The LIBOR Market Model in the South African Setting

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Contents

Chapter 1

Introduction

The LIBOR Market Model has been widely implemented internationally but its implementation has lagged in South Africa. This is due to the very illiquid swaption volatility surface and the fact that forward starting swaps do not actively trade in South Africa. Without a liquid volatility surface, it becomes more difficult to calibrate the LIBOR Market Model and use it to price exotic products. Without forward starting swaps, it becomes difficult to hedge exotic interest rate options.

The LIBOR Market Model is a modelling framework which uses the no-arbitrage pricing theory to work out prices of exotic interest rate options by using as inputs the quoted cap and swaption volatilities. Thus the main benefit of the LIBOR Market Model is that calibration is almost immediate as the calibration inputs are market observable which is not the case in short rate models.

1.1 Subject

This dissertation aims to provide an overview of the theory and application of the LIBOR Market Model in South African interest rate markets. It aims to touch on the most important points in implementing the model in a bank. Emphasis shall be placed on the underlying theory, different ways of calibrating the model, hedging of swaptions and caplets and the pricing of barrier swaptions. Barrier options are very popular in the more liquid South African equity setting and therefore it is proposed that barrier swaptions may enjoy similar popularity as the South African interest rate markets develop.

1.2 Scope and Limitations

The dissertation shall focus on at-the-money interest rate options - the volatility smile shall not be taken into account (Brigo and Mercurio [2006], Meister [2004], Svoboda-Greenwood [2007] and Rebonato [2007] provide in depth discussions of various ways in which the smile can be incorporated into the LIBOR Market Model).

1.3 Plan of Development

The next chapter shall give an overview of the necessary tools and concepts in working with the LIBOR Market Model in the South African setting. The third chapter shall then look at the main approaches to the implementation of the LIBOR Market Model and establish the LIBOR Market Model theory. Given the theory, the fourth chapter shall look at the ways in which the LIBOR Market Model can be calibrated to market data. The fifth chapter provides a comparison of the calibration approaches by looking at how well each method allows the hedging of standard interest rate options on historical data. The sixth chapter shall look at the implementation of the LIBOR Market Model in pricing exotic barrier swaption contracts and also derive a closed form approximation for the barrier swaption price based on the Swap Market Model.

Chapter 2

Preliminaries

This chapter aims to give an overview of the necessary tools and concepts in working with the LIBOR Market Model. A good reference for South African market practice is West [2007].

2.1 Market instruments

Consider a set of dates $\mathcal{T} = \{T_{-1}, T_0, T_1, ..., T_n\}$ where $T_{-1} = 0$ and $\delta_i = T_i - T_{i-1}$. In this dissertation $\mathcal{T}$ will contain dates that are spaced approximately three months apart (the exact spacing will be determined according to the South African swap day schedule which will be explained below).

The 'modified following' rule determines when n months is from today. West [2007] states that according to the modified following rule, n months from today is given by the following criteria:

- It has to be in the month which is n months from the current month.
- It should be the first business day on or after the date with the same day number as the current. But if this contradicts the above rule, we find the last business day of the correct month.

Interest rate instruments are based on nominal amounts. Without loss of generality we will take the nominal amounts to be equal to one.

Definition 2.1.1 *A zero coupon bond is a contract that allows the holder to receive the nominal value of* 1 *at time* T_i. *The value of the contract at time* t *is* $P(t, T_i)$.

Definition 2.1.2 *A forward rate agreement (FRA) is a contract between two parties which fixes an interest rate between* T_α *and* $T_{\alpha+1}$. *One party agrees to pay a floating rate in exchange for a fixed rate at some point in the future. The fixed rate is determined at the outset of the contract such that the contract value is zero. All payments are based on a unit notional amount. The fair value of the fixed rate that will set the contract value to zero at time* t $(t < T_\alpha)$ *is*

$$Fixed = L(t, T_\alpha) = \frac{1}{\delta_{\alpha+1}} \left(\frac{P(t, T_\alpha)}{P(t, T_{\alpha+1})} - 1 \right)$$

and the floating payment (which is only known at time T_α*) is* $L(T_\alpha, T_\alpha)$.

In South Africa, Forward Rate Agreements are settled in advance and the floating payment is based on three month JIBAR. This implies that, the fixed rate payer will receive

$$\frac{\delta_{\alpha+1}L(T_\alpha, T_\alpha)}{1 + \delta_{\alpha+1}L(T_\alpha, T_\alpha)}$$

at time T_α. The floating rate payer will receive

$$\frac{\delta_{\alpha+1}L(t, T_\alpha)}{1 + \delta_{\alpha+1}L(T_\alpha, T_\alpha)}$$

at time T_α. The typical international practice is for the fixed rate payer to receive $\delta_{\alpha+1}L(T_\alpha, T_\alpha)$ at time $T_{\alpha+1}$ and for the floating rate payer to receive $\delta_{\alpha+1}L(t, T_\alpha)$ at time $T_{\alpha+1}$. The FRA price under these two approaches is the same (all else equal).

An $n \times m$ South African FRA which starts at time t, is an FRA with $T_\alpha = \text{modfol}(t, n)$ and $T_{\alpha+1} = \text{modfol}(T_\alpha, m - n)$. South African FRAs are generally based on quarterly forward periods. 3×6, 6×9, 9×12 are typical FRAs in the South African setting.

Definition 2.1.3 *An interest rate swap is an agreement between two parties to exchange floating and fixed payments on a unit notional amount at $T_0, ..., T_\beta$ (the payment dates). The payments are determined at times $T_{-1}, ..., T_{\beta-1}$ (the setting dates). The swap is entered into at time T_{-1}. At time T_i ($i = 0, ..., \beta$) the floating rate payer pays $\delta_i L(T_{i-1}, T_{i-1})$ and the fixed rate payer pays $\delta_i S_{-1,\beta}(T_{-1})$. $S_{-1,\beta}(T_{-1})$ is the fixed rate that sets the swap value equal to zero at time T_{-1}.*

$$S_{-1,\beta}(T_{-1}) = \frac{P(T_{-1}, T_{-1}) - P(T_{-1}, T_\beta)}{G_{0,\beta}(T_{-1})} = \frac{1 - P(T_{-1}, T_\beta)}{G_{0,\beta}(T_{-1})}$$

where

$$G_{0,\beta}(T_{-1}) = \sum_{i=0}^{\beta} \delta_i P(T_{-1}, T_i).$$

In South Africa, payments in a swap agreement are made on a quarterly basis. Suppose that a South African swap (which is entered into at time t) has n payments. The payments will occur at $\text{modfol}(t, 3 \times i)$ where ($i = 1, 2, ..., n$). This is different from the FRA convention used to determine the payment dates. Thus, West [2007] notes that a South African swap is not a strip of South African FRAs.

Definition 2.1.4 *A forward starting interest rate swap is an agreement, at time t ($t < T_\alpha < T_\beta$), between two parties to enter into an interest rate swap at time T_α on a unit notional amount. The reset (or setting) dates of the swap are T_α, $T_{\alpha+1}$,...,$T_{\beta-1}$. The payment dates are $T_{\alpha+1}$, $T_{\alpha+2}$,...,T_β. The floating rate payer pays $\delta_{i+1} \times L(T_i, T_i)$ at time T_{i+1} ($i = \alpha, ..., \beta - 1$). The fixed rate payer pays $\delta_{i+1} \times S_{\alpha,\beta}(t)$ at time T_{i+1} ($i = \alpha, ..., \beta - 1$). The value of the swap rate that will set the value of the forward starting swap to zero at time t is*

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{G_{\alpha+1,\beta}(t)} \tag{2.1}$$

where

$$G_{\alpha+1,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P(t, T_i).$$

Consider the remarks on the swap definition (definition 2.1.3) and the remarks on the FRA definition (definition 2.1.2). The payment dates of a series of quarterly FRAs will not be the same as the payment dates on swaps with quarterly payments. In other words, due to the application of the modified following rule not being the same in the case of the FRAs and swaps, the two instruments will refer to different $\mathcal{T} = \{T_{-1}, T_0, T_1, ..., T_n\}$.

For example, on 15 August 2008, a one period swap will apply between 17 November 2008 and 16 February 2009. The corresponding $3X6$ FRA will apply between 17 November 2008 and 17 February 2009.

For the purposes of calibrating the LIBOR Market Model, it would be very useful if the payment dates of swaps and FRAs are the same. We therefore make the following assumption.

Assumption 2.1.5 *The payment and setting dates of the tradeable South African FRAs can be determined according to the South African swap day count convention.*

The assumption can be justified to the extent that the set of quarterly dates arising from the FRA day count convention is similar to the set of quarterly dates arising from the swap day count convention. We assume that the differences between the two sets of dates is negligible. One of the first useful consequences of this assumption is that we can now represent the forward starting swap rate as a strip of forward rates (and these forward rates underly tradable FRAs). This will prove to be very useful in calibrating the LIBOR Market Model [1]. Equation 2.1 can be manipulated into the following form:

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha}^{\beta-1} \omega_{i+1}(t) L(t, T_i) \tag{2.2}$$

where

$$\omega_i(t) = \frac{\delta_i P(t, T_i)}{G_{\alpha+1,\beta}(t)}$$

and

$$G_{\alpha+1,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P(t, T_i).$$

Definition 2.1.6 *A caplet is an option on the forward rate. The option is on a unit notional amount. An at-the-money caplet applying to the forward rate between time T_α and $T_{\alpha+1}$, has the following payoff*

$$\text{Payoff on caplet} = \frac{\delta_{\alpha+1} \max(L(T_\alpha, T_\alpha) - L(T_{-1}, T_\alpha), 0)}{1 + \delta_{\alpha+1} L(T_\alpha, T_\alpha)}$$

at time T_α.

[1] Proposition 4.2.1 relies on this assumption.

Remark 2.1.7 *In South Africa, caplets obey the FRA day schedule and are settled in advance (the caplet is settled at time T_α as a discounted value to the amount that would be received in an arrears settlement at time $T_{\alpha+1}$).*

Definition 2.1.8 *Consider $\mathcal{T} = \{T_{-1}, T_0, T_1, ..., T_n\}$. A cap is a series of options on the forward rate. The at-the-money cap consists of the following payoffs:*

$$\text{Payoff on cap at time } T_{i+1} \quad = \quad \delta_{i+1}\max(L(T_i, T_i) - S_{0,n}(T_{-1}), 0)$$

for $i = 0, ..., n-1$. The value of the cap is the sum of the set of option values with these payoffs.

Remark 2.1.9 *In South Africa, caps obey the swap day schedule and are settled in arrears.*

Remark 2.1.10 *Since caps follow the swap day count convention and caplets follow the FRA day count convention, we cannot consider a cap as simply being the sum of caplets. Only after we invoke assumption 2.1.5 (we assume that the difference between a set of dates determined according to the swap day count convention and a set of dates determined according to the FRA day count convention is negligible) can we say that a cap is a sum of caplets. The fact that caplets are settled in advance and caps are settled in arrears has no impact on pricing.*

Definition 2.1.11 (Black's Formula for swaptions) *A (payer) swaption has a payoff of $G_{\alpha+1,\beta}(T_\alpha) \times (S_{\alpha,\beta}(T_\alpha) - R)^+$ at time T_α. The volatility quoted in the market is $\nu^M_{\alpha,\beta}(t)$. The value of the swaption at time t ($t < T_\alpha$) is given by*

$$\mathbf{Swaption}_t = G_{\alpha+1,\beta}(t)\left(S_{\alpha,\beta}(t)\Phi(d_+) - R\Phi(d_-)\right)$$

where

$$d_\pm = \frac{\ln\left(\frac{S_{\alpha,\beta}(t)}{R}\right) \pm \frac{(\nu^M_{\alpha,\beta}(t))^2}{2}(T_\alpha - t)}{\nu^M_{\alpha,\beta}(t)\sqrt{T_\alpha - t}}$$

and

$$G_{\alpha+1,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P(t, T_i).$$

Definition 2.1.12 (Black's Formula for caplets)[2] *A caplet has a payoff of $(L(T_{k-1}, T_{k-1}) - R)^+$ at time T_k. The volatility quoted in the market is $\nu^M(t)$. The value of this caplet at time t ($t < T_{k-1}$) is given by*

$$\mathbf{Caplet}_t = P(t, T_k)\left(L(t, T_{k-1})\Phi(d_+) - R\Phi(d_-)\right)$$

where

$$d_\pm = \frac{\ln\left(\frac{L(t,T_{k-1})}{R}\right) \pm \frac{(\nu^M(t))^2}{2}(T - t)}{\nu^M(t)\sqrt{T - t}}.$$

Remark 2.1.13 *Black's formula for swaptions is used as a tool for converting quoted volatilities into prices. The formulae can be justified without some of the crude assumptions that were made by Black when the formulae were first proposed. The justification shall be provided as part of the LIBOR Market Model theory covered in chapter three.*

[2]The formula is presented for a caplet which is settled in arrears. As noted above, the advance/arrears settlement of caplets has no impact on pricing with the appropriate discounting of the payoff.

2.2 The modelling framework

We consider the set of quarterly dates (determined according to the South African swap day schedule): $\mathcal{T} = \{T_{-1}, T_0, T_1, ...T_\alpha, ..., T_\beta, ..., T_n\}$.

This dissertation is concerned with the modelling of LIBOR forward rates ($L(t,T_k)$) and forward swap rates ($S_{\alpha,\beta}(t)$). These stochastic processes exist in a probability space (Ω,$\mathcal{A}$,$\mathbb{P}$) with associated Brownian filtration $\mathcal{F}_t$. The LIBOR forward dynamics shall be presented and manipulated in two ways: vector form and one dimensional form.

2.2.1 LIBOR dynamics in vector form

The vector form is the form which is most suited to the *implementation* of the LIBOR Market Model. Once the parameters of the vector form is known, it is only necessary to generate independent standard normal draws in order to simulate the set of LIBOR forward rates. Consider a set of $m+1$ LIBOR forward rates. These can be modelled as

$$
\begin{aligned}
d\begin{pmatrix} L(t,T_0) \\ \vdots \\ L(t,T_m) \end{pmatrix} &= \begin{pmatrix} L(t,T_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & L(t,T_m) \end{pmatrix} \begin{pmatrix} \mu(t,T_0) \\ \vdots \\ \mu(t,T_m) \end{pmatrix} dt \\
&+ \begin{pmatrix} L(t,T_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & L(t,T_m) \end{pmatrix} \begin{pmatrix} \lambda_{01}(t) & \dots & \lambda_{0d}(t) \\ \vdots & \ddots & \vdots \\ \lambda_{m1}(t) & \dots & \lambda_{md}(t) \end{pmatrix} \begin{pmatrix} dW_t^1 \\ \vdots \\ dW_t^d \end{pmatrix}
\end{aligned}
$$

or

$$\frac{dL(t,T_k)}{L(t,T_k)} = \mu(t,T_k)dt + \lambda(t,T_k)\cdot d\mathbf{W_t} \quad k = 0,1,...,m$$

where $\mathbf{W_t}$ is a $d \times 1$ vector of independent Brownian motions and $\lambda(t,T_k)$ is the $(k+1)^{th}$ row of the volatility matrix. This formulation can also be written more compactly as

$$d\mathbf{L}(t) = D[L(t)]\left[\, \underline{\mu}(t)dt + \underline{\lambda}(t)d\mathbf{W_t} \,\right] \tag{2.3}$$

where $D[L(t)]$ is a $(m+1)\times(m+1)$ diagonal matrix of LIBOR forward rates, $\underline{\mu}(t)$ is an $(m+1)\times 1$ vector and $\underline{\lambda}(t)$ is a $(m+1)\times d$ volatility matrix.

2.2.2 LIBOR dynamics in one dimensional form

In this dissertation, we *calibrate* the LIBOR market model by parameterising the one dimensional form of the forward LIBOR dynamics. The one dimensional form of a set of forward rates is presented in terms of *correlated* Brownian motions:

$$\frac{dL(t,T_k)}{L(t,T_k)} = \mu(t,T_k)dt + \sigma(t,T_k)dB_t^k \quad k = 0,1,...,m \tag{2.4}$$

where $dB_t^i dB_t^j = \rho_{ij}dt$.

2.2.3 The relationship between the vector and the one dimensional forms

The LIBOR Market Model is first calibrated to a data set. From this calibration, the LIBOR forward rate dynamics can be parameterised in terms of equation 2.4 (the one dimensional form). It was noted that it is more convenient to simulate under equation 2.3 (the multi dimensional form). It is therefore necessary to take the instantaneous correlation structure and instantaneous volatility structure of the one dimensional form and manipulate it into a parameterisation of the multi-dimensional form. Notice that

$$d\mathbf{B}_t d\mathbf{B}_t^T = \begin{pmatrix} 1 & \rho_{01} & \cdots & \rho_{m0} \\ \rho_{01} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \rho_{(m-1)(m)} \\ \rho_{(m)(0)} & \cdots & \rho_{(m)(m-1)} & 1 \end{pmatrix} dt := \underline{\rho} dt.$$

Suppose that there is an $(m+1) \times d$ matrix $\mathbf{A}$ such that:

$$\underline{\rho} = \mathbf{A}\mathbf{A}^{\mathbf{T}}$$

then

$$\begin{aligned} (\mathbf{A}d\mathbf{W_t})(\mathbf{A}d\mathbf{W_t})^T &= \mathbf{AIA^T}dt \\ &= \underline{\rho} dt \end{aligned}$$

where $\mathbf{I}$ is the identity matrix. This implies that

$$\mathbf{A}d\mathbf{W_t} = d\mathbf{B}_t$$

and if A_k is the k^{th} row of the matrix $\mathbf{A}$ then

$$\lambda(t, T_k) = \sigma(t, T_k)\mathbf{A}_k \tag{2.5}$$

for $k = 0, ..., m$. Equation 2.5 is the link between the multi-dimensional form and one-dimensional form of the forward LIBOR dynamics. We can now specify equation 2.4 (the one dimensional) form as:

$$\frac{dL(t, T_k)}{L(t, T_k)} = \mu(t, T_k)dt + \sigma(t, T_k)\mathbf{A}_k \cdot d\mathbf{W_t} \tag{2.6}$$

where $\mathbf{W_t}$ is a $d \times 1$ vector of independent standard Brownian motions and $k = 0, ..., m$. This is an extremely useful representation of the forward LIBOR dynamics for two reasons:

- The output from the calibration of the LIBOR Market Model can easily be manipulated into a form which is appropriate for the simulation of LIBOR forward rates.
- $\mathbf{A}$ can be formulated such that the number of columns in $\mathbf{A}$ is less than $m+1$. This allows for computational efficiency.

The key tool which drives the manipulation of the one-dimensional form into the multi dimensional form is the decomposition of $\underline{\rho}$ into $\mathbf{A}\mathbf{A}^{\mathbf{T}}$. If $\mathbf{A}$ is an $(m+1) \times (m+1)$ matrix, then Cholesky factorisation could be used to de-construct $\underline{\rho}$. The methods for constructing $\mathbf{A}$ (if it is an $(m+1) \times d$ matrix where $d < m+1$), are discussed in section 4.4. It is also worth noting that

$$\begin{aligned} \sigma(t,T_k) d\mathbf{B}_t^k &= \lambda(t,T_k) \cdot d\mathbf{W_t} \\ &= \sigma(t,T_k) \frac{\lambda(t,T_k) \cdot d\mathbf{W_t}}{\sigma(t,T_k)} \end{aligned}$$

and that

$$\begin{aligned} d\mathbf{B}_t^i d\mathbf{B}_t^j &= \rho_{ij} dt \\ &= \left(\sigma(t,T_i) \frac{\lambda(t,T_i) \cdot d\mathbf{W_t}}{\sigma(t,T_i)} \right) \left(\sigma(t,T_j) \frac{\lambda(t,T_j) \cdot d\mathbf{W_t}}{\sigma(t,T_j)} \right) \end{aligned}$$

thus,

$$i = j \Rightarrow \rho_{ij} = 1 \Rightarrow \sigma(t,T_i) = \|\lambda(t,T_j)\|$$

and

$$i \neq j \Rightarrow \rho_{ij} = \frac{\lambda(t,T_i) \cdot \lambda(t,T_j)}{\|\lambda(t,T_i)\| \|\lambda(t,T_j)\|}$$

for $i = 0, ..., m$ and $j = 0, ..., m$.

2.3 Changes of numeraire

The change of numeraire technique is presented here as it is a key tool which will be used in the development of the LIBOR Market Model[3]. Derivative pricing happens with respect to a certain numeraire or 'currency'. Appropriate numeraire choices may lead to a simplification of the pricing problem as demonstrated in Geman et al. [1995]. Consider first the asset dynamics with respect to the bank account as numeraire.

$$P_B^i(t) = \frac{P^i(t)}{e^{\int_0^t r_u du}}$$

where P refers to any asset and r is the stochastic short rate. It is shown in section A.2 that in successfully finding a vector, γ_t, such that $\underline{\lambda}\gamma_t = -(\underline{\mu} - \underline{r})$ (and where λ_k is the k^{th} row of $\underline{\lambda}$), Girsanov's theorem (A.2.1) makes it possible to define a new measure $\mathbb{Q}$ such that

$$d\underline{P} = D[P][\underline{r}dt + \underline{\lambda} d\mathbf{W}_t^{\mathbb{Q}}]$$

and which therefore implies that pricing under the $\mathbb{Q}$ expectation leads to arbitrage free prices. It is now natural to ask how the asset dynamics change when the dynamics under other EMMs are considered. First, consider $P^i(t)$ as the numeraire instead of the bank account. A measure $\mathbb{Q}^i$ must be constructed such that it is equivalent to $\mathbb{Q}$ and such that it martingalizes the ratios $\frac{P^j(t)}{P^i(t)}$. By Ito's lemma it can be shown that

$$\frac{d\left(\frac{P^j(t)}{P^i(t)}\right)}{\frac{P^j(t)}{P^i(t)}} = -\lambda_i(\lambda_j - \lambda_i)dt + (\lambda_j - \lambda_i) d\mathbf{W}_t^{\mathbb{Q}}.$$

[3]The notation that will be used here is introduced in the section on no arbitrage pricing in sections A.1 and A.2.

Define $\mathbb{Q}^i$ by

$$\frac{d\mathbb{Q}^{T_i}}{d\mathbb{Q}} = \mathcal{E}\left(\int_0^T u_t \cdot d\mathbf{W}_t^{\mathbb{Q}}\right)$$

where $u_t = \lambda_i$. Girsanov's Theorem (theorem A.2.1) will ensure that

$$dW_t^{\mathbb{Q}^{T_i}} = dW_t^{\mathbb{Q}} - u_t dt \text{ and that } \mathbb{Q}^{T_i} \text{ is equivalent to } \mathbb{Q}$$

and thus

$$\frac{d\left(\frac{P(t,T_j)}{P(t,T_i)}\right)}{\frac{P(t,T_j)}{P(t,T_i)}} = (\lambda_j - \lambda_i) d\mathbf{W}_t^{\mathbb{Q}^{T_i}}.$$

Now, if we change from $\mathbb{Q}^i$ to $\mathbb{Q}^k$ then we must define $\mathbb{Q}^k$ such that it martingalizes the ratios $\frac{P^j(t)}{P^k(t)}$. Under $\mathbb{Q}^i$, $\frac{P^j(t)}{P^k(t)}$ follows

$$\begin{aligned}
d\frac{P^j(t)}{P^k(t)} &= d\left(\frac{P^j(t)}{P^i(t)}\frac{P^i(t)}{P^k(t)}\right) \\
&= \frac{P^j(t)}{P^i(t)} d\left(\frac{1}{\frac{P^k(t)}{P^i(t)}}\right) + \frac{1}{\frac{P^k(t)}{P^i(t)}} d\left(\frac{P^j(t)}{P^i(t)}\right) + d\left(\frac{P^j(t)}{P^i(t)}\right) d\left(\frac{1}{\frac{P^k(t)}{P^i(t)}}\right) \quad \text{(Ito Product rule)} \\
&= -\frac{P^j(t)}{P^k(t)}(\lambda_k - \lambda_i)(\lambda_j - \lambda_k) dt + \frac{P^j(t)}{P^k(t)}(\lambda_j - \lambda_k) d\mathbf{W}_t^{\mathbb{Q}^i}
\end{aligned}$$

and these dynamics can be made driftless under a new measure, $\mathbb{Q}^{T_k}$, by defining $\mathbb{Q}^{T_k}$ as $\frac{d\mathbb{Q}^{T_k}}{d\mathbb{Q}^{T_i}} = \mathcal{E}\left(\int_0^T u_t \cdot d\mathbf{W}_t^{\mathbb{Q}}\right)$ where $u_t = \lambda_k - \lambda_i$. Table 2.1 summarises the technicalities of changing measure.

Measure Change	Girsanov Kernel
$\mathbb{P}$ to $\mathbb{Q}$	γ_t such that $\underline{\lambda}\gamma_t = (\underline{r} - \underline{\mu})$
$\mathbb{Q}$ to $\mathbb{Q}^{T_i}$	λ_i
$\mathbb{Q}^{T_i}$ to $\mathbb{Q}^{T_k}$	$\lambda_k - \lambda_i$

Table 2.1: Girsanov kernels to be used when performing specific measure changes.

Chapter 3

LIBOR Market Model theory

The LIBOR Market Model has its roots in the Heath Jarrow Morton models that are concerned with the modelling of the instantaneous forward rates. The Brace, Gatarek, Musiela (BGM) model is actually a framework for parameterising an HJM model using market quoted cap and swaption volatilities. The later approaches developed by Jamshidian, Musiela and Rutkowski, abandon the idea of modelling the instantaneous forward rates in favour of the modelling of a finite set of LIBOR, non-instantaneous, forward rates. This innovation drastically simplifies the calibration of the LIBOR Market Model. This chapter shall give a brief overview of HJM models and then provide the essentials of the BGM, Musiela and Rutkowski's and Jamshidian's approaches to the modelling of forward rates.

3.1 Heath Jarrow Morton Models

HJM models provide a modelling framework for instantaneous forward rates of all maturities T (less than T^*) under the risk neutral measure (ie the equivalent martingale measure associated with the bank account numeraire). This modelling framework is briefly introduced here as it is this class of models that gave rise to the original BGM model due to Brace et al. [1997]. The HJM framework was proposed by Heath et al. [1992].

Instead of only modelling the short rate, the entire yield curve is modelled over time. Thus, for all $t \leq T \leq T^*$, the instantaneous forward rate is assumed to follow

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T) \cdot d\mathbf{W_t}$$

where $\mathbf{W_t}$ is a d dimensional Brownian motion under the real world probability measure $\mathbb{P}$. If bond prices are modelled as

$$dP(t,T) = P(t,T)\left(m(t,T)dt + v(t,T) \cdot d\mathbf{W_t}\right)$$

then is can be shown (Bingham and Kiesel [2004], p 343) that

$$\begin{aligned} m(t,T) &= f(t,t) + A(t,T) + \frac{1}{2}||S(t,T)||^2 \\ v(t,T) &= S(t,T) \end{aligned}$$

where

$$A(t,T) = -\int_t^T \alpha(t,s)ds \text{ and } S(t,T) = -\int_t^T \sigma(t,s)ds.$$

The intention is to model the instantaneous forward rate under the risk neutral measure. To do so, it should be noted that under the risk neutral measure, $P(t,T)/e^{\int_0^t f(u,u)du}$ should be a martingale. This implies that the drift of $P(t,T)/e^{\int_0^t f(u,u)du}$ under $\mathbb{Q}$ should be zero. By applying Ito's lemma, it can be shown that

$$d\left(\frac{P(t,T)}{e^{\int_0^t f(u,u)du}}\right) = \frac{P(t,T)}{e^{\int_0^t f(u,u)du}}\left(\left(A(t,T)+\frac{1}{2}\|S(t,T)\|^2 - S(t,T)\lambda(t)\right)dt + S(t,T)d\mathbf{W}_{\mathbf{t}}^{\mathbb{Q}}\right)$$

where the Girsanov kernel used to effect this change of measure is $\lambda(t)$. Under this change of measure, the drift must be zero so that

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du + \sigma(t,T)\lambda(t).$$

Now, if the interest rate model was presented under the risk neutral measure from the outset, then $\lambda(t) = 0$ so that

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$$

and this equality is known as the HJM drift condition. Thus, by specifying and parameterising the volatility function $\sigma(t,T)$ or $S(t,T)$, the stochastic dynamics of the instantaneous forward rate would be completely specified. The volatility surface must now be parameterised such that it is consistent with the cap and swaption volatilities quoted in the market. This is where the BGM framework delivers a contribution. However, first the Black formulae shall be revisited.

3.2 Revisiting the Black formulae

Black's formulae were stated in chapter two. These formulae can be justified by making suitable numeraire changes. A lemma shall first be presented which aids us in justifying the Black formulae.

Lemma 3.2.1 *If X is a lognormal random variable such that $\ln X \sim N(\mu, s^2)$ (s is the standard deviation), then*

$$E\left[(X-K)^+\right] = E[X]\,N(d_+) - KN(d_-)$$

where

$$d_\pm = \frac{\ln\left(\frac{E[X]}{K}\right) \pm \frac{1}{2}s^2}{s}.$$

Proposition 3.2.2 *If the LIBOR forward rate is modelled by*

$$\frac{dL(t,T_n)}{L(t,T_n)} = \lambda(t,T_n) \cdot d\mathbf{W}_t^{\mathbb{Q}^{T_{n+1}}}$$

where $\lambda(t,T_n)$ *is a deterministic* $1 \times d$ *function and* $\mathbf{W}_t^{\mathbb{Q}^{T_{n+1}}}$ *is a* $d \times 1$ *vector of independent standard Brownian motions under measure* $\mathbb{Q}^{T_{n+1}}$*, then the value of a caplet with a notional of* 1*, strike rate of* R *and a payoff, at time* T_{n+1}*, of*

$$\delta_{n+1}(L(T_n,T_n) - R))^+$$

is

$$\begin{aligned}
\mathbf{CAPLET_t} &= \delta_{n+1}[L(t,T_n)\Phi(d_1) - R\Phi(d_2)] \\
d_1 &= \frac{\log\left(\frac{L(t,T_n)}{R}\right) + \frac{\nu_n^2(t)(T_n-t)}{2}}{\nu_n(t)\sqrt{T_n - t}} \\
d_2 &= d_1 - \nu_n(t)\sqrt{T_n - t}
\end{aligned}$$

where

$$\nu_n^2(t) = \frac{1}{(T_n - t)} \int_t^{T_n} \|\lambda(u,T_n)\|^2 du.$$

Proof

Under the risk neutral measure, the caplet value is

$$\mathbf{CAPLET_t} = e^{\int_0^t r_u du} \mathbb{E}^{\mathbb{Q}} \left[\frac{\delta_{n+1}(L(T_n,T_n) - R))^+}{e^{\int_0^{T_{n+1}} r_u du}} |\mathcal{F}_t \right]$$

where, r_u is the *stochastic* short rate. Now, change the measure to the EMM associated with the numeraire $P(t,T_{n+1})$. Thus,

$$\begin{aligned}
\mathbf{CAPLET_t} &= P(t,T_{n+1})\mathbb{E}^{\mathbb{Q}^{T_{n+1}}} \left[\frac{\delta_{n+1}(L(T_n,T_n) - R))^+}{P(T_{n+1},T_{n+1})} |\mathcal{F}_t \right] \\
&= P(t,T_{n+1})\mathbb{E}^{\mathbb{Q}^{T_{n+1}}} \left[\delta_{n+1}(L(T_n,T_n) - R))^+ |\mathcal{F}_t \right].
\end{aligned}$$

But, under $\mathbb{Q}^{T_{n+1}}$, $L(t,T_n)$ is a martingale and thus,

$$\frac{dL(t,T_n)}{L(t,T_n)} = \lambda(t,T_n) \cdot d\mathbf{W}_t^{\mathbb{Q}^{T_{n+1}}}.$$

Since $\lambda(t,T_n)$ is assumed to be deterministic

$$dL(s,T_n) = L(t,T_n)e^{\int_t^s \lambda(t,T_n)\cdot d\mathbf{W}_t^{\mathbb{Q}^{T_{n+1}}} - \frac{1}{2}\int_t^s \|\lambda(u,T_n)\|^2 du}$$

and therefore

$$L(s,T_n) \sim LN\left(L(t,T_n) - \frac{1}{2}\int_t^s \|\lambda(u,T_n)\|^2 du, \int_t^s \|\lambda(u,T_n)\|^2 du \right).$$

Thus, by applying lemma 3.2.1, Black's formula results.

