Enlargement of Filtration, Backward Stochastic Differential Equations and Optimal Stopping Problems

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Declaration

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Abstract

This thesis focuses on the application of the enlargement of filtration to backward stochastic differential equations (BSDEs) and optimal stopping problems. In particular, the thesis develops the theory of the progressive enlargement of filtration with multiple random times and their associated marks. Several extensions of the classical progressive enlargement of filtration are derived, including a semimartingale decomposition theorem and a martingale representation theorem. The extensions then allow for the study of BSDEs and optimal stopping problems in an enlarged filtration. BSDEs are a very useful tool in stochastic optimal control and mathematical finance, the usefulness in the latter being that the solutions provide simultaneous calculation of derivative prices and their corresponding hedging strategies. Enlargement of filtration has a very intuitive application to BSDEs in a financial context, it models the effect that additional information has on the valuation of derivatives and their hedging strategies. This thesis develops certain classical results on BSDEs in the context of enlargement of filtration. The thesis then progresses to studying the effect of additional information on the value process of an optimal stopping problem. This again has an intuitive application to finance, as the effect of valuing American contingent claims in the presence of additional information. A very useful decomposition of the Snell envelope is derived. The thesis is rounded out with several applications of certain key results to topical fields in mathematical finance such as utility optimisation, risk metrics and Snell envelopes.

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Contents

Notation

Unless otherwise stated, throughout the thesis, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of right-continuity and $\mathbb{P}$ - completeness. The following list of notation will be assumed throughout the thesis:

- Throughout this thesis, $T > 0$ will be a fixed finite time and $d \in \mathbb{N}$.
- $\mathcal{N}(\mu, \sigma^2)$ is the Gaussian distribution with mean μ and variance σ^2.
- For an arbitrary filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ and a sigma-algebra $\mathcal{A}$, $\mathbb{H} \otimes \mathcal{A}$ is understood as follows,
$$\mathbb{H} \otimes \mathcal{A} = (\mathcal{H}_t \otimes \mathcal{A})_{t\geq 0}\,.$$
- For an arbitrary filtration $\mathbb{H}$, $\mathcal{O}(\mathbb{H})$ (resp. $\mathcal{P}(\mathbb{H})$) is the optional sigma-algebra (resp. predictable sigma-algebra) on $\mathbb{H}$.
- For an arbitrary set $A \subseteq \mathbb{R}^d$, $\mathcal{B}(A)$ denotes the Borel sigma-algebra on A.
- For an arbitrary set $A \subseteq \mathbb{R}^d$, $B(A)$ denotes the set of all real Borel functions on A.
- For an arbitrary sigma-algebra $\mathcal{A}$ and $p \geq 1$,
$$L^p(\mathcal{A}) := \{X\text{: } \mathcal{A}\text{- measurable such that } \mathbb{E}[|X|^p]^{\frac{1}{p}} < \infty\}.$$
- Any statement made about a process or random variable will be understood in the almost sure sense unless the context suggests otherwise.
- The term càdlàg, from the French *continue à droite, limite à gauche*, will refer to a process or function that is right continuous with left limits up to an evanescent set.
- For two vectors x and y in $\mathbb{R}^d$, xy is understood as the dot product $x^T y$.
- For a stochastic process $X : (\omega, t) \mapsto X_t(\omega)$, X will denote the entire process $(X_t)_{t\geq 0}$ while X_t will denote the random variable X_t at some $t \geq 0$. Furthermore, denote the left continuous version of the process X as X_- where, $Xt- := \lim_{s\uparrow t} X_s$ with the convention that $X_{0-} := X_0$. The jump process of X, denoted ΔX can then be defined as $\Delta X := X - X_-$.
- For two stopping times τ and ν, the random set, denoted $[\![\tau, \nu]\!]$ is defined as follows,
$$[\![\tau, \nu]\!] := \{t > 0, w \in \Omega : \tau(\omega) \leq t \leq \nu(\omega)\}.$$
Similarly for the definitions of $(\!(\tau, \nu]\!]$, $[\![\tau, \nu)\!)$ and $(\!(\tau, \nu)\!)$.

- In accordance with the definition 1.15 in Aksamit and Jeanblanc [2017], for two càdlàg local-martingales X and Y, the quadratic covariation process $[X, Y]$ is the unique càdlàg, finite variation process such that $XY - [X, Y]$ is a local martingale. Similarly, the predictable quadratic covariation process $\langle X, Y \rangle$ is, provided it exists, the unique predictable finite variation process such that $XY - \langle X, Y \rangle$ is a local martingale. Furthermore, the jumps of the quadratic covariation are $\Delta[X, Y] = \Delta X \Delta Y$.

- The definition of the quadratic covariation process is extended to the class of semimartingales as follows: For semimartingales X and Y, the quadratic covariation of X and Y is defined as

$$[X, Y] = XY - \int_0^{\cdot} X_{t-} dY_t - \int_0^{\cdot} Y_{t-} dX_t.$$

Chapter 1

Introduction

1.1 Overview

This thesis extends the current, well-published theory of enlargement of filtration with the aim of applying the newly established foundation to two independent fields of mathematics: backward stochastic differential equations and optimal stopping problems. The importance of this work, as well as a clearer definition of the scope, is expanded on in the following paragraphs. While this thesis is purely mathematical in nature, its application and the intuition behind the results can be interpreted in a mathematical finance context. Central to the study of financial markets is the concept of information. More specifically, this thesis focuses on two topics in the context of financial market information: potential asymmetries between two parties to a financial agreement and secondly the amount of financial information that is necessary to act as a rational market participant. Mathematically, these two topics can be modelled using the enlargement of filtration. A filtration is a mathematical construct that models the sequential increase in information through time. For example, financial information arriving in the form of news and stock market prices would be ”added” to the ”market” filtration. This information is freely available to everyone in the market and therefore it can be assumed that all market participants have at least this much information available to them to guide their decisions. An insider trader may have access to potentially more information and therefore a larger filtration. Filtrations can also be generated by more granular forms of financial market data, for example, a filtration can be generated by a particular share price. Filtrations also model the necessity of information, for example, the probability of one option expiring in-the-money may or may not be dependent on whether another option expires in-the-money. This could be determined by assessing whether the filtrations generated by these two informational flows are independent of each other. Central to credit risk modelling is the notion of a credit default event which is triggered when a party to a financial agreement is unable meet their payment obligations. The time at which the credit default event occurs is called a default time. Additional information that may arise at the time of default, for example the loss given default, is captured by the default time’s associated mark. A party conducting many financial agreements is exposed to multiple credit default events. It is in their interest to understand the dependencies between these default events. This thesis aims at modelling multiple default times and their associated marks to enable these dependencies to be incorporated. The inter-dependencies between credit default events are modelled as well the dependencies between these default events and the rest of the financial market. For example, consider a bank which loans money to several different borrowers in different geographic regions and industries. In determining the amount of interest to charge on these loans, the bank would need to take into account the correlation between the default propensities of each of the individual borrowers as well as any correlations between the borrowers' default propensities and other market factors that may affect the bank's liquidity. The aim is to solve this problem by assuming a given set of market information (the market filtration) and enlarging this filtration with progressive knowledge about the default times and associated marks. By assuming a general dependency between the default times, associated marks and the reference filtration, standard quantitative finance techniques such as valuing and hedging financial products are derived.

A central theme of enlargement of filtration is the fundamental question of whether semimartingales in the reference filtration remain semimartingales in the enlarged filtration. Semimartingales play a crucial role in mathematical finance; being good integrators, they enable the valuation of contingent claims and the associated hedging strategies. To enable the enlarged filtration to be treated in a mathematical finance context, it is important that the same semimartingales in the reference filtration remain semimartingales in the enlarged filtration. For example, consider a predictable trading strategy $\phi = (\phi_t)_{0 \le t \le T}$ on a stock price process $S = (S_t)_{0 \le t \le T}$ that are both adapted to the reference filtration. We assume here that S is a semimartingale in the reference filtration. The value of the trading strategy at time T is $V_T := \int_0^T \phi_t dS_t$. In order for this quantity to have meaning in the enlarged filtration, we need (among other things) that S remains a semimartingale in the enlarged filtration. The preservation of semimartingales also has links to the existence of arbitrage opportunities; indeed, under certain conditions, the existence of a semimartingale decomposition formula in the enlarged filtration implies the existence of a risk-neutral measure in the enlarged space. The study of arbitrages and the enlargement of filtration is a well-studied topic and is given attention in the literature review.

Another central theme of this thesis is the predictable representation theorem in the enlarged filtration. A predictable representation theorem with respect to a given filtration and a set of generating martingales allows one to represent a random variable (and hence uniformly integrable martingales) as a stochastic integral of a predictable process with respect to the set of generating martingales. This is closely related to the concept of market completeness in mathematical finance. Indeed, it allows one to derive the hedging strategy and hence the price of any contingent claim in the financial market. It is therefore important to study whether a predictable representation theorem holds in the enlarged filtration if it does hold in the reference filtration. Furthermore, what is the resulting set of generating martingales?

Having introduced the financial intuition behind this research, the mathematical introduction along with a comprehensive literature review is given in the following.

Enlargement of filtration is the study of two filtrations, the reference or market filtration $\mathbb{F}$ and the enlarged filtration $\mathbb{G} := (\mathcal{G}_t)_{t \ge 0}$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \ge 0$, the fundamental question being, *under which conditions do $\mathbb{F}$-martingales remain $\mathbb{G}$-semimartingales?* This study began in the late 70's in the French school of mathematics with the seminal works of Jeulin and Yor [1978],Yor [1978], Yor [1980], Jeulin [1980], Meyer [1980] and Jacod [1985] laying the foundation for further work. Enlargement of filtration has primarily focused on two methods of enlarging the filtration $\mathbb{F}$:

– Initial enlargement of filtration: Given a random variable ξ, $\mathbb{G}$ is the smallest right-continuous filtration such that ξ is $\mathcal{G}_t$ measurable for all $t \geq 0$.

– Progressive enlargement of filtration: Given a non-negative random variable τ, $\mathbb{G}$ is the smallest right-continuous filtration such that τ is a $\mathbb{G}$ stopping time.

Due to the complexity of enlarging a filtration with an arbitrary random variable, the literature has predominantly studied various conditions to be placed on either ξ or τ such that meaningful analysis can be done in the enlarged space (see section 2.5 in Song [2015] and the references within for an overview of these conditions).

1.2 Literature Review

The following subsection presents a review of the current literature on the relevant topics in this thesis, enlargement of filtration, backward stochastic differential equations and optimal stopping problems. The aim of this subsection is to familiarise the reader with the history and current landscape of this research that has inspired this thesis.

1.2.1 Enlargement of Filtration

Enlargement of filtration has since gained a lot of attention in the literature due to its applications to a number of important topics in stochastic analysis and mathematical finance. An enlarged filtration has a straight-forward practical understanding, that being the modelling of two agents in a financial market, one of whom has only the information that the rest of the market has and one of whom has extra information. This intuition naturally raises several questions such as;

1. *By having extra information, may the insider take advantage of any arbitrage opportunities?*
2. *Given that the financial market under consideration is complete, does the insider too enjoy a complete market?*
3. *Given the ability to value contingent claims in the financial market, how may the insider value the claims differently?*
4. *How may the insiders quantify the value of their extra information?*
5. *Having derived mathematical tools for how an insider may react with extra information, can we exploit these to detect insider trading?*

These questions, among others, have gained a significant amount of attention in the literature.

Arbitrages in the insider market are dealt with in Aksamit et al. [2014], Aksamit et al. [2015],

Acciaio et al. [2016] and Aksamit et al. [2017]. Here the authors work under a specific definition of no-arbitrage (see Fontana [2015] for a detailed explanation of different no-arbitrage conditions) and consider under what conditions extra information does not yield any arbitrage opportunities. Thanks to the seminal works of Delbaen and Schachermayer [1994] and Delbaen and Schachermayer [1995], the existence of arbitrages for the insider is closely linked to the fundamental question of whether $\mathbb{F}$ martingales remain $\mathbb{G}$ semimartingales.
Given that the financial market under consideration is complete, meaning that every contingent claim is attainable and can be hedged in an arbitrage free manner, the completion of the insider market is not trivial. Indeed, the class of contingent claims under the insider's information is larger and therefore more care needs to be taken in describing their hedging strategies. This is dealt with in Grorud and Pontier [2001], here the authors show that under certain conditions on the extra information, the insider enjoys a complete market. Completion of financial markets is very closely related to martingale representation. Assuming the financial market enjoys martingale representation, Amendinger [2000], Amendinger et al. [2003], Jeanblanc and Song [2015] and Fontana [2018] study under what conditions will the enlarged filtration enjoy martingale representation. Here the authors show that, under certain assumptions and a given set of generating martingales in the reference space, the enlarged space enjoys martingale representation with respect to a possibly larger set of generating martingales.
An important tool for valuing contingent claims and simultaneous calculation of their hedging strategy in a complete market is backward stochastic differential equations (BSDEs), first considered in a mathematical finance context in El Karoui et al. [1997b]. A particular area of interest in the study of BSDEs is the conditions under which solutions exist and in which these solutions are unique. A progression in the context of enlargement of filtration is, given the existence and uniqueness of solutions in the reference space, under what conditions on the extra information will the solutions exist and remain unique in the insider space? A financial interpretation of this result would be, given that non-insiders are able to price any contingent claim and determine the appropriate hedging strategy, how may an insider do so and, given the presence of additional information, may there be more than one price of a contingent claim? Eyraud-Loisel [2005], Jeanblanc and Le Cam [2009b], Eyraud-Loisel and Royer-Carenzi [2010] and Kharroubi and Lim [2014] study the existence and uniqueness of solutions to BSDEs in the enlarged filtration. Here the authors focus on conditions on the extra information such that solutions remain existent and unique in the enlarged space.
When considering extra information in a financial context, a natural question to ask is how much an individual should be willing to pay for extra information. This equates to quantifying the value of extra information. This too is non-trivial and relies on a subjective measure of additional information. The first consideration of this problem was in Pikovsky and Karatzas [1996]; here the authors consider an optimal investment problem for an insider and a non-insider. Amendinger et al. [1998] consider the additional logarithmic utility that an insider achieves compared with a non-insider when trading a financial derivative; the authors extend their work to the case of a general utility function in Amendinger et al. [2003]. Chau et al. [2020] attempt quantifying the value of extra information by consider-

ing optimal investment and consumption strategies in the insider market. Ankirchner et al. [2006] quantify the value of extra information via entropy-related quantities, linking their results back to the case of a logarithmic utility function. Beissner and Tölle [2018] give a framework for defining the convergence of sigma-algebras; while the authors do not use this metric for the valuation of extra information, their results can be understood as metricising the value of extra information.

Having derived numerous mathematical results for the effect of insider information in a financial market, Grorud and Pontier [1998] propose a statistical test to test for the presence of insider trading.

Other miscellaneous references that the interested reader should consider are Kchia and Protter [2014], here the authors, much like in Beissner and Tölle [2018], consider the convergence of sigma-algebras; using their results they study a filtration enlarged by the natural filtration of a convergent sequence of càdlàg processes. They show that under certain assumptions on the sequence of processes, the question of whether $\mathbb{F}$-martingales remain $\mathbb{G}$-semimartingales can be answered in the affirmative. Kchia et al. [2013] give an interesting account on the link between initial and progressive enlargement of filtration; here the authors define the notion of agreement of sigma-algebras; using this they prove that the initial and progressively enlarged filtrations agree after the random default time. Föllmer and Imkeller [1993] provide an interesting example of when an insider's information is eliminated by a change of measure, however, the resulting enlarged space with new measure does not coincide with the reference space.

1.2.2 Backward Stochastic Differential Equations and Optimal Stopping Problems

Backward stochastic differential equations (BSDEs) were first introduced in the seminal paper by Pardoux and Peng [1990]. Since then, several works have continued this research, predominantly focusing on necessary conditions to ensure the existence and uniqueness of solutions to BSDEs. BSDEs provide an incredibly powerful tool used in mathematical finance and stochastic optimal control.

El Karoui et al. [1997b] provide in depth analysis of the application of BSDEs to finance. Here the authors recall standard results on the existence and uniqueness of solutions; they then analyse the use of BSDEs to valuing contingent claims, the utility of using BSDEs is that the solution not only values the claim but also simultaneously computes the hedging portfolio. The authors then show how BSDEs can be applied to stochastic optimal control problems, focusing on their applications to optimal investment strategies.

The connection between BSDEs and utility maximisation is shown in Hu et al. [2005] where the authors show how to solve a utility maximisation problem using an exponential utility function by correctly parameterising the driver of a BSDE. Of particular importance to this thesis is the effect that an enlarged filtration has on the utility maximisation problem and BSDEs. One way to do this is to consider BSDEs where the terminal time is random. This is dealt with (among others) in Eyraud-Loisel and Royer-Carenzi [2010] and Jeanblanc et al. [2015] where the authors consider BSDEs up to a random time horizon. The former proves

the existence and uniqueness of the BSDE and relates it to the pricing of American-stlye options. The latter article shows how the exponential utility maximisation problem in an enlarged filtration can be formulate using a BSDE with a random time horizon.
Dynamic risk measures have a natural connection to the solutions of a BSDE, first considered in Detlefsen and Scandolo [2005] where the authors define a subset of risk measures called "Dynamic Risk Measures" that are solutions to a BSDE provided certain constraints on the parameters of the BSDE. This then was extended to the case of a progressive enlargement of filtration in Calvia and Gianin [2020] where the author attempts to decompose the risk measure in the enlarged filtration as a sum of two risk measures in the reference filtration.
A particular stochastic optimal control problem that will be relevant in this thesis is optimal stopping problems. This can be interpreted as an agent who can receive a random reward at any particular point in time; her decision is to find the best time to exercise her reward. BSDEs have a very natural application to optimal stopping problems, first introduced by Cvitanic and Karatzas [1996] and El Karoui et al. [1997a], here the authors show that when the solutions of BSDEs are constrained by a reflecting barrier, the value process of the optimal stopping problem and the optimal exercise time can be derived from the solution to such a BSDE. In the context of enlargement of filtration, Esmaeeli and Imkeller [2018] considers how the value process of an optimal stopping problem in an initially enlarged filtration can be written as a parameterised value process in the reference filtration. To our knowledge, this is the only consideration of optimal stopping problems and enlargement of filtration together.

1.3 The Contributions of this Thesis

The overarching contribution of this thesis is the extension of progressive enlargement techniques to the case of multiple default times and their associated marks. These extensions are then applied to two areas of interest in financial mathematics, namely backward stochastic differential equations and optimal stopping problems. The following subsection gives a cursory overview of the presentation of the thesis. The subsequent subsections then detail the layout and contributions of each chapter.

1.3.1 Overview of the thesis

The goal of the thesis is to develop the theory of enlargement of filtration with respect to multiple default times and their associated marks to create a framework for studying backward stochastic differential equations and optimal stopping problems in the presence of multiple defaults. The thesis is structured as follows: chapter 2 introduces the important mathematical preliminaries used throughout the thesis as well as introducing the enlargement of filtration. This chapter presents no original results but is important in laying the foundation for later contributions. Chapter 3 develops the theory of the progressive enlargement of filtration with multiple random times and their associated marks. The chapter contains two main contributions, theorems 3.2.7 and 3.2.10. These two results are used extensively in chapter 4 and 5. Chapter 4 presents a framework for analysing the solutions to backward stochastic differential equations in a progressively enlarged filtration with multiple random times and their marks, extending the work of Eyraud-Loisel and Royer-Carenzi [2010], Kharroubi and Lim [2014] and Calvia and Gianin [2020]. Here the main contributions are theorems 4.1.2, 4.2.1, 4.3.1 and 4.3.2. Finally, chapter 5 studies two methods of solving an optimal stopping problem in the enlarged filtration, the classical supermartingale approach and the reflected BSDE approach. The utility of the results are then demonstrated via an example with a known solution. This chapter presents a novel way of transferring the Snell envelope in the enlarged filtration to a series of Snell envelopes in the reference filtration. The main contributions being theorems 5.2.3, 5.4.2 and 5.4.3.

1.3.2 Chapter 2

Section 2.1 introduces the mathematics and intuition behind the enlargement of filtration. The section begins with a famous example of the semimartingale property not being conserved when moving from the reference filtration to the enlarged filtration, highlighting the foundation of the study of enlargement. By simply considering an arbitrarily enlarged filtration, there is no guarentee that the semimartingale property will be conserved. For this reason, a certain structure or assumption needs to be made on how a filtration is enlarged. The well-known Jacod's density hypothesis, developed by Jean Jacod in 1985, is introduced here and proceeds to be an extremely useful tool throughout the thesis. The consequences of this assumption are then applied to two cases of enlargement of filtration; *initial enlargement of filtration* (section 2.2) and *progressive enlargement of filtration* (section 2.3).

Section 2.2 presents known results on the initial enlargement of filtration. While this section merely acts as a necessary predecessor to section 2.3 and chapter 3, Proposition 2.2.3 is a crucial result and is used extensively throughout the thesis.

Section 2.3 too presents known results on the progressive enlargement of filtration. This section acts as a road-map for chapter 3, meaning chapter 3 largely mirrors section 2.3 extending the results to the case of multiple default times and their associated marks. The main results to be extended are the projection and decomposition formulas (section 2.3.1), martingales in the enlarged filtration (section 2.3.2) and the predictable representation property (section 2.3.3). The only original result (to our knowledge) from this section and the entire chapter is corollary 2.3.2.1, this being a very useful result that is applied in theorems 2.3.6 and 3.2.7.

1.3.3 Chapter 3

Chapter 3 extends the known theory on the one-default progressive enlargement of filtration under a Jacod density hypothesis to the case of multiple defaults with associated marks under Jacod density hypothesis on all default times and marks. This chapter sets up the application-focused chapters 4 and 5.

Section 3.1 sets up the enlarged filtrations that are used throughout chapters 3, 4 and 5. The multiple default and associated mark setup is not an original formulation but is taken from

(among others) Pham [2010].
Next, section 3.2 begins presenting known results in this setup, predominantly having been derived in Pham [2010]. Lemma 3.2.3 is a new contribution of the thesis and both supports the iterative nature of the multiple default enlargement and introduces some key processes used throughout the thesis. Having set the multiple default framework and defined the necessary processes and notation, the chapter proceeds to extend the one-default progressive enlargement of filtration theory, starting with projection formulas in subsection 3.2.1. Here it is shown how to project integrable random variables from the enlarged filtrations to the reference filtration, the main contribution being proposition 3.2.4. Next, subsection 3.2.2 proves a characterisation of martingales in the enlarged filtration in terms of parameterised martingales in the reference filtration, the purpose being to prove that semimartingales are preserved in this setup. The main contributions from this section are theorem 3.2.5, proposition 3.2.6 ultimately leading to corollary 3.2.6.1. Building on these two previous subsections, section 3.2.3 proves that any martingale in the reference filtration indeed remains a semimartingale in the enlarged filtration, answering the fundamental question of enlargement of filtration. The key contributions of this section are theorem 3.2.7, corollary 3.2.7.2 and the instructive argument presented before and including proposition 3.2.8. Subsection 3.2.4 then introduces an important family of jump measures used in representing martingales in the enlarged filtration. This section does not present any original results or derivations but instead defines the measures and their compensators. Finally, subsection 3.2.5 presents the main contribution of chapter 3, utlising the previous four subsections. Here a predictable representation property for martingales in the enlarged filtration is proven in theorem 3.2.10. This result is key in studying backward differential equations in the enlarged filtration.

1.3.4 Chapter 4

Chapter 4 studies backward stochastic differential equations (BSDEs) in the multiple default setup of chapter 3, ultimately extending the work of Kharroubi and Lim [2014] and Calvia and Gianin [2020].
The chapter begins with some useful preliminary results and definitions, adapting the setup of chapter 3 in a Brownian setting. Subsection 4.1 presents the BSDE to be studied along with the key result of the chapter. Assuming Lipschitz-continuity of the BSDE's driver, theorem 4.1.2 proves that the solution to BSDE (4.4) exists and is unique. This result serves as the main contribution of the chapter and extends theorems 3.1 and 4.1 in Kharroubi and Lim [2014]. In completing the study of BSDE (4.4), theorem 4.2.1 of subsection 4.2 briefly proves a comparison theorem for its solutions. Subsection 4.3 progresses to the task of solving BSDE (4.4) in terms of BSDEs in the reference filtration. This is done in two ways, firstly by assuming a simplified representation of the driver of BSDE (4.4); its solution is proved to be a series of parameterisations of a single set of solutions to a BSDE in the reference filtration. Secondly, BSDE (4.4) with only hypothesis 6 assumed for its driver is proved to be solved by parameterising a series of solutions to BSDEs in the reference filtration, the main contributions of this section being theorems 4.3.1 and 4.3.2. Finally, subsection 4.4 rounds out chapter 4 with two brief examples of applying BSDE (4.4) and the

results derived in the previous three subsections. Utility maximisation and risk measures are used to illustrate the power of the previous results. No significant contribution is made in this subsection; the results presented are merely a view in to the utility of the previously derived results. Further research could certainly stem from the more detailed application of these results.

1.3.5 Chapter 5

Chapter 5 studies the effect of enlargement of filtration on optimal stopping problems, the combined study having received very little attention in the literature. The chapter does not attempt to prove any novelties in the field of optimal stopping problems but instead proves a transfer formula for the Snell envelope in the enlarged filtration to a series of processes in the reference filtration, the main contribution of the chapter being theorem 5.2.3 proven in subsection 5.2. The utility of this theorem is then demonstrated by solving a well-known optimal stopping problem, that being the optimal stopping of a Brownian bridge, only using results from 3 and 5. Section 5.3 is devoted to introducing the problem as well as its previously derived solutions and then reproducing the solution using the novel method derived in subsection 5.2. Finally, Section 5.4, similar to chapter 4, studies reflected BSDEs in the enlarged filtration, the connection with optimal stopping problems being detailed at the start of the section. An existence and uniqueness theorem is proved similarly to theorem 4.1.2 in chapter 4. The solution of the reflected BSDE is then proved to be derived from a series of parameterised solutions in the reference filtration, much like theorem 4.3.2. The main contributions of this section are theorems 5.4.2 and 5.4.3.

Chapter 2

One-default Enlargement of Filtration

This chapter introduces the technicalities of enlargement of filtration. No new results are derived in this chapter, it is merely to introduce the techniques used to transfer existing theory of the one-default enlargement of filtration to the case of multiple defaults. The following results can all be found in Callegaro et al. [2013] and Aksamit and Jeanblanc [2017] with the precise references to follow.

2.1 Jacod's Density Hypothesis

Before introducing Jacod's density hypothesis, a well-known example presented in Proposition 1.1 in Ernst et al. [2017] is given below to motivate the need for conditions to be placed on the enlarged filtration $\mathbb{G}$.

Suppose for the moment that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space supporting a 1-dimensional Brownian motion W, such that $\mathbb{F}$ is its natural, completed filtration. Consider for any $\epsilon > 0$, the enlarged filtration $\mathbb{G}$ defined as $\mathcal{G}_t = \mathcal{F}_{t+\epsilon}$ for all $t \geq 0$, clearly, $\mathcal{F}_t \subseteq \mathcal{G}_t$. The fundamental question of enlargement of filtration is *Do $\mathbb{F}$-semimartingales remain semimartingales in the enlarged filtration?* In this current setup, we ask the question of whether the $\mathbb{F}$-Brownian motion W remains a semimartingale in the enlarged filtration $\mathbb{G}$. This is in fact not the case and is shown as follows: for $n \geq 1$, define

$$H_t^n = \frac{\sum_{k=1}^n \text{sign}(\Delta_k^n)}{\sqrt{n}} \mathbb{1}_{\{(k-1)\epsilon < nt \leq k\epsilon\}},$$

where $\Delta_k^n = W_{\frac{k\epsilon}{n}} - W_{\frac{(k-1)\epsilon}{n}}$. Note that H^n is $\mathbb{G}$-predictable and $H^n \to 0$ uniformly. Now consider

$$\int_0^t H_s^n dW_s = \frac{\sum_{k=1}^n |\Delta_k^n|}{\sqrt{n}}.$$

Note that $\sqrt{n}\Delta_k^n \sim \mathcal{N}(0, \epsilon)$, meaning by the law of large numbers $\int_0^t H_s^n dW_s \to \mathbb{E}[|W_\epsilon|]$. Now by the Bichteler-Dellacherie theorem, W is a $\mathbb{G}$-semimartingale if for any $\mathbb{G}$-predictable process H^n such that $H^n \to 0$, then $\int_0^\cdot H_t^n dW_t \to 0$. It is concluded that W is not a $\mathbb{G}$-semimartingale.

This example illustrates the non-triviality of enlargement of filtration, W is clearly $\mathbb{G}$-adapted however the anticipative nature of the enlargement renders W not even a $\mathbb{G}$-semimartingale.
In the case of enlarging $\mathbb{F}$ with a random variable (initially or progressively), Jacod [1985] shows that a way of ensuring the preservation of semimartingales in $\mathbb{G}$ is by assuming an absolute continuity condition on the conditional laws of the random variable with its unconditional law. This is often referred to as Jacod's density hypothesis in the literature. A less general form of Jacod's density hypothesis is presented below:

Definition 2.1.1. *For $d \geq 1$ and a random variable ζ taking values in $A \subseteq \mathbb{R}^d$ and a filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ satisfying the usual conditions, we say that ζ obeys Jacod's density hypothesis with respect to the filtration $\mathbb{H}$ and denote it by $\zeta \ll \mathbb{H}$ if there exists a non-negative $\mathbb{H} \otimes \mathcal{B}(A)$-optional process κ such that for any $t \geq 0$*

$$\mathbb{P}(\zeta \in dz|\mathcal{H}_t) = \kappa_t(z)dz.$$

We call κ the density process of ζ with respect to $\mathbb{H}$ and shall refer to it by $\kappa = \mathcal{T}\{\zeta, \mathbb{H}\}$.

Remark 1. *This definition is significantly stronger than the original formulation in Jacod [1985]. Here the author assumes that the conditional law of ζ is absolutely continuous with respect to some non-random law. From there, it turns out that the non-random law can be chosen to be the unconditional law of ζ. Hypothesis 2.1.1 states that ζ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

This thesis focuses on two constructions of an enlarged filtration, the initial and progressive enlargement of filtration. The former considers enlarging the reference filtration with a random variable, making the enlarged filtration the smallest filtration satisfying the usual conditions such that the random variable is measurable at every point in time. In a financial context, this can model the existence of insider information in a financial market. An insider trader will have information that is always available to her that the market does not. By knowing the state of the random variable at all times, her financial decisions may be different and consequently more beneficial than a regular market participant. For example, consider a filtration generated by a stock price process and enlarging the filtration with some terminal stock price. The insider, who has access to the enlarged filtration will know the current value of the stock price as well as the value at some future point in time, this will surely benefit her decision making.
The progressive enlargement of filtration considers enlarging the reference filtration with a positive random variable in a progressive manner, meaning that the random variable is not measurable in the enlarged filtration at all times but instead a stopping time in the enlarged filtration. This has a particular application to credit risk modelling in finance since it is only necessary to know if a credit default has occurred or not, and if it has occurred, then its exact timing must be known. The benefit of enlargement of filtration is that the financial market need only reference the reference filtration while the credit default event needn't be dependent on the financial market.

2.2 Initial enlargement of filtration

For this subsection, let $d \geq 1$ and ξ be a random variable taking values in $E \subseteq \mathbb{R}^d$. Define the initially enlarged filtration $\mathbb{G}^\xi := \left(\mathcal{G}_t^\xi\right)_{t\geq 0}$ by

$$\mathcal{G}_t^\xi = \cap_{s>t} \left(\mathcal{F}_s \vee \sigma(\xi)\right).$$

Note that the $\mathbb{G}$ is ensured to be right-continuous.
As stated before, the current setup is too general to make any meaningful analysis on the enlarged filtration $\mathbb{G}^\xi$, a certain structure is required on ξ to continue. Throughout the thesis, Jacod's density hypothesis (definition 2.1.1) will be used.

Hypothesis 1. *The random variable ξ satisfies Jacod's density hypothesis with respect to $\mathbb{F}$, i.e. $\xi \ll \mathbb{F}$ and $\alpha^\xi = \mathcal{T}\{\xi, \mathbb{F}\}$.*

Two important consequences with respect to the density process α^ξ are: (see proposition 4.17 and 4.18 in Aksamit and Jeanblanc [2017])

1. For any fixed $e \in E$, $\alpha^\xi(e)$ is an $\mathbb{F}$-martingale.
2. By definition $\alpha^\xi(e) \geq 0$ and $\alpha^\xi(\xi) > 0$ (equation 4.10 in Aksamit and Jeanblanc [2017]).

Having now made a density hypothesis on ξ, some known results on $\mathbb{G}^\xi$ are presented. Firstly, it is shown in lemma 4.20 in Aksamit and Jeanblanc [2017] that the right-continuous regularisation in the definition of $\mathbb{G}^\xi$ is not necessary when ξ satisfies Jacod's density hypothesis.

Lemma 2.2.1. *The definition of $\mathbb{G}^\xi$ can be simplified to*

$$\mathcal{G}_t^\xi = \mathcal{F}_t \vee \sigma(\xi).$$

Next, a useful result on characterising $\mathbb{G}^\xi$ adapted processes is recalled (see proposition 2.7 in Callegaro et al. [2013] or proposition 4.22 in Aksamit and Jeanblanc [2017] for example).

Proposition 2.2.2. *A process Y is $\mathbb{G}^\xi$ adapted if and only if there exists an $\mathbb{F} \otimes \mathcal{B}(E)$ adapted process $\hat{Y}$ such that*

$$Y = \hat{Y}(\xi). \quad i.e.\ Y_t(\omega) = \hat{Y}_t(\omega, \xi(\omega))$$

for all $t \geq 0$ and $\omega \in \Omega$.

Proof. Using lemma 2.2.1, $\mathbb{G}^\xi$ adapted processes are generated by the form $yf(\xi)$, where y is $\mathbb{F}$ adapted and f is a bounded Borel function on E. The result then follows using the monotone class theorem. □

The ability to characterise $\mathbb{G}^\xi$ adapted processes in this parameterised way is extremely important. For example, suppose from Proposition 2.2.2 that $\mathbb{F}$ was a Brownian filtration supporting a d-dimensional Brownian Motion W and Y was an $\mathbb{F}$-martingale. Now suppose, we wanted to compute Y_t^2 for some $t \geq 0$ using Itô's lemma. While this can be done in the filtration $\mathbb{G}$, we need to first prove that W is a $\mathbb{G}$-semimartingale, next we would need to derive a $\mathbb{G}$-Brownian motion and then we could apply Itô's lemma. Instead, we could fix an $e \in E$, and simply apply Itô's lemma on the process $\hat{Y}(e)^2$ in the filtration $\mathbb{F}$. Once we have an expression for this, we could then substitute ξ in place of e. Proposition 2.2.2 allows us to work in the reference filtration $\mathbb{F}$ and then transfer to the enlarged filtration. This subsection is concluded by presenting a result on computing conditional expectations in $\mathbb{G}^\xi$ and $\mathbb{F}$ (see lemma 2.9 in Callegaro et al. [2013] or corollary 4.21 in Aksamit and Jeanblanc [2017] for example).

Proposition 2.2.3. *For $t \leq T$ and any $X \in L^1(\mathcal{F}_T \otimes \mathcal{B}(E))$*

1. $\mathbb{E}[X(\xi)|\mathcal{F}_t] = \mathbb{E}[\int_E X(e)\alpha_T^\xi(e)de|\mathcal{F}_t]$
2. $\mathbb{E}[X(\xi)|\mathcal{G}_t^\xi] = \frac{\mathbb{E}[X(e)\alpha_T^\xi(e)|\mathcal{F}_t]|_{e=\xi}}{\alpha_t^\xi(\xi)}$

Proof. 1. From hypothesis 1, for any $e \in E$

$$\mathbb{P}(\xi \in de|\mathcal{F}_T) = \alpha_T^\xi(e)de.$$

Implying

$$\begin{aligned}\mathbb{E}[X(\xi)|\mathcal{F}_t] =& \mathbb{E}[\mathbb{E}[X(\xi)|\mathcal{F}_T]\mathcal{F}_t] \\ =& \mathbb{E}\left[\int_E X(e)\alpha_T^\xi(e)de|\mathcal{F}_t\right].\end{aligned}$$

2. Consider $F \in \mathcal{F}_t$ and a bounded Borel function f,

$$\begin{aligned}\mathbb{E}[X(\xi)\mathbb{1}_F f(\xi)] =& \mathbb{E}\left[\mathbb{1}_F\mathbb{E}\left[X(\xi)f(\xi)|\mathcal{F}_t\right]\right] \\ =& \mathbb{E}\left[\mathbb{1}_F\mathbb{E}\left[\int_E X(e)f(e)\alpha_T^\xi(e)de|\mathcal{F}_t\right]\right] \quad \text{(Point 1.)} \\ =& \mathbb{E}\left[\int_E f(e)\mathbb{1}_F\mathbb{E}\left[X(e)\alpha_T^\xi(e)|\mathcal{F}_t\right]de\right] \\ =& \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_F f(\xi)\left(\frac{\mathbb{E}[X(e)\alpha_T^\xi(e)|\mathcal{F}_t]|_{e=\xi}}{\alpha_t^\xi(\xi)}\right)|\mathcal{F}_t\right]\right] \quad \text{(Point 1.)} \\ =& \mathbb{E}\left[\mathbb{1}_F f(\xi)\left(\frac{\mathbb{E}[X(e)\alpha_T^\xi(e)|\mathcal{F}_t]|_{e=\xi}}{\alpha_t^\xi(\xi)}\right)\right].\end{aligned}$$

Using the fact that random variables of the form $\mathbb{1}_F f(\xi)$ generate $\mathcal{G}_t^\xi$, the Monotone class theorem can applied to conclude

$$\mathbb{E}[X(\xi)|\mathcal{G}_t^\xi] = \frac{\mathbb{E}[X(e)\alpha_T^\xi(e)|\mathcal{F}_t]|_{e=\xi}}{\alpha_t^\xi(\xi)}.$$

□

Note again the utility of this result, the expectation with respect to the enlarged filtration, is simply an expectation in the reference filtration parameterised at the random variable ξ.

2.3 Progressive enlargement of filtration

For this subsection let τ be an $\mathcal{F}$-random time, i.e. a non-negative $\mathcal{F}$-measurable random variable. Then the progressive enlargement of $\mathbb{F}$ with τ is defined as $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, with

$$\mathcal{G}_t = \cap_{s>t} \left(\mathcal{F}_s \vee \sigma(\tau \wedge s)\right).$$

This is the smallest right-continuous filtration such that τ is a $\mathbb{G}$ stopping time. Recall that two sigma-algebras $\mathcal{X}$ and $\mathcal{Y}$ coincide on a set A if for every $X \in \mathcal{X}$, there exists $Y \in \mathcal{Y}$ such that

$$X \cap A = Y \cap A,$$

and vice versa. Remark 4.41 in Aksamit and Jeanblanc [2017] states that for any $t \geq 0$, $\mathcal{G}_t$ coincides with $\mathcal{F}_t \vee \sigma(\tau)$ on the set $[\tau, \infty)$. This means that for any integrable random variable X,

$$\mathbb{E}[X|\mathcal{G}_t]\mathbb{1}_{\{\tau\leq t\}} = \mathbb{E}[X|\mathcal{F}_t \vee \sigma(\tau)]\mathbb{1}_{\{\tau\leq t\}}.$$

Indeed, by definition, for $F \in \mathcal{F}_t \vee \sigma(\tau)$, there exists $G \in \mathcal{G}_t$ such that

$$F \cap \{\tau \leq t\} = G \cap \{\tau \leq t\},$$

and

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X\mathbb{1}_{\{\tau\leq t\}}|\mathcal{G}_t]\mathbb{1}_F] =& \mathbb{E}[\mathbb{E}[X|\mathcal{G}_t]\mathbb{1}_{F\cap\{\tau\leq t\}}]\\ =& \mathbb{E}[\mathbb{E}[X\mathbb{1}_{G\cap\{\tau\leq t\}}|\mathcal{G}_t]]\\ =& \mathbb{E}[X\mathbb{1}_{G\cap\{\tau\leq t\}}].\end{aligned}$$

Finally, clearly $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau)$ when $\tau \leq t$, implying that $\mathbb{E}[X\mathbb{1}_{\{\tau\leq t\}}|\mathcal{G}_t]$ is a version of $\mathbb{E}[X\mathbb{1}_{\{\tau\leq t\}}|\mathcal{F}_t \vee \sigma(\tau)]$. Similar computations yield that $\mathcal{F}_t$ and $\mathcal{G}_t$ coincide on the set $\{\tau > t\}$ for all $t \geq 0$.

This can be interpreted as saying that knowing time τ has occurred (i.e. $\tau \leq t$) means we know the exact value of τ however knowing that time τ has not occurred (i.e. $\tau > t$) means we do not know its exact value but merely that it has not occurred. This can be interpreted in a financial context, a market participant exposed to the default time τ only cares if the default has occurred or not, if it has occurred she needs to know at what point in time exactly did it occur.

Throughout this subsection, Jacod's density hypothesis on the random variable τ will be assumed.

Hypothesis 2. *The random variable τ satisfies Jacod's density hypothesis with respect to $\mathbb{F}$, i.e. $\tau \ll \mathbb{F}$ and $\alpha^\tau = \mathcal{T}\{\xi, \mathbb{F}\}$.*

Remark 2. *Just as in Lemme 2.2.1 in the initial enlargement of filtration, the right-continuous regularisation of $\mathbb{G}$ is not necessary given hypothesis 2. Furthermore, in this setup, it is noted that hypothesis 2 implies that the conditional law of τ with respect to $\mathbb{F}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$. This implies that the conditional law of τ is non-atomic. Moreover, the non-atomicity of the conditional law of τ means that τ avoids $\mathbb{F}$ stopping times. In other words, for any $\mathbb{F}$ stopping time ν and any $t \geq 0$*

$$\mathbb{P}(\tau = \nu|\mathcal{F}_t) = 0.$$

2.3.1 Projection and Decomposition Formulas

The following proposition utilises the fact that $\mathbb{G}$ and $\mathbb{F} \vee \sigma(\tau)$ coincide on the set $[\![\tau, \infty[\![$, which allows the decomposition of random variables measurable in the enlarged filtration.

Proposition 2.3.1. *For any $t \geq 0$, a random variable Y is $\mathcal{G}_t$-measurable if and only if there exists an $\mathcal{F}_t$-measurable random variable Y^0 and an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable Y^1 such that*

$$Y = Y^0 \mathbb{1}_{\{\tau > t\}} + Y^1(\tau)\mathbb{1}_{\{\tau \leq t\}}.$$

Proof. It is noted that $\mathcal{G}_t$-random variables are generated by random variables of the form

$$Y = y_t f(\tau \wedge t) = y_t f(t)\mathbb{1}_{\{\tau > t\}} + y_t f(\tau)\mathbb{1}_{\{\tau \leq t\}}.$$

Where y_t is an $\mathcal{F}_t$-random variable and f is a bounded Borel function. Then the Monotone class theorem can be applied to yield the result. □

Interestingly, in this setup, there has been substantial research on the decomposition of $\mathbb{G}$-adapted processes, specifically $\mathbb{G}$-optional processes. A famous counter-example in Barlow [1978] has meant that care has to be taken when decomposing $\mathbb{G}$-optional processes. Song [2014] deals specifically with the problem of decomposing $\mathbb{G}$-optional processes and focuses on the conditions on τ when proposition 2.3.1 holds for not only random variables but $\mathbb{G}$-optional processes. In the case of Jacod's density hypothesis on τ, a $\mathbb{G}$-optional decomposition is indeed valid.
A process of utmost importance for future results is the so called Azéma supermartingale of τ with respect to $\mathbb{F}$. Let $G = (G_t)_{t \geq 0}$ be defined by

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t).$$

The following conditional expectation result can be found in lemma 2.5 in Callegaro et al. [2013] or lemma 5.24 in Aksamit and Jeanblanc [2017].

Proposition 2.3.2. *For $t \leq T$ and any $X \in L^1(\mathcal{F}_T \otimes \mathcal{B}(E))$*

$$\mathbb{E}[X(\tau)|\mathcal{G}_t] = \frac{\mathbb{E}\left[\int_t^\infty X(u)\alpha_T^\tau(u)du|\mathcal{F}_t\right]}{G_t}\mathbb{1}_{\{\tau > t\}} + \frac{\mathbb{E}[X(u)\alpha_T^\tau(u)|\mathcal{F}_t]|_{u=\tau}}{\alpha_t^\tau(\tau)}\mathbb{1}_{\{\tau \leq t\}}.$$

Proof. Note that by Proposition 2.3.1, there exists an $\mathcal{F}_t$-random variable Y^0 and an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-random variable Y^1 such that

$$\mathbb{E}[X(\tau)|\mathcal{G}_t] = Y^0\mathbb{1}_{\{\tau > t\}} + Y^1(\tau)\mathbb{1}_{\{\tau \leq t\}}.$$

Implying

$$\mathbb{E}[X(\tau)|\mathcal{G}_t]\mathbb{1}_{\{\tau > t\}} = Y^0\mathbb{1}_{\{\tau > t\}}.$$

Taking conditional expectations on both sides with respect to $\mathcal{F}_t$

$$\mathbb{E}[\mathbb{E}[X(\tau)|\mathcal{G}_t]\mathbb{1}_{\{\tau > t\}}|\mathcal{F}_t]\mathbb{1}_{\{\tau > t\}} = \mathbb{E}[Y^0\mathbb{1}_{\{\tau > t\}}|\mathcal{F}_t]\mathbb{1}_{\{\tau > t\}}.$$

The tower property of conditional expectations then yields

$$\begin{aligned}\mathbb{E}[X(\tau)\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{1}_{\{\tau>t\}} =&\mathbb{E}[Y^0\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{1}_{\{\tau>t\}}\\ \mathbb{E}[\mathbb{E}[X(\tau)\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_T]|\mathcal{F}_t]\mathbb{1}_{\{\tau>t\}} =&\mathbb{E}[Y^0\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{1}_{\{\tau>t\}}.\end{aligned}$$

Finally, Proposition 2.2.3 is used to compute the left hand side and the $\mathcal{F}_t$-measurability of Y^0 on the right hand side means that $\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right]$ is identified as G_t, meaning

$$\mathbb{E}\left[\int_t^\infty X(u)\alpha_T^\tau(u)du|\mathcal{F}_t\right]\mathbb{1}_{\{\tau>t\}} =Y^0G_t\mathbb{1}_{\{\tau>t\}}.$$

Finally, G_t is strictly positive on the set $\{\tau > t\}$, indeed

$$\mathbb{E}[\mathbb{1}_{\{G_t=0\}}\mathbb{1}_{\{\tau>t\}}] = \mathbb{E}[\mathbb{1}_{\{G_t=0\}}G_t] = 0.$$

This implies that

$$Y^0\mathbb{1}_{\{\tau>t\}} = \frac{\mathbb{E}\left[\int_t^\infty X(u)\alpha_T^\tau(u)du|\mathcal{F}_t\right]}{G_t}\mathbb{1}_{\{\tau>t\}}.$$

Next,

$$\mathbb{E}[X(\tau)|\mathcal{G}_t]\mathbb{1}_{\{\tau\le t\}} =Y^1(\tau)\mathbb{1}_{\{\tau\le t\}}.$$

The defintion of $\mathcal{G}_t$ after τ means we can rewrite the left hand side as

$$\mathbb{E}[X(\tau)|\mathcal{F}_t\vee\sigma(\tau)]\mathbb{1}_{\{\tau\le t\}} =Y^1(\tau)\mathbb{1}_{\{\tau\le t\}}.$$

The left hand side is a conditional expectation with respect to the initial enlarged filtration of $\mathbb{F}$ with the random variable τ meaning Proposition 2.2.3 can be used.

$$\frac{\mathbb{E}[X(u)\alpha_T^\tau(u)|\mathcal{F}_t]|_{u=\tau}}{\alpha_t^\tau(\tau)}\mathbb{1}_{\{\tau\le t\}} =Y^1(\tau)\mathbb{1}_{\{\tau\le t\}}$$

Therefore, Y^1 can be chosen as

$$Y^1(u) = \frac{\mathbb{E}[X(u)\alpha_T^\tau(u)|\mathcal{F}_t]}{\alpha_t^\tau(u)}\mathbb{1}_{\{\alpha_t^\tau(u)>0\}}.$$

□

A useful consequence of Proposition 2.3.2 is given in the following corollary.

Corollary 2.3.2.1. *For an $\mathbb{F}$-optional process X, denote its jump process by $\Delta X := X - X_-$, then*

$$\left(\int_0^t \Delta X_s\alpha_s^\tau(s)ds : t\ge 0\right)$$

is an $\mathbb{F}$-martingale.

