-755

Teacher:	For all those five equations your *x* can take any number. It is going from negative infinity to?
Learners:	Positive infinity [some chorus].
Teacher:	Is it clear?
Learners:	Yes Sir.
Teacher:	Who have found that?
Learner:	Yes.
Learner:	Me.
Learner:	All of us.
Teacher:	Why you didn't find that? [Points at learner who indicates 'no']

Extract 7.4 illustrates that the teacher expected all learners to produce the same answer for the domain and range for each of the linear functions provided, i.e. $\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$ for the domain and $\{y: y \in \mathbb{R};\ -\infty < y < \infty\}$ for the range. Observe the teacher's surprise when a learner does not produce the expected outcome, "why you didn't find that?" (S02T04L01: line 756). Each learner produces the same solution for each of the five problems, thus supposedly constituting an "infinite" number of individual examples which "allows" the teacher and learners to claim that the characteristics regarding the domain and range of one linear function applies to all linear functions. This mode of reasoning can be described analytically by appealing to Hegel's *syllogism of induction*[69], which he describes as:

> Induction, therefore, is not the syllogism of mere *perception* or of contingent existence, like the second figure corresponding to it, but the syllogism of *experience* – of the subjective gathering together of

[69] See Davis (2005b) who used Hegel to analyse a textbook which claimed to support an inductive pedagogy.

singulars in the genus, and of the conjoining of the genus with a universal determinateness on the ground that the latter is found in all singulars (Hegel, 1969, pp. 612-613, italics in the original)

The learners are confronted with a particular task, namely, to calculate the domain and range of five linear functions. All the learners produce the same result to the task for each of the five linear functions, thus constituting a series of singulars which mediate between the particular and the universal proposition that the domain of all linear functions is $\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$. The task represents an attempt to move from the particular to the universal through the singulars. The move from particular to singular, and then from singular to universal is contained in the structure of the syllogism of induction, Particular-Singular-Universal (USP or PSU)[70], and is represented by (Hegel, 1969, p. 612) as follows:

s
s
U– –P
s
s
ad
infinitum.

Hegel (1969) argues that induction works as a syllogism because the singular is immediately posited as identical to the universal. The establishment of the proposition, that the domain of all linear functions is $\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$, is only apparently inductive because the very proposition that the task seeks to reveal for the learner is already structured into the task. In other words, the task is pre-figured by the knowledge that learners are meant to "discover" or "notice" and so as long as the learners produce what is expected of them, the outcome of the task is guaranteed. The learners and the teacher therefore 'pretend' to be engaged in a process of producing new knowledge through inductive reasoning but the mathematical truth is presupposed by the task. Hegel warns of the danger of induction because as he indicates inferences that rely on induction are always problematic.

In induction, therefore, there recurs the *progression* into the bad infinity; *singularity* ought to be posited as *identical* with *universality*, but since the *singulars* are equally posited as *immediate*, the intended unity remains only a perpetual *ought*; it is a unity of *likeness*; the terms which are supposed to be identical are at the same time supposed *not* to be identical. The a, b, c, d, e, constitute the genus only further on, in the infinite; they do not yield a complete experience. The *conclusion* of induction thus remains *problematic* (ibid., p613, italics in original).

The proposition that the domain of a linear function is $\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$ is accepted as a mathematical truth in this pedagogic context but we know that this is not necessarily the case for all linear functions. The use of quasi-induction in this pedagogic context constitutes mathematics as an empirical

[70] Some translations of Hegel's work and commentators use the scheme USP.

activity rather than one where mathematical propositions are derived from an axiomatic deductive system. The intelligible nature of the mathematics underlying the topic is replaced with the sensible.

The recognition and realisation rules employed in this evaluative event generate a closed pedagogic text because the underlying fundamental Mathematics definitions and propositions are not drawn on explicitly as computational resources. The presence of the empirical in the absence of the fundamental Mathematics axioms, definitions and propositions is indicative of inductive reasoning and renders the pedagogic text as a weak closed text. Closed pedagogic texts are open to “aberrant decodings”, as evident in the learners’ responses to the problem, calculate the domain and range of the linear functions (see Figure 7.10). The learners all seem to follow the teacher by generating a table of values without realising that they are implicitly deciding which values to use in the table, thus implicitly deciding on the domain. Learner 1, for example, uses the set $\{0; 1; 2; 3; 4; 5; 6; \ldots\}$ for the domain of the function $y = 2x - 1$ and so concludes that the domain is $\{x: x \in \mathbb{N}; x \geq 0\}$. The learner erroneously describes the set as $\mathbb{N}$ rather than $\mathbb{N}_0$. The range is read off the set of y-values $\{-1;\ 1; 3; 5; 7; 9; 11 \ldots\}$, leading to the conclusion that the range is a subset of integers ($\{y: y \in \mathbb{Z}; y \geq -1\}$ presumably because -1 is included in the set. While we can’t be certain what the learners’ recognition and realisation criteria are, it does appear as though they are mimicking the teacher’s solution demonstrated in the bread-buying problem. Here we have a glimpse of the structuring effect of evaluation on the computational performance of learners which is explored in more detail in Chapter 9.

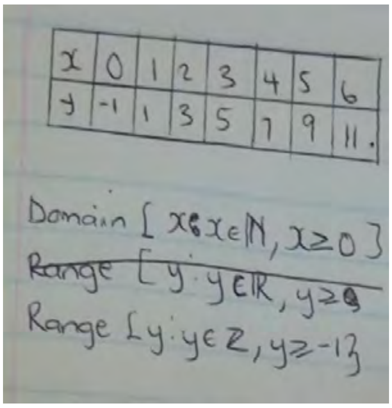


Figure 7.10a. Learner 1

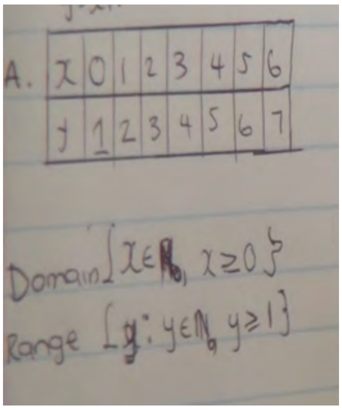


Figure 7.10b. Learner 2

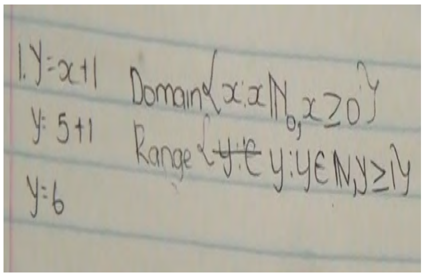


Figure 7.10c. Learner 3

Figure 7.10. “Aberrant decodings”

The model learner implied by this pedagogic text is constructed as one who is not capable of engaging with the content associated with the topic from the point of view of the Mathematics encyclopaedia and instead is only capable of arriving at mathematical ideas “inductively”. Evaluation realises mathematics as an empirical activity in that substitution of values into the equation of the function serves as a primary computational resource. However, as discussed earlier, this weakly closed pedagogic text is accompanied by a strong emphasis on the expressive elements as indicated by the teacher’s use of the domain and range expressed in set-builder notation as a source of data for rewriting the domain and range in interval notation. Below I describe the computational activity involved when producing the domain in set-builder notation and how this is over-determined by the computations involved in producing the domain or range expressed in interval notation.

7.4.3 Weak closed pedagogic text, strong expression-orientation and the use of a character distribution matrix

I focus on the example discussed here instead of exploring the bread-buying example further because the particular set-builder and interval expression are used in six of Jono's evaluative events. Secondly, the example illustrates very clearly what a strongly expression-oriented pedagogic text looks like. Thirdly, it is characteristic of the dominant pedagogic modality in Jono's pedagogic context.

Extract 7.5 picks up the discussion on the domain of the function $y = x + 1$ when Jono had already established that the domain is the set of reals earlier in the evaluative event. The discussion here essentially focuses on how the domain of the function $y = x + 1$ has to be expressed in set-builder notation.

Extract 7.5. S02T04L01 transcript: lines 500 – 509

Teacher:	So we have this equation [refers to /$y = x + 1$/ written on the board] Neliswa! [reprimanding a learner]. Do we have any number that can make this function undefined?
Learner:	No Sir.
Learner:	No.
Teacher:	So our x can go from …
Learners:	[some chorus] Negative infinity to positive infinity.
Teacher:	Negative infinity to positive infinity. Now we have to represent it as …
Learner:	x?
Teacher:	A set .. of numbers. So we put x. We know that our x is element of …
Learner:	rational
Teacher:	Real numbers. Now going from negative infinity to positive infinity. Now we have our x. [completes /$\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$/ on board]

This discussion is extremely curious since the inclusion of $-\infty < x < \infty$ in addition to stating that $x \in \mathbb{R}$ at first seems redundant. In fact, the statement $-\infty < x < \infty$ suggests that there is an upper and lower bound which seems contradictory given that the statement $x \in \mathbb{R}$ suggests that there are no exclusions, x can be any real number. It is also not clear what it means to say that $-\infty < x < \infty$. Such a statement seems to indicate that ∞ and $-\infty$ are numbers which can be excluded from the domain. Thus, Jono appears to present two contradictory conceptions of infinity. I return to this point later.

The statement, $-\infty < x < \infty$, serves the purpose of generating symbols required for writing the domain in interval notation $(-\infty; \infty)$ and suggests the presence of a character distribution matrix (see Figure 7.11).

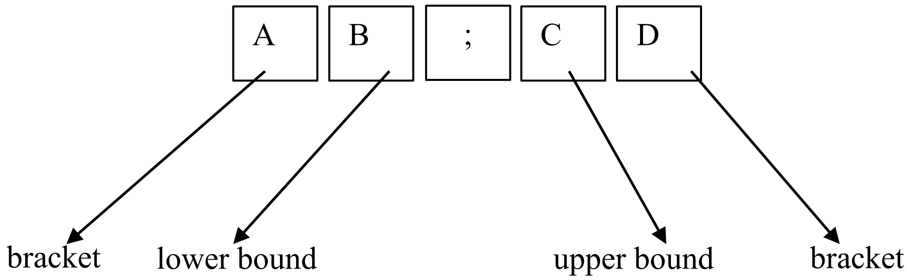


Figure 7.11. Character distribution matrix for generating interval notation for domain and range

B and C are spaces for the lower and upper bounds respectively, which are read directly off the statement $-\infty < x < \infty$. The smaller number written on the left occupies position B and the bigger number written on the right is positioned in space C i.e. $-\infty$ is written in the B space and ∞ in the C space. The spaces, A and D, are reserved for brackets. The rule for deciding whether brackets should be square or round depends on the inequality signs present in the statement. If symbols from the set $\{<;>\}$ appear in the statement then round brackets are used in interval notation i.e. brackets from the set $\{(;)\}$. Round brackets signify that the numbers in B and/or C are excluded from the set. If symbols from the set $\{\leq;\geq\}$ appear in the statement then square brackets are used in interval notation i.e. from the set $\{[;\ ]\}$. Square brackets indicate that the numbers in B and/or C are included in the set. The rules for interval notation were discussed in the introduction of the lesson where the teacher asked learners what (4;16] means, a curious introduction to the topic, domain and range of functions. In fact, it is only at this point in the lesson that the teacher's focus on interval notation as an introduction to a lesson on domain and range becomes clear. It now becomes apparent that the computational activity is over-determined by the need to generate symbols required to occupy the character distribution matrix shown in Figure 7.11. In other words, interval notation has a structuring effect on evaluation and so on the computational activity.

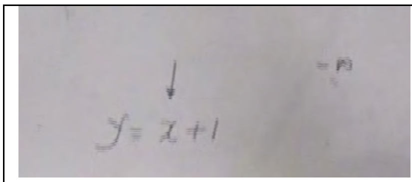
As discussed earlier, Jono's discussion of infinity suggests a contradiction (see Extract 7.6). On the one hand positive and negative infinity seem to be included in the set of real numbers – "negative infinity is the smallest number. Positive infinity is the biggest number but we can't touch this number" (S02T04L01: line 461). On the other hand, he excludes positive and negative infinity from the set of real numbers when he says "it (infinity) is excluded yes. This is excluded yes because it is number. Ja (yes) number that you can't really get or touch" (S02T04L01: lines 481).

Extract 7.6. S02T04L01 transcript

Teacher:	Yes. Where do you say … negative infinity to positive infinity. Okay and this is correct. Because infinity is a number that we can't touch. It is not really quantifiable. It is a big big number but you can't quantify it. (S02T04L01: line 373)
Learner:	Yes Sir. What does this mean [points to round bracket in $(-\infty;\infty)$]? It means that it won't touch the number.
Teacher:	It is excluded yes. This is excluded yes because it is number. Ja (yes) number that you can't really get or touch. Is it clear? (S02T04L01: lines 480 -481)

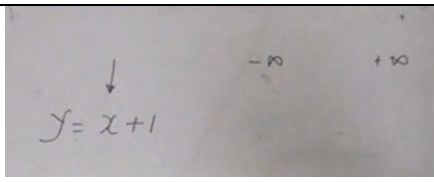
In non-standard analysis, infinitesimals and infinity are smaller than the smallest positive real numbers and bigger than the biggest real number respectively. Infinitesimals and infinity, which have the properties of real numbers are not real numbers and are referred to as hyperreals, which are used in the same way as reals for calculational or proof purposes. For example, the proof that the area of a circle is half the circumference of a circle uses the notion of infinitesimals(Davis, Hersh, & Marchisotto, 1995 p. 238). Mathematicians, however, do not necessarily concur when it comes to infinitesimals and infinity (Davis, Hersh, & Marchisotto, 1995). Euclid, for example, deliberately excluded both infinity and infinitesimals. It appears as though the teacher may be switching between the different positions with respect to infinity in the field of Mathematics. However, the teacher's two conceptions of infinity are not located either within non-standard analysis or standard analysis, they merely contradict each other and indicate an absence of a Mathematics encyclopaedic notion of infinity, which is substituted with an iconic auxiliary description.

A second absence from the point of view of the Mathematics encyclopaedia is the notion of numerical order which is an axiomatic feature of real numbers. At no point in the discussion shown in Extract 7.5 or any time in the three observed lessons does the teacher make the order relations explicit for learners.



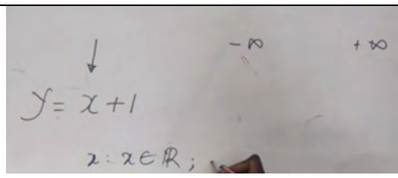
So our x can go from […] negative infinity

[Writes /−∞/ on the left]



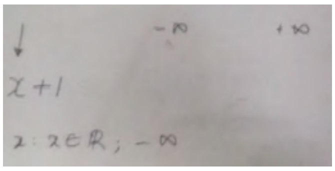
to positive infinity

[Writes /+∞/ on the right]



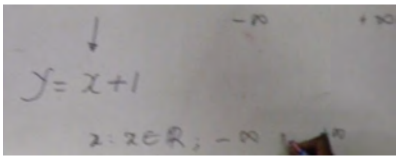
Now we have to represent it as […] a set .. of numbers. So we put x. We know that our x is element of … […] real numbers.

[Writes /x: x ∈ ℝ;/]



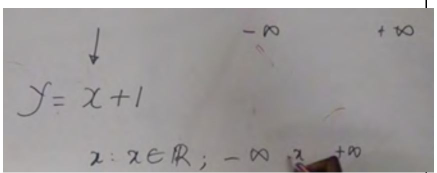
Now going from negative

[Writes /−∞/ on the left]



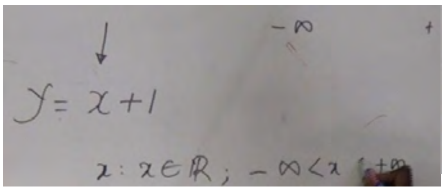
to positive infinity.

[Writes /+∞/ on the right]

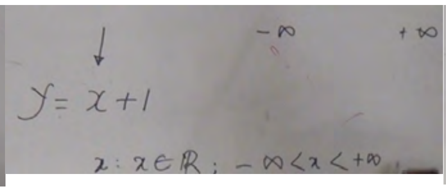


Now we have our x

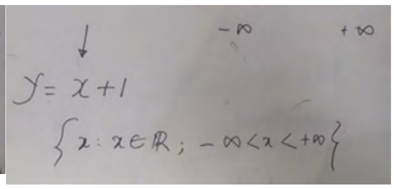
[Writes x between /−∞/ and /+∞/]



[Writes /</ between /−∞/ and /x/]



[Writes /</ between /x/ and /+∞/]



[Inserts /{/ and /}/ after before setbuilder notation

Figure 7.12. Board text accompanying the generation of the expression

A statement such as x is greater than negative infinity and less than infinity foregrounds the order relation, which refers to a comparison between two real numbers. If we compare two real numbers, a and b, and a is less than b, the order relation is represented by $a < b$ or $b > a$. The point here is not whether the order relation exists for the teacher but that the recognition and realisation criteria that he uses render the order relations implicit for the learners. What then takes the place of the order relation? If we observe carefully what the teacher says and does, we notice that the order relation is replaced with spatial order (see Figure 7.12 where the part of Extract 7.5 is matched with the board text which shows the symbols generated to produce the legitimate text).

We observe that Jono provides the description of domain as "x can go from … negative infinity to positive infinity", which is accompanied by him writing $-\infty$ on the left and $+\infty$ on the right, thus generating an implicit number line which is a spatial representation of the reals that encodes the order relation but masks the order of the reals. Expressing $-\infty < x < \infty$ in this way is a convention which matches the spatial representation of real numbers. The statement could be also represented as $\infty > x > -\infty$ but this representation does not observe the convention of the real number line, which has $-\infty$ on the left and $+\infty$ on the right. The set of real numbers, however, does not have spatial order, instead order derives from the axioms of the reals. Jono never states that x is greater than infinity and less than infinity, thus the expression $-\infty < x < \infty$ appears to be a notational representation of "our x goes from negative infinity to positive infinity" - a statement which replaces numerical order with spatial order.

The computational activity displayed in Extract 7.6 and Figure 7.12 illustrates how evaluation functions to produce the particular symbols that have to be realised as part of the legitimate text - a closed pedagogic text. The focus of the computational activity is on the expressive goals which is strongly indicative of an orientation to mathematics which takes the expressive elements as its primary concern. Here mathematics is a set of computational resources that operates directly on the expressive elements and ignores the fundamental Mathematics axioms, propositions and definitions associated with the announced topic. The particular notational targets have a structuring effect on evaluation. The notational features of the pedagogic text take precedence and shape computational choices. The realisation of content appears to be governed by the pedagogic target, that is particular notational forms. In other words, computations on symbols are constituted as the mathematics. Notational devices are required as resources for communicating arguments and ideas or as a means of capturing and expressing thought. However, in this case, the notational symbols become the mathematics – "operations" are performed on symbols. The model learner implied by this pedagogic text is constructed as one who is not capable of engaging with the content associated with the topic from the point of view of the Mathematics encyclopaedia.

The strong expression-orientation displayed in Extract 7.6 and Figure 7.12 is repeated throughout the remainder of Jono's lesson 1 and lessons 2 and 3.

7.4.4 Weak closed pedagogic text, weak expression-orientation

Nine evaluative events analysed across the four pedagogic contexts are classified as weak closed pedagogic texts. Given that T_c^-/O_e^- pedagogic texts are absent in Maya's pedagogic context and only present in one of Jono's evaluative events and that I discussed an example of a weak closed and weak expression-oriented pedagogic text in Sara's pedagogic context (see Chapter 5 Sections 5.3.3 and 5.3.4) where Sara dealt with two different procedures for calculating the x-intercepts of a parabola $y = -(x-3)^2 + 16$. I now focus on an evaluative event from Jada's pedagogic context. In this evaluative event (S01T02L01 EE7), Jada discusses the mathematics problem from the test given in the first observed lesson which was to calculate the equation of the parabola shown in Figure 7.13 in standard form, factorised form and completed square form.

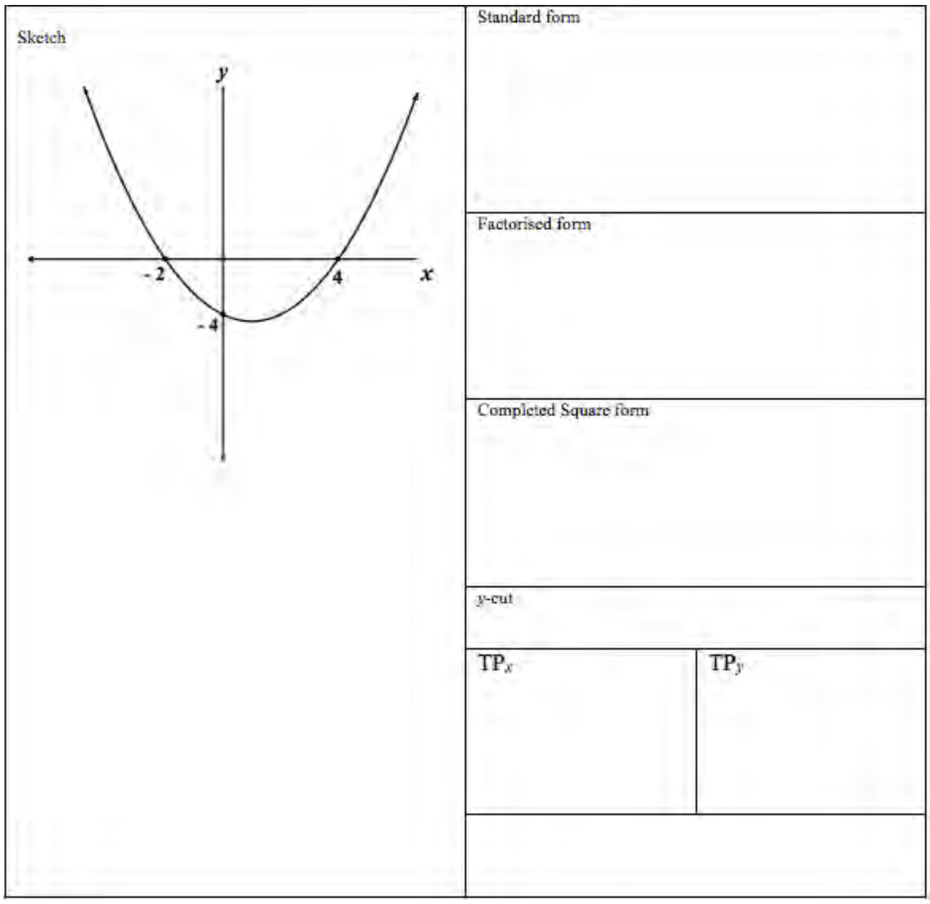


Figure 7.13. Jada's test Question 3

Extract 7.7 displays Jada's procedure for calculating the standard form of the parabola equation. We observe that her procedure entails the use of a proposition "if the answer (x-intercept) is minus two the bracket must be the opposite" (S01T02L01 transcript: line 421), which is effected as an operation-like manipulation that "changes the sign" of the x-intercept. This operation-like manipulation, referred to as ALT from now on, does not use numbers as its domain because it is not possible to "change the sign" of a number. So the domain and codomain of ALT are in fact symbols. As such, ALT is situated external to the Mathematics encyclopaedia and is part of an auxiliary calculus used by the teacher.

Extract 7.7 S01T02L01 transcript: lines 421 – 431

Teacher:	You have x minus four in the one bracket. What's the other bracket?
Learners:	x plus 2.
Teacher:	x plus 2. Because if the answer is minus two the bracket must be the opposite. Now he got that far very well. [referring to $y = (x-4)(x+2)$]. It is not good enough because if I multiply it out I have to get back to the picture. What's this y cut going to be?
Learners:	Four. Eight.
The teacher then discusses how to calculate the y-intercept by substituting x equal to nought into the equation	$y = (x-4)(x+2)$*].*
Teacher:	minus eight. It's not good enough because we are supposed to be getting minus four. Right. So how are we going to fix it?
Learners:	Divide by two. Divide by two.
Teacher:	You want to divide by two.
Learner:	Or times by a half.
Teacher:	There we go. We want to times by half. That's perfect. [Writes /$y = \frac{1}{2}(x-4)(x+2)$/]

The proposition and operation-like manipulation stands in place of the encyclopaedic proposition which states that a parabola equation can be expressed as $y = a(x - x_1)(x - x_2)$ where $a \neq 0$ and x_1 and x_2 are the *x*-intercepts of the parabola. Figure 7.14 shows a structure preservation map in which it is the operation-like manipulation, *string*. which takes a number such as -2 and produces the string /-2/, that enables the substitution of multiplication of reals with the auxiliary operation ALT over characters.

$$
\begin{tikzcd}
-2 \arrow[r, "\text{multiply by -1}"] \arrow[d, "\text{string}"'] & +2 \arrow[d, "\text{string}"] \\
/\text{-}2/ \arrow[r, "\text{ALT}"] & /\text{+}2/
\end{tikzcd}
$$

Figure 7.14. Structure preservation map – "change sign"

The use of ALT and a character distribution matrix produces the equation $y = (x-4)(x+2)$. However, $y = (x-4)(x+2)$ is not the correct equation because the *y*-intercept of the given graph is -4 not -8. The teacher and learners state that the equation has to be "fixed" by dividing by 2 or multiplying by a half to produce the equation of the given function. However, the teacher and learners are actually referring to the expression $(x-4)(x+2)$ rather than the equation $y = (x-4)(x+2)$ because if we were to multiply the equation $y = (x - 4)(x + 2)$ by a half we ought to produce the equation $\frac{1}{2}y = \frac{1}{2}(x - 4)(x + 2)$. This suggests that the teacher and the learners do not distinguish between an equation and an expression in this instance. It is not the case that the teacher necessarily conflates equations with expressions but the recognition and realisation criteria used potentially create the conditions for the learners to do so. The correct equation is established empirically by finding the value required to "adjust" the equation to produce the correct y-intercept. The computational resources substitute for the encyclopaedic proposition that states that the equation of the family of parabolas with x-intercepts x_1 and x_2 is defined by the equation $y = a(x - x_1)(x - x_2)$ where a, the scaling factor, does not equal 0.

Evaluation in this evaluative event produces a closed pedagogic text which is simultaneously expression-oriented because Mathematics encyclopaedic axioms, definitions or propositions are replaced with an auxiliary calculus and a character distribution matrix. However, the closed text and expression-orientation are weakened by the use of empirical computational resources. The model learner implied by the weak closed pedagogic text is constructed as one who is not capable of engaging with content associated with the topic from the point of view of the Mathematics encyclopaedia. The content realised in this evaluative event is a combination of arithmetic over rationals and an auxiliary calculus. Weak closed text and weak expression-oriented pedagogic texts were present in five of Jada's, three of Sara' and one of Jono's evaluative events.

7.4.5 Weak open pedagogic text and weak content-orientation

Twelve evaluative events analysed across the four pedagogic contexts are classified as weak open pedagogic texts. Eight of these evaluative events are located in Sara's lessons and four in Jada's lessons. As discussed in Chapter 5, open pedagogic texts are weakened by the (1) the use of the iconic and empirical (2) fixed selection and sequencing of operations i.e. no mathematical necessity for order of operations; (3) privileging particular solution methods; and (4) expression states such as standard forms of expressions as prompts for particular solution methods. One example of a weak open pedagogic text comes from an evaluative event in Jada's observed lessons where the task was to calculate the equation of the parabola shown in Figure 7.15. This is the same task discussed above but dealt with in the next lesson.

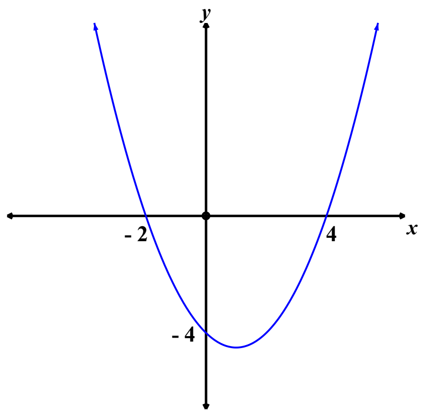


Figure 7.15. Calculating equation of parabola task

Jada starts by asking learners "These x cuts here do not control the equation of the graph .. on their own. They are not powerful enough. It is .. how many other graphs do you think I can draw through those two x cuts?" (S01T02L02 transcript: line 101), thus highlighting a fundamental proposition regarding parabolas - a family of parabolas have the same x-intercepts, so a third known point on the given parabola in addition to the x-intercepts is required to calculate the equation unique to the given parabola (see Extract 7.8) and confirmed later in the lesson by her statement "we have to give you a point in addition to the x cuts" (S01T02L02 transcript: line 113).

Extract 7.8 S01T02L02 transcript: line 107

Teacher:	So I can't just take the x-cuts and just say well here is my equation. I have to control the graph through something else. Now if I put any other point in to play. That point for example. It's absolutely impossible to draw a perfect parabola through those three points but that's different from what I have. I can draw millions (parabolas) through the x cuts. That's what I am worrying about. But once if I put a third point in doesn't matter where there is no other parabola than that one that will go through those three points. So that fixes it. So your job is to understand that this is your starting point. We know if the root is four the bracket has to be x minus four. A number of you very impressively without being taught did this [Refers to $(x-4)(x+2)$] which I was very very impressed by. So this half in the front you must figure out yourself. What you have to do to do that. You take this equation and you write it with an a. You go a and then here you multiplied out. x squared minus two x minus eight. So you take the actual equation which many of you got right and you just times it out. The a in front is what we call our controlling value. If we now have a y value equal to that. You take any point. It doesn't matter which point but not the x cuts because it goes into an identity. Take any other point.

This encyclopaedic proposition, if we are given the x-intercepts and another point on the parabola, then we have sufficient information to calculate the equation of the parabola, serves as a regulative resource underpinning her procedure for calculating the equation of a parabola. So, this evaluative event is coded as an open pedagogic text. However, her solution privileges a fixed selection and sequencing of operations as indicated by her statement "so your job is to understand that this is your starting point. We know if the root is four the bracket has to be x minus four." Here, she seems to be using "reverse" factorisation to produce the starting point of the procedure as $y=(x-4)(x+2)$, which is "adjusted" to $y=a(x-4)(x+2)$, where "the a in front is what we call our controlling value". The insertion of a serves as a means of dealing with the proposition that a family of parabolas has the same x-intercepts but a third point on a given parabola in addition to the x-intercepts is required to calculate the equation unique to the given parabola. Her justification stated later that a is "the multiple that was taken out in the factorising process" (S01T02L02: line 115) together with her procedure for calculating the equation of the parabola discussed above substitutes for another encyclopaedic proposition that states that the equation of the family of parabolas with x-intercepts x_1 and x_2 is defined by the equation $y=a(x-x_1)(x-x_2), a \neq 0$.

Jada's starting point to the solution of the mathematics problem has no mathematical necessity since the problem could also be solved by setting up a system of equations to calculate the values of a, b and c of the general equation of a parabola, $y=ax^2+bx+c$. So, the fixed selection and sequencing of operations forecloses solution methods to one privileged by the teacher for solving the mathematics problem, thus weakening the openness of the pedagogic text. Furthermore, we observe that her procedure entails the use of another proposition established in the previous lesson "if the answer (x-intercept) is minus two the bracket must be the opposite" (S01T02L01 transcript: line 421). So, although not explicitly stated in this lesson, Jada recruits an iconic auxiliary proposition as a computational resource. The use of the iconic alongside the recruitment of fundamental mathematics propositions weakens an open pedagogic text because the content realised in the name of topic represents a hybrid of content typically associated with the topic from the point of view of the Mathematics encyclopaedia together with auxiliary content.

The recognition and realisation rules employed in this evaluative event produce a weak open pedagogic text in which the implied model learner is constructed as one who is capable of engaging with content associated with the topic from the point of view of the Mathematics encyclopaedia and is simultaneously constructed as one who is not capable of selecting appropriate computational resources from the Mathematics encyclopaedia to fashion solution(s) to the mathematics problem, and so a particular solution method with a strict ordering of operations is provided. Similarly, the orientation to mathematics is content-oriented because propositions from the Mathematics encyclopaedia are recruited as support for the computational activity. However, this content-orientation is weakened by the inclusion of the iconic and computational rigidity.

7.4.6 Strong open pedagogic text and strong content-orientation

Strong open pedagogic texts and strong content-orientation are only evident in Sara's lessons and constitute only three (20%) evaluative events. An example of such a pedagogic text is illustrated in Extract 7.9. In this evaluative event, Sara discusses different forms of a parabola equation and the relation between the form of the equation and key points of the parabola such as the turning point, x-intercepts and y-intercept.

Extract 7.9. transcript S01T01L02: lines 63 - 70

Teacher:	Okay and then this [pointing at $y = ax^2 + bx + c$] could then be written in the form x minus x one x minus x two which is the factorised form [points to $y = a(x - x_1)(x - x_2)$]. So we have seen that the parabola can actually be written in three forms. One two three [points at different forms on board]. What did I call this form [referring to $y = ax^2 + bx + c$]?
Learners:	Standard form.
Teacher:	Standard form. This one I called? [pointing at $y = a(x - p)^2 + q$]
Learner:	[indistinct]
Teacher:	Turning point form
Learners:	Turning point form.
Teacher:	Okay. We said *p q*. If it is in that form *p q* is equal to the turning point. So these are our x cuts and this [referring to x1] would be B [pointing to an x-intercept] and this [referring to x_2] would be C [pointing to a second x-intercept]. Okay. when you factorise it. That point [referring to A] is the …
Learner:	y cut.

Sara recruits two encyclopaedic propositions that support the computational activity in this evaluative event. Firstly she refers to the proposition, if $y = a(x - p)^2 + q$ then the turning point is $(p; q)$ and secondly if

$y = a(x - x_1)(x - x_2)$ then x_1 and x_2 are the x-intercepts of the parabola. Notably, this evaluative event does not contain any auxiliary descriptions, auxiliary propositions or auxiliary operations. The only operation referred to in this event is multiplication in connection with transforming the turning point form of the equation, $y = a(x - p)^2 + q,$ to the standard form, $y = ax^2 + bx + c.$ Multiplication is an operation located in the Mathematics encyclopaedia which takes the real numbers as its domain and codomain.

The recognition and realisation rules employed in this evaluative event produce an open pedagogic text which is content-oriented because propositions from the Mathematics encyclopaedia are recruited as computational resources. Secondly, this evaluative event is coded as a strong open pedagogic text and strongly content-oriented because it does not rely on empirical and/or iconic computational resources. The implied model learner generated through the evaluative activity in this evaluative event is one who is considered capable of engaging with content associated with the topic from the point of view of the Mathematics encyclopaedia.

7.5 Conclusion

This chapter discussed the secondary analysis of the recognition and realisation rules, described in terms of the computational activity, evident in the observed lessons with the aim of developing a description of the computational performance of the implied model learner on the one hand and to infer the orientations to mathematics privileged in the observed lessons on the other hand. In so doing, the secondary analysis provides insight into the functioning of evaluation at the level of the instructional discourse in each pedagogic context.

The secondary analysis of the computational activity evident in the observed lessons reveals five pedagogic modalities described in terms of open and closed pedagogic texts and content-oriented and expression-oriented orientations to mathematics across the four pedagogic contexts: (1) strong open pedagogic text accompanied by a strong content-orientation (T_O^+/O_c^+); (2) weak open pedagogic text accompanied by a weak content-orientation (T_O^-/O_c^-); (3) weak closed pedagogic text accompanied by a weak expression-orientation (T_c^-/O_e^-); (4) weak closed pedagogic text accompanied by a strong expression-orientation (T_c^-/O_e^+); and (5) strong closed pedagogic text accompanied by a strong expression-orientation (T_c^+/O_e^+). With the exception of Maya's pedagogic context, all the others are hybrids exhibiting different combinations of the five pedagogic modalities (see Table 7.4).

The pedagogic contexts populated by learners from working-class families (Evergreen High) exhibit pedagogic modalities characterised as closed pedagogic texts accompanied by an orientation to mathematics that is expression-oriented. Evaluation in the working-class pedagogic contexts generates an implied model learner who is constructed by the computational activity as one who is not capable of engaging with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia. The model learner's computational performance is such that the content realised is closed in relation to the announced topic(s) from the point of view of the Mathematics encyclopaedia. The model learner's orientation to mathematics is expression-oriented because the mathematics constituted is expressively convergent with the announced topic but divergent from the content associated with the announced topic from the point of view of the Mathematics encyclopaedia. In Jono's pedagogic context, mathematics is also constituted as an empirical activity in that the pedagogy is reliant on quasi-induction.

Table 7.4. Summary of pedagogic modalities in the four pedagogic contexts

	Prestige College Upper middle-class/elite		Evergreen High Lower working-class	
	Sara	**Jada**	**Maya**	**Jono**
Pedagogic modality	Hybrid Closed and open pedagogic texts combined with content and expression-orientation but leaning towards weak open pedagogic text and weak content-orientation	Hybrid Closed and open pedagogic texts combined with content and expression-orientation but leaning towards strong closed pedagogic text and strong expression-orientation	Pure Strong closed pedagogic text and strong expression-orientation	Hybrid Closed pedagogic text and expression-orientation but features a quasi-inductive pedagogy
Computational performance	Closed and open pedagogic texts but open pedagogic texts in a greater proportion of evaluative events	Closed and open pedagogic texts but closed pedagogic texts in a greater proportion of evaluative events	Closed pedagogic texts	Closed pedagogic texts
Orientation to mathematics	Content-oriented and expression oriented but content oriented in greater proportion of evaluative events	Content-oriented and expression oriented but expression oriented in greater proportion of evaluative events	Expression-oriented	Expression-oriented

In the case of the pedagogic contexts populated by learners from upper-middle-class/elite families (Prestige College), the pedagogic modalities are hybrids of open and closed pedagogic texts accompanied by content- and expression-orientations to mathematics, with Sara leaning towards open pedagogic texts and content-orientation whereas Jada tends towards closed pedagogic texts and expression-orientation. Evaluation therefore produces a hybrid model learner implied by the computational activity. The model learner is, on the one hand, constructed as capable of engaging with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia, to a greater extent in the case of Sara than Jada and on the other hand, the model learner is constructed as incapable of engaging with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia, to a greater extent in the case of Jada than Sara. The model learner's computational performance is such that the content realised is open in relation to the announced topic(s) from the point of view of the Mathematics encyclopaedia to a greater extent with respect to Sara than Jada. Furthermore, the model learner's computational performance is constructed as that which produces content that is closed with respect to the announced topic(s) from the point of view of the Mathematics encyclopaedia to a greater extent with respect to Jada than Sara. The model learner's orientation to mathematics is content-oriented to greater extent in the case of Sara than Jada and is expression-oriented to greater extent in the case of Jada than Sara. In some respects, the implied model learner in the upper-middle-class/elite context is comparable to the implied model learner in the pedagogic contexts populated by learners from working-class families to a greater degree in the case of Jada than Sara.

In chapter 9, I consider the functioning of evaluation and the corresponding computational performance and orientations to mathematics of actual learners when doing mathematics independently of their mathematics teacher. Before doing so, however, Chapter 8 considers what the setting and marking of the texts imply about the computational performance and orientations to mathematics of the model learner.

Chapter 8

The model learner implied by the setting and marking of tests

8.1 Introduction

Chapter 6 provided a description of the computational activity operative in the observed lessons which revealed the structuring effect of evaluation on the content realised in the observed lessons. Chapter 7 focused on what the computational activity in the observed lessons illuminates about the implied model learner's computational performance and orientation to mathematics. This chapter continues the focus on the implied model learner's computational performance and orientations to mathematics as evidenced in the evaluative activity entailed in the setting and marking of a test administered and marked by the teacher in each pedagogic context. What follows is an analysis of the test and the marking of the test used in each pedagogic context.

8.2 Prestige College test (Sara and Jada)

Prestige College wrote a "common" test, which was a test set by one teacher and written by all Grade 10 learners at the school. The test, referred to as *Common Test Functions 2012* (Appendix 8.1), covered the announced topic functions (linear function, parabola, exponential function and hyperbola). The test therefore included announced topics dealt with during the observed lessons as well as announced topics covered after the observed lessons. The test comprised four problems, each containing sub-problems, and was marked out of a total of 45.

Figure 8.1 shows test problem 1, which was used as the basis for the interviews with learners (see Chapter 9) since the topic assessed in Problem 1 corresponds with the announced topic(s) in the observed lessons. All four test problems are classified as multi-topic mathematics problems because they each involve more than one announced topic. Problem 1 references four announced topics: (1) graphing parabola (Questions 1.1, 1.2 and 1.3); (2) the range of functions (Question 1.4); (3) the points of intersection (Question 1.5); and (4) the values of x for which $f(x) < 0$ (Question 1.6).

Grade 10 **Common Test Functions 2012** **Total 45 marks**

1. Given $f(x)=-2x^2+2x+4$ and $g(x)=-8x+4$

1.1 Determine the co-ordinates of the x-intercepts of graph f. (3)

1.2 Determine the co-ordinates of the turning point of f. (3)

1.3 Draw the graphs of $f(x)=-2x^2+2x+4$ and $g(x)=-8x+4$ on the same set of axes on your answer sheet. Label the graphs clearly, including all the intercepts with the axes and the turning point. (5)

1.4 Write down the range of f. (1)

1.5 Determine algebraically the coordinates of the points of intersection of $f(x)$ and $g(x)$ (5)

1.6 Use your graphs to determine for which values of x, $f(x)<0$. (2)

Figure 8.1. Problem 1 of test administered by Prestige College teachers

Problem 1 of the *Common Test Functions 2012* is very similar to Question 2 of the *Parabola worksheet* (see Figure 8.2). The *Parabola worksheet*, which was used during the observed lessons as a classroom task and given as homework, consisted of three multi-topic problems on the parabola and straight line. In addition, learners were given a *Tutorial on graphs* which they were required to submit for marks. The *Tutorial on graphs* covered the same topics as the *Common Test Functions 2012* and consisted of mono-topic problems such as "sketch the graph on separate axes. Show all intercepts with the coordinate axes, turning points, axes of symmetry and asymptotes" as well as multi-topic problems such as those shown in Figure 8.3. The *Tutorial on graphs* therefore served as preparation for the test, which in turn appeared to be intended as preparation for the examination.

Question 2

$f: y=x^2-2x-3$ and $g: y=-3+x$

2.1 Draw neat sketch graphs of f and g on the same system of axes and show the intercepts on both axes as well as the co-ordinates of the turning point of the graph of f. (8)

2.2 Use your graphs to read off the values of x for which $f(x)=g(x)$ (2)

2.3 State the domain and the range of f. (4)

Figure 8.2. Extract from *Parabola worksheet* used by Prestige College teachers

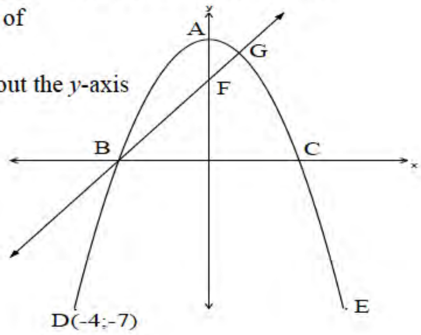
1.5 In the diagram alongside the graphs of $f(x) = -x^2 + 9$ and $g(x) = 2x + 6$ are shown. Write down the coordinates of

(a) A, B and C

(b) E, the point symmetrical to D about the y-axis

(c) F

1.6 Solve the equation $-x^2 + 9 = 2x + 6$ and explain how this helps find two of the points labeled on the graph.



Question 2

Consider the functions $s(x) = x^2 - 9$ and $t(x) = 2x - 6$

2.1 Sketch the graphs of s and t on the <u>same</u> set of axes, showing all intercepts and turning points

2.2 Use your sketch to find the values of x for which

2.2.1 $s(x) = t(x)$

2.2.2 $s(x) > 0$

2.2.3 $s(x) < t(x)$

Figure 8.3. Extract from *Tutorial on graphs* used by Prestige College teachers

Prestige College learners were therefore given considerable practice opportunities by their teachers, mostly independently of the teacher, since the tasks were either given as homework exercises or tutorials that were required to be submitted. In addition, Prestige College learners were exposed to variations in phrasing of mathematics problems. For example, problems on calculating the points of intersection of two functions were posed in different ways in the *Parabola worksheet* and the *Tutorial on graphs*. The problems on intersections of functions were stated as follows: (1) calculate the points of intersection of the functions f and g; or (2) calculate the values for x for which $f(x) = g(x)$; or (3) solve the equation $-x^2 + 9 = 2x + 6$ and explain how this helps find two of the points labelled on the graph or (4) calculate the coordinate of the point T. In the last problem, learners were first expected to work out that point T represents a point of intersection of two functions.

The *Parabola worksheet, Tutorial on graphs* and *the Common test functions* used by Prestige College teachers contained mathematics problems which did not directly name the procedure required to solve the problem. Learners were, therefore, expected to analyse the problem statement in order to decipher the appropriate procedure required. For example, Problem 1.5 (Figure 8.3) requires that learners identify what the coordinates A, B, C and F represent and then select the appropriate procedure to calculate the coordinates. Thus, Prestige College learners were provided with problems that required analysis in order to identify the required procedure. So, evaluation functions in a way that attempts to move beyond recall and rehearsal of procedures for solving particular classes of mathematical problems. Furthermore, the test, like the worksheet and tutorial, encourages inter-topic connectivity in that mathematical problems focus on more than one topic simultaneously and so require learners to select the appropriate computational resources. The evaluative activity instantiated in the test suggests an open pedagogic text as opposed to a closed pedagogic text.

8.3 Marking of Prestige College test (Sara and Jada)

The memorandum of *Common Test Functions 2012* (Appendix 8.2) provides solutions to the test problem as well as the allocation of marks. Figure 8.4 which shows the memorandum for Problem 1. The memorandum seems very specific as suggested by the comment "must give co-ords" in 1.5 and the details of how Problem 1.3 should be marked.

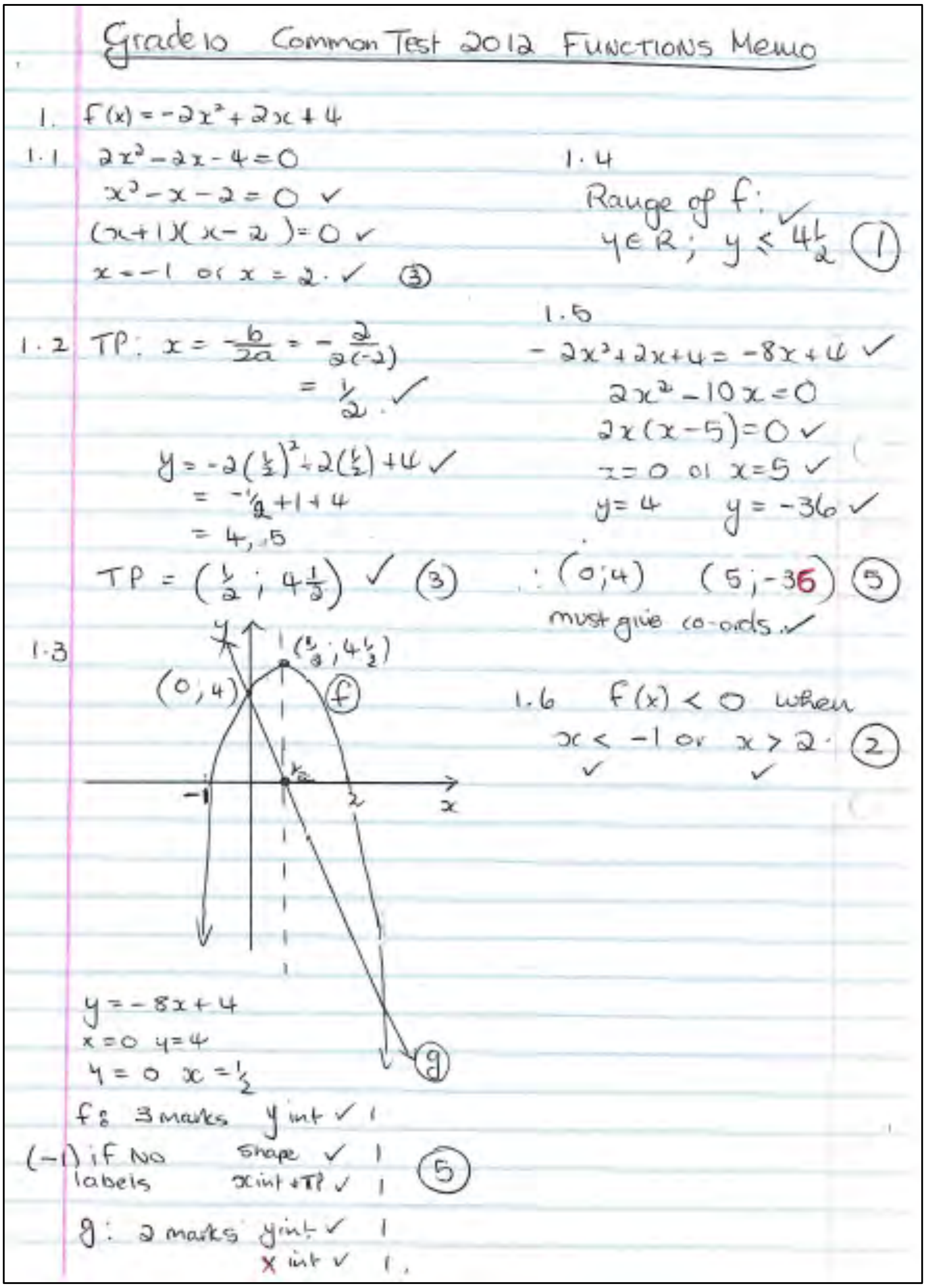


Figure 8.4. Memorandum for Problem 1 of Prestige College test

It is interesting that in Problem 1.1, marks are allocated for factorising the quadratic equation and writing down the x-intercepts but not for setting up the equation, that is, for establishing $f(x) = 0$ which is central to calculating the x-intercepts of the function. The mark allocation suggests evaluative criteria that prioritise obtaining the correct values of the x-intercepts without the notion of an equation serving as a regulative resource. This hypothesis is borne out by the marking of the test scripts by Sara.

Sara mostly makes the evaluative criteria explicit to learners by correcting errors (see Figure 8.5) and/or providing evaluative commentary (see Figure 8.6), except for Ted's (P01) solution to Problem 1.6 where she neglected to point out that that he should have used "OR" rather than "AND" (see Figure 8.7)[71]. Her failure to correct Ted's solution might be an oversight on her part given that her marking of learners' test scripts was generally consistent.

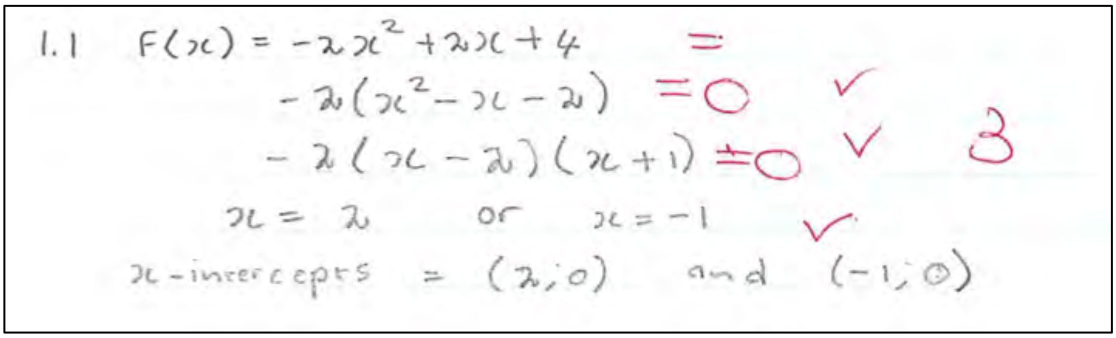

1.1 $F(x) = -2x^2 + 2x + 4 =$
$-2(x^2 - x - 2) = 0$
$-2(x - 2)(x + 1) = 0$ 3
$x = 2$ or $x = -1$
x-intercepts $= (2;0)$ and $(-1;0)$

Figure 8.5. Ted's (P01) marked solution to Problem 1.1

Sara awarded full marks for Problem 1.1 to two learners (Ted and Pat) despite the fact that they generated the correct x-intercepts without setting up an equation (see Figures 8.5 and 8.6). Her marking of Ted's and Pat's solution as correct despite the explicit absence of an equation as a regulative resource might suggest an orientation to mathematics that is expression-oriented rather than content-oriented, that is, an orientation that focuses on producing the correct expressions irrespective of the associated content. However, the fact that she corrected the learner's solution indicates that she does make the evaluative criteria that the learner ought to display explicit.

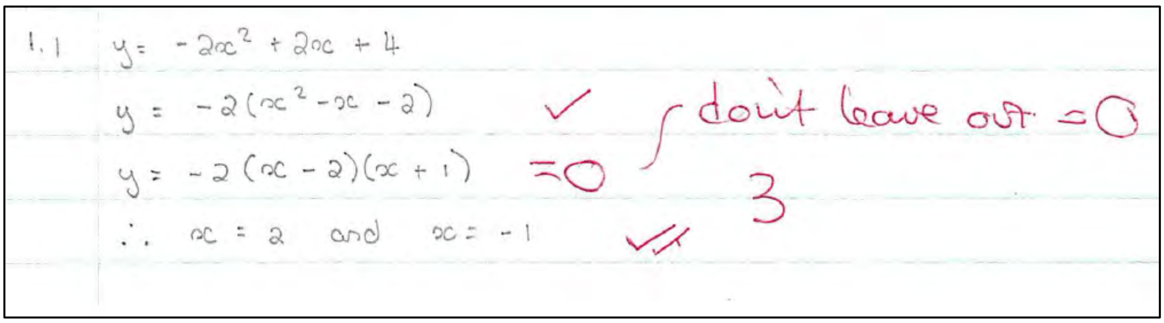

1.1 $y = -2x^2 + 2x + 4$
$y = -2(x^2 - x - 2)$ don't leave out $= 0$
$y = -2(x - 2)(x + 1) = 0$ 3
$\therefore x = 2$ and $x = -1$

Figure 8.6. Pat's (P18) marked solution to Problem 1.1

[71] The use of "OR" and "AND" in capitals signify logical connectives as opposed to ordinary language grammatical conjunctions "or" and "and'.

Similarly, in Sara's marking of Problem 1.6 , she indicates that "AND" is incorrect in Tom's solution (Figure 8.7) but awards full marks. Similarly, Ted's solution (Figure 8.8) is awarded full marks for problem 1.6 even though he used "OR" rather than "AND". So, it appears that the correct usage of the logical connective "OR" rather than "AND" is not deemed important (see Figure 8.7 and Figure 8.8) to Problem 1.6).

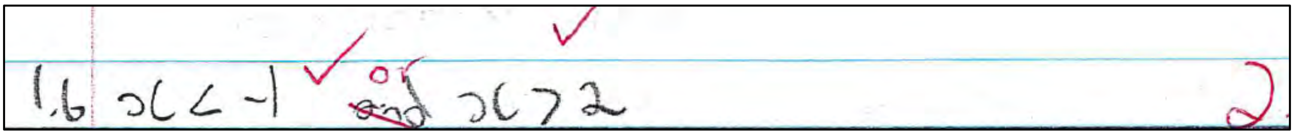


Figure 8.7. Tom's (P10) marked solution to Problem 1.6

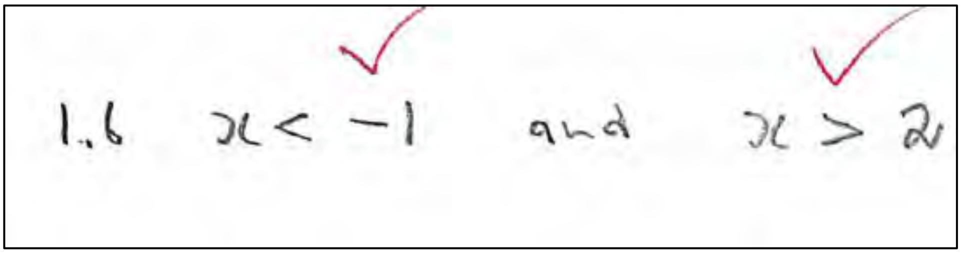


Figure 8.8. Ted's (P01) marked solution to Problem 1.6

It is curious though that Sara is not prepared to accept the solution produced by Wes (see Figure 8.9[72]) as correct. However, the statement $2 < x < -1$ produced by Wes (Figure 8.9) is equivalent to the statement, $x < -1$ and $x > 2$, produced by Tom (Figure 8.7) and Ted (Figure 8.8). Both statements imply that there is a number, *x,* which is simultaneously greater than 2 and less than -1 and so disrupt the order relation.

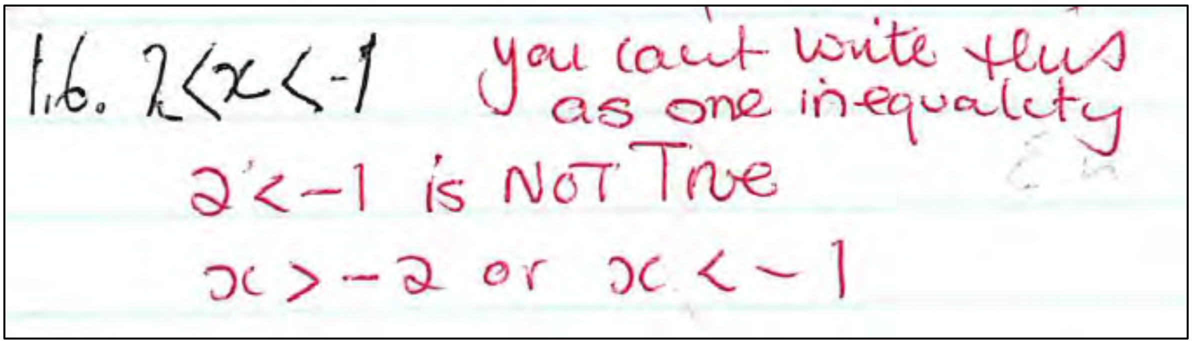


Figure 8.9. Wes; (P19) marked solution to Problem 1.6

Sara's marking of Tom's and Ted's solution to Problem 1.6 suggests that the logical connectives and order relations are not explicitly required as regulative resources and indicates content that diverges both at the level of expression and at the level of content is accepted as correct. Furthermore, her marking indicates that the presence of the expressions $/x < -1/$ and $/x > 2/$ are prioritised over the solution set that satisfies the

[72] Note that Sara makes a mistake when she provides the corrected solution. She stated the solution is $x > -2$ or $x < -1$. The correct solution is $x > 2$ or $x < -1$.

condition that $f(x) < 0$ (see Problem 1.6 in Figure 8.1), suggesting an orientation to mathematics that is expression-centred rather than content-centred.

Jada's marking was consistent across learners' test scripts and she made the evaluative criteria explicit to learners by correcting errors and/or providing evaluative commentary (see Figure 8.10).

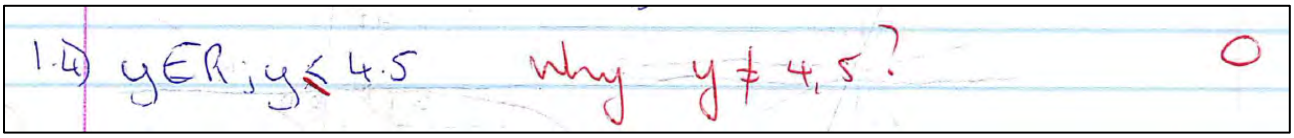


Figure 8.10. Noa's (P12) solution to Problems 1.4

In contrast to Sara, Jada deducted marks for errors committed by learners. For example, she deducted one mark for not equating $f(x)$ with 0 when solving Problem 1.1 (see Figures 8.11 and 8.12). She therefore prioritises the notion of an equation as a central regulative resource in solving Problem 1.1. Her marking of learners' solutions stands in opposition to Sara's marking and appears to be more content-oriented than expression-oriented.

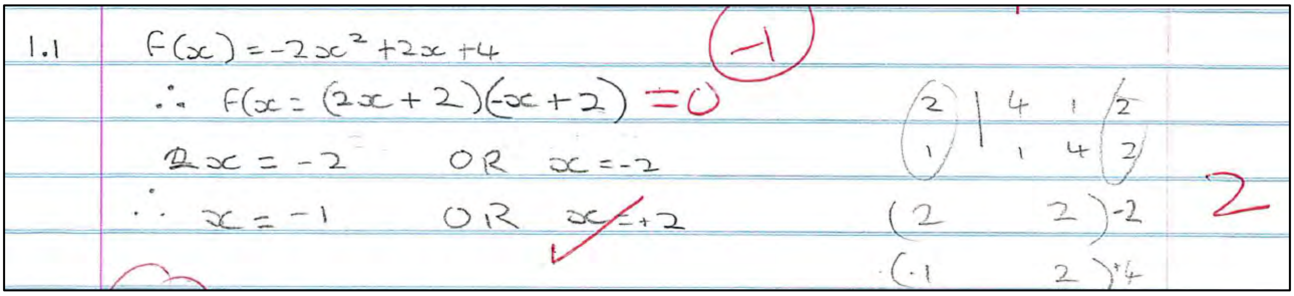


Figure 8.11. Jon's (P24) solution to Problem 1.1

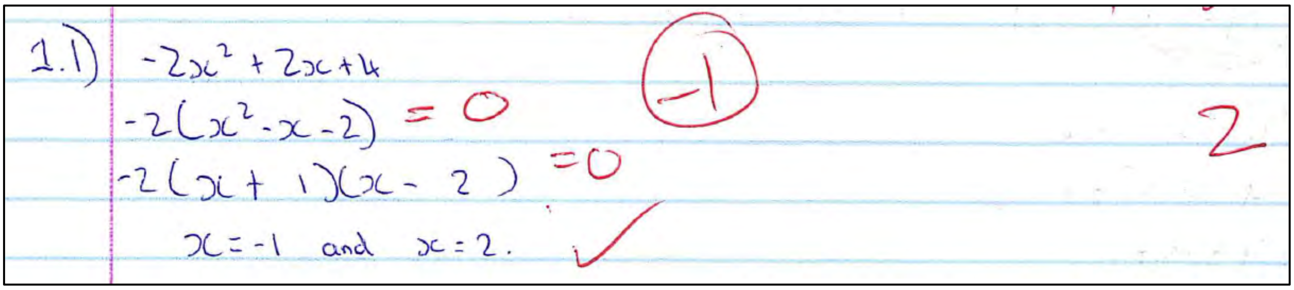


Figure 8.12. Noa's (P12) solution to Problem 1.1

The orientation to mathematics implied by Sara's and Jada's marking is contrary to the orientation to mathematics implied by the computational activity present in the observed lessons, where the orientation to mathematics in Sara's pedagogic context was more content-oriented than in Jada's pedagogic context.

8.3 Evergreen High test (Maya)

At Evergreen High, each teacher set their own test. So, the tests administered by the two teachers at Evergreen High differed in terms of announced topics assessed. Maya's test (Appendix 8.3) covered the topics dealt with during the observed lessons and consisted of four test problems. Problems 1, 2 and 3 entailed finding the equation of a given function provided as a sketch (see Figure 8.13 for Problems 1 and 2) and Problem 4 focused on sketching the graph of the function $y = \frac{-3}{x+1} + 2$.

Problems 1-3 are of the type "calculate the equation of the function" and Problem 4 of the type "sketch the graph of the function", which are the problem types covered in class during the observed lessons. All the test problems are classified as mono-topic problems. The test problem types are the same as the classroom exercises but the examples differed.

The test, like the problems and worksheet (see Figure 8.14), used during the observed lessons directly named the procedure that learners were expected to carry out. Although not stated on the worksheet, learners were instructed to find the equation of the given function. As such, Maya's learners were not expected to analyse problems in order to select a particular procedure for solving a problem. Furthermore, learners in Maya's pedagogic context were not exposed to variations in the phrasing of problem types. The absence of problems that require analysis in order to select appropriate procedures for solving the problem and the lack of variation in problem statements are suggestive of closed pedagogic texts which attempt to elicit precise responses from learners through the rehearsal of particular procedures for solving particular problems. In other words, learners are encouraged to recognise problem types and then match the correct procedure.

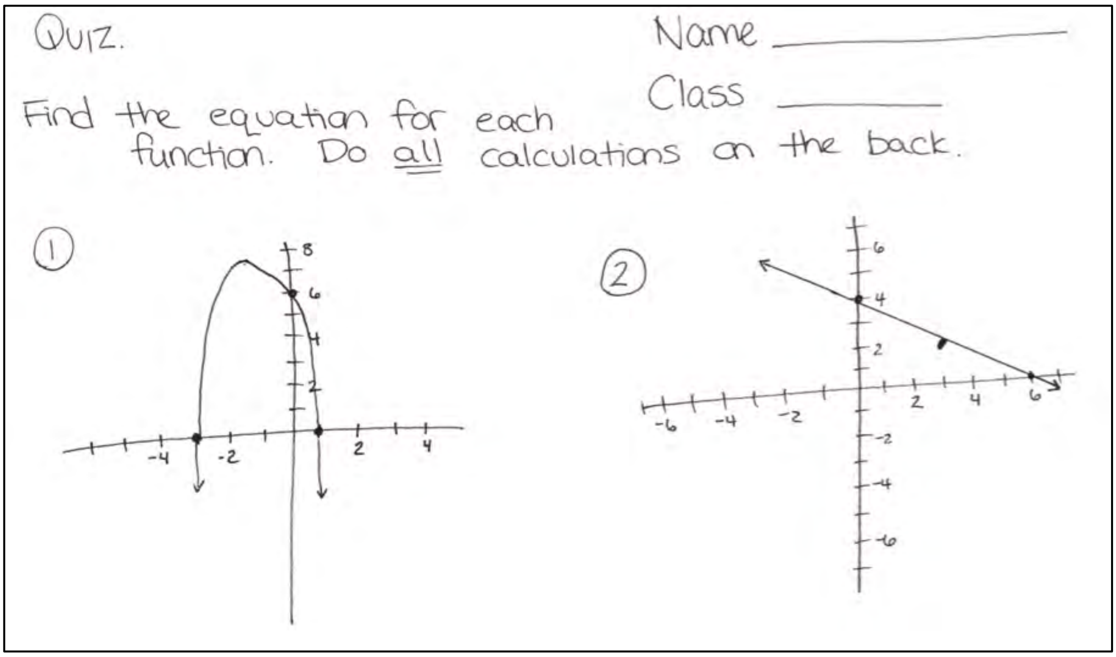


Figure 8.13. Extract of test on functions administered by Maya

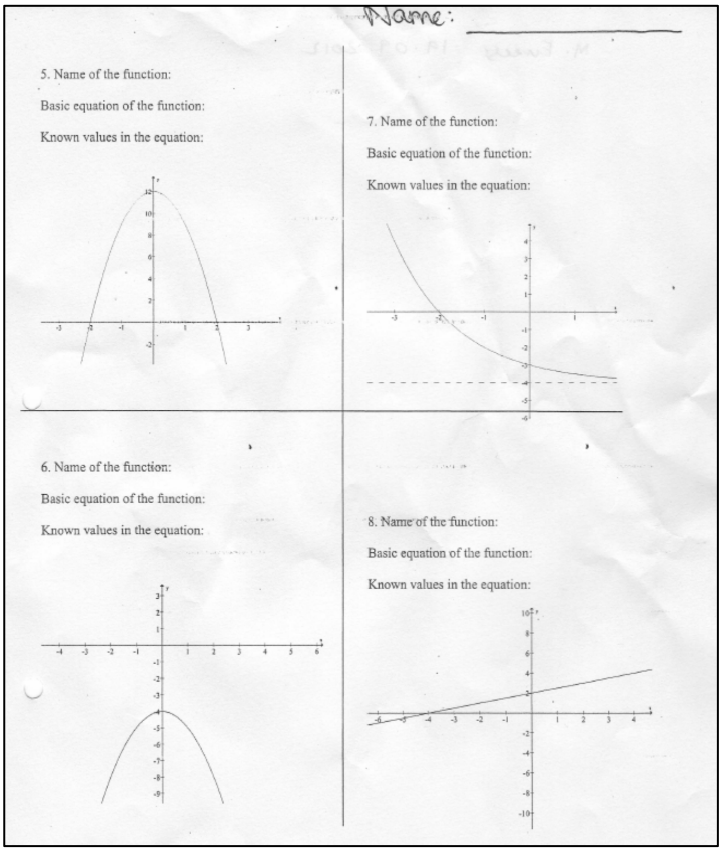


Figure 8.14. Worksheet used by Maya during the observed lessons

Furthermore the test like the worksheet treated topics separately. So therefore, unlike the Prestige College test, the test suggests a lack of inter-topic connectivity.

