Pricing, Calibration and Hedging under the LIBOR model

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

June 18, 2023 Signed by candidate

Abstract

This dissertation reviews work done by Dun *et al.* (2001). We present an algorithm for generating the LIBOR forward rates, which encompasses the functionality for pricing interest rate derivatives. We further generalise the algorithm to implement the predictor-corrector method.

Calibration is carried out to price swaptions using the Black-76 and LIBOR methods, and the hedging strategies implied by both methods are considered. We aim to determine whether the theoretical and computational overhead associated with hedging swaptions using the LIBOR method improves the hedging accuracy over the more straightforward Black-76 method.

The simulation is conducted within the LIBOR model framework. While inconsistent with the model assumptions, the Black method performed equally well as the LIBOR method as we obtained similar hedging profit and loss distributions even at high portfolio rebalancing frequencies.

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Contents

List of Figures

List of Tables

Chapter 1

INTRODUCTION

The LIBOR model, also known as the BGM or BGMJ model after the authors Brace *et al.* (1997) and Jamshidian (1997) who pioneered the approach, which was also popularised by Miltersen *et al.* (1997), and Musiela and Rutkowski (1997), is a so-called *market model*.

It plays a significant role in the pricing and hedging of interest rate derivatives in the financial industry since it models the actual observed traded instruments in the market instead of idealisations like the short rate or continuously compounded forward rates. In particular, all rates are specified using simple compounding. Additionally, it is also used to price and hedge exotic options embedded in insurance contracts, thus the accuracy of this model is paramount to society.

Globally, Interbank Offered Rates (IBORs) such as LIBOR, which set in advance, are being replaced by benchmarks based on new risk-free term rates (RFRs), which set in arrears. In recent years, regulators, market participants, and industry organisations have developed several alternative benchmarking frameworks, such as SOFR (Secure Overnight Funding Rate) in the US; SARON (Swiss Average Rate Overnight) in Switzerland, and TONA (Tokyo Overnight Average Rate) in Japan, to name a few. It is argued that this migration is necessary since according to the UK and US governments, multiple attempts were made by banks to manipulate LIBOR during the 2007-2009 credit crisis.

The LIBOR model remains relevant since Lyashenko and Mercurio (2019) have shown that backward-looking rates (RFRs) and forward-looking rates (IBORs) can be modeled jointly (i.e, with one stochastic process). This results in what they call a generalised forward market model (FMM), which extends and completes the classic single-curve LIBOR market model (LMM) since it provides additional information about the rate dynamics between fixing/payment times.

1.1 Problem Specification

We consider an equi-spaced tenor structure defined by

$$T_i = T_0 + i\delta \text{ for } i = 1, 2, \ldots, n,$$

where $\delta := \delta_i = T_i - T_{i-1}$ is the accrual period (or year fraction) and $T_0 = 0$.

Define $F_i(t) := F(t, T_{i-1}, T_i)$ so that F_1 is fixed and F_i for $i > 1$ is dynamic[1]. The LIBOR model assumes that the dynamics of the LIBOR forward rates, $F_2, F_3, \ldots, F_n$, under the measure $\mathbb{Q}^k$, $1 < k \leq n$, associated with the numeraire asset $P(t, T_k)$ (zero coupon bond), each satisfy the following SDE:

$$dF_i(t) = \mu_i(t)F_i(t)dt + \sigma_i(t)F_i(t)dW_i^k(t),$$

for $t < T_i$, where $\mu_i(t)$ is the drift term which must be derived. We do this later but under the spot measure, we do not derive it for a general measure k. Furthermore, $\sigma_i(t)$ is the instantaneous volatility, and $W_i^k(t)$ is the i^{th} component of the multidimensional $\mathbb{Q}^k$ Brownian motion with instantaneous correlations given by

$$d\langle W_i^k, W_j^k \rangle = \rho_{ij}dt.$$

The correlation matrix formed by the elements ρ_{ij} is denoted ρ, and is assumed to be of rank r. If $r = n - 1$ then ρ is full rank.

Term structure models of this type provide a theoretical justification for analytical pricing of cap and floor contracts using the Black-76 formula since it allows lognormal dynamics of the LIBOR forward rates using deterministic volatilities σ_i.