Proof. From remark 2, τ avoids $\mathbb{F}$ stopping times meaning

$$\{\Delta X \neq 0\} \not\subset \{(\omega, t) : \tau(\omega) = t\}.$$

Further note that for any $u \in \mathbb{R}^+$, $\alpha^\tau(u)$ is an $\mathbb{F}$-martingale.
For any $T \geq 0$, the random variable $Y(u) := \Delta X_u \mathbb{1}_{\{u \leq T\}}$, for $u \geq 0$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, then by Proposition 2.2.3 for any $t \leq T$

$$0 = \mathbb{E}\left[\Delta X_\tau \mathbb{1}_{\{\tau \leq T\}} | \mathcal{F}_t\right] = \mathbb{E}\left[\int_0^T \Delta X_u \alpha_T^\tau(u) du | \mathcal{F}_t\right].$$

Now consider

$$\mathbb{E}\left[\int_t^T \Delta X_u \alpha_u^\tau(u) du | \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T \Delta X_u \mathbb{E}[\alpha_T^\tau(u) | \mathcal{F}_u] du | \mathcal{F}_t\right]$$

this uses the fact that for any $u \in \mathbb{R}^+$, $\alpha^\tau(u)$ is an $\mathbb{F}$-martingale. Finally, the last expectation can be simplified as follows

$$\begin{aligned}\mathbb{E}\left[\int_t^T \Delta X_u \alpha_u^\tau(u) du | \mathcal{F}_t\right] &= \mathbb{E}\left[\int_t^T \Delta X_u \alpha_T^\tau(u) du | \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\Delta X_\tau \mathbb{1}_{\{t < \tau \leq T\}} | \mathcal{F}_t\right] = 0.\end{aligned}$$

□

2.3.2 $\mathbb{F}$-martingales in $\mathbb{G}$

The fundamental question of enlargement of filtration has been whether $\mathbb{F}$-semimartingales remain $\mathbb{G}$-semimartingales. A key tool in answering this question is the ability to characterise $\mathbb{G}$-martingales in terms of $\mathbb{F}$-martingales. The following result can be found in proposition 3.3 in Callegaro et al. [2013] or proposition 5.29 in Aksamit and Jeanblanc [2017].

Proposition 2.3.3. *A process $Y = Y^0 \mathbb{1}_{[\![0,\tau)\!)} + Y^1(\tau)\mathbb{1}_{[\![\tau,\infty)\!)}$ is a $\mathbb{G}$-local martingale if the following are $\mathbb{F}$-local martingales.*

a. $(Y_t^1(u)\alpha_t^\tau(u) : t \geq u)$

b. $(m_t = \mathbb{E}[Y_t | \mathcal{F}_t] : t \geq 0)$

Proof. The case of strict martingales is proven here. To extend to the case of local martingales, it is noted that any $\mathbb{F}$-stopping time is a $\mathbb{G}$-stopping time, therefore localising sequences for the $\mathbb{F}$-local martingales in a. and b. when joined at τ will be a localising sequence for Y.
For $s \leq t$, the goal is to show that $\mathbb{E}[Y_t | \mathcal{G}_s] = Y_s$. This is done in two parts, when $\tau \leq s$ and when $\tau > s$. Firstly,

$$\mathbb{E}[Y_t | \mathcal{G}_s]\mathbb{1}_{\{\tau \leq s\}} = \mathbb{E}[Y_t \mathbb{1}_{\{\tau \leq s\}} | \mathcal{G}_s] = \mathbb{E}[Y_t^1(\tau) | \mathcal{F}_s \vee \sigma(\tau)]\mathbb{1}_{\{\tau \leq s\}}.$$

Proposition 2.2.3 can be used to compute the expectation with respect to the initially enlarged filtration.

$$=\frac{\mathbb{E}[Y_t^1(u)\alpha_t^\tau(u)|\mathcal{F}_s]|_{u=\tau}}{\alpha_s^\tau(\tau)}\mathbb{1}_{\{\tau\le s\}}$$

By assuming property (a), the numerator at a fixed $u \le s$ is an expectation of the martingale $Y^1(u)\alpha^\tau(u)$, therefore

$$=Y_s^1(\tau)\mathbb{1}_{\{\tau\le s\}} \quad \text{(By assumption (a), } u=\tau\le s\le t)$$

Therefore $\mathbb{E}[Y_t|\mathcal{G}_s]\mathbb{1}_{\{\tau\le s\}} = Y_s\mathbb{1}_{\{\tau\le s\}}$.
For the second part, proposition 2.3.1 states that when $\tau > s$, there exists an $\mathcal{F}_s$-random variable y^0 such that

$$y^0\mathbb{1}_{\{\tau>s\}} = \mathbb{E}[Y_t|\mathcal{G}_s]\mathbb{1}_{\{\tau>s\}}.$$

Taking conditional expectations on both sides with respect to $\mathcal{F}_s$ and keeping the indicator $\mathbb{1}_{\{\tau>s\}}$ on both sides

$$y^0\mathbb{E}\left[\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\right]\mathbb{1}_{\{\tau>s\}} = \mathbb{E}[\mathbb{E}[Y_t\mathbb{1}_{\{\tau>s\}}|\mathcal{G}_s]|\mathcal{F}_s]\mathbb{1}_{\{\tau>s\}}.$$

The left hand side's expectation is identified as G_s while the right hand side's expectation is simplified using the Tower propoerty of conditional expectations

$$\begin{aligned} y^0G_s\mathbb{1}_{\{\tau>s\}} &= \mathbb{E}[Y_t\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s]\mathbb{1}_{\{\tau>s\}} \\ &= \left(\mathbb{E}[Y_t|\mathcal{F}_s] - \mathbb{E}[Y_t^1(\tau)\mathbb{1}_{\{\tau\le s\}}|\mathcal{F}_s]\right)\mathbb{1}_{\{\tau>s\}} \\ &= \left(\mathbb{E}[m_t|\mathcal{F}_s] - \mathbb{E}[\mathbb{E}[Y_t^1(\tau)\mathbb{1}_{\{\tau\le s\}}|\mathcal{F}_t]|\mathcal{F}_s]\right)\mathbb{1}_{\{\tau>s\}}. \end{aligned}$$

By condition (b) the first term is an expectation of the martingale m and the second term's inner expectation is simplified using proposition 2.2.3

$$= \left(m_s - \mathbb{E}\left[\int_0^s Y_t^1(u)\alpha_t^\tau(u)du|\mathcal{F}_s\right]\right)\mathbb{1}_{\{\tau>s\}}.$$

Fubini's theorem can now be used on the second term to take the expectation in to the integral. By condition (a) the integrand is an $\mathbb{F}$-martingale.

$$= \left(m_s - \int_0^s Y_s^1(u)\alpha_s^\tau(u)du\right)\mathbb{1}_{\{\tau>s\}}.$$

Finally, proposition 2.2.3) is reversed to identify the second term as

$$\begin{aligned} &= \left(m_s - \mathbb{E}[Y_s^1(\tau)\mathbb{1}_{\{\tau\le s\}}|\mathcal{F}_s]\right)\mathbb{1}_{\{\tau>s\}} \\ &= \mathbb{E}[Y_s^0\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s]\mathbb{1}_{\{\tau>s\}} \\ &= Y_s^0G_s\mathbb{1}_{\{\tau>s\}} \end{aligned}$$

Therefore $\mathbb{E}[Y_t|\mathcal{G}_s]\mathbb{1}_{\{\tau>s\}} = Y_s\mathbb{1}_{\{\tau>s\}}$. □

By assuming an absolutely continuous condition on the random variable τ, i.e. ensuring the density process α^τ is strictly positive, Callegaro et al. [2013] show the reverse implication of proposition 2.3.3 for strict martingales while the result needn't hold for local-martingales. The characterisation of $\mathbb{F}$-martingales is the first step in enabling us to answer the fundamental question of enlargement of filtration. The following two technical lemmata provide the second step to answering this question. Firstly, a particular case of the Itô-Wentzell formula is proven.

Lemma 2.3.4. *Let X be a $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}^+)$-martingale, that is, $X(u)$ is an $\mathbb{F}$-martingale for all $u \in \mathbb{R}^+$ and the map $(\omega, t, u) \to X_t(\omega, u)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable for all $t \geq 0$. Then for $t \geq 0$*

$$d\left(\int_0^t X_t(u)du\right) = X_t(t)dt + \int_0^t dX_t(u)du.$$

Proof. Start with an elementary martingale of the form $X_t(u) = X_t f(u)$ where X is an $\mathbb{F}$-martingale and f is a bounded Borel function. Then

$$\int_0^t X_t(u)du = X_t \int_0^t f(u)du.$$

Using integration by parts, this can be written in differential notation as

$$\begin{aligned} d\left(\int_0^t X_t(u)du\right) &= X_t f(t)dt + \int_0^t f(u)du dX_t \\ d\left(\int_0^t X_t(u)du\right) &= X_t(t)dt + \int_0^t dX_t(u)du. \end{aligned}$$

The result then follows from the monotone class theorem. □

Note that Lemma 2.3.4 is understood as follows: suppose M is a continuous $\mathbb{F}$-martingale and b is an $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}^+)$-predictable process. Define an $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}^+)$-martingale X as follows,

$$dX_t(u) := b_t(u)dM_t,$$

then

$$d\left(\int_0^t X_t(u)du\right) = X_t(t)dt + \int_0^t b_t(u)du dM_t.$$

The next lemma is an application of Fubini's theorem that is useful in proving theorems 2.3.6 and 3.2.7.

Lemma 2.3.5. *For any $\mathbb{F}$-martingale M, the process defined as*

$$\left(\int_0^t \int_u^\infty M_t - M_u dv du : t \geq 0\right)$$

is also an $\mathbb{F}$-martingale.

Proof. Consider $s \leq t$, then

$$\begin{aligned} \mathbb{E}\left[\int_0^t \int_u^\infty M_t - M_u dv du | \mathcal{F}_s\right] &= \int_0^t \int_u^\infty M_s - (M_u \mathbb{1}_{\{u \leq s\}} + M_s \mathbb{1}_{\{u > s\}}) dv du \\ &= \int_0^t \int_u^\infty (M_s - M_u)\mathbb{1}_{\{u \leq s\}} dv du \\ &= \int_0^s \int_u^\infty (M_s - M_u) dv du \end{aligned}$$

□

We now have all the tools at our disposal to answer the fundamental question of enlargement of filtration: do $\mathbb{F}$-local martingales remain semimartingales in the enlarged filtration $\mathbb{G}$?

Theorem 2.3.6. *Any càdlàg $\mathbb{F}$-local martingale X is a $\mathbb{G}$-semimartingale and*

$$\hat{X}_t = X_t - \int_0^{\tau\wedge t} \frac{1}{G_{s-}} d\langle X, G\rangle_s - \int_{\tau\wedge t}^t \frac{1}{\alpha_{s-}^\tau(\tau)} d\langle X, \alpha^\tau(u)\rangle_s|_{u=\tau}$$

is a $\mathbb{G}$-local martingale.

Proof. Firstly, thanks to lemma 1.49 in Aksamit and Jeanblanc [2017], the predictable quadratic covariation $\langle X, G\rangle$ exists. Next from proposition 2.2 in Jeanblanc and Le Cam [2009b], $\langle X, \alpha^\tau(u)\rangle$ exists on the set $\{\alpha^\tau(u) > 0\}$ for almost all $u \in \mathbb{R}^+$. In order to utilise proposition 2.3.3, the process $\hat{X}$ is written in the following form

$$\hat{X}_t = X_t^0 \mathbb{1}_{\{\tau>t\}} + X_t^1(\tau)\mathbb{1}_{\{\tau\le t\}},$$

where

$$\begin{aligned}
X_t^0 =& X_t - \int_0^t \frac{1}{G_{s-}} \mathbb{1}_{\{G_{s-}>0\}} d\langle X, G\rangle_s \\
X_t^1(u) =& X_t - \int_0^u \frac{1}{G_{s-}} \mathbb{1}_{\{G_{s-}>0\}} d\langle X, G\rangle_s - \int_u^t \frac{1}{\alpha_{s-}^\tau(u)} \mathbb{1}_{\{\alpha_{s-}^\tau(u)>0\}} d\langle X, \alpha^\tau(u)\rangle_s.
\end{aligned}$$

Then from proposition 2.3.3, if we can show that

a. $(X_t^1(u)\alpha_t^\tau(u) : t \ge u)$

b. $(m_t = \mathbb{E}[\hat{X}_t|\mathcal{F}_t] : t \ge 0)$

are $\mathbb{F}$-local martingales then the result will hold.

a. Using integration by parts

$$\begin{aligned}
d\left(X_t^1(u)\alpha_t^\tau(u)\right) =& \alpha_{t-}^\tau(u) dX_t^1(u) + X_{t-}^1(u) d\alpha_t^\tau(u) + d[X^1(u), \alpha^\tau(u)]_t \\
=& \alpha_{t-}^\tau(u) \left(dX_t - \frac{1}{\alpha_{t-}^\tau(u)} \mathbb{1}_{\{\alpha_{s-}^\tau(u)>0\}} d\langle X, \alpha^\tau(u)\rangle_t \right) + X_{t-}^1(u) d\alpha_t^\tau(u) + d[X, \alpha^\tau(u)]_t \\
=& \alpha_{t-}^\tau(u) dX_t + X_{t-}^1(u) d\alpha_t^\tau(u) + d\left([X, \alpha^\tau(u)] - \langle X, \alpha^\tau(u)\rangle\right)_t
\end{aligned}$$

which is the sum of $\mathbb{F}$ local-martingales.

b. Firstly, from proposition 2.2.3

$$m_t = X_t^0 G_t + \int_0^t X_t^1(u)\alpha_t^\tau(u) du.$$

Next, using Proposition 5.26 in Aksamit and Jeanblanc [2017] and the definition of G, it can be decomposed as follows

$$\begin{aligned}
G_t =& \mathbb{P}(\tau > t|\mathcal{F}_t) \\
=& \int_t^\infty \alpha_t^\tau(u) du \\
=& \int_0^\infty \alpha_t^\tau(u) du - \int_0^t \alpha_t^\tau(u) du.
\end{aligned}$$

Note that the first term is an integral over the domain of τ with respect to its $\mathcal{F}_t$-conditional law and is thus equal to one.

$$\begin{aligned} G_t =& 1 + \int_0^t \left(\alpha_u^\tau(u) - \alpha_t^\tau(u)\right) du - \int_0^t \alpha_u^\tau(u) du \\ =:& n_t - \int_0^t \alpha_u^\tau(u) du. \end{aligned}$$

The fact that $\alpha^\tau(u)$ is an $\mathbb{F}$-martingale for all $u \geq 0$ coupled with Fubini's theorem imply that n is an $\mathbb{F}$-martingale. Now using lemma 2.3.4

$$\begin{aligned} dm_t =& X_{t-}^0 dG_t + G_{t-} dX_t^0 + d[X^0, G]_t + X_t^1(t)\alpha_t^\tau(t)dt + \int_0^t d\left(X_t^1(u)\alpha_t^\tau(u)\right) du \\ =& X_{t-}^0 dn_t + G_t dX_t + d\left([X,G] - \langle X, G\rangle\right)_t + \left(X_t^1(t) - X_{t-}^0\right)\alpha_t^\tau(t)dt \\ &+ \int_0^t d\left(X_t^1(u)\alpha_t^\tau(u)\right) du. \end{aligned} \tag{2.1}$$

Is it seen from the definition of X^0 and X^1 that $X_t^1(t) - X_{t-}^0 = \Delta X_t^0$, meaning that by corollary 2.3.2.1,

$$\int_0^t \left(X_s^1(s) - X_{s-}^0\right)\alpha_s^\tau(s)ds$$

is an $\mathbb{F}$-martingale. Finally, the last term in equation (2.1) is written in integral form as

$$\begin{aligned} \int_0^t \int_0^s d\left(X_s^1(u)\alpha_s^\tau(u)\right) du ds &= \int_0^t \int_u^t d\left(X_s^1(u)\alpha_s^\tau(u)\right) ds du \\ &= \int_0^t \left(X_t^1(u)\alpha_t^\tau(u) - X_u^1(u)\alpha_u^\tau(u)\right) du. \end{aligned}$$

Therefore by Lemma 2.3.5 the above term has zero expectation and is an $\mathbb{F}$-martingale. We conclude that m is the sum of $\mathbb{F}$-local martingales.

□

2.3.3 Martingale Representation in $\mathbb{G}$

A martingale representation theorem (or predictable representation theorem) states that given a set of generating martingales, every martingale can be written as a stochastic integral with respect to a predictable process and this set of generating martingales. A famous example is the Brownian filtration. A process that is a martingale with respect to the canonical filtration of a Brownian motion can be written as a stochastic integral of a predictable process with respect to the Brownian motion. The aim of this section is to show that if a martingale representation theorem holds in the reference filtration and the generating martingales are continuous, then a martingale representation theorem holds in the enlarged space. It is interesting to note that the number of generating martingales needed in the enlarged space is one more than that needed in the reference space. This is to account for the additional discontinuity introduced by the enlargement with the default time τ.

The preservation of the martingale representativeness in the enlarged filtration can be financially interpreted as the enlarged market (i.e. the financial market with filtration $\mathbb{G}$) being complete if the reference market is also complete given the risky assets are continuous. This is important if the results from the enlargement of filtration are to be utilised in a financial context as market completeness is often crucial in deriving arbitrage-free prices of financial instruments.
As mentioned above, the number of generating martingales used in the enlarged space is one more than the number of generating martingales in the reference space. Below, we begin defining this martingale as a compensated form of the default intensity process. For $t \geq 0$, define the default intensity process

$$H_t := \mathbb{1}_{\{\tau \leq t\}}.$$

The compensator of H is the unique predictable process $\tilde{H}$ such that $\left(H - \tilde{H}\right)$ is a $\mathbb{G}$-martingale. Proposition 2.15 in Aksamit and Jeanblanc [2017] states that

$$\tilde{H}_t = \int_0^{\tau\wedge t} \frac{1}{G_{s-}} dA_s,$$

where A is the $\mathbb{F}$-dual predictable projection of H and recall that G is the $\mathbb{F}$-optional projection of $(1 - H)$, in other words $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$. From the proof of Theorem 2.3.6, it is seen that

$$dA_t = \alpha_t^\tau(t)dt.$$

Therefore

$$M_t = H_t - \int_0^t (1 - H_{s-}) \frac{\alpha_s^\tau(s)}{G_{s-}} ds.$$

The process $(1 - H_-)\frac{\alpha^\tau(\cdot)}{G_-}$ is referred to as the $\mathbb{G}$-compensator of H (or τ). To conclude this subsection it will be assumed that a martingale representation theorem holds in the reference space and that the set of generating martingales in continuous.

Hypothesis 3. *Assume the existence of a continuous $\mathbb{F}$-local martingale S such that for any $\mathbb{F}$-local martingale P, there exists an $\mathbb{F}$-predictable process ϕ such that*

$$P_t = P_0 + \int_0^t \phi_s dS_s.$$

Hypothesis 3 implies that all $\mathbb{F}$-martingales are continuous, meaning that for any $u \in \mathbb{R}^+$, $\alpha^\tau(u)$ is continuous, furthermore due to the fact that τ avoids $\mathbb{F}$ stopping times G is continuous too (see Proposition 3.9 in Aksamit and Jeanblanc [2017]). Following theorem 2.3.6, define the $\mathbb{G}$-martingale

$$S_t^{\mathbb{G}} = S_t - \int_0^{\tau\wedge t} \frac{1}{G_s} d\langle S, G\rangle_s - \int_{\tau\wedge t}^t \frac{1}{\alpha_s^\tau(\tau)} d\langle S, \alpha^\tau(u)\rangle_s|_{u=\tau}.$$

The following theorem can be found in Theorem 2.1 in Jeanblanc and Le Cam [2009a] and Theorem 6.4 in Jeanblanc and Song [2015], the proof is omitted here due to its length and similarity with Theorem 3.2.10 in Chapter 3.

Theorem 2.3.7. *For any $\mathbb{G}$-local martingale N, there exist $\mathbb{G}$-predictable processes γ and β such that*

$$N_t = N_0 + \int_0^t \gamma_s dS_s^{\mathbb{G}} + \int_0^t \beta_s dM_s.$$

2.3.4 Immersion hypothesis

This very brief subsection defines a well known hypothesis on the variable τ. Focus is not put on this hypothesis as it is nowhere used throughout this thesis. We introduce it merely for the sake of completion on the topic of enlargement of filtration with one default time.

Definition 2.3.8. *An immersion hypothesis is said to hold if every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.*

This assertion is very restrictive however it has useful application in credit risk modelling. See Jeanblanc and Le Cam [2009a] for example.

Chapter 3

Progressive Enlargement with Multiple Default Times and Their Marks

This chapter begins developing the main contributions of this thesis. Chapter 2 introduced the mathematics on enlargement of filtration, focusing on initial and progressive enlargement of filtration with one default time or mark. This thesis's contribution is the extension of the enlargement of filtration theory to the case of multiple default times and their associated marks and it's application. This chapter is purely focused on the extension of chapter 2 to the case of multiple default times and their associated marks. The goal of this chapter is to prove that a martingale representation theorem holds in the enlarged filtration if it holds in the reference filtration.

In a similar structure to chapter 2, section 3.1 begins by introducing the structure of the enlarged filtrations using a recursive approach as well as the certain definitions and conventions. Section 3.2 begins by assuming hypothesis 1 from chapter 2 on the defaults times and marks. The first contribution of the chapter is then given in lemma 3.2.3 proving that hypothesis 1 can be be deduced for any subset of default times and marks with respect to the recursively defined enlarged filtrations. Section 3.2.1, similar to section 2.3.1, proves a formula for projecting random variables from the enlarged filtration to the reference filtration, the main contribution being proposition 3.2.4. To prove a martingale representation theorem in the enlarged filtration, three preliminary steps are necessary: firstly a martingale characterisation property needs to be shown for martingales in the enlarged filtration. This is shown in section 3.2.2, theorem 3.2.5 and corollary 3.2.6.1 being the main contributions of this section. Next, the fundamental question of enlargement of filtration is answered in section 3.2.3, theorem 3.2.7 answering this question and giving the Doob-decomposition of any reference filtration martingale in the enlarged filtration. Proposition 3.2.8 being the most important result from this section to be used to prove a martingale representation theorem. Finally, the in order to prove a martingale representation theorem, the measures induced by the random times and their marks is introduced along with certain unoriginal results on their compensators, this is done in section 3.2.4. Having then completed all the necessary steps, a martingale representation theorem is proved in the enlarged filtration assuming it holds in the reference filtration. The main contribution of this section and the chapter is 3.2.10

3.1 Setup

The setup for the multiple default time and associated mark framework is not original and was first introduced in [Pham, 2010]. Begin with $n \in \mathbb{N}$:

- Let $\tau := (\tau_1, \tau_2, ..., \tau_n)$ be a non-decreasing sequence of random times, i.e. $\mathcal{F}$-random variables each taking values in $\mathbb{R}^+$.
- Let $\xi := (\xi_1, \xi_2, ..., \xi_n)$ be a sequence of random variables each taking values in a Borel set $E \subseteq \mathbb{R}^m$ for $m \geq 1$.

Define the following set, for $k \in \{1, 2, ..., n\}$:

$$\Theta_k := \{(u_1, u_2, ..., u_k) \in (\mathbb{R}^+)^k : u_1 \leq u_2 \leq ... \leq u_k\},$$

then each subset $(\tau_1, \tau_2, ..., \tau_k) \in \Theta_k$. We will focus on using a recursive-based approach to the progressive enlargement of $\mathbb{F}$ with the random times τ and their associated marks ξ. This is done as follows:

1. For $k \in \{1, 2, ..., n\}$ let
$$\mathbb{D}^k := \left(\mathcal{D}_t^k\right)_{t\geq 0}, \qquad \mathcal{D}_t^k := \sigma\left(\tau_k \wedge t\right) \vee \sigma\left(\xi_k \mathbb{1}_{\{\tau_k \leq t\}}\right).$$
2. For $k \in \{1, 2, ..., n\}$ let
$$\mathbb{G}^k := \left(\mathcal{G}_t^k\right)_{t\geq 0}, \qquad \mathcal{G}_t^k := \cap_{s>t}\left(\mathcal{F}_s \vee \mathcal{D}_s^1 \vee \mathcal{D}_s^2 \vee ... \vee \mathcal{D}_s^k\right).$$
3. The progressive enlargement of $\mathbb{F}$ with the random times τ and their associated marks ξ is then
$$\mathbb{G} := \left(\mathcal{G}_t\right)_{t\geq 0}, \qquad \mathcal{G}_t := \mathcal{G}_t^n.$$
4. For $k \in \{1, 2, ..., n\}$ it will be useful to define the initial enlargement of $\mathbb{F}$ with the random variables (τ, ξ) as follows:
$$\mathbb{G}^{\tau,\xi,k} := \left(\mathcal{G}_t^{\tau,\xi,k}\right)_{t\geq 0}, \qquad \mathcal{G}_t^{\tau,\xi,k} := \cap_{s>t}\left(\mathcal{F}_s \vee \sigma(\tau_1, \tau_2, .., \tau_k, \xi_1, \xi_2, ...\xi_k)\right).$$
$$\mathbb{G}^{\tau,\xi} := \left(\mathcal{G}_t^{\tau,\xi}\right)_{t\geq 0}, \qquad \mathbb{G}_t^{\tau,\xi} := \mathbb{G}_t^{\tau,\xi,n}.$$

Remark 3. *Using the same reasoning as Section 2.3,we see that for $t \geq 0$ and $k \in \{1, 2, ..., n\}$, $\mathcal{G}_t^k$ agrees with $\mathcal{G}_t^{\tau,\xi,k}$ when $\tau_k \leq t$ in the sense that,*

$$\mathbb{E}[X|\mathcal{G}_t^k]\mathbb{1}_{\{\tau_k \leq t\}} = \mathbb{E}[X|\mathcal{G}_t^{\tau,\xi,k}]\mathbb{1}_{\{\tau_k \leq t\}},$$

for any integrable random variable X. Furthermore, because the random times are ordered,

$$\mathbb{E}[X|\mathcal{G}_t^k]\mathbb{1}_{\{\tau_j \leq t\}} = \mathbb{E}[X|\mathcal{G}_t^{\tau,\xi,k}]\mathbb{1}_{\{\tau_j \leq t\}},$$

for $j \geq k$.

The following definitions and conventions will be used throughout for ease of presentation:

– For any $i, j \in \{1, 2, ..., n\}$ let $\tau^{(i:j)} = (\tau_i, \tau_{i+1}, ..., \tau_j)$ if $i \leq j$ and $\tau^{(i:j)} = (\tau_i, \tau_{i-1}, ..., \tau_j)$ if $i \geq j$. In a similar way, define $\xi^{(i:j)}$. For notational ease $\tau^{(k)}$ and $\xi^{(k)}$ shall be used to denote $\tau^{(1:k)}$ and $\xi^{(1:k)}$ respectively.

– Similarly for any $u \in \Theta_n$, $e \in E^n$ and $i, j \in \{1, 2, ..., n\}$ define $u^{(i:j)}$ and $e^{(i:j)}$ as above.

– A convention used throughout will be $\tau_0 = 0$ and $\tau_{n+1} = \infty$.

– Similarly, ξ_0 and ξ_{n+1} can be thought of as deterministic maps adding no source of randomness.

– Finally, as the definition suggests, $\mathbb{G}^n := \mathbb{G}$ and the convention will be $\mathbb{G}^0 := \mathbb{F}$.

The optional and predictable sigma-algebras play an important role in decomposing $\mathbb{G}$-optional and predictable processes, define the family of sigma-algebras as follows, for $k \in \{1, 2, ..., n\}$:

– $\mathcal{O}(\mathbb{G}^k)$ (resp. $\mathcal{O}(\mathbb{F})$) is the σ-algebra generated by $\mathbb{G}^k$-optional processes (resp. $\mathbb{F}$-optional processes).

– $\mathcal{P}(\mathbb{G}^k)$ (resp. $\mathcal{P}(\mathbb{F})$) is the σ-algebra generated by $\mathbb{G}^k$-predictable processes (resp. $\mathbb{F}$-predictable processes).

– $\mathcal{O}(\mathbb{F}, \Theta_k, E^k) = \mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\Theta_k) \otimes \mathcal{B}(E^k)$.

– $\mathcal{P}(\mathbb{F}, \Theta_k, E^k) = \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Theta_k) \otimes \mathcal{B}(E^k)$.

As it stands, the setup is too general for any meaningful application, some structure needs to be placed on the random variables τ and ξ to continue our analysis. Just as in chapter 2, Jacod's density hypothesis from Definition 2.1.1 is used.

3.2 Multiple Default Enlargement

This section presents the main results from this chapter. Using Jacod's density hypothesis, known results about the one-default enlargement are extended to the case of multiple defaults with random marks. The ultimate goal being to prove a martingale representation theorem in the enlarged filtration.
First, we assume a density hypothesis on the multiple default times and their associated marks and define the density proess, α.

Hypothesis 4. *The random variables (τ, ξ) satisfy Jacod's density hypothesis with respect to $\mathbb{F}$, i.e. $(\tau, \xi) \ll \mathbb{F}$. Let $\alpha = \mathcal{T}\{(\tau, \xi), \mathbb{F}\}$. That is, for any $t \geq 0, u \in \Theta_n$ and $e \in E^n$:*

$$\begin{aligned}\mathbb{P}((\tau, \xi) \in (du, de)|\mathcal{F}_t) =& \alpha_t(u_1, u_2, ..., u_n, e_1, e_2, ..., e_n) du_1 du_2 ... du_n de_1 de_2 ... de_n \\ =& \alpha_t(u, e)\; du\; de.\end{aligned}$$

α will be called the density process of (τ, ξ) with respect to $\mathbb{F}$.

Remark 4. *α is measurable with respect to $\mathcal{O}(\mathbb{F}, \Theta_n, E^n)$, that is, for any $(u, e) \in \Theta_n \times E^n$, $\alpha(u, e) = (\alpha_t(u, e))_{t \geq 0}$ is $\mathbb{F}$-optional and for any $t \geq 0$, the mapping $(\omega, u, e) \to \alpha_t(\omega, u, e)$ is $\mathcal{F}_t \otimes \mathcal{B}(\Theta_n) \otimes \mathcal{B}(E^n)$-measurable.*

We will assume furthermore that the Borel subset E which the random marks take their values in, is bounded according to the Lebesgue measure on R^m. Note that this is assumption is not restrictive as hypothesis 4 could have been stated as the random times and random marks being absolutely continuous with respect to the product of initial laws of τ and ξ. The chosen method is merely for ease-of-notation. The interested reader is encouraged to look at the original formulation of the initial enlargement of filtration in Jacod [1985]. Note that the random times $\{\tau_1, \tau_2, ..., \tau_n\}$ are ordered, this implies that the density process has the following helpful property,

$$\alpha(u, e) = 0$$

if $u_k > u_{k+1}$ for any $k \in \{1, 2, ..., n-1\}$.

In order to study $\mathbb{G}$-adapted processes, we make use of the decomposition of $\mathbb{G}$-optional and predictable processes found in Lemma 2.1 in Pham [2010], see also Theorem 7.5 and Remark 7.6 in Song [2014] for an insightful discussion on decomposing $\mathbb{G}$-optional processes. Note furthermore that the decomposition of $\mathbb{G}$-predictable processes does not require any hypothesis on τ or ξ (see Lemma 4.4 in Jeulin [1980]). The necessity of a hypothesis on τ and ξ for the decomposition of $\mathbb{G}$-optional processes is motivated by the famous counter-example of Barlow [1978].

Proposition 3.2.1. *For $k \in \{1, 2, ..., n\}$*

- *A process Y is $\mathbb{G}^k$-optional if and only if it admits a decomposition*

$$Y_t = \sum_{j=0}^{k-1} Y_t^j(\tau^{(j)}, \xi^{(j)}) \mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}} + Y_t^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \leq t\}},$$

where each Y^j is $\mathcal{O}(\mathbb{F}, \Theta_j, E^j)$-measurable.

- *A process Y is $\mathbb{G}^k$-predictable if and only if it admits a decomposition*

$$Y_t = \sum_{j=0}^{n} Y_t^j(\tau^{(j)}, \xi^{(j)}) \mathbb{1}_{\{\tau_j < t \leq \tau_{j+1}\}} + Y_t^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k < t\}},$$

where each Y^j is $\mathcal{P}(\mathbb{F}, \Theta_k, E^j)$-measurable.

In keeping with the recursive theme of this chapter, a decomposition of $\mathbb{G}^{k+1}$-measurable processes in terms of $\mathbb{G}^k$ and $\mathbb{G}^k \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E)$-measurable processes. This is stated in the following proposition.

Proposition 3.2.2. *For $k \in \{1, 2, ..., n\}$*

- *Every $\mathbb{G}^{k+1}$-optional process Y admits a decomposition*

$$Y_t = Y_t^k \mathbb{1}_{\{\tau_{k+1} > t\}} + \hat{Y}_t^k(\tau_{k+1}, \xi_{k+1}) \mathbb{1}_{\{\tau_{k+1} \leq t\}},$$

where Y^k is $\mathbb{G}^k$-optional and $\hat{Y}^k$ is $\mathcal{O}(\mathbb{G}^k) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E)$-measurable.

– *Every $\mathbb{G}^{k+1}$-predictable process Y admits a decomposition*

$$Y_t = Y_t^k \mathbb{1}_{\{\tau_{k+1}\geq t\}} + \hat{Y}_t^k(\tau_{k+1}, \xi_{k+1})\mathbb{1}_{\{\tau_{k+1}<t\}},$$

where Y^k is $\mathbb{G}^k$-predictable and $\hat{Y}^k$ is $\mathcal{P}(\mathbb{G}^k)\otimes\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{B}(E)$-measurable.

Proof. The result is proved for the optional case only, the predictable case is done similarly. Applying Proposition 3.2.1 to the $\mathbb{G}^{k+1}$-optional process Y, we get the existence of a family $\{\tilde{Y}^0, \tilde{Y}^1, ..., \tilde{Y}^{k+1}\}$ such that:

$$\begin{aligned} Y_t &= \sum_{j=0}^{k} \tilde{Y}_t^j(\tau^{(j)}, \xi^{(j)})\mathbb{1}_{\{\tau_j\leq t<\tau_{j+1}\}} + \tilde{Y}_t^{k+1}(\tau^{(k+1)}, \xi^{(k+1)})\mathbb{1}_{\{\tau_{k+1}\leq t\}} \\ &= \left(\sum_{j=0}^{k-1} \tilde{Y}_t^j(\tau^{(j)}, \xi^{(j)})\mathbb{1}_{\{\tau_j\leq t<\tau_{j+1}\}} + \tilde{Y}_t^k(\tau^{(k)}, \xi^{(k)})\mathbb{1}_{\{\tau_k\leq t\}}\right)\mathbb{1}_{\{\tau_{k+1}>t\}} \\ &\quad + \tilde{Y}_t^{k+1}(\tau^{(k+1)}, \xi^{(k+1)})\mathbb{1}_{\{\tau_{k+1}\leq t\}} \end{aligned}$$

Taking $Y_t^k = \sum_{j=0}^{k-1} \tilde{Y}_t^j(\tau^{(j)}, \xi^{(j)})\mathbb{1}_{\{\tau_j\leq t<\tau_{j+1}\}} + \tilde{Y}_t^k(\tau^{(k)}, \xi^{(k)})\mathbb{1}_{\{\tau_k\leq t\}}$ and $\hat{Y}_t^k(u_{k+1}, e_{k+1}) = \tilde{Y}_t^{k+1}(\tau^{(k)}, u_{k+1}, \xi^{(k)}, e_{k+1})$ the result is shown. □

By making a density hypothesis on the entire sequence of random times and random marks, we can conclude a density hypothesis of each subsequence of random times and marks as well as a density hypothesis on each random time and mark with respect to the progressively enlarged filtration of random times and marks before it. This is formalised in the following lemma.

Lemma 3.2.3. *For $k\in\{1,2,...,n-1\}$:*

(a)

$$\begin{aligned} &(\tau^{(k)}, \xi^{(k)}) \ll \mathbb{F} \text{ and} \\ &\mathcal{T}\{(\tau^{(k)}, \xi^{(k)}), \mathbb{F}\} = \int_{u_k}^{\infty}\int_{u_{k+1}}^{\infty}...\int_{u_{n-1}}^{\infty}\int_{E^{n-k}} \alpha(u,e)de^{(k+1:n)}du^{(n:k+1)} \\ &\qquad =: \alpha^{(k)}(u^{(k)}, e^{(k)}). \end{aligned}$$

(b) For any $k\in\{1,2,...,n\}$, $\alpha^{(k)}(\tau^{(k)}, \xi^{(k)}) > 0$.

(c)

$$\begin{aligned} &(\tau_{k+1}, e_{k+1}) \ll \mathbb{G}^{\tau,\xi,k} \text{ and} \\ &\mathcal{T}\{(\tau^{k+1}, \xi^{k+1}), \mathbb{G}^{\tau,\xi,k}\} = \frac{\alpha^{(k+1)}(\tau^{(k)}, u_{k+1}, \xi^{(k)}, e_{k+1})}{\alpha^{(k)}(\tau^{(k)}, \xi^{(k)})} \\ &\qquad =: \alpha^k(u_{k+1}, e_{k+1}). \end{aligned}$$

(d)

$$\begin{aligned} &(\tau^{(k+1:n)}, \xi^{(k+1:n)}) \ll \mathbb{G}^{\tau,\xi,k} \text{ and} \\ &\mathcal{T}\{(\tau^{(k+1:n)}, \xi^{(k+1:n)}), \mathbb{G}^{\tau,\xi,k}\} = \frac{\alpha(\tau^{(k)}, u^{(k+1:n)}, \xi^{(k)}, e^{(k+1:n)})}{\alpha^{(k)}(\tau^{(k)}, \xi^{(k)})} \\ &\qquad =: \alpha^{k+1:n}(u^{(k+1:n)}, e^{(k+1:n)}). \end{aligned}$$

(Note the difference between $\alpha^{(k)}$ and α^k defined in (a) and (b) respectively.)

Proof. To prove the four assertions, we let $t > 0$ be a fixed finite time.

(a) The first result is obtained by using the fact that

$$\begin{aligned}&\mathbb{P}\left((\tau^{(k)},\xi^{(k)}) \in \left(du^{(k)},de^{(k)}\right)|\mathcal{F}_t\right)\\ =&\int_{u_k}^{\infty}\int_{u_{k+1}}^{\infty}\cdots\int_{u_{n-1}}^{\infty}\int_{E^{n-k}}\mathbb{P}((\tau,\xi)\in(du,de)|\mathcal{F}_t)\\ =&\alpha_t^{(k)}(u^{(k)},e^{(k)})du^{(k)}de^{(k)}\end{aligned}$$

(b) For any $k \in \{1,2...,n\}$, define the stopping time

$$R^{(k)}(u^{(k)},e^{(k)}) = \inf\{t \geq 0 : \alpha_t^{(k)}(u^{(k)},e^{(k)}) = 0\},$$

then we want to show that $R^{(k)}(\tau^{(k)},\xi^{(k)}) = \infty$. To do so, consider the following for any $t \geq 0$:

$$\mathbb{E}\left[\mathbb{1}_{\{R^{(k)}(\tau^{(k)},\xi^{(k)})\leq t\}}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{R^{(k)}(\tau^{(k)},\xi^{(k)})\leq t\}}|\mathcal{F}_t\right]\right].$$

The inner expectation is computed using Proposition 2.2.3 as follows

$$\begin{aligned}=&\mathbb{E}\left[\int_0^{\infty}\int_{u_1}^{\infty}\cdots\int_{u_{n-1}}^{\infty}\int_{E^n}\mathbb{1}_{\{R^{(k)}(u^{(k)},e^{(k)})\leq t\}}\alpha_t(u,e)de\,du^{(n:1)}\right]\\ =&\mathbb{E}\left[\int_0^{\infty}\int_{u_1}^{\infty}\cdots\int_{u_k}^{\infty}\int_{E^{n-k}}\mathbb{1}_{\{R^{(k)}(u^{(k)},e^{(k)})\leq t\}}\alpha_t^{(k)}(u^{(k)},e^{(k)})de^{(k)}du^{(k:1)}\right]\\ =&0.\end{aligned}$$

(c) Note that according to Lemma 4.20 in Aksamit and Jeanblanc [2017], the filtration

$$\mathbb{G}^{\tau,\xi,k,0} := \left(\mathcal{G}_t^{\tau,\xi,k,0}\right)_{t\geq 0}, \quad \mathcal{G}_t^{\tau,\xi,k,0} := \mathcal{F}_t \vee \sigma(\tau^{(k)}) \vee \sigma(\xi^{(k)})$$

is right continuous and hence $\mathbb{G}^{\tau,\xi,k} = \mathbb{G}^{\tau,\xi,k,0}$. Let $C \in \mathcal{B}(\mathbb{R}^+)\otimes\mathcal{B}(E)$ then for a fixed finite time $t > 0$ consider

$$\mathbb{E}[\mathbb{1}_{\{(\tau_{k+1},\xi_{k+1})\in C\}}|\mathcal{G}_t^{\tau,\xi,k}].$$

Firstly note that this is $\mathcal{G}_t^{\tau,\xi,k}$ measurable and hence by Proposition 4.22 in Aksamit and Jeanblanc [2017] there exists a $\mathcal{F}_t \otimes \mathcal{B}(\Theta_k) \otimes \mathcal{B}(E^k)$-measurable random variable x such that

$$\mathbb{E}[\mathbb{1}_{\{(\tau_{k+1},\xi_{k+1})\in C\}}|\mathcal{G}_t^{\tau,\xi,k}] = x(\tau^{(k)},\xi^{(k)}).$$

Secondly, the fact that $\mathcal{G}_t^{\tau,\xi,k} = \mathcal{F}_t \vee \sigma(\tau^{(k)}) \vee \sigma(\xi^{(k)})$ means that the probability density of (τ_{k+1},ξ_{k+1}) conditioned on $\mathcal{G}_t^{\tau,\xi,k}$ is equal to

$$\begin{aligned}&\mathbb{P}((\tau_{k+1},\xi_{k+1}) \in (du_{k+1},de_{k+1})|\mathcal{G}_t^{\tau,\xi,k})\\ =&\frac{\mathbb{P}\left((\tau^{(k+1)},\xi^{(k+1)}) \in (du^{(k+1)},de^{(k+1)})|\mathcal{F}_t \vee \sigma(\tau^{(k)}) \vee \sigma(\xi^{(k)})\right)}{\mathbb{P}\left((\tau^{(k)},\xi^{(k)}) \in (du^{(k)},de^{(k)})|\mathcal{F}_t\right)}\\ =&\frac{\mathbb{P}\left((\tau^{(k+1)},\xi^{(k+1)}) \in (du^{(k+1)},de^{(k+1)})|\mathcal{G}_t^{\tau,\xi,k}\right)}{\mathbb{P}\left((\tau^{(k)},\xi^{(k)}) \in (du^{(k)},de^{(k)})|\mathcal{F}_t\right)}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{(\tau_{k+1},\xi_{k+1})\in C\}}|\mathcal{G}_t^{\tau,\xi,k}] &= \int_C \mathbb{P}((\tau_{k+1},\xi_{k+1})\in(du_{k+1},de_{k+1})|\mathcal{G}_t^{\tau,\xi,k})\\ &= \int_C \frac{\mathbb{P}((\tau^{(k+1)},\xi^{(k+1)})\in(du^{(k+1)},de^{(k+1)})|\mathcal{F}_t)}{\mathbb{P}((\tau^{(k)},\xi^{(k)})\in(du^{(k)},de^{(k)})|\mathcal{F}_t)}\Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}\\ &= \int_C \frac{\alpha_t^{(k+1)}(\tau^{(k)},u_{k+1},\xi^{(k)},e_{k+1})}{\alpha_t^{(k)}(\tau^{(k)},\xi^{(k)})}de_{k+1}du_{k+1}.\end{aligned}$$

Note the evaluation at $\tau^{(k)}$ and $\xi^{(k)}$ in the second equality is possible due to the fact that x is $\mathcal{F}_t\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$ and hence

$$x(\tau^{(k)},\xi^{(k)}) = x(u^{(k)},e^{(k)})|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}.$$

(d) Using a similar argument to (b), if $D\in\mathcal{B}(\Theta_{n-k})\otimes\mathcal{B}(E^{n-k})$ then

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{(\tau^{(k+1:n)},\xi^{(k+1:n)}\in D\}}|\mathcal{G}_t^{\tau,\xi,k}] &= \int_D \mathbb{P}((\tau^{(k+1:n)},\xi^{(k+1:n)})\in(du^{(k+1:n)},de^{(k+1:n)})|\mathcal{G}_t^{\tau,\xi,k})\\ &= \int_D \frac{\mathbb{P}((\tau,\xi)\in(du,de)|\mathcal{F}_t)}{\mathbb{P}((\tau^{(k)},\xi^{(k)})\in(du^{(k)},de^{(k)})|\mathcal{F}_t)}\Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}\\ &= \int_D \frac{\alpha_t(\tau^{(k)},u^{(k+1:n)},\xi^{(k)},e^{(k+1:n)})}{\alpha_t^{(k)}(\tau^{(k)},\xi^{(k)})}de^{(k+1:n)}du^{(k+1:n)}.\end{aligned}$$

□

From Jacod [1985] and Section 2.2 in Chapter 2, we know that for a fixed $(u,e)\in\Theta_n\times E^n$, $\alpha(u,e)$ is an $\mathbb{F}$-martingale. It follows from this fact that for any $k\in\{1,2,...,n\}$ and $(u_{k+1},e_{k+1})\in\Theta_k\times E^k$, $\alpha^k(u_{k+1},e_{k+1})$ is a $\mathbb{G}^{\tau,\xi,k}$-martingale.
Before we continue the analysis of the enlarged filtration $\mathbb{G}$, we introduce the following family of processes which will play a crucial role in future results. For any $k=\{0,1,...,n-1\}$ define

$$G_t^k = \mathbb{P}(\tau_{k+1}>t|\mathcal{G}_t^k). \tag{3.1}$$

Note from Remark 3 that:

$$G_t^k\mathbb{1}_{\{\tau_k\le t\}} = \mathbb{P}(\tau_{k+1}>t|\mathcal{G}_t^{\tau,\xi,k})\mathbb{1}_{\{\tau_k\le t\}} = \int_t^\infty\int_E \alpha_t^k(u_{k+1},e_{k+1})de_{k+1}du_{k+1}\mathbb{1}_{\{\tau_k\le t\}},$$

which implies

$$G_t^k = \mathbb{1}_{\{\tau_k>t\}} + \mathbb{1}_{\{\tau_k\le t\}}\int_t^\infty\int_E \alpha_t^k(u_{k+1},e_{k+1})de_{k+1}du_{k+1}.$$

3.2.1 Projection formulas

This section deals with projection of $\mathbb{G}$-measurable random variables to $\mathbb{F}$. Before being able to do this, a family of processes is defined in the following:

$$\begin{aligned}
\gamma_t^0 &:= = G_t^0 \mathbb{P}(\tau_1 > t|\mathcal{F}_t) = \int_t^\infty \int_{u_1}^\infty \int_{u_2}^\infty \dots \int_{u_{n-1}}^\infty \int_{E^n} \alpha_t(u,e) de du \\
\gamma_t^1(u_1, e_1) &:= \int_t^\infty \int_{u_2}^\infty \int_{u_3}^\infty \dots \int_{u_{n-1}}^\infty \int_{E^{n-1}} \alpha_t(u,e) de^{(2:n)} du^{(n:2)} \\
&\vdots \\
\gamma_t^{n-1}(u^{(n-1)}, e^{(n-1)}) &:= \int_t^\infty \int_E \alpha_t(u,e) de_n du_n \\
\gamma_t^n(u,e) &:= \alpha_t(u,e).
\end{aligned} \tag{3.2}$$

Note the subtle difference between $\gamma_t^k(u^{(k)}, e^{(k)}) = \int_t^\infty \int_{u_{k+1}}^\infty \dots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha(u,e) de^{(k+1:n)} du^{(n:k+1)}$ and $\alpha_t^{(k)}(u^{(k)}, e^{(k)}) = \int_{u_k}^\infty \int_{u_{k+1}}^\infty \dots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha(u,e) de^{(k+1:n)} du^{(n:k+1)}$ means that $\gamma_t^k(t, u^{(k-1)}, e^{(k)}) = \alpha_t^{(k)}(t, u^{(k-1)}, e^{(k)})$.
Now for any fixed $T > 0$ and $k \in \{1, 2, .., n\}$, define the following operator on $L^1(\mathcal{F}_T \otimes \mathcal{B}(\Theta_n) \otimes \mathcal{B}(E^n))$. For $X \in L^1(\mathcal{F}_T \otimes \mathcal{B}(\Theta_n) \otimes \mathcal{B}(E^n))$,

$$\mathcal{E}_{t,T}^k(X)(u^{(k)}, e^{(k)}) := \int_t^\infty \int_{u_{k+1}}^\infty \dots \int_{u_{n-1}}^\infty \int_{E^{n-k}} X(u,e) \alpha_T(u,e) de^{(k+1:n)} du^{(n:k+1)}.$$

We are now in a position to project random variables from the enlarged filtration to the reference filtration. The following proposition shows how to compute conditional expectations in $\mathbb{G}$ in terms of parametric conditional expectations in $\mathbb{F}$.