Only sketches of functions were provided in Problems 1, 2 and 3 of the test (see Figure 8.13), as was the case for the mathematics problems presented during the observed lessons and the worksheet (see Figure 8.14). In other words, the problems did not state what type of function is represented nor was the general equation of the function provided. Learners were thus expected to determine the type of function from the sketch, suggesting that iconic ground functions as a regulative resource because it is the imagistic features of the text that learners are expected to draw on in order to determine the type of function represented.

8.4 Marking of Maya's test

The total mark of the test and mark allocation per problem were not provided to learners and the teacher's memorandum (Appendix 8.4) does not show the mark allocation (see the teacher's memorandum to test problems 1 and 2 shown in Figure 8.15). The teacher's memorandum only shows the answers to the test problems suggesting that the final expression is more important than the method for solving the problem. The presence of final expression without the solution method suggests an expression-orientation.

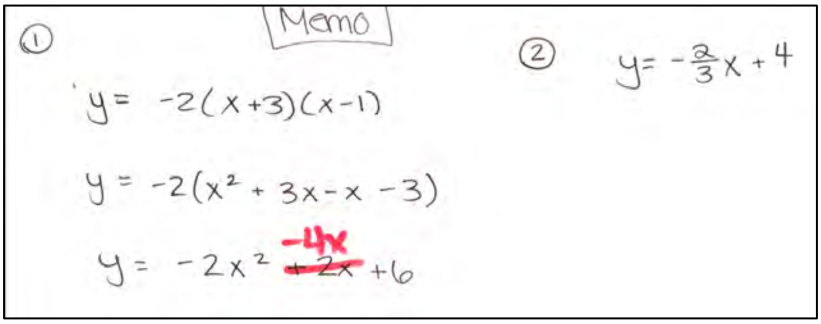


Figure 8.15. Maya's memorandum to test problems 1 and 2

The marked solution of a learner's response to Problem 1 (see Figure 8.16) provides much better insight into the teacher's evaluative criteria. From the marked scripts, it becomes apparent that the teacher allocated four marks per problem bringing the total of the test to 16. We observe that the teacher allocates marks in the following way: one mark was allocated for the general formula, $y = ax^2 + bx + q$, one mark each for the obtaining $a = 2$ and $b = -4$ and one mark for writing the equation as $y = -2x^2 - 4x + 6$.

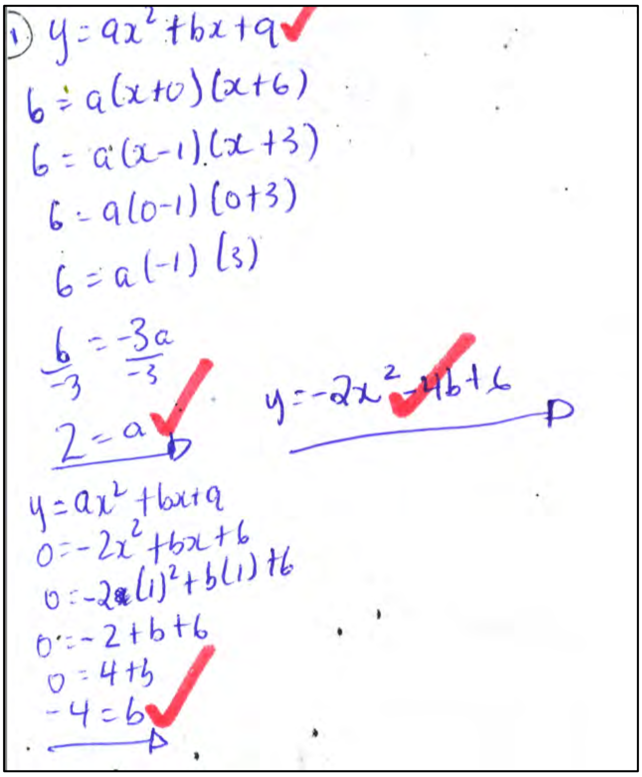


Figure 8.16. Nia's (P02) marked solution to Problem 1

In the marking of learners' test scripts, Maya at times indicated that an error was produced by a learner (see Figure 8.17) where the learner has an incorrect gradient for the straight line and Maya deducts one mark to penalise the learner. On a number of occasions though, errors produced by learners are not highlighted by the teacher and are marked as correct.

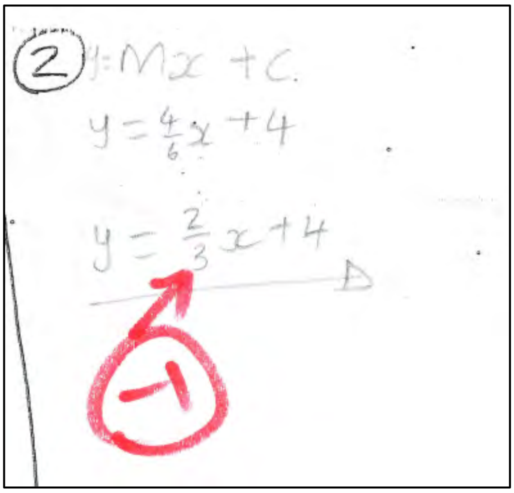


Figure 8.17. Max's (P09) marked solution to Problem 2

For example, in Figure 8.16, Maya awards full marks to Nia for the solution to Problem 1 despite a number of computational inconsistencies with respect to the Mathematics encyclopaedia. Firstly, after writing down the expected general formula $y = ax^2 + bx + q$, Nia produced the expression $6 = a(x + 0)(x + 6)$ which is incorrect but not pointed out by Maya. Secondly, Nia incorrectly computes $\frac{6}{-3}$ as 2 but her value of a is marked as correct by the teacher, presumably because the teacher assumes that Nia has made a computational error which she corrects later in her solution. Thirdly, Nia's final equation $y = -2x^2 - 4b + 6$ is marked as correct even though it is incorrect presumably because Nia positioned the value of b in the "correct place" in the equation. Maya's marking of Nia's solution confirms the strong expression-orientation to mathematics established in Chapter 7 since her assessment of learners' work validates the production of the expected expressions despite divergence from the mathematics content associated with the topic. Further confirmation of the strong expression-orientation cultivated through the evaluative activity of the teacher is evident in her marking of another learner's (Fay) solution shown in Figure 8.18.

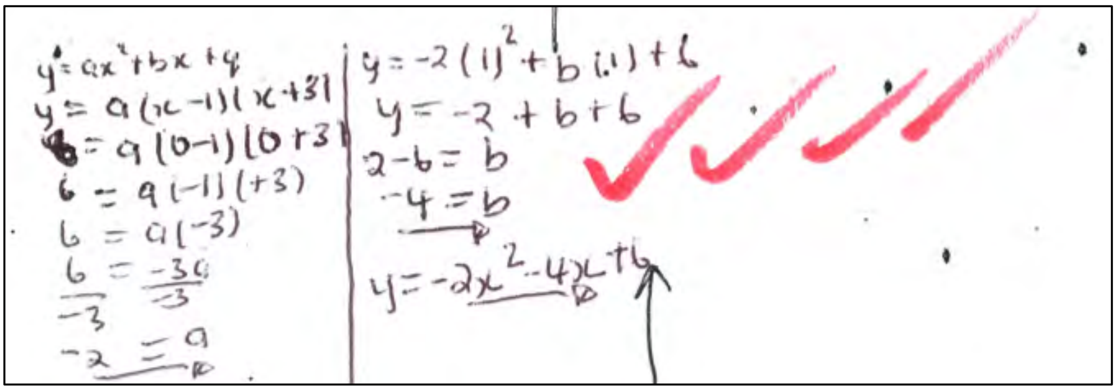


Figure 8.18. Fay's (P01) marked solution to Problem 1

Fay produces the correct value for b even though her solution indicates that the notion of an equation, central to the computation, is absent. She produces the expression $2 - 6 = b$ from the expression $y = -2 + b + 6$, which is incorrect because y simply "disappears". Despite the mathematical inconsistencies produced by Fay, the teacher awards her full marks. (Fay's computational activity is discussed further in Chapter 9.) So, the orientation to mathematics privileged by Maya's evaluation of learners' mathematical work reveals that mathematics constituted in this pedagogic context is primarily a form of mathematical knowledge which diverges from the Mathematics encyclopaedia at the level of expression and at the level of the content associated with the topic. The teacher's marking of learners' solutions suggests that her evaluation cultivates an orientation to mathematics which is expression-oriented and confirms our analysis of the implied model learner in Maya's pedagogic context discussed in Chapter 7.

8.5 Evergreen High test (Jono)

Jono's test (Appendix 8.5) comprises 10 problems requiring learners to "find the domain and range" of functions, two of which are linear functions (Section A), three quadratic functions (Section B), three hyperbolic functions (Section C) and two exponential functions (Section D). The test is an extract of the worksheet used during the observed lessons and resembles the worksheet both in its structure and mathematics problems. In fact, the test constitutes a selection of items from the worksheet evident when we compare Sections A and B of the test (Figure 8.19) to Sections A and B of the worksheet (Figure 8.20).

Gr 10 functions TEST: domain and Range Name: ____________________

70 minutes 10 October 2012

A. Find the **domain** and **range** of following linear functions (write it as **interval** and **set notations**):

1. $\frac{x}{2} - \frac{y}{3} = 1$
2. $y = x + 1$

B. Find the **domain** and **range** of following quadratic functions (write it as **interval** and **set notations**):

1. $y = x^2 - 2$
2. $y = x^2 + 1$
3. $y = -x^2 + 2$

Figure 8.19. Test on domain and range of functions administered by Jono

A. Find the **domain** and **range** of following linear functions (write it as **interval** and **set notations**):

1. y = x+1
2. H(x) = 2x-1
3. 3y-2x = 6
4. X= 3y
5. $\frac{x}{2}-\frac{y}{3}=1$

B. Find the **domain** and **range** of following quadratic functions (write it as **interval** and **set notations**):

1. $y=x^2-2$
2. $y=x^2+1$
3. $f(x)=-x^2+2$
4. $f(x)=x^2$

Figure 8.20. Part A of the worksheet used in Jono's observed lessons

Table 8.1 lists the test problem (see Appendix 8.5) and the corresponding problem in the worksheet (Appendix 6.10) used by Jono during the observed lessons, thus illustrating that all the problems from the test were selected from the worksheet, sometimes in the same order.

Table 8.1. Comparing the test and worksheet problems

Test problem	Worksheet problem
A1	A5
A2	A1
B1	B1
B2	B2
B3	B3
C1	C1
C2	C2
C3	C3
D1	D1
D2	D5

All the test problems are classified as mono-topic mathematics problems. Jono's learners, like Maya's learners, were not expected to analyse problems in order to select a particular procedure for solving a problem. The function type was identified for the learner, thus generating a test of low complexity because learners mostly needed to recall the propositions with respect to each function type established during the observed lessons.

Furthermore, Jono's learners, like Maya's learners, were not exposed to variations in the phrasing of problem types. The absence of problems that required analysis in order to select appropriate procedures for solving them and the lack of variation in problem statements suggest that learners were expected to rehearse and repeat particular procedures for solving particular problems, typical of an expression-orientation to mathematics. Secondly, the learners had seen the test problems and worked through the problems in class. It seems that the test assesses whether learners are able to repeat the texts produced in class under test conditions. In other words, the evaluation encourages learners to reproduce texts, that precisely conform with the text considered as legitimate in the pedagogic context, through repetition and rehearsal. The test is therefore strongly suggestive of an orientation to mathematics which is expression-oriented and thus confirms the dominant pedagogic modality established in Chapter 7.

8.6 Marking of Jono's test

The test memorandum (Appendix 8.6) provides solutions to the test problems but how marks ought to be awarded is not made explicit. Figure 8.21 shows a section of the memorandum, the solution to A1. The equation $\frac{x}{2} - \frac{y}{3} = 1$ however is not equivalent to $y = \frac{3x-3}{2}$. The error, however, has no bearing on the domain and range of the function.

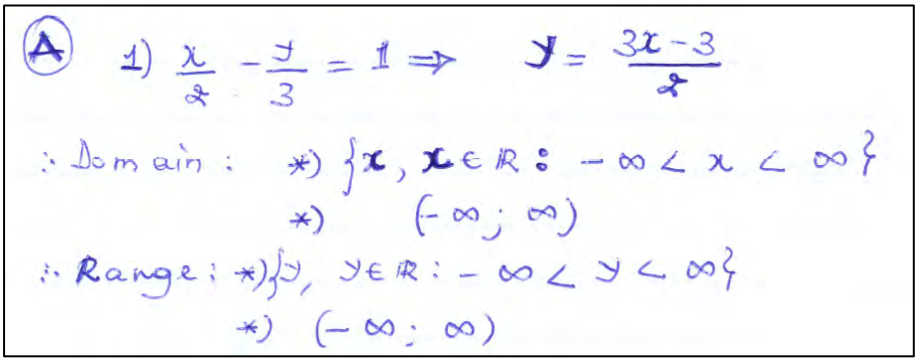


Figure 8.21. Jono's solution to test problem A1 from test memorandum

The marked tests show that two marks were allocated per test problem, half a mark each for the domain and range expressed in set-builder notation and interval notation (see Figure 8.22).

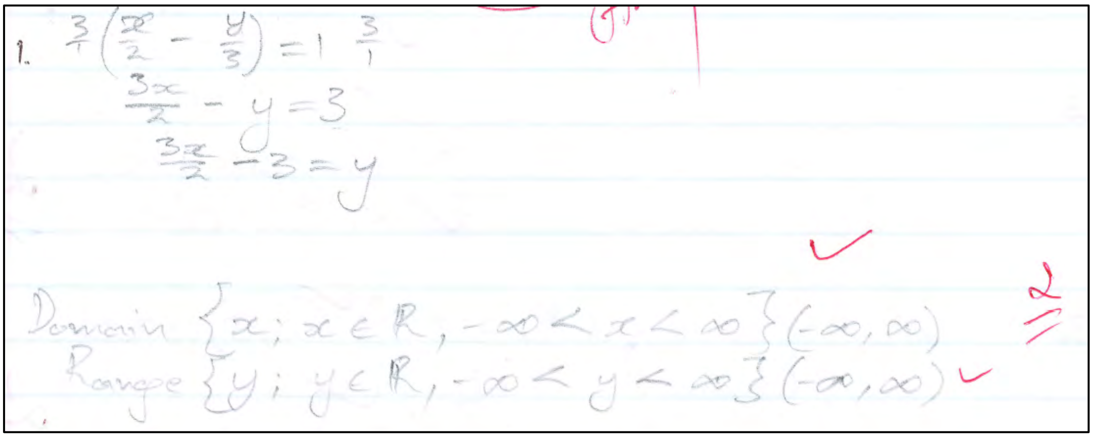


Figure 8.22. Tim's (P01) solution to Problem A1

Table 8.2 shows the solutions to problem A1 produced by learners selected to be interviewed and whether the solution is correct or incorrect according to the teacher's memorandum which indicate what is considered as legitimate in this pedagogic context. Table 8.2 reveals the inconsistencies in the teacher's marking of the learners solution to problem A1 in that incorrect components are treated as though they are correct,

Table 8.2. Solutions to Problem A1 of Jono's learners selected to be interviewed

Name	Code	Domain (set builder)	Domain (interval)	Range (set builder)	Range (interval)	Mark awarded
Tim	P01	$\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$ correct	$(-\infty; \infty)$ correct	$\{y: y \in \mathbb{R};\ -\infty < x < \infty\}$ correct	$(-\infty; \infty)$ correct	2
Ali	P02	$\{x/\ x \in \mathbb{R};\ -\infty > x < \infty\}$ incorrect	$(-\infty; \infty)$ correct	$\{y/\ y \in \mathbb{R};\ -\infty > x < \infty\}$ incorrect	$(-\infty; \infty)$ correct	2
Ozi	P11	$\{x: x \in \mathbb{R};\ -\infty > x > \infty\}$ incorrect	$(\infty; \infty)$ incorrect	$\{y: y \in \mathbb{R};\ -\infty > y > \infty\}$ incorrect	$(-\infty; \infty)$ correct	1,5
Lea	P12	$\{x: x \in \mathbb{R};\ x \neq 0\}$ incorrect	$(0; \infty)$ incorrect	$\{y: y \in \mathbb{R};\ -\infty < x < \infty\}$ incorrect	$(-\infty; \infty)$ correct	1
Ory	P17	$\{x: x \in \mathbb{Z};\ x < 3\}$ incorrect	$(-\infty; \infty)$ correct	$\{y: y \in \mathbb{R};\ y \geq 3\}$ incorrect	$(-\infty; \infty)$ correct	1
Zoe	P18	$\{x: x \in \mathbb{R};\ -\infty;\ \infty\}$ incorrect	$(-\infty; \infty)$ correct	$\{y: y \in \mathbb{R};\ \neq 0\}$ incorrect	$(-\infty; \infty)$ correct	1,5

Ali (P02) produces incorrect statements for the domain and range in set-builder notation but is awarded full marks by the teacher. Similarly, Ozi (see Figure 8.23) and Lea have three parts to the solution to problem A1 incorrect but Ozi is awarded 1,5 marks and Lea is awarded one mark. Similarly, Ory and Zoe both produce the correct domain and range in interval notation despite the fact that both learners obtained the set-builder notation incorrect. Ory was awarded one mark and Zoe was awarded 1,5 marks.

Over and beyond the inconsistencies in the marking of learners' test scripts, Jono at times corrected errors as he does with Ozi's solution to problem A1 (Figure 8.23) and on other occasions he neglected to identify the errors as is evident in his marking of Ali's solution to Problem A1 (see Figure 8.24).

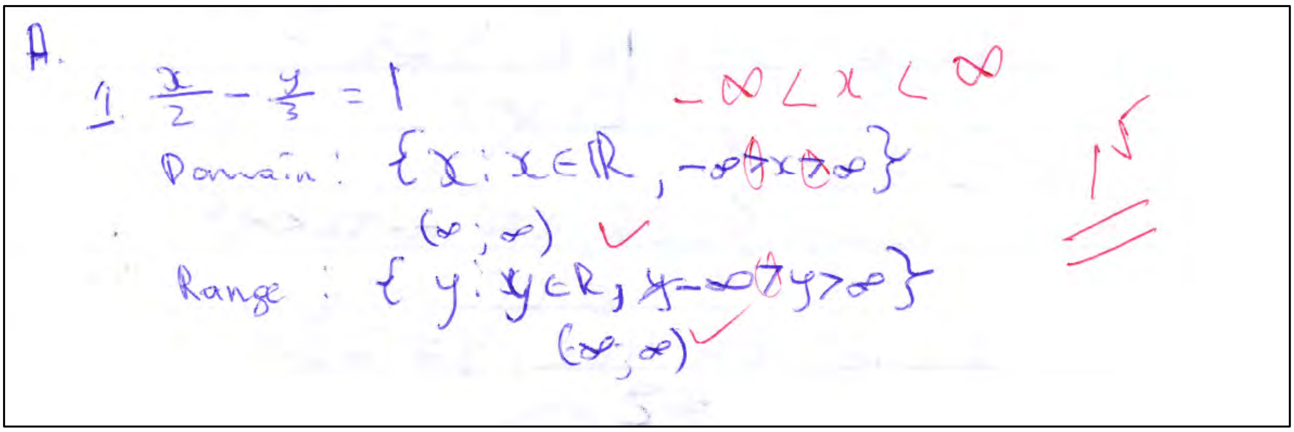


Figure 8.23. Ozi's (P11) solution to Problem A1

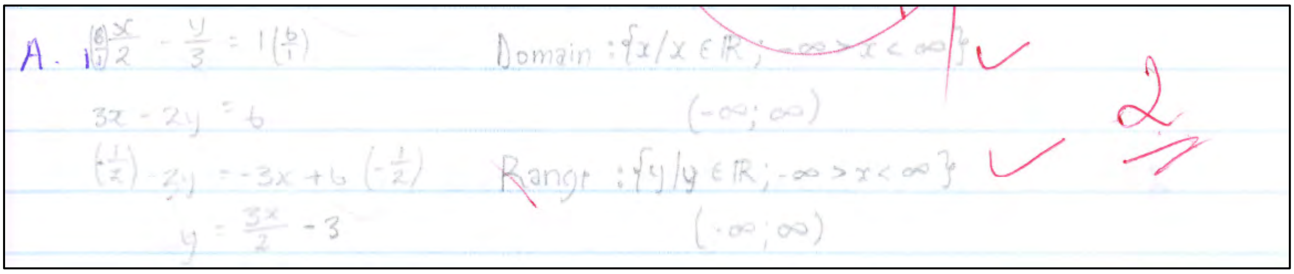


Figure 8.24. Ali's (P02) solution to Problem A1

Ali's statements for the domain, $\{x/\,x \in \mathbb{R};\ -\infty\ >\ x\ <\ \infty\}$, and range, $\{x/x \in \mathbb{R};\ -\infty\ >\ x\ <\ \infty\}$, of the function $\frac{x}{2} - \frac{y}{3} = 1$ in set-builder notation are incorrect but they are not corrected by the teacher. In fact, Ali makes the same mistake throughout the test but Jono does not ever correct the error. The fact that Ali was awarded full marks for his solution to Problem A1 and the fact the Jono does not correct Ali's errors suggest that Jono does not require learners to use order relations as regulative resources. Jono's evaluative activity as instantiated in the marking of learners' test scripts validates content that differs from the Mathematics encyclopaedia at the level of expression as well as the level of content. In addition, his evaluation of learners' test scripts produces a closed pedagogic text and represents an extreme version of expression-orientation in that as long as the expressions produced by learners resemble the correct expression they are accepted as correct.

8.7 Conclusion

Comparing the tests across the two schools reveals differences in the preparation for tests in the four pedagogic contexts. At Prestige College learners' preparation involved independent work on worksheets and tutorials that pose mathematics problems in different ways. Evergreen High learners' preparation seems to be restricted to the mathematics problems encountered during the observed lessons with no variation in the way problems are phrased. Evergreen High tests appear to encourage an expression-orientation given the similarity of the test and worksheets used in the observed lessons, with Jono's test representing an extreme case of expression-orientation because the test is a repetition of the worksheet used during the observed lessons. The test used in Prestige College appears to be more content-oriented

Comparing the marking of the test also reveals differences across the four pedagogic contexts. Maya and Jono's marking included instances where mathematical violations were not corrected by the teacher and so were accepted as correct. Furthermore their marking is inconsistent and at times learners' solutions marked as correct did not match their memoranda. Sara and Jada corrected learner errors, thus making learners' errors explicit to them. Sara, however, at times did not deduct marks even though the solutions contained errors and on one occasion marked an incorrect solution as correct. Thus, Sara's, Jono's and Maya's marking encourages an expression-orientation to mathematics. Jono's and Maya's marking confirm the expression-orientation observed in the observed lessons. However, Sara's marking contradicts the content-orientation evident in the observed lessons. Jada's marking suggests content-orientation which differs from the expression-orientation evident in the observed lessons.

The absence of multi-topic mathematics problems in the Evergreen High tests corresponds with the absence of multi-topic mathematics problems in the observed lessons, which indicates that topics were treated in isolation by Evergreen High teachers thus resulting in a lack of inter-topic connectivity. As such, Evergreen High learners were left to make connections between topics independently of the teacher. In this case, it could be argued that synthesis of school mathematics topics into a coherent whole was made much harder for the working-class learners than the upper-middle-class/elite learners.

Chapter 9

Specialisation of learners' mathematical thought - computational performance and orientation to mathematics

9.1 Introduction

The principal concern of Chapters 6 and 7 was the functioning of evaluation in the instructional discourse. Chapter 8 focused on the evaluative activity evident in the setting of the test and marking of learners' test scripts and the implications for the model learner. The present chapter focuses on the evaluative activity of learners when doing mathematical work independently of the teacher. I present the analysis of learners' recognition and realisation rules, described in terms of computational activity, evident in their solutions to a mathematics test set by their teacher and interviews with selected learners in each pedagogic context on their solutions to selected test problems in order to reveal their specialisation of consciousness, described as: 1) the computational performance of the selected learners; and 2) the orientation to mathematics of selected learners. The computational performance and orientation to mathematics of actual learners will be compared with that of the model learner presupposed by the instructional discourse. Through an analysis of learner test performance and learner interviews, it is possible to consider the extent to which the computational activity in the instructional discourse structures the computational performance of learners and their orientation to mathematics.

The central interest, in relation to the learners' test scripts and the interviews with learners, lies in their computational activity displayed when solving mathematics problems independently of the teacher. In doing so, I am examining the recognition and realisation rules used by learners when doing mathematical work independently of the teacher, bearing in mind that it is possible for recognition and realisation rules to differ across learners and for these to differ from the teacher even though there may be convergence at the level of expression.

In each pedagogic context, the teacher set, administered and marked a test on the announced topic(s) covered during the observed lessons. I interviewed six learners (two top performers, two mid performers and two bottom performers on the test) in each pedagogic context with the exception of Jono's class (where only four learners were interviewed) in order to ascertain the learners' recognition and realisation rules, described in terms of their computational activity. This provides the basis for establishing the nature of the specialisation of consciousness described in terms of their computational performance and the orientation to mathematics displayed by learners. Before doing so, I report on the learners' performance on the tests.

9.2 Learner performance on the tests

The test scores obtained by learners in each pedagogic context on a test set and marked by the teachers are presented graphically as percentages in Figures 9.1 to 9.4.

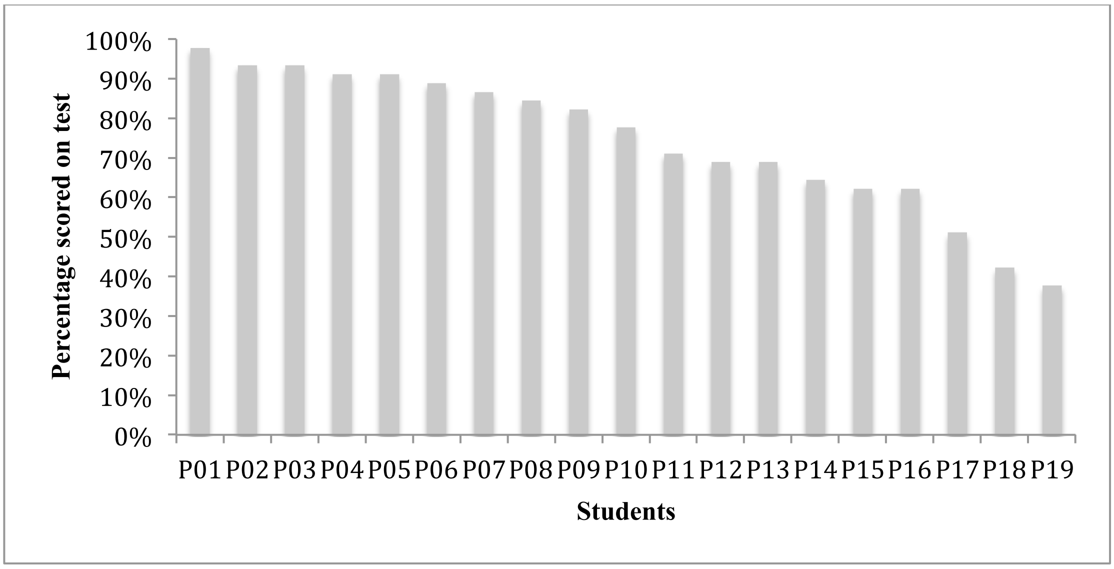


Figure 9.1. Sara's learners' test scores

At Prestige College, 19 of the 21 learners wrote the test in Sara's class and all learners (24) wrote the test in Jada's class. The average score in Sara's class was 75% with the highest score being 98% and the lowest 38% (see Figure 9.1). Jada's learners scored an average of 69% on the same test with the highest result of 96% scored by two learners and the lowest score being 29% (see Figure 9.2). The scores of the two groups of learners are comparable given that the learners wrote the same test. The difference in performance of the two groups of learners on the test is expected since Sara's learners are in the second set and Jada's learners in the third set of the grade, the sets streamed according to Grade 9 performance.

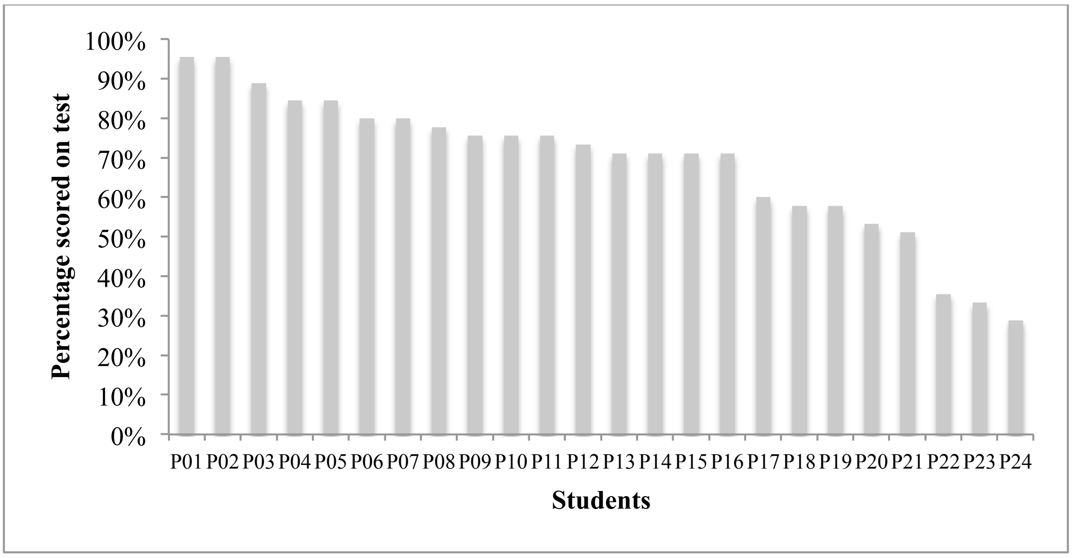


Figure 9.2. Jada's learners' test scores

At Evergreen High, 19 of the 21 learners wrote the test in Maya's class whereas 19 out of 25 learners wrote the test in Jono's class. Maya's learners scored an average of 56% on the test with the highest score of 100% achieved by two learners and the lowest 19% (see Figure 9.3) and Jono's learners scored an average of 60% with the highest score of 100% and the lowest score being 5% (see Figure 9.4). At Evergreen High, learners' performance on the test are not comparable given that tests written by the two groups of learners were different.

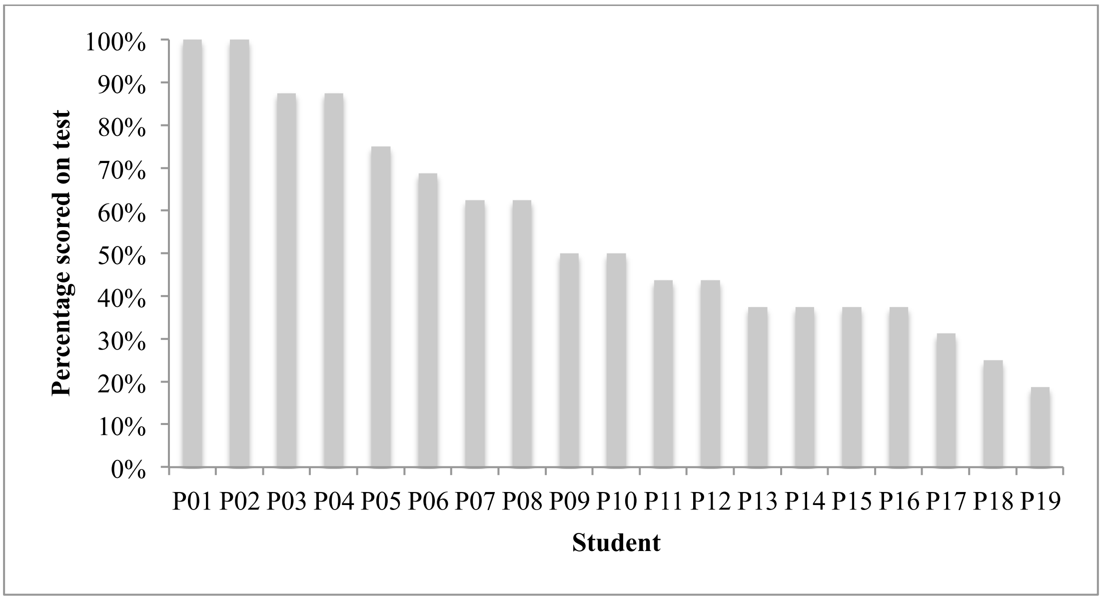


Figure 9.3. Maya's learners' test scores

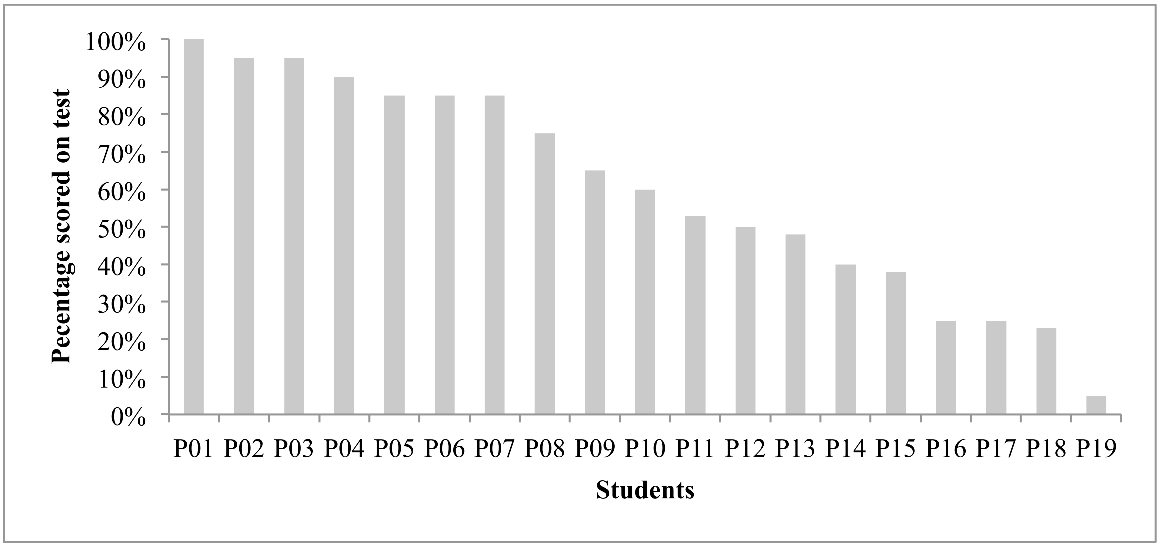


Figure 9.4. Jono's learners' test scores

Table 9.1 summarises the learners' performance on the tests across the four pedagogic contexts. Although learners' performance on the tests across the two schools and within Evergreen High are not comparable, the data shown in Table 9.1 provides some measure of comparison. Table 9.1 shows that a larger proportion of learners at Prestige College scored more than 50% on the test administered to them. In Sara's class, 89,5% of learners who wrote the test scored 50% or more on the test, and 87,5% of Jada's learners who wrote the test scored 50% or more on the test. At Evergreen High, the number of learners who wrote the test scoring 50% or more was considerably lower than the number of learners who wrote the test scoring 50% or more at Prestige College. In Maya's class, 52,6% of learners who wrote the test scored 50% or more and 63,2% of Jono's learners who wrote the test scored 50% or more. If we take into account the inconsistencies in the marking by Evergreen High teachers (discussed in Chapter 8), then the percentage of learners scoring 50% or more at Evergreen High is lower than that reported here. The performance of Evergreen High learners is poor despite the strong similarity between the worksheet used during the observed lessons and the test in Maya's case and the extract of the worksheet used as the test in Jono's case.

Table 9.1. Summary of learner performance on tests across the four pedagogic contexts

Teacher	Learners in class	Wrote test	50% or more	Average score
Sara	21	19	17	75%
Jada	25	24	21	69%
Maya	20	19	10	56%
Jono	25	19	12	60%

The results of learners discussed above merely provide an overview of their performance on the tests but does not provide insight into the recognition and realisation rules employed, which would reveal their specialisation of consciousness (computational performance and orientation to mathematics). It is to this discussion to which we now turn.

9.3 Describing the interviews with learners

From the learners' test scripts, I selected a test problem or problems, that corresponded with the announced topics covered during the observed lessons, as the focus of the interviews with learners. The purpose of the interviews was to ascertain the recognition and realisation rules used by learners when doing mathematics independently of their teacher. In video-recorded individual clinical interviews, I presented each learner with a copy of their marked test script as well as the test paper, and I asked them to explain the methods they used to produce their solutions. I explained that it did not matter whether their solutions were correct or incorrect because I was interested in their reasoning. I also explained that they could provide an alternate solution to the one presented in their test script. During the interview, I probed each learner's reasoning of the computations employed, asking for clarifications where necessary. When the learner presented incorrect mathematical reasoning, I would first attempt to elicit from the learner their reasoning, and then direct the learner towards more mathematically appropriate reasoning. If the learner was unable to self-correct their reasoning, I would point out their error and provide the correct mathematical explanation in order to assist the learner with the particular mathematical problem. Learners were encouraged to write if they needed to.

All the learners' interviews and tests were transcribed and analysed in full. Appendix 9.2 illustrates how this analysis was carried out for one of the interviews (see Appendix 9.1 for transcript of that interview). The findings are presented here. For reasons of limited space, only the work of the top-performing learner in each pedagogic context is discussed in detail in order to illuminate how the analysis was carried out. I focus primarily on the top learner because the top learner is most likely to produce the legitimate text expected by the teacher. I then comment on the computational activity of the other learners interviewed in much less detail.

9.4 Analysing the tests and interviews of Sara's learners

9.4.1 Sara's interviewed learners

The six learners interviewed in Sara's class are referred to as Ted (P01), Tom (P10), Luc (P12), Ray (P13), Pat (P18) and Wes (P19)[73]. A number of learners were away on an exchange trip at the time of the interviews, so the learner selected as the second top learner was replaced with Tom (P10). The learners were interviewed about their solutions to the test problem 1 of the *Common Test Functions 2012* (see Figure 9.5). The number of sub-problems dealt with differed across learners since only half an hour was available per interview. However, all the learners were interviewed about Problems 1.1, 1.2 and 1.3.

Grade 10 Common Test Functions 2012 Total 45 marks

1. Given $f(x)=-2x^2+2x+4$ and $g(x)=-8x+4$

1.1 Determine the co-ordinates of the x-intercepts of graph f. (3)

1.2 Determine the co-ordinates of the turning point of f. (3)

1.3 Draw the graphs of $f(x)=-2x^2+2x+4$ and $g(x)=-8x+4$ on the same set of axes on your answer sheet. Label the graphs clearly, including all the intercepts with the axes and the turning point. (5)

1.4 Write down the range of f. (1)

1.5 Determine algebraically the coordinates of the points of intersection of $f(x)$ and $g(x)$ (5)

1.6 Use your graphs to determine for which values of x, $f(x)<0$. (2)

Figure 9.5. Prestige College test

Table 9.2 displays the interviewed learners' marks awarded by the teacher for Problem 1. All the learners appeared to have no difficulty in answering Problems 1.1-1.3, Pat being the only learner who did not score

73 Learner numbers indicate their ranking, measured in terms of their performance on the test. So, P01 indicates the top-performing learner and P02 the second top-performing learner and so on. In Jada's class, two learners scored the top mark, the learner who scored full marks for Question 1 was labelled as P01. In Maya's class, two learners scored the top mark, the learner who produced the correct equation according to the teacher's memorandum was labelled as P01.

full marks for these three sub-problems because he did not show all his calculations for Problem 1.2. Problems 1.4-1.6 presented difficulty for some learners as reflected in the marks awarded in Table 9.2.

Table 9.2. Distribution of marks awarded by Sara for Problem 1 to interviewed learners

Name	Code	1.1 (3)	1.2 (3)	1.3 (5)	1.4 (1)	1.5 (5)	1.6 (2)	Total (19)
Ted	P01	3	3	5	1	5	2	19
Tom	P10	3	3	5	1	4	2	18
Luc	P12	3	3	5	0	5	0	16
Ray	P13	3	3	5	1	2	2	16
Pat	P18	3	1	5	0	2	2	13
Wes	P19	3	3	5	1	3	0	15

9.4.2 Computational activity of Sara's top learner (Ted)

The analysis focuses primarily on the computational activity of the top learner (Ted) i.e. the recognition and realisation rules displayed by Ted during the test and clinical interview. Brief comments on commonalities and differences with respect to the computational activity of the other interviewed learners follow. Ted scored 98% for the test. His solution to Problem 1.1 (see Figure 9.6) was awarded full marks by the teacher despite of the notion of an equation being absent as a regulative resource and the failure to distinguish between an equation as different from expressions in general.

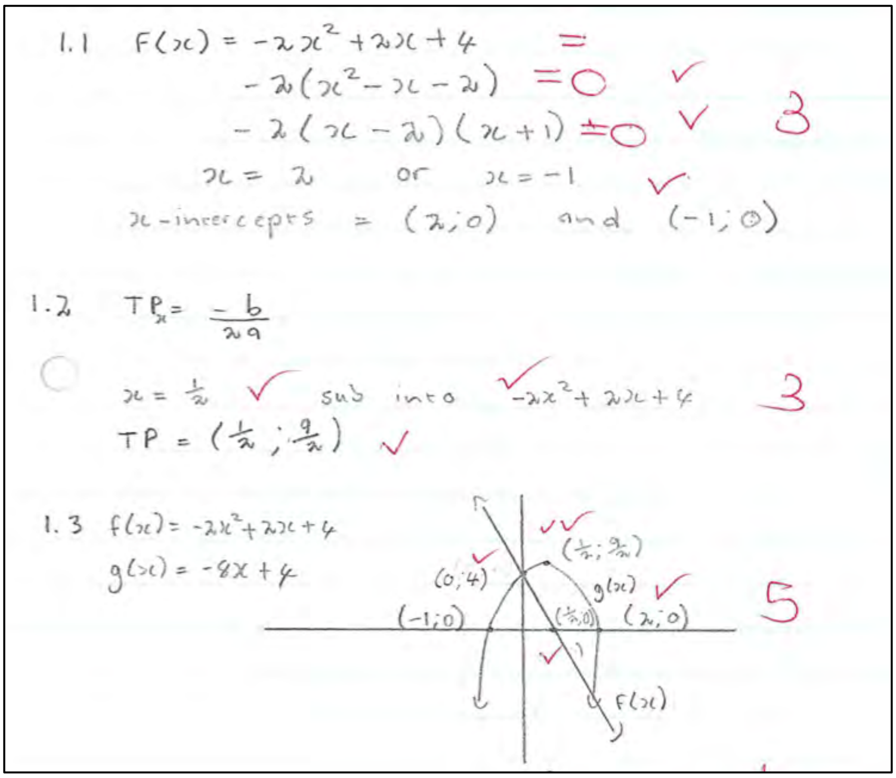


Figure 9.6. Ted's test solution to 1.1-1.3

During the interview, Ted does not explain how he knows that $f(x) = -2x^2 + 2x + 4$ represents a parabola but he states that "if it's [the x^2 term] negative then it [the parabola] faces downwards, if it's positive it faces upwards" (S01T01P01: line 195). Ted's proposition focuses on the iconic features of the parabola and substitutes for the encyclopaedic proposition which states that a parabola $y = ax^2 + bx + c$ has a maximum value if $a < 0$ and a minimum value if $a > 0$. Ted thus recruits an auxiliary proposition as opposed to referencing the general formula of a parabola ($y = ax^2 + bx + c$) as mentioned by Ray, Tom and Pat.

Ted explained that he computed the x-intercepts from $-2(x-2)(x+1)$ by taking "the opposite of negative two so positive two" (S01T01P01: line 20). Thus, Ted uses an operation-like manipulation "take the opposite sign" of the numerals in the brackets. This operation-like manipulation is one that is not a familiar operation in the Mathematics encyclopaedia and is one which entails an existential shift from numbers to characters, since you can't "change" the sign of a number. Such an auxiliary operation can only be performed on characters. The auxiliary operation, "take the opposite sign", substitutes for the zero product property (if $a.b = 0$ then $a = 0$ or $b = 0$) which is fundamental to solving quadratic equations that have been factorised into a product of linear factors and substitutes for the encyclopaedic proposition: if $y = a(x - x_1)(x - x_2)$ then $x_1$and x_2 are the x-intercepts of the parabola. The absence of an equation as a regulative resource and the substitution of the zero product property and encyclopaedic proposition with an auxiliary operation "take the opposite sign" is indicative of an expression-orientation to mathematics. In other words, the correct expression $x = 2$ or $x = -1$ is arrived at by Ted despite content that diverges from the content usually associated with the topic from the point of view of the Mathematics encyclopaedia.

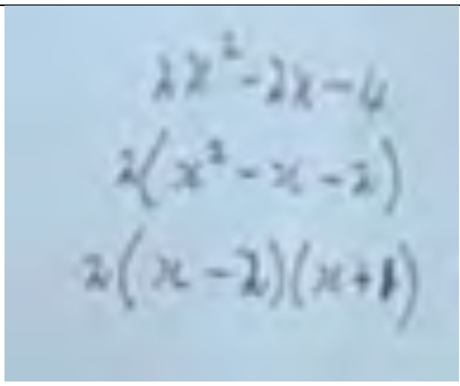
It is interesting that Sara did not use the operation-like manipulation "change signs" in the context of solving a factorised quadratic equation. However, she used an operation-like manipulation, "change signs", in three different computational contexts. Firstly, in the context of calculating the turning point of $y = (x-2)^2$, she concludes inductively using geometry software that "if this [referring the number in the bracket] is minus two it's [referring to x-coordinate of the turning point] is the [opposite] sign. Okay. We call it counter intuitive." (S01T01L01: line 252) and again, when explaining to a learner that the turning point of $y = -(x-3)^2 - 16$ is $(3; -16)$ because "it's opposite signs" (S01T01L01: line 411). Secondly, when calculating the x-intercepts of $y = -x^2 + 6x - 5$ and so solving the equation $-x^2 + 6x - 5 = 0$, she restates a learner's computation, "make that x squared. Take everything to the other side" (S01T01L01: line 377) as "Right. So we have to make x squared positive. Change all the signs. In other words we are multiplying everything by minus one" (S01T01L01: line 378). Thirdly, when explaining to a learner how she obtained $(x-3)^2 = 4$ from $0 = -(x-3)^2 + 4$, she says "it is almost like taking that [pointing at $-(x-3)^2$] over to that side. I could have said minus x minus three squared equals minus four. I just skipped out a step" (S01T01L01: line 400). It is therefore unsurprising that Ted formulated an operation-like manipulation, "take the opposite sign" that substitutes for the encyclopaedic proposition, which states that x_1 and x_2 are the x-intercepts of a parabola $y = a(x - x_1)(x - x_2)$, used by Sara.

Although Ted did not equate $f(x)$ with 0 in the test initially, when questioned about his operation-like manipulation "take the opposite sign", he eventually realised that $f(x) = 0$ because "when it [the graph] intercepts the x line [axis] then y is obviously going to be zero" (S01T01P01: line 34). After much probing, Ted eventually explains that $y = 0$ because the y-coordinates of the x-intercepts are zero (see S01T01P01: lines 22-44). Ted replaces the zero product property with the multiplication fact $0 \times 0 = 0$. In other words, for him each factor must be 0 because the product of 0 and 0 is 0, rather than either one of the factors is 0 or both factors are 0 because the product is zero (see Extract 9.1).

Extract 9.1 S01T01P01 (lines 59 – 62)

Interviewer:	So why does that [referring to $/x - 2/$] have to be zero?
Learner:	Because negative two times zero's zero.
Interviewer:	Ja?
Learner:	And then that times zero is zero. So then you'll get zero here [referring to $-2(x-2)(x+1) = 0$].

The absence of an equation as a regulative resource is confirmed when he calculates the x-intercepts of the reflection of f during the interview as shown in Figure 9.7 (typed version shown alongside) which illustrates the same error produced in the test. He calculates the x-intercepts as $x = 2$ and $x = -1$ from the expression $2(x-2)(x+1)$ (also see S01T01P01: lines 171-175).



$$\boldsymbol{2x^2 - 2x - 4}$$

$$2(\boldsymbol{x^2 - x} - 2)$$

$$2(x-2)(x+1)$$

Figure 9.7. Ted's calculation of the x-intercepts of f 's reflection during the interview

Ted recruits auxiliary operations alongside operations from the field of the reals as well as auxiliary propositions that stand in place of encyclopaedic propositions as part of his computational activity. The use of iconic auxiliary propositions and auxiliary operations grounds his computational activity in the iconic. The recognition and realisation rules used by Ted constitute the content of the announced topic parabola as a combination of an auxiliary calculus on characters together with operations on whole numbers.

9.4.3 Computational activity of Sara's other interviewed learners

When comparing the recognition and realisation rules used by Ted with the other interviewed learners in Sara's class, we note from Table 9.3 that, like Ted, Pat's computational activity also displays an absence of encyclopaedic propositions and the notion of an equation as regulative resources although Pat recognises the function as a parabola from the general formula as well from its iconic features. Wes is able to provide an encyclopaedic explanation for equating $f(x)$ with 0 in Problem 1.1 but the notion of an equation does not feature as a regulative resource for him. In contrast, Tom, Luc and Ray recruit encyclopaedic propositions and explicitly employ the notion of an equation as a regulative resource.

Table 9.3. Encyclopaedic propositions used by Sara's learners[74]

Learner	Encyclopaedic explanation for $f(x) = 0$	Zero product property	Notion of an equation
Ted (P01)	(✓)		absent
Tom (P10)	✓	✓	explicit
Luc (P12)	✓	✓	explicit
Ray (P13)	✓	✓	explicit
Pat (P18)			absent
Wes (P19)	✓		absent

Table 9.4 shows that all of the interviewed learners employed auxiliary operations, auxiliary propositions and/or auxiliary descriptions as their recognition and realisation rules during the interview. This auxiliary calculus coexisted with fundamental propositions for all students, except for Ted and Pat, and to a lesser extent for Wes who does not recruit the notion of an equation (see Table 9.3).

This mean that the recognition and realisation rules used by Ted and Pat realise content associated with the announced topics that can be characterised as ancillary whereas the others use recognition and realisation rules that produce content categorised as symbiotic. So, the computational activity of Tom, Luc. Ray and Wes is regulated by both iconic as well as fundamental ground. In the case of Tom and Ray, empirical ground also featured in the regulative resources employed, evident in the 'trial and error' method of factorising the quadratic trinomial in the case of Tom and calculating the equation of the reflection of f in the case of Ray.

[74] The brackets indicate that the proposition did not function as a regulative resource for the learner.

Table 9.4. Auxiliary operations used by Sara's learners

Learner	Auxiliary operations	Auxiliary propositions/descriptions
Ted (P01)	"opposite of negative two" (L20) "both negatives so they cancel" (L72) "turn into a positive" (159) "times by a negative" (L171)	"If it's (the x^2 term) negative then it (the parabola) faces downwards, if it's positive it faces upwards" (L195) If it's an equation then *y* equals zero (L26-28)
Tom (P10)	"switch around" (L16) "times by a negative" (L16) "turn positives into negatives" (L22, 180)	The axis of symmetry is the "line that the graph turns on" (L68) "Because of the *x* squared and I know this is the formula for a parabola (L114)
Luc (P12)	"swopping signs" (L124) "timesing by a negative" (136)	The axis of symmetry is a "line where if you draw a line there then you can fold there and then it will be exactly the same" (L46) "because firstly you know that it's going downwards because it's a minus at the start and then because you see an *x* squared. You know it's this type of graph. With two arms going downwards" (L66)
Ray (P13)	"swop signs" (L262)	The axis of symmetry is "a point at which it's (parabola) symmetrical so if you fold it over that (arms of the parabola) is equivalent to that" (L94)
Pat (P18)	"swopped the negatives around" (L264) "change the sign" (L298)	Axis of symmetry is the "line that cuts the specific graph into two equal parts" (L50, L60) To calculate *y* you make *x* zero (L118, L128) "If you see anything like *x* squared plus so I kind of guess it's a parabola" (L154) "When it's positive in a kind of in a U shape parabola" (L260)
Wes (P19)	switch the symbols (L18) Make the negative positive (L26) Switched positives and negatives around (L140)	"If it's *x* squared then it's a parabola" (L88) "If it's a negative gradient it's going to go down which it was before. And er if it's a positive gradient it will curve up" (L190)

Figure 9.8 summarises the realised content produced by Sara's learners with respect to the announced topics in terms of the presence/absence of encyclopaedic computational resources and the presence/absence of auxiliary computational resources.

Furthermore Ted, Pat and Wes produce mathematics that diverges from the Mathematics encyclopaedia at the level of expression and at the level of content from the point of view of the Mathematics encyclopaedia. Tom, Ray and Luc, on the other hand, realise mathematics that converges with mathematics at the level of expression and diverges from the Mathematics encyclopaedia at the level of content.

		Encyclopaedic computational resources	
		present	**absent**
Auxiliary computational resources	**absent**	*Canonical*	*Elementary*
	present	*Symbiotic* Tom (P10), Ray (P13) Luc (P12), Wes (P19)	*Ancillary* Ted (P01) Pat (P18)

Figure 9.8. Realisation of content displayed by Sara's learners

The distribution of realised content across the interviewed learners in Sara's pedagogic context to some extent mirrors the realisation of content in the instructional discourse which was a hybrid of canonical, symbiotic and ancillary content types (See Chapter 6.4).

9.4.4 Computational performance and orientation to mathematics (Sara's learners)

The recognition and realisation rules employed by Ted, as illuminated through the analysis of his computational activity discussed above, realise mathematics that is strongly closed with respect to the topic from the point of view of the Mathematics encyclopaedia because of the absence of fundamental axioms, definitions and propositions. Recall that open or closed pedagogic texts are indicators of computational performance. The orientation to mathematics displayed by Ted is strongly expression-oriented because he produced the required solution despite the absence of fundamental computational resources. Pat's computational activity mirrors that of Ted. So Pat's computational performance is also classified as a strongly closed pedagogic text (see Table 9.5). Wes, like Ted and Pat, constitutes the content for the topic parabola as a combination of an auxiliary calculus on symbols together with basic arithmetic but the inclusion of encyclopaedic propositions weakens the closed nature of the pedagogic texts and also weakens the expression-orientation to mathematics.

Table 9.5. Summary of Sara's learners' computational performance and orientation to mathematics

Learner	Computational performance Open text (T_o) /Closed Text (T_c)	Orientation to mathematics Content-orientation (O_c) / Expression-orientation (O_e)
Ted (P01)	T_c^+	O_e^+
Tom (P10)	T_o^-	O_c^-
Luc (P12)	T_o^-	O_c^-
Ray (P13)	T_o^-	O_c^-
Pat (P18)	T_c^+	O_e^+
Wes (P19)	T_c^-	O_e^-

The recognition and realisation rules employed by Ray, Tom and Luc produce mathematics that is considered as open pedagogic texts and an orientation to mathematics that is content-oriented because their computational activity is regulated by fundamental Mathematical ground. i.e. encyclopaedic propositions are recruited. However, the presence of auxiliary operations, auxiliary propositions and/or auxiliary descriptions weakens the openness of the pedagogic text and weakens the content-orientation to mathematics. It is interesting that the computational performance of the top-performer is more like that of the bottom-performer than that of the mid-level performer.

The variation in computational performance and orientation to mathematics (i.e. variation in the specialisation of consciousness) observed in a selection of learners from Sara's class reflects to some extent the hybridity of the pedagogic modality evident in the instructional discourse (cf. Chapter 7). Recall that Sara's pedagogic modality is a hybrid that leans towards an open pedagogic text that is content-oriented with the dominant pedagogic modality realised as T_O^-/O_c^-. The computational performance and orientation to mathematics of three of the interviewed learners are characterised as T_O^-/O_c^-, one as T_c^-/O_e^- and two as T_c^+/O_e^+. However, none of the interviewed learners display computational performances that produce strongly open pedagogic texts and orientations to mathematics that are strongly content-oriented (T_o^+/O_c^+), even though some of Sara's evaluative events were coded as such. Since only a selection of learners were interviewed, we can't establish with certainty whether there are learners with other combinations of computational performances and orientations to mathematics. However, the analysis strongly suggests a correspondence between the functioning of evaluation in the observed lessons and the evaluative activity of the learners despite differences in recognition and realisation rules amongst the learners and when compared to that of the teacher. The computational performances and orientations to mathematics displayed by the actual learners corresponds with the hybrid model learner implied by the computational activity in the observed lessons.

9.5 Analysing the tests and interviews of Jada's learners

9.5.1 Jada's interviewed learners

The learners interviewed in Jada's class are referred to as Rod (P01), Leo (P02), Noa (P12), Jay (P13), Jon (P24) and Gio (P22) who replaced Mat (P23) who was away at the time of the interview. As with Sara's learners, Jada's learners were interviewed about their solutions to Problem 1 of the *Common Test Functions 2012* (see Figure 9.5). The number of sub-questions dealt with differed across learners since only half an hour was available per interview. However, all the learners were interviewed about Problems 1.1, 1.2 and 1.3. Table 9.6 displays the interviewed learners' marks awarded by the Jada for Problem 1 of the test[75].

[75] Gio replaced Mat (P23). Gio's test script, however, was not available at the time of the interview. So his mark allocation for Question 1 is not displayed in the table.

With the exception of Jon, the learners answered Problem 1 fairly well according to the marks awarded by the teacher as shown in Table 9.6. Although Rod and Leo both scored the same total mark for the test, only Rod scored full marks for Problem 1.

Table 9.6. Distribution of marks awarded by Jada for Question 1 to interviewed learners

Name	Code	1.1 (3)	1.2 (3)	1.3 (5)	1.4 (1)	1.5 (5)	1.6 (2)	Total (19)
Rod	P01	3	3	5	1	5	2	19
Leo	P02	3	3	5	1	5	0	17
Noa	P12	2	2	4	0	4	0	12
Jay	P13	3	3	4	0	4	0	14
Jon	P24	2	0	1	0	2	0	5

9.5.2 Computational activity of Jada's top learner (Rod)

The analysis of learners' responses to Problem 1 presented below focuses primarily on the computational activity of the top learner (Rod), i.e. the recognition and realisation rules displayed by Rod during the test and clinical interview. Brief comments on commonalities and differences with respect to the computational activity of the other interviewed learners follow.

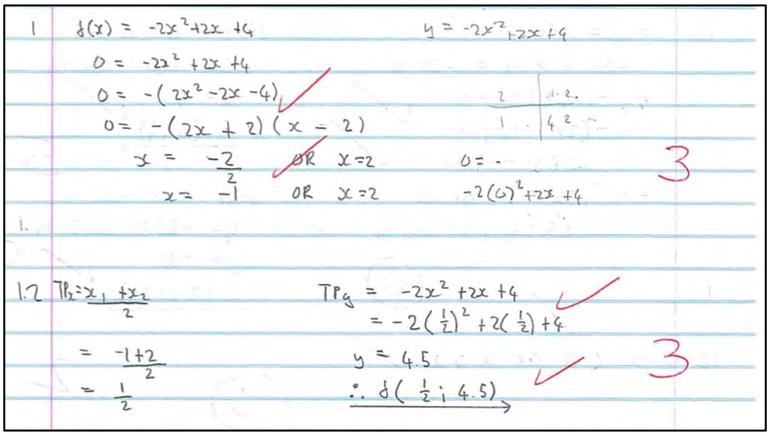


Figure 9.9. Rod's marked solution to Question 1.1 and 1.2

Rod scored 96% for the test. His marked solution to Problems 1.1 and 1.2 is shown in Figure 9.9 and the marked solution to Problem 1.3 is displayed in Figure 9.10.

Rod's solution to Problem 1 was rewarded with a smiley face by his teacher, who seemed pleased by his solution since he scored full marks. However, Rod had great difficulty during the interview when solving Problem 1, particularly with calculating the x-intercepts of f. He claimed that "I haven't done this in a while" (S01T02P01: line 74), a refrain repeated many times during the interview and used by other learners in Jada's class during the interviews, which suggests that these learners tended to memorise procedures for solving particular classes of problems and experienced difficulty when they were not prepared.

Rod identified the function $f(x) = -2x^2 + 2x + 4$ as a parabola because it has the general form $y = ax^2 + bx + c$ (S01T01P01: lines 307 – 310). He was the only learner to refer to the general form of the parabola. However, he also identified the shape of the parabola as follows “negative *x* squared told you that it went downward. *x* squared told you that it went up” (S01T02P01: line 174), thus utilising an iconic auxiliary proposition that stands in place of the encyclopaedic proposition which states that a parabola $y = ax^2 + bx + c$ has a maximum value if $a < 0$ and a minimum value if $a < 0$. When calculating the *x*-intercepts of $f(x)$, Rod explained that $f(x)$ equals 0 because the *x*-intercepts are “where it (the graph) cuts the *x* axis so and then if at that point wait yes at that point it’d be zero. *y* would be zero” (S01T02P01: line 10). Rod was the only learner who identified that $f(x)$ equals 0 because the *y*-coordinate of an *x*-intercept equals 0.

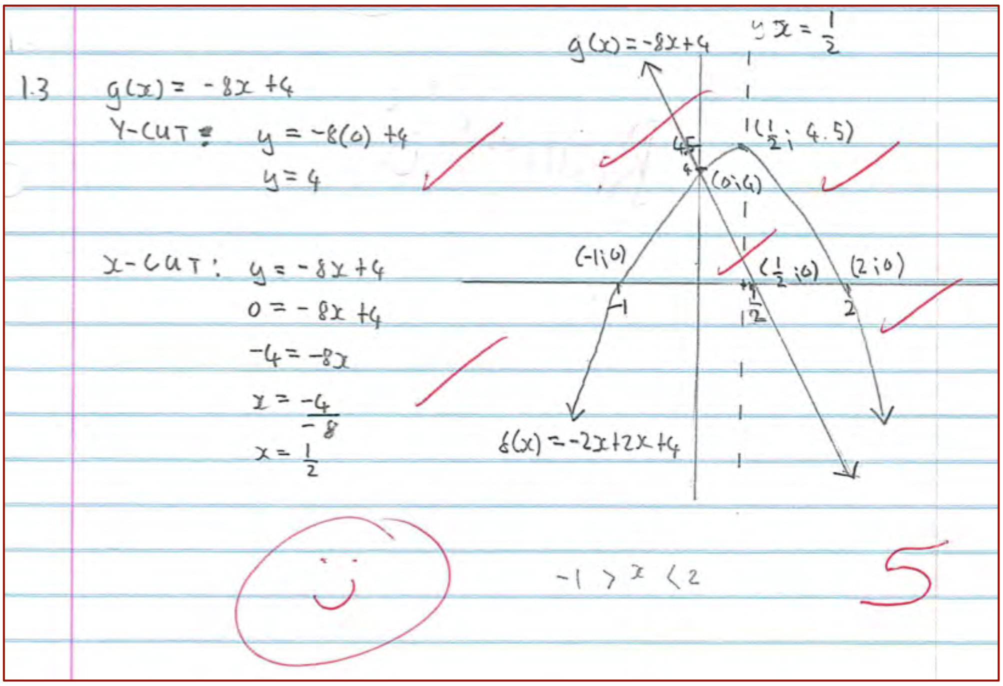


Figure 9.10. Rod’s marked solution to Question 1.3

Although he offered the above as a reason for why $f(x)$ equals 0, he backtracked when asked why he equated $f(x)$ with 0, stating that “um I actually didn’t need to do that. I realise that now” (S01T02P01: line 20). His response suggests uncertainty, as though asking him to explain why $f(x)$ equals 0 indicated to him that he was incorrect. His response to the interviewer suggests that for him necessity is located in the authority of the teacher or interviewer rather than in mathematics.

Rod’s next computational move was to factorise $-2x^2 + 2x + 4$ which he solved differently to the test. In the test he factorised $-2x^2 + 2x + 4 = 0$ to produce – $(2x^2 - 2x - 4) = 0$. In the interview he explained that “I didn’t take out a common factor” (S01T02P01: line 62) referring to his test, which suggests that /-/ in front of the bracket in – $(2x^2 - 2x - 4) = 0$ does not refer to a common factor of -1 but a negative sign. This is confirmed by the explanation he provided in the interview, where he factorised $-2x^2 + 2x + 4$ with great difficulty to produce $-2(x^2 - x - 2)$. He started by stating that “I can take the common factor of 2” but he wrote $-2(x^2 + 2x + 4)$ which suggests that /-2/ is considered as /2/ with an appended negative. This conception of -2 is confirmed when he corrects the expression $-2(x^2 + 2x + 4)$ to produce $-2(x^2 + x + 2)$

when it seems as though he realised that 2 should be "taken out" of all the terms. He makes a further correction when he changes the plus sign appended to constant term from plus to minus, which suggests that having "taken out 2" as a common factor he realised that he had to "take out the negative" as well but that he forgot to "change" the sign associated with the x term. His computations involved in factorising $-2x^2 + 2x + 4$ are illustrated in Extract 9.2.

Extract 9.2 . S01T02P01 (lines 12-16)

Learner:	So then I'd then *fx* means .. is referring to the *y* so I went zero is equal to um wait ... ja zero is equal to minus I can take the common factor here of two. So minus two go *x* squared plus two *x* plus four. Not plus four. Sorry that's two.
Interviewer:	So what have you got now? Two *x* squared
Learner:	minus two then *x* squared plus *x* then that's plus then minus two. Minus two. Okay. And then
Interviewer:	But just check this one. I think you've made a mistake here as well.
Learner:	Ja could be. Ja that's minus.

Rod's procedure for substituting $-2(x^2 - x - 2)$ with $-2(x - 2)(x + 1)$ entailed the use of a character distribution matrix (   )(   ) and included sub-procedures for generating the characters that populate the template. Firstly, he factorised x^2 to produce the factors x and x, then he used a set of auxiliary propositions that enabled him to generate the $/+/$ and $/-/$ signs required for the "brackets" and finally he used the factors of 2 to produce the numerals required for the brackets. The use of the auxiliary proposition stands in place of Viete's theorem which underpins the factorisation of quadratic trinomials. When he was asked to explain his auxiliary propositions used to produce $+$ and $-$ signs required for the "brackets", he explained that "minus 'cause you know it's different signs … so then it's either one's minus and one's plus" (S01T02P01: lines 34-36). In other words, because the last term is a minus, the brackets have opposite signs. When asked to explain why that was the case, he provided a set of propositions for different combinations of the sign of the middle term and the sign of last term. He, however, could not explain why the propositions work in the way that they do. His only explanation was "because that's just the rule" (S01T02P01: line 44), which indicates the absence of fundamental ground when carrying out the procedure for factorising quadratic trinomials. Moreover, Rod's computational activity comprises a combination of whole number arithmetic and an auxiliary calculus, indicative of an iconic grounding. His procedure for factorising a quadratic trinomial mirrored the procedure outlined by his teacher in the observed lessons.

