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TUTTE'S FIRST COLOUR-CYCLE CONJECTURE

by

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PREFACE

This thesis presents a proof of Conjecture I (see Section 35) of W. T. Tutte's paper "A contribution to the theory of chromatic polynomials" [15]. It is believed that this conjecture has not previously been resolved.

Sections 25 and 38 are original. The remainder of the thesis is a summary of the requisite graph theory and matroid theory. Most of the material in this summary is elementary. However, its inclusion makes the presentation self-contained.

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CONTENTS

CHAPTER 1 GRAPHS

(Sections 1 - 17)

This chapter is preliminary and consists of a summary of the graph theory needed in Chapters 2 and 3.

SECTION 1 GRAPHS

DEFINITION 1.1 [17, p. 3]

A *graph* G consists of two finite sets $E(G)$ and $V(G)$ together with a function $I(G,-)$ which associates with each element e of $E(G)$ a set $I(G,e)$ of one or two elements of $V(G)$.

The elements of $E(G)$ are called *edges* of G and the elements of $V(G)$ are called *vertices* of G . The function $I(G,-)$ is called the *incidence function* of G .

If $e \in E(G)$ and $v \in I(G,e)$, then we say that v is a *G-end* of e and we also say that v is *G-incident* with e and that e is *G-incident* with v . Unless confusion is likely to result, we use "*end*" instead of "G-end" and "*incident*" instead of "G-incident".

An edge of G is a *loop* of G if it has just one end and it is a *link* of G if it has two ends.

DEFINITION 1.2 [17, p. 3]

A graph with no edges and no vertices is a null graph.

A graph with no edges and just one vertex v is a vertex-graph and is denoted by $[v]$.

A graph consisting just of a loop and its single end is a loop-graph.

A graph consisting just of a link and its two ends is a link-graph.

SECTION 2 VALENCY

DEFINITION 2.1 [17, p. 4]

If a vertex v of a graph G is incident with m loops and n links of G , then the valency $\mathrm{val}(G,v)$ of v in G is defined by the equation

$$\mathrm{val}(G,v) = 2m + n .$$

DEFINITION 2.2 [17, p. 6]

The number of elements of a finite set S is denoted by $|S|$.

DEFINITION 2.3

Let G be a graph.

If G is non-null, then

$$\sum_{v \in V(G)} \mathrm{val}(G,v)$$

denotes the sum of the valencies of the vertices of G.

If G is null, then we define

$$\sum_{v \in V(G)} \mathrm{val}(G,v)$$

to be zero.

PROPOSITION 2.4 [17, p. 5, 1.21]

If G is a graph, then

$$\sum_{v \in V(G)} \mathrm{val}(G,v) = 2|E(G)| .$$

SECTION 3 SUBGRAPHS

DEFINITION 3.1 [17, p. 5]

A graph H is a subgraph of a graph G if $E(H) \subseteq E(G)$, $V(H) \subseteq V(G)$, and, for every $e \in E(H)$, $I(H,e) = I(G,e)$.

A subgraph of a graph G is a <u>proper subgraph</u> of G if it is not identical to G.

If H is a subgraph (respectively proper subgraph) of a graph G, then we say that H is <u>contained</u> (respectively <u>properly contained</u>) in G and write $H \subseteq G$ (respectively $H \subset G$).

A subgraph H of a graph G is a <u>spanning subgraph</u> of G if $V(H) = V(G)$.

<u>PROPOSITION 3.2</u> [17, p. 5, 1.32]

Let G be a graph.

If A is a subset of $E(G)$ and X is a subset of $V(G)$, then the following two conditions are equivalent.

(1) For every $e \in A$, $I(G,e) \subseteq X$.

(2) There is a subgraph H of G such that $E(H) = A$ and $V(H) = X$.

<u>DEFINITION 3.3</u> [17, p. 5]

Let G be a graph and let A be a subset of $E(G)$.

By Proposition 3.2, there is a subgraph H of G such that $E(H) = A$ and $V(H) = V(G)$. We say that H is the <u>spanning subgraph of</u> G <u>determined by</u> A and denote it by $G:A$.

There is also, by Proposition 3.2, a subgraph K of G such that $E(K) = A$ and $V(K)$ is the set of all vertices of G which are ends of elements of A. We call K the <u>reduction of</u> G <u>to</u> A and denote it by $G \cdot A$.

<u>DEFINITION 3.4</u> [17, p. 5]

Let $H_1, \cdots, H_p$ be subgraphs of a graph G.

By Proposition 3.2, there is a subgraph H of G such that

$$E(H) = \bigcup_{i=1}^{p} E(H_i)$$

and

$$V(H) = \bigcup_{i=1}^{p} V(H_i) .$$

We call H the <u>union</u> of the subgraphs H_i and write

$$H = \bigcup_{i=1}^{p} H_i = H_1 \cup \cdots \cup H_p$$

<u>DEFINITION 3.5</u>

Let $\mathcal{H}$ be a collection of subgraphs of a graph G.

If $\mathcal{H}$ is non-empty, then $\cup \mathcal{H}$ denotes the union of the members of $\mathcal{H}$.

If $\mathcal{H}$ is empty, then $\cup \mathcal{H}$ is defined to be the null subgraph of G .

DEFINITION 3.6 [17, p. 6]

Subgraphs $H_1, \cdots, H_p$ of a graph are said to be disjoint if no two distinct ones have a vertex in common and are said to be edge-disjoint if no two distinct ones have an edge in common.

By Proposition 3.2 , disjoint subgraphs are edge-disjoint.

SECTION 4 PATHS

DEFINITION 4.1 [17, p. 28]

A path α in a graph G is a non-null finite sequence

$$\alpha = (x_0, a_1, x_1, \cdots, a_m, x_m)$$

which satisfies the following three conditions:

(1) for $i \in \{0, \cdots, m\}$, x_i is a vertex of G ;
(2) for $i \in \{1, \cdots, m\}$, a_i is an edge of G ; and
(3) for $i \in \{1, \cdots, m\}$, $I(G,a_i) = \{x_{i-1}, x_i\}$.

We say that the path α is a path <u>from</u> x_0 <u>to</u> x_m .

We write $E(\alpha)$ for the set of edges of G which occur in α and $V(\alpha)$ for the set of vertices of G which occur in α . By Proposition 3.2, $E(\alpha)$ and $V(\alpha)$ determine a subgraph $G(\alpha)$ of G .

<u>DEFINITION 4.2</u> [17, p. 28]

Let

$$\alpha = (x_0, a_1, x_1, \cdots, a_m, x_m)$$

and

$$\beta = (y_0, b_1, y_1, \cdots, b_n, y_n)$$

be paths in a graph G with $x_m = y_0$.

Then

$$(x_0, a_1, x_1, \cdots, a_m, x_m, b_1, y_1, \cdots, b_n, y_n)$$

is a path in G . We denote this path by $\alpha\beta$ and call it the <u>product</u> of the paths α and β .

<u>DEFINITION 4.3</u> [17, p. 30]

A path

$$(x_0, a_1, x_1, \cdots, a_m, x_m)$$

in a graph is <u>simple</u> if it satisfies the following condition:

$$\text{if} \quad 0 \leqslant i < j \leqslant m \text{ , then } \quad x_i \neq x_j \text{ .}$$

PROPOSITION 4.4

Let v and w be vertices of a graph G .

Then: there is a path from v to w in G if and only if there is a simple path from v to w in G .

PROOF

The "if" part is trivial.

The "only if" part:

Suppose that there is a path from v to w in G . Then there is a path

$$\alpha = (x_0, a_1, x_1, \cdots , a_m, x_m)$$

from v to w in G which has the smallest number of terms consistent with the property of being a path from v to w in G .

Suppose that α is not simple. Then there are integers i and j such that $0 \leqslant i < j \leqslant m$ and $x_i = x_j$. Let α' be the subsequence of α which has all the terms of α except those from x_i to a_j .

Then α' is a path from v to w in G which has fewer terms than α has. Thus the choice of α is contradicted.

We conclude that α is simple.

The "only if" part of the proposition follows.

DEFINITION 4.5 [17, p. 30]

A path

$$\alpha = (x_0, a_1, x_1, \cdots, a_m, x_m)$$

in a graph G is a circular path if it satisfies the following four conditions:

(1) $x_0 = x_m$;

(2) if $0 \leqslant i < j \leqslant m - 1$, then $x_i \neq x_j$;

(3) at least one edge of G is a term of α ; and

(4) if $1 \leqslant i < j \leqslant m$, then $a_i \neq a_j$.

SECTION 5 CONNECTION

DEFINITION 5.1 [9, p. 13]

A graph G is connected if it satisfies the following condition:

if v and w are vertices of G , then there is a path from v to w in G .

PROPOSITION 5.2 [17, p. 13]

Every null graph, vertex-graph, link-graph, and loop-graph is connected.

SECTION 6 COMPONENTS

DEFINITION 6.1 [17, p. 13, 2.51]

A maximal non-null connected subgraph of a graph G is called a component of G.

We denote the number of components of a graph G by $c(G)$.

PROPOSITION 6.2 [17, p. 13, 2.53]

The components of a non-null graph G are disjoint and their union is equal to G.

PROPOSITION 6.3 [15, p. 81]

Let v and w be vertices of a graph G.

Then: v and w belong to the same component of G if and only if there is a path from v to w in G.

SECTION 7 PARTITIONS

DEFINITION 7.1 [2, p. 15, Ex. 9]

A partition ρ of a non-empty set S is a collection of disjoint non-empty subsets of S whose union is S.

We call the members of ρ the parts of ρ.

DEFINITION 7.2 [2, p. 15, Ex. 9]

Let ρ and τ be partitions of a non-empty set.

We say that ρ is a refinement of τ and write $\rho \leqslant \tau$ if each part of ρ is contained in some part of τ.

We say that ρ is a proper refinement of τ and write $\rho < \tau$ if $\rho \leqslant \tau$ and $\rho \neq \tau$.

PROPOSITION 7.3 [2, p. 15, Ex. 9 and p. 1, P2]

If ρ and τ are partitions of a non-empty set, then $\rho = \tau$ if and only if $\rho \leqslant \tau$ and $\tau \leqslant \rho$.

SECTION 8 $\pi(G)$

DEFINITION 8.1

If G is a non-null graph, then we define $\pi(G)$ by the equation

$$\pi(G) = \{V(K) : K \text{ a component of } G\} .$$

PROPOSITION 8.2 [15, p. 81]

If G is a non-null graph, then $\pi(G)$ is a partition of $V(G)$.

PROOF

This follows from Proposition 6.2 .

PROPOSITION 8.3 [15, p. 81]

Let v and w be vertices of a graph G .

Then: v and w belong to the same part of $\pi(G)$ if and only if there is a path from v to w in G .

PROOF

This follows from Proposition 6.3 .

PROPOSITION 8.4

Let $K_1, \cdots, K_p$ be the components of a graph G and let A be a subset of $E(G)$.

Then

$$\pi(G:(E(G) - A)) = \bigcup_{i=1}^{p} \pi(K_i:(E(K_i) - A))$$

and

$$c(G:(E(G) - A)) = \sum_{i=1}^{p} c(K_i:(E(K_i) - A)) .$$

PROOF

Since G has at least one component, G is non-null.

By definition

$$V(G:(E(G) - A)) = V(G)$$

and, for each $i \in \{1, \cdots, p\}$,

$$V(K_i:(E(K_i) - A)) = V(K_i) .$$

Thus, since G is non-null, $G:(E(G) - A)$ is non-null. Also, for each $i \in \{1, \cdots, p\}$, K_i is non-null so $K_i:(E(K_i) - A)$ is non-null.

Let

$$\rho = \bigcup_{i=1}^{p} \pi(K_i : (E(K_i) - A)) .$$

By Proposition 8.2 ,

$$\pi(G) = \{V(K_1), \cdots , V(K_p)\}$$

is a partition of

$$V(G) = V(G:(E(G) - A)) ,$$

and, for each $i \in \{1, \cdots , \overset{p}{\cancel{m}}\}$,

$$\pi(K_i : (E(K_i) - A))$$

is a partition of

$$V(K_i : (E(K_i) - A)) = V(K_i) .$$

Thus ρ is a partition of $V(G:(E(G) - A))$.

We show that $\rho = \pi(G:(E(G) - A))$.

Suppose that v and w belong to the same part of ρ . Then there is an $i \in \{1, \cdots , p\}$ such that v and w belong to the same part of $\pi(K_i : (E(K_i) - A))$. Thus, by Proposition 8.3 , there is a path α from v to w in $K_i : (E(K_i) - A)$. But $K_i : (E(K_i) - A)$ is a subgraph of $G:(E(G) - A)$. Thus α is a path from v to w in $G:(E(G) - A)$. So, by Proposition 8.3 , v and w belong to the same part of $\pi(G:(E(G) - A)$. Thus $\rho \leqslant \pi(G:(E(G) - A))$.

On the other hand, suppose that v and w belong to the same part of $\pi(G:(E(G) - A))$. Then, by Proposition 8.3 , there is a path α from v to w in $G:(E(G) - A)$. Since $G:(E(G) - A)$ is a subgraph of G , it follows that α is a path from v to w in some component K_i of G . Moreover, no element of A is a term of α . Thus α is a path from v to w in $K_i:(E(K_i) - A)$. So, by Proposition 8.3 , v and w belong to the same part of $\pi(K_i:(E(K_i) - A))$. So v and w belong to the same part of ρ . Thus $\pi(G:(E(G) - A)) \leqslant \rho$.

Thus, by Proposition 7.3 , $\rho = \pi(G:(E(G) - A))$.

Moreover,

$$c(G:(E(G) - A)) = |\pi(G:(E(G) - A)| = |\rho|$$

$$= \sum_{i=1}^{p} |\pi(K_i:(E(K_i) - A))|$$

$$= \sum_{i=1}^{p} c(K_i:(E(K_i) - A)) .$$

<u>COROLLARY 8.5</u>

Let G be a non-null graph.

If A is a subset of $E(G)$, then

$$\pi(G:(E(G) - A)) \leqslant \pi(G)$$

and

$$c(G:(E(G) - A)) \geqslant c(G) .$$

SECTION 9 CUTSETS

DEFINITION 9.1 [9, p. 38], [21, p. 503]

Let G be a graph.

A subset A of $E(G)$ is a cutset of G if the number of components of $G:(E(G) - A)$ is greater than the number of components of G .

PROPOSITION 9.2

Let G be a non-null graph.

If A is a subset of $E(G)$, then the following four conditions are equivalent.

(1) A is a cutset of G .

(2) $c(G:(E(G) - A)) > c(G)$.

(3) $\pi(G:(E(G) - A)) < \pi(G)$.

(4) For some pair of vertices v and w of G , there is a path from v to w in G but no path from v to w in $G:(E(G) - A)$.

PROOF

(1) $\Longleftrightarrow$ (2): By Definition 9.1 .

(2) $\Longleftrightarrow$ (3): By Definition 8.1 and Corollary 8.5 .

(3) $\Longleftrightarrow$ (4): By Proposition 8.3 and Corollary 8.5 .

PROPOSITION 9.3

Let G be a non-null graph.

If A is a subset of E(G) , then the following four conditions are equivalent.

(1) A is not a cutset of G .

(2) $c(G:(E(G) - A)) = c(G)$.

(3) $\pi(G:(E(G) - A)) = \pi(G)$.

(4) For every pair of vertices v and w of G , if there is a path from v to w in G , then there is a path from v to w in $G:(E(G) - A)$.

PROOF

By Proposition 9.2 and Corollary 8.5 .

PROPOSITION 9.4

Let G be a graph.

Then: a subset A of E(G) is a cutset of G if and only if there is a component K of G such that $A \cap E(K)$ is a cutset of K .

PROOF

A null graph has no cutset and no component so we may assume that G is non-null.

Let $K_1, \cdots, K_p$ be the components of G and let A be a subset of $E(G)$. Then, by Proposition 8.4 ,

$$c(G:(E(G) - A)) = \sum_{i=1}^{p} c(K_i:(E(K_i) - A)) .$$

Thus:

$$c(G:(E(G) - A)) > c(G) = p$$

if and only if, for some $i \in \{1, \cdots, p\}$,

$$c(K_i:(E(K_i) - A)) > 1 = c(K_i) .$$

The proposition follows.

PROPOSITION 9.5

If H is a subgraph of a connected graph G and $\emptyset \subset V(H) \subset V(G)$, then the set L of links of G with just one end in $V(H)$ is a cutset of G .

PROOF

Let H be a subgraph of a connected graph G such that $\emptyset \subset V(H) \subset V(G)$.

Let L be the set of links of G with just one end in $V(H)$.

Since $V(H) \neq \emptyset$, there is a vertex $v \in V(H)$. Since $V(G) - V(H) \neq \emptyset$, there is a vertex $w \in V(G) - V(H)$. Moreover, since G is connected, there is a path from v to w in G .

Let

$$\alpha = (x_0, a_1, x_1, \cdots , a_m, x_m)$$

be a path from v to w in G . Then there is an $i \in \{1, \cdots , m\}$ such that $x_{i-1} \in V(H)$ and $x_i \in V(G) - V(H)$. Thus a_i is a link of G with just one end in $V(H)$. So $a_i \in L$. It follows that α is not a path in $G:(E(G) - L)$.

But, since $G:(E(G) - L) \subseteq G$, each path in $G:(E(G) - L)$ is a path in G .

It follows that there is no path from v to w in $G:(E(G) - L)$.

Thus, by Proposition 9.2 , L is a cutset of G .

SECTION 10 BONDS

DEFINITION 10.1 [16, p. 5]

A bond of a graph G is a minimal cutset of G.

PROPOSITION 10.2

Let G be a graph.

Then: a subset B of $E(G)$ is a bond of G if and only if B is a bond of some component of G.

PROOF

A null graph has no bond and no component so we may assume that G is non-null.

Suppose that A is a cutset of G. Then, by Proposition 9.4, A contains a cutset of some component K of G. Thus A contains a bond B of K. By Proposition 9.4, B is a cutset of G. Since $B \subseteq A$, it follows that A is a bond of G only if $A = B$, i.e. only if A is a bond of K.

