# Études on Fuzzy Geometry and Cosmology

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Thesis submitted for the degree of Doctor of Philosophy
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*To my parents, who gave me life*
*and Amanda who fills it with warmth!*

## Abstract

We investigate various aspects of noncommutative geometry and fuzzy field theory and their relations to string theory. In particular, we study the BPS and non-BPS solutions of the $\mathbb{CP}^N$ nonlinear sigma model on the noncommutative plane in some detail and show among other things that a class of its solitonic excitations may be built from bound states of noncommutative scalar solitons. We then go on to construct a fuzzy extension of the semilocal $SU(N)_G \times U(1)_L$ Yang-Mills-Higgs model. We find that not only does this noncommutative model support a large class of BPS vortex solutions but, unlike in the commutative model, these are exact solutions of the BPS equations. we also study the large coupling limit of the semilocal model and demonstrate conclusively the metamorphosis of the semilocal vortex to an appropriate degree instanton of the fuzzy $\mathbb{CP}^N$ model.

In the second part of this work, we study the perpendicular intersection of $D1-$ and $D7-$branes in type IIB string theory and the fuzzy $6-$sphere that resolves the singularity of the intersection. We demonstrate the equivalence of the $D7$ and dual $D-$string descriptions by computing the energy, charge and radial profiles of the solution in each description.

We conclude the thesis with a foray into cosmology by constructing a realisation of a recently proposed singularity-free inflating universe. We discuss the basic characteristics of this model and show that none are at odds with current observations.

## Acknowledgements

Throughout this often turbulent period in my life, my parents and sister have been pillars of support - for that I am eternally grateful. It is also with deep gratitude that I acknowledge my advisor, George Ellis. More than just a great mentor and teacher, he has been a good friend - being patient with me when progress was slow and driving me on with his seemingly boundless energy when the results were coming in thick and fast. I thank him also for the freedom that he afforded me in my choice of research problems. Consequently, I have found my years as his graduate student immensely rewarding.

I owe a great dept of gratitude to my collaborators; Rory Adams, Paul Cook, George Ellis, Antony Millner, Christos Tsagas, Tony Rothman and especially Robert de Mello Koch. It has been amazing working with such smart people (and, in Robert's case, who dress so well too). Robert's faith in me and his contagious enthusiasm has been nothing short of inspirational - one day, when I grow up, I want to be able to think about physics the way he does.

A large part of the research for this thesis was carried out at the Mathematical Institute, Oxford. From my advisor there, Philip Candelas, I have learned a great deal about being a good scientist. For this, his unwavering support and sage advice, I thank him. To my friends there I owe much. In particular, my "boring married friends", Bogdan and Corina Florea made even the most dull of Oxonian days enjoyable. Olivier, my house mate educated me about the nuances of French cinema and Robert and Marcelle, about good friends, good wine and punting on the Thames.

Arguably, one of the best things about academia (apart from the fact that I don't ever have to wear a tie to work) is the ability to travel. Especially from South Africa, being able to meet and interact with colleagues in different institutions has been invaluable. I have enjoyed many fruitful discussions and warm hospitality at Columbia University; the Institute for Theoretical Physics at Santa Barbara; DAMTP, Cambridge; Imperial College; the University of the Witwatersrand and the Stellenbosch Institute for Advanced Study. I thank Richard Easther, Dan Kabat, Brian Greene, Clifford Johnson, Neil Turok, Bernard Leong, Stefon Alexander, Joao Rodrigues and Robert de Mello Koch for the part they have each played in making these trips possible.

Many people contributed to making my time in Cape Town some of the best years of my life. At the university, I thank Di Loureiro and the wonderful people in Room 319 for going the extra mile for me; the faculty and staff of the Department of Mathematics and Applied Mathematics for providing such a fantastic environment; Zeb Vilakazi for his unwavering (and sometimes unfounded) faith in me; Anslyn John whose life is more interesting than mine and Kevin Colville whose isn't. I owe a great debt of gratitude to the Lindbury trust for keeping me in the money during my PhD studies and to Linda Vranas and Marian Green for making it possible. Many thanks too, to Braam and Kallie at the Coffee Bean for their magnanimous support of my coffee habit. On the topic of addictions...I owe a special word of thanks to my climbing partners over the years. To all the people who've ever held the other end of the rope; thank you! I am especially grateful to the Weltmans, Ivan, Sheina and their children who in the past few years have become my surrogate family. They've taken me into their home and lives and through all the ups and downs have kept me grounded. I can only hope that one day I can repay their generosity.

Speaking of Weltmans...through everything, the good times and the grading, the highs and lows and the frustrations and joys Amanda has been my rock! And as rocks go, Amanda is a diamond. Life is certainly an adventure and I cannot express enough how lucky I am to have her to partner me through it.

# Contents

we find the Hamiltonian

$$H = \frac{1}{2m}\Big(p_x^2 + (p_y - qxB)^2\Big).$$

The spectrum of this Hamiltonian consists of infinitely degenerate Landau levels with eigenenergies $E_n = (\hbar qB/m)(n + \frac{1}{2})$, and a mass gap $\Delta = \hbar qB/m$ between levels $|n, k\rangle$ and $|n + 1, k\rangle$. The physics of the projection onto the lowest Landau level $|0, k\rangle$ is quite straightforward; if the magnetic field is strong enough, all higher states essentially decouple to infinity. From this point of view, the large $B$ limit is equivalent to the small $m$ limit so that the effective Lagrangian describing the particle motion projected onto $|0, k\rangle$ is just the second term of (1.1). Consequently, the momentum canonically conjugate the $y$ coordinate of the particle is

$$p_y = \frac{dL}{d\dot{y}} = qBx$$

and the canonical commutation relations become

$$[x, y] = -i\frac{\hbar}{qB} \tag{1.2}$$

Remarkably, the space seen by the particle restricted to the first Landau level is *no longer commutative*! Parenthetically, an entirely valid question at this point is whether this behaviour is specific to the lowest Landau level. This question was addressed superbly in [64] and we will content ourselves with simply quoting the result. Restricting the particle to, say the lowest $N + 1$ levels, results in the coordinate commutator having nonvanishing matrix elements

$$\langle N, k|[x, y]|N, k'\rangle = -i\frac{\hbar}{qB}(N + 1)\delta(k - k')$$

So, for example, when $N = 1$ the coordinate commutator becomes $[x, y] = \mathrm{diag}(0, -2i\hbar/qB)$. The lesson to be learned here is that the noncommutativity of the spatial coordinates persists for a restriction of the particle motion onto an *arbitrary finite number* of Landau levels. Underlying this is the fact that the restriction to a finite number of Landau levels actually truncates the state space of the particle. The emergent noncommutativity of the spatial coordinates is a consequence of this truncation.

## 1.1 A prehistory

The idea that the continuous, smooth appearance of spacetime may give way to a seething foam-like structure on microscopic scales is certainly not new [68]. Noncommutative geometry, the idea that spacetime itself should become "fuzzy" on sufficiently small scales allows for a novel way of quantifying this spacetime foam. As a subject, noncommutative geometry has enjoyed a long though, perhaps, not so illustrious history. The idea of some minimal length scale of spacetime dates back to the 1930's when Heisenberg and others investigated discretised models of spacetime in the hope of regulating divergences in quantum field theory arising from the assumption of point-like interactions between fields. However it was not until the 1947 work of Snyder [82] that noncommuting coordinates were actually employed to realise a quantum discreteness of spacetime. Since then, interest in the idea of fuzzy geometry has ebbed and flowed but it was not until the seminal work of Seiberg and Witten [81] recognising the emergence of noncommutativity in string theory that the field has really exploded. To even attempt a pedagogical review of the vast body of work that has since emerged on fuzzy geometries would reflect an alarming degree of optimism on my part. Instead, this chapter is intended as a pedestrian walk toward some of the ideas that will prove relevant for the rest of this thesis. For more details I refer the reader to the excellent reviews of Douglas and Nekrasov and Szabo [19, 87] and references therein. Our treatment here closely follows that of [88].

## 1.2 The Landau effect

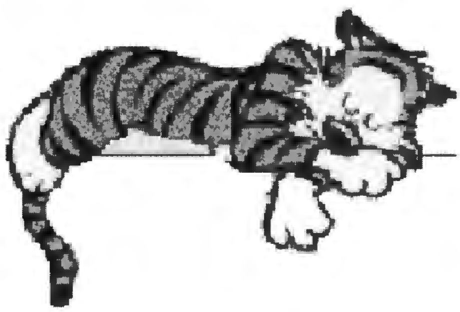
As abstract as it might first appear, the idea of noncommuting (spatial) coordinates actually has a well known physical realisation; the motion of a charged particle in a constant magnetic field, projected onto its lowest Landau level, is effectively confined to a 2−dimensional noncommutative plane $\mathbb{R}^2_\theta$ transverse to the magnetic field. With the foresight of knowing that the motion of the particle is effectively planar anyway, we can fully capture its dynamics by the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}}\cdot\mathbf{A} \qquad (1.1)$$

where $q$ is the charge of the particle, $\dot{\mathbf{x}} = (\dot{x}, \dot{y}, 0)$ and, choosing the magnetic field in the $\hat{z}$ direction, $\mathbf{A} = (0, xB, 0)$. Only a Legendre transformation away,

# Chapter 1

# That fuzzy feeling again all over again



*The idea of noncommutativity is introduced via a Moyal deformation of the algebra of smooth functions over a manifold. By focusing on the example of the Landau problem for a charged particle moving on the plane in the presence of a strong magnetic field, we provide an interpretation of the noncommutative parameter $\vartheta$. This interpretation is then clarified and made precise by reviewing the Seiberg-Witten description of noncommutative gauge theory as the zero slope limit of string theory in an NS-NS $B-$field background. The material in this chapter is intended to be a self-contained introduction to the material used in the rest of the thesis.*

## 1.3 Moyal deformation

The key formula to be distilled from the above argument is the coordinate commutation relation (1.2). More generally, this can be written

$$[x^i, x^j] = i\theta^{ij} \tag{1.3}$$

where $\theta^{ij}$ is an antisymmetric, non-degenerate matrix. In words, this says that the spatial coordinates on some manifold satisfy a Heisenberg-like algebra normally reserved for the canonically conjugate pair $(x^i, p^i)$. Noncommutative geometry is essentially the study of noncommutative algebra. More precisely, the Gelfand-Naimark theorem guarantees that the structure of an ordinary commutative manifold $M$ is encoded in the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on $M$. The algebraic product on $\mathcal{A}$ is just the pointwise multiplication of functions. The transition to a noncommutative manifold $M_\theta$ is made by deforming the algebra $\mathcal{A}$. For concreteness, let us restrict ourselves to considering $M = \mathbb{R}^2$ for the moment.

Of the many possible deformations of $\mathcal{A}$, the one of most interest to us in this work is the so-called *Moyal deformation* where the commutative product of functions is replaced by the $\star$−product

$$(f \star g)(x) \equiv e^{\frac{i}{2}\theta^{ij}\partial_{x^i}\partial_{y^j}} f(x)g(y)\Big|_{x=y}$$

Notice that choosing $f$ and $g$ to be the coordinate functions immediately yields the above coordinate algebra. So much for multiplying functions on $\mathbb{R}^n_\theta$; in order to study physics on noncommutative spaces we need to understand how to differentiate and integrate on these spaces. Fortunately, this task is simplified quite significantly by observing that a judicious choice of coordinates followed by a suitable rescaling reveals that (1.3) satisfies the standard algebra of raising and lowering operators of a harmonic oscillator. As a result, the noncommutative space $\mathbb{R}^2_\theta$ is put into one-to-one correspondence with a one-particle Hilbert space $\mathcal{H}$ of a harmonic oscillator through the so-called Weyl transform $\mathcal{W}$, a nonsingular map that is to noncommutative spaces roughly what the Fourier transform is to commutative spaces. Under $\mathcal{W}$, noncommutative functions become operators acting on $\mathcal{H}$, derivatives become commutators and integrals map to traces over the Hilbert space. These ideas will be elaborated on in greater detail shortly but for now the time has come to talk of branes and strings other things...

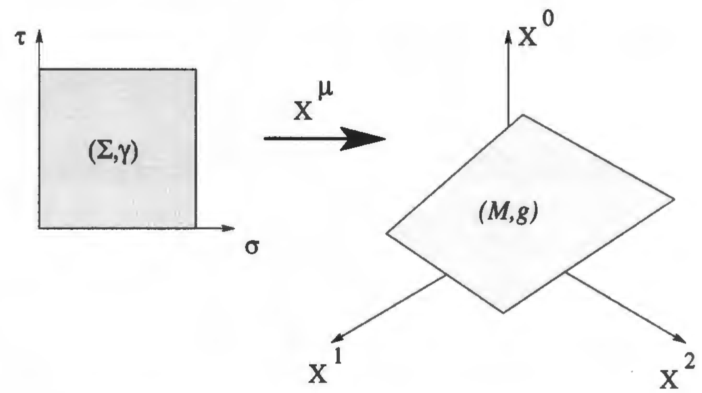



Figure 1.1: An embedding of the string worldsheet into an ambient spacetime. The worldsheet fields are the coordinates of $\Sigma$ in spacetime

## 1.4 Noncommutativity in String Theory

Undoubtedly, the explosion of recent interest in fuzzy geometry can be traced back to the emergence of noncommutativity in string theory. Of the many avenues that lead toward this result (including Matrix theory and the theory of Nonabelian branes [19]) perhaps the most illuminating is the stringy analogue of the Landau problem discussed above.

The worldsheet field theory for open strings attached to a $Dp-$brane (in Bosonic string theory) is described by a sigma model whose fields $X^I$ map the string worldsheet $\Sigma$ into a 26−dimensional target spacetime. The geometry of the target spacetime is characterised by the closed string sector of the string theory. Among the massless fields in this sector we find the symmetric, rank two object $g_{IJ}$ and a non-degenerate two-form field $B_{IJ}$ - the Neveau-Schwarz $B-$field. While the interpretation of $g_{IJ}$ is clear as the spacetime metric, the NS-NS $B-$field is slightly more mysterious. However, it couples directly to the string worldsheet via

$$-\frac{i}{2}\int_{\Sigma} B_{IJ}\, dX^I \wedge dX^J$$

and so is not unlike a magnetic field that couples to a particle worldline; actually, in a sense to be made more precise in chapter 3, the NS-NS $B-$field may be thought of as a magnetic field on the $Dp-$brane. The string action can be written

$$S = \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2z\, g_{IJ}\partial_z X^I \partial_{\bar{z}} X^J - \frac{i}{2}\int_{\Sigma} B_{IJ} dX^I \wedge dX^J \tag{1.4}$$

where, as usual, the Regge slope $\alpha' = l_s^2$. When $B_{IJ}$ is constant, the second term may be integrated by parts to give a boundary action

$$S_{\partial\Sigma} = -\frac{i}{2}\int_{\partial\Sigma} B_{IJ}X^I\dot{X}^J \tag{1.5}$$

where $\partial\Sigma$ denotes the worldsheet boundary and $\dot{X}^J = dX^J/dt$, the tangential derivative along the boundary. In the presence of the $Dp-$brane, this boundary term no longer vanishes. Staring a little at (1.5) reveals its similarity, at least formally, to the action for the planar particle of the Landau problem projected onto the lowest level. However, this is a string theory and not a point particle theory. As such, the first term in (1.4) encoding the closed string sector requires careful consideration. The remarkable insight of Seiberg and Witten [81] was that in the so-called zero slope limit $\alpha' \to 0$ with $B$ fixed, not only do the massive string modes decouple from the theory but so too does the worldsheet dynamics. What remains of the worldsheet theory is described precisely by the boundary action (1.5). Subsequent canonical quantization of the worldsheet field theory yields the commutator

$$[X^I, X^J] = i\theta^{IJ}$$

with the noncommutativity parameter $\theta = 1/B$. The result, as promised, is that the $Dp-$brane worldvolume is now a *noncommutative* space. These results can be neatly encapsulated in the Sieberg-Witten prescription for obtaining the effective worldvolume action for a $Dp-$brane in a constant NS-NS $B-$field:

- Begin with the the $Dp-$brane effective action with zero $B-$field.
- Replace the closed string metric $g^{IJ}$ with the open string metric

$$G^{IJ} = \Big(\frac{1}{g + 2\pi\alpha' B}\Big)^{IJ}_{\rm sym}$$

- Replace the string coupling constant $g_s$ with

$$G_s = g_s\sqrt{\det(2\pi\alpha' B g^{-1})}$$

- Use a star product to multiply fields with

$$\theta^{IJ} = 2\pi\alpha'\Big(\frac{1}{g + 2\pi\alpha' B}\Big)^{IJ}_{\rm antisym}$$

## 1.5 Things to come

A few years ago, while working on the result that eventually grew into the second chapter of this thesis, I picked up a comic book by Bill Watterson for a friend. It was a collection of the final few strips telling of the adventures of a certain irrepressible six year old and his pet tiger [1]. The book ends with Calvin and Hobbes sledding into the horizon with the words "*It's a wonderful world 'ol buddy. Let's go exploring.*" Strangely enough, these words had an enormous impact on the way I've come to view physics. Indeed, if this work seems a little more eclectic than most theses, it is because I have chosen to explore problems that I have found interesting as I have encountered them. Nevertheless, in the interests of coherence, I have had to, regrettably, omit some work in order to attempt to fit the underlying theme of noncommutative geometry.

The thesis can, roughly, be separated into three more-or-less distinct parts. Part I consists of the Introduction and chapters 2 and 3 where we undertake an extension of several result on BPS and non-BPS sigma model solitons. In particular we show that the solution space of solitonic excitations in noncommutative sigma models with Kähler target spaces is significantly larger than the corresponding commutative models. Moreover, we demonstrate how the vector solitons of the nonlinear sigma models may be constructed from bound states of certain noncommutative scalar solitons. In addition, we show how the well known infinite coupling transition of vortex solitons of a gauged linear sigma model to instantons of a nonlinear sigma model on $\mathbb{CP}^n$ persists in the noncommutative limit. More than just an academic exercise, we use this result to identify a possible $D-$brane configuration corresponding to the sigma model instantons. Apart from this Introductory chapter, all the work in this part is original and based on the papers [70] and [71]

Part II is much more string theoretic in nature and is taken up by the study of fuzzy funnel solutions to the non-Abelian $D1-$brane equations of motion. In particular, we identify the configuration of intersecting $D1-$ and $D7-$branes described by the geometry of the funnel and provide a dual description of the configuration from the perspective of the $D7-$brane worldvolume. Our work in this section provides a natural generalisation of the previously studied cases of intersecting $D1-D3$ and $D1-D5$ systems described by fuzzy $2-$ and $4-$spheres respectively. Again, this work is based on the original work contained in [14].

[1] who, the reader will note features quite extensively throughout this work.

Finally, we conclude with a short excursion into inflationary cosmology. Beginning from the fact that current measurements of the Cosmic Microwave Background do not rule out closed universes, we provide a realisation of a singularity-free inflationary universe in the form of a simple cosmological model dominated at early times by a single minimally coupled scalar field with a physically based potential. The universe starts asymptotically from an Einstein static state, enters an expanding phase that leads to inflation followed by reheating and a standard hot Big Bang evolution. We discuss the basic characteristics of this Emergent universe and show that none is at odds with current observations. The results of this chapter are also original and were reported in [23].

So, having dispensed with all the formalities, let's go exploring...

# Chapter 2

# Fuzzy lumps roaming the plane

*We review the derivation of a noncommutative version of the nonlinear sigma model on $\mathbb{CP}^n$ and it's soliton solutions for finite $\vartheta$ emphasizing the similarities it bears to the GMS scalar field theory. It is also shown that unlike the scalar theory, some care needs to be taken in defining the topological charge of BPS solitons of the theory due to nonvanishing surface terms in the energy functional. Finally it is shown that, like its commutative analogue, the noncommutative $\mathbb{CP}^n$-model also exhibits a non-BPS sector. Unlike the commutative case however, there are some surprises in the noncommutative case that merit further study.*

## 2.1 Introduction

Nonlinear sigma models on Kähler and hyper-Kähler target spaces are arguably some of the most important test beds of ideas that invariably find their way into the more daunting arena of physical gauge theories in four dimensional spacetimes. Certainly one of the most favored of such theories is the $d = 2$ sigma model with target space $\mathbb{CP}^n$ - the $n$-dimensional complex projective space. Like the $d = 4$ self-dual Yang-Mills theory it too exhibits asymptotic freedom, conformal invariance and a rich solitonic sector.

A large class of the soliton solutions of the $\mathbb{CP}^n$ model are the finite energy lump-like solutions that correspond to holomorphic functions on the two-dimensional base space. The lumps saturate a BPS bound on the $\mathbb{CP}^n$ energy functional and are consequently stabilized by some finite topological charge. Although these are by far the most well studied, they are by no means the only solitonic solutions exhibited by the $\mathbb{CP}^n$ sigma model. It has been known for some time that certain bound states of such BPS lumps also solve the sigma model equations [16]. These are, however, not solutions of any first order BPS equations and consequently lack the stability properties of the lumps. Nevertheless, there exists a Bäcklund-like solution generating technique for generating general (non-BPS) $\mathbb{CP}^n$ solutions from a given holomorphic BPS soliton [17, 78]. Yet even in the light of such remarkable similarities between the $\mathbb{CP}^n$ sigma model and four-dimensional gauge theories, some differences are quite stark. Chief among these are the lack of a more complete understanding of the soliton moduli space and the absence of a general construction technique like the ADHM method for the $\mathbb{CP}^n$ model. One avenue toward a better understanding of the dynamics of the $\mathbb{CP}^n$ lumps (as encoded in the moduli space) lies in the deformation of the base space on which the lumps move.

Ever since the realization that the low energy effective theory of D-branes in a $B$ field background [81, 98] is a noncommutative field theory, the deformation of choice has become that of the algebra of smooth functions over the base. This yields a noncommutative $\mathbb{CP}^n$ sigma model whose basic solitonic excitations have by now been well documented [58]. In particular, the moduli space metric was explicitly computed for the $1-$ and $2-$soliton solutions and shown to be nonsingular and Kähler in both cases [25]. Moreover, in [26] it was shown that, in stark contrast to the commutative case, the noncommutative $\mathbb{CP}^1$ sigma model contains a non-BPS sector that is closely tied to the scalar solitons of the GMS field theory [29]. The existence of these new non-BPS excitations of the $\mathbb{CP}^1$ model (and, more generally in the $\mathbb{CP}^n$ model) is certainly intriguing. If nothing else, it is a reminder of the fact that the volume of the solution space of the noncommutative theory is significantly larger than the corresponding commutative one. An interesting question then, is whether the known soliton generating technique of [17, 78] probe this sector of the solution space of the noncommutative $\mathbb{CP}^n$ sigma model. As will be demonstrated, this technique is, perhaps surprisingly, deficient in the noncommutative model.

Despite (or perhaps because of) their remarkable simplicity, GMS solitons have had a huge impact on recent literature (see [36] for a recent review). In particular, it was shown in [8, 9] that the algebraic structure of a family of solitonic solutions of the vacuum string field theory equations

$$\Psi_m * \Psi_m = \Psi_m \tag{2.1}$$

is exactly isomorphic to the corresponding one for the GMS solitons of a noncommutative pure scalar field theory. Exploiting this isomorphism leads one to the interpretation of such noncommutative solitons as relics of $D23$-branes in the low energy limit. If, as in [26] (and later on in this ), solitonic excitations of the $\mathbb{CP}^n$ sigma model exist that can be built up of bound states of scalar solitons, it seems natural to ask whether the noncommutative sigma model solitons may have some interpretation as $D$-brane configurations also.

The organization of this chapter is as follows: After a brief description of the $\mathbb{CP}^n$ sigma model and its (commutative) instanton solutions, we proceed to a review of the corresponding noncommutative instantons. While the results in this section are themselves not new, the *formulation* of the noncommutative sigma model is. By focusing on the formal similarity between the sigma model equations and that of the noncommutative scalar field theory, the BPS bound on the energy functional is rewritten to emphasize the subtleties encountered in defining topological charges of noncommutative objects. This section will establish all the necessary formalism required for the main result of this work: the construction of non-BPS solitons of the noncommutative $\mathbb{CP}^n$ sigma model[1]. After the construction of several explicit non-BPS solitons for both the $\mathbb{CP}^1$ and $\mathbb{CP}^2$ sigma models, the following section is devoted to the

[1] As this work was nearing completion we became aware of the work of Foda *et. al.* [24] whose results have significant overlap with our own. The emphasis in [24] is largely on demonstrating that many of the the known results for the construction of general solitonic

comparison of the BPS solitons constructed as holomorphic curves on $\mathbb{CP}^n$ and those obtained from bound states of GMS scalar solitons.

## 2.2 The noncommutative $\mathbb{CP}^n$ soliton

By way of establishing notation and some of the conventions, to be followed for the remainder of this chapter, we begin by reviewing the construction of the noncommutative soliton solutions of the nonlinear sigma model on a $\mathbb{CP}^n$ target space. This section follows closely the recent work of Lee, Lee and Yang[58].

### 2.2.1 Notation

In studying noncommutative nonlinear sigma models we will, for the most part, be interested in maps $u : \mathbb{R}^2_\vartheta \times \mathbb{R} \to M$ with the target, $M$ a Kähler (or hyper-Kähler) manifold. The sigma model field $u$ takes values in the $\vartheta$-deformed algebra of functions over $\mathbb{R}^2_\vartheta$, $\mathcal{A}_\vartheta$, whose elements satisfy

$$f \star g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x)g(x')|_{x=x'} \,, \tag{2.2}$$

where $\theta^{ij} = \vartheta\epsilon^{ij}$ is a nondegenerate, antisymmetric constant matrix and $\vartheta$ is a positive deformation (noncommutativity) parameter of dimension $(mass)^2$. Consequently, coordinates on the noncommutative plane $\mathbb{R}^2_\vartheta$ satisfy the Heisenberg algebra $[x^1, x^2] := x^1 \star x^2 - x^2 \star x^1 = i\vartheta$. Written in terms of the complex coordinates $z := (x^1 + ix^2)/\sqrt{2}$ and $\bar{z} := (x^1 - ix^2)/\sqrt{2}$ on $\mathbb{R}^2_\vartheta$ the commutator becomes $[z, \bar{z}] = \vartheta$ which, up to a rescaling is nothing but the algebra of annihilation and creation operators for the simple harmonic oscillator. By use of the Weyl transform [36], we associate to a function on the noncommutative space an operator acting on an auxiliary Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. In a basis of simple harmonic oscillator eigenstates $\mathcal{H} = \bigoplus_n \mathbb{C}|n\rangle$. The vacuum $|0\rangle$ is defined, as usual, by the action of the annihilation operator $\widehat{z}$ on it as $\widehat{z}|0\rangle = 0$. Further, we have

$$\widehat{z}|n\rangle = \sqrt{\vartheta n}|n-1\rangle \,, \tag{2.3}$$

solutions to the $\mathbb{CP}^n$ sigma model are equally applicable in the noncommutative case. The point of our work however, is to highlight the similarities in the description of the BPS and non-BPS solitons of the noncommutative sigma model to that of the scalar GMS solitons hopefully paving the way for further study into the possible embedding of the former into a stringy framework [90]

$$\widehat{\bar{z}}|n\rangle = \sqrt{\vartheta(n+1)}|n+1\rangle\,. \tag{2.4}$$

This association of functions on the noncommutative space and operators in the Hilbert space is particularly useful in treating differentiation and integration on the noncommutative plane. Under the Weyl map the operations of differentiating and integrating functions over $\mathbb{R}^2_\vartheta$ transform to

$$\partial_i \rightarrow \frac{i}{\vartheta}\epsilon_{ij}[\widehat{x^j},\cdot\,]\,, \tag{2.5}$$

$$\int_{\mathbb{R}^2_\vartheta} d^2x\, f(x^i) \rightarrow 2\pi\vartheta\,\mathrm{Tr}_{\mathcal{H}}\,\widehat{O_f}(\widehat{x^i}) = 2\pi\vartheta\sum_{n\geq 0}\langle n|\widehat{O_f}|n\rangle\,. \tag{2.6}$$

In particular, tracing over the Hilbert space preserves the translational symmetry of the noncommutative plane.