He struggled to calculate the numbers in the brackets. At first he used 1 and 1, perhaps because 1 plus 1 is two. However, he realised his error because he did not produce the same x-intercepts as he did in the test. "Ja basically and then you go x equal to equal to um one I think or x is equal to negative one. But it's wrong. Ja. .. I haven't done this in a while so" (S01T0P01: line 74). He then changed the values in the brackets from 1 and 1 to 2 and 1 and verified that he produced the correct factors as follows "Okay so now it has to times into that so then negative two negative two and it must equal that so negative two times by one […] so then negative two negative two plus one is equal to negative one" (S01T02P02: lines 84-86).

Having factorised $-2(x^2 - x - 2)$ to produce $-2(x-2)(x+1)$, Rod writes down the x-intercepts as $x = 2$ or $x = -1$, suggesting that the notion of an equation is absent as a regulative resource. Even though he identified that $f(x) = 0$ and could explain why $f(x) = 0$, his reasons are not connected to the notion of an equation as the central computational resource underpinning the calculation of x-intercepts of the parabola. Instead, equating $f(x)$ with 0 appeared to be one of the steps in the procedure for calculating x-intercepts, suggesting the presence of algorithmic ground without the support of the fundamental mathematical axioms and propositions. However, when asked to explain how he produced the x-intercepts from $-2(x-2)(x+1)$, he inserts the 0 which produces an equation at the level of an expression but Rod does not refer to the expression as an equation. Instead, it seems as though 0 is required so that he has two sides in order to "move x across" the equal sign: "y equals nought so now you're saying that then x you moving it across so it's actually nought is equal to x minus two so therefore two is equal to x" (S01T02P01: line 100). Thus the insertion of "$= 0$" serves the purpose of establishing a character distribution matrix which creates the spaces to move symbols around in order to produce the required solution.

Extract 9.3 S01T02P02 (line 103 – 110)

Interviewer:	How do you get that statement [referring to $x - 2 = 0$]?
Learner:	What do you mean? This [referring to $x - 2 = 0$] ?
Interviewer:	Yeah
Learner:	Because .. nought .. both these things [points to the brackets]
Interviewer:	Mm
Learner:	equal to nought.
Interviewer:	Okay.
Learner:	So. Ja. So you know that on the graph there will be no *y*. There will be no *y*. It will be nought.

When asked to explain why he can make the statement that $x - 2 = 0$ from $-2(x-2)(x+1)$= 0 he argues that both "brackets" equal zero because $y = 0$ for both brackets. This suggests that he "splits" $(x-2)(x+1) = 0$ into $x - 2 = 0$ and $x + 1 = 0$ when he says "Because .. nought .. both these things" (see Extract 9.3).

The auxiliary operation, "splitting", stands in place of the zero product property which underpins the solution to the quadratic equation required to produce the x-intercepts. Curiously, he says that "there will be no *y*. It will be nought" as though 0 is the same as "nothing". Another example of Rod's auxiliary calculus is revealed when he explains what happens to -2 when producing the x-intercepts from $-2(x-2)(x+1)$= 0 (see Extract 9.4).

Extract 9.4 . S01T02P01 (lines 91-94)

Interviewer:	What's happened to the minus two?
Learner:	Minus two you can leave off.
Interviewer:	Why?
Learner:	I I. My teacher says you can leave it off.

Rod simply "leaves" the -2". His computation has no mathematical necessity. Instead necessity is located in the authority of the teacher. The computational resources supporting his computational activity are not grounded in the fundamental Mathematics axioms, definitions and propositions but in the steps of the procedure outlined by the teacher, suggesting that algorithmic ground with the fundamental ground supports his computational activity.

Rod produces the parabola considered as correct according to the memorandum. However, when asked to explain why the parabola has that shape, he explains that "cause negative *x* squared told you that it went downward (draws parabola with minimum turning point). *x* squared told you that it went up (draws parabola with maximum turning point)" (S01T02P01: line 174). The use of his proposition appeared contradictory at first because according to his proposition the parabola $y = -2x^2 + 2x + 4$ should be "downward" and not "upward" as he drew it. When questioned, however, it was clear that he was referring to the equation in the form $y = -2(x^2 - x - 2)$ as though it were $y = x^2 - x - 2$ (see Extract 9.5).

Extract 9.5. S01T02P01 (lines 175-179)

Learner:	cause negative *x* squared told you that it went downward [draws parabola with minimum turning point]. *x* squared told you that it went up [draws parabola with maximum turning point]. So ja.
Interviewer:	Are you sure?
Learner:	Think so.
Interviewer:	So you've got negative two *x* squared.
Learner:	Yeah but I took out the common factor.
Interviewer:	Okay. So *x* squared.. so you're referring to this *x* squared [pointing at $y = x^2 - x - 2$]?

In fact for him, -2 could be ignored because $y = -2(x^2 - x - 2)$ and $y = x^2 - x - 2$ both represent the same graph (see Extract 9.6).

Extract 9.6 S01T02P01 (lines 189 -190)

Interviewer:	That this is the graph of *y* equals *x* squared minus two *x* plus four or is it the graph of negative two *x* squared plus two *x* plus four?
Learner:	It's the same thing. It's just I took out the common factor. Ja. But the reason I knew it went like that is because I got I worked out that the half and that the four point five the turning points. I know that's a positive then cos it wasn't negative. It wasn't negative four point five. If it was negative four point five it would be down there going like that.

The recognition and realisation rules used by Rod constitute the content of the announced topic parabola as a combination of an auxiliary calculus on characters together with basic arithmetic (operations on whole numbers).

9.5.3 Computational activity of Jada's other learners

From the above discussion we observe that Rod's computational activity, as displayed during the interview, entails the recruitment of auxiliary operations such as "taking out a negative", "splitting" the product of two binomials into two parts, "moving x across" and "dropping the minus" and auxiliary propositions alongside operations from the field of the reals. He was the only learner to recognise the function as a parabola from the general equation of a parabola and the only learner who provided an encyclopaedic proposition as resource for explaining why $f(x) = 0$ (see Table 9.7). However, for Rod and the other interviewed learners the notion of an equation was absent as a computational resource.

Table 9.7. Encyclopaedic propositions used by Jada's learners

Learner	Encyclopaedic explanation for $f(x) = 0$	Zero product property	Notion of an equation
Rod (P01)	✓		absent
Leo (P02)			absent
Noa (P12)			absent
Jay (P13)			absent
Gio (P22)			absent
Jon (P24)			absent

With the exception of the encyclopaedic propositions used by Rod, the computational activity of the other interviewed learners mirrored that of Rod in that they all employed auxiliary operations and auxiliary propositions or auxiliary descriptions (see Table 9.8). Table 9.8 shows similarities and differences in the recognition and realisation rules across the learners. All the learners locate necessity external to mathematics because the primary regulative resources underpinning their computational activity are grounded in the iconic - algorithmic.

The recognition and realisation rules employed by Rod realised content characterised as symbiotic in that he recruited an encyclopaedic proposition alongside the auxiliary calculus. All the other learners produced content described as ancillary given the presence of auxiliary computational resources and the absence of encyclopaedic computational resources (see Figure 9.11). Furthermore, the content constituted for the topic parabola by all the interviewed learners diverges from the content associated with the topic from the point of view of the Mathematics encyclopaedia. However, Rod, Jay and Leo produced mathematical texts that converge with the Mathematics encyclopaedia at the level of expression whereas Noa, Jon and Gio produce mathematics that diverges at the level of expression.

Table 9.8. Auxiliary operations, propositions and descriptions used by Jada's learners

Learner	Auxiliary operations	Auxiliary propositions/descriptions
Rod (P01)	"taking out a minus" (L12) "leave the minus two" (L94, L220) "splitting" the brackets (L101) "moving it (x) across" (L100) times by a negative (L146)	The x-coordinate of the turning point is "halfway" between the two x-intercepts (L132) "because .. um x at this point the y cut there's no x on either side. This is nought" (L162) "cos negative x squared told you that it went downward [draws parabola with minimum turning point]. x squared told you that it went up [draws parabola with maximum turning point] (L162)
Leo (P02)	turn into minuses because I took out the minus (L30) splitting up the equation into two parts (L50) "dropping -2" (L74)	Make it a quadratic equation by making it equal 0, "not to complicate any things and then find different numbers nought is the easiest digit" (Ll42-46) "because um the negative means um that the graph is inverted. It's reflected um it means it's reflected in the x axis" (L134) It's a parabola because " it's the equation is set up in er with er with the x.. with the squared" (L150)
Noa (P12)	"changed the sign" (L12, L159, L191) "opposite signs" (L28)	"for parabolas I always look for the sign over here which will be a negative so it's unhappy" (L42) "symmetry axis is where if it's folded it will be perfectly matching on one side as the other side" (L76)
Jay (P13)	"separating" trinomial into two equations (L16) "you scratch you take the nought out" (L18) "you take the two across to the open side" (L20) "reverse the signs" (L24) "change the sign" (L122)	Make trinomial equal to zero " to get all the $x$es and units on the one side" (L4) The symmetry line is the climax of the parabola (L86)
Gio (P22)	"take the minus away straight away"(L16) "swop the signs" (L18, L32) "dividing by a minus" (L22) "split (trinomial)" (L46)	"To find the y you take you make the x nought" (L114) "to find all the x's you would make y nought" (L126)
Jon (P24)	"switch that around because they are negative" (L66) "swopping things around" (L68) "separate the two brackets into two answers" (L76)	"To find x-intercepts you make the y-intercept zero"

		Encyclopaedic computational resources	
		present	**absent**
Auxiliary computational resources	**absent**	*Canonical*	*Elementary*
	present	*Symbiotic* Rod (P01)	*Ancillary* Leo (P02), Noa (P12) Jay (P13), Gio (P22) Jon (P24)

Figure 9.11. Realisation of content by Jada's learners

The distribution of realised content across the interviewed learners in Jada's pedagogic context to some extent mirrors the realisation of content in the observed lessons which was a hybrid of canonical, symbiotic and ancillary content types (Cf. Section 6.4). In comparing the realised content produced by Sara's learners with that of Jada's learners, we see that a greater proportion of Sara's learners (three out of six) produce symbiotic content than do Jada's learners (one out of six) which corresponds with the observed lessons where Sara's pedagogic context had a greater proportion of symbiotic content than Jada's.

9.5.4 Computational performance and orientation to mathematics (Jada's learners)

From the analysis of the computational activity of learners discussed above, the recognition and realisation rules employed by all the learners (with the exception of Rod) generates mathematics that is strongly closed with respect to the topic from the point of view of the Mathematics encyclopaedia because of the absence of fundamental axioms, definitions and propositions. So the computational performances of learners is such that they produce strongly closed pedagogic texts. The orientation to mathematics displayed by those learners is strongly expression-oriented because they produce the required solution despite the absence of fundamental computational resources. The inclusion of encyclopaedic propositions by Rod represents a weakening of the strong closed pedagogic text and strong expression-orientation exhibited by the other learners. Table 9.9 summarises the computational performances and orientations to mathematics of Jada's interviewed learners.

The variation in computational performances and orientations to mathematics (i.e. variation in the specialisation of consciousness) observed in a selection of learners from Jada's class reflects to some extent the hybridity of the pedagogic modality evident in the observed lessons (cf. Chapter 7). Recall that Jada's pedagogic modality is a hybrid that leans towards a closed pedagogic text that is expression-oriented with the dominant pedagogic modality realised as T_c^+/O_e^+. The computational performance and orientation to mathematics of five of the six interviewed learners are characterised as T_c^+/O_e^+ which reflects the dominant pedagogic modality evident in the observed lessons. The other learner displays computational performance and orientation to mathematics characterised as T_c^-/O_e^-. None of the interviewed learners display computational performances that produce open pedagogic texts and orientations to mathematics that are content-oriented (T_o^+/O_c^+ and T_o^-/O_c^-) even though some of Jada's evaluative events were coded as such.

Table 9.9. Summary of Jada's learners' computational performance and orientation to mathematics

Learner	**Computational performance** Open text (T_o) /Closed Text (T_c)	**Orientation to mathematics** Content-orientation (O_c) / Expression-orientation (O_e)
Rod (P01)	T_c^-	O_e^-
Leo (P02)	T_c^+	O_e^+
Noa (P12)	T_c^+	O_e^+
Jay (P13)	T_c^+	O_e^+
Gio (P22)	T_c^+	O_e^+
Jon (P24)	T_c^+	O_e^+

Since only a selection of learners were interviewed, we can't establish with certainty whether there are learners with other combinations of computational performances and orientations to mathematics. However, the analysis strongly suggests a correspondence between the functioning of evaluation in the instructional discourse and the evaluative activity of the learners despite differences in recognition and realisation rules amongst the learners and that used by the teacher. The computational performances and orientations to mathematics displayed by the interviewed learners corresponds with the two faces of the model learner implied by the computational activity in the instructional discourse.

9.6 Analysing the tests and interviews of Maya's learners

9.6.1 Maya's interviewed learners

The learners interviewed in Maya's class are referred to as Fay (P01), Nia (P02), Max (P09), Pam (P10), Sue (P17) and Sam (P19). The interviews with Maya's learners centred around test problem 1 which focused on calculating the equation of the given function (see Figure 9.12). Problem 1 resembled the type of mathematics problem that learners encountered during the observed lessons (Refer to discussion in Chapter 8).

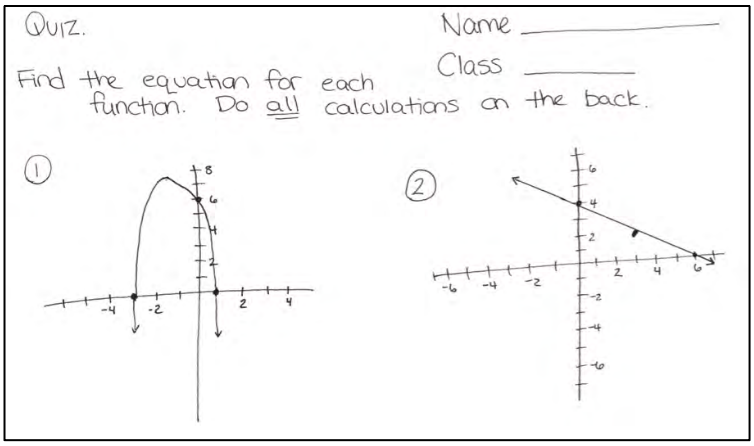


Figure 9.12. Extract of Maya's test

Figure 9.13 displays the class' performances on Problem 1 where the maximum score awarded was four marks. All the learners, barring one, wrote the test and all the learners who wrote the test attempted a solution to Problem 1. Four learners were awarded full marks for their solution to test problem 1 by the teacher, four were awarded two marks each, and eight were awarded one mark each because they correctly identified the function as having the general formula $y = ax^2 + bx + q$. Three learners (P12, P18, P19) were awarded zero for their attempts at solving Problem 1.

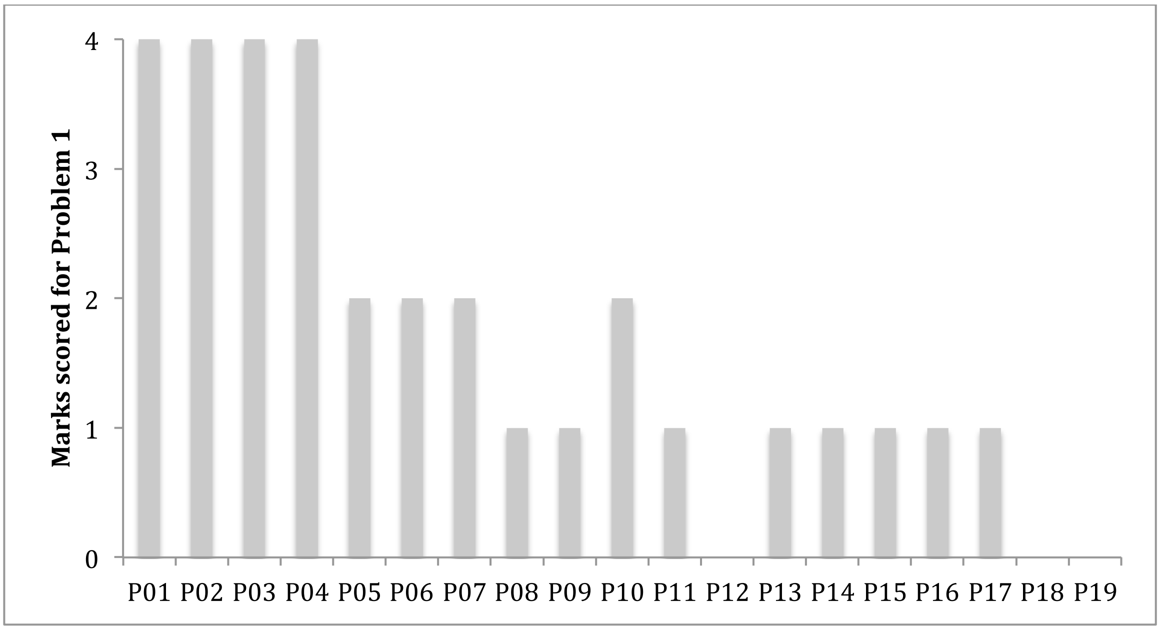


Figure 9.13. Maya's learners performance on Problem 1

9.6.2 Computational activity of Maya's top learner (Fay)

The analysis of learners' responses to Problem 1 presented below focuses primarily on the computational activity of the top learner (Fay), i.e. the recognition and realisation rules employed during the test and clinical interview. Brief comments on commonalities and differences with respect to the computational activity of the other interviewed learners follow. Fay scored full marks for the test and full marks for Problem 1 (see Figure 9.14).

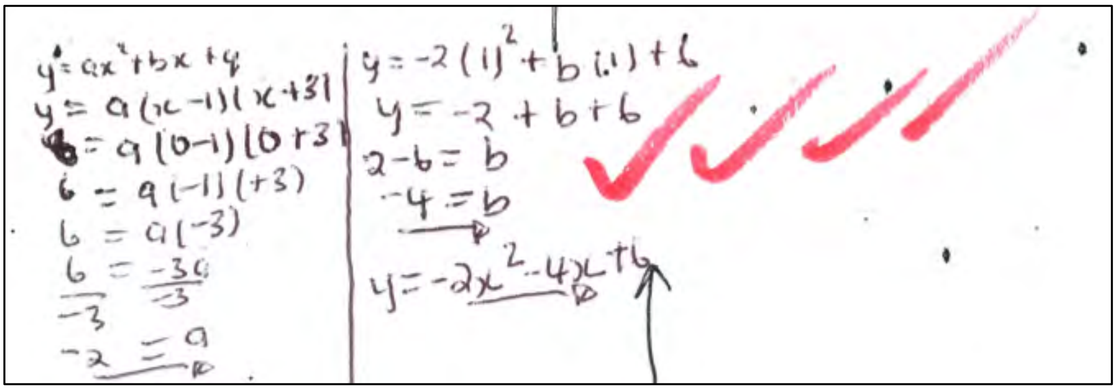


Figure 9.14. Fay's solution to Problem 1

Although, the teacher awarded full marks, Fay's solution is mathematically flawed. Note the error on the right-hand side of Figure 9.14 where Fay computes the value of b. She substitutes $y = -2 + b + 6$ (line 2) with $2 - 6 = b$ by ignoring y and "moving" -2 and b across the = sign. In doing so, y seems to "disappear" and $2 - 6$ is moved into the space previously occupied by y. So she produces $b = -4$, the required value but the notion of an equation does not function as a regulative resource for her. Instead the "=" sign in the expression " $= -2 + b + 6$" is part of character distribution matrix which has spaces for symbols on both sides of the "=" sign that enables the movement of symbols to produce the required solution $b = -4$. Fay's solution to Problem 1 represents a case where the expressions making up the learner's solution to Problem 1 and the content indexed by her solution diverge from the Mathematics encyclopaedia since the notion of an equation is absent as a regulative resource. Fay displays an orientation to mathematics that is expression-centred because she knows what needs to be produced expressively but does so with computations that are auxiliary to the Mathematics encyclopaedia.

Fay, in fact, produced a second solution to Problem 1 in the test (see Figure 9.15) where she did not reproduce the error discussed above. This solution, however, was not marked by the teacher. Here, her solution conforms expressively with the Mathematics encyclopaedia but we cannot tell whether it corresponds with the Mathematics encyclopaedia at the level of the content.

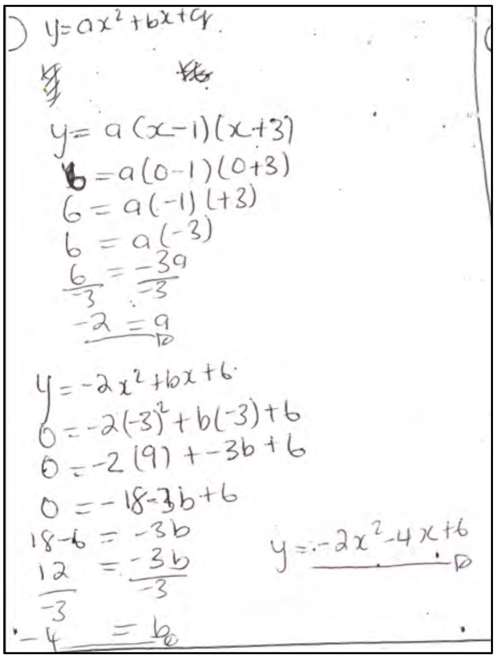


Figure 9.15. Fay's (P01) unmarked solution to problem 1 produced in the test

Fay does not reproduce the error committed during the test during the interview though (see Figure 9.16). In the interview, Fay identifies the function as a parabola with general equation $y = ax^2 + bx + q$ because "the square tell me that there are two positions that's going to be the *x*-axis" (SO2T03P01: Line 8). Her use of an auxiliary proposition relies on the iconic features of the text which she is obliged to use because the type of function represented is not stated in the problem nor are learners provided with the general formula of the function. We know from the analysis of the instructional discourse discussed in Chapter 6 and Chapter 7 that learners are meant to deduce the type of function represented from the image provided and then to assign the appropriate general formula. So, it seems as though the expression-orientation to mathematics exhibited by the learners is shaped by the computational activity observed in the instructional discourse.

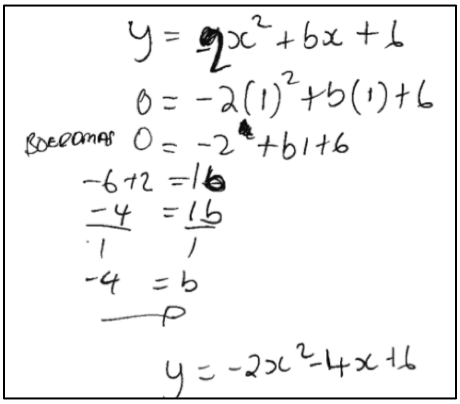


Figure 9.16. Fay's calculation of *b* produced in the interview

The method used by Fay to solve Problem 1 was referred to as the "short cut" method by the teacher. Fay produces the expression $y = a(x-1)(x+3)$ as the first line of her solution but says she cannot really explain why she does so. When faced with another problem of the same type, she first produces the template $y = a(\quad)(\quad)$ and indicates that the brackets are to be used in conjunction with the *x*-intercepts, strongly suggesting the presence of a character distribution matrix which is associated with the *x*-intercepts.

We note that the manner in which the coordinates on the graph are used by Fay during the interview suggests that she treats the information given in the sketch as though they are numbers rather than coordinate pairs. For example, she substitutes $y = 6$ into the equation without simultaneously substituting $x = 0$, which was substituted much later in her solution (see Figure 9.17). Secondly, when asked why she used $x = 0$ and not $x = -3$ or $x = 1$, she does not argue that 0 should accompany 6 since $(0; 6)$ is a given coordinate pair. Instead, she attempted to substitute $x = 1$ but abandoned her attempt presumably because it produced the equation $6 = 0$.

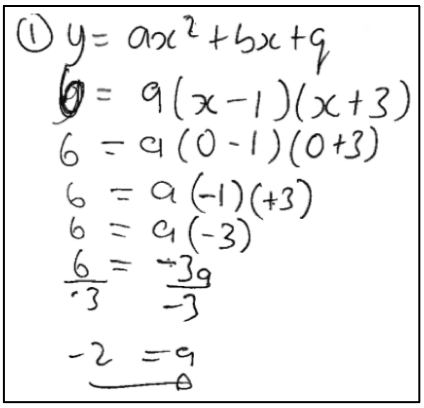


Figure 9.17. Part of Fay's solution to Problem 1 produced during the interview

Fay substitutes $a = -2$ and $q = 6$ into the equation $y = ax^2 + bx + q$ to produce the equation $y = -2x^2 + bx + 6$. Her reason for equating q with 6 is because "it's the highest point" on the graph (S02T03P01: line 70). When the interviewer points out to her that 6 is not the highest point on the graph, she argues that 6 represents the turning point of the graph (see S02T03P01: line 74), thus revealing an incorrect proposition that q is the turning point of a parabola. When she is made aware that the turning point is not 6 but elsewhere on the graph, she claims “cause I can’t really know what this is [points to turning point]” (S02T03P01: line 82), “so I got to work with this one (refers to y-intercept)” (S02T03P01: line 84). Her approach to solving the problem by using whatever values are available can be traced back to the instructional discourse when the teacher encouraged learners to use whatever values were available in order to “get rid of” variables (see Chapter 7).

The recognition and realisation rules used by Fay constitute the content of the announced topic parabola as a combination of an auxiliary calculus on characters together with basic arithmetic (operations on whole numbers).

9.6.3 Computational activity of Maya’s other interviewed learners

The computational activity of Fay is similar to that of the other interviewed learners. Table 9.10 summarises the final equation (computational object) produced by the interviewed learners, the method employed and the mark awarded by the teacher for their solution to test problem 1.

Table 9.10. Solution methods used by interviewed learners

Learner	Code	Final computational object	Solution method	Marks awarded
Fay	P01	$y = -2x^2 - 4x + 6$	“short cut” method	4
Nia	P02	$y = -2x^2 - 4b + 6$	“short cut” method	4
Max	P09	$b = -2a + 6$	“crazy complicated” method	1
Pam	P10	$y = -2x^2 + bx + 6$	“short cut” method but incomplete	2
Sue	P17	$y = 9a + +1b + 6$	“crazy complicated” method.	1
Sam	P19	$y = 2^2 + -10x + 6$	“short cut” method	0

As discussed above, Fay (P01)[76] was the only interviewed learner who produced the correct final computational object expressively but, as discussed, her solution is inconsistent with the Mathematics encyclopaedia. All the interviewed learners used a combination of basic arithmetic and an auxiliary calculus, comprising operation-like manipulations and auxiliary propositions and/or descriptions during the interviews (see Table 9.11).

[76] One other learner (P04) of those who wrote the test produced the correct final computational object expressively but her solution was also mathematically flawed.

Table 9.11. Auxiliary computational resources used by Maya's learners during interview

Learner	Auxiliary operations	Auxiliary propositions/descriptions
Fay (P01)	minus minus is a positive (L248) cross multiply (L366)	The equation of the function is $y = ax^2 + bx + q$ because "the square tell me that there are two positions that's going to be the *x*-axis. (L8) The highest point on the parabola is q (L70) which where the graph turns (L74) "Asymptote is my line is not going to touch or cross this lines(graph)" (L256) If you want to find x, make $y = 0$ (L340), If you want to find y, make $x = 0$ (L357)
Nia (P02)	"swop signs" (L54) "changing signs" (L100, L234) "when you go to the other side you change the sign" (L106)	"it is like a curvy thing and it has two points that are going down so that's a parabola. It's not a linear or exponential graph" (L12) A parabola has general formula $y = ax^2 + bx + q$ as opposed to $y = ax^2 + q$ because "the highest point it's not exactly on the *y* intercept then we know obviously there's *bx*" (L10) "when we're solving for *x* we make *y* zero" (L72) "An asymptote is the highest point or highest point or maximum point where it's the vertex of a parabola" (L88) The parabola will be "a sad face since it will be like this since our *a* is negative) (L310)
Max (P09)	A number times zero produces a void - "it just cancels" (L50) Take over to the other side (L116) when it's changed sides it becomes negative (L148, 154)	"because the sides (*x*-intercepts) are not equal. So the equation is *ax* squared plus *bx* plus *q*" (L8) *q* is the *y* intercept (L29)
Pam (P10)		The function is a parabola "because it is facing like this down and sometimes it it faces up" (L12) "when we want to find the *x*-intercept you must always make y equal to zero" (L60) "A function is I think it is a formula" (L84)
Sue (P17)	Change sign - "It's not going to be negative three it's going to be positive here. (L28)	"if you want *y* you gonna make *x* zero" (L188)
Sam (P19)	Take over and change the sign (L109)	The function is a parabola because "it has two .. it has two points and it must have an asymptote that is on this *y*-intercept " (L18) "A parabola must have a squared" (L32) "I'm factorising because of the squared" (L76) "a must be negative … because it's sad" (L95-99) A parabola has a "sad face" when a is negative and a "smiley-face" when a is positive (L155-157)

The realised content of all the learners is characterised as ancillary (see Figure 9.18). However, the auxiliary calculus used by Fay is such that she produces mathematics that converges at the level of expression but which sometimes diverges at the level of content associated with the topic from the point of view of the Mathematics encyclopaedia. All the other learners, including Fay, employ an auxiliary calculus that produces divergences in expression and content associated with the topic from the point of view of the Mathematics encyclopaedia.

		Encyclopaedic computational resources	
		present	**absent**
Auxiliary computational resources	**absent**	*Canonical*	*Elementary*
	present	*Symbiotic*	*Ancillary* Fay (P01), Nia (P02) Max (P09), Pam (P10) Sue (P17), Sam (P19)

Figure 9.18. Realisation of content by Maya's learners

The high proportion of learner texts that differ from the Mathematics encyclopaedia at the level of content and at the level of expression appears to be a consequence of the strongly closed and strongly expression-oriented pedagogic modality evident in the instructional discourse (discussed in Chapters 6 and 7) as well as the expression-centred orientation to mathematics displayed in the setting of the test and marking of learners' test scripts (discussed in Chapter 8). Here it seems that Eco's (1984) proposition which states that the potential for "aberrant decodings" to emerge from a reading of a closed text is borne out in a pedagogic context where a strongly closed, expression-oriented pedagogic text shapes the computational performance of learners and shapes their orientation to mathematics.

9.6.4 Computational performance and orientations to mathematics (Maya's learners)

As discussed above, the fundamental mathematics axioms and propositions are absent from the computational activity of all learners and are replaced with iconic and algorithmic computational resources. As such the recognition and realisation rules employed by all of Maya's learners constitute mathematics which is strongly closed with respect to the topic from the point of view of the Mathematics encyclopaedia. The presence of the iconic and algorithmic ground and the absence of fundamental ground from the learners' computational activity mirrors that exhibited in the instructional discourse. However, as discussed above the learners' recognition and realisation rules differ from each other and differ in some respects from that exhibited in the instructional discourse. Mathematics is constituted as a combination of a basic arithmetic and an auxiliary calculus that focuses on generating the required expressive elements. The sketch in the mathematics problem is used as a source of data for performing the computations used to generate the required expressions. The notion of a function and the notion of an equation are absent as regulative resources. As such, the orientation to mathematics of all the learners is strongly expression-oriented and the texts produced are strong closed pedagogic texts (see Table 9.12).

Table 9.12. Summary of Maya's learners' computational performance and orientation to mathematics

Learner	**Computational performance** Open text (T_o) /Closed Text (T_c)	**Orientation to mathematics** Content-orientation (O_c) / Expression-orientation (O_e)
Fay (P01)	T_c^+	O_e^+
Nia (P02)	T_c^+	O_e^+
Max (P09)	T_c^+	O_e^+
Pam (P10)	T_c^+	O_e^+
Sue (P17)	T_c^+	O_e^+
Sam (P19)	T_c^+	O_e^+

The lack of variation in computational performance and orientation to mathematics (i.e. sameness in the specialisation of consciousness) observed in a selection of learners from Maya's class does not seem surprising given that all the evaluative events of the Maya's observed lessons were coded as strong closed pedagogic texts and strongly expression-oriented (T_c^+/O_e^+). The analysis strongly suggests a correspondence between the functioning of evaluation in the instructional discourse and the evaluative activity of the learners despite differences in recognition and realisation rules amongst the learners and that used by the teacher. The computational performances and orientations to mathematics displayed by the actual learners correspond with that of the model learner implied the computational activity in the instructional discourse.

9.7 Analysing the tests and interviews of Jono's learners

9.7.1 Jono's interviewed learners

The learners from Jono's class who were interviewed are referred to as Tim (P01), Ali (P02), Ozi (P11) and Lea (P12), with Lea being the only girl amongst the group. Only four of the six selected learners turned up for the interviews. The two bottom performing learners selected for interviewing, Ory (P17) and Zoe (P18), refused to be interviewed and none of the other learners were willing to be interviewed. However, their test scripts were included in the analysis. The absence of interviews for the two learners impacted minimally on the analysis and results.

The interview with Jono's learners centred around test problem A1 (see Figure 9.19) which corresponds with problem A5 in the worksheet used during the observed lessons.

A. Find the **domain** and **range** of following linear functions (write it as **interval** and **set notations**):

1. $\frac{x}{2} - \frac{y}{3} = 1$
2. $y = x + 1$

Figure 9.19. Problems A1 and A2 from Jono's test

As discussed in Chapter 8, the test problems were taken directly from the worksheet used during the observed lessons. Furthermore, the whole of the first observed lesson focused on the domain and range of linear functions. About 20 minutes (approximately 30%) of the first observed lesson was spent on worksheet problem A1, which corresponds with test problem A2, and was briefly reviewed at the start of the second observed lesson. Despite the extended amount of time spent on the sub-topic, only three (P01, P03, P06) of the 19 learners who wrote the test produced the correct solution according to the teacher's memorandum to test problem A1. The learner's responses to test item A1 indicates a high proportion of "aberrant decoding" of the test problem at the level of expression.

Figure 9.20 shows the marks awarded by Jono for test item A1. However, the marks do not accurately reflect the learners' performance on the item because as we recall from Chapter 8, Jono's marking is flawed and inconsistent. The striped bars represent instances where incorrect solutions were awarded marks.

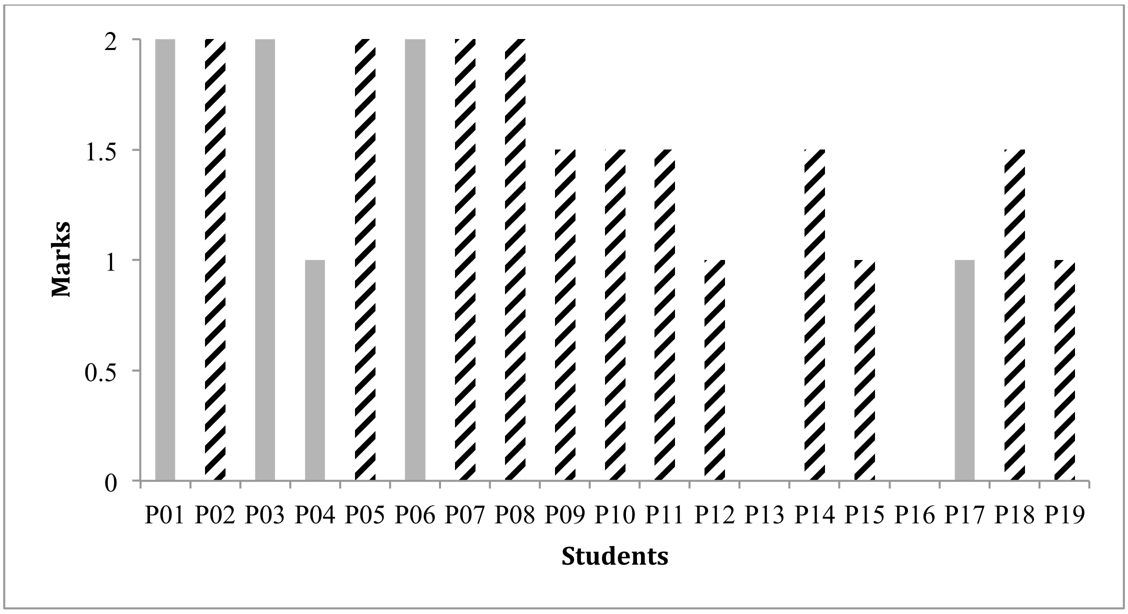


Figure 9.20. Marks awarded by Jono for learners' solution to test item A1

9.7.2 Computational activity of Jono's top learner (Tim)

The analysis of learners' responses to test problem A1 presented below focuses on the computational activity of the top learner (Tim), i.e. the recognition and realisation rules employed in the test and clinical interview. Brief comments on commonalities and differences with respect to the computational activity of the other interviewed learners follow

Tim was the only learner in Jono's class to score full marks for the test and the only interviewed learner who obtained the correct solution to test problem A1 according to the teacher's memorandum. His marked solution to test problem A1 is shown in Figure 9.21.

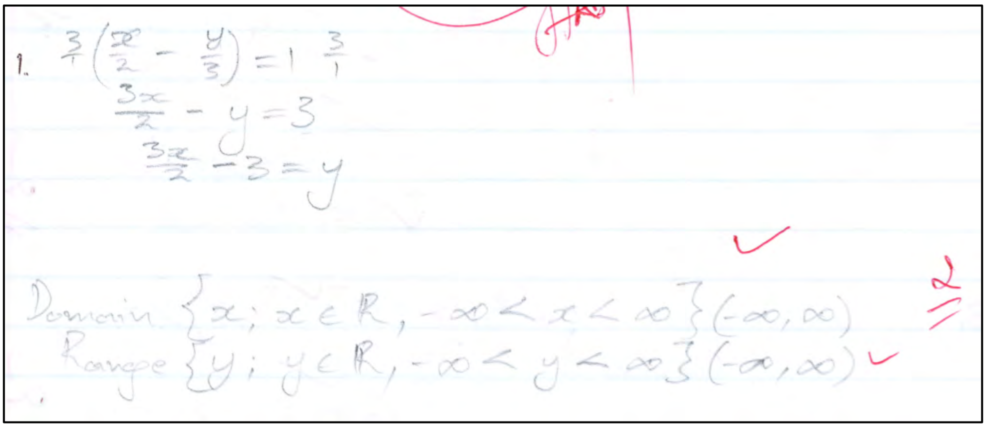


Figure 9.21. Tim's (P01) solution to Problem A1

However, in the interview, Tim could not provide a definition or even a description of the domain of a function. He only knew how to compute the domain of a function (see S02T04P01: lines 11-18). The computations employed by Tim when substituting $\frac{x}{2}-\frac{y}{3}=1$ with a "total equation", $\frac{3x}{2}-3=y$, included the basic arithmetic operations (multiplication and division) on real numbers in combination with the auxiliary operations (transposition or "change sides, change signs") performed on characters. Interestingly, Tim used the operation "add the same thing to both sides of the equation" as well. However, his computations were not grounded by the idea that "adding the same thing to both sides of an equation" preserves identity. Instead, his computations appeared to be grounded by the need to "move" /3/ to the other side of the equation as displayed in Extract 9.7.

Extract 9.7 .S02T04P01 (lines 77 – 80)

Interviewer:	What you doing when you put minus three and minus three there [referring to adding -3 to both sides of the equation $\frac{3x}{2}-y=3$]?
Learner:	Er um I want to take this three to this side [left hand side of equation].
Interviewer:	Oh okay.
Learner:	Since um I can't just move it so I have to kill it Miss. Ja.

It is also interesting that he expresses the need to "kill" or cancel /3/ in order to "move" /3/ across the equal sign but that he does not do so to "move" $/y/$ across the equal sign. The difference in the way that he treats /3/ compared to $/y/$ was not explored in the interview but suggests that the dominant operation is that of spatial displacement of characters, justified by adding the same thing to both sides of the equation.

Extract 9.8 illustrates the proposition used by Tim when computing the domain of a function. His propositions can be stated in the following way: if x is a denominator then the domain $x \neq 0$ but if x is not a denominator, then x is any real number. Thus, Tim's proposition entails a mapping based on what the equation looks like. In other words, iconic ground serves as a regulative resource.

Extract 9.8 S02T04P01 (lines 112 – 120)

Learner:	So um the domain um ....[writes] brackets … x is equal to. So now what we're doing here is we're saying that x x this x over here [referring to $y = \frac{3x}{2} - 3$].
Interviewer:	Yes.
Learner:	can be is an element of any real numbers.
Interviewer:	Okay.
Learner:	So what we can't do is make uh uh if this x was a denominator …
Interviewer:	Yes.
Learner:	What what we have to see first is how can we make this whole equation undefined.
Interviewer:	Okay.
Learner:	So if this x was the denominator and the number divided by zero is undefined.

The strong regulative effect of the iconic on Tim's computational activity is further revealed when he explained during the interview that for the equation $y = \frac{3}{2}$, "there won't be any domain" because "there wasn't any x" in the equation (S02T04P01: lines 248-256). So, the absence or presence of the symbol x in the equation and the location of x in the equation determines for Tim whether the function has a domain and what the domain of the function is.

In order to explore Tim's conception of domain of functions in more depth, I asked him what the domain of the function shown in Figure 9.22 is. This problem was constructed by me and presented to Tim during the interview.

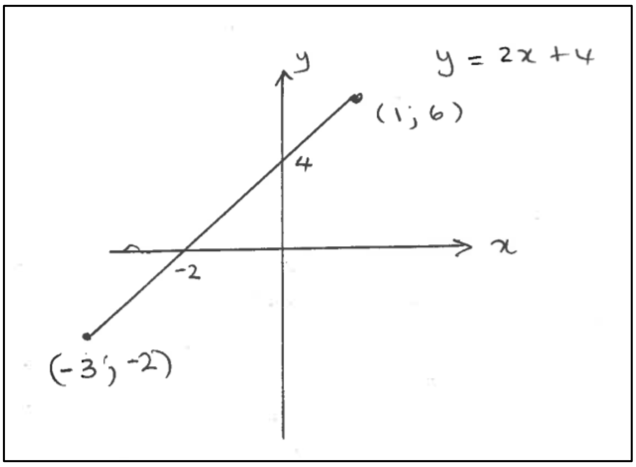


Figure 9.22. Problem used during the interviews with Jono's learners

His first response was that "since this is a linear equation" (S02T04P01: line 284) the domain "is an element of real numbers. Can't be smaller than negative infinity can't be larger than infinity". His response reveals that he uses a proposition that states the domain of all linear functions is the set of real numbers – a proposition which corresponds with the one arrived at "inductively" in class during the observed lessons and discussed in detail in Chapter 7. When it was pointed out to him that the domain of $y = 2x + 4$ is restricted, he changed the domain to $\{x \in \mathbb{Z};\ -3 \leq x \leq 1\}$ (see S02T04P01: lines 342-358).

Extract 9.9 S02T04P01 (lines 387-390)

Interviewer:	Because I don't know why you put .. why did you put er integers now?
Learner:	Integers.
Interviewer:	and not reals?
Learner:	Miss ah ha since um I can see these numbers [Points to the coordinates on the sketch]

His reasons for his solution are presented in Extract 9.9. It seems that the presence of negative numbers provided in Figure 9.22 serves as the ground for stating that the domain is the set of integers, confirming that the iconic operates as a regulative resource. The recognition and realisation rules used by Tim constitute the announced topic, domain and range of linear functions, as a combination of an auxiliary calculus (auxiliary operations and auxiliary propositions) together with arithmetic on integers.

9.7.3 Computational activity of Jono's other interviewed learners

With the exception of Tim, all the learners selected to be interviewed produced incorrect statements with respect to the set-builder notation for the domain and range of the function $\frac{x}{2} - \frac{y}{3} = 1$, test problem A1 (see Table 9.13, previously Table 8.2). Ory and Zoe produced the correct interval notation despite incorrect statements for domain and range in set-builder notation, suggesting that learners were merely recalling solutions produced in class.

Table 9.13. Test solutions to Problem A1 of Jono's learners selected to be interviewed

Name	Code	Domain (set builder)	Domain (interval)	Range (set builder)	Range (interval)	Mark awarded
Tim	P01	$\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$ correct	$(-\infty; \infty)$ correct	$\{y: y \in \mathbb{R};\ -\infty < x < \infty\}$ correct	$(-\infty; \infty)$ correct	2
Ali	P02	$\{x/\ x \in \mathbb{R};\ -\infty > x < \infty\}$ incorrect	$(-\infty; \infty)$ correct	$\{y/\ y \in \mathbb{R};\ -\infty > x < \infty\}$ incorrect	$(-\infty; \infty)$ correct	2
Ozi	P11	$\{x: x \in \mathbb{R};\ -\infty > x > \infty\}$ incorrect	$(\infty; \infty)$ incorrect	$\{y: y \in \mathbb{R};\ -\infty > y > \infty\}$ incorrect	$(-\infty; \infty)$ correct	1,5
Lea	P12	$\{x: x \in \mathbb{R};\ x \neq 0\}$ incorrect	$(0; \infty)$ incorrect	$\{y: y \in \mathbb{R};\ -\infty < x < \infty\}$ incorrect	$(-\infty; \infty)$ correct	1
Ory	P17	$\{x: x \in \mathbb{Z};\ x < 3\}$ incorrect	$(-\infty; \infty)$ correct	$\{y: y \in \mathbb{R};\ y \geq 3\}$ incorrect	$(-\infty; \infty)$ correct	1
Zoe	P18	$\{x: x \in \mathbb{R};\ -\infty;\ \infty\}$ incorrect	$(-\infty; \infty)$ correct	$\{y: y \in \mathbb{R};\ \neq 0\}$ incorrect	$(-\infty; \infty)$ correct	1,5

During the interviews the learners reproduced the same errors for the domain of the function $\frac{x}{2} - \frac{y}{3} = 1$ as they did in the test. Ali produced the statement $-\infty > x < \infty$, Ozi wrote down $-\infty > x > \infty$, and Lea's statement recorded in the interview was $-\infty \ x \ \infty$. Extract 9.10 shows Ali's explanation of his statement, $-\infty > x < \infty$.

Extract 9.10. S02T04P02 (line 66)

Learner:	negative infinity is greater than x and x is less than negative infinity which means that anything between. It's like Miss when you say a number is in between let's say two and four but then it's negative infinity and positive infinity.

Ali describes the inequality statement as "between negative infinity and positive infinity". Lea, on the other hand, describes her statement, $-\infty \ x \ \infty$, as "from negatives to positives" (S02T04P04: line 109). Ozi stated that that "x can be between negative infinity like this .. could be less than .. it is less than negative infinity. Then it is also less than positive infinity" (S02T04P11: line 66) when he describes the domain of the function $y = x + 1$. Ali, Ozi and Lea provide descriptions which rely on spatial order, that is between two numbers or from one number to another. Thus, spatial order takes the place of order relations which are violated by Ali and Ozi and completely absent in the case of Lea and Zoe. The learners' conception of the inequality indicates grounding in spatial order rather than numerical order which mirrors the prioritisation of spatial order over numerical order in the instructional discourse (see Chapter 7). We observe that only Tim produced the correct inequality statement, suggesting that he implicitly recruits order relations as regulative resources which is borne out by his description of the domain of problem A1 in the interview "x is greater than negative infinity also smaller than infinity" (S02T04P01: line 190).

		Encyclopaedic computational resources	
		present	**absent**
Auxiliary computational resources	**absent**	*Canonical*	*Elementary*
	present	*Symbiotic*	*Ancillary* Tim (P01), Ali (P02) Ozi (P11), Lea (P12) Ory (P17). Zoe (P18)

Figure 9.23. Realisation of content by Jono's learners

All of the learners realised content described as ancillary (see Figure 9.23). Tim is the only learner whose recognition and realisation rules produce mathematics that diverges from the Mathematics encyclopaedia at the level of content but converges at the level of expression. All the other learners constitute mathematics that diverges from the Mathematics encyclopaedia at the level of content as well as expression using recognition and realisation rules that differ across learners.

9.7.4 The use of character distribution matrices

The responses of the interviewed learners and an analysis of the test scripts leads me to conclude that the learners in Jono's class constitute the topic (domain and range of functions) as basic arithmetic combined with an auxiliary calculus that enables them to transform a given function into standard form. They then recruit auxiliary propositions that match the equation to an inequality statement representing the domain and range of the given function.

In addition, there seems to be widespread use of a character distribution matrix which has the following form: { x: $\boxed{E1}$;$\boxed{E2}$} (e.g. $\{x: x \in \mathbb{R};\ -\infty < x < \infty\}$. E1 describes the type of number and has the form $x \in \boxed{A}$ where *A* represents a symbol selected from the set $\{\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0\}$. As discussed above, there appears to be a range of different criteria used by learners to decide whether the domain is set of real numbers (ℝ). The domain of a function is said to be real if:

1) equation of the function does not have a variable in the denominator or as a divisor (iconic)
2) the function is a linear function (iconic)
3) by substituting values, you do not produce values that make the function undefined (empirical)
4) you can't "find" a value of *x* for which the function is undefined (algorithmic)

Furthermore the presence of certain numbers suggests particular mappings:

$\{\ldots.-3;\ -2;\ -1; 0; 1; 2; 3 \ldots..\} \rightarrow \text{integers } (\mathbb{Z})$

$\{1; 2; 3; 4; \ldots \ldots \ldots \ldots \ldots \ldots \ldots. \ldots..\} \rightarrow \text{natural numbers } (\mathbb{N})$

$\{0; 1; 2; 3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots..\} \rightarrow \text{whole numbers } (\mathbb{N}_0)$

In other words, there is a mapping from a number set to the set $\{\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0\}$.

The expression E2 stipulates a number range or exclusion of particular values from the domain, for example $x \neq 0$ or $-2 < x < 3$.

E2 takes one of two forms:

(1) $x\ \boxed{B}\ \boxed{C}$ (e.g. $x \neq 0$), where *B* represents a symbol selected from the set $\{\neq; <; >; \leq; \geq\}$ and *C* is an integer deduced from the equation or simply read off the equation or *C* is the set of numerical symbols which includes ∞ and $-\infty$; or

(2) $\boxed{D}\ \boxed{E}\ x\ \boxed{E}\ \boxed{D}$ (e.g. $-2 < x < 3$), where *E* represents a pair of symbols selected from the cross product of the set $F = \{<; >; \leq; \geq\}$ with itself. That is, $F \times F = E = \{(<<); (>,<); (>,>), (<,>); (\leq,\leq); (\geq,\leq); (\geq,\geq); (\leq,\geq); (<,\leq); (>,\leq),; (>,\leq); (<,\geq); (\leq,<); (\geq,<); (\leq,>); (\leq,<)\}$ and *D* is an integer deduced from the equation or simply read off the equation or *D* could be the set of numerical symbols which includes ∞ and $-\infty$.

9.7.5 Computational performance and orientation to mathematics (Jono's learners)

In summary, the fundamental mathematics axioms, definitions and propositions are absent from the computational activity of all learners and are replaced with iconic, empirical and algorithmic computational resources. The presence of the iconic, empirical and algorithmic ground and the absence of fundamental ground from the learners' computational activity mirrors that evident in the instructional discourse as discussed in Chapters 6 and 7. However, as discussed above the learners' recognition and realisation rules differ from each other and differ in some respects from that found in the observed lessons. Mathematics is constituted as a combination of basic arithmetic and an auxiliary calculus that focuses on generating the required expressive elements.

We observe that the learners use symbols as objects of computation, i.e., computation involving symbols is constituted as the mathematics. Notational devices are required as resources for communicating arguments and ideas or as a means of capturing and expressing thought. However, in this pedagogic context, operations on the notational symbols is the mathematics. As such, the orientation to mathematics of all the learners is expression-oriented and the computational performances of all the learners produce mathematics that is strongly closed with respect to the topic from the point of view of the Mathematics encyclopaedia. See Table 9.14 for a summary of the computational performances and orientations to mathematics of the learners in Jono's class who were selected to be interviewed.

Table 9.14. Jono's learners' computational performances and orientations to mathematics

Learner	**Computational performance** Open text (T_o) /Closed Text (T_c)	**Orientation to mathematics** Content-orientation (O_c) / Expression-orientation (O_e)
Tim (P01)	T_c^+	O_e^+
Ali (P02)	T_c^+	O_e^+
Ozi (P11)	T_c^+	O_e^+
Lea (P12)	T_c^+	O_e^+
Ory (P17)	T_c^+	O_e^+
Zoe (P19)	T_c^+	O_e^+

The analysis strongly suggests a correspondence between the functioning of evaluation in the observed lessons and the evaluative activity of the learners despite differences in recognition and realisation rules amongst the learners and that used by the teacher. The computational performances and orientations to mathematics displayed by the actual learners correspond with that of the model learner implied the computational activity in the instructional discourse.

The high proportion of learner texts that differ from the Mathematics encyclopaedia at the level of content and at the level of expression appears to be a consequence of a closed pedagogic text that is strongly expression-oriented and combined with a quasi-inductive pedagogy (see Chapters 6 and 7). The evaluative criteria emergent in the instructional discourse together with the teacher's inconsistent and flawed evaluation of learner productions (cf. Chapter 8) appears to strongly shape the learners' expression-orientation to mathematics. As with Maya's pedagogic context, Eco's (1984) proposition that the potential for "aberrant decodings" to emerge from a reading of a closed text is borne out in a pedagogic context where a strongly expression-oriented pedagogic text constrains the computational performance of learners and shapes their orientation to mathematics. A lack of variation in computational performance and orientation to mathematics (i.e. sameness in the specialisation of consciousness) is observed in a selection of learners from Jono's class.

9.8 Concluding remarks

This chapter considered the analysis of learners' test scripts and learner interviews on selected test problems. Interviews focused on the evaluative activity of learners when doing mathematical work independently of the teacher. The interviews were necessary to elicit the recognition and realisation rules used by learners to solve mathematical problems because the test scripts were insufficient given that it is entirely possible to produce the same expressions using very different recognition and realisation rules. The learners' computational activity evident in their solutions to the test and displayed during the interview were used to read off their specialisation of consciousness, described in terms of computational performance and orientation to mathematics.

Table 9.15 summarises the recognition and realisation rules, described in terms of the computational activity of the interviewed learners, and the realised content across the four pedagogic contexts. At Evergreen High, the school populated by learners from working-class families, the computational activity of all the learners included operations from both the field of the reals as well as auxiliary operations and the use of auxiliary propositions. None of the learners selected for the interviews recruited definitions, axioms or propositions located in the Mathematics encyclopaedia, indicating a marked absence of fundamental ground functioning as regulative resources in the computational activity of all Evergreen High interviewed learners. So, the content realised by all learners is described as ancillary.

Table 9.15. Summary of the recognition and realisation rules employed by interviewed learners

	Prestige College Upper middle class/elite		**Evergreen High Working-class**	
	Sara	**Jada**	**Maya**	**Jono**
Computational activity	All learners recruit operations from both the field of the reals and auxiliary operations but four also recruit encyclopaedic propositions.	All learners recruit operations from both the field of the reals and auxiliary operations except for one who also recruits an encyclopaedic proposition.	All learners recruit operations from both the field of the reals and auxiliary operations. Encyclopaedic propositions absent for all learners.	All learners recruit operations from both the field of the reals and auxiliary operations. Encyclopaedic propositions absent for all learners
Realised content	Symbiotic content (4 learners) Ancillary content (2 learners) Content divergence from announced topic for all learners but expression divergence for 3 and expression convergence for 3	Symbiotic content (1learner) Ancillary content (5 learners) Content divergence from announced topic for all learners but expression divergence for 3 and expression convergence for 3	Ancillary content (all learners) Content divergence from announced topic for all learners but expression divergence for 5 and expression convergence for 1	Ancillary content (all learners) Content divergence from announced topic for all learners but expression divergence for 5 and expression convergence for 1

The recognition and realisation rules employed by all the interviewed learners at Evergreen high realise content that entails a combination of basic arithmetic and an auxiliary calculus on symbols. In other words, the realised content diverges from the content typically associated with the topic from the point of view of the Mathematics encyclopaedia. Furthermore, with the exception of one learner in Maya's class and one learner in Jono's class, the remaining interviewed learners all produced mathematics that diverged from the Mathematics encyclopaedia at the level of expression.

At Prestige College, the school populated by learners from upper-middle-class/elite families, the computational activity of all of the learners included operations from both the field of the reals as well as auxiliary operations and the use of auxiliary propositions but to differing degrees. Four learners from Sara's class and one from Jada's class recruited propositions from the Mathematics encyclopaedia alongside auxiliary operations and propositions, thus producing symbiotic content compared to the others who produced ancillary content.

The realised content elaborated by all the interviewed learners at Prestige College comprised a combination of basic arithmetic and an auxiliary calculus on symbols. However, the computational activity of four learners in Sara's class and one learner in Jada's class was augmented with encyclopaedic propositions to differing degrees. So, the realised content diverged from the content typically associated with the topic from the point of view of the Mathematics encyclopaedia. For three of Sara's learners and three of Jada's learners, convergence at level of expression was achieved despite divergence at the level of content. For three of Sara's learners and three of Jada's learners, both divergence with respect to content as well as expression occurred.

A summary of the specialisation of consciousness of the learners in the four pedagogic contexts is displayed in Table 9.16 which shows the distribution of computational performances across pedagogic contexts and Table 9.17 which displays the distribution of learners' orientations to mathematics across pedagogic contexts.

Table 9.16. Computational performances of interviewed learners across pedagogic contexts

Pedagogic context	**Teacher**	**Strongly closed pedagogic texts**	**Weakly closed pedagogic texts**	**Weakly open pedagogic texts**	**Strongly open pedagogic texts**
Prestige College Upper middle-class/elite	Sara	2 learners	1 learner	3 learners	0 learners
	Jada	5 learners	1 learner	0 learners	0 learners
Evergreen High Working-class	Maya	6 learners	0 learners	0 learners	0 learners
	Jono	6 learners	0 learners	0 learners	0 learners

All the learners in the working-class context demonstrated computational performances that produce strongly closed pedagogic texts, indicating that for those learners the topic is closed with respect to the Mathematics encyclopaedia. All the learners displayed an orientation to mathematics that is expression- oriented, indicating that for these learners operations on the expressions constitute the mathematics. The computational performances and orientations to mathematics of actual learners correspond with the computational performances and orientations to mathematics of the model learners implied by the recognition and realisation rules evident in the instructional discourse in Maya's and Jono's pedagogic contexts.

Table 9.17. Summary of orientations to mathematics of interviewed learners across pedagogic contexts

Pedagogic context	Teacher	Strongly expression-oriented	Weakly expression-oriented	Weakly content-oriented	Strongly content-oriented
Prestige College Upper-middle-class/elite	Sara	2 learners	1 learner	3 learners	0 learners
	Jada	5 learners	1 learner	0 learners	0 learners
Evergreen High Working-class	Maya	6 learners	0 learners	0 learners	0 learners
	Jono	6 learners	0 learners	0 learners	0 learners

The upper-middle class/elite contexts exhibits variation in the specialisation of learners' consciousness, borne out by the distribution of computational performances and orientations to mathematics of the interviewed learners. For two of Sara's learners and five of Jada's learners who produce strongly closed pedagogic texts and display strongly expression-centred orientations to mathematics, their computational performances and orientations to mathematics resemble that of the learners in the working-class context. So for those learners, the topic is closed with respect to the Mathematics encyclopaedia and mathematics is constituted as the operations on expressions. One of Sara's learners and one of Jada's learners produce weakly closed pedagogic texts and their orientations to mathematics is described as weakly expression-oriented. Their computational performances and orientations to mathematics are comparable to that of learners who produce strongly closed pedagogic texts and display strongly expression-centred orientations to mathematics but their computational activity includes encyclopaedic propositions. Three of Sara's learners produce weakly open pedagogic texts and display weakly content-centred orientations to mathematics. For this group of learners, although they recruit auxiliary computational resources, the topic is open with respect to the Mathematics encyclopaedia. The computational performances and orientations to mathematics of actual learners correspond with the computational performance and orientation to mathematics of the hybrid model learners implied by the computational activity in Sara's and Jada's pedagogic contexts.

In summary, we observe sameness with respect to learners' specialisation of learners' consciousness in the pedagogic contexts populated by learners from working-class families. Secondly, there is a strong resemblance between the learners' computational performances and that of the model learner implied by recognition and realisation rules evident in the instructional discourse in Maya's and Jono's pedagogic contexts. This strongly suggests that the dominance of closed pedagogic texts and expression-orientation evident in Maya's and Jono's observed lessons shaped learners' computational performance and orientations to mathematics. In contrast, Prestige College (populated by learners from upper-middle class/elite families) exhibits variation in learners' specialisation of learners' consciousness. The hybridity with respect to the pedagogic modalities evident in the instructional discourse of Sara and Jada seems to play out at the level of the learners' computational performances and orientations to mathematics. Secondly, there is a strong resemblance between the learners' computational performances and that of the model learner implied by the recognition and realisation rules evident in Sara's and Jada's instructional discourse.

Chapter 10

Discussion of findings and conclusion

10.1 Introduction

In this chapter, I summarise the principal findings of the study presented in the preceding four analysis chapters. The study set out to investigate the functioning of evaluation at the level of the instructional discourse in pedagogic contexts in two schools that differ with respect to learners' social class membership. Specifically, the study examines the complexity of the recognition and realisation rules, described in terms of the computational activity employed by pedagogic agents in their productions of mathematics. The study is concerned with what evaluation reveals about the constitution of the content of school mathematics in four cases in two independent schools that differ with respect to the social class membership of their learner populations and the implications for the specialisation of learners' mathematical thought. Two independent secondary schools differentiated with respect to the social class membership of their learner populations were selected as the empirical sites. At each school, two mathematics teachers and their Grade 10 mathematics class of learners constituted the research participants of the study.

I consider the findings in relation to the research hypotheses established in Chapter 3, followed by a discussion of the findings in the light of claims and propositions established in the literature discussed in the thesis. The chapter ends with a consideration of the limitations and potential of the study.

10.2 Considering the research hypotheses

Recall from Chapter 1 that the functioning of evaluation in a pedagogic context reveals the recontextualisation of Mathematics to schooling and the distribution of mathematical knowledge and forms of consciousness (mathematical thought in this case) to different groups of learners. Figure 10.1 shows the relation between evaluation and the recontextualisation and distribution of mathematical knowledge and forms of consciousness. Evaluation is read off the computational activity which in addition gives insight into the implied model learner. The computational performance and orientation to mathematics of the model learner is implied by the computational activity emergent in the instructional discourse. In Chapter 1, I described actual learners' specialisation of consciousness with respect to school mathematics as the computational performance and orientation to mathematics displayed in a test and clinical interview.

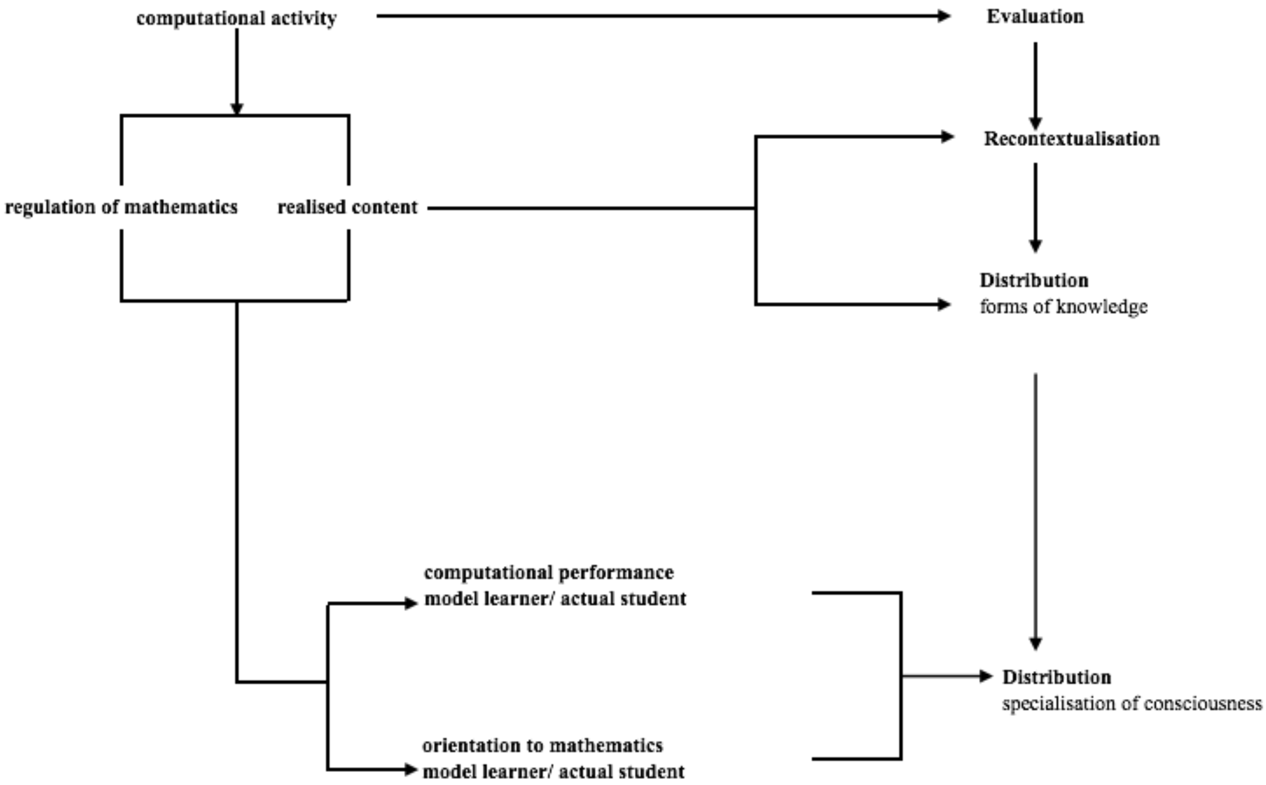


Figure 10.1. Schematic overview of theoretical framework

The results of the analysis discussed in Chapters 6, 7, 8 and 9 are presented in relation to the research hypotheses outlined in Chapter 3. Each sub-section starts with a restatement of the research hypothesis and then considers the findings in relation to the hypothesis.

10.2.1 Research hypotheses 1 and 2 – the realised content in the instructional discourse

RH1: *The realised content associated with announced topic(s) in the instructional discourse diverges from the Mathematics encyclopaedic content associated with the topic(s) in pedagogic contexts populated by learners from working-class families.*

RH2: *The realised content associated with announced topic(s) in the instructional discourse converges with the Mathematics encyclopaedic content associated with the topic(s) in pedagogic contexts populated by learners from upper-middle-class/elite families.*

The first interest of the study, captured in Research hypotheses 1 and 2, was to develop a picture of the effect of evaluation on the content realised in the instructional discourse in each pedagogic context by examining the recognition and realisation rules in terms of the computational activity of teachers and learners. The initial analysis of the computational activity of teachers and learners in the instructional discourse was directed by the following two questions:

- What does the computational activity in the instructional discourse reveal about the content realised with respect to Grade 10 mathematics topics in the instructional discourse in pedagogic contexts that differ with respect to the social class membership of learners?

- How is the realisation of content in the instructional discourse regulated in these pedagogic contexts?