Conversely, suppose that A is a cutset of some component K of G. Then, by Proposition 9.4, A is a cutset of G. So A contains a bond B of G. Then, by the part of the proposition which has already been proved, B is a bond of some component of G.

Moreover, $B \subseteq A \subseteq E(K)$ and so, by Proposition 6.2, B cannot be a bond of a component of G other than K. Thus B is a bond of K. Since $B \subseteq A$, it follows that A is a bond of K only if $A = B$, i.e. only if A is a bond of G.

LEMMA 10.3

Let G be a graph and let A be a subset of $E(G)$.

If e is a link of G whose two ends belong to different components of $G:(E(G) - A)$, then $e \in A$.

PROOF

Suppose that $e \in E(G) - A$. Then $G\cdot\{e\} \subseteq G:(E(G) - A)$. Thus, since $G\cdot\{e\}$ is connected, $G\cdot\{e\}$ is contained in some component K of $G:(E(G) - A)$. So $I(G,e) \subseteq V(K)$.

PROPOSITION 10.4 [16, p. 5]

Let b be an edge of a graph G and let B be a bond of G.

Then: $b \in B$ if and only if b is a link of G whose two ends belong to different components of $G:(E(G) - B)$.

PROOF

The "if" part of the proposition follows directly from Lemma 10.3 .

The "only if" part:

Since B is a bond of G , it follows from Proposition 9.2 that

$$\pi(G:(E(G) - B)) < \pi(G) .$$

Thus there is a component K of $G:(E(G) - B)$ and a component M of G such that $V(K) \subset V(M)$. Since K is a component of $G:(E(G) - B)$, $V(K) \neq \emptyset$. Since K is a non-null connected subgraph of G , K is a subgraph of some component of G . Moreover, since $V(K) \subset V(M)$, it follows from Proposition 6.2 that K is a subgraph of M . Thus K is a subgraph of the connected graph M and $\emptyset \subset V(K) \subset V(M)$.

Let L be the set of links of M with just one end in $V(K)$.

Then, by Proposition 9.5 , L is a cutset of M . So, by Proposition 9.4 , L is a cutset of G .

Moreover, each element of L is a link of G whose two ends belong to different components of $G:(E(G) - B)$. So, by Lemma 10.3 , $L \subseteq B$.

Therefore, since B is a minimal cutset of G ,

$L = B$.

Thus, if $b \in B$, then $b \in L$ and so b is a link of G whose two ends belong to different components of $G:(E(G) - B)$.

PROPOSITION 10.5

If B is a bond of a graph G , then the number of components of $G:(E(G) - B)$ is just one greater than the number of components of G .

PROOF

Let G be a graph.

Since a null graph has no bond, we may assume that G is non-null.

Let B be a bond of G . Then, by Proposition 10.2 , B is a bond of some component of G .

Let $K_1, \cdots, K_p$ be the components of G . We may assume that B is a bond of K_1 . Then $B \subseteq E(K_1)$. So, by Proposition 6.2 ,

$$\text{for } i \in \{2, \cdots, p\} , \quad K_i:(E(K_i) - B) = K_i .$$

Moreover, by Proposition 8.4 ,

$$c(G:(E(G) - B)) = \sum_{i=1}^{p} c(K_i:(E(K_i) - B)) .$$

Thus

$$c(G:(E(G) - B)) = c(K_1:(E(K_1) - B)) + (p - 1) .$$

So we need only show that

$$c(K_1:(E(K_1) - B)) = 2 .$$

Since B is a bond of K_1 and since K_1 is a component,

$$c(K_1:(E(K_1) - B)) > c(K_1) = 1 .$$

Let $b \in B$ and let

$$H = K_1:(E(K_1) - (B - \{b\})) .$$

Since B is a minimal cutset of K_1, $B - \{b\}$ is not a cutset of K_1. Thus, by Proposition 9.3,

$$c(H) = c(K_1) = 1 .$$

So H is connected. Since

$$E(H) - \{b\} = E(K_1) - B ,$$

it follows that

$$H:(E(H) - \{b\}) = K_1:(E(K_1) - B) .$$

Thus

$$c(H:(E(H) - \{b\})) = c(K_1:(E(K_1) - B)) .$$

Therefore

$$c(H:(E(H) - \{b\})) > c(H) .$$

So $\{b\}$ is a bond of H . Thus, by Proposition 10.4 , b is a link of H .

Let y_1 and y_2 be the ends of b . Then y_1 and y_2 are vertices of H .

Let v be a vertex of H . Then, since H is connected, there is a path from v to y_1 in H . Thus, by Proposition 4.4 , there is a simple path α from v to y_1 in H . If the path α includes b , then a subsequence of α is a path from v to y_2 in H which does not include b . Thus, either there is a path from v to y_1 in $H:(E(H) - \{b\})$ or there is a path from v to y_2 in $H:(E(H) - \{b\})$. So, by Proposition 6.3 , either v and y_1 belong to the same component of $H:(E(H) - \{b\})$ or v and y_2 belong to the same component of $H:(E(H) - \{b\})$. Thus it follows from Proposition 6.2 that

$$c(H:(E(H) - \{b\})) \leqslant 2 .$$

Therefore

$$c(H:(E(H) - \{b\})) = 2 .$$

So

$$c(K_1:(E(K_1) - B)) = 2 .$$

SECTION 11 ISTHMUSES

DEFINITION 11.1 [15, p. 82], [17, p. 17]

An edge e of a graph G is an isthmus of G if $\{e\}$ is a bond of G.

PROPOSITION 11.2

If a component of a graph G has an isthmus, then G has an isthmus.

PROOF

This follows directly from Proposition 10.2 .

SECTION 12 CONTRACTIONS

DEFINITION 12.1 [16, p. 6]

Let G be a graph and let A be a subset of $E(G)$.

The contraction $G \text{ ctr } A$ of G to A is a graph whose edges are the elements of A and whose vertices are the components of $G:(E(G) - A)$. The incidence function of $G \text{ ctr } A$ is defined as follows:

for every $e \in A$ and each component K of $G:(E(G) - A)$, K is a $(G \text{ ctr } A)$-end of e if and only if $V(K)$ contains a G-end of e .

PROPOSITION 12.2

If a contraction of a graph G has an isthmus, then G has an isthmus.

PROOF

Let G be a graph.

Since any contraction of a null graph is a null graph and a null graph has no isthmus, we may assume that G is non-null.

Let A be a subset of $E(G)$.

Suppose that e is an edge of G ctr A which is not an isthmus of G .

Let

$$(K_0, a_1, K_1, \cdots, a_m, K_m)$$

be a path in G ctr A .

Suppose that $i \in \{1, \cdots, m\}$. Then, since K_{i-1} is a (G ctr A)-end of a_i , $V(K_{i-1})$ includes a G-end v_{i-1} of a_i . Similarly, $V(K_i)$ includes a G-end w_i of a_i . Thus the sequence $\alpha_i = (v_{i-1}, a_i, w_i)$ is a path in G .

Suppose that $i \in \{1, \cdots, m-1\}$. Then w_i and v_i belong to $V(K_i)$. So, since K_i is a component of $G:(E(G) - A)$, there is a path β_i from

w_i to v_i in $G:(E(G) - A)$. Since $G:(E(G) - A)$ is a subgraph of G , β_i is a path from w_i to v_i in G .

There is thus a path

$$\alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_{m-1} \beta_{m-1} \alpha_m$$

from v_0 to w_m in G .

Since e is not an isthmus of G , it follows from Proposition 9.3 that there is a path

$$(z_0, c_1, z_1, \cdots, c_r, z_r)$$

from v_0 to w_m in $G:(E(G) - \{e\})$.

Let

$$(c_{s_1}, \cdots, c_{s_t})$$

be the subsequence of

$$(c_1, \cdots, c_r)$$

whose terms belong to A .

Then there are components $M_0, \cdots, M_t$ of $G:(E(G) - A)$ such that $M_0 = K_0$, $M_t = K_m$, and, for $i \in \{1, \cdots, t\}$, M_{i-1} and M_i are the $(G \text{ ctr } A)$-ends of c_{s_i} . It follows that the sequence

$$\gamma = (M_0, c_{s_1}, M_1, \cdots, c_{s_t}, M_t)$$

is a path from K_0 to K_m in G ctr A in which e does not occur as a term . Thus γ is a path from K_0 to K_m in (G ctr A):(E(G ctr A) - {e}) .

Using Proposition 9.3 , we conclude that e is not an isthmus of G ctr A .

The proposition follows.

SECTION 13 FORESTS

DEFINITION 13.1 [21, p. 503], [17, p. 17]

A graph F is a forest if there is no circular path in F .

A connected forest is called a tree.

PROPOSITION 13.2 [17, p. 19, 3.31]

Every null graph, vertex-graph, and link-graph is a tree.

PROPOSITION 13.3 [17, p. 18, 3.12]

Every subgraph of a forest is a forest.

PROPOSITION 13.4 [17, p. 17]

Every edge of a forest F is an isthmus of F .

PROOF

Let G be a graph with an edge e which is not an isthmus of G .

If e is a loop of G and v is the end ~~if~~ of e , then (v, e, v) is a circular path in G .

On the other hand, suppose that e is a link of G and that v and w are the ends of e . Then (v, e, w) is a path from v to w in G . Thus, by Propositions 9.3 and 4.4 , there is a simple path α from v to w in $G:(E(G) - \{e\})$. Let β be the path (w, e, v) in G . Since e is not a term of α and since α is simple, it follows that $\alpha\beta$ is a circular path in G .

Thus, in either case, there is a circular path in G .

So G is not a forest.

PROPOSITION 13.5 [17, p. 18, 3.23]

If a graph F is a forest, then

$$|E(F)| = |V(F)| - c(F) .$$

PROOF

Let F be a forest.

If $E(F)$ is empty, then $c(F) = |V(F)|$ and so the proposition holds.

Assume that $E(F)$ is non-empty.

Let

$$E(F) = \{e_1, \cdots, e_m\} .$$

We define a sequence $(F_0, \cdots, F_m)$ of subgraphs of F as follows: $F_0 = F$ and, for $i \in \{1, \cdots, m\}$, $F_i = F_{i-1}:(E(F_{i-1}) - \{e_i\})$. By Proposition 13.3, each F_i is a forest.

Suppose that $i \in \{1, \cdots, m\}$. Then, by Proposition 13.4, e_i is an isthmus of F_{i-1}. Thus, by Proposition 10.5, $c(F_i) = c(F_{i-1}) + 1$.

It follows that

$$c(F_m) = c(F_0) + m .$$

But

$$c(F_m) = |V(F_m)| = |V(F)| ,$$

$$c(F_0) = c(F) , \text{ and}$$

$$m = |E(F)| .$$

Thus

$$|V(F)| = c(F) + |E(F)| .$$

PROPOSITION 13.6   [17, p. 19, 3.35]

Every forest with at least one edge has a vertex of valency one.

PROOF

Let F be a forest with at least one edge.

Let $H = F \cdot E(F)$. Then, for each $v \in V(H)$, $\mathrm{val}(F,v) = \mathrm{val}(H,v) \geq 1$. Moreover, H is non-null and so $c(H) \geq 1$.

By Proposition 13.3 , H is a forest. Thus, by Proposition 13.5 ,

$$|E(H)| = |V(H)| - c(H) .$$

By Proposition 2.4 ,

$$\sum_{v \in V(H)} \mathrm{val}(H,v) = 2|E(H)| .$$

Therefore

$$\sum_{v \in V(H)} \mathrm{val}(H,v) = 2(|V(H)| - c(H)) .$$

Since, for each $v \in V(H)$, $\mathrm{val}(H,v) \geq 1$ and since $c(H) \geq 1$, it follows that some vertex w of H must have valency one. But then w is a vertex of F and $\mathrm{val}(F,w) = \mathrm{val}(H,w) = 1$. So F has a vertex of valency one.

SECTION 14 THE PRINCIPAL FORESTS OF A GRAPH

DEFINITION 14.1 [16, p. 6], [21, p. 503]

Let G be a graph.

A *forest of* G is a subgraph of G which is a forest.

A *spanning tree of* G is a spanning subgraph of G which is a tree.

A *principal forest of* G is a forest of G which contains a spanning tree of each component of G .

PROPOSITION 14.2

A forest F of a graph G is a principal forest of G if and only if $V(F) = V(G)$ and $c(F) = c(G)$.

PROOF

The "only if" part of the proposition follows easily from the definition of a principal forest together with Proposition 8.2 and Corollary 8.5 .

The "if" part:

Let F be a forest of a graph G .

If G is null, then F is a principal forest of G . Thus we may assume that G is non-null.

Suppose that $V(F) = V(G)$ and $c(F) = c(G)$.

Let K be a component of G . Then $K:(E(K) \cap E(F))$ is a spanning subgraph of K . Moreover, since

$$E(K:(E(K) \cap E(F))) = E(K) \cap E(F) \subseteq E(F)$$

and

$$V(K:(E(K) \cap E(F))) = V(K) \subseteq V(G) = V(F) ,$$

it follows that $K:(E(K) \cap E(F))$ is a subgraph of F . Thus, by Proposition 13.3 , $K:(E(K) \cap E(F))$ is a forest.

Since $V(F) = V(G)$,

$$F = G:E(F) = G:(E(G) - (E(G) - E(F))) .$$

Thus, since $c(F) = c(G)$, it follows from Proposition 9.3 that $E(G) - E(F)$ is not a cutset of G . So, by Proposition 9.4 , $(E(G) - E(F)) \cap E(K)$ is not a cutset of K . Thus, by Proposition 9.3 ,

$$c(K:(E(K) \cap E(F)) = c(K) .$$

So $K:(E(K) \cap E(F))$ is connected.

Thus $K:(E(K) \cap E(F))$ is a spanning tree of K contained in F .

It follows that F is a principal forest of G .

PROPOSITION 14.3

If F is a forest of a graph G, then there is a principal forest of G which contains F.

PROOF

Let G be a graph.

Since $E(G)$ and $V(G)$ are finite, we need only show that, if F is a non-principal forest of G, then F is properly contained in some forest of G.

Suppose that F is a non-principal forest of G.

Let

$$H = G:E(F) = G:(E(G) - (E(G) - E(F))) .$$

Then $E(F) = E(H)$ and $V(F) \subseteq V(G) = V(H)$. Thus $F \subseteq H$. Moreover, $H \cdot E(H) \subseteq F$. So, by Proposition 13.3, $H \cdot E(H)$ is a forest. Thus H is a forest of G. Further, by Corollary 8.5, $c(H) \geqslant c(G)$.

If H is a principal forest of G, then $F \neq H$ and so $F \subset H$.

On the other hand, suppose that H is a non-principal forest of G. Then, by Proposition 14.2, $c(H) \neq c(G)$. Thus $c(H) > c(G)$. So, by Proposition 9.2, for some vertices v and w of G, there is a path α from v to w in G but no path from v to w in H. Let

$$\alpha = (x_0, a_1, x_1, \cdots, a_m, x_m).$$

Then, since there is no path from v to w in H, it follows that, for some $i \in \{1, \cdots, m\}$, there is no path from x_{i-1} to x_i in H.

Let

$$K = H \cup G.\{a_i\}.$$

Then $F \subseteq H \subset K$. We show that K is a forest.

Suppose that there is a circular path β in K. Then $V(\beta) \subseteq V(H)$ and $E(\beta) \subseteq E(H) \cup \{a_i\}$. Thus, since there is no circular path in H, a_i is a term of β. But then there is a path from x_{i-1} to x_i in $K:(E(K) - \{a_i\}) = H$. Thus the choice of i is contradicted.

We conclude that there is no circular path in K.

So K is a forest.

<u>COROLLARY 14.4</u> [17, p. 21, 3.41]

Every graph contains a principal forest.

PROPOSITION 14.5 [16, p. 3, 2.11]

If F is a principal forest of a graph G and $e \in (E(G) - E(F))$, then e is a term of a circular path in $F \cup (G \cdot \{e\})$.

PROOF

Suppose that F is a principal forest of a graph G and that $e \in (E(G) - E(F))$.

If e is a loop of G and v is the end of e , then e is a term of the circular path (v, e, v) in $F \cup (G \cdot \{e\})$.

On the other hand, suppose that e is a link of G and that v and w are the ends of e .

Let K be the component of G which contains e and let T be the spanning tree of K contained in F . Then $v, w \in V(H) = V(T)$ and T is connected. Thus there is a path from v to w in T . So, by Proposition 4.4 , there is a simple path α from v to w in T . Since $e \notin E(T)$, e is not a term of α .

Let β be the path (w, e, v) in $G \cdot \{e\}$. Then $\alpha\beta$ is a circular path in

$$T \cup (G \cdot \{e\}) \subseteq F \cup (G \cdot \{e\})$$

and e is a term of α . Thus e is a term of a circular path in $F \cup (G \cdot \{e\})$.

COROLLARY 14.6 [21, p. 503, G3]

No principal forest of a graph G is properly contained in another principal forest of G .

PROPOSITION 14.7 [21, p. 504, G4]

If F_1 and F_2 are principal forests of a graph G and $e_1 \in E(F_1)$, then there is an $e_2 \in E(F_2)$ with the property that

$$G:((E(F_1) - \{e_1\}) \cup \{e_2\})$$

is a principal forest of G .

PROOF

Suppose that F_1 and F_2 are principal forests of a graph G and that $e_1 \in E(F_1)$.

Then, by Proposition 14.2 ,

$$V(F_1) = V(F_2) = V(G)$$

and

$$c(F_1) = c(F_2) = c(G) .$$

Let

$$H = F_1:(E(F_1) - \{e_1\}) = G:(E(F_1) - \{e_1\}) .$$

Then

$$V(H) = V(F_1) = V(G) .$$

By Proposition 13.4 , e_1 is an isthmus of F_1 . Thus, using Proposition 10.3 ,

$$c(H) = c(F_1) + 1 = c(G) + 1 .$$

Let

$$M = H \cup F_2 = G:((E(F_1) - \{e_1\}) \cup E(F_2)) .$$

Then

$$V(M) = V(G) .$$

Moreover, using Corollary 8.5 ,

$$c(G) \leqslant c(M) \leqslant c(F_2) = c(G) .$$

Thus

$$c(M) = c(G) .$$

Since $H \subseteq F_1$, it follows from Proposition 13.3 that H is a forest. Thus, by Proposition 14.3 , H is contained in a principal forest K of M . Using Proposition 14.2 ,

$$V(K) = V(M) = V(G)$$

and

$$c(K) = c(M) = c(G) .$$

Thus, by Proposition 14.2 , K is a principal forest of G .