### 2.2.2 The commutative $\mathbb{CP}^n$ sigma model

In this section we collect some well known results on the classical nonlinear sigma models on a complex projective target space [90, 95, 47] that will prove useful in what follows. The sigma model action is most conveniently formulated in terms of the $\mathbb{CP}^n$ homogeneous coordinates $U = (u_1,\dots,u_{n+1}) \sim (\lambda u_1,\dots,\lambda u_{n+1})$ where $\lambda \in \mathbb{C}^*$ is a nonzero complex number. Defining $DU := dU + iUA$, this is given by

$$S = \int_{\mathbb{R}^{(1,2)}} d^3x\,\eta^{\mu\nu}(D_\mu U)^\dagger D_\nu U\,, \tag{2.7}$$

subject to the constraint $U^\dagger U - 1 = 0$. A few points should be immediately apparent from this formulation; the first being the invariance of the action under global $SU(n+1)$ transformations of the sigma model fields $u_I \rightarrow e^{i\alpha}u_I$. This is merely a reflection of the equivalence relation defining $\mathbb{CP}^n$. The second being the fact that the 'gauge field' is an auxiliary one, completely determined by the sigma model fields $A = iU^\dagger dU$. The corresponding equations of motion written in terms of the (homogeneous) sigma model fields are given by

$$D_\mu D^\mu U + (D_\mu U)^\dagger(D^\mu U)U = 0\,. \tag{2.8}$$

Once again the static energy is bounded by a topological charge $E \geq 2\pi|Q|$ where now

$$Q = \frac{i}{2\pi}\int_{\mathbb{R}^2} d^2x\,\epsilon^{ij}(D_iU)^\dagger(D_jU)\,. \tag{2.9}$$

Reparameterising the sigma model field $U = W/\sqrt{W^\dagger W}$ where $W$ is an $(n+1)$-vector, the energy bound is saturated when the first order BPS equations $\partial_{\bar{z}} W = 0$ or $\partial_z W = 0$ are satisfied. These static solutions of the $(2+1)$-dimensional model are exactly the instanton and anti-instanton solutions of the $(1+1)$-dimensional *Euclidean* $\mathbb{CP}^n$ sigma model, constructed by taking $W$ to be a rational function of $z$ and $\bar{z}$ respectively [2]. The topological charge of the soliton is counted as the highest degree of the rational function components of $W$. Before discussing noncommutative generalizations it is worth noting that the $\mathbb{CP}^n$ sigma model may be formulated completely in terms of the Hermitian projector $P = W(W^\dagger W)^{-1} W^\dagger$ in terms of which the action is given by

$$S = \frac{1}{2} \int_{\mathbb{R}^{(1,2)}} d^3x \ \mathrm{tr} \ \eta^{\mu\nu} \Big( \partial_\mu P \partial_\nu P \Big) \,. \tag{2.10}$$

The 'trace' in the integrand is the usual matrix trace operation and the unitary constraint on the sigma model fields $U(z, \bar{z})$ is reflected in $P^2 = P$. This formulation will prove particularly useful in the construction of non-BPS solitons later.

**Sigma models on Kähler targets**

At first sight, the argument above might seem rather fortuitous, relying as it does on the special properties of the homogeneous coordinates on $\mathbb{CP}^n$. However, it turns out to be quite intimately related to the fact that $\mathbb{CP}^n$ is a *Kähler* manifold. Indeed, if $X : \mathbb{R}^{(1,2)} \to M$ is a map from $(2+1)$-dimensional Minkowski spacetime with standard metric $\eta_{\mu\nu} = diag(-1, +1, +1)$ to a Kähler target manifold with Riemannian metric $g_{IJ}$ then the action for the nonlinear sigma model is

$$S = \frac{1}{2} \int_{\mathbb{R}^{(1,2)}} d^3x \ \eta^{\mu\nu} \partial_\mu X^I \partial_\nu X^J g_{IJ} \,.$$

The Kähler property of the target manifold means that there exists a covariantly constant real $(1,1)$-tensor field (the almost complex structure) $J$ satisfying $J^I_K J^K_L = -\delta^I_L$ and a closed real two form (the Kähler form) $\Omega = \frac{1}{2} J_{IK} dX^I \wedge dX^K$.

[2]Hence, for the rest of this work, we interchange the terms "instanton" and "static soliton" with impunity and beg the reader's patience for this mild abuse of terminology

**Continued...**

In terms of the almost complex structure and the Kähler form, the energy of a static field configuration may be rearranged to give

$$E = \frac{1}{4}\int_{\mathbb{R}^2} d^2x\,(\partial_i X^I \pm \epsilon_i^j J^I_K \partial_j X^K)^2 \mp \underbrace{\frac{1}{2}\int_{\mathbb{R}^2} d^2x\,\Omega_{IK}\epsilon^{lm}\partial_l X^I \partial_m X^K}_{2\pi Q}\,.$$

The second term (the topological charge) is just the integral over $\mathbb{R}^2$ of the pullback of the Kähler form and is a topological invariant as a result of the fact that $\Omega$ is a closed form. This gives the familiar bound on the energy $E \geq 2\pi|Q|$. The energy bound is saturated by configurations that satisfy the BPS equations

$$\partial_i X^I \pm \epsilon_i^j J^I_K \partial_j X^K = 0\,.$$

Since these are the just the Cauchy-Riemann equations, such configurations are nothing but holomorphic curves on the Kähler manifold $M$. Now fix $M$ to be the $n$-dimensional complex projective space $\mathbb{CP}^n = \mathbb{C}^{n+1}/\mathbb{C}^*$. In terms of the sigma model fields $X^I(z,\bar{z})$, $I = 1,\ldots,n$ (the inhomogeneous coordinates on $\mathbb{CP}^n$) $g_{IJ}$ is just the standard Fubini-Study metric on $\mathbb{CP}^n$

$$ds^2 = 4\frac{\delta_{IJ}(1+\overline{X}_K X_K) - \overline{X}_I X_J}{(1+\overline{X}_K X_K)^2} dX^I d\overline{X}^J\,.$$

### 2.2.3 The noncommutative model and its BPS solutions

The transition to a noncommutative $\mathbb{CP}^n$ sigma model is made, following the standard prescription, by replacing all products occurring in the above formulae with Moyal $\star$-products and subsequently by replacing all noncommutative functions with the associated operators on $\mathcal{H}$. As such, the static sub-sector of the $(2+1)$-dimensional sigma model action (2.7) becomes

$$S_\vartheta = \frac{2\pi}{\vartheta}\mathrm{Tr}_{\mathcal{H}}\Big(\delta_{ij}[\widehat{x^i},\widehat{U}^\dagger](1-P)[\widehat{U},\widehat{x^j}]\Big)\,. \qquad (2.11)$$

The unitarity condition on the commutative sigma model fields $U^\dagger U = 1$ becomes an isometry $\widehat{U}^\dagger\widehat{U} = 1$ on $\mathcal{H}$ (see [36] for more details). In deriving (2.11) use was made of the identity $D_i U \to \frac{i}{\vartheta}\epsilon_{ij}(1-P)[\widehat{x^j},\widehat{U}]$ and $P$ is the

Hermitian projector as defined above. As in the commutative case, the static action may be rewritten in completely in terms of $P$ as

$$S_\vartheta = 2\pi \mathrm{Tr}_{\mathcal{H}} \mathrm{tr}\Big( [P, \widehat{a}^\dagger][\widehat{a}, P] \Big) , \tag{2.12}$$

after a further rescaling of the coordinates on $\mathbb{R}^\vartheta_2$ as $\widehat{z} \to \sqrt{\vartheta}\widehat{a}$ and $\widehat{\bar{z}} \to \sqrt{\vartheta}\widehat{a}^\dagger$. It is worth noting at this juncture that the form of the $\mathbb{CP}^n$ sigma model action (2.12) is remarkably similar to the kinetic term of the static energy functional of a $(2+1)$-dimensional noncommutative scalar field (eq.(2.2) of ref[30]). Such noncommutative scalar field theories are known to exhibit a spectrum of localized field configurations (GMS solitons) [29, 30] with several interesting properties. Not least among these is the rich structure of the $k$-soliton moduli space; a Kähler de-singularization of $(\mathbb{R}^2)^k/S_k$, the symmetric product of the single soliton moduli space. It appears though, that this is not unique to the scalar field theory [73, 11]. Indeed a similar resolution of the geometry of the $k$-soliton moduli space, as realized by the noncommutative algebra of projection operators, was demonstrated recently in the noncommutative $\mathbb{CP}^n$ sigma model [25] by explicit computation of the Kähler metric on the one- and two-soliton moduli space. In the light of such remarkable evidence, it seems reasonable to ask if there may be further similarities between GMS solitons and those of the $\mathbb{CP}^n$ sigma model? With this in mind, it will prove useful to proceed in close analogy with the analysis of GMS solitons. Returning to the $\mathbb{CP}^n$ model and equating the energy of the configurations with the static action, it is easily seen that, as in the commutative case, the energy is bounded from below. However, as will be demonstrated shortly, some degree of care must be exercised when dealing with higher rank projectors[3]. Naively following [30] it might seem like the energy may be written as

$$E_\vartheta = 2\pi \mathrm{Tr}_{\mathcal{H}} \mathrm{tr}\Big( 2F_\pm(P)^\dagger F_\pm(P) \pm P \Big) \geq 2\pi \Big| \mathrm{Tr}_{\mathcal{H}} \mathrm{tr} P \Big| , \tag{2.13}$$

where

$$F_\pm(P) = \left\{ \begin{array}{l} (1-P)\widehat{a}P \\ (1-P)\widehat{a}^\dagger P \end{array} \right. \tag{2.14}$$

The inequality would then saturate when the (anti)BPS equations $F_\pm(P) = 0$ are satisfied and the topological charge of the BPS solitons takes on a particularly neat expression, being simply the combined matrix and Hilbert space

[3] We thank Robert de Mello Koch for drawing our attention to this point.

trace of the associated projector. However, this would be too naive! The problem is that $\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}(\mathrm{P})$ is generally infinite and hence cannot represent the soliton charge. Crucial to the resolution of this issue is the understanding that, in the noncommutative case, arguments in the trace may *not* be permuted with impunity. With this in mind, returning to the static energy (2.12) (and focusing on the BPS case for the moment), it may be seen that[4]

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}[P,\widehat{a}^\dagger][\widehat{a},P] &= \mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big(P\widehat{a}^\dagger\widehat{a}P - P\widehat{a}^\dagger P\widehat{a} - \widehat{a}^\dagger P\widehat{a}P + \widehat{a}^\dagger P\widehat{a}\Big) \\ &= \mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big(2F_+(P)^\dagger F_+(P) + P - [\widehat{a},\widehat{a}^\dagger P] + [P,\widehat{a}^\dagger P\widehat{a}]\Big)\,.\end{aligned}$$

A straightforward computation shows that $\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}[P,\widehat{a}^\dagger P\widehat{a}] = 0$ so that the last term may be dropped. Recalling the Weyl prescription mapping functions on a noncommutative space to an auxiliary Hilbert space, the second to last term may be thought of as an"integral of a derivative". As such, this may be evaluated with a noncommutative analogue of Stokes' theorem (see for instance [31, 49])

$$\mathrm{Tr}_{\mathcal{H}}[\widehat{a},\mathcal{O}] = \lim_{M\to\infty}\sqrt{M+1}\langle M+1|\mathcal{O}|M\rangle \tag{2.15}$$

where $\mathcal{O}$ is any appropriately well behaved operator on $\mathcal{H}$. For $\mathcal{O} = \mathrm{tr}\,\widehat{a}^\dagger P$, this term is generally nonvanishing and cannot be neglected. With this in mind the energy functional becomes

$$\begin{aligned}E_\vartheta &= 2\pi\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big(2F_+(P)^\dagger F_+(P) + P - [\widehat{a},\widehat{a}^\dagger P]\Big) \\ &\geq \underbrace{2\pi\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big(P - [\widehat{a},\widehat{a}^\dagger P]\Big)}_{2\pi Q_+}\end{aligned} \tag{2.16}$$

with $F_+(P)$ defined as above. Similarly it may be shown that

$$\begin{aligned}E_\vartheta &= 2\pi\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big(2F_-(P)^\dagger F_-(P) - (P - [\widehat{a},P\widehat{a}^\dagger])\Big) \\ &\geq \underbrace{2\pi\Big|\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big(P - [\widehat{a},P\widehat{a}^\dagger]\Big)\Big|}_{2\pi|Q_-|}\end{aligned} \tag{2.17}$$

[4]We would also like to thank O. Lechtenfeld for bringing to our attention ref.[55, 56] in which it was stressed that $E \neq \mathrm{Tr}_{\mathcal{H}}\mathrm{tr}P$ in the more general setting of a noncommutative $U(n)$-valued field in a modified $(2+1)$-dimensional sigma model.

Again, the energy bound is saturated for configurations for which the (anti)BPS equations $F_\pm(P) = 0$ hold. As shown in [58] such solutions are not hard to find; any Hermitian projector constructed from an $(n+1)$-vector $W$ whose components are (anti)holomorphic polynomials will satisfy the above (anti)BPS equations. These are the noncommutative extensions of the instanton solutions of the conventional $\mathbb{CP}^n$ sigma model. The static 1− and 2−soliton solutions of the noncommutative $\mathbb{CP}^1$ model, for example, are given respectively by

$$W_1 = \begin{pmatrix} a_1 \\ \hat{z} - b_1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 2a_2\hat{z} + b_2 \\ \hat{z}^2 + c_2\hat{z} + e_2 \end{pmatrix}. \tag{2.18}$$

The coefficients $a_1, \ldots, e_2 \in \mathbb{C}$ are chosen to coincide with the standard way of writing the corresponding solitons of the commutative theory [25, 95]. These are the complex moduli of the $\mathbb{CP}^n$ instantons. The solutions are easily visualized in the small $\vartheta$ limit by computing the energy density as an operator on the auxiliary Hilbert space and mapping it back to a function on $\mathbb{R}^2_\vartheta$ by the Weyl correspondence, *i.e.* $\widehat{\mathcal{E}} \mapsto \mathcal{E}_\star = \mathcal{W}^{-1}(\mathcal{E})$. This is exemplified by the simplest instanton solution [58], $W_1 = (1, \hat{z})^T$ for which

$$P = \begin{pmatrix} \frac{1}{1+\hat{\bar{z}}\hat{z}} & \frac{1}{1+\hat{\bar{z}}\hat{z}}\hat{\bar{z}} \\ \hat{z}\frac{1}{1+\hat{\bar{z}}\hat{z}} & \hat{z}\frac{1}{1+\hat{\bar{z}}\hat{z}}\hat{\bar{z}} \end{pmatrix}. \tag{2.19}$$

By way of illustration of the above points it is a useful exercise to compute the topological charge of the above soliton[5]. The trace over $\mathcal{H}$ may be regulated by the introduction of an infrared cutoff $M$ through the restriction to an $M$-dimensional subspace of $\mathcal{H}$ spanned by $\{|0\rangle, |1\rangle, \ldots, |M\rangle\}$ [31]. As such

$$\begin{aligned} Q_+ &= \lim_{M\to\infty} \mathrm{Tr}_{\mathcal{H}_M}\mathrm{tr}P - \mathrm{Tr}_{\mathcal{H}_M}\mathrm{tr}[\hat{a}, \hat{a}^\dagger P] \\ &= \lim_{M\to\infty} \sum_{n=0}^{M} \langle n| \frac{1}{1+\widehat{N}} + \frac{1+\widehat{N}}{2+\widehat{N}} |n\rangle \\ &- \sqrt{M+1}\langle M+1|\hat{a}^\dagger \frac{1}{1+\widehat{N}} + \frac{\widehat{N}}{1+\widehat{N}}\hat{a}^\dagger|M\rangle \\ &= \lim_{M\to\infty} \sum_{n=0}^{M} \Big(\frac{1}{1+n} + \frac{1+n}{2+n}\Big) - \Big(1 + \frac{(M+1)^2}{M+2}\Big) = 1 \end{aligned} \tag{2.20}$$

[5] It is not too difficult to see that the instanton number must be independent of the noncommutativity parameter so in computing $Q_+$, $\vartheta$ may be set to unity without any loss of generality.

In the small $\vartheta$ limit, the energy density of the degree one instanton is written in terms of the noncommuting coordinates on the plane as [58]

$$\mathcal{E}_\star = \frac{1}{1+\bar{z}\star z} \star \frac{1}{1+(\bar{z}\star z+\vartheta)} = \frac{1}{(1+\frac{1}{2}r^2)^2} + \mathcal{O}(\vartheta^2)\,. \tag{2.21}$$

Note that the first corrections to the commutative instanton energy enter only at order $\vartheta^2$ so as long as $\vartheta$ is small the noncommutative instanton energy density may be adequately approximated by the lowest order term in the $\vartheta$ perturbation series. A similar computation for the degree two soliton $W_2$ (with $a_2 = c_2 = 0$) gives the energy density

$$\mathcal{E}_\star = 2\frac{|b_2|^2 r^2}{(|b_2|^2+|e_2|^2+\frac{1}{4}r^4+e_2\bar{z}^2+\overline{e_2}z^2)^2} + \mathcal{O}(\vartheta)\,. \tag{2.22}$$

These solutions are plotted in fig.1 for various values of the complex moduli $b_2$ and $e_2$. It is interesting to note that the low energy scattering of the two degree one instantons is not unlike that of the corresponding configuration of GMS noncommutative scalar solitons [29]. Indeed, this scattering property of was explicitly verified in the $\mathbb{CP}^1$ case in [25] where the metric on the two-soliton moduli space was directly computed.

## 2.3 Non-BPS states

In addition to the simplest solutions of the $\mathbb{CP}^n$ sigma model, the instanton solutions described above; it is also known that these field theories (and their generalizations to sigma models with Grassmannian target $Gr(n,m) = SU(n+m)/S(U(n)\times U(m))$ possess a non-BPS sector consisting essentially of bound states of instantons and anti-instantons [16, 17, 78]. Such solutions solve the ($2nd$ order) equations of motion without saturating any BPS bound on the energy functional and, not protected by supersymmetry, are in general unstable. These classical solutions have also resurfaced recently [47] when it was shown that not only do they solve the $\mathbb{CP}^n$ sigma model equations but that they also solve a Dirac-Born-Infeld (DBI) type action pointing to a bulk interpretation of these solitons as D-brane states although the precise states that they correspond to is not yet clear.

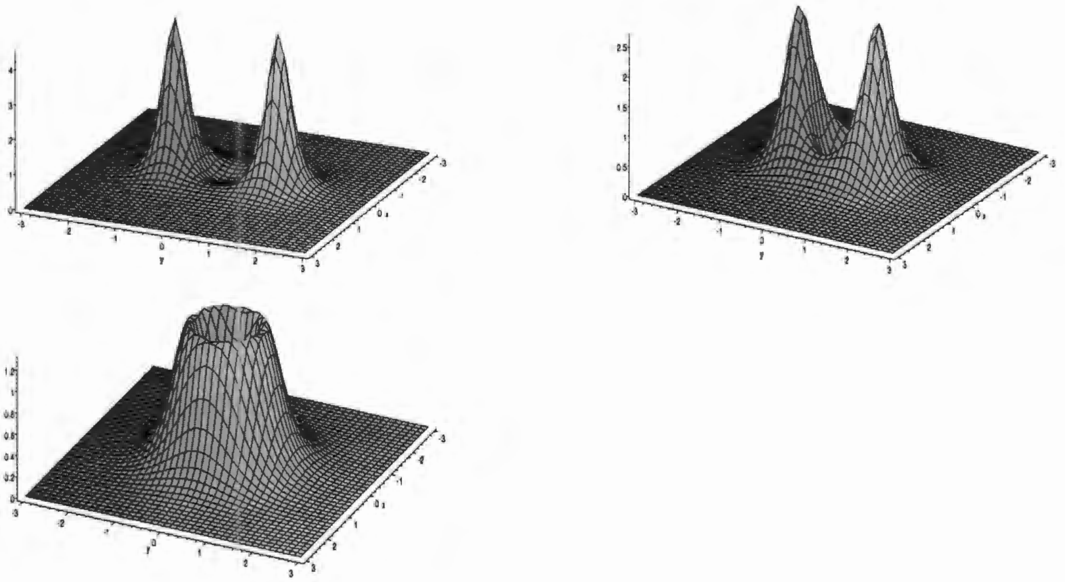


Figure 2.1: Static 2-soliton solution of the $\mathbb{CP}^1$-model with complex moduli $b_2 = 1$, and $e_2 = 1, 0.5$ and 0.

### 2.3.1 Constructing non-BPS states - The commutative case

In this section we aim to continue the analysis carried out in [78] and ask if such non-BPS states persist when the base space of the sigma model is made noncommutative. To this end, we briefly review the elegant construction employed in [78], modifying it to explicitly treat the $\mathbb{CP}^n$ sigma model. The idea behind said approach is rather elementary, demanding only a little linear algebra. Given a holomorphic $(n+1)$-vector $\mathbf{f}$ which characterizes the $\mathbb{CP}^n$ instanton, a set of $n+1$ vectors $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{n+1}\}$ is constructed from $\mathbf{f}$ (as described below) such that $span\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{n+1}\} = \mathbb{C}^{n+1}$. This set may then be orthonormalized by the conventional Gram-Schmidt procedure and, remarkably, any vector in the resulting orthonormal set is a solution of the $\mathbb{CP}^n$ equations of motion. Indeed, this may be seen quite easily as follows; in terms of the complex coordinates $(z, \bar{z})$ on $\mathbb{R}^2$ and the Hermitian projector $P$, the $\mathbb{CP}^n$ sigma model equations of motion may be written as

$$[\partial_z \partial_{\bar{z}} P, P] = 0\,. \tag{2.23}$$

Let $\mathbf{f}$ be some holomorphic $(n+1)$ component vector (any instanton solution will do) and define $\mathbf{f}_1 := \mathbf{f}, \mathbf{f}_2 := \partial_z \mathbf{f}, \ldots, \mathbf{f}_{n+1} := \partial_z^n \mathbf{f}$. Assuming linear independence of the $\mathbf{f}_i$ for $1 \leq i \leq n+1$ means that they span $\mathbb{C}^{n+1}$. This set may be orthonormalized by the usual Gram-Schmidt procedure to give an orthonormal basis for $\mathbb{C}^{n+1}$ as follows: Define $\mathbf{e}'_1 := \mathbf{f}_1$, $\mathbf{e}_1 := \mathbf{e}'_1/(\mathbf{e}'^{\dagger}_1 \cdot \mathbf{e}'_1)^{1/2}$ and

$$\begin{aligned} \mathbf{e}'_i &:= \mathbf{f}_i - \sum_{j=1}^{i-1} \mathbf{e}_j(\mathbf{e}^{\dagger}_j \cdot \mathbf{f}_i) \\ \mathbf{e}_i &:= \frac{\mathbf{e}'_i}{(\mathbf{e}'^{\dagger}_i \cdot \mathbf{e}'_i)^{1/2}}, \end{aligned} \tag{2.24}$$

for $2 \leq i \leq n+1$. It then follows quite straightforwardly that

$$P_i := \mathbf{e}_i \mathbf{e}^{\dagger}_i \quad 1 \leq i \leq n+1 \tag{2.25}$$

is a Hermitian projector. To show that the $\{\mathbf{e}_i\}$ form a set of solutions of the $\mathbb{CP}^n$ model, it suffices to show that the $P_i$ solve the sigma-model equations of motion (2.23) for any $1 \leq i \leq n+1$. To this end, it will prove useful to define the auxiliary matrix variable

$$Q_i := \sum_{j=1}^{i-1} \mathbf{e}_j \mathbf{e}^{\dagger}_j. \tag{2.26}$$

Clearly $Q$ is also a Hermitian projection operator orthogonal to $P$ since (for fixed $i$) $P_i Q_i = Q_i P_i = 0$. A few lines of algebra together with the identities

$$\begin{aligned} \partial_{\bar{z}} \mathbf{e}_i &= \sum_{k=1}^{i} \mathbf{e}_k(\mathbf{e}^{\dagger}_k \partial_{\bar{z}} \mathbf{e}_i) \\ \partial_z \mathbf{e}_i &= \sum_{k=1}^{i+1} \mathbf{e}_k(\mathbf{e}^{\dagger}_k \partial_z \mathbf{e}_i), \end{aligned} \tag{2.27}$$

establishes that[6] $\partial_{\bar{z}} Q_i Q_i = P_i \partial_{\bar{z}} Q_i = \partial_{\bar{z}}(P_i + Q_i)(P_i + Q_i) = \partial_{\bar{z}} P_i Q_i + P_i \partial_{\bar{z}} Q_i = 0$. The last equality of course follows from differentiation of the orthogonality

[6] For a detailed derivation of these properties of the projection operators we refer the interested reader to [78] and relevant references therein

relation satisfied by the $P_i$'s and $Q_i$'s. In much the same way it is also easy to verify that $P_i \partial_z Q_i = \partial_z Q_i$. Combining these gives

$$\partial_{\bar{z}} P_i P_i + \partial_{\bar{z}} Q_i = 0\,, \tag{2.28}$$

and by Hermitian conjugation

$$P_i \partial_z P_i + \partial_z Q_i = 0\,. \tag{2.29}$$

Taking the holomorphic derivative of the former and subtracting the antiholomorphic derivative of the latter gives the desired commutator and completes the proof. In the (commutative) classical $\mathbb{CP}^n$ sigma model, this procedure can be shown [16, 17] to generate *the most general* finite action solutions of the sigma model equations of motion. These solutions are interpreted variously as instantons, anti-instantons or unstable noninteracting mixtures thereof. In recent work [47] it was also shown that these non-BPS solitons of the $\mathbb{CP}^n$ model are not only finite action solutions of the sigma model but are also finite action solutions of a Dirac-Born-Infeld (DBI) model with a $\mathbb{CP}^n$ target space.

### 2.3.2 Constructing non-BPS states - The noncommutative case

Our focus is however on the noncommutative theory and, as such, one question of interest is whether or not the non-BPS construction above extends to the noncommutative $\mathbb{CP}^n$ model. Passing to the noncommutative variables $\widehat{z}$ and $\widehat{\bar{z}}$, results in the equations of motion

$$\Big[[\widehat{\bar{z}},[\widehat{z},P]],P\Big] = 0\,, \tag{2.30}$$

while the BPS and anti-BPS equations are, respectively

$$\begin{aligned} (1-P)\widehat{z}P &= 0 \\ (1-P)\widehat{\bar{z}}P &= 0\,. \end{aligned} \tag{2.31}$$

Any solution of the (anti-)BPS equations is also a solution of the Euler-Lagrange equations of motion; a fact that is obvious when the latter is written as $[\widehat{\bar{z}},(1-P)\widehat{z}P]+[\widehat{z},P\widehat{\bar{z}}(1-P)]=0$ or equivalently $[\widehat{z},(1-P)\widehat{\bar{z}}P]+[\widehat{\bar{z}},P\widehat{z}(1-P)]=0$. The reverse is, of course certainly not true in general and solutions of

(2.30) (if they exist) that do not solve (2.31) are precisely the non-BPS states. We now attempt to find such solutions by adapting the orthonormaization construction of [78]. Let $W$ be a holomorphic $(n+1)$-vector and define

$$\begin{aligned}
\mathbf{f}_1 &:= W \\
\mathbf{f}_2 &:= -\frac{1}{\vartheta}[\hat{\bar{z}}, W] \\
&\vdots \\
\mathbf{f}_k &:= (-1)^{k-1}\frac{1}{\vartheta^{k-1}}\underbrace{[\hat{\bar{z}},\ldots,[\hat{\bar{z}}, W]\ldots]}_{(k-1)\text{ commutators}} \\
&\vdots
\end{aligned} \tag{2.32}$$

The set $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{n+1}\}$ is orthonormalized as follows: Choose $\mathbf{e}_1 = W(1/\sqrt{W^\dagger W})$ and write

$$\begin{aligned}
\mathbf{e}'_2 &= \mathbf{f}_2 - \mathbf{e}_1(\mathbf{e}_1^\dagger \mathbf{f}_2) = -\frac{1}{\vartheta}\Big\{[\hat{\bar{z}}, W] - W\frac{1}{W^\dagger W}W^\dagger[\hat{\bar{z}}, W]\Big\} \\
&= -\frac{1}{\vartheta}(1-P_1)\hat{\bar{z}}W\,,
\end{aligned} \tag{2.33}$$

where $P_1 = W(W^\dagger W)^{-1}W^\dagger$ is a Hermitian projection operator and in the last line, use was made of the fact that $W$ is an eigenvector of $P_1$ with unit eigenvalue. Computing the norm of $\mathbf{e}'_2$ as $(1/\vartheta)W^\dagger\hat{z}(1-P_1)\hat{\bar{z}}W$ allows us to write

$$\mathbf{e}_2 = -(1-P_1)\hat{\bar{z}}W\frac{1}{\sqrt{W^\dagger\hat{z}(1-P_1)\hat{\bar{z}}W}}\,. \tag{2.34}$$

with the associated Hermitian projection operator

$$P_2 = (1-P_1)\hat{\bar{z}}W\frac{1}{W^\dagger\hat{z}(1-P_1)\hat{\bar{z}}W}W^\dagger\hat{z}(1-P_1)\,. \tag{2.35}$$

As in the commutative case, defining $P_j := \mathbf{e}_j\mathbf{e}_j^\dagger$ as the Hermitian projector associated to the $j$'th (orthonormal) basis vector we can by iteration construct

$$\begin{aligned}
\mathbf{e}'_k &:= (-1)^{k-1}\frac{1}{\vartheta^{k-1}}\Big(1-\sum_{j=1}^{k-1}P_j\Big)\Big[\hat{\bar{z}},\ldots,[\hat{\bar{z}}, W]\ldots\Big] \\
&= (-1)^{k-1}\frac{1}{\vartheta^{k-1}}\Big(1-\sum_{j=1}^{k-1}P_j\Big)\hat{\bar{z}}^{k-1}W\,,
\end{aligned} \tag{2.36}$$

where the last equality follows iteratively from the fact that $W \in \ker(1-P_1)$ and $\mathbf{e}_k$ is constructed from $\mathbf{e'}_k$ by the usual normalization. We leave it as a trivial exercise to the reader to verify that the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ is indeed orthonormal. That $\mathbf{e}_k$ as constructed above solves the sigma model equations of motion follows in close analogy to the commutative case. For concreteness though, we show this explicitly for the case $k = 2$. Observe that the projection operators $P_1$ and $P_2$ satisfy the relation

$$[\widehat{z}, P_2]P_2 + [\widehat{z}, P_1]P_2 = 0\,. \tag{2.37}$$

Moreover, the commutative derivative relation $\partial_z P_1 = \mathbf{e}_2\mathbf{e}_2^\dagger(\partial_z \mathbf{e}_1)\mathbf{e}_1^\dagger$ translates in noncommutative coordinates to $[\widehat{\bar{z}}, P_1] = P_2\widehat{\bar{z}}P_1$ so that $P_1$ and $P_2$ further satisfy

$$P_2[\widehat{\bar{z}}, P_1] \;=\; [\widehat{\bar{z}}, P_1]\,. \tag{2.38}$$

Substituting this into eq.(2.37) and applying the commutator $[\widehat{\bar{z}}, \cdot]$ to the resulting equation gives

$$\begin{aligned}\Big[\widehat{\bar{z}}, [\widehat{z}, P_2]P_2\Big] \;&+\; \Big[\widehat{\bar{z}}, [\widehat{z}, P_1]\Big] \\ &= [\widehat{z}, P_2][\widehat{\bar{z}}, P_2] + \Big[\widehat{\bar{z}}, [\widehat{z}, P_2]\Big]P_2 + \Big[\widehat{\bar{z}}, [\widehat{z}, P_1]\Big] = 0\,,\end{aligned} \tag{2.39}$$

where in the last step, we have made use of the Jacobi identity $[A,[B,C]] +$ *cyclic permutations* $= 0$ and the Heisenberg algebra satisfied by the noncommuting coordinates. In the latter form it is clear that the first and third terms in (2.39) are self-adjoint and so the subtraction from (2.39) of its Hermitian conjugate shows that $P_2$ satisfies (2.30) and verifies our claim that $\mathbf{e}_2$ is, in fact an exact solution of the noncommutative $\mathbb{CP}^1$ sigma model. That this is true, in itself should not be surprising given our construction. A further computation shows that

$$(1 - P_2)\widehat{\bar{z}}P_2 = [\widehat{\bar{z}}, (P_1 + P_2)]\,. \tag{2.40}$$

If the commutator on the right hand side of eq.(2.40) vanishes can we conclude that $\mathbf{e}_2$ is a *non-BPS* soliton[7].