Proposition 3.2.4. *Let $T > 0$ be a fixed finite time and $t \leq T$. For any $X \in L^1(\mathcal{F}_T \otimes \mathcal{B}(\Theta_n) \otimes \mathcal{B}(E^n))$,*

$$\mathbb{E}[X(\tau,\xi)|\mathcal{G}_t] = \sum_{k=0}^n \frac{\mathbb{E}\Big[\mathcal{E}_{t,T}^k(X)(u^{(k)}, e^{(k)})|\mathcal{F}_t\Big]\Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}}$$

Proof. Firstly we note that from Proposition 3.2.1 that $\mathbb{E}[X(\tau,\xi)|\mathcal{G}_t]$ admits a decomposition as follows

$$\mathbb{E}[X(\tau,\xi)|\mathcal{G}_t] = \sum_{k=0}^n x_t^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}}, \tag{3.3}$$

where each x^k is $\mathcal{O}(\mathbb{F}, \Theta_k, E^k)$-measurable.
Then to show the result we shall attempt to prove that for $k \in \{0, 1, ..., n\}$:

$$x_t^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}} = \frac{\mathbb{E}\Big[\mathcal{E}_{t,T}^k(X)(u^{(k)}, e^{(k)})|\mathcal{F}_t\Big]\Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}}.$$

Firstly, Equation (3.3) says that

$$x_t^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}} = \mathbb{E}[X(\tau,\xi)|\mathcal{G}_t] \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}}.$$

The agreement of $\mathcal{G}_t^k$ and $\mathcal{G}_t^{\tau,\xi,k}$ when $\tau_k \le t$ means that the conditional expectation on the right side is a version of the conditional expectation with respect to $\mathcal{G}_t^{\tau,\xi,k}$. The term on the left hand side is $\mathcal{G}_t^{\tau,\xi,k}$-measurable except for the factor $\mathbb{1}_{\{\tau_{k+1}>t\}}$, which when conditioned on $\mathcal{G}_t^k$ is identified as G_t^k. Together this means

$$x^k(\tau^{(k)},\xi^{(k)})G_t^k\mathbb{1}_{\{\tau_k\le t\}} = \mathbb{E}[X(\tau,\xi)\mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_t^{\tau,\xi,k}]\mathbb{1}_{\{\tau_k\le t\}}.$$

From Lemma 3.2.3 we know

$$\begin{aligned}&\mathbb{P}((\tau^{(k+1:n)},\xi^{(k+1:n)})\in(du^{(k+1:n)},de^{(k+1:n)})|\mathcal{G}_T^{\tau,\xi,k})\\ =&\alpha_T^{k+1:n}(u^{(k+1:n)},e^{(k+1:n)})du^{(k+1:n)}de^{(k+1:n)}\\ =&\frac{\alpha_T(\tau^{(k)},u^{(k+1:n)},\xi^{(k)},e^{(k+1:n)})}{\alpha_T^{(k)}(\tau^{(k)},\xi^{(k)})}du^{(k+1:n)}de^{(k+1:n)},\end{aligned}$$

then

$$\mathbb{E}\left[X(\tau,\xi)\mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_t^{\tau,\xi,k}\right] = \mathbb{E}\left[\mathbb{E}\left[X(\tau,\xi)\mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_T^{\tau,\xi,k}\right]|\mathcal{G}_t^{\tau,\xi,k}\right].$$

Note that $\tau^{(k)}$ and $\xi^{(k)}$ are $\mathcal{G}_T^{\tau,\xi,k}$-measurable and X is $\mathcal{F}_T\otimes\mathcal{B}(\Theta_n)\otimes\mathcal{B}(E^n)$-measurable, meaning $X(\tau^{(k)},u^{(k+1:n)},\xi^{(k)},e^{(k+1:n)})$ is $\mathcal{G}_T^{\tau,\xi,k}$-measurable. Therefore if we want to compute the inner expectation from above, we need to marginalise $\tau^{(k+1:n)}$ and $\xi^{(k+1:n)}$ using the $\mathcal{G}_T^{\tau,\xi,k}$-distribution.

$$\begin{aligned}&\mathbb{E}[X(\tau,\xi)\mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_T^{\tau,\xi,k}]\\ =&\int_t^\infty\int_{u_{k+1}}^\infty\dots\int_{u_{n-1}}^\infty\int_{E^{n-k}}X(\tau^{(k)},u^{(k+1:n)},\xi^{(k)},e^{(k+1:n)})\alpha_T^{(k+1:n)}(u^{(k+1:n)},e^{(k+1:n)})de^{(k+1:n)}du^{(n:k+1)}\\ =&\frac{\int_t^\infty\int_{u_{k+1}}^\infty\dots\int_{u_{n-1}}^\infty\int_{E^{n-k}}X(\tau^{(k)},u^{(k+1:n)},\xi^{(k)},e^{(k+1:n)})\alpha_T(\tau^{(k)},u^{(k+1:n)},\xi^{(k)},e^{(k+1:n)})de^{(k+1:n)}du^{(n:k+1)}}{\alpha_T^{(k)}(\tau^{(k)},\xi^{(k)})}\\ =&\frac{\mathcal{E}_{t,T}^k(X)(\tau^{(k)},\xi^{(k)})}{\alpha_T^{(k)}(\tau^{(k)},\xi^{(k)})}.\end{aligned}$$

Now from Proposition 2.2.3 we know that for any $Z\in L^1(\mathcal{F}_T\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k))$

$$\mathbb{E}[Z(\tau^{(k)},\xi^{(k)})|\mathcal{G}_t^{\tau,\xi,k}] = \frac{\mathbb{E}\left[Z(u^{(k)},e^{(k)})\alpha_T^{(k)}(u^{(k)},e^{(k)})\right]\Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}}{\alpha_t^{(k)}(\tau^{(k)},\xi^{(k)})}.$$

Now define the $\mathcal{F}_T\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$-measurable random variable Z by

$$Z(u^{(k)},e^{(k)}) := \frac{\mathcal{E}_{t,T}^k(X)(u^{(k)},e^{(k)})}{\alpha_T^{(k)}(u^{(k)},e^{(k)})},$$

then we get

$$\mathbb{E}\left[\frac{\mathcal{E}_{t,T}^k(X)(\tau^{(k)},\xi^{(k)})}{\alpha_T^{(k)}(\tau^{(k)},\xi^{(k)})}|\mathcal{G}_t^{\tau,\xi,k}\right] = \frac{\mathbb{E}\left[\mathcal{E}_{t,T}^k(X)(u^{(k)},e^{(k)})|\mathcal{F}_t\right]\Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}}{\alpha_t^{(k)}(\tau^{(k)},\xi^{(k)})}.$$

Using Lemma 3.2.3 again, we compute

$$G_t^k = \mathbb{P}(\tau_{k+1} > t|\mathcal{G}_t^{\tau,\xi,k}) = \int_t^\infty \int_E \alpha_t^k(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1}$$
$$= \frac{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}{\alpha_t^{(k)}(\tau^{(k)}, \xi^{(k)})},$$

combining this expression with the one for $\mathbb{E}[X(\tau,\xi)\mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_t^{\tau,\xi,k}]$ yields the result. □

3.2.2 Characterisation of $\mathbb{G}$-martingales

Is this section we prove that under Hypothesis 4, we can give a condition for a $\mathbb{G}$-adapted process to be a martingale. This result on its own does not bare a lot of significance however it is useful in answering the fundamental question of enlargement of filtration and deriving the semimartingale decomposition formula, ultimately enabling a martingale representation theorem to be proven. The following theorem is the first of three contributions from this section.

Theorem 3.2.5. *Let $Y = \sum_{k=1}^n Y^k \mathbb{1}_{[\![\tau_k,\tau_{k+1})\!)}$ be a $\mathbb{G}$-adapted process and where each $Y^k \in \mathcal{O}(\mathbb{F}, \Theta_k, E^k)$, then Y is a $\mathbb{G}$-martingale if the following hold:*

a) $(Y_t^n(u,e)\alpha_t(u,e) : t \geq u_n)$ is an $\mathbb{F}$-martingale for all $u \in \Theta_n$ and $e \in E^n$.

b) $\left(m_t^k := \mathbb{E}[Y_t|\mathcal{G}_t^k] : t \geq \tau_k\right)$ is a $\mathbb{G}^k$-martingale for all $k \in \{0, 1, ..., n-1\}$.

Proof. To show that Y is a $\mathbb{G}$-martingale, we need to show that for any $0 \leq s \leq t$, $\mathbb{E}[Y_t|\mathcal{G}_s] = Y_s$. From Proposition 3.2.1 we have that there exist $\mathcal{O}(\mathbb{F}, \Theta_k, E^k)$-measurable processes y^k for $k \in \{0, 1, ..., n\}$ such that the optional projection of Y_t on $\mathcal{G}_s$ is equal to

$$\mathbb{E}[Y_t|\mathcal{G}_s] = \sum_{k=1}^n y_s^k \mathbb{1}_{\{\tau_k \leq s < \tau_{k+1}\}}.$$

We therefore need to show that

$$y_s^k(\tau^{(k)}, \xi^{(k)})\mathbb{1}_{\{\tau_k \leq s < \tau_{k+1}\}} = Y_s^k(\tau^{(k)}, \xi^{(k)})\mathbb{1}_{\{\tau_k \leq s < \tau_{k+1}\}},$$

for all $k \in \{0, 1, ..., n\}$. We do this in three steps:

1. For $k = n$:

$$y_s^n(\tau,\xi)\mathbb{1}_{\{\tau_n \leq s\}} = \mathbb{E}[Y_t|\mathcal{G}_s]\mathbb{1}_{\{\tau_n \leq s\}}$$
$$= \mathbb{E}[Y_t^n(\tau,\xi)|\mathcal{G}_s^{\tau,\xi}]\mathbb{1}_{\{\tau_n \leq s\}}.$$

The right hand side is computed using Proposition 2.2.3 as follows

$$\mathbb{E}[Y_t^n(\tau,\xi)|\mathcal{G}_s^{\tau,\xi}]\mathbb{1}_{\{\tau_n \leq s\}} = \frac{\mathbb{E}\Big[Y_t^n(u,e)\alpha_t(u,e)|\mathcal{F}_s\Big]|_{\substack{u=\tau\\e=\xi}}\mathbb{1}_{\{\tau_n \leq s\}}}{\alpha_s(\tau,\xi)}.$$

The term inside the expectation is assumed to be an $\mathbb{F}$-martingale, meaning

$$\frac{\mathbb{E}\Big[Y_t^n(u,e)\alpha_t(u,e)|\mathcal{F}_s\Big]|_{\substack{u=\tau\\e=\xi}}\mathbb{1}_{\{\tau_n\le s\}}}{\alpha_s(\tau,\xi)} = \frac{Y_s^n(\tau,\xi)\alpha_s(\tau,\xi)\mathbb{1}_{\{\tau_n\le s\}}}{\alpha_s(\tau,\xi)}$$
$$= Y_s^n(\tau,\xi)\mathbb{1}_{\{\tau_n\le s\}}.$$

2. For $k = n-1$:

$$y_s^{n-1}(\tau^{(n-1)},\xi^{(n-1)})\mathbb{1}_{\{\tau_{n-1}\le s<\tau_n\}} = \mathbb{E}[Y_t|\mathcal{G}_s]\mathbb{1}_{\{\tau_{n-1}\le s<\tau_n\}}$$

Taking expectations with respect to $\mathcal{G}_s^{n-1}$ one obtains

$$\mathbb{E}[y_s^{n-1}(\tau^{(n-1)},\xi^{(n-1)})\mathbb{1}_{\{\tau_{n-1}\le s<\tau_n\}}|\mathcal{G}_s^{n-1}] = \mathbb{E}[Y_t\mathbb{1}_{\{\tau_{n-1}\le s<\tau_n\}}|\mathcal{G}_s^{n-1}]$$
$$y_s^{n-1}(\tau^{(n-1)},\xi^{(n-1)})G_s^{n-1}\mathbb{1}_{\{\tau_{n-1}\le s\}} = \mathbb{E}[Y_t\mathbb{1}_{\{\tau_n>s\}}|\mathcal{G}_s^{\tau,\xi,n-1}]\mathbb{1}_{\{\tau_{n-1}\le s\}}.$$

Recall the definition of G^{n-1} from Equation (3.1). Now utilising the fact that $\mathcal{G}_s$ and $\mathcal{G}_s^{\tau,\xi,n-1}$ agree when $\tau_{n-1}\le s$ the right hand side can rewritten as follows

$$= \left(\mathbb{E}\left[Y_t|\mathcal{G}_s^{\tau,\xi,n-1}\right] - \mathbb{E}\left[Y_t\mathbb{1}_{\{\tau_n\le s\}}|\mathcal{G}_s^{\tau,\xi,n-1}\right]\right)\mathbb{1}_{\{\tau_{n-1}\le s\}}$$
$$= \Big(\mathbb{E}\left[Y_t|\mathcal{G}_s^{\tau,\xi,n-1}\right] - \mathbb{E}\Big[\mathbb{E}\Big[Y_t^n(\tau,\xi)\mathbb{1}_{\{\tau_n\le s\}}\Big|\mathcal{G}_t^{\tau,\xi,n-1}\Big]\Big|\mathcal{G}_s^{\tau,\xi,n-1}\Big]\Big)\mathbb{1}_{\{\tau_{n-1}\le s\}}.$$

The first term may be simplified according to assumption (b). The second term is computed using Lemma 3.2.3 in conjunction with Proposition 2.2.3 from Chapter 2 to yield

$$= \left(m_s^{n-1} - \mathbb{E}\left[\int_{\tau_{n-1}}^s\int_E Y_t^n(\tau^{(n-1)},u_n,\xi^{(n-1)},e_n)\alpha_t^{n-1}(u_n,e_n)de_n du_n\Big|\mathcal{G}_s^{\tau,\xi,n-1}\right]\right)\mathbb{1}_{\{\tau_{n-1}\le s\}}.$$

The second term can be further simplified by using Proposition 2.2.3 from Chapter 2 to project the integrand on to $\mathcal{F}_s$. Note that by definition $\alpha^{n-1}(u_n,e_n) = \frac{\alpha(\tau^{(n-1)},u_n,\xi^{(n-1)},e_n)}{\alpha^{(n-1)}(\tau^{(n-1)},\xi^{(n-1)})}$ and that the $\mathbb{G}^{\tau,\xi,n-1}$ density of $\left(\tau^{(n-1)},\xi^{(n-1)}\right)$ is $\alpha^{(n-1)}$, this results in the following simplified expression for the second term

$$= \left(m_s^{n-1} - \frac{\mathbb{E}\Big[\int_{u_{n-1}}^s\int_E Y_t^n(u,e)\alpha_t(u,e)de_n du_n|\mathcal{F}_s\Big]\Big|_{\substack{u^{(n-1)}=\tau^{(n-1)}\\e^{(n-1)}=\xi^{(n-1)}}}}{\alpha_s^{(n-1)}(\tau^{(n-1)},\xi^{(n-1)})}\right)\mathbb{1}_{\{\tau_{n-1}\le s\}}.$$

The numerator is identified as the expectation of an $\mathbb{F}$-martingale, meaning

$$= \left(m_s^{n-1} - \frac{\mathbb{E}\Big[\int_{u_{n-1}}^s\int_E Y_s^n(u,e)\alpha_s(u,e)de_n du_n|\mathcal{F}_s\Big]\Big|_{\substack{u^{(n-1)}=\tau^{(n-1)}\\e^{(n-1)}=\xi^{(n-1)}}}}{\alpha_s^{(n-1)}(\tau^{(n-1)},\xi^{(n-1)})}\right)\mathbb{1}_{\{\tau_{n-1}\le s\}}.$$

Finally, the term inside the expectation is $\mathcal{F}_s$-measurable meaning the second term inside the brackets is simplified using Lemma 3.2.3 as follows

$$
\begin{aligned}
&= \left(m_s^{n-1} - \int_{\tau_{n-1}}^{s} \int_E Y_s^n(\tau^{(n-1)}, u_n, \xi^{(n-1)}, e_n)\alpha_s^{n-1}(u_n, e_n) de_n du_n \right) \mathbb{1}_{\{\tau_{n-1} \le s\}} \\
&= \left(m_s^{n-1} - \mathbb{E}\left[Y_s^n(\tau^{(n)}, \xi^{(n)}) \mathbb{1}_{\{\tau_n \le s\}} | \mathcal{G}_s^{\tau,\xi,n-1} \right] \right) \mathbb{1}_{\{\tau_{n-1} \le s\}} \\
&= \mathbb{E}\left[Y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)}) \mathbb{1}_{\{\tau_{n-1} \le s < \tau_n\}} | \mathcal{G}_s^{\tau,\xi,n-1} \right] \mathbb{1}_{\{\tau_{n-1} \le s\}} \\
&= Y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)}) G_s^{n-1} \mathbb{1}_{\{\tau_{n-1} \le s\}}
\end{aligned}
$$

Note that on the set $\{s < \tau_n\}$, $G_s^{n-1} > 0$ almost surely. Indeed,

$$
\mathbb{E}\left[\mathbb{1}_{\{s<\tau_n\}} \mathbb{1}_{\{G_s^{n-1}=0\}} \right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{s<\tau_n\}} \mathbb{1}_{\{G_s^{n-1}=0\}} | \mathcal{G}_s^{n-1} \right] \right] = \mathbb{E}\left[G_s^{n-1} \mathbb{1}_{\{G_s^{n-1}=0\}} \right] = 0.
$$

Therefore

$$
y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)}) \mathbb{1}_{\{\tau_{n-1} \le s < \tau_n\}} = Y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)}) \mathbb{1}_{\{\tau_{n-1} \le s < \tau_n\}}.
$$

3. For $k = 0, 1, ..., n-2$:

$$
y_s^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}} = \mathbb{E}[Y_t | \mathcal{G}_s] \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}}.
$$

Taking conditional expectations with respect to $\mathcal{G}_s^k$ one obtains

$$
\begin{aligned}
& y_s^k(\tau^{(k)}, \xi^{(k)}) G_s^k \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}} = \mathbb{E}\left[Y_t \mathbb{1}_{\{\tau_{k+1} > s\}} | \mathcal{G}_s^{\tau,\xi,k} \right] \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}} \\
&= \mathbb{E}\left[\mathbb{E}\left[Y_t | \mathcal{G}_s^{k+1} \right] \mathbb{1}_{\{\tau_{k+1} > s\}} | \mathcal{G}_s^{\tau,\xi,k} \right] \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}} \\
&= \mathbb{E}\left[Y_s \mathbb{1}_{\{\tau_{k+1} > s\}} | \mathcal{G}_s^{\tau,\xi,k} \right] \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}} \\
&= Y_s^k(\tau^k, \xi^k) G_s^k \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}}.
\end{aligned}
$$

As before $G_s^k > 0$ when $s < \tau_{k+1}$ implying

$$
y_s^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}} = Y_s^k(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \le s < \tau_{k+1}\}}.
$$

□

Note that had Hypothesis 4 been an equivalence hypothesis i.e. that the density process α be strictly positive, then Theorem 3.2.5 would be an equivalence, meaning we could classify all $\mathbb{G}$-martingales with these characteristics (see Proposition 3.3 in Callegaro et al. [2013] for the one-default case). This equivalence is not needed for future results and so the more flexible absolute continuity hypothesis is incorporated.

Remark 5. *Theorem 3.2.5 holds for local martingales too. Indeed every $\mathbb{G}^k$-stopping time is a $\mathbb{G}$-stopping time so if $(Y_t^n(u,e)\alpha_t(u,e) : t \ge u_n)$ is an $\mathbb{F}$-local martingale and m^k is a $\mathbb{G}^k$-local martingale then Y is a $\mathbb{G}$-local martingale.*

As is the theme of this chapter, we next give a characterisation of $\mathbb{G}^{k+1}$-martingales in terms of $\mathbb{G}^k$-martingales which in turn leads to a practical way of characterising $\mathbb{G}$-martingales only in terms of $\mathbb{F}$-martingales. This is explained in Corollary 3.2.6.1.

Proposition 3.2.6. *For any $k \in \{1,2,...,n-1\}$, a process $Y_t = Y_t^k \mathbb{1}_{\{\tau_{k+1}>t\}} + \hat{Y}_t^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le t\}}$, where Y^k is $\mathbb{G}^k$-adapted and $\hat{Y}^k$ is $\mathbb{G}^k \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E)$-adapted, is a $\mathbb{G}^{k+1}$-martingale if the following two conditions are satisfied:*

- $\left(\hat{Y}_t(u_{k+1},e_{k+1})\alpha_t^k(u_{k+1},e_{k+1}) : t \ge \tau_k\right)$ *is a $\mathbb{G}^k$-martingale.*
- $m_t = \mathbb{E}[Y_t|\mathcal{G}_t^k]$ *is a $\mathbb{G}^k$-martingale for $t \ge \tau_k$.*

Proof. Just as in the proof of Theorem 3.2.5, we want to show that for $0 \le s < t$, $\mathbb{E}[Y_t|\mathcal{G}_s^{k+1}] = Y_s$ when the given conditions are satisfied. From Proposition 3.2.2 we know there exists a $\mathcal{O}(\mathbb{G}^k)$-measurable process y^k and and a $\mathcal{O}(\mathbb{G}^k,\mathbb{R}^+,E)$-measurable process $\hat{y}^k$ such that

$$\mathbb{E}[Y_t|\mathcal{G}_t^{k+1}] = y_t^k \mathbb{1}_{\{\tau_{k+1}>s\}} + \hat{y}_t^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}}. \tag{3.4}$$

We aim to show that $Y_s^k \mathbb{1}_{\{\tau_{k+1}>s\}} = y_s^k \mathbb{1}_{\{\tau_{k+1}>s\}}$ and $\hat{Y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}} = \hat{y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}}$. We start with the second term of Equation (3.4).

$$\hat{y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}} = \mathbb{E}[Y_t|\mathcal{G}_s^{k+1}]\mathbb{1}_{\{\tau_{k+1}\le s\}}.$$

Remark 3 says that $\mathcal{G}_s^{k+1}$ and $\mathcal{G}_s^{\tau,\xi,k}$ agree when $\tau_{k+1} \le s$, meaning

$$\hat{y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}} = \mathbb{E}[\hat{Y}_t^k(\tau_{k+1},\xi_{k+1})|\mathcal{G}_s^{\tau,\xi,k+1}]\mathbb{1}_{\{\tau_{k+1}\le s\}}.$$

From Lemma 3.2.3, $(\tau^{(k+1)},\xi^{(k+1)}) \ll \mathbb{F}$ with $\alpha^{(k+1)} = \mathcal{T}\{(\tau^{(k+1)},\xi^{(k+1)}),\mathbb{F}\}$, this coupled with Proposition 2.2.3 yields

$$\hat{y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}} = \frac{\mathbb{E}\left[\hat{Y}_t^k(u_{k+1},e_{k+1})\alpha_t^{(k+1)}(u^{(k+1)},e^{(k+1)})\middle|\mathcal{F}_t\right]|_{\substack{u^{(k+1)}=\tau^{(k+1)}\\ e^{(k+1)}=\xi^{(k+1)}}}}{\alpha_s^{(k+1)}(\tau^{(k+1)},\xi^{(k+1)})}\mathbb{1}_{\{\tau_{k+1}\le s\}}.$$

Again, Lemma 3.2.3 says that $\alpha^k = \mathcal{T}\{(\tau_{k+1},\xi_{k+1}),\mathbb{G}^{\tau,\xi,k}\}$, this coupled with $\alpha^{(k)} = \mathcal{T}\{(\tau^{(k)},\xi^{(k)}),\mathbb{F}\}$ and Proposition 2.2.3 mean that the numerator is identified as

$$\begin{aligned}\hat{y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}} &= \frac{\mathbb{E}\left[\hat{Y}_t^k(u_{k+1},e_{k+1})\alpha_t^k(u_{k+1},e_{k+1})\middle|\mathcal{G}_s^{\tau,\xi,k}\right]|_{\substack{u_{k+1}=\tau_{k+1}\\ e_{k+1}=\xi_{k+1}}}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})}\mathbb{1}_{\{\tau_{k+1}\le s\}}\\ &= \frac{\mathbb{E}\left[\hat{Y}_t^k(u_{k+1},e_{k+1})\alpha_t^k(u_{k+1},e_{k+1})\middle|\mathcal{G}_s^k\right]|_{\substack{u_{k+1}=\tau_{k+1}\\ e_{k+1}=\xi_{k+1}}}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})}\mathbb{1}_{\{\tau_{k+1}\le s\}}.\end{aligned}$$

Finally, the term inside the expectation is assumed to be a $\mathbb{G}^k$-martingale, meaning

$$\hat{y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}} = \hat{Y}_s^k(\tau_{k+1},\xi_{k+1})\mathbb{1}_{\{\tau_{k+1}\le s\}}.$$

The first term of Equation (3.4) is simplified as follows:

$$y_s^k \mathbb{1}_{\{\tau_{k+1}>s\}} = \mathbb{E}[Y_t|\mathcal{G}_s^{k+1}]\mathbb{1}_{\{\tau_{k+1}>s\}}.$$

Taking expectations with respect to $\mathcal{G}_s^k$, we get

$$\begin{aligned} y_s^k G_s^k \mathbb{1}_{\{\tau_{k+1}>s\}} =& \mathbb{E}[Y_t \mathbb{1}_{\{\tau_{k+1}>s\}} | \mathcal{G}_s^k] \mathbb{1}_{\{\tau_{k+1}>s\}} \\ =& \left(\mathbb{E}[Y_t|\mathcal{G}_s^k] - \mathbb{E}[Y_t \mathbb{1}_{\{\tau_{k+1}\le s\}}|\mathcal{G}_s^k]\right) \mathbb{1}_{\{\tau_{k+1}>s\}}. \end{aligned}$$

Using the Tower property of expectations, the first term inside the brackets is identified as m_s. The second is simplified using the decomposition of Y_t and the fact that $\{\tau_{k+1} \le s\} \subseteq \{\tau_{k+1} \le t\}$ as follows

$$y_s^k G_s^k \mathbb{1}_{\{\tau_{k+1}>s\}} = \left(m_s - \mathbb{E}[\hat{Y}_t^k(\tau_{k+1}, \xi_{k+1}) \mathbb{1}_{\{\tau_{k+1}\le s\}} | \mathcal{G}_s^k]\right) \mathbb{1}_{\{\tau_{k+1}>s\}}.$$

Using Lemma 3.2.3 again, the second term inside the brackets is simplified to

$$y_s^k G_s^k \mathbb{1}_{\{\tau_{k+1}>s\}} = \left(m_s - \mathbb{E}[\int_0^s \int_E \hat{Y}_t^k(u_{k+1}, e_{k+1}) \alpha_t^k(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1} | \mathcal{G}_s^k]\right) \mathbb{1}_{\{\tau_{k+1}>s\}}.$$

The integrand is a $\mathcal{G}^k$-martingale by definition, therefore using Fubini's theorem, we get

$$\begin{aligned} y_s^k G_s^k \mathbb{1}_{\{\tau_{k+1}>s\}} =& \left(m_s - \int_0^s \int_E \hat{Y}_s^k(u_{k+1}, e_{k+1}) \alpha_s^k(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1}\right) \mathbb{1}_{\{\tau_{k+1}>s\}} \\ =& \left(m_s - \mathbb{E}\left[\hat{Y}_s^k(\tau_{k+1}, \xi_{k+1}) \mathbb{1}_{\{\tau_{k+1}\le s\}} | \mathcal{G}_s^{\tau,\xi,k}\right]\right) \mathbb{1}_{\{\tau_{k+1}>s\}} \\ =& \left(m_s - \mathbb{E}\left[\hat{Y}_s^k(\tau_{k+1}, \xi_{k+1}) \mathbb{1}_{\{\tau_{k+1}\le s\}} | \mathcal{G}_s^k\right]\right) \mathbb{1}_{\{\tau_{k+1}>s\}} \\ =& \mathbb{E}\left[Y_s - \hat{Y}_s^k(\tau_{k+1}, \xi_{k+1}) \mathbb{1}_{\{\tau_{k+1}\le s\}} | \mathcal{G}_s^k\right] \\ =& Y_s^k G_s^k \mathbb{1}_{\{\tau_{k+1}>s\}}. \end{aligned}$$

Finally, $G_s^k > 0$ when $\tau_{k+1} < s$, meaning $y_s^k \mathbb{1}_{\{\tau_{k+1}>s\}} = Y_s^k \mathbb{1}_{\{\tau_{k+1}>s\}}$. □

Combining Theorem 3.2.5 and Proposition 3.2.6 we get the following useful corollary.

Corollary 3.2.6.1. *Let $Y = \sum_{k=0}^n Y^k \mathbb{1}_{[\![\tau_k, \tau_{k+1})\!)}$ where each Y^k is $\mathcal{O}(\mathbb{F}, \Theta_k, E^k)$-measurable for $k \in \{0, 1, ..., n\}$, then Y is a $\mathbb{G}$-martingale if the following are all $\mathbb{F}$-martingales for all $u \in \Theta_n$ and $e \in E^n$:*

- $(Y_t^n(u,e)\alpha_t(u,e) : t \ge u_n)$
- $\left(Y_t^{n-1}(u^{(n-1)}, e^{(n-1)})\gamma_t^{n-1}(u^{(n-1)}, e^{(n-1)}) + \int_{u_{n-1}}^t \int_E Y_t^n(u,e)\alpha_t(u,e) de_n du_n : t \ge u_{n-1}\right)$
- $\Big(Y_t^{n-2}(u^{(n-2)}, e^{(n-2)})\gamma_t^{n-2}(u^{(n-2)}, e^{(n-2)})$
  $+ \int_{u_{n-2}}^t \int_{u_{n-1}}^\infty \int_{E^2} Y_t^{n-1}(u^{(n-1)}, e^{(n-1)})\alpha_t(u,e) de_n de_{n-1} du_n du_{n-1}$
  $+ \int_{u_{n-2}}^\infty \int_{u_{n-1}}^t \int_{E^2} Y_t^n(u,e)\alpha_t(u,e) de_n de_{n-1} du_n du_{n-1} : t \ge u_{n-2}\Big)$

  ⋮
- $\Big(Y_t^0 G_t^0 + \int_0^t \int_{u_1}^\infty ... \int_{u_{n-1}}^\infty \int_{E^n} Y_t^1(u_1, e_1)\alpha_t(u,e) de^{(n:1)} du^{(n:1)}$
  $+ \int_0^\infty \int_{u_1}^t \int_{u_2}^\infty ... \int_{u_{n-1}}^\infty \int_{E^n} Y_t^2(u^{(2)}, e^{(2)})\alpha_t(u,e) de^{(n:1)} du^{(n:1)}$
  $+ ... + \int_0^\infty \int_{u_1}^\infty ... \int_{u_{n-1}}^t \int_{E^n} Y_t^n(u,e)\alpha_t(u,e) de^{(n:1)} du^{(n:1)} : t \ge 0\Big),$

where γ^k is defined in Equation (3.2).

Proof. From Theorem 3.2.5, we need to show that $\mathbb{E}[Y_t|\mathcal{G}_t^k]$ is a $\mathbb{G}^k$-martingale for $t \geq \tau_k$ and all $k \in \{0, 1, ..., n-1\}$. Note however that for $t \geq \tau_k$

$$\mathbb{E}[Y_t|\mathcal{G}_t^k] = \mathbb{E}\left[\sum_{j=k}^{n} Y_t^j(\tau^{(j)}, \xi^{(j)})\mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}}|\mathcal{G}_t^k\right]$$

To prove the result, we show when $k = n-1$ and $\tau_{n-1} \leq s \leq t$ that,

$$\begin{aligned}&\mathbb{E}\left[Y_t^{(n-1)}(\tau^{(n-1)}, \xi^{(n-1)})\mathbb{1}_{\{t>\tau_n\}} + Y_t^n(\tau, \xi)\mathbb{1}_{\{\tau_n \leq t\}}|\mathcal{G}_s^{n-1}\right] \\ &= \mathbb{E}\left[Y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)})\mathbb{1}_{\{s<\tau_n\}} + Y_s^n(\tau, \xi)\mathbb{1}_{\{\tau_n \leq s\}}|\mathcal{G}_s^{n-1}\right],\end{aligned}$$

the case when $k \in \{0, 1, ..., n-2\}$, is done similarly. By assumption, we have:

$$\begin{aligned}&\mathbb{E}\left[Y_t^{n-1}(u^{(n-1)}, e^{(n-1)})\gamma_t^{n-1}(u^{(n-1)}, e^{(n-1)}) + \int_{u_{n-1}}^{t}\int_E Y_t^n(u, e)\alpha_t(u, e)de_n du_n|\mathcal{F}_s\right] \\ =&Y_s^{n-1}(u^{(n-1)}, e^{(n-1)})\gamma_s^{n-1}(u^{(n-1)}, e^{(n-1)}) + \int_{u_{n-1}}^{s}\int_E Y_s^n(u, e)\alpha_s(u, e)de_n du_n.\end{aligned}$$

Evaluating at $\tau^{(n-1)}$ and $\xi^{(n-1)}$

$$\begin{aligned}&\mathbb{E}\Bigg[Y_t^{n-1}(u^{(n-1)}, e^{(n-1)})\gamma_t^{n-1}(u^{(n-1)}, e^{(n-1)}) \\ &+ \int_{u_{n-1}}^{t}\int_E Y_t^n(u, e)\alpha_t(u, e)de_n du_n|\mathcal{F}_s\Bigg]\Bigg|_{\substack{u^{(n-1)}=\tau^{(n-1)} \\ e^{(n-1)}=\xi^{(n-1)}}} \\ =&Y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)})\gamma_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)}) \\ &+ \int_{\tau_{n-1}}^{s}\int_E Y_s^n(\tau^{(n-1)}, u_n, \xi^{(n-1)}, e_n)\alpha_s(\tau^{(n-1)}, u_n, \xi^{(n-1)}, e_n)de_n du_n.\end{aligned}$$

From Lemma 3.2.3, $\mathcal{T}\{(\tau_n, \xi_n), \mathbb{G}^{\tau,\xi,n-1}\} = \alpha(\tau^{(n-1)}, \cdot, \xi^{(n-1)}, \cdot)/\alpha^{(n-1)}(\tau^{(n-1)}, \xi^{(n-1)})$. Furthermore, for $\tau_{n-1} \leq s$, $\mathcal{G}_s^{\tau,\xi,n-1}$ agrees with $\mathcal{G}_s^{n-1}$, meaning Proposition 2.2.3 can be used to identify the left hand side as,

$$\begin{aligned}&\alpha_s^{(n-1)}(\tau^{(n-1)}, \xi^{(n-1)})\mathbb{E}[Y_t^{(n-1)}(\tau^{(n-1)}, \xi^{(n-1)})\mathbb{1}_{\{t<\tau_n\}} + Y_t^n(\tau, \xi)\mathbb{1}_{\{\tau_n \leq t\}}|\mathcal{G}_s^{n-1}] \\ =&Y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)})\gamma_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)}) \\ &+ \int_{\tau_{n-1}}^{s}\int_E Y_s^n(\tau^{(n-1)}, u_n, \xi^{(n-1)}, e_n)\alpha_s(\tau^{(n-1)}, u_n, \xi^{(n-1)}, e_n)de_n du_n.\end{aligned}$$

The right hand side when divided by $\alpha_s^{(n-1)}(\tau^{(n-1)}, \xi^{(n-1)})$ is identified as

$$\begin{aligned}&\mathbb{E}\left[Y_t^{(n-1)}(\tau^{(n-1)}, \xi^{(n-1)})\mathbb{1}_{\{t<\tau_n\}} + Y_t^n(\tau, \xi)\mathbb{1}_{\{\tau_n \leq t\}}|\mathcal{G}_s^{n-1}\right] \\ =&\mathbb{E}\left[Y_s^{n-1}(\tau^{(n-1)}, \xi^{(n-1)})\mathbb{1}_{\{s<\tau_n\}} + Y_s^n(\tau, \xi)\mathbb{1}_{\{\tau_n \leq s\}}|\mathcal{G}_s^{n-1}\right].\end{aligned}$$

□

3.2.3 $\mathbb{F}$-martingales in the enlarged filtration

We are now in a position where we can answer the question of whether $\mathbb{F}$-martingales remain semimartingales in the enlarged filtration and if so, how do we decompose them into a $\mathbb{G}$-martingale plus a finite variation process. The following theorem answers this question.

Theorem 3.2.7. *Any càdlàg $\mathbb{F}$-local martingale X is a $\mathbb{G}$-semimartingale and*

$$\hat{X}_t = X_t - \sum_{k=0}^{n} \int_{\tau_k \wedge t}^{\tau_{k+1}\wedge t} \frac{1}{\gamma^k_{s-}(\tau^{(k)}, \xi^{(k)})} d\langle X, \gamma^k(u^{(k)}, e^{(k)})\rangle_s \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}$$

is a $\mathbb{G}$-local martingale.

Note that the predictable quadratic covariation terms $\langle X, \gamma^k(u^{(k)}, e^{(k)})\rangle$ are computed in $\mathbb{F}$.

Proof. Firstly, it is noted that $\hat{X}$ can be decomposed as follows:

$$\hat{X}_t = \sum_{k=0}^{n} Y^k_t(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}},$$

where

$$\begin{aligned}
Y^0_t :=& X_t - \int_0^t \frac{1}{G^0_{s-}} \mathbb{1}_{\{G^0_{s-}>0\}} d\langle X, G^0\rangle_s, \\
Y^1_t(u_1, e_1) :=& X_t - \int_0^{u_1} \frac{1}{G^0_{s-}} \mathbb{1}_{\{G^0_{s-}>0\}} d\langle X, G^0\rangle_s - \int_{u_1}^t \frac{1}{\gamma^1_{s-}(u_1,e_1)} \mathbb{1}_{\{\gamma^1_{s-}(u_1,e_1)>0\}} d\langle X, \gamma^1(u_1,e_1)\rangle_s, \\
Y^2_t(u^{(2)}, e^{(2)}) :=& X_t - \int_0^{u_1} \frac{1}{G^0_{s-}} \mathbb{1}_{\{G^0_{s-}>0\}} d\langle X, G^0\rangle_s - \int_{u_1}^{u_2} \frac{1}{\gamma^1_{s-}(u_1,e_1)} \mathbb{1}_{\{\gamma^1_{s-}(u_1,e_1)>0\}} d\langle X, \gamma^1(u_1,e_1)\rangle_s \\
& - \int_{u_2}^t \frac{1}{\gamma^2_{s-}(u^{(2)},e^{(2)})} \mathbb{1}_{\{\gamma^2_{s-}(u^{(2)},e^{(2)})>0\}} d\langle X, \gamma^2(u^{(2)},e^{(2)})\rangle_s, \\
& \vdots \\
Y^n_t(u^{(n)}, e^{(n)}) :=& X_t - \int_0^{u_1} \frac{1}{G^0_{s-}} \mathbb{1}_{\{G^0_{s-}>0\}} d\langle X, G^0\rangle_s - \dots - \int_{u_n}^t \frac{1}{\alpha_{s-}(u,e)} \mathbb{1}_{\{\alpha_{s-}(u,e)>0\}} d\langle X, \alpha(u,e)\rangle_s.
\end{aligned}$$

Using Theorem 3.2.5, we shall attempt to show that both

- $(Y^n_t(u,e)\alpha_t(u,e) : t \ge u_n)$ is an $\mathbb{F}$-local martingale.
- For all $k \in \{0, 1, ..., n-1\}$, $\left(m^k_t = \mathbb{E}[\hat{X}_t | \mathcal{G}^k_t] : t \ge \tau_k\right)$ is a $\mathbb{G}^k$-local martingale.

Firstly,

$$\begin{aligned}
d\left(Y^n_t(u,e)\alpha_t(u,e)\right) =& Y^n_{t-}(u,e) d\alpha_t(u,e) + \alpha_{t-}(u,e) dY^n_t + d[Y^n, \alpha(u,e)]_t \\
=& Y^n_{t-}(u,e) d\alpha_t(u,e) + \alpha_{t-}(u,e) dX_t + d\left([X, \alpha(u,e)] - \langle X, \alpha(u,e)\rangle\right)_t
\end{aligned}$$

which is the sum of $\mathbb{F}$-local martingales.
For the second condition we shall use Corollary 3.2.6.1 and backward induction to show that m^k is a $\mathbb{G}^k$-local martingale for all $k \in \{0, 1, ..., n-1\}$.

Base Case: $k = n-1$ We need to show that

$$n_t^{n-1}(u^{(n-1)}, e^{(n-1)}) := Y_t^{n-1}(u^{(n-1)}, e^{(n-1)})\gamma_t^{n-1}(u^{(n-1)}, e^{(n-1)}) + \int_0^t \int_E Y_t^n(u,e)\alpha_t(u,e)de_n du_n$$

is an $\mathbb{F}$-martingale for all $(u^{(n-1)}, e^{(n-1)}) \in \Theta_{n-1} \times E^{n-1}$. First note the following decomposition of γ^{n-1}:

$$\begin{aligned}\gamma_t^{n-1}(u^{(n-1)}, e^{(n-1)}) &= \int_t^\infty \int_E \alpha_t(u,e)de_n du_n \\ &= \int_0^\infty \int_E \alpha_t(u,e)de_n du_n - \int_0^t \int_E \left(\alpha_t(u,e) - \alpha_{u_n}(u,e)\right) de_n du_n \\ &\quad - \int_0^t \int_E \alpha_{u_n}(u,e)de_n du_n \\ &= \mu_t^{n-1}(u^{(n-1)}, e^{(n-1)}) - \int_0^t \int_E \alpha_{u_n}(u,e)de_n du_n\end{aligned}$$

where $\mu_t^{n-1}(u^{(n-1)}, e^{(n-1)}) = \int_0^\infty \int_E \alpha_t(u,e)de_n du_n - \int_0^t \int_E \left(\alpha_t(u,e) - \alpha_{u_n}(u,e)\right) de_n du_n$ is an $\mathbb{F}$-martingale by the fact that $\alpha(u,e)$ is an $\mathbb{F}$-martingale for all $(u,e) \in \Theta_n \times E^n$ and Fubini's theorem. For ease of notation we shall omit the dependence on $(u^{(n-1)}, e^{(n-1)})$ for this part of the proof. Then using Lemma 2.3.4, we get

$$\begin{aligned}dn_t^{n-1} &= \gamma_{t-}^{n-1} dY_t^{n-1} + Y_{t-}^{n-1} d\mu_t^{n-1} - Y_{t-}^{n-1} \int_E \alpha_t(t, e_n)de_n dt + d[Y^{n-1}, \gamma^{n-1}]_t \\ &\quad + \int_E Y_t^n(t, e_n)\alpha_t(t, e_n)de_n dt + \int_0^t \int_E d\left(Y_t^n(u_n, e_n)\alpha_t(u_n, e_n)\right) de_n du_n \\ &= \gamma_{t-}^{n-1} dX_t + Y_{t-}^{n-1} d\mu_t^{n-1} + d\left([X, \gamma^{n-1}] - \langle X, \gamma^{n-1}\rangle\right)_t \\ &\quad + \int_E \left(Y_t^n(t, e_n) - Y_{t-}^{n-1}\right) \alpha_t(t, e_n)de_n dt + \int_0^t \int_E d\left(Y_t^n(u_n, e_n)\alpha_t(u_n, e_n)\right) de_n du_n.\end{aligned} \tag{3.5}$$

Note that, by definition of Y^{n-1} and Y^n

$$Y_t^n(t, e_n) - Y_{t-}^{n-1} = \Delta Y_t^{n-1}$$

Then by Corollary 2.3.2.1 in Chapter 2, $\int_0^t \int_E \left(Y_s^n(s, e_n) - Y_{s-}^{n-1}\right) \alpha_s(s, e_n)de_n ds$ is an $\mathbb{F}$-martingale. Finally, the last term in equation (3.5) can be written in integral form as

$$\begin{aligned}&\int_0^t \int_0^s \int_E d\left(\int_E Y_s^n(u_n, e_n)\alpha_s(u_n, e_n)\right) de_n du_n \\ &= \int_0^t \int_{u_n}^t \int_E d\left(\int_E Y_s^n(u_n, e_n)\alpha_s(u_n, e_n)\right) de_n du_n \\ &= \int_0^t \int_E \left(Y_t^n(, e_n)\alpha_t(u_n, e_n) - Y_{u_n}^n(, e_n)\alpha_{u_n}(u_n, e_n)\right) de_n du_n.\end{aligned}$$

By assumption $Y^n(, e_n)\alpha(u_n, e_n)$ is an $\mathbb{F}$-martingale meaning the above has zero expectation and is an $\mathbb{F}$-martingale. Therefore n^{n-1} is the sum of $\mathbb{F}$-martingales.

Inductive Step: We assume that

$$
\begin{aligned}
&n_t^{k+1}(u^{(k+1)}, e^{(k+1)}) = Y_t^{k+1}(u^{(k+1)}, e^{(k+1)})\gamma_t^{k+1}(u^{(k+1)}, e^{(k+1)}) \\
&+ \int_0^t \int_t^\infty \dots \int_{U_{n-1}}^\infty \int_{E^{n-k-1}} Y_t^{k+2}(u^{(k+2)}, e^{(k+2)})\alpha_t(u,e) de^{(n:k+2)} du^{(n:k+2)} \\
&+ \int_{u_{k+1}}^\infty \int_{u_{k+2}}^t \dots \int_{u_{n-1}}^\infty \int_{E^{n-k-1}} Y_t^{k+3}(u^{(k+3)}, e^{(k+3)})\alpha_t(u,e) de^{(n:k+2)} du^{(n:k+2)} \\
&+ \dots + \int_{u_{k+1}}^\infty \int_{u_{k+2}}^\infty \dots \int_{u_{n-1}}^t \int_{E^{n-k-1}} Y_t^{n}(u,e)\alpha_t(u,e) de^{(n:k+2)} du^{(n:k+2)}
\end{aligned}
$$

is an $\mathbb{F}$-martingale. We need to show that

$$
\begin{aligned}
&n_t^{k}(u^{(k)}, e^{(k)}) = Y_t^{k}(u^{(k)}, e^{(k)})\gamma_t^{k}(u^{(k)}, e^{(k)}) \\
&+ \int_0^t \int_t^\infty \dots \int_{u_{n-1}}^\infty \int_{E^{n-k}} Y_t^{k+1}(u^{(k+1)}, e^{(k+1)})\alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&+ \int_{u_k}^\infty \int_{u_{k+1}}^t \dots \int_{u_{n-1}}^\infty \int_{E^{n-k}} Y_t^{k+2}(u^{(k+2)}, e^{(k+2)})\alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&+ \dots + \int_{u_k}^\infty \int_{u_{k+1}}^\infty \dots \int_{u_{n-1}}^t \int_{E^{n-k}} Y_t^{n}(u,e)\alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)}
\end{aligned}
$$

is an $\mathbb{F}$-martingale.
Using the definition of γ^k and the fact that the random times are ordered, in other words $\alpha_t(u,e) = 0$ if $u_k > u_{k+1}$, the integrals may be rewritten as follows

$$
\begin{aligned}
n_t^{k}(u^{(k)}, e^{(k)}) =& Y_t^{k}(u^{(k)}, e^{(k)})\gamma_t^{k}(u^{(k)}, e^{(k)}) \\
&+ \int_0^t \int_t^\infty \dots \int_{u_{n-1}}^\infty \int_{E^{n-k}} Y_t^{k+1}(u^{(k+1)}, e^{(k+1)})\alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&+ \int_0^t \int_{u_{k+1}}^t \dots \int_{u_{n-1}}^\infty \int_{E^{n-k}} Y_t^{k+2}(u^{(k+2)}, e^{(k+2)})\alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&+ \dots + \int_0^t \int_{u_{k+1}}^\infty \dots \int_{u_{n-1}}^t \int_{E^{n-k}} Y_t^{n}(u,e)\alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
=& Y_t^{k}(u^{(k)}, e^{(k)})\gamma_t^{k}(u^{(k)}, e^{(k)}) \\
&+ \int_0^t \int_E Y_t^{k+1}(u^{(k+1)}, e^{(k+1)})\gamma_t^{k+1}(u^{(k+1)}, e^{(k+1)}) de_{k+1} du_{k+1} \\
&+ \int_0^t \int_0^t \int_{E^2} Y_t^{k+2}(u^{(k+2)}, e^{(k+2)})\gamma_t^{k+2}(u^{(k+2)}, e^{(k+2)}) de_{k+2} de_{k+1} du_{k+2} du_{k+1} \\
&+ \int_0^t \int_0^t \dots \int_0^t \int_{E^{n-k}} Y_t^{n}(u,e)\alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)},
\end{aligned}
$$

Using the definition of n^{k+1}, this is identified as

$$
n_t^{k}(u^{(k)}, e^{(k)}) = Y_t^{k}(u^{(k)}, e^{(k)})\gamma_t^{k}(u^{(k)}, e^{(k)}) + \int_0^t \int_E n_t^{k+1}(u^{(k+1)}, e^{(k+1)}) de_{k+1} du_{k+1}.
$$

It therefore follows that $n_t^k(u^{(k)}, e^{(k)})$ is an $\mathbb{F}$-martingale by the same reasoning as that of $n^{(n-1)}(u_n, e_n)$. □

The above theorem enables us to decompose $\mathbb{G}$-martingales in terms of $\mathbb{F}$-martingales, to continue the theme of this paper, we give the decomposition of an $\mathbb{F}$-martingale in terms of a $\mathbb{G}^k$-martingale and a recursive decomposition of $\mathbb{G}^{k+1}$-martingales in terms of $\mathbb{G}^k$-martingales, these are presented in the following two corollaries.

Corollary 3.2.7.1. *For $k \in \{1, 2, ..., n\}$, any $\mathbb{F}$-martingale X is a $\mathbb{G}^k$-semimartingale and*

$$\begin{aligned}\hat{X}_t =& X_t - \int_0^{\tau_1 \wedge t} \frac{1}{G^0_{s-}} d\langle X, G^0\rangle_s \\ &- \int_{\tau_1\wedge t}^{\tau_2 \wedge t} \frac{1}{\gamma^1_{s-}(\tau_1, \xi_1)} d\langle X, \gamma^1(u_1, e_1)\rangle_s \Big|_{\substack{u_1=\tau_1\\ e_1=\xi_1}} \\ &- \int_{\tau_2\wedge t}^{\tau_3 \wedge t} \frac{1}{\gamma^2_{s-}(\tau^{(2)}, \xi^{(2)})} d\langle X, \gamma^2(u^{(2)}, e^{(2)})\rangle_s \Big|_{\substack{u^{(2)}=\tau^{(2)}\\ e^{(2)}=\xi^{(2)}}} \\ &- \dots - \int_{\tau_k\wedge t}^{t} \frac{1}{\alpha^{(k)}_{s-}(\tau^{(k)}, \xi^{(k)})} d\langle X, \alpha^{(k)}(u^{(k)}, e^{(k)})\rangle_s \Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}\end{aligned}$$

is a $\mathbb{G}^k$-martingale.