By Proposition 13.5 ,

$$|E(K)| = |V(K)| - c(K)$$

and

$$|E(H)| = |V(H)| - c(H) .$$

Thus

$$|E(K)| = |V(K)| - c(K) = |V(G)| - c(G)$$

$$= |V(H)| - (c(H) - 1) = |E(H)| + 1 .$$

So there is an $e_2 \in E(F_2)$ such that

$$E(K) = (E(F_1) - \{e_1\}) \cup \{e_2\} .$$

Since $V(K) = V(G)$, we have

$$K = G:((E(F_1) - \{e_1\}) \cup \{e_2\}) .$$

The proposition follows.

SECTION 15 THE RANK OF A GRAPH

DEFINITION 15.1 [21, p. 504]

The rank $r(G)$ of a graph G is defined by the equation

$$r(G) = |V(G)| - c(G) .$$

PROPOSITION 15.2 [17, p. 18, 3.23]

A graph F is a forest if and only if $r(F) = |E(F)|$.

PROOF

The "only if" part of the proposition follows directly from Proposition 13.5 .

The "if" part:

Suppose that G is a graph which is not a forest. By Corollary 14.4 , G contains a principal forest F . By Proposition 14.2 , $V(F) = V(G)$ and $c(F) = c(G)$. Since G is not a forest, $F \subset G$. Thus, since $V(F) = V(G)$, it follows that $E(F) \subset E(G)$. So $|E(F)| < |E(G)|$. By Proposition 13.5 , $|E(F)| = |V(F)| - c(F)$.

Therefore

$$r(G) = |V(G)| - c(G) = |V(F)| - c(F)$$

$$= |E(F)| < |E(G)| .$$

PROPOSITION 15.3 [21, p. 504]

If F is a principal forest of a graph G, then $r(F) = r(G)$.

PROOF

This follows directly from Proposition 14.2 .

SECTION 16 POLYGONS

DEFINITION 16.1 [17, p. 30, 4.32 and p, 31, 4.36]

A graph P is a *polygon* if there is a circular path α in P such that $P = G(\alpha)$.

PROPOSITION 16.2 [17, p. 26]

Every loop-graph is a polygon.

PROPOSITION 16.3 [17, p. 27, 3.75 and p. 18, 3.23]

A graph is a forest if and only if it has no subgraph which is a polygon.

PROPOSITION 16.4 [17, p. 26, 3.71; p. 27, 3.75; and p. 18, 3.23]

Every proper subgraph of a polygon is a forest.

PROOF

Let H be a proper subgraph of a polygon P. Then there is an $e \in E(P)$ such that $H \subseteq P:(E(P) - \{e\})$.

Since P is a polygon, $|E(P)| = |V(P)|$ and $P:(E(P) - \{e\})$ is connected. Therefore

$$\begin{aligned}
&r(P:(E(P) - \{e\})) \\
&= |V(P:(E(P) - \{e\}))| - c(P:(E(P) - \{e\})) \\
&= |V(P)| - 1 \\
&= |E(P)| - 1 \\
&= |E(P:(E(P) - \{e\}))|
\end{aligned}$$

Thus, by Proposition 15.2, $P:(E(P) - \{e\})$ is a forest.

So, by Proposition 13.3, H is a forest.

SECTION 17 THE POLYGONS OF A GRAPH

DEFINITION 17.1

A polygon of a graph G is a subgraph of G which is a polygon.

PROPOSITION 17.2 (Petersen) [9, p. 90, Thm 9.9]

If G is a graph, then the following two conditions are equivalent.

(1) Every vertex of G has even valency.

(2) There is a collection $\mathscr{P}$ of edge-disjoint polygons of G such that $E(\cup \mathscr{P}) = E(G)$.

PROOF

(1) $\Rightarrow$ (2):

Suppose that G is a graph with no edges. Let $\mathscr{P}$ be the empty collection of polygons of G . Then $\mathscr{P}$ is a collection of edge-disjoint polygons and $E(\cup \mathscr{P}) = \emptyset = E(G)$. Thus G satisfies (2).

Assume that, if a graph G satisfies (1) and G has fewer than $n > 0$ edges, then G satisfies (2).

Suppose that G is a graph which satisfies (1) and has n edges. Since G has at least one edge and no vertex of G has valency one, it follows from

Proposition 13.6 that G is not a forest. So G must contain a polygon P .

Let

$$H = G:(E(G) - E(P)) .$$

Then H has fewer than n edges.

Suppose that $v \in V(H)$. If $v \in V(P)$, then $\mathrm{val}(H,v) = \mathrm{val}(G,v) - 2$. If $v \notin V(P)$, then $\mathrm{val}(H,v) = \mathrm{val}(G,v)$. Thus, in either case $\mathrm{val}(H,v)$ is even. So every vertex of H has even valency.

Thus, by the induction hypothesis, there is a collection $\mathscr{P}$ of edge-disjoint polygons of G such that $E(\cup \mathscr{P}) = E(H)$.

Since $E(P) \cap E(H) = \emptyset$ and each polygon of H is a polygon of G , $\{P\} \cup \mathscr{P}$ is a collection of edge-disjoint polygons of G . Since $E(P) \cup E(H) = E(G)$, $\cup (\{P\} \cup \mathscr{P}) = E(G)$. Thus G satisfies (2).

So, by induction, (1) implies (2).

(2) $\Rightarrow$ (1):

Let $\mathscr{P}$ be a collection of edge-disjoint polygons of a graph G such that $E(\cup \mathscr{P}) = E(G)$.

If $\mathscr{P}$ is empty, then $E(G) = E(\cup \mathscr{P}) = \emptyset$ and so every vertex of G has even valency. Thus we may assume that $\mathcal{P}$ is non-empty.

Let $\mathcal{P} = \{P_1, \cdots, P_m\}$ and, for each $i \in \{1, \cdots, m\}$, let $K_i = G{:}E(P_i)$. Then the K_i are edge-disjoint and

$$\bigcup_{i=1}^{m} E(K_i) = E(G) .$$

Suppose that $v \in V(G)$. Then

$$\mathrm{val}(G,v) = \sum_{i=1}^{m} \mathrm{val}(K_i,v) .$$

But, for each $i \in \{1, \cdots, m\}$, if $v \in V(P_i)$, then $\mathrm{val}(K_i,v) = 2$; and, if $v \notin V(P_i)$, then $\mathrm{val}(K_i,v) = 0$. Thus $\mathrm{val}(G,v)$ is even.

Therefore each vertex of G has even valency.

Thus (2) implies (1).

<u>PROPOSITION 17.3</u> [10, p. 163]

If $Q_1, \cdots, Q_n$ are polygons of a graph G, then there is a collection $\mathcal{P}$ of edge-disjoint polygons of G such that

$$E(\cup \mathcal{P}) = E(Q_1) + \cdots + E(Q_n)$$

where $E(Q_1) + \cdots + E(Q_n)$ is the symmetric difference of the $E(Q_i)$.

PROOF

Let $Q_1, \cdots, Q_n$ be polygons of a graph G.

Let $H = G:(E(Q_1) + \cdots + E(Q_n))$ and, for each $i \in \{1, \cdots, n\}$, let $K_i = G:E(Q_i)$.

Suppose that $v \in V(H)$. Then

$$\mathrm{val}(H,v) \equiv \sum_{i=1}^{n} \mathrm{val}(K_i,v) \pmod 2 .$$

But, for each $i \in \{1, \cdots, n\}$, if $v \in V(\overset{Q_i}{\cancel{P_i}})$, then $\mathrm{val}(K_i,v) = 2$; and, if $v \notin V(\overset{Q_i}{\cancel{P_i}})$, then $\mathrm{val}(K_i,v) = 0$. Thus

$$\sum_{i=1}^{n} \mathrm{val}(K_i,v)$$

is even. So $\mathrm{val}(H,v)$ is even.

Therefore each vertex of H has even valency. Thus, by Proposition 17.2, there is a collection $\mathscr{P}$ of edge-disjoint polygons of H such that $E(\cup\, \mathscr{P}) = E(H)$. But then $\mathscr{P}$ is a collection of edge-disjoint polygons of G such that

$$E(\cup\, \mathscr{P}) = E(Q_1) + \cdots + E(Q_n) .$$

PROPOSITION 17.4

If F is a principal forest of a graph G, then there is a collection $\mathscr{P}$ of edge-disjoint polygons of G such that

$$E(\cup \mathscr{P}) \supseteq (E(G) - E(F)) .$$

PROOF

Let F be a principal forest of a graph G.

Suppose that $E(G) - E(F)$ is empty and let $\mathscr{P}$ be the empty collection of polygons of G. Then $\mathscr{P}$ is a collection of edge-disjoint polygons of G and $E(\cup \mathscr{P}) = \emptyset = E(G) - E(F)$.

On the other hand, suppose that $E(G) - E(F)$ is non-empty and let $E(G) - E(F) = \{e_1, \cdots, e_n\}$.

Suppose that $i \in \{1, \cdots, n\}$. Then, since F is a principal forest of G and $e_i \in E(G) - E(F)$, it follows from Proposition 14.5 that there is a circular path α_i in G such that $e_i \in E(\alpha_i) \subseteq E(F) \cup \{e_i\}$. Let $Q_i = G(\alpha_i)$. Then Q_i is a polygon of G and $e_i \in E(Q_i) \subseteq E(F) \cup \{e_i\}$.

Thus, if $i, j \in \{1, \cdots, n\}$ and $i \neq j$, then $e_i \notin E(Q_j)$. So

$$\{e_1, \cdots, e_n\} \subseteq E(Q_1) + \cdots + E(Q_n) .$$

Thus

$$E(G) - E(F) \subseteq E(Q_1) + \cdots + E(Q_n) .$$

Therefore, using Proposition 17.3 , there is a collection $\mathcal{P}$ of edge-disjoint polygons of G such that

$$E(\cup \mathcal{P}) = E(Q_1) + \cdots + E(Q_n) \supseteq E(G) - E(F) .$$

CHAPTER 2 MATROIDS AND GRAPHS

(Sections 18 - 25)

The proof of Conjecture 1, given in the next chapter, makes use of the following proposition:

(Proposition 25.3)

If every bond of a graph G has at least three edges, then there are collections $\mathscr{P}_1$, $\mathscr{P}_2$, and $\mathscr{P}_3$ of edge-disjoint polygons of G such that

$$\bigcup_{i=1}^{3} E(\cup \mathscr{P}_i) = E(G) .$$

The purpose of the present chapter is to prove this result. The proof involves applying a theorem of Edmonds' (Proposition 24.3) to the bond matroid of a graph (Definition 23.7). Thus: Sections 18 - 22 outline the required elements of matroid theory, Section 23 is concerned with the polygon and bond matroids of a graph, in Section 24 Edmonds' theorem is proved, and Section 25 gives a proof of Proposition 25.3 .

For further information about matroids the reader is referred to the expository articles of Wilson [21] and Harary and Welsh [10] and to the texts of Tutte [19], Crapo and Rota [6], and Mirsky [11]. Matroids are called pregeometries in [6] and independence spaces in [11].

SECTION 18 MATROIDS

DEFINITION 18.1 [21, p. 506]

A matroid M consists of a finite set $E(M)$ together with a collection $\mathcal{B}(M)$ of subsets of $E(M)$ which satisfies the following three conditions:

(1) $\mathcal{B}(M)$ is non-empty;

(2) no member of $\mathcal{B}(M)$ properly contains another member of $\mathcal{B}(M)$; and

(3) if B_1 and B_2 are members of $\mathcal{B}(M)$ and $b_1 \in B_1$, then there is an element $b_2 \in B_2$ such that

$$(B_1 - \{b_1\}) \cup \{b_2\}$$

is a member of $\mathcal{B}(M)$.

The matroid M is said to be on the set $E(M)$.

The elements of $E(M)$ are called elements of M .

The members of $\mathcal{B}(M)$ are called bases of M .

PROPOSITION 18.2 [21, p. 506]

All the bases of a matroid have the same number of elements.

PROOF

Suppose that B_1 and B_2 are bases of a matroid M and $|B_1| \leqslant |B_2|$.

Choose a base B_3 of M such that $|B_3| = |B_2|$ and

$$|B_3 \cap B_1| = \max\{|B \cap B_1| : B \text{ a base of } M \text{ and } |B| = |B_2|\} .$$

We show that $B_3 \subseteq B_1$.

Suppose that $B_3 \nsubseteq B_1$. Let $b_3 \in B_3 - B_1$. Then, by 18.1 (3) there is an element $b_1 \in B_1$ such that the set

$$B_4 = (B_3 - \{b_3\}) \cup \{b_1\}$$

is a base of M . If $b_1 \in B_3$, then $B_4 \subset B_3$ contradicting 18.1 (2). On the other hand, if $b_1 \notin B_3$ then

$$|B_4 \cap B_1| > |B_3 \cap B_1|$$

contradicting the choice of B_3 .

It follows that $B_3 \subseteq B_1$. Thus

$$|B_1| \geq |B_3| = |B_2| .$$

Therefore, since $|B_1| \leq |B_2|$, it follows that $|B_1| = |B_2|$.

SECTION 19 INDEPENDENT SETS

DEFINITION 19.1 [21, p. 506]

Let M be a matroid.

A subset of $E(M)$ is said to be M-independent if it is contained in some base of M . Unless confusion is likely to result, we use "independent" instead of "M-independent" .

We also say that an independent subset of $E(M)$ is an independent set of M .

PROPOSITION 19.2 [21, p. 507]

If X and Y are independent sets of a matroid M and $|X| < |Y|$, then there is an element $y \in Y - X$ such that $X \cup \{y\}$ is an independent set of M .

PROOF

Suppose that X and Y are independent sets of a matroid M and $|X| < |Y|$.

Since X and Y are independent sets of M , each is contained in some base of M . Let B_2 be a base of M containing Y . Choose a base B_1 of M containing X and such that

$$|B_1 \cap B_2| = \max\{|B \cap B_2| : B \text{ a base of } M \text{ and } X \subseteq B\} .$$

We show that $B_1 \cap (Y - X) \neq \emptyset$.

Suppose that $B_1 \cap (Y - X) = \emptyset$. Then $(B_1 - X) \cap Y = \emptyset$. Since $|X| < |Y|$ and since, by Proposition 18.2 , $|B_1| = |B_2|$, it follows that $|B_1 - X| > |B_2 - Y|$. Thus $B_1 - X \not\subseteq B_2 - Y$. Since $(B_1 - X) \cap Y = \emptyset$, it follows that $B_1 - X \not\subseteq B_2$. Let $b_1 \in (B_1 - X) - B_2 = B_1 - (X \cup B_2)$. Then, by 18.1 (3) , there is a $b_2 \in B_2$ such that

$$B_3 = (B_1 - \{b_1\}) \cup \{b_2\}$$

is a base of M . If $b_2 \in B_1$, then $B_3 \subset B_1$ contradicting 18.1 (2) . On the other hand, if $b_2 \notin B_1$, then $|B_3 \cap B_2| > |B_1 \cap B_2|$ contradicting the choice of B_1 since $X \subseteq B_3$. We conclude that $B_1 \cap (Y - X) \neq \emptyset$.

Let $y \in B_1 \cap (Y - X)$. Then $X \cup \{y\} \subseteq B_1$. Thus $X \cup \{y\}$ is an independent set of M .

PROPOSITION 19.3 [21, p. 507]

Let M be a matroid and let A be a subset of $E(M)$.

If X and Y are maximal independent subsets of A , then $|X| = |Y|$.

PROOF

Suppose that X and Y are independent subsets of A and $|X| < |Y|$. Then, by Proposition 19.2 , There is a $y \in Y - X$ such that $X \cup \{y\}$ is an independent set of M . But $X \subset X \cup \{y\} \subseteq A$. Thus X is not a maximal independent subset of A .

SECTION 20 RANK

DEFINITION 20.1 [7, p. 68]

Let M be a matroid.

If A is a subset of $E(M)$, then the rank $r_M(A)$ of A in M is defined to be the cardinality of a maximal independent subset of A .

Proposition 19.3 ensures that r_M is well-defined.

SECTION 21 CIRCUITS

DEFINITION 21.1 [21, p. 507]

Let M be a matroid.

A subset of E(M) is said to be M-dependent if it is not independent. Unless confusion is likely to result, we use "dependent" instead of "M-dependent".

We also say that a dependent subset of E(M) is a dependent set of M .

A circuit of M is a minimal dependent subset of E(M) .

PROPOSITION 21.2 [7, p. 70, Prop. 2]

Let M be a matroid.

Then: a subset of E(M) is independent if and only if it contains no circuit of M .

PROPOSITION 21.3 [7, p. 70, Prop. 1]

If X is an independent set of a matroid M and e is an element of M, then $X \cup \{e\}$ contains at most one circuit.

PROOF

Let X be an independent set of a matroid M and let e be an element of M.

Suppose that C_1 and C_2 are distinct circuits of M contained in $X \cup \{e\}$. Since C_1 and C_2 are distinct, there is an element $c_1 \in C_1 - C_2$. Note that $C_2 \subseteq (X \cup \{e\}) - \{c_1\}$. Since C_1 is a circuit of M contained in $X \cup \{e\}$, it follows that $C_1 - \{c_1\}$ is an independent set contained in $X \cup \{e\}$. Thus $C_1 - \{c_1\}$ is contained in a maximal independent subset Y of $X \cup \{e\}$. Since Y is independent, $C_1 \nsubseteq Y$. So $c_1 \notin Y$. Thus $Y \subseteq (X \cup \{e\}) - \{c_1\}$. Since X is a maximal independent subset of $X \cup \{e\}$, it follows from Proposition 19.3 that $|Y| = |X|$. Thus $Y = (X \cup \{e\}) - \{c_1\}$. But then $C_2 \subseteq Y$ and so, by Proposition 21.2, the fact that Y is independent is contradicted.