[7] This is slightly abusive terminology since (2.40) would ensure only that $\mathbf{e}_2$ is not an anti-BPS soliton. We shall take "non-BPS" to mean both equations in (2.31) are nonvanishing.

### 2.3.3 Examples

This noncommutative modification of the Sasaki-Din-Zakrewski (SDZ) construction is perhaps best illustrated by some examples.

- $\underline{\mathbb{CP}^1}$: To begin with, let us consider the case $n = 1$. It is a well known fact [16, 17, 100] that for the commutative $\mathbb{CP}^1$ sigma model the SDZ construction maps instantons directly to their corresponding anti-instanton solutions. Since the construction yields a complete set of finite action solutions to the sigma model equations of motion it follows then that the commutative $\mathbb{CP}^1$ sigma model *does not possess a non-BPS spectrum.* One might naturally ask if the same is true for the noncommutative $\mathbb{CP}^1$ sigma model. It was already shown in [58] that the simplest BPS solution of the noncommutative $\mathbb{CP}^1$ model is the $Q = 1$ instanton with $W = (1, z)^T$ and associated projector (2.19). Substituting this into the expression for $\mathbf{e}_2$ in (2.34), simplifying the resulting 2-vector and relabelling the solitonic configuration by $\widetilde{W_2}$ we get

$$\widetilde{W_2} = \begin{pmatrix} -\frac{1}{1+\widehat{\bar{z}}\widehat{z}}\widehat{\bar{z}} \\ \frac{1}{1+\widehat{\bar{z}}\widehat{z}} \end{pmatrix} \sqrt{1+\widehat{\bar{z}}\widehat{z}} = \begin{pmatrix} -\widehat{\bar{z}} \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+\widehat{\bar{z}}\widehat{z}}}. \tag{2.41}$$

  This is precisely the normalized anti-holomorphic vector corresponding to the anti-instanton solution expected of the SDZ construction for $\mathbb{CP}^1$. This is easily verified by noting that $P_1 + P_2 = \mathbb{1}$ so that the commutator on the right hand side of eq.(2.40) vanishes. However, concluding from this that, as in the commutative case, the noncommutative $\mathbb{CP}^1$ sigma model does not possess a non-BPS sector would be at best premature (and certainly in this case erroneous). In a remarkable recent work [26] a large class of non-BPS configurations were constructed from meta-stable bound states of solitons and anti-solitons of the *GMS* noncommutative scalar field theory [29],[30]. The construction of [26] hinges on the fact that in a basis that diagonalizes the $(2 \times 2)$ Hermitian projector $P$ associated to a solution of the $\mathbb{CP}^1$ sigma model equations, the diagonal entries $\phi_1(\widehat{z}, \widehat{\bar{z}})$ and $\phi_2(\widehat{z}, \widehat{\bar{z}})$ will also solve (2.30). In particular if $\phi_1(\phi_2)$ are taken to be GMS (anti)solitons satisfying $(1 - \phi_1)\widehat{z}\phi_1 = 0$ and $(1-\phi_2)\widehat{\bar{z}}\phi_2 = 0$ respectively, then $P$ does not solve either of the equations in (2.31) and the corresponding field configuration $W$ is a non-BPS soliton of the $\mathbb{CP}^1$ sigma model. From this example it is alarmingly clear that the SDZ construction does not saturate the set of solutions of the noncommutative $\mathbb{CP}^1$ sigma model.

- $\underline{\mathbb{CP}^2}$: Having shown that the modified SDZ construction is insensitive to the non-BPS spectrum of the noncommutative $\mathbb{CP}^1$ sigma model we now consider the $n = 2$ case. Analysis of these solutions will prove useful in facilitating comparison with the work of [26]. The simplest instanton solution of the $\mathbb{CP}^2$ sigma model is $W = (1, \widehat{z}, \widehat{z}^2)^T$. The corresponding Hermitian projector is computed to be

$$P_1 := W\frac{1}{W^\dagger W}W^\dagger = \begin{pmatrix} \frac{1}{A} & \frac{1}{A}\widehat{\bar{z}} & \frac{1}{A}\widehat{\bar{z}}^2 \\ \widehat{z}\frac{1}{A} & \widehat{z}\frac{1}{A}\widehat{\bar{z}} & \widehat{z}\frac{1}{A}\widehat{\bar{z}}^2 \\ \widehat{z}^2\frac{1}{A} & \widehat{z}^2\frac{1}{A}\widehat{\bar{z}} & \widehat{z}^2\frac{1}{A}\widehat{\bar{z}}^2 \end{pmatrix},$$

where $A(\widehat{\bar{z}}\widehat{z}) = 1 + \widehat{\bar{z}}\widehat{z} - \vartheta\widehat{\bar{z}}\widehat{z} + (\widehat{\bar{z}}\widehat{z})^2$ is the square modulus of $W$. Using the relations $\widehat{\bar{z}}f(\widehat{\bar{z}}\widehat{z}) = f(\widehat{\bar{z}}\widehat{z} - \vartheta)\widehat{\bar{z}}$ and $\widehat{z}f(\widehat{\bar{z}}\widehat{z}) = f(\widehat{\bar{z}}\widehat{z} + \vartheta)\widehat{z}$ we find

$$\begin{aligned} \widetilde{W_2} &:= \mathbf{e}_2 = -(1 - P_1)\widehat{\bar{z}}W\frac{1}{\sqrt{W^\dagger\widehat{z}(1 - P_1)\widehat{\bar{z}}W}} \\ &= \begin{pmatrix} -\widehat{\bar{z}}(1 + 2\widehat{\bar{z}}\widehat{z}) \\ 1 - \vartheta\widehat{\bar{z}}\widehat{z} - (\widehat{\bar{z}}\widehat{z})^2 \\ \widehat{z}(\widehat{\bar{z}}\widehat{z} + 2) \end{pmatrix}\frac{1}{\sqrt{B(\widehat{\bar{z}}\widehat{z})}}, \end{aligned} \tag{2.42}$$

with $B(\widehat{\bar{z}}\widehat{z}) = 1 + \vartheta + (5 + 6\vartheta + \vartheta^2)\widehat{\bar{z}}\widehat{z} + (6 + 6\vartheta + \vartheta^2)(\widehat{\bar{z}}\widehat{z})^2 + (5 + 2\vartheta)(\widehat{\bar{z}}\widehat{z})^3 + (\widehat{\bar{z}}\widehat{z})^4$. It is straightforward (but tedious) to compute $P_2 = \widetilde{W_2}^\dagger\widetilde{W_2}$ and check that the commutator $[\widehat{z}, (P_1 + P_2)]$ is nonvanishing and so conclude that $\widetilde{W_2}$ is a genuine non-BPS soliton of the $\mathbb{CP}^2$ sigma model. As a check we find that in the $\vartheta \to 0$ limit $\widetilde{W_2}$ becomes

$$\begin{pmatrix} -\bar{z}(1 + r^2) \\ 1 - \frac{1}{4}r^4 \\ z(\frac{1}{2}r^2 + 2) \end{pmatrix}\frac{1}{\sqrt{1 + \frac{5}{2}r^2 + \frac{3}{2}r^4 + \frac{5}{8}r^6 + \frac{1}{16}r^8}}, \tag{2.43}$$

in complete agreement with [16].

- $\underline{\mathbb{CP}^2}$: As a final illustration of the construction technique we start with the $\mathbb{CP}^2$ instanton $W = (\widehat{z}^2 + 1, \widehat{z}^2 - 1, 2\widehat{z})^T$. With $A(\widehat{\bar{z}}\widehat{z}) = 2 + (4 - 2\vartheta)\widehat{\bar{z}}\widehat{z} + 2(\widehat{\bar{z}}\widehat{z})^2$, the corresponding projection operator is written

$$P_1 = \begin{pmatrix} (\widehat{z}^2 + 1)\frac{1}{A}(\widehat{\bar{z}}^2 + 1) & (\widehat{z}^2 + 1)\frac{1}{A}(\widehat{\bar{z}}^2 - 1) & 2(\widehat{z}^2 + 1)\frac{1}{A}\widehat{\bar{z}} \\ (\widehat{z}^2 - 1)\frac{1}{A}(\widehat{\bar{z}}^2 + 1) & (\widehat{z}^2 - 1)\frac{1}{A}(\widehat{\bar{z}}^2 - 1) & 2(\widehat{z}^2 - 1)\frac{1}{A}\widehat{\bar{z}} \\ 2\widehat{z}\frac{1}{A}(\widehat{\bar{z}}^2 + 1) & 2\widehat{z}\frac{1}{A}(\widehat{\bar{z}}^2 - 1) & 4z\frac{1}{A}\widehat{\bar{z}} \end{pmatrix}.$$

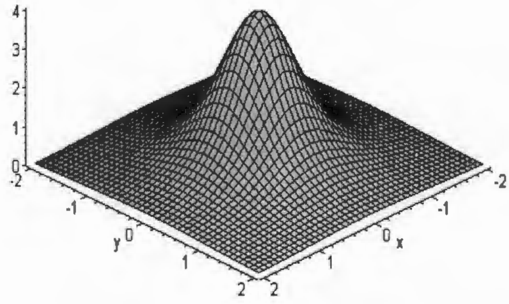



Figure 2.2: A $\mathbb{CP}^2$ bound state consisting of two instantons and two anti-instantons all coincident at the origin with total energy density four times that of a single instanton.

If we define $B(\hat{\bar z}\hat z) = 1 + 2\vartheta + (4 + 6\vartheta + 2\vartheta^2)\hat{\bar z}\hat z + (6 + 6\vartheta + \vartheta^2)(\hat{\bar z}\hat z)^2 + (4+2\vartheta)(\bar z z)^3 + (\hat{\bar z}\hat z)^4$ then the (normalized) non-BPS soliton constructed from $W$ may be written

$$\widetilde{W_2} = \left[ \begin{pmatrix} \hat z - \hat{\bar z} \\ \hat z + \hat{\bar z} \\ 1 - \hat{\bar z}\hat z \end{pmatrix} (1 + \hat{\bar z}\hat z) + \vartheta \begin{pmatrix} \hat z \\ \hat z \\ -\hat{\bar z}\hat z \end{pmatrix} \right] \frac{1}{\sqrt{B(\hat{\bar z}\hat z)}}. \tag{2.44}$$

In this form the commutative limit is very conveniently investigated. Sending $\vartheta$ to zero and noticing that $B(\hat{\bar z}\hat z) \to (1 + \bar z z)^4$ reduces $\widetilde{W_2}$ to

$$\begin{pmatrix} z - \bar z \\ z + \bar z \\ 1 - \frac{1}{2}r^2 \end{pmatrix} \frac{1}{1 + \frac{1}{2}r^2}. \tag{2.45}$$

Again, this corresponds exactly to what is expected in the commutative limit. As can be seen from the soliton energy density (see Fig.2) the non-BPS state is formed from a bound state of two degree-1 $\mathbb{CP}^2$ instantons and two anti-instantons all coincident at the origin.

## 2.4 $\mathbb{CP}^n$ solitons and GMS solitons

The existence of a non-BPS spectrum of the noncommutative $\mathbb{CP}^1$ model is intriguing [26]; even more so since the construction of non-BPS solitonic configurations is so intimately connected to the noncommutative scalar solitons of [29]. It seems only natural then to try and probe this connection further in the hope of a deeper understanding of the space of solutions to the noncommutative $\mathbb{CP}^n$ sigma model. Returning to the BPS solitons of Sec.(2.3), it may be immediately seen that an alternative construction of degree $k$ solitons would be to take the Hermitian projector $P$ in the diagonal representation (as in [26])

$$P = \begin{pmatrix} \widehat{\phi}_1 & 0 & \dots & 0 \\ 0 & \widehat{\phi}_2 & 0\dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \widehat{\phi}_{n+1} \end{pmatrix}, \tag{2.46}$$

where each of the $\widehat{\phi}$'s satisfy[8]

$$(1-\widehat{\phi}_i)\widehat{a}\widehat{\phi}_i = 0 \tag{2.47}$$

and $\sum_i \mathrm{Rank}(\widehat{\phi}_i) = k$. This necessarily implies that $(1-P)\widehat{a}P = 0$ – precisely the condition that $P$ corresponds to a BPS $\mathbb{CP}^n$ configuration. In general, solutions to (2.47) are parameterized by $k_i$ complex coherent state vectors $|z_i^a\rangle := \exp(z_i^a\widehat{a}^\dagger)|0\rangle$ with

$$\widehat{\phi}_i = \sum_{a,b=1}^{k_i} |z_i^a\rangle \frac{1}{\langle z_i^a|z_i^b\rangle}\langle z_i^b| \tag{2.48}$$

These are, of course, the noncommutative scalar solitons of the GMS model[29, 30]. Apparently then, in addition to the standard construction of $k$-soliton solutions of the noncommutative $\mathbb{CP}^n$ sigma model [58, 25, 26], such solutions may also be constructed from stacking GMS solitons of appropriate charge on the plane. The emergence of a non-BPS sector of the $\mathbb{CP}^1$ model for finite $\vartheta$ via such a construction lends much weight in favor of this claim. It will be shown in this section however that, at least for (anti-)BPS solutions, such an 'alternative' construction should have been expected since (2.46) is just the diagonal representation of the usual Hermitian projector associated to BPS solutions of the $\mathbb{CP}^n$ sigma model. This is most easily illustrated for the case of the static 1-soliton solution of the noncommutative $\mathbb{CP}^1$ model for which $P$

[8]For concreteness, we shall focus only on the BPS solutions with the understanding that similar relations hold for the anti-BPS case.

is given by

$$P = \begin{pmatrix} \frac{1}{1+\widehat{N}} & \frac{1}{1+\widehat{N}}\widehat{a}^\dagger \\ \widehat{a}\frac{1}{1+\widehat{N}} & \frac{1+\widehat{N}}{2+\widehat{N}} \end{pmatrix} \tag{2.49}$$

after setting $\vartheta = 1$. Denoting $|I\rangle = (1,0)^T$ and $|II\rangle = (0,1)^T$, an eigenvector of $P$ with eigenvalue $\lambda$ may be expanded as

$$|\Psi\rangle = |\psi_1\rangle \otimes |I\rangle + |\psi_2\rangle \otimes |II\rangle \tag{2.50}$$

where $|\psi_i\rangle \in \mathcal{H}$ may be expanded in the harmonic oscillator basis as $|\psi_i\rangle = \sum_{n=0}^{\infty} c_{n,i}|n\rangle$. The action of $P$ on the basis elements is easily determined to be

$$\begin{aligned} P|n\rangle \otimes |I\rangle &= \frac{1}{1+n}|n\rangle \otimes |I\rangle + \frac{\sqrt{n}}{1+n}|n-1\rangle \otimes |II\rangle \\ P|n\rangle \otimes |II\rangle &= \frac{\sqrt{1+n}}{2+n}|n+1\rangle \otimes |I\rangle + \frac{1+n}{2+n}|n\rangle \otimes |II\rangle \end{aligned} \tag{2.51}$$

so that in terms of the expansion coefficients $c_{n,1}$ and $c_{n,2}$ the eigenvalue equation for $P$ becomes

$$\begin{aligned} &\sum_{n=0}^{\infty} \Big(\frac{c_{n,1} - \lambda(n+1)c_{n,1} + \sqrt{n}c_{n-1,2}}{n+1}\Big)|n\rangle \otimes |I\rangle \\ + &\sum_{n=0}^{\infty} \Big(\frac{c_{n+1,1}\sqrt{n+1} + (n+1)c_{n,2} - \lambda(n+2)c_{n,2}}{n+2}\Big)|n\rangle \otimes |II\rangle = 0 \end{aligned} \tag{2.52}$$

Since $P$ is a projection operator, $\lambda = 0$ or $1$. Choosing first $\lambda = 1$ reduces (2.52) to

$$c_{n,2} = \sqrt{n+1}c_{n+1,1} \tag{2.53}$$

which fixes completely the $c_{n,2}$ coefficients in terms of the $c_{n,1}$'s and gives

$$\begin{aligned} |\psi_1\rangle &= \sum_{n=0}^{\infty} c_{n,1}|n\rangle \\ |\psi_2\rangle &= \sum_{n=0}^{\infty} \sqrt{n+1}c_{n+1,1}|n\rangle \end{aligned} \tag{2.54}$$

In an orthonormal eigenbasis $\{|\chi_1\rangle, |\chi_2\rangle\}$ the diagonal representation of the $2 \times 2$ matrix $P$ is

$$P = |\chi_1\rangle\langle\chi_1| \otimes |I\rangle\langle I| + |\chi_2\rangle\langle\chi_2| \otimes |II\rangle\langle II| \tag{2.55}$$

It remains only to fix the $c_{n,1}$ coefficients. This may be done by noting that $P$ is a solution of the $\mathbb{CP}^n$ BPS equations only if $(1 - |\chi_i\rangle\langle\chi_i|)\widehat{a}|\chi_i\rangle\langle\chi_i| = 0$ *i.e.*, if $|\chi_i\rangle$ is an eigenstate of $\widehat{a}$. Written in terms of the expansion coefficients, this condition reads

$$c_{n,1} = \frac{\chi_1^n}{\sqrt{n!}} c_{0,1} \tag{2.56}$$

where $\chi_1$ is the eigenvalue corresponding to $|\chi_1\rangle$. This is, of course, expected of a coherent state in a harmonic oscillator basis. It may also quite easily be established that the $\lambda = 0$ case is trivial, yielding $|\chi_1\rangle = |\chi_2\rangle = 0$. A straightforward application of Gram-Schmidt orthonormalization finds

$$|\chi_1\rangle = \sum_{n=0}^{\infty} \frac{c_{n,1}}{\sqrt{\sum_{m\geq 0} |c_{m,1}|^2}} |n\rangle , \qquad |\chi_2\rangle = 0 \tag{2.57}$$

leaving only the first term in (2.55). Without loss of generality, the residual degree of freedom in (2.56) may be fixed by choosing $c_{0,1} = 1$. The eigenvalues $\chi_1$ are then interpreted as the complex location moduli of the solitons. For example, the simplest choice of $\chi_1 = 0$ produces a degree 1 soliton localized at the origin,

$$P = \begin{pmatrix} |0\rangle\langle 0| & 0 \\ 0 & 0 \end{pmatrix} \tag{2.58}$$

The end result then is that in a diagonal representation the $\mathbb{CP}^1$ 1-soliton solution is nothing but a unit rank GMS soliton. These results are easily extended to show that the $k$-soliton solution of the $\mathbb{CP}^n$ sigma model in a diagonalizing basis may be written in the form (2.46). The interpretation here is that any degree $k$ $\mathbb{CP}^n$ soliton may be built up of appropriate rank GMS solitons. Note, however, that the diagonalization is non unitary - given a rank $k$ matrix of the form (2.46) it is not possible in general to associate to it a unique (non-diagonal) Hermitian projection matrix that is also a solution of the sigma model equations. The set of solutions to the sigma model equations that are of the form (2.46) is considerably larger that those formed by adapted commutative constructions. As such, it is not surprising that the solution space of the noncommutative $\mathbb{CP}^n$ sigma model is much larger than the corresponding commutative theory. In particular, as shown in [26], certain quasi-stable configurations of GMS solitons and anti-solitons form non-BPS states of of the noncommutative $\mathbb{CP}^n$ sigma model that have zero size in the vanishing $\vartheta$ limit. Such solutions cannot be realized as the diagonalization of any non-diagonal solution of the sigma model equations.

**GMS solitons**

Sometimes it is, in fact, possible to get something for nothing, or at least for very little. Consider the rather trivial noncommutative static action

$$S = \int_{\mathbb{R}^2_\vartheta} dx_1 dx_2\, V_\star(\phi)$$

consisting only of a potential that is polynomial in the noncommutative field. Assuming that the potential is at least quadratic in the field, $\phi$ may always be chosen such that the $V_\star$ takes the form

$$V_\star(\phi) = \frac{m^2}{2}\phi \star \phi + c_1 \phi \star \phi \star \phi + \dots$$

Translating to the operator formulation, the action becomes

$$S = 2\pi\vartheta \mathrm{Tr}_{\mathcal{H}} V(\hat{\phi})$$

The equation of motion resulting from the variation of $S$ can be written in the form

$$c\hat{\phi} \prod_{k=1}^{n} (\hat{\phi} - \lambda_k) = 0$$

where the $\lambda_i$ are critical points of the potential and $c$ is a (commuting) number. Clearly, if this were a commutative equation, the only solutions would be the trivial ones $\phi = \lambda_i$. Not so in the noncommutative case! Here non-trivial solutions can be constructed by using the projection operator $P = |n\rangle\langle n|$. In particular $\hat{\phi} = \lambda_i P$ is a solution of the equation of motion.

**Continued**

To see that this is indeed the case, notice that, on substituting into the equation of motion, there will always be a product of terms $P(1-P)$ which vanishes by the properties of projectors. Indeed this means that if $\lambda_i P$ is a solution, so too is $\lambda_i(1-P)$. These are the so-called Gopakumar-Minwalla-Strominger (GMS) solitons. Their realisation on the noncommutative plane is perhaps best illustrated by choosing $P = |0\rangle\langle 0|$, the projector onto the harmonic oscillator ground state. In this case, an application of the inverse Weyl transform shows that $|0\rangle\langle 0| \to 2e^{-(x_1^2+x_2^2)/\vartheta}$, a Gaussian lump with finite energy

$$E = 2\pi\vartheta \mathrm{Tr}_{\mathcal{H}} V(\hat{\phi}) = 2\pi\vartheta V(\lambda_i)$$

## 2.5 Summary and discussion

In trying to understand the connection (if any) between solitonic excitations of the noncommutative sigma model on $\mathbb{CP}^n$ and $D$-brane configurations in string theory, we have reformulated the noncommutative $\mathbb{CP}^n$ model of [58] in a way that makes manifest the similarities (and differences) with the GMS scalar field theory. In doing so it becomes evident that the BPS solitons of the sigma model are no more immune from problems in the definition of the topological charge than any of the higher codimension solitons of, say, four-dimensional noncommutative gauge theory [48]. In this case, the naive calculation of the topological charge is in fact incorrect and must be supplemented by the addition of a nonvanishing "surface term" of the form $\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}[\widehat{a},\cdot]$. Such terms vanish for GMS solitons and are consequently dropped in that case.

We have also extended the SDZ construction for non-BPS solitons of the $\mathbb{CP}^n$ model from known holomorphic (BPS) lumps and constructed explicit solutions for the case of $\mathbb{CP}^1$ and $\mathbb{CP}^2$ . Unlike the commutative case though, the noncommutative SDZ construction does not yield the most general solitonic solutions of the sigma model equations. This incompleteness is due largely to the emergence of a new length scale in the problem as set by the noncommutativity parameter $\vartheta$. Evidently, as shown in [26], the solution space of the noncommutative sigma model is significantly larger than the corresponding commutative one with the additional (non-BPS) solitons made up of quasi-stable bound states of GMS solitons. While this construction might seem independent of the standard one, we have shown that, at least for the case of BPS solitons, they arise in the diagonalization of the Hermitian projector associated to a given BPS soliton.

The (commutative) $\mathbb{CP}^n$ model is well known to arise as the low energy (infinite coupling) limit of a gauged linear sigma model with Fayet-Illiopolous $D$-term [80]. One class of solitonic excitations of this model are the vortex solutions of the first order BPS equations

$$\begin{aligned} F_{12} + e^2(\phi^\dagger\phi - 1) &= 0 \\ D_{\bar{z}}\phi &= 0 \\ \Phi = \int d^2x F_{12} &= 2\pi k > 0 \end{aligned} \tag{2.59}$$

In the $e^2 \to \infty$ limit these vortex solitons descend to the usual $\mathbb{CP}^n$ lumps (when $\phi$ is an $(n+1)$-component complex vector). It was recently demonstrated that the noncommutative version of the linear sigma model in question also exhibits vortex excitations that are solutions of the noncommutative vortex equations [3, 4, 49, 89]

$$\begin{aligned} 1 + [C^\dagger, C] &= \gamma(\phi^\dagger\phi - 1) \\ \phi a + C\phi &= 0 \\ \mathrm{Tr}_{\mathcal{H}}\Big(1 - [C^\dagger, C]\Big) &= -k \end{aligned} \tag{2.60}$$

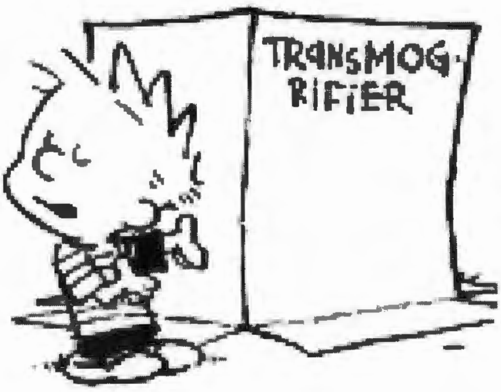
where $\gamma = \vartheta e^2$ is a dimensionless parameter and $C$ is effectively the (noncommutative) Abelian gauge field. The exact vortex solutions of [3, 49, 89] manifest in the $\gamma \to \infty$ limit (in which the vortex equations become tractable). Usually this limit is taken by sending $\vartheta \to \infty$ but clearly may also arise in the infinite coupling limit; the vortex equations are insensitive to which. As such, it is not unreasonable to expect that the exact vortex solutions descend to the lump solutions of the noncommutative $\mathbb{CP}^n$ sigma model. Moreover, the noncommutative Abelian Higgs model may be embedded in a $(5+1)$-dimensional, $\mathcal{N} = 1$ supersymmetric theory so that the vortices of the former become BPS 3-branes which preserve half of the supersymmetries. So, more than just another academic exercise, the study of the infinite coupling limit of the noncommutative Abelian Higgs model may provide valuable insight into a string theoretic interpretation of the noncommutative $\mathbb{CP}^n$ lumps. These issues will be addressed in future work.