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Remark 3.2.3 *Notice how the change of numeraire technique has played a crucial role. It has allowed a reduction in the dimension of the problem which has allowed the price to be represented as the expectation of one random variable instead of two.*

Proposition 3.2.4 *Consider a swaption which has reset dates $T_\alpha, T_{\alpha+1}, ..., T_{\beta-1}$ $(t < T_0)$, payment dates of $T_{\alpha+1}, T_{\alpha+2}, ..., T_\beta$, a strike rate of R and in which the swap rate, $S_{\alpha,\beta}(t)$, has dynamics*

$$\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} = \lambda_{\alpha,\beta}(t) \cdot d\mathrm{W}_t^{\mathbb{Q}^{G_{\alpha+1,\beta}}}$$

where $\lambda_{\alpha,\beta}(t)$ is a $1 \times d$ vector of deterministic functions and $\mathrm{W}_t^{\mathbb{Q}^{G_{\alpha+1,\beta}}}$ is a $d \times 1$ vector of independent standard Brownian motions under $\mathbb{Q}^{G_{\alpha+1,\beta}}$. The value, at time t, of this swaption is

$$\mathbf{Swaption}_t = G_{\alpha+1,\beta}(t)\left(S_{\alpha,\beta}(t)\Phi(d_1) - R\Phi(d_2)\right)$$

where

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_{\alpha,\beta}(t)}{R}\right) + \frac{\nu^2_{\alpha,\beta}(t)(T_\alpha - t)}{2}}{\nu_{\alpha,\beta}(t)\sqrt{T_\alpha - t}} \\ d_2 &= d_1 - \nu_{\alpha,\beta}(t)\sqrt{T_\alpha - t} \\ G_{\alpha+1,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} \delta_i P(t, T_i) \\ S_{\alpha,\beta}(t) &= \frac{P(t,T_\alpha) - P(t,T_\beta)}{G_{\alpha+1,\beta}(t)} \end{aligned}$$

and where

$$\nu^2_{\alpha,\beta}(t) = \frac{1}{(T_\alpha - t)} \int_t^{T_\alpha} \|\lambda_{\alpha,\beta}(u)\|^2 du.$$

Proof

The proof is very similar to the proof for Black's formula for caplets. In the risk neutral valuation formula, the swaption payoff value at time T_α is considered. The measure change is now from $\mathbb{Q}$ (the EMM associated with the bank account) to the equivalent martingale measure $\mathbb{Q}^{G_{\alpha+1,\beta}}$ (the EMM associated with the annuity factor $G_{\alpha+1,\beta}(t)$).

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Remark 3.2.5 *Black's formulae for swaptions and caplets are mutually incompatible. The basic reason for this is that the swap rate can be expressed as the weighted sum of forward rates (see equation 2.2 and the corresponding assumption 2.1.5). Thus, if the forward rates are assumed* *to be lognormal, then the swap rate is the weighted sum of lognormal random variables. The sum of lognormal random variables is not lognormal. Hence, Black's formulae for swaptions and Black's formulae for caplets are incompatible. The LIBOR Market Models allows for the recovery of Black's formulae for caplets whereas the Swap Market Model allows for the recovery of Black's formula for swaptions. The two modelling frameworks are therefore incompatible. The rest of this chapter is concerned with the LIBOR Market Model. In chapter 6, a closed form approximation to a barrier swaption price is derived and this is, more correctly, termed as being in the Swap Market Model framework.*

3.3 The Brace Gatarek Musiela model

Brace et al. [1997] cast their model within the HJM framework and therefore work under the risk neutral measure (as opposed to the spot or terminal measures that which will be introduced later). Thus, instead of having a discrete set of tenor dates, they model all instantaneous forward rates between 0 and T^*. The dynamics of the instantaneous forward rates (or, equivalently, the zero coupon bond prices) are determined in terms of market observables such as caplet and swaption volatilities. Thus the basic plan is to:

1. Determine the driftless dynamics of the LIBOR forward rate in terms of bond price volatilities and use this to determine an expression for the bond volatilities.
2. Use the HJM drift conditions to completely specify a model for the instantaneous forward rate or the bond prices under the risk neutral measure.

The BGM model provides a *framework for parameterising an HJM model.* Firstly note that

$$L(t, T_{i-1}) = \frac{1}{\delta_i}\left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right).$$

Since $L(t, T_{i-1})$ is a martingale under $\mathbb{Q}^{T_i}$, the martingale representation theorem (theorem A.2.2) can be invoked to show that for some pre-visible process $\lambda(t, T_{i-1})$,

$$dL(t, T_{i-1}) = L(t, T_{i-1})\lambda(t, T_{i-1}) \cdot d\mathbf{W}_t^{\mathbb{Q}^{T_i}}.$$

Under the assumption that $\lambda(t, T_{i-1})$ is deterministic, this implies a lognormal model for the forward LIBOR rate and therefore that the Black formulae hold. Secondly, note that

$$\begin{aligned} dL(t, T_{i-1}) &= \frac{1}{\delta_i} d\left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right) \\ &= \frac{1}{\delta_i} d\left(\frac{P(t,T_{i-1})}{P(t,T_i)}\right) \\ &= \frac{1}{\delta_i}\frac{P(t,T_{i-1})}{P(t,T_i)}\left(S(t,T_{i-1}) - S(t,T_i)\right) \cdot d\mathbf{W}_{\mathbf{t}}^{\mathbb{Q}^{\mathbf{T_i}}} \end{aligned}$$

where the last line follows by applying Ito's formula to the bond dynamics

$$dP(t,T) = P(t,T)\left(m^{\mathbb{Q}^{T_i}}(t,T)dt + v(t,T) \cdot d\mathbf{W}_{\mathbf{t}}^{\mathbb{Q}^{\mathbf{T_i}}}\right).$$

Now relate the two expressions for $dL(t, T_{i-1})$ to see that

$$S(t, T_{i-1}) - S(t, T_i) = \frac{\lambda(t, T_{i-1}) L(t, T_{i-1})}{1 + \delta_i L(t, T_{i-1})}.$$

The following crucial assumption is now made

$$S(t, T_i) = 0 \text{ for } 0 \leq T_i - t \leq \delta_i$$

or if $\tau = \inf\{n : T_n \geq t\}$ then $S(t, T_\tau) = 0$. The assumption being made is that zero coupon bonds that are sufficiently close to maturity have zero volatility. To a certain extent this assumption may be plausible as one would expect a reduction in volatility as the bonds converge to their known redemption value. This assumption allows $S(t, T_i)$ to be written as

$$\begin{aligned} S(t, T_i) &= -(S(t, T_{i-1}) - S(t, T_i)) - (S(t, T_{i-2}) - S(t, T_{i-1})) \\ &- \ldots \\ &- (S(t, T_{\tau+1}) - S(t, T_{\tau+2})) - (S(t, T_\tau) - S(t, T_{\tau+1})) \\ &= -\sum_{k=\tau}^{i-1} (S(t, T_k) - S(t, T_{k+1})) \\ &= -\sum_{k=\tau}^{i-1} \frac{\lambda(t, T_k) L(t, T_k)}{1 + \delta_k L(t, T_k)}. \end{aligned}$$

The structure of λ will be specified by choosing an instantaneous volatility function and then fitting the chosen function to the available caplet volatility data by making use of

$$(\nu_n^M(t))^2 = \int_t^{T_n} \|\lambda(u, T_n)\|^2 du$$

where $\nu_n^M(t)$ is the market quoted volatility for a caplet expiring at time T_n. Thus, the formula for $S(t, T)$ can be fully specified in terms of the approximate λ function. The risk neutral dynamics of $f(t, T)$ is therefore

$$df(t, T) = S(t, T) \frac{\partial S(t, T)}{\partial T} dt - \frac{\partial S(t, T)}{\partial T} d\mathbf{W}_t^{\mathbb{Q}}$$

and has been calibrated to the available caplet volatilities.

3.4 Forward LIBOR dynamics under the terminal measure

The lognormal specification of the simple forward rates have allowed prices to be derived for caplets and swaptions. A natural question to ask is therefore how to price other interest rate derivatives using a model that can be calibrated to the available volatility quotes. The BGM approach of the last section really required volatilities for every maturity between 0 and T^* so as to determine a volatility function for the instantaneous forward rate. The approaches presented in this and the next section shows how modelling under measures other than the risk neutral measure allows the model to overcome the large input data requirement. This section presents the derivation of the forward LIBOR dynamics under a single measure - the *terminal* measure

- which is in the vein of the derivation given by Musiela and Rutkowski [1997]. We consider a tenor structure

$$T_{-1} = 0 < T_0 < T_1 < ... < T_N$$

where $\delta_j = T_j - T_{j-1}$. The objective is to construct a series of measures $(\mathbb{Q}^{T_j}, \mathbb{Q}^{T_{j+2}}, ..., \mathbb{Q}^{T_{i-1}})$ relative to a measure $\mathbb{Q}^{T_i}$ such that $L(t, T_k)$ is a martingale under $\mathbb{Q}^{T_{k+1}}$ $(k = j-1, j..., i-1)$. Having constructed these measures in this way, we would have defined the *measure changes* (or equivalently, the *Girsanov* kernels of the measure changes). Since we know the Girsanov kernels, we can therefore determine the dynamics of $L(t, T_{j-1}), L(t, T_{j+1}), ..., L(t, T_{i-1})$ under $\mathbb{Q}^{T_i}$. This captures the essence of the derivation of the forward rate dynamics under the terminal measure. So, define

$$\begin{aligned} U_n(t, T_k) &= \frac{P(t, T_k)}{P(t, T_{n+1})} \\ &= \frac{P(t, T_k)/P(t, T_{n+2})}{P(t, T_{n+1})/P(t, T_{n+2})} \\ &= \frac{U_{n+1}(t, T_k)}{1 + \delta_{n+2} L(t, T_{n+1})}. \end{aligned}$$

Now, $U_{n+1}(t, T_k)$ and $L(t, T_{n+1})$ are martingales under $\mathbb{Q}^{T_{n+2}}$. The following lemma shall now be used.

Lemma 3.4.1 *If* $dX_t = \alpha_t dW_t$ *and* $dY_t = \beta_t dW_t$ *and define* $D_t = \frac{1}{1+Y_t}$ *then*

$$d(D_t Y_t) = D_t(\alpha_t - D_t X_t \beta_t) \cdot (dW_t - D_t \beta_t dt).$$

Proof
The proof follows by a direct application of Ito's lemma.

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Therefore,

$$dU_n(t, T_k) = \eta_t \cdot (dW_t^{\mathbb{Q}^{T_{n+2}}} - \frac{\delta_{n+2} L(t, T_{n+1})}{1 + \delta_{n+2} L(t, T_{n+1})} \lambda(t, T_{n+1}) dt).$$

Now, we need to define $\mathbb{Q}^{T_{n+1}}$ relative to $\mathbb{Q}^{T_{n+2}}$ such that $U_n(t, T_k)$ is a martingale under $\mathbb{Q}^{T_{n+1}}$. Thus, let the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{T_{n+1}}}{d\mathbb{Q}^{T_{n+2}}}$$

be defined by the Girsanov kernel

$$\frac{\delta_{n+2} L(t, T_{n+1})}{1 + \delta_{n+2} l(t, T_{n+1})} \lambda(t, T_{n+1}).$$

Girsanov's theorem then shows that

$$W_t^{\mathbb{Q}^{T_{n+1}}} = W_t^{\mathbb{Q}^{T_{n+2}}} - \int_0^t \frac{\delta_{n+2}L(u,T_{n+1})}{1+\delta_{n+2}l(u,T_{n+1})}\lambda(u,T_{n+1})du.$$

Thus, the drift coefficient of $dU_n(t,T_k)$ collapses to zero and hence it is a local martingale. As a corollary to the above development, it can be seen that

$$\frac{d\mathbb{Q}^{T_{n+2}}}{d\mathbb{Q}^{T_{n+1}}} = -\frac{\delta_{n+2}L(t,T_{n+1})}{1+\delta_{n+2}L(t,T_{n+1})}\lambda(t,T_{n+1}).$$

Thus, we have the following *rules* for changing between adjacent measures:

Radon-Nikodym derivative	Girsanov kernel
$\frac{d\mathbb{Q}^{T_{k-1}}}{d\mathbb{Q}^{T_k}}$	$\frac{\delta_k L(t,T_{k-1})}{1+\delta_k l(t,T_{k-1})}\lambda(t,T_{k-1})$
$\frac{d\mathbb{Q}^{T_{k+1}}}{d\mathbb{Q}^{T_k}}$	$-\frac{\delta_{k+1}L(t,T_k))}{1+\delta_{k+1}l(t,T_k)}\lambda(t,T_k)$

Consider now $L(t,T_{k-1})$ which is a martingale under $\mathbb{Q}^{T_k}$. If the volatility coefficient is deterministic, then this implies that Black's formula for caplets is recoverable from this modelling framework.

Suppose $i<k$, then

$$\begin{aligned}
\frac{dL(t,T_{k-1})}{L(t,T_{k-1})} &= \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^{T_k}} \\
&= \frac{\lambda(t,T_{k-1})\delta_k L(t,T_{k-1})\lambda(t,T_{k-1})dt}{1+\delta_k L(t,T_{k-1})}dt + \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^{T_{k-1}}} \\
&= \left(\frac{\lambda(t,T_{k-1})\delta_k L(t,T_{k-1})\lambda(t,T_{k-1})}{1+\delta_k L(t,T_{k-1})} + \frac{\lambda(t,T_{k-2})\delta_{k-1} L(t,T_{k-2})\lambda(t,T_{k-2})}{1+\delta_{k-1} L(t,T_{k-2})}\right)dt \\
&+ \lambda(t,T_{k-2})dW_t^{\mathbb{Q}^{T_{k-2}}} \\
&= \ldots \\
&= \lambda(t,T_{k-1})\sum_{s=i}^{k-1}\frac{\delta_{s+1}L(t,T_s)}{1+\delta_{s+1}L(t,T_s)}\lambda(t,T_s)dt + \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^{T_i}}.
\end{aligned}$$

Now, suppose $i>k$, then

$$\begin{aligned}
\frac{dL(t,T_{k-1})}{L(t,T_{k-1})} &= \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^{T_k}} \\
&= -\frac{\lambda(t,T_k)\delta_{k+1} L(t,T_k)\lambda(t,T_k)dt}{1+\delta_{k+1} L(t,T_k)}dt + \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^{T_{k+1}}} \\
&= -\left(\frac{\lambda(t,T_k)\delta_{k+1} L(t,T_k)\lambda(t,T_k)dt}{1+\delta_{k+2} L(t,T_{k+1})} + \frac{\lambda(t,T_{k+1})\delta_{k+2} L(t,T_{k+1})\lambda(t,T_{k+1})}{1+\delta_{k+2} L(t,T_{k+1})}\right)dt \\
&+ \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^{T_{k+1}}} \\
&= \ldots \\
&= -\lambda(t,T_{k-1})\sum_{s=k}^{i-1}\frac{\delta_{s+1}L(t,T_s)}{1+\delta_{s+1}L(t,T_s)}\lambda(t,T_s)dt + \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^{T_i}}.
\end{aligned}$$

3.5 The spot measure

This approach is an alternative perspective on the modelling of forward LIBOR rates and is due to Jamshidian [1997]. This is the approach that will be used in chapter 6 to price barrier swaptions under the LIBOR Market Model. Instead of considering the dynamics of forward LIBOR under the EMM associated with a zero coupon bond (as was done in the previous section), the EMM associated with a *rolling cd* (rolling certificate of deposit) bond is considered. The rolling cd bond is an initial investment of one in the zero coupon bond with maturity equal to the next tenor date. Upon reaching each subsequent tenor date, the amount redeemed is immediately reinvested in the zero coupon bond associated with the next tenor date. Thus, let

$$m(t) \quad = \quad \inf\{n : T_n > t\}$$

and

$$\begin{aligned} N(t) &= \prod_{k=0}^{m(t)-1} \frac{1+\delta_k L(T_{k-1},T_{k-1})}{P(T_{m(t)-1},T_{m(t)})} P(t,T_{m(t)}) \\ &= \prod_{k=0}^{m(t)} (1+\delta_k L(T_{k-1},T_{k-1})) P(t,T_{m(t)}). \end{aligned}$$

Notice that $N(t)$ is in the form of (*number of bonds purchased at time* $T_{m(t)-1}$)$\times$(*value of* $T_{m(t)}$ *bond at time* t). Consider the following lemma which shall be used in the derivation which is to follow.

Lemma 3.5.1 *Suppose that* $R = \prod_{k=m(t)}^{i-1} \frac{1}{(1+\delta_{k+1}L(t,T_k))} = \prod_{k=m(t)}^{i-1} D_k$ *is a martingale under a measure* $\mathbb{Q}^S$*, then*

$$\frac{d\left(\prod_{k=m(t)}^{i-1} \frac{1}{(1+\delta_{k+1}L(t,T_k))}\right)}{\prod_{k=m(t)}^{i-1} \frac{1}{(1+\delta_{k+1}L(t,T_k))}} \quad = \quad - \sum_{k=m(t)}^{i-1} \frac{\delta_{k+1}L(t,T_k)\lambda(t,T_k)}{(1+\delta_{k+1}L(t,T_k))} dW_t^{\mathbb{Q}^S}.$$

Proof

Since R is a martingale under $\mathbb{Q}^S$, it must be driftless under this measure (ie no dt terms under this measure). The lemma now follows from repeated use of Ito's formula.

$$\begin{aligned} dR &= d\left(\prod_{k=m(t)}^{i-1} D_k\right) \text{ minus dt terms} \\ &= (dD_{i-1}) \prod_{k=m(t)}^{i-2} D_k + D_{i-1} d\left(\prod_{k=m(t)}^{i-2} D_k\right) \text{ minus dt terms} \\ &= (dD_{i-1}) \prod_{k=m(t)}^{i-2} D_k + D_{i-1}(dD_{i-2}) \prod_{k=m(t)}^{i-3} D_k + D_{i-1}D_{i-2} d\left(\prod_{k=m(t)}^{i-3} D_k\right) \text{ minus dt terms} \\ &= \ldots \\ &= (dD_{i-1}) \prod_{k=m(t)}^{i-2} D_k + D_{i-1}(dD_{i-2}) \prod_{k=m(t)}^{i-3} D_k + \ldots \text{ minus dt terms} \end{aligned}$$

$$
\begin{aligned}
&= \frac{R}{D_{i-1}} dD_{i-1} + \frac{R}{D_{i-2}} dD_{i-2} + \frac{R}{D_{i-3}} dD_{i-3} + \ldots + \frac{R}{D_{m(t)}} dD_{m(t)} \;\; \textit{minus dt terms} \\
&= \sum_{k=m(t)}^{i-1} \frac{R}{D_k} dD_k \;\; \textit{minus dt terms}
\end{aligned}
$$

Now, by Ito's formula

$$
\begin{aligned}
dD_k &= d\left(\frac{1}{(1+\delta_{k+1}L(t,T_k))}\right) \\
&= -\frac{\delta_{k+1}}{(1+\delta_{k+1}L(t,T_k))^2} d(L(t,T_k)) + \frac{\delta_{k+1}^2 d(L(t,T_k))}{(1+\delta_{k+1}L(t,T_k))}^{3} \\
&= -\frac{\delta_{k+1}L(t,T_k)\lambda(t,T_k)}{(1+\delta_{k+1}L(t,T_k))^2} dW_t^{\mathbb{Q}^S} + \;\; \textit{dt terms}
\end{aligned}
$$

and thus

$$
\frac{d\left(\prod_{k=m(t)}^{i-1} \frac{1}{(1+\delta_{k+1}L(t,T_k))}\right)}{\prod_{k=m(t)}^{i-1} \frac{1}{(1+\delta_{k+1}L(t,T_k))}} = -\sum_{k=m(t)}^{i-1} \frac{\delta_{k+1}L(t,T_k)\lambda(t,T_k)}{(1+\delta_{k+1}L(t,T_k))} dW_t^{\mathbb{Q}^S}.
$$

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Now, notice that $L(t,T_{k-1})\frac{P(t,T_k)}{N(t)}$ and $\frac{P(t,T_k)}{N(t)}$ are both martingales under the spot measure $\mathbb{Q}^S$. It is clear that $\frac{P(t,T_k)}{N(t)}$ is the ratio of a tradeable to the numeraire tradeable and that

$$
L(t,T_{k-1})\frac{P(t,T_k)}{N(t)} = \frac{1}{\delta_k}\left(\frac{P(t,T_{k-1}) - P(t,T_k)}{N(t)}\right).
$$

The above expression is the ratio of a portfolio of bonds to the numeraire asset. The bond portfolio consists of a long position (with notional of $\frac{1}{\delta_k}$) in the T_{k-1} maturity bond and a short position in the T_k maturity bond (with notional of $\frac{1}{\delta_k}$). Now, first consider $\frac{P(t,T_k)}{N(t)}$. This expression can be split into an $\mathcal{F}_t$ measurable (non-random) part and an $\mathcal{F}_t$ non-measurable (random) part.

$$
\begin{aligned}
\frac{P(t,T_k)}{N(t)} &= \frac{\prod_{s=m(t)}^{k-1} \frac{1}{(1+\delta_{s+1}L(t,T_s))P(t,T_{m(t)})}}{\prod_{k=0}^{m(t)}(1+\delta_s L(T_{s-1},T_{s-1}))P(t,T_{m(t)})} \\
&= \alpha \times \prod_{s=m(t)}^{k-1} \frac{1}{(1+\delta_{s+1}L(t,T_s))}
\end{aligned}
$$

where α is the measurable (non-random) part. Using the above lemma, it can be shown that under $\mathbb{Q}^S$,

$$
\begin{aligned}
& d\left(\alpha \times \prod_{s=m(t)}^{k-1} \frac{1}{(1+\delta_{s+1}L(t,T_s))}\right) \\
= \;\; & -\alpha \times \left(\prod_{s=m(t)}^{k-1} \frac{1}{(1+\delta_{s+1}L(t,T_s))}\right) \sum_{s=m(t)}^{k-1} \frac{\delta_{s+1}\lambda(t,T_s)L(t,T_s)}{1+\delta_{s+1}L(t,T_s)} dW^{\mathbb{Q}_t^S} \quad .
\end{aligned}
$$

Now, by Ito's formula

$$
\begin{aligned}
d\left(L(t,T_{k-1})\frac{P(t,T_k)}{N(t)}\right) &= L(t,T_{k-1})d\left(\frac{P(t,T_k)}{N(t)}\right) \\
&+ \frac{P(t,T_k)}{N(t)}dL(t,T_{k-1}) \\
&+ d\left(L(t,T_{k-1})\right)d\left(\frac{P(t,T_k)}{N(t)}\right) \\
&= \alpha \times \prod_{s=m(t)}^{k-1} \frac{L(t,T_{k-1})}{(1+\delta_{s+1}L(t,T_s))} \\
&\times \left(\mu(t,T_{k-1}) - \lambda(t,T_{k-1})\sum_{s=m(t)}^{k-1}\frac{\delta_{s+1}\lambda(t,T_s)L(t,T_s)}{1+\delta_{s+1}L(t,T_s)}\right)dt \\
&+ dW^{\mathbb{Q}_t^S}\ terms.
\end{aligned}
$$

As $\left(L(t,T_{k-1})\frac{P(t,T_k)}{N(t)}\right)$ is driftless under $\mathbb{Q}^S$, the drift term must be zero and thus,

$$
\mu(t,T_{k-1}) = \lambda(t,T_{k-1})\sum_{s=m(t)}^{k-1}\frac{\delta_{s+1}\lambda(t,T_s)L(t,T_s)}{1+\delta_{s+1}L(t,T_s)}.
$$

The dynamics of $L(t,T_{k-1})$ under $\mathbb{Q}^S$ are therefore

$$
\frac{dL(t,T_{k-1})}{L(t,T_{k-1})} = \lambda(t,T_{k-1})\sum_{s=m(t)}^{k-1}\frac{\delta_{s+1}\lambda(t,T_s)L(t,T_s)}{1+\delta_{s+1}L(t,T_s)}dt + \lambda(t,T_{k-1})dW_t^{\mathbb{Q}^S}.
$$

Chapter 4

Calibration

The purpose of this chapter is to describe ways of calibrating the LIBOR Market Model and to explain which of these methods is most suited to the South African setting.

Section 2.2 pointed out that the calibration of the LIBOR Market Model involves parameterisation of the one dimensional form of the LIBOR forward rate dynamics. This equation is restated below.

$$\frac{dL(t,T_k)}{L(t,T_k)} = \mu(t,T_k)dt + \sigma(t,T_k)dB_t^k \quad k = 0,1,...,m \tag{4.1}$$

where $dB_t^i dB_t^j = \rho_{ij}dt \ (i,j = 0,1,...,m)$.

Thus, in terms of equation 4.1, calibration of the LIBOR Market Model assigns values to the instantaneous volatility structure, $\sigma(t,T_k)$ $(k = 0,1,...,m)$, and the instantaneous correlation structure (the matrix of ρ_{ij}'s $(i,j = 0,1,...,m)$).

The process of calibrating the LIBOR Market Model is summarised below.

- Determine a formulaic structure for the instantaneous volatility and instantaneous correlation functions. The structure will typically have an economic foundation.
- Obtain the market data to which the LIBOR Market Model will be calibrated.
- Use a formula to obtain a theoretical expression of the data in terms of the instantaneous correlation and instantaneous volatility functions. This formula is the link between the data and the instantaneous correlation and instantaneous volatility functions.
- Choose the parameters of the instantaneous volatility and instantaneous correlation functions such that the distance (in some sense) between the market data and the formulaic link to the market data is minimised.

The second point in the list above refers to 'market data'. *Market data* refers to quoted swaption volatilities and observed market forward rates. Section 4.1 will describe the form of the South African market data to which calibration takes place. Section 4.1 will also define the swaption volatility matrix. The swaption volatility matrix is the key data object in the calibration of the

LIBOR Market Model.

Section 4.2 is central to this chapter as it explains how the entries in the swaption volatility matrix can be represented in terms of the instantaneous volatility and correlation structures of equation 2.4. This is the theoretical link referred to above and will take the form of a formula which will often be referred to as Rebonato's formula (proposition 4.2.1).

The first point in the list refers to the structure of instantaneous volatilities and correlations. The *method* of calibrating the LIBOR Market Model refers (in part) to the choice of formulaic structure for the instantaneous volatilities and correlations. Section 4.6 presents a number of ways of formulating the instantaneous correlation and instantaneous volatility structures. Rebonato's formula (proposition 4.2.1) is expressed in terms of each of these formulations.

This chapter will demonstrate the process of calibrating the various volatility and correlation structures to South African market data. All calibrations use the South African swaption volatility matrix and South African yield curve of 2 May 2007. The next section will describe how the data of 2 May 2007 has been manipulated in order to arrive at the calibration data input.

4.1 Data

Yield curve data and swaption volatility data are required in order to undertake a calibration of the LIBOR Market Model. The yield curve data has to be manipulated by two interlinked processes: bootstrapping and interpolation.

4.1.1 Bootstrapping and interpolation

Bootstrapping and interpolation of yield curves is an extensive area of research. This section will summarise the requirements of bootstrapping and interpolation procedures. This section will also present the method in which the South African yield curve data has been manipulated in this project.

Consider again the set of quarterly maturities determined according to the South African swap day schedule: $\mathcal{T} = (T_{-1}, T_0, T_1, ..., T_\beta)$. Suppose that we have swap rates for each of these maturities (we have $S_{-1,0}(T_{-1}), S_{-1,1}(T_{-1}), ..., S_{-1,\beta}(T_{-1})$). We are able to back out the discount factors that apply to these maturities by using the following equation recursively:

$$\begin{aligned} P(T_{-1}, T_n) &= \frac{P(T_{-1}, T_{-1}) - \sum_{i=0}^{n-1} \delta_i P(T_{-1}, T_i)}{1 + \delta_i S_{-1,n}} \\ &= \frac{1 - \sum_{i=0}^{n-1} \delta_i P(t, T_i)}{1 + \delta_i S_{-1,n}}. \end{aligned}$$

The NACC spot rate ($r_{T_n}(t)$) and NACA ($R_{T_n}(t)$) spot rates can be backed out from this set of zero coupon bond prices using the following relationship:

$$r_{T_n}(t) = -\frac{1}{T_n} \ln P(t, T_n)$$

and

$$R_{T_n}(t) = e^{r_{T_n}(t)} - 1.$$

The schedule of these rates versus their maturity is known as *the term structure of interest rates.* In practice we encounter two problems:

- Swap rates are not available at every quarterly maturity.
- FRAs do not follow the same day schedule as the swaps. This complicates the procedure when the calibration happens to swap rates and forward rates.

Suppose that we have a set of swaps that do not expire at every quarterly maturity. $(T_{-1}, T_0, T_1, ..., T_{\beta-1})$ represents the setting dates of the available swaps. $\mathcal{T_P} = (T_0, T_1, ..., T_\beta)$ represents the payment dates of the available swaps ($\mathcal{T_P}$ represents every quarterly date up to and including T_β). Also suppose that $\mathcal{C} = (C_0, C_1, ..., C_m)$ is a subset of $\mathcal{T_P}$ and that it represents the quoted maturities of the swaps that are available now. Hagan and West [2006] describes the following iterative procedure to deal with the holes in the swap term structure:

- For each of the quoted expiries in $\mathcal{C}$ we guess spot rates $(r_{C_0}(T_{-1}), r_{C_1}(T_{-1}), ..., r_{C_m}(T_{-1}))$.
- We interpolate between these rates to find spot rates for all points in $\mathcal{T_P}$.
- We now use the equation

  $$r_{C_n}(T_{-1}) = -\frac{1}{T_n} \ln\left(\frac{1 - \sum_{i=0}^{n-1} \delta_i P(T_{-1}, T_i)}{1 + \delta_i S_{-1,n}(T_{-1})}\right)$$

  to find new estimates of the spot rates in $\mathcal{C}$.

Hagan and West [2006] states that convergence using this procedure is fast over the entire yield curve.

Thus, the bootstrapping and interpolation procedures are inextricably linked since the bootstrap proceeds with incomplete information and this information is completed by the interpolation procedure (Hagan and West [2008]).

Hagan and West [2006] specifies the desirable features of bootstrapping and interpolation procedures. These are summarised below:

1. The curve construction procedure should be rapid and with a small degree of error.
2. The instantaneous forward rates should be continuous and positive.
3. The interpolation method should be local - changes in the inputs in one location do not affect the value of the curve at other locations.
4. Stability of forward rates - changes in the forward rates are proportionate to changes in the inputs into the yield curve.

5. The hedges constructed for a particular interest rate derivative should be reasonable and stable when the bootstrapped curve is perturbed.