Proof. Here we use the fact that Theorem 3.2.5 can be generalised to $\mathbb{G}^k$-martingales i.e. $Y = \sum_{j=0}^{k-1} Y^j(\tau^{(j)}, \xi^{(j)})\mathbb{1}_{[\![\tau_j,\tau_{j+1})\!)} + Y^k(\tau^{(j)}, \xi^{(j)})\mathbb{1}_{[\![\tau_k,\infty)\!)}$ is a $\mathbb{G}^k$-martingale if

- $(Y^k_t(u^{(k)}, e^{(k)})\alpha^{(k)}_t(u^{(k)}, e^{(k)}) : t \geq u_k)$ is an $\mathbb{F}$-martingale.
- $m^j_t = \mathbb{E}[Y_t|\mathcal{G}^j_t]$ is a $\mathbb{G}^j$-martingale for $t \geq \tau_j$ and $j \in \{1, 2, ..., k-1\}$.

Then using the same proof as Theorem 3.2.7, the result holds. □

Corollary 3.2.7.2. *Any $\mathbb{G}^k$-martingale X is a $\mathbb{G}^{k+1}$-semimartingale and*

$$\hat{X}_t = X_t - \int_{\tau_k\wedge t}^{\tau_{k+1}\wedge t} \frac{1}{G^k_{s-}} d\langle X, G^k\rangle_s - \int_{\tau_{k+1}\wedge t}^{t} \frac{1}{\alpha^k_{s-}(\tau_{k+1}, \xi_{k+1})} d\langle X, \alpha^k(u_{k+1}, e_{k+1})\rangle_s \Big|_{\substack{u_{k+1}=\tau_{k+1}\\ e_{k+1}=\xi_{k+1}}}$$

is a $\mathbb{G}^{k+1}$-martingale.

Proof. The result follows from Proposition 3.2.6 and a similar reasoning as in Equation (3.5) in the proof of Theorem 3.2.7. □

The above decompositions will be of particular use in proving a martingale representation theorem in the enlarged filtration. A natural question arises from the above decompositions:

1. Start with an $\mathbb{F}$-martingale X.
2. Define the $\mathbb{G}^k$ and $\mathbb{G}^{k+1}$-martingales from Corollary 3.2.7.1, call them $\hat{X}^k$ and $\hat{X}^{k+1}$ respectively.
3. Now using $\hat{X}^k$ define the $\mathbb{G}^{k+1}$-martingale from Corollary 3.2.7.2.
4. Does the resultant martingale coincide with $\hat{X}^{k+1}$?

The following proposition answers this question.

Proposition 3.2.8. *Starting with an $\mathbb{F}$-martingale X, define the following according to corollary 3.2.7.1:*

$$\begin{aligned}\hat{X}_t^k =& X_t - \int_0^{\tau_1 \wedge t} \frac{1}{G_{s-}^0} d\langle X, G^0 \rangle_s - \int_{\tau_1 \wedge t}^{\tau_2 \wedge t} \frac{1}{\gamma_{s-}^1(\tau_1, \xi_1)} d\langle X, \gamma^1(u_1, e_1) \rangle \Big|_{\substack{u_1 = \tau_1 \\ e_1 = \xi_1}} \\ & - \dots - \int_{\tau_k \wedge t}^{t} \frac{1}{\alpha_{s-}^{(k)}(\tau^{(k)}, \xi^{(k)})} d\langle X, \alpha^{(k)}(u^{(k)}, e^{(k)}) \rangle_s \Big|_{\substack{u^{(k)} = \tau^{(k)} \\ e^{(k)} = \xi^{(k)}}}\end{aligned} \tag{3.6}$$

$$\begin{aligned}\hat{X}_t^{k+1} =& X_t - \int_0^{\tau_1 \wedge t} \frac{1}{G_{s-}^0} d\langle X, G^0 \rangle_s - \int_{\tau_1 \wedge t}^{\tau_2 \wedge t} \frac{1}{\gamma_{s-}^1(\tau_1, \xi_1)} d\langle X, \gamma^1(u_1, e_1) \rangle \Big|_{\substack{u_1 = \tau_1 \\ e_1 = \xi_1}} \\ & - \dots - \int_{\tau_{k+1} \wedge t}^{t} \frac{1}{\alpha_{s-}^{(k+1)}(\tau^{(k+1)}, \xi^{(k+1)})} d\langle X, \alpha^{(k+1)}(u^{(k+1)}, e^{(k+1)}) \rangle_s \Big|_{\substack{u^{(k+1)} = \tau^{(k+1)} \\ e^{(k+1)} = \xi^{(k+1)}}}.\end{aligned} \tag{3.7}$$

Then the $\mathbb{G}^{k+1}$-martingale $\hat{X}^{k+1}$ coincides with the martingale in Corollary 3.2.7.2 starting with $\hat{X}^k$. That is,

$$\begin{aligned}\hat{X}_t^{k+1} =& \hat{X}_t^k - \int_{\tau_k \wedge t}^{\tau_{k+1} \wedge t} \frac{1}{G_{s-}^k} d\langle \hat{X}^k, G^k \rangle_s \\ & - \int_{\tau_{k+1} \wedge t}^{t} \frac{1}{\alpha_{s-}^k(\tau_{k+1}, \xi_{k+1})} d\langle \hat{X}^k, \alpha^k(u_{k+1}, e_{k+1}) \rangle_s \Big|_{\substack{u_{k+1} = \tau_{k+1} \\ e_{k+1} = \xi_{k+1}}}\end{aligned} \tag{3.8}$$

Proof. To prove the result, we first note a technicality about predictable quadratic variations. For non-zero semi-martingales, U, V and W, such that $\langle V, V \rangle$,$\langle u, W \rangle$, $\langle U, V \rangle$, $\langle V, W \rangle$ exists, the following formula for the product $\left(\frac{U_t}{V_t} W_t\right)$ can be derived. This is first done by treating this as a product of $\frac{U_t}{V_t}$ and W_t as follows:

$$\begin{aligned}d\left(\frac{U_t}{V_t} W_t\right) =& \frac{U_{t-}}{V_{t-}} dW_t + W_{t-} d\left(\frac{U_t}{V_t}\right) + d\left[\frac{U}{V}, W\right]_t \\ =& \frac{U_{t-}}{V_{t-}} dW_t + \frac{W_{t-}}{V_{t-}} dU_t - \frac{U_{t-} W_{t-}}{(V_{t-})^2} dV_t + \frac{U_{t-} W_{t-}}{(V_{t-})^3} d[V, V]_t \\ & - \frac{W_{t-}}{(V_{t-})^2} d[U, V]_t + d\left[\frac{U}{V}, W\right]_t\end{aligned}$$

Next we treat the product as the product of all three terms U_t, $\frac{1}{V_t}$ and W_t as follows:

$$\begin{aligned}d\left(\frac{U_t}{V_t} W_t\right) =& \frac{W_{t-}}{V_{t-}} dU_t + \frac{U_{t-}}{V_{t-}} dW_t - \frac{U_{t-} W_{t-}}{(V_{t-})^2} dV_t + \frac{U_{t-} W_{t-}}{(V_{t-})^3} d[V, V]_t \\ & + \frac{1}{V_{t-}} d[U, W]_t - \frac{W_{t-}}{(V_{t-})^2} d[U, V]_t - \frac{U_{t-}}{(V_{t-})^2} d\,[V, W]_t\end{aligned}$$

Combing these two formulas for $d\left(\frac{U_t}{V_t} W_t\right)$, we get

$$d\Big[\frac{U}{V}, W\Big]_t = \frac{1}{V_{t-}} d[U, W]_t - \frac{U_{t-}}{V_{t-}^2} d[V, W]_t.$$

Finally, noting that the dual predictable projection of the quadratic covariation process is the predictable quadratic covariation process (see Chapter 8 in He et al. [1992]), we get the following helpful formula:

$$d\Big\langle \frac{U}{V}, W\Big\rangle_t = \frac{1}{V_{t-}} d\langle U, W\rangle_t - \frac{U_{t-}}{V_{t-}^2} d\langle V, W\rangle_t. \tag{3.9}$$

We now recall the definition of α^k and G^k

$$\alpha_t^k(u_{k+1}, e_{k+1}) = \frac{\alpha_t^{(k+1)}(\tau^{(k+1)}, \xi^{(k+1)})}{\alpha_t^{(k)}(\tau^{(k)}, \xi^{(k)})}$$

$$G_t^k = \mathbb{P}(\tau_{k+1} > t | \mathcal{G}_t^k) = \mathbb{1}_{\{\tau_k > t\}} + \mathbb{1}_{\{\tau_k \le t\}} \frac{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}{\alpha_t^{(k)}(\tau^{(k)}, \xi^{(k)})}$$

Therefore, for $t \ge \tau_k$

$$d\langle \hat{X}^k, G^k\rangle_t = d\Big\langle X, \frac{\gamma_t^k(u^{(k)}, e^{(k)})}{\alpha_t^{(k)}(u^{(k)}, e^{(k)})}\Big\rangle_t \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}.$$

Using Equation (3.9) this is expanded as

$$\begin{aligned} d\langle \hat{X}^k, G^k\rangle_t =& \frac{1}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})} d\langle X, \gamma^k(u^{(k)}, e^{(k)})\rangle_t \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}} \\ &- \frac{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})^2} d\langle X, \alpha^{(k)}(u^{(k)}, e^{(k)})\rangle_t \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}. \end{aligned}$$

Now using the definition of G_s^k, the second term can be written as

$$\begin{aligned} d\langle \hat{X}^k, G^k\rangle_t =& \frac{1}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})} d\langle X, \gamma^k(u^{(k)}, e^{(k)})\rangle_t \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}} \\ &- \frac{G_{t-}^k}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})} d\langle X, \alpha^{(k)}(u^{(k)}, e^{(k)})\rangle_t \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}, \end{aligned}$$

and

$$d\langle \hat{X}^k, \alpha^k(u_{k+1}, e_{k+1})\rangle_t \Big|_{\substack{u_{k+1}=\tau_{k+1} \\ e_{k+1}=\xi_{k+1}}} = d\Big\langle X, \frac{\alpha_t^{(k+1)}(u^{(k+1)}, e^{(k+1)})}{\alpha_t^{(k)}(u^{(k)}, e^{(k)})}\Big\rangle_t \Big|_{\substack{u^{(k+1)}=\tau^{(k+1)} \\ e^{(k+1)}=\xi^{(k+1)}}}$$

Again, using Equation (3.9), this is expanded as

$$\begin{aligned} d\langle \hat{X}^k, \alpha^k(u_{k+1}, e_{k+1})\rangle_t \Big|_{\substack{u_{k+1}=\tau_{k+1} \\ e_{k+1}=\xi_{k+1}}} =& \frac{1}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})} d\langle X, \alpha^{(k+1)}(u^{(k+1)}, e^{(k+1)}\rangle_t|_{\substack{u^{(k+1)}=\tau^{(k+1)} \\ e^{(k+1)}=\xi^{(k+1)}}} \\ &- \frac{1}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})^2} d\langle X, \alpha^{(k)}(u^{(k)}, e^{(k)})\rangle_t \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}. \end{aligned}$$

Using the definition of α_t^k, this is simplified to

$$\begin{aligned} d\langle \hat{X}^k, \alpha^k(u_{k+1}, e_{k+1})\rangle_t \Big|_{\substack{u_{k+1}=\tau_{k+1} \\ e_{k+1}=\xi_{k+1}}} =& \frac{1}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})} d\langle X, \alpha^{(k+1)}(u^{(k+1)}, e^{(k+1)}\rangle_t|_{\substack{u^{(k+1)}=\tau^{(k+1)} \\ e^{(k+1)}=\xi^{(k+1)}}} \\ &- \frac{\alpha_{t-}^k(\tau_{k+1}, \xi_{k+1})}{\alpha_{t-}^{(k)}(\tau^{(k)}, \xi^{(k)})} d\langle X, \alpha^{(k)}(u^{(k)}, e^{(k)})\rangle_t \Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}. \end{aligned}$$

We now show that the right hand side of equation (3.8) is equal to the decomposition in equation (3.7).

$$\begin{aligned}\hat{X}_t^{k+1} =& \hat{X}_t^k - \int_{\tau_k\wedge t}^{\tau_{k+1}\wedge t} \frac{1}{G_{s-}^k} d\langle \hat{X}^k, G^k\rangle_s \\ &- \int_{\tau_{k+1}\wedge t}^{t} \frac{1}{\alpha_{s-}^k(\tau_{k+1},\xi_{k+1})} d\langle \hat{X}^k, \alpha^k(u_{k+1},e_{k+1})\rangle_s \Big|_{\substack{u_{k+1}=\tau_{k+1}\\ e_{k+1}=\xi_{k+1}}}.\end{aligned}$$

The computation of $\langle \hat{X}^k, G^k\rangle$ and $\langle \hat{X}^k, \alpha^k\rangle$ are introduced and $\hat{X}^k$ is expanded to yield

$$\begin{aligned}=& X_t - \int_0^{\tau_1\wedge t} \frac{1}{G_{s-}^0} d\langle X, G^0\rangle_s - \ldots - \Bigg(\int_{\tau_k\wedge t}^{t} \frac{1}{\alpha_{s-}^{(k)}(\tau^{(k)},\xi^{(k)})} d\langle X, \alpha^{(k)}(u^{(k)},e^{(k)})\rangle_s \Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}} \\ &- \int_{\tau_k\wedge t}^{\tau_{k+1}\wedge t} \frac{1}{\gamma_{s-}^k(\tau^{(k)},\xi^{(k)})} d\langle X, \gamma^k(u^{(k)},e^{(k)})\rangle_s \Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}} \\ &+ \int_{\tau_k\wedge t}^{\tau_{k+1}\wedge t} \frac{1}{\alpha_{s-}^{(k)}(\tau^{(k)},\xi^{(k)})} d\langle X, \alpha^{(k)}(u^{(k)},e^{(k)})\rangle_s \Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}} \\ &- \int_{\tau_{k+1}\wedge t}^{t} \frac{1}{\alpha_{s-}^{(k+1)}(\tau^{(k+1)},\xi^{(k+1)})} d\langle X, \alpha^{(k+1)}(u^{(k+1)},e^{(k+1)})\rangle_s \Big|_{\substack{u^{(k+1)}=\tau^{(k+1)}\\ e^{(k+1)}=\xi^{(k+1)}}} \\ &+ \int_{\tau_{k+1}\wedge t}^{t} \frac{1}{\alpha_{s-}^{(k)}(\tau^{(k)},\xi^{(k)})} d\langle X, \alpha^{(k)}(u^{(k)},e^{(k)})\rangle_s \Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}. \Bigg)\end{aligned}$$

It is noticed that inside the brackets, the first, third and fifth terms sum to zero. The leftover terms are then

$$\begin{aligned}=& X_t - \int_0^{\tau_1\wedge t} \frac{1}{G_{s-}^0} d\langle X, G^0\rangle_s - \ldots - \int_{\tau_k\wedge t}^{\tau_{k+1}\wedge t} \frac{1}{\gamma_{s-}^k(\tau^{(k)},\xi^{(k)})} d\langle X, \gamma^k(u^{(k)},e^{(k)})\rangle_s \Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}} \\ &- \int_{\tau_{k+1}\wedge t}^{t} \frac{1}{\alpha_{s-}^{(k+1)}(\tau^{(k+1)},\xi^{(k+1)})} d\langle X, \alpha^{(k+1)}(u^{(k+1)},e^{(k+1)})\rangle_s \Big|_{\substack{u^{(k+1)}=\tau^{(k+1)}\\ e^{(k+1)}=\xi^{(k+1)}}} \\ =& \hat{X}_t^{k+1}.\end{aligned}$$

□

3.2.4 Random jump measures and their compensators

We devote this subsection to introducing a very important family of processes. There are no original results here but simply a presentation of the random jump measures induced by τ and ξ and their compensators.
For $k \in \{1, 2, ..., n\}$ let

$$\mu^k([0,t]\times A) := \mathbb{1}_{\{\tau_k\le t\}}\mathbb{1}_{\{\xi_k\in A\}}$$

for all $t \in \mathbb{R}^+$ and $A \in \mathcal{B}(E)$.
We define the total jump measure as follows:

$$\mu([0,t]\times A) = \sum_{k=1}^n \mu^k([0,t]\times A).$$

To remain consistent with the notation defined in previous sections, we define the cumulative total jump measure as follows:

$$\mu^{(k)}([0,t]\times A)=\sum_{i=1}^{k}\mu^{i}([0,t]\times A).$$

A key component that is needed for the application of jump measures to stochastic calculus is their compensators. The following comes from Proposition 2.1 in Kharroubi and Lim [2014]:

Proposition 3.2.9. *The random jump measure μ admits a $\mathbb{G}$-compensator $\lambda_t(e)\,de\,dt$ where*

$$\lambda_t(e)=\sum_{i=1}^{n}\lambda_t^k(e),\qquad \lambda_t^k(e)=\frac{\gamma_t^k(\tau^{(k-1)},t,\xi^{(k-1)},e)}{\gamma_t^{k-1}(\tau^{(k-1)},\xi^{(k-1)})}\mathbb{1}_{\{t\le\tau_k\}}.$$

That is, for any $A\in\mathcal{B}(E)$,

$$\left(\mu([0,t]\times A)-\int_0^t\int_A\lambda_s(e)\,de\,ds:t\geq 0\right)$$

is a $\mathbb{G}$-martingale.

A useful byproduct of this proposition is the $\mathbb{G}^k$-compensator of the jump measures μ^k and $\mu^{(k)}$, this is presented in the following corollary.

Corollary 3.2.9.1. *For any $k\in\{1,2,...,n\}$ the jump measures μ^k and $\mu^{(k)}$ admit $\mathbb{G}^k$-compensators $\lambda_t^k(e)\,de\,dt$ and $\lambda_t^{(k)}(e)\,de\,dt$ respectively where*

$$\lambda_t^{(k)}(e)=\sum_{i=1}^{k}\lambda_t^i(e),$$

and $\lambda_t^k(e)$ is defined in Proposition 3.2.9.

Remark 6. *The fact that the random times are ordered implies that $\gamma_t^k(\tau^{(k-1)},t,\xi^{(k-1)},e)=0$ for $t<\tau_{k-1}$. Meaning the compensator of μ^k can be written $\lambda_t^k(e)\mathbb{1}_{\{\tau_{k-1}\le t\le\tau_k\}}$.*

Proof. From Lemma 3.2.3, we know that $(\tau^{(k)},\xi^{(k)})\ll\mathbb{F}$, furthermore, from remark 6, the compensator of μ is a series of disjoint processes. It therefore follows that Proposition 2.1 in Kharroubi and Lim [2014] can be applied to the filtration $\mathbb{G}^k$ and the jump measure $\mu^{(k)}$. □

In what follows we shall denote by $\tilde{\mu}$, $\tilde{\mu}^k$ and $\tilde{\mu}^{(k)}$ the compensated measures of μ, μ^k and $\mu^{(k)}$ respectively.

3.2.5 Martingale Representation in $\mathbb{G}$

This section provides the main contribution of this chapter: we prove that when the reference filtration $\mathbb{F}$ enjoys martingale representation then under certain conditions, all $\mathbb{G}$-martingales are represented by an integral with respect to a continuous martingale and a sequence of purely-discontinuous martingales. We begin by defining the following

– For an arbitrary filtration $\mathbb{H}$, define $\mathcal{M}^2(\mathbb{H})$ to be the space of all $\mathbb{H}$-martingales, M such that $\mathbb{E}[|[M,M]_\infty|] < \infty$.

– For any $M \in \mathcal{M}^2(\mathbb{H})$, define $L^2(M) := \{\phi : \phi \in \mathcal{P}(\mathbb{H}), \mathbb{E}\left[\int_0^t |\phi_s|^2 d[M,M]_s\right] < \infty\}$.

Let S be an $\mathbb{F}$-martingale. For the rest of this chapter we shall assume that $\mathbb{F}$ enjoys martingale representation with respect to S, that is, for any $M \in \mathcal{M}^2(\mathbb{F})$ there exists $\phi \in L^2(S)$ such that

$$M_t = M_0 + \int_0^t \phi_s dS_s,$$

for all $t \geq 0$.

Our goal is to prove that given this structure on $\mathbb{F}$, $\mathbb{G}$ enjoys martingale representation too. To do so we make a continuity assumption on the martingale S, this may seem restrictive but martingale representation theorems are most useful when proving the existence and uniqueness of stochastic differential equations, which are often driven by a continuous Brownian motion.

Hypothesis 5. *The martingale S is continuous.*

Note that this assumption coupled with the fact that each τ_k avoids $\mathbb{F}$-stopping times (a consequence of the density hypothesis 4 according to remark 2 in Chapter 2)) implies that all $\mathbb{F}$-martingales are continuous, in particular for any $u \in \Theta_n$ and $e \in E^n$, $\alpha(u,e)$ is a continuous martingale, hence α^k, $\alpha^{(k)}$, G^k, and γ^k are all continuous for all $k \in \{1,2,...,n\}$. In essence, what will be evident is that all $\mathbb{G}$-martingales are continuous on the time interval $[\tau_k, \tau_{k+1})$. Define the following $\mathbb{G}$-martingale in accordance with Theorem 3.2.7,

$$\begin{aligned} S_t^{\mathbb{G}} =& S_t - \int_0^{\tau_1\wedge t} \frac{1}{G_s^0} d\langle S, G^0\rangle_s - \int_{\tau_1\wedge t}^{\tau_2\wedge t} \frac{1}{\gamma_s^1(\tau_1,\xi_1)} d\langle S, \gamma^1(u_1,e_1)\rangle_s\Big|_{\substack{u_1=\tau_1\\ e_1=\xi_1}} \\ &- \int_{\tau_2\wedge t}^{\tau_3\wedge t} \frac{1}{\gamma_s^2(\tau^{(2)},\xi^{(2)})} d\langle S, \gamma^2(u^{(2)},e^{(2)})\rangle_s\Big|_{\substack{u^{(2)}=\tau^{(2)}\\ e^{(2)}=\xi^{(2)}}} - \dots - \int_{\tau_n\wedge t}^{t} \frac{1}{\alpha_s(\tau,\xi)} d\langle S, \alpha(u,e)\rangle_s|_{\substack{u=\tau\\ e=\xi}}. \end{aligned}$$

The following theorem is the main contribution of this chapter.

Theorem 3.2.10. *The filtration $\mathbb{G}$ enjoys martingale representation with respect to $\{S^{\mathbb{G}}, \tilde{\mu}^1, \tilde{\mu}^2, ..., \tilde{\mu}^n\}$, that is, for any $M \in \mathcal{M}^2(\mathbb{G})$ there exists $\phi \in L^2(S^{\mathbb{G}})$ and $\beta^k \in L^2(\tilde{\mu}^k)$[1], $k = 1,2,...,n$ such that*

$$M_t = M_0 + \int_0^t \phi_s dS_s^{\mathbb{G}} + \sum_{k=1}^n \int_0^t \int_E \beta_s^k(e) \tilde{\mu}^k(ds,de).$$

Proof. The goal is to show that for all $k = 1,2..,n$, $\mathbb{G}^k$ enjoys martingale representation hence concluding the result for $k = n$. Define the following family of processes in accordance with Corollary 3.2.7.1

$$\begin{aligned} S_t^{\mathbb{G},k} =& S_t - \int_0^{\tau_1} \frac{1}{G_s^0} d\langle S, G^0\rangle_s \\ &- \int_{\tau_1}^{\tau_2} \frac{1}{\gamma_s^1(\tau_1,\xi_1)} d\langle S, \gamma^1(u_1,e_1)\rangle_s\Big|_{\substack{u_1=\tau_1\\ e_1=\xi_1}} \\ &- \dots - \int_{\tau_k}^{t} \frac{1}{\alpha_s^{(k)}(\tau^{(k)},\xi^{(k)})} d\langle S, \alpha^{(k)}(u^{(k)},e^{(k)})\rangle_s\Big|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}} \end{aligned}$$

[1] This is understood as the space $\{\beta : \beta \in \mathcal{O}(\mathbb{G}) \otimes \mathcal{B}(E), \mathbb{E}\left[\int_0^t \int_E |\beta_s(e)|^2 \lambda_s^k(e) deds\right] < \infty\}$

then we will show that for every $M^k \in \mathcal{M}^2(\mathbb{G}^k)$ there exists $\phi^k \in L^2(S^{\mathbb{G},k})$ and $\beta^{i,k} \in L^2(\tilde{\mu}^i)$, $i = 1, 2, ..., k$ such that

$$M_t^k = M_0^k + \int_0^t \phi_s^k dS_s^{\mathbb{G},k} + \sum_{i=1}^{k} \int_0^t \int_E \beta_s^{i,k}(e)\tilde{\mu}^i(ds, de).$$

This will be done using induction. It is important to note that under Hypothesis 5, $S^{\mathbb{G},k}$ is a continuous $\mathbb{G}^k$-martingale.

Base Case $k = 1$**:** The proof for the standard one-default progressive enlargement is done in Theorem 2.1 in Jeanblanc and Le Cam [2009a] under the same assumptions and is omitted from this proof.

Inductive Step: We assume that $\mathbb{G}^k$ enjoys martingale representation with respect to $\{S^{\mathbb{G},k}, \tilde{\mu}^1, \tilde{\mu}^2, ..., \tilde{\mu}^k\}$ for some $k \in \{1, 2, ..., n\}$. We want to prove that $\mathbb{G}^{k+1}$ enjoys martingale representation with respect to $\{S^{\mathbb{G},k+1}, \tilde{\mu}^1, \tilde{\mu}^2, ..., \tilde{\mu}^{k+1}\}$. Firstly, by Corollary 3.2.7.2 and Proposition 3.2.6, we have that

$$S_t^{\mathbb{G},k+1} = S_t^{\mathbb{G},k} - \int_{\tau_k}^{\tau_{k+1}\wedge t} \frac{1}{G_s^k} d\langle S^{\mathbb{G},k}, G^k\rangle_s \tag{3.10}$$

$$- \int_{\tau_{k+1}\wedge t}^{t} \frac{1}{\alpha_s^k(\tau_{k+1}, \xi_{k+1})} d\langle S^{\mathbb{G},k}, \alpha^k(u_{k+1}, e_{k+1})\rangle_s \Big|_{\substack{u_{k+1}=\tau_{k+1} \\ e_{k+1}=\xi_{k+1}}}. \tag{3.11}$$

Secondly, by the inductive hypothesis, we define the integral representations of G^k and α^k, recalling their continuity

$$\begin{aligned}
G_t^k &= \int_t^\infty \int_E \alpha_t^k(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1} \\
&= \int_0^\infty \int_E \alpha_t^k(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1} + \int_t^\infty \int_E \Big(\alpha_{u_{k+1}}^k(u_{k+1}, e_{k+1}) \\
&\quad - \alpha_t(u_{k+1}, e_{k+1})\Big) de_{k+1} du_{k+1} - \int_0^t \int_E \alpha_{u_{k+1}}(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1} \\
&= N_t^k - \int_0^t \int_E \alpha_{u_{k+1}}(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1}
\end{aligned}$$

where

$$\begin{aligned}
N_t^k &= \int_0^\infty \int_E \alpha_t^k(u_{k+1}, e_{k+1}) de_{k+1} du_{k+1} \\
&\quad + \int_t^\infty \int_E \Big(\alpha_{u_{k+1}}^k(u_{k+1}, e_{k+1}) - \alpha_t(u_{k+1}, e_{k+1})\Big) de_{k+1} du_{k+1}
\end{aligned}$$

It is noted by the fact that $\alpha^k(u_{k+1}, e_{k+1})$ is a continuous $\mathbb{G}^k$-martingale and by Fubini's theorem N_t^k is too a continuous $\mathbb{G}^k$-martingale and therefore the induction hypothesis implies the existence of a process $n^k \in L^2(S^{\mathbb{G},k})$ such that

$$N_t^k = N_0^k + \int_0^t n_s^k dS_s^{\mathbb{G},k}.$$

For the remainder of this proof, any newly defined martingales will be denoted with a capital letter and their integral representations in lower case.
As noted previously, for any $(u_{k+1}, e_{k+1}) \in \mathbb{R}^+ \times E$, the process $\alpha^k(u_{k+1}, e_{k+1})$ defined in Lemma 3.2.3 is a $\mathbb{G}^{\tau,\xi,k}$, meaning it is a $\mathbb{G}^k$-martingale on $[\![\tau_k, \infty)\!)$. By the inductive hypothesis, there exists $a^k(u_{k+1}, e_{k+1}) \in L^2(S^{\mathbb{G},k})$ such that

$$\alpha_t^k(u_{k+1}, e_{k+1}) = \alpha_0^k(u_{k+1}, e_{k+1}) + \int_0^t a_s^k(u_{k+1}, e_{k+1}) dS_s^{\mathbb{G},k}.$$

To show that $\mathbb{G}^{k+1}$ enjoys martingale representation, we may restrict our attention to only uniformly integrable martingales of the form

$$M_t = \mathbb{E}[M_T | \mathcal{G}_t^{k+1}],$$

the main result can then be extended in the limit. Using a monotone class argument we may consider random variables of the form

$$M_T = Z_T h(\tau_{k+1} \wedge T) g(\xi_{k+1} \mathbb{1}_{\{\tau_{k+1} \leq T\}}),$$

where h and g are bounded Borel functions and Z_T is a $\mathcal{G}_T^k$-measurable random variable.
Now using Proposition 3.2.2, we can decompose M into the sum of three terms as follows:

$$\begin{aligned} M_t =& \mathbb{E}[Z_T h(\tau_{k+1} \wedge T) g(\xi_{k+1} \mathbb{1}_{\{\tau_{k+1} \leq T\}}) | \mathcal{G}_t^{k+1}] \\ =& \mathbb{E}[Z_T h(T) g(0) \mathbb{1}_{\{\tau_{k+1} > T\}} | \mathcal{G}_t^{k+1}] + \mathbb{E}[G_T h(\tau_{k+1}) g(\xi_{k+1}) \mathbb{1}_{\{t < \tau_{k+1} \leq T\}} | \mathcal{G}_t^{k+1}] \\ & + \mathbb{E}[Z_T h(\tau_{k+1}) g(\xi_{k+1}) \mathbb{1}_{\{\tau_{k+1} \leq t\}} | \mathcal{G}_t^{k+1}]. \end{aligned}$$

Using Proposition 2.2.3, the three terms above are simplified to

$$\begin{aligned} M_t =& \frac{\mathbb{E}[Z_T h(T) g(0) \mathbb{1}_{\{\tau_{k+1} > T\}} | \mathcal{G}_t^k]}{G_t^k} \mathbb{1}_{\{\tau_{k+1} > t\}} + \frac{\mathbb{E}[Z_T h(\tau_{k+1}) g(\xi_{k+1}) \mathbb{1}_{\{t < \tau_{k+1} \leq T\}} | \mathcal{G}_t^k]}{G_t^k} \mathbb{1}_{\{\tau_{k+1} > t\}} \\ & + \frac{\mathbb{E}[Z_T h(u_{k+1}) g(e_{k+1}) \alpha_T^k(u_{k+1}, e_{k+1}) | \mathcal{G}_t^k] \Big|_{\substack{u_{k+1} = \tau_{k+1} \\ e_{k+1} = \xi_{k+1}}}}{\alpha_t^k(\tau_{k+1}, \xi_{k+1})} \mathbb{1}_{\{\tau_{k+1} \leq t\}} \\ =& M_t^{(1)} + M_t^{(2)} + M_t^{(3)} \end{aligned}$$

We therefore proceed with the proof in three parts, the goal being to show that M can be written as a stochastic integral with respect to $\{S^{\mathbb{G},k+1}, \tilde{\mu}^1, \tilde{\mu}^2, ..., \tilde{\mu}^{k+1}\}$.
For the sake of brevity the decompositions of $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$ are broken up into the following three lemmas.
Recall the definition of H^k for $k \in \{1, 2, ..., n\}$ and $t \geq 0$

$$H_t^k = \mathbb{1}_{\{\tau_k \leq t\}}.$$

The first lemma gives the decomposition of $M^{(1)}$.

Lemma 3.2.11. *Let $P_t := \mathbb{E}[Z_T h(T)g(0)\mathbb{1}_{\{\tau_{k+1}>T\}}|\mathcal{G}_t^k]$, then P is a $\mathbb{G}^k$-martingale and so by the inductive hypothesis, there exists $\{p, \hat{p}^1, \hat{p}^2, ..., \hat{p}^k\}$ such that*

$$P_t = P_0 + \int_0^t p_s dS_s^{\mathbb{G},k} + \sum_{i=1}^k \int_0^t \int_E \hat{p}_s^i(e)\tilde{\mu}^i(ds, de),$$

then

$$\begin{aligned} M_t^{(1)} =& h(T)g(0)\Big(M_0^{(1)} + \int_0^t \frac{1-H_{s-}^{k+1}}{\left(G_s^k\right)^2}\left(p_s G_s^k - P_{s-}n_s^k\right) dS_s^{\mathbb{G},k+1} \\ &+ \sum_{i=1}^k \int_0^t \int_E \left(\frac{\hat{p}_s^i(e)(1-H_{s-}^{k+1})}{G_s^k}\right)\tilde{\mu}^i(ds, de) + \int_0^t \int_E \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k}\tilde{\mu}^{k+1}(ds, de)\Big). \end{aligned}$$

Proof. Firstly, note from Corollary 3.2.9.1

$$H_t^{k+1} = \int_0^t \int_E \left(\tilde{\mu}^{k+1}(de, ds) + \lambda_s^{k+1}(e)de\ ds\right). \tag{3.12}$$

Now using the definition of P and H^{k+1}, $M^{(1)}$ is written as

$$\frac{M_t^{(1)}}{h(T)g(0)} = \frac{P_t(1-H_t^{k+1})}{G_t^k}.$$

Now using Itô's formula with jumps, we expand the right hand side as follows:

$$\begin{aligned} \frac{M_t^{(1)}}{h(T)g(0)} =& \frac{M_0^{(1)}}{h(T)g(0)} + \int_0^t \frac{1-H_{s-}^{k+1}}{G_s^k} dP_s - \int_0^t \frac{P_{s-}}{G_s^k} dH_s^{k+1} - \int_0^t \frac{P_{s-}(1-H_{s-}^{k+1})}{\left(G_s^k\right)^2} dG_s^k \\ &- \int_0^t \frac{1-H_{s-}^{k+1}}{\left(G_s^k\right)^2} d\langle P^c, G^k\rangle_s + \int_0^t \frac{1-H_{s-}^{k+1}}{\left(G_s^k\right)^3} d\langle G^k, G^k\rangle_s \\ &+ \sum_{s\le t}\Big[\frac{P_s(1-H_s^{k+1})}{G_s^k} - \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} - \frac{1-H_{s-}^{k+1}}{G_s^k}\Delta P_s + \frac{P_{s-}}{G_s^k}\Delta H_s^{k+1}\Big], \end{aligned}$$

where the sum ranges over jump times of the processes P and H^{k+1}, i.e. $\{\tau_1, \tau_2, ..., \tau_{k+1}\}$ provided they occur before time t. Note that H^{k+1} only jumps at time τ_{k+1}, meaning

$$\begin{aligned}\frac{M_t^{(1)} - M_0^{(1)}}{h(T)g(0)} &= \int_0^t \frac{1-H_{s-}^{k+1}}{G_s^k} p_s dS_s^{\mathbb{G},k} + \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{G_s^k} \hat{p}_s^i \tilde{\mu}^i(ds,de) \\ &- \int_0^t \frac{P_{s-}(1-H_{s-}^{k+1})}{(G_s^k)^2} n_s^k dS_s^{\mathbb{G},k} + \int_0^t \int_E \frac{P_{s-}(1-H_{s-}^{k+1})}{(G_s^k)^2} \alpha_s^k(s,e)deds \\ &- \int_0^t \frac{1-H_{s-}^{k+1}}{(G_s^k)^2} p_s n_s^k d\langle S^{\mathbb{G},k}, S^{\mathbb{G},k}\rangle_s + \int_0^t \frac{1-H_{s-}^{k+1}}{(G_s^k)^3} \left(n_s^k\right)^2 d\langle S^{\mathbb{G},k}, S^{\mathbb{G},k}\rangle_s \\ &+ \sum_{i=1}^k \int_0^t \left(\frac{P_s(1-H_{s-}^{k+1})}{G_s^k} - \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} \right) dH_s^i \\ &+ \int_0^t \left(\frac{P_{s-}(1-H_s^{k+1})}{G_s^k} - \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} \right) dH_s^{k+1} \\ &- \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{G_s^k} \hat{p}_s^i(e) \mu^i(ds,de) + \int_0^t \frac{P_{s-}}{G_s^k} dH_s^{k+1}.\end{aligned}$$

Note the following

a) From Lemma 3.2.3 and Proposition 3.2.9, we see that

$$\lambda_t^{k+1}(e) = \frac{\alpha_t^k(t,e)}{G_t^k}(1-H_{t-}^{k+1}).$$

b) From Equation (3.12),

$$\int_0^t \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} dH_s^{k+1} = \int_0^t \int_E \left(\frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} \tilde{\mu}^{k+1}(ds,de) + \frac{P_{s-}(1-H_{s-}^{k+1})}{(G_s^k)^2} \alpha_s^k(s,e)deds \right).$$

b) The integrator dH_s^{k+1} is supported by the set $\{\tau_{k+1} \leq s\}$, therefore

$$\int_0^t \frac{P_s(1-H_s^{k+1})}{G_s^k} dH_s^{k+1} = 0.$$

These three facts combine to yield the following:

$$\begin{aligned}\frac{M_t^{(1)} - M_0^{(1)}}{h(T)g(0)} &= \int_0^t \frac{1-H_{s-}^{k+1}}{(G_s^k)^2} \left(p_s G_s^k - P_{s-} n_s^k\right) dS_s^{\mathbb{G},k} - \int_0^t \frac{n_s^k(1-H_{s-}^{k+1})}{(G_s^k)^3} \left(p_s G_s^k - P_{s-} n_s^k\right) d\langle S^{\mathbb{G},k}, S^{\mathbb{G},k}\rangle_s \\ &- \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{G_s^k} \hat{p}_s^i(e) \lambda_s^i(e) deds + \int_0^t \int_E \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} \tilde{\mu}^{k+1}(ds,de) \\ &+ \sum_{i=1}^k \int_0^t \left(\frac{P_s(1-H_{s-}^{k+1})}{G_s^k} - \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} \right) dH_s^i.\end{aligned}$$

From Equation (3.10) and the dynamics of G^k, we see that

$$(1-H^{k+1}_{t-})dS^{\mathbb{G},k+1}_t = (1-H^{k+1}_{t-})\left(S^{\mathbb{G},k}_t - \frac{n^k_t}{G^k_t}d\langle S^{\mathbb{G},k}, S^{\mathbb{G},k}\rangle_s\right),$$

meaning

$$\begin{aligned}\frac{M^{(1)}_t - M^{(1)}_0}{h(T)g(0)} &= \int_0^t \frac{1-H^{k+1}_{s-}}{(G^k_s)^2}\left(p_s G^k_s - P_{s-}n^k_s\right)dS^{\mathbb{G},k+1}_s - \sum_{i=1}^k \int_0^t\int_E \frac{1-H^{k+1}_{s-}}{G^k_s}\hat{p}^i_s(e)\lambda^i_s(e)deds \\ &+ \int_0^t\int_E \frac{P_{s-}(1-H^{k+1}_{s-})}{G^k_s}\tilde{\mu}^{k+1}(ds,de) \\ &+ \sum_{i=1}^k\int_0^t\left(\frac{P_s(1-H^{k+1}_{s-})}{G^k_s} - \frac{P_{s-}(1-H^{k+1}_{s-})}{G^k_s}\right)dH^i_s.\end{aligned}$$

The fourth term can be extended to an integral over the measure μ^i. Furthermore, from the definition, the jump of P at τ_i is $\hat{p}^i_{\tau_i}(e)$, this implies

$$\begin{aligned}\frac{M^{(1)}_t - M^{(1)}_0}{h(T)g(0)} &= \int_0^t \frac{1-H^{k+1}_{s-}}{(G^k_s)^2}\left(p_s G^k_s - P_{s-}n^k_s\right)dS^{\mathbb{G},k+1}_s - \sum_{i=1}^k \int_0^t\int_E \frac{1-H^{k+1}_{s-}}{G^k_s}\hat{p}^i_s(e)\lambda^i_s(e)deds \\ &+ \int_0^t\int_E \frac{P_{s-}(1-H^{k+1}_{s-})}{G^k_s}\tilde{\mu}^{k+1}(ds,de) + \sum_{i=1}^k\int_0^t\int_E\left(\frac{\hat{p}^i_s(e)(1-H^{k+1}_{s-})}{G^k_s}\right)\mu^i(ds,de).\end{aligned}$$

Combining the second and fourth term together, we get

$$\begin{aligned}&= \int_0^t \frac{1-H^{k+1}_{s-}}{(G^k_s)^2}\left(p_s G^k_s - P_{s-}n^k_s\right)dS^{\mathbb{G},k+1}_s + \sum_{i=1}^k\int_0^t\int_E\left(\frac{\hat{p}^i_s(e)(1-H^{k+1}_{s-})}{G^k_s}\right)\tilde{\mu}^i(ds,de) \\ &+ \int_0^t\int_E \frac{P_{s-}(1-H^{k+1}_{s-})}{G^k_s}\tilde{\mu}^{k+1}(ds,de).\end{aligned}$$

□

The next lemma shows the decomposition of $M^{(2)}$.

Lemma 3.2.12. *For $(u,e)\in\mathbb{R}^+\times E$, let $P_t(u,e) := \mathbb{E}[Z_T h(u)g(e)\alpha^k_T(u,e)|\mathcal{G}^k_t]$, then by the inductive hypothesis there exists $\{p(u,e),\hat{p}^1(u,e),\hat{p}^2(u,e),...,\hat{p}^k(u,e)\}$ such that*

$$P_t(u,e) = P_0(u,e) + \int_0^t p_s(u,e)dS^{\mathbb{G},k}_s + \sum_{i=1}^k\int_0^t\int_E \hat{p}^i_s(u,e,e')\tilde{\mu}^i(ds,de'),$$

then just as in the decomposition of G^k_t define,

$$\begin{aligned}Q_t := \int_t^T\int_E P_t(u,e)dedu &= \int_0^T\int_E P_t(u,e)dedu + \int_0^t\int_E\left(P_u(u,e)-P_t(u,e)\right)dedu \\ &- \int_0^t\int_E P_u(u,e)dedu \\ &= \int_0^t q_s dS^{\mathbb{G},k}_s + \sum_{i=1}^k\int_0^t\int_E \hat{q}^i_s(e)\tilde{\mu}^i(de,ds) - \int_0^t\int_E P_u(u,e)dedu,\end{aligned}$$

then

$$M_t^{(2)} = M_t^{(0)} + \int_0^t \frac{1-H_{s-}^{k+1}}{(G_s^k)^2}\left(q_s G_s^k - Q_{s-} n_s^k\right) dS_s^{\mathbb{G},k+1}$$
$$+ \sum_{i=1}^k \int_0^t \int_E \left(\frac{\hat{q}_s^i(e)(1-H_{s-}^{k+1})}{G_s^k} \right) \tilde{\mu}^i(ds,de) + \int_0^t \int_E \frac{Q_{s-}(1-H_{s-}^{k+1})}{G_s^k} \tilde{\mu}^{k+1}(ds,de)$$
$$- \int_0^t \int_E \frac{P_s(s,e)(1-H_{s-}^{k+1})}{G_s^k} de\, ds$$

Proof.

$$M_t^{(2)} = \frac{\mathbb{E}[Z_T h(\tau_{k+1}) g(\xi_{k+1}) \mathbb{1}_{\{t<\tau_{k+1}\le T\}} | \mathcal{G}_t^k]}{G_t^k} \mathbb{1}_{\{\tau_{k+1}>t\}}$$
$$= \frac{\mathbb{E}[\int_t^T \int_E Z_T h(u) g(e) \alpha_T^k(u,e) de du | \mathcal{G}_t^k] H_t^{k+1}}{G_t^k}.$$

Using Fubini's Theorem, this is written in terms of Q as

$$M_t^{(2)} = \frac{Q_t(1-H_t^{k+1})}{G_t^k}.$$

Using Lemma 2.3.4 in Section 2.3 in Chapter 2, we have

$$M_t^{(2)} - M_0^{(2)} = \int_0^t \frac{(1-H_{s-}^{k+1})}{G_s^k} dQ_s - \int_0^t \frac{Q_{s-}}{G_s^k} dH_s^{k+1} - \int_0^t \frac{Q_{s-}(1-H_{s-}^{k+1})}{(G_s^k)^2} dG_s^k$$
$$- \int_0^t \frac{(1-H_{s-}^{k+1})}{(G_s^k)^2} d\langle Q, G^k\rangle_s + \int_0^t \frac{Q_{s-}(1-H_{s-}^{k+1})}{G_s^k} d\langle G^k, G^k\rangle_s$$
$$+ \sum_{s\le t} \Big[\frac{Q_s(1-H_s^{k+1})}{G_s^k} - \frac{Q_{s-}(1-H_{s-}^{k+1})}{G_s^k} - \frac{1-H_{s-}^{k+1}}{G_s^k}\Delta Q_s + \frac{Q_{s-}}{G_s^k}\Delta H_s^{k+1} \Big].$$

It is clear from the decomposition of $M^{(1)}$ in Lemma 3.2.11 that the decomposition of $M^{(2)}$ will be very similar with the addition of one finite variation term, that is

$$M_t^{(2)} - M_0^{(2)} = \int_0^t \frac{1-H_{s-}^{k+1}}{(G_s^k)^2}\left(q_s G_s^k - Q_{s-} n_s^k\right) dS_s^{\mathbb{G},k+1}$$
$$+ \sum_{i=1}^k \int_0^t \int_E \left(\frac{\hat{q}^i(e)_s(1-H_{s-}^{k+1})}{G_s^k} \right) \tilde{\mu}^i(de,ds) + \int_0^t \int_E \frac{Q_{s-}(1-H_{s-}^{k+1})}{G_s^k} \tilde{\mu}^{k+1}(ds,de)$$
$$- \int_0^t \int_E \frac{P_s(s,e)(1-H_{s-}^{k+1})}{G_s^k} deds.$$

□

Finally, the decomposition of $M^{(3)}$ is given.

Lemma 3.2.13. *Recall the integral representation of $P(u,e)$ for $(u,e) \in \mathbb{R}^+ \times E$,*

$$P_t(u,e) = P_0(u,e) + \int_0^t p_s(u,e)dS_s^{\mathbb{G},k} + \sum_{i=1}^k \int_0^t \int_E \hat{p}_s^i(e)\tilde{\mu}^i(ds,de),$$

then the decomposition of $M^{(3)}$ is as follows

$$\begin{aligned}
M_t^{(3)} =& \int_0^t \frac{H_{s-}^{k+1}}{\left(\alpha_s^k(\tau_{k+1},\xi_{k+1})\right)^2} \left(p_s(\tau_{k+1},\xi_{k+1})\alpha_s^k(\tau_{k+1},\xi_{k+1}) - P_{s-}(\tau_{k+1},\xi_{k+1})a_s(\tau,\xi)\right) dS_s^{\mathbb{G},k+1} \\
&+ \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})}\hat{p}_s^i(\tau_{k+1},\xi_{k+1},e)\tilde{\mu}^i(ds,de) \\
&+ \int_0^t \int_E \left(\frac{P_s(s,e)(1-H_{s-}^{k+1})}{\alpha_s^k(s,e)}\right)\tilde{\mu}^{k+1}(ds,de) + \int_0^t \int_E \frac{P_s(s,e)(1-H_{s-}^{k+1})}{G_s^k} de\ ds
\end{aligned}$$

Proof.

$$\begin{aligned}
M_t^{(3)} =& \frac{\mathbb{E}\Big[Z_T h(u_{k+1})g(e_{k+1})|\mathcal{G}_t^k\Big]\Big|_{\substack{u_{k+1}=\tau_{k+1}\\ e_{k+1}=\xi_{k+1}}}}{\alpha_t^k(\tau_{k+1},\xi_{k+1})} \mathbb{1}_{\{\tau_{k+1}\le t\}} \\
=& \frac{P_t(\tau_{k+1},\xi_{k+1})H_t^{k+1}}{\alpha_t^k(\tau_{k+1},\xi_{k+1})}
\end{aligned}$$

Again, using integration by parts we get

$$\begin{aligned}
&\frac{P_s(u,e)H_s^{k+1}}{\alpha_s^k(u,e)} - \frac{P_0(u,e)H_0^{k+1}}{\alpha_0^k(u,e)} \\
=& \int_0^t \frac{H_{s-}^{k+1}}{\alpha_s^k(u,e)} dP_s(u,e) + \int_0^t \frac{P_{s-}(u,e)}{\alpha_s^k(u,e)} dH_s^{k+1} - \int_0^t \frac{P_{s-}(u,e)H_{s-}^{k+1}}{\left(\alpha_s^k(u,e)\right)^2} d\alpha_s^k(u,e) \\
&- \int_0^t \frac{H_{s-}^{k+1}}{\left(\alpha_s^k(u,e)\right)^2} d\langle P(u,e),\alpha^k(u,e)\rangle_s + \int_0^t \frac{P_{s-}(u,e)H_{s-}^{k+1}}{\left(\alpha_s^k(u,e)\right)^3} d\langle \alpha^k(u,e),\alpha^k(u,e)\rangle_s \\
&+ \sum_{s\le t}\Big[\frac{P_s(u,e)H_s^{k+1}}{\alpha_s^k(u,e)} - \frac{P_{s-}(u,e)H_{s-}^{k+1}}{\alpha_s^k(u,e)} - \frac{H_{s-}^{k+1}}{\alpha_s^k(u,e)}\Delta P_s(u,e) - \frac{P_{s-}(u,e)}{\alpha_s^k(u,e)}\Delta H_s^{k+1}\Big] \\
=& \int_0^t \frac{H_{s-}^{k+1}}{\left(\alpha_s^k(u,e)\right)^2}\left(p_s(u,e)\alpha_s^k(u,e) - P_{s-}(u,e)a_s(u,e)\right) dS_s^{\mathbb{G},k} + \int_0^t \frac{P_{s-}(u,e)}{\alpha_s^k(u,e)} dH_s^{k+1} \\
&- \int_0^t \frac{H_{s-}^{k+1}a_s(u,e)}{\left(\alpha_s^k(u,e)\right)^3}\left(p_s(u,e)\alpha_s^k(u,e) - P_{s-}(u,e)a_s(u,e)\right) d\langle S^{\mathbb{G},k},S^{\mathbb{G},k}\rangle_s \\
&+ \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{\alpha_s^k(u,e)}\hat{p}_s^i(u,e,e')\tilde{\mu}^i(ds,de') \\
&+ \sum_{i=1}^k \int_0^t \left(\frac{P_s(u,e)H_{s-}^{k+1}}{\alpha_s^k(u,e)} - \frac{P_{s-}(u,e)H_{s-}^{k+1}}{\alpha_s^k(u,e)}\right) dH_s^i + \int_0^t \left(\frac{P_s(u,e)H_s^{k+1}}{\alpha_s^k(u,e)} - \frac{P_s(u,e)H_{s-}^{k+1}}{\alpha_s^k(u,e)}\right) dH_s^{k+1} \\
&- \sum_{i=1}^k \frac{H_{s-}^{k+1}}{\alpha_s^k(u,e)}\hat{p}_s^i(u,e,e')\mu^i(ds,de') - \int_0^t \frac{P_{s-}(u,e)}{\alpha_s^k(u,e)} dH_s^{k+1}.
\end{aligned}$$

From equation (3.10), we see that $H_t^{k+1}dS_t^{\mathbb{G},k+1} = H_t^{k+1}\left(S_t^{\mathbb{G},k} - \frac{a_t(\tau_{k+1},\xi_{k+1})}{\alpha_t^k(\tau_{k+1},\xi_{k+1})} d\langle S^{\mathbb{G},k},S^{\mathbb{G},k}\rangle_s\right)$.