Yet another intriguing avenue for a stringy interpretation of the BPS solitonic excitations of the $\mathbb{CP}^n$ sigma model is that offered by the work of [57].

Drawing on the (tree level) equivalence of $N = 2$ open string theory and self-dual Yang-Mills theory in (2+2)-dimensions [75] it was argued that the effective field theory induced by open $N = 2$ strings in a Kähler $B$-field background on the worldvolume of $n$ coincident $D2$-branes is a modified $U(n)$ sigma model. The latter was also shown to exhibit solitonic solutions which were elegantly constructed using a 'dressing method' [55, 56]. From this perspective, a string theoretic interpretation already exists: an $m$-soliton solution to the noncommutative $\mathbb{CP}^n$-sigma model should correspond to $m$ $D0$-branes inside $(n+1)$ coincident $D2$-branes. A positive identification of the solitonic excitations of the sigma model with the $D0 - D2$ system would, however, require more than just a matching of the energies of the two systems; it remains to compute the fluctuation spectra around the respective configurations. This is certainly an exciting avenue and warrants further research.

# Chapter 3

# Transmogrifying fuzzy vortices




*The construction of vortex solitons of the noncommutative Abelian-Higgs model is extended to a critically coupled gauged linear sigma model with Fayet-Illiopolous D-terms. Like its commutative counterpart, this fuzzy linear sigma model has a rich spectrum of BPS solutions. We offer an explicit construction of the degree−$k$ static semilocal vortex and study in some detail the infinite coupling limit in which it descends to a degree−$k$ $\mathbb{CP}^N$ instanton. This relation between the fuzzy vortex and noncommutative lump is used to suggest an interpretation of the noncommutative sigma model soliton as tilted D-strings stretched between an NS5-brane and a stack of D3-branes in type IIB superstring theory.*

## 3.1 Introduction

A little more than a decade ago, the study of electroweak strings in a modified Abelian-Higgs theory initiated in [93] revealed a curious new vortex solution. As the story goes, vortices are indeed enigmatic objects [89] and the *semilocal vortices* found in [93] are no exception. Firstly, standard lore holds that a non-simply-connected vacuum manifold is a necessary condition for the existence of stable, finite energy cosmic string solutions. If this is anything to go by, the very existence of these semilocal vortices should be called into question since the vacuum manifold of the modified Abelian-Higgs theory is $S^3$. Yet exist they do. Consequently, a more consistent condition was offered in [44]. Semilocal vortices (actually, this holds for other defects as well) form in theories exhibiting spontaneous symmetry breaking and whose vacuum manifold is fibred by the action of the gauge group in some non-trivial way. In this same work it was realised also that the low momentum dynamics of these vortices bear a striking resemblance to the 2−dimensional lump solutions of the $\mathbb{CP}^N$ nonlinear sigma model. Since then, this similarity between the modified Abelian-Higgs theory (a.k.a gauged linear sigma model) and the $\mathbb{CP}^N$ (or, more generally, Grassmannian) sigma model has found itself the subject of much attention [34, 80, 99]. Nevertheless, much of what is known about the semilocal vortex is only asymptotic. Even its descent to the lump in the infinite coupling limit is only exact at spatial infinity and suffers Skyrme term corrections at smaller radial distances. This is the allure and frustration of vortices; as simple as their defining equations seem, they are also remarkably unyielding.

Until a short time ago, the only avenue toward tractable vortex equations was a curvature deformation of the background space in which the vortices live [86, 97]. These are, of course, not without their own puzzles. The recent renaissance in noncommutative geometry (due in no small part to the seminal work of [81]) offers new recourse. *Fuzzy deformations* of the background space have, in only a few years, not only yielded a wealth of new solitonic solutions but also several new insights into old solutions to a host of field theories (see [19, 36, 87] for excellent reviews). The *noncommutative* Abelian-Higgs model, for example, exhibits exact vortex solutions [3, 4, 61, 49] whose moduli space metric can be computed explicitly in the large noncommutativity limit [89]. In this work we extend this idea to the $(2+1)$−dimensional, critically coupled, gauged linear sigma model with an $N+1$ component Higgs field. The BPS spectrum of the resulting fuzzy theory is studied and, like its commutative counterpart, shown to have quite rich structure. In particular, we use an extension of the computational technique of [61] to explicitly construct a family of exact semilocal vortices. As expected, our family contains the Abelian-Higgs vortices of [3, 61, 49] as well as the fluxons of [38] as special cases. As suggested by the title, the metamorphosis of the semilocal vortex is of central importance in this chapter. By turning up the gauge coupling, we demonstrate conclusively, at the level of the solutions, the descent of the semilocal vortex into the instanton solution of the fuzzy $\mathbb{CP}^N$ model of the same degree. Interestingly, unlike the commutative case, this "transmogrification" of the vortex is exact at a certain point in the parameter space of the theory. Finally, we turn our attention toward the physical[1] interpretation of the $k-$lump solution of the noncommutative $\mathbb{CP}^N$ model of [58]. Without much additional work, the brane configuration in type II-B string theory that realises the fuzzy lump may be read off from the construction of [34] as tilted $D-$strings suspended between an $NS5-$ and $D3-$brane. We conclude, as is conventional, with the conclusions.

## 3.2 The gauged linear sigma model

### 3.2.1 Definitions

Among the many extensions to the Abelian-Higgs model, one of the most natural is the gauged linear sigma model with Fayet-Illiopolous D-terms [80, 99]. This is certainly true if the aim is the construction of a model that supports solitonic excitations saturating BPS-like bounds. With its $\mathbb{C}^{N+1}$-valued scalar fields and $U(1)^{N+1}$ gauge symmetry, the linear sigma model is a natural springboard for our discussion of the relation between noncommutative semilocal vortices, fuzzy sigma model lumps and the braney systems they are associated with. To this end then it will prove useful to briefly review some of the ideas and notation used to extract the vortex excitations from the solution spectrum of the semilocal model. Following [80] we write the linear sigma model action as

$$S_{SL} = -\int_{\mathbb{R}^{(1,2)}} d^3x \Big[(D_\mu\Phi)(D^\mu\Phi)^\dagger + \sum_{a=1}^{N+1}\frac{1}{4e_a^2}({F^a}_{\mu\nu})^2 + \sum_{a=1}^{N}\frac{e_a^2}{2}(R_a - \Phi\tau_a\Phi^\dagger)^2\Big] \tag{3.1}$$

[1] By which we mean 'stringy'.

The dynamical degrees of freedom in this model are encoded in a $\mathbb{C}^{N+1}$-valued spacetime scalar $\Phi = (\phi_1, \ldots, \phi_{N+1})$ and the $N+1$ $U(1)$-valued connection forms $A^a = A^a{}_\mu dx^\mu$ with associated curvature 2-forms $F^a = dA^a$. The $\tau_a$ are the $N+1$ generators of $U(1)^{N+1}$. The gauge covariant derivative we will take as

$$D := d - i \sum_{a=1}^{N+1} \tau_a A^a \tag{3.2}$$

There are two sets of parameters in the theory; the $N+1$ coupling constants $e_a$ of dimension $(mass)^{1/2}$ and $N+1$ Fayet-Illiopolous (FI) parameters $R_a$ - effectively the vacuum expectation values of the components of $\Phi$. Without loss of generality (and because we can always re-scale the fields to absorb them anyway) we set the latter to unity. The coupling constants we retain because they control the energy scales of the model.

In the temporal gauge, the static energy corresponding to the action (3.1) is

$$E = \int_{\mathbb{R}^2} d^2x \left[ (D_i\Phi)(D_i\Phi)^\dagger + \sum_{a=1}^{N+1} \frac{1}{4e_a^2}(F^a{}_{ij})^2 + \sum_{a=1}^{N+1} \frac{e_a^2}{2}(\Phi\tau_a\Phi^\dagger - 1)^2 \right] \tag{3.3}$$

For instance, in the case $N = 1$, following [80] the energy functional becomes (in exhaustive detail)

$$\begin{aligned} E = \int_{\mathbb{R}^2} d^2x \quad \Big[ & (D_1\Phi)(D_1\Phi)^\dagger + (D_2\Phi)(D_2\Phi)^\dagger + \frac{1}{4e_1^2}(F_{ij})^2 + \frac{1}{4e_2^2}(G_{ij})^2 \\ + & \frac{e_1^2}{2}(\Phi\tau_1\Phi^\dagger - 1)^2 + \frac{e_2^2}{2}(\Phi\tau_2\Phi^\dagger - 1)^2 \Big] \end{aligned} \tag{3.4}$$

where the $GL(2,\mathbb{R})$-valued connections $A = \tau_1 A^1$ and $B = \tau_2 A^2$ are associated to the curvature forms $F = dA$ and $G = dB$ respectively. For our purposes, it will suffice to turn off $B$ and $e_2$ and take $\tau_1 = \mathbf{1}_2$ giving

$$E = \int_{\mathbb{R}^2} d^2x \left[ (D_1\Phi)(D_1\Phi)^\dagger + (D_2\Phi)(D_2\Phi)^\dagger + \frac{1}{2e_1^2}(F_{ij})^2 + \frac{e_1^2}{4}(\Phi\Phi^\dagger - 1)^2 \right] \tag{3.5}$$

Even under such restricted circumstances, the resulting linear sigma model is still remarkably rich [80, 99], exhibiting a wealth of solitonic structure and enjoying intimate relations with nonlinear sigma models on toric varieties.

In what follows, it will prove convenient to rewrite the energy in terms of the complex coordinates $z := (x^1 + ix^2)/\sqrt{2}$ and $\bar{z} := (x^1 - ix^2)/\sqrt{2}$. This particular normalization means that

$$\partial_z := \frac{\partial}{\partial z} = \frac{1}{\sqrt{2}}\Big(\partial_1 - i\partial_2\Big) \qquad \partial_{\bar{z}} := \frac{\partial}{\partial \bar{z}} = \frac{1}{\sqrt{2}}\Big(\partial_1 + i\partial_2\Big) \tag{3.6}$$

This in turn induces a complexification of the gauge covariant derivative so that

$$D_z := \frac{1}{\sqrt{2}}\Big(D_1 - iD_2\Big) \qquad D_{\bar{z}} := \frac{1}{\sqrt{2}}\Big(D_1 + iD_2\Big). \tag{3.7}$$

when $A_z := (A_1 - iA_2)/\sqrt{2}$ and $A_{\bar{z}} := (A_1 + iA_2)/\sqrt{2}$. These are of course now $GL(2,\mathbb{C})$-valued objects. With these definitions,

$$E = \int_{\mathbb{C}} d^2z \left[(D_z\Phi)(D_z\Phi)^\dagger + (D_{\bar{z}}\Phi)(D_{\bar{z}}\Phi)^\dagger + \frac{1}{2e^2}(F_{12})^2 + \frac{e^2}{2}(\Phi\Phi^\dagger - 1)^2\right] \tag{3.8}$$

### 3.2.2 Solitons on the plane

To see the emergence of the semilocal vortex in the spectrum of the gauged linear sigma model, the usual method of "completing the square" in the energy functional may be followed. After some straightforward manipulations, (3.8) may be put into the form

$$\begin{aligned} E &= \int_{\mathbb{C}} d^2z \left[2(D_{\bar{z}}\Phi)(D_{\bar{z}}\Phi)^\dagger + \frac{1}{2e^2}\Big|F_{12} + e^2(\Phi\Phi^\dagger - 1)\Big|^2\right] \\ &+ \int_{\mathbb{C}} d^2z\, T + \underbrace{\int_{\mathbb{C}} d^2z\, F_{12}}_{2\pi k} \end{aligned} \tag{3.9}$$

where $T = \partial_{\bar{z}}(\Phi D_z \Phi^\dagger) - \partial_z(\Phi D_{\bar{z}}\Phi^\dagger)$. As such, the second to last term is a total derivative whose integral vanishes. Consequently, a nonvanishing lower bound of $E \geq 2\pi k$ is established on finite energy field configurations. As usual, the bound saturates when the first order system

$$\begin{aligned} D_{\bar{z}}\Phi &= 0 \\ F_{12} &= e^2(\Phi\Phi^\dagger - 1) \\ \int_{\mathbb{C}} d^2z\, F_{12} &= 2\pi k \end{aligned} \tag{3.10}$$

is satisfied. The first of these is, of course, really two equations, one for each component of the $\mathbb{C}^2$-valued field $\Phi$. The equations in (3.11) form a closed system whose solutions are precisely the semilocal vortices of [43, 80, 93].

Although such solitonic solutions are vortex-like in many respects, a little analysis soon reveals that their asymptotic behavior is very different from the exponential falloff of Abelian-Higgs vortices [43, 44]. In fact the fields of the semilocal model exhibit a distinctive power-law behavior at spatial infinity, a symptom of the fact that the width of the flux tube is an arbitrary parameter of the theory. This should be contrasted with Abelian-Higgs vortices where the width is controlled by the Compton wavelength of the gauge boson. In this sense, these vortex solutions are rather reminiscent of $\mathbb{CP}^N$ instantons. This is no mere coincidence. In fact, the correspondence can be made precise in the large coupling limit in which the semilocal vortices of the $U(1)^{N+1}$-gauged linear sigma model descend to the instanton solutions of a $\mathbb{CP}^N$ nonlinear sigma model [80, 99]. While this is quite clear at the levels of the action and equations of motion, its realization at the level of the solutions is marginally obscured by the fact that only the asymptotic forms of the vortex solutions are known to exist. This is not unlike the situation with the conventional Nielsen-Olesen vortex. However this particular hurdle was recently surmounted in [3, 4, 61, 49] where a noncommutative deformation of the two-dimensional configuration space of the Abelian-Higgs model allows for the construction of *exact vortex solutions*. The fact that the noncommutative version of the theory seems so much richer than its commutative counterpart is by now not surprising [29, 30]. It would seem then, that a noncommutative deformation of the base space of the two-dimensional gauged linear sigma model might offer, if nothing else, an interesting avenue to explore the construction of exact semilocal vortices.

## 3.3 And then everything became fuzzy...

Recall from the previous chapter that conventionally, a noncommutative deformation of the two-dimensional configuration space is imposed by a Moyal-deformation of the algebra of functions over $\mathbb{R}^2$ and implemented by replacing ordinary pointwise multiplication of functions by Moyal $*$-multiplication. Consequently, coordinates on the noncommutative plane $\mathbb{R}^2_\theta$ satisfy the Heisenberg algebra

$$[x^i, x^j] = x^i * x^j - x^j * x^i = i\theta^{ij}.$$

Once again, in terms of the complex coordinates on $\mathbb{R}^2_\vartheta$, the commutator $[z,\bar{z}] = \vartheta$ is easily seen to be isomorphic to the algebra of annihilation and creation operators for the simple harmonic oscillator. Thus any appropriately well behaved function on the noncommutative space may be Weyl transformed [36], to an operator acting on an auxiliary one-particle Hilbert space $\mathcal{H} = \bigoplus_n \mathbb{C}|n\rangle$ built from harmonic oscillator eigenstates. We refer the reader to the relevant section in the previous chapter for a translation of various operations in the noncommutative space to the Fock space.

### 3.3.1 The noncommutative semilocal model

With this Weyl prescription at hand, the noncommutative semilocal energy functional (3.8) can be written as

$$E_\vartheta = 2\pi\vartheta \mathrm{Tr}_{\mathcal{H}}\Big[(\widehat{D_z\Phi})(\widehat{D_z\Phi})^\dagger + (\widehat{D_{\bar{z}}\Phi})(\widehat{D_{\bar{z}}\Phi})^\dagger + \frac{1}{2e^2}\widehat{F_{12}}^2 + \frac{e^2}{2}(\widehat{\Phi}\widehat{\Phi}^\dagger - 1)^2\Big] \tag{3.11}$$

where, now $\widehat{D_z\Phi} = (\widehat{\Phi}\widehat{a}^\dagger + \widehat{C}^\dagger\widehat{\Phi})/\sqrt{\vartheta}$, $\widehat{D_{\bar{z}}\Phi} = -(\widehat{\Phi}\widehat{a} + \widehat{C}\widehat{\Phi})/\sqrt{\vartheta}$ and the gauge field is parameterized as $\widehat{A}_z = (i/\sqrt{\vartheta})(\widehat{a}^\dagger + \widehat{C}^\dagger)$. As in the commutative case, this can be massaged into a Bogomol'nyi form which is saturated when the BPS equations

$$\begin{aligned} \widehat{\Phi}\widehat{a} + \widehat{C}\widehat{\Phi} &= 0 \\ 1 + \Big[\widehat{C}^\dagger, \widehat{C}\Big] &= \vartheta e^2(\widehat{\Phi}\widehat{\Phi}^\dagger - 1) \\ \mathrm{Tr}_{\mathcal{H}}\Big(1 + \Big[\widehat{\mathrm{C}}^\dagger, \widehat{\mathrm{C}}\Big]\Big) &= -k \end{aligned} \tag{3.12}$$

are satisfied. As in the commutative case, this is a system of three first order equations, subject to the flux constraint. Solutions of this system will be the noncommutative generalizations of the semilocal vortex of [44]. In the spirit of [4, 61], we begin with an *ansatz* for the Higgs doublet and the gauge field. To this end the most general vortex-like solution of the BPS equations which maintain the cylindrical symmetry is of the form[2]

$$\widehat{\Phi} = \widehat{\phi}_1 \otimes \langle \mathrm{I}| + \widehat{\phi}_2 \otimes \langle \mathrm{II}| \tag{3.13}$$

[2]The generalization to an $N+1$ component Higgs field is quite straightforward so we persist in restricting our attention to the $N=1$ case for the moment.

where $\langle \mathrm{I}| = (1,0)$, $\langle \mathrm{II}| = (0,1)$ and

$$\widehat{\phi}_i = \sum_{m=0}^{\infty} f_m^{(i)} |m\rangle\langle m + q^{(i)}| \tag{3.14}$$

where $\{q^{(1)}, q^{(2)}\}$ is a set of integers related to the topological charge and, respectively, angular momentum quantum number of the vortex as we show below. For the $U(1)$ gauge field we take the cylindrically symmetric ansatz

$$\widehat{C} = \sum_{m=0}^{\infty} g_m |m\rangle\langle m + 1|. \tag{3.15}$$

Without loss of generality all coefficients are taken to be real. The construction of exact vortex solutions to the semilocal model now hinges on determining the various coefficients in the above *ansatze* that satisfy the appropriate boundary conditions. In terms of the coefficients $f_n \equiv f_n^{(1)}$ and $h_n \equiv f_n^{(2)}$, the first of eqs.(3.13) become

$$\begin{aligned} f_m \sqrt{m+q+1} + g_m f_{m+1} &= 0 \\ h_m \sqrt{m+1} + g_m h_{m+1} &= 0 \end{aligned} \tag{3.16}$$

with the choice $q^{(1)} = q$ and $q^{(2)} = 0$. An explanation for this will follow in due course. For the moment though, notice that eqs.(3.17) mean that the coefficients of each of the components of the Higgs doublet are not independent. Indeed

$$h_{m+1} = \sqrt{\frac{m+1}{m+q+1}} \Big(\frac{f_{m+1}}{f_m}\Big) h_m. \tag{3.17}$$

This is a simple recurrence relation which is easily solved for $h_m$ to give

$$h_m = \sqrt{\frac{m!q!}{(m+q)!}} \kappa f_m \tag{3.18}$$

with $\kappa = h_0/f_0$ determining the relationship between the initial conditions of each coefficient sequence. With the convenient definitions of $Q_n \equiv f_n^2$ and $P_n \equiv h_n^2$, this may be combined with the second of the BPS equations to give

$$
\begin{aligned}
Q_1 &= \frac{(q+1)Q_0}{1+\gamma-\gamma(1+\kappa^2)Q_0} \\
Q_{m+1} &= \frac{(m+q+1)Q_m^2}{Q_m+(m+q)Q_{m-1}-\gamma Q_m\Big[\Big(1+\frac{m!q!}{(m+q)!}\kappa^2\Big)Q_m-1\Big]} \qquad m>0
\end{aligned}
\tag{3.19}
$$

where, following [89] the dimensionless combination of $\vartheta e^2$ is hereafter christened $\gamma$. In principle then, the noncommutative vortex solution of the critically coupled linear sigma model may be determined by solving the recurrence relation (3.20) and consequently (3.18) subject to the "boundary conditions" $f_n \to 1$, $h_n \to 0$ as $n \to \infty$. Well, almost. The attentive reader would of course have noticed that there is still the matter of the arbitrary integer $q$. Fortunately, there is also the third of the BPS equations, the flux constraint. Using (3.17) and the *ansatz* for the gauge field it may be shown that

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\Big(1+[\widehat{C}^\dagger,\widehat{C}]\Big) &= \mathrm{Tr}_{\mathcal{H}}\Big(\sum_{m=0}^{\infty}\Big(1+g_{m-1}^2-g_m^2\Big)|m\rangle\langle m|\Big) \\
&= \sum_{l,m=0}^{\infty}\Big(1+g_{m-1}^2-g_m^2\Big)\langle l|m\rangle\langle m|l\rangle \\
&= \lim_{M\to\infty}\Big[M+1-(M+q+1)\frac{Q_M}{Q_{M+1}}\Big].
\end{aligned}
\tag{3.20}
$$

In the last step, the cutoff of [49] was employed to regulate the trace. In the large $M$ limit, the convergence of the coefficient sequence means that the ratio of successive $Q$'s approaches unity. Consequently, the flux constraint equation implies that $q = k$. Indeed, a quick comparison with the analogous commutative result confirms that this is the only physically meaningful conclusion; the index of $\phi_1$ is equal to the topological number of the vortex. Interestingly enough, choosing $q^{(2)} \neq 0$ does not affect this conclusion. Again, this is not altogether unexpected since $q^{(2)}$ is just the angular momentum quantum number of the vortex [43]. Convergence of the coefficient sequence for $\phi_2$ bounds the angular momentum quantum number to the range $0 \leq q^{(2)} < k$. However, since none of the arguments presented here depends essentially on $q^{(2)}$ we can, without any loss of generality, set $q^{(2)} = 0$. It is also worth noting that when $q = 0$, $h_m = \kappa f_m$ and both boundary conditions can only be simultaneously satisfied if $h_m \equiv 0$ which reduces to the $k = 0$ vortex of the noncommutative Abelian-Higgs model [3, 4, 61, 89, 49]. Instead of solving eqs.(3.20) in full generality, it is perhaps more illuminating to focus on a few examples.

### 3.3.2 Examples

1. To begin with we consider the case $h_m = 0$ for all $m$. In this case the Higgs doublet $\Phi = (\phi_1, 0)$ satisfies exactly the equations of motion of the noncommutative Abelian-Higgs model and it is quite easy to check that the solutions of (3.20) reduce to the degree$-k$ vortices found in [61] for which

$$\begin{aligned} Q_1 &= \frac{(q+1)Q_0}{1+\gamma(1-Q_0)} \\ Q_{m+1} &= \frac{(m+q+1)Q_m^2}{Q_m + (m+q)Q_{m-1} - \gamma Q_m\Big(Q_m - 1\Big)} \qquad m > 0 \end{aligned} \tag{3.21}$$

This set of equations has been studied extensively and numerically shown to exhibit regular vortex solutions with $+k$ units of magnetic flux for a large $\gamma$ range. In particular, for small $\vartheta$ (and consequently $\gamma$) the regular commutative Neilsen-Olesen vortex solutions of [15, 74] are obtained. In addition, an obvious solution to (3.22) that satisfies the boundary conditions of the semilocal model is $Q_m \equiv 1$. As noted in [61], these are exactly the fluxon solutions of [38].

2. Moving on now to the more interesting case of non-vanishing $h_m$, it will suffice to restrict our attention to $k = 1$ for which the BPS recurrence relations become

$$\begin{aligned} P_m &= \frac{\kappa^2}{m+1}Q_m \\ Q_1 &= \frac{2Q_0}{1+\gamma-\gamma(1+\kappa^2)Q_0} \\ Q_{m+1} &= \frac{(m+2)Q_m^2}{Q_m + (m+1)Q_{m-1} - \gamma Q_m\Big[\Big(1+\frac{\kappa^2}{m+1}\Big)Q_m - 1\Big]} \qquad m > 0. \end{aligned} \tag{3.22}$$

The vortex solutions of the noncommutative semilocal model are constructed by solving eqs.(3.23) subject to the constraint $(P_m, Q_m) \to (0, 1)$

as $n \to \infty$. From the first of these it is clear that when the $Q_m$ sequence converges and $\kappa$ is of order unity, $P_m \sim 1/m$ for large $m$. Again, this remains true for any fixed value of the angular momentum quantum number. We solve the above system numerically using a double precision, split-step shooting algorithm. At first glance, the shooting-parameter space looks to be two-dimensional (corresponding to the different values of the pair $(P_0, Q_0)$) but a prescient choice of $\kappa^2 = 1/\vartheta$ fixes one of these parameters in terms of the other and reduces the dimension to one. With the initial value $Q_0$ as the shooting parameter, we solve (3.23) for various values of $\gamma$ and tabulate our results below.

| $\vartheta$ | $e^2$ | $\gamma$ | $Q_0$ |
|---|---|---|---|
| 0.2 | 1 | 0.2 | 0.099732894 |
| 0.2 | 4 | 0.8 | 0.140471163 |
| 0.2 | 16 | 3.2 | 0.158732886334 |
| 0.2 | 36 | 7.2 | 0.16297094403243935 |
| 0.5 | 1 | 0.5 | 0.215729007 |
| 0.5 | 4 | 2 | 0.2895665841653 |
| 0.5 | 16 | 8 | 0.32043540606185 |
| 0.5 | 36 | 18 | 0.32737242959721649 |

Each of these initial values for the $Q_m$ results in a coefficient sequence that converges (with varying degrees of accuracy) to one. Once determined, the $P_m$ and $Q_m$ may then be used to compute other characteristic quantities associated with the semilocal vortex. For example, the magnetic field of the semilocal vortex may easily be computed as

$$\widehat{B} = \frac{\gamma}{\vartheta}\sum_{n=0}^{\infty}\Big[1 - \Big(\frac{n+1+\vartheta^{-1}}{n+1}\Big)Q_n\Big]|n\rangle\langle n|. \qquad (3.23)$$

Substituting this, together with the covariant derivative

$$\begin{aligned}\widehat{D}_z\widehat{\Phi} &= \sum_{m=0}^{\infty}\frac{1}{\sqrt{\vartheta}}\Big(f_m\sqrt{m+1} + g_{m-1}f_{m-1}\Big)|m\rangle\langle m| \otimes \langle \mathrm{I}| \\ &+ \sum_{m=0}^{\infty}\frac{1}{\sqrt{\vartheta}}\Big(h_{m+1}\sqrt{m+1} + g_m f_m\Big)|m+1\rangle\langle m| \otimes \langle \mathrm{II}|\end{aligned}$$

into eq.(3.11) allows for the energy density of the vortex to be computed quite straightforwardly as

$$\begin{aligned}\widehat{\mathcal{E}} &= \frac{1}{\vartheta}\sum_{m=0}^{\infty}\Big[\frac{m+1}{Q_m}(Q_m - Q_{m-1})^2 + \frac{m}{P_m}(P_m - P_{m-1})^2 \\ &+ \gamma\Big(1+(\frac{m+1+\vartheta^{-1}}{m+1})Q_m\Big)^2\Big]|m\rangle\langle m|. \end{aligned} \qquad (3.24)$$

It may be verified numerically that up to the first few hundred terms the above expression for the energy density sums to $1/(2\pi\vartheta)$ to within a few percent as is expected for the 1−vortex solution. To make contact with the primary aim of this work, it will be convenient to visualize the profile of the vortex, especially as $\gamma$ is turned up. However, both eqs.(3.23) and (3.24) are Fock space representations. Fortunately, these can be turned into (noncommutative) coordinate space representations relatively easily with the inverse Weyl map. In fig.1 we plot the magnetic field as a function of $r$ for various values of the dimensionless parameter $\gamma$. Fig.2. contains a series of snapshots of the energy profile of the vortex as gamma increases from 0.2 to 28.8.

**There and back again...**

In order to understand what the semilocal vortices look like on the noncommutative plane, we need to find the inverse Weyl transform of their energy profile. As operators on the Hilbert space go, the magnetic field and energy density above are not too bad. As diagonal operators on $\mathcal{H}$, to find their representation on $\mathbb{R}^2_\vartheta$ all we have to do is understand how the operator $\widehat{P}_n = |n\rangle\langle n|$ transforms under the inverse Weyl transform. The route from $R^2_\vartheta$ to $\mathcal{H}$ is well mapped [79]: Given a function $f(z,\bar{z})$, the associated operator

$$\begin{aligned}\widehat{O}_f(\hat{a},\hat{a}^\dagger) &= \mathcal{W}[f(z,\bar{z})] \\ &= \int \frac{d^2k}{4\pi^2}\widetilde{f}(k,\bar{k})e^{-i(\bar{k}\hat{a}+k\hat{a}^\dagger)}\end{aligned}$$

where

$$\widetilde{f}(k,\bar{k}) = \int d^2z\, f(z,\bar{z})e^{i(\bar{k}z+k\bar{z})/\vartheta}$$

is the Fourier transform of $f$.