Hagan and West [2006] and Hagan and West [2008] scrutinises a number of methods using these criteria. 'Linear' interpolation methods such as

- linear interpolation on continuously compounded rates;
- linear interpolation on continuously compounded log rates;
- linear interpolation on discount factors;
- linear interpolation on the log of the discount factors and
- piecewise linear forward

are generally deemed unsuitable. The first and second methods are rejected as they lead to jumps in the instantaneous forward rates and also the possibility of negative instantaneous forward rates (arbitrage). The third method is rejected due to instantaneous forward rates exhibiting jump behaviour. The second last method corresponds to an interpolation method in which forward rates are modelled as being piecewise constant. Continuity of the instantaneous forward rates is a problem here as well. The last method attempts to remedy the defect in the second last method by imposing a piecewise linear form between forward rates. This method is rejected as it leads to implausible yield curve structures and due to the method not being very local. Of the 'linear' interpolation methods, the linear interpolation on the log of the discount factors is the most attractive.

The quadratic spline interpolation on rates is rejected as it leads to similar behaviour of the yield curve as in piecewise linear forward method. The cubic spline and the quartic spline interpolation methods are also surveyed but are rejected due to the oscillatory behaviour that may be observed in the resulting yield curve and also due to poor performance under the 'localness' requirement.

Hagan and West [2006] and Hagan and West [2008] introduce the monotone convex method for interpolation and show that this method performs well under all of the listed requirements. VBA code for this method is provided on the second author's website[1].

4.1.2 Obtaining the term structure

Yield Curve Data was obtained in the form of BEASSA stripped zero coupon bond curves. The yield curves were obtained by applying the BEASSA stripping algorithm to South African swap market data. The second column of table 4.1 shows the zero-coupon discount rates quoted as semi-annually compounded rates [2]. The Nominal Annual Compounded Annually ($NACA$) and Nominal Annual Compounded Continuously ($NACC$) rates were obtained by converting the Nominal Annual Compounded Semi-Annually ($NACS$) rate as follows: $NACArate = (1 + (NACSrate)/2)^2$ and $NACCrate = \ln(1 + NACArate)$. The BEASSA yield curves were

[1] `http://www.finmod.co.za/monotoneconvex.xls`

[2] The yield curve data (in a semi annual format) was obtained from Old Mutual's Asset Liability Management Unit. The data provided by Old Mutual took the form of the first two columns of table 4.1.

obtained for every business day from 16 January 2006 to 23 September 2008.

Time To Maturity (Days)	NACS Rate	NACA Rate	NACC Rate
1	8.7556%	8.9472%	8.5693%
92	9.3131%	9.5299%	9.1028%
184	9.3550%	9.5738%	9.1428%
278	9.3465%	9.5649%	9.1347%
366	9.3237%	9.5410%	9.1129%
460	9.2790%	9.4942%	9.0702%
551	9.2260%	9.4388%	9.0195%
642	9.1668%	9.3768%	8.9629%
733	9.1049%	9.3121%	8.9037%
1097	8.8559%	9.0520%	8.6655%
1462	8.6627%	8.8504%	8.4804%
1827	8.5299%	8.7118%	8.3530%
2192	8.4222%	8.5996%	8.2497%
2557	8.3330%	8.5066%	8.1641%
2924	8.2536%	8.4239%	8.0879%
3289	8.1932%	8.3610%	8.0298%
3653	8.1370%	8.3025%	7.9758%
4383	7.9763%	8.1353%	7.8213%
5479	7.8114%	7.9639%	7.6627%
7306	7.3561%	7.4913%	7.2240%
10960	6.7783%	6.8932%	6.6660%

Table 4.1: The BEASSA yield curve on 2 May 2007. Similar yield curves were obtained for every business day from 16 January 2006 to 23 September 2008.

In Chapter 5, daily hedging will take place. It is necessary to have stripped rates based on a much finer 'day-mesh' than the twenty one dates presented in table 4.1.

The Monotone Convex method is used to perform interpolation between the first fourteen rates in table 4.1. The code provided by the second author in Hagan and West [2006] and Hagan and West [2008] is used to perform the monotone convex interpolation. The interpolated curve will be referred to as the HMC (hybrid monotone convex) curve.

The immediate criticism of this approach is that *the interpolation has been separated from the bootstrapping procedure.* The monotone convex code is intended to be used in the iterative procedure described in section 4.1.1, but here it is used to interpolate between rates that have been stripped using the BEASSA stripping procedure. The monotone convex interpolation method has been superimposed on the results of a BEASSA stripping algorithm. It is almost certain that some traded instruments which depend on rates which are not the product of the BEASSA stripping algorithm, will be mispriced under the HMC curve.

For example, suppose that we attempt to price a four year swap under the interpolated yield curve. The fair value of the fixed and floating payments that occur after year two will be determined by rates that have been interpolated (using the monotone convex method) between the two and three year rate and the three and four year rate. It is almost certain that these rates will not be the same as that given by the BEASSA stripping/interpolation method. Thus, the practice of superimposing the monotone convex method on the results of a BEASSA stripping algorithm leads to a mispricing of the four year swap.

However, the HMC curve has advantages that will suite our purposes:

- The curve is theoretically arbitrage free in that instantaneous forward rates are always positive[3].
- The HMC will price the instruments dependent on the first eight quarters in an arbitrage free manner as these rates were inputs into the HMC interpolation.

Chapter 5 will make extensive use of the HMC. It is noted that the HMC approach has flaws. But, these flaws are countered, to a degree, by the advantages that have been highlighted above. Given the data set as presented in table 4.1, the HMC approach was deemed the best approach with which to continue.

4.1.3 The swaption volatility matrix

The swaption volatility matrix is the key input into a calibration of the LIBOR Market Model. Table 4.2 gives an example of the structure of a typical swaption volatility matrix. This is the form of the swaption volatility matrix that appears in most textbooks. West [2009] (p 39) provides an example of a South African swaption volatility matrix - this matrix is the transpose of the format shown in table 4.2.

	Swap Length							
First Setting Date	0.25	0.5	0.75	1	1.25	1.5	1.75	2
0.25	X	X	X	X	X	X	X	X
0.5	X	X	X	X	X	X	X	X
0.75	X	X	X	X	X	X	X	X
1	X	X	X	X	X	X	X	X
1.25	X	X	X	X	X	X	X	X
1.5	X	X	X	X	X	X	X	X
1.75	X	X	X	X	X	X	X	X
2	X	X	X	X	X	X	X	X

Table 4.2: The structure of a typical at-the-money swaption volatility matrix.

The entry corresponding to a first setting date of 1.5 years and a swap length of 2 years is the volatility of the swap rate with its first setting date being in 1.5 years time and with quarterly payments after that until 3.5 years from now. Thus, the rows of the matrix correspond to option maturities and the columns correspond to forward tenors (in terms of the notation

[3]For each of the BEASSA curves from 16 January 2006 to 23 September 2008, the HMC was calculated. Each HMC was calculated such that it gave values for every day up until a maturity of 2557 days. It was confirmed that the 2556 discount factors for each of these curves were strictly decreasing with respect to maturity.

of definition 2.1.4, the rows correspond to T_α and the columns correspond to $T_\beta - T_\alpha$ and not T_β).

The first column of the matrix represents the volatilities of the one period swap rates. Ignoring day count differences between FRAs and swaps (assumption 2.1.5), the first column will represent the caplet volatilities.

In all the subsequent sections, the notation $\nu^M_{\alpha,\beta}(t)$ shall be used to describe entries in the swaption volatility matrix - the volatility of the quarterly swap rate with first setting date at T_α and final payment date at time T_β.

In the South African setting there are two issues to deal with in order to obtain a swaption volatility matrix.

- The volatility matrix is very sparsely populated.
- Caplet volatilities are not quoted and have to be inferred from cap volatilities.

The second issue will be addressed first.

4.1.4 Stripping caplet volatilities

In order to strip the caplet volatilities it is necessary to view the cap as a sum of caplets. Remark 2.1.10 explained how this is possible in the South African setting.

Consider again the tenor structure $\mathcal{T} = (T_{-1}, T_0, T_1, ..., T_n)$ (quarterly dates are determined according to assumption 2.1.5) and define the following:

- ν_i^{caplet} is the *spot* volatility of a forward rate between T_i and T_{i+1}.
- $\nu_{0,k}^{cap}$ is the *flat*[4] volatility of a forward rates between T_i and T_{i+1} for $i = 0, ..., k-1$.

The inputs into the stripping process are the at-the-money cap volatilities. The value of the cap with caplet setting dates at times $T_0, ..., T_{n-1}$ is

$$\text{Cap}(t, S_{0,n}(t), \nu_n^{cap}) = \sum_{i=0}^{n-1} \text{Caplet}(t, L(t, T_i), S_{0,n}(t), \nu_n^{cap})$$

so that caplet i is evaluated at a strike of $S_{0,n}(t)$, a spot forward rate of $L(t, T_i)$ and a volatility of ν_n^{cap}.

Now, suppose that volatilities (ν_k^{cap}) are available for $k = 0, 1, 2, ..., n-1$. Since we are not pricing the volatility skew, this allows us to back out the spot volatilities. We effectively have a system of n equations and n unknowns.

$$\begin{aligned}
\text{Cap}(t, S_{0,1}(t), \nu_{0,1}^{cap}) &= \text{Caplet}(t, L(t, T_0), S_{0,1}(t), \nu_0^{caplet}) \\
\text{Cap}(t, S_{0,2}(t), \nu_{0,2}^{cap}) &= \text{Caplet}(t, L(t, T_0), S_{0,2}(t), \nu_0^{caplet}) + \text{Caplet}(t, L(t, T_1), S_{0,2}(t), \nu_1^{caplet}) \\
\vdots \quad &= \quad \vdots \\
\text{Cap}(t, S_{0,n-1}(t), \nu_{0,n-1}^{cap}) &= \text{Caplet}(t, L(t, T_0), S_{0,n-1}(t), \nu_0^{caplet}) + \ldots + \text{Caplet}(t, L(t, T_{n-2}), S_{0,n-1}(t), \nu_{n-2}^{caplet}) \\
\text{Cap}(t, S_{0,n}(t), \nu_{0,n}^{cap}) &= \text{Caplet}(t, L(t, T_0), S_{0,n}(t), \nu_0^{caplet}) + \ldots + \text{Caplet}(t, L(t, T_{n-1}), S_{0,n}(t), \nu_{n-1}^{caplet})
\end{aligned}$$

[4] The use of the terms spot and flat volatilities appear in Hull [2004].

This system of equations can easily be solved by solving the topmost equation first (it is clear that $\nu_{0,1}^{cap} = \nu_0^{caplet}$) and then proceeding down the array. Thus, only one variable is solved for at a time as we move down the array.

A complication arises when quotes on South African data are considered as cap maturities do not increase in quarterly steps. The array of equations, in the South African setting, has more unknowns than equations. To overcome this problem it is necessary to make an inference as to what the missing cap volatilities are.

i	T_i	Quoted $\nu_{0,i+1}^{cap}$	Interpolated $\nu_{0,i+1}^{cap}$	ν_i^{caplet}
0	0.25	-	8.2000%	8.2000%
1	0.5	-	8.2000%	8.2000%
2	0.75	8.2%	8.2000%	8.2000%
3	1	-	8.6352%	9.5646%
4	1.25	-	9.1000%	10.4663%
5	1.5	-	9.5500%	11.1750%
6	1.75	10.0%	10.0000%	11.9187%
7	2	-	10.4986%	12.9713%
8	2.25	-	10.9973%	13.8003%
9	2.5	-	11.4959%	14.5904%
10	2.75	12.0%	12.0000%	15.5638%
11	3	-	12.1233%	12.7965%
12	3.25	-	12.2479%	13.0298%
13	3.5	-	12.3740%	13.2945%
14	3.75	12.5%	12.5000%	13.5831%
15	4	-	12.6233%	13.7875%
16	4.25	-	12.7479%	14.0362%
17	4.5	-	12.8740%	14.3036%
18	4.75	13.0%	13.0000%	14.5927%
19	5	-	13.0489%	13.5643%
20	5.25	-	13.0989%	13.6881%
21	5.5	-	13.1489%	13.7787%
22	5.75	13.2%	13.2000%	13.9779%
23	6	-	13.2717%	14.4411%
24	6.25	-	13.3475%	14.6448%
25	6.5	-	13.4250%	14.8909%
26	6.75	13.5%	13.5000%	15.0531%

Table 4.3: Cap volatilities, interpolated cap volatilities and stripped caplet volatilities on 2 May 2007.

The issue of missing cap volatilities was dealt with by interpolating linearly between known cap volatilities.

Monthly cap volatilities were obtained from 1 September 2006 to 29 February 2008. The obtained cap volatilities are mid rates. The volatilities were provided by Cadiz FSG. Cadiz FSG sourced the data from Rand Merchant Bank. Table 4.3 shows the interpolation and stripping procedure applied to the market data obtained on 2 May 2007. The first three interpolated cap volatilities have been set equal to the first available cap volatility. The missing cap volatilities are then interpolated between available cap volatilities. The result is that there are now as many equations as unknowns and this allows the spot volatilities (the final column of table 4.3) to be solved for.

4.1.5 Populating the swaption volatility matrix

The raw swaption volatility matrix can now be constructed as in the form of table 4.2. The output from the previous section appears in the left hand column of table 4.4. Monthly swaption volatilities were obtained from 1 September 2006 to 29 February 2008. Swaption volatilities were obtained from Cadiz FSG. Cadiz FSG obtained the data from Standard Bank. The swaption volatilities are mid rates. The matrix of obtained swaption volatilities is very sparsely populated as table 4.4 demonstrates.

	Swap Length							
First Setting Date	0.25	0.5	0.75	1	1.25	1.5	1.75	2
0.25	8.2000%	-	-	-	-	-	-	-
0.5	8.2000%	-	-	9.9606%	-	-	-	10.5106%
0.75	8.2000%	-	-	-	-	-	-	-
1	9.5646%	-	-	12.1000%	-	-	-	12.6500%
1.25	10.4663%	-	-	-	-	-	-	-
1.5	11.1750%	-	-	-	-	-	-	-
1.75	11.9187%	-	-	-	-	-	-	-
2	12.9713%	-	-	-	-	-	-	-
2.25	13.8003%	-	-	-	-	-	-	-
2.5	14.5904%	-	-	-	-	-	-	-
2.75	15.5638%	-	-	-	-	-	-	-
3	12.7965%	-	-	-	-	-	-	-
3.25	13.0298%	-	-	-	-	-	-	-
3.5	13.2945%	-	-	-	-	-	-	-
3.75	13.5831%	-	-	-	-	-	-	-
4	13.7875%	-	-	-	-	-	-	-
4.25	14.0362%	-	-	-	-	-	-	-
4.5	14.3036%	-	-	-	-	-	-	-
4.75	14.5927%	-	-	-	-	-	-	-
5	13.5643%	-	-	-	-	-	-	-
5.25	13.6881%	-	-	-	-	-	-	-
5.5	13.7787%	-	-	-	-	-	-	-
5.75	13.9779%	-	-	-	-	-	-	-
6	14.4411%	-	-	-	-	-	-	-
6.25	14.6448%	-	-	-	-	-	-	-

Table 4.4: The raw, un-interpolated swaption volatility matrix on 2 May 2007.

In section 4.6.2 (when the Cascade Calibration methods are considered), it will be necessary to have a fully populated swaption volatility matrix. One way to obtain an approximation of what this matrix could be is to perform interpolation inside the table 4.4. The product of such an interpolation is shown in table 4.5. Linear interpolation was carried out between the values in rows relating to 0.5 years and 1 years to maturity. To obtain the values in the row relating to 0.75 years to maturity, linear interpolation was carried out vertically between the two previously mentioned rows. All other values were obtained by scaling (vertically) an interpolated or known value by the proportional change in the caplet volatilities. For example, the volatility of a swap with a length of 1.25 years and maturity of 1.25 years was obtained by the following calculation

$$13.39 = 12.24 \times \frac{10.47}{9.46}.$$

This interpolation scheme is crude. It will only be used in demonstrating the Cascade Calibration Algorithm.

	Swap Length							
First Setting Date	0.25	0.5	0.75	1	1.25	1.5	1.75	2
0.25	8.20%	8.79%	9.37%	9.96%	10.10%	10.24%	10.37%	10.51%
0.5	8.20%	8.79%	9.37%	9.96%	10.10%	10.24%	10.37%	10.51%
0.75	8.20%	9.60%	10.31%	11.03%	11.17%	11.31%	11.44%	11.58%
1	9.56%	10.41%	11.25%	12.10%	12.24%	12.38%	12.51%	12.65%
1.25	10.47%	11.39%	12.32%	13.24%	13.39%	13.54%	13.69%	13.84%
1.5	11.17%	12.16%	13.15%	14.14%	14.30%	14.46%	14.62%	14.78%
1.75	11.92%	12.97%	14.02%	15.08%	15.25%	15.42%	15.59%	15.76%
2	12.97%	14.12%	15.26%	16.41%	16.60%	16.78%	16.97%	17.16%

Table 4.5: Interpolation on a portion of the raw swaption volatility matrix presented in table 4.4. The values that appear in red are non-interpolated values.

4.2 The link between the swaption volatility matrix and the instantaneous volatility and correlation structures

We wish to parameterise the instantaneous volatility parameters and the instantaneous correlation parameters of the equations 2.4 and 4.1 which is restated here as

$$\frac{dL(t,T_k)}{L(t,T_k)} = \mu(t,T_k)dt + \sigma(t,T_k)dB_t^k \quad k = 0,1,...,m$$

where $dB_t^i dB_t^j = \rho_{ij}dt$.

The LIBOR Market Model takes as inputs the instantaneous volatilities and the instantaneous correlation structures of forward rates. Section 4.1 shows that the data to which we have to calibrate (the swaption volatility matrix) is in the form of annualised volatilities. It is therefore necessary to establish a link between the annualised swaption volatilities and the instantaneous volatility and correlation structures.

The link between the instantaneous volatility structures and the caplet volatilities is immediately observable from proposition 3.2.2.

$$\begin{aligned} \nu_n^2(t) &= \frac{1}{(T_n - t)} \int_t^{T_n} \|\lambda(u, T_n)\|^2 du \\ &= \frac{1}{(T_n - t)} \int_t^{T_n} \sigma(u, T_n)^2 du. \end{aligned}$$

In the next section, some specific structures of $\sigma(u, T_n)$ will be considered.

As remark 3.2.5 pointed out, the LIBOR Market Model and the Swap Market Model are inconsistent. Therefore it is not possible to find an exact relationship between the swaption volatilities and the instantaneous correlation and volatility structures.

Joshi [2003] and Brigo and Mercurio [2006] give overviews of approaches to pricing swaptions approximately using inputs from the LIBOR Market Model. The approximation to the swaption volatility that appears most widely was introduced by Rebonato [1998]. Given the LIBOR Market Model parameters (instantaneous volatility structures and instantaneous correlations) a formula is derived for the approximate corresponding volatility of the swap rate.

Proposition 4.2.1 (Rebonato's Formula) *In the Swap Market Model framework, the swap rate follows, under $\mathbb{Q}^{G_{\alpha+1,\beta}}$, the following process*

$$\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} = \lambda_{\alpha,\beta}(t) dW_t^{\mathbb{Q}^{G_{\alpha+1,\beta}}}.$$

An approximate formula for the instantaneous volatility of the swap rate can be calculated using the inputs of the LIBOR Market Model and the following equation

$$\nu_{\alpha,\beta}^2(t) = \frac{\sum_{k=\alpha}^{\beta-1} \sum_{h=\alpha}^{\beta-1} \omega_{k+1}(t)\omega_{h+1}(t) L(t, T_k) L(t, T_h) \rho_{kh} \int_t^{T_\alpha} \sigma(u, T_k)\sigma(u, T_h) du}{S_{\alpha,\beta}(t)^2 \times (T_\alpha - t)}.$$

Derivation

In Black's formula for swaptions, it is known that

$$\begin{aligned} \nu_{\alpha,\beta}^2(t) &= \frac{1}{T_\alpha - t} \int_t^{T_\alpha} \|\lambda_{\alpha,\beta}(u)\|^2 du \\ &= \frac{1}{T_\alpha - t} \int_t^{T_\alpha} d(\ln S_{\alpha,\beta}(u)) d(\ln S_{\alpha,\beta}(u)). \end{aligned}$$

Since we work in the South African setting, we invoke assumption 2.1.5 so that we can represent a swap rate (applying to a tradable swap) as the weighted sum of forward rates (applying to tradable FRAs). Thus we can represent the swap rate in terms of equation 2.2 so that

$$S_{\alpha,\beta}(u) = \sum_{i=\alpha+1}^{\beta} \omega_i(u) L(u, T_{i-1}).$$

Now, assume that the weights remain constant at their initial values (ie at time t), then

$$dS_{\alpha,\beta}(u) \approx \sum_{i=\alpha+1}^{\beta} \omega_i(t) dL(u, T_{i-1}).$$

The quadratic variation is approximately

$$dS_{\alpha,\beta}(u)dS_{\alpha,\beta}(u) \approx \sum_{i=\alpha+1}^{\beta}\sum_{j=\alpha+1}^{\beta} \omega_j(t)L(u,T_{j-1})\omega_i(t)L(u,T_{i-1})\sigma(u,T_i)\sigma(u,T_j)\rho_{ij}du.$$

Since

$$d(\ln S_{\alpha,\beta}(u))d(\ln S_{\alpha,\beta}(u)) = \frac{d(S_{\alpha,\beta}(u))}{S_{\alpha,\beta}(u)}\frac{d(S_{\alpha,\beta}(u))}{S_{\alpha,\beta}(u)}$$

we have (after freezing forward rates and swap rates at their initial values)

$$\int_t^{T_\alpha} d(\ln S_{\alpha,\beta}(u))d(\ln S_{\alpha,\beta}(u)) \approx \frac{\sum_{i=\alpha}^{\beta-1}\sum_{j=\alpha}^{\beta-1}\omega_{j+1}(t)L(t,T_j)\omega_{i+1}(t)L(t,T_i)\rho_{ij}\int_t^{T_\alpha}\sigma(u,T_i)\sigma(u,T_j)du}{S_{\alpha,\beta}^2(t)}$$

and Rebonato's approximation follows.

Ξ

It is worth highlighting the assumptions that have been made in this derivation.

- Since we are working in the South African context, we cannot simply represent a swap rate as a weighted sum of forward rates. We therefore assumed that FRAs obey the swap day schedule (assumption 2.1.5).
- The ω's were set equal to their values at the start of the period. The implication is that the swap rate is treated as a *constant* linear combination of the forward rates until the first setting date.
- After the quadratic variation has been computed it is assumed that the forward rates and the swap rates are set equal to their initial values at time t.

Brigo and Mercurio [2006] notes that proposition 4.2.1 is quite an accurate approximation despite the assumptions that are made.

Proposition 4.2.1 is a key tool in the calibration of the LIBOR Market model as it provides the link between each entry in the swaption volatility matrix and the instantaneous correlation and volatility structures.

On examining the formula in proposition 4.2.1, it can be seen that the formula depends on market quoted quantities ($S_{\alpha,\beta}(u)$ and $L(t,T_i)$ $(i = \alpha, ..., \beta - 1)$) and quantities that will be specified in terms of formulas or exogenously (ρ_{ij} and $\sigma(u,T_i)$ $(i,j = \alpha, ..., \beta - 1)$).

The instantaneous volatility and correlation structures are given general formulaic structures. Different structures lead to different forms of proposition 4.2.1. Hence different instantaneous volatility and correlation structures imply different approaches to the calibration of the LIBOR Market Model. The aim of sections 4.3 and 4.4 is as follows:

- Show how the components (instantaneous volatility and correlation structures) of the formula in proposition 4.2.1 will be specified.
- Present the formula in proposition 4.2.1 in terms of the specified instantaneous correlation and volatility structures.

4.3 Instantaneous volatility structures

The instantaneous volatility structures are either specified as piecewise constant or as parametric functions.

4.3.1 Piecewise constant instantaneous volatility structures

Consider the tenor structure $\mathcal{T} = (T_{-1}, T_0, ..., T_n)$. A piecewise constant instantaneous volatility function, $\sigma(t, T_k)$, is constant for $T_i < t < T_{i+1}$ $(i = 0, ..., k-1)$. Thus, a piecewise constant function is a 'step' function. Two piecewise constant formulations will be considered[5].

Functional form dependent on current time, time to maturity and time of maturity

This functional form will be used in the Cascade Calibration Algorithm and is the most general piecewise constant volatility specification[6]. Table 4.6 shows how the instantaneous volatility changes over time.

	Current time				
	Time $t \in (0, T_0]$	$t \in (T_0, T_1]$	$t \in (T_1, T_2]$	...	$t \in (T_{N-2}, T_{N-1}]$
$L(t, T_0)$	$\sigma_{1,1}$	dead	dead	...	dead
$L(t, T_1)$	$\sigma_{2,1}$	$\sigma_{2,2}$	dead	...	dead
$\vdots$	...	...	...	...	...
$L(t, T_N)$	$\sigma_{N,1}$	$\sigma_{N,2}$	$\sigma_{N,3}$	...	$\sigma_{N,N}$

Table 4.6: The most general piecewise constant instantaneous volatility formulation dependent on current time, time to maturity and time of maturity.

This functional form implies a non-homogeneous evolution of the volatility term structure (ie forward rate volatilities at different times but with the same time to maturity may differ). Under this functional form, caplet volatilities can be presented as follows:

$$\nu_k^2(T_{-1}) \quad = \quad \nu_k^2(0)$$

[5]Brigo and Mercurio [2006] presents four other piecewise constant functional forms.
[6]This formulation is presented on page 210 of Brigo and Mercurio [2006].

$$
\begin{aligned}
&= \frac{1}{T_k}\int_0^{T_k}\sigma(u,T_k)^2du \\
&= \frac{1}{T_k}\sum_{i=1}^{k+1}\delta_i\sigma_{k+1,j}^2.
\end{aligned}
$$

This volatility structure will be applied to the formula in proposition 4.2.1. Under this volatility formulation, the swaption volatility given by the proposition 4.2.1 can be expressed as

$$
(\nu_{\alpha,\beta}(t))^2 = \frac{\sum_{i=\alpha+1}^{\beta}\sum_{j=\alpha+1}^{\beta}\omega_i(t)\omega_j(t)L(t,T_{i-1})L(t,T_{j-1})\rho_{i-1,j-1}\sum_{h=0}^{\alpha}\delta_h\sigma_{i,h+1}\sigma_{j,h+1}}{(T_\alpha - t)(S_{\alpha,\beta}(t))^2}.
$$

The suitability of this functional form in the South African setting is directly linked to the suitability of the Cascade Calibration Algorithms in the South African setting. As will be explained later, the Cascade Calibration Algorithms is probably not the best calibration procedure to use on the South African swaption volatility matrix.

Functional form dependent on time of maturity

An instantaneous volatility formulation which is only dependent on the time *of* maturity is considered [7]. Table 4.7 shows the evolution of this volatility structure over time. This functional form simplifies many calculations. However, it is a time-inhomogeneous form since the instantaneous volatility is not necessarily constant for different points in time and the same time to maturity. Under this functional form, caplet volatilities can be presented as follows:

	Current time				
	Time $t \in (0, T_0]$	$t \in (T_1, T_2]$	$t \in (T_2, T_3]$	...	$t \in (T_{N-2}, T_{N-1}]$
$L(t, T_0)$	$\sigma_{1,.}$	dead	dead	...	dead
$L(t, T_1)$	$\sigma_{2,.}$	$\sigma_{2,.}$	dead	...	dead
$\vdots$	...	...	...	...	...
$L(t, T_N)$	$\sigma_{N,.}$	$\sigma_{N,.}$	$\sigma_{N,.}$	...	$\sigma_{N,.}$

Table 4.7: The piecewise constant instantaneous volatility formulation in which the instantaneous volatility is only dependent on the maturity date.

$$
\begin{aligned}
\nu_k^2(T_{-1}) &= \nu_k^2(0) \\
&= \frac{1}{T_k}\int_0^{T_k}\sigma(u,T_k)^2du \\
&= \frac{1}{T_k}\sum_{i=1}^{k+1}\delta_i\sigma_{k+1}^2 \\
&= \sigma_{k+1}^2.
\end{aligned}
$$

This volatility formulation does not model the term structure of volatility and as such is less suitable than the parametric form.

[7]This formulation is presented on page 211 of Brigo and Mercurio [2006].

4.3.2 Parametric instantaneous volatility structures

The most common approach to fitting a parametric form to the volatility structure is

$$\sigma(t, T_k) = [a(T_k - t) + d]e^{-b(T_k - t)} + c. \tag{4.2}$$

Note that this specification allows the volatility structure to be completely time-homogeneous. However, it also leads to an imperfect fit of the caplet volatilities. In this model the caplet volatility is given as

$$\begin{aligned}\nu_k^2(T_{-1}) &= \nu_k^2(0) \\ &= \frac{1}{T_k}\int_0^{T_k} ([a(T_k - u) + d]e^{-b(T_k - u)} + c)^2 du \\ &= \frac{1}{T_k} I^2(T_k; a, b, c, d).\end{aligned}$$

The swaption volatility is given by proposition 4.2.1

$$\nu_{i,j}^2(t) \times (T_i - t) \times S_{i,j}(t) = \sum_{k=i}^{j-1}\sum_{h=i}^{j-1} \omega_{k+1}(t)\omega_{h+1}(t)L(t, T_k)L(t, T_h)\rho_{ij}\int_t^{T_i} \sigma(u, T_k)\sigma(u, T_h)du$$

where

$$\sigma(u, T_k) = [a(T_k - u) + d]e^{-b(T_k - u)} + c.$$

Joshi [2003] notes that by allowing for some time-inhomogeneity in the volatility structure, a perfect fit can be obtained to the caplet prices (the swaption prices won't be fitted perfectly). The parameters $K_0, ..., K_{N-1}$ are introduced and the new parametric specification is

$$\sigma(t, T_k) = K_k\left([a(T_k - t) + d]e^{-b(T_k - t)} + c\right) \tag{4.3}$$

where

$$K_k = \frac{(\nu_{k,k+1}^M(0))^2}{\frac{1}{T_k} I^2(T_k; a, b, c, d)}.$$

Thus,

$$\nu_k^2(T_{-1}) = (\nu_{k,k+1}^M(0))^2$$

where $k = 0, ..., N - 1$. If all the K_i's are close to one, then the first parametric volatility structure (equation 4.2) provides a very good fit on its own. The more widely the K_i's are dispersed around one, the greater the amount of time-inhomogeneity introduced into the model for instantaneous volatility.