We shall use this lognormal forward LIBOR model under the spot measure to price and hedge swaptions, but in this case, the pricing does not match the market standard Black-76 swaption formula as argued by Dun *et al.* (2001). This happens because the forward LIBOR and swap rates cannot be considered lognormal simultaneously. Thus, the Black-76 formula can be applied consistently either to caps/floors or swaptions. Black's formula for swaptions is supported by a model assuming log-normality for the swap rates as proposed by Jamshidian (1997).

The difference between these two models is that the lognormal forward LIBOR model assumes that the LIBOR forward rates are log-normally distributed, which in turn, given that the swap rates will be implied from these forward rates, does not allow for log-normality of the swap rates. In contrast, the model proposed by Jamshidian (1997) assumes that the swap rate dynamics are log-normally distributed.

[1] Here the LIBOR forward rates are indexed based on the end of the accrual period. The market generally indexes it based on the start of the accrual period, which differs slightly from this.

Thus the difference between these models goes beyond the pricing equations and implies different methods for hedging swaptions. Nevertheless, Brace *et al.* (2001) argue that in practice, the LIBOR model yields swaption prices that are close to those given by the Black-76 formula (independently, Rebonato (1999) provides a similar conclusion).

In this dissertation, we wish to determine whether this *closeness* applies to hedging as well. To experiment, we simulate swaption hedges using both the LIBOR and Black methods to unpack the degree to which the former statement is accurate. We aim to test the effectiveness of hedging swaptions as prescribed by the lognormal forward LIBOR model (LFM, alternatively LMM) against the simpler and well-understood market practitioners' hedging approach using the Black-76 formula.

To justify whether there is any rationale in favouring the more complicated approach to swaption hedging as prescribed by the LFM over the simpler Black type hedging, we simulate both hedging strategies within the framework of the LIBOR model. That is, we favour the LFM by assuming that it describes the *true* term structure dynamics underlying our simulation.

Due to discretisation, either the replicating or the self-financing property will be lost to some extent since the simulated hedging strategies will be an approximation of the continuous, replicating, and self-financing strategies implied by the models. We employ the predictor-corrector method as proposed by Hunter *et al.* (2001) in the log-discretisation simulation of the forward rates since this method has been proven to produce a more accurate estimate of the drift term required over the update/accrual period.

Suppose the results demonstrate that LIBOR model swaptions can be hedged through Black hedging methods. In that case, this will strengthen the notion that these models are close while simultaneously simplifying the use and implementation of the LIBOR model. However, the study results may suggest that LIBOR-based swaption hedging offers something extra to practitioners in terms of swaption hedging if results indicate that Black hedging is not adequate. Hence, if the effectiveness of the LFM hedging is not significantly improved over the Black hedge, we shall conclude that the additional complexity that the LFM hedging demands is not worth the effort.

1.2 Literature Review

A great deal of work considered in the dissertation has previously been conducted by Dun *et al.* (2001), where they perform simulation under the LIBOR model framework for a range of swaptions and volatility structures. They show that the Black method performs well comparatively when compared to the LIBOR model, despite incompatibilities with the model assumptions. This is because it yields similar hedging profit and loss distributions, even at high rehedging frequencies. In addition to demonstrating the robustness of the Black hedging method, this result suggests that, due to its simplicity and better understanding by financial practitioners, it would be a more effective method in practice.

An extension of the LFM called the displaced forward LIBOR model (DLFM) was considered by Van Appel and McWalter (2020) where they presented an algorithm to approximate moments for forward rates. This, in turn, provides an improved swaption volatility approximation, and we have made use of some of the results from this paper. In particular, the implementation of the Kawai approximation proposed by Kawai (2003) is used.

1.2.1 Short-rate models

Brigo and Mercurio (2006) reveal that the theory of interest-rate modeling was originally based on the assumption of specific one-dimensional dynamics for the instantaneous short rate process r_t. Direct modeling of such dynamics is very convenient since all fundamental quantities (rates and bonds) can be defined by no-arbitrage arguments and as the expectation of a functional of the process r_t.

We consider below one popular short rate model called the Vasicek (1977) model, which we shall later use as input rates for the initial term structure as outlined in Section 2.2.1.

Vasicek Model

A general form of the term structure of interest rates is derived under two main assumptions:

1. r_t follows a diffusion process.
2. r_t is mean-reverting.

The model of Vasicek (1977) describes the short rate using the Ornstein-Uhlenbeck process

$$dr_t = \alpha(b - r_t)dt + \sigma_v dW_t,$$

where W_t is a standard Brownian motion, and $\alpha, b, \sigma_v \in \mathbb{R}^+$ represent the *rate of mean reversion*, the *mean reversion level* and the *volatility* respectively.