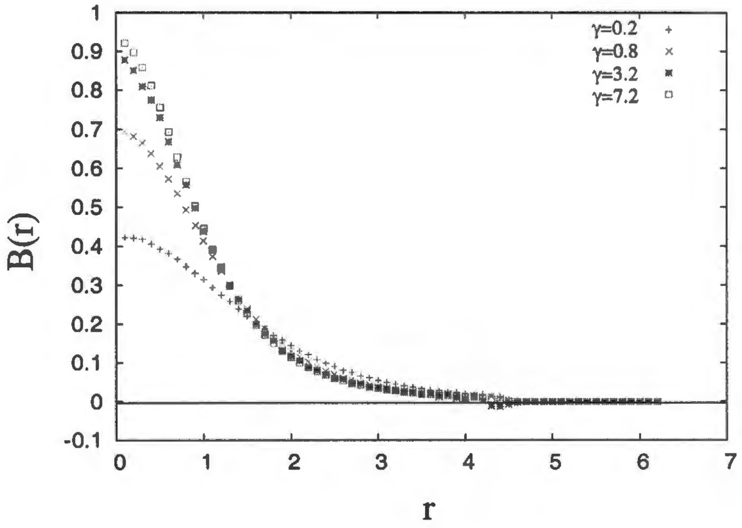



Figure 3.1: The magnetic field trapped in the vortex core for varying $\gamma$

**Continued...**

To invert $\widehat{P}_n$, notice that, in terms of the normal ordered ladder operators,

$$\begin{aligned} |n\rangle\langle n| &= :\frac{1}{m!}\hat{a}^{\dagger n}e^{-\hat{a}^\dagger\hat{a}}\hat{a}^n: \\ &= \int\frac{d^2k}{4\pi^2}\widetilde{f}(k,\bar{k}):e^{-i(\bar{k}\hat{a}+k\hat{a}^\dagger)}: \end{aligned}$$

where $\widetilde{f}(k,\bar{k}) = 2\pi\exp(-k^2/4)L_n(k^2/2)$ and $L_n$ is the $n^{th}$ Laguerre polynomial. Inverse Fourier transforming this final expression and using the result $\bar{z}z = r^2/2$ gives the final result

$$|n\rangle\langle n| \to 2(-1)^n\exp(-\frac{r^2}{\vartheta})L_n\Big(\frac{2r^2}{\vartheta}\Big) \tag{3.25}$$

## 3.4 The large coupling limit

Having presented a general algorithm for the construction of degree$-k$ semilocal vortex solutions of the gauged noncommutative linear sigma model and explicitly constructed the $1-$vortex solution we proceed now to study one of the more interesting limits of the semilocal model: its large coupling limit. At the level of the action (3.11), the $e^2 \to \infty$ limit decouples the gauge field dynamics and any finite energy static solution has

$$E = 2\pi\vartheta \mathrm{Tr}_{\mathcal{H}}\Big[(\widehat{D}_z\widehat{\Phi})(\widehat{D}_z\widehat{\Phi})^\dagger + (\widehat{D}_{\bar{z}}\widehat{\Phi})(\widehat{D}_{\bar{z}}\widehat{\Phi})^\dagger\Big] \qquad (3.26)$$

subject to the constraint $\widehat{\Phi}\widehat{\Phi}^\dagger = 1$. In this limit, the gauge field is relegated to an auxiliary field, completely determined by $\Phi$. Recalling that $\Phi$ is an $(n+1)-$component complex vector leads to the conclusion that this is, of course, nothing but the noncommutative version of the $\mathbb{CP}^N$ sigma model. At the level of the action this observation is certainly not new; in the commutative case[3], this relation has been commented on by several authors in many different contexts [43, 44, 80, 99]. However, it remains to be seen whether this correspondence persists at the level of the solutions. If it does we will have produced an explicit descent from the vortices of the fuzzy linear sigma model to the instantons of the noncommutative $\mathbb{CP}^N$ model. In the interests of self-containment, we review now the derivation of the lump solutions of the sigma model.

With eq.(3.26) as a starting point, we relabel and reparameterize the $(N+1)-$component Higgs field as $\widehat{\Phi} \to \widehat{U} = (1/\sqrt{\widehat{W}\widehat{W}^\dagger})\widehat{W}$. A subsequent definition of the Hermitian projector $P \equiv \widehat{W}^\dagger(\widehat{W}\widehat{W}^\dagger)^{-1}\widehat{W}$ allows for the static energy (or two-dimensional action) to be written as

$$E = 2\pi\, \mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big([P,\widehat{a}^\dagger][\widehat{a},P]\Big). \qquad (3.27)$$

In this form, the $\mathbb{CP}^N$ energy is remarkably similar to the kinetic term of the static energy of a $(2+1)-$dimensional noncommutative scalar field (see eq.(2.2) of ref.[30]) with the crucial difference of the additional matrix trace in eq.(3.27). Indeed it was shown in [56, 57, 70] that the quantity $\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}[\widehat{a},\widehat{a}^\dagger P]$ contributes a nonvanishing boundary term to the energy and some care needs

[3]Indeed, even in the noncommutative case it has not gone entirely unnoticed. In [89] a formal $2k$-parameter solution to the vortex equations of the noncommutative Abelian-Higgs model was found to all orders in $\gamma^{-1}$ and, in particular, the metric on the moduli space of vortices explicitly computed in the limit $\gamma \to \infty$. There it was also noted that while this limit is usually taken to mean $\vartheta \to \infty$, it could equally well correspond to the large coupling limit. It is this latter view that we advocate.

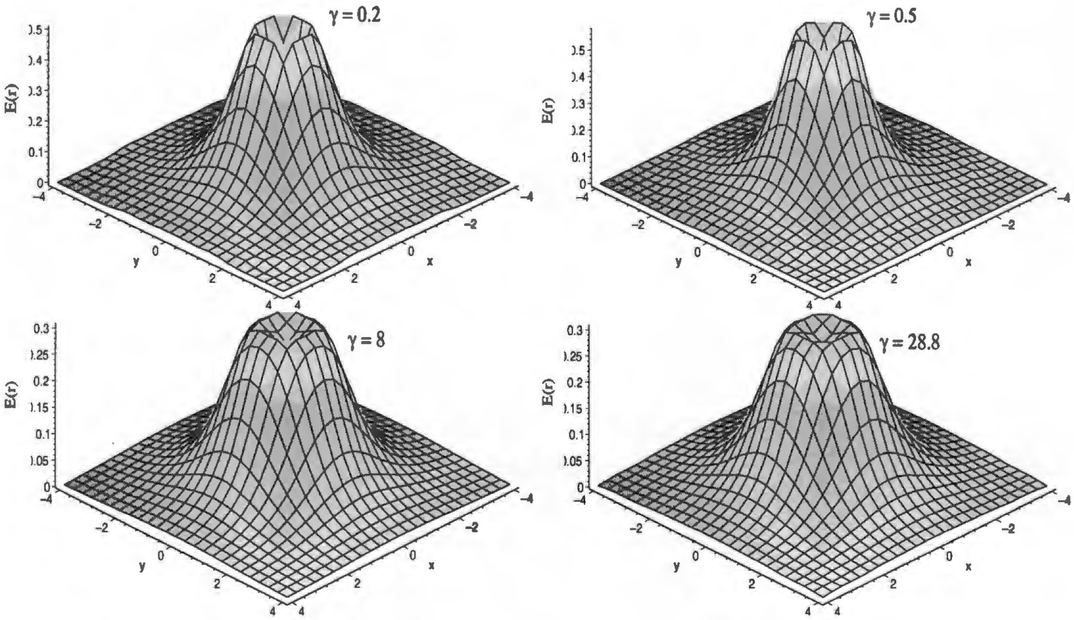



Figure 3.2: The metamorphosis of the semilocal vortex into the $\mathbb{CP}^N$ lump

to be exercised in the derivation of the noncommutative Bogomol'nyi bound. With this in mind, the energy may correctly be written as

$$E = 2\pi\,\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}\Big(2F_+(P)^\dagger F_+(P)\Big) + 2\pi Q_+ \geq 2\pi Q_+ \tag{3.28}$$

with the topological charge $Q_+ \equiv \mathrm{Tr}_{\mathcal{H}}\mathrm{tr}(P - [\widehat{a}, \widehat{a}^\dagger P])$ and $F_+(P) \equiv (1-P)\widehat{a}P$. A similar expression holds for the anti-BPS states. Focusing on the BPS states though, saturation of the bound on the energy is obtained when $F_+(P) = 0$. As first shown in [58], solutions are not difficult to find; any Hermitian projector constructed from an $(n+1)-$vector $\widehat{W}$ whose components are holomorphic polynomials in $\widehat{z}$ will satisfy the above BPS equation. These are precisely the noncommutative extension of the instanton solutions of the conventional $\mathbb{CP}^N$ sigma model. For example, the static, $1-$ and $2-$lump solutions of the noncommutative $\mathbb{CP}^1$ model are given by

$$\widehat{W}_1 = \Big(\widehat{z} - a_1\,,\, b_1\Big) \qquad \widehat{W}_2 = \Big(\widehat{z}^2 - a_2\,,\, 2b_2\widehat{z} + c_2\Big) \tag{3.29}$$

where the soliton parameters $a_1, ..., d_2 \in \mathbb{C}$ are chosen to coincide with the standard way of writing the solutions in the commutative theory [95]. These are the complex moduli of the $\mathbb{CP}^1$ instanton. To facilitate comparison with the vortices, these may be written in the harmonic oscillator basis so that, for example, the 1−lump solution becomes

$$\widehat{U}_1 = \sum_{n=0}^{\infty} \sqrt{\frac{\vartheta(n+1)}{\vartheta(n+1)+1}} |n\rangle\langle n+1| \otimes \langle \mathrm{I}| + \sum_{n=0}^{\infty} \sqrt{\frac{1}{\vartheta(n+1)+1}} |n\rangle\langle n| \otimes \langle \mathrm{II}|. \tag{3.30}$$

Returning to the degree−$k$ semilocal vortex of the last section, notice that eq.(3.20) may be recast as

$$\Big(1 + \frac{n!k!}{(n+k)!}\kappa^2\Big) Q_n - 1 + \frac{1}{\gamma}\Big((n+k+1)\frac{Q_n}{Q_{n+1}} - (n+k)\frac{Q_{n-1}}{Q_n} - 1\Big) = 0$$

In the infinite coupling limit $e^2 \to \infty$ (or equivalently $\gamma \to \infty$), the above recurrence relation may be be solved exactly to give

$$Q_n = \Big(1 + \frac{n!k!}{(n+k)!}\kappa^2\Big)^{-1}. \tag{3.31}$$

In particular, for $k = 1$ we find

$$Q_n = \frac{n+1}{n+1+\kappa^2} \qquad P_n = \frac{\kappa^2}{n+1+\kappa^2}. \tag{3.32}$$

Finally, matching coefficients to all orders in eqs.(3.30) and (3.32) means that the descent from noncommutative vortex to fuzzy lump only occurs when $\kappa^2 = 1/\vartheta$. Indeed, this is exactly the choice we made in our numerical computations to reduce the dimension of the shooting-parameter space. As a check, we expect that for a fixed value of $\vartheta$, $Q_0 \to \vartheta/(\vartheta+1)$ as $\gamma \to \infty$. A quick glance at the table of our numerical results verifies that this is indeed the case for $\vartheta = 0.2$ and 0.5. Moreover, hindsight reveals that the set of energy densities in figure 2. is in fact a series of snapshots of the $k = 1$ vortex of the noncommutative semilocal model morphing into a fuzzy $\mathbb{CP}^1$ 1−lump. The case $k = 2$ is no less straightforward. With its center of mass localised at the origin, the $\mathbb{CP}^1$ 2−lump in eq.(3.29) can be written as

$$\begin{aligned} \widehat{U}_2 &= \sum_{n=0}^{\infty} \sqrt{\frac{\vartheta^2(n+1)(n+2)}{\vartheta^2(n+1)(n+2)+1}} |n\rangle\langle n+2| \otimes \langle \mathrm{I}| \\ &+ \sum_{n=0}^{\infty} \sqrt{\frac{1}{\vartheta^2(n+1)(n+2)+1}} |n\rangle\langle n| \otimes \langle \mathrm{II}| \end{aligned} \tag{3.33}$$

when $b_2$, the frozen out modulus [25] is set to vanish. A comparison with the general expression for the infinite coupling coefficients (3.31) reveals a matching at all levels only if $\kappa^2 = 1/2\vartheta$. Generalisation to larger $k$ follows in much the same way so no further attention is paid to it here.

At this juncture, a few comments are in order. The Bogomol'nyi equations of the commutative gauged linear sigma model admit a one parameter family of vortex solutions [44]. This single complex parameter $w$ is to the commutative theory what the ratio of initial coefficients $\kappa$ is to our noncommutative model with $w = 0$ corresponding to the conventional Neilsen-Olesen string. One of the distinguishing characteristics of the $w \neq 0$ semilocal vortices is the power law behavior exhibited by the scalar and gauge fields as they relax to their respective vacuum values. Consequently, the magnetic field[4] $B \sim 2|w|^2/\xi^4$ and the width of the flux tube trapped in the vortex core is an arbitrary parameter instead of the Compton wavelength of the vector boson as in the Neilsen-Olesen vortex. In the noncommutative model we once again find a one parameter family of vortices only now the parameter, $\kappa$, is not at all arbitrary. Indeed, we find that there exists a point in the $\kappa$ parameter space dependent on the degree of the vortex and the deformation parameter $\vartheta$ at which the semilocal vortex *exactly* descends to the corresponding noncommutative $\mathbb{CP}^N$ lump. Correspondingly, the width of the magnetic flux tube associated with the semilocal vortex is set by the scale of noncommutativity. This observed exact metamorphosis of the vortex into the lump should be compared to the results of section 3. of [44]. There an expansion of the 1−instanton solution of the commutative $\mathbb{CP}^N$ model in powers of $|w|/|z|$ was used to establish that the vortex-instanton matching was exact at spatial infinity with differences emerging at $\mathcal{O}(|w|^4/|z|^4)$ in this expansion.

## 3.5 Brane realisations

Quite apart from their intrinsic field theoretic value [93, 43, 44], the vortices of gauged linear sigma models also have a remarkably rich stringy structure. Beginning with the ground-breaking work of [35] in which the $(2+1)$−dimensional, $\mathcal{N} = 4$ $U(N)$ Yang-Mills-Higgs theory was recognised as the worldvolume theory on a stack of $N$ $D3$−branes suspended between two parallel $NS5$−branes, an intricate tapestry of ideas can be woven, leading inexorably to a realisation of the noncommutative semilocal vortex as a $D$−brane

[4] Following [44] $\xi$ is a dimensionless radial variable on the plane.

configuration in type IIB string theory [34]. In this section, we review some of these ideas and cast them into a form that better facilitates comparison with our results.

As in [34] the description of the system begins with a $(2+1)-$dimensional, $\mathcal{N}=4$, $U(N)$ Yang-Mills-Higgs theory. The field content of the theory consists of a $U(N)$ vector multiplet made up of a gauge field $A_\mu$ and a triplet of adjoint scalars $\phi^r$ together with their fermionic super partners. Coupled to these are $N$ fundamental hypermultiplets each of which contain a doublet of complex scalars $q$ and $\widetilde{q}$ and their super partners. The Lagrangian for the theory is endowed with a global $SU(N+M)$ flavour symmetry as well as a local $U(N)$ gauge symmetry. Consequently, under these two groups and with $N_f \equiv N+M$ denoting the number of flavours, $q$ and $\widetilde{q}$ transform as $(\mathbf{N},\overline{\mathbf{N_f}})$ and $(\overline{\mathbf{N}},\mathbf{N_f})$ respectively; the fundamental scalars are represented by $N\times(N+M)$ matrices. The dynamical content of the bosonic sector of the theory is contained in the Lagrangian

$$\begin{aligned} \mathcal{L} &= -\mathrm{Tr}\Big[\frac{1}{4e^2}F^2+\frac{1}{2e^2}(D\phi^r)^2+(Dq)^2+(D\widetilde{q})^2+e^2|q\widetilde{q}|^2 \\ &+ \frac{1}{2e^2}[\phi^r,\phi^s]^2+(q^2+\widetilde{q}^2)\phi^r\phi^r+\frac{e^2}{2}(q^2-\widetilde{q}^2-\zeta\mathbf{1}_N)\Big]. \end{aligned} \tag{3.34}$$

where the Fayet-Illiopolous (FI) parameter, $\zeta$, in the final D-term in (3.34) is chosen to be positive. This theory exhibits a Higgs branch of vacua which possess BPS vortices only if $\widetilde{q}$ and $\phi^r$ both vanish. This constraint defines a so-called reduced Higgs branch, $\mathcal{N}_{N,M}=Gr(N,N+M)$, the Grassmannian manifold of $N-$dimensional hyperplanes in $\mathbb{C}^{N+M}$. A particular vacuum choice[5] is made by picking

$$q_{\rm vac} = \begin{cases} \sqrt{\zeta}\delta^a_i & a,i=1,...,N \\ 0 & i=N+1,...,N+M \end{cases} \tag{3.35}$$

In our abelian case, for example, $N=1$, the reduced Higgs branch $\mathcal{N}_{1,M}=Gr(1,1+M)\simeq\mathbb{CP}^M$ and $q_{\rm vac}=(\sqrt{\zeta},0)$. Relabeling $q\to\Phi$, setting the FI parameter $\zeta=1$ and restricting to time-independent solutions trivially establishes the equivalence of the action in this branch with the static energy (3.5). As discussed earlier, the spectrum of solutions of this theory is rich with BPS vortices. The brane realisation of these vortices is built up from the $U(N)$

[5]Since the Grassmannian is, after all, a symmetric space, no generality is lost in this choice.

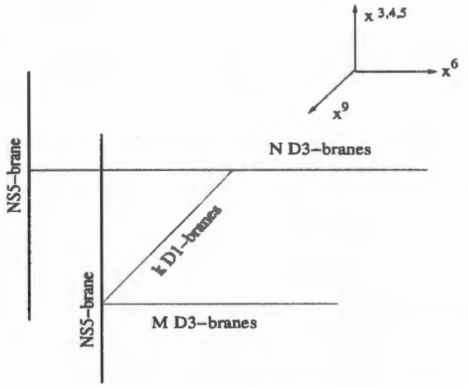



Figure 3.3: The degree$-k$ BPS vortex as $k$ stretched D-strings.

Yang-Mills-Higgs described in (3.34). It consists of $N$ $D3-$branes suspended between two parallel $NS5-$branes and a further $N+M$ $D3$'s attached to the right hand $NS5-$brane to add flavour (see figure 3).

In the Higgs branch, one of the $NS5-$branes is separated from the others. This separation is proportional to the FI parameter $\zeta$. The degree$-k$ BPS vortices manifest as $k$ $D-$strings stretched between the $D3-$branes and the separated $NS5-$brane - an identification made on the basis of the fact that the stretched $D1-$branes are the *only* BPS states of the brane configuration with the correct mass. More than just a pretty picture, the geometry of the $D-$brane configuration in figure 3 encodes vital information about the FI parameter, $\zeta$ as well as the gauge coupling $e$ as

$$\frac{1}{e^2} = \frac{\Delta x^6}{2\pi g_s}; \quad \zeta = \frac{\Delta x^9}{4\pi^2 g_s l_s^2} \tag{3.36}$$

where $l_s$ and $g_s$ are the string length and coupling respectively and $\Delta x^6$ and $\Delta x^9$ are the separation distances between the $NS5-$branes defined as in figure 1. It is now clear that the sigma model limit ($e^2 \to \infty$) of the vortex occurs precisely when the separation of the $NS5-$branes in the $6-$direction vanishes. The configuration that realises the $k-$lump solution of the (commutative) $\mathbb{CP}^M$ nonlinear sigma model then is as above only with $N=1$. In string theory the transition from commutative to noncommutative worldvolume theories is achieved by turning on an NS-NS $B-$field in the appropriate direction [81]. In the present context, the transition from the semilocal action (3.5) to its noncommutative counterpart (3.11) translates into turning on a constant NS-NS $B-$field $B_{12} = \vartheta\, dx^1 \wedge dx^2$ in the $(1,2)-$directions in a background of two $NS5-$branes with a $D3-$brane stretched between them and a further $M+1$ $D3$'s attached to the right hand $NS5-$brane. What of the vortices? The effect of the $B-$field on the $D-$strings stretched between the $NS5-$brane and the $D3$ is quite remarkable. The basic physics is analogous to the situation of a $D-$string suspended between two $D3-$branes studied in [41] and was first described for the vortex case in [34] (see also the insert following). The NS-NS 2$-$form manifests on the $D3-$worldvolume as a constant magnetic flux $\mathcal{F}_{12}$ while the $D-$string endpoint appears as a magnetic source. Since on the 4$-$dimensional worldvolume of the $D3-$brane $\mathcal{F}_{12} = \star\mathcal{F}_{06}$, the magnetic endpoint of the $D1-$brane feels the same force as an electric charge in a constant electric field in the 6$-$direction. However, as other end of the $D-$string remains married to the $NS5-$brane, the $D-$string responds to this force by tilting as in figure 2.4.

**Computing $D-$brane tilt**

The effect of the tilting of $D-$strings stretched between $D3-$branes was investigated in [41] by studying the $D-$string Born-Infeld action at weak string coupling. When the $D3$'s are separated by a distance of $\Delta x^9$ the Born-Infeld action takes the form

$$S = \frac{1}{2\pi l_s^2}\int_0^{\Delta x^9} dx^9 \left(\frac{1}{g_s}\sqrt{1+\left(\frac{dx^6}{dx^9}\right)^2} + A_{06}\frac{dx^6}{dx^9}\right).$$

The NS-NS field $\mathcal{B}_{12} = -B\,dx^1\wedge dx^2$ manifests by inducing a Ramond-Ramond (RR) 2$-$form $A_{06} = \frac{1}{g_s}\sqrt{\frac{B^2}{1+B^2}}$ which couples to the $D-$string worldvolume. Under the influence of the NS-NS field, the $D-$string reacts by seeking out it's minimal energy configuration. Thus, the extent to which it tilts can be found by minimising the energy of the system encoded in the Born-Infeld action. A straightforward calculation establishes that

$$\frac{\delta S}{\delta\frac{dx^6}{dx^9}} = 0 \quad \Rightarrow \frac{dx^6}{dx^9} = -B$$

Substituting this back into the action and integrating with respect to $x^9$ gives the minimum energy

$$\mathcal{E} = \frac{\Delta x^9}{2\pi\alpha' g_s}\left(\sqrt{1+B^2} - \frac{B}{\sqrt{1+B^2}}\right).$$

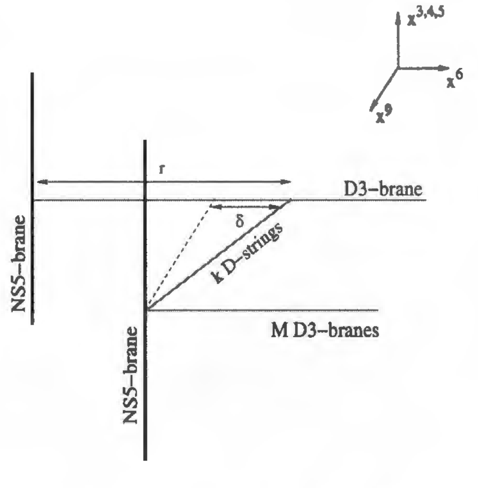



Figure 3.4: The magnetic field induced on the $D3$–worldvolume causes the $D$–strings to tilt.

**Continued...**

It is precisely when this energy is reached that the string tension balances out the magnetic force on the $D$–string endpoint and the string comes to rest. To compute the amount of tilting, notice that in the $(x^6, x^9)$–plane, the $D$–string is essentially a straight line with slope

$$\frac{dx^6}{dx^9} = -B = \frac{\delta}{\Delta x^9}$$

from which we conclude that $\delta = (\vartheta \Delta x^9)/(2\pi l_s^2)$.

With some straightforward algebra, the distance between the $D$–string endpoint and the left $NS5$–brane can be computed. With the choice of $\zeta = 1$ for the FI parameter, the result is

$$r = 2\pi g_s \Big(\frac{1}{e^2} + \vartheta\Big). \tag{3.37}$$

This distance is, in fact, the FI parameter of the theory living on the $D1$–branes (see [34] for a lucid discussion of this aspect). Having fixed $\Delta x^9$ with the choice $\zeta = 1$ the magnitude of $r$ is completely determined by the size of the gauge coupling as determined by the $NS5$–brane separation in the 6–direction and the noncommutativity. Since the latter is also fixed, the transition from vortex to lump can be studied by changing the distance between the $NS5$–branes. As

$\Delta x^6$ is decreased to zero, the separation between the $D-$string endpoint and the left $NS5-$brane decreases to $r_* = 2\pi g_s \vartheta$. It is this configuration of the $k$ tilted $D-$strings stretched between the (formerly right hand) $NS5-$brane and the $D3-$brane that realises the degree$-k$ instanton of the $\mathbb{CP}^M$ sigma model.

More than just an academic exercise, this identification of the semilocal vortex and $\mathbb{CP}^M$ instanton has proven invaluable in the understanding of the low energy dynamics of both the vortex and instanton as encoded in the geometry of their respective moduli spaces [89]. We refer the interested reader to [34] for a nice discussion of the structure of the moduli spaces and content ourselves with merely summarising some of their most pertinent results. The moduli space of degree$-k$ semilocal vortices $\hat{\mathcal{V}}_{k,(1,M)}$ is a $2k(1+M)-$dimensional space with a natural Kähler metric defined by the overlap of zero modes. However, this metric is afflicted with some non-normalisable zero modes that, classically, correspond to the moduli with infinite moments of inertia and that make the quantum mechanical treatment of these solitonic objects quite subtle. Fortunately these subtleties may be circumvented with a little help from the branes. A study of the theory on the $D1-$brane predicts that the Higgs branch, $\hat{\mathcal{M}}_{k,(1,M)}$, constructed by a $U(k)$ Kähler quotient of $\mathbb{C}^{k(1+M+k)}$ is isomorphic to the moduli space $\hat{\mathcal{V}}_{k,(1,M)}$. While the metric on $\hat{\mathcal{M}}_{k,(1,M)}$ retains all the symmetries of the Kähler metric on the vortex moduli space, it is finite and suffers from none of the non-normalisablity problems of the latter. Consequently, the study of the quantum theory of semilocal vortices may be simplified somewhat by replacing the natural metric on the vortex moduli space with the metric on the Higgs branch of the $D-$string theory inherited from the Kähler quotient construction of [34].

## 3.6 Discussions

The primary concern of this work has been the construction and study of a noncommutative extension of $(2+1)-$dimensional critically coupled, gauged linear sigma model. Like its commutative counterpart this theory possesses a rich spectrum of BPS solutions. By extending the systematic construction of [61] we have explicitly constructed a family of vortex solutions to the BPS equations (3.13) for arbitrary positive values of the noncommutativity parameter $\vartheta$. As expected, these fuzzy vortices reduce to the exact Neilsen-Olesen strings of the noncommutative Abelian-Higgs model [3, 4, 61, 49] on the codimension one surface $\kappa = 0$ of the parameter space. Despite retaining many of the properties of their commutative cousins [44, 93], the introduction of a new length scale set by the noncommutativity parameter $\vartheta$ induces several remarkable differences. Among these we find that the width of the magnetic flux tube trapped in the vortex core no longer exhibits the characteristic arbitrariness of the commutative semilocal vortex. In the noncommutative model, this width is set by the scale of the noncommutativity.

The detailed investigation of the large coupling ($e^2 \to \infty$) regime of the $\vartheta-$deformed gauged linear sigma model carried out in section 4. confirms, both numerically and analytically, the commutative intuition of the vortex morphing into a lump of the (fuzzy) $\mathbb{CP}^M$ sigma model. Additionally, while the agreement between vortex and lump in the $\vartheta = 0$ case is precise only asymptotically [44], we find an *exact* matching at all levels of the harmonic oscillator expansion at finite $\vartheta$. Indeed, insisting that this agreement holds selects a preferred set of values for $\kappa$, dependent on the scale of noncommutativity and the degree of the vortex. This effectively reduces the dimension of the parameter space by one. While we have explicitly constructed solutions for the $1-$ and $2-$vortex cases, the construction of higher degree solutions follows in much the same way and we do not expect any further surprises.