Thus $X \cup \{e\}$ contains at most one circuit.

PROPOSITION 21.4 [10, p. 160, Thm 7]

If B is a base of a matroid M and e is an element of M which does not belong to B, then there is a unique circuit $J(B,e)$ of M such that $e \in J(B,e) \subseteq B \cup \{e\}$.

PROOF

Suppose that B is a base of a matroid M and $e \in E(M) - B$.

Since no base of M properly contains another base of M, $B \cup \{e\}$ is dependent. Thus $B \cup \{e\}$ contains a circuit C of M. If $e \notin C$, then $C \subseteq B$. But B is independent and so contains no circuit of M. Thus $e \in C$. Moreover, by Proposition 21.3, C is the only circuit contained in $B \cup \{e\}$.

SECTION 22 ORTHOGONALITY

PROPOSITION 22.1 [21, p. 516]

If M is a matroid, then

$$\{E(M) - B : B \text{ a base of } M\}$$

is the collection of bases of a matroid M^* on $E(M)$.

We say that M^* is the matroid <u>orthogonal</u> to M .

PROOF

Let M be a matroid and let

$$\mathcal{B}(M^*) = \{E(M) - B : B \in \mathcal{B}(M)\} .$$

Since $\mathcal{B}(M)$ is non-empty, $\mathcal{B}(M^*)$ is non-empty.

Since no member of $\mathcal{B}(M)$ properly contains another member of $\mathcal{B}(M)$, no member of $\mathcal{B}(M^*)$ properly contains another member of $\mathcal{B}(M^*)$.

Suppose that B_1 and B_2 are members of $\mathcal{B}(M^*)$ and that $b_1 \in B_1$. Then $E(M) - B_1$ is a base of M and $b_1 \notin E(M) - B_1$. Thus, by Proposition 21.4 , there is a unique circuit C of M such that $b_1 \in C \subseteq (E(M) - B_1) \cup \{b_1\}$. Since $E(M) - B_2$ is M-independent, we have $C \not\subseteq E(M) - B_2$. Thus there is an element $b_2 \in C \cap B_2$. Let

$$X = ((E(M) - B_1) \cup \{b_1\}) - \{b_2\} .$$

Since C is the only circuit of M contained in $(E(M) - B_1) \cup \{b_1\}$ and C is not contained in X , it follows that X is M-independent. But $|X| = |E(M) - B_1|$. Thus, using Proposition 18.2 , it follows that X is a base of M . Now

$$X = \Big((E(M) - B_1) \cup \{b_1\}\Big) - \{b_2\}$$

$$= \Big((E(M) - B_1) \cup \{b_1\}\Big) \cap \Big(E(M) - \{b_2\}\Big)$$

$$= \Big(E(M) - (B_1 \cap (E(M) - \{b_1\}))\Big) \cap \Big(E(M) - \{b_2\}\Big)$$

$$= E(M) - \Big((B_1 \cap (E(M) - \{b_1\})) \cup \{b_2\}\Big)$$

$$= E(M) - \Big((B_1 - \{b_1\}) \cup \{b_2\}\Big)$$

Thus $(B_1 - \{b_1\}) \cup \{b_2\}$ is a member of $\mathcal{B}(M^*)$.

PROPOSITION 22.2 [21, p. 516]

If M is a matroid, then the rank function r_{M^*} of the matroid M^* orthogonal to M satisfies the following condition:

for every subset A of $E(M^*) = E(M)$,

$$r_{M^*}(A) = |A| + r_M(E(M) - A) - r_M(E(M)) .$$

PROOF

Let M be a matroid and let A be a subset of $E(M^*) = E(M)$.

Suppose that X is a maximal M^*-independent subset of A . Then, since X is M^*-independent, X is contained in some base B of M^* . So $X \subseteq A \cap B$. Since $A \cap B$ is an M^*-independent subset of A and since X is a maximal

M^*-independent subset of A, $|A \cap B| \leqslant |X|$. Thus $X = A \cap B$. Therefore

$$r_{M^*}(A) = \max\{|A \cap B| : B \text{ a base of } M^*\}.$$

Similarly

$$
\begin{aligned}
& r_M(E(M) - A) \\
&= \max\{|(E(M) - A) \cap B| : B \text{ a base of } M\} \\
&= \max\{|(E(M) - A) \cap (E(M) - B)| : B \text{ a base of } M^*\}.
\end{aligned}
$$

Let B be a base of M^*. Then

$$
\begin{aligned}
& |A \cap B| \\
&= |A| + |B| - |A \cup B| \\
&= |A| + (|E(M)| - |A \cup B|) - (|E(M)| - |B|) \\
&= |A| + |E(M) - (A \cup B)| - |E(M) - B| \\
&= |A| + |(E(M) - A) \cap (E(M) - B)| - r_M(E(M)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
r_{M^*}(A) &= \max\{|A \cap B| : B \text{ a base of } M^*\} \\
&= \max\{|A| + |(E(M) - A) \cap (E(M) - B)| - r_M(E(M)) : \\
&\qquad B \text{ a base of } M^*\}
\end{aligned}
$$

$$= |A| + \max\{|(E(M) - A) \cap (E(M) - B)| : B \text{ a base of } M^*\} - r_M(E(M))$$

$$= |A| + r_M(E(M) - A) - r_M(E(M)) .$$

SECTION 23 THE POLYGON MATROID AND THE BOND MATROID OF A GRAPH

PROPOSITION 23.1 [21, p. 509, (2)]

If G is a graph, then the edge-sets of the principal forests of G are the bases of a matroid $M(G)$ on the set $E(G)$.

PROOF

This proposition is an easy consequence of Corollaries 14.4 and 14.6 and Proposition 14.7 .

PROPOSITION 23.2 [21, p. 509, (2)]

If G is a graph, then the edge-sets of the polygons of G are the circuits of the matroid $M(G)$.

PROOF

Let G be a graph.

Suppose that P is a polygon of G. By Proposition 16.3, $E(P)$ is not contained in the edge-set of any principal forest of G. Thus $E(P)$ is a dependent set of $M(G)$. By Proposition 16.4, every proper subset of $E(P)$ is the edge-set of a forest of G. Thus, by Proposition 14.3, every proper subset of $E(P)$ is contained in the edge-set of some principal forest of G and so is an independent set of $M(G)$. Thus $E(P)$ is a circuit of $M(G)$.

Conversely, suppose that C is a circuit of $M(G)$. Then C is not contained in the edge-set of any principal forest of G. Thus, by Proposition 14.3, C is not contained in the edge-set of any forest of G. Thus $G \cdot C$ is not a forest. So, by Proposition 16.3, $G \cdot C$ contains a polygon P of G. But then $E(P)$ is a circuit of $M(G)$ and $E(P) \subseteq C$. Thus, since C is a circuit of (G), $E(P) = C$.

<u>DEFINITION 23.3</u> [21, p. 509, (2)]

The matroid $M(G)$ is called the <u>polygon matroid</u> of the graph G.

PROPOSITION 23.4 [21, p. 509, (2)]

The rank function $r_{M(G)}$ of the polygon matroid $M(G)$ of a graph G satisfies the following condition:

for every subset A of $E(G)$,

$$r_{M(G)}(A) = r(G:A) .$$

PROOF

Let G be a graph and let A be a subset of $E(G)$.

Suppose that X is a maximal $M(G)$-independent subset of A . We show that $G:X$ is a principal forest of $G:A$. Since $X \subseteq A$, $G:X$ is a subgraph of $G:A$. Since X is $M(G)$-independent, it follows from Propositions 23.1 and 14.3 that $G:X$ is a forest. Thus $G:X$ is a forest of $G:A$. So, by Proposition 14.3 , $G:X$ is contained in a principal forest F of $G:A$. But $G:A \subseteq G$. So F is a forest of G . Thus, by Proposition 14.3 , F is contained in a principal forest of G . Thus $E(F)$ is $M(G)$-independent. But $X \subseteq E(F) \subseteq A$ and X is a maximal $M(G)$-independent subset of A . Thus $X = E(F)$. Moreover, using Proposition 14.2 , $V(F) = V(G:A) = V(G)$. So $G:X = F$. Thus $G:X$ is a principal forest of $G:A$.

Therefore, using Propositions 15.2 and 15.3 ,

$$r_{M(G)}(A) = |X| = |E(G:X)| = r(G:X) = r(G:A) .$$

<u>DEFINITION 23.5</u> [21, p. 509, (3)]

Let G be a graph.

We denote the matroid orthogonal to $M(G)$ by $M^*(G)$.

<u>PROPOSITION 23.6</u> [21, p. 509, (3)]

If G is a graph, then the bonds of G are the circuits of the matroid $M^*(G)$.

<u>PROOF</u>

Let G be a graph and let A be a subset of $E(G) = E(M^*(G))$.

Suppose that A is not a cutset of G . Then, by Proposition 9.3 , $c(G:(E(G) - A)) = c(G)$. Let F be a principal forest of $G:(E(G) - A)$. Then F is a forest of G . Moreover, using Proposition 14.2 , we have

$$V(F) = V(G:(E(G) - A)) = V(G)$$

and

$$c(F) = c(G:(E(G) - A)) = c(G) .$$

Thus, by Proposition 14.2 , F is a principal forest of G . Since $E(F) \subseteq E(G) - A$, $A \subseteq E(G) - E(F)$. Thus A is $M^*(G)$-independent .

Conversely, suppose that A is $M^*(G)$-independent . Then there is a principal forest F of G such that $A \subseteq E(G) - E(F)$. Thus $E(F) \subseteq E(G) - A$. Since F is a principal forest of G , it follows from Proposition 14.2 that $V(F) = V(G)$ and $c(F) = c(G)$. Since $V(F) = V(G)$ and $E(F) \subseteq E(G) - A$,

$$F = G:(E(F)) \subseteq G:(E(G) - A) \subseteq G .$$

Thus, using Corollary 8.5 ,

$$c(F) = c(G:E(F)) \leqslant c(G:(E(G) - A)) \leqslant c(G) .$$

But $c(F) = c(G)$. So

$$c(G:(E(G) - A)) = c(G) .$$

Thus, by Proposition 9.3 , A is not a cutset of G .

Therefore: A is a cutset of G if and only if A is $M^*(G)$-dependent.

Since the circuits of $M^*(G)$ are the minimal

$M^*(G)$-dependent subsets of $E(M^*(G)) = E(G)$, the proposition follows.

DEFINITION 23.7 [21, p. 509, (3)]

The matroid $M^*(G)$ is called the bond matroid of the graph G.

SECTION 24 EDMONDS' THEOREM

LEMMA 24.1 [16, p. 3, 2.22]

If B is a base of a matroid M and e is an element of M which does not belong to B, then, for each element c of $J(B,e)$, $(B - \{c\}) \cup \{e\}$ is a base of M.

PROOF

Suppose that B is a base of a matroid M and $e \in E(M) - B$.

Let $c \in J(B,e)$.

If $c = e$, then $(B - \{c\}) \cup \{e\} = B$ and so $(B - \{c\}) \cup \{e\}$ is a base of M.

On the other hand, suppose that $c \neq e$. Then

$J(B,e) \not\subseteq (B - \{c\}) \cup \{e\}$. By Proposition 21.4 , $J(B,e)$ is the only circuit of M contained in $B \cup \{e\}$. Thus, since $(B - \{c\}) \cup \{e\} \subseteq B \cup \{e\}$, it follows that $(B - \{c\}) \cup \{e\}$ contains no circuit of M . So $(B - \{c\}) \cup \{e\}$ is an independent set of M . Thus $(B - \{c\}) \cup \{e\}$ is contained in a base of M . But, since $|(B - \{c\}) \cup \{e\}| = |B|$, it follows from Proposition 18.2 that no base of M properly contains $(B - \{c\}) \cup \{e\}$. So $(B - \{c\}) \cup \{e\}$ is a base of M .

LEMMA 24.2

Let B be a base of a matroid M .

Let $\{a_1, \cdots , a_n\}$ be a subset of $E(M)$ which is disjoint from B .

Let $\{b_1, \cdots , b_n\}$ be a subset of B which satisfies the following two conditions:

(1) if $1 \leqslant i \leqslant n$, then $b_i \in J(B,a_i)$; and

(2) if $1 \leqslant i < j \leqslant n$, then $b_j \notin J(B,a_i)$.

Then

$$(B - \{b_1, \cdots , b_n\}) \cup \{a_1, \cdots , a_n\}$$

is a base of M .

PROOF

By Lemma 24.1 , the proposition holds if $n = 1$.

Assume that $n > 1$.

By Lemma 24.1 , the set

$$B' = (B - \{b_n\}) \cup \{a_n\}$$

is a base of M .

Suppose that $1 \leqslant i \leqslant n-1$. Then, since $b_n \notin J(B,a_i) \subseteq B \cup \{a_i\}$, we have $J(B,a_i) \subseteq B' \cup \{a_i\}$. But, since $a_i \notin B'$, it follows from Proposition 21.4 that $J(B',a_i)$ is the only circuit contained in $B' \cup \{a_i\}$. Thus $J(B',a_i) = J(B,a_i)$.

Thus $\{b_1, \cdots , b_{n-1}\}$ is a subset of B' which satisfies the conditions obtained from Conditions (1) and (2) by replacing "n" with "n - 1" and "B" with "B' " .

Moreover, B' is a base of M and $\{a_1, \cdots , a_{n-1}\}$ is a subset of $E(M)$ which is disjoint from B' .

Thus, since $n - 1 < n$, we may assume as an induction hypothesis that

$$(B' - \{b_1, \cdots, b_{n-1}\}) \cup \{a_1, \cdots, a_{n-1}\}$$

is a base of M. But

$$(B - \{b_1, \cdots, b_n\}) \cup \{a_1, \cdots, a_n\}$$

$$= (B' - \{b_1, \cdots, b_{n-1}\}) \cup \{a_1, \cdots, a_{n-1}\} .$$

So

$$(B - \{b_1, \cdots, b_n\}) \cup \{a_1, \cdots, a_n\}$$

is a base of M.

PROPOSITION 24.3 (Edmonds) [7, p. 69, Thm 1]

The set $E(M)$ of elements of a matroid M can be expressed as the union of as few as k bases of M if and only if,

for every subset A of $E(M)$,

$$|A| \leq k \cdot r_M(A) .$$

PROOF

Let M be a matroid. Then: $E(M)$ can be expressed as the union of no bases of M if and only if $E(M)$ is empty. Thus we may assume that $k > 0$.

The "only if" part: [7, p. 69]

Suppose that $B_1, \cdots, B_k$ are bases of M and

$$E(M) = \bigcup_{i=1}^{k} B_i .$$

Let A be a subset of $E(M)$. Then

$$A \subseteq \bigcup_{i=1}^{k} B_i .$$

So

$$|A| \leqslant \sum_{i=1}^{k} |A \cap B_i| .$$

Moreover, for $i \in \{1, \cdots, k\}$, $A \cap B_i$ is an independent subset of A and so

$$|A \cap B_i| \leqslant r_M(A) .$$

Thus

$$|A| \leqslant k \cdot r_M(A) .$$

The "if" part:

The proof which follows is an adaptation of proofs of Bruno and Weinberg [3, p. 29, 5.2 and p. 34, 5.3]

Let $(B_1, \cdots, B_k)$ be a k-tuple of bases of M such that

$$\left| E(M) - \bigcup_{i=1}^{k} B_i \right|$$

is minimal.

We show that, if

$$\left| E(M) - \bigcup_{i=1}^{k} B_i \right| \geq 1 ,$$

then there is a subset A_t of $E(M)$ such that

$$|A_t| > k \cdot r_M(A_t) .$$

Suppose that

$$\left| E(M) - \bigcup_{i=1}^{k} B_i \right| \geq 1 .$$

Let

$$A_0 = E(M) - \bigcup_{i=1}^{k} B_i .$$

For $n \geq 0$, we define A_{n+1} in terms of A_n by the equation

$$A_{n+1} = \bigcup_{i=1}^{k} \left(\cup \{J(B_i,e) : e \in A_n \text{ and } e \notin B_i\} \right)$$

Since no member of A_0 belongs to any B_i, it follows that, for $n \geqslant 0$, $A_n \subseteq A_{n+1} \subseteq E(M)$. Thus, since $E(M)$ is finite, there is a least positive integer t such that, for $n \geqslant t$, $A_n = A_t$.

Let

$$X = \{e \in E(M) : e \text{ is an element of at least two coordinates of } (B_1, \cdots, B_k)\}.$$

Then $A_0 \cap X = \emptyset$. We show that $A_t \cap X = \emptyset$.

Suppose that there is a least positive integer s such that $A_s \cap X \neq \emptyset$. Let $e_s \in A_s \cap X$. Then, by the construction of the sequence A_n there is a sequence

$$(e_0, \cdots, e_s)$$

of elements of $E(M)$ and a sequence

$$(D_1, \cdots, D_s)$$

of coordinates of $(B_1, \cdots, B_k)$ such that

(1) if $0 \leqslant i \leqslant s$, then $e_i \in A_i$; and

(2) if $1 \leqslant i \leqslant s$, then $e_{i-1} \notin D_i$ and $e_i \in J(D_i, e_{i-1})$.

Suppose that there are integers i and j such that $1 \leqslant i < j \leqslant s$ and $e_j \in J(D_i, e_{i-1})$. Then there is an $n < s$ such that $e_s \in A_n$. But then $e_s \in A_n \cap X$, contradicting the supposition that s is the least integer such that $A_s \cap X \neq \emptyset$.

Thus, if $1 \leqslant i < j \leqslant s$, then $e_j \notin J(D_i, e_{i-1})$. It follows that the e_i are distinct and that, for $1 \leqslant i \leqslant s$, $e_i \in D_i$.

We proceed to construct a new k-tuple $(B'_1, \cdots, B'_k)$ of bases of M.

Suppose that $1 \leqslant i \leqslant k$.

If no term of $(D_1, \cdots, D_s)$ is equal to B_i, then let $B'_i = B_i$.