The results of the analysis, discussed in Chapter 6, show what is realised as the content associated with announced topics and how the realised content is regulated in the instructional discourse across the four pedagogic contexts. The results illustrate differences and similarities between what is constituted as mathematics between social class contexts and within social class contexts. In so doing, differences and similarities in the functioning of evaluation, specifically the recognition and realisation rules which are described in terms of computations employed when doing mathematics, are demonstrated. In this way, the analysis of the computational activity reveals both what is recontextualised and to whom different forms of knowledge and consciousness are distributed. A key finding is that substitution of Mathematics encyclopaedic content takes place in all the pedagogic contexts but to differing degrees. Thus, content substitution appears to be a school mathematics phenomenon rather than a function of the social class membership of a pedagogic context's learner population. This hypothesis requires further investigation in a larger sample of pedagogic contexts.

In the working-class pedagogic contexts, the recognition and realisation rules employed realise content that diverges from the Mathematics encyclopaedia in all the evaluative events. The content realised in relation to the announced topics is described as ancillary in that auxiliary computational resources (auxiliary descriptions, auxiliary propositions and/or auxiliary operations over the domains of characters) are recruited together with arithmetic over whole numbers and, occasionally, fractions to constitute content associated with respect to announced topics. Notably, the realised content in those pedagogic contexts is marked by an absence of encyclopaedic computational resources (definite descriptions and/or propositions from the Mathematics encyclopaedia and/or encyclopaedic operations over the domain of real numbers). So, the principal computational resources regulating the computational activity are located elsewhere. The iconic is recruited as the dominant form of regulation of the computational activity and, in the case of Jono, empirical ground is relied on extensively to support the computational activity evident in his pedagogic context.

In the upper-middle-class/elite pedagogic contexts, the recognition and realisation rules employed realise content that partially converges with the Mathematics encyclopaedia to a greater extent in Sara's pedagogic context (set-2) than in Jada's (set-3) and corresponds in part with the content realised in the working-class context to greater extent with respect to set-3 than set-2. The similarities with the working-class context relate to the presence of ancillary content in the upper-middle-class context, with Jada displaying the use of more ancillary content than Sara. The upper-middle-class context differs from the working-class context in the occurrence of canonical content and symbiotic content, both of which comprise encyclopaedic computational resources and, in the case of symbiotic content, the co-presence and/or interchangeable recruitment of auxiliary computational resources. Set-2 displays a greater proportion of canonical and symbiotic content than does set-3. It should be noted that although fundamental ground regulates the computational activity to differing degrees in the two upper-middle-class/elite pedagogic contexts, definite descriptions are absent as computational resources, as is the case in the working-class school. Secondly, the iconic regulates the computational activity in both Sara's and Jada's pedagogic context but is offset by the inclusion of fundamental ground, more so in set-2 than in set-3.

The constitution of mathematics in the pedagogic contexts populated by learners from working-class families differs from that constituted in the pedagogic contexts populated by learners from upper-middle-class/elite families. Learners in the working-class context are essentially engaging in content comprised of arithmetic over whole numbers mostly and, occasionally, fractions in combination with an auxiliary calculus. In the upper-middle-class/elite contexts, learners also engage in content comprised of arithmetic over whole numbers and, occasionally, fractions in combination with an auxiliary calculus, but this content is augmented with fundamental propositions from the field of Mathematics to a greater extent in set-2 than in set-3. This finding illustrates that different forms of knowledge are distributed along social class lines, a finding which concurs with claims made by others (Dowling, 1998; Hoadley, 2005, 2007; O'Halloran, 1996), who also argue that different forms of knowledge are distributed along social class lines. However, this study shows differences in what is constituted as mathematics *within* the upper-middle-class/elite context and similarities *between* what is constituted as mathematics in the upper-middle-class elite context and what comes to be constituted as mathematics in the pedagogic contexts populated by learners from working-class families.

Thus, the analysis confirms Research hypothesis 1 and partially confirms Research hypothesis 2. Research hypothesis 2 is reformulated as follows: The realised content associated with announced topic(s) in the instructional discourse partially converges with and partially diverges from the Mathematics encyclopaedic content associated with the topic(s) in pedagogic contexts populated by learners from upper-middle-class/elite families.

10.2.1 Research hypotheses 3 and 4 – the model learner

RH3: *Pedagogic contexts populated by learners from working-class families construct model learners of closed pedagogic texts that are expression-oriented.*

RH4: *Pedagogic contexts populated by learners from upper-middle-class/elite families construct model learners of open pedagogic texts that are content-oriented.*

Research hypotheses 3 and 4 capture the implications of evaluation for the model learner presupposed by the computational activity in the instructional discourse. A secondary analysis of the recognition and realisation rules, described in terms of the computational activity evident in the observed lessons, was presented in Chapter 7. In this chapter, the dominant pedagogic modalities in relation to what the computational activity in each pedagogic context implies about the model learner's computational performance and orientation to mathematics are described. Two questions steered the secondary analysis of the computational activity discussed in Chapter 6:

- What does the computational activity elaborated in the instructional discourse imply about the computational performance of the model learner constructed in pedagogic contexts that differ with respect to the social class membership of learners?

- What orientations to mathematics are implied by the computational activity emerging in the instructional discourse in pedagogic contexts that differ with respect to the social class membership of learners?

Furthermore, an analysis of the tests constructed by the teachers and their marking of learners' test scripts (discussed in Chapter 8) contributes further to our understanding of the construction of the model learner in each pedagogic context.

The secondary analysis of the computational activity evident in the instructional discourse revealed five pedagogic modalities described in terms of open/closed pedagogic texts and content/expression-orientations to mathematics across the four pedagogic contexts: (1) strong open pedagogic text accompanied by a strong content-orientation (T_o^+/O_c^+); (2) weak open pedagogic text accompanied by a weak content-orientation (T_o^-/O_c^-); (3) weak closed pedagogic text accompanied by a weak expression-orientation (T_c^-/O_e^-); (4) weak closed pedagogic text accompanied by a strong expression-orientation (T_c^-/O_e^+); and (5) strong closed pedagogic text accompanied by a strong expression-orientation (T_c^+/O_e^+).

The working-class pedagogic contexts (Evergreen High) display pedagogic modalities characterised as closed pedagogic texts accompanied by an orientation to mathematics that is expression-oriented, with Maya's pedagogic texts being strongly closed and the orientation to mathematics strongly expression-oriented whereas Jono's pedagogic context is a hybrid of weakly closed and strongly closed pedagogic texts and weak and strong expression-orientations to mathematics. Evaluation in the working-class pedagogic contexts generates an implied model learner who is constructed by the computational activity as one who is not capable of engaging with the content associated with the announced topic(s) from the point of view of the Mathematics encyclopaedia. The model learner's computational performance exhibits content that is closed with respect to the announced topic(s) from the point of view of the Mathematics encyclopaedia. The model learner's orientation to mathematics is expression-oriented because the mathematics constituted is expressively convergent with the announced topic but divergent from the content associated with the announced topic from the point of view of the Mathematics encyclopaedia. The operations on expressions constitute the mathematics in the working-class pedagogic contexts. In Jono's pedagogic context, mathematics is also constituted as an empirical activity due to the quasi-inductive pedagogy.

In the case of the pedagogic contexts populated by learners from upper-middle-class/elite families (Prestige College), the pedagogic modalities are hybrids of open/closed pedagogic texts accompanied by content/expression-orientations to mathematics, with Sara leaning towards open pedagogic texts and content-orientation and Jada tending towards closed pedagogic texts and expression-orientation. Evaluation therefore produces a hybrid model learner implied by the computational activity. The model learner is, on the one hand, constructed as one who *is capable* of engaging with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia, to a greater extent in set-2 than in set-3 and, on the other hand, the model learner is constructed as one who *is not capable* of engaging with the content associated with the announced topic from the point of view of the Mathematics encyclopaedia, to a greater extent in the case of set-3 than of set-2. The model learner's computational performance is such that the content realised is open in relation to the announced topic(s) from the point of view of the Mathematics encyclopaedia to a greater extent with respect to Sara than Jada. Furthermore, the model learner's computational performance is constructed as that which exhibits content that is closed with respect to the announced topic(s) from the point of view of the Mathematics encyclopaedia to a greater extent with respect to Jada than Sara.

The orientation to mathematics of the model learner is both content-oriented (to a greater extent in set-2) and expression-oriented (to a greater extent in set-3).

The analysis confirms Research hypothesis 3 and partially confirms Research hypothesis 4. Research hypothesis 4 is reformulated as follows: The model learner implied by the computational activity in the instructional discourse exhibits open pedagogic texts that are content-oriented and closed pedagogic texts that are expression-oriented in pedagogic contexts populated by learners from upper-middle-class/elite families.

10.2.3 Research hypotheses 5 and 6 – specialisation of learners' consciousness

RH6: *The computational performance of learners from working-class families exhibits closed pedagogic texts and their orientation to mathematics is expression-oriented.*

RH5: *The computational performance of learners from upper-middle-class/elite families exhibits open pedagogic texts and their orientation to mathematics is content-oriented.*

The second interest of the study, highlighted in Research hypotheses 5 and 6, was to examine the evaluative activity of actual learners in order to ascertain the recognition and realisation rules described in terms of computations employed when doing mathematics independently of the teacher. The computational performances and orientations to mathematics are compared with the computational performance and orientation to mathematics of the model learner presupposed by the computational activity in each pedagogic context. The analysis of the computational activity of learners in tests and clinical interviews is discussed in Chapter 9. Two questions framed this analysis:

- What does learners' computational activity imply about their computational performance in pedagogic contexts that differ with respect to their social class membership?
- What does learners' computational activity imply about their orientation to mathematics in pedagogic contexts that differ with respect to their social class membership?

Computational performance was categorised in terms of the production of open/closed pedagogic texts with weak or strong variants in each case. Orientation to mathematics was categorised as either expression-oriented or content-oriented, with weak or strong variants in each case.

In the working-class context, the computational activity of all the learners included operations from both the field of the reals as well as auxiliary operations, propositions and descriptions. None of the learners selected for the interviews recruited definitions, axioms or propositions located in the Mathematics encyclopaedia, indicating a marked absence of fundamental ground functioning as a regulative resource in the computational activity of all interviewed learners at Evergreen High. The content realised by all learners is described as ancillary. The realised content elaborated by all the interviewed learners at Evergreen high diverged from the content typically associated with the topic from the point of view of the Mathematics encyclopaedia. Furthermore, with the exception of one learner in Maya's class and one learner in Jono's class, the remaining interviewed learners all produced mathematics that diverged from the Mathematics encyclopaedia at the level of expression, indicating a prevalence of the "aberrant decodings" that are an effect of closed pedagogic texts.

All the learners in the working-class context displayed computational performances that exhibit strongly closed pedagogic texts, indicating that for those learners the topic is closed with respect to the Mathematics encyclopaedia. All the learners displayed an orientation to mathematics that is expression-oriented, indicating that operations on the expressions is what is constituted as the mathematics. The computational performances and orientations to mathematics of actual learners correspond with the computational performances and orientations to mathematics of the model learners implied by the computational activity in Maya's and Jono's pedagogic contexts, although the recognition and realisation rules used by learners in the interviews were not necessarily those employed in the instructional discourse.

In the upper-middle-class/elite context, the computational activity of all of the learners included operations from both the field of the reals as well as auxiliary operations and the use of auxiliary propositions but to differing degrees. Four learners from Sara's class (set-2) and one from Jada's class (set-3) recruited propositions from the Mathematics encyclopaedia alongside auxiliary operations and propositions, thus producing symbiotic content compared to the others who produced ancillary content. For three of Sara's learners and three of Jada's learners, convergence at level of expression was achieved despite divergence at the level of content. For the other three of Sara's learners and three of Jada's learners, both divergence with respect to content as well as expression occurred.

In the upper-middle-class/elite contexts, we see variation in the computational performances and orientations to mathematics of the interviewed learners. Sara's two learners and Jada's five learners who produced ancillary content exhibit strongly closed pedagogic texts and display strong expression-orientations to mathematics. Their computational performances and orientations to mathematics resemble that of the learners in the working-class context. So for those learners, the topic is closed with respect to the Mathematics encyclopaedia and mathematics is constituted as operations on expressions. One of Sara's learners and one of Jada's learners who produced symbiotic content exhibit weakly closed pedagogic texts and their orientations to mathematics are described as weakly expression-oriented. Their computational performances and orientations to mathematics are comparable to those who produce strongly closed pedagogic texts and display strong expression-orientations to mathematics but their computational activity includes encyclopaedic propositions. Three of Sara's learners who produced symbiotic content display weakly open pedagogic texts and weak content-orientations to mathematics. This group of learners recruited auxiliary computational resources in conjunction with encyclopaedic propositions. So for them, the topic is partially open with respect to the Mathematics encyclopaedia and expressions are secondary, a means to communicate mathematics rather than constituting the mathematics. The computational performances and orientations to mathematics of learners in the upper-middle-class/elite pedagogic contexts correspond with the computational performances and orientations to mathematics of the hybrid model learners implied by the computational activity in the instructional discourse in Sara's and Jada's pedagogic contexts.

In summary, we observe sameness with respect to learners' computational performances and orientations to mathematics in the pedagogic contexts populated by learners from working-class families. The learners' mathematical thought is specialised in the same way i.e. all learners display computational performances exhibiting closed pedagogic texts and orientations to mathematics that are expression-oriented. There is a strong resemblance between the learners' computational performances and orientations to mathematics and that of the model learners implied by the computational activity evident in Maya's and Jono's instructional discourse. This strongly suggests that the dominance of closed pedagogic texts and expression-orientation evident in Maya's and Jono's instructional discourse shaped learners' computational performances and orientations to mathematics.

In contrast, Prestige College (populated by learners from upper-middle-class/elite families) exhibits variation in learners' computational performances and orientations to mathematics. The hybridity with respect to the pedagogic modalities and the model learners evident in the instructional discourse of Sara and Jada reflects the learners' computational performance and orientations to mathematics. Here, we observe variation in the specialisation of mathematical thought of learners in the upper-middle-class contexts.

In summary, the analysis firstly reveals differential distribution of the specialisation of consciousness (computational performance and orientation to mathematics) to learners across social class contexts and within social class contexts. Furthermore , the analysis illustrates the structuring of learners' non-core domain computational performances by the mathematics experienced in the instructional discourse - the effect of second factor contextual data on the genetically endowed core domain computational competence of learners. Moreover, the analysis confirms Research hypothesis 5 and partially confirms Research hypothesis 6. Research hypothesis 6 is reformulated as follows: The computational performances of learners from upper-middle-class/elite families exhibit open pedagogic texts that are content-oriented and closed pedagogic texts that are expression-oriented.

Next, I reflect on the literature discussed in Chapter 2 in the light of the findings of the study.

10.3 Revisiting the literature

In Chapter 2, I discussed propositions based on the academic-everyday distinction, the explicit-implicit evaluative criteria distinction and the procedural-conceptual opposition/distinction recruited by mathematics education scholars to explain differential achievement in school mathematics along social class lines. The first two oppositions/distinctions are employed in the Bernsteinian and neo-Bernsteinian mathematics education literature. The conceptual-procedural opposition/distinction, on the other hand, is more widely used in the broader mathematics education field. In Chapter 2, I argued that the use of the oppositions/distinctions tends to produce a reading of the empirical that overlooks or distorts the mathematics emerging in pedagogic situations. The misreading of the empirical with respect to the reproduction of school mathematics generates questionable propositions regarding descriptions of the constitution of mathematics along social class lines.

Scholars using methodological resources to operationalise the above-mentioned distinctions tend to read the content associated with mathematics topics in pedagogic situations as that announced by teachers and learners or the content usually associated with familiar signifiers in mathematics. Given the methodological shortcomings outlined in Chapters 1 and 2, I chose, following Davis (2013a, 2013b), a methodological approach based on a computational theory of mind (see Chomsky, 2006, 2007; Fodor, 1998, 2010; Gallistel & King, 2010; Pinker, 1995, 1997, 2007), which posits that thought is computational. The methodology for the study, discussed in Chapters 3 and 5, employed resources recruited from Davis (2013a, 2013b) and developed additional methodological resources in order to address the research problem of the study. The methodology entails deploying resources for describing what emerges as mathematics in a particular pedagogic context instead of basing decisions regarding the content associated with topic names on *a priori* notions of mathematics content indexed by topic names.

Below, I return to the propositions based on the aforementioned oppositions/distinctions and other propositions regarding social class differences with respect to mathematics present in the literature in the light of the findings of this study.

10.3.1 The procedural-conceptual opposition/distinction

Recall from Chapter 2 that the procedural-conceptual opposition/distinction is widely deployed by researchers to describe the constitution of mathematics in pedagogic contexts, either in terms of 'understanding' or the nature of mathematics realised in pedagogic situations (e.g. Baroody, Feil, & Johnson, 2007; Hiebert & Lefevre, 1986; Skemp, 1976). Conceptual knowledge is associated with "meaning" or "sense-making" (Hierbert & Lefevre, 1986, p8). Procedural knowledge, on the other hand, is likened to "rules without reason" (Skemp, 1976, p2), described as "concept free" (Kieran, 2013, p169) or characterised as "meaningless" (Hierbert & Lefevre, p8). Furthermore, many studies claim that procedural knowledge is dominant in contexts populated by learners from working-class families (e.g. Anyon, 1980, 1981; Atweh & Cooper, 1995; Atweh et al., 1998; Hoadley, 2007; Rubel & Chu, 2010; Swanson, 2002, 2006; Taylor & Vinjevold, 1999), thus partially attributing the poor performance of learners from working-class families compared to learners from middle-class/elite families to the absence of conceptual knowledge.

In Chapter 2, I argued that construing procedural knowledge as aconceptual is problematic since concepts, principles or propositions are necessary components of thought (Chomsky, 2006; Pinker, 1997; Vergnaud, 1998). Secondly, a number of scholars argue that the dichotomy between conceptual knowledge and procedural knowledge is a false one (Kieran, 2013; Star, 2005; Wu, 1999). I argued further that procedural knowledge is just another form of sense-making or meaning and that sense or meaning is to be found in the recognition and realisation rules described in terms of the computations performed by pedagogic agents. This concern was taken up methodologically by examining the computational activity, and so recognition and realisation rules of teachers and learners in terms of their use of definitions or descriptions of mathematical terms, propositions and the operations or operation-like manipulation and associated domains.

The analysis of the use of pedagogic time across the four pedagogic contexts in this study shows that a substantial proportion of mathematical activity time was spent on elaboration of procedures as worked examples by teachers and as practice exercises or assessment tasks by learners. The time spent on procedural evaluative events ranged from 93% of mathematical activity time to 77% of mathematical activity time (see Chapter 6). However, a closer examination of the computational activity of teachers and learners reveals the use of concepts, principles or propositions to fashion procedures for solving particular classes of mathematical problems (see Chapter 6, 7 and 9), where the concepts, principles or propositions are recruited from the Mathematics encyclopaedia and/or are auxiliary to the Mathematics encyclopaedia.

The analysis (see Chapters 6, 7 and 9) illustrates that auxiliary propositions, particularly iconic ones, were employed in all four pedagogic contexts, suggesting an hypothesis that the use of auxiliary propositions is a general feature of school mathematics. This hypothesis requires further investigation in a larger sample of schools because it is not possible to generalise the findings of this study to pedagogic situations in different contexts. In the literature, the use of auxiliary computational resources such as "change sides, change signs" used to solve equations are referred to as procedural knowledge or procedural understanding (e.g. Capraro & Joffrion, 2006; Lima & Tall, 2008).

Encyclopaedic propositions were used only in the pedagogic contexts populated by learners from upper-middle-class/elite families, where encyclopaedic computational resources were used in the absence of auxiliary computational resources (canonical realised content) or coexisted with auxiliary computational resources (symbiotic realised content). Sara was the only teacher to use auxiliary propositions interchangeably with their encyclopaedic correlates. Definite descriptions were absent from all the pedagogic contexts. As stated earlier, this is not surprising given a curriculum context which downplays formal definitions. The absence of encyclopaedic propositions in pedagogic contexts populated by learners from working-class families is perhaps a contributory factor in the social class aligned difference in performance in school mathematics. This remains a hypothesis to be tested in a larger study.

Despite the limitations of the study, the findings demonstrate empirically that "procedural" knowledge, or, better still, knowledge of the use of procedures does not exist without a conceptual base. Furthermore, the findings raise questions regarding claims in the literature which attribute the social class aligned difference in performance to the absence of conceptual or principled knowledge in pedagogic contexts populated by learners from working-class families and the presence of conceptual or principled knowledge in the case of pedagogic contexts populated by learners from upper-middle-class/elite families. Concepts and procedures are distinct forms of knowledge and both are required to do mathematics. However, they are not in opposition to each other. As illustrated, there are distinct differences between the concepts and propositions recruited in the working-class pedagogic contexts compared to the middle-class pedagogic contexts but there are also similarities.

In Section 10.6, I reflect on the procedural-conceptual opposition in relation to core and non-core domain knowledge.

10.3.2 The academic-everyday distinction

As discussed in Chapter 2, mathematics education scholars are divided when it comes to the inclusion of the 'everyday' into school mathematics. Those against the inclusion of the 'everyday' in the school curriculum argue that the 'everyday' prevents learners' access into mathematics (e.g., Cooper & Dunne, 2000; Dowling, 1998; Straehler-Pohl, 2010; Straehler-Pohl, Fernández, Gellert, & Figueiras, 2014). Those promoting the inclusion of the 'everyday' contend that it is the 'everyday, particularly the pedagogic use of the everyday lives of learners or the cultural practices and artefacts that gives working-class learners a greater chance of succeeding in mathematics (Gerdes, 2011; Gutstein, 2003, 2006). Hoadley (2016, p. 46) recently argued that "everyday knowledge and specialised knowledge should be seen relationally rather than dichotomously" in that "everyday knowledge (is seen) as providing a conduit to specialised knowledge", an argument initially made by Dowling (1998, p. 145) in his claim that the public domain serves as a portal to the esoteric domain. Recall from Chapter 2 that Dowling's claim is contradictory because the public domain cannot simultaneously act as a portal and deny learners access to the esoteric domain. It appears that Dowling's (2009, 2013) restructuring of the public domain as containing all four domains is an attempt to deal with this contradiction.

In the present study, there were very few instances of teachers or learners using what might be considered as the 'everyday'. Sara referred to the "arms" of a parabola pointing upwards or downwards; she also referred to "smiley-faced"/"sad-faced" parabolas, and so did Jada and Maya. These descriptions of a parabola represent auxiliary propositions standing in place of the encyclopaedic propositions which state that a parabola $y = ax^2 + bx + c$ has a maximum or minimum value if $a < 0$ or $a > 0$ respectively. Jada used the metaphor of skipping to explain the reflection of a parabola in the x-axis. I have shown in Chapters 5 and 6, that we can construct homomorphisms (structure-preserving maps) which illustrate how the auxiliary resources used by teachers or learners substitute for the encyclopaedic resources. The auxiliary resources, therefore, constitute the mathematics in the pedagogic situations rather than constituting the 'everyday'.

Jono used what Dowling would refer to as public domain activities. He used the context of bread-buying to explain the concepts of domain and range of a function and the spread of AIDS to explain the notion of a function. In case of the bread-buying context, the computational objects are loaves of bread, taxi fare, and total cost. Here Jono derived the notion of domain and range of a function empirically. A table of values was used to read off the domain and range of the function. With the AIDS example, the objects operated with are number of days and number of instances of AIDS. Recall that objects such as loaves of bread are considered as computational objects in mathematics given that it does not matter whether the objects are numbers or loaves of bread since mathematics is agnostic about its objects (Whitehead, 1911). The proposition that the use of the 'everyday' is implicated in the failure of learners from working-class families is questionable because it has been shown that 'everyday' objects can function as computational objects in mathematics.

In section 10.6, I argue that the recruitment of the 'everyday' by teachers and learners relates to the use of core domain knowledge for the acquisition of non-core domain knowledge.

10.3.3 Explicit-implicit evaluative criteria

Recall from Chapter 1 that a key proposition emerging from studies employing Bernstein's work locates differential achievement between learners from middle-class families and learners from working-class families in the explicitness or implicitness of evaluative criteria in the instructional discourse. They argue that pedagogies that are successful in making specialised knowledge available to working-class learners are marked by explicit evaluative criteria i.e., strong framing over the evaluative criteria (see Hoadley, 2005, 2007; Morais, 2002; Morais & Neves, 2001; Morais, Neves, & Pires, 2004; Rose, 2004; Straehler-Pohl, 2010).

In Chapter 2, I discussed the methodological insufficiency of the use of Bernstein's notion of framing with respect to evaluative criteria. I illustrated that the use of the explicit-implicit distinction tends to produce a reading of the empirical that distorts the mathematics emerging in pedagogic situations and so raises questions regarding the validity of the above-mentioned proposition. Davis & Johnson (2007, p. 130) argue that explicit evaluative criteria, as used by studies recruiting Bernstein, refer to the "legitimate reproduction of science (knowledge) in a principled fashion". In other words, Davis & Johnson (ibid) contend that explicit evaluative criteria with respect to school mathematics refer to mathematics that is aligned with the Mathematics encyclopaedia. The present study concurs with Davis & Johnson (ibid) that "such criteria need not be present in a given empirical pedagogic context and are, therefore, externally defined […] as necessary". However, I argue further that explicit evaluative criteria as used by scholars recruiting Bernstein need not refer to 'principled' knowledge or rather knowledge aligned with the Mathematics encyclopaedia.

The findings of the present study illustrate that legitimate texts may align expressively with the Mathematics encyclopaedia but need not do so in relation to content. Content convergence with the Mathematics encyclopaedia only occurs in the instructional discourse of the middle-class pedagogic contexts but to a limited extent (20% of evaluative events for Sara and 11% of evaluative events for Jada). Content divergence with respect to the Mathematics encyclopaedia occurs in 91% of evaluative events of the instructional discourse across all four pedagogic contexts, indicating that content divergence with respect to the Mathematics encyclopaedia is not necessarily correlated with learners' social class membership. Furthermore, there were no instances of what might be referred to as implicit evaluative criteria in the four pedagogic contexts.

The study, therefore, illustrates that the explicit-implicit evaluative criteria distinction is not adequate to describe the complexity of the evaluative criteria circulating in the instructional discourse. Furthermore, the study demonstrates the utility of the fine-grained investigation of evaluation in terms of the computational activity of teachers and learners. The benefits of the methodology employed in this study for examining the functioning of evaluation is further illuminated by the analysis of the recognition and realisation rules employed by learners when doing mathematics independently of the teacher.

10.3.4 Recognition and realisation rules

Recognition and realisation rules in Bernstein's theory of pedagogic discourse are transmitted by teachers and acquired by learners. In Chapter 1, I argued, drawing on Davis (2013a), that the transmission and acquisition of recognition and realisation rules is more complex than simply a transfer from teacher to learner. In mathematics, it is always possible to substitute a function rule with another function rule. As such it is possible, as has been demonstrated in this study, for teachers and learners to substitute encyclopaedic computational resources with auxiliary computational resources. So it is entirely possible for teachers and learners to use different computational resources, so that different recognition and realisation rules produce the same outcome expressively. Chapter 9, in particular, illustrated how different learners solving the same mathematics problem and producing the same outcome, could do so using different computational resources and so different recognition and realisation rules. At times, learners' recognition and realisation rules differed from the recognition and realisation rules used by the teacher in the instructional discourse, illustrating that it is not simply a case of transfer of recognition and realisation rules. Learners and teachers are guided by the expressive outcome and so fashion computational resources to produce legitimate mathematical expressions. The computational resources that they employ may produce expressions that are associated with content that is aligned with the Mathematics encyclopaedic content or content that diverges from the Mathematics encyclopaedic content.

Revealing the complexity of the recognition and realisation rules used by teachers and learners is made possible through the methodology employed. Firstly, the methodology employs a proposition which states that language does not have a reference function, that is, there are no *a priori* direct word-object relations. This means that teachers and learners may use the same signifiers to refer to very different signifieds. Secondly, given the aforementioned proposition, it was not possible to infer the recognition and realisation rules used by learners directly from test scripts. Instead, clinical interviews were required to probe the recognition and realisation rules employed by learners by exploring the computations they used in their productions of mathematics.

10.4 Orientation to meaning and the construction of 'ability'

The literature, discussed in Chapters 1 and 2, draws attention to features of socialisation that prepare learners from working-class families and learners from middle-class families differently for school (see Bernstein 1975, 1990, Holland, 1981; Heath, 1982; 1983; Painter, 1985). The literature claims that learners from working-class families are socialised into a restricted orientation to meaning whereas learners from middle-class families are more likely to privilege an elaborated orientation to meaning in pedagogic contexts. While it is the case that middle-class children have greater opportunities than their working-class peers to learn school ways of thinking in the home prior to entering school given the educational, cultural, linguistic and economic resources of middle-class families, the distribution of social class opportunities to learn school ways of knowing (Hasan, 2002; Heath, 1982, 1983; Painter, 1999) differs substantially from attributing modes of cognition on the basis of an individual's social class membership.

The Holland (1981) experiment, which is used by Bernstein and recruited by other scholars (e.g. Cooper & Dunne, 2000; Hoadley, 2005; Hoadley & Ensor, 2009; Lerman, 2014; Lubienski, 2000, 2004; Skerrit, 2017) as a key empirical study confirming the social class basis of orientations to meaning, established a correlation between social class and particular modes of thought but her study is interpreted as setting up a causal relation between social class and semantic orientation. Furthermore, Bernstein's notion of orientation to meaning does not distinguish between group-level correlations and individual characteristics. The fact that a particular orientation to meaning correlates with social class in one study does not mean that all working-class learners think in the same way.

In contrast, Dowling (1998, pp. 292-293) argues that schooling constructs differences in 'ability' through the forms of knowledge distributed to learners. The present study concurs with Dowling's (1998) claim that 'ability' is constructed by schooling. The findings show a strong similarity between the computational performances and orientations to mathematics exhibited in the instructional discourse and the computational performances and orientations to mathematics displayed by learners when working independently of the teacher in the context of the clinical interview and the test. Dowling, however, does not provide any direct empirical data for his claim that high 'ability' textbooks are distributed to middle-class learners and that low 'ability' textbooks are distributed to working-class learners. Rather, high 'ability' is mapped to middle-class and low 'ability' to working-class by examining the positioning strategies used in the texts. The mapping is based on Sohn-Rethel's (1978) use of the intellectual-manual opposition as an organising principle of social class and Tunstall's analysis of the readership of "quality and popular press" (Dowling, 1993, p. 320).

Dowling's proposition that schooling constructs 'ability' aligns with a fundamental premise declared at the outset of the thesis, that is that all learners are endowed with exactly the same core domain knowledge/systems regardless of their social class membership. So, variation in performance in school mathematics (non-core domain knowledge) must be a consequence of what Chomsky (2005) refers to as second factor. In other words, learners' mathematical experiences are structured by genetically endowed core domain knowledge which constitutes contextual computational data (what is presented as mathematics). Computational data in different pedagogic contexts vary, thereby producing variation in the constitution of non-core domain computational resources.

This study illustrates the structuring of learners' computational performances and orientations to mathematics by the computational activity evident in the instructional discourse, bearing in mind the cumulative effect of schooling on learners' computational performances given that most learners were already in their 10th year of schooling.

At Evergreen High, there appears to be a strong correlation between closed pedagogic texts and expression-orientation to mathematics emerging in the instructional discourse and the closed pedagogic texts and expression-orientation to mathematics displayed by the interviewed learners, although there are differences between the recognition and realisation rules used by the learners and teacher. At Prestige College, hybrid pedagogic modalities are evident in both Sara's and Jada's pedagogic contexts and reflected in the range of computational performances and orientations to mathematics exhibited by the interviewed learners. In Jada's lessons, closed pedagogic texts and expression-orientations to mathematics are dominant, reflecting the computational performances and orientations to mathematics displayed by most interviewed learners. With respect to Sara, however, weak open pedagogic texts and weak content-orientation were dominant in the observed lessons and displayed by some of the learners. The computational activity of the other learners reflected the less dominant pedagogic modalities present in the observed lessons. A better picture would have been possible if interviews were conducted with all the learners to check the spread of computational performances and orientations to mathematics. However this was not possible given the constraints of a doctoral study and the logistics of organising interviews with all learners.

Given the differential performance in mathematics along social class lines and claims in the literature regarding the social class basis of orientations to meaning as an explanatory factor, there is an expectation of marked differences between learners' computational performances and orientations to mathematics in the two social class contexts. However, the current study shows similarities in the computational performances and orientations to mathematics of some learners in middle-class/elite contexts to learners in the working-class contexts. In both social class contexts, learners produce closed pedagogic texts and exhibit an orientation to mathematics that is expression-oriented, suggestive of an "orientation to meaning" that is restricted with respect to the Mathematics encyclopaedia. The current study thus raises questions regarding the validity of the social class basis of orientation to meaning. Relatedly, Jaffer & Davis (2012) report on learners from working-class families in one school displaying an "orientation to meaning" that is elaborated with respect to the Mathematics encyclopaedia and learners from middle-class/elite families exhibiting a more restricted "orientation to meaning" with respect to the Mathematics encyclopaedia.

10.5 Social class aligned achievement gap

Chapter 1 highlighted the potential for social class differences in the constitution of mathematics given the social class and 'racial' achievement gap in South African school mathematics as revealed in a number of national and international assessments (Bloch, 2009; Fleisch, 2008; Reddy, van der Berg, Lebani, & Berkowitz, 2006; Spaull, 2013; van der Berg, 2007). In order to situate the schools and learners participating in this study in the achievement profile of the country, I refer to the 2014 National Senior Certificate (NSC) results (see Appendix 10.1) and the 2014 NSC Mathematics and Mathematical literacy results (see Appendix 10.2) of the two schools participating in this study. The 2014 NSC results include the scores of learners who participated in this study. Those learners were in Grade 10 in 2012 when the information archive of the study was compiled and would have sat the NSC examination in 2014 if they were successful in Grades 10 and 11.

The differences with respect to computational performance and orientation to mathematics displayed by the interviewed learners at both schools point to a possible explanation for variation in pedagogic outcomes along social class lines. Although some Prestige College learners displayed computational performances and orientations to mathematics that mirrored those found at Evergreen High, Prestige College learners were more adept at producing mathematics that conformed with mathematics at the level of expression, that is, producing mathematics that converged with the Mathematics encyclopaedia expressively even though for some learners it diverged at the level of content. Note that an examiner only has direct access to the expressive level in tests and examinations and not to the content referred to by the expressions. Recall that it is entirely possible for different content to be associated with the same expression given the arbitrary relation between signifier and signified (Saussure, 1983) and the fact that it is always possible in mathematics to substitute operations with different operations and produce the same outcome.

The possible reasons for greater success of Prestige College learners perhaps lie in differences in the kinds of mathematics problems presented to learners across the two social class contexts (discussed in Chapter 6). Both Prestige College teachers, in contrast to Evergreen High teachers, encouraged inter-topic connectivity by presenting learners with mathematics problems that required procedures introduced in the observed lessons as well as procedures encountered in topics dealt with previously. In contrast, the lack of inter-topic connectivity evident in the observed lessons and tests administered at Evergreen High strongly suggests that those learners were left to make connections between topics independently of the teacher. Thus, connections between topics and synthesis of school mathematics topics into a coherent whole was made much harder for the working-class learners than the upper-middle-class/elite learners. Furthermore, Prestige College learners were provided with in-class opportunities for analysing mathematics problems in that some of the problems presented to them did not explicitly reference the procedure to be carried out whereas all the mathematics problems presented to Evergreen High learners such as classwork or homework exercises and those that appeared in the class test were all of the type which explicitly referred to the procedure required to solve the mathematics problem.

Secondly, Prestige College learners were given considerable practice opportunities by their teachers through homework tasks and tutorials and were exposed to variations in the phrasing of mathematics problems. In contrast, the learners at Evergreen High were not exposed to variations in problem types during the observed lessons and the tests administered by the teachers, indicating that learners were expected to rehearse and repeat particular procedures for solving particular problems, typical of an expression-orientation to mathematics. In the case of Jono, the test problems were extracted from a worksheet used during the observed lessons, thus encouraging the repetition of texts produced in class, strongly suggestive of an expression-orientation to mathematics resonating with closed pedagogic texts which lend themselves to "aberrant decodings".

Thirdly, in contrast to Evergreen High learners, Prestige College learners were provided with multi-topic mathematics problems that resembled typical examination-type problems in class, as homework exercises, in tutorials and tests. Thus, it appears that Prestige College learners were provided with more opportunities to prepare for examination-type problems than their counterparts at Evergreen High.

10.6 Returning to core and non-core domain knowledge

The analysis in Chapter 6 shows that 74% of evaluative events across the observed lessons in the four pedagogic contexts were coded as realising content described as ancillary (auxiliary computation resources only) and 18% of evaluative events were coded as symbiotic (auxiliary computational resources in combination with encyclopaedic computational resources). This means that 92% of evaluative events across the four pedagogic contexts involved auxiliary computational resources. In Chapter 9, we observed that all the interviewed learners recruited an auxiliary calculus to various degrees. Thus, the analysis of the functioning of evaluation reveals the pervasive use of an auxiliary calculus that sits alongside basic arithmetic operations on whole numbers across all four pedagogic contexts.

This finding concurs with previous research which also found that mathematics produced in schools entails substantial use of an alternate calculus on characters (see Arendse, 2013; Chitsike, 2011b; Davis, 2010a, 2010b, 2010c, 2011a, 2011b, 2011c, 2012, 2013a, 2013b, 2014; Jaffer, 2009, 2010a, 2010b, 2011a, 2012; Johnson & Davis, 2010). However, the aforementioned studies were restricted to pedagogic contexts populated by learners from working-class families. This study extends the finding to pedagogic contexts populated by learners from upper-middle-class/elite families and resonates with studies involving interviews with a small number of learners from upper-middle-class elite families (see Davis & Gripper (2012a) and Jaffer & Davis (2012)). It is not possible to generalise the findings to other pedagogic contexts given the small number of cases presented in this study. Further research is required to establish whether the same phenomenon is present in other pedagogic contexts. However, research studies using different theoretical and methodological resources report on similar alternate computational resources in a range of pedagogic contexts (e.g. de Lima & Tall, 2010; Sfard, 2007; Staats & Batteen, 2009; Wittmann, Flood, & Black, 2013). The findings of the current study together with the aforementioned studies raise questions about why auxiliary computational resources are so prevalent in pedagogic contexts. A tentative explanation for the extensive presence of auxiliary computational resources in pedagogic contexts is located in the distinction between *core domain* and *non-core domain* knowledge, introduced in Chapter 1.

A number of scholars distinguish genetically endowed knowledge referred to as *core domain* knowledge from *non-core domain* knowledge (Gelman, 2009a, 2009b, 2015; Spelke, 2000). The central distinction between the two orders of knowledge is that individuals learn core domain knowledge “on the fly” using innate mental structures whereas non-core domain knowledge requires the establishment of new mental structures and considerable effort, often accompanied by explicit instruction from a more knowledgeable other (Butterworth, 1999; Dehaene, 1997; Gelman, 2009a, 2015).

Core domain knowledge structures our experiences by selecting information relevant to the specific cognitive structure from the myriad of sensory data we are exposed to continuously. The notion that human experience of their environment is structured by basic categories (space, time and number) originates in

Immanuel Kant's *Critique of Pure Reason* and has been the subject of much philosophical discussion (see Peirce, 1992; Pinker, 2007). It is important to note that core domains are not complete at the time of birth, but equip humans to be attuned to relevant data from the environment (Gelman, 2009a, 2015). That core domains become knowledge-rich over time rather than starting off being complete is captured by Gelman's (2009b, p. 226) metaphor of core domains as "skeletal". Core domains direct attention to domain-specific sensory data from the environment, putting flesh onto the bone, so to speak, without the child realising that s/he is learning[77]. As such, the notion of core domains addresses the standard circular argument in the literature "that selective attention is due to salience and salience directs attention" (Gelman, 2009b, p. 228).

Recall from Chapter 3 that Gallistel & King (2010) argue that structure preservation mappings are the fundamental learning mechanisms. The human mind is continuously filtering contextual data and selecting isomorphic data that is mapped to the relevant core domain structure (Gelman, 2009b, p. 230). According to Gelman (2009b), learning in this way need not occur in one event. Individuals see patterns in particular examples selected from the contextual data and generate an initial hypothesis which is confirmed or rejected as more data becomes available. Over time, the mental structure becomes populated with a growing category of relevant data[78]. In this way a coherent core knowledge domain becomes organised according to the principles of the domain. The domain principles, however, are implicit since babies and young children, and many adults cannot state them. Core domains, generally accepted in the literature, is a relatively small set and include those that "govern the perception of and reasoning about objects, natural numbers, causality, the animate vs. inanimate, language, and sociality" (Gelman, 2015, p. 186). A number of research studies have shown that pre-verbal children regard objects in the world as permanent and solid, they distinguish between inanimate objects and animate organisms, they extract numerical and quantity information from sensory data and they reason causally about events in the world (ibid.).

In spite of continued debates about the nature of core domain knowledge with respect to natural number in the literature and on-going research in this field, broad consensus on genetically endowed knowledge of number seems to have been established: (1) pre-verbal children can discriminate sets of objects with cardinalities of up to about four; (2) pre-verbal children and adults possess an *approximate number system* that enables them to estimate the cardinality[79] of larger discrete sets; (3) pre-verbal children can add, subtract and order sets of small cardinality and like adults can perform approximate addition, subtraction and ordering on sets of larger cardinalities; and (4) pre-verbal children possess innate capacity to deal with continuous quantity (Butterworth, 2005, 2010; Dehaene, 1997; Gelman, 2009a, 2015; Spelke, 2000; Spelke & Kinzler, 2007). The innate knowledge of number present in pre-verbal children forms the basis for the development of children's verbal counting and arithmetic (Gelman, 2015, p. 195).

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[77] The development of core domains as learning on the fly seems to capture what is called "learning through play" in the literature.

[78] See Peirce (1931) whose notion of *abduction* resonates with current cognitive scientists conception of learning. "Abduction, in the sense I give the word, is any reasoning of a large class of which the provisional adoption of an explanatory hypothesis is the type." (Peirce, 1931, 4.541)

[79] Butterworth (2005) uses the term *numerosity* as the cognitive analogue of *cardinality* used by mathematicians.

Gelman and Gallistel (1986, p. 243-245) maintain that learning to count is guided by innate counting principles: “one-to-one correspondence principle” (each object must only be counted once); the “cardinal-word principle” (last number represents the cardinality of the set); the “order-irrelevance principle” (the order of counting does not matter); the “abstractness-principle” (does not matter what the objects are); and the “stable-order principle” (counting words must be in a fixed order). This does not mean that children instinctively know the correct counting words and the correct counting sequence. Children learn counting words and sequence over time, depending on the counting system used in the cultural context.

> To appeal to universal innate principles is *not* to assume that learning does not take place. Instead, it forces us to ask what kind of theory of learning we need to account for early learnings and the extent to which these serve as bridges or barriers to later learnings (Gelman, 2015, p. 228, italics in orginal).

However, Gelman & Gallistel’s position is contested in the literature. Wynn (1990, 1992) and Fuson (1988) argue that the counting principles emerge in stages and Fuson maintains that children learn to count by mimicking adults through recitation of number words, initially as one long string before they learn to parse the string correctly and say number words in the correct sequence. Irrespective of the contestation regarding exactly how children learn to count, there is consensus that humans are endowed with innate knowledge of number that enables us to count and do basic arithmetic on natural numbers (positive integers) presented in symbolic form.

> A quantitative representation, inherited from our evolutionary past, underlies our intuitive understanding of numbers. If we did not already possess some internal nonverbal representation of the quantity “eight,” we would probably be unable to attribute a meaning to the digit 8. We would then be reduced to purely formal manipulations of digital symbols, in exactly the same way that a computer follows an algorithm without ever understanding its meaning (Dehaene, 1997, p. 75).

It has also been shown that pre-verbal children possess a “mental number line” with smaller magnitudes which develops rapidly between the ages of three and eight and develops further with explicit teaching to include larger numbers. This mental number line is spatially configured with smaller numbers located on the left and larger numbers on the right for people living in cultures with left-to-right writing practices and vice versa for people living in cultures with right-to-left writing practices (Dehaene, 1997). Furthermore, research shows that adults, when faced with Arabic numerals, convert the symbol to “an internal analogical magnitude that preserves the proximity relations between quantities” (Dehaene, 1997, p. 75), showing core domain knowledge persists into adulthood.

As stated earlier, Gelman and Gallistel’s (1986) list of computational competences are not all evident at the time of birth but the counting principles are present. Counting, for healthy children, develops before school-going age. Counting bridges the genetically endowed mathematical knowledge with arithmetic over natural numbers presented symbolically, which in turn serves as the basis for the acquisition of advanced mathematical knowledge. The mental number line is restricted to positive integers, which leads Dehaene (ibid.) to argue that our intuitive grasp of positive integers perhaps explains the historical difficulty with acceptance of other numbers such as negative numbers, irrational numbers and complex numbers in

Mathematics and perhaps explains the difficulty that children experience at school when encountering rational numbers (fractions and decimals) and negative integers (see also Gallistel & Gelman, 2005; Gelman, 2015; Sfard, 1991; Smith, Solomon, & Carey, 2005). Whole-number thinking dominates learners' treatment of fractions (Basbozkurt, 2010a, 2010b; Lortie-Forgues, Tian, & Siegler, 2015; Smith et al., 2005) or algebraic equations (see Gelman, 2015). Similarly, negative numbers are treated as whole numbers with appended signs (e.g. Davis, 2010b, 2013a; Sfard, 2007).

> I would like to suggest that these mathematical entities are so difficult for us to accept, and so defy intuition, because they do not correspond to any preexisting category in our brain. Positive integers naturally find an echo in the innate mental representation of numerosity; hence, a 4-year-old can understand them. Other sorts of numbers, however, do not have any direct analogue in the brain. To really understand them, one must piece together a novel mental model that provides for intuitive understanding. This is exactly what teachers do when they introduce negative numbers with such metaphors as temperatures below zero, money borrowed from the bank, or simply a left ward extension of the number line (Dehaene, 1997, p. 76).

Dehaene's proposition that new mental models are required to deal with numbers other than positive integers (see also Gelman, 2015) provides a starting point for a speculative explanation of the ubiquity of auxiliary computational resources employed in all the pedagogic contexts involved in the current study. Dehaene (1997) and Gelman (2015) argue that new mental models are required to cope with the acquisition of non-core domain knowledge. On reflection though, the metaphors mentioned by Dehaene such as temperature or borrowing money enable teachers and learners to use their knowledge of operations over positive integers to deal with a problem involving negative integers, thus utilising core domain knowledge for non-core domain mathematical topics. The same can be said about the auxiliary computational resources employed by teachers and learners in the current study.

Recall the extensive discussion of Sara's procedure for calculating the x-intercepts of a parabola $y = -x^2 + 6x - 5$ in Chapter 5 (see Section 5.2.3.4). The procedure involves "changing" $-x^2 + 6x - 5 = 0$ to $x^2 - 6x + 5 = 0$ so that the x^2 term is positive using auxiliary operations such as "change sign" or "take over and change the sign" or the encyclopaedic operation multiplication by -1. It is much easier to factorise $x^2 - 6x + 5 = 0$ than $-x^2 + 6x - 5 = 0$ because positive integers "naturally find an echo in the innate mental representation of numerosity" (Dehaene, 1997, p. 76). In chapter 5, I illustrated the morphism relating the structure, multiplication by minus one over the reals $(\mathbb{R}, \times(-1))$ with the structure, "changing signs" performed on characters ($\mathbb{X}$, ALT) (see Figure 5.6) and the morphism relating the learner's "take over and change the sign" or transposition (TRP) over characters ($\mathbb{X}$, TRP) and the teacher's "multiplying by minus one" over reals $(\mathbb{R}, \times(-1))$ (see Figure 5.7). The morphisms show that the same outcome can be achieved with completely different operations or operation-like manipulations. Recall that homomorphisms are central to human thought because mental representations "require structure-preserving mappings (homomorphisms) from states of the world (the represented system) to symbols in the brain (the representing system)" (Gallistel & King, 2010, p. x).

In Chapter 5, I also discussed the procedure for solving $x^2 - 6x + 5 = 0$ which involves the use of a Character Distribution Matrix (CDM). The production of symbols to populate the spatial template (CDM) includes auxiliary operations and arithmetic over the positive integers. Of importance here is that integers are conceived of as whole numbers with appended signs. This example illustrates the fashioning of procedures that entail a shift of domain from the reals to positive integers through the employment of auxiliary computational resources, thus demonstrating an attempt to return to analogues of core domain knowledge.

Approximately 90% of all evaluative events across the four pedagogic contexts rely on iconic auxiliary computational resources, which include iconic auxiliary descriptions, iconic auxiliary propositions and auxiliary operations. Iconic auxiliary descriptions are explanations of mathematical terms that treat mathematical objects as though they are physical objects or an image of a physical object and those propositions that focus on imagistic features are referred to as iconic auxiliary propositions. An example of an iconic proposition used by Jada involves the metaphor of skipping as an analogy for reflecting a parabola in the *x*-axis. In this case, the teacher exploits core domain knowledge regarding objects and their motion. Humans are born with a knowledge system that enables them to perceive objects and the motion of objects (Gelman, 2009b; Spelke, 1994). The use of the skipping metaphor, therefore, draws on learners' core domain knowledge to assist them with developing a mental model for thinking about reflections of functions in the *x*-axis.

Auxiliary operations are operation-like manipulations that take symbols or characters as their domains and codomains. Auxiliary operations such as "change sides, change signs", "cross multiply", "drop the negative" used by teachers and learners in the current study treat parts of expressions as objects that can be "moved" to different locations to produce the desired mathematical expression. Ascribing imaginary movement to stationary objects has been posited by Talmy (2000) as a feature of the cognitive structure of language. The English language is replete with expressions exhibiting what he refers to as *fictive motion* such as "The scenery rushed past us as we drove along" which attributes motion to the fence when we are aware that the fence is in fact stationary (Talmy, 2000, p. 99). Fictive motion perhaps explains computations that effectively involve the movement of symbols from one location to another (e.g. one side of an equation to the other side). Talmy's (2000) notion of fictive motion has been used by scholars in mathematics education, particularly those located in the field of embodied cognition[80] to describe learners' mathematical thought (see Ferrara, 2003; Radford & Puig, 2007; Staats & Batteen, 2009, 2010; Wittmann et al., 2013). Genetically endowed Kantian categories such as time and space structure our experience and also structure language (Pinker, 2007, p. 105). Space acts as a conceptual vehicle for state change. The English language treats a changing entity in the same way as a moving entity, where motion is at times metaphorical. For example, in the statement "a traffic light can go from green to red", the change in state (green to red) simultaneously ascribes metaphorical motion to the traffic light (Pinker, 2007, p. 42). Similarly, auxiliary operations such as

[80] Lakoff and Núñez (2000, p. 39) adopt an embodied cognition approach which denies the body/mind duality. They claim that concepts are based in our perceptual motor system and that conceptual metaphors and image schemas, components of metaphorical thinking, are central to abstract thought.

"change sides, change signs" entail a state change from positive to negative or vice versa accompanied by metaphorical motion from one side of an equation to the other. Auxiliary operations, like our innate knowledge of language, are structured by time and space and thus appear to reside in core domain knowledge, which might explain why teachers and learners spontaneously resort to auxiliary computational resources and possibly explains the ubiquity of auxiliary computational resources.

In the mathematics education literature, auxiliary computational resources are described in terms of the academic-everyday distinction e.g. descriptions or propositions which reference 'everyday' terms such as "sad-faced" parabola or skipping to describe reflections of functions; or auxiliary computational resources are described in terms of the conceptual-procedural opposition e.g. a learner who uses the auxiliary operation "change sides change signs" to solve an equation is said to have a procedural understanding or is using procedural knowledge. Given the problems with the academic-'everyday' distinction and the conceptual-procedural opposition outlined earlier in this Chapter and in Chapters 1 and 2, the core domain and non-core domain distinction is proposed as a possible alternative. The tentative hypothesis put forward here is that auxiliary computational resources recruit or enable the use of core domain knowledge to solve non-core domain mathematical problems, either by directly recruiting core domain knowledge or by utilising auxiliary computational resources that enable teachers and learners to shift the domain to positive integers. An extensive exposition of the relation between core domain knowledge and auxiliary computational resources is beyond the scope of the current study. Sketched above is a limited, speculative reflection on the analysis undertaken in the current study in relation to core domain knowledge. A more fully worked out explanation is left for further study.

10.7 Limitations and potential of the study

10.7.1 Small scale case study research

The study was set up as exploratory and small-scale with the aim of developing a model for analysing the functioning of evaluation at the level of the instructional discourse and the complexity of the recognition and realisation rules used by teachers and learners to produce mathematics in pedagogic situations differentiated with respect to social class. The study was not designed to be representative of schools with particular social class membership, so no generalisations can be derived from a few cases. This problem is dealt with by employing a rigorous methodology that enables moving from the theory to the specific cases and back to the theory again. The potential for generalisability of the findings is reliant on extending the methodology to large-scale research or combining research results from numerous small-scale case studies.

Despite the limitation with respect to the size of the study and the concomitant problem associated with generalising the findings of the study, its potential lies precisely in its small scale which has enabled an in-depth analysis of observed lessons and interviews with learners, thus providing opportunities for using and developing methodological resources for analysing the functioning of evaluation at the level of the instructional discourse. The contribution of the study lies in the model for analysing the functioning of evaluation at the level of the instructional discourse rather than in the specific findings. Further research is required to test whether the findings are representative.

10.7.2 Selection of cases

The study selected four cases, two populated by learners from working-class families and two populated by learners from upper-middle-class/elite families. The purposively selected cases were based on an assumption that differential achievement along social class lines potentially represents differences in the constitution of mathematics. The cases are situated in two schools differentiated in terms of social class and performance in mathematics. However, the selection of schools was limited to one school type in that both schools are classified as independent schools. The selection of cases could involve different configurations specified in terms of the social class membership of learner populations, Apartheid department of education designation, geographical location or 'racial' composition of learner populations. Studies focusing on the constitution of mathematics across the range of different school types would greatly improve our understanding of the functioning of evaluation at the level of the instructional discourse, and what teachers and learners constitute as mathematics and how such constitution takes place, and is thus left for further study.

10.7.3 Methodological and empirical significance

I argued, in Chapter 2, that researchers explain social class differences in mathematics performances by employing methodologies that produce reading of the empirical which ignores or misrepresents the mathematics emerging in pedagogic contexts. Broadly, the methodological significance of the study lies in its potential to more adequately describe the functioning of evaluation at the level of the instructional discourse by describing the recognition and realisation rules used by teachers and learners in computational terms thus enabling a better description of what is realised as mathematics in pedagogic contexts.

The methodology draws substantially from the work of Davis (2013a, 2013b). The current study has extended the methodology through the development of additional methodological resources and the construction of an analytic framework, outlined in detail in Chapter 5, for producing and analysing data from the information archive. Specifically, an extensive framework for analysing the computational activity of teachers and learners draws on Eco's semiotic concepts, particularly *topic* and *isotopy* and his notions of the *Model Reader* and of *closed/open* texts for analysing mathematics pedagogic texts; Davis' (2011a) reconfiguration of Lotman's concepts, *grammar/text-orientation* and *content/expression-orientation* for describing orientations to mathematics were adapted for this study, and coding categories for analysing the realised content were developed. Furthermore, the current study extends the empirical sites, albeit in a limited way, beyond those considered thus far in the body of research using the methodology developed by Davis (2013a, 2013b) to empirical sites populated by learners from upper-middle-class/elite families.

The specific methodological significance of the study is two-fold. Firstly, the study's attention on what content is actually realised in the name of mathematics topics enriches our understanding of whether or not the mathematics constituted by teachers and learners involves substitution of content with respect to the

Mathematics encyclopaedia and what the content substitutions are. Thus, both the state and the broader mathematics education community could be better positioned to design targeted interventions to improve learners' and teachers' understanding of mathematics rather than focusing narrowly on improving examination/test results. The study demonstrates that it is possible for individuals to produce the same expressions with entirely different recognition and realisation rules. The use of clinical interviews has underscored the importance of such interviews for reading learners' recognition and realisation rules, which are not always evident from their written solutions to tests and examinations (see Section 10.3.4).

Secondly, the study's employment of methodological resources that describe the empirical more adequately in terms of mathematics has revealed *differences* but also *similarities* between the realised content and the specialisation of learners' mathematical thought in pedagogic contexts differentiated with respect to the social class membership of learners. This finding challenges the stark partitioning of schools and learners along social class lines pervasive in the literature (e.g., Anyon, 1980, 1981; Atweh & Cooper, 1995; Atweh et al., 1998; Cooper & Dunne, 1998, 2000; Hoadley, 2005, 2007; Lubienski, 2004; O'Halloran, 1996, 2004).

Thirdly, the methodological resources which were used to analyse the teaching and learning of specific announced topics, broadly referred to as functions in CAPS FET, can potentially be employed to analyse the teaching and learning of other topic areas in school mathematics at different grade levels.

10.7.4 Mathematics knowledge of teachers

The study did not explicitly set out to examine teachers' knowledge of mathematics but this does emerge as a concern. As pointed out in the analysis discussed in Chapters 6 and 7, there were definitely instances where the teacher's knowledge of the topic did not align with the Mathematic encyclopaedia. This suggests that there might be a correspondence between a teacher's knowledge of mathematics and what is constituted as mathematics in a pedagogic context. However, it is quite possible for there to be a disjuncture between the teacher's mathematical knowledge and the mathematics constituted in a pedagogic context. The relationship between constitution of mathematics in pedagogic contexts and teacher's knowledge of mathematics requires further investigation. Chitsike (2011b), who investigated co-constitution of mathematics and learner identity, also identified the need for research that explores the relationship between teacher's knowledge of mathematics and the constitution of mathematics in pedagogic contexts.

10.7.5 Higher education-schooling interface

A number of studies highlight the difficulties experienced by learners and their general under-preparedness to cope with the demands of first-year mathematics in South Africa (e.g. Engelbrecht, Harding, & Phiri, 2010; Wolmarans, Smit, Collier-Reed, & Leather, 2010) and elsewhere (e.g. Bergsten & Jablonka, 2015; Hourigan & O'Donoghue, 2007). Hourigan & O'Donoghue (2007), conducting a study in Ireland, argue that "under-preparedness" does not correspond directly to final school-examination results, emphasising that even learners who perform very well on the school leaving examination are classified as at-risk because of their poor knowledge of mathematics. Many studies compare learners' scores on final school-leaving examinations to first-year test or examination scores (e.g. Engelbrecht et al., 2010; Hunt, Ntuli, Rankin, Schöer, & Sebastiao, 2011). The methodology used in this study illustrates that test scores conceal the detail of the computational performance of learners and so do not provide sufficient information regarding what mathematics learners actually know. The study demonstrated that even those scoring high marks on a mathematics test display computational performances that exhibited closed pedagogic texts and expression-orientations to mathematics. Clinical interviews conducted with learners proved to be an invaluable means of collecting information on learners' mathematical thought (see Section 10.3.3). The school-higher education interface is therefore another potential area for further investigation.

10.8 Concluding remarks

This chapter reiterated the central focus of the study, that is an analysis of the functioning of evaluation in pedagogic contexts. In particular, the study examined teachers' and learners' recognition and realisation rules, described computationally, when doing mathematical work in pedagogic contexts differentiated with respect to learners' social class membership. The research problem stems from a hypothesis that difference at the level of learner performance in mathematics suggests difference in the functioning of evaluation and so difference in what is constituted as mathematics and how mathematics is constituted.

This chapter highlighted the insights gained through analysing the computational activity of teachers and learners and so extends our understanding of what is constituted as mathematics in pedagogic contexts differentiated with respect to social class, and how mathematics is constituted beyond what is currently described in the literature. The methodology adopts an Integrated Causal Model, described in Chapter 1, which integrates theoretical and methodological resources across a number of disciplines including cognitive science, semiotics, philosophy, and mathematics to address the research problem. As such the methodology significantly advances on methodologies currently employed in mathematics education, enabling a more descriptively adequate account of the functioning of evaluation at the level of the instructional discourse in pedagogic contexts, particularly with respect to mathematics and of the recognition and realisation rules employed by teachers and learners. The methodological framework captures a range of computational activities not visible in the work of Davis (2011a, 2013a) which served as the methodological basis for this study, highlighting the contribution of the study to the field of mathematics generally and to Bernstein's theory of pedagogic discourse specifically. In particular, the study highlights the fecundity of the methodology to capture the complexity of the evaluative criteria employed by teachers and learners and as such contributes to Bernstein's notion of evaluation.

The study questions the validity of a number of propositions employing the academic-everyday distinction, the conceptual-procedural distinction or the explicit-implicit evaluative criteria distinction as explanations for the differential achievement along social class lines. Although not intended, the methodology provides a mathematically-attuned explanation for variation in pedagogic outcomes along social class lines, providing a more descriptively adequate account of what is realised as the content associated with topic names and the specialisation of mathematical thought than current literature.

The study highlighted the ubiquity of auxiliary computational resources, particularly iconic auxiliary computational resources, across all pedagogic contexts irrespective of the social class membership of learners. A speculative explanation for the pervasiveness of auxiliary computational resources was located in the recruitment of core domain knowledge. I tentatively argue that auxiliary computational resources recruit or enable the use of core domain knowledge to solve non-core domain mathematical problems by either directly recruiting core domain knowledge or by utilising auxiliary computational resources that enable teachers and learners to shift the domain to positive integers.

The study reveals the following: (1) the commonly used descriptions of evaluative criteria as explicit/implicit are analytically blunt and consequently mask the complexity of criteria operative in pedagogic contexts; (2) differences as well as strong similarities in the functioning of evaluation and, therefore, differences and similarities in what is constituted as mathematics are evident in pedagogic situations differentiated with respect to social class; (3) an orientation to mathematics that constitutes mathematics as computations on the typographical elements of mathematical expressions is common to pedagogic situations involving learners from both upper-middle-class/elite families and working-class families; and (4) greater variation and inter-connectedness in computational resources is realised in pedagogic situations involving learners from upper-middle-class/elite families than in those involving learners from working-class families, where computational resources are relatively restricted and weakly connected. The differences between the two types of situation appear to be enabling of greater flexibility in mathematical thought and action for upper-middle-class/elite learners, on the one hand, and restricting for working-class learners, on the other.

The contribution of the thesis is four-fold. The study: (1) provides a methodology for exploring the complexity of pedagogic evaluation by describing the computations performed by learners and teachers in mathematical terms, thus contributing to Bernstein's account of pedagogic discourse as it applies to the teaching and learning of mathematics; (2) contributes to our understanding of the structuring effect of evaluation on learners' mathematical thought; (3) contributes to the methodological resources developed by Davis for describing the constitution of mathematics in pedagogic situations; and (4) extends analyses of the constitution of mathematics in pedagogic situations to those populated by learners from upper middle-class/elite families in the South African context, albeit in a limited way.

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Appendix 1
The evolution of Bernstein's code theory

The construct *orientation to meaning* is situated in Bernstein's code theory that developed over time in relation to developments in his theory and related empirical research. Orientation to meaning refers to "the selection and organisation of meaning, of what is seen as relevant and taken as the focus of attention in any situation and the way in which these meanings are organised in practical discourse" (Holland, 1981, p. 1). Bernstein argued that orientation to meaning is related to the social division of labour.

> Orientation to meaning refers to *privileged* and *privileging* referential relations. 'Privileged' refers to the priority of meanings within a context. 'Privileging' refers to the power conferred upon the speaker as a consequence of the selected meanings. Now the source of power and its legitimation, from our perspective, does not arise out of the social relationship within the context, but out of the social base external to the context. (Bernstein, 1990, pp. 15-16; italics in orginal)

While the definition of code changed several times during the development of Bernstein's theory, code as "a regulative principle, tacitly acquired, which selects and integrates relevant meanings, forms of realizations, and evoking contexts" is a delineation that was eventually fixed in the theory (Bernstein, 1990, p. 101).

The origins of the concepts *orientation to meaning* and *code* can be traced to Bernstein's socio-linguistic thesis where he distinguished between *public language* and *formal language,* antecedents of later concepts *restricted code* and *elaborated code* respectively (Bernstein, 1958, 1959). Bernstein described *public language* as one that is characterised by short, grammatically simple sentences where meaning is implicit whereas *formal language* exhibits grammatically accurate complex sentences used to convey meaning explicitly (Bernstein, 1959). When describing a *public language* Bernstein stated that

> a correlate of this linguistic form is a low level of conceptualization – an orientation to a low order of causality, a disinterest in processes, a preference to be aroused by and respond to that which is immediately given (Bernstein, 1959, p. 318).

The above quote suggests that Bernstein read differences in social class aligned speech patterns as differences in social class aligned cognitive capabilities because as he states "a correlate of this linguistic form is a low level of conceptualization". The strong association between language and thought reflects the influence of the linguistic determinism of the Sapir-Whorf hypothesis[81] and particularly its weaker version of linguistic relativity (language structures thought) on Bernstein's theory[82]. Labov (1972) like Bernstein,

[81] The Sapir-Whorf hypothesis can easily be refuted as Pinker illustrates. "The idea that thought is the same thing as language is an example of what can be called a conventional absurdity: […] And if thoughts depended on words, how could a new word ever be coined? How could a child learn a word to begin with? How could translation from one language to another be possible?" (Pinker, 1995, p. 57)

[82] See Bernstein (1971) where he acknowledges the influence of the linguists Cassirer, Sapir and Whorf on his early

found linguistic differences between working-class and middle-class individuals[83]. However, he attributed the speech differences to differences in the linguistic genres of Non-standard English and Standard English rather than the cognitive capabilities of individuals.

Assigning particular linguistic forms together with particular attitudes, behaviour and cognitive ability to particular social class groups has the effect of essentialising working-class and middle-class individuals. (1991, p. 53) argued that Bernstein constituted the elaborated code "as the norm of all linguistic practices" thus measuring working-class linguistic practices in terms of the linguistic practices of the dominant middle-class. Bernstein's socio-linguistic thesis received considerable criticism from other linguists like Labov (1972) who characterised Bernstein's socio-linguistic thesis as a deficit theory. Bernstein (1990), however, vociferously maintained that his theory was 'misrecognised' as a deficit theory and "never conceded that his critics certainly not Stubbs (1976) and Gordon (1981) had weakened the theory in any respect" (Nash, 2006, p. 541).

The introduction of *codes* in papers written in 1962, according to Bernstein, resulted from his encounter with the work of Vygotsky and Luria for whom speech was "an orientating and regulative system." (Bernstein, 1971, p. 6). Although the introduction of code appeared to signal a disassociation from the Sapir-Whorf hypothesis, code was essentially an attempt to define the regulative principle underlying the linguistic forms, public and formal language, thus emphasising the social structuring of language and thought. Code was defined as "the ease or difficulty of predicting the syntactic alternatives taken up to organize meaning." (Bernstein, 1990, p. 96). Elaborated code and restricted code were distinguished in terms of the combinatorial possibilities of the syntactical resources used by the individual.

> In an elaborated code, relative to a restricted code, the speakers explore more fully the resources of the grammar and therefore I considered there were more possibilities of combination. (Bernstein, 1971, p. 8)

Describing codes in terms of the predictability of syntactical resources proved difficult to operationalise in analysis of data. Consequently, Bernstein found it necessary to distinguish between the speech variant (the patterning of speech) and the code regulating speech. Code was re-described in terms of different orders of meaning (ibid, p195). Restricted code was associated with context-dependent, implicit and particularistic meanings and elaborated code entailed more context-independent, explicit and universalistic meanings (Bernstein, 1990, p. 96). Bernstein emphasised that language was merely a linguistic realisation of codes. This shift from defining code in terms of language to code in terms of orders of meaning can be read as an attempt by Bernstein to mask the Sapir-Whorf hypothesis underpinning his socio-linguistic codes.