Note also that

$$\int_0^t \left(\frac{P_s(\tau_{k+1},\xi_{k+1})H_s^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} - \frac{P_s(\tau_{k+1},\xi_{k+1})H_{s-}^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} \right) dH_s^{k+1}$$
$$= \int_0^t \int_E \left(\frac{P_s(\tau_{k+1},\xi_{k+1})}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} - \frac{P_s(\tau_{k+1},\xi_{k+1})H_{s-}^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} \right) \mu^{k+1}(ds,de)$$
$$= \int_0^t \int_E \left(\frac{P_s(s,e)}{\alpha_s^k(s,e)} - \frac{P_s(s,e)H_{s-}^{k+1}}{\alpha_s^k(s,e)} \right) \mu^{k+1}(ds,de).$$

Altogether this implies

$$\frac{P_t(\tau_{k+1},\xi_{k+1})H_t^{k+1}}{\alpha_t^k(\tau_{k+1},\xi_{k+1})} - \frac{P_0(\tau_{k+1},\xi_{k+1})H_0^{k+1}}{\alpha_0^k(\tau_{k+1},\xi_{k+1})}$$
$$= \int_0^t \frac{H_{s-}^{k+1}}{\left(\alpha_s^k(\tau_{k+1},\xi_{k+1})\right)^2} \left(p_s(\tau_{k+1},\xi_{k+1})\alpha_s^k(\tau_{k+1},\xi_{k+1}) - P_{s-}(\tau_{k+1},\xi_{k+1})a_s(\tau,\xi)\right) dS_s^{\mathbb{G},k+1}$$
$$+ \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} \hat{p}_s^i(\tau_{k+1},\xi_{k+1},e)\tilde{\mu}^i(de,ds)$$
$$+ \int_0^t \int_E \left(\frac{P_s(s,e)}{\alpha_s^k(s,e)} - \frac{P_s(s,e)H_{s-}^{k+1}}{\alpha_s^k(s,e)} \right) \mu^{k+1}(de,ds).$$

Expanding μ^{k+1}, we get

$$= \int_0^t \frac{H_{s-}^{k+1}}{\left(\alpha_s^k(\tau_{k+1},\xi_{k+1})\right)^2} \left(p_s(\tau_{k+1},\xi_{k+1})\alpha_s^k(\tau_{k+1},\xi_{k+1}) - P_{s-}(\tau_{k+1},\xi_{k+1})a_s(\tau,\xi)\right) dS_s^{\mathbb{G},k+1}$$
$$+ \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} \hat{p}_s^i(\tau_{k+1},\xi_{k+1},e)\tilde{\mu}^i(de,ds)$$
$$+ \int_0^t \int_E \left(\frac{P_s(s,e)(1-H_{s-}^{k+1})}{\alpha_s^k(s,e)} \right) \tilde{\mu}^{k+1}(de,ds) + \int_0^t \int_E \left(\frac{P_s(s,e)(1-H_{s-}^{k+1})}{\alpha_s^k(s,e)} \right) \lambda_s^{k+1}(e)de\, ds$$

Using the definition, $(1-H_{s-}^{k+1})\lambda_s^{k+1} = (1-H_{s-}^{k+1})\alpha_s^k(s,e)/G_s^k$, the last term is simplified to

$$= \int_0^t \frac{H_{s-}^{k+1}}{\left(\alpha_s^k(\tau_{k+1},\xi_{k+1})\right)^2} \left(p_s(\tau_{k+1},\xi_{k+1})\alpha_s^k(\tau_{k+1},\xi_{k+1}) - P_{s-}(\tau_{k+1},\xi_{k+1})a_s(\tau,\xi)\right) dS_s^{\mathbb{G},k+1}$$
$$+ \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} \hat{p}_s^i(\tau_{k+1},\xi_{k+1},e)\tilde{\mu}^i(de,ds) + \int_0^t \int_E \left(\frac{P_s(s,e)(1-H_{s-}^{k+1})}{\alpha_s^k(s,e)} \right) \tilde{\mu}^{k+1}(de,ds)$$
$$+ \int_0^t \int_E \frac{P_s(s,e)(1-H_{s-}^{k+1})}{G_s^k} deds.$$

□

The final representations of $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$ are

$$
\begin{aligned}
M_t^{(1)} =& M_0^{(1)} + h(T)g(0)\Big(\int_0^t \frac{1-H_{s-}^{k+1}}{(G_s^k)^2} \left(p_s G_s^k - P_{s-} n_s^k\right) dS_s^{\mathbb{G},k+1} \\
&+ \sum_{i=1}^k \int_0^t \int_E \left(\frac{\hat{p}_s^i(e)(1-H_{s-}^{k+1})}{G_s^k} \right) \tilde{\mu}^i(ds,de) + \int_0^t \int_E \frac{P_{s-}(1-H_{s-}^{k+1})}{G_s^k} \tilde{\mu}^{k+1}(ds,de) \Big). \\
M_t^{(2)} =& M_t^{(0)} + \int_0^t \frac{1-H_{s-}^{k+1}}{(G_s^k)^2} \left(q_s G_s^k - Q_{s-} n_s^k\right) dS_s^{\mathbb{G},k+1} \\
&+ \sum_{i=1}^k \int_0^t \int_E \left(\frac{\hat{q}_s^i(e)(1-H_{s-}^{k+1})}{G_s^k} \right) \tilde{\mu}^i(ds,de) + \int_0^t \int_E \frac{Q_{s-}(1-H_{s-}^{k+1})}{G_s^k} \tilde{\mu}^{k+1}(ds,de) \\
&- \int_0^t \int_E \frac{P_s(s,e)(1-H_{s-}^{k+1})}{G_s^k} de\ ds \\
M_t^{(3)} =& \int_0^t \frac{H_{s-}^{k+1}}{(\alpha_s^k(\tau_{k+1},\xi_{k+1}))^2} \left(p_s(\tau_{k+1},\xi_{k+1})\alpha_s^k(\tau_{k+1},\xi_{k+1}) - P_{s-}(\tau_{k+1},\xi_{k+1}) a_s(\tau,\xi)\right) dS_s^{\mathbb{G},k+1} \\
&+ \sum_{i=1}^k \int_0^t \int_E \frac{1-H_{s-}^{k+1}}{\alpha_s^k(\tau_{k+1},\xi_{k+1})} \hat{p}_s^i(\tau_{k+1},\xi_{k+1},e) \tilde{\mu}^i(ds,de) \\
&+ \int_0^t \int_E \left(\frac{P_s(s,e)(1-H_{s-}^{k+1})}{\alpha_s^k(s,e)} \right) \tilde{\mu}^{k+1}(ds,de) + \int_0^t \int_E \frac{P_s(s,e)(1-H_{s-}^{k+1})}{G_s^k} de\ ds
\end{aligned}
$$

Combining these three terms, we see that indeed, M is decomposed as a sum of stochastic integrals with respect to $S^{\mathbb{G},k+1}$ and $\{\tilde{\mu}^1, \tilde{\mu}^2, ..., \tilde{\mu}^{k+1}\}$.
Finally, the following Lemma asserts that the integrands in the above integral representations belong to $L^2(S^{\mathbb{G},k+1})$ and $L^2(\tilde{\mu}^i)$ for $i \in \{1, 2..., k+1\}$.

Lemma 3.2.14. *Define the following for $t \geq 0$ and $i \in \{1, 2, ..., k\}$,*

$$
\begin{aligned}
\phi_t^{k+1} &= \left(1-H_{t-}^{k+1}\right) \frac{h(T)g(0)\left(p_t G_t^k - P_{t-} n_t^k\right) + \left(q_t G_t^k - Q_{t-} n_t^k\right)}{(G_t^k)^2} \\
&\quad + H_{t-}^{k+1} \frac{\left(p_s(\tau_{k+1},\xi_{k+1})\alpha_s^k(\tau_{k+1},\xi_{k+1}) - P_{s-}(\tau_{k+1},\xi_{k+1}) a_s(\tau,\xi)\right)}{(\alpha_s^k(\tau_{k+1},\xi_{k+1}))^2}, \\
\beta_t^{i,k+1}(e) &= \left(1-H_{t-}^{k+1}\right) \left(\frac{\hat{p}_t^i(e) + \hat{q}_t^i(e)}{G_t^k} + \frac{\hat{p}_t^i(\tau_{k+1},\xi_{k+1},e)}{\alpha_t^k(\tau_{k+1},\xi_{k+1})} \right), \\
\beta_t^{k+1,k+1}(e) &= \left(1-H_{t-}^{k+1}\right) \left(\frac{P_{t-} + Q_{t-}}{G_t^k} + \frac{P_t(t,e)}{\alpha_t^k(t,e)} \right),
\end{aligned}
$$

then $\phi^{k+1} \in L^2(S^{\mathbb{G},k+1})$ and $\beta^{i,k+1} \in L^2(\tilde{\mu}^i)$ for all $i \in \{1, 2, ..., k+1\}$ and

$$
M_t = M_0 + \int_0^t \phi_s^{k+1} dS_s^{\mathbb{G},k+1} + \sum_{i=1}^{k+1} \int_0^t \int_E \beta_s^{i,k+1}(e) \tilde{\mu}^i(ds,de).
$$

Proof. The integral representation comes from Lemmas 3.2.11, 3.2.12 and 3.2.13.
By the inductive hypothesis, p, q, n^k and $a^k(u,e)$ belong to $L^2(S^{\mathbb{G},k})$ which implies they belong to $L^2(S^{\mathbb{G},k+1})$ too. Furthermore, the boundedness of Z_T, h and g means P is bounded.

Note also that

$$
\begin{aligned}
H_{t-}^{k+1}P_t(\tau_{k+1},\xi_{k+1}) =& H_t^{k+1}\alpha_t^k(\tau_{k+1},\xi_{k+1})\mathbb{E}\left[Z_T h(\tau_{k+1})g(\xi_{k+1})|\mathcal{G}_t^{k+1}\right].\\
(1-H_{t-}^{k+1})Q_t =&(1-H_{t-}^{k+1})\int_t^T\int_E P_t(u,e)de\,du\\
=&(1-H_{t-}^{k+1})\mathbb{E}\left[\frac{P_t(\tau_{k+1},\xi_{k+1})}{\alpha_t^k(\tau_{k+1},\xi_{k+1})}\mathbb{1}_{\{t<\tau_{k+1}\le T\}}\right].
\end{aligned}
$$

Finally, the fact that $0<G_t^k\le 1$ when $t>\tau_{k+1}$, means that $\phi^{k+1}\in L^2(S^{\mathbb{G},k+1})$.
The fact that $\beta^{i,k+1}\in L^2(\tilde{\mu}^i)$ for $i\in\{1,2,...,k\}$ follows because $\hat{p}$, $\hat{q}$ and $\hat{p}(\tau_{k+1},\xi_{k+1})$ all belong to $L^2(\tilde{\mu}^i)$.
Note the following

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t\int_E\left(\frac{P_s(s,e)}{\alpha_s^k(s,e)}\right)^2\lambda_s^{k+1}(e)de\,ds\right]=&\mathbb{E}\left[\int_0^t\int_E\left(\frac{P_s(s,e)}{\alpha_s^k(s,e)}\right)^2\mu^{k+1}(ds,de)\right]\\
=&\mathbb{E}\left[\left(\frac{P_{\tau_{k+1}}(\tau_{k+1},\xi_{k+1})}{\alpha_{\tau_{k+1}}^k(\tau_{k+1},\xi_{k+1})}\right)^2\right]<\infty.
\end{aligned}
$$

This implies that $\beta^{k+1,k+1}\in L^2(\tilde{\mu}^{k+1})$. □

By the induction conclusion, we can now conclude that $\mathbb{G}^k$ enjoys martingale representation for all $k=1,2...,n$. □

3.3 Remarks

This chapter has derived results using a progressive enlargement of filtration with a sequence of ordered random times and their associated marks. The purpose of including random times and random marks is, in part, to allow one to transfer between traditional progressive and initial enlargement theory.
Indeed, one can recover a traditional progressive enlargement by setting all random marks to some deterministic set of functions on $(\Omega,\mathcal{F},\mathbb{P})$. One can recover an initial enlargement by setting all random times to 0.

Chapter 4

Backward Stochastic Differential Equations in an Enlarged Filtration

This chapter will focus on the application of chapter 3 to the solutions of backward stochastic differential equations (BSDEs). BSDEs are a useful tool in mathematical finance, the seminal work of El Karoui et al. [1997b] giving an overview of their applications to finance. In essence, a BSDE can model the value of a hedging portfolio for a derivative contract that settles at maturity.
The BSDEs in this chapter will reference a Brownian Motion, we therefore set $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space supporting a d-dimensional Brownian motion W for the remainder of this chapter. Let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ be the completed, natural filtration of W. The notation of chapter 3 is kept the same. Before continuing we recall two helpful results for the study of backward stochastic differential equations, namely the Burkholder-Davis-Gundy inequality (see Theorem 48, Chapter 4 in Protter [2005] for example) and Banach fixed point theorem(see Lemma 1.5.18 in Cohen and Elliott [2015]). The Burkholder-Davis-Gundy inequality will be used to control the norms of semimartingales.

Proposition 4.0.1. *For $1 \leq p < \infty$, there exist constants c_p and C_p such that for any local martingale X and stopping time ν*

$$c_p \mathbb{E}\left[[X,X]_\nu^{p/2}\right] \leq \mathbb{E}\left[\left(\sup_{t\leq\nu} X_t\right)^p\right] \leq C_p \mathbb{E}\left[[X,X]_\nu^{p/2}\right].$$

The Banach fixed point theorem will be used to prove the uniqueness of solutions to BSDEs in the enlarged filtration.

Proposition 4.0.2. *A function $F : X \to X$ defined on a non-empty metric space (X,d) is a contraction if there exists a $c \in [0,1)$ such that for any $x_1, x_2 \in X$, $d(F(x_2), F(x_1)) \leq cd(x_2, x_1)$. If (X,d) is a complete metric space then F admits a unique fixed point, meaning a point $X^* \in X$ such that $F(x^*) = x^*$.*

The enlarged filtration $\mathbb{G}$, is setup as it is in Section 3.1 in Chapter 3. To setup the backward stochastic differential equation in $\mathbb{G}$, we recall the decomposition of the family γ^k defined in Equation (3.2) in Subsection 3.2.1 of Chapter 3, as follows, for $k \in \{0, 1, ..., n-1\}$ and

$(u^{(k)}, e^{(k)}) \in \Theta_k \times E^k$

$$\begin{aligned}
\gamma_t^k(u^{(k)}, e^{(k)}) &= \int_t^\infty \int_{u_{k+1}}^\infty \cdots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&= \int_0^\infty \int_{u_{k+1}}^\infty \cdots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&\quad - \int_0^t \int_{u_{k+1}}^\infty \cdots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&= \int_0^\infty \int_{u_{k+1}}^\infty \cdots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha_t(u,e) de^{(n:k+1)} du^{(n:k+1)} \\
&\quad - \int_0^t \int_{u_{k+1}}^\infty \cdots \int_{u_{n-1}}^\infty \int_{E^{n-k}} (\alpha_t(u,e) - \alpha_{u_{k+1}}(u,e)) de^{(n:k+1)} du^{(n:k+1)} \\
&\quad - \int_0^t \int_{u_{k+1}}^\infty \cdots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha_{u_{k+1}}(u,e) de^{(n:k+1)} du^{(n:k+1)}.
\end{aligned} \tag{4.1}$$

Then using Lemma 2.3.5 from Chapter 2 it is seen that the first two terms are $\mathbb{F}$-martingales and therefore by the martingale representation theorem for Brownian motion, there exists $a^k \in \mathcal{P}(\mathbb{F}, \Theta_k, E^k)$ such that

$$\begin{aligned}
\gamma_t^k(u^{(k)}, e^{(k)}) =& \gamma_0^k(u^{(k)}, e^{(k)}) + \int_0^t a_s^k(u^{(k)}, e^{(k)}) dW_s \\
& - \int_0^t \int_{u_{k+1}}^\infty \cdots \int_{u_{n-1}}^\infty \int_{E^{n-k}} \alpha_{u_{k+1}}(u,e) de^{(n:k+1)} du^{(n:k+1)}. \\
=& \gamma_0^k(u^{(k)}, e^{(k)}) + \int_0^t a_s^k(u^{(k)}, e^{(k)}) dW_s - \int_0^t \int_E \gamma_{u_{k+1}}^{k+1}(u^{(k+1)}, e^{(k+1)}) de_{k+1} du_{k+1}
\end{aligned} \tag{4.2}$$

Then using Theorem 3.2.7 from chapter 3, we see

$$W_t^{\mathbb{G}} = W_t - \sum_{k=0}^n \int_{\tau_k \wedge t}^{\tau_{k+1} \wedge t} \frac{a_s^k(\tau^{(k)}, \xi^{(k)})}{\gamma_s^k(\tau^{(k)}, \xi^{(k)})} ds. \tag{4.3}$$

is a $\mathbb{G}$-martingale, furthermore from Lévy's characterisation of Brownian motion, $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion.

In view of Theorem 3.2.10, all martingales in $\mathbb{G}$ are generated by $W^{\mathbb{G}}$ and the family of compensated jump measures $(\tilde{\mu}^k)_{k \in \{1,2,\ldots,n\}}$. Before introducing the BSDE, the components which generate the solutions are introduced. The terminal condition X and the driver f are defined below:

a) The terminal condition $X \in L^2(\mathcal{G}_T)$.

b) The driver $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times B(E) \to \mathbb{R}$ is such that for any fixed $(y,z,u) \in \mathbb{R} \times \mathbb{R}^d \times B(E)$[1], $(f(t,y,z,u))_{t \geq 0}$ is $\mathbb{G}$-predictable and $f(\cdot,0,0,0)$ is a $\mathbb{P}$-square integrable $\mathbb{G}$-predictable process.

[1] Recall that the set $B(E)$ is defined as all Borel measurable functions on E.

The set (X, f) will be referred to as the data generating the BSDE in $\mathbb{G}$.

Let $\beta \geq 0$ and define the following spaces:

a) $\mathbb{S}^{2,\beta}_{\mathbb{G}} := \{\phi \in \mathcal{P}(\mathbb{G})$, possibly multi-dimensional :

$$\|\phi\|_{\mathbb{S}^{2,\beta}_{\mathbb{G}}} := \mathbb{E}\left[\sup_{t\in[0,T]}\left(e^{\beta t}|\phi_t|^2\right)\right]^{\frac{1}{2}} < \infty\}.$$

b) $\mathbb{H}^{2,\beta}_{\mathbb{G}} := \{\phi \in \mathcal{P}(\mathbb{G})$, possibly multi-dimensional :

$$\|\phi\|_{\mathbb{H}^{2,\beta}_{\mathbb{G}}} := \mathbb{E}\left[\int_0^T e^{\beta s}|\phi_s|^2 ds\right]^{\frac{1}{2}} < \infty\}.$$

c) $\mathbb{H}^{2,\beta}_{\mathbb{G},E} := \{\psi \in \mathcal{P}(\mathbb{G} \otimes \mathcal{B}(E))$, possibly multi-dimensional :

$$\|\psi\|_{\mathbb{H}^{2.\beta}_{\mathbb{G},E}}$$

$:= \mathbb{E}\left[\int_0^T \int_E e^{\beta s}\lambda_s(e)|\psi_s(e)|^2 de\ ds\right]^{\frac{1}{2}} < \infty\}$. The spaces $\mathbb{S}^2_{\mathbb{G}}, \mathbb{H}^2_{\mathbb{G}}$ and $\mathbb{H}^2_{\mathbb{G},E}$ will refer to the above spaces with $\beta = 0$.

Note that the assumption that E has finite Lebesgue measure implies that for any $\psi \in \mathbb{H}^2_{\mathbb{G},E}$ and $t \in [0, T]$

$$\int_E \sqrt{\lambda_t(e)}|\psi_t(e)|de > \infty.$$

Indeed,

$$\mathbb{E}\left[\int_0^T \int_E \lambda_s(e)|\psi_s(e)|^2 de\ ds\right]^{\frac{1}{2}} \geq \mathbb{E}\left[\left(\int_0^T \int_E \lambda_s(e)|\psi_s(e)|^2 de\ ds\right)^{\frac{1}{2}}\right].$$

This then implies that for all $t \in [0, T]$

$$\left(\int_E \lambda_t(e)|\psi_t(e)|^2 de\right)^{\frac{1}{2}} < \infty.$$

Finally, the assumption that E has finite Lebesgue measure means that Jensen's inequality can be applied to the above integal to imply

$$\int_E \sqrt{\lambda_t(e)}|\psi_t(e)|de < \infty.$$

We consider the following BSDE in $\mathbb{G}$, where the solution triple $\left(Y, Z, \left(U^k\right)_{k\in\{1,2,\ldots,n\}}\right) \in \mathbb{S}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$ solves the following backward equation

$$Y_t = X + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW^{\mathbb{G}}_s - \sum_{k=1}^n \int_t^T \int_E U^k_s(e)\tilde{\mu}^k(ds, de), \tag{4.4}$$

where $U := \sum_{k=1}^{n} U^k \mathbb{1}_{[\![\tau_{k-1},\tau_k))}$.
Note that the process Y jumps at each τ_k, and that each pure jump component U^k may be chosen to vanish strictly after τ_k. In other words, the process Y is continuous on each interval $[\tau_k, \tau_{k+1})$ at which point it jumps by an amount $U^{k+1}_{\tau_{k+1}}(\xi_{k+1})$ at time τ_{k+1} provided τ_{k+1} occurs before time T and after time t. After time τ_{k+1}, the process U^{k+1} does not jump at any further $(\tau_j : j > k+1)$ and so the process can be chosen to be zero after τ_{k+1}.

4.1 Existence and Uniqueness of Solutions

The following subsection will investigate the existence and uniqueness of solutions of BSDE (4.4). In order to prove uniqueness of a solution to BSDE (4.4), a continuity assumption on the driver f will be made.

Hypothesis 6. *For all $t \geq 0$, there exists $C > 0$ such that*

$$|f(t,y,z,u) - f(t,y',z',u')| \leq C\left(|y-y'| + |z-z'| + \sum_{k=1}^{n}\int_E \sqrt{\lambda_t^k(e)}|u^k(e) - u'^k(e)|de\right)$$

for all $\left(y,z,(u^k)_{k\in\{1,2,\ldots,n\}}\right) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{H}^2_{\mathbb{G},E}$ and $\left(y',z',(u'^k)_{k\in\{1,2,\ldots,n\}}\right) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{H}^2_{\mathbb{G},E}$, where $u := \sum_{k=1}^{n} u^k \mathbb{1}_{[\![\tau_{k-1},\tau_k))}$ and $u' := \sum_{k=1}^{n} u'^k \mathbb{1}_{[\![\tau_{k-1},\tau_k))}$.

With hypothesis 6, it will be possible to show that the solution to BSDE (4.4) exists and is unique. This effectively means that the norms of the solutions Y, Z and $\left(U^k\right)_{k\in\{1,2,\ldots,n\}}$ can be controlled given the data (X, f). The first step in proving this is to show that we can restrict our attention to $Y \in \mathbb{H}^2_{\mathbb{G}}$, allowing an easier computation of the norms of the solution triple. Thanks to Doob's maximal inequality, we get the first lemma:

Lemma 4.1.1. *If f satisfies Hypothesis 6 and $\left(Y,Z,(U^k)_{k\in\{1,2,\ldots,n\}}\right) \in \mathbb{H}^2_{\mathbb{G}}\times\mathbb{H}^2_{\mathbb{G}}\times\mathbb{H}^2_{\mathbb{G},E}$. Then $Y \in \mathbb{S}^2_{\mathbb{G}}$.*

Proof. Firstly, the Cauchy-Schwartz inequality says that for for $a,b,c,d \in \mathbb{R}$,

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2),$$

then squaring either side of BSDE (4.4), taking suprema and expectations gives

$$\mathbb{E}\left[\sup_{t\leq T} Y_t^2\right] \leq 4\mathbb{E}\Bigg[X^2 + \sup_{t\leq T}\left(\int_t^T f(s,Y_s,Z_s,U_s)ds\right)^2 + \sup_{t\leq T}\left(\int_t^T Z_s dW_s^{\mathbb{G}}\right)^2 + \sum_{k=1}^{n}\sup_{t\leq T}\left(\int_t^T\int_E U_s^k(e)\tilde{\mu}^k(ds,de)\right)^2\Bigg].$$

The last two terms in the expectation on the right hand side are identified as the suprema of local martingales, therefore the Burkholder-Davis-Gundy inequality (Proposition 4.0.1) can be used with $p = 2$ and $\nu = T$ to ensure the existence of a constant K such that

$$\mathbb{E}\left[\sup_{t\leq T} Y_t^2\right] \leq K\mathbb{E}\Bigg[X^2 + \int_0^T f(s,Y_s,Z_s,U_s)^2 ds + \left[\int_0^{\cdot} Z_s dW_s, \int_0^{\cdot} Z_s dW_s\right]_T + \sum_{k=1}^{n}\left[\int_0^{\cdot}\int_E U_s^k(e)\tilde{\mu}^k(ds,de), \int_0^{\cdot}\int_E U_s^k(e)\tilde{\mu}^k(ds,de)\right]_T\Bigg].$$

The second to last term is computed using Itô-isometry and the last term is the quadratic variation of a purely-discontinuous martingale, meaning

$$\mathbb{E}\left[\sup_{t\leq T} Y_t^2\right] \leq K\mathbb{E}\left[X^2 + \int_0^T f(s,Y_s,Z_s,U_s)^2 ds + \int_0^T |Z_s|^2 ds + \sum_{k=1}^n \int_0^T \int_E U_s^k(e)^2 \lambda_s^k(e) deds\right].$$

Hypothesis 6 coupled with the fact that $f(\cdot,0,0,0)$ is square integrable yields a new constant K' such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq T} Y_t^2\right] &\leq K'\mathbb{E}\Bigg[X^2 + \int_0^T \left(f(s,0,0,0)^2 + Y_s^2 + |Z_s|^2 + \int_E \sum_{k=1}^n \lambda_s^k(e) U_s^k(e)^2 de\right) ds \\ &\quad + \int_0^T |Z_s|^2 ds + \sum_{k=1}^n \int_0^T \int_E U_s^k(e)^2 \lambda_s^k(e) deds\Bigg] \\ &\leq K'\left(\|f(\cdot,0,0,0)\|^2_{\mathbb{H}^2_{\mathbb{G}}} + \|X\|^2_{L^2(\mathcal{G}_T)} + \|Y\|^2_{\mathbb{H}^2_{\mathbb{G}}} + \|Z\|^2_{\mathbb{H}^2_{\mathbb{G}}} + \sum_{k=1}^n \|U^k\|^2_{\mathbb{H}^2_{\mathbb{G},E}}\right) < \infty\end{aligned}$$

□

We are now in a position to extend the results of Kharroubi and Lim [2014] to the case without immersion (See section 2.3.4 for a definition). The following theorem proves the existence and uniqueness to the solutions of BSDE (4.4).

Theorem 4.1.2. *Suppose the driver f satisfies Hypothesis 6, then for $\beta \geq 0$ BSDE (4.4) admits a unique solution $\left(Y, Z, (U^k)_{k\in\{1,2,\dots,n\}}\right) \in \mathbb{S}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$ such that $U_t^k = 0$ for $t > \tau_k$ for $k \in \{1,2,...,n\}$.*

Proof. The proof will be broken down in to the following three lemmas:

1. Lemma 4.1.3 proves the existence of a solution to BSDE (4.4).
2. Lemma 4.1.4 derives estimates of the solution to be used to prove their uniqueness.
3. Lemma 4.1.5 concludes the proof of Theorem 4.1.2 by utilising the Banach fixed point theorem (see Proposition 4.0.2).

For $\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right) \in \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$, let $u := \sum_{k=1}^n u^k \mathbb{1}_{[\![\tau_{k-1},\tau_k)\!)}$ and consider the BSDE

$$Y_t = X + \int_t^T f(s,y_s,z_s,u_s)ds - \int_t^T Z_s dW_s^{\mathbb{G}} - \sum_{k=1}^n \int_t^T \int_E U_s^k(e)\tilde{\mu}^k(ds,de). \tag{4.5}$$

The outline of the entire proof is as follows:

1. First show that for a fixed $\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right)$, a solution to BSDE 4.5 exists.
2. Next, imply a mapping from $\mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$ to itself where a fixed $\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right)$ is inputted to BSDE 4.5 and the output is a solution $\left(Y, Z, (U^k)_{k\in\{1,2,\dots,n\}}\right)$

3. Show that this mapping admits a fixed point therefore implying a unique solution.

The first lemma proves the existence of such a mapping.

Lemma 4.1.3. *There exists a mapping* $\Phi : \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E} \to \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$ *such that for* $\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right) \in \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$, $\left(Y, Z, (U_k)_{k\in\{1,2,\dots,n\}}\right) := \Phi\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right)$ *is a solution to* (4.5)

Proof. This is done by first noting that the driver does not depend on the solution $(Y, Z, (U_k)_{k\in\{1,2,\dots,n\}})$, define the martingale

$$M_t := \mathbb{E}\left[X + \int_0^T f(s, y_s, z_s, u_s)ds|\mathcal{G}_t\right]$$

by Theorem 3.2.10 there exist processes Z and $\left(U^k\right)_{k\in\{1,2,\dots,n\}}$ such that

$$M_t = M_0 + \int_0^t Z_s dW^{\mathbb{G}}_s + \sum_{k=1}^n \int_0^t \int_E U^k_s(e)\tilde{\mu}^k(ds, de).$$

Now define $Y_t := M_t - \int_0^t f(s, y_s, z_s, u_s)ds, \quad Y_T = X$ and we see that Y solves BSDE (4.5). This implies the existence of Φ, a mapping on $\mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$ to $\mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$ which maps $\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right) \in \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$ to a solution $\left(Y, Z, (U^k)_{k\in\{1,2,\dots,n\}}\right) = \Phi\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right)$ according to BSDE (4.5).

□

The next step is deriving a control on the norms of the solution triple $\left(Y, Z, (U^k)_{k\in\{1,2,\dots,n\}}\right)$ in terms of the input triple $\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right)$. To do so we introduce a second set of inputs and outputs to the mapping ϕ, superscripted by a dash. For $\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right)$ and $\left(y', z', (u^{k\prime})_{k\in\{1,2,\dots,n\}}\right)$, define $\left(Y, Z, (U^k)_{k\in\{1,2,\dots,n\}}\right) = \Phi\left(y, z, (u^k)_{k\in\{1,2,\dots,n\}}\right)$ and $\left(Y', Z', (U_k')_{k\in\{1,2,\dots,n\}}\right) = \Phi\left(y', z', (u^{k\prime})_{k\in\{1,2,\dots,n\}}\right)$ and let

$$\begin{aligned}
\delta Y &:= Y - Y' \\
\delta Z &:= Z - Z' \\
\delta U^k &:= U^k - U^{k\prime} \\
\delta f(t, y, z, u, y', z', u') &:= \delta f(t) = f(t, y, z, u) - f(t, y', z', u')
\end{aligned}$$

The next lemma proves an inequality on the norms of the solution triple in terms of the norms of the input triple. This inequality will then be used to show that ϕ is a contraction mapping.

Lemma 4.1.4. *There exists a constant* K *such that*

$$\|\delta Y\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}} + \|\delta Z\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}} + \sum_{k=1}^n \|\delta U^k\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}} \leq \frac{K^2}{\beta}\left(\|y\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}} + \|\delta z\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}} + \sum_{k=1}^n \|\delta u^k\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}}\right)$$

Proof. Now Itô's formula applied to $e^{\beta t}(\delta Y_t)^2$ yields.

$$d\left(e^{\beta t}(\delta Y_t)^2\right) = \beta e^{\beta t}(\delta Y_t)^2 dt + 2e^{\beta t}(\delta Y_{t-})d(\delta Y_t) + e^{\beta t} d[(\delta Y),(\delta Y)]_t.$$

Using BSDE (4.5), we get the dynamics of (δY_t) to be

$$d(\delta Y_t) = -f(t)dt + \delta Z_s dW_t^{\mathbb{G}} + \sum_{k=1}^{n}\int_E \delta U_t^k(e)\tilde{\mu}^k(dt,de).$$

This means that the quadratic variation of (δY_t) can be written as

$$d[\delta Y,\delta Y]_t = (\delta Z_t)^2 dt + \sum_{k=1}^{n}\int_E (\delta U_t^k(e))^2 \mu^k(dt,de).$$

Furthermore, the compensator of μ^k defined in from Proposition 3.2.9, Corollary 3.2.9.1 and Remark 6 in Chapter 3 means we can rewrite the quadratic variation as follows

$$d[\delta Y,\delta Y]_t = (\delta Z_t)^2 dt + \sum_{k=1}^{n}\int_E (\delta U_t^k(e))^2 \tilde{\mu}^k(dt,de) + \sum_{k=1}^{n}\int_E (\delta U_t^k(e))^2 \lambda_t^k(e)de\, dt$$

Substituting this term back in to the equation for $d\left(e^{\beta t}(\delta Y_t)^2\right)$ above and using the dynamics of δY_t, we get the following equation

$$\begin{aligned}
&e^{\beta t}(\delta Y_t)^2 + \int_t^T e^{\beta s}|\delta Z_s|^2 ds + \sum_{k=1}^{n}\int_t^T\int_E e^{\beta s}\lambda_s^k(e)\left(\delta U_s^k(e)\right)^2 deds \\
&= \int_t^T e^{\beta s}\left(2\delta Y_{s-}\delta f(s) - \beta(\delta Y_{s-})^2\right)ds - 2\int_t^T e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}} \\
&- \sum_{k=1}^{n}\int_t^T\int_E e^{\beta s}\left(2\delta Y_{s-}U_s^k(e) - \left(\delta U_s^k(e)\right)^2\right)\tilde{\mu}^k(ds,de).
\end{aligned}$$

Isolating $2\delta Y_s\delta f(s)$ in the first integral on the right hand side, using Hypothesis 6, we get

$$2\delta Y_s\delta f(s) \le 2\delta Y_s C\left(|\delta y_s| + |\delta z_s| + \sum_{k=1}^{n}\int_E |\delta u_s^k(e)|\sqrt{\lambda_s^k(e)}de\right)$$

Now using the Cauchy-Schwartz inequality agin, there exists a constant η such that

$$2\delta Y_s\delta f(s) \le \frac{(\delta Y_s)^2}{\eta^2} + \frac{C^2}{\eta^2}\left(|\delta y_s| + |\delta z_s| + \sum_{k=1}^{n}\int_E |\delta u_s^k(e)|\sqrt{\lambda_s^k(e)}de\right)^2.$$

Using Cauchy-Schwartz again, we get

$$2\delta Y_s\delta f(s) \le \frac{(\delta Y_s)^2}{\eta^2} + \frac{3C^2}{\eta^2}\left(|\delta y_s|^2 + |\delta z_s|^2 + \left(\sum_{k=1}^{n}\int_E |\delta u_s^k(e)|\sqrt{\lambda_s^k(e)}de\right)^2\right).$$

Note that the integral on the right hand side is finite by assumption, meaning Jensen's inequality can be used, meaning there exists another constant K such that

$$2\delta Y_s \delta f(s) \leq \frac{(\delta Y_s)^2}{\eta^2} + \frac{K^2}{\eta^2}\left(|\delta y_s|^2 + |\delta z_s|^2 + \left(\sum_{k=1}^{n}\int_E |\delta u_s^k(e)|^2 \lambda_s^k(e) de\right)\right.$$

Adding this inequality back to the original equation, we get

$$\begin{aligned}
&e^{\beta t}(\delta Y_t)^2 + \int_t^T e^{\beta s}|\delta Z_s|^2 ds + \sum_{k=1}^{n}\int_t^T \int_E e^{\beta s}\lambda_s^k(e)\left(\delta U_s^k(e)\right)^2 deds \\
&\leq \int_t^T e^{\beta s}\left(\frac{(\delta Y_{s-})^2}{\eta^2} + \eta^2 K^2\left((\delta y_s)^2 + |\delta z_s|^2 + \sum_{k=1}^{n}\int_E \lambda_s^k(e)\left(\delta u_s^k(e)\right)^2 de\right) - \beta(\delta Y_{s-})^2\right) ds \\
&-2\int_t^T e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}} - \sum_{k=1}^{n}\int_t^T \int_E e^{\beta s}\left(2\delta Y_{s-}U_s^k(e) - \left(\delta U_s^k(e)\right)^2\right)\tilde{\mu}^k(ds, de).
\end{aligned} \tag{4.6}$$

The last two terms have zero expectation, being expectations of increments of martingales. The fact that they are martingales is thanks to the Burkholder-Davis-Gundy inequality, indeed it ensures the existence of a constant K such that

$$\mathbb{E}\left[\sup_{t\leq T}|\int_0^t e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}}|\right] \leq K\mathbb{E}\left[\left(\int_0^T e^{2\beta s}(\delta Y_{s-})^2(\delta Z_s)^2 ds\right)^{\frac{1}{2}}\right].$$

Because $Y \in \mathbb{H}_{\mathbb{G}}^{2,\beta}$, Lemma 4.1.1 says that $Y \in \mathbb{S}_{\mathbb{G}}^{2,\beta}$. The integrand may then be simplified as follows:

$$\mathbb{E}\left[\sup_{t\leq T}|\int_0^t e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}}|\right] \leq K\mathbb{E}\left[\sup_{0\leq s\leq T}\left(e^{\beta s}(\delta Y_{s-})^2\right)\right]^{\frac{1}{2}}\mathbb{E}\left[\int_0^T e^{\beta s}(\delta Z_s)^2 ds\right]^{\frac{1}{2}} < \infty \tag{4.7}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq T}|\sum_{k=1}^{n}\int_t^T \int_E 2e^{\beta s}\delta Y_{s-}\delta U_s^k(e)\tilde{\mu}^k(ds, de)|\right] \\
&\leq K\sum_{k=1}^{n}\mathbb{E}\left[\left(\int_0^T \int_E e^{2\beta s}\left(\delta Y_{s-}^2\right)\delta U_s^k(e)^2 \mu^k(ds, de)\right)^{\frac{1}{2}}\right] \\
&\leq K\sum_{k=1}^{n}\mathbb{E}\left[\sup_{t\leq T}\left(e^{\beta t}\delta Y_t\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\int_0^T \int_E e^{\beta s}\delta U_s(e)^2 \lambda_s^k(e) deds\right]^{\frac{1}{2}} < \infty
\end{aligned} \tag{4.8}$$

Furthermore, the fact that $(U^k)_{k\in\{1,2,\dots,n\}} \in \mathbb{H}_{\mathbb{G},E}^{2,\beta}$ implies that $\sum_{k=1}^{n}\int_0^T \int_E e^{\beta s}\delta U_s^k(e)^2\tilde{\mu}^k(ds, de)$ has zero expectation.

Therefore by taking expectations in equation (4.6)

$$\mathbb{E}[e^{\beta t}(\delta Y_t)^2]+\mathbb{E}\left[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\right]+\mathbb{E}\left[\sum_{k=1}^n\int_t^T\int_E e^{\beta s}\lambda_s^k(e)\left(\delta U_s^k(e)\right)^2 deds\right]$$
$$\leq\left(\frac{1}{\eta^2}-\beta\right)\mathbb{E}\left[\int_t^T e^{\beta s}(\delta Y_{s-})ds\right]$$
$$+\eta^2K^2\mathbb{E}\left[\int_t^T e^{\beta s}\left((\delta y_s)^2+|\delta z_s|^2+\sum_{k=1}^n\int_E\lambda_s^k(e)\left(\delta u_s^k(e)\right)^2 de\right)ds\right].$$

Conveniently choosing $\eta^2=\frac{1}{\beta}$, the first term on the right hand side disappears. Next, the first term on the left hand side is non-negative, meaning we can conclude

$$\mathbb{E}\left[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\right]+\mathbb{E}\left[\sum_{k=1}^n\int_t^T\int_E e^{\beta s}\lambda_s^k(e)\left(\delta U_s^k(e)\right)^2 deds\right]$$
$$\leq\frac{K^2}{\beta}\mathbb{E}\left[\int_t^T e^{\beta s}\left((\delta y_s)^2+|\delta z_s|^2+\sum_{k=1}^n\int_E\lambda_s^k(e)\left(\delta u_s^k(e)\right)^2 de\right)ds\right].$$

Setting $t=0$, we get the following inequality

$$\|\delta Z\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\sum_{k=1}^n\|\delta U^k\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}}\leq\frac{K^2}{\beta}\left(\|\delta y\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\|\delta z\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\sum_{k=1}^n\|\delta u^k\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}}\right).$$

A similar argument yields

$$\mathbb{E}[e^{\beta t}(\delta Y_t)^2]\leq\frac{K^2}{\beta}\mathbb{E}\left[\int_t^T e^{\beta s}\left((\delta y_s)^2+|\delta z_s|^2+\sum_{k=1}^n\int_E\lambda_s^k(e)\left(\delta u_s^k(e)\right)^2 de\right)ds\right].$$

Intregating both sides with respect to t and changing the order of integration yields

$$T\|\delta Y\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}\leq\frac{K^2}{\beta}\mathbb{E}\left[\int_0^T se^{\beta s}\left((\delta y_s)^2+|\delta z_s|^2+\sum_{k=1}^n\int_E\lambda_s^k(e)\left(\delta u_s^k(e)\right)^2 de\right)ds\right].$$
$$\leq\frac{TK^2}{\beta}\left(\|\delta y\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\|\delta z\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\sum_{k=1}^n\|\delta u^k\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}}\right).$$

Combining both inequalities yields

$$\|\delta Y\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\|\delta Z\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\sum_{k=1}^n\|\delta U^k\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}}\leq\frac{K^2}{\beta}\left(\|y\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\|\delta z\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G}}}+\sum_{k=1}^n\|\delta u^k\|^2_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}}\right)$$

□

Lemma 4.1.5. *The mapping Φ admits a unique fixed point $\left(Y,Z,\left(U^k\right)_{k=1,2,\dots,n}\right)$. In other words, the solution $\left(Y,Z,\left(U^k\right)_{k=1,2,\dots,n}\right)$ to BSDE (4.4) is unique.*

Proof. Choosing $\beta > K^2$ the map Φ is contracting. Using the Banach fixed point theorem (Proposition 4.0.2), Φ admits a unique fixed point $\left(Y, Z, \left(U^k\right)_{k\in\{1,2,\ldots,n\}}\right) \in \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G}} \times \mathbb{H}^{2,\beta}_{\mathbb{G},E}$. Finally, using lemma 4.1.1, $Y \in \mathbb{S}^{2,\beta}_{\mathbb{G}}$. □

This completes the proof of Theorem 4.1.2. □

The following theorem allows us to compute the solution of a linear BSDE in $\mathbb{G}$ by the use of an adjoint process. This will be of particular use when comparing two solution triples.

Theorem 4.1.6. *For bounded processes a, b and h, assume the existence of a family of $\mathbb{G} \otimes \mathcal{B}(E)$-measurable process $\left(c^k\right)$ such that $(\int_E c^k_t(e)\sqrt{\lambda^k_t(e)}de)_{t\geq 0}$ is bounded for all $k \in \{1, 2, ..., n\}$. The BSDE*

$$\begin{aligned} Y_t =& X + \int_t^T \left(a_sY_s + b_sZ_s + \sum_{k=1}^n \int_E c^k_s(e)\lambda^k_s(e)U^k_s(e)de + h_s\right) ds - \int_t^T Z_s dW^{\mathbb{G}}_s \\ &- \sum_{k=1}^n \int_t^T \int_E U^k_s(e)\tilde{\mu}^k(ds, de), \end{aligned}$$

admits a unique solution (Y, Z, U) such that $U^k_t(e) = 0$ for $t > \tau_k$ and $e \in E$. Furthermore for $t \leq T$, define the adjoint process $(\Gamma^t_s)_{t\leq s\leq T}$ by the forward SDE

$$\Gamma^t_s = 1 + \int_t^s \Gamma^t_u a_u du + \int_t^s \Gamma^t_u b_u dW^{\mathbb{G}}_u + \sum_{k=1}^n \int_t^s \int_E \Gamma^t_u c^k_u(e)\tilde{\mu}^k(du, de) \tag{4.9}$$

Then

$$Y_t = \mathbb{E}\left[X\Gamma^t_T + \int_t^T \Gamma^t_s h_s ds|\mathcal{G}_t\right].$$

Proof. Note for $t \geq 0$, $(y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times H^2_E$, the driver $f(t, y, z, u) := a_ty + b_tz + \sum_{k=1}^n \int_E c^k_t(e)\lambda^k_t(e)u^k(e)de + h_t$ satisfies the Hypothesis 6 since $(\int_E c^k_t(e)\sqrt{\lambda^k_t(e)}de)_{t\geq 0}$ is bounded for all $k \in \{1, 2, ..., n\}$.
Using Itô's Lemma

$$\begin{aligned} (\Gamma^t_t Y_t) =& \Gamma^t_T X + \int_t^T \Gamma^t_s h_s ds - \int_t^T \left(Y_{s-}\Gamma^t_s b_s + \Gamma^t_{s-} Z_s\right) dW^{\mathbb{G}}_s \\ &- \sum_{k=1}^n \int_t^T \int_E \left(Y_{s-}\Gamma^t_s c^k_s(e) + \Gamma^t_{s-} c^k_s(e)U^k_s(e) + \Gamma^t_{s-} U^k_s(e)\right)\tilde{\mu}^k(ds, de) \end{aligned}$$

Using the fact that $(Y, Z, U) \in \mathbb{S}^2_{\mathbb{G}} \times \mathbb{H}^2_{\mathbb{G}} \times \mathbb{H}^2_{\mathbb{G},E}$ and the processes $a, b, \int_E c^k(e)\sqrt{\lambda^k(e)}de$ and h are bounded, a similar argument to that of equations (4.7) and (4.8) in the proof of theorem 4.1.2 we get that the last two terms have zero conditional expectation. Therefore

$$Y_t = \mathbb{E}\left[\Gamma^t_T X + \int_t^T \Gamma^t_s h_s ds|\mathcal{G}_t\right].$$

□

4.2 Comparison of Solutions

The first solution component of BSDE (4.4) is a semimartingale with drift governed by the driver f and martingale parts governed by Z and U. It is useful to state, whether a larger drift and terminal condition will result in a larger solution. This is referred to as a comparison theorem in the literature. The answer to such a question is given below.

Theorem 4.2.1. *Let $\bar{f}$ and f be two drivers satisfying hypothesis 6 and $\bar{X}$ and X be terminal values belonging to $L^2(\mathcal{G}_T)$. Let $(\bar{Y}, \bar{Z}, \bar{U})$ and (Y, Z, U) be the two solutions associated with $(\bar{f}, \bar{X})$ and (f, X). Suppose that $\bar{X} \geq X$ almost surely and $\bar{f}(Y_t, Z_t, U_t) \geq f(Y_t, Z_t, U_t)$ $d\mathbb{P} \otimes dt$ almost surely. Assume furthermore that there exists a family of non-negative processes (c^k) such that $\left(\int_E c_t^k(e)\sqrt{\lambda_t^k(e)}de\right)_{t\leq T}$ is bounded for all $e \in E$ and $k \in \{1, 2, ..., n\}$ and*

$$\bar{f}(t, Y_t, Z_t, \bar{U}_t) - \bar{f}(t, Y_t, Z_t, U_t) \leq \sum_{k=1}^{n} \int_E c_t^k(e)\lambda_t^k(e)\delta U_t^k(e)de$$

Then $\bar{Y}_t \geq Y_t$ for all $t \leq T$ (Note the right-hand side of the above condition is well defined because $\int_E c^k(e)\sqrt{\lambda^k(e)}de$ is bounded and $U^k \in \mathbb{H}_E^2$ for all $k \in \{1, 2..., n\}$).