In order to use equations 4.2 and 4.3 to calculate the swaption volatility (in terms of proposition 4.2.1), it is necessary to compute the following integral:

$$\int_t^{T_i} \sigma(u, T_k)\sigma(u, T_h)du = I(T_i; T_k, T_h, a, b, c, d) - I(t; T_k, T_h, a, b, c, d).$$

By noting that

$$\int t^n e^{xt} dt = \sum_{i=0}^{n} \frac{(-1)^i \prod_{j=1}^{i}(n-j+1)}{x^{i+1}} e^{xT} T^{n-i} + Constant$$

it can be seen that

$$\begin{aligned} I(y; T_k, T_h, a, b, c, d) &= q\left(\frac{e^{xy}y^2}{x} - \frac{2e^{xy}y}{x^2} + \frac{2e^{xy}}{x^3}\right) \\ &+ r\left(\frac{e^{xy}y}{x} - \frac{e^{xy}}{x^2} + \frac{1}{x^2}\right) \\ &+ s\left(\frac{e^{xy}}{x} - \frac{1}{b}\right) \\ &+ u\left(\frac{e^{by}y}{b} - \frac{1}{b}\right) \\ &+ v\left(\frac{e^{by}y}{b} - \frac{e^{by}}{b^2} + \frac{1}{b^2}\right) \\ &+ wy \\ &+ Constant \end{aligned}$$

where

$$\begin{aligned} x &= 2b \\ q &= B1 \times B2 \\ r &= -A1 \times B2 - B1 \times A2 \\ s &= A1 \times A2 \\ u &= A1 \times c + A2 \times c \\ v &= -B1 \times c - B2 \times c \\ w &= c^2 \end{aligned}$$

and

$$\begin{aligned} A1 &= (aT_h + d)e^{-bT_h} \\ B1 &= ae^{-bT_h} \\ A2 &= (aT_k + d)e^{-bT_k} \\ B2 &= ae^{-bT_k}. \end{aligned}$$

Equations 4.2 and 4.3 are probably the most suitable instantaneous volatility structures to employ in the South African setting because of three reasons:

- These specifications allow for a term structure of volatility.
- It is not necessary to have a fully populated swaptions volatility matrix in order to parameterise the models.
- By introducing the K parameters, an exact fit can be achieved to the caplet data.

4.4 The instantaneous correlation structure

The previous section dealt with the specification of the instantaneous volatility in proposition 4.2.1. This section will deal with the specification of the instantaneous correlation structure (ρ_{ij}). There are two choices in modelling ρ_{ij}.

- Let ρ_{ij} be specified by an exogenously specified correlation matrix.
- Model ρ_{ij} using a parametric structure and parameterise it as part of the calibration to the swaption volatility matrix.

4.4.1 Properties of correlation matrices

Brigo and Mercurio [2006] points out that there are three basic expectations of the correlations between forward rates that should, ideally, also be present in the model of the instantaneous correlations.

1. The correlations between forward rates must be positive.
2. Forward rates with closer maturities have higher correlation.
3. Forward rates with longer maturities are more correlated.

Furthermore, a correlation matrix is a symmetric positive definite matrix [8].

4.4.2 Full rank versus reduced rank

Consider the dynamics

$$\frac{dL(t,T_k)}{L(t,T_k)} = \mu(t,T_k)dt + \sigma(t,T_k)dB_t^k \quad k \; = \; 0,1,...,m \quad (4.4)$$

where $dB_t^i dB_t^j = \rho_{ij}dt$. If $\mathbf{B_t}$ is the $(m+1) \times 1$ vector of correlated Brownian motions then

$$d\mathbf{B}_t d\mathbf{B}_t^T = \begin{pmatrix} 1 & \rho_{01} & \cdots & \rho_{m0} \\ \rho_{01} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \rho_{(m-1)(m)} \\ \rho_{(m)(0)} & \cdots & \rho_{(m)(m-1)} & 1 \end{pmatrix} dt := \underline{\rho} dt.$$

In section 2.2.3 it was noted that a key manipulation to be carried out was to find an $(m+1) \times d$ matrix $\mathbf{A}$ such that

$$\underline{\rho} \;=\; \mathbf{AA^T}$$

since this allows us to rewrite the forward LIBOR dynamics in a form conducive to simulation. The smaller d is, the fewer independent Brownian motions have to be simulated. Hence the simulation process is less computationally intensive. We are therefore interested in minimising

[8] This implies that ρ can be written as XDX^T where X is an orthogonal matrix and D is the diagonal matrix with entries equal to the eigenvalues of matrix ρ. Let Γ be the diagonal matrix such that $\Gamma\Gamma = D$ then it can be seen that there exists an $(m+1) \times (m+1)$ matrix $B = \Gamma X$ such that $BB^T = \Gamma X(\Gamma X)^T = X\Gamma\Gamma^T X^T = XDX^T$

d whilst still maintaining an accurate correlation structure. We can achieve $d < m + 1$ in a number of ways[9]:

1. Specify an exogenous correlation matrix $\underline{\rho}$ which has rank equal to $d < m + 1$.
2. Specify a parametric correlation structure for $\underline{\rho}$ which is structured such that it has rank $d < m+1$. Parameterise the correlation structure as part of the calibration to the swaption volatility matrix.
3. If the exogenously specified correlation matrix is of full rank then approximate this matrix with a reduced rank form and simulate using the reduced rank form.
4. Specify a full rank parameterisation for the correlation matrix. Parameterise the correlation matrix as part of the calibration to the swaption volatility matrix. Approximate this matrix with a reduced rank form and simulate using the reduced rank form.

This project will focus on the third and the fourth approaches. Thus, it is necessary to introduce a full rank parametric form for the correlation matrix. It is also necessary to introduce a reduced rank approximation to a full rank matrix.

4.4.3 Full rank parameterisations

A full rank parameterisation presented in Joshi [2003] is

$$\rho_{ij} = e^{-\beta|T_i - T_j|} \tag{4.5}$$

such that ρ_{ij} is the instantaneous correlation between $L(t, T_i)$ and $L(t, T_j)$. β is a parameter which can be determined in the calibration process. Joshi [2003] notes that correlations modelled as

$$\rho_{ij} = e^{-0.1|T_i - T_j|} \tag{4.6}$$

tend to fit the market quite well. Thus in all subsequent sections correlation structures will be treated as follows:

1. The exogenous correlation structure will be assumed to be of the above functional form with $\beta = 0.1$.
2. Alternatively, β will be a free parameter which will be parameterised in the calibration to the swaption volatility matrix.

Thus, the question as to how ρ_{ij} will be treated in proposition 4.2.1 has now been answered. Figure 4.1 shows the form of the correlation structure when it is specified as Joshi [2003] recommends.
The following section will provide an overview of two ways in which to approximate the above full rank parameterisation with reduced rank formulations. As noted above, obtaining the approximate reduced rank correlation matrix is essential for computationally efficient simulation

[9] A full rank decomposition can be accomplished through a Cholesky factorisation (Glasserman [2003] and Joshi [2003] give descriptions of Cholesky factorisation). But this implies that $d = m + 1$ and this will typically be too computationally expensive for use in a simulation of forward rates.

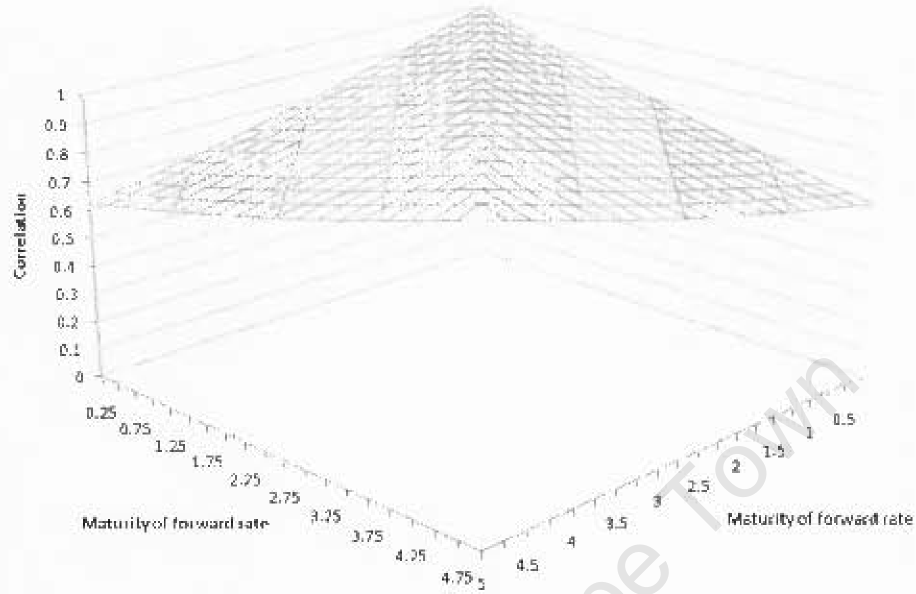

Figure 4.1: The instantaneous correlation between forward rates where the correlation is specified as $\rho_{ij} = e^{-0.1|T_i - T_j|}$. The horizontal axes show the setting dates of forward rates. The correlation matrix has been presented for the first twenty rates (quarterly forward rates with first setting dates being 0.25,0.5,...,4.75,5).

of the system of forward rates.

Schoenmaker and Coffey [2000], Rebonato [1999a] and Rebonato [1999b] propose other full rank parameterisations. Brigo and Mercurio [2006] also provide a review of these methods.

4.4.4 Reduced rank correlation matrices

Two approaches shall be considered in order to approximate the full rank correlation matrix:

- Rebonato's angle parameterisation.
- Eigenvalue zeroing.

Pietersz and Groenen [2004] explains that there are five methods by which to find reduced rank approximations of the correlation matrix. In addition to the above two methods there is the geometric programming approach of Grubisic and Pietersz [2004], the Lagrange Multiplier method of Zhang and Wu [2003] and lastly the majorisation method Pietersz and Groenen [2004].

Rebonato's angle parameterisation

The key points in this parameterisation are set out below.

- $\underset{\sim}{\rho}$ is not directly specified but is assumed to be of rank d.
- $\mathbf{A}$ is given a parametric structure such that $\underset{\sim}{\rho}$ retains all its essential properties (positive semi-definite and unit diagonals).
- The parameters of the reduced rank matrix are estimated by calibrating the parametric correlation matrix to swaption prices or by solving for the parameters that will minimise the distance between the exogenously specified matrix and the approximate correlation matrix.

The approach presented here is reviewed in Brigo and Mercurio [2006]. The entries in matrix $\mathbf{A}$ are specified as follows:

$$\begin{aligned} a_{i,1} &= \cos(\theta_{i,1}) \\ a_{i,k} &= \cos(\theta_{i,k})\sin(\theta_{i,1})...\sin(\theta_{i,k-1}) \quad 1 < k < d \\ a_{i,d} &= \sin(\theta_{i,1})...\sin(\theta_{i,d-1}) \end{aligned}$$

or

$$A = \begin{pmatrix} \cos(\theta_{1,1}) & \cos(\theta_{1,2})\sin(\theta_{1,1}) & \dots & \sin(\theta_{1,1})...\sin(\theta_{1,d-1}) \\ \cos(\theta_{2,1}) & \cos(\theta_{2,2})\sin(\theta_{2,1}) & \dots & \sin(\theta_{2,1})...\sin(\theta_{2,d-1}) \\ \cos(\theta_{3,1}) & \cos(\theta_{3,2})\sin(\theta_{3,1}) & \dots & \sin(\theta_{3,1})...\sin(\theta_{3,d-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\theta_{N+1,1}) & \cos(\theta_{N+1,2})\sin(\theta_{N+1,2}) & \dots & \sin(\theta_{N+1,1})...\sin(\theta_{N+1,d-1}) \end{pmatrix}.$$

Also note that the diagonal elements of the resulting matrix $\underset{\sim}{\rho}$ are

$$\begin{aligned} \rho_{ii} &= \|A_i\|^2 \\ &= \cos^2(\theta_{i,1}) + \cos^2(\theta_{i,2})\sin^2(\theta_{i,1}) + ... + \sin(\theta_{i,1}) \times ... \times \sin(\theta_{i,d-1}) \\ &= 1 - \sin^2(\theta_{i,1})(1 - \cos^2(\theta_{i,2})) + ... + \sin(\theta_{i,1}) \times ... \times \sin(\theta_{i,d-1}) \\ &= 1 - \sin^2(\theta_{i,1})\sin^2(\theta_{i,2})(1 - \cos^2(\theta_{i,2})) + ... + \sin(\theta_{i,1}) \times ... \times \sin(\theta_{i,d-1}) \\ &= ... \\ &= 1 - \sin(\theta_{i,1}) \times ... \times \sin(\theta_{i,d-2})(1 - \cos^2(\theta_{i,d-1})) + ... + \sin(\theta_{i,1}) \times ... \times \sin(\theta_{i,d-1}) \\ &= 1 - \sin(\theta_{i,1}) \times ... \times \sin(\theta_{i,d-1}) + \sin(\theta_{i,1}) \times ... \times \sin(\theta_{i,d-1}) \\ &= 1. \end{aligned}$$

The values of the θ_{ij}'s are now determined by calibrating the formula in proposition 4.2.1 (with the ρ_{ij}'s specified using the above form) to the swaption volatility matrix. Alternatively (and this is the approach that this dissertation will take) the distance between an exogenously specified correlation matrix and the parametric reduced rank correlation matrix will be minimised. The error function that will be minimised is the Frobenius norm.

$$\begin{aligned} \sum_{i=0}^{m}\sum_{j=0}^{m}(\underset{\sim}{\rho}_{ij} - \tilde{\underset{\sim}{\rho}}_{ij})^2 &= \text{trace}((\underset{\sim}{\rho} - \tilde{\underset{\sim}{\rho}})(\underset{\sim}{\rho} - \tilde{\underset{\sim}{\rho}})^T) \\ &= \|(\underset{\sim}{\rho} - \tilde{\underset{\sim}{\rho}})\|_{\mathbf{F}} \end{aligned}$$

This is the sum of squared differences between the entries in the reduced rank formulation and the entries in the exogenously specified correlation matrix.

Eigenvalue zeroing

This approach follows the reverse direction of the angles parameterisation. Instead of using $\mathbf{A}$ as the starting point, the full rank, exogenously specified $\underline{\rho}$ is decomposed so as to obtain $\mathbf{A}$. $\underline{\rho}$ is written as XDX^T where X is an orthogonal matrix and D is a diagonal matrix with entries all equal to the eigenvalues of the correlation matrix. The columns and rows corresponding to the $(m+1)-d$ smallest eigenvalues are now removed so that $\hat{D}$ (which is now a $d \times d$ matrix) is obtained and the resulting square root matrix of $\hat{D}$ is $\hat{\Gamma}$. The corresponding columns in X are also removed. Thus, the $(m+1) \times d$ matrix $\mathbf{A} = \hat{X}\hat{\Gamma}$ is obtained. Now, the act of removing eigenvalues has led to $G = \mathbf{A}\mathbf{A}^{\mathbf{T}}$ no longer being a correlation matrix (the diagonal elements may no longer be one). It is therefore suggested that $\hat{\rho}$ is used as the approximation to $\underline{\rho}$ where the elements of $\hat{\rho}$ are specified as

$$\hat{\rho}_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}}.$$

And so, the new pseudo square root matrix has ijth entry

$$\hat{\mathbf{A}}_{ij} = \frac{\mathbf{A_{ij}}}{\sqrt{G_{ii}}}.$$

The result is that the full rank correlation matrix ρ is approximated by $\hat{\rho}$ which is a correlation matrix that has been modified by removing d of the smallest eigenvalues and then re-scaling the matrix entries so as to ensure unit diagonals.

Comparison of angles parameterisation and eigenvalue zeroing for the rank reduction of an exogenously specified correlation matrix

Consider again the exogenously specified correlation presented in figure 4.1.

The correlations between each of the twenty rates is now approximated by a two factor angles approximation and a two factor eigenvalue zeroing approximation. For each of the two approximations, the absolute difference between the entries in the correlation matrix and the entries in the approximation to the correlation matrix is determined. The better the approximation, the smaller the absolute difference. The points at which the eigenvalue approach provides a better approximation than Rebonato's angle approach are noted.

Table 4.8 shows a grid comparing the approximations: a point at which the eigenvalue approach gives a better approximation than the angle approach is denoted by a 1. Note that in the grid, the top left hand corner represents the approximations based on the correlation between quarterly forward rates with first setting dates both being 0.25 years. The bottom right hand corner represents the approximations based on the correlation between quarterly forward rates with first setting dates both being 5 years.

Table 4.9 presents a grid in which the results from a similar comparison between the four factor angles and eigenvalue zeroing approximations have been performed.

In general, table 4.8 and table 4.9 are dominated by blank spaces. This indicates that for most entries, the angle approximation serves as a better approximation to the correlation matrix than the eigenvalue zeroing approximation.

D																1	1	1	1
	D															1	1	1	1
		D				1	1	1								1	1	1	1
			D	1	1	1	1	1								1	1	1	1
			1	D	1	1	1	1											
			1	1	D	1	1	1											
		1	1	1	1	D	1												
		1	1	1	1	1	D												
		1	1	1	1			D											
									D										
										D									
											D			1	1	1	1		
												D	1	1	1	1	1		
												1	D	1	1	1	1		
											1	1	1	D	1	1			
											1	1	1	1	D	1			
1	1	1	1								1	1	1	1	1	D			
1	1	1	1								1	1	1				D		
1	1	1	1															D	
1	1	1	1																D

Table 4.8: Comparison of the two factor angle and eigenvalue zeroing approximations to the correlation matrix presented in figure 4.1. A '1' denotes the case in which the absolute difference between the angle formulation and the correlation matrix is greater than the absolute difference between the eigenvalue zeroing approach and the correlation matrix. The 'D' denotes the diagonal elements in which there is no difference between the two approaches.

D								1				1	1	1				1	1
	D						1	1				1	1	1				1	1
		D										1							
			D		1														
				D	1								1	1					
			1	1	D								1	1	1			1	1
						D						1	1	1	1		1	1	1
	1						D			1	1	1	1				1	1	1
1	1							D	1	1	1	1							
								1	D	1	1	1							
							1	1	1	D	1								
							1	1	1	1	D								
1	1	1				1	1	1	1			D						1	1
1	1			1	1	1	1						D						
1	1			1	1	1								D	1				
					1	1								1	D				
																D			
						1	1										D		
1	1				1	1	1					1						D	
1	1				1	1	1					1							D

Table 4.9: Comparison of the four factor angle and eigenvalue zeroing approximations to the correlation matrix presented in figure 4.1. A '1' denotes the case in which the absolute difference between the angle formulation and the correlation matrix is greater than the absolute difference between the eigenvalue zeroing approach and the correlation matrix. The 'D' denotes the diagonal elements in which there is no difference between the two approaches.

4.5 Calibration methods

Calibration of the LIBOR Market Model can occur in a number of ways. To price instruments where the payoff depends on a number of forward rates (such as barrier or trigger swaptions), calibration must occur with a specified correlation matrix or give a correlation matrix as an output. The centrality of Rebonato's swaption volatility formula (proposition 4.2.1) has consistently been highlighted. Sections 4.4 and 4.3 emphasised the fact that the modeler must specify the instantaneous volatility structures and the instantaneous correlation structure since the formula of 4.2.1 depends on these two structures.

The aim of calibration is to parameterise the chosen instantaneous volatility and correlation structures such that the error between model swaption prices and actual swaption prices is minimised whilst maintaining a parsimonious parameter set. Models in which the error is not minimised are open to arbitrage. Models which do not have a parsimonious parameter set are typically over parameterised and will perform poorly when applied to data which have not been used in the parameter estimation.

Glasserman [2003] notes that simulation can be used for model calibration. For different sets of parameter values calculate the swaption prices implied by the model. Choose the parameter set that results in swaption prices that most accurately approximate the prices observed in the market. This approach is slow and other approaches will probably be more efficient.

Brigo and Mercurio [2006] show that calibration to the swaptions matrix using a piecewise constant volatility formulation which depends on the current time and the time of maturity of the forward rate together with angles parameterisation of the swaptions matrix produces a very irregular evolution of the volatility term structure. The same results are observed for other piecewise constant instantaneous volatility forms and also for parametric instantaneous volatility forms when the instantaneous correlation structure is treated as an output of the optimisation process.

Brigo and Mercurio [2006] analyse the calibration outputs of the closed form methods known as the Cascade Calibration Algorithm (CCA) and Rectangular Cascade Calibration Algorithms (RCCA) (the RCCA is laid out in Brigo and Morini [2002]). It is noted that linear interpolation of missing values in the swaptions volatility matrix leads to the output of the calibration algorithms producing negative and imaginary numbers in some instances. Brigo and Mercurio [2006] show that the smoothing of the volatility matrix using a parametric form can lead to this problem being overcome to some degree. The problem of an extremely erratic term structure of volatility evolution still remains.

Brigo and Mercurio [2006] investigate the CCA under various correlations and ranks and find that, for the data set investigated, there appears to be a trade off between the regularity of the evolution of the term structure and the qualitative acceptability of the terminal correlations[10] (the greater the rank of the reduced rank correlation matrix, the more irregular the evolution of the volatility term structure). Brigo and Morini [2003] extends the RCCA to cope with

[10] Terminal correlation is a measure of the correlation, at time 0, between the two random variables $L(t,T_i)$ and $L(t,T_j)$ where $0 < t < T_i < T_j$. This is different from the instantaneous correlation that has been considered thus far which is the correlation between $dL(t,T_i)$ and $dL(t,T_j)$.

missing volatilities (without having to resort to linear interpolation) in the Rectangular Cascade Calibration with Endogenous Interpolation Algorithm (RCCAEI).

4.6 Calibration

The calibration of the LIBOR Market Model involves the specification of the instantaneous volatility and correlation structures with respect to a swaptions volatility matrix and a yield curve. There are two general approaches for calibrating the LIBOR Market Model:

1. Optimisation.
2. Cascade Calibration.

Rebonato's formula (proposition 4.2.1) will be the theoretical link between the data and the models. The model volatility is $\nu_{\alpha,\beta}(t)$ ($\nu_{\alpha,\beta}(t)$ is dependent on the chosen volatility and correlation structure). The corresponding data entry in the volatility matrix is $\nu^M_{\alpha,\beta}(t)$ (the volatility of the swap rate with first setting date at time T_α and final payment at T_β and which is found in the volatility matrix in the row corresponding to T_α and the column corresponding to $T_\beta - T_\alpha$).

Calibrating the LIBOR Market Model through optimisation involves minimising the following sum over a portion or all of the swaption volatility matrix:

$$\sum_i \sum_j \left(\nu^M_{i,j}(t) - \nu_{i,j}(t)\right)^2.$$

Optimisation is the calibration approach associated with the parametric instantaneous volatility and correlation structures. This provides an approximate fit to the swaptions volatility matrix.

The Cascade Calibration approach involves a procedure for solving a series of one dimensional equations so as to parameterise the instantaneous volatility structure presented in table 4.6. It provides an exact fit to the swaption volatility matrix.

All calibrations are performed with the use of Excel and VBA using a 1.59 GHz Mobile AMD Sempron 3400+ with 2 GB of RAM. Where optimisation is required, Excel Solver is used.

4.6.1 Calibration through optimisation

A number of different instances and combinations of the parametric forms for instantaneous correlation and volatility are considered. Parameters are estimated based on a calibration to the swaption volatility matrix presented in table 4.4. The forward rates that are used in the parameterisations is presented in table B.1. In order to parameterise the model the following sum is minimised:

$$\begin{aligned} SS &= \sum_{i=0}^{24}(\nu^M_{i,i+1}(0) - \nu_{i,i+1}(0))^2 \\ &+ (\nu^M_{1,5}(0) - \nu_{1,5}(0))^2 \\ &+ (\nu^M_{3,7}(0) - \nu_{3,7}(0))^2 \end{aligned}$$

$$
\begin{aligned}
&+ \quad (\nu_{1,9}^M(0) - \nu_{1,9}(0))^2 \\
&+ \quad (\nu_{3,11}^M(0) - \nu_{3,11}(0))^2
\end{aligned}
$$

where $\nu_{i,j}(0)$ is given by Rebonato's formula in proposition 4.2.1. The following sections will look at how the calibrations vary when different instantaneous parametric volatility and correlation structures are used in proposition 4.2.1.

Rebonato [2002] (p 168) suggests that the following constraints be observed: $c+d > 0$, $c > 0$ and $b > 0$. These constraints were not applied in the calibrations that appear in the next section.

Parametric volatility structure and exogenous correlation

We calibrate to the swaption volatility matrix presented in table 4.4 and the forward rates presented in table B.1. The volatility structure is

$$
\sigma(t, T_k) = \left([a(T_k - t) + d]e^{-b(T_k - t)} + c\right)
$$

and the correlation structure is

$$
\rho_{ij} = e^{-0.1|T_i - T_j|}.
$$

There are four free parameters: a, b, c, d. β is set equal to 0.1. Tables 4.10 and 4.11 summarise the calibration output from minimising SS. The maximum absolute difference between the modelled volatility and the actual volatility is between $\nu_{10,11}^M(0)$ and $\nu_{10,11}(0)$. Figure 4.2 shows the modelled caplet volatilities and the actual caplet volatilities (the actual caplet volatilities are obtained through the procedure described in section 4.1.4) after calibration. The erratic behaviour of the actual caplet volatilities leads to the fairly large difference between $\nu_{10,11}^M(0)$ and $\nu_{10,11}(0)$.

This parameterisation is appealing from a modelling perspective as it implies a time homogenous evolution of the instantaneous volatility structure (the time homogeneity is a trait of a parsimonious model and is therefore appealing). This is shown in figure 4.3: the instantaneous volatility structures do not change as time progresses (the structure applicable in 1.5 years is the same as the structure applicable today).

a	0.118946
b	0.287499
c	-0.03073
d	0.091358
β	0.1

Table 4.10: The parameter values estimated from calibration to table 4.4 using a parametric volatility structure and an exogenous correlation matrix.

SS	0.00164
Maximum Absolute Difference	2.180%
Total Time Taken	5.7 minutes

Table 4.11: Calibration statistics from calibration to table 4.4 using a parametric volatility structure and an exogenous correlation matrix.

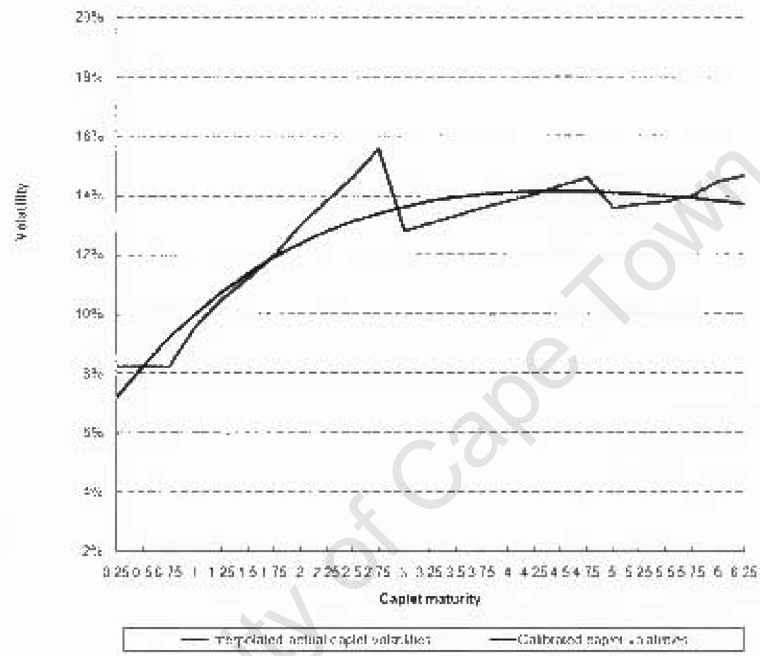

Figure 4.2: The market caplet volatilities and the calibrated caplet volatilities. The calibration took place with a, b, c, d as free parameters and with β equal to 0.1.

Parametric volatility structure, exogenous correlation, perfect fit to caplet volatilities

We calibrate to the swaption volatility matrix presented in table 4.4 and the forward rates presented in table B.1. The volatility structure is

$$\sigma(t, T_k) = K_k \left([a(T_k - t) + d] e^{-b(T_k - t)} + c \right)$$

and the correlation structure is

$$\rho_{ij} = e^{-0.1|T_i - T_j|}.$$

Once again, there are four free parameters: a, b, c, d. β is set equal to 0.1. a, b, c, d is estimated as in the previous calibration procedure. Thus, these parameters take the same values as in table 4.10. $K_0, ..., K_{24}$ is introduced so as to obtain a perfect fit to the market caplet volatilities. The

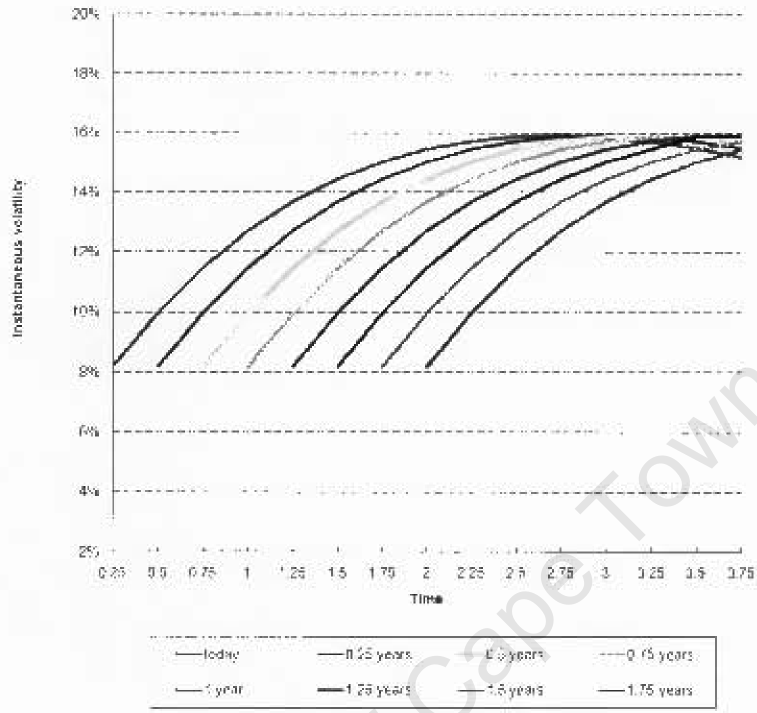

Figure 4.3: The evolution of the instantaneous volatility. The calibration took place with a, b, c, d as free parameters and with β equal to 0.1.

maximum absolute error is now between $\nu_{1,7}^{M}(0)$ and $\nu_{1,7}(0)$. The overall fit has improved (SS has dropped to 0.00032 in table 4.12 compared to 0.0016 in table 4.11). The computation time is virtually unchanged compared to the previous calibration. The trade off occurs in terms of the time homogeneity of the evolution of the instantaneous term structure of volatility. Figure 4.4 shows that the introduction of the K parameters has destroyed the time homogenous evolution of the term structure observable in figure 4.3. Notice that the kink observable in figure 4.2 has contributed to the large degree of irregularity at the 2.75 and 3 year points in figure 4.4.

SS	0.00032
Maximum Absolute Difference	1.459%
Total Time Taken	5.7 minutes

Table 4.12: Calibration statistics from calibration to table 4.4 using a parametric volatility structure and an exogenous correlation matrix. K parameters have been introduced so as to obtain a perfect fit to the caplet volatilities.

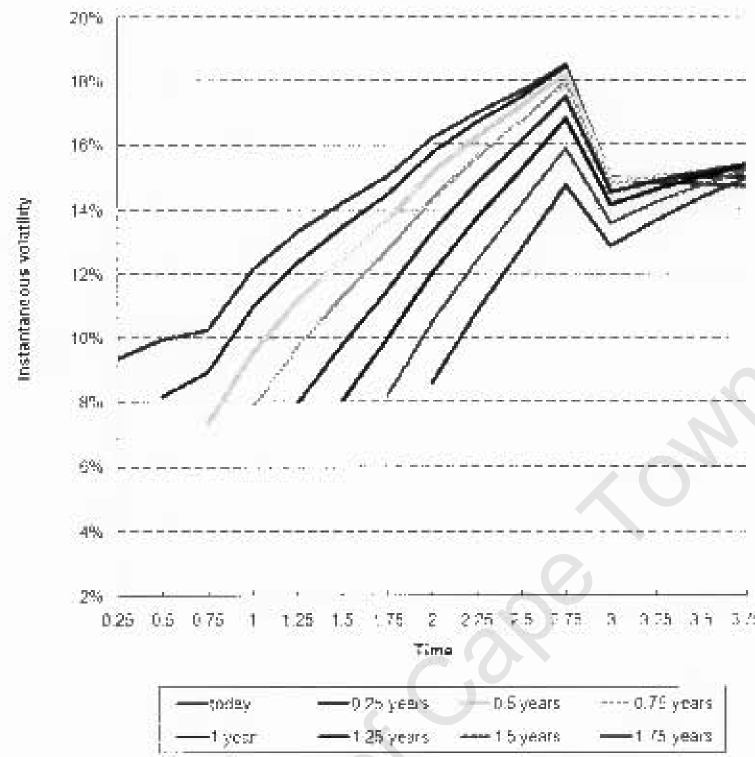

Figure 4.4: The evolution of the instantaneous volatility when caplet volatilities are perfectly fitted. The calibration took place with a, b, c, d as free parameters and with β equal to 0.1.