Otherwise, let

$$(D_{i_1}, \cdots, D_{i_m})$$

be the maximal subsequence of $(D_1, \cdots, D_s)$ all of whose terms are equal to B_i. Then

$$\{e_{i_1-1}, \cdots, e_{i_m-1}\} \subseteq E(M)$$

and

$$\{e_{i_1-1}, \cdots, e_{i_m-1}\} \cap B_i = \emptyset .$$

Moreover,

$$\{e_{i_1}, \cdots, e_{i_m}\}$$

is a subset of B_i such that

(3) if $1 \leqslant p \leqslant m$, then $e_{i_p} \in J(B_i, e_{i_p - 1})$; and

(4) if $1 \leqslant p < q \leqslant m$, then $e_{i_q} \notin \cancel{J(B_{i_p - 1})}$. $J(B_i, e_{i_p - 1})$

Thus it follows from Lemma 24.2 that the set

$$B'_i = (B_i - \{e_{i_1}, \cdots, e_{i_m}\}) \cup \{e_{i_1 - 1}, \cdots, e_{i_m - 1}\}$$

is a base of M .

Consider the k-tuple

$$(B'_1, \cdots, B'_k) .$$

It follows from the construction of the B'_i that

$$\left(\left(\bigcup_{i=1}^{k} B_i\right) - \{e_1, \cdots, e_s\}\right) \cup \{e_0, \cdots, e_{s-1}\}$$

$$\subseteq \bigcup_{i=1}^{k} B'_i$$

Moreover, there is only one i such that $e_s \in B_i$ and $e_s \notin B'_i$. But $e_s \in X$ and so e_s belongs to at least two coordinates of $(B_1, \cdots, B_k)$. Thus

$$e_s \in \bigcup_{i=1}^{k} B'_i .$$

So

$$\left(\bigcup_{i=1}^{k} B_i \right) \cup \{e_0\} \subseteq \bigcup_{i=1}^{k} B'_i .$$

But $e_0 \in A_0$ and so

$$e_0 \notin \bigcup_{i=1}^{k} B_i .$$

Thus

$$\bigcup_{i=1}^{k} B_i \subset \bigcup_{i=1}^{k} B'_i .$$

Therefore

$$\left| E(M) - \bigcup_{i=1}^{k} B'_i \right| < \left| E(M) - \bigcup_{i=1}^{k} B_i \right| .$$

But this contradicts the minimality of

$$\left| E(M) - \bigcup_{i=1}^{k} B_i \right| .$$

So we conclude that, for every $n \geqslant 0$, $A_n \cap X = \emptyset$.

Thus $A_t \cap X = \emptyset$.

It follows that the sets

$$A_t \cap B_1 , \cdots , A_t \cap B_k$$

are disjoint subsets of A_t . So

$$(5) \qquad |A_t| = \left(\sum_{i=1}^{k} |A_t \cap B_i| \right) + \left| E(M) - \bigcup_{i=1}^{k} B_i \right| .$$

We show that, for $1 \leqslant i \leqslant k$, $A_t \cap B_i$ is a maximal independent subset of A_t .

Suppose that $1 \leqslant i \leqslant k$.

Since B_i is a base of M and $A_t \cap B_i \subseteq B_i$, it follows that $A_t \cap B_i$ is an independent subset of A_t .

Suppose that $e \in A_t - (A_t \cap B_i)$. Then, by the construction of the sequence A_n , $J(B_i,e) \subseteq A_{t+1}$. But, by the choice of t , $A_{t+1} = A_t$. So $J(B_i,e) \subseteq A_t$. Since $J(B_i,e) \subseteq B_i \cup \{e\}$, it follows that

$$J(B_i,e) \subseteq A_t \cap (B_i \cup \{e\}) = (A_t \cap B_i) \cup \{e\} .$$

Thus $(A_t \cap B_i) \cup \{e\}$ is not an independent set of M .

Thus, for $1 \leqslant i \leqslant k$, $A_t \cap B_i$ is a maximal independent subset of A_t .

Therefore

$$k \cdot r_M(A_t) = \sum_{i=1}^{k} |A_t \cap B_i| .$$

So, using (5),

$$|A_t| = k \cdot r_M(A_t) + \left| E(M) - \bigcup_{i=1}^{k} B_i \right| .$$

But, by supposition,

$$\left| E(M) - \bigcup_{i=1}^{k} B_i \right| \geqslant 1 .$$

So

$$|A_t| > k \cdot r_M(A_t) .$$

SECTION 25 AN APPLICATION OF EDMONDS' THEOREM TO GRAPH THEORY

PROPOSITION 25.1

If G is a connected graph and every bond of G has at least three edges, then there are spanning trees T_1, T_2, and T_3 of G such that

$$\bigcup_{i=1}^{3} (E(G) - E(T_i)) = E(G) .$$

PROOF

Let G be a connected graph.

Suppose that G is null. Then G is a spanning tree of itself. Let $T_1 = T_2 = T_3 = G$. Then

$$\bigcup_{i=1}^{3} (E(G) - E(T_i)) = \emptyset = E(G) .$$

Thus we may assume that G is non-null.

Consider the following five conditions on G :

(1) Every bond of G has at least three edges.

(2) For every subset A of $E(G)$,

$$3(c(G:(E(G) - A)) - 1) \leqslant 2|A| .$$

(3) For every subset A of $E(M^*(G))$,

$$|A| \leq 3 \cdot r_{M^*(G)}(A) .$$

(4) $E(M^*(G))$ can be expressed as the union of as few as three bases of $M^*(G)$.

(5) There are spanning trees T_1, T_2, and T_3 of G such that

$$\bigcup_{i=1}^{3} (E(G) - E(T_i)) = E(G) .$$

We show that $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5)$.

$(1) \Rightarrow (2)$:

Let A be a subset of $E(G)$ and let $m = c(G:(E(G) - A))$. Since G is non-null, $m \neq 0$. If $m = 1$, then we have

$$3(c(G:(E(G) - A)) - 1) = 3(m - 1) = 0 \leq 2|A| .$$

and so, in this case, (1) implies (2). Thus we may assume that $m \geq 2$.

Consider the contraction $G \text{ ctr } A$ of G to A. By the definition of a contraction,

$$E(G \text{ ctr } A) = A$$

and

$$V(G \text{ ctr } A) = \{K_1, \cdots, K_m\}$$

where $K_1, \cdots, K_m$ are the components of $G:(E(G) - A)$.

Suppose that $1 \leqslant i \leqslant m$. Since components are non-null, $\emptyset \subset V(K_i)$. Since $G:(E(G) - A)$ has at least two components, it follows from Proposition 6.2 that $V(K_i) \subset V(G:(E(G) - A) = V(G)$. Thus $\emptyset \subset V(K_i) \subset V(G)$. Let L_i be the set of links of G with just one end in K_i . Then, by Proposition 9.5 , L_i is a cutset of G . Thus L_i contains a bond of G . So $|L_i| \geqslant 3$. Suppose that $e \in L_i$. Then e is a link with one end in $V(K_i)$ and the other end not in $V(K_i)$. Thus, by Lemma 10.3 , $e \in A$. So e is an edge of G ctr A and K_i is a $(G \text{ ctr } A)$-end of e . Thus

$$\text{val}(G \text{ ctr } A, K_i) \geqslant |L_i| \geqslant 3 .$$

Therefore, using Proposition 2.4 , we have

$$\begin{aligned} 3(c(G:(E(G) - A)) - 1) &= 3(m - 1) \leqslant 3m \\ &\leqslant \sum_{K_i \in V(G \text{ ctr } A)} \text{val}(G \text{ ctr } A, K_i) \\ &= 2|E(G \text{ ctr } A)| = 2|A| . \end{aligned}$$

(2) $\Longleftrightarrow$ (3):

By the definition of $M^*(G)$, $E(M^*(G)) = E(G)$. Let A be a subset of $E(M^*(G))$. By Proposition 22.2 ,

$$r_{M^*(G)}(A) = |A| + r_{M(G)}(E(G) - A) - r_{M(G)}(E(G)) .$$

By Proposition 23.4 ,

$$r_{M(G)}(E(G) - A) = r(G{:}(E(G) - A))$$

$$= |V(G{:}(E(G) - A)| - c(G{:}(E(G) - A))$$

$$= |V(G)| - c(G{:}(E(G) - A))$$

and

$$r_{M(G)}(E(G)) = r(G{:}(E(G)))$$

$$= r(G) = |V(G)| - c(G) .$$

But G is connected so $c(G) = 1$. Thus

$$r_{M(G)}(E(G)) = |V(G)| - 1 .$$

Therefore:

$$|A| \leqslant 3 \cdot r_{M^*(G)}(A)$$

if and only if

$$|A| \leqslant 3\Big(|A| + r_{M(G)}(E(G) - A) - r_{M(G)}(E(G))\Big)$$

if and only if

$$3\Big(r_{M(G)}(E(G)) - r_{M(G)}(E(G) - A)\Big) \leqslant 2|A| .$$

if and only if

$$3\Big((|V(G)| - 1) - (|V(G)| - c(G:(E(G) - A))\Big) \leqslant 2|A| .$$

if and only if

$$3\Big(c(G:(E(G) - A)) - 1\Big) \leqslant 2|A| .$$

(3) $\Longleftrightarrow$ (4):

This is a corollary of Edmonds' theorem (Proposition 24.3).

(4) $\Rightarrow$ (5):

Let B be a base of $M^*(G)$, then $E(G) - B$ is a base of $M(G)$. By the definition of $M(G)$, there is thus a principal forest F of G such that $E(F) = E(G) - B$. By Proposition 14.2 , $V(F) = V(G)$ and $c(F) = c(G)$. Since $c(G) = 1$, $c(F) = 1$. Thus F is a spanning tree of G . Therefore: if B is a base of $M^*(G)$, then there is a spanning tree T of G such that $E(G) - E(T) = B$. Thus (4) implies (5).

PROPOSITION 25.2

If every bond of a graph G has at least three edges, then there are principal forests F_1, F_2, and F_3 such that

$$\bigcup_{i=1}^{3} (E(G) - E(F_i)) = E(G) .$$

PROOF

Let G be a graph with n components.

If $n = 0$ or $n = 1$, then G is connected and the proposition follows directly from Proposition 25.1 .

Suppose that $n \geqslant 2$ and that every bond of G has at least three edges.

Let $K_1, \cdots, K_n$ be the components of G .

Suppose that $1 \leqslant i \leqslant n$. Then, since K_i is a component of G, K_i is connected. Moreover, by Proposition 10.2 , every bond of K_i is a bond of G . Thus K_i satisfies the conditions of Proposition 25.1 . So there are spanning trees T_{i1}, T_{i2}, and T_{i3} of K_i such that

$$\bigcup_{j=1}^{3} (E(K_i) - E(T_{ij})) = E(K_i) .$$

Suppose that $1 \leqslant j \leqslant 3$. Let

$$F_j = \bigcup_{i=1}^{n} T_{ij} .$$

Then

$$V(F_j) = \bigcup_{i=1}^{n} V(T_{ij}) = \bigcup_{i=1}^{n} V(K_i) = V(G) .$$

Moreover,

$$F_j = G:\left(\bigcup_{i=1}^{n} E(T_{ij})\right)$$

and

$$T_{ij} = K_i:E(T_{ij}) .$$

So, by Proposition 8.4 ,

$$c(F_j) = \sum_{i=1}^{n} c(T_{ij}) .$$

But the T_{ij} are connected and so it follows that

$$c(F_j) = c(G) .$$

By Proposition 6.2 , the K_i are disjoint. Thus the T_{ij} are disjoint. So

$$|E(F_j)| = \sum_{i=1}^{n} |E(T_{ij})|$$

and

$$|V(F_j)| = \sum_{i=1}^{n} |V(T_{ij})| .$$

Therefore, using Proposition 15.2 ,

$$|E(F_j)| = \sum_{i=1}^{n} |E(T_{ij})|$$

$$= \sum_{i=1}^{n} (|V(T_{ij})| - c(T_{ij}))$$

$$= |V(F_j)| - c(F_j) .$$

Thus, by Proposition 15.2 , F_j is a forest. But $F \subseteq G$, $V(F_j) = V(G)$, and $c(F_j) = c(G)$, so, by Proposition 14.2 , F_j is a principal forest of G .

Since the K_i are disjoint and, for $1 \leqslant i \leqslant n$, $E(T_{ij}) \subseteq E(K_i)$, we have, for $1 \leqslant j \leqslant 3$,

$$\bigcup_{i=1}^{n} (E(K_i) - E(T_{ij}))$$

$$= \left(\bigcup_{i=1}^{n} E(K_i) \right) - \left(\bigcup_{i=1}^{n} E(T_{ij}) \right) .$$

Moreover,

$$\bigcup_{i=1}^{n} E(K_i) = E(G)$$

and, for $1 \leqslant j \leqslant 3$,

$$\bigcup_{i=1}^{n} E(T_{ij}) = E(F_j) .$$

Thus

$$E(G) = \bigcup_{i=1}^{n} E(K_i)$$

$$= \bigcup_{i=1}^{n} \left(\bigcup_{j=1}^{3} (E(K_i) - E(T_{ij})) \right)$$

$$= \bigcup_{j=1}^{3} \left(\bigcup_{i=1}^{n} (E(K_i) - E(T_{ij})) \right)$$

$$= \bigcup_{j=1}^{3} \left(\left(\bigcup_{i=1}^{n} E(K_i) \right) - \left(\bigcup_{i=1}^{n} E(T_{ij}) \right) \right)$$

$$= \bigcup_{j=1}^{3} (E(G) - E(F_j)) .$$

PROPOSITION 25.3

If every bond of a graph G has at least three edges, then there are collections $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ of edge-disjoint polygons of G such that

$$\bigcup_{i=1}^{3} E(\cup \mathcal{P}_i) = E(G) .$$

PROOF

Let G be a graph.

Suppose that every bond of G has at least three edges. Then, by Proposition 25.2, there are principal forests F_1, F_2, and F_3 such that

$$\bigcup_{i=1}^{3} (E(G) - E(F_i)) = E(G) .$$

It follows from Proposition 17.4 that, for each $i \in \{1, 2, 3\}$, there is a collection $\mathcal{P}_i$ of edge-disjoint polygons of G such that

$$E(\cup \mathcal{P}_i) \supseteq E(G) - E(F_i) .$$

Thus

$$\bigcup_{i=1}^{3} E(\cup \mathcal{P}_i) \supseteq \bigcup_{i=1}^{3} (E(G) - E(F_i)) = E(G) .$$

So

$$\bigcup_{i=1}^{3} E(\cup \mathscr{P}_i) = E(G) .$$

CHAPTER 3 CYCLES AND COLOUR-CYCLES

(Sections 26 - 38)

Sections 26 - 35 are concerned with the content of pages 80 - 84 of W. T. Tutte's paper [15]. In the last of these sections Tutte's conjectures are discussed and it is shown that in order to prove Conjecture 1 it is sufficient to prove the following proposition:

(Proposition 38.4)

If a graph G has no isthmus, then it has a colour-cycle over Z_8 .

In Sections 36 and 37 two results concerning the existence of colour-cycles are proved. These results are used in Section 38 together with Proposition 25.3 to prove Proposition 38.4 .

SECTION 26 R , Z , AND Z_n

In the sequel

R denotes a commutative ring with a unit element;

Z denotes the ring of integers; and

Z_n denotes the ring of integers modulo n, where n is an integer greater than or equal to 2. We consider Z_n to consist of the subset $\{0, \cdots, n-1\}$ of Z together with integral addition modulo n and integral multiplication modulo n.

SECTION 27 ORIENTATIONS

DEFINITION 27.1 [15, p. 81]

Let G be a graph.

An orientation of G over R is a map η from $E(G) \times V(G)$ to R which satisfies the following two conditions:

(1) for $e \in E(G)$ and $v \in V(G)$, if e is a loop of G or if v is not an end of e, then $\eta(e,v) = 0$; and

(2) for $e \in E(G)$ and $v, w \in V(G)$, if e is a link of G and v and w are its ends, then either $\eta(e,v) = 1$ and $\eta(e,w) = -1$ or $\eta(e,v) = -1$ and $\eta(e,w) = 1$.

PROPOSITION 27.2

If G is a graph with no edges, then there is just one orientation of G over R .

PROOF

Let G be a graph with no edges. Then $E(G) \times V(G)$ is empty and so there is just one map from $E(G) \times V(G)$ to R . Moreover, Conditions (1) and (2) of Definition 27.1 are vacuous when $E(G)$ is empty and so this map is an orientation of G over R .

PROPOSITION 27.3

If G is a graph, then there is an orientation of G over R .

SECTION 28 CYCLES

DEFINITION 28.1

Let f be a map from a finite set E to R .

If A is a non-empty subset of E , then

$$\sum_{a \in A} f(a)$$

denotes the sum of the images of the elements of E .

If A is the empty subset of E , then we define

$$\sum_{a \in A} f(a)$$

to be the zero element of R .

<u>DEFINITION 28.2</u> [15, p. 82]

Let G be a graph and let η be an orientation of G over R .

An <u>η-cycle</u> of G over R is a map f from $E(G)$ to R which satisfies the following condition:

for every $v \in V(G)$,

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) = 0 .$$

<u>DEFINITION 28.3</u> [15, p. 82]

Let G be a graph.

A <u>cycle</u> of G over R is a map from $E(G)$ to R which is an η-cycle of G over R for some orientation η of G over R .

PROPOSITION 28.4

If G is a graph with no edges, then there is just one cycle of G over R .

SECTION 29 SUMS OF CYCLES

PROPOSITION 29.1 [18, p. 16]

Let E be a set.

If $f_1, \cdots, f_m$ are maps from E to R and if $\rho_1, \cdots, \rho_m$ are elements of R , then there is just one map f from E to R which satisfies the condition:

for every $e \in E$,

$$f(e) = \rho_1 f_1(e) + \cdots + \rho_m f_m(e) .$$

We denote this map f by $\rho_1 f_1 + \cdots + \rho_m f_m$.

PROPOSITION 29.2 [18, p. 16]

Let G be a graph and let η be an orientation of G over R .

If $f_1, \cdots, f_m$ are η-cycles of G over R and if $\rho_1, \cdots, \rho_m$ are elements of R , then $\rho_1 f_1 + \cdots + \rho_m f_m$ is an η-cycle of G over R .