Proof. Define $\delta Y := \bar{Y} - Y$, $\delta Z := \bar{Z} - Z$ and $\delta U := \bar{U} - U$ as before in the proof of Theorem 4.1.2. Consider the following processes

$$\begin{aligned} a_t =& \frac{\bar{f}(t, \bar{Y}_t, \bar{Z}_t, \bar{U}_t) - \bar{f}(t, Y_t, \bar{Z}_t, \bar{U}_t)}{\delta Y_t} \\ b_t =& \frac{\bar{f}(t, Y_t, \bar{Z}_t, \bar{U}_t) - \bar{f}(t, Y_t, Z_t, \bar{U}_t)}{\delta Z_t} \end{aligned}$$

Note by Hypothesis 6, a and b are bounded processes. Let $h_s = \bar{f}(s, Y_s, Z_s, U_s) - f(s, Y_s, Z_s, U_s)$, rearranging BSDE (4.4), we get

$$\begin{aligned} \delta Y_t =& \delta X + \int_t^T \left(a_s \delta Y_s + b_s \delta Z_s + \bar{f}(s, Y_s, Z_s, \bar{U}_s) - \bar{f}(s, Y_s, Z_s, U_s) + h_s\right) ds \\ & - \int_t^T \delta Z_s dW_s^{\mathbb{G}} - \sum_{k=1}^{n} \int_t^T \int_E \delta U_s^k(e)\tilde{\mu}^k(ds, de) \end{aligned}$$

Define the adjoint process $(\Gamma_s^t)_{t\leq s\leq T}$ by the following forward SDE

$$\Gamma_s^t = 1 + \int_t^s a_u \Gamma_u^t du + \int_t^s b_u \Gamma_u^t dW_u^{\mathbb{G}} + \sum_{k=1}^{n} \int_t^s \int_E c_u^k(e)\Gamma_u^t \tilde{\mu}^k(du, de)$$

Then just as in the proof of theorem 4.1.6

$$\begin{aligned}\delta Y_t =&\Gamma_T^t \delta X + \int_t^T \Gamma_s^t \left(\bar{f}(s, Y_s, Z_s, \bar{U}_s) - f(s, Y_s, Z_s, U_s) - \sum_{k=1}^n \int_E c_s^k(e)\lambda_s^k(e)\delta U_s^k(e)de + h_s \right) ds \\ &- \int_t^T \left(\Gamma_{s-}^t \delta Z_s + \delta Y_{s-} b_s \Gamma_s^t\right) dW_s^{\mathbb{G}} \\ &- \sum_{k=1}^n \int_t^T \int_E \left(\Gamma_{s-}^t \delta U_s^k(e) + \delta Y_{s-} c_s^k(e)\Gamma_s^t + c_s^k(e)\Gamma_s^t \delta U_s(e)\right) \tilde{\mu}(ds, de)\end{aligned}$$

Now by assumption the bracketed term in the first integral is less than zero, meaning

$$\delta Y_t \geq \mathbb{E}\left[\Gamma_T^t \delta X + \int_t^T \Gamma_s^t h_s ds\right] \geq 0.$$

□

4.3 Decomposition of $\mathbb{G}$-adapted Solutions

Considering the fact that any $\mathbb{G}$-optional (resp. predictable) process can be decomposed in to the sum of $\mathbb{F}\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$-optional (resp predictable) processes (see Proposition 3.2.1), we therefore pose the question of how the solutions to BSDE (4.4) can be decomposed. We begin by dealing with a simpler version of BSDE (4.4), one where the data is $\mathbb{F}$-adapted. This in turn means that the driver cannot depend on the solution. Remarkably, in this setup, the solution of BSDE (4.4) is determined by one parameterised BSDE in $\mathbb{F}$; this is presented in theorem 4.3.1. When the data is $\mathbb{G}$-adapted, the problem of decomposing the solution is more cumbersome: we prove a decomposition result similar to that of Theorem 3.1 in Kharroubi and Lim [2014], the difference being that BSDE (4.4) is driven by a $\mathbb{G}$-Brownian motion and the family of compensated jump measures $\{\tilde{\mu}^k : k = 1, 2, ..., n\}$ as opposed to an $\mathbb{F}$-Brownian motion and the uncompensated measures $\{\mu^k : k = 1, 2, ..., n\}$.

4.3.1 BSDE (4.4) with $\mathbb{F}$-adapted data

In this subsection we study BDSE (4.4) with $\mathbb{F}$-adapted data. Keeping in mind that the solution $(Y, Z, (U^k)_{k\in\{1,2,...,n\}})$ will still be $\mathbb{G}$-adapted, we are forced to consider $\mathbb{F}$-predictable drivers of the form $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, in other words the driver f does not depend on the solution triple, but just on t. The choice of f as $\mathbb{F}$-predictable is motivated by the aim to reduce the solution of BSDE 4.4 to a series of solutions in $\mathbb{F}$, therefore we start with the simplest form of driver. Suppose furthermore that f is square integrable to ensure the existence all solutions defined in the sequel. BSDE (4.4) then becomes

$$Y_t = X + \int_t^T f(s)ds - \int_t^T Z_t dW_s^{\mathbb{G}} - \sum_{k=1}^n \int_t^T \int_E U_s^k(e)\tilde{\mu}^k(ds, de) \tag{4.10}$$

Remarkably, in this setup the solution to BSDE (4.10) is fully determined by one parameterised BSDE in the reference filtration.
Recall the following two definitions

1. For any fixed $T > 0$, $k \in \{1, 2, .., n\}$ and $X \in L^1(\mathcal{F}_T \otimes \mathcal{B}(\Theta_n) \otimes \mathcal{B}(E^n))$, the operator $\mathcal{E}^k_{t,T}$ is defined as follows

$$\mathcal{E}^k_{t,T}(X)(u^{(k)}, e^{(k)}) := \int_t^\infty \int_{u_{k+1}}^\infty ... \int_{u_{n-1}}^\infty \int_{E^{n-k}} X(u, e)\alpha_T(u, e)de^{(k+1:n)}du^{(n:k+1)}.$$

For ease of notation, we let $\mathcal{E}^k_t = \mathcal{E}^k_{t,t}$.

2. The decomposition of γ^k in equation (4.2),

$$\gamma^k_t(u^{(k)}, e^{(k)}) = \gamma^k_0(u^{(k)}, e^{(k)}) + \int_0^t a^k_s(u^{(k)}, e^{(k)})dW_s - \int_0^t \int_E \gamma^{k+1}_{u_{k+1}}(u^{(k+1)}, e^{(k+1)})de_{k+1}du_{k+1}.$$

Theorem 4.3.1. *Suppose the following BSDE in $\mathbb{F}$ has a solution $(Y^n(u, e), Z^n(u, e))$ for all $(u, e) \in \Theta_n \times E^n$*

$$Y^n_t(u, e) = X\alpha_T(u, e) + \int_t^T f(s)\alpha_s(u, e)ds - \int_t^T Z^n_s(u, e)dW_s, \tag{4.11}$$

and such that Y^n and Z^n are $\mathcal{O}(\mathbb{F}, \Theta_n, E^n)$-measurable. Then

$$Y_t = \sum_{k=1}^n \frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)}, \xi^{(k)})}{\gamma^k_t(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}},$$

$$Z_t = \sum_{k=1}^n \frac{\mathcal{E}^k_t\left(\frac{Z^n_t\gamma^k_t - Y^n_t a^k_t}{\alpha_t}\right)(\tau^{(k)}, \xi^{(k)})}{\gamma^k_t(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}},$$

$$U^k_t = \left(\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k-1)}, t, \xi^{(k-1)}, \cdot)}{\gamma^k_t(\tau^{(k-1)}, t, \xi^{(k-1)}, \cdot)} - \frac{\mathcal{E}^{k-1}_t(Y^n_t/\alpha_t)(\tau^{(k-1)}, \xi^{(k-1)})}{\gamma^{k-1}_t(\tau^{(k-1)}, \xi^{(k-1)})}\right) \mathbb{1}_{\{\tau_{k-1} \le t < \tau_k\}},$$

is a solution to BSDE (4.10).

Proof. It will be shown that Y, Z and $(U^k)_{k\in\{1,2,...,n\}}$ given above solve BSDE (4.10). To do this we compute for each $k \in \{1, 2, ..., n\}$ the BSDE solved by each $\frac{\mathcal{E}^k_t(Y^n_t)(\tau^{(k)}, \xi^{(k)})}{\gamma^k_t(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}$ and $\frac{\mathcal{E}^k_t(\frac{Z^n_t\gamma^k_t - Y^n_t a^k_t}{\alpha_t})(\tau^{(k)}, \xi^{(k)})}{\gamma^k_t(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}$. This is done in two parts, first beginning with $k = n$ and then for any $k \in \{1, 2, ..., n-1\}$. For any $k \in \{1, 2..., n\}$, let the default indicator process be

$$H^k_t = \mathbb{1}_{\{\tau_k \le t\}},$$

then from Corollary 3.2.9.1,

$$dH^k_t = \int_E \tilde{\mu}^k(dt, de) + \int_E \lambda^k_t(e)dedt.$$

The proof may now proceed:

For $k = n$: In this case, since $\gamma^n = \alpha$ and the operator $\mathcal{E}$ simplifies as follows

$$\frac{\mathcal{E}_t^n(Y_t^n/\alpha_t)(\tau,\xi)}{\gamma_t^n(\tau,\xi)}\mathbb{1}_{\{\tau_n\le t\}} = \frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)}$$
$$\frac{\mathcal{E}_t^n(\frac{Z_t^n\gamma_t^n - Y_t^n a_t^n}{\alpha_t})(\tau,\xi)}{\gamma_t^n(\tau,\xi)}\mathbb{1}_{\{\tau_n\le t\}} = \frac{Z_t^n(\tau,\xi)\alpha_t(\tau,\xi) - Y_t^n(\tau,\xi)a_t(\tau,\xi)}{\alpha_t(\tau,\xi)^2}H_t^n$$

Using Itô's Lemma consider,

$$d\left(\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)}\right) = \frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dH_t^n + H_{t-}^n\Big[\frac{1}{\alpha_t(\tau,\xi)}dY_t^n(\tau,\xi) - \frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2}d\alpha_t(\tau,\xi)$$
$$- \frac{1}{\alpha_t(\tau,\xi)^2}d\langle\alpha(u,e),Y^n(u,e)\rangle_t|_{\substack{u=\tau\\ e=\xi}} + \frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^3}d\langle\alpha(u,e),\alpha(u,e)\rangle_t|_{\substack{u=\tau\\ e=\xi}}\Big] \tag{4.12}$$

To simplify this equation, we present each term inside the brackets separately. From equation (4.11), the first term is

$$\frac{1}{\alpha_t(\tau,\xi)}dY_t(\tau,\xi) = \frac{1}{\alpha_t(\tau,\xi)}\left[-f(t)\alpha_t(\tau,\xi)dt + Z_t^n(\tau,\xi)dW_t\right]$$
$$= -f(t)dt + \frac{Z_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dW_t.$$

Note the integrand in the second term is $\mathbb{G}$-adapted, meaning the second term is understood as a stochastic integral with respect to the $\mathbb{G}$-semimartingale W. Recall the dynamics the dynamics of the $\mathbb{F}$-martingale $\alpha(u,e)$: $d\alpha_t(u,e) = a_t^n(u,e)dW_t$. Meaning the second term inside the brackets in equation (4.12) is

$$\frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}d\alpha_t(\tau,\xi) = \frac{Y_t^n(\tau,\xi)a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dW_t.$$

The dynamics of Y^n and α, mean the last two terms inside the brackets in equation (4.12) are

$$\frac{1}{\alpha_t(\tau,\xi)^2}d\langle\alpha(u,e),Y^n(u,e)\rangle_t|_{\substack{u=\tau\\ e=\xi}} = \frac{1}{\alpha_t(\tau,\xi)^2}\left[a_t^n(u,e)Z_t^n(u,e)\right]|_{\substack{u=\tau\\ e=\xi}}dt$$
$$= \frac{a_t^n(\tau,\xi)Z_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2}dt.$$

And

$$\frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^3}d\langle\alpha(u,e),\alpha(u,e)\rangle_t|_{\substack{u=\tau\\ e=\xi}} = \frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^3}\left[a_t^n(u,e)^2\right]|_{\substack{u=\tau\\ e=\xi}}dt$$
$$= \frac{Y_t^n(\tau,\xi)a_t^n(\tau,\xi)^2}{\alpha_t(\tau,\xi)^3}dt$$

Putting this all together, we proceed with the proof:

$$
\begin{aligned}
d\left(\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)}\right) =&\frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dH_t^n + H_{t-}^n\Big[-f(t)dt + \frac{Z_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dW_t - \frac{Y_t^n(\tau,\xi)a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2}dW_t \\
&- \frac{a_t(\tau,\xi)Z_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2}dt + \frac{Y_t^n(\tau,\xi)a_t^n(\tau,\xi)^2}{\alpha_t(\tau,\xi)^3}dt\Big] \\
=&\frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dH_t^n + H_{t-}^n\Big[-f(t)dt \\
&+ \frac{Z_t^n(\tau,\xi)\alpha_t(\tau,\xi) - Y_t^n(\tau,\xi)a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2}\left(dW_t - \frac{a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dt\right)\Big].
\end{aligned}
$$

From equation (4.3) it is seen that $H_{t-}^n dW_t^{\mathbb{G}} = H_{t-}^n\left(dW_t - \frac{a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dt\right)$, meaning

$$
d\left(\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)}\right) =\frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dH_t^n + H_{t-}^n\Big[-f(t)dt + \frac{Z_t^n(\tau,\xi)\alpha_t(\tau,\xi) - Y_t^n(\tau,\xi)a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2}dW_t^{\mathbb{G}}\Big].
$$

Integrating both sides from t to T yields

$$
\begin{aligned}
\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)} =&XH_T^n + \int_t^T H_{s-}^n f(s)ds - \int_t^T H_{s-}^n\frac{Z_s^n(\tau,\xi)\alpha_s(\tau,\xi) - Y_s^n(\tau,\xi)a_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)^2}dW_s^{\mathbb{G}} \\
&- \int_t^T \frac{Y_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)}dH_s^n.
\end{aligned}
$$

The last term's integrand has a dependence on τ_n which when integrating with respect to H_s^n may be replaced by a dependence on s, in other words

$$
\begin{aligned}
\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)} =&XH_T^n + \int_t^T H_{s-}^n f(s)ds - \int_t^T H_{s-}^n\frac{Z_s^n(\tau,\xi)\alpha_s(\tau,\xi) - Y_s^n(\tau,\xi)a_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)^2}dW_s^{\mathbb{G}} \\
&- \int_t^T \frac{Y_s^n(\tau^{(n-1)},s,\xi)}{\alpha_s(\tau^{(n-1)},s,\xi)}dH_s^n.
\end{aligned}
$$

Again the last term has a dependence on ξ_n which may be rewritten as an integral with respect to $\mu^n(de,ds)$,

$$
\begin{aligned}
\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)} =&XH_T^n + \int_t^T H_{s-}^n f(s)ds - \int_t^T H_{s-}^n\frac{Z_s^n(\tau,\xi)\alpha_s(\tau,\xi) - Y_s^n(\tau,\xi)a_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)^2}dW_s^{\mathbb{G}} \\
&- \int_t^T\int_E \frac{Y_s^n(\tau^{(n-1)},s,\xi^{(n-1)}),e)}{\alpha_s(\tau^{(n-1)},s,\xi^{(n-1)},e)}\mu^n(de,ds).
\end{aligned}
$$

Lastly, from Proposition 3.2.9, Corollary 3.2.9.1 and Remark 6 in Chapter 3, the compensator of the random measure μ^n is $\frac{\alpha(\tau^{(n-1)},\cdot,\xi^{(n-1)},\cdot)}{\gamma^{n-1}(\tau^{(n-1)},\xi^{(n-1)})}\left(H^{n-1} - H_-^n\right)$, meaning

$$
\begin{aligned}
\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)} =&XH_T^n + \int_t^T H_{s-}^n f(s)ds - \int_t^T H_{s-}^n\frac{Z_s^n(\tau,\xi)\alpha_s(\tau,\xi) - Y_s^n(\tau,\xi)a_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)^2}dW_s^{\mathbb{G}} \\
&- \int_t^T\int_E \frac{Y_s^n(\tau^{(n-1)},s,\xi^{(n-1)}),e)}{\alpha_s(\tau^{(n-1)},s,\xi^{(n-1)},e)}\tilde{\mu}^n(de,ds) \\
&- \int_t^T\int_E \frac{Y_s^n(\tau^{(n-1)},s,\xi^{(n-1)}),e)}{\gamma_s^{n-1}(\tau^{(n-1)},s,\xi^{(n-1)},e)}(H_s^{n-1} - H_{s-}^n)deds.
\end{aligned}
$$

For $k \in \{1, 2, ..., n-1\}$**:** Before continuing, for ease of notation, we let

$$\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)}) = \int_t^\infty X_t(u_{k+1})du_{k+1}$$

where

$$X_t(u_{k+1}) = \int_{u_{k+1}}^\infty \int_{u_{k+2}}^\infty ... \int_{u_{n-1}}^\infty \int_{E^{n-k}} Y_t^n(\tau^{(k)}, u^{(k+1:n)}, \xi^{(k)}, e^{(k+1:n)})de^{(k+1:n)}du^{(n:k+2)}.$$

Helpfully, it is noted that $X_t(t) = \int_E \mathcal{E}_t^{k+1}(Y_t^n/\alpha_t)(\tau^{(k)}, t, \xi^{(k)}, e)de$. This can be seen by the definition of the operator $\mathcal{E}_t$. Now, using Lemma 2.3.4 from Chapter 2 in its backward version with the newly defined process X we get

$$d\left(\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)})\right) = -X(t)dt + \int_t^\infty dX_t(u_{k+1})du_{k+1}.$$

The first term is simplified using the the definition of $X_t(t)$ from above. Using Fubini's theorem, the second term simplifies to

$$\begin{aligned}
&= -\int_E \mathcal{E}_t^{k+1}(Y_t^n)(\tau^{(k)}, t, \xi^{(k)}, e)de\ dt \\
&\quad + \int_t^\infty \int_{u_{k+1}}^\infty ... \int_{u_{n-1}}^\infty \int_{E^{n-k}} dY_t^n(\tau^{(k)}, u^{(k+1:n)}, \xi^{(k)}, e^{(k+1:n)})de^{(k+1:n)}du^{(n:k+1)}
\end{aligned}$$

Now using the dynamics of Y^n from equation (4.11), we get

$$\begin{aligned}
&= -\int_E \mathcal{E}_t^{k+1}(Y_t^n)(\tau^{(k)}, t, \xi^{(k)}, e)de\ dt \\
&\quad - \int_t^\infty \int_{u_{k+1}}^\infty ... \int_{u_{n-1}}^\infty \int_{E^{n-k}} f(t)\alpha_t(\tau^{(k)}, u^{(k+1:n)}, \xi^{(k)}, e^{(k+1:n)})de^{(k+1:n)}du^{(n:k+1)}dt \\
&\quad + \int_t^\infty \int_{u_{k+1}}^\infty ... \int_{u_{n-1}}^\infty \int_{E^{n-k}} Z_t^n(\tau^{(k)}, u^{(k+1:n)}, \xi^{(k)}, e^{(k+1:n)})de^{(k+1:n)}du^{(n:k+1)}dW_t
\end{aligned}$$

The second term is simplified using the fact that $f(t)$ has no dependence on $(u^{(k)}, e^{(k)})$. The third term is identified as the operator $\mathcal{E}$ acting on Z_t^n/α_t.

$$= -\int_E \mathcal{E}_t^{k+1}(Y_t^n/\alpha_t)(\tau^{(k)}, t, \xi^{(k)}, e)de\ dt - f(t)\gamma_t^k(\tau^{(k)}, \xi^{(k)})dt + \mathcal{E}_t^k(Z_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)})dW_t$$

Again, using Itô's Lemma, consider

$$\begin{aligned}
&d\left(\frac{\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}(H_t^k - H_t^{k+1})\right) \\
=&\frac{\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}(dH_t^k - dH_t^{k+1}) \\
&+ (H_{t-}^k - H_{t-}^{k+1})\Big[\frac{1}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}d\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)}) - \frac{\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}d\gamma_t^k(\tau^{(k)}, \xi^{(k)}) \\
&- \frac{1}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}d\langle\mathcal{E}^k(Y^n/\alpha_t)(u^{(k)}, e^{(k)}), \gamma^k(u^{(k)}, e^{(k)})\rangle_t\big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}} \\
&+ \frac{\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^3}d\langle\gamma^k(u^{(k)}, e^{(k)}), \gamma^k(u^{(k)}, e^{(k)})\rangle_t\big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}\Big]
\end{aligned} \tag{4.13}$$

For the reader's sake, just as it was done before, each in the brackets above will be presented separately. Firstly, the derivation of the dynamics of $\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})$ implies the first term is written as

$$\frac{1}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}d\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)}) = -\frac{\int_E \mathcal{E}^{k+1}_t(Y^n_t/\alpha_t)(\tau^{(k)},t,\xi^{(k)},e)de}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}dt - f(t)dt + \frac{\mathcal{E}^k_t(Z^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}dW_t$$

The dynamics of γ^k from equation (4.2) imply the second term inside the brackets of equation (4.13) simplifies to

$$\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}d\gamma^k_t(\tau^{(k)},\xi^{(k)}) = \frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})a^k_t(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}dW_t - \frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})\int_E \gamma^{k+1}_t(\tau^{(k)},t,\xi^{(k)},e)de}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}dt$$

Finally, the dynamics of $\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})$ and γ^k combine the yield the last two terms inside the brackets in equation (4.13)

$$\begin{aligned}&\frac{1}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}d\langle\mathcal{E}^k(Y^n/\alpha_t)(u^{(k)},e^{(k)}),\gamma^k(u^{(k)},e^{(k)})\rangle_t|_{\substack{u^{(k)}=\tau^{(k)}\\e^{(k)}=\xi^{(k)}}}\\
=&\frac{1}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}\left[\mathcal{E}^k_t(Z^n_t/\alpha)(u^{(k)},e^{(k)})a^k_t(u^{(k)},e^{(k)})\right]|_{\substack{u^{(k)}=\tau^{(k)}\\e^{(k)}=\xi^{(k)}}}dt\\
=&\frac{\mathcal{E}^k_t(Z^n_t/\alpha)(\tau^{(k)},\xi^{(k)})a^k_t(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}dt.\end{aligned}$$

And

$$\begin{aligned}&\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^3}d\langle\gamma^k(u^{(k)},e^{(k)}),\gamma^k(u^{(k)},e^{(k)})\rangle_t|_{\substack{u^{(k)}=\tau^{(k)}\\e^{(k)}=\xi^{(k)}}}\\
=&\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^3}\left[a^k_t(u^{(k)},e^{(k)})^2\right]|_{\substack{u^{(k)}=\tau^{(k)}\\e^{(k)}=\xi^{(k)}}}dt\\
=&\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})a^k_t(\tau^{(k)},\xi^{(k)})^2}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^3}\end{aligned}$$

Putting this altogether allows us to continue the proof.

$$\begin{aligned}&d\left(\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}(H^k_t - H^{k+1}_t)\right)\\
=&\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}(dH^k_t - dH^{k+1}_t)\\
&+(H^k_{t-}-H^{k+1}_{t-})\Big[-\frac{\int_E \mathcal{E}^{k+1}_t(Y^n_t/\alpha_t)(\tau^{(k)},t,\xi^{(k)},e)de}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}dt\\
&-f(t)dt+\frac{\mathcal{E}^k_t(Z^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}dW_t-\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)}))a^k_t(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}dW_t\\
&+\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})\int_E \gamma^{k+1}_t(\tau^{(k)},t,\xi^{(k)},e)de}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}dt-\frac{\mathcal{E}^k_t(Z^n_t/\alpha_t)(\tau^{(k)},\xi^{(k)})a^k_t(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}dt\\
&+\frac{\mathcal{E}^k_t(Y^n_t/\alpha_t)a^k_t(\tau^{(k)},\xi^{(k)})^2}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^3}dt\Big].\end{aligned}$$

Similarly for the case $k = n$, we see from equation (4.3) that $(H_t^k - H_t^{k+1})dW_t^{\mathbb{G}} = (H_t^k - H_t^{k+1})\left(dW_t - \frac{a_t^k(\tau^{(k)},\xi^{(k)})}{\gamma_t^k(\tau^{(k)},\xi^{(k)})}dt\right)$. Furthermore the compensators of μ^k and μ^{k+1} from from Proposition 3.2.9, Corollary 3.2.9.1 and Remark 6 in Chapter 3 are

$$\mu^k(dt,de) = \mu^k(dt,de) - \frac{\gamma_t^k(\tau^{(k-1)},t,\xi^{(k-1)},e)}{\gamma_t^{k-1}(\tau^{(k-1)},\xi^{(k-1)})}de\ dt$$
$$\mu^{k+1}(dt,de) = \mu^{k+1}(dt,de) - \frac{\gamma_t^{k+1}(\tau^{(k)},t,\xi^{(k)},e)}{\gamma_t^k(\tau^{(k)},\xi^{(k)})}de\ dt.$$

Then integrating both sides with respect from t to T yields

$$\begin{aligned}
&\frac{\mathcal{E}_t^k(Y_t^n/\alpha_t)(\tau^{(k)},\xi^{(k)})}{\gamma_t^k(\tau^{(k)},\xi^{(k)})}(H_t^k - H_t^{k+1})\\
=&X(H_T^k - H_T^{k+1}) + \int_t^T f(s)(H_s^k - H_s^{k+1})ds\\
&-\int_t^T (H_s^k - H_s^{k+1})\frac{\mathcal{E}_s^k(Z_s^n/\alpha_s)(\tau^{(k)},\xi^{(k)})\gamma_s^k(\tau^{(k)},\xi^{(k)}) - \mathcal{E}_s^k(Y_s^n/\alpha_s)(\tau^{(k)},\xi^{(k)})a_s^k(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})^2}dW_s^{\mathbb{G}}\\
&+\int_t^T\int_E \frac{\mathcal{E}_s^k(Y_s^n/\alpha_s)(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})}\tilde{\mu}^{k+1}(ds,de) - \frac{\mathcal{E}_s^k(Y_s^n/\alpha_s)(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma_s^k(\tau^{(k-1)},s,\xi^{(k-1)},e)}\tilde{\mu}^k(de,ds)\\
&+\int_t^T\int_E \frac{\mathcal{E}_s^{k+1}(Y_s^n/\alpha_s)(\tau^{(k)},s,\xi^{(k)},e)}{\gamma_s^k(\tau^{(k)},\xi^{(k)})}(H_s^k - H_s^{k+1})deds\\
&-\int_t^T\int_E \frac{\mathcal{E}_s^k(Y_s^n/\alpha_s)(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma_s^{k-1}(\tau^{(k-1)},\xi^{(k-1)})}(H_s^{k-1} - H_s^k)deds
\end{aligned}$$

summing over all k yields the result. □

4.3.2 BSDE (4.4) with $\mathbb{G}$-adapted data

In this section we decompose the solution of BSDE (4.4) when the driver is $\mathbb{G}$-predictable and the terminal condition is $\mathcal{G}_T$-measurable. Here we may take the driver to be dependent on Y, Z and U. A similar result is derived in Kharroubi and Lim [2014], here the authors consider a BSDE driven by the $\mathbb{F}$-Brownian motion W and the uncompensated jump measure μ. The authors show that given the existence of $n+1$ solutions to parameterised BSDEs in $\mathbb{F}$, the solution of the BSDE in $\mathbb{G}$ is found by combining these parameterised solutions as in proposition 3.2.1. The following differs in that the BSDE is driven by the $\mathbb{G}$-Brownian motion $W^{\mathbb{G}}$ and the family of compensated jump measures $\{\tilde{\mu}^1, ..., \tilde{\mu}^n\}$. In this setup, the driver f and terminal condition X, admit the following decompositions, for $(Y, Z, U) \in \mathbb{R} \times \mathbb{R}^d \times B(E)$

$$f(t,Y,Z,U) = \sum_{k=0}^n f^k(t,Y,Z,U,\tau^{(k)},\xi^{(k)})\mathbb{1}_{\{\tau_k < t \le \tau_{k+1}\}}$$
$$X = \sum_{k=0}^n X^k(\tau^{(k)},\xi^{(k)})\mathbb{1}_{\{\tau_k \le T < \tau_{k+1}\}}.$$

Theorem 4.3.2. *Suppose the following BSDE in $\mathbb{F}$ has a solution for all $(u, e) \in \Theta_n \times E^n$,*

$$\begin{aligned} Y_t^n(u,e) =& X^n(u,e)\alpha_T(u,e) + \int_t^T f^n(s, Y_s^n(u,e), Z_s^n(u,e), U = 0, u, e)\alpha_s(u,e)ds \\ & - \int_t^T Z_s^n(u,e)dW_s, \end{aligned}$$

and that the following BSDEs have solutions for all $k \in \{0, 1, ..., n-1\}$ and all $(u^{(k)}, e^{(k)}) \in \Theta_k \times E^k$,

$$\begin{aligned} Y_t^k(u^{(k)}, e^{(k)}) =& X^k(u^{(k)}, e^{(k)})\gamma_T^k(u^{(k)}, e^{(k)}) \\ & + \int_t^T f^k(s, Y^k(u^{(k)}, e^{(k)}), Z^k(u^{(k)}, e^{(k)}), Y_s^{k+1}(u^{(k)}, s, e^{(k)}, \cdot) \\ & - Y_s^k(u^{(k)}, e^{(k)}), u^{(k)}, e^{(k)})\gamma_s^k(u^{(k)}, e^{(k)})ds - \int_t^T Z_s^k(u^{(k)}, e^{(k)})dW_s \\ & + \int_t^T \int_E Y_s^{k+1}(u^{(k)}, s, e^{(k+1)})de_{k+1}ds. \end{aligned}$$

and that for all $k \in \{0, 1, ..., n\}$, Y^k is $\mathcal{O}(\mathbb{F}, \Theta_k, E^k)$-measurable. Then

$$\begin{aligned} Y_t &= \sum_{k=0}^n \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} \\ Z_t &= \sum_{k=0}^n \frac{Z_t^k(\tau^{(k)}, \xi^{(k)})\gamma_t^k(\tau^{(k)}, \xi^{(k)}) - a_t^k(\tau^{(k)}, \xi^{(k)})Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2} \mathbb{1}_{\{\tau_k < t \le \tau_{k+1}\}} \\ U_t^k(e) &= \left(Y_t^{k+1}(\tau^{(k)}, t, \xi^{(k)}, e) - Y_t^k(\tau^{(k)}, \xi^{(k)})\right) \mathbb{1}_{\{\tau_{k-1} \le t < \tau_k\}} \end{aligned}$$

is a solution to BSDE (4.4).

Remark 7. *This results proves a similar result to Theorem 3.1 in Kharroubi and Lim [2014]. The differences being that the BSDE considered in Kharroubi and Lim [2014] is driven by the $\mathbb{F}$-Brownian motion W, and the uncompensated jump measure μ. This is not considered in this thesis as neither W nor μ are $\mathbb{G}$-martingales and therefore the BSDE would not be applicable to financial mathematics applications such as stochastic optimal control, optimal stopping or hedging in the enlarged filtration $\mathbb{G}$ without an Immersion hypothesis. The BSDE presented in Kharroubi and Lim [2014] can be derived from BSDE 4.4 by expanding the $\mathbb{G}$-martingale decomposition of $W^{\mathbb{G}}$ and $\tilde{\mu}$.*

Proof. The proof is very similar to that of Theorem 4.3.1. We need to show that the stated Y,Z and U solve BSDE (4.4). To do this we derive the dynamics of YH^k for all $k \in \{0, 1, ..., n\}$ and then sum the results to yield the dynamics of Y.

For $k = n$: In this case

$$Y_t H_t^n = \frac{Y_t^n(\tau, \xi) H_t^n}{\alpha_t(\tau, \xi)}.$$

$$
\begin{aligned}
d\left(\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)}\right) =& \frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dH_t^n + H_{t-}^n\Bigg(\frac{1}{\alpha_t(\tau,\xi)}dY_t^n(\tau,\xi) - \frac{Y_t^n(\tau,\xi)}{\alpha_t^2(\tau,\xi)}d\alpha_t(\tau,\xi) \\
&- \frac{1}{\alpha_t(\tau,\xi)^2}d\langle\alpha(u,e),Y^n(u,e)\rangle_t|_{\substack{u=\tau\\e=\xi}} + \frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^3}d\langle\alpha(u,e),\alpha(u,e)\rangle_t|_{\substack{u=\tau\\e=\xi}}\Bigg).
\end{aligned}
$$

Similarly to the proof of Theorem 4.3.1, given the assumed dynamics of Y^n and the dynamics of α, we conclude the following (see the proof of Theorem 4.3.1 for more detail)

$$
\begin{aligned}
d\left(\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)}\right) =& \frac{Y_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dH_t^n + H_{t-}^n\Bigg(- f^n(t,Y_t^n(\tau,\xi),Z_t^n(\tau,\xi),0,\tau,\xi)dt \\
&+ \left(\frac{Z_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)} - \frac{Y_t^n(\tau,\xi)a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2}\right)dW_t - \Big(\frac{Z_t^n(\tau,\xi)a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)^2} \\
&- \frac{Y_t^n(\tau,\xi)a_t^n(\tau,\xi)^2}{\alpha_t(\tau,\xi)^3}\Big)dt\Bigg).
\end{aligned}
$$

Equation (4.3) states that $H_t^n dW_t^{\mathbb{G}} = H_t^n\left(dW_t - \frac{a_t^n(\tau,\xi)}{\alpha_t(\tau,\xi)}dt\right)$. Now integrating both sides yields

$$
\begin{aligned}
\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)} =& XH_T^n + \int_t^T H_{s-}^n f^n(s,Y_s^n(\tau,\xi),Z_s^n(\tau,\xi),0,\tau,\xi)ds \\
&- \int_t^T H_{s-}^n\frac{Z_s^n(\tau,\xi)\alpha_s(\tau,\xi) - Y_s^n(\tau,\xi)a_s(\tau,\xi)}{\alpha_s(\tau,\xi)^2}dW_s^{\mathbb{G}} \\
&- \int_t^T \frac{Y_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)}dH_s^n.
\end{aligned}
$$

The last term's integrand contains a dependence on the random variables τ_n and ξ_n, which when expanding the integral with respect to μ^n can be omitted as follows

$$
\int_t^T \frac{Y_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)}dH_s^n = \int_t^T\int_E \frac{Y_s^n(\tau^{(n-1)},s,\xi^{(n-1)},e)}{\alpha_s(\tau^{(n-1)},s,\xi^{(n-1)},e)}\mu^n(de,ds).
$$

From Proposition 3.2.9 and Corollary 3.2.9.1 in Chapter 3, the compensator of the random measure μ^n is $\frac{\alpha(\tau^{(n-1)},\cdot,\xi^{(n-1)},\cdot)}{\gamma^{n-1}(\tau^{(n-1)},\xi^{(n-1)})}\left(H_-^{n-1} - H_-^n\right)$, meaning

$$
\begin{aligned}
\int_t^T \frac{Y_s^n(\tau,\xi)}{\alpha_s(\tau,\xi)}dH_s^n =& \int_t^T\int_E \frac{Y_s^n(\tau^{(n-1)},s,\xi^{(n-1)},e)}{\alpha_s(\tau^{(n-1)},s,\xi^{(n-1)},e)}\tilde{\mu}^n(de,ds) \\
&- \int_t^T \frac{Y_s^n(\tau^{(n-1)},s,\xi^{(n-1)},e)}{\gamma_s^{n-1}(\tau^{(n-1)},\xi^{(n-1)})}\left(H_{s-}^{n-1} - H_{s-}^n\right)ds.
\end{aligned}
$$

Combining this with the previous computations yields

$$\begin{aligned}\frac{Y_t^n(\tau,\xi)H_t^n}{\alpha_t(\tau,\xi)} =& XH_T^n + \int_t^T H_{s-}^n f^n(s, Y_s^n(\tau,\xi), Z_s^n(\tau,\xi), 0, \tau, \xi)ds \\ &- \int_t^T H_{s-}^n \frac{Z_s^n(\tau,\xi)\alpha_s(\tau,\xi) - Y_s^n(\tau,\xi)a_s(\tau,\xi)}{\alpha_s(\tau,\xi)^2} dW_s^{\mathbb{G}} \\ &- \int_t^T \int_E \frac{Y_s^n(\tau^{(n-1)}, s, \xi^{(n-1)}), e)}{\alpha_s(\tau^{(n-1)}, s, \xi^{(n-1)}, e)} \tilde{\mu}^n(de, ds) \\ &- \int_t^T \int_E \frac{Y_s^n(\tau^{(n-1)}, s, \xi^{(n-1)}), e)}{\gamma_s^{n-1}(\tau^{(n-1)}, s, \xi^{(n-1)}, e)} (H_{s-}^{n-1} - H_{s-}^n) de\, ds.\end{aligned}$$

For $k \in \{0, 1, ..., n-1\}$: In this case,

$$Y_t H_t^k = \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})H_t^k}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}.$$

Using a similar reasoning as the case when $k = n$, the differences being the additional term in the BSDE for Y^k and the fact that

$$d\gamma_t^k(\tau^{(k)}, \xi^{(k)}) = a_t^k(\tau^{(k)}, \xi^{(k)})dW_t^{\mathbb{G}} - \int_E \gamma_t^{k+1}(\tau^{(k)}, t, \xi^{(k)}, e)dedt.$$

See the proof of Theorem 4.3.1 for a more detailed derivation.

$$\begin{aligned}&\frac{Y_t^k(\tau^{(k)}, \xi^{(k)})(H_t^k - H_t^{k+1})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})} \\ =& X(H_T^k - H_T^{k+1}) + \int_t^T f^k(s, Y_s^k(\tau^{(k)}, \xi^{(k)}), Z_s^k(\tau^{(k)}, \xi^{(k)}), Y_s^{k+1}(\tau^{(k)}, s, \xi^{(k)}, \cdot) \\ &- Y_s^k(\tau^{(k)}, \xi^{(k)}), \tau^{(k)}, \xi^{(k)})(H_s^k - H_s^{k+1})ds \\ &- \int_t^T (H_s^k - H_s^{k+1}) \frac{Z_t^k(\tau^{(k)}, \xi^{(k)})\gamma_t^k(\tau^{(k)}, \xi^{(k)}) - a_t^k(\tau^{(k)}, \xi^{(k)})Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2} dW_s^{\mathbb{G}} \\ &+ \int_t^T \frac{Y_s^k(\tau^{(k)}, \xi^{(k)})}{\gamma_s^k(\tau^{(k)}, \xi^{(k)})}(dH_s^{k+1} - dH_s^k) + \int_t^T \int_E (H_{s-}^k - H_{s-}^{k+1}) \frac{Y_s^{k+1}(\tau^{(k)}, s, \xi^{(k)}, e)}{\gamma_s^k(\tau^{(k)}, \xi^{(k)})} deds. \\ &+ \int_t^T \int_E \frac{Y_s^k(\tau^{(k)}, \xi^{(k)})\gamma_s^{k+1}(\tau^{(k)}, s, \xi^{(k)}, e)}{\gamma_s^k(\tau^{(k)}, \xi^{(k)})^2} \left(H_{s-}^k - H_{s-}^{k+1}\right) deds.\end{aligned}$$

Just as in the $k = n$ case, the third to last integrand contains a dependence on τ_k and ξ_k which can be omitted when expanding the integrator as follows

$$\begin{aligned}&\int_t^T \frac{Y_s^k(\tau^{(k)}, \xi^{(k)})}{\gamma_s^k(\tau^{(k)}, \xi^{(k)})}(dH_s^{k+1} - dH_s^k) \\ =& \int_t^T \int_E \frac{Y_s^k(\tau^{(k)}, \xi^{(k)})}{\gamma_s^k(\tau^{(k)}, \xi^{(k)})} \mu^{k+1}(de, ds) - \int_t^T \int_E \frac{Y_s^k(\tau^{(k-1)}, s, \xi^{(k-1)}, e)}{\gamma_s^k(\tau^{(k-1)}, s, \xi^{(k-1)}, e)} \mu^k(de, ds).\end{aligned}$$

From Proposition 3.2.9 and Corollary 3.2.9.1 in Chapter 3, the compensator of the random measures μ^{k+1} and μ^k are $\frac{\gamma^{k+1}(\tau^{(k)},\cdot,\xi^{(k)},\cdot)}{\gamma^k(\tau^{(k)},\xi^{(k)})}(H^k_- - H^{k+1}_-)$ and $\frac{\gamma^k(\tau^{(k-1)},\cdot,\xi^{(k-1)},\cdot)}{\gamma^{k-1}(\tau^{(k-1)},\xi^{(k-1)})}(H^{k-1}_- - H^k_-)$ respectively.

$$
\begin{aligned}
&\int_t^T \frac{Y^k_s(\tau^{(k)},\xi^{(k)})}{\gamma^k_s(\tau^{(k)},\xi^{(k)})}(dH^{k+1}_s - dH^k_s)\\
=&\int_t^T\int_E \frac{Y^k_s(\tau^{(k)},\xi^{(k)})}{\gamma^k_s(\tau^{(k)},\xi^{(k)})}\tilde{\mu}^{k+1}(de,ds) - \int_t^T\int_E \frac{Y^k_s(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma^k_s(\tau^{(k-1)},s,\xi^{(k-1)},e)}\tilde{\mu}^k(de,ds)\\
-&\int_t^T\int_E \frac{Y^k_s(\tau^{(k)},\xi^{(k)})\gamma^{k+1}_s(\tau^{(k)},s,\xi^{(k)},e)}{\gamma^k_s(\tau^{(k)},\xi^{(k)})^2}\left(H^{k-1}_{s-} - H^k_{s-}\right) de\, ds\\
&-\int_t^T\int_E \frac{Y^k_s(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma^{k-1}_s(\tau^{(k-1)},\xi^{(k-1)})}\left(H^{k-1}_{s-} - H^k_{s-}\right) de\, ds.
\end{aligned}
$$

Combining all terms we get

$$
\begin{aligned}
&\frac{Y^k_t(\tau^{(k)},\xi^{(k)})(H^k_t - H^{k+1}_t)}{\gamma^k_t(\tau^{(k)},\xi^{(k)})}\\
=&X(H^k_T - H^{k+1}_T) + \int_t^T f^k(s, Y^k_s(\tau^{(k)},\xi^{(k)}), Z^k_s(\tau^{(k)},\xi^{(k)}), Y^{k+1}_s(\tau^{(k)},s,\xi^{(k)},\cdot)\\
&- Y^k_s(\tau^{(k)},\xi^{(k)}),\tau^{(k)},\xi^{(k)})(H^k_s - H^{k+1}_s)ds\\
&-\int_t^T (H^k_s - H^{k+1}_s)\frac{Z^k_t(\tau^{(k)},\xi^{(k)})\gamma^k_t(\tau^{(k)},\xi^{(k)}) - a^k_t(\tau^{(k)},\xi^{(k)})Y^k_t(\tau^{(k)},\xi^{(k)})}{\gamma^k_t(\tau^{(k)},\xi^{(k)})^2}dW^{\mathbb{G}}_s\\
&+\int_t^T\int_E \frac{Y^k_s(\tau^{(k)},\xi^{(k)})}{\gamma^k_s(\tau^{(k)},\xi^{(k)})}\tilde{\mu}^{k+1}(de,ds) - \int_t^T\int_E \frac{Y^k_s(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma^k_s(\tau^{(k-1)},s,\xi^{(k-1)},e)}\tilde{\mu}^k(de,ds)\\
&+\int_t^T\int_E (H^k_{s-} - H^{k+1}_{s-})\frac{Y^{k+1}_s(\tau^{(k)},s,\xi^{(k)},e)}{\gamma^k_s(\tau^{(k)},\xi^{(k)})}deds\\
&-\int_t^T\int_E (H^{k-1}_{s-} - H^k_{s-})\frac{Y^k_s(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma^{k-1}_s(\tau^{(k-1)},\xi^{(k-1)})}deds.
\end{aligned}
$$

Again, summing over all k, yields the result. □

4.4 Applications of BSDE (4.4)

In this brief section we present various applications of BSDE (4.4) to financial mathematics. The purpose being to give the reader an idea of how this chapter may applied. It is however emphasised that this section is not a main contribution of the thesis. Having proven an existence and uniqueness result for BSDE (4.4), we are able to extend the work of Kharroubi and Lim [2014] and Calvia and Gianin [2020]. Both pairs of authors consider a form of BSDE (4.4) coupled with an immersion hypothesis (see section 2.3.4 for a definition), they show an application to exponential utility maximisation and risk measures respectively. We show that given theorems 3.2.10 and 4.1.2, applying BSDE (4.4) to utility maximisation and risk measures is possible without an immersion hypothesis. For the ease-of-notation, assume for the remainder of this section that W is a one-dimensional Brownian motion (i.e. $d = 1$).

4.4.1 Utility Maximisation

This minor subsection is devoted to extending the results of Kharroubi and Lim [2014] to the case without immersion. All proofs are suspended to the appendix due to their similarity with the original results of Hu et al. [2005], Morlais [2009] and Kharroubi and Lim [2014]. See also Romo [2016] and Jeanblanc and Wu [2017] for a similar derivation.

Assume for now that our probability space is embedded in a financial market model with a risk-free interest rate $(b_t)_{t\geq 0}$ and $\mathbb{P}$ is the unique risk-neutral probability measure. Let S be the value of a risky asset with dynamics

$$dS_t = b_t S_t dt + \sigma_t S_t dW_t$$

where the coefficients b and σ are assumed to be $\mathcal{P}(\mathbb{F})$-measurable and $\sigma_t > 0$ for all $t \geq 0$. For each $t > 0$, consider an investor with initial wealth x who invests a total amount π_t of her wealth in the risky asset, S_t at time t. Her resultant wealth $X^{\pi,x}$ will then have dynamics

$$dX_t^{\pi,x} = \pi_t b_t dt + \pi_t \sigma_t dW_t.$$

Consider a contingent claim with payoff C at time T. Assume that C is $\mathcal{F}_T$-measurable and bounded a.s. Consider two optimisation problems, one where the investor needs to hedge her exposure to the contingent claim C with the highest expected utility by considering $\mathbb{F}$-adapted trading strategies and the second where the investor may consider $\mathbb{G}$-adapted trading strategies. Firstly, a formal definition of a trading strategy is given below.

Definition 4.4.1. *Define the following two spaces of $\mathbb{F}$- and $\mathbb{G}$-admissible strategies*

$$\mathcal{A}^{\mathbb{F}} := \{\pi\text{: } \mathcal{P}(\mathbb{F})\text{-measurable such that } \mathbb{E}\left[\int_0^T (\pi_t\sigma_t)^2\, dt\right] < \infty\}$$

$$\mathcal{A}^{\mathbb{G}} := \{\pi\text{: } \mathcal{P}(\mathbb{G})\text{-measurable such that } \mathbb{E}\left[\int_0^T (\pi_t\sigma_t)^2\, dt\right] < \infty\}.$$

For $\eta > 0$, we will consider the problem of the investor maximising her exponential utility. This equates to the investor having a utility function $U(x) := (1 - \frac{e^{-\eta x}}{\eta})$ and trying to maximise this utility with respect to her choice of investment in S. The object of interest being the increase in optimal utility when an investor can choose from the possibly wider range of investment strategies in $\mathcal{A}^{\mathbb{G}}$. Mathematically, maximising $(1 - \frac{e^{-\eta x}}{\eta})$ over some subset of x values, is no different to maximising $-\frac{e^{-\eta x}}{\eta}$, furthermore, the denominator may be excluded since we are interested in the relative difference in optimal utilities. We then consider the following two optimisation problems:

$$V^{\mathbb{F}}(x) := \sup_{\pi\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\left[-\exp(-\eta(X_T^{\pi,x} - C))\right]$$

$$V^{\mathbb{G}}(x) := \sup_{\pi\in\mathcal{A}^{\mathbb{G}}} \mathbb{E}\left[-\exp(-\eta(X_T^{\pi,x} - C))\right]$$

the goal being to explicitly show the additional utility achieved by optimising over the possibly larger set $\mathcal{A}^{\mathbb{G}}$.

The methodology used to solve these problems follows that of Hu et al. [2005] and is unoriginal.
Define the following families of processes $(R^{\pi,\mathbb{F}})_{\pi\in\mathcal{A}^{\mathbb{F}}}$ and $(R^{\pi,\mathbb{G}})_{\pi\in\mathcal{A}^{\mathbb{G}}}$ such that

1. $R_T^{\pi,\mathbb{F}} = R_T^{\pi,\mathbb{G}} = -\exp(-\eta(X_T^{\pi,x} - C))$.
2. $R_0^{\pi,\mathbb{F}}$ and $R_0^{\pi,\mathbb{G}}$ are independent of the choice of π.
3. $R^{\pi,\mathbb{F}}$ and $R^{\pi,\mathbb{G}}$ are $\mathbb{F}$- and $\mathbb{G}$-supermartingales respectively and there exists $\pi^{\mathbb{F}} \in \mathcal{A}^{\mathbb{F}}$ and $\pi^{\mathbb{G}} \in \mathcal{A}^{\mathbb{G}}$ such that $R^{\pi^{\mathbb{F}},\mathbb{F}}$ and $R^{\pi^{\mathbb{G}},\mathbb{G}}$ are $\mathbb{F}$- and $\mathbb{G}$-martingales respectively.