Parametric volatility structure, parametric correlation

We calibrate to the swaption volatility matrix presented in table 4.4 and the forward rates presented in table B.1. The volatility structure is

$$\sigma(t, T_k) = \left([a(T_k - t) + d]\, e^{-b(T_k - t)} + c\right)$$

and the correlation structure is

$$\rho_{ij} = e^{-\beta|T_i - T_j|}.$$

There are five free parameters: a, b, c, d and β. By minimising SS, the parameter values presented in table 4.13 are obtained.
The impact of the introduction of the additional β parameter is observable when the figures in table 4.11 and table 4.14 are compared. As expected, the computation time has increased by about 2 minutes. The fit is marginally better - the introduction of the beta parameter has led to SS dropping to 0.00149 from 0.0016 and the maximum absolute difference dropping to 2.007 % from 2.18%. The maximum absolute difference between the estimated volatility and the actual volatility is still between $\nu^{M}_{10,11}(0)$ and $\nu_{10,11}(0)$. The evolution of the instantaneous term

a	0.124171
b	0.313268
c	-0.0173
d	0.076788
β	0.352235

Table 4.13: The parameter values estimated from calibration to table 4.4 using a parametric volatility structure and a parametric correlation matrix

SS	0.00149
Maximum Absolute Difference	2.007%
Total Time Taken	7.98 minutes

Table 4.14: Calibration statistics from calibration to table 4.4 using a parametric volatility structure and a parametric correlation matrix

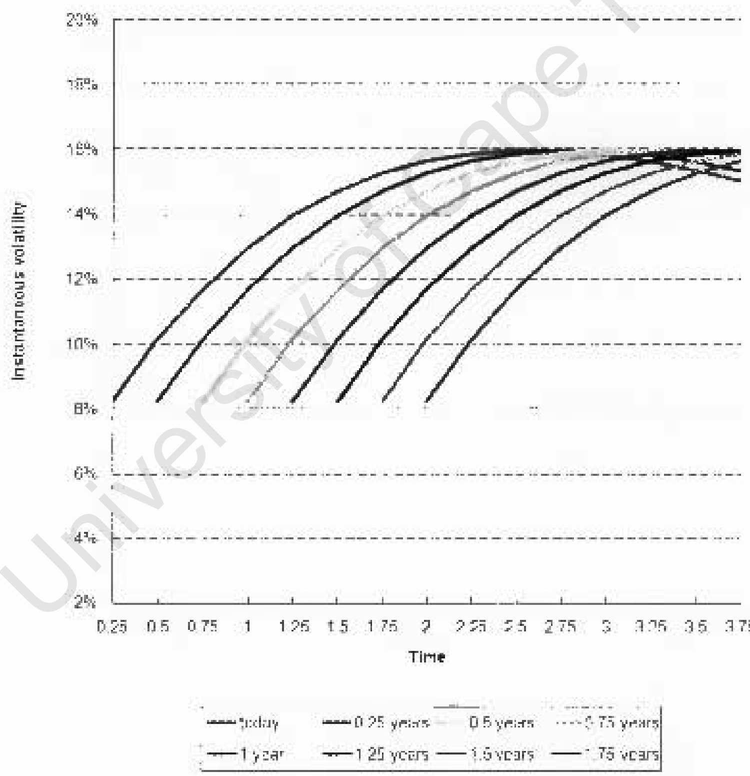

Figure 4.5: The evolution of the instantaneous volatility function for a calibration with five free parameters.

structure of volatility, figure 4.5, also appears very similar to the evolution of the instantaneous term structure of volatility presented in figure 4.3 (the introduction of the β parameter does not affect the time homogeneity of the evolution of $\sigma(t, T_k)$ over time).

Parametric volatility structure, parametric correlation, perfect fit to caplet volatilities

We calibrate to the swaption volatility matrix presented in table 4.4 and the forward rates presented in table B.1. The volatility structure is

$$\sigma(t, T_k) = K_k \left([a(T_k - t) + d]e^{-b(T_k - t)} + c \right)$$

and the correlation structure is

$$\rho_{ij} = e^{-\beta|T_i - T_j|}.$$

There are five free parameters: a, b, c, d and β. These parameters are estimated first and are thus equal to the parameters presented in table 4.13. $K_0, ..., K_{24}$ are then estimated so as to ensure that model caplet volatilities are equal to the actual caplet volatilities.

As before, the introduction of the K parameters has led to a large reduction in the size of SS at no additional computational cost (as shown in table 4.15). However, the K parameters have also led to the instantaneous volatility function changing over time (figure 4.6).

SS	0.00017
Maximum Absolute Difference	1.076%
Total Time Taken	7.98 minutes

Table 4.15: Calibration statistics from calibration to table 4.4 using a parametric volatility structure and an exogenous correlation matrix. K parameters have been introduced.

Summary of key observations in calibrating through optimisation

Four approaches to calibrating the LIBOR Market Model through optimisation have been considered. From the calibration results the following points are noted:

- Model parsimony is traded for model fit. This is evident when perfect fit to the caplet volatilities is achieved. A perfect fit to the caplet volatilities is only achieved through the introduction of irregularity in the evolution of the term structure of instantaneous volatility.

- The time taken to accomplish each of the four calibrations is fairly similar. When choosing a calibration procedure in chapter 5, time taken will therefore be less of a concern.

- We have calibrated to the South African swaption volatility matrix without having to interpolate between entries in the swaption volatility matrix (except in obtaining the stripped caplet volatilities). The ability to calibrate without interpolation is a key consideration in this dissertation when choosing between an optimisation approach and a Cascade Calibration approach in the South African setting.

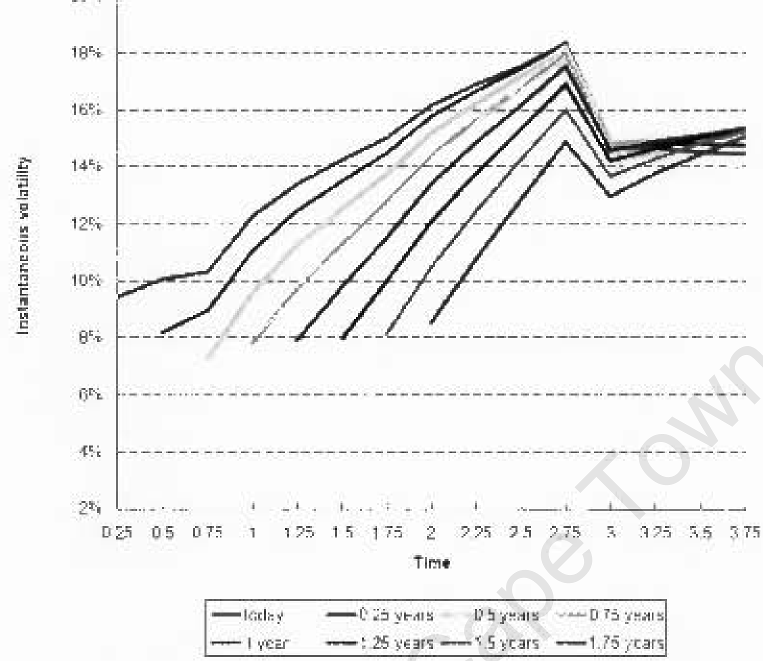

Figure 4.6: The evolution of the instantaneous volatility function for a calibration with five free parameters. Additional K parameters were estimated in order to ensure that the caplet volatilities were fitted perfectly.

4.6.2 Cascade Calibration

Consider the general piecewise constant formulation of the instantaneous volatility dynamics presented in table 4.6 in section 4.3. If the formula of proposition 4.2.1 is presented with this volatility structure it takes the following form:

$$(\nu_{\alpha,\beta}(t))^2 = \frac{\sum_{i=\alpha+1}^{\beta}\sum_{j=\alpha+1}^{\beta}\omega_i(t)\omega_j(t)L(t,T_{i-1})L(t,T_{j-1})\rho_{i-1,j-1}\sum_{h=0}^{\alpha}\delta_h\sigma_{i,h+1}\sigma_{j,h+1}}{(T_\alpha - t)(S_{\alpha,\beta}(t))^2}.$$

This formula can be rewritten as follows:

$$A_{\alpha,\beta}\sigma_{\beta,\alpha+1}^2 + B_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C_{\alpha,\beta} = 0 \tag{4.7}$$

where

$$\begin{aligned}
A_{\alpha,\beta} &= \omega_\beta(t)^2 L(t,T_{\beta-1})^2(T_\alpha - T_{\alpha-1}) \\
B_{\alpha,\beta} &= 2\sum_{j=\alpha+1}^{\beta-1}\omega_\beta(t)\omega_j(t)L(t,T_{\beta-1})L(t,T_{j-1})\rho_{\beta-1,j-1}(T_\alpha - T_{\alpha-1})\sigma_{j,\alpha+1} \\
C_{\alpha,\beta} &= \sum_{i=\alpha+1}^{\beta-1}\sum_{j=\alpha+1}^{\beta-1}\omega_i(t)\omega_j(t)L(t,T_{i-1})L(t,T_{j-1})\rho_{i-1,j-1}\sum_{h=0}^{\alpha}(T_h - T_{h-1})\sigma_{i,h+1}\sigma_{j,h+1}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\sum_{j=\alpha+1}^{\beta-1} \omega_\beta(t)\omega_j(t)L(t,T_{\beta-1})L(t,T_{j-1})\rho_{\beta-1,j-1}\sum_{h=0}^{\alpha-1}(T_h - T_{h-1})\sigma_{j,h+1}\sigma_{\beta,h+1} \\
&+ \quad \omega_\beta(t)^2 L(t,T_{\beta-1})^2\sum_{h=0}^{\alpha-1}(T_h - T_{h-1})\sigma_{\beta,h+1}^2 - (T_\alpha - t)S_{\alpha,\beta}(t)^2(\nu_{\alpha,\beta}^M(t))^2.
\end{aligned}
$$

Now assume an exogenous correlation matrix where each entry in the correlation matrix is given by equation 4.6. We can now calibrate the LIBOR Market Model by solving a series of quadratic equations in order to obtain an exact fit to the swaption volatility matrix. The following algorithm (which appears in Brigo and Mercurio [2006]) captures the process.

Algorithm 4.6.1 *The Cascade Calibration Algorithm is as follows:*

1. *Select the number, s, of rows in the swaption matrix that are of interest for the calibration.*
2. *Set* $\alpha = 0$.
3. *Set* $\beta = \alpha + 1$.
4. *Solve for* $\sigma_{\beta,\alpha+1}$ *in equation 4.7. Since both* $A_{\alpha,\beta}$ *and* $B_{\alpha,\beta}$ *are strictly positive, if we assume positive instantaneous correlations, 4.7 has at most one positive solution, namely*

$$
\sigma_{\beta,\alpha+1} = \frac{-B_{\alpha,\beta} + \sqrt{B_{\alpha,\beta}^2 - 4A_{\alpha,\beta}C_{\alpha,\beta}}}{2A_{\alpha,\beta}}
$$

   if and only if $C_{\alpha,\beta} < 0$.

5. *Increase* β *by one. If* β *is smaller than or equal to s, go back to 4, otherwise increase* α *by one.*
6. *If* $\alpha < s$ *go back to 3, otherwise stop.*

The algorithm will be demonstrated in the following worked example.

A worked example of the Cascade Calibration Algorithm

Consider again table 4.5 which is presented here as table 4.16. Only the upper triangular portion of table 4.5 is needed for the Cascade Calibration Algorithm. Table 4.17 serves as a reference to table 4.16 so that the reader can easily reference the values in the swaption volatility matrix that will be mentioned. In this worked example $T_i = 0.25 \times (i+1)$. The Cascade Calibration algorithm will attempt to parameterise table 4.18.

At the start of algorithm 4.6.1, we have $\alpha = 0$ and $\beta = 1$(steps 1 and 2 of algorithm 4.6.1). In step 3 of algorithm 4.6.1 we solve equation 4.7 for the first time to find that

$$
\begin{aligned}
\sigma_{1,1} &= \nu_{0,1}^M(0) \\
&= 8.2\%.
\end{aligned}
$$

	Swap Length							
First Setting Date	0.25	0.5	0.75	1	1.25	1.5	1.75	2
T_0	8.20%	8.79%	9.37%	9.96%	10.10%	10.24%	10.37%	10.51%
T_1	8.20%	8.79%	9.37%	9.96%	10.10%	10.24%	10.37%	
T_2	8.20%	9.60%	10.31%	11.03%	11.17%	11.31%		
T_3	9.56%	10.41%	11.25%	12.10%	12.24%			
T_4	10.47%	11.39%	12.32%	13.24%				
T_5	11.17%	12.16%	13.15%					
T_6	11.92%	12.97%						
T_7	12.97%							

Table 4.16: The upper triangular portion of the interpolated swaption volatility matrix obtained on 2 May 2007.

	Swap Length							
First Setting Date	0.25	0.5	0.75	1	1.25	1.5	1.75	2
T_0	$\nu^M_{0,1}(0)$	$\nu^M_{0,2}(0)$	$\nu^M_{0,3}(0)$	$\nu^M_{0,4}(0)$	$\nu^M_{0,5}(0)$	$\nu^M_{0,6}(0)$	$\nu^M_{0,7}(0)$	$\nu^M_{0,8}(0)$
T_1	$\nu^M_{1,2}(0)$	$\nu^M_{1,3}(0)$	$\nu^M_{1,4}(0)$	$\nu^M_{1,5}(0)$	$\nu^M_{1,6}(0)$	$\nu^M_{1,7}(0)$	$\nu^M_{1,8}(0)$	
T_2	$\nu^M_{2,3}(0)$	$\nu^M_{2,4}(0)$	$\nu^M_{2,5}(0)$	$\nu^M_{2,6}(0)$	$\nu^M_{2,7}(0)$	$\nu^M_{2,8}(0)$		
T_3	$\nu^M_{3,4}(0)$	$\nu^M_{3,5}(0)$	$\nu^M_{3,6}(0)$	$\nu^M_{3,7}(0)$	$\nu^M_{3,8}(0)$			
T_4	$\nu^M_{4,5}(0)$	$\nu^M_{4,6}(0)$	$\nu^M_{4,7}(0)$	$\nu^M_{4,8}(0)$				
T_5	$\nu^M_{5,6}(0)$	$\nu^M_{5,7}(0)$	$\nu^M_{5,8}(0)$					
T_6	$\nu^M_{6,7}(0)$	$\nu^M_{6,8}(0)$						
T_7	$\nu^M_{7,8}(0)$							

Table 4.17: Reference to table 4.16

	Current time							
	$(0,T_0]$	$(T_0,T_1]$	$(T_1,T_2]$	$(T_2,T_3]$	$(T_3,T_4]$	$(T_4,T_5]$	$(T_5,T_6]$	$(T_6,T_7]$
$L(t,T_0)$	$\sigma_{1,1}$	dead	dead	dead	dead	dead	dead	dead
$L(t,T_1)$	$\sigma_{2,1}$	$\sigma_{2,2}$	dead	dead	dead	dead	dead	dead
$L(t,T_2)$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$	dead	dead	dead	dead	dead
$L(t,T_3)$	$\sigma_{4,1}$	$\sigma_{4,2}$	$\sigma_{4,4}$	$\sigma_{4,4}$	dead	dead	dead	dead
$L(t,T_4)$	$\sigma_{5,1}$	$\sigma_{5,2}$	$\sigma_{5,3}$	$\sigma_{5,4}$	$\sigma_{5,5}$	dead	dead	dead
$L(t,T_5)$	$\sigma_{6,1}$	$\sigma_{6,2}$	$\sigma_{6,3}$	$\sigma_{6,4}$	$\sigma_{6,5}$	$\sigma_{6,6}$	dead	dead
$L(t,T_6)$	$\sigma_{7,1}$	$\sigma_{7,2}$	$\sigma_{7,3}$	$\sigma_{7,4}$	$\sigma_{7,5}$	$\sigma_{7,6}$	$\sigma_{7,7}$	dead
$L(t,T_7)$	$\sigma_{8,1}$	$\sigma_{8,2}$	$\sigma_{8,3}$	$\sigma_{8,4}$	$\sigma_{8,5}$	$\sigma_{8,6}$	$\sigma_{8,7}$	$\sigma_{8,8}$

Table 4.18: The most general piecewise constant instantaneous volatility formulation that will be parameterised by calibrating to table 4.16

We now assess the condition in step 4 of algorithm 4.6.1 and this sends us back to step 3 in which we solve the equation

$$\begin{aligned} S_{0,2}(0)^2(\nu^M_{0,2}(0))^2 &= \omega_1(0)^2 L(0,T_0)\sigma_{1,1}^2 + \omega_2(0)^2 L(0,T_1)\sigma_{2,1}^2 \\ &+ 2\rho_{1,2}\omega_1(0)L(0,T_0)\omega_2(0)L(0,T_1)\sigma_{1,1}\sigma_{2,1}. \end{aligned}$$

This implies that

$$\sigma_{2,1} \quad = \quad 9.5021\%.$$

The condition in step 4 is once again assessed and this implies that we now solve the following equation:

$$\begin{aligned} S_{0,3}(0)^2(\nu^M_{0,3}(0))^2 &= \omega_1(0)^2L(0,T_0)\sigma^2_{1,1} + \omega_2(0)^2L(0,T_1)\sigma^2_{2,1} + \omega_3(0)^2L(0,T_2)\sigma^2_{3,1} \\ &+ 2\rho_{1,2}\omega_1(0)L(0,T_0)\omega_2(0)L(0,T_1)\sigma_{1,1}\sigma_{2,1} \\ &+ 2\rho_{1,3}\omega_1(0)L(0,T_0)\omega_3(0)L(0,T_2)\sigma_{1,1}\sigma_{3,1} \\ &+ 2\rho_{2,3}\omega_2(0)L(0,T_1)\omega_3(0)L(0,T_2)\sigma_{2,1}\sigma_{3,1}. \end{aligned}$$

Thus,

$$\sigma_{3,1} \quad = \quad 10.8087\%.$$

Similarly, we solve for $\sigma_{4,1}$, $\sigma_{5,1}$, $\sigma_{6,1}$, $\sigma_{7,1}$ and $\sigma_{8,1}$. Each equation that we solve will have a greater number of terms but will still be quadratic. We have found that we have worked across the first row of table 4.17 and table 4.16 and down the first column of table 4.18 in solving for $\sigma_{1,1}$ to $\sigma_{8,1}$.

Once we have solved for $\sigma_{8,1}$, we assess the condition in step 4. This time it implies that α is increased by one ($\alpha = 1$). In step 3, β is set equal to 2. This puts us in the second row of table 4.17 and table 4.16 and in the second column of table 4.18. The equation which we now solve only depends on $\nu^M_{1,2}$, $\sigma_{2,1}$ and $\sigma_{2,2}$ and is

$$T_1(\nu^M_{1,2}(0))^2 = T_0\sigma_{2,1} + (T_1 - T_0)\sigma^2_{2,2}$$

which implies that

$$\sigma_{2,1} = 6.6475\%.$$

Again the algorithm sends the process across the second row of table 4.17 and table 4.16 and down the second column of table 4.18 in solving for $\sigma_{2,2}$ to $\sigma_{2,8}$. Each time the equation to be solved is quadratic and depends on values that have been calculated earlier on in the process and the relevant entry in the swaption volatility matrix.

After $\sigma_{2,8}$ has been calculated, the algorithm sends the process across the third row of table 4.17 and table 4.16 and down the second column of table 4.18 in solving for $\sigma_{3,3}$ to $\sigma_{8,3}$. Table 4.19 presents the progress that has been made so far.

The algorithm then attempts to solve for $\sigma_{4,4}$. At this point a problem is encountered as the discriminant in the quadratic equation is

$$\Delta = B^2_{3,4} - 4A_{3,4}C_{3,4}$$

and in this case, it is less that zero. This implies that the equation 4.7 has no real valued solution for $\sigma_{4,4}$. Hence the Cascade Calibration procedure cannot proceed beyond the results that have

	Current time							
	$(0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$
$L(t, T_0)$	8.20%	dead	dead	dead	dead	dead	dead	dead
$L(t, T_1)$	9.50%	6.64%	dead	dead	dead	dead	dead	dead
$L(t, T_2)$	10.80%	7.99%	4.58%	dead	dead	dead	dead	dead
$L(t, T_3)$	12.16%	9.28%	12.72%	$\sigma_{4,4}$	dead	dead	dead	dead
$L(t, T_4)$	11.07%	13.70%	11.73%	$\sigma_{5,4}$	$\sigma_{5,5}$	dead	dead	dead
$L(t, T_5)$	11.49%	10.66%	18.93%	$\sigma_{6,4}$	$\sigma_{6,5}$	$\sigma_{6,6}$	dead	dead
$L(t, T_6)$	11.92%	11.06%	13.46%	$\sigma_{7,4}$	$\sigma_{7,5}$	$\sigma_{7,6}$	$\sigma_{7,7}$	dead
$L(t, T_7)$	12.35%	11.48%	13.90%	$\sigma_{8,4}$	$\sigma_{8,5}$	$\sigma_{8,6}$	$\sigma_{8,7}$	$\sigma_{8,8}$

Table 4.19: Progress made by the Cascade Calibration Algorithm immediately before it fails.

been generated in table 4.19. Brigo and Mercurio [2006] attributes this feature to the practice of filling the gaps in the swaptions volatility matrix by linear interpolation.

Brigo and Mercurio [2006] suggests that the Cascade Calibration proceeds more successfully if the gaps are filled by fitting a smoothed parametric surface to the sparsely populated swaptions volatility matrix and then performing Cascade Calibration on this surface. Following this idea, suppose that we perform Cascade Calibration on the smoothed surface given by the parameters in table 4.10. Thus, we perform Cascade Calibration on a parametric surface with the following parameters:

$$\begin{aligned} a &= 0.118945943214265 \\ b &= 0.287499275459561 \\ c &= -0.0307325360359084 \\ d &= 0.0913581099426343 \\ \beta &= 0.1. \end{aligned}$$

The smoothed swaption volatility surface is now given by table 4.20. The Cascade Calibration algorithm now runs to completion and produces the output displayed in table 4.21.

	Swap Length							
First Setting Date	0.25	0.5	0.75	1	1.25	1.5	1.75	2
T_0	7.18%	8.12%	8.90%	9.61%	10.20%	10.71%	11.13%	11.49%
T_1	8.21%	9.04%	9.76%	10.38%	10.90%	11.34%	11.72%	
T_2	9.17%	9.91%	10.53%	11.07%	11.53%	11.92%		
T_3	9.99%	10.62%	11.18%	11.66%	12.06%			
T_4	10.71%	11.28%	11.77%	12.19%				
T_5	11.35%	11.85%	12.27%					
T_6	11.89%	12.33%						
T_7	12.37%							

Table 4.20: The matrix produced by fitting a smoothed parametric function to the swaption volatility surface of 2 May 2007.

	Current time							
	$(0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$
$L(t, T_0)$	7.18%	dead	dead	dead	dead	dead	dead	dead
$L(t, T_1)$	9.19%	7.11%	dead	dead	dead	dead	dead	dead
$L(t, T_2)$	10.73%	9.23%	7.21%	dead	dead	dead	dead	dead
$L(t, T_3)$	12.19%	10.80%	9.20%	7.03%	dead	dead	dead	dead
$L(t, T_4)$	13.21%	12.13%	10.81%	9.12%	7.22%	dead	dead	dead
$L(t, T_5)$	14.08%	13.22%	12.15%	10.78%	9.30%	6.98%	dead	dead
$L(t, T_6)$	14.75%	14.09%	13.25%	12.12%	10.81%	9.31%	6.93%	dead
$L(t, T_7)$	15.24%	14.76%	14.11%	13.20%	12.16%	10.80%	9.31%	7.05%

Table 4.21: The output produced by the Cascade Calibration Algorithm when applied to table 4.20. This is the swaption volatility matrix implied by a parametric volatility and correlation structure with parameter values as in table 4.10.

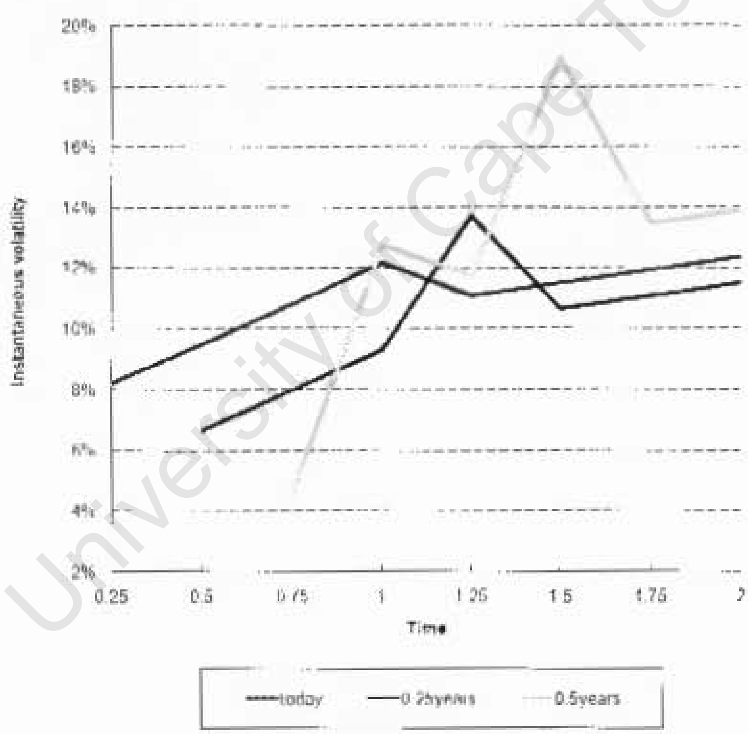

Figure 4.7: A graphical representation of the evolution of the term structure of instantaneous volatility presented in table 4.19. Since the Cascade Calibration was implemented on a linearly interpolated surface, the evolution is clearly time inhomogeneous.

The output produced in table 4.19 and 4.21 represents the evolution of the instantaneous volatility structure over time. The Cascade Calibration approach to parameterisation does not produce

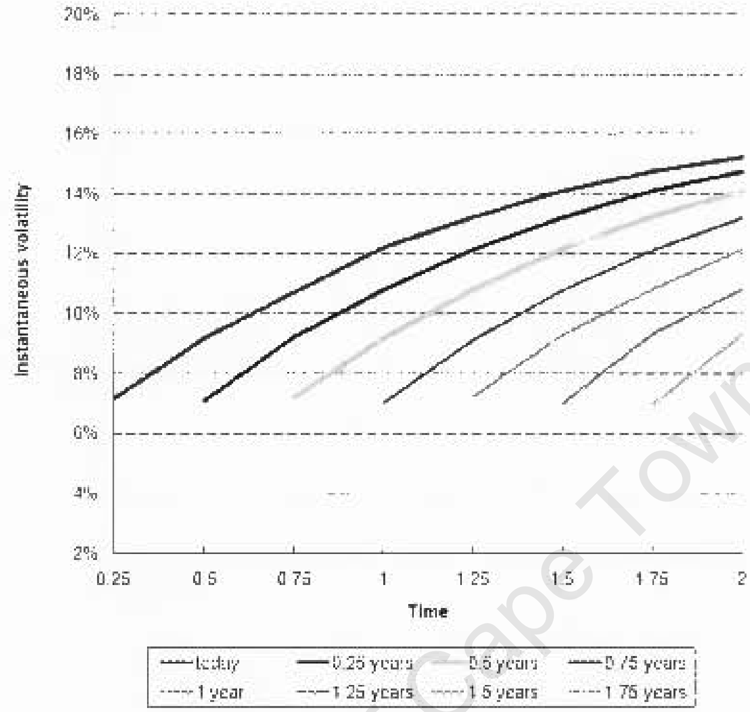

Figure 4.8: A graphical representation of the evolution of the term structure of instantaneous volatility presented in table 4.21. Since the Cascade Calibration was implemented on a smoothed surface (and the smoothed surface is generated by a *time homogenous* instantaneous volatility structure) the time inhomogeneity of the term structure evolution is much less visible than in figure 4.7.

a time homogenous volatility structure. This can be seen from figure 4.7[11]. Figure 4.8 presents the evolution of the term structure of volatility when the Cascade Calibration is performed on the smoothed swaption volatility matrix. Since this matrix was generated by a time homogenous instantaneous volatility structure, the time inhomogeneity in this figure is much less visible.

Suitability of the Cascade Calibration in the South African setting

The problem with applying the Cascade Calibration to South African data is that the swaptions volatility matrix is very sparsely populated. It is necessary to perform interpolation on the volatility matrix before it is ready to be used as a source for Cascade Calibration. This leads to two problems.

- The linear interpolation leads to an evolution of the instantaneous volatility term structure which is far from time homogenous.

[11] In terms of tables 4.19 and 4.21 the time inhomogeneity is indicated by values along the diagonals (running from the top left of the table to the bottom right of the table) not being equal.

- The linear interpolation makes it likely that the algorithm will fail to calibrate completely to the volatility matrix due to the appearance of imaginary numbers.

One solution to these problems lie in performing Cascade Calibration on a smoothed surface which has been fitted to the volatility matrix. This leads to the Cascade Calibration running to completion. It also leads to a large reduction in the time inhomogeneity of the evolution of the term structure of instantaneous volatility.

This dissertation argues that this solution is not a practical approach for calibration in the South African setting. Instead of first fitting a parametric surface and then performing a Cascade Calibration, the smoothed fitted surface can just be used to obtain the instantaneous volatility and instantaneous correlation functions. Thus, instead of using the parameterisation implied by table 4.21, just use the parameterisation implied by table 4.10.

4.6.3 The Rectangular Cascade Calibration Algorithm

Consider table 4.22 which shows a 2×2 swaption volatility matrix (this is the upper left hand corner of table 4.16). Table 4.23 shows the notation that will be used to refer to each of the entries in 4.22. The previous section showed how to calibrate to the upper triangular portion of this matrix. This section will show how to calibrate to the entire matrix by using the assumptions of the Rectangular Cascade Calibration Algorithm. In doing so, table 4.24 will be parameterised.

	Swap Length	
Time Until First Setting Date	0.25	0.5
T_0	8.20%	8.79%
T_1	8.20%	8.79%

Table 4.22: The swaptions volatility matrix. This is the upper 2×2 portion of table 4.16.

	Swap Length	
Time Until First Setting Date	0.25	0.5
T_0	$\nu_{0,1}^M(0)$	$\nu_{0,2}^M(0)$
T_1	$\nu_{1,2}^M(0)$	$\nu_{1,3}^M(0)$

Table 4.23: Reference to table 4.22.