Adlam (1977, pp. 13-14) noted that the terms, *context-dependent/independent*, were used to describe both meanings and speech. Meanings were described as particularistic/universalistic and speech was described in terms of implicitness/explicitness. She directed attention to the danger of simplistically aligning

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work.

[83] Although Labov (1972) used 'race' rather than class in his study, it could be argued that the inner city Black participants in his study were working class.

universalistic meanings with explicit speech and particularistic meanings with implicit speech[84]. For example, in Lineker's (1977) study of children's description of the hide-and-seek game, some children provided particularistic descriptions of the game using explicit speech. Adlam's model of the relationship between speech and meaning used in the studies, published in *Code in Context,* linked universalistic meanings with explicit speech, but maintained that particularistic meanings can be expressed in both explicit language and implicit language (Adlam, 1977, pp. 14-15). In spite of Adlam's recognition of the conflation of language and meaning in Bernstein's theory, she curiously confirmed Bernstein's proposition:

> For this reason Bernstein can say that explicit speech is universalistic in nature in the sense that meaning is more universally available, is available to more listeners. And implicit speech can be called particularistic in the sense that particular others – those with whom the speaker has certain shared knowledge either in terms of their common history, or in terms of the immediate context – can fully decode the message (Adlam, 1977, p. 15).

Despite extensive modifications to the theory that stressed that language was merely a linguistic realisation of codes, linguistic forms remained the indices of meaning and therefore code. The linguistic relativist influence of the Sapir-Whorf hypothesis persisted in the theory. The series of studies documented in Adlam, Turner, and Linekar (1977) served to highlight the methodological problem of reading code off speech in different contexts. A further distinction was made between speech variant (the patterning of speech in a particular context) and the code regulating speech in diverse contexts. From Halliday's (1973) description of seven functional linguistic contexts (as cited in Adlam et al., 1977), Bernstein constructed a description of four primary socialising contexts: regulative context, instructional context, imaginative context and inter-personal context.

A study conducted by Holland (1981) and supervised by Bernstein and Adlam (Adlam et al., 1977) was central in the reformulation of Bernstein's early notion of sociolinguistic codes to its current formulation in terms of semantic orientation. Holland's (1981) study was a replication of aspects of Luria's (1976) study, conducted in Uzbekistan in the early 1930s, a time marked by shifts in the mode of production. He employed clinical interviews to determine individuals' modes of thought. The clinical interviews took the form of lengthy discussions with adults who were given, amongst others, classification tasks in order to ascertain the form of thought commonly used by them. His research showed that the way in which an individual construed objects in the world was related to the mode of production. He concluded that schooled individuals such as teachers or farm managers were able to think about the world in general terms, in contrast to unschooled illiterate peasants, who tended to think about experience in largely situated ways.

Holland's (1981) study, in contrast to that of Luria, entailed an experiment with eight-year-old second year working-class and middle-class students, where students were asked to group pictures of food items in any way they desired and to provide reasons for their groupings. They were then asked to repeat the sorting task

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[84] "Implicit speech" is contradictory because speech by nature can only be explicit. What is meant here is speech that contains shifters. So, "implicit" seems to refer to speech in which the referents are not explicitly mentioned.

and to provide different reasons for their groupings. The study concluded that working-class children generally used context-dependent sorting principles in that their groupings referred to everyday use (e.g. 'I like those things.'; 'That's what we have for Sunday dinner.', 'These are supper time foods.') (Holland, 1981, p. 8).

Reasons privileged by the working-class children were categorised by Holland (1981) as context-dependent and as exhibiting particularistic meanings. According to Holland (1981), working-class children predominantly used a context-dependent orientation to meaning in both sorting tasks whereas middle-class children used general principles (e.g. a food category) in the first task and more personal, everyday reasons in the second task. Holland concluded that middle-class children displayed two semantic orientations, context-independent and context-dependent. Holland's conclusions confirmed earlier work in sociolinguistic codes by Bernstein and his colleagues. The conclusions of Holland's study were read by Bernstein in terms of his code theory – working-class students were described as possessing a *restricted code* that entailed context-dependent, implicit and particularistic meanings and middle-class students possessed an *elaborated code* that entailed more context-independent, explicit and universalistic meanings (Bernstein, 1990, p. 96).

> Prior to the above model the code thesis distinguished between coding orientations, elaborated/restricted in terms of implicit/explicit, context-dependent/context-independent meaning. [...] Basically, there has been a movement from the giving of definitions in terms of general linguistic indices (which proved impossible to operationalize [...] to the giving of definitions in terms of a generating contextually specific semantic. However in all definitions the underlying semantic was considered to be the regulator of linguistic realizations. (Bernstein, 1990, p. 101)

Holland's study marked a change from linguistic form as an index of orientation to meaning to a focus on the academic-everyday distinction as the discriminator of orders of meaning.

Holland (1981), in contrast to Luria (1976), did not conduct clinical interviews that probed the reasoning of students to establish the necessary cognitive capabilities of her research subjects. Luria persistently questioned his research subjects until he was certain that they were unable to move beyond the explanations they provided. For example, in an interview with an illiterate peasant presented with the task of identifying the object that did not belong to the group, a hammer, a saw, a log, and a hatchet, the peasant insisted that the four objects belonged together. Luria repeatedly questioned the peasant using a number of different examples and ways of asking the same question, for example deliberately stating that the log did not belong to evoke a response from the peasant. The peasant insisted that all four objects belonged together and did not shift his reasoning for grouping the objects. In contrast, Holland did not employ the same mode of interviewing students to explore whether they were capable of thinking differently. Students were asked for reasons for grouping items but their reasons were not probed. For example, a student who grouped the items roast beef, peas and potatoes together because "that's what we have for Sunday dinner" was not asked whether the peas and potatoes could be grouped together or whether the roast beef was the odd item. Consequently, it appears that Holland's (ibid.) research focused on individuals' habits of thought rather than testing limits to the children's cognitive abilities. Her research, however, is used as the empirical basis of

Bernstein's proposition regarding the semantic orientation of individuals based on their social class membership and continues to be invoked by researchers to support Bernstein's proposition (see Cooper & Dunne, 1998, 2000; Hoadley, 2005, 2007; Hoadley & Ensor, 2009; Lerman & Zevenbergen, 2004; Lubienski, 2004; Muller & Taylor, 1995, 2000; Taylor, 1999, 2000).While Luria's study shows that exposure to very little schooling makes a difference to the way in which individual's organise the world, Holland's study paints a very grim picture of the possibilities for schooled children from working class families.

Furthermore, it appears that in Holland's study Luria's description of the differences in the semantic orientations of rural peasants and the urban working class was mapped onto the semantic orientations of the urban working-class and the urban middle-class school children. The underlying assumption in Holland's study and consequently Bernstein's proposition employed by the studies cited above is that contemporary working-class individuals are cognitively very similar to Luria's rural peasants, and that contemporary middle-class individuals are cognitively very similar to Luria's urban working class (Jaffer & Davis, 2012).

The above discussion raises a number of concerns related to Bernstein's proposition on semantic orientation. Firstly, Bernstein's notion of orientation to meaning distributes particular semantic orientations to individuals on the basis of their social class membership, thus essentialising middle-class and working-class students. Secondly, Bernstein's strong alignment with the Sapir-Whorf hypothesis reveals the linguistic determinism underpinning his orientation to meaning proposition despite attempts to mask the emphasis on language through reformulations of the proposition. Consequently, Bernstein's orientation to meaning proposition and its deployment in mathematics education research generates problematic expectations with respect to the performance of working-class students in school mathematics as compared to their middle-class counterparts. In many research studies, working-class students are expected to perform poorly because of their purported restricted orientation to meaning. In contrast, Dowling (1998) argues that it is schooling that constructs ability, distributing low-ability to working-class students and high-ability to middle-class students, through the form of knowledge made available to students.

The concerns raised about Bernstein's proposition regarding the semantic orientation to meaning is not to deny that given the educational and economic resources of middle-class families, middle-class children have greater opportunities than their working-class counterparts to learn school ways of thinking in the home prior to entering school (see Heath, 1982; Heath, 1983; Painter, 1999) and are consequently better positioned to succeed at school. However, the distribution of social class opportunities to learn school ways of knowing differs from attributing semantic orientations on the basis of an individual's social class membership.

Contrary to the expectations set up by the literature, a recent study Jaffer and Davis (2012) on the computational resources used by top-performing students found many similarities in the mathematical work of middle-class students at a private school and working-class students from an ex-DET school. Students from both schools tended to perform computations or manipulations without taking into account the global features of the expressions being transformed, as well as failing to distinguish between equations and non-equations. In addition, students from both schools deployed repertoires of alternate syntactical rules, such as ‘change sides, change signs’, or operation-like manipulations, such as ‘cancelling’, to transform expressions, at times mimicking mathematical operations by producing mathematically correct outcomes and at other times revealing their ‘mal-rule’[85] nature by producing incorrect mathematical outcomes.

In contrast, Davis and Gripper (2012a) and Jaffer and Davis (2012) found that working-class students at an independent school appeared to operate in a much more mathematically attuned manner. These students focused attention on the operational implications of the global features of expressions being transformed. They distinguished between equations and non-equations and transformed expressions on the basis of mathematical principles such as “what you do on the left you also do it on the right, when you are doing equations” and using additive inverses to solve equations. Although two students employed alternate syntactical rules such as ‘change sides, change signs’ when solving simple linear equations, Jaffer and Davis (2012) found no evidence of the ‘mal-rules’ that were prevalent in the computational activity of the private school middle-class students and the ex-DET school working-class students.

Davis and Gripper (2012a) and Jaffer and Davis (2012) claim that their findings contradict the prevailing explanations of social class differences in performance in schooling in general and mathematics in particular. The studies raise questions regarding explanations based on the purported “restricted orientation to meaning” of working-class students and the ostensible “elaborated orientation to meaning” of middle-class students. The studies present a case of middle-and upper middle-class students performing their mathematical work in rather mathematically restricted ways, and working-class students working in a much more mathematically-attuned or elaborated fashion.

It is important to note that Bernstein’s notion of orientation to meaning conflates correlation with causation. The Holland experiment established a correlation between social class and particular modes of thought but her study is interpreted as setting up a causal relation between social class and semantic orientation. Secondly, Bernstein’s notion of orientation to meaning does not distinguish between group-level correlations and individual characteristics. The fact that a particular cognitive mode correlates with social class in one study, does not mean that all working-class students think in the same way.

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[85] See Sleeman (1984) for his discussion of mal-rules.

Appendix 2
The case for the inclusion of the 'everyday' into school mathematics

Calls for the inclusion of the 'everyday' in school mathematics emanate from a number of mathematics education scholars, particularly from the subfields *ethnomathematics* (e.g., d'Ambrosio, 1985, 2007, 2016; Gerdes, 1985, 1988, 2011), *critical mathematics education* (e.g., Powell & Frankenstein, 1997; Skovsmose, 2013; Skovsmose & Borba, 2004), *mathematics for social justice* (e.g., Gutstein, 2003, 2006) and *culturally relevant pedagogies* (e.g., Enyedy & Mukhopadhyay, 2007; Ladson-Billings, 1997; Ladson-Billings & Tate, 1995; Leonard, Napp, & Adeleke, 2009). Despite differences across the aforementioned subfields, they share a common emancipatory goal of improving marginalised learners' access to and success with school mathematics by drawing on learners' cultural backgrounds as a curriculum resource. So, in direct contrast to the research discussed above that eschews the incorporation of the 'everyday' into school mathematics, scholars located in the above-mentioned subfields regard the integration of the 'everyday' and school mathematics as a favourable and necessary mechanism to induct learners, particularly marginalised learners, into mathematics. Below I briefly, discuss ethnomathematics and critical mathematics education, the latter subsuming mathematics for social justice and culturally relevant pedagogies.

Ethnomathematics

Ethnomathematics, a term coined by D'Ambrosio (1985), refers to a body of work that evolved in the former colonies to challenge the dominance of Eurocentric conceptions of mathematics and the history of mathematics. Following D'Ambrosio (1985), a plethora of studies in the ethnomathematics tradition emerged and continue to flourish through research of diverse cultural practices (e.g., Arisetyawan, Suryadi, Herman, & Rahmat, 2014; Gerdes, 1985, 1988, 2011; Knijnik, 1993, 2002, 2012; Pinxten & François, 2011; Powell & Frankenstein, 1997). Ethnomathematics attempts to deconstruct the Eurocentric myth that propagates the idea that mathematics was created by European males and counters hegemonic positions that devalue and ignore the contributions of colonized people to the body of mathematics knowledge by treating their cultural practices as non-mathematical (Mukhopadhyay, 2013; Powell & Frankenstein, 1997). Scholars in the field of ethnomathematics claim that cultural practitioners such as basket weavers and boatmakers are engaged in doing mathematics. Similarly, Bishop (1988) claims that mathematics is a '"pan-human" activity and argues that mathematical activities such as counting, measuring, locating, designing, playing and explaining that lead to the development of mathematics span all cultural groups. Counting and measuring involve concepts of number, the first discrete and the second continuous. Both locating and designing involve geometric concepts involving space, shape and direction. Locating focuses on the topographical and cartographical aspects of the environment, and designing involves spatial objects. The mathematical activities referred to by Bishop (1988) describe the computations engaged in by basket weavers and boatmakers when producing cultural artefacts. Bishop's universal mathematical activities are in fact core domain knowledge, which is innate to all humans, rather than non-core domain knowledge as suggested by Bishop (1988) and Gerdes (1986, 2011).

Ethnomathematics scholars claim that the mathematics 'frozen' in cultural artefacts should be revealed to learners as a means of conscientising learners. Cultural practices such as hut building, basket and button weaving are redescribed in terms of mathematics. In this way, ethnomathematics seeks to provide cultural affirmation and to establish cultural confidence amongst previously colonized groups (Gerdes, 1988, pp. 139-140). The pedagogic activity therefore focuses on developing non-core domain knowledge such as the Theorem of Pythagoras through using cultural artefacts such as woven buttons, produced through the core domain knowledge of the designers and producers of cultural artefacts. A number of critical reviews argue that the intended emancipatory ideals of ethnomathematics are in fact inadvertently subverted by the very propositions of the ethnomathematics project (see Dowling, 1998; Hottinger, 2016; Jaffer, 2013; Pais, 2011).

While Ethnomathematics offers a means for addressing the social class-aligned achievement gap, it does not really provide methodological resources for analysing what emerges as mathematics in pedagogic contexts. The focus instead is on casting a mathematical gaze on cultural artefacts and so re-describing cultural artefacts in terms of mathematics.

Criticalmathematics Education

Critical theory, originating from the Frankfurt School, entails an examination and critique of society and culture; and forms the basis of Critical pedagogy. Freire's *Pedagogy of the Oppressed* exhibits a genre of Critical pedagogy and underpins the body of mathematics education research referred to as Criticalmathematics Education (CmE) Several studies located within CmE research attempt to address issues of equity in mathematics education on the grounds of learners' gender, race, language, and social class etcetera. Typical examples of this literature include the work of (Frankenstein, 1983, 1995; Gutiérrez, 2013; Julie, 2004; Skovsmose, 1994, 2013; Skovsmose & Borba, 2004)

Following Freire, CmE's central aims are to develop learners' political awareness of their positions within society and in history, to enable them to identify the injustices of society and to motivate learners to transform society. Mathematics serves as a tool for exposing and analyzing injustices in society. The concept of power, derived from Marxism and Critical theory, stresses critique of the role of mathematics and mathematics education in creating and maintaining social inequality as an essential component of CmE.

The work of Gutstein (2003, 2006, 2016) has influenced a minor variant of CmE referred to as teaching mathematics for social justice that emphasizes the role of mathematics to empower learners to confront issues of social inequality but simultaneously aims to provide learners with mathematics-specific knowledge (see also Burton, 2003). Both Frankenstein in CmE and Gutstein in mathematics for social justice have shown as teacher researchers how it is possible to change learners' perceptions of mathematics and their success in school mathematics (Diversity in Mathematics Education Center for Learning and Teaching, 2007, p. 420). However, Bartell (2013) in a study of eight secondary teachers' implementation of a social justice curriculum found that teachers in attempting to implement a social justice curriculum had difficulty in achieving the dual aim of mathematics acquisition and social justice. Some teachers emphasized social justice issues at the expense of mathematics acquisition while others focused on the teaching of mathematics without addressing the social justice goals. Similarly, Brantlinger (2011, 2013) discusses the difficulties of incorporating CmE into the teaching of geometry.

Research deriving its conceptual basis from Critical Race Theory (Ladson-Billings, 1995, 2014; Ladson-Billings & Tate, 1995; Tate, 1997) share the goals of CmE but does not refer to itself as such[86]. Critical Race Theory is a loosely unified body of work that focuses on the analysis of the way in which racial inequalities particularly in relation to African Americans, Latin@s[87] and American Indians and white supremacy are reproduced and examines ways of achieving racial emancipation (Ladson-Billing & Tate, 1995). Ladson-Billings coined the term *culturally relevant pedagogy* to describe a form of pedagogy that develops learners' cultural competence and academic success simultaneously. Cultural competence entails acknowledging the diversity of learners' cultural backgrounds and using their cultural histories as a basis for learning new knowledge. Furthermore culturally relevant pedagogy equips learners to understand how the social order is structured, how it shapes their lives and how to challenge the existing social structure (Amidon, 2010; Erchick, Dornoo, Joseph, & Brosnan, 2010; Rubel & Chu, 2010).

Criticalmathematics Education and Ethnomathematics portray the class struggle as a series of partial hegemonic struggles such as racism, sexism etc. that accepts capitalism unquestioningly as the only viable economic system without acknowledging its structuring effect on all aspects of social life (Žižek, 2000). Here we observe how through a Marxist's notion of power social class features implicitly in Criticalmathematics Education, but what is absent is an analysis of social class. See also (Dowling, 2010; Pais, Fernandez, Matos, & Alves, 2012) for a critical review of CmE.

Like Ethnomathematics, Criticalmathematics Education offers a means for addressing the social class-aligned achievement gap, but it does not really provide methodological resources for analysing what emerges as mathematics in pedagogic contexts. In this set of literature, the inclusion of the 'everyday' in mathematics curricula is viewed as a means of enabling access into mathematics for marginalized learners.

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[86] Racism remains a concern in the United States. Anderson and Tate (2008) notes a trend of increasing resegregation of American schools in spite of the 1954 Supreme Court's school desegregation decision Brown vs. Board of Education.
[87] Gutiérrez (2013, p. 5) uses the term Latin@ refer to both male (Latino) and females (Latina) to identify with individuals who self- identify as lesbian, gay, bisexual, transgender, questioning and queer.

Appendix 3
The constitution of mathematics – an anthropological approach

This section examines literature that considers the constitution of school mathematics from an anthropological orientation (see Bosch, Chevallard, and Gascón, 2006; Barbé, Bosch, Espinoza, and Gascón, 2005) for a summary of this research. This literature differs from the anthropological studies of Lave (1998) who focused on the relationship between "mathematical" practices of individuals in non-pedagogic contexts e.g. grocery shoppers and dieters rather than on what is constituted as mathematics in pedagogic contexts.

The anthropology of mathematics education has largely been influenced by the seminal work of Chevallard's anthropological theory of didactics (ATD) (1985, 1992, 1999 as cited in Bosch et al. (2006)) and Brousseau's theory of didactics (Brousseau, 1977). Chevallard first proposed the theory of *didactic transposition* that focuses on the transformation of mathematics from its production by mathematicians to knowledge actually learnt by students. Thus didactic transposition distinguishes between (1) 'original' or 'scholarly' mathematical knowledge as it is produced by mathematicians; (2) knowledge to be taught officially prescribed by the curriculum; (3) knowledge as it is actually taught by teachers in their classrooms and (4) knowledge as it is actually learnt by students Bosch et al. (2006, p. 4).

At one level, Chevallard's (1985, 1992) theory of didactic transposition resonates with Bernstein's (1996, 2000) notion of recontextualisation of knowledge, that is, the transformation of knowledge as it relocates from the field of production, through the field of recontextualisation, to the field of reproduction. However, Chevallard and Bernstein differ epistemologically as well as methodologically, with Bernstein located in the field of sociology and Chevallard in the sub-field of anthropology of mathematics education. Furthermore, in contrast to Bernstein's theory (1996, 2000) that attempts to offer an explanation for the differential performance of students from different social class backgrounds, Chevallard's (1992) theory of *didactic transposition* is not an explanatory theory but a theory that offers a means to describe mathematics constituted in institutions.

Chevallard's (1992) anthropological theory of didactics emphasises "the institutional relativity of knowledge and situates didactic problems at an institutional level" (Bosch, Chevallard & Gascón, 2005: 4). Thus, for Chevallard, what mathematics *is* in a particular institution such as the mathematical community, the educational system or the classroom can only be established empirically. In other words, what constitutes mathematics in the pedagogic situations of schooling emerges from the empirical. In this respect Davis' notion of the constitution of mathematics in pedagogic situations of schooling (Davis 2010a, 2010b, 2010c, 2011a, 2011b, 2011c) corresponds with that proposed by Chevallard. However, Davis' methodology explained briefly in Chapter 3 focuses on teachers and students' computational activity that entails examining the objects and operations deployed by teachers and students.

Appendix 4.1
School questionnaire

School questionnaire

Thank you for agreeing to the interview. Please note that the information obtained during this interview is completely anonymous. Your responses will not be shown to any teachers or Government officials. Please answer the questions as accurately as possible.

General information

1. Name of the school: ____
2. How long has school been in existence (including this year)? _________________
3. What are the current school fees per annum? __________________________

Facilities

4. How many of the following facilities do you have at your school?
   Write a number for each one.
   a) Rooms used for teaching and learning (including specialist rooms such as laboratories, libraries etc.)

      ____
   b) libraries ____
   c) science/biology laboratories _________________________________
   d) computer laboratories _____________________________________
   e) staff rooms ____
   f) offices ____
   g) halls ____
   h) telephones ____
   i) photocopying machines_____________________________________
   j) fax machines ____

5. If you have a library,

a) approximately how many books are in the library?

____

b) what type of books are in the library?

____

c) how often is the library used by learners?

____

d) for what purpose is the library used by learners?

____

e) how often is the library used by teachers?

____

f) for what purpose is the library used by teachers?

____

6. If you have a computer laboratory,

a) how many working computers are in the laboratory? _______________________

b) how often is the computer laboratory used by learners?

____

c) for what purpose is the computer laboratory used by learners?

____

d) how often is the computer laboratory used by teachers?

____

e) for what purpose is the computer laboratory used by teachers?

____

7. If you have a science/biology laboratory,

f) does the laboratory have equipment that can be used?_____________________

g) how often is the laboratory used by learners?

____

h) for what purpose is the laboratory used by learners?

___

i) how often is the laboratory used by teachers?

___

j) for what purpose is the laboratory used by teachers?

___

8. Does your school have a shortage or inadequacy of any of the following?
Tick one box in each line.

	Yes	No
a) electricity		
b) teachers		
c) budget for covering running costs and maintenance		
d) supplies of paper, pencils, notebooks etc.		
e) classrooms		
f) chairs, desks, tables		
g) classroom size		
h) textbooks		
i) calculators for mathematics instruction		
j) library materials relevant for teaching		
k) audio-visual resources/ equipment for teaching (e.g. overhead projectors, data projectors, electronic whiteboards)		
l) computers and computer software		
m) facilities for duplicating worksheets		

Learners

9. What is the total school enrolment (number of learners)? ___

10. What is the breakdown of learner enrolment per grade?
Write a number for each one.

Grade 8	**Grade 9**	**Grade 10**	**Grade 11**	**Grade 12**

11. How many classes are there in each grade?

Write a number in each case.

Grade 8	Grade 9	Grade 10	Grade 11	Grade 12

12. What is the average class size in each grade?
Write a number in each case.

Grade 8	Grade 9	Grade 10	Grade 11	Grade 12

13. Over the past week, how many learners were absent from school?

____

14. Is this the usual pattern of attendance? (*yes/no*) ___________________________

15. Does the school have a one or more feeder areas? _________________________

16. If yes, what are the feeder areas? ______________________________________

____

17. On what basis are learners admitted to Grade 8 at your school?

Circle either yes or no in each case.

a) residence in a particular area Yes No

b) preference given to learners whose primary language is the same as the majority of learners at the school Yes No

c) learner's academic performance Yes No

d) preference given according to date of application Yes No

e) no admission criteria Yes No

f) Other criteria, specify ……………….. Yes No

18. In your opinion, roughly how many learners at your school:
Tick one box in each line.

	0%	< 25%	25-50%	50-75%	>75%	100%
a) Come from poverty-stricken backgrounds						
b) Come from homes where their parents/ main caregivers did not receive more than primary schooling						
c) Come from homes which do not have electricity						

d) Come from homes that do not have running water						
e) Have health or nutritional problems						

19. In your opinion, roughly how many parents of learners at your school:
Tick one box in each line

	0%	< 25%	25-50%	50-75%	>75%	100%
a) Are in professional, high skill jobs						
b) Are in semi-skilled jobs						
c) Are in unskilled jobs						
d) Are unemployed						

20. In your opinion, roughly how many parents of learners at your school:
Tick one box in each line.

	0%	< 25%	25-50%	50-75%	>75%	100%
a) Have no schooling						
b) Have primary schooling only						
c) Have primary and secondary schooling						
d) Have university or college education						

21. How many learners at the school are:
Write a number in each case.

a) 'African'*	
b) 'Coloured'*	
c) 'Indian'*	
d) 'White'*	
e) Other	

**Note: Current research conventions recognise the use of racial classifications for analytic purposes only recognising that such racial classifications were constructed under Apartheid law as part of oppressive socio-economic practices.*

Staffing

22. What is the total number of staff in the school? ____________________________

23. How many of the following are on full-time staff at the school?
Write a number in each case.

a) Principals	
b) Deputy principals	
c) Heads of department	

d) Active subject teachers	
e) Cleaners, gardeners, guards etc.	
f) Administrative staff (secretaries etc.)	

24. How many staff at the school are:
Write a number in each case.

a) 'African'*	
b) 'Coloured'*	
c) 'Indian'*	
d) 'White'*	
e) Other	

* *Note: Current research conventions recognise the use of racial classifications for analytic purposes only recognising that such racial classifications were constructed under Apartheid law as part of oppressive socio-economic practices.*

****

Appendix 4.2
Principal interview

About the principal

1. Describe the management structure?
2. Since when have you been the head of this school?
3. Have you had other headship positions? At which schools?
4. How long have you been in the teaching profession? What have you taught and for how long?

About the school

5. What do you think distinguishes this school from other schools?
6. In your opinion, what do you think contributes to the academic success of the school?
7. What proportion of your budget for running the school comes from the state, parent contributions, fundraising activities?

About your teachers

8. What proportion of your teaching posts is governing body posts?
9. What is the average age of teachers are on your staff? Are there more younger teachers than older teachers?
10. On what basis do you select teachers for your school? Do you have specific criteria for selection of teachers?
11. How would you describe your management style? Do you have direct management and monitoring of all teachers or do you devolve this responsibility to your heads of department?
12. What support is provided to new teachers?

About the students

13. How many Grade 8 places are there at the school?
14. Approximately how many Grade 8 applicants do you receive each year?
15. Describe the selection process of Grade 8 students?
16. How often do you have school assemblies? Grade assemblies?
17. What for you is the main purpose of school assemblies?

About your parents

18. To what extent are parents involved with the school?
19. What form does parent involvement in the school take?
20. How do you communicate with parents?
21. How often do you have parents' meetings? What is the purpose of these meetings?

Appendix 4.3
Teacher interview

About the teacher

1. How long have you been teaching at this school?
2. How did you come to be employed at this school?
3. How long have you been teaching mathematics? Which grades have you taught?
4. How many years have you taught Grade 10 mathematics?
5. Where did you study to become a teacher?
6. What did you study?
7. What mathematics did you study after school?
8. Where did you go to school?
9. Do you attend professional development activities? If so, explain.

About the school

10. How many mathematic teachers at the school?
11. How many Grade 10 mathematics classes?
12. Are students streamed into ability groups? If so, on what basis is streaming done?
13. How many Grade 10 mathematics literacy classes are there?

About the curriculum

14. To what extent do you follow the sequence of topics suggested by CAPS?
15. To what extent do you follow content specified by CAPS?
16. How many periods of mathematics are allocated to Grade 10 mathematics?
17. How are these periods used?
18. What support do you receive from curriculum advisors?
19. What form, if any, monitoring does the WCED of your school's mathematics curriculum and your teaching?

Lesson planning, resources teaching and assessment

20. What resources such as textbooks, worksheets etc. do you draw on to plan your lessons?
21. Do all the grade 10 teachers use the same teaching resources (textbooks, worksheets)?
22. If you have a common curriculum who is responsible for developing the curriculum?
23. What resources such as textbooks or worksheets are provided to students?
24. How often do you give students homework?
25. Do you have grade meetings? If so, what is the nature of such meetings?
26. How often do you assess students?
27. What form do the assessments take?
28. Do all the Grade 10 teachers use the same assessment tasks?

Student support

29. Do you provide additional mathematics support for your students?
30. If so, what form of additional support is provided to students?
31. Do your students attend private mathematics tuition? Approximately, how many of your students do so?

Appendix 4.4
Learner questionnaire

Name: ………………………………………………………………………… Age:………………… years

Name of your Grade 10 Mathematics teacher: ………………………………………………………….

Name of school: ……………………………………………………………………………………………………..

General Directions

In this questionnaire we are going to ask you questions about yourself. The information you give us will be kept confidential. We will not show your responses to any teachers or any other students at your school.

Some of the questions will be followed by choices indicated by a letter next to or below it. For these questions, circle the letter next to or below your choice as shown in Example 1.

Example 1

1. I attend school

   Yes………………………………………………………………………………………………… A

   No ………………………………………………………………………………………………… B

The letter 'A' has been circled because you attend school.

If you decide to change your answer to a question, put an X over your first choice and then put a circle around you new choice as shown in Example 2.

Example 2

1. I like chicken

   Yes ………………………………………………………………………………………………… A

   No ………………………………………………………………………………………………… B

For other questions you will be asked to write a word, number or date in the space provided. Please make sure that your handwriting is clear.

Please answer the questions as accurately and carefully as possible and ask for help if you do not understand a question or are not sure how to answer.

ABOUT YOU

1. On what date were you born?
Write in the day, month and year.

………………… Day …………………………. Month …………………………… Year

2. Are you a girl or a boy?
Circle either A or B.

Girl ……………………………………………………………………………………………………A

Boy…………………………………………………………………………………………………..B

ABOUT YOUR FAMILY

3. Who is your primary caregiver?
Circle just one letter.

Mother ………………………………………………………………………………………………A

Father ………………………………………………………………………………………………B

Grandmother…………………………………………………………………………………………C

Grandfather………………………………………………………………………………………..D

Another person (aunt etc.) …………………………………………………………………………E

No-one, I take care of myself……………………………………………………………………..F

4. Did your primary caregiver finish high school?
Circle just one letter.

Yes
……………………………………………………………………………………………………A

No ……………………………………………………………………………………………B

I don't know…………………………………………………………………………………...C

5. Did your primary caregiver study further after high school?
Circle just one letter.

Yes, at a university ……………………………………………………………………………..A

Yes, at a college ………………………………………………………………………………B

Yes, at a technikon………………………………………………………………………………C

Yes, on the job training ………………………………………………………………………D

Yes, at other post-school institutions ………………………………………………………...E

No……………………………………………………………………………………………F

I don't know ……………………………………………………………………………… G

6. What language do you mainly speak with your family at home?
Circle just one letter.

Afrikaans………………………………………………………………………………………A

English……………………………………………………………………………………………………B

IsiXhosa…………………………………………………………………………………………………..C

Other………………………………………………………………………………………………………D

7. Does your primary caregiver work?
Circle just one letter.

Yes, s/he now works full time………………………………………………………………………A

Yes, s/he now works part time ……………………………………………………………………..B

No, s/he is now looking for work …………………………………………………………………C

No, s/he is now a full time caregiver ……………………………………………………………..D

No, s/he is not working at present (unemployed) ………………………………………………E

No, s/he is not working at present (unemployed, retired) ………………………………………F

8. If you answered A or B in Question 7, write down the type of job your caregiver does?
Write in the type of job. Describe it as clearly as possible.

………

………

ABOUT YOUR HOUSE

9. In which area do you live?
Write down the name of the area.

………

10. What type of building do you live in?
Circle just one letter.

Brick or concrete house on separate plot ……………………………………………………………A

Informal shack ………………………………………………………………………………………..B

Apartment or flat ……………………………………………………………………………………...C

House/flat/room in backyard…………………………………………………………………………D

Wooden house in backyard……………………………………………………………………………E

Other……………………………………………………………………………………………………F

11. How many bedrooms are there in your house?
Circle just one letter.

0………A

1………B

2………C

3………D

4 …………..…………………………………………………………………………………………E

5 or more……………………………………………………………………………………………..F

12. Including yourself, how many people live in your house?
Circle just one letter.

1………A

2……B

3……C

4………………….……………………………………………………………………………………..D

5 more……………….………………………………………………………………………………….E

13. Does your house have the following?
Circle either A or B each time.

	Yes	No
Electricity ………………………………………………………………..	A	B
Running tap water …………………………………………………………	A	B
Hot running water …………………………………………………………	A	B
Water flushed toilet…………………………………………………………	A	B

14. Approximately how many books are there in your home? (Do NOT count magazines, newspapers and school books)?
Circle just one letter.

Less than a bookshelf (0 to 25 books) ………………………………………………………… A

One or several bookshelves (25 to 100 books) ………………………………………………..B

One or more full bookshelves (over 100 books) ……………………………………. …….C

15. Do you have the use of a computer at home?
Circle either A or B.

Yes……………………………………………………………………………………………………A

No ……………………………………………………………………………………………………B

16. Do you have internet access at home?
Circle either A or B.

Yes
…………………………………………………………………………………………………..A

No …………………………………………………………………………………………………...B

17. Do you have a cellphone?
Circle either A or B.

Yes …………………………………………………………………………………………..A

No ……………………………………………………………………………………………B

ABOUT YOUR MATHEMATICS

18. Do you get homework from your mathematics teacher?
Circle either A or B.

Yes
……………………………………………………………………………………………………….A

No
……………………………………………………………………………………………………….B

19. How often do you get mathematics homework?
Circle just one letter.

never ……….………………………………………………………………………………………...A

once a week………………………..………………………………………….………………………B

twice a week………………………..……………………………………………………………...C

thrice a week………………………..……………………………………………………………..D

almost everyday ………………………..…………………………………………………………E

everyday ………………………..…………………………………………………………………F

20. When you do your mathematics homework, how do you do it?
Circle just one letter.

On your own
……………………………………………………………………………………………………A

With help from your friends ……………………………………………………………………..B

With help from your caregivers …………………………………………………………………C

With help from your brother/sister ………………………………………………………………D

21. Do you attend mathematics classes in addition to those offered at school?
Circle either A or B.

Yes…………………………………………………………………………………………………A

No…………………………………………………………………………………………………..B

22. If you answered 'yes' to Question 20, where do you go for additional mathematics hrlp?
Circle just one letter.

Private tutor ……………………………………………………………………………………A

Institution e.g. MasterMaths …………………………………………………………………B

Other (describe) …………………………………………………………………………………C

23. How often do you attend additional mathematics classes referred to in Question 22?
Circle just one letter.

Once per week………………………………………………………………………………..A

Twice per week ……………………………………………………………………………………B

More than twice per week ………………………………………………………………………C

Thank you for your time and effort taken to answer these questions.

Appendix 4.5
Transcription conventions

T	Teacher
I	Interviewer
L	Learner
Ls	Learners
/../	Short pause
/…/	Long pause
[ ]	Text inserted by the author, comments about the transcription e.g.[inaudible] or comments not captured in the speech e.g. capturing text written on board or worksheet that teacher or learner refers to in speech e.g. [Teacher writes /$y = 2x$/ on the board]
[indistinct]	can hear the person speaking but can't make out what they are saying
[inaudible]	can see the person speaking but can't hear what they are saying
Underlined text	Non-English word or phrase e.g. haai (no)
( )	translations of words e.g. e.g. haai (no)
Italicised letters	Mathematical symbols e.g. if I take *y* equals two to the two *x*
Text in italics	Indicates observer description or field notes as opposed to verbatim dialogue

Appendix 5
Auxiliary operations

The following auxlliary operations used in the thesis are defined as foolows:

(1)" STR(μ) returns a character string, /μ/, derived from a discursive object, μ. A character string is a sequence of alphanumeric characters. For example, the word "dog" can be considered as a sequence of letters (alphanumeric characters) "d", "o", "g". Such strings are indicated by a pair of forward slashes: /μ/. STR(-7 + 2) returns the alphanumeric string /-7 + 2/, consisting of the characters "-", "7", "+", "2". The spaces between the characters listed here would be recognised as alphanumeric characters in certain other contexts, like computer programming, but are excluded as characters here. Once a discursive object is rendered as a character string each of the individual characters, or combinations of them, are available to operations or operation-like manipulations." (Davis, 2010b, pp.106-107)

(2) "NUM(/λ/,A), where /λ/ is a alphanumeric string and A ⊆ ¡ returns the value λ ∈A. This restricts the composition of /λ/ to concatenations of certain combinations of elements of, at least, the list {/-/,/+/,/,/0/,/1/,/2/,/3/,/4/,/5/,/6/,/7/,/8/,/9/,/,/./,/·/}. If additional alphanumeric characters are needed to generate elements of ¡ they can be included in the list." (Davis, 2013a, p. 11)

(3) "SUN(/λ/) sunders an alphanumeric string, /λ/, into a list of two or more alphanumeric strings (/λ1/,…,/λn/), $n \in \mathrm{N}_w, n \geq 2$. So, SUN(/-7/) returns the list of alphanumeric strings (/-/,/7/), while SUN(/-7 + 2/) could return the list (/-/,/7 + 2/), or even the list (/-/,/7/,/+ 2/), or even (/-/,/7/,/+/,/2/), or any other combination of alphanumeric strings derivable from /-7 + 2/. Clearly the result of SUN(/λ/) is not unique, its output being contingent on the decision of the agent effecting the sundering." (Davis, 2010b, pp.107)

(4) "CON(/λ1/, … ,/λn/) returns the concatenation of a list of strings (/λ1/, … ,/λn/) to produce the alphanumeric string /λ1λ2…λn/. For example, CON(/-/,/5/) returns /-5/." (Davis, 2010b, pp.107)

(5) ALT maps $/+/$ to $/-/$ or vice versa. In other words, ALT: $(+, a) \rightarrow (-, a)$ and ALT: $(-, a) \rightarrow (+, a)$ where $a \in C$. Sometimes, the output requires the operation-like manipulation, concatenation, which combines the sign and the alphanumeric character to produce $+a$ or $-a$, which represents the required outcome.

Appendix 6.1
Lesson transcript: Sara Lesson 1 (S01T01L01)

Non-Evaluative Event Segment (NES)

Time	#	Speech
00:01	01	[students enter chatting]
00:38	02	T: Right. Good afternoon gentlemen.
00:40	03	L: Good afternoon Mam.
00:41	04	T: Please sit down. We are going to be using our laptops today so can you please take them out.
	05	L: [Inaudible] [seems to indicate that laptop forgotten]
00:48	06	T: I did tell you. You will just have to share. Who has not got their laptop here? [learner talk]
	07	T: Daniel where is your laptop?
01:08	08	L: It is here.
	09	T: Get it out please. Graham have you got your laptop?
	10	L: [chatter as laptops are started]
	11	T: Right. [moves to front as organisational chatter continues]
	12	T: Now before we start or before we start formerly I am going to be handing out consent forms. Please understand that this there is an anonymity clause. Do you know what anonymity means?
	13	L: I have no idea.
	14	T: Okay no one will know who you are or who I am. They will never show the video. So you are quite safe. Nothing is going to happen. It is just purely for research and if they do ever show it will be the just classroom or just voices or something. You will never … so its part of but we still have to get you to sign the form. Right? Okay. Right so let's start. I just want to hand out this sheet.
02:05	15	L: [inaudible]
	16	T: Ja I know. They have got two minutes. Okay.
	17	L: [inaudible]
	18	T: Where are they?
	19	L: They are at Simply Blue auditions.
	20	T: Simply Blue Auditions?
	21	T: Make sure that you have got Geogebra open. A'right I am going to hand out the sheets. Please make sure that you all have a pen and a pencil. [hands out

Time	#	Speech
		sheets amidst chatter] Where is Benson? Is he coming?
	22	L: Daniel.
	23	R: I don’t think that this is working because this is not getting current either.

EE1: Revision of basic functions

Time	#	Speech
03:10	24	T: Okay let’s start. It doesn’t matter that the others aren’t here. Um right. .. You all got your laptops open and you have all got your *Geogebra* on. Okay.
	25	L: [aside talk]
	26	T: Okay. Now. Up until now we have been working out of our textbook and we have looked at the following graphs. [students arrive] Come guys why you late?
	27	L: We were waiting for …
	28	L: You were waiting for [indistinct]
	29	L: Sorry.
	30	T: Right. We are going to be working on our laptops. Please make sure you get them out. So I am just going to quickly put you in the picture of what we have covered so far. Uhm Collin. Attention please. Right. So we have looked at the graphs of y equals a x plus p. We have looked at the graph of y equals a over x plus p. We have looked at the graph of y equals a to the x plus p and we have looked at the graph of y equals a x squared plus p. What do we call these graphs? We gave them a special name. The whole family of graphs.
	31	L: Parabola
	32	T: No.
	33	L: [indistinct]
	34	T: All the graphs together. We said we were looking at functions.
	35	L: Oh. [discussion]
	36	T: Anybody remember what a function is?
	37	L: Definition of a function is …
	38	L: Like f of x
	39	T: f of x okay that is another name that we could call this f of x. .. [Writes on the board] Okay. And we said that a function is something that for every x there is one corresponding ..
	40	Ls: y value

Time	#	Speech
	41	T: y value and we see the shapes and I said to you that you have to be able to look. Yes gentlemen why you late?
	42	L: Sorry Mam. I lost track of time.
	43	T: Boarders!
	44	L: [comments indistinct]
05:00	45	T: Alright. Uhm. Now. What do we call a and p?
	46	Ls: [No response]
	47	T: I gave them a special name. We called them?
	48	Ls: [No response]
	49	T: It starts with a p …
	50	Ls: [No response]
	51	T: Parameters. Okay and we looked at all the simple graphs and we said the mother graph or whatever was y equals x. This one was y equals one over x. This one was y equals a to the x. Two to the x or something. And this one was y equals x squared. And I hope in your own minds you all have a picture of what these graphs look like? Can you all picture them?
	52	L: Yes Mam.
	53	L: Yes Mam.
	54	T: Okay who wants to tell me what this one is?
	55	L: A straight line.
	56	L: A straight line.
	57	T: Right. A straight line going through the ... [T draws graph on board]
	58	L: The origin.
	59	T: Through the origin. Right what shape is this?
	60	Ls: Hyperbola. Hyperbola
	61	L: A hyperbola.
	62	T: What does it look like?
	63	L: That's a U Ma.
	64	T: Not a U. I think you do all know. Right. It looks like that [draws]. What does y equal a to the x look like?
06:00	65	L: A parabola … [offers made but nor audible]
	66	T: What is the name of the graph?
	67	L: It is a hyperbola.
	68	T: No that’s the hyperbola.
	69	L: I mean a parabola.

	70	T: Exponential graph and this one is a …
	71	L: A parabola.
	72	T: A parabola. Right. Now in all three graphs *p* does the same thing and *a* does sort of the same thing. Okay. So let's start with *p*. What does *p* do to all of those graphs?
	73	L: Shifts them up or down.
06:25	74	T: Shifts them up or .. down. What does *p* have to be for the graph to shift up?
	75	L: Positive.
	76	T: Positive. Okay. And for the graph to shift down *p* has to be …
	77	L: Negative.
	78	T: Negative. Good you know that. Now what effect does *a* have on this graph? [points to $y=ax+p$]
	79	L: The gradient. The gradient.
	80	T: The gradient. So it is almost like a vertical stretch we could say cause if I take *y* equals two to the two *x* I mean. The graph is a bit steeper. So it is almost like we've stretched that point up looking a bit like that. Okay. What does *y* equal three over *x* or six over *x* look like compared to. What does the *a* do there [referring to hyperbola?
07:08	81	L: Changes the distance from the .. origin.
	82	T: Okay. This line has an axis of …
	83	L: Symmetry.
	84	T: Symmetry. [draws dotted line] We know all these features. Okay and that *a* or three moves it out or in [gestures with hands] closer to this point here. And if *a* is negative what happens to the graph?
	85	L: Switches round.
	86	T: Flips into the other two quadrants. Good. Right. *y* equals *x* squared. Okay. What does the *a* do?
	87	L: [No comment]
	88	T: If what does *y* equals two *x* squared
	89	L: [lots of comment]
	90	T: and *y* equals a half *x* squared look like.
	91	L: Average gradient!
	92	L: It changes the [indistinct]
	93	T: The steepness of the graph. So if *y* equals *x* squared is like this two *x* squared is a bit steeper. [uses arms to show]
	94	L: Mam, ss it like correct to say uhm that it is like a relative gradient? Like an average gradient.
	95	T: Average gradient changes. Yes. We haven't really dealt with that yet but yes that does come into it. Okay. So you know all this don't you. Yesterday in

		quite a big rush we got out our *Geogebra* and we started looking at *y* equals *a* to the *x*. All right now we must know the features. What is domain? What is range? Alright.
	96	L: [mutters]
	97	T: What is an asymptote? Which of these two graphs have asymptotes?
	98	L: Exponential.
08:26	99	L: Um *a x* squared …
	100	L: a to the *x*
	101	L: The two last ones.
	102	L: The two …
	103	T: This one [pointing to hyperbola] has asymptotes. How many asymptotes does it have?
	104	L: Two
	105	T: Two. And this graph has an asymptote.
08:37	106	L: One.
	107	T: okay. And what is the asymptote of this function [Refers to the exponential function]?
	108	L: Zero.
	109	L: *y* equals one.
	110	T: In this case if I have *y* equals zero. If I have *y* equals two to the *x*. But if I have *y* equals two to the *x* plus one what does the asymptote become?
	111	L: One.
	112	L: *y* equals one.
	113	T: Right. Now you know that. Okay. Now that is basically your grade ten work. Now we are an Admaths class so we are going to go on and do grade eleven work.

EE2.1: Setting up sliders in *Geogebra*

Time	#	Speech
09:08	114	T: So what we are going to do today is some grade eleven work. Okay. This will be in your Admaths paper in September. And we are going to now look. We are going to just concentrate on the parabola today or the next few days. And we are going to have *y* equals *a x* minus *p* squared plus *q*. Now I know in this one I put the p at the end. I could have changed it but it doesn't really matter. Okay. So what I want us to do is I want you to get our Geogebra out okay. And I want you to type in this equation. I'll show you how to do it. I think you do know. Then we are going to look at what the *p* does here [referring to $y = a(x-p)^2 + q$] and we know the *q* moves it up. Now we are going to look at the effect of that and then we will go on and look at the parabolas *y* equals. Sorry the hyperbola *y* equals *a* over *x* minus *p* plus *q* and uhm we will also look at how I won't put the *q* in quite yet over there. So you've each got a sheet here okay and it says here so far you have sketched *y* equals *a x* squared plus *q* and I want you to start this sheet with the aid of your Geogebra. This [points to sheet] you should

		be able to do without Geogebra and here you can either if you don't know how to set up a slider I'll show you quickly. Who doesn't know how to set up a slider? Okay let's just show you.
	115	L: [talk indistinct]
	116	T: Just give this a jiggle here.
	117	Ls: [talk indistinct]
	118	T: Okay so now let's watch. Let's show you how you do a slider. In your *Geogebra* there is this little uhm icon. You drop it down and it says slider. Alright. And you drag it across and that comes up. Now you will notice it says *b* there but I haven't got a *b* here. But that's fine my slider *b* is going to come up and I want it to go from minus five to five is enough and our increment we don't want it to be nought comma one. We just going to make it one every time and we put apply and then there is your little slider. Now every time in Geogebra that you want to move this you just do that and then you can drag this around and change *b*. Now nothing is happening because there is no *b* in the graph. I can move the whole panel. You can move the whole thing away. It doesn't seem to be working. Okay. Now at the bottom here you've got the input bar. Most of you know so I am going to type in *y* equals *b x* and I press the enter button. Oh!
11:40	119	L: You have to give a value for *b* Mam.
	120	L: Ja.
	121	L: Ja. *b* has to have a value.
	122	L: It is an unknown value.
	123	T: But I set up my slider. I think nothing was [indistinct] Right lets try again. I am going to try *f* of *x* equals *b*. I think it probably wants bracket *x*. Okay. Right so there is my graph [refers to the graph of $y = bx$.on Geogebra] It says input there sometimes want brackets. Now if I click here and I change my slider. The moment *b* is one you can see that the graph is changing around. Okay and that is changing the value of b and *b* is negative. Okay now what has happened to my other graph? The whole thing disappeared.
	124	L: [muttering]
	125	T: What happened to my parabola? Oh dear. Anyway. Okay. So now we are going to type in here. You are going to type in *y* equals. Didn't I have a parabola up there a few minutes ago? I don't know why it has disappeared? Now you are going to type in this [refers to $y = a(x-p)^2 + q$.] We are going to play a bit and I will get my one right and you are going to draw those graphs and I am going to give you about three or four minutes or maybe a bit longer for you to work out for yourself what the effect of this *p* is. How the graph is shifted. Okay. Yes um Raoul?
12:53	126	L: I think that your parabola escaped because you made some of those values zero.
	127	T: Aha. Yes you are quite right and they are equal to one. Thank you. I'll go and change that now. Okay. Right. So. We are going to if you can't if you can't manage with the slider then you just type in the equations of these graphs and see what the different ones are. Okay? Sometimes it helps to type it in. So lets do that for a few minutes and we do page one and then we will discuss it. Yes?

EE2.2: investigating $y = a(x-p)^2 + q$ using Geogebra to establish the effect of a, p and q on the basic parabola

Time	#	Speech
13:20	128	L: Are we getting our tut today?
	129	T: Are you getting your tut today? No the tut's only due on Thursday. Okay?
	130	L: What?
	131	L: Is it due on Thursday!
	132	T: Excuse me. No it is not due. Sorry. It is being handed out on Thursday. It is due by the teacher.
	133	L: For?
	134	T: For the section on graphs
	135	L: I'm leaving. Can I have it early?
13:41	136	T: When are you leaving?
	137	L: [inaudible]
	138	T: Then I will give it to you this week Thursday.
	139	L: Yay. Can we have it now?
	140	T: Mr M hasn't set it.
	141	L: But can we hand it in by [indistinct] because I leave next Friday.
	142	T: [raises her hands] Anyway. I don't know.
	143	L: [muttering]
	144	T: It's up on the board there.
	145	L: [muttering]
	146	T: Right. Come on now. Get started?
	147	L: Mam, can you keep our worksheets until we get back?
	148	T: Yes, I will. [Moves to assist a student]
	149	L: Sorry Mam [Inaudible]
	150	T: I don't know what is going on. Maybe it doesn't like a dot. Maybe you mustn't put the dot in.
	151	L: [talk amongst themselves] You put the gradient there.
	152	T: Move your thing down.
	153	L: [inaudible]
	154	T: Okay. And what does it do? …
	155	L: [inaudible]
	156	T: Okay. You must decide which way it moves. Does it move positive? Does

		it move negative?
	157	L: [inaudible]
	158	T: You figure it out and fill in these things. [moves to another S]
	159	L: The gradient … [discussion continues]
	160	T: Is it working Graham? [background comment] You've put a point on the graph there. You don't want that. Where is your? You see you need to type in here. Okay. Now type in *y* equals *a x* squared. Now you can go and move your slider
	161	L: Must I press enter?
	162	T: You must press enter. You probably got to do a bracket for the *x* squared. So say bracket *x* squared. Okay I think when you put the parameters … [background talk dominates] … brackets …
	163	L: You were saying that the function like here *x* [indistinct[Should I not put a number in it?
	164	T: You can rather if you want to and then just change the number. Okay. My slider was working. I don't know if there is something funny. [moves off]
	165	T: Okay is yours working? You managed to do *a x* squared?
	166	L: [chatter]
	167	T: Okay. We will make sure that it is on. [works with cable]
	168	L: [chatter]
	169	T: You are supposed to get parabolas.
	170	L: [indistinct]
	171	T: No but you shouldn't get this. You haven't got a square. You are typing in a whole lot of straight lines. These are not straight lines.
16:40	172	L: How do I enter [inaudible]
	173	T: Some of us are struggling here with the parameters. Okay. If you figure out how to do it. Whether you put a bracket or a dot. Otherwise just type in those equations and see what they all do. Yes, T?
	174	L: Mam …
	175	T: Yes [goes to assist a student and looks at screen] …
	176	L: [explains problem but inaudible]
	177	T: Okay now you need to [indistinct] Have you set up your slider? You need a slider. So let's do another slider [indistinct] It just changes that value. Now do another slider … *q* just change it. Now. .. *h* is one. *p* is one … [inaudible]
	178	L: [talking to another learner] Hey how did you do that graph?
18:02	179	T: I am going to make *a* one and I am going to make *q* nought.
	180	L: [general discussion about problem] *a* is greater than nought and *q* is greater than nought
	181	L: [discuss problem] *a* is greater than nought and *q* is …

	182	T: [continues to assist L who tries as well] Okay lets just change this to *q* and then just make our increment one. Okay.
	183	L: *q* is greater than nought.
18:47	184	T: Have you guys figured it out Wes?
	185	L: Uhmm. Yes.
	186	L: The sliders?
	187	T: Yes.
	188	L: No. When I type in the gradient … the same … [inaudible]
	189	L: [another S offers an idea]
	190	T: Just type type in the equations. Maybe that's easier. I am not quite sure [indistinct].
	191	L: You know what it looks like.
	192	T: If you roughly know what it looks like that is also fine. Yes. Check on your thing by just typing in. That's fine. Okay. [moves away] Right. Are we getting it to work? I want the increment to be one. Just tell me. You can even [indistinct] Can I do another one? [assists talk inaudible] Type a number … okay now you are going to type *y* equals *a x* minus *b* … plus *c*. … Now what we want to do is put it in a brackets … Now see if that works …Aha! Ah! Right.
20:04	193	T: Okay. It seems like if you are going to use your parameters uhmm I think that you must do this. [writes on board *y* = (*a*)(*x*-*p*)^2+*q*] You must type in *y* equals bracket *a* uhm *x* minus *p* and then it seems to work. [Indistinct] Have you tried that? I know that your mac machine does different things.
	194	L: [comment on this] yeah … [students discuss]
	195	L: [asks question but indistinct]
	196	T: We haven't worked with these graphs. You are discovering something new.
	197	L: [general comment] But it is so …
	198	T: It is similar!
	199	L: [S comments]

EE2.3: the effect of a, p and q for the function $y = a(x-p)^2 + q$

Time	#	Speech
20:59	200	T: Okay. Is anybody ready to tell us uhm what they've discovered? Yes. Right. T.
21:06	201	L: If you change the *b* value the graph gets wider or
	202	T: The *b* value?
	203	L: Or …

	204	T: In other words this one over here. [referring to p in $y = a(x - p)^2 + q$]
	205	L: The *a* value sorry.
	206	T: The a.
	207	L: Ja.
	208	T: Right Tim says okay if you change the *a* value the graph gets wider or …
	209	L: Narrower.
	210	T: Narrower. That we knew from our previous graphs. Am I correct? Okay and let's see if it's is correct. I am now going to change *a* .. and it does this [shows on graph]. Doesn't it??
	211	L: Yes.
	212	T: Okay so that is fine and we notice if *a* is negative what happens?
	213	L: It flips.
	214	L: It flips.
	215	T: It goes. It's reflected. The original graph is reflected about?
	216	L: The *x* axis.
	217	T: The x axis right. So that one we know. Now let's change q. See if it applies the same to this graph as it did to that other one. Okay.
	218	L: It shifts left to right.
	219	T: No first we have to look at *q*. Change *q* over there. What's *q* going to do?
	220	L: It goes up or down.
	221	T: I don't know what. Oh *c* is nought so let's just make *a* one. Okay. So now lets try *q* and we see that the graph goes up or down. [demonstrates using Geogebra]
	222	L: Down.
	223	T: Are you happy with that?
	224	L: Ja.
22:20	225	T: Right. Now. What does *p* do?
	226	L: It shifts it left or right.
	227	T: It shifts it left or …
	228	L: Right.
	229	T: Right.
	230	L: Yes.
	231	T: Now we are not quite sure which way it goes left or right depending on *p*. This is quite a tricky one so watch carefully. Okay. Let's make *p* equal to uhm at the moment *p* is nought. Now I am going to make *p* one. Okay.
	232	L: Ja.
	233	T: Now notice that graph becomes *y* equals *x* minus one squared plus nought.

What is the turning point of this graph?

234 L: One.

235 T: It is one nought. Okay? Right. Now if I have *y* equals. Can you see this? These pens are so awful. I will use a darker one. *y* equals *x* minus two squared. What do you expect the point the turning point to be?

236 L: Two Mam.

237 T: Two. So this is two nought and if we take *y* equals *x* plus three squared what do we expect for the turning point?

238 L: Mam wouldn't it be four?

239 L: Negative three nought.

240 T: Okay. Let's see if we are correct.

241 L: But in the second one when you square the bracket out don't you do two squared.

23:45 242 T: Well let's see what happens. Okay just hang on.

243 L: If my [indistinct]

244 T: Okay. Now I will answer your question in a minute but what are we looking at? *p*. So now I make it two and there it is. Now what has become four? Not the turning point but the .?

245 L: *y* intercept.

246 T: *y* intercept. Now. We know that this graph. Okay Another way of writing this is *y* equals what?

247 L: *x* squared…

248 T: *x* squared minus four *x* plus four. So here I am showing you are different format seeing that you happened to mention it of this graph okay and we are going to. Jumping the gun a bit but it is *y* equals *a x* squared plus *b x* plus *c*. That is called the standard form of a parabola. Okay. And this we call the turning point form [points to board]. If it is in this form *p* and *q* I am not going to spoil it for you represents something. Okay. So. Let's just try making *p* equal to minus three. Okay let's do that. And we can see there if I make it minus. No hang on something's gone wrong with mine.

249 L: Make it minus three then the turning point is at minus three. [silence]

250 T: Okay. I just want to change this input. Okay. Can you notice here that it gives me this [uses curser to point]? Alright and I don't want it in that form [meaning standard form]. So to get rid of that what you do is go *f* of *x* instead of *y*. Okay. Now something is going wrong here. *f* of *x* equals um *a* .. let's just do this .. *a* bracket *x* minus .. what have we got now? *p* bracket

251 L: [indistinct]

252 T: Okay. Now it's got it's like this. .. I know from *Geogebra* if you click in *y* equals alright then it gives it in the *a x* squared plus *b x* plus *c* form. But we not looking at that at the moment. We're looking at it in this form. So it is better to type in *f* of *x*. Then it keeps it in that form [referring to turning point form]. Okay. Now. Notice here [referring to $y = (x-1)^2$] I have got minus one and the turning point becomes plus one. I have got to key in *x* minus *p* squared not *x* plus *p* then it does something different. Okay there is that one. And now if I make *x* minus three [T means p]. Okay. Uhm let's type in we wanted to make *x* minus three [Moves the slider for *p*. T means making *p*

		equal to -3]. Hang on this isn't behaving the way I was expecting it. .. Oh no that's correct. Okay. If I type in minus three okay then what happens? It becomes plus three. Doesn't it? Okay. Err if I make *p* equal to minus three then I get *x* minus minus three which is *x* plus three. You happy with that? Okay. And so if that is like that then the turning point is minus three nought. Okay. So. Who can tell me alright what is happening? Maybe I should just say it. If this is minus two it's the (opposite) sign. Okay. We call it counter intuitive. You might think that when *p* is two the graph is shifting …
	253	L: To the right.
	254	T: If it says minus two then the graph is not shifting to the left it is shifting to the?
	255	L: Right.
	256	T: Right okay. Now if you think about it this is where the minimum value or maximum value occurs and what is the minimum value that this [refers to $y = (x-2)^2$] can be if that expression is squared. The minimum value is?
27:40	257	L: Two
	258	T: No.
	259	L: One zero
	260	T: Is zero. And for that to be zero what does *x* have to be?
	261	L: Two.
	262	T: *x* has to be two. Okay. So that's why the cursor is at *x* equals to two. So when you see uhm *y* equals *a x* squared plus *p*. If it is three it goes up three but for left and right shifting it is almost the opposite. Am I confusing you or do you understand that?
	263	L: I understand.
	264	T: Okay I hope so. Right. Now. Let's look at another graph. Let's look at. And hopefully now you can predict. If I have two *x* minus three squared plus four. Okay I said that is called the turning point form of the graph. What is the turning point of this graph? Who wants to tell me?
	265	L: Four.
	266	L: Four.
	267	L: Which one?
	268	L: Four three.
	269	T: No the turning point has a *x* coordinate and a *y*.
	270	L: Which one?
	271	T: This graph here [points to equation $y = 2(x-3)^2 + 4$]. Who wants to tell me what that graph is?
	272	L: Ummm
	273	T: The turning point is …
	274	L: Three
	275	T: Three.

	276	L: Four.
	277	T: Four. Okay. So we can say that in the original graph .if we go back to y equals x squared What has happened. The original y equals x squared has shifted how many. .. How many?
	278	L: The original is set at four mam.
	279	T: No it is two. It is three four. How many has it shifted up?
	280	L: Four.
	281	T: Four. And how many has it shifted left or right?
	282	L: Three
	283	L: Three
	284	T: Three. Has it shifted left or right?
	285	L: Right.
	286	T: It has shifted right. Okay. And the two makes it. So let's check if we are right there. So we're going to start with p being three. Okay. And we making q equal to four. So there you can see the graph shifts up and if I make a equal to two. Now my picture is disappearing off my screen but it just makes it like that. Okay. Now what if I make a minus two? What's going to happen?
	287	L: Going to go the other way.
29:40	288	T: It is going to go the other way round. [demonstrates using *Geogebra*] Okay. Now.
	289	L: [question indistinct ignored by teacher]
	290	T: There's quite a nice picture there. Let's make a minus one so we can see the x intercepts. Alright. Now you know that when you have to draw a parabola you not going to have *Geogebra* with you. I don't mind if you keep it open and we use it to check when we drawing these things. But how am I actually going to draw this whole graph now. Okay. You need to. When you draw a parabola like this you need to put in the y intercept the x intercepts and the ?
	291	L: Turning point [muttered]
	292	T: turning point. Okay. So let me show you how we do that and then we can keep working through our worksheet. Are you all happy with this left and right shift?
	293	L: Mam.
	294	T: Yes.
	295	L: When a graph is now negative will the shift be opposite? Will it be [indistinct]
	296	T: No. It will be exactly the same. Okay so now we have got the graph of y equals minus x minus three squared plus four. Okay now I am not going to say anything. I want you. You can see the picture there but I want you to actually do the working out for me. Okay. On a piece of paper. I know that I planned this lesson and now we are doing something else but it doesn't matter. I want you to tell me. Okay. Are you all happy that three four is the turning point?
	297	L: Yip

	298	L: Yes mam.
31:00	299	T: okay. Now how do you think you are going to get the *x* intercept and how do you think you are going to get the *y* intercept? I am going to give you a few minutes. I am not going to say anything. I don't want you to just draw your *Geogebra* graph and get it. Let's see if you can come up with how we are going to find the *x* and the *y* intercept. So let's try that quickly.
	300	L: Oh.
	301	T: And then we are going to draw some graphs. Okay.
31:23	302	L: Haven't you given us the *y*?
	303	T: No they haven't given you the *y*.
	304	L: Oh.
	305	T: Okay. The *y* is down here somewhere. We can't see it. Okay. Alright. It's minus five.
	306	L: Uhm. Ja. Minus five.
	307	T: Okay come on you have got to do some algebra here.
	308	L: [discuss]
31:48	309	T: Do you know what to do S?
	310	L: [inaudible]
	311	T: Well you start by writing down the equation of the graph.
	312	L: *x* is five.
	313	T: *x* is five?
	314	L: Ja.
	315	L: [another S] Minus five.
	316	T: No. What do you mean?
	317	L: *x* is five.
	318	T: And one. But how did you get that?
	319	L: [comment]
	320	T: Well write it down for me.
	321	L: [comments inaudible]
	322	T: [shakes her head]
	323	L: [discuss values] Find out the value of *c*? [discussion continues]
33:13	324	T: Okay guys. Who can tell me what we are going to do here? Are you still busy?
	325	L: [discussion continues] *p* equals …
	326	L: Then that is *a* *b* and *c*. [discuss]
	327	T: Do you know what you are doing?

	328	L: Four …
	329	L: *y* equals five
	330	L: I got *y* equals one.
	331	L: Okay right cause *a* is …
	332	L: Must you put the negative in first before you multiply?
	333	L: You did that first before you made … if you times that you get a negative.
	334	L: Ja.
	335	L: [discussion continues]
	336	L: Let me see. Got it.
	337	L: Mam how can the *x* [indistinct]
	338	L: Now you have the *a* [discussion continues].
34:28	339	T: Okay. But I don't know how you have done it here.
	340	L: [Indistinct] *y* equals nought right
	341	T: uh mm. [Shaking head in affirmative]
	342	L: And then you do that
	343	T: uh mm. [Shaking head in affirmative]
	344	L: and then take the *x* minus three …
	345	T: Plus or minus
	346	L: Plus or minus
	347	T: Okay. [T and L discuss but inaudible]
	348	L: Oh ja [chatter]
	349	T: That's perfect. There's another way you could do that. [Walks around]
	350	T: That's five and then you have minus five. [Rest of discussion with student is inaudible]
	351	L: [discussion continues with greater intensity]
	352	L: The *y* equals negative five …
	353	L: The second one?
	354	L: That was minus three.
	355	T: The *y* cuts at negative five.
	356	L: [discussion]
35:37	357	T: Beautiful. Right. Now last night I spent hours typing up these things and I think I printed my old notes for you. Half the stuff I intended to be on there is not there. It doesn't matter. We'll write up the equations on the board.
35:49	358	T: Okay. So when you have to plot a parabola you are going to be given it either in this form [pointing to turning point form] or the standard form. If it in this form it is quite nice because you can read off what the turning point is. All right. So we know the turning point. Now what do we have to do here?

We have to multiply out okay to get the y intercept. So this is minus x squared minus six x plus nine plus four and you get. Sorry let's put y equals minus x squared plus six x minus nine plus four and the equation is minus x squared plus six x minus five. Are you all happy with that? And so if I make x nought what is y equal to?

359 L: Minus five.

36:40 360 T: Minus five. So are you all happy that the y intercept is nought minus five. Okay. Now how do we find the x intercept?

361 L: You make y zero.

362 T: You make y zero. Now you have got two options here for y equal to zero. I have been walking around watching you. Some of you've done it one way and some of you have done it another way. Alright. So you can make y zero here [points to std. form] or you can make y zero over here [points to TP form]. How many of you made y zero over here and solved this equation [TP form]?

363 L: [No response]

364 T: Okay so lets have a look. You've got minus x minus three squared plus four is nought. These pens! [changes pen]. Okay. That's the same as saying x minus three squared equals four. You happy with that?