Then for any $\pi \in \mathcal{A}^{\mathbb{F}}$

$$\mathbb{E}[-\exp(-\eta(X_T^{\pi,x} - C))] \leq R_0^{\pi,\mathbb{F}} = \mathbb{E}[-\exp(-\eta(X_T^{\pi^{\mathbb{F}},x} - C))] = V^{\mathbb{F}}(x).$$

Similarly, for any $\pi \in \mathcal{A}^{\mathbb{G}}$

$$\mathbb{E}[-\exp(-\eta(X_T^{\pi,x} - C))] \leq R_0^{\pi,\mathbb{G}} = \mathbb{E}[-\exp(-\eta(X_T^{\pi^{\mathbb{G}},x} - C))] = V^{\mathbb{G}}(x).$$

To construct such a family we let

$$\begin{aligned} R_t^{\pi,\mathbb{F}} &= -\exp(-\eta(X_t^{\pi,x} - Y_t^{\mathbb{F}})) \\ R_t^{\pi,\mathbb{G}} &= -\exp(-\eta(X_t^{\pi,x} - Y_t^{\mathbb{G}})) \end{aligned}$$

where $Y^{\mathbb{F}}$ and $Y^{\mathbb{G}}$ solve the following BSDEs

$$\begin{aligned} Y_t^{\mathbb{F}} &= C + \int_t^T f^{\mathbb{F}}(s, Z_s^{\mathbb{F}})ds - \int_t^T Z_s^{\mathbb{F}} dW_s \\ Y_t^{\mathbb{G}} &= C + \int_t^T f^{\mathbb{G}}(s, Z_s^{\mathbb{G}}, U_s) - \int_t^T Z_s^{\mathbb{G}} dW_s^{\mathbb{G}} - \sum_{k=1}^n \int_t^T \int_E U_s^k(e)\tilde{\mu}^k(de, ds). \end{aligned}$$

$f^{\mathbb{F}}$ and $f^{\mathbb{G}}$ will be chosen such that the necessary conditions of $R^{\mathbb{F}}$ and $R^{\mathbb{G}}$ are satisfied. In view of their definition, hypothesis 6 will not hold but instead a quadratic growth assumption would be needed. From here on, we assume that the above BSDEs omit solutions that are unique. The existence and uniqueness of BSDEs with quadratic growth is a very well studied topic, see for example Theorem 2.1 in Du and Chen [2014].
The solution to $f^{\mathbb{F}}$ is well known and was first derived in Theorem 7, Hu et al. [2005].

$$f^{\mathbb{F}}(t,z) = \frac{\eta}{2} \inf_{\pi\in\mathcal{A}^{\mathbb{F}}} \left(\pi_t\sigma_t - \left(\frac{b_t}{\eta\sigma_t} + z\right)\right)^2 - \frac{b_t^2}{2\eta\sigma_t^2} - \frac{b_t z}{\sigma_t},$$

for $t \geq 0$ and $z \in \mathbb{R}$.
In order to find $f^{\mathbb{G}}$, recall from Theorem 3.2.7, there exists a $\mathbb{G}$ predictable process ϕ such that

$$W_t^{\mathbb{G}} = W_t - \int_0^t \phi_s ds.$$

Using a similar methodology to Hu et al. [2005], the driver $f^{\mathbb{G}}$ is found to be (see Appendix 6):

$$\begin{aligned} f^{\mathbb{G}}(t,z,u) =& \frac{\eta}{2} \inf_{\pi\in\mathcal{A}^{\mathbb{G}}} \left(\pi_t\sigma_t - \left(\frac{b_t}{\eta\sigma_t} + \frac{\phi_t}{\eta} + z\right)\right)^2 - \frac{\eta}{2}\sum_{k=1}^n \int_E u^k(e)^2 \lambda_t^k(e)de \\ & - \frac{1}{2\eta}\left(\frac{b_t}{\sigma_t} + \phi_t\right)^2 - \frac{b_t z}{\sigma_t} - \phi_t z, \end{aligned}$$

for $t \geq 0$, $z \in \mathbb{R}$ and $u^k \in H^2_E$. For the remainder of this subsection assume

$$\mathbb{E}\left[\int_0^T \phi_t^2 dt\right] < \infty.$$

This will ensure the existence of a $\mathbb{G}$-optimal portfolio. This assumption is crucial for the remaining analysis and extends beyond utility maximisation. The link between arbitrage opportunities and enlargement of filtration often hinges around this assumption, indeed under the present assumption there exists a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the enlarged space $(\mathbb{G}, \tilde{\mathbb{P}})$ is arbitrage free (see for example Amendinger et al. [1998], Amendinger [2000] and Acciaio et al. [2016]).
Using this assumption, it is clear from the definition of $f^{\mathbb{F}}$ and $f^{\mathbb{G}}$ that the optimal portfolio weights $\pi^{\mathbb{F}}$ and $\pi^{\mathbb{G}}$ are

$$\pi^{\mathbb{F}} = \frac{b + \eta\sigma Z^{\mathbb{F}}}{\eta\sigma^2} \in \mathcal{A}^{\mathbb{F}}$$

and

$$\pi^{\mathbb{G}} = \frac{b + \sigma\phi + \eta\sigma Z^{\mathbb{G}}}{\eta\sigma^2} \in \mathcal{A}^{\mathbb{G}}.$$

Implying

$$\begin{aligned} f^{\mathbb{F}}(t,z) =& -\frac{b_t^2}{2\eta\sigma_t^2} - \frac{b_t z}{\sigma_t}, \\ f^{\mathbb{G}}(t,z,u) =& -\frac{\eta}{2}\sum_{k=1}^n \int_E u^k(e)^2 \lambda_t(e) de - \frac{1}{2\eta}\left(\frac{b_t}{\sigma_t} + \phi_t\right)^2 \\ & - \frac{b_t z}{\sigma_t} - \phi_t z. \end{aligned}$$

Proposition 4.4.2. *The value functions $V^{\mathbb{F}}(x)$ and $V^{\mathbb{G}}(x)$ have solutions*

$$\begin{aligned} V^{\mathbb{F}}(x) &= -\exp(-\eta(x - Y_0^{\mathbb{F}})) \\ V^{\mathbb{G}}(x) &= -\exp(-\eta(x - Y_0^{\mathbb{G}})), \end{aligned}$$

where $Y^{\mathbb{F}}$ and $Y^{\mathbb{G}}$ solve the following BSDEs in $\mathbb{F}$ and $\mathbb{G}$ respectively

$$\begin{aligned} Y_t^{\mathbb{F}} =& C + \int_t^T f^{\mathbb{F}}(s, Z_s^{\mathbb{F}}) ds - \int_t^T Z_s^{\mathbb{F}} dW_s, \\ Y_t^{\mathbb{G}} =& C + \int_t^T f^{\mathbb{G}}(s, Z_s^{\mathbb{G}}, U_s) ds - \int_t^T Z^{\mathbb{G}} dW_s^{\mathbb{G}} - \sum_{k=1}^n \int_t^T U_s^k(e)\tilde{\mu}^k(ds, de). \end{aligned}$$

Proof. The solution of $V^{\mathbb{F}}$ is well known and is proven in Hu et al. [2005] and Morlais [2009] among others.
The solution of $V^{\mathbb{G}}$ is shown in Proposition 3.2.9 Romo [2016]. It just remains to show that $f^{\mathbb{G}}$ satisfies Hypothesis 6. This is so due to the integrability constraints imposed on b, σ and ϕ. □

Having now presented the value functions in $\mathbb{F}$ and $\mathbb{G}$, we aim to quantify the gain that an investor having access to the information in $\mathbb{G}$ has compared with the investor who only has access to $\mathbb{F}$. To do so we shall utilise Theorem 3.1.5 in Romo [2016] and Proposition 1.2.1 in Jeanblanc and Wu [2017] in the following proposition:

Proposition 4.4.3. *Define the* $\mathbb{F}$*-optional projections* $\hat{Y}_t = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t]$ *and* $\hat{Z}_t = \mathbb{E}[Z_t^{\mathbb{G}}|\mathcal{F}_t]$*, then*

$$\hat{Y}_t = C + \int_t^T \mathbb{E}[f^{\mathbb{G}}(s, Z_s^{\mathbb{G}}, U_s)|\mathcal{F}_s]ds - \int_t^T (\hat{Z}_s + \mathbb{E}[Y_{s-}^{\mathbb{G}}\phi_s|\mathcal{F}_s])dW_s \tag{4.14}$$

This allows us to define $\hat{Y}$ in terms of $Y^{\mathbb{F}}$ using the fact that BSDE solutions are unique in $\mathbb{F}$ provided certain constraints on the driver.

Proposition 4.4.4. *The process* $\hat{Y}$ *can be decomposed via the following forward SDE*

$$\begin{aligned}\hat{Y}_t =& Y_t^{\mathbb{F}} - \int_0^t \Big(\frac{\eta}{2}\sum_{k=1}^n \mathbb{E}\left[\int_E U_s^k(e)^2\lambda_s^k(e)de|\mathcal{F}_s\right] + \frac{b_s}{\eta\sigma_s}\mathbb{E}[\phi_s|\mathcal{F}_s] + \frac{1}{2\eta}\mathbb{E}[\phi_s^2|\mathcal{F}_s]\\ &+ \mathbb{E}[\phi_s Z_s^{\mathbb{G}}|\mathcal{F}_s]\Big)ds - \int_0^t \mathbb{E}[Y_{s-}^{\mathbb{G}}\phi_s|\mathcal{F}_s]dW_s\end{aligned}$$

Proof. Firstly we note that

$$\begin{aligned}\mathbb{E}[f^{\mathbb{G}}(t, Z_t^{\mathbb{G}}, U_t)|\mathcal{F}_t] =& -\frac{\eta}{2}\sum_{k=1}^n \mathbb{E}\left[\int_E U_t^k(e)^2\lambda_t^k(e)de|\mathcal{F}_t\right] - \frac{1}{2\eta}\frac{b_t^2}{\sigma_t^2} - \frac{b_t}{\eta\sigma_t}\mathbb{E}[\phi_t|\mathcal{F}_t] - \frac{1}{2\eta}\mathbb{E}[\phi_t^2|\mathcal{F}_t]\\ &- \frac{b_t}{\sigma_t}\hat{Z}_t - \mathbb{E}[\phi_t Z_t^{\mathbb{G}}|\mathcal{F}_t]\\ =& f^{\mathbb{F}}(t,\hat{Z}_t) - \frac{\eta}{2}\sum_{k=1}^n \mathbb{E}\left[\int_E U_t^k(e)^2\lambda_t^k(e)de|\mathcal{F}_t\right] - \frac{b_t}{\eta\sigma_t}\mathbb{E}[\phi_t|\mathcal{F}_t] - \frac{1}{2\eta}\mathbb{E}[\phi_t^2|\mathcal{F}_t]\\ &- \mathbb{E}[\phi_t Z_t^{\mathbb{G}}|\mathcal{F}_t].\end{aligned}$$

Rewriting equation (4.14)

$$\begin{aligned}\hat{Y}_t =& \left(C + \int_t^T f^{\mathbb{F}}(s,\hat{Z}_s)ds - \int_t^T \hat{Z}_s dW_s\right) - \int_t^T \left(\frac{\eta}{2}\sum_{k=1}^n \mathbb{E}\left[\int_E U_s^k(e)^2\lambda_s^k(e)\mathbb{1}_{\{\tau_{k-1}<s\le\tau_k\}}de|\mathcal{F}_s\right]\right.\\ &\left.+ \frac{b_s}{\eta\sigma_s}\mathbb{E}[\phi_s|\mathcal{F}_s] + \frac{1}{2\eta}\mathbb{E}[\phi_s^2|\mathcal{F}_s] + \mathbb{E}[\phi_s Z_s^{\mathbb{G}}|\mathcal{F}_s]\right)ds - \int_t^T \mathbb{E}[Y_{s-}^{\mathbb{G}}\phi_s|\mathcal{F}_s]dW_s.\end{aligned}$$

The first bracketed term is a BSDE in $\mathbb{F}$ with the same driver and terminal condition as the BSDE for $(Y^{\mathbb{F}}, Z^{\mathbb{F}})$, implying that the solutions are the same (see Proposition 2.2 in El Karoui et al. [1997b] for example). □

This allows us to derive a lower bound for the value of additional information in this context

Proposition 4.4.5. *The expected proportional value of additional information*

$$\mathbb{E}\left[\frac{V^{\mathbb{G}}(x)}{V^{\mathbb{F}}(x)}\right] \ge \exp\left(-\frac{1}{2}\mathbb{E}\left[\int_0^t \phi_s^2 ds\right] - \mathbb{E}\left[\int_0^t \phi_s Z_s^{\mathbb{G}} ds\right] - \frac{\eta^2}{2}\sum_{k=1}^n \mathbb{E}\left[\int_0^t\int_E U_s^k(e)^2\lambda_s^k(e)deds\right]\right)$$

for all $t \in [0,T]$.

Proof. Firstly we note, using Jensen's inequality

$$\begin{aligned}\mathbb{E}\left[\frac{V^{\mathbb{G}}(x)}{V^{\mathbb{F}}(x)}\right] &\geq \exp\left(\mathbb{E}\left[\log\left(\frac{V^{\mathbb{G}}(x)}{V^{\mathbb{F}}(x)}\right)\right]\right)\\ &= \exp\left(\mathbb{E}[\eta(Y_t^{\mathbb{G}} - Y_t^{\mathbb{F}})]\right)\\ &= \exp\left(\mathbb{E}[\eta(\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] - Y_t^{\mathbb{F}})]\right)\\ &= \exp\left(\mathbb{E}\left[-\eta\left(\int_t^T \left(\frac{\eta}{2}\sum_{k=1}^n \mathbb{E}\left[\int_E U_s^k(e)^2\lambda_s^k(e)de|\mathcal{F}_s\right] + \frac{b_s}{\eta\sigma_s}\mathbb{E}[\phi_s|\mathcal{F}_s]\right.\right.\right.\right.\\ &\quad \left.\left.\left.\left. + \frac{1}{2\eta}\mathbb{E}[\phi_s^2|\mathcal{F}_s] + \mathbb{E}[\phi_s Z_s^{\mathbb{G}}|\mathcal{F}_s]\right)ds\right)\right]\right).\end{aligned}$$

Finally,

$$\mathbb{E}\left[\int_0^t \frac{b_s\phi_s}{\eta\sigma_s}ds\right] = \mathbb{E}\left[\int_0^t \frac{b_s}{\eta\sigma_s}dW_s\right] + \mathbb{E}\left[\int_0^t \frac{b_s}{\eta\sigma_s}dW_s^{\mathbb{G}}\right] = 0$$

Using the fact that b and σ are $\mathbb{F}$-adapted and therefore also $\mathbb{G}$-adapted.

□

4.4.2 Risk Measures

This very short subsection is devoted to extending one result of Calvia and Gianin [2020]. To begin we define the notion of a risk measure.

Definition 4.4.6. *For an arbitrary filtration $\mathbb{H} = (\mathcal{H}_t)$, a fixed finite time T and any $t \in [0,T]$, a function $\rho_t : L^2(\mathcal{H}_T) \to L^2(\mathcal{H}_t)$ is called an $\mathbb{H}$-dynamic risk measure if*

$$\rho_T(X) = -X \text{ for all } X \in L^2(\mathcal{H}_T).$$

It is well known (see for example Peng [2004] or Gianin [2006]) that in this setup, the solutions to BSDEs induce dynamic risk measures. Indeed, for a given $\mathbb{G}$-adapted driver f satisfying hypothesis 6 and $0 \leq t \leq T$, the function $\rho_t : L^2(\mathcal{G}_T) \to L^2(\mathcal{G}_t)$ defined by $\rho_t(X) := Y_t^{-X}$, where Y^{-X} solves BSDE (4.4) with driver f and terminal condition $-X$ is a $\mathbb{G}$-dynamic risk measure. This fact means that for any driver satisfying Hypothesis 6, a $\mathbb{G}$-dynamic risk measure is induced.
For a $\mathbb{G}$-adapted driver f satisfying hypothesis 6, recall its decomposition for $(Y,Z,U) \in \mathbb{R}\times\mathbb{R}\times B(E)$,

$$f(\cdot,Y,Z,U) = \sum_{k=1}^n f^k(\cdot,Y,Z,U,\tau^{(k)},\xi^{(k)})\mathbb{1}_{(\!(\tau_k,\tau_{k+1}]\!]}.$$

Proposition 4.4.7. *Suppose the conditions of Theorem 4.3.2 are satisfied. For any $k \in \{1,2..,n\}$ and $(u^{(k)},e^{(k)}) \in \Theta_k \times E^k$, let $\rho^k(u^{(k)},e^{(k)})$ be the $\mathbb{F}$-dynamic risk measure induced by the driver $f^k(\cdot,\cdot,\cdot,\cdot,u^{(k)},e^{(k)})\gamma^k(u^{(k)},e^{(k)})$, then the $\mathbb{G}$-dynamic risk measure ρ induced by the driver f, can be decomposed as follows*

$$\rho_t(X) = \sum_{k=1}^n \frac{\rho_t^k(\tau^{(k)},\xi^{(k)})\left(X^k(\tau^{(k)},\xi^{(k)})\gamma_T^k(\tau^{(k)},\xi^{(k)})\right)}{\gamma_t^k(\tau^{(k)},\xi^{(k)})}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}},$$

where $X := \sum_{k=1}^{n} X^k(\tau^{(k)}, \xi^{(k)})\mathbb{1}_{\{\tau_k \le T < \tau_{k+1}\}} \in L^2(\mathcal{G}_T)$.

Proof. The result is a direct consequence of theorem 4.3.2. □

Note the utility of this result is that for any $k \in \{1, 2, ..., n\}$, the $\mathbb{F}$-dynamic risk measure $\rho^k(u^{(k)}, e^{(k)})$ is computed in $\mathbb{F}$ with a fixed $u^{(k)}$ and $e^{(k)}$. In other words, if we assume that the computation of risk measures in $\mathbb{F}$ is possible, however with the enlargement of $\mathbb{F}$ with τ and ξ, this is no longer possible, then Proposition 4.4.7 enables one to compute risk measures in the enlarged filtration.

Chapter 5

Optimal Stopping in an enlarged filtration

This chapter focuses on the application of enlargement of filtration with multiple random times and their marks to optimal stopping problems. Optimal stopping and enlargement of filtration have a very intuitive relationship. We are interested in studying how additional information may effect the optimal stopping decision and to quantify the effect on the value process. Before setting up the problem, the following subsection defines the relevant notation and conventions to be used throughout this chapter, the goal being to derive a transfer formula for essential suprema in the enlarged filtration to the reference filtration. Remarkably, optimal stopping in $\mathbb{G}$ is done by only considering stopping times in $\mathbb{F}$. Intuitively this means that the addition of (τ, ξ) does not add any additional degrees of freedom when choosing when to stop in an optimal stopping problem. Note that while the selection of stopping times is only in $\mathbb{F}$, the resulting optimal stopping time may be a $\mathbb{G}$-stopping time. In view of Proposition 3.2.4 in Chapter 3, the essential supremum in $\mathbb{G}$ will be a series of essential suprema in $\mathbb{F}$. It is noted that this chapter merely focuses on the interaction between enlargement of filtration and optimal stopping problems, optimal stopping is a broad topic with a rich bank of literature and is considered a very complex and rigorous field of mathematics. It is well known that the method of solution to optimal stopping problems is done in two ways; the Markovian-PDE and the Martingale-SDE approach. By nature of the fact that this thesis deals with an arbitrary reference filtration and an arbitrary set of random variables (τ, ξ), we will make use of the Martingale-SDE approach.
Optimal stopping problems in the context of enlargement of filtration have been considered in the literature. Esmaeeli and Imkeller [2018] consider a general optimal stopping in an initial enlargement of filtration setting, here the authors prove that the Snell envelope in $\mathbb{G}$ can be written as a parameterised Snell envelope in $\mathbb{F}$. Much like the structure of this chapter, the authors then prove that the $\mathbb{G}$-RBSDE which solves the optimal stopping problem can also be written as a parameterised RBSDE in $\mathbb{F}$. Bayraktar and Zhou [2017] consider Shiryaev's optimal stopping for an insider who can see arbitrarily long in to the future. This set up is outside the scope of this thesis as it is showed in Chapter 2 that preservation of semimartingales in this enlarged filtration is not upheld in this setup.

5.1 Setup

We begin by recalling the definition of the essential supremum.

Definition 5.1.1. *Given a possibly uncountable set of random variables $\{X_i : i \in I\}$ each taking values in the extended real line $\mathbb{R} \cup \{\pm\infty\}$, the essential supremum of $\{X_i : i \in I\}$, denoted $\operatorname{ess\,sup}_{i\in I} X_i$ is the $\mathbb{P}$-a.s. unique random variable such that*

- $\operatorname{ess\,sup}_{i\in I} X_i \geq X_j$, *$\mathbb{P}$-a.s for all $j \in I$.*
- *If Y is another random variable such that $Y \geq X_i$, $\mathbb{P}$-a.s for all $i \in I$ then $Y \geq \operatorname{ess\,sup}_{i\in I} X_i$.*

An interesting case of the essential supremum is the Snell envelope, first introduced by Mertens [1972] in reference to the work of Snell [1952]. Let R be a positive and continuous $\mathbb{F}$-adapted process such that

$$\mathbb{E}\left[\sup_{\nu\in\mathcal{T}_{0,T}(\mathbb{F})} |R_\nu|\right] < \infty.$$

Definition 5.1.2. *For $T > 0$ and for an arbitrary filtration $\mathbb{H} = (\mathcal{H}_t)$, the Snell envelope of R with respect to $\mathbb{H}$ on $[0, T]$ is*

$$\operatorname*{ess\,sup}_{\nu\in\mathbb{T}_{t,T}(\mathbb{H})} \mathbb{E}[R_\nu|\mathcal{H}_t],$$

for $t \in [0, T]$, where $\mathbb{T}_{t,T}(\mathbb{H})$ is the set of all $\mathbb{H}$-stopping times valued in $[t, T]$. It can be shown that $\left(\operatorname{ess\,sup}_{\nu\in\mathbb{T}_{t,T}(\mathbb{H})} \mathbb{E}[R_\nu|\mathcal{H}_t] : t \in [0, T]\right)$ is the smallest right-continuous $\mathbb{H}$-supermartingale which dominates R.

Intuitively, we can think of R as a payoff process, an investor must decide when the best time is to devest and receive her payoff. Her stopping strategy may depend on the level of the payoff process R and potentially other information available to her. She will therefore not only consider fixed times but the set of stopping times available to her. Define the following Snell envelopes of R in the filtrations $\mathbb{F}$ and $\mathbb{G}$.

$$V_t^{\mathbb{F}} = \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}[R_\nu|\mathcal{F}_t],$$
$$V_t^{\mathbb{G}} = \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{G})} \mathbb{E}[R_\nu|\mathcal{G}_t].$$

5.2 Optimal Stopping in $\mathbb{G}$

In what follows, it will be useful for the reader to assume that the computation of Snell envelopes in $\mathbb{F}$ is possible, meaning that given any appropriately defined payoff process R, $V^{\mathbb{F}}$ can be computed directly. While that is no longer the case with the addition of (τ, ξ) in $\mathbb{G}$, meaning it is not guaranteed that $V^{\mathbb{G}}$ can be computed for every $\mathbb{F}$-adapted R. While this is seldom the case and we do not assume this, it is to useful for the interpretation of the following results.

The goal of the remainder of this section is to find a way to compute $V^{\mathbb{G}}$ in terms of quantities that can be computed in $\mathbb{F}$. This will be done by utilising results from Chapter 3 and additional results on the enlarged filtration, ultimately developing a transfer formula for

the $\mathbb{G}$-Snell envelope to $\mathbb{F}$.
We begin with the following useful proposition about stopping times in $\mathbb{G}$. This is seen as an extension of Proposition 3.4 in Esmaeeli and Imkeller [2018]. Before continuing, we recall that for $k \in \{1, 2, ..., n\}$, an $\mathbb{F}\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$-stopping time $\hat{\nu}$, is an $\mathcal{F}\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$-positive random variable such that for any $(u^{(k)}, e^{(k)}) \in \Theta_k \times E^k$, $\hat{\nu}(u^{(k)}, e^{(k)})$ is an $\mathbb{F}$-stopping time.

Proposition 5.2.1. *For any $\mathbb{G}$-stopping time ν there exist $\mathbb{F}\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$-stopping times $\hat{\nu}^k$ for $k \in \{1, 2, ..., n\}$ such that*

$$\begin{aligned}\tau_{k+1} \wedge \nu =& \tau_{k+1} \wedge \hat{\nu}^k(\tau^{(k)}, \xi^{(k)}), \quad k \in \{0, 1, ..., n-1\},\\ \tau_k \vee \nu =& \tau_k \vee \hat{\nu}^k(\tau^{(k)}, \xi^{(k)}), \quad k \in \{1, 2, ..., n\}.\end{aligned}$$

Proof. For any $\mathbb{G}$-stopping time ν, we clearly have

$$\nu = \inf\{t \geq 0 : \mathbb{1}_{\{\nu \geq t\}} = 0\}.$$

From Proposition 3.2.1 there exists $J^k \in \mathcal{P}(\mathbb{F}, \Theta_k, E^k)$, $k \in \{0, 1, 2, ..., n\}$ such that

$$\begin{aligned}\mathbb{1}_{\{\nu\geq t\}} =& J^0_t \mathbb{1}_{\{\tau_1 \geq t\}} + J^1_t(\tau_1, \xi_1)\mathbb{1}_{\{\tau_1 < t \leq \tau_2\}} + ... + J^n_t(\tau, \xi)\mathbb{1}_{\{\tau_1 \leq t\}}\\ \mathbb{1}_{\{\nu\geq t\}}\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}} =& J^k_t(\tau^{(k)}, \xi^{(k)})\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}}\end{aligned}$$

this implies that $J^k_t(\tau^{(k)}, \xi^{(k)}) \in \{0, 1\}$ when $\tau_k \leq t < \tau_{k+1}$. Define the $\mathbb{F}\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$-stopping time

$$\hat{\nu}^k(\cdot, \cdot) = \inf\{t \geq 0 : J^k_t(\cdot, \cdot) = 0\}.$$

Then

$$J^k_t(\tau^{(k)}, \xi^{(k)})\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}} = \mathbb{1}_{\{\hat{\nu}^k(\tau^{(k)}, \xi^{(k)}) \geq t\}}\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}}.$$

By definition of J^k, this is written as

$$\mathbb{1}_{\{\nu \geq t\}}\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}} = \mathbb{1}_{\{\hat{\nu}^k(\tau^{(k)}, \xi^{(k)}) \geq t\}}\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}}.$$

Taking compliments on both sides of the above yields

$$\mathbb{1}_{\{\nu < t\}}\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}} = \mathbb{1}_{\{\hat{\nu}^k(\tau^{(k)}, \xi^{(k)}) < t\}}\mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}}.$$

Together, this implies

$$\begin{aligned}\nu \wedge \tau_{k+1} =& \hat{\nu}^k(\tau^{(k)}, \xi^{(k)}) \wedge \tau_{k+1},\\ \nu \vee \tau_k =& \hat{\nu}^k(\tau^{(k)}, \xi^{(k)}) \vee \tau_k.\end{aligned}$$

□

Before continuing, we recall a part of Proposition 3.2.4 from Chapter 3, in particular the expectation of a random variable $X \in L^1(\mathcal{G}_T)$ before τ_1.

$$\mathbb{E}[X|\mathcal{G}_t]\mathbb{1}_{\{\tau_1>t\}} = \frac{\mathbb{E}\left[X\mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t\right]}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}.$$

The following lemma is found in in Proposition 2.18 in Aksamit and Jeanblanc [2017] and will be of use when we transfer the $\mathbb{G}$-Snell envelope to a Snell envelope in $\mathbb{F}$.

Lemma 5.2.2. *For any $\mathbb{F}$-supermartingale Y, $\frac{Y}{G^0}\mathbb{1}_{[\![0,\tau_1[\![}$ is a $\mathbb{G}$-supermartingale.*

Proof. For $s \le t$, using Proposition 3.2.4

$$\begin{aligned}
\mathbb{E}\left[\frac{Y_t}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}|\mathcal{G}_s\right] &= \mathbb{E}\left[\frac{Y_t}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}|\mathcal{G}_s\right]\mathbb{1}_{\{\tau_1>s\}} \\
&= \frac{\mathbb{E}\left[\frac{Y_t}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_s\right]}{G_s^0}\mathbb{1}_{\{\tau_1>s\}} \\
&= \frac{\mathbb{E}\left[\mathbb{E}\left[\frac{Y_t}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t\right]|\mathcal{F}_s\right]}{G_s^0}\mathbb{1}_{\{\tau_1>s\}} \\
&= \frac{\mathbb{E}\left[\frac{Y_t}{G_t^0}\mathbb{E}\left[\mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t\right]|\mathcal{F}_s\right]}{G_s^0}\mathbb{1}_{\{\tau_1>s\}} \\
&= \frac{\mathbb{E}[Y_t|\mathcal{F}_s]}{G_s^0}\mathbb{1}_{\{\tau_1>s\}} \\
&\le \frac{Y_s}{G_s^0}\mathbb{1}_{\{\tau_1>s\}}.
\end{aligned}$$

□

Proposition 5.2.1 and Lemma 5.2.2 can now be used to transfer the $\mathbb{G}$-Snell envelope to a series of $\mathbb{F}$-Snell envelopes.

Theorem 5.2.3. *For any $t \in [0,T]$, the $\mathbb{G}$-Snell envelope $V^{\mathbb{G}}$ can be decomposed as follows*

$$V_t^{\mathbb{G}} = \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{G})} \mathbb{E}[R_\nu|\mathcal{G}_t] = \sum_{k=0}^{n} \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \frac{\mathbb{E}\left[R_\nu\mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_t^k\right]}{G_t^k}\mathbb{1}_{\{\tau_k\le t<\tau_{k+1}\}}.$$

Proof. We start by showing that the first term is

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_1>t\}} = \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \frac{\mathbb{E}[R_\nu\mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t]}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}.$$

Reversing Proposition 3.2.4, we see that the right hand side is exactly

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}[R_\nu|\mathcal{G}_t]\mathbb{1}_{\{\tau_1>t\}}.$$

Therefore, from the definition of the essential supremum over $\mathbb{G}$-stopping times,

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_1>t\}} \ge \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \frac{\mathbb{E}[R_\nu\mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t]}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}.$$

Let $Y_t := \operatorname{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}[R_\nu \mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t]$. Note that Y is an $\mathbb{F}$-supermartingale, and by Lemma 5.2.2, $\frac{Y_t}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}$ is a $\mathbb{G}$-supermartingale. Now note that by definition $Y_t \geq \mathbb{E}[R_\nu \mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t]$ for all $\nu \in \mathcal{T}_{t,T}(\mathbb{F})$, in particular $Y_t\mathbb{1}_{\{\tau_1>t\}} \geq R_t G_t^0 \mathbb{1}_{\{\tau_1>t\}}$. Meaning, $\frac{Y}{G^0}\mathbb{1}_{[\![0,\tau_1)\!)}$ is a $\mathbb{G}$-supermartingale dominating $R\mathbb{1}_{[\![0,\tau_1)\!)}$, so by definition of the essential supremum, $\frac{Y_t}{G_t^0}\mathbb{1}_{\{\tau_1>t\}} \geq V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_1>t\}}$. Therefore

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_1>t\}} = \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \frac{\mathbb{E}[R_\nu \mathbb{1}_{\{\tau_1>t\}}|\mathcal{F}_t]}{G_t^0}\mathbb{1}_{\{\tau_1>t\}}.$$

We now show that for $k \in \{1, ..., n\}$

$$\begin{aligned} V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}} &= \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \frac{\mathbb{E}[R_\nu \mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_t^k]}{G_t^k}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}} \\ &= \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[R_\nu \mathbb{1}_{\{\tau_{k+1}>t\}}|\mathcal{G}_t\right]\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}}. \end{aligned}$$

To do this, we let ν^* be the $\mathbb{G}$-optimal time such that $V_t^{\mathbb{G}} = \mathbb{E}[R_{\nu^*}|\mathcal{G}_t]$. The existence of ν^* is due to the fact that $\mathbb{E}\left[\int_0^T R_t^2 dt\right] < \infty$. Then from Proposition 5.2.1 there exists an $\mathbb{F}\otimes\mathcal{B}(\Theta_k)\otimes\mathcal{B}(E^k)$-stopping time $\hat{\nu}^k$ such that $\tau_k \vee \nu^* = \tau_k \vee \hat{\nu}^k(\tau^{(k)}, \xi^{(k)})$. Restricting the Snell envelope on $\{\tau_k \leq t < \tau_{k+1}\}$, i.e.

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}} = \mathbb{E}[R_{\nu^*}|\mathcal{G}_t]\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}}.$$

The stopping time ν^* can be replaced by $\hat{\nu}^k$ when $\tau_k \leq t$ since $\nu^* \geq t$. Furthermore Proposition 3.2.4 is used to yield

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}} = \frac{\mathbb{E}\left[R_{\hat{\nu}^k(\tau^{(k)},\xi^{(k)})}\mathbb{1}_{\{t<\tau_{k+1}\}}|\mathcal{G}_t^{k,\tau,\xi}\right]}{G_t^k}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}}$$

$\tau^{(k)}$ and $\xi^{(k)}$ are $\mathcal{G}_t^{\tau,\xi,k}$-measurable, therefore

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}} = \frac{\mathbb{E}\left[R_{\hat{\nu}^k(u^{(k)},e^{(k)})}\mathbb{1}_{\{t<\tau_{k+1}\}}|\mathcal{G}_t^{k,\tau,\xi}\right]|_{\substack{u^{(k)}=\tau^{(k)}\\ e^{(k)}=\xi^{(k)}}}}{G_t^k}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}}.$$

By definition, for any $(u^{(k)}, e^{(k)})$, $\hat{\nu}^k(u^{(k)}, e^{(k)})$ is an $\mathbb{F}$-stopping time, meaning the numerator must be inferior to the essential supremum over all $\mathbb{F}$-stopping times, in other words

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}} \leq \frac{\left(\operatorname{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[R_\nu \mathbb{1}_{\{t<\tau_{k+1}\}}|\mathcal{G}_t^{k,\tau,\xi}\right]\right)}{G_t^k}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}}.$$

Finally, Proposition 3.2.4 is reversed to identify the right hand side as

$$V_t^{\mathbb{G}}\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}} \leq \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}[R_\nu|\mathcal{G}_t]\mathbb{1}_{\{\tau_k\leq t<\tau_{k+1}\}}.$$

The right hand side contains an essential supremum only over $\mathbb{F}$-stopping times implying the reverse inequality, the result is then shown. □

Remark 8. *Note that the formula in Theorem 5.2.3 reduces the computation of an essential supremum in* $\mathbb{G}$ *to* $n+1$ *essential suprema in* $\mathbb{F}$. *Indeed for each* $k \in \{0, 1, ..., n\}$, *by Proposition 3.2.4*

$$V_t^{\mathbb{G}} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \frac{\mathbb{E}[R_\nu \mathbb{1}_{\{\tau_{k+1} > t\}} | \mathcal{G}_t^k]}{G_t^k} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}$$

$$= \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \frac{\mathbb{E}\left[R_\nu \gamma_\nu^k(u^{(k)}, e^{(k)}) | \mathcal{F}_t\right] \big|_{\substack{u^{(k)} = \tau^{(k)} \\ e^{(k)} = \xi^{(k)}}}}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})} \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}.$$

Theorem 5.2.3 says that optimal stopping in $\mathbb{G}$ is achieved by only optimising over $\mathbb{F}$-stopping times. This, however, does not imply that the optimal time for which the Snell envelope achieves its essential supremum (nu^* in the proof of Theorem 5.2.3) is an $\mathbb{F}$-stopping time. Moreover, the projection of the $\mathbb{G}$-Snell envelope is not simply the $\mathbb{F}$-Snell envelope as classical results on essential supremum suggest. This is because the set $\{\mathbb{E}[R_\nu|\mathcal{G}_t] : \nu \in \mathcal{T}_{t,T}(\mathbb{F})\}$ does not have a lattice property, meaning there needn't exist a sequence of $\mathbb{F}$-stopping times (ν_n) such that $V_t^{\mathbb{G}} = \lim_{n\to\infty} \mathbb{E}[R_{\nu_n}|\mathcal{G}_t]$. The benefit of such a result is that we may just consider the optimal stopping problem in $\mathbb{G}$ as a series of optimal stopping problems in $\mathbb{F}$. Supposing we were able to compute Snell envelopes in $\mathbb{F}$, this would allow us to compute the $\mathbb{G}$-Snell envelope. The power of this result will become clear in Section 5.3 when we apply Theorem 5.2.3 to an example with a known solution.

5.3 Application to Brownian Bridges

Having now established a method of transferring the $\mathbb{G}$-Snell envelope to a series of $\mathbb{F}$-Snell envelopes, we now present a simple example of how the formula may be used. Note, that up to now, we have considered the progressive enlargement of filtration with multiple random times and marks. As remarked in Section 5.5, the reason for including the marks, among others, is to allow one to simultaneously work under an initial enlargement and traditional progressive enlargement. Indeed, all results presented in this paper apply to the initial enlargement of filtration if all default times are set to zero. Similarly, one can recover the traditional progressive enlargement of filtration by setting all marks to deterministic functions on E. We shall assume the former and apply Theorem 5.2.3 to the Brownian bridge. Note that this problem has twice been solved in the literature, once by Shepp [1969] using a change of time and classical results on the Snell envelope and then by Ekström and Wanntorp [2009] who used PDE methods to solve the optimal stopping problem explicitly. We present here a third way, using the initial enlargement of filtration.

Suppose for simplicity's sake that W is a 1-dimensional Brownian Motion and that $k = 1$ and $\tau = 0$. This is the classical setup for the initial enlargement of filtration. Now for a fixed, finite $T > 0$, let $\xi = W_T$. In this setup, W is a $\mathbb{G}$-Brownian bridge, meaning a Brownian Motion W, whose end point or pinning point, W_T is known. Note that, Hypothesis 4 is not satisfied at time T. However the techniques developed in previous sections will still be applicable before T. The problem of Shepp [1969] and Ekström and Wanntorp [2009] was to find the value process when optimally stopping a Brownian bridge. In our setup this corresponds to computing:

$$V_t^{\mathbb{G}} = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{G})} \mathbb{E}[W_\nu | \mathcal{G}_t].$$

Taking in to account the fact that Hypothesis 4 is not satisfied at time T and the $\mathcal{G}_t$-measurability of W_T, the value process may be rewritten using Theorem 5.2.3.

$$\begin{aligned} V_t^{\mathbb{G}} &= W_T + \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{G})} \mathbb{E}[W_\nu - W_T | \mathcal{G}_t], \\ &= W_T + \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{G})} \mathbb{E}[(W_\nu - W_T)\mathbb{1}_{\{\nu < T\}} | \mathcal{G}_t], \\ &= W_T + \frac{\left(\operatorname{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[(W_\nu - u)\alpha_\nu(u)\mathbb{1}_{\{\nu<T\}} | \mathcal{F}_t\right]\right)|_{u=W_T}}{\alpha_t(W_T)}, \end{aligned} \tag{5.1}$$

where the density process α can be explicitly stated as follows

$$\begin{aligned} \alpha_t(u) &= \frac{\mathbb{P}(W_T \in du | \mathcal{F}_t)}{\mathbb{P}(W_T \in du)}, \qquad (u,t) \in \mathbb{R} \times [0,T), \\ &= \sqrt{\frac{T}{T-t}} \exp\left(-\frac{(W_t - u)^2}{2(T-t)} + \frac{u^2}{2T}\right). \end{aligned}$$

meaning the numerator in equation (5.1) becomes

$$\begin{aligned} &\operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[(W_\nu - u)\alpha_\nu(u)\mathbb{1}_{\{\nu<T\}} | \mathcal{F}_t\right] |_{u=W_T} \\ =& \sqrt{T} \exp\left(\frac{u^2}{2T}\right) \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[\frac{W_\nu - u}{\sqrt{T-\nu}} \exp\left(-\frac{(W_\nu - u)^2}{2(T-\nu)}\right) \mathbb{1}_{\{\nu<T\}} | \mathcal{F}_t\right] |_{u=W_T}. \end{aligned} \tag{5.2}$$

Let $Y_s = \frac{W_s - u}{\sqrt{T-s}}$ for $s < T$ and

$$\begin{aligned} V_t^* &= \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[\frac{W_\nu - u}{\sqrt{T-\nu}} \exp\left(-\frac{(W_\nu - u)^2}{2(T-\nu)}\right) \mathbb{1}_{\{\nu<T\}} | \mathcal{F}_t\right], \\ &= \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[Y_\nu \exp\left(-\frac{Y_\nu^2}{2}\right) \mathbb{1}_{\{\nu<T\}} | \mathcal{F}_t\right]. \end{aligned}$$

It is important to note that Y and V^* both depend on u, for simplicity's sake, the dependence is omitted. This omission continues throughout this section with newly defined quantities.

Lemma 5.3.1. *For $c \in \mathbb{R}$, define the $\mathbb{F}$-stopping time by $\nu_t^c := \inf\{s \geq t : Y_s \geq c\} \wedge T\}$ and $V_t^c = \mathbb{E}\left[Y_{\nu_t^c} \exp\left(-\frac{Y_{\nu_t^c}^2}{2}\right) \mathbb{1}_{\{\nu_t^c < T\}} | \mathcal{F}_t\right]$. Then,*

$$\mathbb{P}(\nu_t^c < T | \mathcal{F}_t) = \frac{\Phi(Y_t)}{\Phi(c)}$$

and

$$V_t^c = \begin{cases} c \exp\left(-\frac{c^2}{2}\right) \Phi(Y_t)\Phi(c)^{-1}, & Y_t < c \\ Y_t \exp\left(-\frac{Y_t^2}{2}\right), & Y_t \geq c \end{cases}$$

where $\Phi(\cdot)$ is the normal distribution function.

Proof. First note that by the definition of Y,

$$\nu_t^c = \inf\{s \geq t : W_s \geq u + c\sqrt{T-s}\} \wedge T.$$

Consider the following

$$\mathbb{P}(W_T > u|\mathcal{F}_t) = \mathbb{P}\left(W_T > u|\{\nu_t^c < T\}, \mathcal{F}_t\right)\mathbb{P}(\nu_t^c < T|\mathcal{F}_t).$$

Therefore,

$$\mathbb{P}(\nu_t^c < T|\mathcal{F}_t) = \frac{\Phi\left(\frac{W_t - u}{\sqrt{T-t}}\right)}{\mathbb{P}\left(W_T > u|\{\nu_t^c < T\}, \mathcal{F}_t\right)}.$$

Note that $\mathcal{F}_t \vee \{\nu_t^c < T\} = \{A \cap \{\nu_t^c < T\} : A \in \mathcal{F}_t\} \subseteq \mathcal{F}_{\nu_t^c}$ and so

$$\mathbb{P}\left(W_T > u|\{\nu_t^c < T\}, \mathcal{F}_t\right) = \mathbb{E}[\mathbb{P}(W_T > u|\mathcal{F}_{\nu_t^c})|\{\nu_t^c < T\}, \mathcal{F}_t].$$

Using the Tower property of conditional expectations

$$\mathbb{P}\left(W_T > u|\{\nu_t^c < T\}, \mathcal{F}_t\right) = \mathbb{E}[\mathbb{P}(W_T - W_{\nu_t^c} > u - (u + c\sqrt{T-\nu_t^c})|\mathcal{F}_{\nu_t^c})|\{\nu_t^c < T\}, \mathcal{F}_t]$$
$$\mathbb{P}\left(W_T > u|\{\nu_t^c < T\}, \mathcal{F}_t\right) = \mathbb{E}\left[\mathbb{P}\left(\frac{W_T - W_{\nu_t^c}}{\sqrt{T-\nu_t^c}} > -c|\mathcal{F}_{\nu_t^c}\right)|\{\nu_t^c < T\}, \mathcal{F}_t\right].$$

Conditional on $\mathcal{F}_{\nu_t^c}$, $W_T \sim \mathcal{N}(W_{\nu_t^c}, T - \nu_t^c)$, this implies the probability statement is simplified to a standard Normal probability

$$\mathbb{P}\left(W_T > u|\{\nu_t^c < T\}, \mathcal{F}_t\right) = \mathbb{E}[\Phi(c)|\{\nu_t^c < T\}, \mathcal{F}_t]$$
$$\mathbb{P}\left(W_T > u|\{\nu_t^c < T\}, \mathcal{F}_t\right) = \Phi(c).$$

Finally, Y_t is $\mathcal{F}_t$ measurable but is not fixed and therefore we need to account for the possibilities $\nu_t^c = t$ and $\nu_t^c > t$ when computing V_t^c. For $\nu_t^c > t$, in other words, for $Y_t < c$:

$$\begin{aligned} V_t^c &= \mathbb{E}\left[Y_{\nu_t^c}\exp\left(-\frac{Y_{\nu_t^c}^2}{2}\right)\mathbb{1}_{\{\nu_t^c < T\}}|\mathcal{F}_t\right] \\ &= c\exp\left(-\frac{c^2}{2}\right)\mathbb{P}(\nu_t^c < T|\mathcal{F}_t) \\ &= \frac{c\exp\left(-\frac{c^2}{2}\right)\Phi(Y_t)}{\Phi(c)}. \end{aligned}$$

The second equality following by continuity of Y. Now for $Y_t \geq c$, $\nu_t^c = t$ and so $V_t^c = Y_t\exp\left(-\frac{Y_t^2}{2}\right)$. Therefore

$$V_t^c = \begin{cases} c\exp\left(-\frac{c^2}{2}\right)\Phi(Y_t)\Phi(c)^{-1}, & Y_t < c \\ Y_t\exp\left(-\frac{Y_t^2}{2}\right), & Y_t \geq c. \end{cases}$$

□

We would like to find a value for c that maximises V_t^c, this is equivalent to maximising V_t^c for $Y_t < c$, that is, maximising

$$\frac{c \exp\left(-\frac{c^2}{2}\right)\Phi(Y_t)}{\Phi(c)}.$$

with respect to c. Differentiating the above quantity and setting it equal to zero yields the following equation:

$$(1-c^2)\Phi(c) = c\phi(c).$$

All that remains to show is that for this value of c, ν_t^c is an $\mathbb{F}$-optimal time for $Y \exp\left(-\frac{Y^2}{2}\right)\mathbb{1}_{[\![0,T)\!)}$.

Proposition 5.3.2. *For c defined above, $V_t^c = V_t^*$ and ν_t^c is $\mathbb{F}$-optimal for $Y \exp\left(-\frac{Y^2}{2}\right)\mathbb{1}_{[\![0,T)\!)}$.*

Proof. Let $R := Y \exp\left(-\frac{Y^2}{2}\right)\mathbb{1}_{[\![0,T)\!)}$. This proof will use the fact that V^* is the smallest $(\mathbb{F},\mathbb{P})$ - supermartingale which dominates R. Our aim is to show that V^c is an $\mathbb{F}$-supermartingale which dominates R and therefore is superior to V^* but V_t^c is just an expectation of R at a particular stopping time, meaning by definition of the essential supremum, V^c must be inferior to V^* and therefore equal. First we show that V^c dominates R.

By definition of c, the function $f(x) = \frac{x\exp\left(-\frac{x^2}{2}\right)}{\Phi(x)}$ acheives its maximum at c. Therefore

$$\frac{x \exp\left(-\frac{x^2}{2}\right)}{\Phi(x)} \leq \frac{c \exp\left(-\frac{c^2}{2}\right)}{\Phi(c)} = \sqrt{2\pi}(1-c^2),$$
$$\therefore x \exp\left(-\frac{x^2}{2}\right) \leq \sqrt{2\pi}(1-c^2)\Phi(x), \quad \forall x.$$

Therefore, when $Y_t < c$, $V_t^c = \sqrt{2\pi}(1-c^2)\Phi(Y_t) \geq R_t$. and when $Y_t \geq c$, $V_t^c = Y_t \exp\left(-\frac{Y_t^2}{2}\right)$. Next we show that V_t^c is an $\mathbb{F}$-supermartingale. To do this, we first show that the process $\left(Y_t \exp\left(-\frac{Y_t^2}{2}\right)\right)_{t<T}$ is an $\mathbb{F}$-supermartingale. This is proved in Appendix 6. Note also from the proof of Lemma 5.3.1 $\Phi(Y_t) = \mathbb{P}(W_T > u|\mathcal{F}_t)$ and so $(\Phi(Y_t))_{t<T}$ is a martingale. An application of Problem 7.3 in Karatzas and Shreve [1998] shows that for the function

$$f(x) = \begin{cases} c \exp\left(-\frac{c^2}{2}\right)\Phi(x)\Phi(c)^{-1}, & x < c \\ x \exp\left(-\frac{x^2}{2}\right), & x \geq c. \end{cases}$$

$f(Y_t)$ is a supermartingale. □

We now return to equation (5.1). For $W_t < u + c\sqrt{T-t}$:

$$\begin{aligned} &\operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[(W_\nu - u)\alpha_\nu(u)\mathbb{1}_{\{\nu<T\}}|\mathcal{F}_t\right] \\ =&\sqrt{T}\exp\left(\frac{u^2}{2T}\right)\sqrt{2\pi}(1-c^2)\Phi(Y_t). \end{aligned}$$

and for $W_t \geq u + c\sqrt{T-t}$:

$$\operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[(W_\nu - u)\alpha_\nu(u)\mathbb{1}_{\{\nu<T\}}|\mathcal{F}_t\right] = (W_t - u)\alpha_t(u).$$

Therefore, setting $u = W_T$, we get

$$V_t^{\mathbb{G}} = \begin{cases} W_T + \frac{(\sqrt{T}\exp\left(\frac{u^2}{2T}\right)\sqrt{2\pi}(1-c^2)\Phi(Y_t))|_{u=W_T}}{\alpha_t(W_t)}, & W_t < W_T + c\sqrt{T-t} \\ W_T + \frac{((W_t-u)\alpha_t(u))|_{u=W_T}}{\alpha_t(W_T)}, & W_t \geq W_T + c\sqrt{T-t} \end{cases}$$
$$= \begin{cases} W_T + \sqrt{2\pi(T-t)}(1-c^2)\exp\left(\frac{(W_T-W_t)^2}{2(T-t)}\right)\Phi\left(\frac{W_t-W_T}{\sqrt{T-t}}\right), & W_t < W_T + c\sqrt{T-t} \\ W_t, & W_t \geq W_T + c\sqrt{T-t} \end{cases}$$

This value process conquers with the solution found in Shepp [1969] and Ekström and Wanntorp [2009].