	Current time	
	$(0, T_0]$	$(T_0, T_1]$
$L(t, T_0)$	$\sigma_{1,1}$	dead
$L(t, T_1)$	$\sigma_{2,1}$	$\sigma_{2,2}$
$L(t, T_2)$	$\sigma_{2,1}$	$\sigma_{2,2}$

Table 4.24: The most general piecewise constant instantaneous volatility formulation that will be parameterised by using the Rectangular Cascade Calibration procedure to calibrate to table 4.22.

We again assume an exogenous correlation structure given by equation 4.6 and that $T_i = 0.25 \times (i+1)$. By entering these formulations into Rebonato's formula in proposition 4.2.1, we find that we repeat the process in the Cascade Calibration over the upper triangular portion of table 4.22. Thus, expressing $\nu_{0,1}(0)$ in terms of the proposition 4.2.1 implies that

$$\begin{aligned}\sigma_{1,1} &= \nu_{0,1}^M(0)\\ &= 8.2\%.\end{aligned}$$

Expressing $\nu_{0,2}(0)$ in terms of the proposition 4.2.1 implies that

$$\begin{aligned}S_{0,2}(0)^2(\nu_{0,2}^M(0))^2 &= \omega_1(0)^2L(0,T_0)\sigma_{1,1}^2 + \omega_2(0)^2L(0,T_1)\sigma_{2,1}^2\\ &+ 2\rho_{1,2}\omega_1(0)L(0,T_0)\omega_2(0)L(0,T_1)\sigma_{1,1}\sigma_{2,1}\\ \Longleftrightarrow \sigma_{2,1} &= 9.5021\%.\end{aligned}$$

Expressing $\nu_{1,2}(0)$ in terms of proposition 4.2.1 implies that

$$\begin{aligned}T_1(\nu_{1,2}^M(0))^2 &= T_0\sigma_{2,1} + (T_1 - T_0)\sigma_{2,2}^2\\ \Longleftrightarrow \sigma_{2,1} &= 6.6475\%.\end{aligned}$$

Up until this point, the procedure has been identical to the Cascade Calibration procedure. We now depart from the Cascade Calibration and attempt to calibrate to $\nu_{1,3}(0)$. When expressed in terms of proposition 4.2.1, the formula for $\nu_{1,3}(0)$ is as follows:

$$\begin{aligned}T_1S_{1,3}(0)^2(\nu_{1,2}^M(0))^2 &= \omega_2(0)^2L(0,T_1)^2(\delta_0\sigma_{2,1}^2 + \delta_1\sigma_{2,2}^2)\\ &+ \omega_3(0)^2L(0,T_2)^2(\delta_0\sigma_{3,1}^2 + \delta_1\sigma_{3,2}^2)\\ &+ 2\rho_{2,3}\omega_2(0)L(0,T_1)\omega_3(0)L(0,T_2)(\delta_0\sigma_{2,1}\sigma_{3,1} + \delta_1\sigma_{2,2}\sigma_{3,2}).\end{aligned}$$

This equation has infinitely many solutions since there are two unknowns: $\sigma_{3,1}$ and $\sigma_{3,2}$. In order to cope with this, Brigo and Morini [2002] sets all unknowns equal to each other when the need arises. Doing this allows us to once again solve the following quadratic equation.

$$\begin{aligned}T_1S_{1,3}(0)^2(\nu_{1,2}^M(0))^2 &= \omega_2(0)^2L(0,T_1)^2(\delta_0\sigma_{2,1}^2 + \delta_1\sigma_{2,2}^2)\\ &+ \omega_3(0)^2L(0,T_2)^2(\delta_0 + \delta_1)\sigma_{3,1}^2\\ &+ 2\rho_{2,3}\omega_2(0)L(0,T_1)\omega_3(0)L(0,T_2)(\delta_0\sigma_{2,1} + \delta_1\sigma_{2,2})\sigma_{3,1}\end{aligned}$$

and this implies that

$$\sigma_{3,1} = \sigma_{3,2} = 9.5712\%.$$

The output of this short calibration process is presented in table 4.25. Notice that the implication of calibrating to the entire swaptions volatility matrix instead of only to the upper triangular portion is that the number of forward rates that can be modelled has risen.

The more general presentation of the Rectangular Cascade Calibration Algorithm is shown in algorithm 4.6.2 (this algorithm appears in Brigo and Mercurio [2006]).

	Current Time	
	$(0, T_0]$	$(T_0, T_1]$
$L(t, T_0)$	8.20%	dead
$L(t, T_1)$	9.50%	6.64%
$L(t, T_2)$	9.57%	9.57%

Table 4.25: The output from applying the Rectangular Cascade Calibration Algorithm to table 4.22.

Algorithm 4.6.2 *Modify the Cascade Calibration Algorithm as follows. At point 5 the condition is no longer $\beta < s$, but $(\beta - \alpha) < s$. Furthermore, in case $\beta = s + \alpha$, i.e. when one reaches one entry on the last column (with the exception of the first one), the new point 4 requires to assume all the unknowns to be equal to the standard unknown $\sigma_{\beta,\alpha+1}$:*

$$\sigma_{\beta,\alpha+1} = \sigma_{\beta,\alpha} = \ldots = \sigma_{\beta,1} \quad for \quad \beta = s + \alpha.$$

Hence the new equation to solve is

$$A^*_{\alpha,\beta}\sigma^2_{\beta,\alpha+1} + B^*_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C^*_{\alpha,\beta},$$

where

$$\begin{aligned}
A^*_{\alpha,\beta} &= \omega_\beta(t)^2 L(t, T_{\beta-1})(T_\alpha - T_{\alpha-1}) + \omega_\beta(t)^2 L(t, T_{\beta-1})^2 \sum_{h=0}^{\alpha-1}(T_h - T_{h-1}) \\
B^*_{\alpha,\beta} &= 2\sum_{j=\alpha+1}^{\beta-1} \omega_\beta(t)\omega_j(t)L(t, T_{\beta-1})L(t, T_{j-1})\rho_{i-1,j-1}(T_\alpha - T_{\alpha-1})\sigma_{j,\alpha+1} \\
&+ 2\sum_{j=\alpha+1}^{\beta-1} \omega_\beta(t)\omega_j(t)L(t, T_{\beta-1})L(t, T_{j-1})\rho_{i-1,j-1}\sum_{h=0}^{\alpha-1}(T_h - T_{h-1})\sigma_{j,h+1} \\
C^*_{\alpha,\beta} &= \sum_{i=\alpha+1}^{\beta-1}\sum_{j=\alpha+1}^{\beta-1} \omega_i(t)\omega_j(t)L(t, T_{i-1})L(t, T_{j-1})\rho_{i-1,j-1}\sum_{h=0}^{\alpha}(T_h - T_{h-1})\sigma_{i,h+1}\sigma_{j,h+1} \\
&- (T_\alpha - t)S_{\alpha,\beta}(t)^2 V^2_{\alpha,\beta}(t).
\end{aligned}$$

The rest of the algorithm is unchanged.

Application of the Rectangular Cascade Calibration Algorithm to the fully populated versions of tables 4.16 and 4.20

When applying the Rectangular Cascade Calibration Algorithm to table 4.16 we expect to run into the same trouble as when we applied the Cascade Calibration algorithm to table 4.16. This is because the Rectangular Cascade Calibration is identical to the Cascade Calibration when the algorithm runs over the upper triangular portion of the swaption volatility matrix. Table 4.26 shows the output from this calibration. Figure 4.9 presents table 4.26 graphically.

The Rectangular Cascade Calibration Algorithm is implemented on the smoothed volatility surface of table 4.20. As with the Cascade Calibration Algorithm, the Rectangular Cascade

Calibration Algorithm now runs to completion. Table 4.27 shows the output. Figure 4.10 presents table 4.27 graphically. Notice that the assumption of equality of instantaneous volatilities (in order to calibrate to the full swaption volatility matrix) has the effect of introducing the volatility structure which is only dependent on time *of* maturity over sections of the table 4.27.

	Current time							
	$(0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$
$L(t, T_0)$	8.20%	dead	dead	dead	dead	dead	dead	dead
$L(t, T_1)$	9.50%	6.64%	dead	dead	dead	dead	dead	dead
$L(t, T_2)$	10.80%	7.99%	4.58%	dead	dead	dead	dead	dead
$L(t, T_3)$	12.16%	9.28%	12.72%	$\sigma_{4,4}$	dead	dead	dead	dead
$L(t, T_4)$	11.07%	13.70%	11.73%	$\sigma_{5,4}$	$\sigma_{5,5}$	dead	dead	dead
$L(t, T_5)$	11.49%	10.66%	18.93%	$\sigma_{6,4}$	$\sigma_{6,5}$	$\sigma_{6,6}$	dead	dead
$L(t, T_6)$	11.92%	11.06%	13.46%	$\sigma_{7,4}$	$\sigma_{7,5}$	$\sigma_{7,6}$	$\sigma_{7,7}$	dead
$L(t, T_7)$	12.35%	11.48%	13.90%	$\sigma_{8,4}$	$\sigma_{8,5}$	$\sigma_{8,6}$	$\sigma_{8,7}$	$\sigma_{8,8}$
$L(t, T_8)$	12.37%	12.37%	14.32%	$\sigma_{9,4}$	$\sigma_{9,6}$	$\sigma_{9,7}$	$\sigma_{9,8}$	$\sigma_{9,9}$
$L(t, T_9)$	13.51%	13.51%	13.51%	$\sigma_{10,4}$	$\sigma_{10,5}$	$\sigma_{10,6}$	$\sigma_{10,7}$	$\sigma_{10,8}$
$L(t, T_{10})$	$\sigma_{11,1}$	$\sigma_{11,2}$	$\sigma_{11,3}$	$\sigma_{11,4}$	$\sigma_{11,5}$	$\sigma_{11,6}$	$\sigma_{11,7}$	$\sigma_{11,8}$
$L(t, T_{12})$	$\sigma_{12,1}$	$\sigma_{12,2}$	$\sigma_{12,3}$	$\sigma_{12,4}$	$\sigma_{12,5}$	$\sigma_{12,6}$	$\sigma_{12,7}$	$\sigma_{12,8}$
$L(t, T_{12})$	$\sigma_{13,1}$	$\sigma_{13,2}$	$\sigma_{13,3}$	$\sigma_{13,4}$	$\sigma_{13,5}$	$\sigma_{13,6}$	$\sigma_{13,7}$	$\sigma_{13,8}$
$L(t, T_{14})$	$\sigma_{14,1}$	$\sigma_{14,2}$	$\sigma_{14,3}$	$\sigma_{14,4}$	$\sigma_{14,5}$	$\sigma_{14,6}$	$\sigma_{14,7}$	$\sigma_{14,8}$
$L(t, T_{15})$	$\sigma_{15,1}$	$\sigma_{15,2}$	$\sigma_{15,3}$	$\sigma_{15,4}$	$\sigma_{15,5}$	$\sigma_{15,6}$	$\sigma_{15,7}$	$\sigma_{15,8}$

Table 4.26: The output from the Rectangular Cascade Calibration Algorithm when applied to the fully populated version of table 4.16.

	Current time							
	$(0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$
$L(t, T_0)$	7.18%	dead	dead	dead	dead	dead	dead	dead
$L(t, T_1)$	9.19%	7.11%	dead	dead	dead	dead	dead	dead
$L(t, T_2)$	10.73%	9.23%	7.21%	dead	dead	dead	dead	dead
$L(t, T_3)$	12.19%	10.80%	9.20%	7.03%	dead	dead	dead	dead
$L(t, T_4)$	13.21%	12.13%	10.81%	9.12%	7.22%	dead	dead	dead
$L(t, T_5)$	14.08%	13.22%	12.15%	10.78%	9.30%	6.98%	dead	dead
$L(t, T_6)$	14.75%	14.09%	13.25%	12.12%	10.81%	9.31%	6.93%	dead
$L(t, T_7)$	15.24%	14.76%	14.11%	13.20%	12.16%	10.80%	9.31%	7.05%
$L(t, T_8)$	15.43%	15.43%	14.78%	14.06%	13.25%	12.13%	10.79%	9.17%
$L(t, T_9)$	15.59%	15.59%	15.59%	14.73%	14.12%	13.21%	12.11%	10.76%
$L(t, T_{10})$	15.39%	15.39%	15.39%	15.39%	15.89%	14.04%	13.19%	12.11%
$L(t, T_{11})$	15.70%	15.70%	15.70%	15.70%	15.70%	14.75%	14.08%	13.21%
$L(t, T_{12})$	15.73%	15.73%	15.73%	15.73%	15.73%	15.73%	14.75%	14.07%
$L(t, T_{13})$	15.71%	15.71%	15.71%	15.71%	15.71%	15.71%	15.71%	14.75%
$L(t, T_{14})$	15.63%	15.63%	15.63%	15.63%	15.63%	15.63%	15.63%	15.63%

Table 4.27: The output produced by the Rectangular Cascade Calibration Algorithm when applied to the fully populated version of table 4.20.

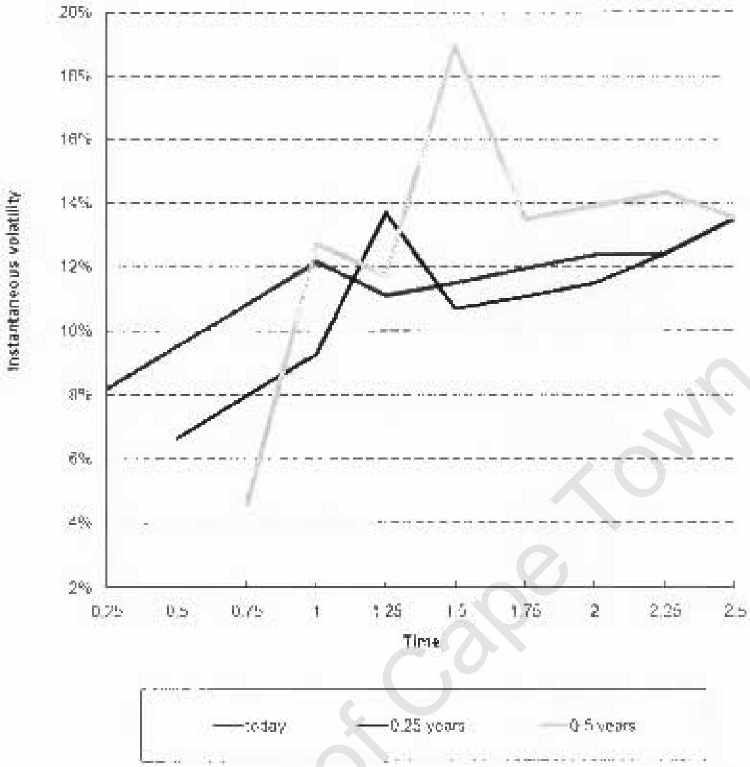

Figure 4.9: A graphical representation of the evolution of the term structure of instantaneous volatility presented in table 4.26. Since the Rectangular Cascade Calibration was implemented on a linearly interpolated surface, the evolution is clearly time inhomogeneous.

Suitability of the Rectangular Cascade Calibration in the South African setting

The comments that applied to the Cascade Calibration Algorithm applies to the Rectangular Cascade Calibration Algorithm as well. This dissertation argues that these calibration methods are unsuitable to the South African setting as it requires interpolation of the swaptions volatility matrix. The interpolation, in turn, has a tendency to lead to an evolution of the instantaneous volatility structure which is far from time homogenous. It also has a tendency to sabotage the Calibration process through the appearance of imaginary numbers.

One of the reasons for using the Cascade Calibration procedures is that they are very quick to run. This benefit is largely negated if one has to fit a surface (through minimisation of sum of squared errors) to the swaptions volatility matrix first.

The other reason for using Cascade Calibration algorithms may be that they achieve a perfect fit to the swaption volatility matrix. In the South African context, this 'greater degree' of fit is a misnomer since the calibration happens with respect to a swaption volatility matrix that has

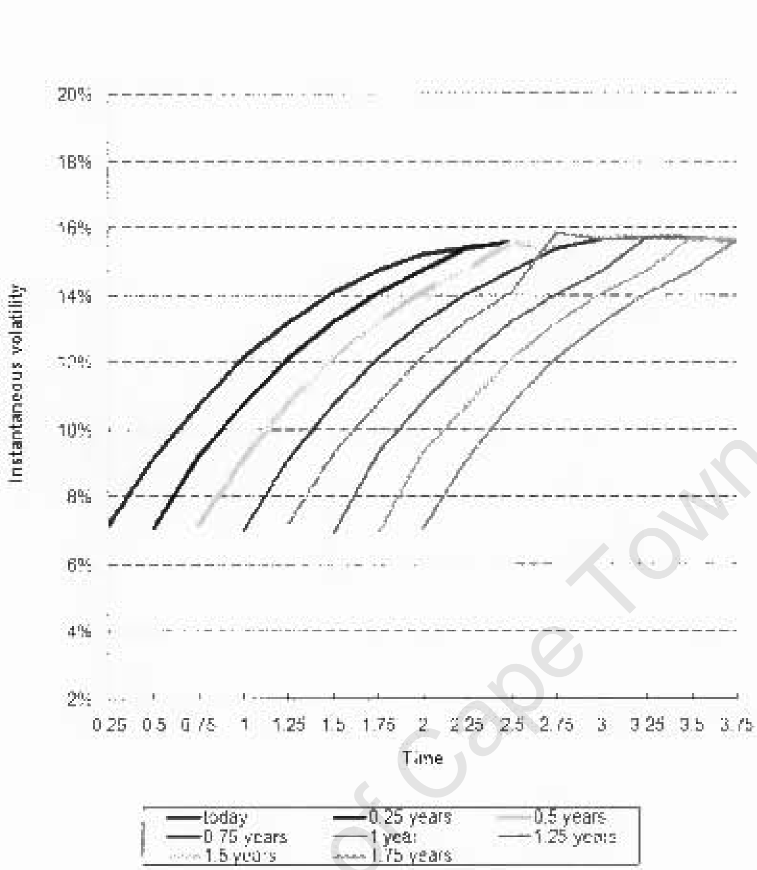

Figure 4.10: A graphical representation of the evolution of the term structure of instantaneous volatility presented in table 4.27. Since the Rectangular Cascade Calibration was implemented on a smoothed volatility surface, the evolution is fairly time homogeneous.

largely been estimated in any case because of the necessity of interpolation.

Chapter 5

Hedging

The process of hedging is the means by which no arbitrage prices are enforced. In a complete market, the payoff of the derivative can always be replicated by trading in the underlying and the numeraire. In an incomplete market, hedging can be used to enforce a range of arbitrage free prices.

In the previous chapter, it was pointed out that the parametric methods should be favoured above the closed form (or Cascade Calibration) methods in the South African setting. The aim of this chapter is to compare the parametric calibration methods further. Interest rate options will be delta hedged under each of the calibration methods. The hedge quality achieved under each of the calibration methods shall be compared.

5.1 Hedging an interest rate derivative in a complete market

Hedging an interest rate derivative is slightly more complicated than hedging a derivative where the underlying is a tradeable asset (for example hedging an equity option using the underlying share). This is because forward rates and swap rates are not tradeable and it is the FRAs or swaps that will have to be used to hedge the derivative. This section aims to show the key issues behind finding a replicating strategy for an interest rate derivative and so will skirt over some of the more technical issues.

Suppose we have a derivative with a payoff of X_T at time T which has a swap rate $S_{\alpha,\beta}(t)$ $(t \leq T)$ as the underlying source of randomness. Since the swap rate is not tradeable, we need to trade in the most basic derivative dependent on the swap rate - ie a swap (with value $\mathbf{Swap}_{\alpha,\beta}(t)$). Set up a portfolio $V(t)$ consisting of $\mathbf{Swap}_{\alpha,\beta}(t)$ and $G_{\alpha+1,\beta}(t)$ and take $G_{\alpha+1,\beta}(t)$ to be the numeraire. Also consider the following process:

$$M_t = \mathbb{E}^{\mathbb{Q}^{G_{\alpha+1,\beta}(t)}}[\frac{X_T}{G_T}|\mathcal{F}_t].$$

By the tower property of conditional expectation, M_t is a martingale under the $\mathbb{Q}^{G_{\alpha+1,\beta}(t)}$ measure. Furthermore, the ratio $\frac{\mathbf{Swap}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}$ is also a martingale under $\mathbb{Q}^{G_{\alpha+1,\beta}(t)}$. Therefore, theorem A.2.2 (the Martingale Representation Theorem) can be used to find a predictable process ϕ_t such that

$$\begin{aligned} dM_t &= \phi_t d\left(\frac{\mathbf{Swap}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}\right) \\ &\text{or} \\ M_t &= M_0 + \int_0^t \phi_u d\left(\frac{\mathbf{Swap}_{\alpha,\beta}(u)}{G_{\alpha+1,\beta}(u)}\right) \end{aligned}$$

Thus, when measuring quantities under the numeraire $G_{\alpha+1,\beta}(t)$, we have:

1. A value process, M_t, that replicates the derivative value since $M_T = \frac{X_T}{G_{\alpha+1,\beta}(T)}$.
2. A self financing trading strategy, ϕ_t, that allows us to construct a portfolio, $V(t)$, that takes on the value of M_t at all t $(0 \leq t \leq T)$.

By the assumption of no arbitrage, we have

$$V^G(t) = \frac{V_t}{G_{\alpha+1,\beta}(t)} = \frac{X_t}{G_{\alpha+1,\beta}(t)}$$

since

$$V^G(T) = M_T = \frac{X_T}{G_{\alpha+1,\beta}(T)}.$$

Applying Ito's lemma to

$$\begin{aligned} V^G(t) &= V^G(0) + \int_0^t \phi(u) d\left(\frac{\mathbf{Swap}_{\alpha,\beta}(u)}{G_{\alpha+1,\beta}(u)}\right) \\ \rightarrow \phi(t) &= \frac{\partial V^G(t)}{\partial\left(\frac{\mathbf{Swap}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}\right)}. \end{aligned}$$

Thus, the self financing trading strategy, ϕ_t, replicates the derivative payoff.

5.1.1 The practical implementation of delta hedging in the case of swaptions

Consider a swaption (a caplet is equivalent to a swaption with a single period swap as the underlying since we make assumption 2.1.5) with the value given by definition 2.1.11. The delta of this swaption is

$$\begin{aligned} \frac{\partial V^G(t)}{\partial\left(\frac{\mathbf{Swap}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}\right)} &= \frac{\partial\left(\frac{\mathbf{Swaption}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}\right)}{\partial\left(\frac{\mathbf{Swap}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}\right)} \\ &= \frac{\partial\left(\frac{G_{\alpha+1,\beta}(t)[S_{\alpha,\beta}(t)\Phi(d_1) - R\Phi(d_2)]}{G_{\alpha+1,\beta}(t)}\right)}{\partial\left(\frac{G_{\alpha+1,\beta}(t)[S_{\alpha,\beta}(t) - R]}{G_{\alpha+1,\beta}(t)}\right)} \\ &= \frac{\partial\left([S_{\alpha,\beta}(t)\Phi(d_1) - R\Phi(d_2)]\right)}{\partial\left(S_{\alpha,\beta}(t)\right)} \\ &= \Phi(d_1) \\ &:= \Delta_t \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{S_{\alpha,\beta}(t)}{K}\right) + \frac{\nu^2_{\alpha,\beta}(t)(T_\alpha - t)}{2}}{\nu_{\alpha,\beta}(t)\sqrt{T_\alpha - t}},$$

$$G_{\alpha+1,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P(t, T_i)$$

and

$$\nu^2_{\alpha,\beta}(t) = \frac{1}{(T_\alpha - t)} \int_t^{T_\alpha} \|\lambda_{\alpha,\beta}(u)\|^2 du.$$

Suppose that a bank writes a swaption. The bank receives a premium for writing the swaption which it invests in the numeraire asset. At the same time the delta hedging strategy is started. This implies that the bank enters into Δ_t number of swaps (at zero cost to the bank). Suppose that the bank decides to re-hedge its position $m+1$ times over the period $[0, T_\alpha]$ and that re-balancings occur at $t_0, t_1, t_2, ..., t_m = T_\alpha$ and $t_{-1} = 0$. At time T_α, the value of this hedging process, in terms of the $G_{\alpha+1,\beta}(t)$ numeraire, is equal to H where H is

$$\begin{aligned} H &= (S_{\alpha,\beta}(t_m) - S_{\alpha,\beta}(t_{-1})) \times \Delta_{t_{-1}} \\ &+ (S_{\alpha,\beta}(t_m) - S_{\alpha,\beta}(t_0)) \times (\Delta_{t_0} - \Delta_{t_{-1}}) \\ &+ (S_{\alpha,\beta}(t_m) - S_{\alpha,\beta}(t_1)) \times (\Delta_{t_1} - \Delta_0) \\ &+ \ldots \\ &+ (S_{\alpha,\beta}(t_m) - S_{\alpha,\beta}(t_{m-1})) \times (\Delta_{t_{m-1}} - \Delta_{t_{m-2}}) \\ &+ (S_{\alpha,\beta}(t_m) - S_{\alpha,\beta}(t_m)) \times (\Delta_{t_m} - \Delta_{t_{m-1}}). \end{aligned}$$

And so, the hedging error (the net amount that the option writer is left with at maturity after hedging) is

$$\left(\text{H} + \text{premium} \times \frac{1}{G_{\alpha+1,\beta}(t_{-1})}\right) \times G_{\alpha+1,\beta}(t_m) - \text{payoff on swaption}.$$

Note that the hedging error and the invested premium have been stated in terms of the numeraire $G_{\alpha+1,\beta}(t)$ and thus, it is necessary to multiply by the value of the numeraire at expiry of the option.

The relative Hedging error will be defined as follows:

$$RHE_{\alpha,\beta}(T_\alpha) = \frac{\left(\text{H} + \textbf{Swaption}_{\alpha,\beta}(t_{-1}) \times \frac{1}{G_{\alpha+1,\beta}(t_{-1})}\right) \times G_{\alpha+1,\beta}(t_m) - \text{payoff on swaption}}{\textbf{Swaption}_{\alpha,\beta}(t_{-1})} \quad (5.1)$$

and this is the absolute hedging error divided by the premium. If the Swaption is un-hedged, the relative error is defined as follows

$$UHE_{\alpha,\beta}(T_\alpha) = \frac{\left(\textbf{Swaption}_{\alpha,\beta}(t_{-1}) \times \frac{1}{G_{\alpha+1,\beta}(t_{-1})}\right) \times G_{\alpha+1,\beta}(t_m) - \text{payoff on swaption}}{\textbf{Swaption}_{\alpha,\beta}(t_{-1})}. \quad (5.2)$$

Note that if the swaption expires out of the money, then the payoff on the swaption = 0 and the un-hedged error is just $UHE_{\alpha,\beta}(T_\alpha) - G_{\alpha+1,\beta}(t_{-1})/G_{\alpha+1,\beta}(t_m)$ which will be the same no matter what the swaption premium is.

5.2 Hedging under different calibrations

5.2.1 The options to be hedged

Swaptions with six months to maturity will be hedged. Swaptions will be hedged over three different non-overlapping sixth month periods. Swaptions based on forward starting swaps (forward staring in six months time) with varying number of resets shall be considered. The swaptions that will be hedged are summarised in table 5.1.

Maturity	Writing Date	Number Of Resets	Number of Business Days
28 February 2007	01 September 2006	1	127
28 February 2007	01 September 2006	4	127
28 February 2007	01 September 2006	8	127
28 February 2007	01 September 2006	12	127
31 August 2007	01 March 2007	1	128
31 August 2007	01 March 2007	4	128
31 August 2007	01 March 2007	8	128
31 August 2007	01 March 2007	12	128
29 February 2008	03 September 2007	1	125
29 February 2008	03 September 2007	4	125
29 February 2008	03 September 2007	8	125
29 February 2008	03 September 2007	12	125

Table 5.1: Details of the swaptions that will be hedged. The 'Number Of Resets' refers to the number of quarterly reset dates in the forward starting swap underlying each swaption. The 'Number of Business days' refers to the number of business days between the writing of the option and the maturity of the option.

5.2.2 Re-calibration during the hedging process

The hedge on each option will be re-balanced on every business day. The daily re-balancing will be based on monthly re-calibrations to the swaptions volatility matrix. Table 5.2 summarises the re-calibration dates. The form of the swaption volatility matrices to which calibration will take place is given in table 5.3.

5.2.3 Perfect Parameters

$S^M_{FWi}(t)$ is the notation that will be used to refer to the forward swap rate with

- first setting date being equal to F (F = 28 February 2007, 31 August 2007, 29 February 2008);

Calibration dates		
written on 01 September 2006 maturity on 28 February 2007	written on 01 March 2007 maturity on 31 August 2007	written on 01 September 2006 maturity on 28 February 2007
01-Sep-06	02-Feb-07	02-Aug-07
02-Oct-06	08-Mar-07	03-Sep-07
01-Nov-06	10-Apr-07	01-Oct-07
05-Dec-06	02-May-07	01-Nov-07
15-Jan-07	04-Jun-07	03-Dec-07
02-Feb-07	03-Jul-07	08-Jan-08
	02-Aug-07	01-Feb-08

Table 5.2: **The** calibration dates that are relevant to the hedging of **each** swaption.

	Swap Length							
Time Until First Setting Date	0.25	0.5	0.75	1	1.25	1.5	1.75	2
0.25	$\nu^M_{0,1}$	-	-	-	-	-	-	-
0.5	$\nu^M_{1,2}$	-	-	$\nu^M_{1,5}$	-	-	-	$\nu^M_{1,9}$
0.75	$\nu^M_{2,3}$	-	-	-	-	-	-	-
1	$\nu^M_{3,4}$	-	-	$\nu^M_{3,7}$	-	-	-	$\nu^M_{3,11}$
1.25	$\nu^M_{4,5}$	-	-	-	-	-	-	-
1.5	$\nu^M_{5,6}$	-	-	-	-	-	-	-
1.75	$\nu^M_{6,7}$	-	-	-	-	-	-	-
2	$\nu^M_{7,8}$	-	-	-	-	-	-	-
2.25	$\nu^M_{8,9}$	-	-	-	-	-	-	-
2.5	$\nu^M_{9,10}$	-	-	-	-	-	-	-
2.75	$\nu^M_{10,11}$	-	-	-	-	-	-	-
3	$\nu^M_{11,12}$	-	-	-	-	-	-	-
3.25	$\nu^M_{12,13}$	-	-	-	-	-	-	-
3.5	$\nu^M_{13,14}$	-	-	-	-	-	-	-
3.75	$\nu^M_{14,15}$	-	-	-	-	-	-	-
4	$\nu^M_{15,16}$	-	-	-	-	-	-	-
4.25	$\nu^M_{16,17}$	-	-	-	-	-	-	-
4.5	$\nu^M_{17,18}$	-	-	-	-	-	-	-
4.75	$\nu^M_{18,19}$	-	-	-	-	-	-	-
5	$\nu^M_{19,20}$	-	-	-	-	-	-	-
5.25	$\nu^M_{20,21}$	-	-	-	-	-	-	-
5.5	$\nu^M_{21,22}$	-	-	-	-	-	-	-
5.75	$\nu^M_{22,23}$	-	-	-	-	-	-	-
6	$\nu^M_{23,24}$	-	-	-	-	-	-	-
6.25	$\nu^M_{24,25}$			-	-	-	-	-

Table 5.3: The general form of the swaptions volatility matrix which will be calibrated to on a monthly basis.

- W is the option writing date (W = 01 September 2006, 01 March 2007, 03 September 2007) and

- with i reset periods ($i = 1,4,8,12$).

The superscript M refers to the fact that this rate has been obtained from the yield curve on day t. This yield curve has been stripped using the HMC method described in section 4.1.