PROOF

Let $f_1, \cdots, f_m$ be η-cycles of G over R and let $\rho_1, \cdots, \rho_m$ be elements of R.

If G has no edge, then $\rho_1 f_1 + \cdots + \rho_m f_m$ is the unique cycle of G over R.

Suppose that $E(G) \neq \emptyset$. Then $V(G) \neq \emptyset$. Let $v \in V(G)$. Then

$$\sum_{e \in E(G)} \eta(e,v)\, (\rho_1 f_1 + \quad + \rho_m f_m)(e)$$

$$= \sum_{i=1}^{m} \rho_i \left(\sum_{e \in E(G)} \eta(e,v)\, f_i(e) \right)$$

$$= 0 .$$

Thus $\rho_1 f_1 + \cdots + \rho_m f_m$ is an η-cycle of G over R.

SECTION 30 CIRCULAR PATHS AND CYCLES

DEFINITION 30.1

If v is a vertex of a graph G, then we define $L(G,v)$ to be the set of links of G which are incident with v.

PROPOSITION 30.2 [12, p. 364]

Let G be a graph, let η be an orientation of G over R, and let f be a map from $E(G)$ to R.

If v is a vertex of G, then

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) = \sum_{e \in L(G,v)} \eta(e,v)\, f(e) .$$

PROPOSITION 30.3

Let G be a graph and let η be an orientation of G over R.

Let

$$\alpha = (x_0, a_1, x_1, \cdots, a_m, x_m)$$

be a circular path in G with at least two edge-terms.

Define a map f from $E(G)$ to R by the following two conditions:

(1) for $i \in \{1, \cdots, m\}$, $f(a_i) = \eta(a_i, x_i)$; and

(2) for $e \in E(G) - E(\alpha)$, $f(e) = 0$.

Then f is an η-cycle of G over R.

PROOF

Let $i \in \{1, \cdots, \overset{m-1}{\cancel{m}}\}$. Then it follows from the definition of a circular path that a_i and a_{i+1} are links of G incident with x_i . Thus $\{a_i, a_{i+1}\} \subseteq L(G,x_i)$. So

$$\sum_{e \in L(G,x_i)} \eta(e,x_i)\, f(e)$$

$$= \sum_{e \in L(G,x_i) - \{a_i, a_{i+1}\}} \eta(e,x_i)\, f(e)$$

$$+ \eta(a_i,x_i)\, f(a_i) + \eta(a_{i+1},x_i)\, f(a_{i+1}) .$$

Now, since a_i and a_{i+1} are the only edges in $E(\alpha)$ which are incident with x_i ,

$$L(G,x_i) \cap E(\alpha) \subseteq \{a_i, a_{i+1}\} .$$

Thus, if $e \in L(G,x_i) - \{a_i, a_{i+1}\}$, then $f(e) = 0$. So

$$\sum_{e \in L(G,x_i) - \{a_i, a_{i+1}\}} \eta(e,\cancel{x})\, f(e) = 0 .$$

x_i

Moreover

$$\eta(a_i,x_i)\, f(a_i) = \eta(a_i,x_i)\, \eta(a_i,x_i) = 1$$

and

$$\eta(a_{i+1},x_i)\, f(a_{i+1}) = \eta(a_{i+1},x_i)\, \eta(a_{i+1},x_{i+1}) = -1 .$$

Thus

$$\sum_{e \in L(G,x_i)} \eta(e,x_i)\, f(e) = 0 .$$

So, by Proposition 30.2 ,

$$\sum_{e \in E(G)} \eta(e,x_i)\, f(e) = 0 .$$

Similarly

$$\sum_{e \in E(G)} \eta(e,x_0)\, f(e) = 0 .$$

Finally, suppose that $v \in V(G) - V(\alpha)$. Then $L(G,v) \cap E(\alpha) = \emptyset$. Thus, if $e \in L(G,v)$, then $f(e) = 0$. So

$$\sum_{e \in L(G,v)} \eta(e,v)\, f(e) = 0 .$$

Thus, by Proposition 30.2 ,

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) = 0 .$$

SECTION 31 COLOUR-CYCLES

DEFINITION 31.1 [18, p. 82]

Let G be a graph and let η be an orientation of G over R .

An *η-colour-cycle* of G over R is an η-cycle of G over R such that, for every $e \in E(G)$, $f(e) \neq 0$.

DEFINITION 31.2 [18, p. 82]

Let G be a graph.

A *colour-cycle* of G over R is a map from $E(G)$ to R which is an η-colour-cycle of G over R for some orientation η of G over R .

PROPOSITION 31.3

If G is a graph with no edges, then the unique cycle of G over R is a colour-cycle of G over R .

SECTION 32 A ONE-ONE CORRESPONDENCE

PROPOSITION 32.1 [18, p. 82]

Let G be a graph and let η and λ be orientations of G over R.

Then there is a one-one correspondence from the set of η-cycles of G over R to the set of λ-cycles of G over R.

Moreover, the appropriate restriction of this correspondence is a one-one correspondence from the set of η-colour-cycles of G over R to the set of λ-colour-cycles of G over R.

PROOF

Let f be a map from $E(G)$ to R. Then there is just one map g from $E(G)$ to R which satisfies the following two conditions:

(1) if e is a loop of G, then $g(e) = f(e)$; and

(2) if e is a link of G and v and w are the ends of e, then

$$g(e) = \lambda(e,v)\,\eta(e,v)\,f(e)$$

$$= \lambda(e,w)\,\eta(e,w)\,f(e) .$$

Suppose that $v \in V(G)$. Then, by Proposition 30.2 ,

$$\sum_{e \in E(G)} \lambda(e,v)\, g(e) = \sum_{e \in L(G,v)} \lambda(e,v)\, g(e) .$$

Now, if $e \in L(G,v)$, then e is a link of G and v is an end of e . Thus

$$\sum_{e \in L(G,v)} \lambda(e,v)\, g(e)$$

$$= \sum_{e \in L(G,v)} \lambda(e,v)\Big(\lambda(e,v)\, \eta(e,v)\, f(e)\Big)$$

$$= \sum_{e \in L(G,v)} \eta(e,v)\, f(e) .$$

Finally, using Proposition 30.2 again,

$$\sum_{e \in L(G,v)} \eta(e,v)\, f(e) = \sum_{e \in E(G)} \eta(e,v)\, f(e) .$$

It follows that g is a λ-cycle of G over R if and only if f is an η-cycle of G over R .

If e is a link of G and v is an end of e , then either $\lambda(e,v) = 1$ or $\lambda(e,v) = -1$, and either $\eta(e,v) = 1$ or $\eta(e,v) = -1$. Thus, if e is a link of G and v is an end of e , then either $\lambda(e,v)\, \eta(e,v) = 1$ or $\lambda(e,v)\, \eta(e,v) = -1$. So, for every $e \in E(G)$, either $g(e) = f(e)$ or

$g(e) = -f(e)$. Therefore, for every $e \in E(G)$, $g(e) = 0$ if and only if $f(e) = 0$.

It follows that g is a λ-colour-cycle of G over R if and only if f is an η-colour-cycle of G over R .

SECTION 33 EXISTENCE OF COLOUR-CYCLES 1. ISTHMUSES

PROPOSITION 33.1 (Tutte) [18, p. 82]

If a graph G has an isthmus, then there is no colour-cycle of G over R .

PROOF

Let b be an isthmus of a graph G . Then $\{b\}$ is a bond of G .

Let

$$H = G:(E(G) - \{b\}) .$$

Then, by Proposition 10.4 , b is a link of G whose two ends belong to different components of H .

Let y be an end of b and let K be the component of H which includes y .

Let η be an orientation of G over R and let f be an η-cycle of G over R. Then, for every $v \in V(G)$,

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) = 0 .$$

Thus

$$\sum_{v \in V(K)} \sum_{e \in E(G)} \eta(e,v)\, f(e) = 0 .$$

So

$$\sum_{e \in E(G)} \left(f(e) \sum_{v \in V(K)} \eta(e,v) \right) = 0 .$$

Suppose that $e \in E(G) - \{b\}$. Then $e \in E(H)$. Thus, since K is a component of H, it follows from Propositions 6.2 and 3.2 that either $I(G,e) \cap V(K) = \emptyset$ or $I(G,e) \subseteq V(K)$. In either case it follows from the definition of an orientation that

$$\sum_{v \in V(K)} \eta(e,v) = 0 .$$

Therefore

$$f(b) \sum_{v \in V(K)} \eta(b,v) = 0 .$$

But, if $v \in V(K) - \{y\}$, then v is not an end of b and so $\eta(b,v) = 0$. Thus

$$f(b)\, \eta(b,y) = 0 .$$

However, b is a link of G and y is an end of b so, by the definition of an orientation, either $\eta(b,y) = 1$ or $\eta(b,y) = -1$. So

$$f(b) = 0 .$$

Thus f is not an η-colour-cycle of G over R .

SECTION 34 EXISTENCE OF COLOUR-CYCLES 2. $Z_n - Z - Z_{n+1}$

REMARK 34.1

In this section the concept of an integral flow in a graph is introduced and used to prove a result of Tutte's (Proposition 34.13).

The word "flow" is used by Berge [1, p. 76] and Rota [12, p. 364] to mean cycle . This is not the sense in which it is used here. In defence of this nonconformity, the usage of Ford and Fulkerson [8, p. 4] may be cited.

DEFINITION 34.2

An integral flow (η,f) in a graph G consists of an orientation η of G over Z together with a map f from $E(G)$ to Z .

DEFINITION 34.3 [8, p. 4]

If (η,f) is an integral flow in a graph G, then an <u>(η,f)-source</u> is a vertex v of G such that

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) > 0 .$$

and an <u>(η,f)-sink</u> is a vertex v of G such that

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) < 0 .$$

PROPOSITION 34.4

Let (η,f) be an integral flow in a graph G.

Then: there is an (η,f)-source if and only if there is an (η,f)-sink.

PROOF

Let

$$S = \{v \in V(G) : v \text{ is an } (\eta,f)\text{-source}\}$$

and

$$T = \{v \in V(G) : v \text{ is an } (\eta,f)\text{-sink}\} .$$

Then S and T are disjoint. So

$$\sum_{v \in S} \sum_{e \in E(G)} \eta(e,v)\, f(e)$$

$$+ \sum_{v \in T} \sum_{e \in E(G)} \eta(e,v)\, f(e)$$

$$+ \sum_{v \in V(G) - (S \cup T)} \sum_{e \in E(G)} \eta(e,v)\, f(e)$$

$$= \sum_{v \in V(G)} \sum_{e \in E(G)} \eta(e,v)\, f(e)$$

$$= \sum_{e \in E(G)} \left(f(e) \sum_{v \in V(G)} \eta(e,v) \right) .$$

But, for every $v \in V(G) - (S \cup T)$,

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) = 0 .$$

Further, by the definition of an orientation, for every $e \in E(G)$,

$$\sum_{v \in V(G)} \eta(e,v) = 0 .$$

Thus

$$\sum_{v \in S} \sum_{e \in E(G)} \eta(e,v)\, f(e)$$

$$= - \sum_{v \in T} \sum_{e \in E(G)} \eta(e,v)\, f(e) .$$

So S is non-empty if and only if T is non-empty.

DEFINITION 34.5

Let (η,f) be an integral flow in a graph G.

A path

$$(x_0, a_1, x_1, \cdots, a_m, x_m)$$

in G is an (η,f)-path if, for every $i \in \{1, \cdots, m\}$,

$$\eta(a_i,x_i)\, f(a_i) < 0 .$$

PROPOSITION 34.6

If (η,f) is an integral flow in a graph G and v is an (η,f)-source, then there is a simple (η,f)-path in G from v to some (η,f)-sink.

PROOF

Let G be a graph and let η be an orientation of G over Z.

Let F' be the set of all maps f' from $E(G)$ to Z which satisfy the following condition:

> there is an (η,f')-source v with the property that there is no simple (η,f')-path in G from v to an (η,f')-sink.

Let F be the subset of F' which satisfies the following condition:

if $f \in F$ and $f' \in F'$, then

$$\sum_{e \in E(G)} |f(e)| \leqslant \sum_{e \in E(G)} |f'(e)| .$$

Then: F' is empty if and only if F is empty.

We show that F is empty.

Suppose that $f \in F$. Then there is an (η,f)-source x_0 with the property that there is no simple (η,f)-path in G from x_0 to an (η,f)-sink.

Since x_0 is an (η,f)-source,

$$\sum_{e \in E(G)} \eta(e,x_0)\, f(e) > 0 .$$

So there is a link a_1 of G such that x_0 is an end of a_1 and

$$\eta(a_1,x_0)\, f(a_1) > 0 .$$

Let x_1 be the other end of a_1. Then

$$\eta(a_1,x_1)\, f(a_1) < 0 .$$

Thus the sequence

$$\alpha_1 = (x_0,\ a_1,\ x_1)$$

is a simple (η,f)-path in G from x_0 .

Suppose that $m \geqslant 1$ and that the sequence

$$\alpha_m = (x_0, a_1, x_1, \cdots , a_m, x_m)$$

is a simple (η,f)-path in G from x_0 . Then

$$\eta(a_m,x_m) f(a_m) < 0 .$$

Moreover, it follows from the choice of x_0 that x_m is not an (η,f)-sink. Thus there is a link a_{m+1} of G such that x_m is an end of a_{m+1} and

$$\eta(a_{m+1},x_m) f(a_{m+1}) > 0 .$$

Let x_{m+1} be the other end of a_{m+1} . Then

$$\eta(a_{m+1},x_{m+1}) f(a_{m+1}) < 0 .$$

It follows that the sequence

$$\alpha_{m+1} = (x_0, a_1, x_1, \cdots , a_m, x_m, a_{m+1}, x_{m+1})$$

is an (η,f)-path in G from x_0 .

Suppose that, for every $k \in \{0, \cdots , m - 1\}$, $x_{m+1} \neq x_k$. Then, since α_m is simple, it follows that α_{m+1} is simple. Thus α_{m+1} is a simple (η,f)-path in G from x_0 and $V(\alpha_m) \subset V(\alpha_{m+1})$.

On the other hand, suppose that there is a $k \in \{0, \cdots , m - 1\}$ such that $x_{m+1} = x_k$. Let

$\alpha^k_{m+1} = (x_k, a_{k+1}, x_{k+1}, \cdots, a_m, x_m, a_{m+1}, x_{m+1})$.

Then, since α_{m+1} is an (η,f)-path in G , it follows that α^k_{m+1} is an (η,f)-path in G . Since α_m is simple we have:

$$\text{for } k \leqslant i < j \leqslant m , \quad x_i \neq x_j \text{ ; and,}$$

$$\text{for } k + 1 \leqslant i < j \leqslant m , \quad a_i \neq a_j .$$

Moreover,

for $k + 1 \leqslant i \leqslant m - 1$, x_m is not an end of a_i .

But x_m is an end of a_{m+1} . Thus,

$$\text{for } k + 1 \leqslant i \leqslant m - 1 , \quad a_{m+1} \neq a_i .$$

Further, since

$$\eta(a_m,x_m)\, f(a_m) < 0$$

and

$$\eta(a_{m+1},x_m)\, f(a_{m+1}) > 0 ,$$

it follows that

$$a_{m+1} \neq a_m .$$

Thus α^k_{m+1} is a circular (η,f)-path in G with at least two edge-terms.

Since, for any path α in G, $V(\alpha) \subseteq V(G)$ and since $V(G)$ is finite, we conclude that there is a circular (η,f)-path

$$\beta = (y_0, b_1, y_1, \cdots, b_n, y_n)$$

in G with at least two edge-terms.

Let g be the map from $E(G)$ to Z such that

for $i \in \{1, \cdots, n\}$, $g(b_i) = \eta(b_i,y_i)$; and,

for $e \in E(G) - E(\beta)$, $g(e) = 0$.

Then, by Proposition 30.3, g is an η-cycle of G over Z.

Let $h = f + g$. Then, since g is an η-cycle, it follows that, for every $v \in V(G)$,

v is an (η,h)-source (respectively (η,h)-sink)
if and only if
v is an (η,f)-source (respectively (η,f)-sink).

Suppose that $i \in \{1, \cdots, n\}$. Then, since β is an (η,f)-path,

$$\eta(b_i,y_i)\, f(b_i) < 0 .$$

Thus

$$|f(b_i) + \eta(b_i,y_i)| < |f(b_i)| .$$

So

$$|h(b_i)| < |f(b_i)| .$$

Moreover, for $e \in E(G) - E(\beta)$,

$$h(e) = f(e) .$$

Therefore

$$\sum_{e \in E(G)} |h(e)| < \sum_{e \in E(G)} |f(e)| .$$

Thus, since $f \in F$, it follows that $h \notin F'$.

So, since x_0 is an (η,h)-source, there is a simple (η,h)-path

$$\gamma = (z_0, c_1, z_1, \cdots , c_p, z_p)$$

in G from x_0 to an (η,h)-sink.

Suppose that $c_i = b_j$ for some $i \in \{1, \cdots , p\}$ and $j \in \{1, \cdots , n\}$. Then either $z_i = y_{j-1}$ or $z_i = y_j$.

Suppose that $z_i = y_{j-1}$. Then

$$\eta(c_i,z_i) = \eta(b_j,y_{j-1}) = -\eta(b_j,y_j)$$

and

$$h(c_i) = h(b_j) = f(b_j) + g(b_j)$$

$$= f(b_j) + \eta(b_j,y_j) .$$

Thus

$$\eta(c_i,z_i)\ h(c_i) \quad = \quad -\eta(b_j,y_j)\ f(b_j) - 1 \ .$$

But β is an (η,f)-path so

$$\eta(b_j,y_j)\ f(b_j) \quad < \quad 0 \ .$$

Thus

$$\eta(c_i,z_i)\ h(c_i) \quad \geqslant \quad 0 \ .$$

However, this contradicts the fact that γ is an (η,h)-path.