365 L: Ja.

366 T: Okay. Then how do I get rid of the squared?

367 L: Square root.

368 T: You square root it. So you get x minus three equals plus or minus?

369 L: Two.

370 T: Two. Don't forget to put plus or minus and so you get x equals three plus two or x equals three minus two and so x is five or x is

371 L: one

372 T: one and then we can see from our graph x is five or x is ..

373 L: One.

374 T: one. Right? Now some of you might have taken this expression our standard form expression. Minus x squared plus six x minus five is nought. How do we solve that?

375 L: x equal five and minus five

376 T: No. I know that's the answer.

377 L: [Various answers] Make it x squared minus six x Make it plus x squared .So take everything to the other side.

378 T: Right. So we have to make x squared positive. Change all the signs. In other words we are multiplying everything by minus one. And now what do? It's a trinomial

379 L: We try to factorise.

380 T: Factorise. Right x x five one minus and minus. So x is one or x is five. Right. So that is how you find the x intercept. So if I just gave you that graph and I said draw the graph you would go through these steps writing down the turning point. Why are we doing this? Okay. Finding the y intercept and

		finding the *x* intercept.
39:08	381	T: What I am going to do is I am just going to. Again I am very upset because I have changed my original notes here and I see half the stuff I typed up last night is not here. So I must have printed the wrong thing. I am going to just write. Just hang on a second. I am going to just write two equations on the board quickly in this form and you are going to do them and when you have done that you can then start working on this sheet over here on page two. Have you all finished this one?
	382	L: Mam [indistinct]
39:37	383	L: Ja.
39:38	384	T: Because you can. I haven't done the answers yet because I expected you to check with *Geogebra*. Quick little graphs you should be able to look and do them.
	385	L: [chatter]
	386	T: The stuff on the board you can do on the back. Right. So here are two graphs to do please. Right. y equals two [writes equations on board $y = (x-1)^2 - 8$ and $y = (x-3)^2 - 16$]
	387	L: [chat]
40:01	388	T: Right off you go. And tomorrow we are going to do it the other way around. We are going to give you *a x* squared plus *b x* squared plus *c* and then how do you get the turning point from that. We will do it the other way round.
	389	L: Mam [indistinct]
	390	T: You are now going to plot these two graphs. You going to find the turning point and *x*
	391	L: From where Mam?
	392	T: Anywhere. On your book or on the back of the sheet or whatever.
	393	L: Mam is this Admaths or normal maths?
	394	T: This is grade eleven maths.
	395	L: So the other sets aren't doing it?
	396	T: Set three are doing it and set one.
	397	L: But Mam it won't be in our err normal maths exam?
	398	T: [shakes her head]
	399	L: What happened to the minus sign?
	400	T: It is almost like taking that [Pointing to $-(x-3)^2$] over to that side. I could have said minus three *x* squared equals minus four. I just skipped out a step.
	401	L: [discussion with T but not audible]
41:04	402	T: With this one that's what I was doing here. You get a minus *x* minus two [inaudible amidst the chatter]. You factorise that *x* squared minus two *x* and you get [inaudible] …
	403	T: [indistinct]

	404	L: Is four the *y*-intercept?
	405	T: It's not the *y*-intercept. It is the y-coordinate. Okay.
	406	T: That's good K.
	407	T: S. How are you getting on here?
	408	L: [inaudible]
	409	T: [moves to another student] I know. I printed out the wrong page. So I want you to do those three over there.
	410	L: [inaudible]
	411	T: I have done the one. [indistinct] The *y* equals two *x* minus one squared minus eight and the other one I've got there is *x* minus three squared plus four. They are not on this sheet. Unfortunately I printed the wrong one. [explains what is to be done indistinct]. Okay. You are going to have *y* equals minus *x* minus three squared plus four and *y* equals two [indistinct] Straight away you can get write down the turning point. Can't you? Three minus sixteen. It's opposite signs. [indistinct] And then you need to times out to get the *x*-intercepts. Okay.
43:08	412	T: Right. Now Benson. What have you been doing the whole lesson?
	413	L: Mam. I'm busy. I'm trying to work out this relationship. If you have opposite [indistinct]
	414	T: Okay good. Okay.
	415	T: [next S] Have you figured this out now?
	416	L: Ja.
	417	T: What did you have to do?
	418	L: [Explanation not audible]
	419	T: Brackets
	420	L: Yes. Can you have two *y*s two y-intercepts and the value of …
	421	T: Uhm. Okay. You can have two *y*s but then the graph is going to be around this way and that's is an inverse graph. Okay.
	422	L: Do we do that?
	423	T: We do that but not quite yet. Then you can have one intercept. Where the graph shifts. Where would you have one intercept? It will look like this. It will just touch. Isn't. Let me show you. [indistinct] Then we talk about two equal roots. Okay. [T explains drawing graph on paper] And then a graph going this way round is going to have two *y*-intercepts.
	424	L: [indistinct]
	425	T: You can have no you can because there is an equation with no *x*-intercepts
	426	L: [comment]
	427	T: No because you always have one or the other. Okay.
	428	T: Right. Graham how are you getting on?
	429	L: No I am good. I am a bit confused though. You know how you get nought.

		You change the *x* to nought …
	430	T: You can do that. That is lovely. You don't have to multiply out.
	431	L: You come here and that is minus …
	432	T: Alright then multiply out. You know how to solve something that doesn't factorise. What do you do?
	433	L: You …
	434	T: You use the …
	435	L: … factorise
	436	T: If it doesn't factorise nicely.
	437	L: Oh yes you have to use that …
	438	T: Formula [emphasises the word].
45:00	439	L: [discussion continues]
	440	T: [addressing another L] Alright. The turning point you just have to use that and that …
	441	S; [inaudible]
	442	T: So its one and negative eight.
	443	L: Why is it two zero.
	444	T: Why because this is always positive. For that to have a minimum value *x* has to be one. There is a bit of the shifting thing alright. If this is minus one it actually goes to plus one. Okay.
	445	L: What about the two?
	446	T: The two just makes it. The turning point no matter what *a* is the turning point could be here it could be here. It could be here [Gestures] The two makes it wider or narrower. The two just affects the [uses hands] The two doesn't affect the coordinates of the turning point.
	447	L: Thanks Mam.
	448	L: [asks question but not audible]
	449	T: Not quite because you have an *x* –intercept. You've got a *y*-intercept. What about your turning point. Your turning point is one minus eight. [indistinct] It goes like this. [T draws graph]
	450	L: [inaudible]
	451	L: I have got a point on the … [Another L discusses]
	452	T: [moves to answer a question where the L had his hand up] *y* is negative …
	453	L: [S discussion intensifies] … minus *x*.
	454	L: Let me see your *x* values. How you do the *x* …
	455	L: Cool.
46:32	456	T: [assisting another S] One minus …
	457	L: [Discussion continues] … minus three squared …

	458	L: Straight straight straight … the project [general discussion].
	459	T: I'll show you now. Okay. So the turning point. If we look at this form of the point then x is one and y is minus eight. We call that p q. And the way you can think of this. You can either think of my original graph goes down eight and the one is the opposite sign [points to /-1/ in $y = 2(x-1)^2 - 8$] or you can say this is always positive. That has a minimum value of nought. And for this to be nought x must be one. That's the opposite sign okay. Good. Right.
47:22	460	T: Right Ryan are you winning? Okay
	461	T: Right guys!
	462	L: Hold on maam.
	463	T: Just hang on.
	464	T: It is nearly the end of the lesson. How many of you have got no clue what's going on?
	465	L: Ja [indistinct]
	466	T: Sorry?
	467	L: Are we going to do more of this stuff?
	468	T: We are going to do a lot more. You will get it one day don't worry.
	469	T: okay. Right. Now for homework tonight. What I have done is I have shown you the graph in this form and how to plot it. I hope you have written these two down.
	470	L: Yes mam.
	471	T: I am going to email you the other sheet. In my haste when I quickly printed the stuff I prepared last night it didn't print. On Thursday we don't have a lesson tomorrow. We will look at this next sheet over here. So. For homework tonight. Are you all listening?
	472	L: Yip.
	473	L: Yes mam.
	474	T: Tom. ..Your homework tonight is please to finish this sheet. I know I rushed things a bit. Some of you have finished it. And to plot these two graphs. Okay. You can also work on your exponents. Who didn't do homework last night?
	475	L: Maam. I don't understand it.
	476	T: Well you can read it in the textbook
	477	L: [general discussion as class ends and laptops are packed away and S leave]
	478	T: I hope that was alright … ha ha.
		Lesson ends at 48:36

Appendix 6.2

Lesson Analysis: Sara Lesson 1 (S01T01L01)

A general description of the lesson

The teacher starts the lessons with revision of the basic functions $y = x; y = \frac{1}{x}; y = 2^x; y = x^2$. The revision entails associating names and graphs with equations of the basic functions and discussing the effect of the parameter a and q on the basic graphs. The rest of the lesson focuses on the parabola. Learners explore the effects of the parameters a, p and q on the basic function $y = x^2$ using the computer software, *Geogebra*. The teacher introduces learners to the turning point form of the parabola equation $y = a(x-p)^2 + q$ and the standard form $y = ax^2 + bx + c$. They then generate graphs from the turning point equation $y = a(x-p)^2 + q$.

Segmenting the lesson [duration: 48 min 36s]

Table 1 and Table 2 show how time was spent during the lesson. The teacher and her learners spent about 93% of the lesson on mathematics. The remaining time was spent on learners setting up their lap tops which were to be used in the lesson.

Table 1. Evaluative events and sub-events (EE) S01T01L01

EE	Start Time	Duration	Transcript lines	Activity	Type
NES	00:00	03 min 10s		Settling in to class, setting up lap tops.	
EE1	03:10	05 min 58s		Revision of basic functions $y = x$; $y = \frac{1}{x}$; $y = a^x$; $y = x^2$	EXP
EE2	09:08			Investigating $y = a(x-p)^2 + q$	
EE2.1	09:08	04 min 12s		Setting up sliders in *Geogebra*	EXP
EE2.2	13:20	07 min 39s		Learners investigate $y = a(x-p)^2 + q$ using *Geogebra* to establish the effect of *a*, *p* and *q* on the basic parabola	CWK
EE2.3	20:59	07 min 08s		Discussing the effect of *a*, *p* and *q* for the function $y = \mathrm{a}(x-p)^2 + q$	EXP
EE2.4	28:07	02 min 26s		Finding the turning point from $y = a(x-p)^2 + q$	EXP
EE3	30:33	18 min 03		Sketching $y = -(x-3)^2 + 4$	
EE3.1	30:33	05 min 04s		Drawing the graph of $y = -(x-3)^2 + 4$	CWK
EE3.2	35:37	04 min 09s		Drawing the graph of $y = -(x-3)^2 + 4$	EXP
EE3.3	39:46	08 min 50s		Drawing the graph of $y = (x-1)^2 + 8$ and $y = (x-3)^2 - 16$	CWK

Table 2 indicates approximately two thirds (28 minutes 19s or 62,3%) of the lesson was spent on exposition by the teacher and a third (17 minutes 7 seconds) on learners working on mathematical tasks.

Table 2. Evaluative events and sub-events (EE) S01T01L01

EE	Start Time	Duration	Transcript lines	Activity	Type
NES	00:00	03 min 10s		Settling in to class, setting up lap tops.	PO*
EE1	03:10	05 min 58s		Revision of basic functions $y = x$; $y = \frac{1}{x}$; $y = a^x$; $y = x^2$	NPE
EE2	09:08				NPE
EE2.1	09:08	04 min 12s		Setting up sliders in *Geogebra*	
EE2.2	13:20	07 min 39s		Learners investigate $y = a(x-p)^2 + q$ using *Geogebra* to establish the effect of a, p and q on the basic parabola	
EE2.3	20:59	07 min 08s		Discussing the effect of a, p and q for the function $y = \mathrm{a}(x-p)^2 + q$	
EE2.4	28:07	02 min 26s		Finding the turning point from $y = a(x-p)^2 + q$	
EE3	30:33	18 min 03		Sketching	PE
EE3.1	30:33	05 min 04s		Drawing the graph of $y = a(x-p)^2 + q$	
EE3.2	35:37	04 min 09s		Drawing the graph of $y = a(x-p)^2$	
EE3.	39:46	08 min 50s		Drawing the graph of $y = a(x-p)^2$	

*PO refers pedagogic organisation, PE to procedural evaluative event and NPE to non-procedural evaluative event

Consulting the curriculum

The content covered can be broadly aligned with the FET topic Functions. An overview of the topic is shown below (FET CAPS: 12).

3.1.1 Overview of topics

1. FUNCTIONS		
Grade 10	**Grade 11**	**Grade 12**
Work with relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and some quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.	Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.	Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear, quadratic and some cubic polynomial functions, exponential and logarithmic functions, and some rational functions.
Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effect of the parameter which results in a vertical shift and that which results in a vertical stretch and /or a reflection about the *x* axis.	Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the *y* axis.	The inverses of prescribed functions and be aware of the fact that, in the case of many-to-one functions, the domain has to be restricted if the inverse is to be a function.
Problem solving and graph work involving the prescribed functions.	Problem solving and graph work involving the prescribed functions. Average gradient between two points.	Problem solving and graph work involving the prescribed functions (including the logarithmic function).

Figure 1. Extract from FET CAPS on functions (DBE, 2011, p.12)

The curriculum expects learners to have covered linear functions in Grade 9. Functions considered in grade 10 include the straight-line, parabola, hyperbola and exponential function. The curriculum advocates an inductive approach to functions and recommends the use of technology as an additional resource. Through point-by-point plotting of the basic functions, $y = x^2, y = \frac{1}{x}$ and $y = b^x$ $(b > 0, b \neq 1)$, where applicable learners are expected to make conjectures regarding the shape, domain, range, axes of symmetry, asymptotes and intercepts on axes for each of the basic functions. In addition, learners are to investigate the effects of parameters a and q on the graphs defined by $y = a.f(x) + q$. In other words, learners are expected to stretch graphs of functions vertically, reflect in the x-axis and vertically translate functions (FET CAPS, p.12 & p.24).

It appears that the curriculum favours an approach that focuses on a general study of functions rather than the study of specific functions such as the parabola, hyperbola or exponential functions. In other words, the curriculum is organised around discovering features of functions shape, domain, range, axes of symmetry, asymptotes and intercepts on axes and the effect of parameters on functions. The specific function types therefore serve to illustrate general ideas about functions.

The teacher, however, approaches functions by focusing on particular function types rather than using the general approach to functions advocated by the curriculum. Table 3 and Table 4 show the sub-topics for the topic functions in Grade 10 and Grade 11 respectively. As discussed earlier, the teacher did not start with the Grade 10 content with regards to the parabola. That is, she did not start with specific case $y = ax^2 + q$ but with the general parabola $y = ax^2 + bx + c$ and $y = a(x + p)^2 + q$. So, the sub-topics dealt with in Lesson 1 are shown in Table 2 below.

Table 3. CAPS Grade 10 Function sub-topics covered in Lesson 1

Grade 10 sub-topics
Function concept - using different representations such as tables, graphs, words and formulae to investigate how output values depend on how input values vary.
Convert flexibly between representations such as tables graphs, words and formulae
Features (shape, domain, range, axes of symmetry, asymptotes and intercepts on axes) of basic graph $y = x^2$ established through point-by-point plotting
Features (shape, domain, range, axes of symmetry, asymptotes and intercepts on axes) of basic graph $y = \frac{1}{x}$ established through point-by-point plotting
Features (shape, domain, range, axes of symmetry, asymptotes and intercepts on axes) of basic graph $y = b^x, b > 0, b \neq 1 \text{established through point} - \text{by} - \text{point plotting}$
Effect of a and q on the graphs defined by $y = a.f(x) + q$ where $f(x) = x, f(x) = x^2, f(x) = \frac{1}{x}$ and $f(x) = b^x, b > 0, b \neq 1$
Reflecting graphs defined by $y = a.f(x) + q$ in the x-axis
Translating graphs defined by $y = a.f(x) + q$ vertically

Sketching graphs defined by $y = a.f(x) + q$
Finding equations of given graphs defined by $y = a.f(x) + q$
Interpreting graphs defined by $y = a.f(x) + q$

Table 4. CAPS Grade 11 Function sub-topics covered in Lesson 1

Grade 11 sub-topics	**Lesson 1**
Investigating the effect of p on the graphs of the functions defined by: $f(x) = a(x+p)^2 + q, f(x) = \frac{a}{x+p} + q$ and $f(x) = ab^{x+p} + q, b > 0, b \neq 1$	x
Sketching graphs of functions defined by $f(x) = a(x+p)^2 + q$*	x
Sketching graphs of functions defined by $f(x) = \frac{a}{x+p} + q$	
Sketching graphs of functions defined by $f(x) = ab^{x+p} + q, b > 0, b \neq 1$	
Finding equations of functions defined by $f(x) = a(x+p)^2 + q$*	x
Finding equations of functions defined by $f(x) = \frac{a}{x+p} + q$	
Finding equations of functions defined by $f(x) = ab^{x+p} + q, b > 0, b \neq 1$	
Interpreting graphs functions defined by: $f(x) = ax + q, f(x) = a(x+p)^2 + q, f(x) = \frac{a}{x+p} + q$ and $f(x) = ab^{x+p} + q, b > 0, b \neq 1$	

* This implies working with $f(x) = ax^2 + bx + c$ because learners are expected to be able to complete the square to convert $ax^2 + bx + c$ to $a(x + p)^2 + q$.

Prior knowledge required for sketching parabolas of the equations in the form of $y = ax^2 + bx + c$ and $y = a(x+p)^2 + q$ include factorising trinomials, solving quadratic equations, plotting points on the Cartesian plane and substituting values into an expression. Completing square is required to convert $y = ax^2 + bx + c$ to the form $y = a(x+p)^2 + q$.

According to the Grade 10 pace-setter, the topic Functions is scheduled for the first five weeks of the second term, with the fifth week set aside for trigonometric functions. The teacher therefore does not follow the CAPS pace-setter nor the sub-topics stipulated by CAPS.

Consulting the Mathematics encyclopaedia

Functions

Saunders Mac Lane (1986, p.129) formally defines a function as follows:

> A *function f* on the set X to the set Y is a set $S \subset X \times Y$ of ordered pairs which to each $x \in X$ contains exactly one ordered pair <*x*, *y*> with first component *x*. The second component of this pair is the value of the function *f* at the argument *x*, written *f*(*x*). We call *X* the *domain* and *Y* the *codomain* of the function *f*.

This definition, Mac Lane argues, covers all pre-formal definitions of a function: a formula, a rule, a graph, a table of values and functional dependence - each of which are deficient in some way (see Mac Lane (1986: 126-128)). CAPS (FET) focus on the pre-formal definitions described by Mac Lane and delays the formal definition of a function to Grade 12.

Schooling deals with the set of Reals as the domain and codomain of functions. So functions considered in CAPS are sub-sets of mappings from $\mathbb{R}$ to $\mathbb{R}$ i.e. a function is a subset of $\mathbb{R} \times \mathbb{R}$. The particular functions considered in Grade 10 are the parabola, hyperbola and exponential functions that are particular regularities of subsets of $\mathbb{R}^2$.

Parabola

2.3.2 Quadratic Polynomial

The *polynomial of second degree*

$$y = ax^2 + bx + c \tag{2.41}$$

(quadratic polynomial) defines a *parabola* with a vertical axis of symmetry at $x = -\dfrac{b}{2a}$ **(Fig. 2.12)**. For $a > 0$ the function is first decreasing, it has a minimum, then it is increasing again. For $a < 0$ first

it is increasing, it has a maximum, then it is decreasing again. The intersection points A_1, A_2 with the x-axis, if any, are at $\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right)$, the intersection point B with the y-axis is at $(0, c)$. The extremum point of the curve is at $C\left(-\dfrac{b}{2a}, \dfrac{4ac - b^2}{4a}\right)$ (for more details about the parabola see 3.5.2.8, p. 203).

(Bronshtein et al., 2007. pp. 62-63)

For Archimedes, graphs of the parabola, hyperbola and circle emerged from conic sections. The parabola is a conic section created from the intersection of a cone and a plane that is not perpendicular to the axis of the cone. Thus the shape of a parabola is a consequence of the shape produced from the intersection of a cone and a plane that is not perpendicular to the axis of the cone.

A parabola can also be considered as the locus of points that are equidistant from both the directrix and the focus point. The line perpendicular to the directrix and passing through the focus (that is, the line that splits the parabola through the middle) is called the "axis of symmetry". The point on the axis of symmetry that intersects the parabola is called the "vertex", and it is the point where the curvature is greatest. (Wikipedia entry)

According to Fermat, any graph can be described in terms of an equation. In $\mathbb{R}^2$, all parabolas can be described by a quadratic equation of the form $P = \{(x; y) | y = ax^2 + bx + c\}$ with fixed real coefficients a, b and c such that $a \neq 0$. Important properties of parabolas are their x-intercepts, y-intercept and turning point.

The school curriculum does not deal with conic sections or calculus as resources to generate graphs of functions. So mathematical necessity is compensated for in some or other way. The curriculum advocates an inductive approach to functions. Through point-by point plotting, the shape of the graph is meant to emerge.

> The *polynomial* of *second degree* $\mathrm{y} = \mathrm{ax}^2 + \mathrm{bx} + \mathrm{c}$ (quadratic polynomial) defines a parabola with a vertical axis of symmetry at $\mathrm{x} = \frac{-\mathrm{b}}{2\mathrm{a}}$. For $\mathrm{a} > 0$ the function is first decreasing, it has a minimum, then it is increasing again. For $\mathrm{a} < 0$ the function is first increasing, it has a maximum, then it is decreasing again. (Bronshtein et al., 2007, pp. 62-63)

Translations

> [A] geometric transformation is a FUNCTION that associates with each point in the plane with some other point in the plane. […] Any geometric transformation that preserves distances between points in the plane (and hence the shape and size of geometric figures) is called an isometry or a rigid motion. One that multiplies all distances between points by a constant factor (called the dilation factor) is called a similitude, and a transformation that takes straight lines to straight lines is called a LINEAR TRANSFORMATION. (Tainton, 2005: 224, capitals original)

> A geometric transformation that moves all points in the plane a fixed distance in a fixed direction is called a translation. […] In a Cartesian coordinate system, a translation takes a point with coordinates $(x; y)$ to the point $(x + a; y + b)$ for some fixed values *a* and *b*. (Tainton, 2005, p. 225)

Reflections

> Given a line *l* in the plane, a reflection in this line takes a point P on one side of *l* to the corresponding point P' on the opposite side of *l* such that the segment connecting P to P is PERPENDICULAR to *l* and bisected by it. The points on *l* itself are left unmoved. Any reflection is an isometry that transforms geometric figures to their mirror images. The line used in performing the reflection is called the line of reflection.

> In a CARTESIAN COORDINATE system, a reflection about the *x*-axis takes a point with coordinates (*x*,*y*) to the point (*x*,–*y*), and a refection about the *y*-axis changes the sign of the *x*-coordinate: (*x*,*y*) becomes (–*x*,*y*). (Tanton, 2005: 224 - 225, capitals original)

Dilation

> A dilation with center O and dilation factor k > 1 is the geometric transformation that leaves O fixed, and moves any point P further away from O, by a factor *k*, along the ray from O through P. Thus a dilation stretches figures uniformly outward from O. It is possible, for example, to convert a square into a rectangle via a dilation. (A dilation with dilation factor *k* between O and 1 “shrinks” all points closer to O.) A dilation is not an isometry. (Tanton, 2005:225)

Consulting the textbook

The teacher and learners use the textbook series *Platinum maths*. However, the content covered in the three lessons form part of the Grade 11 CAPS.

Describing the computational activity: EE1

In Evaluative event 1, the teacher revises content taught previously. She starts by reminding learners of the following general equations: $y = ax + p$ (straight line), $y = \frac{a}{x} + p$ (hyperbola); $y = a^x + p$ (exponential function); $y = ax^2 + p$ (parabola). She reminds them that the equations are all functions and asks them to define the term *function*. The learners do not recall what a function is. One learner refers to a function as "*f* of *x*" (S01T01L01: line 38). Thus merely associating the notation with the definition of a function. The teacher "defines" a function as "something that for every *x* there is one corresponding *y*-value" (S01T01L01: line 39). Her "definition" of a function is an imprecisely stated version of Saunders Mac Lane's definition of a function, so therefore does not function as a definite description.

> A *function f* on the set *X* to the set *Y* is a set $S \subset X \times Y$ of ordered pairs which to each $x \in X$ contains exactly one ordered pair <*x*, *y*> with first component *x*. The second component of this pair is the value of the function *f* at the argument *x*, written *f*(*x*). We call *X* the *domain* and *Y* the *codomain* of the function *f*. (Mac Lane, 1986, p.129)

She identifies equations of what she refers to as "simple" graphs or "mother" graphs of the function types identified above: $y = x$; $y = \frac{1}{x}$; $y = 2^x$; $y = x^2$. She draws rough sketches of the functions and asks learners to recall the names of the functions. The equations are identified as straight line, hyperbola, exponential graph and parabola respectively.

The teacher identifies the letters a and p as parameters but does not define what a parameter is. She reminds learners that the parameter p has the effect of shifting the "mother graph" of the function up or down. So a parameter seems to mean a value in the equation that effects some sort of transformation of the basic function. The teacher's use of the term parameter substitutes for the definition of a parameter as a "variable that can be varied or changed" (Wolfram Mathworld website). The teacher's propositions can be stated as follows: If p is positive the "mother graph" moves up and if p is negative then the "mother graph" moves down. The teacher's propositions regarding the vertical shift of graphs stands in place of the formal notion of a translation which is a function that maps all the coordinate points (*x*;*y*) of the function with a new set of points (*x*; *y*+*p*) for $p \in \mathbb{R}$. The teacher refers to the effect of p as a translation of the graph itself. Thus the operation takes the graphical image *f* as its input and produces the graphical image f' as its output.

She describes the effect of the parameter a as a "vertical stretch". Thus the effect of a, is to "move" the graph "in or out" . The teacher' proposition regarding the effect of a substitutes for formal notion of dilation which is a function that maps all the coordinate points (*x*;*y*) of the function to a new set of points $(x; ay)$ for $a \in \mathbb{R}$. The teacher refers to the effect of a as a "vertical stretch" of the graph itself. Thus the operation takes the graphical image *f* as its input and produces the graphical image f' as its output.

If $a < 0$, the teacher states the hyperbola "flips into the other two quadrants". The teacher's proposition substitutes for formal notion of reflection in the x-axis which is a function that maps all the coordinate points $(x; y)$ of the function to a new set of points $(x; -y)$ for $a \in \mathbb{R}$. She reminds learners that the hyperbola and exponential graph have asymptotes. They do not define what an asymptote is (presumably covered previously) but identify what the asymptotes for each function type are.

Although the teacher provides a description of a function, it is not clear that she uses the notion of a function. Functions are not discussed as subsets of $\mathbb{R}\times\mathbb{R}$ but rather as graphs with equations. However, the notion of a function as a subset of $\mathbb{R}\times\mathbb{R}$ falls outside the notion of function used in school mathematics. Secondly, the transformational effects of the parameters a and p on the "mother graphs" as treated by the teacher differs that of Tainton (2005)

> [A] geometric transformation is a FUNCTION that associates with each point in the plane with some other point in the plane. […] Any geometric transformation that preserves distances between points in the plane (and hence the shape and size of geometric figures) is called an isometry or a rigid motion. One that multiplies all distances between points by a constant factor (called the dilation factor) is called a similitude, and a transformation that takes straight lines to straight lines is called a LINEAR TRANSFORMATION. (Tainton, 2005: 224, capitals original)

> A geometric transformation that moves all points in the plane a fixed distance in a fixed direction is called a translation. […] In a Cartesian coordinate system, a translation takes a point with coordinates $(x; y)$ to the point $(x + a; y + b)$ for some fixed values *a* and *b*. (Tainton, 2005, p. 225)

Tainton (2005) conceives of transformations as functions or mappings from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$ whereas the teacher's propositions suggests that transformations are physical actions on the graphs themselves. The parameter p has the effect of shifting the "mother graph" vertically and the parameter a results in the "mother graph" "moving in or out". School mathematics does not treat transformations as mappings from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$. Instead transformations are considered as operations on the objects themselves such as translating, reflecting or rotating a geometric object.

Primary data production for EE1

Descriptions

In the teacher's discussion of functions, she 'defines' a function as "something that for every x there is one corresponding y-value" (S01T01L01: line 39). Her "definition' functions description rather than as a definite description. This means she is employing a **non-iconic auxiliary description**.

Propositions:

1. For the function $y = a.f(x) + p$, the parameter p causes the graph $f(x)$ to shift vertically. If $p > 0$ then $f(x)$ moves p units upwards and if $p < 0$ then $f(x)$ moves p units downwards. (**encyclopaedic proposition**)

2. For the function $y = a.f(x) + p$, the parameter a stretches the graph of $f(x)$. The value of a results in $f(x)$ becoming “steeper” or “flatter”. (**iconic auxiliary proposition**)
3. If $a < 0$ then $f(x)$ “flips” in the x-axis. (**encyclopaedic proposition**)

Summary of computational activity

Descriptions	**Procedure**	**Propositions**	**Domain, codomain & operations**
non-iconic auxiliary description	No procedures	Encyclopaedic and iconic auxiliary propositions	None

Secondary data production for EE1

Realised content	**Ground/regulation**	**Closed/open pedagogic texts**	**Orientation to mathematics**
Symbiotic Encyclopaedic propositions and auxiliary descriptions and propositions are used alongside each other.	Iconic fundamental	Text is open because encyclopaedic propositions recruited. Openness weakened by the inclusion of the iconic (T_O^-)	Content-oriented because encyclopaedic propositions recruited but weakened by the inclusion of the iconic (O_c^-)

Describing the computational activity: EE2

In this event the teacher focuses on the parabola. She asserts that the equation $y = a(x - p)^2 + q$ represents a parabola and that the task is to investigate the effect of p, given that they already know that q has the effect of moving the graph up or down.

In Evaluative event 2.1 the teacher explains how to set up a slider in *Geogebra* to explore the effects of parameters on the graphs of functions. She starts with the function $y = bx$ and sets up a slider for the parameter b shown in Figure 2.

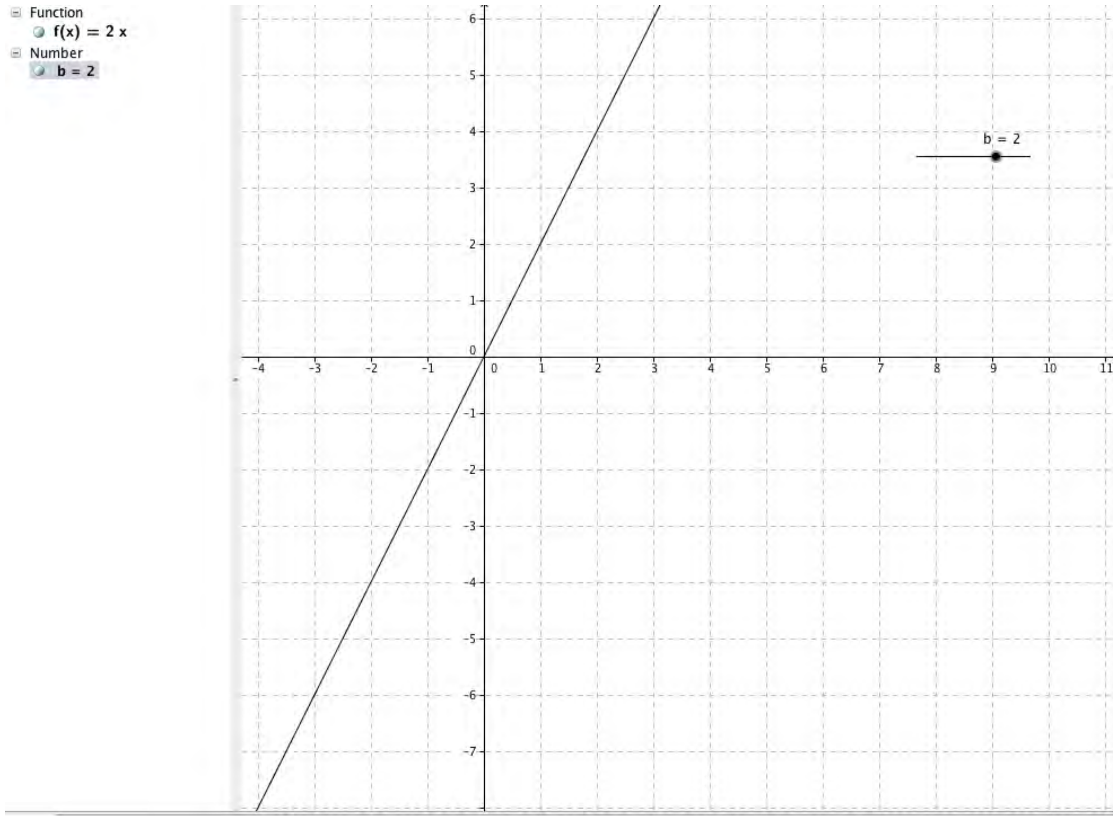


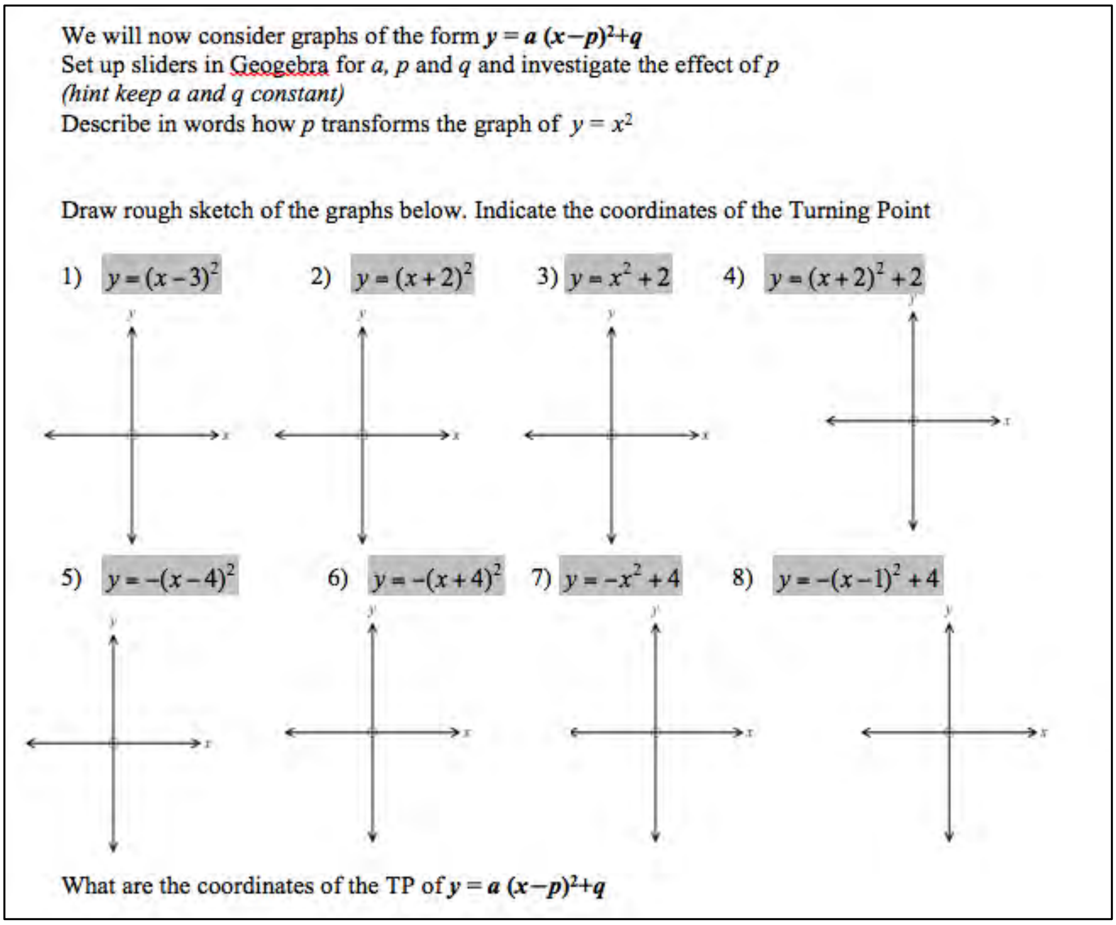
Figure 2. Geogebra screenshot of the function $y = bx$

The teacher demonstrates that by dragging the slider, the value of b changes. She states that when b is one the graph "changes around" which refers to the gradient of the graph changing from positive to negative. However, the graph does not "change" when b is one. Given that the teacher set the slider's increment to 1, the graph $y = 0$ is displayed when b = 0 and a graph $y = -x$ is displayed when b = -1.

The purpose of the example is essentially to demonstrate how to set up a slider to explore the effect of p in $y = a(x - p)^2 + q$.

In EE2.2, learners use *Geogebra* to explore the effect of a, p *and* q. The worksheet (Figure 3) provided by the teacher is shown in Figure 2. She uses an inductive pedagogy where learners have to the "discover" the effects of the parameters. She restricts a to 1 and -1 therefore only focusing on reflecting the parabola in the x-axis.

Many of the learners struggle with *Geogebra's* syntax, so have difficulty in setting up sliders to vary the parameters. The teacher advises those learners to plot each graph separately.



We will now consider graphs of the form $y = a\,(x-p)^2+q$
Set up sliders in Geogebra for a, p and q and investigate the effect of p
(hint keep a and q constant)
Describe in words how p transforms the graph of $y = x^2$

Draw rough sketch of the graphs below. Indicate the coordinates of the Turning Point

1) $y=(x-3)^2$ 2) $y=(x+2)^2$ 3) $y=x^2+2$ 4) $y=(x+2)^2+2$

5) $y=-(x-4)^2$ 6) $y=-(x+4)^2$ 7) $y=-x^2+4$ 8) $y=-(x-1)^2+4$

What are the coordinates of the TP of $y = a\,(x-p)^2+q$

Figure 3. Worksheet on parabola $\boldsymbol{y = a(x-p)^2 + q}$

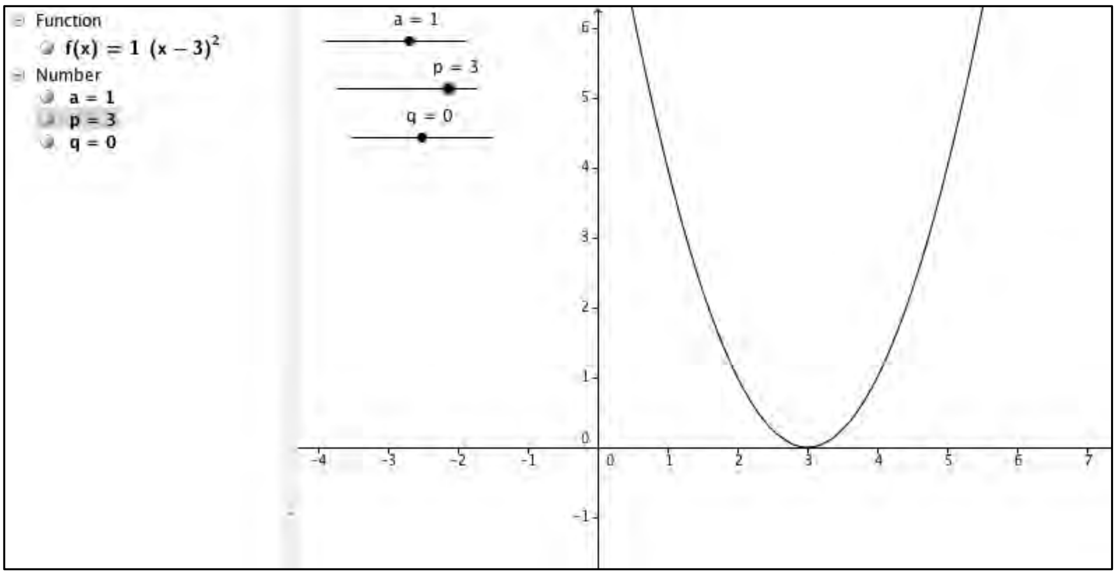


Figure 4. *Geogebra* screenshot of $\boldsymbol{y = (x-3)^2}$

In evaluative event 2.3 the teacher elicits the learners' "discoveries". They establish that changing parameter a results in the graph becoming "wider" or "narrower" and that when a is negative, the graph is reflected in the x-axis. The teacher uses the *Geogebra* slider which varies the value of a to verify the proposition suggested by a learner and herself. Similarly, she establishes that the parameter q results in the graph moving up or down. She reminds learners that they have already established the effects of a and q in previous graph work.

The teacher then moves on to discuss the effects of the parameter p. The learners state that if p varies, the graph shifts to the left or right. In order to work out when the graph moves to the right and when the graph moves to the left, the teacher focuses on setting p equal to different values using *Geogebra*. Starting with the function $y = x^2$, that is with $a = 1$, $q = 0$ and $p = 0$, she generates a graph by changing the value of p to 1 and keeping the other parameters constant. She also writes the equation of the function when $p = 1$ on the board, that is, $y = (x - 1)^2 + 0$ and asks learners for the turning point which they read from the graph. The learners state that the turning point is 1 indicating that they are not identifying points on the graph as coordinate pairs. They are merely reading the value associated with the turning point shown on the graph. The teacher does not explicitly correct the learners. She merely states that the turning point is (1;0). She then asks the learners to identify the turning point for the graph $y = (x - 2)^2$ without showing the graph. Learners state that the turning point is 2. Again, the teacher does not correct them but provides the correct answer as (2;0). With the third example $y = (x + 3)^2$ a learner correctly states the turning point as (-3;0). It appears that the teacher has shifted attention away from the value of p since she does not mention the value of p in the latter two examples.

A learner questions whether the turning point of the graph $y = (x - 2)^2$ should be four. His confusion indicates that the rule he established for the first example was obtained by squaring the value of p, which coincidentally produced the correct answer in the first example but failed in the second example. The teacher uses *Geogebra* to verify that if $p = 2$, the turning point is (2;0) and to illustrate to the learner that it is the y-intercept that is 4. She thus uses an empirical approach to 'prove' to the learner that his rule is incorrect and that $y = (x - 2)^2$ can be written as $y = x^2 - 4x + 4$ and that the y-intercept is in fact 4 not the turning point. The teacher uses the opportunity to show that $y = a(x - p)^2 + q$ which she refers to as the turning point form of the equation can also be written as $y = ax^2 + bx + c$ referred to as the standard form.

The teacher's attempt to develop a procedure for working out whether the graph moves right or left and what the turning point of the parabola is illustrated by the transcript below.

T: If I type in minus three okay then what happens? It becomes plus three. Doesn't it? Okay. Err if I make p equal to minus three then I get x minus minus three which is x plus three. You happy with that? Okay. And so if that is like that then the turning point is minus three nought. Okay. So. Who can tell me alright what is happening? Maybe I should just say it. If this [pointing /-2/ in $y = (x - 2)^2$] is minus two it's [referring to the turning point] the opposite sign. Okay. We call it counter intuitive. You might think that when p is two the graph is shifting …

S: To the right.

T: If it says minus two then the graph is not shifting to the left it is shifting to the?

S: Right.

T: Right okay. Now if you think about it this is where the minimum value or maximum value occurs and what is the minimum value that this [refers to $y = (x-2)^2$] can be if that expression is squared. The minimum value is?

S: Two

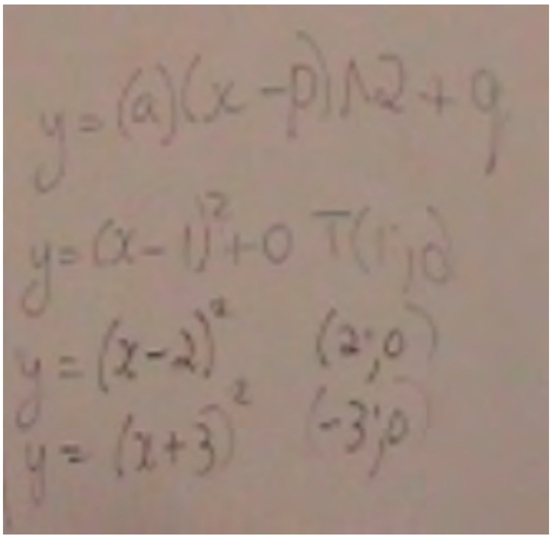
T: No.

S: One zero

T: Is zero. And for that to be zero what does x have to be?

S: Two.

T: *x* has to be two. Okay. So that's why the cursor is at *x* equals to two. So when you see uhm *y* equals *a x* squared *plus* p. If it is three it goes up three but for left and right shifting it is almost the opposite. Am I confusing you or do you understand that? (S01T01L01, lines 252 – 262)



It now becomes clear, that p is used to generate the parabola using *Geogebra* which already has the propositions for generating the graph encoded into its software and to produce the corresponding equation in the form $y = a(x-p)^2 + q$ from which an operation or operation-like manipulation is performed to produce the x-coordinate of the turning point: - the x-coordinate of the turning point is the "opposite sign" of the number in the bracket. So, the mathematical proposition which states that the function $y = a(x-p)^2 + q$ has a minimum or maximum value of q when $x = q$ or that the turning point of the parabola is $(p; q)$ is replaced with an operation-like manipulation which takes as its input the number and its sign in the bracket and changes the sign to the opposite sign. The teacher thus uses a quasi-inductive approach to develop an operation-like manipulation for determining the turning point from an equation of the form $y = a(x-p)^2 + q$. The approach is quasi-inductive rather than a case of true induction because the result is known upfront and is used to structure the mathematical activity.

With respect to the horizontal translation of the parabola, the teacher avoids focusing on the value of p as indicated in the transcript above. She states that "If it says minus two then the graph is not shifting to the left it is shifting to the…". The "it" here refers to the number in the bracket rather than the value of p and thus the shift is "counterintuitive" because as she states that they would expect that if the number in the bracket is negative that the graph should shift to the left but in fact the shift is to the right. Her proposition regarding the horizontal shift of the parabola is thus an altered version of the mathematical proposition which states that for the function $y = a(x - p)^2 + q$, p effects a horizontal shift of the graph with a shift to the right if $p > 0$ and a shift to the left if $p < 0$.

The transformations involved identifying how the original graph $y = x^2$ has shifted from the equation $y = (x - 3)^2$ is shown in Table 8.

Table 5. Transformations involved in the translation of $y = x^2$ to $y = (x - 3)^2$

T	Input	Domain	Operation	Output	Codomain
1	$y = (x - 3)^2$	$\mathbb{R}$	Identify number in bracket	-3	$\mathbb{R}$
2	-3	$\mathbb{R}$	String	/-3/	$\mathbb{X}$
3	/-3/	$\mathbb{R}$	Sunder	/-/,/3/	$\mathbb{X}$
4	/-/	$\mathbb{X}$	Move "original" graph right	translated graph	$\mathbb{X}$
5	/3/	$\mathbb{X}$	Number	3	$\mathbb{R}$
6	3	$\mathbb{R}$	Move "original" graph 3 units right	translated graph	$\mathbb{R}$

The focus of evaluative event 2.4 is on determining the turning point of the graph $y = 2(x - 3)^2 + 4$.

Teacher: So we can say that in the original graph if we go back to *y* equals *x* squared What has happened. The original *y* equals *x* squared has shifted how many. .. How many?

Learner: The original is set at four Mam.

Teacher: No it is two. It is three four. How many has it shifted up?

Learner: Four.

Teacher: Four. And how many has it shifted left or right?

Learner: Three

Learner: Three

Teacher: Three. Has it shifted left or right?

Learner: Right.

Teacher: It has shifted right. Okay. And the two makes it. So let's check if we are right there. So we're going to start with p being three. Okay. And we making q equal to four. So there you can see the graph shifts up and if I make *a* equal to two. Now my picture is disappearing off my screen but it just makes it like that. Okay. Now what if I make *a* minus two? What's going to happen?

Learner: Going to go the other way.

(S01T01L01: lines 277 – 287)

Using proposition which involves the operation-like manipulation developed in the previous evaluative event, they conclude that the turning point is (3;4). The teacher explains using translations that the "original graph", that is $y = x^2$, has shifted 4 units up and three units to the right. So translations are used to justify that the turning point is (3;4). The transformations involved are shown in Table 9. She then uses *Geogebra* to verify the upward shift and shift to the right and to verify the turning point.

Table 6. Transformations involved in the translation of $\boldsymbol{y = x^2}$ to $\boldsymbol{y = 2(x-3)^2 + 4}$

T	Input	Domain	Operation	Output	Codomain
1	$y = 2(x-3)^2 + 4$	$\mathbb{R}$	Reading value of *q*	$q = 4$	$\mathbb{R}$
2	Graph	$\mathbb{R}$	Move up 4 units	vertically translated graph	$\mathbb{R}$
3	$y = 2(x-3)^2 + 4$	$\mathbb{R}$	Reading value in bracket	-3	$\mathbb{R}$
4	-3	$\mathbb{R}$	String	/-3/	$\mathbb{X}$
5	/-3/	$\mathbb{R}$	Sunder	/-/,/3/	$\mathbb{X}$
6	/-/	$\mathbb{X}$	Move vertically translated graph right	Horizontally translated graph	$\mathbb{X}$
7	/3/	$\mathbb{X}$	Number	3	$\mathbb{R}$
8	3	$\mathbb{R}$	Move vertically translated graph 3 units right	translated graph	$\mathbb{R}$
9	horizontally translated graph	$\mathbb{R}$	Read off turning point	(3;4)	$\mathbb{R}$

Propositions derived for the parabola $y = a(x-p)^2 + q$:

1. The parameter q causes the graph to shift vertically. If $q > 0$ then the graph moves q units upwards and if $q < 0$ then $f(x)$ moves q units downwards. (**Encyclopaedic proposition**)
2. The parameter *a* stretches the graph. The value of *a* results in the graph becoming "wider" or "narrower". (***Iconic auxiliary proposition***)
3. If *a* is negative, then the graph is reflected about the *x*-axis (**Encyclopaedic proposition**)
4. For the function $y = a(x-p)^2 + q$, if the number in the bracket (i.e. –*p*) is negative the graph of $y = x^2$ shifts right and left if the number in the bracket is positive (Iconic proposition). This proposition substitutes for the mathematical proposition (encyclopaedic proposition) which states that for the function $y = a(x-p)^2 + q$, the parameter *p* indicates a horizontal shift of $y = x^2$to the right by *p* units if $p > 0$ and to the left by p units if $p < 0$. ***(Iconic auxiliary proposition)***

Procedures

Procedure 1 Calculating TP using translations

1. Read off the value of q from the equation $y = 2(x-3)^2 + 4$. The value of $q = 4$.
2. Move the original graph $y = x^2$ upwards by 4 units.
3. Read off the value of the number in the bracket from the equation $y = 2(x-3)^2 + 4$. The number in the bracket is -3
4. Move the vertically translated graph 3 units to the right.
5. Read off the turning point of the graph.

The value of a is not used to determine the turning point since a has no computational effect on the turning point

Procedure 2 Calculating TP using operation "opposite sign" (ALT)

1. Read the number in the bracket: -3
2. Change the sign of /-3/ to produce /+3/
3. The x-coordinate of the turning point is the opposite sign, that is, +3
4. The y-coordinate of the turning point is 4.
5. The turning point is (3;4).

Structure: Operations, Domain and codomain

Operations: 1) move up/down 2) stretch in/out 3) shift left/right

The operation "move up/down" used by the teacher appears to be a synonym for the mathematical operation *translate vertically* (*translate$_v$*). The operation *translate$_v$* takes every point of the parabola with coordinates $(x; y)$ to the point $(x; y + a)$ where $a \in \mathbb{R}$. Thus, *translate$_v$* is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. Both *translate$_v$* and "move up/down" referred to as M_v from now onwards have the effect of translating $y = x^2$ a specified distance up or down in relation to the position of $y = x^2$. M_v, however, functions differently to *translate*.

The input elements of M_v derive from the constant term in the equation in the parabola $y = a(x-p)^2 + q$. For example /+4/ in $y = 2(x-3)^2 + 4$ is construed as a sign with a number, where the sign ($+$ or $-$) refers to the direction of the vertical shift and the number indicates the distance that the graph of $y = x^2$ is shifted up/down. The set of signs $\{+; -\}$ will be referred to as S and the set of natural numbers as $\mathbb{N}$. The input elements for M_v derive from set obtained from the cross product $S \times \mathbb{N}$. The set $P\ \{up, down\}$ refers to the direction of the move. The output elements for M_v are derived from the cross product of $\mathbb{N} \times P$. So $M_v: S \times \mathbb{N} \to \mathbb{N} \times P$. The application of M_v effects a translation of $y = x^2$ vertically, which can be represented as follows: $M_v: (+, a) \to (a, up)$ or $M_v: (-, a) \to (a, down)$ where $a \in \mathbb{N}$.

The operation "move in/out" has the same effect as the mathematical operation *dilate* or *compress* respectively but behave very differently. The operation *dilate* takes every point of the parabola with coordinates $(x; y)$ to the point $(x; ay)$ where $a > 1$ and the operation *compress* takes every point of the parabola with coordinates $(x; y)$ to the point $(x; ay)$ where $0 < a < 1$. In other words, *dilate* and *compress* are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. In contrast, the operation-like manipulations "move in" and "move out" treat their input objects as though one can physically move the arms of the parabola in or out.

The teacher's operation "move left/right" is underpinned by the proposition - for the function $y = a(x - p)^2 + q$, the graph $y = x^2$ shifts to the right if the sign in the bracket is negative and shifts left if the sign in the bracket is positive. The teacher's proposition functions as an alternate proposition to the mathematical proposition which states that for the function $y = a(x - p)^2 + q$, the parameter p indicates a horizontal shift of $y = x^2$ to the right by p units if $p > 0$ and to the left by p units if $p < 0$. The operation "move left/right" like the operation "move up/down" is a synonym for the mathematical operation *translate* but the direction of the translation is horizontal rather than vertical. The operation $translate_h$ is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ i.e. it takes every point of the parabola with coordinates $(x; y)$ to the point $(x + a; y)$ where $a \in \mathbb{R}$.

Both $translate_v$ and "move up/down" referred to as M_h, have the effect of translating $y = x^2$ a specified distance left or right in relation to the position of $y = x^2$. M_h, however, functions differently to $translate_h$. The input elements of M_v derive from the number together with its sign in the bracket of the equation in the parabola $y = a(x - p)^2 + q$. For example -3 in $y = 2(x - 3)^2 + 4$ is construed as a sign with a number, where the sign ($+$ or $-$) refers to the direction of the horizontal shift and the number indicates the distance that the graph of $y = x^2$ is shifted left/right. The set of signs $\{+;\ -\}$ will be referred to as S and the set of numerals as K. The input elements for M_h derive from set obtained from the cross product $S \times K$. The set P $\{left, right\}$ refers to the direction of the move. The output elements for M_h are derived from the cross product of $K \times P$. So $M_h: S \times K \to K \times P$. The application of M_v effects a translation of $y = x^2$ vertically, which can be represented as follows: $M_v: (+, a) \to (a, left)$ or $M_v: (-, a) \to (a, right)$ where $a \in \mathbb{N}$.

The use of an auxiliary operation**, "opposite sign" (ALT)** which implies an auxiliary operation, *sundering* which splits the character string. Sundering in turn implies an existential shift from numbers to characters so involves an auxiliary operation, *string.*

In summary auxiliary operations used along with operations from the field of the reals $(\mathbb{R}, +, \times)$

Primary data production for EE2

Descriptions	Procedure	Propositions	Domain, codomain & operations
None	See above	Encyclopaedic propositions and auxiliary propositions. (See below)	Auxiliary operations where domain and codomain are character strings $(\mathbb{R}, +, \times)$

Secondary data production for EE2

Realised content	Ground/regulation	Closed/open pedagogic texts	Orientation to mathematics
Symbiotic Encyclopaedic propositions and auxiliary propositions are used alongside each other.	Iconic Fundamental Empirical (computer software used inductively to derive general propositions)	Text is open because encyclopaedic propositions recruited. Openness weakened by the reliance on the empirical and iconic (T_O^-)	Content-oriented because encyclopaedic propositions recruited but weakened by the intrusion of the empirical and iconic (O_c^-)

Describing the computational activity: EE3

The focus of evaluative event 3.1 is on calculating the key points of a parabola (y-intercept, x-intercepts and turning point. She generates the graph of $y = -(x-3)^2 + 4$ using *Geogebra* and explains that the turning point can be read off the equation as $(3;4)$. The learners' task is to calculate the intercepts. Learners are left to figure out how to calculate the *x*-intercepts and *y*-intercept independently of the teacher using the computer generated sketch as a calculator checker.

In EE3.2, the teacher focused on procedures for calculating the *x* and *y*-intercepts of a parabola.

T: Now what do we have to do here? We have to multiply out okay to get the *y* intercept. So this is minus *x* squared minus six *x* plus nine plus four and you get. Sorry let's put *y* equals minus *x* squared plus six *x* minus nine plus four and the equation is minus *x* squared plus six *x* minus five. Are you all happy with that? And so if I make *x* nought what is *y* equal to?

L: Minus five. (S01T01L01: lines 358 – 359)

Her method for calculating the y-intercept of $y = -(x-3)^2 + 4$ entailed converting the equation to standard form before substituting *x* = 0. Here we have a case of the teacher regulating the learner. There is no computational necessity for converting the equation to standard form first to calculate the y-intercept except perhaps to minimise calculation errors on the part of the learner. The computational transformations involved in calculating the y-intercept is shown in Table 7.

Table 7. Calculating the y-intercept

T	Input	Domain	Operation	Output	Codomain
1	$(x-3)^2$	$\mathbb{R}$	Squaring	$x^2 - 6x + 9$	$\mathbb{R}$
2	$x^2 - 6x + 9$;-1	$\mathbb{R}$	Multiplication	$-x^2 + 6x - 9$	$\mathbb{R}$
3	-9;4	$\mathbb{R}$	Addition	-5	$\mathbb{R}$
4	$-x^2 + 6x - 5$	$\mathbb{R}$	Substitute $x = 0$	$-(0)^2 + 6(0) - 9$	$\mathbb{R}$
5	$(0)^2$	$\mathbb{R}$	Squaring	0	$\mathbb{R}$
6	0, -1	$\mathbb{R}$	Multiplication	0	$\mathbb{R}$
7	6, 0	$\mathbb{R}$	Multiplication	0	$\mathbb{R}$
8	0, 0	$\mathbb{R}$	Addition	0	$\mathbb{R}$

9	0, -5	ℝ	Addition	-5	ℝ
10	0,-5	ℝ	Pair	(0;-5)	ℝ

Table 8 shows the transformation involved in calculating the x-intercept from the turning point form of the equation (Method 1)

Table 8. Calculating the *x*-intercept (Method 1)

T	Input	Domain	Operation	Output	Codomain
1	$y = -(x-3)^2 + 4$	ℝ	Substitute $y = 0$	$0 = -(x-3)^2 + 4$	ℝ
2	$0 = -(x-3)^2 + 4$	ℝ	??	$(x-3)^2 = 4$	ℝ
3	$(x-3)^2$	ℝ	Square root	$x - 3$	ℝ
4	4	ℝ	Square root	± 2	ℝ
5	$x - 3 = 2$	ℝ	??	$x = 3 + 2$	ℝ
6	$x - 3 = -2$	ℝ	??	$x = 3 - 2$	ℝ
7	3,2	ℝ	Addition	5	ℝ
8	3,-2	ℝ	Addition	1	ℝ

The teacher asks how do we "get rid of the squared" to which the learner answers square root. The notion of an equation is absent as a regulative a resource, instead the steps in the procedure for getting $x = \cdots$ functions as the dominant regulative resource.

Although the teacher does not explicitly state the operation involved in transforming the equations in lines 2, 5 and 6. I suspect from statements involving equations later in the pedagogic script that she and her learners are implicitly using an operation-like manipulation "change side, change signs". See lines 399 - 400 when a learner asks the teacher to explain how she obtained $(x-3)^2 = 4$ from $0 = -(x-3)^2 + 4$.

Learner: What happened to the minus sign?

Teacher: It is almost like taking that [Pointing to $-(x-3)^2$] over to that side. I could have said minus three x squared equals minus four. I just skipped out a step.

This interaction between the teacher and the learner confirms that she uses an operation-like manipulation "take over, change sign" which she likely used in lines 5 and 6 in Table 8 to solve the two equations.

The values obtained for the x-intercepts by calculation are verified by the graph generated through *Geogebra*. The software is thus used as a calculation checking device. Necessity is thus located in the software.

Method 2 involved using the standard form to calculate the x-intercepts, that is using the equation $y = -x^2 + 6x - 5$. Here, the first step is to "make the x^2 positive" as a condition for solving the equation is shown in the transcript below. However, again there is no computational necessity for making x^2 positive before solving the equation. It is possible to factorise $-x^2 + 6x - 5$. Making x^2 positive appears to be motivated by transforming the expression into one which is more easily factorisable. Thus necessity lies external to the Mathematics encyclopaedia.

Teacher: Now some of you might have taken this expression our standard form expression. Minus x squared plus six x minus five is nought. How do we solve that?

Learner: x equal five and minus five.

Teacher: No. I know that's the answer.

Learner: Make that x squared. Take everything to the other side.

Teacher: Right. So we have to make x squared positive. Change all the signs. In other words we are multiplying everything by minus one. And now what do? It's a trinomial.

Learner: We try to factorise.

Teacher: Factorise. Right x x five one minus and minus. So x is one or x is five.

(S01T0101: lines 374 – 379)

It is interesting that the teacher suggests that to make x^2 positive they should "change all signs" as the operation in response to a learner's suggestion that they should "take everything to the other side". The operation-like manipulation "change all signs" is the same as the operation-like manipulation ALT described in EE2. ALT is seen as equivalent to the operation-like manipulation "change sides, change signs" by the learners. The teacher reminds learners that the operation "change all signs" (ALT) is the same as "multiplying everything by minus one". We have a case here where three different operations or operation-like manipulations are used equivalently. They are seen as equivalent because all three produce the same outcome. Furthermore, we observe how the teacher utilises auxiliary operations or operation-like manipulations in tandem with operations located in the field of the reals. So necessity is simultaneously located internal to mathematics as well as external to mathematics.

Table 9 shows the transformations involved in the calculation of x-intercepts from the standard form using mathematical operations and Table 10 shows the transformations involved in the calculation of x-intercepts from the standard form using auxiliary operations

Table 9. Calculating the x-intercept (Method 2) using mathematical operations

T	Input	Domain	Operation	Output	Codomain
1	$y = -x^2 + 6x - 5$	$\mathbb{R}$	Substitute $y = 0$	$-x^2 + 6x - 5 = 0$	$\mathbb{R}$
2	$-x^2 + 6x - 5 = 0$	$\mathbb{R}$	Multiply by -1	$x^2 - 6x + 5 = 0$	$\mathbb{R}$
3	$x^2 - 6x + 5 = 0$	$\mathbb{R}$	Factorise	$(x - 1)(x - 5) = 0$	$\mathbb{R}$
4	$(x - 1)(x - 5) = 0$	$\mathbb{R}$	??	$x = 1$ or $x = 5$	$\mathbb{R}$

Line 2 in Table 8 as discussed above shows the transformation as located in the field of the reals. The auxiliary counter-part of this operation is shown in Lines 2 and 3 of Table 9. The operation used in line 4 is not clear

Table 10. Calculating the *x*-intercept (Method 2) using auxiliary operations

T	Input	Domain	Operation	Output	Codomain
1	$y = -x^2 + 6x - 5$	$\mathbb{R}$	Substitute $y = 0$	$-x^2 + 6x - 5 = 0$	$\mathbb{R}$
2	$-x^2 + 6x - 5 = 0$	$\mathbb{R}$	String	$/-x^2 + 6x - 5 = 0/$	$\mathbb{X}$
3	$/-x^2 + 6x - 5 = 0/$	$\mathbb{X}$	"change sides, change sign" or "change all signs"	$/x^2 - 6x + 5 = 0/$	$\mathbb{X}$
4	$/x^2 - 6x + 5 = 0/$	$\mathbb{X}$	Sundering	$/x^2/$, $/-6x/$, $/5/$	$\mathbb{X}$
5	$/x^2/$	$\mathbb{X}$	Number	x^2	$\mathbb{R}$
6	x^2	$\mathbb{R}$	Factorising	x, x	$\mathbb{R}$
7	x, x	$\mathbb{R}$	String	$/x/,/x/$	$\mathbb{X}$
8	/5/	$\mathbb{X}$	Existential shift	5	$\mathbb{R}$
9	5	$\mathbb{R}$	Factorising	5, 1	$\mathbb{R}$
10	5, 1	$\mathbb{R}$	String	/5/,/1/	$\mathbb{X}$
11	/-/, /+/	$\mathbb{X}$	Sign	/-/, /-/	$\mathbb{X}$
12	$/(x-1)(x-5) = 0/$	$\mathbb{X}$	Number	$(x-1)(x-5) = 0$	$\mathbb{R}$
13	$(x-1)(x-5) = 0$	$\mathbb{R}$	??	$x = 1$ or $x = 5$	$\mathbb{R}$

Factorisation of the trinomial indicated in Table 9 line 3 involves a series of transformations which is elaborated in lines 4 to 11 in Table 10. As discussed, trinomial factorisation involves a number of auxiliary operations and a character distribution matrix or spatial template

Figure 5 shows the spatial frame marked with the spaces A to F which serve as placeholders for particular symbols or characters.

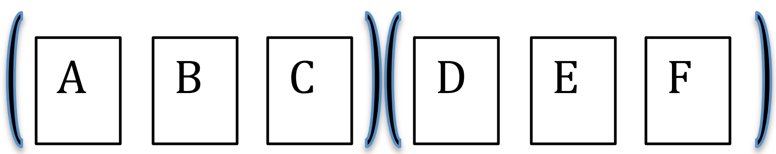


Figure 5. Quadratic trinomial template

A and D are placeholders for the symbol x, B and C are placeholders for signs + or – and C and F are places for numerals, in this case 5 and 1. So, there are three sets of characters that are required to populate the spatial frame, that is a three-stage process of generating characters to occupy spaces in the spatial frame. The generation of these symbols is elaborated in lines 4 to 11 in Table 11.