For each of the forward starting swap rate series, a *perfect volatility parameter* will be estimated. This is effectively a volatility parameter that is estimated in hindsight. We assume the simple volatility structure presented in table 4.7. We define the perfect volatility parameter as $\nu^P_{F,W,i}$.

Let the function $n(F,W)$ count the number of business days between F and W and define $t_0, ..., t_{n(F,W)}$ such that $t_0 = W$ and $t_{n(F,W)} = F$ and t_i $(0 < i < n(F,W))$ is a unique business day between t_0 and $t_{n(F,W)}$ then

$$\nu^P_{F,W,i} = \sqrt{252} \times \sqrt{\frac{\sum_{j=0}^{n(F,W)-1} \left(\ln\left(\frac{S^M_{F,W,i}(t_{j+1})}{S^M_{F,W,i}(t_j)} \right) - \hat{\mu} \right)^2}{n(F,W)-1}}$$

where

$$\hat{\mu} = \frac{\sum_{j=0}^{n(F,W)-1} \left(\ln\left(\frac{S^M_{F,W,i}(t_{j+1})}{S^M_{F,W,i}(t_j)} \right) \right)}{n(F,W)}.$$

Thus, $\nu^P_{F,W,i}$ is the estimated standard deviation of the 'log returns' of each of the swap rate series.

5.3 Results

Section 4.6.1 described four calibration procedures based on optimisation and parametric instantaneous volatility and correlation structures. This section will try to distill the quality of each of these calibration procedures. Every swaption presented in table 5.1 is delta hedged under each of the four different calibration procedures. The hedging error resulting from each of these procedures is compared. The following abbreviations will be used to refer to the different parametric calibration methods.

- Four: a, b, c, d are free parameters. $\beta = 0.1$.
- Five: a, b, c, d and β are all free parameters.
- Four K: a, b, c, d are free parameters. $\beta = 0.1$. Caplets are fitted perfectly.
- Five K: a, b, c, d and β are all free parameters. Caplets are fitted perfectly.
- Perfect: The hedging error achieved by using a volatility parameter estimated as described in section 5.2.3.
- Unhedged: The profit or loss resulting from simply investing the premium and not engaging in any hedging. The profit or loss is stated in terms of equation 5.2.

The tables 5.4, 5.5, 5.6, 5.7, 5.8 and 5.9 present hedging results in the form of equation 5.1 and equation 5.2. The next section will discuss the results presented here.

Swap Resets (i)	Calibrations				Control
	Four	Five	Four K	Five K	Perfect
1	42.45%	42.71%	44.87%	44.78%	-15.26%
4	54.10%	52.75%	54.09%	52.24%	4.86%
8	36.24%	34.13%	36.31%	33.99%	2.03%
12	21.89%	16.93%	22.05%	16.62%	3.07%

Table 5.4: The hedging results for the swaptions that were written on 1 September 2006 and with first setting date on 28 February 2008. The entries in this table are the relative hedging errors defined in equation 5.1. Values in red show the lowest hedging error achieved amongst the four calibration procedures (the lowest error across a row of the table).

Swap Resets (i)	Four	Five	Four K	Five K
1	104.29%	104.29%	104.29%	104.29%
4	104.35%	104.35%	104.35%	104.35%
8	104.61%	104.61%	104.61%	104.61%
12	105.02%	105.02%	105.02%	105.02%

Table 5.5: Swaptions were written on 1 September 2006 and with first setting date on 28 February 2008. The entries in this table represent the un-hedged errors defined according to equation 5.2. Note that the entries are all the same across rows. This implies that all of the swaptions expired out of the money and were therefore not exercised.

Swap Resets	Calibrations				Control
	Four	Five	Four K	Five K	Perfect
1	40.04%	40.36%	33.69%	33.95%	-20.88%
4	36.03%	36.34%	34.41%	34.90%	-17.94%
8	32.33%	33.09%	30.38%	31.42%	-11.15%
12	28.50%	29.64%	27.14%	28.44%	-3.91%

Table 5.6: The hedging results for the swaptions that were written on 1 March 2007 and with first setting date on 31 August 2007. The entries in this table are the relative hedging errors defined in equation 5.1. Values in red show the lowest hedging error achieved amongst the four calibration procedures (the lowest error across a row of the table).

Swap Resets (i)	Four	Five	Four K	Five K
1	-282.87%	-284.60%	-316.59%	-316.71%
4	-373.96%	-372.11%	-398.33%	-402.64%
8	-380.69%	-374.11%	-390.18%	-399.12%
12	359.38%	-348.79%	-355.13%	-367.60%

Table 5.7: Swaptions were written on 1 March 2007 and with first setting date on 31 August 2007. The entries in this table represent the un-hedged errors defined according to equation 5.2.

Swap Resets	Calibrations Four	Five	Four K	Five K	Control Perfect
1	28.81%	28.63%	24.06%	23.70%	31.39%
4	34.68%	33.03%	32.06%	28.95%	-18.80%
8	38.31%	33.79%	38.67%	33.17%	-7.00%
12	41.74%	36.66%	41.74%	36.14%	-0.09%

Table 5.8: The hedging results for the swaptions that were written on 3 September 2007 and with first setting date on 29 February 2008. The entries in this table are the relative hedging errors defined in equation 5.1. Values in red show the lowest hedging error achieved amongst the four calibration procedures (the lowest error across a row of the table).

Swap Resets (i)	Four	Five	Four K	Five K
1	-183.74%	-184.40%	-202.27%	-202.45%
4	-174.37%	-170.00%	-184.37%	-193.47%
8	-145.01%	-131.35%	-129.99%	-146.89%
12	-77.09%	-60.38%	-59.59%	-78.14%

Table 5.9: Swaptions were written on 3 September 2007 and had first setting date on 29 February 2008. The entries in this table represent the un-hedged errors defined according to equation 5.2.

5.4 Discussion of results

When comparing the hedging output to the control cases (the perfect parameter hedging and the unhedged hedging errors), we seem to find what we expect:

- In all cases, the absolute hedging error is smaller under the parametric calibrations than under the un-hedged output. This is expected since this is the purpose of hedging.
- On the other hand, in most cases the perfect parameter hedging produces hedging errors that are much closer to zero than the hedging errors obtained under the parametric calibration procedures. This is also expected since the 'perfect' parameters were estimated in hindsight.

It does appear as if model fit dominates model parsimony in the hedging results that have been produced. Tables 5.4, 5.6 and 5.8 show that the calibration procedures which make use of more parameters produce better hedging errors fairly consistently. In fact, the 'Five K' calibration procedure gives the lowest absolute hedging error in most cases. The results therefore appear to favour the Five K and Four K hedging procedures.

The hedging errors are positive fairly consistently positive. This points to the possibility of the implied volatility being bigger that the realised volatility (option writers make money if the implied volatility is greater than the realised volatility). This seems to be confirmed when the hedging that was carried out between 1 September 2006 and 28 February 2007 is considered (table 5.4). The error resulting from the Five K monthly calibration procedure is 52.24% whereas the perfect parameter hedging error is 4.86%. Figure 5.1 presents the volatility that was used as inputs into the Five K and the Perfect Parameter hedging procedures. Figure 5.1 shows that the Five K calibration procedure does indeed result in an implied volatility that is greater than

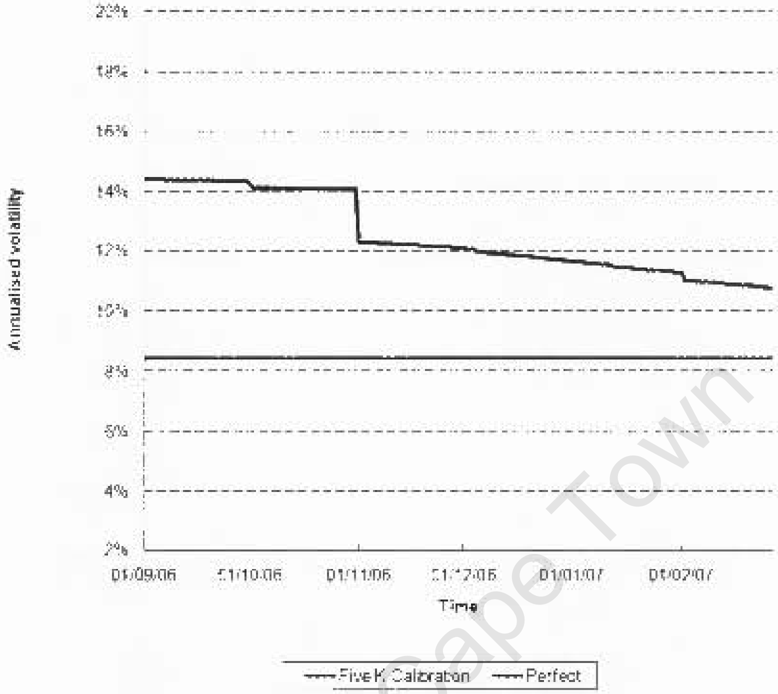

Figure 5.1: The estimated volatility over a six month period between 1 September 2006 and 28 February 2007 that has been used as the input into hedging procedures for swaptions with maturity of half a year and with an underlying forward starting swap with length of one year.The red graph represents the implied volatility resulting from the Five K calibration. Since the re-calibration occurs on a monthly basis, the graph has a 'step' appearance.

the perfect parameter volatility.

This elevated volatility may be evidence of a 'volatility premium' being charged in the market. Option writers may recognise the failure of the Black Scholes assumptions of log normality, absence of transaction costs and taxes and the ability to continuously re-hedge. The volatility premium will emerge in the form of a general upward migration of the bid and offer (and hence also the mid - on which this study is based) volatilities.

Note that the market volatility (as given by the Five K calibration) does appear to approach the 'true' volatility over time in figure 5.1.

The results also bring into question an assumption which underlies this analysis. This assumption is as follows:

Implied volatilities are accurate predictors of future volatilities.

The existence of an erratic 'volatility premium' invalidates this assumption. However, it reasonable to assume that the volatility premium will in general be positive and that hedging based on implied volatilities will in general produce profits (of course this cannot always be the case).

As a caveat, it must be noted that the hedging results displayed here have to be seen in the context of the data that generated the results. The likelihood that these hedging errors could have been replicated in practice is slim. This is because the hedging results were generated on the assumption that forward starting swaps are liquid and tradable enough such that daily re-balancing of the hedging portfolio is possible.

It is necessary to comment on the bootstrapping and interpolation procedure that was used to perform the hedging. To perform daily hedging it was necessary to have bootstrapped curves out to a maximum of 3.5 years. In section 4.1.2 it was mentioned that the HMC curves should price instruments dependent on the first eight quarters in an arbitrage free manner. For maturities greater than 2 years, the zero coupon bonds expiring at years 3 and 4 would be accurately priced. There is less confidence in the pricing of the zero coupon bonds expiring at the quarters in between years 2 and 3 and years 3 and 4. However, since the HMC curve was theoretically arbitrage free (non-negative instantaneous forward rates), the hedging errors produced under the forward starting swaps with forward tenors of 8 and 12 quarters should still be reasonably accurate (since the values of the zero coupon bonds in between years 2 and 3 and years 3 and 4 were constructed such that the curve as a whole would imply non-negative forward rates).

Chapter 6

Application - the pricing of barrier swaptions

In the South African equity market, barrier options are popular products from the perspective of institutional clients such as asset managers or pension funds. This is because they are considerably cheaper than vanilla options. In the relatively illiquid (but growing) market for South African exotic interest rate derivatives, barrier swaptions appear to be the most natural first step into the exotic interest rate derivatives market.

For example, in return for a steady stream of premiums, a life insurer has agreed to provide a guaranteed annuity rate on a known lump sum at some time in the future. If interest rates fall, then the swap rate also falls and the insurer is at risk that the available swap rate is below the rate that it has guaranteed to its policyholders. Thus, to protect against this fall in interest rates the insurer may buy the option to enter into a receive fixed swap. So if the swap rate falls below the guaranteed annuity rate, the insurer can enter into a swap in which it still receives the guaranteed annuity rate and it pays the floating rate on the lump sum. If the swap rate rises above the annuity rate, then the option falls away and the insurer enters into the receive-fixed swap as it is traded on the market.

However, the insurer may feel that the forward swap rate will not rise or fall above or below certain levels over the course of the option. If the insurer feels that the utility gained by saving in the swaption premium is greater than the utility lost from having protection for all forward swap rate levels over the duration of the option then a barrier swaption is suitable.

The first objective of this chapter is to derive a closed form formula for a barrier swaption by making use of two techniques: the change of numeraire technique used to derive closed form swaption prices and the technique of using the *stopped* asset process to derive closed form solutions for barrier options. The second objective is to present the main ideas behind Monte Carlo pricing in the LIBOR Market Model framework. The third objective is to show that the closed form (Swap Market Model framework) formula for a barrier swaption price is a reasonable approximation to the LIBOR Market Model Monte Carlo price.

6.1 Pricing of barrier swaptions in the Swap Market Model framework using analytical formulae

The formulae for barrier options in the equity setting is well known. This section will derive a formula for *barrier swaptions*. The derivation of an analytical formula for the barrier swaption is set in the Swap Market Model framework. The approach to the *barrier* feature in the barrier swaption shall follow the (equity option) approach laid out in Ouwehand [2006] and Bjork [1998] quite closely.

6.1.1 Up and out contracts

Proposition 6.1.1 *The price of an up and out barrier swaption with a unit notional amount is*

$$\mathbf{UOBS}(t) = G_{\alpha+1,\beta}(t)\mathbf{E}^{Q_{\alpha+1,\beta}}\left[\left(S^{\tau}_{\alpha,\beta}(T) - R\right)^{+} I_{\{S^{\tau}_{\alpha,\beta}(T)<L\}}|\mathcal{F}_t\right]$$

where

$$\tau = \inf\left\{s \in (t,T] : S^{\star}_{\alpha,\beta}(s) \geq L\right\}$$

and $S^{\star}_{\alpha,\beta}(T)$ is the maximum value reached by $S_{\alpha,\beta}(s)$ over the period $(t,T]$ and where

$$S^{\tau}_{\alpha,\beta}(T) = \begin{cases} S_{\alpha,\beta}(T) \text{ if } S^{\star}_{\alpha,\beta}(T) < L \\ S_{\alpha,\beta}(\tau) \text{ if } S^{\star}_{\alpha,\beta}(T) > L \end{cases}.$$

Proof
Under the equivalent martingale measure $Q_{\alpha+1,\beta}$ associated with the numeraire $G_{\alpha+1,\beta}(t)$, the price of a barrier swaption is

$$\begin{aligned} & G_{\alpha+1,\beta}(t)\mathbb{E}^{Q_{\alpha+1,\beta}}\left[\frac{G_{\alpha+1,\beta}(T)(S_{\alpha,\beta}(T)-R)^{+}}{G_{\alpha+1,\beta}(T)} I_{\{S^{\star}_{\alpha,\beta}(T)<L\}}|\mathcal{F}_t\right] \\ &= G_{\alpha+1,\beta}(t)\mathbf{E}^{Q_{\alpha+1,\beta}}\left[(S_{\alpha,\beta}(T) - R)^{+} I_{\{S^{\star}_{\alpha,\beta}(T)<L\}}|\mathcal{F}_t\right]. \end{aligned}$$

If $\omega \subset A = \left\{\omega : S^{\star}_{\alpha,\beta}(T,\omega) < L\right\}$ then

$$\begin{aligned} S^{\tau}_{\alpha,\beta}(T,\omega) &= S_{\alpha,\beta}(T,\omega) \\ \Rightarrow (S_{\alpha,\beta}(T,\omega) - R)^{+} I_{\{S^{\star}_{\alpha,\beta}(T)<L\}} &= \left(S^{\tau}_{\alpha,\beta}(T,\omega) - R\right)^{+} I_{\{S^{\tau}_{\alpha,\beta}(T,\omega)<L\}}. \end{aligned}$$

On the other hand, if $\omega \subset B = \left\{\omega : S^{\star}_{\alpha,\beta}(T,\omega) \geq L\right\}$

$$S^{\tau}_{\alpha,\beta}(T,\omega) = L$$

$$\Rightarrow (S_{\alpha,\beta}(T,\omega) - R)^{+} I_{\{S^{\star}_{\alpha,\beta}(T)<L\}} = \left(S^{\tau}_{\alpha,\beta}(T,\omega) - R\right)^{+} I_{\{S^{\tau}_{\alpha,\beta}(T,\omega)<L\}} = 0.$$

Since $A \cup B = \Omega$,

$$\mathbf{UOBS}(t) = G_{\alpha+1,\beta}(t)\mathbf{E}^{Q_{\alpha+1,\beta}}\left[\left(S^{\tau}_{\alpha,\beta}(T) - R\right)^{+} I_{\{S^{\tau}_{\alpha,\beta}(T)<L\}}|\mathcal{F}_t\right].$$

Thus, to calculate an analytical formula for the above barrier swaption, it is necessary to determine the distribution of $S^{\tau}_{\alpha,\beta}(T)$.

Ξ

6.1.2 Up and in contracts

The value of an up and in barrier swaption follows from a simple put call parity relationship and is

$$\mathbf{UIBS}(t) = \mathbf{PSO}(t) - \mathbf{UOBS}(t)$$

where $\mathbf{PSO}(t)$ is the value of the vanilla swaption.

6.1.3 Down and out contracts

Proposition 6.1.2 *The price of a down and out barrier swaption is*

$$\mathbf{DOBS}(t) = G_{\alpha+1,\beta}(t)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[\left(S^{\tau}_{\alpha,\beta}(T) - R\right)^{+} I_{\left\{S^{\tau}_{\alpha,\beta}(T)>H\right\}}|\mathcal{F}_t\right]$$

where

$$\tau = \inf\left\{s \in (t,T] : S^{\downarrow}_{\alpha,\beta}(s) \leq H\right\}$$

and $S^{\downarrow}_{\alpha,\beta}(T)$ *is the minimum value reached by* $S_{\alpha,\beta}(s)$ *over the period* $(t,T]$ *and where*

$$S^{\tau}_{\alpha,\beta}(T) = \begin{cases} S_{\alpha,\beta}(T) \text{ if } S^{\downarrow}_{\alpha,\beta}(T) > H \\ S_{\alpha,\beta}(\tau) \text{ if } S^{\downarrow}_{\alpha,\beta}(T) \leq H \end{cases}.$$

Proof
Under the equivalent martingale measure $\mathbb{Q}_{\alpha+1,\beta}$ associated with the numeraire $G_{\alpha+1,\beta}(t)$, the price of a barrier swaption is

$$\begin{aligned} & G_{\alpha+1,\beta}(t)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[\frac{G_{\alpha+1,\beta}(T)\left(S_{\alpha,\beta}(T)-R\right)^{+}}{G_{\alpha+1,\beta}(T)} I_{\left\{S^{\downarrow}_{\alpha,\beta}(T)>H\right\}}|\mathcal{F}_t\right] \\ &= G_{\alpha+1,\beta}(t)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[\left(S_{\alpha,\beta}(T) - R\right)^{+} I_{\left\{S^{\downarrow}_{\alpha,\beta}(T)>H\right\}}|\mathcal{F}_t\right]. \end{aligned}$$

If $\omega \in A = \left\{\omega : S^{\downarrow}_{\alpha,\beta}(T,\omega) > H\right\}$ then

$$\begin{aligned} S^{\tau}_{\alpha,\beta}(T,\omega) &= S_{\alpha,\beta}(T,\omega) \\ \Rightarrow \left(S_{\alpha,\beta}(T,\omega) - R\right)^{+} I_{\left\{S^{\downarrow}_{\alpha,\beta}(T,\omega)>H\right\}} &= \left(S^{\tau}_{\alpha,\beta}(T,\omega) - R\right)^{+} I_{\left\{S^{\tau}_{\alpha,\beta}(T,\omega)>H\right\}}. \end{aligned}$$

On the other hand, if $\omega \in B = \left\{\omega : S^{\downarrow}_{\alpha,\beta}(T,\omega) \leq H\right\}$

$$S^{\tau}_{\alpha,\beta}(T,\omega) = H$$

$$\Rightarrow \left(S_{\alpha,\beta}(T,\omega) - R\right)^{+} I_{\left\{S^{\downarrow}_{\alpha,\beta}(T,\omega)\leq H\right\}} = \left(S^{\tau}_{\alpha,\beta}(T,\omega) - R\right)^{+} I_{\left\{S^{\tau}_{\alpha,\beta}(T,\omega)\leq H\right\}} = 0 \quad .$$

Since $A \cup B = \Omega$,

$$\mathbf{DOBS}(t) = G_{\alpha+1,\beta}(t)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[\left(S^{\tau}_{\alpha,\beta}(T) - R\right)^{+} I_{\left\{S^{\tau}_{\alpha,\beta}(T)>H\right\}}|\mathcal{F}_t\right].$$

Thus, to calculate an analytical formula for the above barrier swaption, it is necessary to determine the distribution of $S^{\tau}_{\alpha,\beta}(T)$.

Ξ

6.1.4 Down and in contracts

The value of a down and in barrier swaption follows from a simple put call parity relationship and is

$$\mathbf{DIBS}(t) = \mathbf{PSO}(t) - \mathbf{DOBS}(t)$$

where $\mathbf{PSO}(t)$ is the value of the vanilla swaption.

6.1.5 The density of stopped arithmetic Brownian motion

An arithmetic, X_t, Brownian motion is

$$X_t = \alpha + \mu t + \sigma W_t$$

where, α is the starting point, μ is the drift rate, σ is the variance rate and W_t is a standard $\mathbb{P}$-Brownian motion.

Definition 6.1.3 *τ_β is a hitting time and is defined as*

$$\tau_\beta = \inf\{t : X_t \geq \beta\}.$$

The stopped arithmetic Brownian motion associated with this hitting time is

$$X_t^{\tau_\beta} = \begin{cases} X_t & \text{if } X_t^{\star} < \beta \\ X_{\tau_\beta} & \text{if } X_t^{\star} \geq \beta \end{cases}$$

where $X_t^{\star}$ is the running maximum of X_t.

The following results are found in Ouwehand [2006] and Bjork [1998]. These results lead up to the derivation of the density of $X_T^{\tau_\beta}$.

Proposition 6.1.4 *The joint distribution, $F_t(x,y)$, of $(W_t, W_t^{\star})$ where W_t is standard Brownian motion and $W_t^{\star}$ is its running maximum is*

$$F_t(x,y) = \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x-2y}{\sqrt{t}}\right).$$

Proposition 6.1.5 *Let $(X_t)_{t\geq 0}$ be, under the measure $\mathbb{P}$, an arithmetic Brownian motion with drift rate μ and variance rate σ, starting at α. Then the joint distribution of X_t and its running maximum $X_t^\star$ is given by*

$$\mathbb{P}(X_t \leq x, X_t^\star \leq y) = \Phi\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu(y-\alpha)}{\sigma^2}}\Phi\left(\frac{x+\alpha-2y-\mu t}{\sigma\sqrt{t}}\right).$$

Corollary 6.1.6 *The distribution function of the running maximum of an arithmetic Brownian motion X_t with drift rate μ, variance rate σ, starting at α is*

$$\mathbb{P}(X_t^\star \leq x) = \Phi\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) + e^{\frac{2\mu(x-\alpha)}{\sigma^2}}\Phi\left(\frac{-x+\alpha-\mu t}{\sigma\sqrt{t}}\right).$$

Proposition 6.1.7 *The density, $f_t(x;\alpha,\beta)$, of $X_t^{\tau_\beta}$ is*

$$f_t(x;\alpha,\beta) = \varphi(x;\alpha+\mu t,\sigma^2 t) - e^{\frac{2\mu(x-\alpha)}{\sigma^2}}\varphi(x;-\alpha+2\beta_\mu t,\sigma^2 t)$$

where $\varphi(x;\mu,\sigma^2)$ is the Gaussian density with mean μ and variance of σ^2 evaluated at x.

6.1.6 The closed form formula for the up and out barrier swaption

To determine an analytical formula for the up and out barrier swaption, it is necessary to determine the distribution of $S^\tau_{\alpha,\beta}(T)$ under the measure $\mathbb{Q}_{\alpha+1,\beta}$ given that it is known that

$$\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} = \lambda_{\alpha,\beta}(t)dW_t^{\mathbb{Q}^{G_{\alpha+1,\beta}}}$$

and

$$S_{\alpha,\beta}(T) = e^{\ln(S_{\alpha,\beta}(t))-\frac{1}{2}\int_t^T\lambda^2_{\alpha,\beta}(u)du+\int_t^T\lambda_{\alpha,\beta}(u)dW_u^{\mathbb{Q}^{G_{\alpha+1,\beta}}}}.$$

A key simplifying assumption that will be made is that the instantaneous volatility function, $\lambda_{\alpha,\beta}(t)$, of the swap rate, $S_{\alpha,\beta}(t)$, is constant (ie $\lambda_{\alpha,\beta}(u) = \lambda$ for all $u \in (t, T_\alpha]$) so that

$$S_{\alpha,\beta}(T) = e^{\ln(S_{\alpha,\beta}(t))-\frac{1}{2}\lambda^2(T-t)+\lambda W_{T-t}^{\mathbb{Q}^{G_{\alpha+1,\beta}}}}.$$

Without loss of generality, take $t = 0$ and consider

$$\begin{aligned} S_{\alpha,\beta}(s) &= e^{\alpha+\mu s+\sigma W_s^{\mathbb{Q}^{G_{\alpha+1,\beta}}}} \\ &= e^{X_s} \end{aligned}$$

where

$$\begin{aligned} \alpha &= \ln(S_{\alpha,\beta}(0)) \\ \mu &= -\frac{1}{2}\lambda^2 \\ \sigma &= \lambda. \end{aligned}$$

Since $S^{\tau}_{\alpha,\beta}(t) = e^{X^{\tau}_s}$ (and $\tau = \inf\left\{t : S^{\star}_{\alpha,\beta}(t) \geq L\right\} = \inf\{t : X^{\star}_s \geq \ln(L)\}$), and since X^{τ}_s has a density

$$f(x) = \varphi(x; M1, s) - \frac{S_{\alpha,\beta}(0)}{L}\varphi(x; M2, s)$$

where

$$\begin{aligned} M1 &= \ln(S_{\alpha,\beta}(0)) - \frac{1}{2}\lambda^2 T \\ M2 &= -\ln(S_{\alpha,\beta}(0)) + 2\ln(L) - \frac{1}{2}\lambda^2 T \\ &= \ln(\frac{L^2}{S_{\alpha,\beta}(0)}) - \frac{1}{2}\lambda^2 T \\ s &= \lambda\sqrt{T} \end{aligned}$$

and

$$\varphi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

the formula for the up and out call can be written as

$$\begin{aligned} \mathbf{UOBS}(t) &= G_{\alpha+1,\beta}(0)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[\left(S^{\tau}_{\alpha,\beta}(T) - R\right)^+ I_{\left\{S^{\tau}_{\alpha,\beta}(T)<L\right\}} | \mathcal{F}_t\right] \\ &= G_{\alpha+1,\beta}(0)\int_{-\infty}^{\ln(L)} (e^x - R)^+ f(x) I_{\{x<\ln(L)\}} dx \\ &= G_{\alpha+1,\beta}(0)\int_{\ln(R)}^{\ln(L)} (e^x - R) f(x) dx \\ &= G_{\alpha+1,\beta}(0)\int_{\ln(R)}^{\ln(L)} e^x f(x) dx - G_{\alpha+1,\beta}(0) R \int_{\ln(R)}^{\ln(L)} f(x) dx \\ &= G_{\alpha+1,\beta}(0)\int_{\ln(R)}^{\ln(L)} e^x \varphi(x; M1, s) dx - G_{\alpha+1,\beta}(0)\frac{S_{\alpha,\beta}(0)}{L}\int_{\ln(R)}^{\ln(L)} e^x \varphi(x; M2, s) dx \\ &- G_{\alpha+1,\beta}(0) R \int_{\ln(R)}^{\ln(L)} \varphi(x; M1, s) dx + G_{\alpha+1,\beta}(0)\frac{S_{\alpha,\beta}(0) R}{L}\int_{\ln(R)}^{\ln(L)} \varphi(x; M2, s) dx. \end{aligned}$$

Note that

$$\int_a^b \varphi(x; \mu, \sigma) dx = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{b-\mu}{\sigma}\right)$$

and

$$\int_a^b e^x \varphi(x; \mu, \sigma) dx = e^{\mu + \frac{\sigma^2}{2}}\left(\Phi\left(\frac{b-\mu-\sigma^2}{\sigma}\right) - \Phi\left(\frac{b-\mu-\sigma^2}{\sigma}\right)\right)$$

and so

$$\mathbf{UOBS}(0) = G_{\alpha+1,\beta}(0) S_{0,n}(0)\left(\Phi(d^{\frac{L}{S}-}) - \Phi(d^{\frac{R}{S}-})\right)$$

$$
\begin{aligned}
&- \quad G_{\alpha+1,\beta}(0)L\left(\Phi(d^{\frac{S}{L}-}) - \Phi(d^{\frac{RS}{L^2}-})\right)\\
&- \quad G_{\alpha+1,\beta}(0)R\left(\Phi(d^{\frac{L}{S}+}) - \Phi(d^{\frac{R}{S}+})\right)\\
&+ \quad G_{\alpha+1,\beta}(0)\frac{SR}{L}\left(\Phi(d^{\frac{S}{L}+}) - \Phi(d^{\frac{RS}{L^2}+})\right)
\end{aligned}
$$

where

$$
d^{\frac{x}{y}\pm} = \frac{\ln\left(\frac{x}{y}\right) \pm \frac{\lambda^2 T}{2}}{\lambda\sqrt{T}}.
$$

Notice that as $L \longrightarrow \infty$

$$
\mathbf{UOBS}(0) \longrightarrow G_{\alpha+1,\beta}(0)S_{\alpha,\beta}(0)\Phi(d^{\frac{S}{R}+}) - G_{\alpha+1,\beta}R\Phi(d^{\frac{S}{R}-})
$$

which is the value of a swaption.

The delta for the up and out contract can be calculated as

$$
\begin{aligned}
\frac{\partial V^G(t)}{\partial\left(\frac{\mathbf{Swap}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}\right)} &= \frac{\partial\frac{\mathbf{UOBS}(t)}{G_{\alpha+1,\beta}(t)}}{\frac{\mathbf{Swap}_{\alpha,\beta}(t)}{G_{\alpha+1,\beta}(t)}}\\
&= \Phi(d^{\frac{L}{S}-}) - \Phi(d^{\frac{R}{S}-})\\
&- \frac{\phi(d^{\frac{L}{S}-})}{\lambda\sqrt{T_\alpha}} - \frac{L}{S}\frac{\phi(d^{\frac{S}{L}-})}{\lambda\sqrt{T_\alpha}}\\
&+ \frac{R}{L}\Phi(d^{\frac{S}{L}+}) - \frac{R}{L}\Phi(d^{\frac{RS}{L^2}+})\\
&+ \frac{R}{S\lambda\sqrt{T_\alpha}}\phi(d^{\frac{L}{S}+}) + \frac{R}{L\lambda\sqrt{T_\alpha}}\phi(d^{\frac{S}{L}+}).
\end{aligned}
$$

The formula for the price of the up and out barrier swaption is based on the assumption that the forward swap rate follows a lognormal distribution. It will therefore not be the same as the price obtained from the LIBOR Market Model. Section 6.2 will introduce the method by which LIBOR Market Model prices are calculated.

6.2 Monte Carlo pricing of barrier swaptions in the LIBOR Market Model framework

Monte Carlo simulation is a relatively simple (yet computationally intense) way to approximate expectations (and therefore calculate derivative prices). This section will outline the main considerations in using Monte Carlo simulation in conjunction with the LIBOR Market Model in order to calculate interest rate derivative prices. Glasserman [2003] provides an in depth study of the application of Monte Carlo methods in finance.