We conclude that $z_i = y_j$. Then

$$\eta(c_i,z_i) \quad = \quad \eta(b_j,y_j)$$

and

$$h(c_i) \quad = \quad h(b_j) \quad = \quad f(b_j) + g(b_j)$$

$$= \quad f(b_j) + \eta(b_j,y_j)$$

Thus

$$\eta(c_i,z_i)\ h(c_i) \quad = \quad \eta(b_j,y_j)\ f(b_j) + 1 \ .$$

So

$$\eta(c_i,z_i)\ f(c_i) \quad = \quad \eta(b_j,y_j)\ f(b_j)$$

$$= \quad \eta(c_i,z_i)\ h(c_i) - 1 \quad < \quad 0 \ .$$

On the other hand, if $c_i \notin E(\beta)$, then $h(c_i) = f(c_i)$ and so

$$\eta(c_i,z_i)\, f(c_i) = \eta(c_i,z_i)\, h(c_i) < 0 .$$

Thus γ is an (η,f)-path in G from x_0 to an (η,f)-sink. But this contradicts the choice of x_0 .

We conclude that F is empty.

DEFINITION 34.7

If (η,f) is an integral flow in a graph G , then we define $S(\eta,f)$ and $\sigma(\eta,f)$ as follows:

$$S(\eta,f) = \{v \in V(G) : v \text{ is an } (\eta,f)\text{-source}\}$$

and

$$\sigma(\eta,f) = \sum_{v \in S(\eta,f)} \sum_{e \in E(G)} \eta(e,v)\, f(e) .$$

PROPOSITION 34.8

If (η,f) is an integral flow in a graph G and $\sigma(\eta,f) = 0$, then f is an η-cycle of G over Z .

PROOF

Let (η,f) be an integral flow in a graph G .

Suppose that $\sigma(\eta,f) = 0$. Then $S(\eta,f)$ is empty. That is, there are no (η,f)-sources. Thus, by Proposition 34.4 , there are no (η,f)-sinks. Thus f must be an η-cycle.

PROPOSITION 34.9

Let λ be an orientation of a graph G over Z and let n be an integer greater than or equal to 2 .

If there is a map g from $E(G)$ to Z such that

(1) for every $v \in V(G)$,

$$\sum_{e \in E(G)} \lambda(e,v)\, g(e) \equiv 0 \pmod{n} ;$$

(2) for every $e \in E(G)$, $|g(e)| < n$; and

(3) $\sigma(\lambda,g) > 0$;

then there is a map h from $E(G)$ to Z such that

(4) for every $v \in V(G)$,

$$\sum_{e \in E(G)} \lambda(e,v)\, h(e) \equiv 0 \pmod{n} ;$$

(5) for every $e \in E(G)$, $h(e) \equiv g(e) \pmod{n}$;

(6) for every $e \in E(G)$, $|h(e)| < n$; and

(7) $\sigma(\lambda,g) > \sigma(\lambda,h)$.

PROOF

Let g be a map from $E(G)$ to Z which satisfies Conditions (1), (2), and (3). Then, since $\sigma(\lambda,g) > 0$, there is a (λ,g)-source x_0 . Thus, by Proposition 34.6 , there is a simple (λ,g)-path

$$\alpha = (x_0, a_1, x_1, \cdots , a_m, x_m)$$

in G from x_0 to a (λ,g)-sink x_m .

Define a map h from $E(G)$ to Z as follows:

for $i \in \{1, \cdots , m\}$,
$h(a_i) = g(a_i) + \lambda(a_i,x_i)\, n$; and,

for $e \in E(G) - E(\alpha)$,
$h(e) = g(e)$.

Then it is clear that h satisfies Conditions (5) and (4).

If $e \in E(G) - E(\alpha)$, then $h(e) = g(e)$ and so $|h(e)| < n$.

Suppose that $i \in \{1, \cdots , m\}$. Then, since α is a (λ,g)-path,

$$\lambda(a_i,x_i)\, g(a_i) < 0 .$$

Thus $g(a_i)$ and $\lambda(a_i,x_i)$ are of opposite sign. Since $|g(a_i)| < n$, it follows that

$$|g(a_i) + \lambda(a_i,x_i)\ n| \ < \ n\ .$$

So $|h(a_i)| < n$.

Thus h satisfies Condition (6).

If $v \in V(G) - V(\alpha)$, then $L(G,v) \cap E(\alpha) = \emptyset$ and so, using Proposition 30.2 ,

$$\sum_{e \in E(G)} \lambda(e,v)\ h(e) \ = \ \sum_{e \in E(G)} \lambda(e,v)\ g(e)\ .$$

Suppose that $i \in \{1, \cdots , m-1\}$. Then $L(G,x_i) \cap E(\alpha) = \{a_i, a_{i+1}\}$. Thus

$$\begin{aligned} &\sum_{e \in L(G,x_i) - \{a_i,\ a_{i+1}\}} \lambda(e,x_i)\ h(e) \\ &= \sum_{e \in L(G,x_i) - \{a_i,\ a_{i+1}\}} \lambda(e,x_i)\ g(e)\ . \end{aligned}$$

But

$$\begin{aligned} &\lambda(a_i,x_i)\ h(a_i) + \lambda(a_{i+1},x_i)\ h(a_{i+1}) \\ &= \lambda(a_i,x_i)\Big(g(a_i) + \lambda(a_i,x_i)\ n\Big) \\ &\quad + \lambda(a_{i+1},x_i)\Big(g(a_{i+1}) + \lambda(a_{i+1},x_{i+1})\ n\Big) \\ &= \Big(\lambda(a_i,x_i)\ g(a_i) + n\Big) \\ &\quad + \Big(\lambda(a_{i+1},x_i)\ g(a_{i+1}) - n\Big) \\ &= \lambda(a_i,x_i)\ g(a_i) + \lambda(a_{i+1},x_i)\ g(a_{i+1})\ . \end{aligned}$$

So

$$\sum_{e \in L(G,x_i)} \lambda(e,x_i)\ h(e) = \sum_{e \in L(G,x_i)} \lambda(e,x_i)\ g(e)$$

Thus, by Proposition 30.2 ,

$$\sum_{e \in E(G)} \lambda(e,x_i)\ h(e) = \sum_{e \in E(G)} \lambda(e,x_i)\ g(e)\ .$$

Now, $L(G,x_0) \cap E(\alpha) = \{a_1\}$. Thus

$$\sum_{e \in L(G,x_0) - \{a_1\}} \lambda(e,x_0)\ h(e)$$

$$= \sum_{e \in L(G,x_0) - \{a_1\}} \lambda(e,x_0)\ g(e)\ .$$

Moreover,

$$\lambda(a_1,x_0)\ h(a_1)$$

$$= \lambda(a_1,x_0)\ \Big(g(a_1) + \lambda(a_1,x_1)\ n\Big)$$

$$= \lambda(a_1,x_0)\ g(a_1) - n\ .$$

Thus

$$\sum_{e \in E(G)} \lambda(e,x_0)\ h(e) = \Big(\sum_{e \in E(G)} \lambda(e,x_0)\ g(e)\Big) - n\ .$$

Since g satisfies Conditions (1) and (3), it follows that

$$\sum_{e \in E(G)} \lambda(e,x_0)\, g(e) \;>\; \sum_{e \in E(G)} \lambda(e,x_0)\, h(e) \;\geqslant\; 0 .$$

Similarly

$$\sum_{e \in E(G)} \lambda(e,x_m)\, g(e) \;<\; \sum_{e \in E(G)} \lambda(e,x_m)\, h(e) \;\leqslant\; 0 .$$

Therefore

$$\sigma(\lambda,g) \;>\; \sigma(\lambda,h) .$$

Thus h satisfies Condition (7) .

PROPOSITION 34.10

Let λ be an orientation of a graph G over Z and let n be an integer greater than or equal to 2 .

If there is a map g from $E(G)$ to Z such that

(1) for every $v \in V(G)$,

$$\sum_{e \in E(G)} \lambda(e,v)\, g(e) \;\equiv\; 0 \pmod{n} \text{ ; and}$$

(2) for every $e \in E(G)$, $|g(e)| < n$;

then there is a λ-cycle h of G over Z such that

(3) for every $e \in E(G)$, $h(e) \equiv g(e) \pmod{n}$ and $|h(e)| < n$.

PROOF

If (η,f) is an integral flow in G, then $\sigma(\eta,f)$ is finite and ~~positive~~ non-negative. Thus the proposition follows easily from Propositions 34.9 and 34.8 .

PROPOSITION 34.11 (Tutte) [15, p. 83, Thm] and [14, p. 478, Thm IV]

Let G be a graph and let n be an integer greater than or equal to 2 .

If there is a cycle f of G over Z_n , then there is a cycle h of G over Z such that, for every $e \in E(G)$, $h(e) \equiv f(e) \pmod n$ and $|h(e)| < n$.

PROOF

Let f be a cycle of G over Z_n . Then f is an η-cycle of G over Z_n for some orientation η of G over Z_n .

Let i be the inclusion map from Z_n to Z . Let $\lambda = i\eta$ and $g = if$. Then λ and g satisfy the conditions of Proposition 34.10 .

The conclusion of the proposition follows easily.

PROPOSITION 34.12

Let G be a graph and let n be an integer greater than or equal to 2.

Then: there is a colour-cycle of G over Z_n if and only if there is a colour-cycle h of G over Z such that, for every $e \in E(G)$, $|h(e)| < n$.

PROOF

The "only if" part of the proposition follows directly from Proposition 34.11.

The "if" part:

Let h be a colour-cycle of G over Z such that, for every $e \in E(G)$, $|h(e)| < n$. Then there is a map f from $E(G)$ to Z_n which satisfies the following condition:

for every $e \in E(G)$,
if $0 < h(e) < n$, then $f(e) = h(e)$; and,
if $-n < h(e) < 0$, then $f(e) = h(e) + n$.

Moreover, since h is a colour-cycle of G over Z, f is a colour-cycle of G over Z_n.

PROPOSITION 34.13 (Tutte) [15, p. 83, Thm]

Let G be a graph and let n be an integer greater than or equal to 2.

If there is a colour-cycle of G over Z_n, then there is a colour-cycle of G over Z_{n+1}.

PROOF

Suppose that there is a colour-cycle of G over Z_n. Then, by Proposition 34.12, there is a colour-cycle h of G over Z such that, for every $e \in E(G)$, $|h(e)| < n < n + 1$. Thus, again by Proposition 34.12, there is a colour-cycle of G over Z_{n+1}.

SECTION 35 TUTTE'S CONJECTURES

Let G be a graph, let n be an integer greater than or equal to 2, and let η be an orientation of G over Z_n.

We define $\phi(G,n)$ to be the number of η-colour-cycles of G over Z_n.

It follows from Proposition 32.1 that $\phi(G,n)$ does not depend on the orientation η used in the definition and, moreover, $\phi(G,n) > 0$ if and only if there is a colour-cycle of G over Z_n. Thus it follows from Proposition 33.1 that $\phi(G,n) = 0$ if G has an isthmus.

The following two conjectures concerning $\phi(G,n)$ have been advanced by W. T. Tutte [15, p. 83].

CONJECTURE I

There exists a positive integer m such that $\phi(G,n) > 0$ whenever $n \geqslant m$ and G has no isthmus.

CONJECTURE II

$\phi(G,n) > 0$ whenever $n \geqslant 5$ and G has no isthmus.

H. H. Crapo [5, p. 16] refers to Conjecture I as follows:

> "Tutte's conjecture, that every graph is n-cocolorable, has remained untouched for 15 years now. The statement is false for $n = 4$ (consider the Petersen graph). But it is conceivably true for $n = 5$."

In an earlier paper [4, p. 212] Crapo says that Conjecture II "··· has now remained unsettled for fifteen years."

G.-C. Rota [13, p. 231] mentions a similar conjecture.

Tutte [15, p. 83, Thm] has shown that, if $\phi(G,n) > 0$, then $\phi(G,n+1) > 0$. (i.e., Proposition 34.13 .) Thus, in order to prove Conjecture I it is sufficient to exhibit an integer m such that $\phi(G,m) > 0$ whenever G has no isthmus. We shall show that $\phi(G,8) > 0$ whenever G has no isthmus. In other words: we shall show that, if a graph G has no isthmus, then there is a colour-cycle of G over Z_8 .

SECTION 36 EXISTENCE OF COLOUR-CYCLES 3. SUBGRAPHS

PROPOSITION 36.1

Let $H_1, \cdots, H_m$ be edge-disjoint subgraphs of a graph G such that

$$\bigcup_{i=1}^{m} E(H_i) = E(G) .$$

If, for each $i \in \{1, \cdots, m\}$, there is a colour-cycle of H_i over R , then there is a colour-cycle of G over R .

PROOF

It is clear that the proposition holds if $E(G)$ is empty. Thus we may assume that $E(G)$ is not empty.

Let η be an orientation of G over R.

Suppose that $i \in \{1, \cdots, m\}$. Then, since H_i is a subgraph of G, we may define an orientation η_i of H_i over R as follows:

for $e \in E(H_i)$ and $v \in V(H_i)$,
$\eta_i(e,v) = \eta(e,v)$.

Then, since there is a colour-cycle of H_i over R, it follows from Proposition 32.1 that there is an η_i-colour-cycle h_i of H_i over R. We define a map f_i from $E(G)$ to R as follows:

for $e \in E(H_i)$, $f_i(e) = h_i(e)$; and,

for $e \in E(G) - E(H_i)$, $f_i(e) = 0$.

Then, for every $v \in V(H_i)$,

$$\sum_{e \in E(G)} \eta(e,v)\, f_i(e)$$

$$= \sum_{e \in E(H_i)} \eta_i(e,v)\, h_i(e)$$

$$= 0 .$$

If $v \in V(G) - V(H_i)$ and $e \in E(H_i)$, then, by Proposition 3.2, v is not an end of e and so $\eta(e,v) = 0$. Thus, for every $v \in V(G) - V(H_i)$,

$$\begin{aligned} &\sum_{e \in E(G)} \eta(e,v)\, f_i(e) \\ &= \sum_{e \in E(G) - E(H_i)} \eta(e,v)\, f_i(e) \\ &= 0 . \end{aligned}$$

Thus f_i is an η-cycle of G over R.

Let

$$f = f_1 + \cdots + f_m .$$

Then, by Proposition 29.2, f is an η-cycle of G over R.

But the H_i are edge-disjoint, and

$$\bigcup_{i=1}^{m} E(H_i) = E(G) .$$

Thus, if $e \in E(G)$, then there is an $i \in \{1, \cdots, m\}$ such that

$$f(e) = f_i(e) = h_i(e) .$$

Therefore, since, for every $i \in \{1, \cdots, m\}$, h_i is a colour-cycle of H_i over R , it follows that f is an η-colour-cycle of G over R .

SECTION 37 EXISTENCE OF COLOUR-CYCLES 4. POLYGONS

REMARK 37.1

The source of the main result of this section (Proposition 37.5) is to be found in discussions of the critical problem by Crapo and Rota [6, Section 16] (see also Wilson [21, p. 524]) and Tutte [18] .

PROPOSITION 37.2 [9, p. 38], [1, p. 90, Cor.]

Let G be a graph and let η be an orientation of G over R .

If $\mathscr{P}$ is a collection of edge-disjoint polygons of G , then there is an η-cycle f of G over R such that:

(1) for $e \in E(\cup \mathscr{P})$, either $f(e) = 1$ or $f(e) = -1$; and

(2) for $e \in E(G) - E(\cup \mathscr{P})$, $f(e) = 0$.

PROOF

Let $\mathcal{P}$ be a collection of edge-disjoint polygons of G .

Suppose that $\mathcal{P}$ is empty. Then $E(\cup \mathcal{P})$ is empty. Let f be the map from $E(G)$ to R such that, for every $e \in E(G)$, $f(e) = 0$. Then f is an η-cycle of G over R which satisfies Conditions (1) and (2).

Suppose that $\mathcal{P}$ is not empty.

Then, for each $P \in \mathcal{P}$, we may define an η-cycle f_P of G over R as follows.

(a) Suppose that $P \in \mathcal{P}$ and P is a loop-graph. Let f_P be the map from $E(G)$ to R such that

for $e \in E(P)$, $f_P(e) = 1$; and,

for $e \in E(G) - E(P)$, $f_P(e) = 0$.

Then it is easily shown that f_P is an η-cycle of G over R .

(b) Suppose that $P \in \mathcal{P}$ and P is not a loop-graph. Then there is a circular path α in G such that α has at least two edge-terms and $G(\alpha) = P$. Let f_P be the η-cycle of G over R determined by α according to

Proposition 30.3 . Then, for $e \in E(P)$, either $f_P(e) = 1$ or $f_P(e) = -1$.

Let

$$f = \sum_{P \in \mathcal{P}} f_P .$$

Then, by Proposition 29.2 , f is an η-cycle of G over R . Moreover, since the members of $\mathcal{P}$ are edge-disjoint, f satisfies Conditions (1) and (2) .

PROPOSITION 37.3

Let G be a graph.

If f is a cycle of G over Z , then there is a collection $\mathcal{P}$ of edge-disjoint polygons of G such that, for every $e \in E(G)$, $e \in E(\cup \mathcal{P})$ if and only if $f(e)$ is odd.

PROOF

Let f be a cycle of G over Z . Then f is an η-cycle of G over Z for some orientation η of G over Z .

Let

$$H = G:\{e \in E(G) : f(e) \text{ is odd}\} .$$

Suppose that v is a vertex of H. Then

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) = 0 .$$

Thus, by Proposition 30.2,

$$\sum_{e \in L(G,v)} \eta(e,v)\, f(e) = 0 .$$

But

$$L(H,v) = \{e \in L(G,v) : f(e) \text{ is odd}\} .$$

So

$$\sum_{e \in L(H,v)} \eta(e,v)\, f(e)$$

is even. Thus $|L(H,v)|$ is even. So, by the definition of valency, $\mathrm{val}(H,v)$ is even.

Therefore every vertex of H has even valency.

Thus, by Proposition 17.2, there is a collection $\mathcal{P}$ of edge-disjoint polygons of H such that $E(\cup \mathcal{P}) = E(H)$.

Since H is a subgraph of G, $\mathcal{P}$ is a collection of edge-disjoint polygons of G. And, by the definition of H, for every $e \in E(G)$, $e \in E(H) = E(\cup \mathcal{P})$ if and only if $f(e)$ is odd.