Finally, we reviewed the elegant constructions of [34] that lead to a realisation of the noncommutative $\mathbb{CP}^M$ $k-$lump as $k$ tilted $D-$strings stretched between an isolated $NS5-$brane (on which a stack of $M$ semi-infinite $D3-$branes end) and a semi-infinite $D3$ whose one endpoint ends on a second $NS5$ (see figure 4). This identification is built on the foundation of a study of the $\mathcal{N} = 4$ $U(N)$ Yang-Mills-Higgs $D3-$worldvolume theory hinges on the metamorphosis of vortices into lumps. Of course, to be sure that this configuration really does correspond to the lump solution requires more work than just a comparison of the masses of both configurations; the spectrum of fluctuations around each object needs to be computed and compared. This is a more difficult endeavor which, together with a more thorough investigation of the spectrum of BPS objects of the noncommutative gauged linear sigma model is left to future work. Curiously, this realisation of fuzzy $\mathbb{CP}^M$ lumps is not unique, at least for $M = 1$. Drawing on the tree level equivalence between $\mathcal{N} = 2$ open string theory and self-dual Yang-Mills theory in $(2+2)-$dimensions [75], it was argued in [56, 57] that the effective field theory induced on the worldvolume of $N$ $D2-$branes by $\mathcal{N} = 2$ open strings in a Kähler $B-$field background is a noncommutative $U(N)$ sigma model. Using a modified "method of dressing" soliton solutions of the latter were constructed and their various scattering properties investigated. In this context, the $k-$lump solution of the $\mathbb{CP}^1$ sigma model may be interpreted as $k$ $D0-$branes in the worldvolume of a stack of $D2-$branes [57, 46]. Again, while this assertion needs to be tested beyond the level of a mass comparison, the possibility of a duality between $\mathcal{N}=2$ open string theory and the type II-B superstring is, to say the least, intriguing and certainly deserves more attention.

# Chapter 4

# Fuzzy funnels and BIonic branes

*"Fuzzy funnel" solutions to the non-Abelian equations of motion of the D−string are studied. Our funnel describes $n^6/360$ coincident D−strings ending on $n^3/6$ D7−branes, in terms of a fuzzy six-sphere which expands along the string. We also provide a dual description of this configuration in terms of the worldvolume theory of the D7−branes. Our work makes use of an interesting non-linear higher dimensional generalization of the instanton equations.*

## 4.1 Introduction

Many new results in string theory have been obtained by studying the low energy world volume theory of $D-$branes[76]. A fascinating example is the appearance of non-commutative geometry. In particular, an interesting class of solutions has been obtained by studying a set of $D1-$branes that end on an orthogonal $D3-$brane[12] or on an orthogonal $D5-$brane[13]. These fuzzy funnel solutions consist of a fuzzy sphere geometry which expands along the length of the string.

Fuzzy spheres themselves are a fascinating example of non-commutative geometry. They arise as solutions to matrix brane actions[50, 12, 13] and may also play a role in a spacetime explanation of the stringy exclusion principle[65]. The geometry of even dimensional fuzzy spheres has been investigated in [45] and the detailed $SO(m)$ decomposition of the matrix algebras of the fuzzy spheres has been given in [77]. For fuzzy spheres $S^m$ with $m > 2$, it turns out that the matrix algebras contain more representations than is needed to describe functions on the sphere. In fact, in the classical limit (limit of large matrices), the matrix algebras related to even dimensional fuzzy spheres approach the algebra of functions of the higher dimensional space $SO(2k+1)/U(k)$. It has been argued that the appearance of these extra dimensions is a consequence of the Myers effect[52].

In this chapter we study "fuzzy funnel" solutions to the non-Abelian equations of motion of the $D-$string. Our funnel describes $n^6/360$ coincident $D-$strings ending on $n^3/6$ $D7-$branes. The geometry of our solution is that of a fuzzy $S^6$ which expands along the string. This connection between the number of $D-$strings and the number of $D7-$branes has also been obtained directly from the non-commutative geometry of the $S^6$. This solution is a natural generalization of the $D1 \perp D3$ [12] and the $D1 \perp D5$[13] solutions which made use of the fuzzy $S^2$ and fuzzy $S^4$ respectively. We also provide a dual description of this configuration in terms of the world volume theory of the $D7-$branes. The $D7-$brane theory gauge field configurations have non-vanishing third Chern character on the six sphere surrounding the endpoints of the $D-$strings. The energy, charge and radial profile of our solution computed in the two descriptions agree exactly.

Since our solution makes use of the fuzzy $S^6$, we review the relevant matrix algebra in section 3.2. In section 3.3. we develop the description of our system using the low energy $D-$string theory while in section 3.4. we recover the same results using the low energy $D7-$brane theory. In section 3.5. some consideration is given to the simplest fluctuations on the fuzzy funnel solution and we close the chapter with some comments on the domains of validity of both the $D-$string and the $D7-$brane theories.

## 4.2 Fuzzy six-sphere

In this section we review the construction of the fuzzy six-sphere. This is done to establish notation and to derive a number of identities that will be used in later sections. In preparing this section we found [2] helpful. To construct the fuzzy six-sphere, we need to construct solutions to the equation

$$\sum_{i=1}^{7} X^i X^i = c\mathbf{1}, \tag{4.1}$$

with $X^i$ a matrix, $\mathbf{1}$ the identity matrix and $c$ a constant. Schur's lemma can be used to obtain a simple construction of the matrices $X^i$. Toward this end, consider the Clifford algebra

$$\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}, \qquad i, j = 1, 2, ..., 7 \tag{4.2}$$

Denote the space on which the $\Gamma^i$ matrices act by $V$. The $n$-fold tensor product of $V$ is written as $V^{\otimes n}$. The $X^i$ are now obtained by taking

$$X^i = \Big(\Gamma^i \otimes 1 \otimes ... \otimes 1 + 1 \otimes \Gamma^i \otimes ... \otimes 1 + ... + 1 \otimes 1 \otimes ... \otimes \Gamma^i\Big)_{\mathrm{st}}. \tag{4.3}$$

The subscript $st$ is to indicate that the above $X^i$ are to be restricted to the completely symmetric and traceless tensor product space [1]. To prove that the above $X^i$ do indeed provide coordinates for the fuzzy six-sphere, one shows that $\sum_{i=1}^{7} X^i X^i$ commutes with the generators of $SO(7)$

$$X^{kl} = \frac{1}{2}\Big([\Gamma^k, \Gamma^l] \otimes 1 \otimes ... \otimes 1 + 1 \otimes [\Gamma^k, \Gamma^l] \otimes ... \otimes 1 + ... + 1 \otimes 1 \otimes ... \otimes [\Gamma^k, \Gamma^l]\Big)_{\mathrm{st}}$$

The result (4.1) now follows from Schur's lemma. Further, using the Clifford algebra we easily find $c = n(n+6)$. The $X^{ij}$ matrices generate the $SO(7)$ Lie algebra. The matrix algebra associated with the fuzzy $S^6$ includes both the $X^i$ and the $X^{ij}$. Together these matrices generate the $SO(7,1)$ Lie algebra.

[1] This restriction is important if one is to obtain an irreducible representation, which is assumed in the application of Schur's lemma.

The symmetric traceless representation we work with has dimension

$$N = \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5), \tag{4.4}$$

which identifies the representation generated by the $X^{ij}$ as the $\vec{r} = (\frac{n}{2}, \frac{n}{2}, \frac{n}{2})$ irreducible representation of $SO(7)$. Using the above definitions and the Clifford algebra, it is straightforward to derive the following identities (as usual, repeated indices are summed)

$$\begin{aligned}
[X^{ij}, X^{kl}] &= 2\delta^{jk}X^{il} - 2\delta^{ik}X^{jl} + 2\delta^{jl}X^{ki} - 2\delta^{il}X^{kj} \\
[X^{ij}, X^{k}] &= 2(\delta^{jk}X^{i} - \delta^{ik}X^{j}) \\
X^{i}X^{i} &= c\mathbf{1}, \\
X^{ij}X^{j} &= 6X^{i} = X^{j}X^{ji} \\
X^{j}X^{jk}X^{kl}X^{l} &= 6^2 c\mathbf{1} \\
X^{j}X^{jk}X^{kl}X^{lm}X^{mn}X^{n} &= 6^4 c\mathbf{1} \\
X^{j}X^{jk}X^{kl}X^{lm}X^{mn}X^{np}X^{pq}X^{q} &= 6^6 c\mathbf{1} \\
X^{ij}X^{jl} &= 6X^{il} - X^{i}X^{l} + c\delta^{il}\mathbf{1}, \\
X^{ij}X^{ji} &= 6c\mathbf{1} \\
X^{ij}X^{jk}X^{kl}X^{li} &= 6c^2\mathbf{1} \\
X^{ij}X^{jk}X^{kl}X^{lm}X^{mn}X^{ni} &= 6c^3\mathbf{1} \\
\epsilon^{ijklmnq}X^{i}X^{j}X^{k}X^{l}X^{m}X^{n} &= i(384 + 288n + 48n^2)X^{q}.
\end{aligned}$$

The geometry of the fuzzy six sphere has been studied in detail in [45]. These authors argue that the fuzzy six sphere is a bundle over the sphere $S^6$. In the classical limit the fibre over the sphere is the symmetric space $SO(6)/U(3)$. The result of relevance to us, following from this geometrical analysis, is that one can identify points in the base, and as a consequence it is possible to read off the 6−brane charge as $\frac{1}{6}(n+1)(n+2)(n+3)$. We will see that it is possible to reproduce this purely non-commutative geometric derivation of the charge using either a dynamical analysis based on the non-Abelian Born-Infeld description of $N$ coincident $D$−strings in the large $N$ limit, or by using the non-Abelian Born-Infeld description of $n^3/6$ $D7$−branes, in the large $n$ limit. The $D6$−brane charge will correspond to a $D7$−brane charge in our T-dual description. In the remainder of this chapter we work in the large $n$ limit. Consequently we use $N = n^6/360$, $c = n^2$ and take the 6−brane charge to be $n^3/6$.

## 4.3 What the $D$-strings see

In this section we study the fuzzy geometry of the $D7 \perp D1$ system, using the non-Abelian theory describing $N$ coincident $D-$strings. Our construction employs the fuzzy six-sphere to construct a fuzzy funnel in which the D-strings expand into orthogonal $D7-$branes. We use an approach based on minimizing the energy[27], which generalizes the results obtained in [12] for D1-branes expanding into orthogonal $D3-$branes and the results in [13] for $D1-$branes expanding into orthogonal $D5-$branes.

The low energy effective action for $N$ $D-$strings is given by the non-Abelian Born-Infeld action[72, 91]

$$\begin{aligned} S &= -T_1 \int d^2\sigma STr\sqrt{-\det\begin{pmatrix} \eta_{ab} & \lambda\partial_a\Phi^j \\ -\lambda\partial_b\Phi^i & Q^{ij}\end{pmatrix}} \\ &\equiv -T_1 \int d^2\sigma STr\sqrt{-\det M} \end{aligned} \tag{4.5}$$

where $Q^{ij} = \delta^{ij} + i\lambda[\Phi^i, \Phi^j]$ and $\lambda = 2\pi l_s^2$. The symmetrized trace prescription [92] (indicated by $STr$ in the above action) instructs us to symmetrize over all permutations of $\partial_a\Phi^i$ and $[\Phi^i, \Phi^j]$ within the trace over the gauge group indices, after expanding the square root. We are using a static gauge so that the worldsheet coordinates are identified with spacetime coordinates as $\tau = x^0$ and $\sigma = x^9$. The transverse coordinates are now the non-Abelian scalars $\Phi^i$, $i = 1, ..., 8$. These scalars are $N \times N$ matrices transforming in the adjoint representation of the $U(N)$ gauge symmetry present on the worldsheet of the $D1$'s. We seek static solutions with seven of the scalars excited. It is a tedious but straightforward exercise to show that it is consistent with the equations of motion to make use of a static ansatz that involves seven of the scalars, at the level of the action. With this ansatz and a rather lengthly calculation we obtain

$$\begin{aligned} -\det(M) &= 1 + \frac{\lambda^2}{2}\Phi^{ij}\Phi^{ji} + \frac{\lambda^4}{8}(\Phi^{ij}\Phi^{ji})^2 - \frac{\lambda^4}{4}\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{li} \\ &+ \lambda^6\Big(\frac{(\Phi^{ij}\Phi^{ji})^3}{48} - \frac{\Phi^{mn}\Phi^{nm}\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{li}}{8} + \frac{\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{lm}\Phi^{mn}\Phi^{ni}}{6}\Big) \\ &+ \lambda^2\partial_\sigma\Phi^i\partial_\sigma\Phi^i + \lambda^4\Big(\frac{\partial_\sigma\Phi^k\partial_\sigma\Phi^k\Phi^{ij}\Phi^{ji}}{2} - \partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k\Big) \\ &- \lambda^6\Big(\frac{\partial_\sigma\Phi^m\partial_\sigma\Phi^m\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{li}}{4} - \frac{\partial_\sigma\Phi^i\partial_\sigma\Phi^i(\Phi^{ij}\Phi^{ji})^2}{8} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k\Phi^{ml}\Phi^{lm}}{2} - \partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{lm}\partial_\sigma\Phi^m\Big) \\
&- \lambda^8\Big( - \frac{\partial_\sigma\Phi^k\partial_\sigma\Phi^k(\Phi^{ij}\Phi^{ji})^3}{48} + \frac{\partial_\sigma\Phi^p\partial_\sigma\Phi^p\Phi^{ij}\Phi^{ji}\Phi^{kl}\Phi^{lm}\Phi^{mn}\Phi^{nk}}{8} \\
&- \frac{\partial_\sigma\Phi^p\partial_\sigma\Phi^p\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{lm}\Phi^{mn}\Phi^{ni}}{6} + \frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k(\Phi^{ml}\Phi^{lm})^2}{8} \\
&- \frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k\Phi^{ml}\Phi^{ln}\Phi^{np}\Phi^{pm}}{4} - \frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{ln}\partial_\sigma\Phi^n\Phi^{mp}\Phi^{pm}}{2} \\
&+ \partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{ln}\Phi^{np}\Phi^{pm}\partial_\sigma\Phi^m\Big)
\end{aligned}
$$

after a convenient definition of $\Phi^{ij} = [\Phi^i, \Phi^j]$. Our ansatz for the funnel solution is given by

$$\Phi^i = R(\sigma)X^i \tag{4.6}$$

We have checked that the equation determining $R(\sigma)$ obtained by substituting this ansatz into the equations of motion (following from (4.5)) agree with the equations obtained by inserting this ansatz into the action (4.5) and varying with respect to $R(\sigma)$. Following this second procedure, inserting the above ansatz into (4.5) we obtain

$$S = -T_1 \int d^2\sigma STr\sqrt{\left(1+\left(\frac{d\bar{R}}{d\sigma}\right)^2\right)(1+f(\bar{R}))}, \tag{4.7}$$

$$f(\bar{R}) = 12\frac{\bar{R}^4}{c\lambda^2} + 48\frac{\bar{R}^8}{c^2\lambda^4} + 64\frac{\bar{R}^{12}}{c^3\lambda^6} \tag{4.8}$$

where we have introduced the physical radius $\bar{R} = \sqrt{c}\lambda R$.

In obtaining this result, use has been made of the identities listed in section 2. The formula (4.8) is not exact - it catches only the leading large $N$ contribution. If we expand the square root in (4.5) and implement the symmetrization of the trace for each term in the expansion, we find corrections to (4.8) of order $1/c$ relative to the leading term. Thus, our results are only valid for large $N$. Since this is a static configuration, it is easy to obtain the following expression for the energy of our solution

$$E = NT_1 \int d\sigma\sqrt{\left(1+\left(\frac{d\bar{R}}{d\sigma}\right)^2\right)(1+f(\bar{R}))}$$

$$
\begin{aligned}
&= NT_1 \int d\sigma \sqrt{\left(\frac{d\bar{R}}{d\sigma} \pm \sqrt{f(\bar{R})}\right)^2 + \left(1 \mp \frac{d\bar{R}}{d\sigma}\sqrt{f(\bar{R})}\right)^2} \\
&\geq NT_1 \int d\sigma \Big(1 \mp \frac{d\bar{R}}{d\sigma}\sqrt{f(\bar{R})}\Big)
\end{aligned} \tag{4.9}
$$

The above inequality is saturated when

$$
0 = \frac{d\bar{R}}{d\sigma} \pm \sqrt{\frac{12\bar{R}^4}{c\lambda^2} + \frac{48\bar{R}^8}{c^2\lambda^4} + \frac{64\bar{R}^{12}}{c^3\lambda^6}}. \tag{4.10}
$$

For small $\bar{R}$ it is simple to obtain

$$
\frac{d\bar{R}}{d\sigma} = \mp\frac{2\sqrt{3}\bar{R}^2}{\sqrt{c}\lambda} \qquad \Rightarrow \qquad \bar{R} = \pm\frac{\sqrt{c}\lambda}{2\sqrt{3}(\sigma - \sigma_0)}. \tag{4.11}
$$

This is the same behavior as was found in both the $D3-$brane funnel[12], and the $D5-$brane funnel[13]. We have reproduced the expected behaviour for any $D-$string funnel in the region where the funnel is well approximated by the $D-$string. Consider now the large $\bar{R}$ region. If our funnel is to expand into an orthogonal $D7-$brane at large $\bar{R}$, the expansion must be given by an harmonic function in seven spatial dimensions. At large $\bar{R}$ we find

$$
\frac{d\bar{R}}{d\sigma} = \mp\frac{8\bar{R}^6}{c^{\frac{3}{2}}\lambda^3} \qquad \Rightarrow \qquad \sigma - \sigma_0 = \pm\frac{c^{3/2}\lambda^3}{40\bar{R}^5}, \tag{4.12}
$$

which is indeed the correct harmonic behaviour needed for a $D7-$brane to appear at $\sigma = \sigma_0$. Further evidence that we have a funnel expanding into coincident $D7-$branes is provided by computing the RR charge and energy of this solution. The energy of our solution is

$$
\begin{aligned}
E &= NT_1 \int d\sigma \left(1 + \frac{d\bar{R}}{d\sigma}\sqrt{\frac{12\bar{R}^4}{c\lambda^2} + \frac{48\bar{R}^8}{c^2\lambda^4} + \frac{64\bar{R}^{12}}{c^3\lambda^6}}\right) \\
&= NT_1 \int_0^\infty d\sigma + NT_1 \int_0^\infty d\bar{R}\sqrt{\frac{12\bar{R}^4}{c\lambda^2} + \frac{48\bar{R}^8}{c^2\lambda^4} + \frac{64\bar{R}^{12}}{c^3\lambda^6}}.
\end{aligned} \tag{4.13}
$$

The first term is easily identified as the energy of $N$ semi-infinite $D-$strings stretching from $\sigma = 0$ to $\sigma = \infty$. Now consider the second term. We compute this term for large $\bar{R}$, where we expect that the funnel is expanding into a number of coincident $D7-$branes. Using the identities $N = n^6/360$ and $c = n^2$, which are valid for large $n$, as well as the known relation between the tension of the $D-$string and the $D7-$brane and of the $D-$string and the $D3-$brane

$$T_7 = \frac{T_1}{(2\pi l_s)^6}, \qquad T_3 = \frac{T_1}{(2\pi l_s)^2}, \tag{4.14}$$

it is straightforward to obtain the following result for the energy

$$\begin{aligned} E &= \overbrace{NT_1 \int_0^\infty d\sigma}^{\text{D-strings}} + \overbrace{\frac{n^3}{6}T_7\Big(\frac{16\pi^3}{15}\int d\bar{R}\,\bar{R}^6\Big)}^{\text{D7-branes}} \\ &+ \underbrace{\frac{n^5}{240}T_3\int d\bar{R}\;4\pi\bar{R}^2}_{\text{D3-branes}} + \Delta E, \end{aligned} \tag{4.15}$$

where the binding energy

$$\begin{aligned} \Delta E &= NT_1 c^{\frac{1}{4}}\sqrt{\frac{\lambda}{2}}\int_0^\infty \Big[\sqrt{u^{12}+3u^8+3u^4} - u^6 - \frac{3}{2}u^2\Big]du \\ &\approx (0.2629...)NT_1 c^{\frac{1}{4}}\sqrt{\lambda}. \end{aligned} \tag{4.16}$$

The second term in (4.15) is precisely the energy of $n^3/6$ $D7-$branes, so that we have reproduced the noncommutative geometric derivation of the charge given in [45]. The two terms given provide the analog of the two terms providing the total energy of the supersymmetric $D3 \perp D1$ system[12]. The fact that there are further contributions to the energy matches what one finds in the analysis of the $D5 \perp D1$ system[13]. In the $D5 \perp D1$ context, this was interpreted as a consequence of the fact that the system is not supersymmetric. The third term in (4.15) is apparently the energy of $n^5/240$ $D3-$branes. Recall that the zero scale size limit of an instanton in a $Dp-$brane corresponds to a $D(p-4)-$brane bound to the $Dp-$brane[18]. Thus, this term is naturally interpreted as an instanton contribution in the $D7-$brane theory. It is interesting to note that the corresponding term in the $D5 \perp D1$ system arises from a $D1$ contribution, which can be interpreted as an instanton contribution in the $D5-$brane theory. It would be interesting to understand the physical origin of this term, perhaps as a consequence of the Myers effect.

We have evidence that our solution describes a funnel expanding into a number of coincident $D7-$branes located at $\sigma = 0$. The $D7-$branes expand to fill the $X^i$, $i = 1, 2, ..., 7$ directions. If this is indeed the case, this configuration should be a source for the eight-form RR-potential $C^{(8)}_{012345678}$. We check this, providing a further check of the $D7-$brane charge computed by studying the energy of our configuration. The relevant source term comes from the following contribution to the non-Abelian Wess-Zumino action

$$S_{WZ} = -i\frac{\lambda^3}{6}\mu_1 \int STr\ P\Big[(\mathrm{i}_\Phi \mathrm{i}_\Phi)^3 C^{(8)}\Big]. \tag{4.17}$$

Evaluating the value of this term for our solution

$$\begin{aligned} S_{WZ} = & -i\frac{\lambda^4}{6}\mu_1 \int d\sigma d\tau C^{(8)}_{01234567}\ STr\Big(\epsilon^{ijklmnp}\Phi^i\Phi^j\Phi^k\Phi^l\Phi^m\Phi^n\partial_\sigma\Phi^p\Big) \\ = & -i\frac{\lambda^4\mu_1}{6\lambda^7 c^{7/2}} \int d\sigma d\tau C^{(8)}_{01234567}\ STr\Big(\epsilon^{ijklmnp}G^iG^jG^kG^lG^mG^nG^p\Big)\bar{R}^6\frac{d\bar{R}}{d\sigma} \end{aligned}$$

using the identities given in section 2, the relation between $D7$ and $D1$ charges $\mu_7 = \mu_1/(2\pi l_s)^6$, and working in the large $n$ limit, we obtain

$$S_{WZ} = \frac{n^3}{6}\mu_7\Big(\frac{16\pi^3}{15}\int d\bar{R}\ C^{(8)}_{01234567}\ \bar{R}^6\Big) \tag{4.18}$$

This is exactly the seven-brane source term we would expect to get, if we have $n^3/6$ $D7-$branes, in complete agreement with our energy computation. Up to now, we have obtained solutions by employing a method which minimizes the energy. We end this section with a direct analysis of the equations of motion. Requiring that (4.8) is stationary with respect to variations of $\bar{R}$, we obtain the following equation of motion

$$\sqrt{1+\Big(\frac{d\bar{R}}{d\sigma}\Big)^2}\frac{d\sqrt{1+f(\bar{R})}}{d\bar{R}} = \frac{d}{d\sigma}\Bigg(\sqrt{\frac{1+f(\bar{R})}{1+(\frac{d\bar{R}}{d\sigma})^2}}\frac{d\bar{R}}{d\sigma}\Bigg). \tag{4.19}$$

After some straightforward manipulations, this equation of motion can be written as

$$\Big(\frac{d\bar{R}}{d\sigma}\Big)^{-1}\frac{d}{d\sigma}\Bigg(\sqrt{\frac{1+f(\bar{R})}{1+(\frac{d\bar{R}}{d\sigma})^2}}\Bigg) = 0, \tag{4.20}$$

which is easily integrated to give

$$\frac{d\bar{R}}{d\sigma} = \pm\sqrt{kf(\bar{R})-1}, \tag{4.21}$$

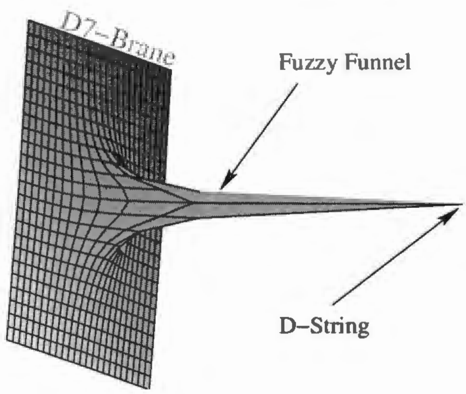



Figure 4.1: $D-$strings intersecting a stack of $D7-$branes.

where $k$ is a non negative dimensionless constant of integration. For $k = 1$, we reproduce the energy we obtained above by minimizing the energy. For $0 \leq k \leq 1$, the solution reaches $\bar{R} = 0$ at finite value of $\sigma$ so that the funnel "pinches" off. As explained in [12] this solution can naturally be continued past $\bar{R} = 0$, by matching to a second pinched off funnel. This configuration provides the description of two parallel sets of coincident $D7-$branes, joined by $N$ finite length $D-$strings. If $k > 1$, the solution reaches $\frac{d\bar{R}}{d\sigma} = 0$ at finite $\sigma$ and terminates. Again [12], this solution is naturally continued by matching to a second funnel. In this case, the double funnel describes $N$ finite $D-$strings joining a set of coincident anti$-D7$ branes with a set of parallel coincident $D7-$branes. This concludes our discussion of the $D-$string theory. In the next section, we turn to a dual description of the same configuration, which employs the non-Abelian world volume theory of the coincident $D7-$branes.