6.2.1 Basic principles of Monte Carlo simulation

As was shown earlier, the value of the up and out swaption can be expressed as

$$\begin{aligned}
\mathbf{UOBS}(t) &= G_{\alpha+1,\beta}(t)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[(S_{\alpha,\beta}(T)-R)^{+} I_{\{S^{\star}_{\alpha,\beta}(T)<L\}}|\mathcal{F}_t\right] \\
&= G_{\alpha+1,\beta}(t)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[\left(\sum_{i=\alpha+1}^{\beta}\omega_i(t)L(t,T_{i-1})-R\right)^{+} I_{\{S^{\star}_{\alpha,\beta}(T)<L\}}|\mathcal{F}_t\right] \\
&= G_{\alpha+1,\beta}(t)\mathbb{E}^{\mathbb{Q}_{\alpha+1,\beta}}\left[X_t|\mathcal{F}_t\right].
\end{aligned}$$

By the central limit theorem, it is known that

$$\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

where $\xrightarrow{d}$ denotes convergence in distribution and $\bar{X}_n = \sum_{i=1}^{n} X_i/n$ where X_i is a realisation (independent of X_j, $j \neq i$) of the random variable X. If

$$s_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}$$

(which is a consistent estimator of σ) then

$$\bar{X}_n \xrightarrow{d} N(\mu, \frac{s_n}{\sqrt{n}})$$

which implies that for every estimate of X, a (random) confidence interval (an interval which should contain the true parameter a certain percentage of times over a number of times in which the confidence interval is recalculated) can be calculated. Thus, for large n, $\mu \approx \bar{X}$ and

$$\left(\bar{X}-\gamma_{1-\alpha}\frac{s_n}{\sqrt{n}}, \bar{X}+\gamma_{\alpha}\frac{s_n}{\sqrt{n}}\right)$$

is an α percentage confidence interval for X where $\gamma_\alpha = \Phi^{-1}(\alpha)$. The interest rate derivative can therefore be valued using Monte Carlo simulation which involves generating many realisations (with increasing accuracy as the number of realisations increases) of the payoff/numeraire ratio and then calculating an average.

6.2.2 Simulation and discretization of the forward LIBOR rates and swap rates in the LIBOR Market Model

When using the LIBOR Market Model to value a path dependent interest rate derivative which depends on the swap rate, $S_{\alpha,\beta}(t)$, it is necessary to propagate the forward rate vector

$$\mathbf{L}(t) = \begin{pmatrix} L(t,T_\alpha) \\ L(t,T_{\alpha+1}) \\ \vdots \\ L(t,T_\beta) \end{pmatrix}$$

over $\mathcal{T} = \{T_{-1}, T_0, T_1, ..., T_{\alpha-1}, T_\alpha\}$.

Suppose that $g(i)+1$ steps are simulated between time T_{i-1} and T_i. Thus simulation takes place at $T_{i-1} = t_0^i, t_1^i, ..., t_{g(i)}^i, t_{g(i)+1}^i = T_i$ $(i = 0, ..., \alpha)$.

Under the spot measure presented in section 3.5 (Jamshidian's approach), the forward rates follow the dynamic,

$$\begin{aligned} L(t_{p+1}^i, T_k) &= L(t_p^i, T_k)\exp\left(\left(\mu(t_p^i, T_k) - \frac{\lambda(t_p^i, T_k)\cdot\lambda(t_p^i, T_k)}{2}\right)(t_{p+1}^i - t_p^i)\right) && (6.1) \\ &\times \exp\left(\sqrt{(t_{p+1}^i - t_p^i)}\lambda(t_p^i, T_k)\cdot \mathbf{Z}^{\mathbb{Q}^S}\right) && (6.2) \end{aligned}$$

where

$$\mu(t_p^i, T_k) = \sum_{s=i}^{k} \frac{\delta_{s+1} L(t_p^i, T_k)}{1 + \delta_{s+1} L(t_p^i, T_k)} \lambda(t_p^i, T_k)\cdot\lambda(t_p^i, T_s)$$

and where $\mathbf{Z}^{\mathbb{Q}^{T_i}}$ is a standard normal d-dimensional vector. More refined discretization schemes are possible and are described in Hunter et al. [2001] (the Predictor Corrector method) and Glasserman and Zhao [2000].

In order to calculate the payoff of a barrier swaption, it is necessary to simulate the swap rate. The swap rate at any time $t \in (T_{-1}, T_\alpha)$ is

$$\begin{aligned} S_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} \omega_i(t) L(t, T_{i-1}) \\ &= \sum_{i=\alpha+1}^{\beta} \frac{\delta_i P(t, T_i) L(t, T_{i-1})}{\sum_{j=\alpha+1}^{\beta} \delta_j P(t, T_j)} \end{aligned}$$

which implies that the zero coupon bond values are required to determine the swap rate. The zero coupon bond values can be calculated from the vector of forward rates at dates that are in $\mathcal{T}$. Calculating the values of the zero coupon bonds in between tenor dates requires approximation.

We will not attempt to calculate the zero coupon bond values at dates that are in $\mathcal{T}$. Instead we note that most of the variability in the swap rate is due to the variability of the $L(t, T_{i-1})$'s rather than variability of the $\omega_i(t)$'s[1]. Thus, to ensure continuity in the simulated forward starting swap rate, we propose that the $\omega_i(t)$'s have the following form at $t \in (T_{p-1}, T_p)$:

$$\omega_i(t) = \omega_i(T_{p-1}) + \frac{(t - T_{p-1})}{(T_p - T_{p-1})}(\omega_i(T_p) - \omega_i(T_{p-1})). \qquad (6.3)$$

Thus, simulate the vector of LIBOR forward rates using the procedure outlined in 6.1. Then use equation 6.3 to calculate the corresponding simulation of the forward starting swap rate.

[1]Brigo and Mercurio [2006] mentions this in the treatment of Rebonato's formula (proposition 4.2.1) for the volatility of the swap rate.

6.3 Comparison of the LIBOR Market Model price and Swap Market Model prices for barrier swaptions

This section will price an up and out barrier option with the following parameters:

- Strike rate = 8%.
- Time to maturity is one year.
- The underlying is the 1 into 3 year forward starting swap rate (i.e. the 3 year swap starting in one year's time).
- The 'out' barrier will vary.

The forward rates used as pricing inputs appear in table B.3 and the instantaneous volatility structure used in the simulation appears in table B.2. The exogenous correlation matrix entries are taken to be $\rho_{i,j} = \exp(-0.1\,|T_i - T_j|)$ (where T_i and T_j are quarterly dates). The Monte Carlo price (the LIBOR Market Model price) was calculated using a two factor model where the correlation decomposition was accomplished using Rebonato's angle parameterisation (see section 4.4.4) in order to make simulation efficient.

Figure 6.1 shows the prices for a range of barrier levels of an up and out payer swaption. Each Monte Carlo price was produced by 1000 Monte Carlo simulations and each simulation consisted of 24 simulated points (equally spaced over one year). For each price the green band denotes the associated 95 percent confidence interval.

There are a number of areas which may lead to discrepancies arising between the Monte Carlo and the closed form prices where these discrepancies are due to *bias* rather than to *error*[2]. These can be summarised as:

- Imperfect, non-arbitrage free discretization of the LIBOR forward rate stochastic differential equation when doing the Monte Carlo path simulation.
- Discretization of continuous dynamics implies that some instances in which the barriers may have been crossed (had the continuous case been considered) are not observed in the case of a discrete simulation of the forward rate. This puts an upward bias on the Monte Carlo Price compared to the closed form price.
- The closed form price assumes that the forward swap rate is lognormal whereas the Monte Carlo method assumes that each individual forward rate is lognormal (a mutually incompatible situation).
- The error introduced by Rebonato's volatility approximation (proposition 4.2.1) is compounded by the fact that the forward swap rate volatility is assumed to be constant whereas the forward rate volatilities are not.

[2]If there is no bias there will still be error due to random variation of the Monte Carlo price. Bias is a systematic under or over pricing of the Monte Carlo price compared to the closed form price.

- The closed form method takes the real correlations as inputs whereas a reduced factor LIBOR Market Model will make use of approximate correlations due to the decomposition and approximation of the correlations matrix.
- The linear approximation (equation 6.3) used to calculate the weight values of the swap rates in between tenor dates may introduce bias.

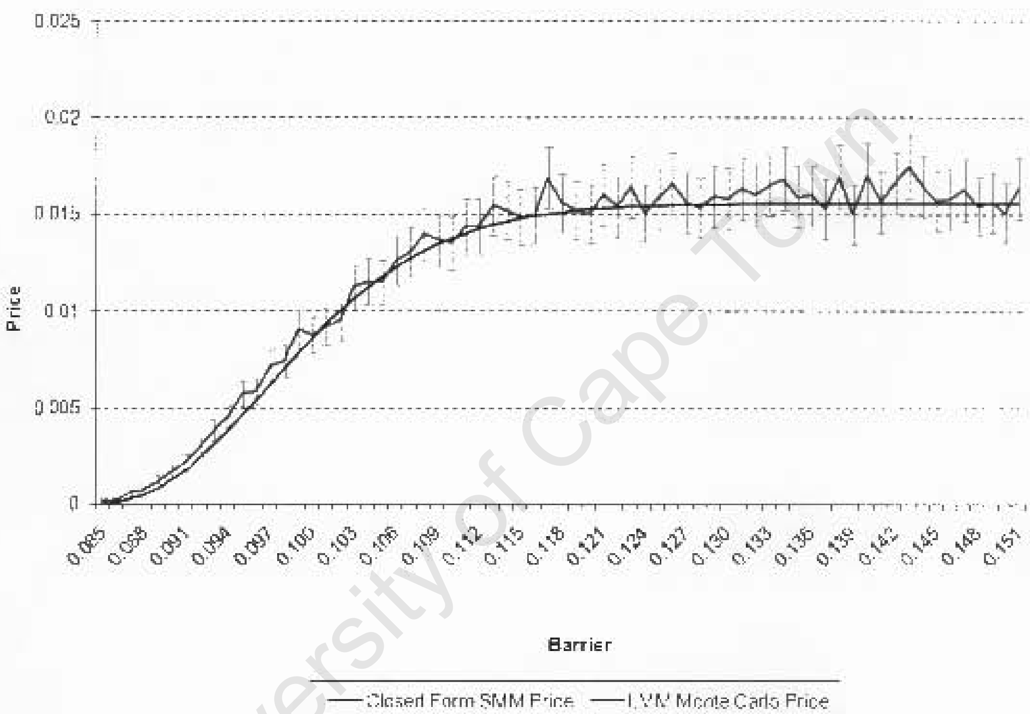

Figure 6.1: Monte Carlo LIBOR Market Model barrier swaption prices versus closed form Swap Market Model barrier swaption prices.

If there is no discrepancy between the two pricing approaches then it would be expected that on average (in the spirit of the frequentist school of thought) for every 100 Monte Carlo prices, 5 of the true option prices would lie outside of the associated 95 percent confidence interval. Figure 6.1 shows that of the 67 Monte Carlo prices calculated, 15 of the closed form prices lie below the lower bound of the 95 percent confidence intervals. 11 of these cases happen at the very low barrier levels. This seems to indicate that Monte Carlo approach to pricing has an upward bias when compared to the closed form price. Nevertheless, despite this upward bias the Monte Carlo price does seem to track the closed form price quite closely in figure 6.1.

Chapter 7

Conclusion

This dissertation has given an overview of the application of the LIBOR Market Model in the South African setting.

It was noted that a number of approximations and assumptions have to be made in order to apply the LIBOR Market Model. Notably, the assumption had to be made that the South African swap day schedule applies to South African Forward Rate Agreements in order to represent a cap as the sum of caplets and to represent a swap rate as a weighted sum of forward rates (two approximations that are critical in the successful application of the LIBOR Market Model in the South African setting).

It was stated that the LIBOR Market Model can be calibrated through either optimisation or through Cascade Calibration. Cascade Calibration was deemed unsuitable in the South African setting as is was concluded that the South African swaption volatility matrix is too sparsely populated in order to have confidence that the algorithm would run to completion. The parametric methods which rely on calibration through optimisation were deemed the most suitable.

In order to examine the suitability of each of the parametric calibration methods in greater depth, a number of interest rate options were hedged under each of the four examined methods. It was suggested that better calibration methods would result in lower hedging errors. The results leaned in favour of the parameter rich calibration approaches rather than the parsimonious approaches.

The last chapter looked at an application of the LIBOR Market Model. It was pointed out that to apply the LIBOR Market Model, it is necessary to perform Monte Carlo simulations. A Swap Market Model approximation to a LIBOR Market Model up and out payer swaption price was derived and it was concluded that the Swap Market Model Price offered a fairly good approximation.

There have been a number of factors that have stalled the implementation of the LIBOR Market Model in South Africa. The two main factors have been the lack of a liquid market for caps and swaptions and the lack of tradeable forward starting swaps. The first factor implies that calibration of the LIBOR Market Model is difficult whereas the second factor implies that there is a lack of instruments with which to hedge exotic interest rate products. This dissertation has shown that, given the hedging instruments, it becomes possible to effectively (to varying degrees) hedge optionality on swaps and forward rates. With the development of the interest rate market (and the corresponding increase in liquidity of instruments that can be used to construct hedges) the applicability of the LIBOR Market Model to the South African interest rate market also increases.

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Appendix A

No arbitrage pricing

This section aims to highlight the most important points in no arbitrage pricing. More detailed approaches can be found in Bingham and Kiesel [2004] and Hunt and Kennedy [2005] amongst others.

A.1 The General Theory

Definition A.1.1 *The Value of a portfolio is $V(t)$ where*

$$V(t) = \sum_{i=0}^{m} P^i(t)\varphi_i(t).$$

$P^i(t)$ is the value of a general asset in the market and $\varphi_i(t)$ is the quantity held at time t.

Definition A.1.2 *The trading strategy $(\varphi_i(t))_{i=0,\ldots,m}$ is said to be self financing if*

$$V(t) = V(0) + \sum_{i=0}^{m} \int_0^t \varphi_i(u)dP^i(u).$$

Remark A.1.3 *Saying that the trading strategy is self financing is equivalent to saying that all changes in the value of the portfolio are due to changes in the value of the hedging assets and not because of cash injections into or cash extractions from the portfolio.*

Definition A.1.4 *A contingent claim is said to be attainable if there exists a self financing trading strategy such that the value of the self financing portfolio is equal to the value of the claim at expiry.*

Definition A.1.5 *A numeraire $N(t)$ is a strictly positive price process (of a tradeable asset) over the interval $t \in [0, T]$.*

Proposition A.1.6 *Self financing portfolios remain self financing under different numeraires.*

Proof
Consider the infinitesimal change in the portfolio $V(t)$ under the numeraire $N^2(t)$ and suppose that

$$d\frac{V(t)}{N^1(t)} = \sum_{i=0}^{m} \varphi_i(u) d\frac{P^i(u)}{N^1(t)}$$

then, using Ito's Product rule

$$\begin{aligned}
d\left(\frac{V(t)}{N^2(t)}\right) &= d\left(\frac{V(t)}{N^1(t)}\frac{N^1(t)}{N^2(t)}\right) \\
&= \frac{N^1(t)}{N^2(t)} d\frac{V(t)}{N^1(t)} + \frac{V(t)}{N^1(t)} d\frac{N^1(t)}{N^2(t)} + d\frac{V(t)}{N^1(t)} d\frac{N^1(t)}{N^2(t)} \\
&= \sum_{i=0}^{m} \varphi_i(t) \left(\frac{N^1(t)}{N^2(t)} d\frac{P^i(t)}{N^1(t)} + \frac{P^i(t)}{N^1(t)} d\frac{N^1(t)}{N^2(t)} + d\frac{N^1(t)}{N^2(t)} d\frac{P^i(t)}{N^1(t)}\right) \\
&= \sum_{i=0}^{m} \varphi_i(t) d\left(\frac{N^1(t)}{N^2(t)}\frac{P^i(t)}{N^1(t)}\right) \\
&= \sum_{i=0}^{m} \varphi_i(t) d\left(\frac{P^i(t)}{N^2(t)}\right).
\end{aligned}$$

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Corollary A.1.7 *If a contingent claim is attainable in one numeraire then it is also attainable in any other numeraire and the replicating strategy remains the same.*

Proof
The corollary follows immediately from the previous proposition.

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Definition A.1.8 *The portfolio $V(t)$ is an arbitrage portfolio and the corresponding trading strategy $(\varphi_i(t))_{i=0,\ldots,m}$ is said to be an arbitrage strategy if:*

1. $V(0) = 0$.
2. $\mathbb{P}(V(T) \geq 0) = 1$.
3. $\mathbb{P}(V(T) > 0) > 0$.

Definition A.1.9 *The measure $\mathbb{Q}$ is said to be an Equivalent Martingale Measure (EMM) if:*

1. *$\mathbb{Q}$ is equivalent to $\mathbb{P}$.*
2. *The asset/numeraire ratios $(P^i(t)/N(t))$ are $\mathbb{P}-$local martingales.*

Theorem A.1.10 *The market model does not admit arbitrage if and only if there is an Equivalent Martingale Measure.*

Remark A.1.11 *This is a fundamental theorem of asset pricing. Proving that the existence of equivalent martingale measures imply no arbitrage in the model is straight forward. However, proving the converse requires the No Free Lunch With Vanishing Risk (NFLVR) condition. Bingham and Kiesel [2004] (p 234) notes a number of complications and details.*

Definition A.1.12 *The market model is said to be complete if every contingent claim is attainable through a unique self financing trading strategy.*

Theorem A.1.13 *A market is said to be arbitrage free and complete if and only if there exists a unique equivalent martingale measure.*

Theorem A.1.14 *In an arbitrage free and complete market the unique (time t) price of a contingent claim, X, at time T is*

$$X(t) = N(t)\mathbb{E}^{\mathbb{Q}^N}\left[\frac{X(T)}{N(T)} \mid \mathcal{F}_t\right]$$

where $N(t)$ is a numeraire with associated equivalent martingale measure $\mathbb{Q}^N$.

Proof
The market is complete so it is possible to set up a portfolio, $V(t)$, driven by a self financing trading strategy, $\varphi_i(t)$, that will replicate the contingent claim $X(T)$ at time T. If we work under the numeraire $N(t)$ with associated equivalent martingale measure $\mathbb{Q}^N$, then

$$\frac{X(T)}{N(T)} = \frac{V(T)}{N(T)} = \frac{V(t)}{N(t)} + \sum_{i=0}^{m}\int_t^T \varphi_i(u)d\frac{P^i(u)}{N(u)}$$

which is guaranteed by the completeness of the market. Now take expectations, under the equivalent martingale measure, on both sides and (skirting over some technicalities)

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^N}\left[\frac{X(T)}{N(T)} \mid \mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}^N}\left[\frac{V(t)}{N(t)} + \sum_{i=0}^{m}\int_t^T \varphi_i(u)d\frac{P^i(u)}{N(u)} \mid \mathcal{F}_t\right] \\
&= \frac{V(t)}{N(t)} + \sum_{i=0}^{m}\int_t^T \varphi_i(u)d\mathbb{E}^{\mathbb{Q}^N}\left[\frac{P^i(u)}{N(u)} \mid \mathcal{F}_t\right] \\
&= \frac{V(t)}{N(t)} + \sum_{i=0}^{m}\int_t^T \varphi_i(u)d\frac{P^i(t)}{N(t)} \\
&= \frac{V(t)}{N(t)} + 0 \\
&= \frac{V(t)}{N(t)}.
\end{aligned}$$

Since $V(t)$ replicates the portfolio value at time T, it must be that $V(t) = X(t)$ otherwise there would be arbitrage. Therefore

$$X(t) = N(t)\mathbb{E}^{\mathbb{Q}^N}\left[\frac{X(T)}{N(T)} \mid \mathcal{F}_t\right].$$

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A.2 The Black Scholes model

The Black Scholes model *is essentially* the geometric Brownian motion model for asset prices. When risk neutral valuation is applied to this model, then closed form solutions are obtained for option prices. Thus, assume that there are $m+1$ assets and that they live in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the associated filtration $\mathcal{F}_t$. The market dynamics is modelled as

$$
\begin{aligned}
d\begin{pmatrix} P^0(t) \\ \vdots \\ P^m(t) \end{pmatrix} &= \begin{pmatrix} P^0(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & P^m(t) \end{pmatrix} \begin{pmatrix} \mu(t,0) \\ \vdots \\ \mu(t,m) \end{pmatrix} dt \\
&+ \begin{pmatrix} P^0(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & P^m(t) \end{pmatrix} \begin{pmatrix} \lambda_{01}(t) & \dots & \lambda_{0d}(t) \\ \vdots & \ddots & \vdots \\ \lambda_{N1}(t) & \dots & \lambda_{Nd}(t) \end{pmatrix} \begin{pmatrix} dW_t^1 \\ \vdots \\ dW_t^d \end{pmatrix}.
\end{aligned}
$$

The aim of the rest of this section is to show that theorem A.1.14 can be used to price a derivative given this model of a market. Thus, it will be shown that these market dynamics imply an arbitrage free and complete market. Consider the asset dynamics with respect to the bank account as numeraire such that

$$P_B^i(t) = \frac{P^i(t)}{B_t} = \frac{P^i(t)}{e^{\int_0^t r_u du}}$$

where r_u is the instantaneous short rate. By Theorem A.1.10, we know that the market model is arbitrage free if we can find a measure under which the asset ratios are martingales. An application of Ito's Lemma yields

$$
\begin{aligned}
d\begin{pmatrix} P_B^0(t) \\ \vdots \\ P_B^m(t) \end{pmatrix} &= \begin{pmatrix} P_B^0(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & P_B^m(t) \end{pmatrix} \begin{pmatrix} \mu(t,0) - r_t \\ \vdots \\ \mu(t,m) - r_t \end{pmatrix} dt \\
&+ \begin{pmatrix} P_B^0(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & P_B^m(t) \end{pmatrix} \begin{pmatrix} \lambda_{01}(t) & \dots & \lambda_{0d}(t) \\ \vdots & \ddots & \vdots \\ \lambda_{N1}(t) & \dots & \lambda_{Nd}(t) \end{pmatrix} \begin{pmatrix} dW_t^1 \\ \vdots \\ dW_t^d \end{pmatrix}
\end{aligned}
$$

which is the dynamics of the asset ratios under $\mathbb{P}$. The first of two crucial theorems is now stated.

Theorem A.2.1 (Girsanov's theorem) *Let $\mathbf{W}_t$ be a d-dimensional $\mathbb{P}$-Brownian Motion and $Y_t(\omega)$ be an $m+1$-dimensional stochastic process with respect to the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ $(\omega \in \Omega)$ with associated filtration $\mathcal{F}_t$ and*

$$dY_t(\omega) = \mu(t,\omega)dt + \sigma(t,\omega)d\mathbf{W}_t$$

where μ is an $m+1$-dimensional vector and σ is an $(m+1) \times d$ dimensional matrix. Suppose that there exists a d-dimensional vector process $\gamma(t,\omega)$ such that

$$\sigma\gamma = \theta - \mu.$$

Assuming sufficient integrability ($\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \|\gamma_u\|^2 du}\right] < \infty$), then there exists a measure $\mathbb{Q}$ such that:

1. $\mathbb{Q}$ *is equivalent to* $\mathbb{P}$.

2. $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \gamma \cdot d\mathbf{W}_t^{\mathbb{P}}\right) = e^{\int_0^T \gamma \cdot d\mathbf{W}_t^{\mathbb{P}} - \frac{1}{2}\int_0^T \|\gamma\|^2 \cdot dt}$.

3. $W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \int_0^t \gamma_u du$.

$W_t^{\mathbb{Q}}$ *is a* $\mathbb{Q}$*-Brownian motion and the dynamics of* Y_t *under* $\mathbb{Q}$ *is*

$$dY_t(\omega) = \theta(t,\omega)dt + \sigma(t,\omega)dW_t^{\mathbb{Q}}.$$

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Now, if we can find a value for the vector, γ_t, such that $\lambda_t \gamma_t = -(\underline{\mu} - \underline{r})$ then we can define a new measure $\mathbb{Q}$ such that

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} &= \mathcal{E}\left(\int_0^T \gamma \cdot d\mathbf{W}_t^{\mathbb{P}}\right) \\ &= e^{\int_0^T \gamma \cdot d\mathbf{W}_t^{\mathbb{P}} - \frac{1}{2}\int_0^T \|\gamma\|^2 \cdot dt} \end{aligned}$$

and Girsanov's Theorem will ensure (subject to the technical condition that $\mathbb{E}^{\mathbb{Q}}[e^{\frac{1}{2}\int_0^T \|\gamma_t\|^2 dt}] < \infty$) that

$$\begin{aligned} dW_t^{\mathbb{Q}} &= dW_t^{\mathbb{P}} - \gamma_t dt \\ &\text{and} \\ \mathbb{Q} &\text{ is equivalent to } \mathbb{P} \end{aligned}$$

which results in

$$d\underline{P} = D[P][\underline{r}dt + \underline{\lambda} d\mathbf{W}_t^{\mathbb{Q}}]$$

and which therefore implies that pricing under the $\mathbb{Q}$ expectation leads to arbitrage free prices since the asset ratios are then martingales under $\mathbb{Q}$. Thus Girsanov's theorem has allowed us to show that the model is arbitrage free. We now show that the model is also complete and to do so the second of the two crucial theorems is stated.

Theorem A.2.2 (Martingale Representation Theorem) *Suppose that* M_t *is an* $(F_t)_t$*-martingale, where* $(F_t)_t$ *is an augmented Filtration generated by a d-dimensional Brownian motion* $(W_t)_t$. *If* $E(M_T)^2 < \infty$ *for some* $T > 0$, *then there is a unique d-dimensional predictable process* ϕ *such that*

$$M_t = M_0 + \int_0^t \phi_u \cdot dW_u.$$

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Thus, by the principle of risk neutral valuation, it is known that an arbitrage free price for a derivative with payoff X_T at time T is

$$X(t) = B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{X(T)}{B(T)} \mid \mathcal{F}_t\right].$$

The martingale representation theorem is used to show that this price is arbitrage free *and also* unique. By the tower property of conditional expectation $M_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(T)}{B(T)} \mid \mathcal{F}_t\right]$ is a martingale under $\mathbb{Q}$. Since P_u^B is also a martingale under $\mathbb{Q}$, by the theorem A.2.2, we have that

$$\begin{aligned} M_T &= M_0 + \int_0^T \phi_u \cdot dW_u^{\mathbb{Q}} \\ &= M_0 + \int_0^T H_u \cdot dP_u^B \end{aligned}$$

where the $m+1$ dimensional process, H_u, is

$$H_u = \phi_t \underline{\lambda}^{-1} D[P^B]^{-1}$$

and where $\underline{\lambda}^{-1}\underline{\lambda}$ is a $d \times d$ identity matrix and $D[P^B]^{-1}D[P^B]$ is the $(m+1) \times (m+1)$ identity matrix. Thus, by setting up a portfolio with value M_0 at time 0 and then following the self financing trading strategy H_u the above equation shows that any contingent claim can be replicated. Thus, the Martingale representation theorem has shown that the market is complete and Girsanov's theorem has shown that the market is arbitrage free. Thus, Theorem A.1.14 can be used to find the unique arbitrage free price - the Black Scholes Price.

Appendix B

Data

Rate	Value
$L(0, T_0)$	9.2080%
$L(0, T_1)$	9.2900%
$L(0, T_2)$	9.2267%
$L(0, T_3)$	9.1433%
$L(0, T_4)$	9.0093%
$L(0, T_5)$	8.8610%
$L(0, T_6)$	8.7143%
$L(0, T_7)$	8.5774%
$L(0, T_8)$	8.4339%
$L(0, T_9)$	8.2967%
$L(0, T_{10})$	8.2022%
$L(0, T_{11})$	8.1475%
$L(0, T_{12})$	8.0828%
$L(0, T_{13})$	8.0031%
$L(0, T_{14})$	7.9661%
$L(0, T_{15})$	7.9601%
$L(0, T_{16})$	7.9537%
$L(0, T_{17})$	7.9357%
$L(0, T_{18})$	7.9111%
$L(0, T_{19})$	7.8796%
$L(0, T_{20})$	7.8479%
$L(0, T_{21})$	7.8178%
$L(0, T_{22})$	7.7945%
$L(0, T_{23})$	7.7696%

Table B.1: Forward rates on 2 May 2007. Note that $T_i = \text{modfol}(2 \text{ May } 2007, 3 \times i)$ and that the forward rates are determined as one period swaps (i.e. we make assumption 2.1.5 so that we apply the swap day schedule to FRAs). The forward rates were calculated from the HMC curve obtained from the BEASSA stripped data provided by Old Mutual's Asset Liability Management Unit.

	Current Time				
	$(0,T_0]$	$(T_0,T_1]$	$(T_1,T_2]$	$(T_2,T_3]$	$(T_3,T_4]$
$L(0,T_0)$	7.447206%				
$L(0,T_1)$	8.140776%	8.791148%			
$L(0,T_2)$	8.821318%	10.312528%	8.515465%		
$L(0,T_3)$	9.527795%	11.800108%	12.298509%	10.048137%	
$L(0,T_4)$	10.256624%	13.321437%	11.465785%	12.402412%	7.333105%
$L(0,T_5)$	11.010684%	10.415745%	16.781752%	12.555140%	14.845424%
$L(0,T_6)$	11.783844%	10.125770%	13.492516%	18.983138%	10.554683%
$L(0,T_7)$	12.570387%	9.785432%	13.778097%	15.361135%	17.892862%
$L(0,T_8)$	11.374552%	11.374552%	14.082291%	15.713357%	14.079068%
$L(0,T_9)$	12.645418%	12.645418%	12.645418%	16.101826%	14.272237%
$L(0,T_{10})$	13.911649%	13.911649%	13.911649%	13.911649%	14.539116%
$L(0,T_{11})$	14.294580%	14.294580%	14.294580%	14.294580%	14.294580%
$L(0,T_{12})$	14.692831%	14.692831%	14.692831%	14.692831%	14.692831%
$L(0,T_{13})$	15.086957%	15.086957%	15.086957%	15.086957%	15.086957%
$L(0,T_{14})$	15.490356%	15.490356%	15.490356%	15.490356%	15.490356%

Table B.2: The instantaneous volatility structure used in producing figure 6.1.

Rate	Value
$L(0,T_0)$	9.208000%
$L(0,T_1)$	9.301720%
$L(0,T_2)$	9.227419%
$L(0,T_3)$	9.164880%
$L(0,T_4)$	9.028159%
$L(0,T_5)$	8.887833%
$L(0,T_6)$	8.734237%
$L(0,T_7)$	8.601180%
$L(0,T_8)$	8.501896%
$L(0,T_9)$	8.397450%
$L(0,T_{10})$	8.295789%
$L(0,T_{11})$	8.196799%
$L(0,T_{12})$	8.169889%
$L(0,T_{13})$	8.083645%
$L(0,T_{14})$	7.999271%
$L(0,T_{15})$	7.916705%
$L(0,T_{16})$	8.004790%
$L(0,T_{17})$	7.938621%
$L(0,T_{18})$	7.873243%
$L(0,T_{19})$	7.808655%
$L(0,T_{20})$	7.889152%
$L(0,T_{21})$	7.834101%
$L(0,T_{22})$	7.779278%
$L(0,T_{23})$	7.724709%

Table B.3: The forward rates used in producing figure 6.1.