5.4 Reflected BSDEs with $\mathbb{F}$-adapted data

We now turn our attention towards another method for solving optimal stopping problems, reflected backward stochastic differential equations (RBSDEs). To do so for the rest of this section, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered Brownian probability space endowed with a d-dimensional Brownian Motion W, such that $\mathbb{F}$ is its completed natural filtration. Just as in Chapter 4, it is noted that α, G^k and γ^k for all $k \in \{0, 1, ..., n\}$ are continuous. Recall from Chapter 4 that

$$W_t^{\mathbb{G}} = W_t - \sum_{k=0}^{n} \int_{\tau_k \wedge t}^{\tau_{k+1} \wedge t} \frac{a_s^k(\tau^{(k)}, \xi^{(k)})}{\gamma_s^k(\tau^{(k)}, \xi^{(k)})} ds,$$

is a $\mathbb{G}$-Brownian motion. Before introducing the RBSDE in $\mathbb{G}$, we recall Grönwall's inequality (see Theorem A.43 in Schilling and Partzsch [2014]), a very useful result for controlling the norms of semimartingales.

Proposition 5.4.1. *Let $f, g : [0, T] \to \mathbb{R}$ be positive, integrable functions and $\beta \geq 0$ such that for any $t \leq T$,*

$$f(t) \leq g(t) + \beta \int_0^t f(s) ds.$$

Then

$$f(t) \leq g(t) + \beta \int_0^t e^{\beta(t-s)} g(s) ds.$$

Furthermore, if g is non-decreasing then

$$f(t) \leq g(t) e^{\beta t}.$$

The RBSDE in $\mathbb{G}$ is now setup as follows: the following components define the RBSDE and generate the solutions:

- The terminal condition $X \in L^2(\mathcal{G}_T)$.
- The driver $f : \Omega \times [0, T] \to \mathbb{R}$ such that $f(\cdot) \in \mathbb{H}_{\mathbb{G}}^{2,\beta}$.
- The continuous obstacle process $L \in \mathbb{S}_{\mathbb{G}}^{2,\beta}$.

A quadruplet $\left(Y, Z, K, (U^k)_{k\in\{1,2,\dots,n\}}\right)$ solves a RBSDE in $\mathbb{G}$ with terminal condition X, driver f and obstacle L if

$$\begin{aligned} Y_t =& X + \int_t^T f(s)ds + K_T - K_t - \int_t^T Z_s dW_s^{\mathbb{G}} - \sum_{k=1}^n \int_t^T \int_E U_s^k(e)\tilde{\mu}^k(ds, de) \\ Y_t \geq& L_t, \qquad \int_0^T (Y_t - L_t)dK_t = 0, \end{aligned} \tag{5.3}$$

where K is a non-decreasing, $\mathbb{G}$-optional process.
Note the lack of dependence of f on the solutions Y, Z and $(U^k)_{k\in\{1,2,\dots,n\}}$. Section 5.2 was concerned with finding a transfer formula for the Snell envelope in the enlarged filtration to a series of Snell envelopes in the reference filtration. This was possible when we considered the payoff process, R to be $\mathbb{F}$-adapted. One way of maintaining $\mathbb{F}$-adaptation of the stopping process in an RBSDE setup is to ensure that the driver f cannot not depend on Y, Z or U. We therefore restrict our attention to homogeneous drivers of the form $f : \Omega \times [0, T] \to \mathbb{R}$.
Recall for $\beta > 0$, the definitions of $\mathbb{S}^{2,\beta}_{\mathbb{G}}$, $\mathbb{H}^{2,\beta}_{\mathbb{G}}$ and $\mathbb{H}^{2,\beta}_{\mathbb{G},E}$

a) $\mathbb{S}^{2,\beta}_{\mathbb{G}} = \{\phi \in \mathcal{P}(\mathbb{G})$, possibly multi-dimensional : $\|\phi\|_{\mathbb{S}^{2,\beta}_{\mathbb{G}}} = \mathbb{E}\left[\sup_{t\in[0,T]}\left(e^{\beta t}|\phi_t|^2\right)\right] < \infty\}$

b) $\mathbb{H}^{2,\beta}_{\mathbb{G}} = \{\phi \in \mathcal{P}(\mathbb{G})$, possibly multi-dimensional : $\|\phi\|_{\mathbb{H}^{2,\beta}_{\mathbb{G}}} = \mathbb{E}\left[\int_0^T e^{\beta s}|\phi_s|^2 ds\right] < \infty\}$.

c) $\mathbb{H}^{2,\beta}_{\mathbb{G},E} = \{\psi \in \mathcal{P}(\mathbb{G} \otimes \mathcal{B}(E))$, possibly multi-dimensional : $\|\psi\|_{\mathbb{H}^{2,\beta}_{\mathbb{G},E}}$ $= \mathbb{E}\left[\int_0^T \int_E e^{\beta s}\lambda_s(e)|\psi_s(e)|^2 de\ ds\right] < \infty\}$.

5.4.1 Existence and Uniqueness of RBSDE (5.3)

In order to continue the study of RBSDE (5.3), we need to prove that a solution actually exists and if it is unique. The following theorem is similar to Theorem 4.1.2 in chapter 4.

Theorem 5.4.2. *There exists a unique solution quadruplet* $\left(Y, Z, K, (U_k)_{k\in\{1,2,\dots,n\}}\right)$ *to RBSDE* (5.3) *such that* $Y \in \mathbb{S}^{2,\beta}_{\mathbb{G}}$,$Z \in \mathbb{H}^{2,\beta}_{\mathbb{G}}$, *$K$ non-decreasing with* $K_T \in L^2(\mathcal{G}_T)$ *and* $(U^k)_{k\in\{1,2,\dots,n\}} \in \mathbb{H}^{2,\beta}_{\mathbb{G},E}$.

Proof. The proof begins with existence. Let

$$Y_t = \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{G})} \mathbb{E}\left[\int_t^\nu f(s)ds + L_\nu \mathbb{1}_{\{\nu<T\}} + X\mathbb{1}_{\{\nu=T\}}|\mathcal{G}_t\right].$$

then $Y + \int_0^\cdot f(s)ds$ is the $\mathbb{G}$-Snell envelope of the payoff process $R = \int_0^\cdot f(s)ds + L\mathbb{1}_{[0,T)} + X\mathbb{1}_{[T]}$, note by Theorem 5.2.3 and the continuity of L, $Y + \int_0^\cdot f(s)ds$ is a càdlàg $\mathbb{G}$-supermartingale. Moreover

$$|Y_t| \leq \left|\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{t,T}(\mathbb{G})} \mathbb{E}\left[\int_t^\nu f(s)ds + L_\nu \mathbb{1}_{\{\nu<T\}} + X\mathbb{1}_{\{\nu=T\}}|\mathcal{G}_t\right]\right|.$$

The triangle inequality then yields

$$|Y_t| \leq \mathbb{E}\left[|X| + \int_0^T |f(s)|ds + \sup_{0\leq s\leq T} |L_s|\Big|\mathcal{G}_t\right].$$

Noting the integrand on the right hand side does not depend on t, the Cauchy-Schwartz inequality implies the existence of $C \in \mathbb{R}$, such that

$$\mathbb{E}\left[\sup_{0\leq s\leq T} Y_t^2\right] \leq C\mathbb{E}\left[|X|^2 + \int_0^T f^2(s)ds + \sup_{0\leq s\leq T} L_s^2\right] < \infty$$

which proves that $Y + \int_0^{\cdot} f(s)ds$ is a square-integrable $\mathbb{G}$-supermartingale. Therefore by the Doob-Meyer decomposition, there exists a uniformly integrable $\mathbb{G}$-martingale M and an increasing process of finite variation K with $K_0 = 0$ such that

$$Y_t = M_t - \int_0^t f(s)ds - K_t.$$

Theorem 3.2.10 says that there exist $Z \in \mathbb{H}^2_{\mathbb{G}}$ and $(U^k)_{k\in\{1,2,\ldots,n\}} \in \mathbb{H}^2_{\mathbb{G},E}$, such that

$$M_t = \int_0^t Z_s dW_s^{\mathbb{G}} + \sum_{k=1}^n \int_0^t \int_E U_s^k(e)\tilde{\mu}^k(ds, de).$$

Now let $\nu_t^* = \inf\{s \geq t : Y_s = L_s\} \wedge T$, then it is well known (see for example Proposition 5.1 in El Karoui et al. [1997a]),

$$Y_t = \mathbb{E}\left[\int_t^{\nu_t^*} f(s)ds + L_{\nu_t^*}\mathbb{1}_{\{\nu_t^*<T\}} + X\mathbb{1}_{\{\nu_t^*=T\}}|\mathcal{G}_t\right]. \tag{5.4}$$

But

$$Y_t = M_t - \int_0^t f(s)ds - K_t.$$

Writing this in forward notation and taking expectations, yields

$$Y_t = \mathbb{E}\left[Y_{\nu_t^*} + \int_t^{\nu_t^*} f(s)ds + K_{\nu_t^*} - K_t|\mathcal{G}_t\right].$$

Using Equation (5.4), $Y_{\nu^*} = L_{\nu^*}\mathbb{1}_{\{\nu^*<T\}} + X\mathbb{1}_{\{\nu^*=T\}}$, therefore by the Optional sampling theorem

$$Y_t = \mathbb{E}\left[L_{\nu_t^*}\mathbb{1}_{\{\nu_t^*<T\}} + X\mathbb{1}_{\{\nu_t^*=T\}} + \int_t^{\nu_t^*} f(s)ds + K_{\nu_t^*} - K_t|\mathcal{G}_t\right].$$

The first three terms define Y_t, meaning

$$\mathbb{E}\left[K_{\nu_t^*} - K_t|\mathcal{G}_t\right] = 0.$$

K is a $\mathbb{G}$-adapted, non-decreasing process meaning

$$K_{\nu_t^*} = K_t.$$

Seeing as K is non-decreasing and $Y_{\nu^*} = L_{\nu^*}$, this condition is equivalently written as, $\int_0^T (Y_t - L_t) dK_t = 0$. Therefore the quadruplet $\left(Y, Z, K, (U^k)_{k\in\{1,2,\ldots,n\}}\right)$, solve RBSDE (5.3). To prove uniqueness consider two solutions (Y, Z, K, U^k) and $(Y', Z', K', (U^k)')$ from data (X, f, L) and (X', f', L'). Define the following

$$\begin{aligned} &\delta Y = Y - Y', \quad \delta Z = Z - Z', \quad \delta U^k = U^k - (U^k)' \quad \delta K = K - K' \\ &\delta X = X - X', \quad \delta f = f - f', \quad \delta L = L - L'. \end{aligned}$$

Using Itô's formula, we see that

$$\begin{aligned} &e^{\beta t}(\delta Y_t)^2 + \int_t^T e^{\beta s}|\delta Z_s|^2 ds + \sum_{k=1}^n \int_t^T \int_E e^{\beta s} \delta U_s^k(e)^2 \lambda_s^k(e) de ds \\ =& e^{\beta T}\delta X^2 + 2\int_t^T e^{\beta s}\delta Y_{s-}\delta f(s) ds + 2\int_t^T e^{\beta s}\delta Y_{s-} d(\delta K_s) - \int_t^T \beta e^{\beta s}\delta Y_{s-}^2 ds \\ &- 2\int_t^T e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}} - \sum_{k=1}^n \int_t^T \int_E e^{\beta s}\left(2\delta Y_{s-}\delta U_s^k(e) - \delta U_s^k(e)^2\right)\tilde{\mu}^k(ds, de). \end{aligned}$$

Now using the Cauchy-Schwartz inequality, the second term can be split up as follows for $\beta > 0$

$$\begin{aligned} &\leq e^{\beta T}\delta X^2 + \int_t^T \beta e^{\beta s}\delta Y_{s-}^2 ds + \int_t^T \frac{1}{\beta} e^{\beta s}\delta f(s)^2 ds + 2\int_t^T e^{\beta s}\delta Y_{s-} d(\delta K_s) - \int_t^T \beta e^{\beta s}\delta Y_{s-}^2 ds \\ &- 2\int_t^T e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}} - \sum_{k=1}^n \int_t^T \int_E e^{\beta s}\left(2\delta Y_{s-}\delta U_s^k(e) - \delta U_s^k(e)^2\right)\tilde{\mu}^k(ds, de). \end{aligned}$$

The second and and fifth term sum to zero

$$\begin{aligned} =& e^{\beta T}\delta X^2 + \int_t^T \frac{1}{\beta} e^{\beta s}\delta f(s)^2 ds + 2\int_t^T e^{\beta s}\delta Y_{s-} d(\delta K_s) - 2\int_t^T e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}} \\ &- \sum_{k=1}^n \int_t^T \int_E e^{\beta s}\left(2\delta Y_{s-}\delta U_s^k(e) - \delta U_s^k(e)^2\right)\tilde{\mu}^k(ds, de) \end{aligned}$$

The last two terms being increments of martingales have zero expectation, indeed by the Burkholder-Davisy-Gundy inequality (Proposition 4.0.1):

$$\begin{aligned} \mathbb{E}\left[\sup_{t\leq T} \left| \int_0^t e^{\beta s}\delta Y_{s-}\delta Z_s dW_s^{\mathbb{G}}\right|\right] \leq& C\mathbb{E}\left[\left(\int_0^T e^{2\beta s}\left(\delta Y_{s-}\right)^2\left(\delta Z_s\right)^2 ds\right)^{\frac{1}{2}}\right] \\ \leq& C\mathbb{E}\left[\sup_{0\leq s\leq T}\left(e^{\beta s}\left(\delta Y_{s-}\right)^2\right)\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^T e^{\beta s}\left(\delta Z_s\right)^2 ds\right]^{\frac{1}{2}} < \infty \end{aligned}$$

The last inequality follows by from the fact that Y and Y' both belong to $\mathbb{S}^{2,\beta}_{\mathbb{G}}$ and that Z and Z' both belong to $\mathbb{H}^{2,\beta}_{\mathbb{G}}$. This inequality shows that $\int_0^t e^{\beta s}\delta Y_{s-}\delta Z_s dW^{\mathbb{G}}_s$ is a uniformly intergrable martingale. Futhermore,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T}|\sum_{k=1}^{n}\int_t^T\int_E 2e^{\beta s}\delta Y_{s-}\delta U^k_s(e)\tilde{\mu}^k(ds,de)|\right]\\
\le& C\sum_{k=1}^{n}\mathbb{E}\left[\left(\int_0^T\int_E e^{2\beta s}\delta Y^2_{s-}\delta U^k_s(e)^2\mu^k(ds,de)\right)^{\frac{1}{2}}\right]\\
\le& C\sum_{k=1}^{n}\mathbb{E}\left[\sup_{t\le T}\left(e^{\beta t}\delta Y_t\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\int_0^T e^{\beta s}\delta U_s(e)^2\lambda^k_s(e)deds\right]^{\frac{1}{2}}<\infty
\end{aligned}$$

Furthermore, the fact that $(U^k)_{k\in\{1,2,\ldots,n\}}\in\mathbb{H}^{2,\beta}_{\mathbb{G},E}$ implies that $\sum_{k=1}^n\int_t^T\int_E e^{\beta s}\delta U^k_s(e)^2\tilde{\mu}^k(ds,de)$ has zero expectation.
Then it follows that

$$\begin{aligned}
&\mathbb{E}\left[(\delta Y_t)^2+\int_t^T e^{\beta s}|\delta Z_s|^2ds+\sum_{k=1}^{n}\int_t^T\int_E e^{\beta s}\delta U^k_s(e)^2\lambda^k_s(e)deds\right]\\
=&\mathbb{E}\left[e^{\beta T}\delta X^2+\frac{1}{\beta}\int_t^T e^{\beta s}\delta f(s)ds+2\int_t^T e^{\beta s}\delta Y_{s-}d\left(\delta K_s\right)\right].
\end{aligned}$$

The conditions on Y, L and K imply that $\int_0^T\left(\delta Y_t-\delta L_t\right)d\left(\delta K_t\right)\le 0$, meaning

$$\le\mathbb{E}\left[e^{\beta T}\delta X^2+\frac{1}{\beta}\int_t^T e^{\beta s}\delta Y_{s-}\delta f(s)ds+2\int_t^T e^{\beta s}\delta L_s d\left(\delta K_s\right)\right].$$

The estimates of the norms of Z and $(U^k)_{k\in\{1,2,\ldots,n\}}$ then follow

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^{\beta s}|\delta Z_s|^2ds\right]\le&\mathbb{E}\left[e^{\beta T}\delta X^2+\frac{1}{\beta}\int_0^T e^{\beta s}\delta f(s)^2ds+2\int_0^T e^{\beta s}\delta L_s d\left(\delta K_s\right)\right]\\
\mathbb{E}\left[\sum_{k=1}^{n}\int_0^T\int_E\delta U^k_s(e)^2\lambda_s(e)deds\right]\le&\mathbb{E}\left[e^{\beta T}\delta X^2+\frac{1}{\beta}\int_0^T e^{\beta s}\delta f(s)^2ds+2\int_0^T e^{\beta s}\delta L_s d\left(\delta K_s\right)\right].
\end{aligned}\tag{5.5}$$

Again using Itô's Lemma, we write $e^{\beta T}\delta K_T$ as

$$\begin{aligned}
e^{\beta T}\delta K_T=&\delta Y_0-e^{\beta T}\delta X-\int_0^T e^{\beta s}\delta f(s)ds+\int_0^T e^{\beta s}\delta Z_s dW^{\mathbb{G}}_s+\sum_{k=1}^{n}\int_0^T\int_E e^{\beta s}\delta U^k_s(e)\tilde{\mu}^k(de,ds)\\
&+\beta\int_0^T e^{\beta s}\delta K_s ds.
\end{aligned}$$

Taking expectations and applying the Cauchy-Schwartz inequality again, we get

$$\begin{aligned}\mathbb{E}[e^{\beta T}\delta K_T^2] \leq & C\mathbb{E}\Bigg[\delta Y_0^2 + e^{\beta T}\delta X^2 + \int_0^T e^{\beta s}\delta f(s)^2 ds + \int_0^T e^{\beta s}|Z_s|^2 ds \\ & + \sum_{k=1}^n \int_0^T \int_E e^{\beta s}\delta U_s^k(e)^2 \lambda_s^k(e) de ds + \int_0^T e^{\beta s}\delta K_s^2 ds\Bigg].\end{aligned}$$

Combining the inequalities in Equation (5.5) we get

$$\mathbb{E}\left[\delta K_T^2\right] \leq C\mathbb{E}\left[e^{\beta T}\delta X^2 + \int_0^T e^{\beta s}\delta f(s)^2 ds + 2\int_0^T e^{\beta s}\delta L_s d\left(\delta K_s\right) + \int_0^T e^{\beta s}\delta K_s^2 ds\right].$$

Using the Cauchy-Schwartz inequality again coupled with the fact that K is non-decreasing

$$\begin{aligned}\mathbb{E}\left[\delta K_T^2\right] \leq & C\mathbb{E}\left[e^{\beta T}\delta X^2 + \int_0^T e^{\beta s}\delta f(s)^2 ds + \int_0^T e^{\beta s}\delta K_s^2 ds\right] + 2C^2\mathbb{E}\left[\sup_{s\leq T}\left(e^{\beta s}\delta L_s\right)^2\right] \\ & + \frac{1}{2}\mathbb{E}\left[e^{\beta s}\delta K_T^2\right].\end{aligned}$$

The last term can be absorbed in to the left hand side as follows

$$\mathbb{E}\left[\delta K_T^2\right] \leq C\mathbb{E}\left[e^{\beta T}\delta X^2 + \int_0^T e^{\beta s}\delta f(s)^2 ds + \sup_{s\leq T}\left(e^{\beta s}\delta L_s\right)^2 + \int_0^T e^{\beta s}\delta K_s^2 ds\right].$$

Now Grönwall's inequality is applied to $\mathbb{E}[\delta K^2]$, i.e. from Proposition 5.4.1 we set $f(T) = \mathbb{E}[\delta K_T^2]$ and $g(T) = \mathbb{E}\left[e^{\beta T}\delta X^2 + \int_0^T e^{\beta s}\delta f(s)^2 ds + \sup_{s\leq T}\left(e^{\beta s}\delta L_s\right)^2\right]$, which then implies

$$\mathbb{E}\left[\delta K_T^2\right] \leq C\mathbb{E}\left[e^{\beta T}\delta X^2 + \int_0^T e^{\beta s}\delta f(s)^2 ds + \sup_{s\leq T}\left(e^{\beta s}\delta L_s\right)^2\right]$$

Combining the estimates for $Y, Z, (U_k)_{k\in\{1,2,\ldots,n\}}$ and K, we get

$$\begin{aligned}&\mathbb{E}\left[e^{\beta t}\left(\delta Y_t\right)^2 + \int_0^T e^{\beta s}|\delta Z_s|^2 ds + \sum_{k=1}^n \int_0^T \int_E e^{\beta s}\delta U_s^k(e)^2 \lambda_s(e) de ds + e^{\beta T}K_T^2\right] \\ \leq & C\mathbb{E}\left[\delta X^2 + \int_0^T \delta f(s)^2 ds + \sup_{s\leq T}\left(\delta L_s^2\right)\right].\end{aligned}$$

Integrating both sides with respect to t, we get

$$\|\delta Y\|_{\mathbb{H}_{\mathbb{G}}^{2,\beta}} + \|\delta Z\|_{\mathbb{H}_{\mathbb{G}}^{2,\beta}} + \sum_{k=1}^n \|\delta U^k\|_{\mathbb{H}_{\mathbb{G},E}^{2,\beta}} + \|\delta K_T\|_{L^2(\mathcal{G}_T)} \leq C\left(\|\delta X\|_{L^2(\mathcal{G}_T)} + \|\delta f(\cdot)\|_{\mathbb{H}_{\mathbb{G}}^{2,\beta}} + \|\delta L\|_{\mathbb{S}_{\mathbb{G}}^{2,\beta}}\right)$$

Therefore for $X' = X$, $f' = f$ and $L' = L$, $\left(Y, Z, K, (U^k)_{k\in\{1,2,\ldots,n\}}\right)$ is a unique solution. Following a similar reasoning as that of Lemma 4.1.1, Y is unique in $\mathbb{S}_{\mathbb{G}}^{2,\beta}$. □

5.4.2 Decomposition of Solutions

Section 5.2 was devoted to finding a transfer formula for the value process of an optimal stopping problem. We introduce a similar narrative in this section. In what follows, it will be useful for the reader to assume that solutions to RBSDEs can be explicitly computed in the reference filtration however with the progressive enlargement of τ and ξ, solutions are no longer able to be computed explicitly. While we don't assume this, it is useful for the interpretation of the following result. Recall the definition of the family of processes $\left(\gamma^k : k \in \{0, 1, ..., n\}\right)$ from equation (3.2) and their decomposition and dynamics from equation (4.2).

Theorem 5.4.3. *Suppose the following RBSDE in $\mathbb{F}$ admits a solution $(Y^n(u,e), Z^n(u,e), K^n(u,e))$ for all $(u,e) \in \Theta_n \times E^n$*

$$
\begin{aligned}
Y^n_t(u,e) =& X\alpha_T(u,e) + \int_t^T f(s)\alpha_s(u,e)ds + K^n_T(u,e) - K^n_t(u,e) - \int_t^T Z^n_s(u,e)dW_s \\
Y^n_t(u,e) \geq& L_t\alpha_t(u,e), \qquad \int_0^T \left(Y^n_t(u,e) - L_t\alpha_t(u,e)\right) dK^n_t(u,e) = 0,
\end{aligned}
\tag{5.6}
$$

and for $k \in \{0, 1, ..., n-1\}$, the following RBSDEs in $\mathbb{F}$ admit solutions $(Y^k(u^{(k)}, e^{(k)}), Z^k(u^{(k)}, e^{(k)}), K^k(u^{(k)}, e^{(k)}))$ for all $(u^{(k)}, e^{(k)}) \in \Theta_k \times E^k$

$$
\begin{aligned}
Y^k_t(u^{(k)}, e^{(k)}) =& X\gamma^k_T(u^{(k)}, e^{(k)}) + \int_t^T f(s)\gamma^k_s(u^{(k)}, e^{(k)})ds + K^k_T(u^{(k)}, e^{(k)}) - K^k_t(u^{(k)}, e^{(k)}) \\
&- \int_t^T Z^k_s(u^{(k)}, e^{(k)})dW_s + \int_t^T \int_E Y^{k+1}_s(u^{(k)}, s, e^{(k+1)})de_{k+1}ds \\
Y^k_t(u^{(k)}, e^{(k)}) \geq& L_t\gamma^k_t(u^{(k)}, e^{(k)}), \qquad \int_0^T \left(Y^k_t(u^{(k)}, e^{(k)}) - L_t\gamma^k_t(u^{(k)}, e^{(k)})\right) dK^k_t(u^{(k)}, e^{(k)}) = 0.
\end{aligned}
\tag{5.7}
$$

Then

$$
\begin{aligned}
Y_t &= \sum_{k=0}^n \frac{Y^k_t(\tau^{(k)}, \xi^{(k)})}{\gamma^k_t(\tau^{(k)}, \xi^{(k)})} Y^k_t(\tau^{(k)}, \xi^{(k)}) \mathbb{1}_{\{\tau_k \leq t < \tau_{k+1}\}} \\
Z_t &= \sum_{k=0}^n \frac{Z^k_t(\tau^{(k)}, \xi^{(k)})\gamma^k_t(\tau^{(k)}, \xi^{(k)}) - a^k_t(\tau^{(k)}, \xi^{(k)})Y^k_t(\tau^{(k)}, \xi^{(k)})}{\gamma^k_t(\tau^{(k)}, \xi^{(k)})^2} \mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}} \\
K_t &= \sum_{k=0}^n \int_0^t \mathbb{1}_{\{\tau_k < s \leq \tau_{k+1}\}} \frac{dK^k_s(\tau^{(k)}, \xi^{(k)})}{\gamma^k_s(\tau^{(k)}, \xi^{(k)})} \\
U^k_t(e) &= \left(Y^{k+1}_t(\tau^{(k)}, t, \xi^{(k)}, e) - Y^k_t(\tau^{(k)}, \xi^{(k)})\right) \mathbb{1}_{\{\tau_{k-1} \leq t < \tau_k\}}
\end{aligned}
\tag{5.8}
$$

is a solution to RBSDE (5.3).

Proof. We show that $\left(Y, Z, K, \left(U^k\right)_{k \in \{1,2,...,n\}}\right)$ from equation (5.8) solve RBSDE (5.3). Similarly to the proof of Theorems 4.3.1 and 4.3.2 this is done in two parts, for $k = n$ and $k \in \{0, 1, ..., n-1\}$.

For $k = n$:

$$d\left(\frac{Y^n_t(\tau,\xi)H^n_t}{\alpha_t(\tau,\xi)}\right) = \frac{Y_t(\tau,\xi)}{\alpha_t(\tau,\xi)}dH^n_t + H^n_{t-}\Bigg(\frac{1}{\alpha_t(\tau,\xi)}dY^n_t(\tau,\xi) - \frac{Y^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)^2}d\alpha_t(\tau,\xi)$$
$$- \frac{1}{\alpha_t(\tau,\xi)^2}d\langle\alpha(u,e),Y^n(u,e)\rangle|_{\substack{u=\tau\\e=\xi}} + \frac{Y^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)^3}d\langle\alpha(u,e),\alpha(u,e)\rangle|_{\substack{u=\tau\\e=\xi}}\Bigg).$$

From RBSDE (5.6), the dynamics of $Y^n(\tau,\xi)$ are

$$dY^n_t(\tau,\xi) = -f(t)dt + Z^n_t(\tau,\xi)dW_t - dK^n_t(\tau,\xi)$$

this coupled with the dynamics of α yield

$$d\langle Y^n(u,e),\alpha(u,e)\rangle_t = Z^n_t(u,e)\alpha_t(u,e)dt$$
$$d\langle\alpha(u,e),\alpha(u,e)\rangle_t = a^n_t(u,e)^2dt$$

Combining these quantities, we get the following

$$d\left(\frac{Y^n_t(\tau,\xi)H^n_t}{\alpha_t(\tau,\xi)}\right) = \frac{Y_t(\tau,\xi)}{\alpha_t(\tau,\xi)}dH^n_t + H^n_t\Bigg(-f(t)dt - \frac{dK^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)} + \frac{Z^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)}dW_t$$
$$- \frac{Y^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)^2}a^n_t(\tau,\xi)dW_t - \frac{a^n_t(\tau,\xi)Z^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)^2}dt + \frac{Y^n_t(\tau,\xi)a^n_t(\tau,\xi)^2}{\alpha_t(\tau,\xi)^3}dt\Bigg)$$
$$= \frac{Y_t(\tau,\xi)}{\alpha_t(\tau,\xi)}dH^n_t + H^n_{t-}\Bigg(-f(t)dt - \frac{dK^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)} + \frac{Z^n_t(\tau,\xi)\alpha_t(\tau,\xi) - Y^n_t(\tau,\xi)a^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)^2}$$
$$\left(dW_t - \frac{a^n_t(\tau,\xi)}{\alpha_t(\tau,\xi)}dt\right)\Bigg).$$

Noting that $H^n_t dW^{\mathbb{G}}_t = H^n_t\left(dW_t - \frac{a_t(\tau,\xi)}{\alpha_t(\tau,\xi)}\right)dt$. Just as in Theorems 4.3.1 and 4.3.2 integrating with respect to H^n may be expanded to a double integral with respect to μ^n. Using these two facts and integrating between t and T

$$\frac{Y^n_t(\tau,\xi)H^n_t}{\alpha_t(\tau,\xi)} = XH^n_T + \int_t^T H^n_{s-}f(s)ds + \int_t^T H^n_{s-}\frac{dK_s(\tau^{(k)},\xi^{(k)})}{\alpha_s(\tau^{(k)},\xi^{(k)})}$$
$$- \int_t^T H^n_{s-}\frac{Z^n_s(\tau,\xi)\alpha_s(\tau,\xi) - Y^n_s(\tau,\xi)a^n_s(\tau,\xi)}{\alpha_s(\tau,\xi)^2}dW^{\mathbb{G}}_s$$
$$- \int_t^T\int_E \frac{Y^n_s(\tau^{(n-1)},s,\xi^{(n-1)}),e)}{\alpha_s(\tau^{(n-1)},s,\xi^{(n-1)},e)}\mu^n(ds,de).$$

Finally, using the compensator of μ^n from Corollary 3.2.9.1 in Chapter 3.

$$\frac{Y^n_t(\tau,\xi)H^n_t}{\alpha_t(\tau,\xi)} = XH^n_T + \int_t^T H^n_{s-}f(s)ds + \int_t^T H^n_{s-}\frac{dK_s(\tau^{(k)},\xi^{(k)})}{\alpha_s(\tau^{(k)},\xi^{(k)})}$$
$$- \int_t^T H^n_{s-}\frac{Z^n_s(\tau,\xi)\alpha_s(\tau,\xi) - Y^n_s(\tau,\xi)a^n_s(\tau,\xi)}{\alpha_s(\tau,\xi)^2}dW^{\mathbb{G}}_s$$
$$- \int_t^T\int_E \frac{Y^n_s(\tau^{(n-1)},s,\xi^{(n-1)}),e)}{\alpha_s(\tau^{(n-1)},s,\xi^{(n-1)},e)}\tilde{\mu}^n(ds,de)$$
$$- \int_t^T\int_E \frac{Y^n_s(\tau^{(n-1)},s,\xi^{(n-1)}),e)}{\gamma^{n-1}_s(\tau^{(n-1)},s,\xi^{(n-1)},e)}(H^{n-1}_{s-} - H^n_{s-})deds.$$

Similarly for $k \in \{0, 1, ..., n-1\}$

$$
\begin{aligned}
&d\left(\frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}(H_t^k - H_t^{k+1})\right) \\
=&\frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}(dH_t^k - dH_t^{k+1}) + (H_{t-}^k - H_{t-}^{k+1})\Bigg(\frac{1}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}dY_t^k(\tau^{(k)}, \xi^{(k)}) \\
&- \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}d\gamma_t^k(\tau^{(k)}, \xi^{(k)}) - \frac{1}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}d\langle\gamma^k(u^{(k)}, e^{(k)}), Y^k(u^{(k)}, e^{(k)})\rangle_t\Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}} \\
&+ \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^3}d\langle\gamma^k(u^{(k)}, e^{(k)}), \gamma^k(u^{(k)}, e^{(k)})\rangle_t\Big|_{\substack{u^{(k)}=\tau^{(k)} \\ e^{(k)}=\xi^{(k)}}}\Bigg).
\end{aligned}
$$

Equations 5.7 and 4.1 from Chapter 4 give the dynamics of $\gamma^k(u^{(k)}, e^{(k)})$ and $Y^k(u^{(k)}, e^{(k)})$. These are then use to expand the above as follows:

$$
\begin{aligned}
&d\left(\frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}(H_t^k - H_t^{k+1})\right) \\
=&\frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}(dH_t^k - dH_t^{k+1}) + (H_{t-}^k - H_{t-}^{k+1})\Bigg(- f(t)dt - \frac{dK_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})} + \frac{Z_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}dW_t \\
&- \frac{\int_E Y_t^{k+1}(\tau^{(k)}, t, \xi^{(k)}, e)de}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}dt - \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}a_t^k(\tau^{(k)}, \xi^{(k)})dW_t \\
&- \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})\int_E \gamma_t^{k+1}(\tau^{(k)}, t, \xi^{(k)}, e)de}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}dt - \frac{Z_t^k(\tau^{(k)}, \xi^{(k)})a_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}dt \\
&+ \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})a_t^k(\tau^{(k)}, \xi^{(k)})^2}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^3}dt\Bigg) \\
=&\frac{Y_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}(dH_t^k - dH_t^{k+1}) + (H_t^k - H_t^{k+1})\Bigg(- f(t)dt - \frac{dK_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})} \\
&+ \frac{Z_t^k(\tau^{(k)}, \xi^{(k)})\gamma_t^k(\tau^{(k)}, \xi^{(k)}) - Y_t^k(\tau^{(k)}, \xi^{(k)})a_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}\Bigg(dW_t \\
&- \frac{a_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}dt\Bigg) + \frac{Y_t^k(\tau^{(k)}, \xi^{(k)})\int_E \gamma_t^{k+1}(\tau^{(k)}, t, \xi^{(k)}, e)de}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})^2}dt - \frac{\int_E Y_t^{k+1}(\tau^{(k)}, t, \xi^{(k)}, e)dedt}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}\Bigg).
\end{aligned}
$$

Just as in the proof of Theorems 4.3.1 and 4.3.2, we identify $(H_t^k - H_t^{k+1})\left(dW_t - \frac{a_t^k(\tau^{(k)}, \xi^{(k)})}{\gamma_t^k(\tau^{(k)}, \xi^{(k)})}dt\right) = (H_t^k - H_t^{k+1})dW_t^{\mathbb{G}}$. The integrals with respect to H^k and H^{k+1} are also expanded as integrals with respect to μ^k and μ^{k+1}. Now, integrating both sides between t and T yields

$$
\begin{aligned}
&\frac{Y_t^k(\tau^{(k)},\xi^{(k)})}{\gamma_t^k(\tau^{(k)},\xi^{(k)})}(H_t^k - H_t^{k+1}) \\
=&X\left(H_T^k - H_T^{k+1}\right) + \int_t^T f(s)\left(H_{s-}^k - H_{s-}^{k+1}\right)ds \\
&+\int_t^T \left(H_{s-}^k - H_{s-}^{k+1}\right)\frac{dK_s^k(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})} \\
&-\int_t^T \frac{Z_s^k(\tau^{(k)},\xi^{(k)})\gamma_s^k(\tau^{(k)},\xi^{(k)}) - Y_s^k(\tau^{(k)},\xi^{(k)})a_s^k(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})^2}dW_s^{\mathbb{G}} \\
&+\int_t^T\int_E \frac{Y_s^k(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})}\mu^{k+1}(ds,de) - \int_t^T\int_E \frac{Y_s^k(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma_s^k(\tau^{(k-1)},s,\xi^{(k-1)},e)}\mu^k(ds,de) \\
&+\int_t^T\int_E \left(H_{s-}^k - H_{s-}^{k+1}\right)\frac{Y_s^{k+1}(\tau^{(k)},s,\xi^{(k)},e)}{\gamma_s^k(\tau^{(k)},\xi^{(k)})}deds \\
&-\int_t^T\int_E \left(H_{s-}^{k-1} - H_{s-}^{k}\right)\frac{Y_s^k(\tau^{(k)},\xi^{(k)})\gamma_s^{k+1}(\tau^{(k)},s,\xi^{(k)},e)}{\gamma_s^k(\tau^{(k)},\xi^{(k)})^2}deds
\end{aligned}
$$

Finally, the compensators of μ^k and μ^{k+1} from Corollary 3.2.9.1 in Chpater 3 are used to yield

$$
\begin{aligned}
&\frac{Y_t^k(\tau^{(k)},\xi^{(k)})}{\gamma_t^k(\tau^{(k)},\xi^{(k)})}(H_t^k - H_t^{k+1}) \\
=&X\left(H_T^k - H_T^{k+1}\right) + \int_t^T f(s)\left(H_{s-}^k - H_{s-}^{k+1}\right)ds \\
&+\int_t^T \left(H_{s-}^k - H_{s-}^{k+1}\right)\frac{dK_s^k(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})} \\
&-\int_t^T \frac{Z_s^k(\tau^{(k)},\xi^{(k)})\gamma_s^k(\tau^{(k)},\xi^{(k)}) - Y_s^k(\tau^{(k)},\xi^{(k)})a_s^k(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})^2}dW_s^{\mathbb{G}} \\
&+\int_t^T\int_E \frac{Y_s^k(\tau^{(k)},\xi^{(k)})}{\gamma_s^k(\tau^{(k)},\xi^{(k)})}\tilde{\mu}^{k+1}(ds,de) - \int_t^T\int_E \frac{Y_s^k(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma_s^k(\tau^{(k-1)},s,\xi^{(k-1)},e)}\tilde{\mu}^k(ds,de) \\
&+\int_t^T\int_E \left(H_{s-}^k - H_{s-}^{k+1}\right)\frac{Y_s^{k+1}(\tau^{(k)},s,\xi^{(k)},e)}{\gamma_s^k(\tau^{(k)},\xi^{(k)})}deds \\
&-\int_t^T\int_E \left(H_{s-}^{k-1} - H_{s-}^{k}\right)\frac{Y_s^k(\tau^{(k-1)},s,\xi^{(k-1)},e)}{\gamma_s^{k-1}(\tau^{(k-1)},\xi^{(k-1)})}deds
\end{aligned}
$$

Summing over all $k \in \{0, 1, ..., n\}$ then yields the result. □

5.5 Remarks

This chapter has studied optimal stopping problems in the enlarged filtration $\mathbb{G}$. For a given $\mathbb{F}$-adapted payoff process, the $\mathbb{G}$-Snell envelope was shown to be a series of $\mathbb{F}$-Snell envelope-like processes. Importantly, the $\mathbb{F}$-adapted terms in the transfer formula in theorem 5.2.3 are not Snell envelopes become of the dependence of t inside the expectation. It is left as an open problem as to whether theorem 5.4.3 can be applied to show a similar transfer formula for the value process Y. Lastly, the emphasis that the payoff process R must be $\mathbb{F}$-adapted ensures the existence of the transfer formula. Without this assumption, theorem 5.2.3 would not hold.

Chapter 6

Conclusion

This thesis extended the existing theory of enlargement of filtration, to the case of a progressive enlargement of filtration with multiple default times and their associated marks.
The thesis begins with an overview of the existing results and theory on the one default initial and progressive enlargement of filtration. Thereafter, chapter 3 extends these results to the case of multiple default times and their associated marks. The main contribution of this chapter is a martingale representation theorem in the enlarged filtration. This is proved in theorem 3.2.10.
Chapter 4 and 5 then aim to apply the newly derived results to two particular cases used in mathematical finance - backward stochastic differential equations (BSDEs) and optimal stopping problems.
In chapter 4, the existence and uniqueness of solutions to a BSDE in the enlarged filtration are proven - in theorem 4.1.2. These solutions are then shown to admit two decompositions in the enlarged filtration, depending on which filtration the data is adapted to. The chapter finishes with two brief examples of how the solutions to the BSDE can be applied in a mathematical finance context.
Chapter 5 focuses on a general optimal stopping problem with a payoff process adapted to the reference filtration. In theorem 5.2.3 it is shown that the Snell envelope in the enlarged filtration can be decomposed into a series of optimal stopping problems in the reference filtration. The utility of this result is shown by solving a well studied problem, that being the optimal stopping of a Brownian bridge. Using enlargement of filtration, the solution is derived and agrees with previously proven solutions. The chapter is finished by studying a reflected BSDE in the enlarged filtration. The existence and uniqueness to the solution of the reflected BSDE is proven in theorem 5.4.2. Finally, a decomposition of the solution is provided in 5.4.3, and relating this decomposition to theorem 5.2.3 is left as an open problem.

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Appendix 1

For the sake of simplicity, we omit the superscripts of $X^{\pi,x}$ and $Y^{\mathbb{G}}$. So that

$$R_t^{\pi,x} = -\exp\left(\eta(X_t - Y_t)\right).$$

Then to find the driver $f^{\mathbb{G}}$, we need to compute the drift of $R^{\pi,x}$ and choose $f^{\mathbb{G}}$ such that the drift is negative. This is done as follows:

$$\begin{aligned}
-dR_t^{\pi,x} =& -\eta R_{t-}^{\pi,x}dX_t + \eta R_{t-}^{\pi,x}dY_t + \frac{1}{2}\eta^2 R_{t-}^{\pi,x}d[X,X]_t + \frac{1}{2}\eta^2 R_{t-}^{\pi,x}d[Y,Y]_t - \eta^2 R_{t-}^{\pi,x}d[X,Y]_t \\
=& R_{t-}^{\pi,x}\Bigg(-\eta\left(\pi_t(b_t + \phi_t\sigma_t) + \pi_t\sigma_t dW_t^{\mathbb{G}}\right) + \eta\Bigg(-f(t, Z_t^{\mathbb{G}}, U_t)dt + Z_t^{\mathbb{G}}dW_t^{\mathbb{G}} \\
&+ \sum_{k=1}^{n}\int_E U_t^k(e)\tilde{\mu}^k(dt,de)\Bigg) + \frac{1}{2}\eta^2\pi_t^2\sigma_t^2 dt \\
&+ \frac{1}{2}\eta^2 Z^{\mathbb{G}2}_t + \frac{1}{2}\eta^2\sum_{k=1}^{n}U_t^k(e)^2\mu^k(dt,de) - \eta^2\sigma_t\pi_t Z_t^{\mathbb{G}}dt\Bigg) \\
=& R_{t-}^{\pi,x}\Bigg(-\eta\pi_t(b_t + \phi_t\sigma_t) - \eta f(t, Z_t^{\mathbb{G}}, U_t) + \frac{1}{2}\eta^2\pi_t^2\sigma_t^2 + \frac{1}{2}\eta^2 Z_t^{\mathbb{G}2} - \frac{1}{2}\eta^2\sum_{k=1}^{n}\int_E U_t^k(e)^2\lambda_t(e)de \\
&- \eta^2\sigma_t\pi_t Z_t^{\mathbb{G}}\Bigg)dt + R_{t-}^{\pi,x}\Bigg(\eta\left(Z_t^{\mathbb{G}} - \pi_t\sigma_t\right)dW_t^{\mathbb{G}} + \eta\sum_{k=1}^{n}\int_E\left(U_t^k(e) + \frac{1}{2}\eta U_t^k(e)^2\right)\tilde{\mu}^k(dt,de)\Bigg).
\end{aligned}$$

We set the drift term to be negative to yield

$$\begin{aligned}
&R_{t-}^{\pi,x}\Bigg(-\eta\pi_t(b_t + \phi_t\sigma_t) - \eta f(t, Z_t^{\mathbb{G}}, U_t) + \frac{1}{2}\eta^2\pi_t^2\sigma_t^2 + \frac{1}{2}\eta^2 Z_t^{\mathbb{G}2} - \frac{1}{2}\eta^2\sum_{k=1}^{n}\int_E U_t^k(e)^2\lambda_t(e)de \\
&- \eta^2\sigma_t\pi_t Z_t^{\mathbb{G}}\Bigg) \leq 0.
\end{aligned}$$

$R^{\pi,x}$ is negative meaning the bracketed factor must be positive. This implies

$$\begin{aligned}
\eta f(t, Z_t^{\mathbb{G}}, U_t) \leq& -\eta\phi_t(b_t + \phi_t\sigma_t) + \frac{1}{2}\eta^2\pi_t^2 Z_t^{\mathbb{G}2} - \frac{1}{2}\eta^2\sum_{k=1}^{n}\int_E U_t^k(e)^2\lambda_t(e)de - \eta^2\sigma_t\pi_t Z_t^{\mathbb{G}} \\
=& \frac{1}{2}\eta^2\sigma_t^2\pi_t^2 - 2\eta\pi_t\left(b_t + \phi_t\sigma_t + \eta\sigma_t Z_t^{\mathbb{G}}\right) + \frac{1}{2}\eta^2 Z_t^{\mathbb{G}2} - \frac{1}{2}\eta^2\sum_{k=1}^{n}\int_E U_t^k(e)^2\lambda_t(e)de \\
=& \frac{1}{2}\left(\eta\pi_t\sigma_t - \left(\frac{b_t}{\sigma_t} + \phi_t + \eta Z_t^{\mathbb{G}}\right)\right)^2 - \frac{1}{2}\eta^2\sum_{k=1}^{n}\int_E U_t^k(e)^2\lambda_t^k(e)de - \frac{1}{2}\left(\frac{b_t}{\sigma_t} + \phi_t\right)^2 \\
&- \eta\frac{b_t}{\sigma_t}Z_t^{\mathbb{G}} + \eta\phi_t Z_t^{\mathbb{G}}
\end{aligned}$$

Therefore we choose

$$\begin{aligned}
f^{\mathbb{G}}(t,z,u) =& \frac{\eta}{2}\inf_{\pi\in\mathcal{A}^{\mathbb{G}}}\left(\pi_t\sigma_t - \left(\frac{b_t}{\eta\sigma_t} + \frac{\phi_t}{\eta} + Z_t^{\mathbb{G}}\right)\right)^2 - \frac{\eta}{2}\sum_{k=1}^{n}\int_E U_t^k(e)^2\lambda_t^k(e)de \\
&- \frac{1}{2\eta}\left(\frac{b_t}{\sigma_t} + \phi_t\right)^2 - \frac{b_t Z_t^{\mathbb{G}}}{\sigma_t} - \phi_t Z_t^{\mathbb{G}}.
\end{aligned}$$

Appendix 2

For a normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$\begin{aligned}\mathbb{E}\left[X \exp\left(-\frac{X^2}{2}\right)\right] &= \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(\sigma^2 x + x^2 - 2x\mu + \mu^2\right)\right) dx\end{aligned}$$

The inner bracket inside the exponent is computed as follows:

$$\begin{aligned}\sigma^2 x + x^2 - 2x\mu + \mu^2 &= (1+\sigma^2)\left(x^2 - 2\frac{\mu}{1+\sigma^2}x\right) + \mu^2 \\ &= (1+\sigma^2)\left(x - \frac{\mu}{1+\sigma^2}\right)^2 - \frac{\mu^2}{1+\sigma^2} + \mu^2 \\ &= (1+\sigma^2)\left(x - \frac{\mu}{1+\sigma^2}\right)^2 + \frac{\sigma^2\mu^2}{1+\sigma^2}\end{aligned}$$

Putting this expression in to the integral above

$$\begin{aligned}\mathbb{E}\left[X \exp\left(-\frac{X^2}{2}\right)\right] &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1+\sigma^2}{2\sigma^2}\left(x - \frac{\mu}{1+\sigma^2}\right)^2\right) \exp\left(-\frac{\mu^2}{2(1+\sigma^2)}\right) dx \\ &= \frac{1}{\sqrt{1+\sigma^2}} \exp\left(-\frac{\mu^2}{2(1+\sigma^2)}\right) \int_{-\infty}^{\infty} \frac{x\sqrt{1+\sigma^2}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1+\sigma^2}{2\sigma^2}\left(x - \frac{\mu}{1+\sigma^2}\right)^2\right) dx \\ &= \frac{1}{\sqrt{1+\sigma^2}} \exp\left(-\frac{\mu^2}{2(1+\sigma^2)}\right) \frac{\mu}{1+\sigma^2}\end{aligned}$$

Recall $Y_t = \frac{W_t - u}{\sqrt{T-t}}$ so for $t \leq s < T$, $Y_s|\mathcal{F}_t$ is normally distributed with mean $\frac{W_t - u}{\sqrt{T-s}}$ and variance $\frac{s-t}{T-s}$. Therefore,

$$\begin{aligned}\mathbb{E}\left[Y_s \exp\left(-\frac{Y_s^2}{2}\right)|\mathcal{F}_t\right] &= \frac{\sqrt{T-s}}{\sqrt{T-t}} \exp\left(-\frac{1}{2}\frac{(W_t-u)^2}{T-s}\frac{T-s}{T-t}\right) \frac{W_t-u}{\sqrt{T-s}}\frac{T-s}{T-t} \\ &= Y_t \exp\left(-\frac{Y_t^2}{2}\right)\frac{T-s}{T-t} \leq Y_t \exp\left(-\frac{Y_t^2}{2}\right)\end{aligned}$$

Since $\frac{T-s}{T-t} \leq 1$. Therefore $\left(Y_t \exp\left(-\frac{Y_t^2}{2}\right)\right)_{t<T}$ is a supermartingale.