## 4.4 The $D7-$brane worldview

In the previous section we have argued that our funnel describes $N = n^6/360$ $D-$strings expanding into $n^3/6$ $D7-$branes. Consequently the $D7-$brane world volume theory is a $7+1$ dimensional non-Abelian Born-Infeld theory with gauge group $U(n^3/6)$. Further, to describe the $D-$strings, we will also have to excite one of the transverse scalars. This scalar has to reside in the overall $U(1)$ component of the $U(n^3/6)$ gauge group, since it describes a deformation of the geometry of all of the $D7-$branes. Consequently, we consider the action

$$S = -T_7 \int d^8\sigma STr\sqrt{-\det(G_{ab} + \lambda^2\partial_a\phi\partial_b\phi + \lambda F_{ab})} \tag{4.22}$$

We employ spherical coordinates on the D7 worldvolume

$$ds^2 = G_{ab}d\sigma^a d\sigma^b = -dt^2 + dr^2 + r^2 g_{ij}d\alpha^i d\alpha^j, \tag{4.23}$$

with $g_{ij}$ the metric on the six sphere of unit radius, $r$ is the radial coordinate and $\alpha^i$ the angles. In analogy to the $D5 \perp D1$ system [13], we make the following ansatz for the scalar and gauge fields

$$\phi = \phi(r), \qquad A_r = 0, \qquad A_{\alpha^i} = A_{\alpha^i}(\alpha^j). \tag{4.24}$$

Once again we have examined the full equations of motion and have verified that this is indeed a consistent ansatz. Inserting this ansatz into the above action and defining $d \equiv \det g_{ij}$, we obtain

$$\begin{aligned} S_7 &= -T_7 \int d^8\sigma \sqrt{\left(1 + \lambda^2\left(\frac{d\phi}{dr}\right)^2\right)} g STr\sqrt{h(r)}, \\ &= -T_7 \int d^8\sigma L_7 \end{aligned} \tag{4.25}$$

$$\begin{aligned} h(r) &= r^{12} + \frac{1}{2}r^8\lambda^2 F^{ij}F_{ij} + \frac{1}{128}r^4\lambda^4\epsilon_{ijklmn}\epsilon^{ijopqr}F^{kl}F^{mn}F_{op}F_{qr} \\ &+ \frac{1}{2304}\lambda^6(\epsilon_{ijklmn}F^{ij}F^{kl}F^{mn})^2. \end{aligned} \tag{4.26}$$

In the above expression, $F_{ij} \equiv F_{\alpha^i\alpha^j}$, indices on the field strength are raised and lowered with the metric $g_{ij}$, and $\epsilon_{123456} = g$. The equation of motion for the scalar is

$$\frac{d}{dr}\left(\frac{\partial L_7}{\partial \phi'}\right) = 0, \qquad \phi' = \frac{d\phi}{dr}. \tag{4.27}$$

As in the $D1$−brane theory, this is easily integrated to obtain

$$\frac{\lambda^2\phi'}{\sqrt{1+\lambda^2(\partial_r\phi)^2}} = \frac{f(\alpha^i)}{\sqrt{g}STr\sqrt{h(r)}}, \tag{4.28}$$

where $f(\alpha^i)$ is an arbitrary function of integration depending only on the angles $\alpha^i$. The left hand side of the above equation is independent of the $\alpha^i$, so we must have $STr\sqrt{h(r)}$ independent of the angles and further,

$$f(\alpha^i) = \frac{\sqrt{g}\lambda^4}{b}, \tag{4.29}$$

with $b$ a dimensionless constant. With this choice we obtain

$$\lambda\phi' = \pm\frac{1}{\sqrt{\frac{b^2[STr(\sqrt{h(r)})]^2}{\lambda^6} - 1}}. \tag{4.30}$$

After identifying $\sigma = \lambda\phi$ we have

$$\lambda\frac{d\phi}{dr} = \frac{d\sigma}{dr}. \tag{4.31}$$

With this identification and $r = \bar{R}$, the radial profile (4.30) can be matched to the result we obtained from the $D-$string world volume theory (4.21) by setting

$$kf(r) = \frac{[STr\sqrt{h(r)}]^2 b^2}{\lambda^6}. \tag{4.32}$$

This last condition can be satisfied by choosing

$$\begin{aligned} F^{ij}F_{ij} &= \frac{3c}{2}\mathbf{1}, \\ \epsilon_{ijklmn}\epsilon^{ijopqr}F^{kl}F^{mn}F_{op}F_{qr} &= 24c^2\mathbf{1}, \\ (\epsilon_{ijklmn}F^{ij}F^{kl}F^{mn})^2 &= 36c^3\mathbf{1}, \end{aligned} \tag{4.33}$$

where $\mathbf{1}$ is the $(n^3/6)\times(n^3/6)$ unit matrix. It is interesting to note that these last three identities reduce to a single independent equation if one chooses

$$8\sqrt{c}F^{ij} = \epsilon^{ijklmn}F_{kl}F_{mn}. \tag{4.34}$$

This last equation provides an interesting non-linear higher dimensional generalization of the instanton equation. This relation is also suggested by the $D-$string description[45]. In Matrix Theory, the commutator $X^{\mu\nu} = i[X^\mu, X^\nu]$ of the matrix valued coordinates is naturally interpreted as a field strength. The state for which

$$X^7|s\rangle = -n|s\rangle, \qquad X^i|s\rangle = 0, \quad i < 7, \tag{4.35}$$

corresponds to a point at the north pole of the sphere. Locally at the north pole, directions $i$ with $i < 7$ correspond to the $\alpha^i$ directions. Acting on this state, we find that the only non-zero "field strengths" are

$$i[X^1, X^2]|s\rangle = 2n|s\rangle, \quad i[X^3, X^4]|s\rangle = -2n|s\rangle, \quad i[X^5, X^6]|s\rangle = -2n|s\rangle.$$

Since $\sqrt{c} = n$, we see that the field strengths at the north pole do indeed satisfy (4.34). In the remainder of this section we will assume that our field strengths satisfy (4.34) and (4.34). We will not address the issue of obtaining an actual gauge field solution from which we can compute these field strengths. Note that the above field strengths satisfy

$$\frac{1}{48\pi^3}\int \mathrm{Tr}\Big(\frac{\epsilon_{ijklmn}F^{ij}F^{kl}F^{mn}}{8}\Big)\sqrt{g}d^6\alpha = \frac{n^6}{360} = N, \tag{4.36}$$

exactly as one would expect for any six sphere surrounding the $D$−string endpoints.

We now turn to a computation of the energy of this solution. To compare to the energy of the configuration that saturated the energy bound, we now set $k = 1$. The energy is

$$E = T_7\int \sqrt{g}d^6\alpha dr\sqrt{1+\lambda^2\Big(\frac{d\phi}{dr}\Big)^2}\frac{n^3}{6}\sqrt{r^{12}+\frac{3r^8\lambda^2 c}{4}+\frac{3r^4\lambda^4c^2}{16}+\frac{\lambda^6c^3}{64}}.$$

After using (4.30) this becomes

$$\begin{aligned} E &= T_1\frac{n^6}{360}\int_0^\infty dr\lambda\frac{d\phi}{dr} \\ &+ T_7\frac{16\pi^3}{15}\frac{n^3}{6}\int_0^\infty dr\sqrt{r^{12}+\frac{3r^8\lambda^2c}{4}+\frac{3r^4\lambda^4c^2}{16}} \\ &= NT_1\int_0^\infty d\sigma + \frac{n^3}{6}T_7\Big(\frac{16\pi^3}{15}\int d\bar{R}\,\bar{R}^6\Big) \\ &+ \frac{n^5}{240}T_3\int d\bar{R}\;4\pi\bar{R}^2 + \Delta E, \end{aligned}$$

where again

$$\begin{aligned} \Delta E &= NT_1c^{1/4}\sqrt{\frac{\lambda}{2}}\int_0^\infty\Big[\sqrt{u^{12}+3u^8+3u^4}-u^6-\frac{3}{2}u^2\Big]du \\ &\approx (0.2629...)NT_1c^{1/4}\sqrt{\lambda}. \end{aligned}$$

This exactly matches the energy computed using the $D$−string description. Thus, the energy, radial profile of the funnel and charge computed using the $D7$ world volume theory is in exact agreement with the calculations performed using the $D$−string world volume theory.

## 4.5 Fluctuating about the $D1 \perp D7$ background

In this section we study the propagation of fluctuations on the fuzzy funnel solution obtained in section 3. For a similar analysis of fluctuations for the $D3 \perp D1$ and the $D5 \perp D1$ systems see [12] and [13] respectively.

Since our funnel has the topology $\mathbb{R} \times S^6$, the fluctuations of this geometry are naturally decomposed in terms of the spherical harmonics on the $S^6$. Of course, we have a fuzzy $S^6$, so it is natural to expand the fluctuations in terms of traceless symmetric products of the $X^i$, which provide the deformation of the usual algebra of functions on $S^6$. One consequence of the fact that we use a fuzzy sphere is simply that there is a highest angular momentum $l \leq l_{max} = n$. Concretely, we consider fluctuations of the form

$$\delta\Phi^8 = \mathcal{C}_{i_1 i_2 \ldots i_n}(\tau,\sigma) X^{i_1} X^{i_2} \ldots X^{i_n}, \qquad \delta\Phi^i = 0, \quad i < 8.$$

where $\mathcal{C}_{i_1 i_2 \ldots i_n}(\tau,\sigma)$ is required to be a traceless symmetric tensor. Our goal in this section is simply to show that these modes, which correspond to partial waves of angular momentum $n$, see the correct angular momentum barrier. The lowest order equation of motion is

$$(-\partial_\tau^2 + \partial_\sigma^2)\Phi^i = [\Phi^j, [\Phi^j, \Phi^i]].$$

This equation of motion is valid for small $\Phi^j$ and hence corresponds to the region of small $R(\sigma)$. The linearized equation for the fluctuation following from this lowest order equation of motion is

$$(-\partial_\tau^2 + \partial_\sigma^2)\delta\Phi^8 = [\delta\Phi^j, [\Phi^j, \Phi^8]] + [\Phi^j, [\delta\Phi^j, \Phi^8]] + [\Phi^j, [\Phi^j, \delta\Phi^8]].$$

Since $\Phi^8 = 0$ and $\delta\Phi^j = 0$ for $j < 8$, this simplifies to

$$(-\partial_\tau^2 + \partial_\sigma^2)\delta\Phi^8 = [\Phi^j, [\Phi^j, \delta\Phi^8]]. \tag{4.37}$$

To evaluate the right hand side, we need to use the result

$$\begin{aligned} [\Phi^j, [\Phi^j, \delta\Phi^8]] &= R^2(\sigma)\, \mathcal{C}^{i_1 i_2 \ldots i_n} [G^j, [G^j, G^{i_1} G^{i_2} \ldots G^{i_n}]] \\ &= 4n(n+4) R^2(\sigma)\, \mathcal{C}^{i_1 i_2 \ldots i_n} G^{i_1} G^{i_2} \ldots G^{i_n}. \end{aligned}$$

Identifying $R^2(\sigma) = 1/(12\sigma^2)$ which is valid when $R(\sigma)$ is small, we obtain

$$\Big(\partial_\tau^2 - \partial_\sigma^2 + \frac{n(n+4)}{3\sigma^2}\Big)\mathcal{C}^{i_1 i_2 \ldots i_n}(\tau,\sigma) = 0. \tag{4.38}$$

Thus, the double commutator on the right hand side of (4.37) has indeed reproduced the correct angular momentum barrier.

## 4.6 Some thoughts on stability

Since the $D1 \perp D5$ system is supersymmetric, it is only natural to ask if our solution for a $D1$ ending on a $D7$ preserves some supersymmetry too. To avoid any undue suspense, we find that not only is it not supersymmetric but it also unstable. This instability in the intersecting configuration is a consequence of the fact that we can *lower* the energy of our solution, something that is not possible for the $D1 \perp D5$ system. To reproduce the correct $D1$ charge, we need to consider a field strength in the $D7$ worldvolume theory which satisfies

$$\int_{S^6} F \wedge F \wedge F \sim N$$

The magnitude of a field strength supported only on a volume $V$ on the $S^6$, can be estimated to be

$$F \sim \left(\frac{N}{V}\right)^{1/3}$$

so that after expanding the Born-Infeld action and keeping only the term quadratic in the field strength, we obtain the following formula for the energy per unit length:

$$\int_{S^6} F^2 \sim V^{1/3} N^{2/3}. \tag{4.39}$$

Clearly, minimising the volume $V$ on which the field strength is nonvanishing lowers the energy per unit length. Consequently, under the assumptions stated, the configuration with homogeneous field strength over the full $S^6$ must be unstable. To compare this to the $D1 \perp D5$ system, note that the field strength on the $D5$ worldvolume that correctly reproduces the $D1$ charge satisfies

$$\int_{S^6} F \wedge F \sim N$$

Correspondingly, we estimate $F \sim (N/V)^{1/2}$. Hence the energy per unit length of the configuration

$$\int_{S^6} F^2 \sim N. \tag{4.40}$$

is independent of $V$.

## 4.7 Final Comments

We have obtained a description of the $D1 \perp D7$ system in terms of a fuzzy six-sphere which expands along the string. We have studied the energy, charge and radial profile of this configuration using the non-Abelian equations of motion of the $D-$string and also by using the dual description provided by the world volume theory of the $D7-$branes. Our analysis is limited to the low energy world volume theory in each case. The agreement between descriptions is perfect. Further, we have found that the configuration describes $n^6/360$ coincident $D-$strings ending on $n^3/6$ $D7-$branes. This relation between the number of $D-$strings and $D7-$branes has also been obtained from a direct study of the non-commutative geometry of the fuzzy $S^6$.

This precise agreement between the two descriptions is also a feature of the $D1 \perp D3$ and $D1 \perp D5$ systems. For the system we have studied in this chapter, we'd expect the $D7-$brane world volume theory will provide a reliable description for those regions of the funnel that have opened up to fill out a seven dimensional spatial volume and are hence well approximated as a $D7-$brane. The $D-$string world volume theory should provide a reliable description of the funnel in the regions where the funnel is very thin and hence well approximated by a $D-$string. Thus we have two complementary descriptions of the $D1 \perp D7$ system. How are we to understand the agreement between the two descriptions of the $D1 \perp D7$ system?

There are two potential sources of corrections to both descriptions. There are both higher derivative corrections and higher order commutator corrections. Following [13] we assume that we can ignore higher derivative corrections when $l_s|\partial^2\Phi| \ll |\partial\Phi|$. For the $D-$string theory, we easily find that this condition implies that $r \ll (n^3\pi^3/12)^{1/5}l_s$. For the $D7-$brane theory this condition implies that $r \gg 2l_s$. Thus, for large $n$ there is a significant region ($2l_s \ll r \ll (n^3\pi^3/12)^{1/5}l_s$) where both descriptions do not receive higher derivative ($\alpha'$) corrections. A conservative bound for the region in which higher commutator terms are avoided is obtained by requiring that the Taylor expansion of the square root in the $D-$string action should converge very rapidly. This implies that $r \ll \sqrt{n}l_s$. For large $n$ we have $\sqrt{n} \gg (n^3\pi^3/12)^{\frac{1}{5}}$, so that this is actually less restrictive than what we obtained above. We have not established the analogous region in which higher commutator terms are avoided in the $D7-$brane theory.

# Chapter 5

# And now for something completely different...

*We provide a realisation of a singularity-free inflationary universe in the form of a simple cosmological model dominated at early times by a single minimally coupled scalar field with a physically based potential. The universe starts asymptotically from an initial Einstein static state, which may be large enough to avoid the quantum gravity regime. It enters an expanding phase that leads to inflation followed by reheating and a standard hot Big Bang evolution. We discuss the basic characteristics of this Emergent model and show that none is at odds with current observations.*

## 5.1 Introduction

The idea that the universe we inhabit might be in a state of eternal inflation is not new. In fact it was realised quite soon after its initial proposal that inflation is usually future eternal; that is, in most cases there will always be regions of spacetime that are inflating in the future. So a natural question arises as to whether the universe was always inflating in the past; could inflation be *past eternal* also? Recent work by several authors [10] seems to indicate that the answer to this intriguing question is decidedly "no"! Using Penrose-Hawking-Geroch techniques, it is argued in detail that a spacetime that (*i*) is past causally simple, (*ii*) is open, (*iii*) is described by Einstein's equations with a matter source that obeys the weak energy condition, and (*iv*) allows for inflation to be future eternal, *cannot be past null complete.* Crucial to the argument is the assumption that the universe is open or at least flat. As is often pointed out though, we are emerging into an era of "precision cosmology" and measurements of temperature anisotropies in the Microwave Background are able to place greater constraints on the curvature of the universe than ever before [84]. The recent WMAP data point to a universe that is close to (but not quite) flat, with a best fit to the total density of $\Omega_{tot} = 1.02 \pm 0.02$. This, taken at its face value, suggests a closed ($k = +1$) model. This would seem to offer a promising avenue around the arguments of [10]. If we live in a universe that is closed (albeit only marginally so) today, then it was always closed, and perhaps inflation is past-eternal after all.

Recently arguments were put forward for several inflationary cosmologies that were past-eternal while avoiding any quantum gravity regime [21]. Each of the proposed models is spatially closed and described only with general relativity, ordinary matter, and minimally coupled scalar fields. Their existence argues against the suggestions of [10] that inflationary universes are necessarily bounded in the past, and shows that a quantum gravity dominated era for the universe may not be inevitable; something that has also been noted in [69]. The Eddington-Lemaître cosmology is a well-known example of a universe that is not past geodesic-incomplete, because of its positively curved spatial sections. Harrison has also given an exact solution with similar properties [39]. His is a radiation-dominated closed universe with a positive cosmological constant. It starts from an Einstein static state, with a radius determined by the value of $\Lambda$, before entering a never-ending period of de Sitter expansion. However both these models do not exit inflation. Here, following [21], we consider a universe filled with a dynamical scalar field, which is past asymptotic to an

Einstein static model with a radius determined by the field's kinetic energy. This model enters a period of de Sitter inflation that comes naturally to an end as the scalar field starts oscillating around the minimum of the potential, before entering the standard hot Big-Bang expansion phase. Thus these are singularity-free inflationary universe models[1], by-passing the restrictions of the singularity theorems mentioned above [2]. They are finely tuned in terms of the initial conditions, although one can use entropy arguments to favor an initial Einstein Static phase for our universe [28]. We consider possible implications of this fine-tuning at the end of this chapter.

## 5.2 An exact solution

Consider a Friedmann-Robertson-Walker (FRW) universe containing a minimally coupled scalar field $\phi$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{5.1}$$

where the last equality follows from assuming spatial homogeneity of the $\phi$ field *i.e.* $\phi = \phi(t)$. With this assumption, the stress-energy tensor takes the form of a perfect fluid with energy density and pressure

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned} \tag{5.2}$$

[1]One objection that could be raised at this point is the persistence of closed trapped surfaces in this cosmological model and the consequent implications for the existence of a cosmological singularity. Indeed such ***past closed trapped surfaces*** do exist but as shown in [20] do not imply the existence of a cosmological singularity in the model.

[2]The assumptions of the singularity theorems were later relaxed [10] and it was pointed out that essentially the same argument holds for any spacetime in which null geodesics do not recross after traversing the entire universe. This is not true for the Einstein static spacetime so the models presented here safely fall out of the domain of the Borde-Vilenkin theorem.

respectively [3]. The classical equation of motion for $\phi$ that follows from variation of the action $S = \int \mathrm{d}^4x \sqrt{-g}\mathcal{L}$ is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0\,, \tag{5.3}$$

where the Hubble parameter $H := \dot{a}(t)/a(t)$. The Raychaudhuri field equation for the FRW model with scalar field matter source and its first integral, the Friedmann equation are

$$3\dot{H} + 3H^2 = 8\pi G\Big(V(\phi) - \dot{\phi}^2\Big) \tag{5.4}$$

and

$$3H^2 + 3\frac{k}{a^2} = 8\pi G\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big) \tag{5.5}$$

respectively. These equations, together with the Klein-Gordon equation form a closed dynamical system from which the evolution of the universe model is determined. However, it is important to note that the Klein-Gordon equation is *auxiliary* in the sense that any solution of (5.4) and (5.5) with nonvanishing $\dot{\phi}$ will necessarily satisfy the Klien-Gordon equation so that the dynamical system in fact contains only *two* independent equations [4]. These may be combined [22] to give the (more convenient) equivalent set of equations

$$\begin{aligned} V(\phi(t)) &= \frac{1}{8\pi G}\Big(\dot{H} + 3H^2 + 2\frac{k}{a^2}\Big) \\ \dot{\phi}^2(t) &= \frac{1}{4\pi G}\Big(\frac{k}{a^2} - \dot{H}\Big). \end{aligned} \tag{5.6}$$

With these equations at hand, the potential $V(\phi)$ is constructed by

- Specifying the constant $k$ and a particular (monotonic) function $a(t)$ and computing the associated Hubble parameter $H$ and $\dot{H}$.

---

[3] More generally, the FRW universe may also be assumed to contain some noninteracting perfect fluid with energy density $\rho$ and pressure $p = w\rho$ with $-\frac{1}{3} < w \leq 1$. The inclusion of such additional matter sources is however unnecessary for the purpose of this work.

[4] If $\dot{\phi}$ does indeed vanish then the Klein-Gordon equation is no longer a consequence of the other two equations but may nevertheless be easily solved to yield a constant potential $V(\phi) = const.$

- Checking that the constraint $k/a^2 - \dot{H} \geq 0$ is met. This assures the positivity of $\dot{\phi}^2$ as is necessary for a neutral scalar field.
- Specifying an initial condition $\phi_0$ for $\phi(t)$ and integrating the second of eqs.(5.6) to get $\phi(t)$. This is then inverted (where possible) to give $t(\phi)$.
- Substituting into the first of eqs.(5.6) to obtain $V(t) = V(t(\phi))$ and subsequently $V(\phi)$.

Thus, as advertised, this algorithm associates to a specified scale factor $a(t)$ a potential $V(\phi)$ to give a model that *exactly* solves the classical inflationary field equations. At this point some comments are in order: First, the reconstruction of the potential is manifestly non-unique as should be clear from the sign ambiguity in the choice of root in the $\dot{\phi}$ equation (5.6). Secondly, it is not clear how sensitive this algorithm is to the choice of initial conditions for $\phi$ *i.e.* on where on the potential the scalar field resides when it begins to roll. While general shape should be independent it might be expected that the details of the potential (location of extrema etc.) would vary with different initial conditions. This, however, does not affect the argument.

The *emergent model* is essentially a modified version of the Eddington-Lemaître universe - $k = +1$, past-asymptotically Einstein static, singularity-free, without particle horizons and ever-inflating (see [21] for more details). Before applying the potential reconstruction technique to this model, a few general observations about the scenario beg attention. The Einstein static state containing matter with energy density $\rho_i$ and pressure $p_i = w_i\rho_i$ $(-1/3 < w_i < 1)$ is characterized by

$$\begin{aligned} \frac{1}{2}(1 - w_i)\rho_i + V(\phi) &= \frac{1}{4\pi G a_0^2}, \\ (1 + w_i)\rho_i + \dot{\phi}_i^2 &= \frac{1}{4\pi G}, \end{aligned} \tag{5.7}$$

where $a_0$ is the radius of the $S^3$ spatial sections of the universe. If the sole content of this universe is the scalar field, as is the case in this note, $\rho_i = 0$ and the second of eq.(5.7) requires that the scalar field have non-vanishing (but constant) kinetic energy. In this model with no matter initially present, the mechanism envisaged in [21] has $\phi$ rolling at a constant speed along a flat potential ($V = V_i$) from $\phi = -\infty$ at $t = -\infty$ to $\phi = 0$ at $t = 0$ where the potential first rises (and inflation is initiated) and then drops to a minimum at $\phi = \phi_f$ where the value of the potential is $V_f = \Lambda/8\pi G \ll V_i$. The field is

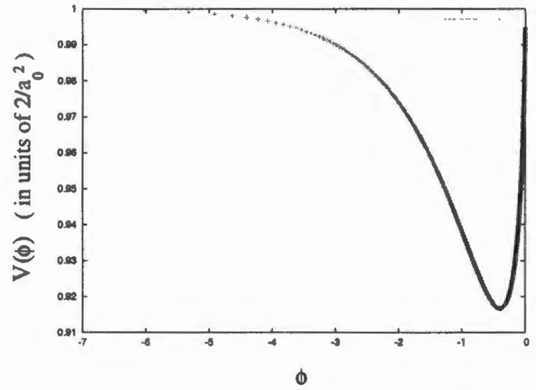



Figure 5.1: A parametric plot of the re-constructed potential corresponding to the *exact solution* $a(t) = a_0 + \exp(ht)$ normalized with respect to the size of the Einstein static universe. This re-constructed potential exhibits the same shape as the effective potential of the $R^2$ model in the Einstein frame but is plagued by a rather shallow minimum (only about 91% of its asymptotic value.)

carried over the hill by its non-zero kinetic energy and slow rolls toward the minimum where its damped oscillations reheats the universe. What follows is an attempt to realise this scenario.

To this end, and to facilitate numerical computations it will prove useful to rescale quantities of interest as

$$\begin{aligned} V(\phi) \to M_{Pl}^4 V(\phi) \quad &, \quad \phi(t) \to M_{Pl}\phi(t), \\ t \to t/M_{Pl} \quad &, \quad a(t) \to a(t)/M_{Pl}, \end{aligned} \tag{5.8}$$

where $M_{Pl}^2 := 8\pi G$. In these units the Raychaudhuri equation and its first integral, the Friedmann equation become

$$\begin{aligned} 3\dot{H} + 3H^2 &= V(\phi) - \dot{\phi}^2, \\ 3H^2 + 3\frac{1}{a^2} &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \end{aligned} \tag{5.9}$$

which, together with the Klein-Gordon equation form the dynamical system governing the evolution of the closed scalar field dominated FRW universe. The corresponding equations (5.6) for the potential and $\dot{\phi}(t)$ that are the starting point for the reconstruction of the potential take the form

$$\begin{aligned} V(\phi(t)) &= \dot{H} + 3H^2 + \frac{2}{a^2} \\ \dot{\phi}^2(t) &= 2\Big(\frac{1}{a^2} - \dot{H}\Big). \end{aligned} \tag{5.10}$$

Following [21] consider the scale factor

$$a(t) = A + B\exp(ht)\,, \tag{5.11}$$

where $A, B, h$ are all positive constants. This universe is past asymptotic to an Einstein static phase, since $a(t) \to A$ as $t \to -\infty$. Thus, $A$ is identified with the radius $a_0$ of the Einstein static universe. At late times, on the other hand, $a(t) \to B\exp(ht)$ and the model approaches a de Sitter expansion phase. The second of eqs. (5.10) then gives

$$\phi(b) = \sqrt{2}\int \frac{db}{hb}\sqrt{\frac{1-h^2a_0b}{(a_0+b)^2}}, \tag{5.12}$$

where $b := B\exp(ht)$. The integral is easily evaluated to give

$$\begin{aligned} \phi(b) &= \frac{\sqrt{2}}{ha_0}\ln\Bigg[\Bigg(\frac{\sqrt{1+h^2a_0^2}+\sqrt{1-h^2a_0b}}{\sqrt{1+h^2a_0^2}-\sqrt{1-h^2a_0b}}\Bigg)^{\sqrt{1+h^2a_0^2}}\cdot \\ &\quad \cdot\Bigg(\frac{1-\sqrt{1-h^2a_0b}}{1+\sqrt{1-h^2a_0b}}\Bigg)\Bigg], \end{aligned} \tag{5.13}$$

and with the $b$ parameterization the potential is written

$$V(b) = \frac{3(hb)^2 + h^2a_0b + 2}{(a_0+b)^2}. \tag{5.14}$$

At this stage, an expression for $V(\phi)$ is, in principle, obtained by inverting eq. (5.13) and substituting $b(\phi)$ into eq.(5.14). However the above form of $\phi(b)$ precludes such a simple treatment. Nevertheless, the general shape of $V(\phi)$ is easily determined by a parametric plot. This is given in Fig.5.1

## 5.3 A potential Emergent potential

Slightly more than a decade ago it was shown [22] that by an inversion of the conventional viewpoint (beginning with a scalar field whose self interaction is dictated by some underlying particle physics considerations and subsequently determining the evolution of the universe) scalar field dynamics could be explicitly accounted for without a *slow-roll approximation.* Indeed a scalar field potential $V(\phi)$ could quite easily be 'reverse engineered' for almost any desired behaviour of the scale factor $a(t)$. Several examples were explicitly computed and are summarized:

| $a(t)$ | $V(\phi)$ |
|---|---|
| $A\exp(\omega t)$ | $3\kappa^{-1}\omega^2 + \omega^2(\phi - \phi_0)^2$ |
| $A\sinh(\omega t)$ | $3\kappa^{-1}\omega^2 + B^2\sinh^2\Big(2\omega(\phi - \phi_0)/B\Big)$ |
| $A\cosh(\omega t)$ | $3\kappa^{-1}\omega^2 + B^2\sin^2\Big(2\omega(\phi - \phi_0)/B\Big)$ |
| $At^n$ | $(3n-1)B^2\exp\Big(\pm 2(\phi - \phi_0)/B\Big)/2$ |

where $\kappa = 8\pi G$. The above respectively correspond to de Sitter exponential expansion, de Sitter expansion from a singularity, de Sitter expansion without a singularity, and power-law expansion respectively [22]. The *eternal emergent universe*, a nonsingular model past asymptotic to an Einstein static universe with topology $\mathbb{R}\times S^3$ and radius $a_0 \gg L_{Pl}$, can be treated in this way too. For a prescribed scale factor behaviour of $a(t) \sim a_0 + \exp(ht)$ an associated potential is not too difficult to compute. While it certainly produces the desired early time behaviour, the potential does not have a definitely zero minimum. This is an undesirable feature as it points to a rather large cosmological constant, although there are ways around it. Also, because of the limits on the integration in the reverse-engineering construction, a graceful exit from the inflationary regime is not always guaranteed. Consequently we need to find a universe with a potential that is essentially the same as that in the eternal emergent universe at very early times but then goes to zero at some finite value of the field. That is, we need to look for a potential $V(\phi)$ that matches onto the reconstructed potential (Fig. 5.1) as $t \to -\infty$, with a long flat plateau, but then has a vanishing minimum value.

Fortunately we need not look too far. Inflationary models based on higher derivative curvature terms go back to the remarkably prescient work of Starobinsky [85] in which the de Sitter phase was driven by the trace anomaly of the energy momentum tensor (see also [42]). Among the variants on the original Starobinsky model, $R^2$-inflation based on a Lagrangian of the form $\mathcal{L} = R + \alpha R^2$ exhibits a particularly elegant implementation of a de Sitter phase with a linearly decaying Hubble parameter, $H$. The $R^2$ term in this action is effectively an additional scalar degree of freedom which may be absorbed by the introduction of a (non-dynamical) scalar field [94].[5] Einstein gravity is restored by an appropriate conformal transformation but at the expense of a *dynamical* scalar field with an interesting potential [96, 63, 51, 5].

**From Jordan to Einstein.**

Since we build our model from a potential quite closely related to the that of $R^2$ gravity, it is worth seeing how the effective potential in the latter arises from a conformal transformation between Jordan and Einstein frames.