The operation-like manipulation involved in generating the signs which occupy places B and F have their roots in the axioms associated with addition and multiplication of integers. The set of integers together with binary operations addition and multiplication is considered to be a ring (Stewart & Tall, 1977, pp.172-174) . A ring has the following operatory properties:

Table 11. Operatory properties of the ring $(\mathbb{Z},+,\times)$ (Adapted from Stewart & Tall, 1977, p.172

Axioms	Properties
$\forall a,b \in \mathbb{Z}, a+b=b+a$	Commutativity of addition
$\forall a,b,c \in \mathbb{Z}, a+(b+c)=(a+b)+c$	Associativity of addition
$\exists 0 \in \mathbb{Z}$ such that $\forall a \in \mathbb{Z}, a+0=a=0+a$	Additive identity
$\forall a \in \mathbb{Z}, \exists(-a) \in \mathbb{Z}$ such that $a+(-a)=0$	Additive inverse
$\forall a,b \in \mathbb{Z}, a\times b=b\times a$	Commutativity of multiplication
$\forall a,b,c \in \mathbb{Z}, a\times(b\times c)=(a\times b)\times c$	Associativity of multiplication
$\forall a \in \mathbb{Z}, a\times 1=a=1\times a$	Multiplicative identity
$\forall a,b,c \in \mathbb{Z}, a\times(b+c)=(a\times b)+(b\times c)$	Distributivity of multiplication over addition

Using the operatory properties associated with $(\mathbb{Z},+,\times)$, the following propositions regarding multiplication of integers, where $a,b>0,$ can be derived:

Proposition 1: $\forall a,b \in \mathbb{Z}, a\times b=ab$

Proposition 2: $\forall a,b \in \mathbb{Z}, -a\times b=-ab$

Proposition 3: $\forall a,b \in \mathbb{Z}, a\times -b=-ab$

Proposition 4: $\forall a,b \in \mathbb{Z}, -a\times -b=ab$

Proposition 5: $\forall a,b \in \mathbb{Z}, a+b>0$

Proposition 6: $\forall a,b \in \mathbb{Z}, (-a)+(-b)<0$

Proposition 7: $\forall a,b \in \mathbb{Z}, (-a)+(b)<0$ if $b<|-a|$

Proposition 8: $\forall a,b \in \mathbb{Z}, (-a)+(b)>0$ if $b>|-a|$

Proposition 1 forms the basis for the rule a positive 'times' a positive is a positive $(+\times+=+)$, Propositions 4 the basis for the rule a negative 'times' a negative is a positive $(-\times-=+)$ and Propositions 2 and 3 serve as the basis of the rule a positive 'times' a negative is a negative $(+\times-=-)$ or $(-\times+=-)$. The operation 'times' used in these rules are not equivalent to the mathematical operation multiplication which is a binary operation that takes two numbers as its input and generates a number as its output. The operation-like manipulation 'times' used in the sign rules discussed above uses the symbols for its domain and codomain. The domain obtained from the cross product of the set D = $\{+, -\}$ with itself. And the codomain is the set D = $\{+, -\}$.

In the teacher's procedure for factorising a trinomial of the form $x^2 + bx + c$, the users rules for generating the signs required to populate the factorisation brackets are in fact implicitly based on the combination of one of the Propositions 1 to 4 and one of the Propositions 5 to 8. [Expand this discussion]

The teacher ends this evaluative event by summarising the steps for drawing a graph given in turning point form $y = a(x - p)^2 + q$.

1. Write down the turning point.
2. Calculate the *y*-intercept.
3. Calculate the *x*-intercept.

In EE3.3 Learners work on sketching graphs of $y = (x - 1)^2 - 8$ and $y = (x - 3)^2 - 16$ using the method elaborated in EE3.2 and the teacher assists learners.

When discussing the x-intercepts of $y = (x - 1)^2 - 8$, the teacher advises a group of learners to use the formula to calculate the *x*-intercepts "if the equation doesn't factorise nicely".

> Teacher: Alright then multiply out. You know how to solve something that doesn't factorise. What do you do?
>
> Learner: You …
>
> Teacher: You use the …
>
> Learner: … factorise
>
> Teacher: If it doesn't factorise nicely.
>
> Learner: Oh yes you have to use that …
>
> Teacher: Formula [emphasises the word]. (S01T01L01: lines 432-438)

When explaining why the turning point of $y = (x - 1)^2 - 8$ is $(1; 8)$ the teacher provides an explanation involving the translation of the "original graph" as well as a deductive argument which is based on the minimum value of $y = (x - 1)^2 - 8$.

> Teacher: I'll show you now. Okay. So the turning point. If we look at this form of the point then *x* is one and *y* is minus eight. We call that *p q*. And the way you can think of this. You can either think of my original graph goes down eight opposite sign, Or you can say this is always positive. That has a minimum value of nought. And for this to be nought *x* must be one. That's the opposite sign okay. Good. Right.((S01T01L01: line 459)

Summary

Procedures

Procedure for sketching a parabola $y = x^2 + q$ using transformations

1. Draw the mother graph $y = x^2$.
2. If q > 0, move the graph up. If q < 0, move the graph down.

Procedure for sketching parabola $y = a(x-p)^2 + q$:

1. Write down the turning point.
2. Calculate the *y*-intercept by first multiplying out and substituting $x = 0$.
3. Calculate the *x*-intercept (method 1* or method 2**)

**Sub-procedure for calculating x-intercept using TP form of equation (Method 1)*

1. Substitute $y = 0$.
2. Change sides, change sign to produce an equation in the form $(x-p)^2 = q$
3. Take square roots on both sides of the equation
4. Solve the two linear equations

***Sub-procedure for calculating x-intercept using standard form of equation (Method 2)*

1. Substitute $y = 0$.
2. Make the x^2 term positive
3. Factorise trinomial***
4. Solve equation

****Sub-procedure for trinomial factorisation $x^2 + bx + c$*

1. Open brackets
2. Factorise x^2 into x and x
3. Factorise the value of *c* without its sign
4. Determine the signs in the brackets using propositions regarding 'multiplication' of signs

Propositions regarding 'multiplication of signs'

1. If *c* is positive and *b* is positive in $x^2 + bx + c$, the signs in the brackets are both +
2. If *c* is positive and *b* is negative in $x^2 + bx + c$, the signs in the brackets are both -
3. If *c* is negative in $x^2 + bx + c$, the sign in one brackets is + and the other bracket is -
4. If *c* is negative and *b* is negative in $x^2 + bx + c$, the negative signs is assigned to the bigger factor of *c* without its sign
5. If *c* is negative and *b* is positive in $x^2 + bx + c$, the positive signs is assigned to the bigger factor of *c* without its sign

Other propositions

1. For the function $y = a(x-p)^2 + q$, *a* makes it wider or narrower **(Iconic auxiliary)**

2. For the function $y = a(x-p)^2 + q$, p moves graph left/right left if sign in bracket is /+/ and right if sign in bracket is /-/. **(Iconic auxiliary)**

3. If $y = a(x-p)^2 + q$, then the maximum or minimum value of the parabola is q when $x = p$ i.e. that the turning point of the parabola is $(p; q)$. **(encyclopaedic)**

Structure: Operations, domain and codomain

Operations involve both those located in the field of the reals and auxiliary operation-like manipulations.

Operations from the field of the reals include: addition, multiplication, substitution, square roots.

Auxiliary operations

1. Shift graph vertically
2. Stretch graph in/out.
3. Change signs (ALT) which is used simultaneous to multiply by -1.
4. Change sides, change sign involved in solving equations or "take over and change the sign".

Transposition (Take over and change sign)

The operational features of operation-like manipulation "take over and change the sign" (transposition) described by Davis (2013) is the same as the operation-like manipulation ALT discussed above. Davis (2013) discusses transposition as a binary operation and neglects to factor in location as a component of the operation. If we consider location of the transposed terms, then we have to conceive of transposition as a ternary operation. My adaptation of Davis' description of the transposition operation as follows

The transposition of added/subtracted terms is used by this teacher in the context of solving equations. For example, the equation $x - 5 = 0$ has the solution $x = 5$, which is obtained by transposing -5 from the left-hand side (LHS) of the equation to the right-hand side (RHS) of the equation. The operation-like manipulation, transposition, is referred to as T_{+}. The objects that serve as arguments for T_{+} cannot be real numbers because one cannot move and change the sign of real numbers. We can, however, perform T_{+}on numerical and literal symbols. The operation-like manipulation T_{+} therefore implies that an existential shift that transforms the number -5 to symbol /-5 / has to take place. In other words, /-5/ has to be conceived of as a sign (negative) and a numeral. The set of signs $\{+; -\}$ will be referred to as *S*, the set of numerals and letters will be called K and the set $P\ \{Left, Right\}$ refers to the initial and final positions of the transposed term i.e. LHS or RHS of the equation. The input elements for T_{+} derive from set obtained from the cross product $S \times K \times P$. These input elements are mapped to elements in $S \times K \times P$. So $S \times K \times P$ constitutes both the domain and codomain of T_{+}. The application of T_{+} "converts" $+$ to – or – to $+$ and moves terms from the LHS to RHS of the equation of vice versa. In other words, $T_{+}: (+, a, Left) \rightarrow (-\ , a, Right)$ or

$T_{+}: (-, a, Left) \rightarrow (+, a, Right)$ or $T_{+}: (+, a, Right) \rightarrow (-, a, Left)$ or $T_{+}: (-, a, Right) \rightarrow (+, a, left)$ where $a \in K$. The output requires an operation-like manipulation, concatenation, which combines the sign and the numeral to produce $+a$ or $-a$, which represents the term. So for the to solve $x - 5 = 0$ using the operation T_{+} requires splitting apart the sign from the numeral and taking into account the position of -5 to produce $(-,5, left)$ as the input for T_{+} which generates $(+,3, right)$ as the output. Combining the sign and the numeral results in the required transposed term +5 on the RHS of the equation..

"Change the signs" or ALT

The operation ALT is referred to as state changing operation-like manipulation. The teacher uses an operation-like manipulation referred to as "change the signs" of $-x^2 + 6x - 5 = 0$. If we refer to this operation-like manipulation as ALT. The objects that serve as arguments for ALT cannot be real numbers because one can not change the sign of real numbers. We can, however, perform the "operation'" ALT on numerical and literal symbols. The operation-like manipulation ALT therefore implies that an existential shift has occurred that transforms the number -3 to symbol /-3/. In other words, /-3/ has to be conceived of as a sign, positive or negative, and a numeral. The set of signs $\{+; -\}$ will be referred to as S and the set of numerals will be called *C*. The input elements for ALT is contained in set obtained from the cross product $S \times C$. These input elements are mapped to elements in $S \times C$. So $S \times C$ constitutes both the domain and codomain of ALT. The application of ALT "converts" $+$ to $-$ or $-$ to $+$. In other words, ALT: $(+, a) \rightarrow (-, a)$ or ALT: $(-, a) \rightarrow (+, a)$ where $a \in C$ .The output requires an operation-like manipulation, concatenation, which combines the sign and the numeral to produce $+a$ or $-a$.

Primary data production for EE2: Summary of computational activity

Descriptions	Procedure	Propositions	Domain, codomain & operations
None	See above	Encyclopaedic propositions and auxiliary propositions.	Auxiliary operations with Domain and codomain are character strings Encyclopaedic - Field of the reals $((\mathbb{R}, +, \times)$

Secondary data production for EE2

Realised content	Ground/regulation	Closed/open pedagogic texts	Orientation to mathematics
Symbiotic Encyclopaedic propositions and auxiliary propositions are used alongside each other.	Fundamental Iconic Empirical (computer software used inductively to derive general propositions) Algorithmic	Text is open because grounded in encyclopaedic propositions and multiple methods for *x*-intercepts but weakened by fixed selection and sequencing of operations, insistence on standard form (T_O^-)	Content-oriented because encyclopaedic propositions recruited but weakened by the use of auxiliary calculus for *x*-intercepts . Auxiliary calculus not dominant and at times accompanied by encyclopaedic correlates(O_c^-)

Appendix 6.3
Lesson analysis summary - Sara

S01T01L01

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground/regulation	Closed/open pedagogic texts Orientation to mathematics
1	Revision of basic functions : Effects of parameters a and p $y = ax + p$ $y = ax^2 + p$ $y = \frac{a}{x} + p$ $y = a^x + q$	No procedures	For function $y = af(x) + p$, if $p > 0$ the graph $f(x)$ shifts up p units and if $p < 0$, then $f(x)$ shifts down p units. (E) For the function $y = af(x) + p$ $y = ax^2$, a "moves graph in/out (I) For the function $y = ax^2$, graph "flips"/reflects in x-axis if a is negative (E)	None	Graphs are shapes which can be shifted vertically or stretched. **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. Transformations are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ *Symbiotic*	Iconic fundamental Description of function provided rather than definite description	Text is open because encyclopaedic propositions recruited. Openness weakened by the inclusion of the iconic (T_O^-) Content-oriented because encyclopaedic propositions recruited but weakened by the inclusion of the iconic (O_c^-)
2.1	Setting up *Geogebra* $(y = bx)$	Slider set up in *Geogebra*		$(\mathbb{R}, +, \times)$ using a slider to "substitute" values	Graphs are shapes which can be shifted vertically or stretched.	Empirical Necessity located in software $y = a(x-p)^2 + q$ form verified by software	Text is open because encyclopaedic propositions recruited a. Openness weakened by the reliance on the empirical and iconic (T_O^-) Content-oriented because encyclopaedic propositions recruited but weakened by the intrusion of the empirical and iconic (O_c^-)
2.2	Investigating $y = a(x-p)^2 + q$ (see worksheet)	Students set up sliders to investigate parameters or input equations into *Geogebra* to generate graphs		$(\mathbb{R}, +, \times)$ using a slider to "substitute" values	Graphs are shapes which can be shifted vertically or stretched.	Empirical Necessity located in software – mathematical necessity implicit	
2.3	$y = a(x-p)^2 + q$: effects of a, p and q $y = (x-1)^2 + 0$ $y = (x-2)^2$ $y = (x+3)^2$	Shifting graphs vertically Stretching graphs Shifting graphs horizontally	For the function $y = a(x-p)^2 + q$, the parameter q shifts the graph $y = x^2$ up or down. If $q > 0$, the graph $y = x^2$ shifts up and if $q < 0$, the graph $y = x^2$ shifts down (P4 – E) For the function $y = a(x-p)^2 + q$ if the parameter $a < 0$ the graph of $y = x^2$ is reflected about the x-axis (P6 – E) For the function $y =$	Graph moves to right if number in bracket is negative; graph moves to the left if number in bracket is positive. It's the opposite Sundering $(\mathbb{R}, +, \times)$ - minimum value of expression $y = a(x-p)^2 + q$ TPx is the opposite sign of number in	Graphs are shapes which can be shifted vertically or stretched. Derivation of a rule for finding TP from TP form **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. Transformations are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ *Symbiotic*	Empirical Iconic fundamental Verifies rule; 1. *Geogebra* to generate graphs 2. explains minimum value of expression (local deductive) Necessity located in software	

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground/regulation	Closed/open pedagogic texts Orientation to mathematics
			$a(x-p)^2+q$, a makes it wider or narrower (I) For the function $y = a(x-p)^2+q$, p moves graph left/right left if sign in bracket is + and right if sign in bracket is – (I)	bracket –auxiliary operation			
2.4	Calculating TP of $y = 2(x-3)^2+4$	Original graph $y = x^2$ Moves up 4 units up 3 units to the right Translations and then *Geogebra* used to justify/authorise solution	For the function $y = a(x-p)^2+q$, p moves graph left/right left if sign in bracket is + and right if sign in bracket is – (I)	$(\mathbb{R},+,\times)$	Graphs are shapes which can be shifted vertically or stretched. **Substituted content** Functions as subsets of $\mathbb{R}\times\mathbb{R}$. Transformations are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$	Empirical Iconic Necessity located in software	
3.1	Calculating key points of $y = -(x-3)^2+4$ (CWK)	Reading TP from equation in TP form. Calculating x-intercepts – make x^2positive first 2 methods of calculating x-intercepts Calculating y-intercept-	If $y = a(x-p)^2+q$, then the maximum or minimum value of the parabola is q when $x = p$ i.e. that the turning point of the parabola is $(p;q)$ **(P9 – E)**	$(\mathbb{R},+,\times)$ auxiliary operations (trinomial factorisation) transposition change signs used with multiplication	Graph is a shape. with specific "dimensions" (intercepts and TP) **Substituted content** Viete theorem and fundamental theorem of algebra Notion of equation absent	Empirical Algorithmic fundamental Necessity external to mathematics. Necessity located in software	Text is open because grounded in encyclopaedic propositions and multiple methods for x-intercepts but weakened by fixed selection and sequencing of operations, insistence on standard form (T_O^-)
3.2	Calculating key points of $y = a(x-p)^2+q$ $y = -(x-3)^2+4$ (EXP)	Reading TP from equation in TP form. Calculating x-intercepts Calculating y-intercept	If $y = a(x-p)^2+q$, then the maximum or minimum value of the parabola is q when $x = p$ i.e. that the turning point of the parabola is $(p;q)$. **(P9 – E)** used interchangeably with auxiliary operation $y = a(x-p)^2+q$	$(\mathbb{R},+,\times)$ auxiliary operations & $(\mathbb{R},+,\times)$ transposition get rid of square – square root $(\mathbb{R},+,\times)$ TPx is the opposite sign of number in bracket	Graph is a shape. with specific "dimensions" (intercepts and TP) **Substituted content** Functions as subsets of $\mathbb{R}\times\mathbb{R}$. *Symbiotic*	Fundamental Algorithmic Iconic Necessity external to mathematics in procedure asserted by teacher- must convert to std form for y-intercept and must get rid of minus	Content-oriented because encyclopaedic propositions recruited but weakened by the use of auxiliary calculus for x-intercepts . Auxiliary calculus not dominant and at times accompanied by encyclopaedic correlates(O_c^-)
3.3	Sketching parabola $y = a(x-p)^2+q$ $y = (x-1)^2-8$ $y = (x-3)^2-16$	Reading TP from equation in TP form. Calculating x-intercepts Calculating y-intercept	If $y = a(x-p)^2+q$, then the maximum or minimum value of the parabola is q when $x = p$ i.e. that the turning point of the parabola is $(p;q)$. **(P9 – E)**	$(\mathbb{R},+,\times)$ auxiliary operations & $(\mathbb{R},+,\times)$ $(\mathbb{R},+,\times)$	Graph is a shape. with specific "dimensions" (intercepts and TP) **Substituted content** Functions as subsets of $\mathbb{R}\times\mathbb{R}$.	Fundamental Algorithmic Iconic Necessity external to mathematics minimum value of expression (local deductive)	

S01T01L02

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground	Closed/Open pedagogic text Orientation to mathematics
1	Comparing different forms of parabola equation	None	If $y = a(x-p)^2 + q$ then the turning point is $(p;q)$ (E) if $y = a(x-x_1)(x-x_2)$ then x_1 and x_2 are the *x*-intercepts of the parabola. (E)	$(\mathbb{R},+,\times)$	*canonical*	fundamental algorithmic – form of expression regulates procedure . TP form associated with transformations or calculation of key points	Text is open because encyclopaedic propositions recruited and absence of iconic and empirical (T_O^+) Content-oriented because encyclopaedic propositions recruited and absence of iconic and empirical (O_c^+)
2.1	Finding TP of $y = ax^2 + q$ using transformations	TP by 1. Shifting graphs vertically 2. Converting equation to TP form $y = a(x-0)^2 + q$	For function $y = af(x) + p$, if $p > 0$ the graph $f(x)$ shifts up *p* units and if $p < 0$, then $f(x)$ shifts down *p* units. (P1 – E) If $y = a(x-p)^2 + q$ then the turning point is $(p;q)$ (E)	$(\mathbb{R},+,\times)$	Graphs are shapes which can be shifted vertically or stretched. **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. Transformations are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$	fundamental	Text is open because encyclopaedic propositions recruited and deductive justifications provided but openness weakened by the use of auxiliary operations (T_O^-) Content-oriented because encyclopaedic propositions and deductive recruited but weakened by the use of the iconic (O_c^-)
2.2	Sketching $y = -(x-1)^2 + 4$ using transformations (no intercepts)	Shifting graphs vertically Shifting graphs horizontally Stretching graphs	If $a < 0$, the parabola has a minimum and if $a > 0$ the parabola has a maximum (E) $-(x-1)^2 + 4$ has a maximum of 4 when $(x-1)^2 = 0$, i.e. when *x* = 1 (E)	Graphs treated as input and output objects. +4 “move up” & -1 shifts +1 to the right (opposite sign) Minus reflects graph-implies sundering	Graphs are shapes which can be shifted vertically or stretched. **Substituted content -** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. Transformations are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ *symbiotic*	Iconic Fundamental fundamental – maximum value of $-(x-1)^2$ is 0 when x =1	
3	Sketching parabola $y = a(x-p)^2 + q$	Reading TP from TP form of equation	If $y = a(x-p)^2 + q$ then the turning point is $(p;q)$	$(\mathbb{R},+,\times)$	Graphs are shapes which have a specific position on the	Algorithmic Fundamental	Text is open because encyclopaedic propositions

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground	Closed/Open pedagogic text Orientation to mathematics
	$y = (x-3)^2 - 16$	Calculating y-intercept (multiply out first) Calculating x-intercepts from standard form rather than turning point form indicating preferred method	(E) the axis of symmetry is halfway between x-intercepts (NI)	auxiliary operations –trinomial factorisation implicit & $(\mathbb{R}, +, \times)$ does not show alternate methods	Cartesian plane – the position is determined by calculating key points. **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. *symbiotic*	Iconic - implicit Seems to be following a strict order which has no mathematical necessity – necessity located in procedure – standard form necessary to calculate y-intercept	recruited but weakened by fixed selection and sequencing of operations, insistence on standard form and implicit auxiliary calculus (T_O^-) Content-oriented because encyclopaedic proposition recruited (O_c^-) - auxiliary calculus for trinomial factorisation not made explicit
4.1	Calculating TP of $y = ax^2 + bx + c$ $y = x^2 - 6x - 7$	Adding x-intercepts and divide by 2	The axis of symmetry of a parabola is halfway between the x-intercepts (I)	$(\mathbb{R}, +, \times)$	notion of symmetry used to regulate procedure. But notion of symmetry just asserted not established or supported through formal definitions	algorithmic - notion of symmetry asserted	Text is open because grounded in encyclopaedic propositions but weakened by iconic correlates (T_O^-) Content-oriented because encyclopaedic propositions recruited but weakened by iconic correlates (O_c^-)
4.2	Sketching $y = ax^2 + bx + c$ Classwork examples: $y = x^2 + 2x$ $y = x^2 - 6x + 8$ Classwork	Calculating x-intercepts Calculating y-intercept Calculate TP	If $a < 0$, the parabola has a minimum and if $a > 0$ the parabola has a maximum (E) If a is negative, the parabola's "arms go down" and if a is positive the parabola's "arms go up" (I) If a is negative, the parabola has a "sad face" and if a is positive the parabola's a "smiley face". (I)	$(\mathbb{R}, +, \times)$ minimum – arms go up – smiley maximum – arms go down - sad	Graphs are shapes which have a specific position on the Cartesian plane – the position is determined by calculating key points. Main content – substitution & solving equations **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. *imagistic*	fundamental Algorithmic Iconic	
4.3	Sketching $y = ax^2 + bx + c$ y= $x^2 + 2x$ (teacher explanation)	Calculating x-intercepts Calculating y-intercept Calculate TP	If a is negative, the parabola's "arms go down" and if a is positive the parabola's "arms go up" (I)	$(\mathbb{R}, +, \times)$	Graphs are shapes which have a specific position on the Cartesian plane – the position is determined by calculating key	Algorithmic iconic	

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground	Closed/Open pedagogic text Orientation to mathematics
			The axis of symmetry of a parabola is halfway between the x-intercepts (NI)		points. Main content – substitution & solving equations **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. *imagistic*		
5.1	Converting standard form to TP form ($y = x^2 + 2x$)	Completing the square	If $y = a(x-p)^2 + q$ then the turning point is $(p;q)$ (E)	$(\mathbb{R}, +, \times)$		Fundamental Algorithmic	Text is open because encyclopaedic propositions recruited but weakened by fixed selection and sequencing of operations, insistence on standard form and auxiliary calculus for x-intercepts(T_O^-) Content-oriented because encyclopaedic proposition recruited weakened by auxiliary calculus for trinomial factorisation (O_c^-)
5.2	Sketching parabola (root method and completing square method) $y = x^2 - 6x + 8$	Calculating x-intercepts Calculating y-intercept Calculate TP through midpoints or completing the square Multiple methods	If $y = a(x-p)^2 + q$ then the turning point is $(p;q)$ (E)	auxiliary operations – trinomial factorisation & $(\mathbb{R}, +, \times)$ $(\mathbb{R}, +, \times)$ $(\mathbb{R}, +, \times)$	Graphs are shapes which have a specific position on the Cartesian plane –position determined by key points. Main content – substitution & solving equations **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. *imagistic*	fundamental Algorithmic iconic	

S01T01L03

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground	Closed/open pedagogic text
1	Calculating parabola's key points (marking homework)	Calculating x-intercepts Calculating y-intercept Calculating TP through midpoints or completing the square	If $y = a(x-p)^2 + q$ then the turning point is $(p;q)$ (E)	auxiliary operations (trinomial factorisation) & $(\mathbb{R}, +, \times)$ $(\mathbb{R}, +, \times)$ $(\mathbb{R}, +, \times)$	Graphs are shapes which have a specific position on the Cartesian plane – the position is determined by calculating key points. **Substituted content**	fundamental Algorithmic iconic	Text is open because encyclopaedic proposition recruited but weakened by fixed selection and sequencing of operations, insistence on standard form and auxiliary calculus for x-intercepts(T_O^-)

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground	Closed/open pedagogic text
					Functions as subsets of $\mathbb{R} \times \mathbb{R}$. *Symbiotic*		Content-oriented because encyclopaedic proposition recruited weakened by auxiliary calculus for trinomial factorisation (O_c^-)
2	Calculating TP using formula Example: $y = x^2 - 4x - 1$	Calculating TP using formula Derivation of TP_x formula ($x = \frac{-b}{2a}$) Derivation based on an asserted notion of symmetry.	the axis of symmetry is halfway between x-intercepts (Iconic)	$(\mathbb{R}, +, \times)$	Derivation of TP_x formula ($x = \frac{-b}{2a}$) The line perpendicular to the directrix and passing through the focus is called the "axis of symmetry". *Ancillary*	algorithmic Necessity based on ease of calculation – external to mathematics	Text is closed because absence fundamental ground "proof" weakens closed-nature of text (T_c^-) Expression-oriented because absence of fundamental ground but proof weakens expression orientation (O_e^-)
3	Interpreting graphs – calculating key points $f(x) = -x^2 + 2x + 8$	Calculating x-intercepts- change sign to + Calculating y-intercept Calculating TP through midpoints, completing the square or formula Multiple methods for calculating TP – formula, x-intercepts and completing square		auxiliary operations- minus minus is plus minus times minus is plus trinomial factorisation drop minus & $(\mathbb{R}, +, \times)$ $(\mathbb{R}, +, \times)$ $(\mathbb{R}, +, \times)$	Graphs are shapes which have a specific position on the Cartesian plane – the position is determined by calculating key points. **Substituted content** Functions as subsets of $\mathbb{R} \times \mathbb{R}$. *Ancillary*	Algorithmic Iconic Necessity located external to mathematics: 1) removing potential of error - make x^2 term positive; (2) quicker to use a/s formula than completing square method	Text is closed because fixed selection and sequencing of operations, insistence on standard form and auxiliary calculus but multiple methods weakens closed nature of text (T_C^-) Expression- oriented because auxiliary calculus with respect to x-intercepts but weakened by multiple methods for TP (O_e^-)

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground	Closed/open pedagogic text
4	Calculating equation of straight line	Calculating equation of straight line	slope to the left so gradient is negative (Iconic)	auxiliary operations & $(\mathbb{R}, +, \times)$ gradient = 4 cross and 8 up over for divide i.e. 8 over 4 “dropping the minus” concatenation	*Ancillary*	Iconic algorithmic	Text is closed because strong focus on auxiliary calculus (T_C^+) Expression- oriented because auxiliary propositions recruited (O_e^+)
5	Calculating lengths	Vertical lengths – substitute *y* values and subtract Horizontal lengths : substitute *x* values and subtract Sloping lines – Pythagoras or distance formula	For an equation if you know *x* you know *y* **(E)**	$(\mathbb{R}, +, \times)$	Connections made evident for students *canonical*	Algorithmic fundamental	Text is open because encyclopaedic proposition used (T_O^+) Content-oriented because fundamental ground and connections to other sub-topics made (O_c^+)
6	Calculating points of intersection	Point of intersection same as $f(x) = g(x)$ – makes connections for students – different ways of asking the same question Equate functions then solve equation		$(\mathbb{R}, +, \times)$ auxiliary operation – trinomial factorisation	*ancillary*	Algorithmic iconic Quadratic equation -make it equal 0 Always make x^2 positive	Text is closed because fixed selection and sequencing of operations, insistence on standard form and recruitment of the iconic but weakened by making different ways of asking the same question (T_C^-) Expression- oriented because auxiliary calculus recruited but weakened by connections to different ways of asking the same question (O_e^-)

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content	Ground	Closed/open pedagogic text
7.1	Solving inequalities graphically For which values of x is $f(x) \geq 0$ or $-x^2 + 2x + 8 \geq 0$	1. Identify x-intercepts. 2. Mark part of graph above x-axis 3. Read off corresponding x-values. different ways of asking the same question		$(\mathbb{R}, +, \times)$	No content substitution *canonical*	algorithmic	Text is open because no content substitution (T_O^+) Content-oriented because no content substitution and the expressive is secondary (O_c^+)
7.2	Solving inequalities graphically For which values of x is $f(x) > g(x)$	1. Identify points of intersection 2. Mark f above g 3. Read off corresponding x-values. different ways of asking the same question		$(\mathbb{R}, +, \times)$	No content substitution *encyclopaedic*	Algorithmic	

Appendix 6.4

Lesson analysis summary - Jada

S01T02L01

EE	Topic/sub-topic	Procedure	Propositions	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1.1	Calculating equation of symmetry axis of $y = x^2 - 7x + 12$	Calculate x-intercepts Add x-intercepts Divide by 2	P1: The symmetry axis of a parabola is "bang in the middle of the x-cuts (I) if you fold it it will be perfectly symmetrical" (SO1T02L01: line 89) (I)	$(\mathbb{R}, +, \times)$		Iconic Proposition that parabola is symmetrical is **asserted**	Closed text – focus on iconic but weakened by inclusion of encyclopaedic proposition (T_C^-)
1.2	Factorising $y = x^2 - 7x + 12$	last term positive, factors: $(x + \square)(x + \square)$ or $(x - \square)(x - \square)$ last term negative, factors $(x + \square)(x - \square)$ or $(x - \square)(x + \square)$		Auxiliary operations: Sign operations (characters, symbols "operations") R1: For trinomial factorisation, if the last term is positive then the signs in the brackets are the same (x + □)(x + □) or (x - □)(x - □) (I) R3: For trinomial factorisation, if the last term is negative then the signs in the brackets are opposite (x + □)(x - □) or (x - □)(x + □) (I)	character distribution matrix: **Substituted content:** Fundamental theorem of arithmetic Viete's formula: the factors of $ax^2 + bx + c$ are $(mx + p)(nx + q)$ where $mn = a$, $pq = c$ and $pn + mq = b$	algorithmic iconic	Expression- oriented because auxiliary calculus and iconic propositions recruited but weakened by inclusion of encyclopaedic propositions (O_e^-)

1.3	Calculating TPy	Substitute symmetry axis into equation If I have an x-value on a graph, find the y-value by substituting the x-value into the graph's equation or vice versa (E)	P1: The symmetry axis of a parabola is "bang in the middle of the x-cuts (I)	$(\mathbb{R},+,\times)$		Algorithmic fundamental	
1.4	Calculating intercepts of $y = x^2 - 7x + 12$	y-intercept – make $x = 0$ x-intercepts: make $y = 0$ trinomial factorisation one brackets 0 or the other brackets 0 solve linear equations	P1: If $(x - x_1)(x - x_2) = 0$, then the one bracket is zero or the other bracket has a zero value "either one bracket has a zero value or the other bracket has a zero value." (S01T02L01: line 87) (E) P3: If you fold it (parabola), it will be symmetrical (uses gesture to demonstrate). A parabola can be folded so that the two bars fit over each other. (I)	$(\mathbb{R},+,\times)$ Auxiliary operations: (characters, symbols "operations")	character distribution matrix absent zero product proposition Substituted content *Symbiotic*	algorithmic iconic fundamental	
1.5	Identifying critical points of parabola	Plot: y-intercept, x-intercepts & turning point Strict order of plotting - Do not plot TP first.		Non-computational		Algorithmic **Necessity external to mathematics** - Strict order of plotting	

2.1	Calculating intercepts of $y = 2x^2 - 4x + 6$	y-intercept – make $x = 0$ x-intercepts: make $y = 0$ trinomial factorisation solve linear equations First write down y-cut. Strict order of operations $2(x-3)(x+1) = 0$ 2 is irrelevant i.e. 2 can just disappear rather than $(x-3)(x+1) = 0$ can be obtained		$(\mathbb{R}, +, \times)$ $\mathbb{X}$ (characters, symbols "operations") character distribution matrix – trinomial factorisation R3: For trinomial factorisation, if the last term is negative then the signs in the brackets are opposite (x + □)(x - □) or (x - □)(x + □) (I) Auxiliary operation - "drop 2"	*Ancillary*	algorithmic iconic Necessity external to mathematics – Strict order of operations First write down y-cut.	Closed text – fixed selection and sequencing of operations and strong focus on iconic(T_C^+) Expression- oriented because auxiliary calculus recruited (O_e^+)
2.2	Calculating turning point of $y = 2x^2 - 4x + 6$	Calculate symmetry-axis Substitute symmetry -axis	The symmetry axis of a parabola is "bang in the middle of the x-cuts" (Where is my symmetry line?) (I)	$(\mathbb{R}, +, \times)$		Algorithmic	
3.1	Reflecting $y = 2x^2 - 4x + 6$ in the x-axis	Skipping as a metaphor for reflecting in x-axis Rule – "change every sign to the opposite sign"	Reflecting a parabola in the x-axis is like skipping with x-axis acting as a handle [and the parabola the rope] (I)	Auxiliary operations: ALT - Change signs to opposite sign Sundering concatenation	*Ancillary*	Iconic Algorithmic Empirical Rule asserted but also justified empirically through geometry software	Closed text – auxiliary calculus and iconic propositions weakened by inclusion of the empirical (T_C^-) Expression-orientation – rules for operating on expressive elements weakened by empirical (O_e^-)
3.2	Reflecting $y = 2x^2 - 4x + 6$ in the y-axis	"change every x to the opposite sign"		Auxiliary operations: Change sign of every x to opposite sign Does "change" mean substitute		Rule asserted but also justified empirically through geometry software	

3.3	Translating $y = 2x^2 - 4x + 6$ up 2 units and 1 unit across	Not explained. EE 4.1 deals with shifting parabolas horizontally		Not explained		n/a	
4.1	Turning point of $y = a(x-p)^2 + q$ $y = (x-1)^2 - 4$	TP is (1;-4) Why? The smallest value of the bracket is zero so the smallest value of the equation is -4 when $x = 1$. TP read off graph generated through software (empirical)		$(\mathbb{R}, +, \times)$		Empirical Deductive argument using a specific example Necessity located in software	Closed text absence of fundamental ground weakened by inclusion of the empirical (T_C^-) Expression-oriented – rules for operating on expressive elements but weakened by deductive argument((O_e^-)
4.2	Using *Geogebra* to determine turning point $y = (x+1)^2 - 4$ $y = (x+3)^2 - 4$ $y = (x+5)^2 - 4$ $y = (x-5)^2 - 4$	TP*x* is the opposite sign of what I am putting in (I)	P5: For the function $y = a(x-p)^2 + q$, the turning point has the opposite sign of what I am putting in i.e if the number in the bracket (i.e. –*p*) is negative the graph of $y = x^2$ shifts right and left if the number in the bracket is positive. (I)	$\mathbb{X}$ (characters, symbols "operations") Opposite sign of the bracket	*Ancillary*	Empirical Iconic Necessity located in software- inductive reasoning	
5	Reflecting parabolas in the *y*-axis and x-axis $y = x^2 - 5x - 6$	"change every *x* to the opposite sign"		$\mathbb{X}$ (characters, symbols "operations") ALT - Change signs to opposite sign Change sign of every *x* to opposite sign	*Ancillary*	Iconic algorithmic	Closed text – strong focus on imagistic (T_C^+) Expression-oriented – rules for operating on expressive elements (O_e^+)

6	Sketching graph of $y = x^2 - 7x + 12$	Plot *x*-cuts, *y*-cut Calculate TP by adding *x*-cuts and dividing by 2, then sub into equation then insert symmetry axis and plot TP. Connect points	P1: The symmetry axis of a parabola is "bang in the middle of the *x*-cuts (I)	$(\mathbb{R}, +, \times)$	*Ancillary*	Algorithmic	Open text because no content substitution weakened by fixed selection and sequencing of operations focus on iconic absent (T_O^-) Content-oriented – expressive elements secondary O_c^-)
7.1	Calculate equation of graph in standard form	Write down brackets which is the opposite sign of *x*-intercepts Multiply out Compare *y*-intercept of graph with *y*-intercept of equation Calculate ratio of graph y-intercept to equation y-intercept Adjust equation by multiplying by ratio	P6: If the *x*-cut is *p*, then the factorisation bracket is (*x*-p) (I)	$(\mathbb{R}, +, \times)$ ALT – the bracket must have the opposite sign to the *x*-cuts Character distribution matrix	*Ancillary*	Iconic Algorithmic Empirical Reverse factorisation rule asserted Justification lies in procedure Adjustment based on what graph looks like	Closed text – strong focus on auxiliary calculus weakened by the empirical(T_C^-) Expression-oriented – rules for operating on expressive elements weakened by the empirical(O_e^-)
7.2	Writing parabola's equation in completed square form	TP is (1; -4,5) and a = ½ $p = 1, q = -4{,}5$ and $a = ½$ Sub into $y = a(x-p)^2 + q$		$(\mathbb{R}, +, \times)$ substitution ALT – opposite sign of TP*x*	*Ancillary*	Iconic algorithmic	

S01T02L02

EE	Topic/sub-topic	Proposition	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1.1	Calculating *x*-intercepts of parabola $y = x^2 - 7x + 12$		Trinomial factorisation to convert standard form to factorised form- purpose to calculate *x*-cuts.	No computations		Algorithmic Strict order. No mathematical necessity for factorised form first.	Closed text – strong focus on auxiliary calculus, fixed selection and sequencing of

1.2	Calculating turning points from completed square form $y = (x - 3{,}5)^2 - 0{,}25$	P5: For the function $y = a(x - p)^2 + q$, the turning point has the opposite sign of what I am putting in i.e if the number in the bracket (i.e. $-p$) is negative the graph of $y = x^2$ shifts right and left if the number in the bracket is positive.(I) Uses proposition established empirically and deductively (local) in lesson 1	Completed square form $y = a(x - p)^2 + q$ - read off TP using auxiliary operation	opposite sign $\mathbb{X}$ (characters, symbols "operations")	*Ancillary*	Algorithmic iconic	operations (T_C^+) Expression-oriented – rules for operating on expressive elements (O_e^+)
1.3	Calculating turning points using x-intercepts $y = x^2 - 7x + 12$			$(\mathbb{R}, +, \times)$		Algorithmic	

2	Calculating turning points using formula $y = x^2 + 3x + 5$ $y = x^2 - 7x + 12$ $y = 2x^2 + 5x - 3$		R1 For trinomial factorisation, if the last term is positive then the signs in the brackets are the same (x + □)(x + □) or (x - □)(x - □) For the trinomial $x^2 + bx + c$, if c is negative and b is negative then the "bigger factor" of $\|c\|$ is assigned to the minus factorisation bracket (the brackets are $(- \quad)(+ \ )$. Standard form: TPx - a/s: $x = \frac{-b}{2a}$ (asserted) TPy: substitute a/s	$(\mathbb{R}, +, \times)$ $\mathbb{X}$ (characters, symbols "operations") character distribution matrix	Not the first option. Use a check method *Ancillary*	Algorithmic- TP formula asserted but motivated by equations that 'can't be factorised" Iconic involved in motivation for TP formula Empirical – **verified through calculation**	Open text because no content substitution weakened by use of iconic (T_O^-) Content-oriented – expressive elements secondary weakened by iconic O_c^-)
3	Converting completed square form to standard form $y = (x-3)^2 - 16$ Factorising $y = x^2 - 6x - 7$		Multiply out R1 For trinomial factorisation, if the last term is positive then the signs in the brackets are the same (x + □)(x + □) or (x - □)(x - □)	$(\mathbb{R}, +, \times)$ $\mathbb{X}$ (characters, symbols "operations") character distribution matrix	*Ancillary*	Algorithmic iconic	Closed text – strong focus on auxiliary calculus, fixed selection and sequencing of operations (T_C^+) Expression-oriented – rules for operating on expressive elements (O_e^+)

4	Calculate equation of graph	P7: An infinite number of parabola can be drawn through x-intercepts but a unique parabola can be drawn through three points (E) P6: If the x-cut is p, then the factorisation bracket is (x-p), the opposite sign (I)	Write down brackets which is opposite sign of x-intercepts Multiply out Compare y-intercept of graph with y-intercept of equation Substitute a point other than x-cuts to calculate a	$(\mathbb{R}, +, \times)$ $\mathbb{X}$ (characters, symbols "operations") opposite sign	Reverse factorisation $y = a(x - x_1)(x - x_2)$ where a represents controlling variable and a represents the multiple that taken out during factorisation *Ancillary*	fundamental Algorithmic Iconic Mathematical necessity located in reverse factorisation	Open text – recruits fundamental proposition but weakened by the iconic (T_O^-) Content-oriented – presence of fundamental ground weakened rules for operating on expressive elements (O_c^-)
5.1	Calculating $f(x)$ for a given x value		Substitute value of x into equation	$(\mathbb{R}, +, \times)$	$f(x)$ is the name of the function $f(1)$ – what is the value of the function when $x = 1$	Algorithmic	Closed text – strong focus on auxiliary calculus, fixed selection and sequencing of operations (T_C^+) Expression-oriented – rules for operating on expressive elements (O_e^+)
5.2	Calculate x if $f(x)$= 8 for		Solve equation	$(\mathbb{R}, +, \times)$ - auxiliary operations trinomial factorisation	*Ancillary*	Algorithmic iconic	

6.1	Calculating length of AB		Calculate x-intercepts: Take out negative Ignore negative Solve equation Trinomial factorisation	$(\mathbb{R}, +, \times)$ Auxiliary operations: Ignore negative Trinomial factorisation For the trinomial $x^2 + bx + c$, if c is negative and b is negative then the "bigger factor" of $\|c\|$ is assigned to the minus factorisation bracket (the brackets are $(-\quad)(+\quad)$.	We can ignore the negative. Substitutes for multiplying or dividing by -1.	Iconic Algorithmic Take negative out first then factorise. Order of steps has no mathematical necessity.	Closed text – strong focus on auxiliary calculus, fixed selection and sequencing of operations (T_C^+) Expression-oriented – rules for operating on expressive elements (O_e^+)
6.2	Calculate TP	P1: The symmetry axis of a parabola is "bang in the middle of the *x*-cuts" (Where is my symmetry line?) (I) P8: If parabola is not a smiley face, then its negative (I)	TPx - a/s: $x = \frac{x_1 + x_2}{2}$ TPy: substitute a/s	$(\mathbb{R}, +, \times)$	*Ancillary*	Algorithmic iconic Do not use completing the square. Only 2 marks. Necessity located in exams	

S01T02L03

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1	Equation of straight line	P9: If the top of the straight line is in line with negative x-axis, then the gradient is negative. (I) Rise comes before run. You have to rise to run. So rise is on top and run is at the bottom (I)	Calculate gradient: • Direction - negative • Gradient = rise/run • Paste sign and rise/run answer for gradient (m) • Read off y-intercept (c) Equation – sub m and c into standard form of linear function equation $y = mx + c$	$(\mathbb{R}, +, \times)$ Auxiliary operations: Direction – (You see the negative x-axis, Look at the top) Rise comes before run. concatenation	*Ancillary*	Iconic Algorithmic	Closed text – strong focus on auxiliary calculus,(T_C^+) Expression-oriented – rules for operating on expressive elements (O_e^+)
2	Calculate vertical distance	P10: If a line is vertical, then the x is constant . (E)	Substitute $x1$ to calculate $y1$ Substitute $x2$ to calculate $y2$ Subtract y-values	$(\mathbb{R}, +, \times)$	*canonical*	Algorithmic Fundamental	Open text – recruits fundamental proposition but weakened by the iconic (T_O^+) Content-oriented because encyclopaedic propositions recruited (O_c^+)

3	Calculate points of intersection		Equate equations Solve equation Trinomial factorisation	$(\mathbb{R}, +, \times)$ substitution Auxiliary operations: Take over, change sign (move everything to the right to get everything positive) Trinomial factorisation For the trinomial $x^2 + bx + c$, if c is negative and b is negative then the "bigger factor" of $\lvert c \rvert$ is assigned to the minus factorisation bracket (the brackets are $(\ - \quad)(\ + \ )$.	move everything to the right to get everything positive. No mathematical necessity *Ancillary*	Algorithmic Iconic	Closed text – strong focus on imagistic, fixed selection and sequencing of operations (T_C^+) Expression-oriented – rules for operating on expressive elements (O_e^+)
4.1	Sketching exponential graph	P11: The graph of an exponential function is not a straight line. The graph goes on forever but I can't cross this line (x-axis). It is called a "turning line" (I) An asymptote is a line you can't cross (I)	Find *y*-intercept make *x* = 0 Can't calculate *x*-cut because $2^x \neq 0$. T sub's in values getting closer and closer to *y* = 0 Choose *x* = -1, 0, 1.	$(\mathbb{R}, +, \times)$	*Ancillary*	Iconic Algorithmic Empirical – software used to establish necessity and subbing particular values	Closed text – strong focus on imagistic, weakened by the empirical (T_C^-) Expression- oriented because trong focus on imagistic, weakened by the empirical (O_e^-)
4.2	Reflecting exponential graph		Reflect individual points in *x*-axis to obtain reflected graph	*Geogebra* used to verify shape and calculations		Empirical iconic	

5	Hyperbola-	P13: Both axes are asymptotes of an hyperbola (E)	Sub $x = 0$ into $y = \frac{4}{x}$ And $y = 0$ into $x = \frac{4}{y}$	Geogebra used to verify shapes and asymptotes	*canonical*	fundamental Empirical- software used to establish necessity	Open text – recruits fundamental proposition but weakened by the empirical (T_O^-) Content-oriented because encyclopaedic propositions recruited but weakened by the intrusion of the empirical (O_c^-)

Appendix 6.5
Lesson analysis summary - Maya

S02T03L01

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1.1	Drawing straight-line and parabola graphs		Sketching straight lines Sketching parabolas Students appear to calculate intercepts then connecting intercepts with either a straight line or "curve"	$(\mathbb{R},+,\times)$ $(\mathbb{X},*)$	Basic arithmetic and auxiliary calculus Graphs are shapes that can be obtained by "connecting the dots" **Substituted content:** functions as subsets of $\mathbb{R} \times \mathbb{R}$	Algorithmic iconic	Closed text- strong focus on the iconic T_c^+ Closed text- strong focus on the iconic O_e^+
1.2	Comparing $y = 2x - 8$ and $y = 2x^2 - 8$	P1: presence of the 'square' in $y = 2x^2 - 8$ indicates parabola and absence of the 'square' in $y = 2x - 8$ indicates straight line. (I) P2: If the coefficient of x^2 term is positive, the graph has a "smiley face" and a "grumpy face" if coefficient of x^2 term is negative (I)		$(\mathbb{X},*)$	Graphs are "shapes" determined by the expressions. **Substituted content:** functions as subsets of $\mathbb{R} \times \mathbb{R}$ If $a > 0$ then $y = ax^2 + bx + c$ has a min and if $a < 0$ then $y = ax^2 + bx + c$ has a max *Ancillary*	Iconic – focusing on parts of the expression and focus on what the expression looks like	
2.0	Graphing parabola and hyperbola & finding equation of graphs (Group work)	P4 : If the graph is a line then it is a linear function then it has the formula $y = mx + c$ (I) P6: If a graph has "smiley-	Sketching parabolas Sketching hyperbolas Calculating equations (S work in groups)	$(\mathbb{R},+,\times)$ $(\mathbb{X},*)$	Basic arithmetic and auxiliary calculus Images of graphs mapped to general formula **Substituted content**	Iconic –graphical image serves as input object	Closed text- strong focus on the iconic T_c^+ Closed text- strong focus on the iconic O_e^+

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
		face" or "sad-face" the graph is a parabola (I)			functions as subsets of $\mathbb{R} \times \mathbb{R}$ general equations of functions *Ancillary*		
3.1	Describing features of linear functions	P3 If the graph is a line then it is a linear function (I) Arrows on graphs are used to substitute for the domain as $x \in \mathbb{R}$	Selecting appropriate formula for graph	$(\mathbb{X},*)$ Operation - selection	A linear function includes any image that looks like a line including graphs defined by $x = k$, which are not linear functions. **Substituted content** linear function as a subset of $\mathbb{R} \times \mathbb{R}$ General equation of a linear function *Ancillary*	Iconic – focusing on the graphical image	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+
3.2	Finding the equation of a straight line function	P4 : If the graph is a line then it is a linear function then it has the formula $y = mx + c$ (I)	Calculating equation of straight line Graph is used to read off data required to populate the general formula which is used as a template *y, x, c, m* are all treated as variables. Teacher does not distinguish between parameters and variables.	$(\mathbb{R},+,\times)$ $(\mathbb{X},*)$ - transposition	Basic arithmetic and auxiliary calculus Formula as a CDM **Substituted content** the idea that two points defines a unique line Right cancellation theorem *Ancillary*	Algorithmic Iconic	
4.0	Discussing features of parabola functions	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola	Selecting appropriate formula for graph	Non-computational	mathematical objects treated as images **Substituted content**	Iconic –graphical image serves as input object	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
					parabola as a subset of $\mathbb{R}\times\mathbb{R}$ General equation of a parabola *Ancillary*		on the iconic O_e^+

S02T03L02

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1.1	Finding equation of $y = ax^2 + q$	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola (I) P9: If a parabola has the "same" x-intercepts then the formula is $y = ax^2 + q$ (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus Formula as a CDM **Substituted content:** parabola as a subset of $\mathbb{R}\times\mathbb{R}$ *Ancillary*	Iconic Algorithmic	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+
1.2	Finding equation of $y = ax^2 + q$	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola (I) P9: If a parabola has the "same" x-intercepts then the formula is $y = ax^2 + q$ (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus, Formula as a CDM **Substituted content:** parabola as a subset of $\mathbb{R}\times\mathbb{R}$ *Ancillary*	Iconic Algorithmic	
2	Comparing graphs of $y = ax^2 + q$ and $y = ax^2 + bx + q$	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola (I) P9: If a parabola has the "different" x-intercepts then the formula is $y = ax^2 + bx + q$ (I) P8: If the graph is a 'shifted" parabola then the formula is $y = ax^2 + bx + q$ (I)		Non-computational	mathematical objects treated as images **Substituted content:** parabola as a subset of $\mathbb{R}\times\mathbb{R}$ *Ancillary*	Iconic	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+
3.1	Finding equation of $y = ax^2 + bx + q$	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus	Iconic Algorithmic	Closed text- strong focus on the iconic T_C^+

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
		P9: If a parabola has the "different" x-intercepts then the formula is $y = ax^2 + bx + q$ (I) P8: If the graph is a 'shifted" parabola then the formula is $y = ax^2 + bx + q$ (I)			Formula as a CDM **Substituted content** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ **Substituted content** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ *Ancillary*		Closed text- strong focus on the iconic O_e^+
3.2	Finding equation of $y = ax^2 + bx + q$ – "crazy complicated" method	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola (I) P9: If a parabola has the "different" x-intercepts then the formula is $y = ax^2 + bx + q$ (I) P8: If the graph is a 'shifted" parabola then the formula is $y = ax^2 + bx + q$ (I)	Finding equation of $y = ax^2 + bx + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus Formula as a CDM **Substituted content:** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ *Ancillary*	Iconic Algorithmic Empirical – just try any point	
4	Finding equation of $y = ax^2 + bx + q$ - "shortcut" method	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola (I) P9: If a parabola has the "different" x-intercepts then the formula is $y = ax^2 + bx + q$ (I) P8: If the graph is a 'shifted" parabola then the formula is $y = ax^2 + bx + q$ (I)	Finding equation of $y = ax^2 + bx + q$ T establishes template/CDM $y = a(x - x_1)(x - x_2)$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus Formula as a CDM **Substituted content:** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ *Ancillary*	Iconic Algorithmic	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+

S02T03L03

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1.1	Finding equation of parabola problem 1	P6: If a graph has “smiley-face” or “sad-face” the graph is a parabola (I) P9: If a parabola has the “same” x-intercepts then the formula is $y = ax^2 + q$ (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus, Formula as a CDM **Substituted content:** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ *Ancillary*	Iconic Algorithmic	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+
1.2	Finding equation of parabola problem 1 - explanation	P6: If a graph has “smiley-face”/ “sad-face”, graph is a parabola (I) P9: If a parabola has the “same” x-intercepts then the formula is $y = ax^2 + q$ (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$		Iconic Algorithmic	
2.0	Common mistakes			$(\mathbb{X}, *)$ –sign in bracket opposite sign to x-intercept	auxiliary calculus *Ancillary*	iconic	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+
3.1	Finding equation of parabola problem 2	P6: If a graph has “smiley-face” or “sad-face” the graph is a parabola (I) P9: If a parabola has the “different” x-intercepts then the formula is $y = ax^2 + bx + q$ (I) P8: If the graph is a ‘shifted” parabola then the formula is $y = ax^2 + bx + q$ (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus, Formula as a CDM **Substituted content:** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ *Ancillary*	Iconic Algorithmic	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+
3.2	Finding equation of parabola problem 2 - marking	P6: If a graph has “smiley-face” or “sad-face” the graph is a parabola (I) P9: If a parabola has the “different” x-intercepts then the formula is $y = ax^2 + bx + q$ (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$		Iconic Algorithmic	

EE	Topic/sub-topic	Propositions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
		P8: If the graph is a 'shifted" parabola then the formula is $y = ax^2 + bx + q$ (I)					
4.0	Finding equation of straight line and parabola - review	P6: If a graph has "smiley-face" or "sad-face" the graph is a parabola (I) P9: If a parabola has the "different" x-intercepts then the formula is $y = ax^2 + bx + q$, if has the "same" x-intercepts then the formula is $y = ax^2 + q$ P8: If the graph is a 'shifted" parabola then the formula is $y = ax^2 + bx + q$ (I)	Finding equation of $y = ax^2 + q$	$(\mathbb{R}, +, \times)$ $(\mathbb{X}, *)$	Basic arithmetic and auxiliary calculus Formula as a CDM **Substituted content** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ **Substituted content** parabola as a subset of $\mathbb{R} \times \mathbb{R}$ *Imagistic* *Ancillary*	Iconic Algorithmic	Closed text- strong focus on the iconic T_C^+ Closed text- strong focus on the iconic O_e^+

Appendix 6.6

Lesson analysis summary - Jono

S02T04L01

EE	Topic/sub-topic	Propositions/Descriptions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1	What is a function?	A function is a relation between two variables x is input, x is the independent variable. Input is the independent variable (I) y is output, y is the dependent variable. Output is the dependent variable (I) Natural kind descriptions and propositions: Function as a "box" (I)	none	No computations	Auxiliary description of function as a relation between two variables – independent variable (input or x) and dependent variable (output or y) Describing relations rather than functions. So category shift (natural kinds) *Ancillary*	Iconic	Closed text- strong focus on the iconic T_C^+ Expression- oriented because iconic propositions recruited (O_e^+)
2.0	Domain and range of linear functions $y = 8x + 6$ (buying bread)	The set of values for independent variable represents the domain. (NI) The set of values for the dependent variable represents the range. (NI) Natural kind descriptions:	Substitute values for x start with $x = 0$ to produce values of y Read off the type of numbers from table of values Determine lower bound and upper bound- used as characters fir CDM For interval notation, use round brackets if number is excluded and square brackets if number is included	$(\mathbb{R}, +, \times)$ Substitution, multiplication, addition Character distribution matrix (CDM) – $[0; \infty)$ $[0; \infty)$ which incorrectly represents the set {0; 1; 2; 3; 4: ….} since interval notion can only be used for real numbers iconic mapping	The domain is the outcome of calculation rather than decided upfront and embedded in the problem Auxiliary descriptions *Ancillary*	Empirical iconic Table of values serves as regulative resource Calculation serves to establish necessity.	Closed text- absence of propositional but weakened by empirical used to establish the notion of domain and range T_C^- Expression- oriented symbols generated to populate CDM (O_e^+)

3.1	Domain and range of linear functions $y = x + 1$ (CWK)		Substitute values for x to produce values of y Read off the type of numbers from table of values Determine lower bound and upper bound For interval notation, use round brackets if number is excluded and square brackets if number is included	$(\mathbb{R}, +, \times)$ Substitution, multiplication, addition	The domain is the outcome of calculation rather than decided upfront	Empirical Table of values serves as regulative resource	Closed text- absence of propositional and presence of the iconic weakened by empirical T_C^- Expression- oriented because iconic propositions recruited weakened by empirical (O_e^+)
3.2	Domain and range of linear functions $y = x + 1$ (EXP)	If there is no value of x for which the function is undefined, then the domain is the set of real numbers (NI) If the variable x is not assigned anything (e.g. loaves of bread), then the domain is the set of real numbers (NI) Real numbers are any numbers. Real numbers include fractions, negative numbers, rational numbers and irrational numbers.(NI)	Is there any value that this variable can take and make this function undefined" (L309) If the answer is no, then x can take any number. So x is any real number	$(\mathbb{R}, +, \times)$	The domain is the outcome of calculation rather than decided upfront. L328: T has to admit that if x i.e. domain is not stipulated as in the bread problem, then x can be any real number.	empirical "you can put negative, you can put positive numbers. You add one but you have something here" (L324)	

3.3	Domain and range of linear functions (CWK) $y = x + 1$ and the rest of Worksheet Section A	Infinity is a number that we can't touch. It is not really quantifiable. It is a big big number but you can't quantify it. (I) Negative infinity is the smallest number and positive infinity is the biggest number (NI) If the variable x is not assigned anything (e.g. loaves of bread), then the domain is the set of real numbers (NI)	"x goes from negative infinity to positive infinity" Write smallest number on the left. Write biggest number on the right Insert x in the middle Insert inequality signs	Auxiliary operations Emphasis on a character distribution matrix From here to there (spatial ordering)	Spatial order replaces numerical order (imagistic because negative infinity must be on the left) $\infty < x < +\infty$ *Ancillary*	Iconic Algorithmic	
3.4	Domain and range of linear functions (MRK) $y = x + 1$	If there is no value of x for which the function is undefined, then the domain is the set of real numbers (NI) Infinity is a number that we can't touch (I)	Is there any value that this variable can take and make this function undefined" (L309) If the answer is no, then x can take any number. So x is any real number ($x \in \mathbb{R}$), x goes from negative infinity to positive infinity	Auxiliary operations Spatial order Emphasis on a character distribution matrix (see 3.3) Character distribution matrix (CDM) – $(-\infty; \infty)$	Spatial order *Ancillary*	Iconic Algorithmic	
4.1	Domain and range of linear functions (CWK) the rest of Worksheet Section A	If there is no value of x for which the function is undefined, then the domain is the set of real numbers (NI)	Isolate y for domain Isolate x for range "Is there any value that this variable can take and make this function undefined" If the answer is no, then x can take any number. So x is any real number ($x \in \mathbb{R}$), x goes from negative infinity to positive infinity	Auxiliary operations: "kill" replaced by divide (I) Spatial order (I) Take over, change sign (I) Emphasis on a character distribution matrix (see 3.2)	Spatial order *Ancillary*	Iconic algorithmic	Closed text- strong focus on the iconic weakened by empirical T_C^- Expression- oriented – auxiliary calculus to generate symbols for CDM, iconic propositions established (O_e^+)

4.2	Summarising: domain and range of linear functions	The domain of a linear function is the set of real numbers from negative infinity to positive infinity (I)	The function $y = \frac{2}{x-1}$ is given to contrast with the 6 linear functions dealt with. For $y = \frac{2}{x-1}$, domain is all real numbers except $x = 1$.	Auxiliary operations: Spatial order	General proposition: Domain of all linear functions is "x can take any number from negative infinity to positive infinity" (L745-748) – arrived at inductively – quasi –inductive activity False proposition based on what equation looks like *Ancillary*	Empirical (quasi-inductive) Iconic algorithmic	
5.0	Types of functions – quadratic, power and exponential	A quadratic function has 2 as its highest power. A quadratic function is a power function with the highest power being 2. The latter is incorrect. (I) T describes $x^{16} + 8x^{10}$ as a power function but this is incorrect. For an exponential function, the base is a number and the power is the variable (I)	T distinguishes between quadratic, power and exponential functions	n/a	Focus on what equations look like. *Ancillary*	iconic and incorrect	Closed text- strong focus on the iconic T_C^+ Expression- oriented because strong focus on iconic (O_e^+)

6.0	Domain and range of quadratic functions	If there is no value of *x* for which the function is undefined, then the domain is the set of real numbers (NI)	Domain: “Is there any number that *x* if our *x* take our *y* become undefined” T construct table but does not fill in values but points to table when asking if there any values of *x* that makes *y* undefined. no, then *x* can take any number. So *x* is any real number ($x \in \mathbb{R}$), *x* goes from negative infinity to positive infinity. Range: *y* is always positive.$\{y: y \in \mathbb{R}; 0 \leq y \leq \infty\}$ incorrect statement not corrected the next day.	Auxiliary operations: Spatial order imagistic	The domain is the outcome of calculation rather than decided upfront. Range is incorrect *Ancillary*	Iconic algorithmic Implied empirical (T constructs table) for domain *y* is always positive from 0 to infinity established deductively	Closed text- strong focus on the iconic weakened by deductive reasoning T_C^- Expression- oriented because strong focus on iconic weakened by deductive reasoning (O_e^+)

S02T04L02

EE	Topic/sub-topic	Propositions/Descriptions	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1	Domain and range of linear functions (revision) $y = x + 1$ $y = 8x + 6$ (buying bread)	If there is no value of x for which the function is undefined, then the domain is the set of real numbers (NI) Infinity is a number that we can't touch. It is not really quantifiable. It is a big big number but you can't quantify it. (I)	"Can we find any value here that makes this function undefined?" (L39) no, then x can take any number. So x is any real number ($x \in \mathbb{R}$), x goes from negative infinity to positive infinity. For interval notation: Determine lower bound and upper bound use round brackets if number is excluded and square brackets if number is included	**Auxiliary operations:** Spatial order Emphasis on a character distribution matrix (see L01 EE3.2)	*Ancillary*	Iconic Empirical – bread problem	Closed text- strong focus on the iconic weakened by empirical T_C^- Expression- oriented because strong focus on iconic weakened by empirical (O_e^+)
2.1	Domain and range of quadratic functions $y = x^2 + 2$	The codomain is the range (NI) The domain of a linear function is the set of real numbers from negative infinity to positive infinity (I) The domain of a quadratic function is the set of real numbers from negative infinity to positive infinity (I)	"Is there any number that our x can take so this function becomes undefined?" (L158) no, then x can take any number. So x is any real number ($x \in \mathbb{R}$), x goes from negative infinity to positive infinity. Generalises domain for linear and quadratic equation	Squaring, Addition **Auxiliary operations:** Spatial order Emphasis on a character distribution matrix (see L01 EE3.2)	Spatial order T restates students numerical order in terms of spatial order *Ancillary*	Iconic Algorithmic Empirical – range general - anything squared is positive so 0 is the smallest value therefore y greater than or equal to two – deductive explanation	Closed text- strong focus on the iconic weakened by deductive reasoning and empirical T_C^- Expression- oriented because focus on iconic weakened by local deductive explanation

2.2	Domain and range of quadratic functions $y = x^2 - 2$	The domain of a linear function is the set of real numbers from negative infinity to positive infinity (I) The domain of a quadratic function is the set of real numbers from negative infinity to positive infinity (I)	Generalises domain for linear and quadratic equation Does not determine range	**Auxiliary operations:** Spatial order Emphasis on a character distribution matrix (see L01 EE3.2)	Ancillary	Iconic Empirical	and empirical (O_e^+)
3.1	Domain and range of hyperbolic functions $y = \frac{1}{x} + 2$		"Do we have a number that makes this function undefined?" (L410) Yes, zero because dividing by zero is undefined. For range, "isolate" *x* then determine the value of *y* which makes the function undefined by working out what makes the denominator zero. T's procedure serves as a justification that the horizontal and vertical asymptotes are the values excluded from the domain and range.	Division, multiplication **Auxiliary operations:** Spatial order, Take over, change signs, Cross multiply Emphasis on a character distribution matrix (see L01 EE3.2)	*Ancillary*	Iconic Algorithmic Domain and range arrived at through calculation	Closed text- strong focus on the iconic T_C^+ Expression- oriented because strong focus on iconic (O_e^+)
3.2	Domain and range of hyperbolic functions $y = \frac{2}{x-3} + 4$			Multiply, divide **Auxiliary operations:** Spatial order, Take over, change signs, Cross multiply Emphasis on a character distribution matrix (see L01 EE3.2) *Ancillary*		Iconic Algorithmic Domain and range arrived at through calculation	

3.3	Domain and range of hyperbolic functions Worksheet addendum		Students develop shortcut rule inductively – for $y = \frac{a}{x-p} + q$, p is the value excluded for *domain* and q is the value excluded for *range*	**Auxiliary operations:** Spatial order Emphasis on a character distribution matrix (see L01 EE3.2) *Ancillary*		Iconic Algorithmic	

S02T04L03

EE	Topic/sub-topic	Computational activity	Procedure	Domain, codomain & operations	Realised content Content Substituted	Ground	Closed/open text
1.1	Domain and range of exponential functions (EXP)	codomain is range (NI) If there is no value of x for which the function is undefined, then the domain is the set of real numbers (NI) The domain of an exponential function is the set of real numbers from negative infinity to positive infinity (I) The domain of a linear function is the set of real numbers from negative infinity to positive infinity (I) The domain of a quadratic function is the set of real numbers from negative infinity to positive infinity (I) we start from smallest then go to biggest (497)	Domain: "Can we find a value of x that can make that function undefined?" (164) No, then x can take any number. So x is any real number ($x \in \mathbb{R}$), x goes from negative infinity to positive infinity. Establishes general proposition for domain of all exponential functions - ($x \in \mathbb{R}$), x goes from negative infinity to positive infinity. (L341) Range: Find the asymptote of the function, exclude this value from the range. T justifies this by substituting asymptote for y. (L343) General for hyperbola – asymptotes are excluded from domain and range (L341)	**Auxiliary operations:** Spatial order Emphasis on a character distribution matrix (see L01 EE3.2)	*Ancillary*	Iconic Empirical algorithmic	Closed text- strong focus on the iconic weakened by empirical T_C^- Expression- oriented because strong focus on iconic weakened by empirical (O_e^+)

1.2	Domain and range of exponential functions (CWK)	The domain of an exponential function is the set of real numbers from negative infinity to positive infinity	Uses general rule for domain of all exponential functions - $(x \in \mathbb{R})$, x goes from negative infinity to positive infinity. (L341) Range: Uses general rule: asymptote of the function, exclude this value from the range. General for hyperbola –	**Auxiliary operations**: Spatial order Emphasis on a character distribution matrix (see L01 EE3.2)	*Ancillary*	Iconic algorithmic	
2	Domain and range of trigonometric functions (EXP)	A function is a relationship between two variables (NI)	Over as a synonym for divide	Over as a synonym for divide	y = sin x is function but sin x is not – focus on what the expression looks like (I) *Ancillary*	iconic	Closed text- strong focus on the iconic T_C^+ Expression- oriented because strong focus on iconic (O_e^+)
3	Aids problem	A function is a relationship between two variables (NI)	Notion of a function as a relation – non-iconic auxiliary description	(R. +, x)	Shows what a function is by using an example of the number of AIDS cases doubling each day. Notion of a function as relation between two variables. So function thought of as a relation. The essential aspect of function i.e. unique output for every input is absent *Ancillary*	empirical	Closed text- absence of the fundamental weakened by the empirical T_C^- Expression- oriented because absence of fundamental weakened by empirical (O_e^-)

Appendix 6.7
Graph tutorial (Prestige College)

Grade 10 **Tutorial on Graphs**

1. Hand out: Thursday July 26th
2. Due Date: Monday 6th August
3. **Validation Test** on Tuesday 7th August
4. Scoring less than 50% on the VT will mean that your punctuality, completeness will count for 0. So make sure you understand what
5. Marking rubric – hand this in with your tutorial answers:

Name:	……………………..	Maximum	M obta
Tutorial	Punctuality	2	
	Neatness and Layout	3	
	Completeness	5	
Validation Test	Out of	30	
	Total for Tutorial (%)	100	

Question 1

1.1 Sketch the following graphs on separate sets of axes.
Show all intercepts with the coordinate axes, turning points, axes and asymptotes:

(a) $y=-x+5$ (b) $y=-x^2+25$

(c) $xy=5$ (d) $y=\frac{-5}{x}+1$

(e) $5y=x$ (f) $y=-5^x$

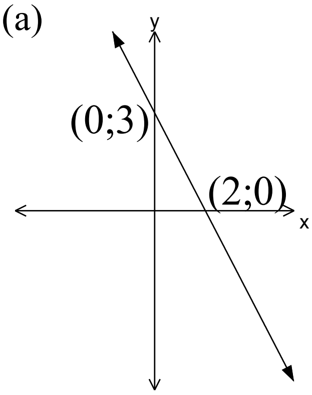
(g) $y=5^{-x}$ (h) $y^2=x$

1.2 State the Domain and Range for each of the graphs (a) – (h) above

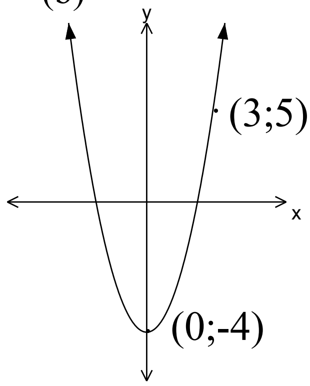
1.3 If each of the graphs (a), (c), (e) and (g) above is translated up 2 units write down the <u>new</u> equation for each one in the form *f(x) =………*

1.4 Find the equations of the following graphs

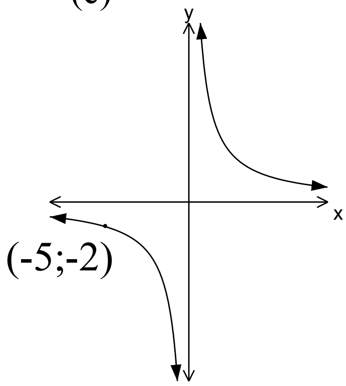
(a)



(b)



(c)

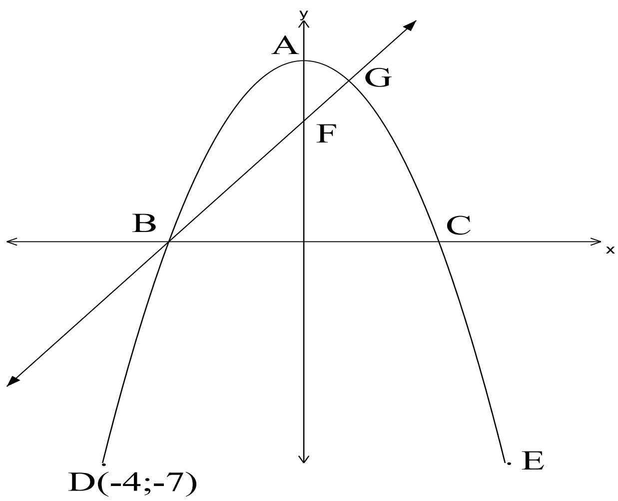


1.5 In the diagram alongside the graphs of $f(x) = -x^2 + 9$ and $g(x) = 2x + 6$ are shown. Write down the coordinates of

(a) A, B and C

(b) E, the point symmetrical to D about the y-axis

(c) F



1.6 Solve the equation $-x^2 + 9 = 2x + 6$ and explain how this helps find two of the points labeled on the graph.

Question 2

Consider the functions $s(x) = x^2 - 9$ and $t(x) = 2x - 6$

2.1 Sketch the graphs of s and t on the <u>same</u> set of axes, showing all intercepts and turning points

2.2 Use your sketch to find the values of x for which

2.2.1 $s(x) = t(x)$

2.2.2 $s(x) > 0$

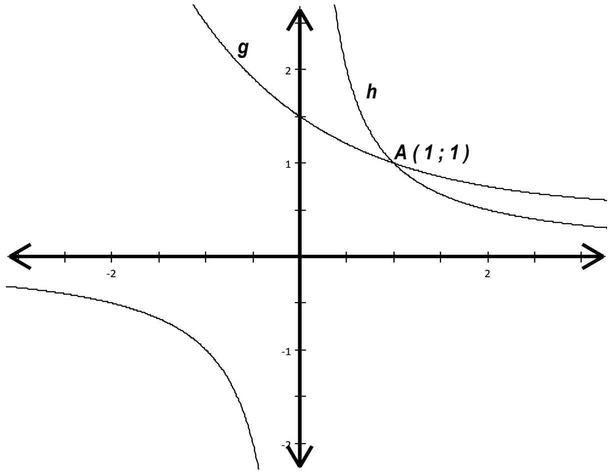
2.2.3 $s(x) < t(x)$

2.3 Write down the equation of $q(x)$ which is the result of shifting $s(x)$ 2 units up.

2.4 Write down the equation of $f(x)$, the result of reflecting $s(x)$ in the x-axis.

Question 3

$g(x) = b^x + c$ and $h(x) = \dfrac{k}{x}$ are shown. A(1 ; 1) is the point of intersection.