PROPOSITION 37.4

Let G be a graph and let m be a positive integer.

If f is a cycle of G over Z such that, for every $e \in E(G)$, $|f(e)| < 2^m$, then there are collections $\mathcal{P}_1, \cdots, \mathcal{P}_m$ of edge-disjoint polygons of G such that, for every $e \in E(G)$,

$$e \in \bigcup_{i=1}^{m} E(\cup \mathcal{P}_i) \quad \text{if and only if} \quad f(e) \neq 0 .$$

PROOF

Note that we do not require the collections $\mathcal{P}_i$ to be non-empty or to be distinct.

Let f be a cycle of G over Z such that, for every $e \in E(G)$, $|f(e)| < 2^m$. Then f is an η-cycle of G over Z for some orientation η of G over Z.

We define a sequence

$$(f_1, \cdots, f_m)$$

of η-cycles of G over Z and a sequence

$$(\mathcal{P}_1, \cdots, \mathcal{P}_m)$$

of collections of edge-disjoint polygons of G such that, for $i \in \{1, \cdots, m\}$ and $e \in E(G)$,

$$e \in E(\cup \mathcal{P}_i) \text{ if and only if } f_i(e) \text{ is odd .}$$

Let $f_1 = f$.

Suppose that $i \in \{1, \cdots, m\}$ and that f_i has been defined. Then, by Proposition 37.3, there is a collection $\mathcal{P}_i$ of edge-disjoint polygons of G such that, for every $e \in E(G)$,

$$e \in E(\cup \mathcal{P}_i) \text{ if and only if } f_i(e) \text{ is odd .}$$

Suppose that $i \in \{1, \cdots, m - 1\}$ and that f_i and $\mathcal{P}_i$ have been defined. Then, by Proposition 37.2, there is an η-cycle g_i of G over Z such that,

for $e \in E(\cup \mathcal{P}_i)$, either $g_i(e) = 1$ or $g_i(e) = -1$; and,

for $e \in E(G) - E(\cup \mathcal{P}_i)$, $g_i(e) = 0$.

Now, by Proposition 29.2, $f_i + g_i$ is an η-cycle of G over Z . Moreover, for every $e \in E(G)$, $(f_i + g_i)(e)$ is even. Thus $\frac{1}{2}(f_i + g_i)$ is an η-cycle of G over Z . Let $f_{i+1} = \frac{1}{2}(f_i + g_i)$. Note that, for $e \in E(G)$, if $f_i(e)$ is even then $f_{i+1}(e) = \frac{1}{2}f_i(e)$.

We show that the sequence $(\mathcal{P}_1, \cdots, \mathcal{P}_m)$ satisfies the condition:

for every $e \in E(G)$,

$$e \in \bigcup_{i=1}^{m} E(\cup \mathcal{P}_i) \quad \text{if and only if} \quad f(e) \neq 0 .$$

Let $e \in E(G)$.

Suppose that $f(e) \neq 0$. Then, since $|f(e)| < 2^m$, there is a $k \in \{1, \cdots, \overset{m}{\cancel{m-1}}\}$ such that $f(e)$ is divisible by 2^{k-1} but not divisible by 2^k . Since $f(e)$ is divisible by 2^{k-1} , it follows that $f_k(e) = (\frac{1}{2})^{k-1} f(e)$. But $(\frac{1}{2})^{k-1} f(e)$ is odd and so $f_k(e)$ is odd. Thus $e \in E(\cup \mathcal{P}_k)$. So

$$e \in \bigcup_{i=1}^{m} E(\cup \mathcal{P}_k) .$$

On the other hand, suppose that $f(e) = 0$. Then, for $i \in \{1, \cdots, m\}$, $f_i(e) = 0$ and so $e \notin E(\cup \mathcal{P}_i)$. Thus

$$e \notin \bigcup_{i=1}^{m} E(\cup \mathcal{P}_i) .$$

PROPOSITION 37.5

Let G be a graph and let m be a positive integer.

Then: there is a colour-cycle f of G over Z such that, for every $e \in E(G)$, $|f(e)| < 2^m$ if and only if there are collections $\mathcal{P}_1, \cdots, \mathcal{P}_m$ of edge-disjoint polygons of G such that

$$\bigcup_{i=1}^{m} E(\cup \mathcal{P}_i) = E(G) .$$

PROOF

The "only if" part of the proposition follows directly from Proposition 37.4 .

The "if" part:

Let $\mathcal{P}_1, \cdots, \mathcal{P}_m$ be collections of edge-disjoint polygons of G such that

$$\bigcup_{i=1}^{m} E(\cup \mathcal{P}_i) = E(G) .$$

Let η be an orientation of G over Z .

Suppose that $i \in \{1, \cdots, m\}$. Then, by Proposition 37.2 , there is an η-cycle f_i of G

over Z such that:

for $e \in E(\cup \mathcal{P}_i)$, either $f_i(e) = 1$ or $f_i(e) = -1$; and,

for $e \in E(G) - E(\cup \mathcal{P}_i)$, $f_i(e) = 0$.

Let

$$f = \sum_{i=1}^{m} 2^{i-1} f_i .$$

Then, by Proposition 29.2 , f is an η-cycle of G over Z .

Suppose that $e \in E(G)$. Then, since

$$\bigcup_{i=1}^{m} E(\cup \mathcal{P}_i) = E(G) ,$$

it follows that e belongs to at least one of the $E(\cup \mathcal{P}_i)$. Thus there is an integer k such that

$$k = \max \{ i : 1 \leqslant i \leqslant m \text{ and } e \in E(\cup \mathcal{P}_i) \} .$$

Therefore

$$|f(e)| = \left| \sum_{i=1}^{m} 2^{i-1} f_i(e) \right|$$

$$= \left| \sum_{i=1}^{k} 2^{i-1} f_i(e) \right|$$

$$\geqslant \quad |2^{k-1} f_k(e)| - \left| \sum_{i=1}^{k-1} 2^{i-1} f_i(e) \right|$$

$$\geqslant \quad 2^{k-1} - \sum_{i = \cancel{1}\, 0}^{k-2} 2^{i} = 1 > 0 \; .$$

Moreover,

$$|f(e)| = \left| \sum_{i=1}^{m} 2^{i-1} f_i(e) \right|$$

$$\leqslant \quad \sum_{i=1}^{m} 2^{i-1} = 2^m - 1 < 2^m \; .$$

Thus f is a colour-cycle of G over Z such that, for every $e \in E(G)$, $|f(e)| < 2^m$.

PROPOSITION 37.6

Let G be a graph and let m be a positive integer.

Then: there is a colour-cycle of G over Z_{2^m} if and only if there are collections $\mathscr{P}_1, \cdots, \mathscr{P}_m$ of edge-disjoint polygons of G such that

$$\bigcup_{i=1}^{m} E(\cup \mathscr{P}_i) = E(G) \; .$$

PROOF

This proposition follows directly from Propositions 34.12 and 37.5 .

SECTION 38 EXISTENCE OF COLOUR-CYCLES OVER Z_8

DEFINITION 38.1

We say that a graph G is vertex-critical if it satisfies the following two conditions:

(1) G has no isthmus and there is no colour-cycle of G over Z_8 ; and

(2) if H is a graph which has fewer vertices than G has, then either H has an isthmus or there is a colour-cycle of H over Z_8 .

PROPOSITION 38.2

If a graph is vertex-critical, then it is connected.

PROOF

Suppose that G is a vertex-critical graph which is not connected.

Let $K_1, \cdots, K_m$ be the components of G. Then, by Proposition 6.2, the K_i are edge-disjoint and

$$\bigcup_{i=1}^{m} E(K_i) = E(G) .$$

Suppose that $i \in \{1, \cdots, m\}$. Then, since G has at least two components and since the components of a graph are non-null and disjoint, $|V(K_i)| < |V(G)|$. Also, since G has no isthmus, it follows from Proposition 11.2 that K_i has no isthmus. Thus, since G is vertex-critical, there is a colour-cycle of K_i over Z_8.

Thus it follows from Proposition 36.1 that G has a colour-cycle over Z_8. But this contradicts the supposition that G is vertex-critical.

We conclude that every vertex-critical graph is connected.

PROPOSITION 38.3

Every bond of a vertex-critical graph has at least three edges.

PROOF

Let B be a bond of a vertex-critical graph G .

Since a bond must have at least one edge and since G has no isthmus, it follows that B has at least two edges.

Suppose that B has just two edges, a and b say.

Since B is a bond of G , it follows from Proposition 10.5 that the number of components of $G:(E(G) - B)$ is just one greater than the number of components of G . But, by Proposition 38.2 , G is connected. Thus $G:(E(G) - B)$ has just two components, M_1 and M_2 say.

By Proposition 10.4 , both a and b are links of G and neither a nor b has both of its G-ends contained in the same component of $G:(E(G) - B)$.

Let x_1 (respectively x_2) be the G-end of a contained in M_1 (respectively M_2) .

Let y_1 (respectively y_2) be the G-end of b contained in M_1 (respectively M_2) .

Note that, in the argument which follows, we make no assumption as to whether or not $x_1 = y_1$ (respectively $x_2 = y_2$) .

Let

$$H = G \text{ ctr } (E(G) - \{a\}) .$$

Then $E(H) = E(G) - \{a\}$. Moreover, the vertices of H are the components of $G:\{a\}$, that is,

$$V(H) = \Big\{[v] : v \in V(G) - \{x_1, x_2\}\Big\} \cup \Big\{G \cdot \{a\}\Big\} .$$

Note that $|V(H)| < |V(G)|$.

Since G has no isthmus, it follows from Proposition 12.2 that H has no isthmus. Further, $|V(H)| < |V(G)|$ and G is vertex-critical. Thus there is a colour-cycle of H over Z_8 .

Let η be an orientation of G over Z_8 . Since H is a contraction of G , e is a link of H and K is an H-end of e only if e is a link of G and just one G-end of e is contained in K . Thus we may define an orientation λ of H over Z_8 as follows:

for $e \in E(H)$ and $K \in V(H)$,

if e is a loop of H or if K is not an H-end of e , then $\lambda(e,K) = 0$; and,

if e is a link of H and K is an H-end of e , then $\lambda(e,K) = \eta(e,v)$ where v is the G-end of e which is contained in K .

Since there is a colour-cycle of H over Z_8, it follows from Proposition 32.1 that there is a λ-colour-cycle g of H over Z_8.

Define a map f from $E(G)$ to Z_8 as follows:

for $e \in E(G) - \{a\} = E(H)$, $f(e) = g(e)$; and

$f(a) = -\eta(a,x_1)\,\eta(b,y_1)\,f(b) = -\eta(a,x_2)\,\eta(b,y_2)\,f(b)$.

We shall show that f is an η-colour-cycle of G over Z_8 and so obtain a contradiction.

Let $v \in V(G) - \{x_1, x_2\}$. If $e \in L(G,v)$, then $I(G,e) \neq \{x_1, x_2\}$. But, if e is a link of G and a loop of H , then $I(G,e) = \{x_1, x_2\}$. Thus, if $e \in L(G,v)$, then $e \in L(H,[v])$. On the other hand, if $e \in L(H,[v])$, then $e \in L(G,v)$. Therefore $L(G,v) = L(H,[v])$. It follows that, if $e \in L(G,v)$, then $\eta(e,v) = \lambda(e,[v])$. Moreover, since $I(G,a) = \{x_1, x_2\}$, $a \notin L(G,v)$ and so, if $e \in L(G,v)$, then $f(e) = g(e)$. Thus

$$\sum_{e \in L(G,v)} \eta(e,v)\, f(e) = \sum_{e \in L(H,[v])} \lambda(e,[v])\, g(e) .$$

Therefore, using Proposition 30.2 and the fact that g is a λ-cycle of H over Z_8 ,

$$\sum_{e \in E(G)} \eta(e,v)\, f(e) = \sum_{e \in E(H)} \lambda(e,[v])\, g(e) = 0 .$$

It remains to show that

$$(1) \qquad \sum_{e \in E(G)} \eta(e,x_1)\, f(e) = 0$$

and that

$$(2) \qquad \sum_{e \in E(G)} \eta(e,x_2)\, f(e) = 0 .$$

We show that (1) holds. If $v \in V(M_1) - \{x_1\}$, then $v \in V(G) - \{x_1, x_2\}$. So

$$\sum_{v \in V(M_1) - \{x_1\}} \sum_{e \in E(G)} \eta(e,v)\, f(e) = 0 .$$

Thus

$$\sum_{e \in E(G)} \eta(e,x_1)\, f(e)$$

$$= \sum_{e \in E(G)} \eta(e,x_1)\, f(e)$$

$$+ \sum_{v \in V(M_1) - \{x_1\}} \sum_{e \in E(G)} \eta(e,v)\, f(e)$$

$$= \sum_{v \in V(M_1)} \sum_{e \in E(G)} \eta(e,v)\, f(e)$$

$$= \sum_{e \in E(G)} \sum_{v \in V(M_1)} \eta(e,v)\, f(e) .$$

Now, $E(G)$ is the union of the disjoint sets $E(M_1)$, $E(M_2)$, and B. If $e \in E(M_1)$, then

$I(G,e) \subseteq V(M_1)$. Thus, if $e \in E(M_1)$, then

$$\sum_{v \in V(M_1)} \eta(e,v)\, f(e) = 0 \, .$$

If $e \in E(M_2)$ and $v \in V(M_1)$, then e is not G-incident with v and so $\eta(e,v) = 0$. Thus, if $e \in E(M_2)$, then

$$\sum_{v \in V(M_1)} \eta(e,v)\, f(e) = 0 \, .$$

Therefore

$$\begin{aligned}
&\sum_{e \in E(G)} \sum_{v \in V(M_1)} \eta(e,v)\, f(e) \\
&= \sum_{e \in E(M_1)} \sum_{v \in V(M_1)} \eta(e,v)\, f(e) \\
&\quad + \sum_{e \in E(M_2)} \sum_{v \in V(M_1)} \eta(e,v)\, f(e) \\
&\quad + \sum_{e \in B} \sum_{v \in V(M_1)} \eta(e,v)\, f(e) \\
&= \sum_{e \in B} \sum_{v \in V(M_1)} \eta(e,v)\, f(e) \\
&= \sum_{v \in V(M_1)} \eta(a,v)\, f(a) \\
&\quad + \sum_{v \in V(M_1)} \eta(b,v)\, f(b) \, .
\end{aligned}$$

Now

$$V(M_1) \cap I(G,a) = \{x_1\}$$

and

$$V(M_1) \cap I(G,b) = \{y_1\} .$$

Thus

$$\sum_{v \in V(M_1)} \eta(a,v)\, f(a) + \sum_{v \in V(M_1)} \eta(b,v)\, f(b)$$

$$= \eta(a,x_1)\, f(a) + \eta(b,y_1)\, f(b) .$$

But, by the definition of f ,

$$f(a) = -\eta(a,x_1)\, \eta(b,y_1)\, f(b) .$$

So

$$\eta(a,x_1)\, f(a) = -\eta(b,y_1)\, f(b) .$$

Thus

$$\eta(a,x_1)\, f(a) + \eta(b,y_1)\, f(b) = 0 .$$

We have thus shown that (1) holds.

A similar argument, with x_1 , y_1 , M_1 , and M_2 replaced by x_2 , y_2 , M_2 , and M_1 respectively, shows that (2) holds.

Therefore f is an η-cycle of G over Z_8 .

Now, if $e \in E(G) - \{a\} = E(H)$, then $f(e) = g(e) \neq 0$. Moreover; either $\eta(a,x_1) = 1$ or $\eta(a,x_1) = -1$, and either $\eta(b,y_1) = 1$ or $\eta(b,y_1) = -1$. Thus either $\eta(a,x_1)\,\eta(b,y_1) = 1$ or $\eta(a,x_1)\,\eta(b,y_1) = -1$. So, by the definition of f , either $f(a) = f(b)$ or $f(a) = -f(b)$. Thus $f(a) \neq 0$.

Thus f is an η-colour-cycle of G over Z_8 . But this contradicts the fact that G is vertex-critical. So we conclude that B has at least three edges.

PROPOSITION 38.4

If a graph G has no isthmus, then there is a colour-cycle of G over Z_8 .

PROOF

If the proposition is false, then there is a vertex-critical graph.

Suppose that G is a vertex-critical graph. Then, by Proposition 38.3 , every bond of G has at least three edges. Thus, by Proposition 25.3 , there are collections $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ of edge-disjoint polygons of G such that

$$\bigcup_{i=1}^{3} E(\cup \mathcal{P}_i) = E(G) .$$

So, by Proposition 37.6 , there is a colour-cycle of G over Z_8 . But this contradicts the supposition that G is vertex-critical

Thus there is no vertex-critical graph and so the proposition holds.

REMARK 38.5

By virtue of the considerations of Section 35, it follows that Conjecture I is true.

As a bonus we obtain a strengthening of Proposition 25.3 , namely:

PROPOSITION 38.6

If a graph G has no isthmus, then there are collections $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ of edge-disjoint polygons of G such that

$$\bigcup_{i=1}^{3} E(\cup \mathcal{P}_i) = E(G) .$$

PROOF

This follows from Propositions 38.4 and 37.6 .

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LIST OF SYMBOLS

1.1	$E(G)$
	$V(G)$
	$I(G,-)$
1.2	$[v]$
2.1	$\mathrm{val}(G,v)$
2.2	$\lvert S \rvert$
2.3	$\sum_{v \in V(G)} \mathrm{val}(G,v)$
3.1	$\subseteq$ (subgraphs)
	$\subset$ (subgraphs)
3.3	$G:A$
	$G \cdot A$
3.4	$\bigcup_{i=1}^{p} H_i$ (subgraphs)
	$H_1 \cup \cdots \cup H_p$ (subgraphs)
3.5	$\bigcup \mathcal{H}$ (subgraphs)
4.1	$E(\alpha)$
	$V(\alpha)$
	$G(\alpha)$
4.2	$\alpha\beta$

30.1 $L(G,v)$

34.2 (η,f)

34.7 $S(\eta,f)$

$\sigma(\eta,f)$

35 $\phi(G,n)$

INDEX OF DEFINITIONS