Starting from an $R^2$-modified action

$$\begin{aligned} S &= \int \mathrm{d}^4x\,\sqrt{-g}\Big[R+\alpha R^2\Big] \\ &= \int \mathrm{d}^4x\,\sqrt{-g}\Big[(1+2\alpha R)R-\alpha R^2\Big], \end{aligned} \tag{5.15}$$

define $\Omega^2 := 1+2\alpha R$ and make the conformal transformation $g_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ so that $\sqrt{-\widetilde{g}} = \Omega^4\sqrt{-g}$ and

$$\widetilde{R} = \frac{1}{\Omega^2}\Big[R - 6g^{\mu\nu}\nabla_\mu\nabla_\nu(\ln\Omega) - 6g^{\mu\nu}\nabla_\mu(\ln\Omega)\nabla_\nu(\ln\Omega)\Big]$$

Substituting this into (5.15) gives

$$S = \int \mathrm{d}^4x\sqrt{-\widetilde{g}}\Big[\widetilde{R} - \frac{\alpha}{\Omega^4}R^2\Big] + 6\int \mathrm{d}^4x\sqrt{-g}\Omega\nabla_\mu\Big(g^{\mu\nu}\partial_\nu\Omega\Big)$$

The last term may be evaluated by noting that the divergence $\nabla_\mu X^\mu = (1/\sqrt{-g})\partial_\mu(\sqrt{-g}X^\mu)$. With this,

$$\int \mathrm{d}^4x\sqrt{-g}\Omega\nabla_\mu\Big(g^{\mu\nu}\partial_\nu\Omega\Big) = -\int \mathrm{d}^4x\sqrt{-\widetilde{g}}\frac{1}{\Omega^2}\widetilde{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega$$

after a boundary term is discarded.

[5] This scalar field is in fact an auxiliary one whose 'equation of motion' may be trivially solved to show that it is nothing but the scalar curvature in disguise.

**Continued....**

The action for the $R^2$ model in the Einstein frame is then written as

$$S = \int d^4x\sqrt{-\tilde{g}}\Big\{\tilde{R} - \frac{6\alpha^2}{(1+2\alpha R)^2}\Big[\tilde{g}^{\mu\nu}\partial_\mu R\partial_\nu R + +\frac{1}{6\alpha}R^2\Big]\Big\} \tag{5.16}$$

To make contact with a canonical form for scalar field actions [5], it is usual to rewrite the scalar degree of freedom offered by the scalar curvature as $\varphi := \sqrt{3}\ln(1+2\alpha R)$ so that

$$S = \int d^4x\sqrt{-\tilde{g}}\Big\{\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{4\alpha}(e^{-\varphi/\sqrt{3}} - 1)^2\Big\}.$$

The effective potential above is just the reflection of that in Fig. 5.2 about $\phi = 0$. Note that the parameters describing the potential are fairly rigidly constrained (with respect to $\alpha$) by the conformal transformation.

Since it's conception the $R^2$ model has received a significant amount of attention, largely as a result of the fact that the scalar field driving the de Sitter phase arises so naturally and is not inserted "by hand" solely to provide the inflationary dynamics. Indeed it was shown in [63] that the model quite naturally supports a transient period of inflation followed by a FRW universe. Constraints on the coupling constant $\alpha$ are imposed by requiring that density perturbations be of an appropriate magnitude. Consequently, $10^{12}M_{Pl}^{-2} \lesssim \alpha \lesssim 10^{16}M_{Pl}^{-2}$. This in turn determines the height of the plateau of the potential to be of the order $10^{-13} - 10^{-17} \times M_{Pl}^4$ [67]. This form of the potential will form the cornerstone of our construction of the Emergent universe.

## 5.4 Determination of the parameters in the potential

Emergent type universes can be realised by relaxing the constraints on the $R^2$ effective potential and considering a spacetime filled with a minimally coupled, single scalar field $\phi$ with a potential of the general form

$$V = V(\phi) = \left(A\,e^{B\phi} - C\right)^2 + D\,, \tag{5.17}$$

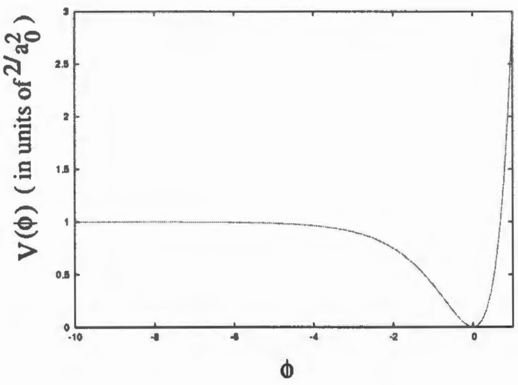



Figure 5.2: The Emergent potential for the parameter choice $B = 1$ is identical up to an overall rescaling to the effective potential for the scalar field in the Einstein frame of the $R^2$-driven inflation model after relabeling $\phi$ as $-\varphi$.

where $A$, $B$, $C$ and $D$ are constants to be determined by the specific properties of the emergent universe. Then,

$$V'(\phi) = 2AB\left(A\,\mathrm{e}^{B\phi} - C\right)\mathrm{e}^{B\phi}\,, \tag{5.18}$$

and

$$V''(\phi) = 2AB^2\left(2A\mathrm{e}^{B\phi} - C\right)\mathrm{e}^{B\phi}\,, \tag{5.19}$$

where a prime indicates a derivative with respect to $\phi$. Therefore, the above potential has a minimum at $\phi_0 = (1/B)\ln(C/A)$ with $V_0 = V(\phi = \phi_0) = D$. Consequently, if we want to set the minimum of the potential at the origin of the axes, we must choose $A = C$ and $D = 0$. Note that zero minimum for $V(\phi)$ guarantees that there is no residual cosmological constant. Then, expression (5.17) reduces to

$$V(\phi) = A\left(\mathrm{e}^{B\phi} - 1\right)^2\,. \tag{5.20}$$

By definition, the emergent universe corresponds to a past-asymptotic Einstein-static (ES) model [21]. This means that

$$V(\phi \to -\infty) = \frac{2}{\kappa a_0^2}\,, \tag{5.21}$$

where $a_0$ is the radius of the initial static model [6]. It is then clear that for $a_0 \gg L_{Pl}$, where $L_{Pl}$ is the Planck length, the model can avoid the quantum regime. To determine an additional parameter recall that an ES universe filled with a single scalar field satisfies the condition $V(\phi) = 2/\kappa a_0^2 = \dot{\phi}$, where $\kappa = 8\pi G$ [6]. Hence, for an initially ES state we require that $A = 2/\kappa a_0^2$, which brings the expression of the potential down to

$$V(\phi) = \frac{2}{\kappa a_0^2}\left(e^{B\phi} - 1\right)^2 . \tag{5.22}$$

Therefore, the basic properties of the emergent universe have already fixed three of the four parameters in the original potential given by Eq. (5.17). The remaining parameter ($B$) will be determined by considering other aspects of the model and in particular by looking into the density perturbation spectrum (see Sec. VIII below). Nevertheless, we can still determine, the sign of $B$ by demanding, without loss of generality, that

$$V' = \frac{4B}{\kappa a_0^2}\left(e^{B\phi} - 1\right)e^{B\phi} < 0 \tag{5.23}$$

for $-\infty < \phi < 0$. This means that $e^{B\phi} - 1 < 0$ and consequently that $B > 0$. Incidentally, on comparison with the corresponding potential of the $R^2$ action, this is seen to correspond to a choice of negative coupling constant $\alpha$ as required to avoid manifesting tachyons and singular perturbative behaviour in the model [5].

## 5.5 Leaving the Einstein-static regime

As the universe leaves the ES state the evolution of the scalar field is determined by the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \tag{5.24}$$

where $H = \dot{a}/a$ is the Hubble parameter. Therefore, given that $\dot{\phi} > 0$, we have expansion only if $\ddot{\phi} < -V'$. Since the potential drops with increasing $\phi$, we may consider the following two alternative cases:

(i) $\ddot{\phi} < 0$, namely a decelerated scalar field, where the friction for the deceleration comes from the expansion. In this case one expects the kinetic energy of $\phi$ drops. Then, since $\dot{\phi}^2 \simeq V$ initially and $V' \simeq 0$, a slow-rolling period with $\dot{\phi}^2 \ll V$ seems likely.

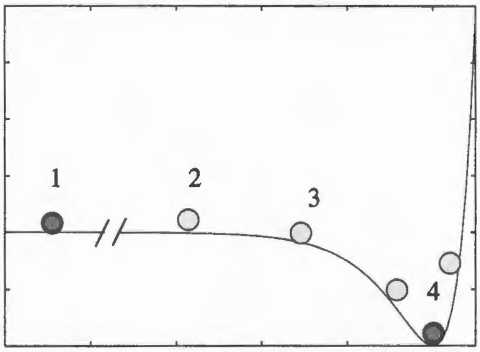



Figure 5.3: A schematic illustration of the scalar field evolution in the Emergent universe. After leaving its initial static state **(1)** the model enters a slow-rolling regime (see case (ia)), or it goes through an intermediate pre-slow-roll phase (see case(ib)) - **(2)**. In either case the scale factor grows sufficiently quickly to mitigate neglecting the curvature effects. A period of slow-roll **(3)** inflation is followed by a re-heating phase **(4)**and then by the standard hot Big Bang evolution.

(ii) $0 \leq \ddot{\phi} < -V'$, that is a "slowly" accelerating scalar field. Then, one expects the kinetic energy of the scalar field to increase. Under these conditions, a slow-rolling regime seems unlikely. Therefore, we are left with case (i), which splits further into two subcases:

(ia) $V' < \ddot{\phi} < 0$, namely a "weakly" decelerated scalar field, monitored by the familiar expression associated with slow-roll inflation

$$3H\dot{\phi} + V' = 0\,, \tag{5.25}$$

since $|\ddot{\phi}| < |V'|$; and

(ib) $\ddot{\phi} < V' < 0$, that is a "strongly" decelerated scalar field, described by the following form of the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} = 0\,, \tag{5.26}$$

since $|\ddot{\phi}| > |V'|$. So, as the universe leaves the ES state, it may evolve in a number of ways depending on the relation between $\ddot{\phi}$ and $V'$. Cases (i) and

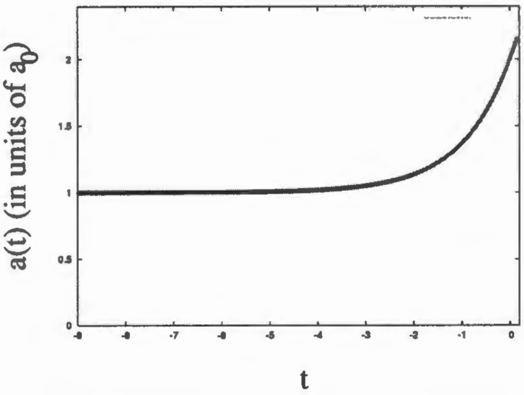



Figure 5.4: The initial evolution of the scale factor in the Emergent universe.

(ii) lead to expanding models, of which (i) is the alternative corresponding to the Emergent Universe. Of the two possible subcases of (i), the first leads immediately to the standard slow-rolling inflationary regime. In the next sections we will concentrate primarily on this particular case. Before we proceed further, however, we should make the following comment with regard to case (ib). It involves a strongly decelerated, that is a very slow-rolling scalar field. Nevertheless, Eq. (5.26) also implies that $\dot{\phi} \propto a^{-3}$. The latter suggests that, as $\phi$ drops rapidly, subcase (ib) could also lead to the familiar slow-rolling inflation. We may test this possibility by considering the reverse engineered potential above. (see Eqs. (5.13), (5.14)), where $\ddot{\phi}$ dominates over $V'$ just like in (ib). Then, the point where the two potentials deviate should determine the initial conditions for the Einstein field equations that describe subcase (ib).

## 5.6 The duration of the slow-roll regime

We solve numerically and plot the results for the scale factor in Fig. 5.4. The graph clearly shows that $a$ has the familiar exponential increase associated with standard slow-rolling inflationary models while at earlier times approaches a constant non-zero value.

The duration of the slow-rolling inflationary regime, in a spatially flat model, is determined by the usual slow-roll parameters [59]

$$\epsilon(\phi) = \tfrac{1}{2} M_{Pl}^2 \left(\frac{V'}{V}\right)^2 \qquad \text{and} \qquad \eta(\phi) = M_{Pl}^2 \frac{V''}{V}, \tag{5.27}$$

where $M_{Pl}$ is the Planck mass. Throughout slow-roll approximation, the above satisfy the constraints $\epsilon(\phi) \ll 1$ and $|\eta(\phi)| \ll 1$, which provide the limits of the slow-roll regime. Applied to the potential of the emergent universe (given by Eq. (5.22), with $B > 0$ and for $-\infty < \phi < 0$), the above constraints read

$$\epsilon(\phi) = \frac{2B^2 e^{2B\phi}}{\left(e^{B\phi} - 1\right)^2} \ll 1 \tag{5.28}$$

and

$$|\eta(\phi)| = \left|\frac{2B^2\left(2e^{B\phi} - 1\right)e^{B\phi}}{\left(e^{B\phi} - 1\right)^2}\right| \ll 1\,, \tag{5.29}$$

respectively, where we have set $M_{Pl} = 1$ for simplicity. It should be emphasised that, although our model is spatially closed, the effect of the curvature (which is dominant at very early times) becomes negligible after few e-foldings, and only re-emerges in the recent universe.

After a rather lengthy, but fairly straightforward analysis, one can show that neither constraint provides a lower bound for $\phi$. Therefore, the slow-roll regime starts at an arbitrarily small value of $\phi$. In practice, this means that the emergent model starts slow rolling at a few e-foldings after leaving its initial ES state, at a finite $\phi_i$, when the curvature effects have become negligible. Given that the static regime corresponds to $\phi \to -\infty$, $\phi_i$ is very small and the available number of e-foldings can be very large.

Note that, by employing (5.28) and (5.29), we can also show that for any positive value of $B > 0$ there is always a negative value for $\phi$ at which the slow-rolling regime ends *i.e.*, where $\epsilon$, $\eta \simeq 1$. For example, when $B = 1$ we find that $\phi < -\ln(1+\sqrt{2})$ (for $M_{Pl} = 1$).

## 5.7 The number of e-foldings

For standard slow-roll inflation the number of e-foldings is given by the expression [53]

$$N(\phi_i \to \phi_f) = \int_{t_i}^{t_f} H \mathrm{d}t\,, \tag{5.30}$$

where $\phi_i$, $\phi_f$ and $t_i$, $t_f$ are the initial and final values of $\phi$ and $t$ respectively, and $H$ is the Hubble parameter. The latter is generally given by [6, 21]

$$H^2 = \tfrac{1}{3}\kappa\left(\tfrac{1}{2}\dot{\phi} + V\right) - \frac{k}{a^2}\,, \tag{5.31}$$

where $k = 0, \pm 1$ is the 3-curvature index. After few e-foldings, however, the effect of the curvature term becomes negligible, while for slow-roll inflation we have $\dot{\phi}^2 \ll V$. Thus, throughout the slow-rolling regime $H = \sqrt{\kappa V/3}$. Combining expression (5.30) with this result we obtain

$$N = -\kappa \int_{\phi_i}^{\phi_f} \frac{V}{V'} \mathrm{d}\phi \,. \tag{5.32}$$

given that $\mathrm{d}t = \mathrm{d}\phi/\dot{\phi}$ and that $\dot{\phi} = -V'/\sqrt{3\kappa V}$, as the slow-rolling version of the Klein-Gordon equation (*i.e.* with $\ddot{\phi} \simeq 0$) guarantees.

Applying the above to the case of the emergent universe, namely inserting the potential (5.22), we arrive at the expression

$$N = -\frac{\kappa}{2B} \int_{\phi_i}^{\phi_f} \frac{\mathrm{e}^{B\phi} - 1}{\mathrm{e}^{B\phi}} \,, \tag{5.33}$$

which provides the number of e-foldings associated with the emergent universe as a function of the yet undetermined parameter $B$. This is easily computed numerically, and in Fig. 5.5 we plot $N$ against the value of $\phi$ at the start of the slowroll period for various values of the parameter $B$. Clearly, depending on the value of $\phi_i$, sufficient e-folds are easily obtainable in this model. Note that $N$, while certainly very large, is nevertheless finite since the value of $\phi_i$ can be stretched back only to the point where the curvature effects become appreciable. From a different point of view, the reason there are only a finite number of efoldings is because the initial value of the scale-factor is non-zero (unlike the standard inflationary model with $K = 0$).

## 5.8 Scale factor evolution

Starting from the Friedmann equation of the slow-rolling inflationary regime (see Eq. (5.31)) we have

$$\frac{\dot{\phi}}{a} \frac{\mathrm{d}a}{\mathrm{d}\phi} = \sqrt{\tfrac{1}{3}\kappa V} \,. \tag{5.34}$$

At the same time, the Klein-Gordon equation gives $\dot{\phi} = -V'/\sqrt{3\kappa V}$, having set $\ddot{\phi} \simeq 0$. Combining the two we arrive at

$$\frac{1}{a}\mathrm{d}a = -\frac{\kappa V}{V'}\mathrm{d}\phi \,. \tag{5.35}$$

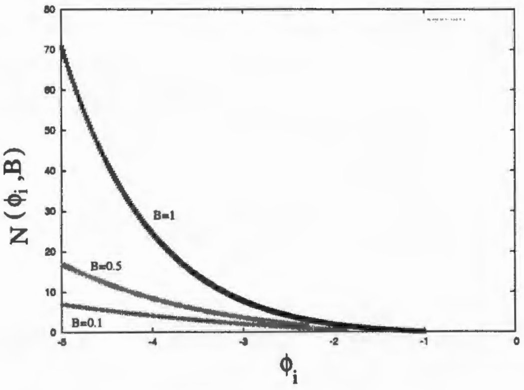



Figure 5.5: The number of e-folds obtained during a slowroll regime in the Emergent model plotted against $\phi_i$ and for various values of the parameter $B$.

Applied to the emergent universe, by using expressions (5.22) and (5.23), the above lead to the differential equation

$$\frac{1}{a}\mathrm{d}a = \frac{\kappa}{2B}\frac{1-\mathrm{e}^{B\phi}}{\mathrm{e}^{B\phi}}\mathrm{d}\phi\,, \tag{5.36}$$

which integrated gives

$$\ln\left(\frac{a}{a_i}\right) = -\frac{\kappa}{2B^2}\left(\mathrm{e}^{-B\phi}-\mathrm{e}^{-B\phi_i}\right) - \frac{\kappa}{2B}(\phi-\phi_i)\,. \tag{5.37}$$

So, the value of the scale factor at the end of the slow-rolling regime depends crucially on $\phi_i$, namely on the value of $\phi$ at the onset of slow-roll inflation. As expected, the smaller $\phi_i$ is the larger the final value of $a$.

## 5.9 The density spectrum

For structure formation purposes it is crucial to determine the density contrast at horizon crossing 50 e-foldings before the end of inflation. Following [53] we have

$$\left(\frac{\delta\rho}{\rho}\right)_{Hor} \simeq \left(\frac{H^2}{\dot{\phi}}\right)_{N=50} \simeq -\left(\frac{3H^3}{V'}\right)_{N=50}\,. \tag{5.38}$$

During the slow-rolling regime the Hubble parameter is given by Eq. (5.31) and the above takes the form

$$\left(\frac{\delta\rho}{\rho}\right)_{Hor} \simeq -\sqrt{\tfrac{1}{3}\kappa^3}\left(\frac{V^{3/2}}{V'}\right)_{N=50}. \tag{5.39}$$

Substituting the potential of the emergent universe (see Eq. (5.22)), we obtain the following expression for the density contrast associated with the model

$$\left(\frac{\delta\rho}{\rho}\right)_{Hor} \simeq \frac{\kappa}{\sqrt{6}Ba_0}\left[\frac{\left(e^{B\phi}-1\right)^2}{e^{B\phi}}\right]_{N=50}, \tag{5.40}$$

which also depends on the yet undetermined parameter $B$. In view of the COBE observations the above is constraint by

$$\frac{\kappa}{\sqrt{6}Ba_0}\left[\frac{\left(e^{B\phi}-1\right)^2}{e^{B\phi}}\right]_{N=50} \simeq 10^{-5}. \tag{5.41}$$

Consequently, the CMB anisotropy limits allow us to express $B$ as a function of $a_0$, the radius of the initial ES state, which therefore becomes the key parameter of the emergent model. In Fig. 5.6 we plot $\delta\rho/\rho$ against $a_0$ for several values of the parameter $B$. The appropriate value of the Einstein radius is then read off from the intersection of the perturbation curves with $\delta\rho/\rho = 10^{-5}$. For example, the $B = 1$ curve satisfies the density perturbation requirement when $a_0 \simeq 10^6\ L_{Pl}$. Again a comparison with the corresponding $R^2$ potential shows that $\alpha \sim a_0^2 \sim 10^{12}M_{Pl}^{-2}$, within the required range for that parameter [67]. This result fixes the remaining parameter in our initial potential (see Eq. (5.17)) leaving us with just one more free parameter; the scalar field value when slow-roll commences $\phi_i$.

With this value for the Einstein static radius, we anticipate that the CMB anisotropy spectrum will have the usual Sachs-Wolfe plateau and peaks as confirmed by current observations [6]. Moreover, once the scalar field potential is fixed we have a residual *one-parameter family* of models - parameterised by $\phi_i$ - all asymptotic to the same Einstein static universe and within which is contained a model with any $\Omega_{tot} > 1$. This includes in particular, a model for which $\Omega_{tot} = 1.02$.

[6]Indeed the following observation is relevant: the model proposed here can have the standard $S^3$ topology for the positively curved spatial surfaces, but that is not obligatory; they could for example be Poincaré dodecahedral spaces with positive spatial curvature, as suggested by Luminet et al [62]. In that case they might give a better fit to the WMAP data at large angular scales than other models [62].

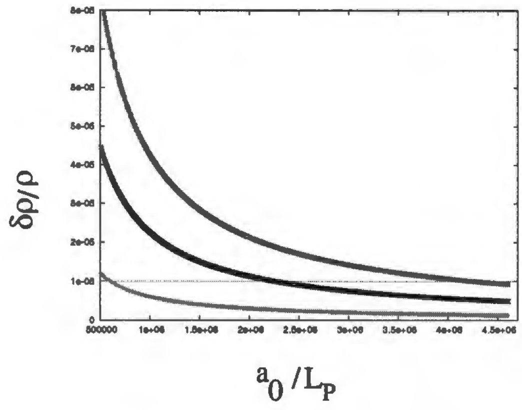



Figure 5.6: Plots of $\delta\rho/\rho$ against the size of the Einstein static universe $a_0$ for various values of the parameter $B$. As $B$ increases from $B = 0.1$ through $B = 1$ the value of $a_0$ for which $\delta\rho/\rho \sim 10^{-5}$ increases from $a_0 \sim 6.02 \times 10^5 L_{Pl}$ to $a_0 \sim 0.42 \times 10^7 L_{Pl}$

## 5.10 Available energy for reheating

Inflation starts at $V(\phi \to -\infty) = 2/\kappa a_0^2$ and ends at $\phi = 0$ with $V_0 = 0$, which means that the maximum energy "stored" in the inflaton field is

$$V(\phi \to \infty) = \frac{2}{\kappa a_0^2} \simeq \frac{10^{37} GeV^2}{a_0^2}\,. \tag{5.42}$$

In other words, the maximum energy available for reheating is inversely proportional to the square of the radius of the initial ES state. If $a_0 = 10^6 L_{Pl}$ we find

$$V(\phi \to \infty) \simeq 10^{63} GeV^4\,. \tag{5.43}$$

Therefore, provided that thermal equilibrium has been achieved and that the reheating process is efficient, the temperature at the beginning of the standard Big Bang evolution can be as high as

$$T_{RH} \simeq 10^{16} GeV\,. \tag{5.44}$$

Obviously, for less efficient reheating $T_{RH} < 10^{16} GeV$.

## 5.11 Discussion

Closed inflationary models have not had a happy time of late. Indeed many authors have taken the latest measurements of Microwave Background isotropies, in particular the WMAP best fit estimate of $\Omega_{tot} = 1.02 \pm 0.02$ to signal a flat infinite universe. Yet there are two ways to interpret these observations. It could be that the universe *is* in fact flat and more precise experiments in the future will eventually whittle away at the error bars until $\Omega_{tot} = 1$ to within, say, one part in a million. On the other hand, a comparison of BOOMERANG data [7] (with a best fit of $\Omega_{tot} = 1.02^{+0.06}_{-0.05}$) with WMAP data would seem to indicate a convergence on $\Omega_{tot} = 1.02 > 1$ with increasing resolution. Admittedly, the data set from which we draw this conclusion is rather limited and it is with much eagerness that we await results from the European Space Agency's PLANCK satellite. However, until this debate is settled one way or the other, one is forced to take closed models seriously. In stark contrast to some claims in recent literature [60], it is not too difficult to construct consistent, single field inflationary models in a closed universe. We give one such construction here (but invite the reader to see [54] for another, simple yet remarkably elegant model).

The "Emergent Universe" proposed in [21] is a simple closed inflationary model in which the universe emerges from an Einstein static state with radius $a_0 \gg L_{Pl}$, inflates and is then subsumed into a hot Big Bang era. The attractiveness of the proposed model is that one can avoid an initial quantum-gravity stage if the Einstein Static radius is larger than the Planck length. One might then ask whether such a model has a simple representation and whether it lies within the boundaries of current observations. In this chapter, we provide a first explicit construction of such a universe. As such, it is a *manifestly non-singular* closed inflationary cosmology that begins from a meta-stable Einstein static state and decays into a de Sitter phase and subsequently into standard hot Big Bang evolution. Inspired by an exact reverse-engineered potential constructed as in [22], this phenomenological model employs a single scalar field with a potential very similar to that arising in conformally transformed $R^2$-inflation only with a relaxing of some of the rigidity of that potential. In particular, beginning with a four-parameter potential (5.17), we can immediately fix two of the parameters by fixing the origin of the potential and requiring vanishing cosmological constant. By requiring this potential to match onto the exact reverse-engineered one at early times ($|\phi| \gg 1$), a third parameter of the potential is related to the radius of the initial Einstein universe by $A^2 = 2/\kappa a_0^2$.

The remaining two-parameter model is then shown to exhibit all the desired properties of the Emergent universe model. Among others, these include a sufficient number of e-folds to solve the late-time flatness problem, a spectrum of density fluctuations with magnitude of the order of $10^{-5}$ and sufficient energy in the scalar field to allow for adequate post-inflationary reheating. Of these we find that the spectrum of density fluctuations provides another constraint that allows us to relate the remaining two parameters so that effectively, the only parameter in the model is the size of the Einstein static universe. As for the re-heating energy, we show that for an initial radius of $a_0 = 10^6 L_{Pl}$ we can obtain re-heating temperatures of around $10^{16}\, GeV$.

The attractive aspects of the Einstein Static solution as a preferred initial state for our universe have been considered in the past. In fact, Gibbons has argued for the higher probability of an Einstein Static initial phase based on the model's maximal entropy [28]. Crucially, once the universe finds itself near the static state it could remain there for an undetermined amount of time. This is guaranteed by the neutral stability of the Einstein Static model against inhomogeneous (either pure fluid or pure scalar field) perturbations [40, 28, 6]. Typically, expansion away from the static solution will lead to inflation followed by the standard Hot Big Bang evolution. However, the reader will probably by now have noted the large degree of fine-tuning that went into setting up the initial state from which the universe emerges. Indeed, the emergent model is a very special trajectory in the space of possible inflationary evolutions. We have shown existence of such models, but not that they are probable.

Some may regard this as a deadly blow to these models, but we believe the case is wide open. Firstly, we note that although the idea that the universe should be probable (and so not fine-tuned) is the dominant paradigm in cosmology at present, there is no scientific proof that this has to be the case. This is an unproven and indeed unprovable philosophical assumption, which may or may not be true [7]. It is equally conceivable that - as was taken for granted in the past - whatever process causes the universe to come into being prefers a high-symmetry state. It is then relevant that the Einstein-static model is the highest symmetry non-empty Robertson-Walker universe, and so would be preferred by such a process.

Secondly, the models presented here show one can avoid the initial singularity if initial conditions were fine tuned in the way remarked on above. We believe it likely that this is a generic result: that given the usual physics

[7] The proposal of an actually existing multiverse, which could possibly be used to justify the idea that the universe must indeed be probable, is also unprovable.

of inflationary fields in the early universe (i.e. avoiding the introduction of 'shadow matter' which violates the weak energy condition), there is either a singularity at the start of the universe [8] or a fine-tuned initial state. This may be the real philosophical choice facing us: to decide which is worse, a space-time singularity, with all that that entails, or a fine tuning of initial conditions. It certainly seems very difficult to (phenomenologically, at least) construct a model that avoids both, and it is useful to recall Wheeler's characterisation of space time singularities caused by gravitational collapse as the worst crisis facing theoretical physics. Nowadays we do not perhaps take such singularities seriously enough.

Models of the kind presented here are useful in terms of making clear the alternatives facing us: we can indeed avoid both a singularity and the quantum gravity regime, without introducing any exotic physics; but there is a price to pay in terms of fine-tuning. From some philosophical standpoints the high symmetry of the initial state may even be an advantage.

[8] or at least non-avoidance of the quantum gravity domain, when quantum gravity processes may possibly avoid the singularity.

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