3.1 Find the values of k, c and b

3.2 What is the equation of the asymptote of $g(x)$?

3.3 What is the range of $g(x)$?

3.4 What is the equation of $f(x)$, the reflection of $g(x)$ in the y-axis?

Question 4

If $f(x) = 2x + 2$ and $g(x) = \dfrac{4}{x}$ and $h(x) = -x^2 + 2$, answer the following questions:

4.1 Determine the values of the following:

(a)	$f(0)$	(b)	$f(-2)$
(c)	x if $f(x) = 4$	(d)	$g(-2)$
(e)	$g(1)$	(f)	$h(0)$
(g)	x if $h(x) = f(x)$	(h)	$h(-\sqrt{2})$

4.2 Describe the type of function that is defined in each case

4.3 Draw a sketch graph showing these functions showing all asymptotes, axes of symmetry and intercepts clearly marked. You should use the values in 4.1 to assist you.

4.4 Determine the domain and range of the functions f, g and h.

Appendix 6.8
Common Test Functions 2012 (Prestige College)

Grade 10 Common Test Functions 2012 Total 45 marks

1. Given $f(x) = -2x^2 + 2x + 4$ and $g(x) = -8x + 4$

1.1 Determine the co-ordinates of the x-intercepts of graph f. (3)

1.2 Determine the co-ordinates of the turning point of f. (3)

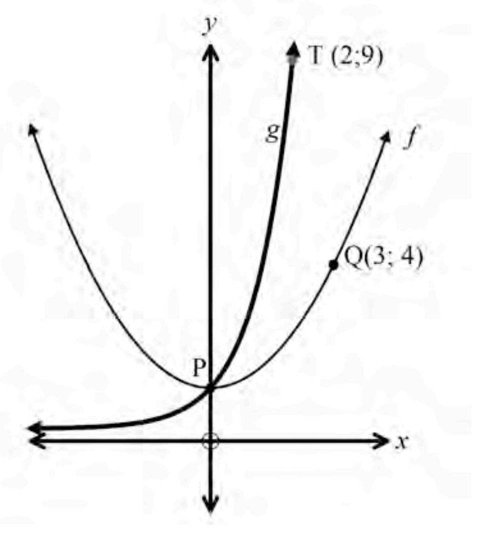
1.3 Draw the graphs of $f(x) = -2x^2 + 2x + 4$ and $g(x) = -8x + 4$ on the same set of axes on your answer sheet. Label the graphs clearly, including all the intercepts with the axes and the turning point. (5)

1.4 Write down the range of f. (1)

1.5 Determine algebraically the coordinates of the points of intersection of $f(x)$ and $g(x)$ (5)

1.6 Use your graphs to determine for which values of x , $f(x) < 0$. (2)

2. The graphs of $f(x) = ax^2 + p$ and $g(x) = b^x$ are drawn below.



2.1 Write down the coordinates of P, the *y* intercept of *f* and *g* (2)

2.2 If Q (3 ; 4) is a point on f, determine the equation of $f(x)$ (3)

2.3 If T (2; 9) is a point on $g(x)$, find the value of b (2)

2.4 If $g(x)$ is translated 2 units down to become $h(x)$, write down the equation of $h(x)$ in the form $y = \ldots.$ (1)

2.5 If the graph of $f(x)$ is reflected in the x-axis to become $s(x)$, write down the equation of $s(x)$ in the form $y = \ldots..$ (1)

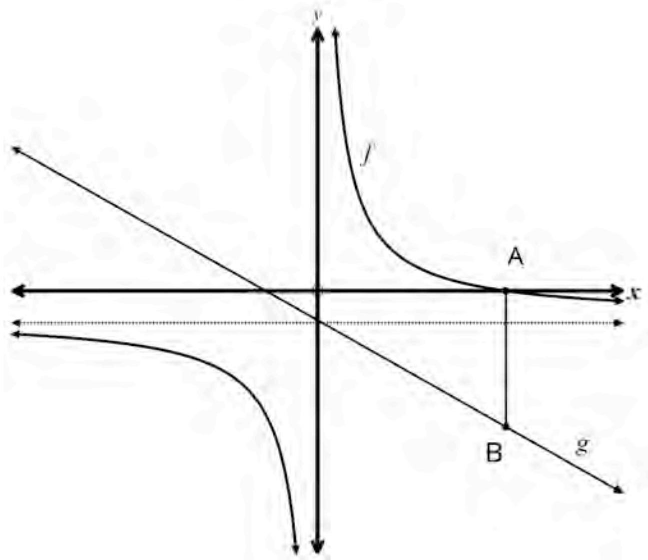
3. Given $f(x)=\left(\frac{1}{2}\right)^x - 2$ and $g(x)=\frac{-2}{x}-2$

3.1 Draw the graphs $f(x)=\left(\frac{1}{2}\right)^x - 2$ and $g(x)=\frac{-2}{x}-2$

on the same set of axes on your answer sheet (8)

3.2 Read off from your graph one point of intersection of $f(x)$ and $g(x)$ (1)

4 The graphs of $f(x)=\frac{4}{x}-1$ and $g(x)$, an axis of symmetry of $f(x)$, are drawn below A is the x-intercept of $f(x)$ and B lies directly below A on g.



4.1 Write down the domain of $f(x)$ (2)

4.2 Determine the equation of $g(x)$ (2)

4.3 Determine the length of AB (4)

Appendix 6.9

Parabola worksheet (Prestige College)

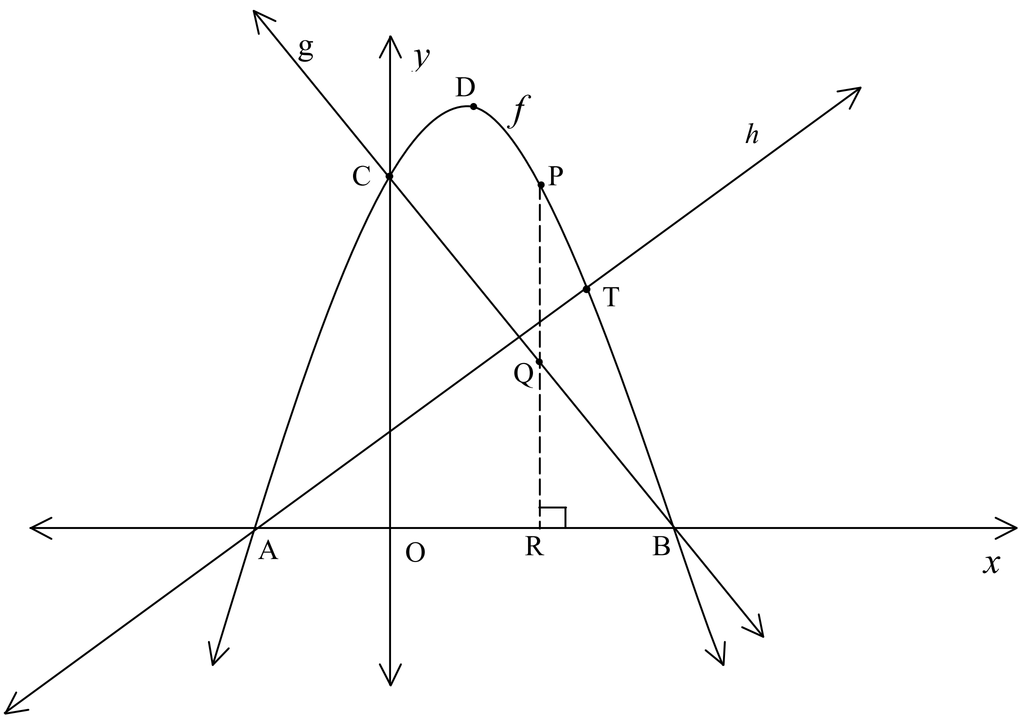
Parabola Worksheet

Question 1

In the accompanying sketch, which is not drawn to scale, *f* and *g* intersect at points B and C.

f and *h* intersect at points A and T. $f(x) = -x^2 + 2x + 8$ and $h(x) = x + 2$.

The parabola intersects the x-axis at points A and B and the *y*-axis at C.



1.1 Determine the length of AB. (4)

1.2 Write down the equation of the axis of symmetry of *f*. (1)

1.3 Determine the co-ordinates of D, the turning point of *f*. (2)

1.4 Determine the equation of g in the form $y = mx + c$. (3)

1.5 Calculate the length of PQ if $\text{OR} = 2$ units. (3)

1.6 Determine, by calculation, the co-ordinates of point T. (6)

Question 2

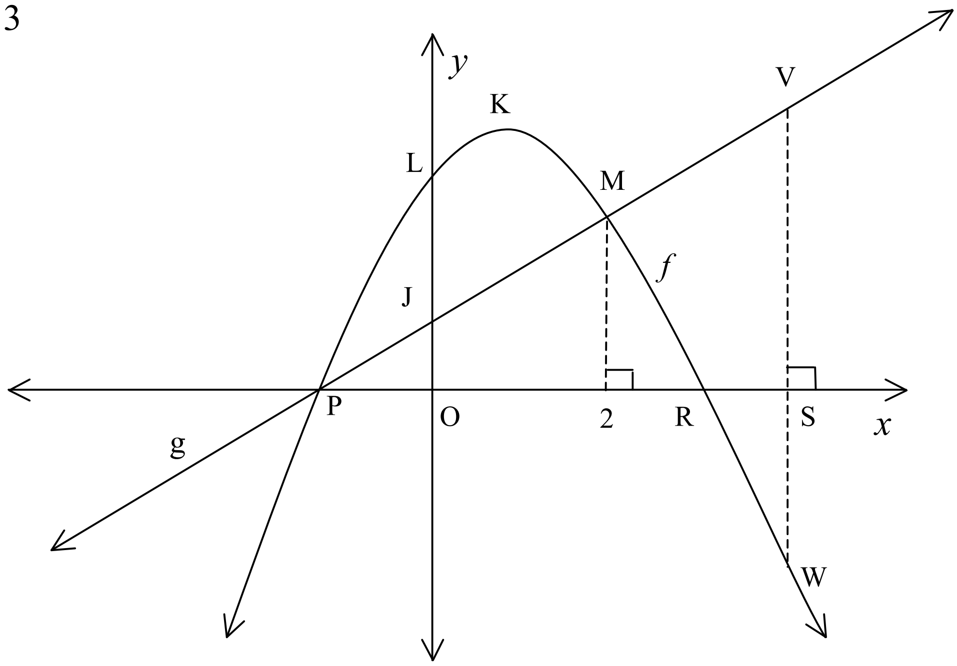
$f: y = x^2 - 2x - 3$ and $g: y = -3 + x$

2.1 Draw neat sketch graphs of f and g on the same system of axes and show the intercepts on both axes as well as the co-ordinates of the turning point of the graph of *f*. (8)

2.2 Use your graphs to read off the values of x for which $f(x) = g(x)$ (2)

2.3 State the domain and the range of f. (4)

Question 3



The accompanying graph represents the following functions:

$f: \quad y = -x^2 + x + 6$ and $g: \quad y = mx + c$

Determine:

3.1 the lengths of

3.1.1 OP and OR (4)

3.1.2 OL (1)

3.2 the co-ordinates of

3.2.1 the turning point K (3)

3.2.2 M (2)

3.3 the values of m and c (3)

3.4 the length of VW if OS = 4 units (3)

3.5 State the domain and the range of *f*. (4)

3.6 Write down the equation of *h* if *h* is obtained by translating f 2 units to the right and 1 unit up. (4)

3.7 Without doing any calculations write down the co-ordinates of the turning point of *h*. (2)

3.8 State the domain and the range of *h*. (4)

Appendix 6.10

Domain and range worksheet (Jono)

A. Find the **domain** and **range** of following linear functions (write it as **interval** and **set notations**):

1. y = x+1
2. H(x) = 2x-1
3. 3y-2x = 6
4. X= 3y
5. $\frac{x}{2}-\frac{y}{3}=1$

B. Find the **domain** and **range** of following quadratic functions (write it as **interval** and **set notations**):

1. $y=x^2-2$
2. $y=x^2+1$
3. $f(x)=-x^2+2$
4. $f(x)=x^2$

C. Find the **domain** and **range** of following hyperbolic functions (write it as **interval** and **set notations**):

1. $y=\frac{1}{x}-2$
2. $y=\frac{-2}{x}+2$
3. $y=\frac{2}{x-3}+2$
4. $y=\frac{6}{x+2}-4$

D. Find the **domain** and **range** of following exponential functions (write it as **interval** and **set notations**):

1. $y=\left(\frac{1}{2}\right)^x-1$
2. $y=3.2^x-3$
3. $y=2.3^{-x}$
4. $f(x)=2^x$
5. $f(x)=2^x-1$

Hyperbola functions Exercises

A. Write down the domain and codomain of the following functions:

1. $Y = \frac{2}{x-3} + 4$
2. $Y = -\frac{3}{x} - 9$
3. $Y = \frac{4}{x+1} + 2$
4. $Y = -\frac{3}{x+2}$
5. $Y = \frac{12}{x}$
6. $Y = \frac{4}{x+2} - 3$

Appendix 8.1

Prestige College test

Grade 10 **Common Test Functions 2012** **Total 45 marks**

1. Given $f(x)=-2x^2+2x+4$ and $g(x)=-8x+4$

1.1 Determine the co-ordinates of the x-intercepts of graph f. (3)

1.2 Determine the co-ordinates of the turning point of f. (3)

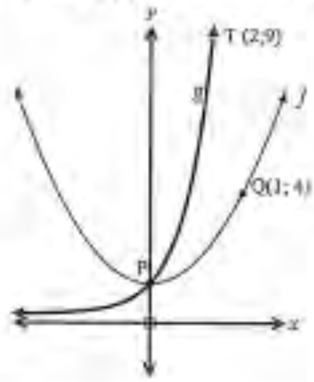
1.3 Draw the graphs of $f(x)=-2x^2+2x+4$ and $g(x)=-8x+4$ on the same set of axes on your answer sheet. Label the graphs clearly, including all the intercepts with the axes and the turning point. (5)

1.4 Write down the range of f. (1)

1.5 Determine algebraically the coordinates of the points of intersection of $f(x)$ and $g(x)$ (5)

1.6 Use your graphs to determine for which values of x, $f(x)<0$. (2)

2. The graphs of $f(x)=ax^2+p$ and $g(x)=b^x$ are drawn below.



2.1 Write down the coordinates of P, the y intercept of f and g (2)

2.2 If Q (1 ; 4) is a point on f, determine the equation of $f(x)$ (3)

2.3 If T (2; 9) is a point on $g(x)$, find the value of b (2)

2.4 If $g(x)$ is translated 2 units down to become $h(x)$, write down the equation of $h(x)$ in the form $y=$ .... (1)

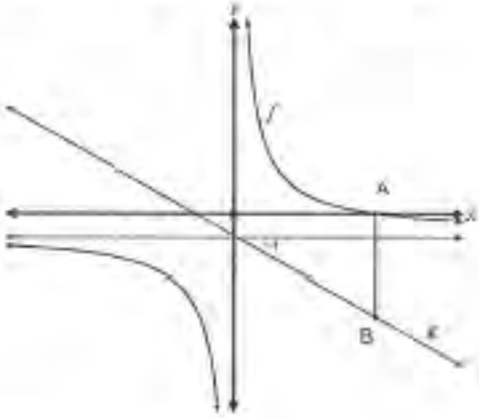
2.5 If the graph of $f(x)$ is reflected in the x-axis to become $s(x)$, write down the equation of $s(x)$ in the form $y=$ .... (1)

3. Given $f(x)=\left(\frac{1}{2}\right)^x-2$ and $g(x)=\frac{-2}{x}-2$

3.1 Draw the graphs $f(x)=\left(\frac{1}{2}\right)^x-2$ and $g(x)=\frac{-2}{x}-2$ on the same set of axes on your answer sheet (8)

3.2 Read off from your graph one point of intersection of $f(x)$ and $g(x)$ (1)

4 The graphs of $f(x)=\frac{4}{x}-$ and $g(x)$, an axis of symmetry of $f(x)$, are drawn below. A is the x-intercept of $f(x)$ and B lies directly below A on g.



4.1 Write down the domain of $f(x)$ (2)

4.2 Determine the equation of $g(x)$ (2)

4.3 Determine the length of AB (4)

Appendix 8.2

Prestige College test memo

Grade 10 Common Test 2012 Functions Memo

1. $f(x) = -2x^2 + 2x + 4$

1.1 $2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$ ✓

$(x+1)(x-2) = 0$ ✓

$x = -1$ or $x = 2$. ✓ (3)

1.2 TP: $x = -\frac{b}{2a} = -\frac{2}{2(-2)}$

$= \frac{1}{2}$. ✓

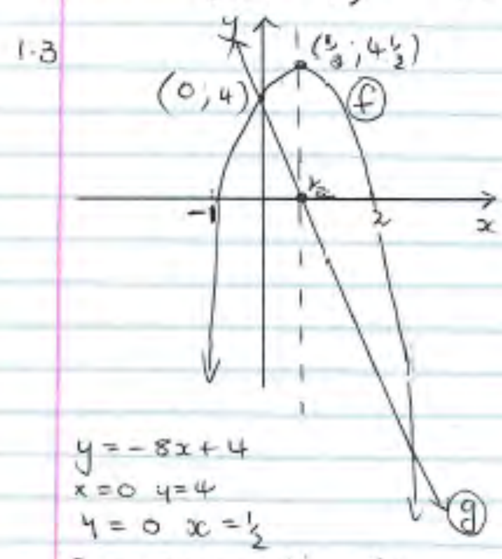
$y = -2(\frac{1}{2})^2 + 2(\frac{1}{2}) + 4$ ✓

$= -\frac{1}{2} + 1 + 4$

$= 4,5$

TP $= (\frac{1}{2}; 4\frac{1}{2})$ ✓ (3)

1.3



$y = -8x + 4$

$x = 0 \quad y = 4$

$y = 0 \quad x = \frac{1}{2}$

f: 3 marks y int ✓ 1

(−1) if no labels shape ✓ 1

x int + TP ✓ 1 (5)

g: 2 marks y int ✓ 1

x int ✓ 1.

1.4

Range of f: ✓

$y \in R;\ y \leq 4\frac{1}{2}$ (1)

1.5

$-2x^2 + 2x + 4 = -8x + 4$ ✓

$2x^2 - 10x = 0$

$2x(x-5) = 0$ ✓

$x = 0$ or $x = 5$ ✓

$y = 4 \qquad y = -36$ ✓

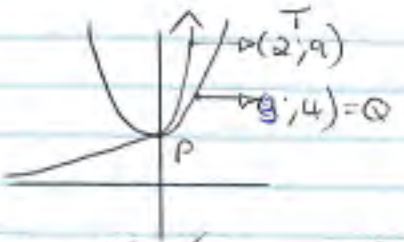
$\therefore (0;4) \quad (5;-36)$ (5)

must give co-ords ✓

1.6 $f(x) < 0$ when

$x < -1$ ✓ or $x > 2$. ✓ (2)

2.



2.1 $P = (0;1)$ ✓✓ (2)

2.2 $y = ax^2 + 1$ ✓

sub $(3;4)$

$4 = a(3)^2 + 1$ ✓

$9a = 3$ ✓ (3)

$a = \frac{1}{3}$

$\therefore y = \frac{1}{3}x^2 + 1$

2.3 $T(2;9)$ is on $g(x)$

$y = b^x$

$9 = b^2$ ✓

$b = 3$ (2)

$\therefore y = 3^x$ ✓

2.4 $h(x)$: $y = 3^x - 2$ ✓ (1)

2.5 $s(x)$: $y = -\frac{1}{3}x^2 - 1$ (1)

4.1 $f(x) = \frac{4}{x} - 1$.

Domain x:

$x \in R$ ✓ $x \neq 0$ ✓ (2)

4.2 $g(x) = -x - 1$ ✓✓ (2)

4.3 $A = x$ int

$\frac{4}{x} - 1 = 0$ ✓

$\frac{4}{x} = 1$

$x = 4$ ✓

At B $x = 4$

$g(4) = -4 - 1 = -5$ ✓

$\therefore AB = 5$ ✓ (4)

3.1 $f(x) = \left(\frac{1}{2}\right)^x - 2$ (4)

$x = 0$ $y = -1$

$y = 0$ $\frac{1}{2}^x = 2$

$2^{-x} = 2^1$

$x = -1$

asymp $y = -2$ ✓

y int $(0;-1)$ ✓

x int $(-1;0)$ ✓

shape ✓

$x = -2$ $y = 2$

(4)

$g(x) = -\frac{2}{x} - 2$ asympt $y = -2$ ✓

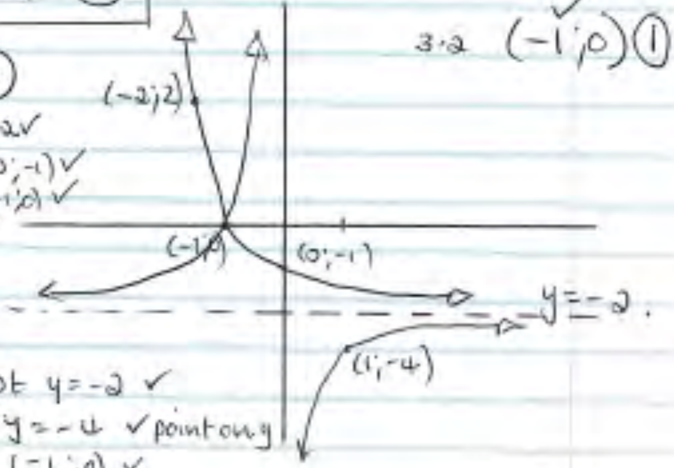
$y = 0$ $\frac{2}{x} = -2$

$x = -1$

$x = 1$ $y = -4$ ✓ point on g

x int $(-1;0)$ ✓

shape ✓



3.2 $(-1;0)$ ✓ (1)

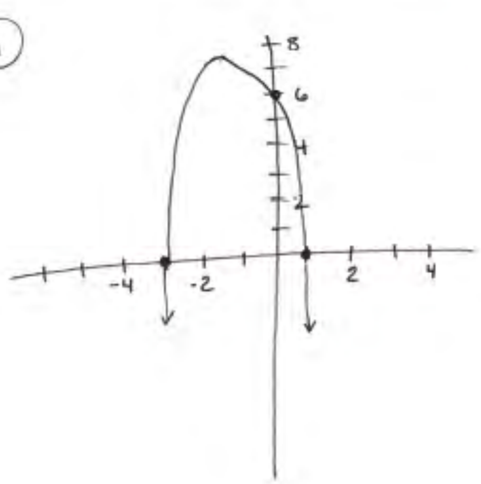
Appendix 8.3
Maya's test

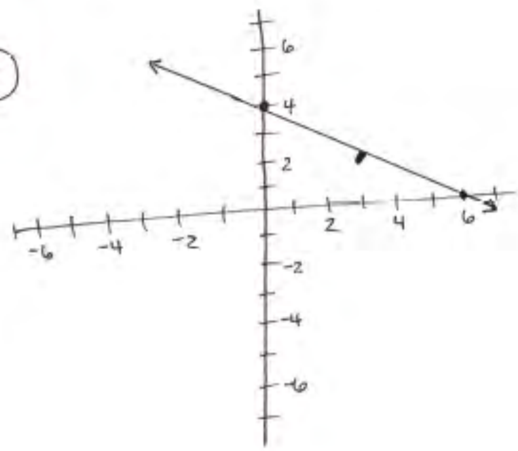
Quiz.

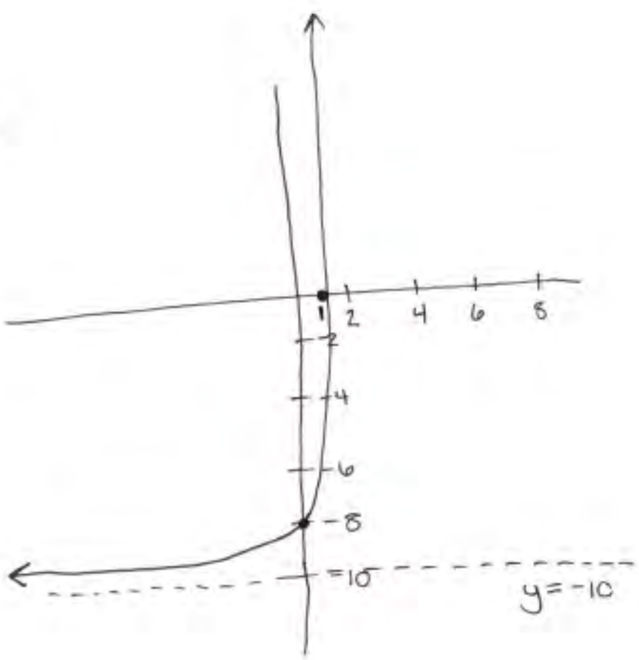
Name ____________

Class __________

Find the equation for each function. Do <u>all</u> calculations on the back.

① 

② 

③ 

④ Sketch this graph on the back.

$$y = \frac{-3}{x+1} - 2$$

Appendix 8.4

Maya’s test memorandum

$\boxed{\text{Memo}}$

1

$y = -2(x+3)(x-1)$

$y = -2(x^2 + 3x - x - 3)$

$y = -2x^2 \overset{-4x}{\cancel{+2x}} + 6$

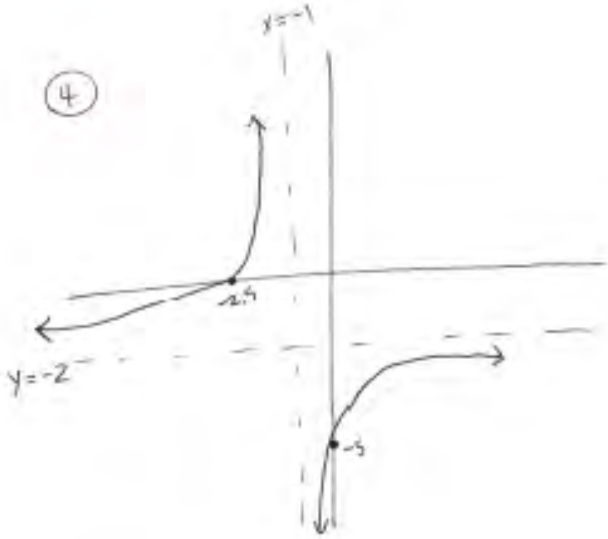
2

$y = -\frac{2}{3}x + 4$

3

$y = 2 \cdot 5^x - 10$

4



$y = \frac{-3}{0+1} - 2$

$y = -3 - 2$

$y = -5$

$(0;-5)$

$0 = \frac{-3}{x+1} - 2$

$2 = \frac{-3}{x+1}$

$2x + 2 = -3$

$2x = -5$

$x = -2{,}5$

$(-2{,}5;0)$

Appendix 8.5
Jono's test

Gr 10 functions TEST: domain and Range Name: ______________

70 minutes **10 October 2012**

A. Find the **domain** and **range** of following linear functions (write it as **interval** and **set notations**):

1. $\frac{x}{2}-\frac{y}{3}=1$
2. $y=x+1$

B. Find the **domain** and **range** of following quadratic functions (write it as **interval** and **set notations**):

1. $y=x^2-2$
2. $y=x^2+1$
3. $y=-x^2+2$

C. Find the **domain** and **range** of following hyperbolic functions (write it as **interval** and **set notations**):

1. $y=\frac{1}{x}-2$
2. $y=\frac{-2}{x}+2$
3. $y=\frac{2}{x-3}+2$

D. Find the **domain** and **range** of following exponential functions (write it as **interval** and **set notations**):

1. $y=\left(\frac{1}{2}\right)^x-1$
2. $y=2^x-1$

Appendix 8.6
Jono's test memorandum

(A) 1) $\frac{x}{2} - \frac{y}{3} = 1 \Rightarrow \quad y = \frac{3x-3}{2}$

$\therefore$ Domain : *) $\{x,\ x \in \mathbb{R} : -\infty < x < \infty\}$

*) $(-\infty ; \infty)$

$\therefore$ Range : *) $\{y,\ y \in \mathbb{R} : -\infty < y < \infty\}$

*) $(-\infty ; \infty)$

2) $y = x + 1$

$\therefore$ D: *) $\{x,\ x \in \mathbb{R} : -\infty < x < \infty\}$

*) $(-\infty ; \infty)$

$\therefore$ R: *) $\{y,\ y \in \mathbb{R} : -\infty < y < \infty\}$

*) $(-\infty ; \infty)$

(B) 1) $y = x^2 - 2$

$\therefore$ D *) $\{x,\ x \in \mathbb{R} : -\infty < x < \infty\}$

*) $(-\infty ; \infty)$

$\therefore$ R *) $\{y,\ y \geqslant -2\}$ or $-2 \leq y < \infty$

*) $[-2 ; \infty)$

2) $y = x^2 + 1$

$\therefore$ D *) $\{x,\ x \in \mathbb{R} : -\infty < x < \infty\}$

*) $(-\infty ; \infty)$

$\therefore$ R *) $\{y,\ y \geqslant 1\}$ or $1 \leq y < \infty$

*) $[1 ; \infty)$

3) $y = -x^2 + 2$

$\therefore$ D *) $\{x,\ x \in \mathbb{R} : -\infty < x < \infty\}$

*) $(-\infty ; \infty)$

$\therefore$ R *) $\{y,\ y \leq 2\}$ or $-\infty < y \leq 2$

*) $(-\infty ; 2]$

Appendix 9.1
Clinical interview transcript

Lines	**Learner: Leo** **School: Prestige College** **Teacher: Jada**
01	I: So as I said to you Leo I want you to try and explain to me what you did in question um .. one .. hey!
02	S: Okay.
03	I: You can look at your answers. You got everything right. You got the answer completely right I think.
04	S: Ja.
05	I: Full marks you got for the test.
06	S: No well.
07	I: No. I meant for that question.
08	S: Oh ja ja ja.
09	I: I'm focusing on question one.
10	S: Okay. Um ja so we have to find the x intercepts of er .. the .. parabola
11	I: Yes.
12	S: So what I did was I took the equation of the graph which is minus two x squared plus two x um plus four. And um what I did is I took out a common factor. I want to want to factorise the uh trinomial now. So I took out a common factor of minus two. x squared plus x plus two. And then I factorised .. so .. er .... …. [Factorises equation silently]
13	I: So talk me through how you're factorising.
14	S: Um sorry I'm just trying to remember .. umm ..so trying to get .. so um you need plus one in the middle. .. Well it would just be plus one plus two [Fills in values in brackets]
15	I: How did you do that?
16	S: Um cos x squared er so we have the x squared there. Then x times x um no wait sorry. Um I'm wrong. .. Er ..
17	I: Why are you wrong. Yes you are wrong but do you know why you're wrong?
18	S: Sorry. It's been a while since I've factorised.
19	I: Hey?
20	S: Um ja sorry I ..
21	I: Tell me why you are wrong. You are wrong.
22	S: The thing is …
23	I: What was it that you realised .. I'm just trying to see right again. What is it what is it that made you realise that you were wrong?

24 S: Because um I realised that x that um that two x plus one x is three x and that's um I'm trying to find one x.

25 I: Oh!

26 S: So it's going to have to be a minus x

27 I: but your mistake isn't here. I mean you're realising your mistake is in the second line already.

28 S: Of course ja.

29 I: hmm?

30 S: Ja. Ja I know. Um because I took out the minus. I'm not paying attention. Um so these would all turn into minuses because I took out the minus. Ja.

31 I: So you see your mistake?

32 S: Okay. So. Let me just do that again ...

33 I: Okay so where do these come from? [indistinct]

34 S: Umm the x the x squared

35 I: Okay

36 S: um and then so I'll try to um I'm going to try get this minus x here so uh I would have to minus two plus one so I can do minus two there and plus one there. So ..

37 I: So how do you know those are . How did you know you should write down plus one and minus two there?

38 S: So cos x times minus two is minus two x and then um plus one times x is x minus and then um minus two x plus x is minus x which is what I'm looking for. And then plus one times minus two is minus two which I'm looking for.

39 I:uh huh

40 S: so um ja. So ja so now I have a quadratic equation and I can find the x intercepts by saying that x is equal to minus one and x is equal to two.

41 I: Okay. Can you hold on? I see a couple of things here. So firstly where does the nought come from?

42 S: Well um you make it into I made it into a quadratic equation. I should have actually made it into a quadratic equation over here. So that's basically

43 I: so to make it into an equation you put equal to zero?

44 S: Ja

45 I: Why do you put it into zero? Could it be any other number?

46 S: Um um ja it could but um to find um y to find not to complicate any things and then find different numbers nought is the easiest digit

47 I: Okay.

48 S: to put there and so we can find our intercepts.

49 I: Okay. So you got that. And then how do you get x equals minus one and x equals two from that statement?

50 S: So you take um so you you take each bracket at a time so you get x plus one is equal to nought. So you just basically divide like splitting up the equation so that

51 I: Why do you say that? Why can you make that statement? Why can you make this statement from that statement?

52 S: Um because well we learnt it this way cos it’s how I understand a quadratic equation to be like um .. well ja pretty much like um .. ja that’s how I learnt

53 I: Okay. That’s fine. There are no right or wrong answers. I’m just trying to find out

54 S: I know like um I’m quite a .. I like to make um like I like to recognise things. I like to know that it’s a quadratic equation and

55 Hey?

56 S: I’m quite a .. I like to make um like I like to recognise things. I like to know that it’s a quadratic equation and

57 I: Yes.

58 S: I see the brackets and I see the nought so I just know what to do from there.

59 I: Alright.

60 S: But ja I don’t really know like how how I got it.

61 I: Okay.

62 S: [indistinct]

63 Fine. That’s fine. So now you’ve found .. oh yes another question. What happened to the minus two?

64 S: Where?

65 I: Here.

66 S: Here?

67 I: You had it there and you had it there but it’s no longer here. What happened to it?

68 S: Oh I um well it had no part in finding like .. it has no part in finding the rest of the um in finding the final answer.

69 I: Yes.

70 S: It was just there so that I could make my um factorising easier.

71 I: Okay

72 S: so um I took my common factor out. I got x squared minus x minus two. I showed it here just to show how how I took it out as a common factor.

73 I: mm

74 S: And then once it had no more part of my answer I can I just dropped it. Because I don’t need it in my final answer.

75 I: Okay. So can I ask you if I had .. if I wasn’t solving an equation but I was just factorising right

76 S: Okay.

77 I: like you did there hey?

78 S: Okay

79 I: So .. is it minus x minus two?

80 S: Yeah

81 I: Am I able to just .. is this correct if I just drop it in the next line?

82 S: Um .. if we're just doing a straight question then no. Um I think I'd get a mark deducted cos I haven't shown like the full .. like the full answer.

83 I: Okay. So here I'm allowed to drop it.

84 S: Cos I think because because we're working with a graph and I'm not actually doing it just straight like factorising I'm trying to find something else which doesn't relate at all to the minus two.

85 I: Okay.

86 S: Then that's why. That's why you can drop it. Ja. That's what I that's what I learnt.

87 I: Okay. Cool.

88 S: Okay. Um one point two. One point two. Determine the coordinates of the turning point of f. So the er .. ja. The turning point formula is um minus b over two a. Um .. sorry I just need to .. I just need to familiarise myself again over this. Um [indistinct mutters to self]. Ah! Um .. so minus b over two a would be … [indistinct whispers to self]. Ja okay. So this would be .. minus two over two minus two .. is equal to so the turning point would be equal to .. a half.

89 I: Okay. Now where does this come from? This minus four.

90 S: Ja

91 I: What is it?

92 S: um. Ja. So the equation for the like the standard way to set up a parabola would be y equals um a x plus b x plus c.

93 I: hmm

94 S: So what's this asking for is this number that which is before the second x and this is before the first x. So um in the equation the b so minus b is um two there

95 I: Yes

96 S: and then the a is the minus two over there. So I just substitute that into the equation and I work it out and I get a half.

97 I: Okay. But what is that a half though? That you found

98 S: It's um it's the coordinate of the x the x value and the coordinate of the turning value.

99 I: mm

100 S: Ja. So what it looks like now is I've got the turning point the x value

101 I: Yes

102 S: but now I don’t know the y value. So to get the y value I have to substitute the .. a half um into the equation.

103 I: Okay.

104 S: So I’ll say the f of x is equal to um minus two. Half I’m substituting instead of the x um plus two half plus four and um ja then I’ll just work it out from there and I can get the y value.

105 I: Okay.

106 S: So nine over two. You don’t mind me

107 I: No it’s fine you can

108 S: like straight from my

109 I: .. you just made a mistake hey.

110 S: did I?

111 I: You didn’t notice that. You made the same mistake

112 S: Oh. Sorry. Sorry. Ja

113 I: That’s an x squared.

114 S: Ja. Sorry. Um ja. So that’s squared. Sorry. Um so then ja so then the turning point .. the overall turning point would be a half nine over two.

115 I: Okay. Now tell me why do you substitute the half into the equation?

116 S: ja. Um because if you want to find the y value um for this by substituting you’re saying when x when the x coordinate is a half the y value will have to be this.

117 I: Okay.

118 S: Does that makes sense?

119 I: Yes it does. It makes absolute sense.

120 S: Cool. next question.

121 I: Ja. Okay. What have you done now? Okay can you draw the graph? Cos I think that was what you

122 S: One point three ja.

123 I: that’s what you need to do. You need to draw the graph.

124 S: Okay. [draws graph] … … Okay so we know the two x co ordinates of um This is minus one .. and two and we know the tuning poi the and the y cut is four

125 I: How do you know that?

126 S: Um because the c value is always the y cut of the graph

127 I: mm

128 S: [indistinct - calculates using calculator] four point five. Here’s four. Here’s the turning point.

129 I: mm

130 S: which is a half nine over two. Ja. So then this is enough to make my parabola. So ... [draws parabola]

131 I: Okay. Now how do you know Leo that the graph of this function

132 S: Ja

133 I: looks like this?

134 S: Um because um the negative means um that the graph is inverted. It's reflected um it means it's reflected in the *x* axis.

135 I: Okay.

136 S: Because um ... um I think it's right. Um I think I think what I can remem ja so um you you you make the *y* negative and then it causes a reflection in the *x* axis. I think I think that's why.

137 I: Okay

138 S: Cos I know I know that a negative there means that you invert the .. graph

139 I: the graph. Okay. Okay. So I'm asking you a more general question.

140 S: Okay.

141 I: which is how do you know this graph doesn't do something like this? How do you know it looks like that? Okay I think you're trying to answer the question like why does it look

142 S: why does it

143 I: like that [maximum turning point] or why does it look like that [minimum turning point].

144 S: ja ja ja

145 I: Okay. Which is one part of the answer. You're pointing to the negative in front of the two *x* the *x* squared.. that coefficient. My my question is also a more general question which is how do you know the graph does this after the two.. that this is what it's doing?

146 S: oh ja ja

147 I: And it doesn't do this.

148 S: um .. parabolas I'm pretty sure parabolas in general once they um once they cross the graph they keep going they keep going in their true direction and there's no deviation.

149 I: Okay. So what tells you that this is a parabola?

150 S: Um it's the equation is set up in er with er with the *x*.. with the squared

151 I: mm

152 S: there and then um I know it's a trinomial so it won't be.. so the turning point will be off the er *x* axis. But if it was um I would know it's a parabola if it was just like a perfect square combination.

153 I: Okay.

154 S: I would also know that it's a parabola so ja it's mostly about the square on the first *x*.

155 I: Okay. So if I gave you this equation ... what kind of graph would I ...

156 S: Well that would be five *x* plus five so you'd have a straight line graph.

157 I: Okay. But this is also a trinomial.

158 S: Ja. Ja I know.

159 I: And that's a trinomial.

160 S: Ja.

161 I: so when you say trinomial what do you mean?

162 S: I mean that um you can't simplify the equation further.

163 I: Okay. So this is not an example of a trinomial?

164 S: Well I um I .. no. I don't think so. Because I can three *x* plus two *x* is five *x* so I can simplify it further.

165 I: Okay. So this won't be a graph like this.

166 S: No it will just be just a straight line.

167 I: Okay. All right.

168 S: Okay

169 I: Where are we?

170 S: So um …

171 I: Just hold on. I want to ask you a question. I want to ask you about about the reflection. So now let's say we took this graph cos you were talking about reflections right?

172 S: Okay

173 I: We took this function *f* and we reflected it in the *x* axis what would its equation be?

174 S: Um .. two *x* squared minus two *x* minus four.

175 I: How do you know that?

176 S: Because um ..

177 I: What are you doing?

178 S: I'm taking away the minus and by doing that the graph obviously flipped and um but then the *y* cut cannot be plus four any more.

179 I: mm

180 S: So it has to be minus four and um the turning point will also have to be negative. So I think um ja

181 I: Okay. So what would its what would its turning point be? I mean you said its *y* cut was minus four. What would its turning point be?

182 S: um .. the negative of this which is minus half minus nine over two

183 I: Okay. So how would your graph look?

184 S: So .. Can I draw on this axis?

185 I: Hmm

186 S: So um .. sort of like … that.

187 I: Okay. So this point? aAe you now saying this point is now minus four?

188 S: Ja

189 I: And this point would be?

190 S: um

191 I: the turning point

192 S: Ja minus a half. Half minus nine over two. So that would be minus four.

193 I: So this point will be minus a half?

194 S: Ja

195 I: Okay

196 S: Ja it's not very accurate like

197 I: No no it's fine. But are you sure it's minus a half?

198 S: ... um ..

199 I: so this point here would be? .. You're saying it's minus a half. Are you saying minus half minus four and a half?

200 S: Ja

201 I: And why? Cos you multiplying everything by a minus? .. hm?

202 S: Ja. I'm pretty sure.

203 I: Okay. And so why are these staying the same then? ..

204 S: [no response]

205 I: Why are your *x* intercepts still in the same place?

206 S: ...um... well... um ... pretty sure if I did the calculation that they'd like still be the same.

207 I: uh

208 S: but you want another reason.

209 I: Now you're pretty sure that if you did the calculation.

210 S: If I did the calculation …

211 I: I'm just trying to follow your logic.

212 S: Ja

213 I: I'm just pushing you with your logic.

214 S: Ja. Okay okay.

215 I: You were just saying that I multiply everything by minus one. That's what you said.

216 S: Okay. Well I

217 I: that point there goes

218 S: What I see with the reflection is that everything is just the other way round.

219 I: Okay.

220 S: Like when I'm standing in front of a mirror. I'm there but like I'm

221 I: Yes.

222 S: turned around sort of. So …

223 I: And this is the mirror almost [pointing to x-axis]. That's what you're saying hey?

224 S: Ja. Ja. And so like I don't see why they would change at all. .. I think they would be the same.

225 I: So this point's minus four and that's correct I think. This point however is not it can't be minus half. It's sitting on this side of the x axis.

226 S: ... of course

227 I: Hey!

228 S: .. Ja. It has to still be a half.

229 I: That's it

230 S: But that would be minus nine over two. Ja. Sorry. Sorry. Ja.

231 I: Okay.

232 S: Um So it would still be a half.

233 I: Okay. So we got that.

234 S: Ja

235 I: So that's right [changes x coordinate of turning point to1/2]?

236 S: Okay

237 I: Okay. So why do these values stay the same?

238 S: .. um because I think the .. I I .. I see the graph as like turning like those are the pivots

239 I: mm

240 S: turning on .. it's turning on the pivots so

241 I: they gonna stay the same

242 S: Ja they stay the same.

243 I: Okay. Alright. Now if we took f. We turned it upside down hey.

244 S: Yes.

245 I: Reflected it on the x axis but if you took f same graph and you shifted it one unit to the left what would its equation be?

246 S: one unit to the left

247 I: mm

248 S: um ... there would be a new turning point which would be um minus half.

249 I: uh hm

250 S: Minus half nine over two

251 I: uh hm

252 S: um ... ah ... well it would be ... er umm ...

253 I: What did you do here? You shifted that point that one that way. Right?

254 S: Ja. I can use the ... that um turning point thing [writes speech indistinct] . ... No. I don't know.

255 I: What you doing to try and work out what b and [indistinct] hey?

256 S: try to work out [indistinct] That won't work though.

257 I: Hey?

258 S: it didn't work though. [indistinct]

259 I: Okay. Okay. So what would happen if you shifted that point what would happen to these points?

260 S: They would also shift.

261 I: Okay.

262 S: .. Ja. So that would be minus two. And that would be .. one [referring to x-intercepts]

263 I: mm

264 S: So [writes] ... … …

265 I: Okay. So what are you doing?

266 S: x minus one. x minus two. Sorry.

267 I: mm. ... What you doing here?

268 S: Um I'm sort of reversing it. You know how I factorised it?

269 I: oh ja ja.

270 S: So I'm reversing

271 I: You're doing the reverse. Okay.

272 S: Ja I'm reversing the graph ... So ... um ... [writes and whispers indistinct] Minus two. … So .. I think .. just from this

273 I: mm

274 S: I think the equation would be .. minus two x squared minus two x minus .. minus two. .. Or no! ... um ..

275 I: You're on the right track.

276 S: Umm ja.

277 I: You're on the right track except you made a mistake here [points to error].

278 S: Well ..

279 I: You see if you were going to go back hey?

280 S: I'm not sure about this.

281 I: Hey you not sure about this?

282 S: No I'm not sure what or where I made the mistake.

283 I: Oh. What where you made a mistake?

284 S: [indistinct]

285 I: Okay look here. So here you have .. you started .. you factorised. You took out the negative two.. you got that. And then you got the factors of the the quadratic trinomial there. Right?

286 S: Yes.

287 I: And then you got the roots. Okay?

288 S: Okay.

289 I: So if you were to go back. That's what you doing. Your roots was *x* minus one. Your solution was *x* minus one and *x* equals two. Then your factors would be *x* plus one *x* minus two…

290 S: Ohhh!

291 I: See the mistake you made.

292 S: Oh yes of course. So this is plus. That would be minus.

293 I: ja

294 S: um ja. So ... (whispers) minus *x* plus so that would be plus *x* and

295 I: minus two

296 S: minus two.

297 I: Okay so you're right to say that. What does this *a* mean when you put the *a* in front of it?

298 S: Ja this *a* means that um tsk can't. ... It's basically the number in front of this and it can influence the equation.

299 I: mm

300 S: How I remembered when my teacher explained it to me was when there could be .. like .. there could be so many more more graphs that have *x* squared plus *x* minus two. But this number makes it exact and it like .. you can't you can't just say that the equation of the graph is that because there could be some other value.

301 I: That's right

302 S: um ja .. so and then you would work that out by doing the equation

303 I: Uhmm

304 S: um with with *y*. But in this case I don't know where.. the *y* would be .. It would still be .. four right? Ja it would. So still

305 I: How do you know that?

306 S: Oh no. No. Sorry.

307 I: Why you changing your mind? Firstly how do you know that? And when I asked you the question you immediately thought it was wrong hey.

308 S: I’m just thinking cos the *y* ...

309 I: Well this one was four. Right?

310 S: ja. Wouldn’t I say .. the *y* here .. wouldn’t I say that’d be the *y* cut? That’s what I’m thinking.

311 I: Yeah. It could be the *y* cut.

312 S: ja. Ah. I don’t know.

313 I: But do you have the *y* cut?

314 S: oh ja I do. It’s four. And that wouldn’t change if we shifted it left.

315 I: uh hm

316 S: So .. um ..

317 I: Why wouldn’t it change? If the turning point is changing why would the *y* intercept not change? Why would the *y* cut not change?

318 S: .. um because you’re not moving up or down. You’re just moving it

319 I: across

320 S: across. So you um .. ja. But ...

321 I: So this point? I mean …

322 S: You still ..

323 I: that point shifted there. Right?

324 S: Ja

325 I: cos that’s the new point now. This point shifted somewhere here. You said one

326 S: one. Ja.

327 I: you had. And this point you said shifted there.

328 S: Ja.

329 I: So why would this point not shift?

330 S: No it would.

331 I: Oh I see

332 S: but it would still be four.

333 I: Yes?

334 S: It would shift but it would .. the um

335 I: so what would its new position be?

336 S: it would be it would be oh um minus one four.

337 I: minus one four

338 S: It would not be be nought four?

339 I: But is that a y cut then?

340 S: .. um ... no [laughs]

341 I: Hey? [Laughs]

342 S: No. Shees I'm just ja ...

343 I: No it's fine. I'm just .. we don't have to go further with this but you've got it. You see you've got y you've got x and you've got a. You've got three unknowns here.

344 S: Yeah yeah yeah.

345 I: And you were right. To find a point so that you can have only this unknown to solve it.

346 S: Of course ja then we .. ja then we have to substitute. Then we take a point

347 I: That's right.

348 S: on the graph and then we sub in

349 I: that's it. And which is the only other point that you know? On the graph.

350 S: It will be one and four. Minus one and four.

351 I: minus one and four. You could use those.

352 S: or you could do um one and nought.

353 I: One and nought .. oh okay.

354 S: Cos it's on the graph. So ja

355 I: Ja ja anyway. Okay. You don't have to. We've done it.

356 S: Ja sorry I just ah

357 I: that's fine. That's fine.

358 S: I don't know. I haven't like er .. when I learn for exams like I have to revise before I like I properly understand the work again. Otherwise I just

359 I: you forget again.

360 S: get careless and forget.

361 S: But like I do understand it properly.

362 I: You do. You have. I mean you did very well.

363 S: [laughs]

364 I: One last question. We were speaking about functions all the time hey?

365 S: Ja.

366 I: Okay now what in your understanding is a function?

367 S: .. a function?

368 I: mm

369 I: Like in this sense? The mathematics sense?

370 I: mm mmm

371 S: Well .. um.. it's a .. it's an equation .. it's basically an equation that can be .. relayed into a um .. points on a graph.

372 I: Okay.

373 S: And a um .. that's basically it.

374 I: So can you give an example of something that's not a function?

375 S: ... um ... one plus one

376 I: is not a function?

377 S: is not a function.

378 I: Why's it not a function?

379 S: Well because I don't know how we'd find an x or y value from one plus one.

380 I: hmm

381 S: Ja

382 I: So function's an expression that's got x and y in it.

383 S: Ja well that's what I sort of understand it to be.

384 I: So is this a function?

385 S: .. well ja. This

386 I: Is this a function? The red one [points to graph drawn by student earlier].

387 S: Ja. Well it has the equation which would be the function.

388 I: Okay. Great. Thanks Luca.

389 S: Is that it?

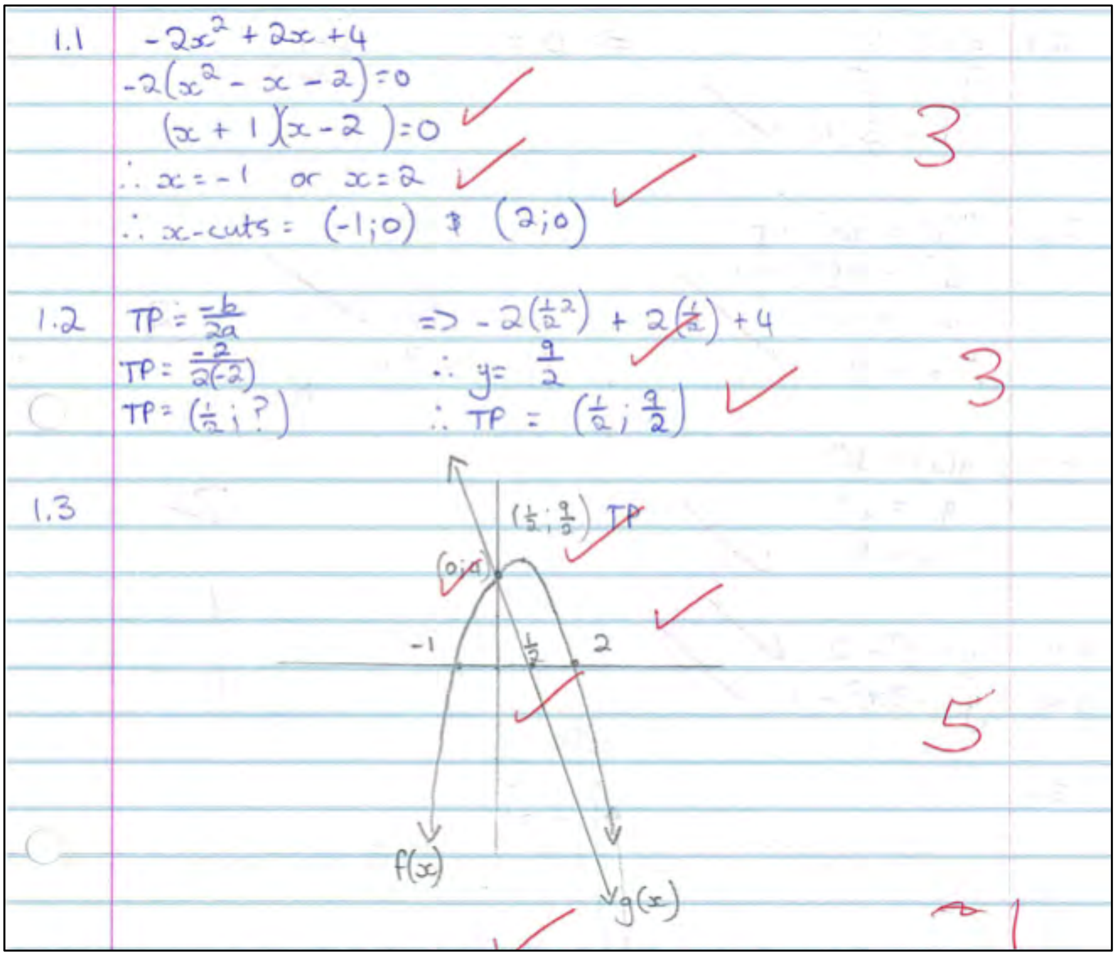
390 I: that's it.

391 S: thanks.

Appendix 9.2
Analysis of a clinical interview

Description of computational activity

Leo scored 17 out of 19 for Question 1 which suggests that he is able to produce the legitimate text expressively i.e. he is able to produce mathematical expressions that converge with the Mathematics encyclopaedia expressively. However, we need to examine his computational activity in order to ascertain the extent to which the content realised converges with the Mathematics encyclopaedia.



Leo identifies the function as a parabola because “the equation is set up in er with er with the x.. with the squared” and “I know it’s a trinomial”(S01T02P01: Lines 150-152) and “I would also know that it’s a parabola so ja it’s mostly about the square on the first x.” (S01T02P01: line 154). This suggests that his computational activity is regulated by iconic ground. The presence of the ”squared” and “trinomial” indicates to him that the equation represents a parabola. The propositions that he uses stand in place of the general formula of a parabola, $y = ax^2 + bx + c$.

Secondly, he concludes that the parabola is "inverted" or "reflected" cause "Um because um the negative means um that the graph is inverted. It's reflected um it means it's reflected in the x axis." (S01T02P01: line 134). Again, the use of an iconic auxiliary proposition, "because um the negative means um that the graph is inverted", which stands in place of the encyclopaedic proposition: $y = ax^2 + bx + c$ has a maximum value if $a < 0$.

In calculating the x-intercepts. He starts by factorising $-2x^2 + 2x + 4$. He states that he has taken out -2 but produces $-2(x^2 + x + 2)$ which suggests that he has in fact divided the expression $-2x^2 + 2x + 4$ by 2 rather than -2. When he realises his error, he states that "I took out the minus [...] so these would all turn into minuses", which suggests that he conceives of -2 as 2 with a negative sign appended to it rather than as an integer. He also uses auxiliary operations which splits the /-2/ into /-/ and /2/ as well as another auxiliary operation which "turns" positives into negatives.

He realises that he needs an equation in order to calculate the value of x, he equates $f(x)$ with 0. His reasons for equation $f(x)$ with zero is because "not to complicate any thing and then find different numbers. Nought is the easiest digit ... to put there" (S02T02P01: Line 46 - 48). He states that $f(x)$ could equal any number but that 0 is the easiest. This proposition stands in place of the notion that the x-intercept lies on the x-axis and any point on the x-axis has a y-coordinate of 0.

S: Well um you make it into I made it into a quadratic equation. I should have actually made it into a quadratic equation over here. So that's basically

I: So to make it into an equation you put equal to zero?

S: Ja

I: Why do you put it into zero? Could it be any other number?

S: Um um ja it could but um to find um y to find not to complicate any thing and then find different numbers. Nought is the easiest digit.

I: Okay.

S: to put there and so we can find our intercepts. (S02T02P01: Lines 42- 48).

So, Leo knows that he needs an equation in order to solve for x, which means that the notion of an equation acts as a regulative resource. He, however, does not make any connection between the x-intercepts and the value of y. It seems as though all equations for him equals 0.

He then "take(s) each bracket at a time so you get x plus one is equal to nought. So you just basically divide like splitting up the equation" (S01T02P01: Lines 50) "because well we learnt it this way cos it's how I understand a quadratic equation to be like um .. well ja pretty much like um .. ja that's how I learnt it" (S01T02P01: Lines 52). So the zero product property which underpins the solution of a quadratic equation factored into linear factors is replaced with an operation-like manipulation which allows Leo to "split" the equation into two parts. So although it appeared earlier that the notion of an equation acts as a regulative resource, it seems as though the 0 was required to ”split” the expression into two parts. So, the notion of an equation is in fact absent.

Leo ignores -2 in the equation $-2(x+1)(x-2)=0$ because "it has no part in finding the rest of the um in finding the final answer … It was just there so that I could make my um factorising easier … And then once it had no more part of my answer I can I just dropped it. Because I don't need it in my final answer.." (S01T02P01: Lines 68 -72). In other words, Leo does not recognise that the equation $-2(x+1)(x-2)=0$ is equivalent to the equation $(x+1)(x-2)=0$ because both equations have the same solution and that the second equation an be derived from the first if we divide $-2(x+1)(x-2)=0$ by -2 to produce $(x+1)(x-2)=0$. The /-2/ is "dropped" because it is not "it had no part in finding like .. it has no part in finding the rest of the um in finding the final answer […] It was just there so that I could make my um factorising easier". (S01T02P01: line 68-70). So, "dropping" is another auxiliary operation in use.

Primary data: Summary of computational activity

Absence of encyclopaedic computational resources - the notion of an equation is absent and the zero product property is absent.

Auxiliary propositions:

Make it a quadratic equation by making it equal 0, "not to complicate any things and then find different numbers nought is the easiest digit" (Ll42-46)

"because um the negative means um that the graph is inverted. It's reflected um it means it's reflected in the *x* axis" (L134)

It's a parabola because " it's the equation is set up in er with er with the *x*.. with the squared" (L150)

Auxiliary operations:

turn into minuses because I took out the minus (L30)

splitting up the equation into two parts (L50)

"dropping -2" (L74)

Secondary data

Realised content: Ancillary because of the presence of auxiliary operations and auxiliary propositions and the absence of encyclopaedic computational resources.

Computational performance: strongly closed pedagogic texts

Orientation to mathematics: strongly expression-oriented because the strong focus on expressions as the primary computational id offset by his empirical approach to factorising the quadratic trinomial

Appendix 10.1
2014 National Senior Certificate performance of Evergreen High and Prestige College

NSC passes are categorised in terms of the type of Higher Education programme that the pass gives a student access to. There are four categories of passes: (1) no access to further study, (2) access to a Higher Certificate, (3) access to a Diploma, and (4) access to a Bachelors degree. A Higher Certificate and a Diploma are both vocational qualifications pegged at exit level 5 and exit level 7 on the National Qualification Framework (NQF) respectively. A Degree is an academic qualification set at exit level 7. The distribution of pass categories for the students at Prestige College and Evergreen High who sat the 2014 NSC examination is displayed in Table 1.

Table 1. NSC performance by level of pass

School	Wrote	Passed	Failed	Bachelors	Diploma	Higher certificate
Prestige College	149	149 (100%)	0 (0%)	148 (99,3%)	1 (0,7%)	0 (0%)
Evergreen High	82	77 (93,9%)	5 (6,1%)	49 (59,7%)	16 (19,5%)	12 (14,6%)

All the students at Prestige College who wrote the 2014 NSC examination passed at a level that provided access to Higher education. In contrast, 93,9% of students who wrote the NSC exam at Evergreen High passed but only 59,7% achieved a pass allowing entry into a Bachelor's degree compared to 99,3% of students at Prestige College. Thus, almost all the students at Prestige College achieved an NSC pass which enabled access to an academic Higher Education qualification and therefore potentially to occupations that are mapped to upper-middle-class/elite and middle-class locations (see social class locations discussed in Chapter 4). In comparison, about 40% of Evergreen High's students failed to achieve a pass that allows access to an academic Higher Education qualification and are therefore potentially excluded from occupying upper-middle-class/elite or middle-class positions in society. So, although not much of a difference is evident if we compare the two schools in terms of overall number of passes, there is a substantial difference in the quality of passes at the two schools.

Appendix 10.2

2014 National Senior Certificate Mathematics and Mathematical literacy results

An examination of student performance on the 2014 NSC Mathematics/Mathematical Literacy examinations (see Table 2) provides a further perspective on the disparity in performance of students at the two schools. Students are expected to elect Mathematics or Mathematical Literacy in order to meet the requirements of the NSC. None of the students at Evergreen High who sat the 2014 NSC examinations wrote Mathematical literacy.

Table 2. NSC 2014 Mathematics and Mathematical literacy percentage distribution

School	Wrote	Passed	Failed	80-100	70-79	60-69	50-59	40-49	30-39	0-29
Prestige College (Mathematics)	130	130 (100%)	0 (0%)	80	23	18	7	2	0	0
Prestige College (Math Literacy)	19	19 (100%)	0 (0%)	11	7	1	0	0	0	0
Evergreen High (Mathematics)	82	65 (79,3 %)	17 (20,7%)	0	4	7	19	13	22	17

The results shown in Table 2 illustrates that students at Prestige College achieved a greater pass rate in Mathematics and a greater proportion scoring 50% or above when compared to Evergreen High. The proportion achieving 60% or more at Prestige College is even greater than that at Evergreen High, where only 13,4% scored 60% or more. As such, the results reflect the long standing national disparity in performance in Mathematics along social class lines discussed above.