The Vasicek (1977) model is an example of an *affine term structure* (ATS) model, which means that bond prices are given by

$$P(t,T) = e^{A(t,T)+B(t,T)r(t)},$$

where $A(t,T)$ and $B(t,T)$ are (sufficiently regular) deterministic functions.

Ouwehand (2021) reveals that the advantages of short rate models are that, specifying r_t as the solution of an SDE allows us to use Markov theory which leads to PDEs via no-arbitrage, the Feynman-Kac theorem, or the Kolmogorov forward and backward equations for example. Also, it is often possible for analytical formulas for bond and bond options to be obtained, which is helpful for calibration.

However, the disadvantages are that, it is unreasonable to regard the short rate as the only explanatory variable. Also, it is difficult to incorporate views about different times in the future. Another issue is that it can be quite difficult to fit a realistic volatility structure and for the model to have even a remote chance of being correct, it is necessary to invert the yield curve, which can be difficult.

Ouwehand (2021) further argues that multifactor models overcome some of these disadvantages but at the cost of reduced intuition. That is, the economic meaning of the underlying factors is opaque.

1.2.2 The Heath-Jarrow-Morton Framework

The Heath-Jarrow-Morton (HJM) approach overcomes some of these difficulties by specifying dynamics for the entire (uncountable) family of forward rates. The idea is to proceed as follows:

For each $T \geq 0$, assume that the forward rates $f(t,T)$ have "real-world" dynamics given by

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW_t \quad \text{for } 0 \leq t \leq T, \\ f(0,T) &= f^*(0,T) \quad \text{(i.e., observed forward rate/initial term structure)}, \end{aligned}$$

where W_t is a finite-dimensional Brownian motion under the real world measure $\mathbb{P}$, and $\alpha(t,T)$ and $\sigma(t,T)$ are adapted (and sufficiently regular, e.g. bounded and jointly measurable in the t- and T- variables).

Here it is possible to specify an arbitrage-free drift condition known as the HJM drift conditions for $\alpha(t,T)$. This is achieved by assuming that the bond market is arbitrage-free in the following strong sense:

There is a risk-neutral measure for bonds of all maturities.

Then there is a (multidimensional) process $\lambda(t)$ such that, for all maturities T, we have

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma^{\intercal}(t,s)ds + \sigma(t,T)\lambda(t).$$

Under the risk-neutral measure $\mathbb{Q}$, the market price of risk is $\lambda = 0$. Hence, under $\mathbb{Q}$, the drifts $\alpha(t,T)$ are completely determined by the volatility surface $\sigma(t,T)$, which we can estimate or otherwise specify.

The advantages of this approach are that, the initial term structure is fitted automatically as an initial condition. Also, It is easy to incorporate views about different maturities because we have many different SDEs[2], one for every T. Unfortunately, having many SDEs is also a disadvantage.

It is crucial to note that the HJM is not a model but a framework of models for the bond market. Also, Ouwehand (2021) argues that we can easily let α and σ depend on history. Hence, in contrast to short rate diffusion models, HJM models need not be Markov.

1.2.3 LIBOR market model

The London Inter-Bank Offered Rate (LIBOR) is determined by averaging estimates of interest rates offered by various top-tier banks in London to each other, based on their outlooks on economic conditions for different loan maturities. LIBOR has served as a benchmark for setting interest rates for adjustable-rate loans, mortgages, and corporate bonds.

Short rate models and HJM models deal with instantaneous rates (which hold over infinitesimal periods). These models are easy to handle mathematically. However, they are mathematical artifacts that are not observable in real life, therefore, they are not easy to estimate. Furthermore, it is generally not easy to calibrate volatility under these models to cap or swaption data.

It may be better to model directly market observable/quoted rates (e.g. LIBOR rates and swap rates). According to Ouwehand (2021), the market practice had been to value caps/floors and swaptions using Black-76 models, which Black obtained by first assuming that the random variable in the payoff (LIBOR or swap rate) is lognormal under the risk-neutral measure. From this, one can discount outside the risk-neutral expectation, that is, we essentially assume that the bank account and the LIBOR/swap rate are uncorrelated. Then finally, one can approximate the risk-neutral expectation of the LIBOR/swap rate by the corresponding forward rate.

[2] However, we have finitely many noise sources, so the SDEs are manageable.

Unfortunately, these prices are inconsistent with those obtained in the short rate and HJM models. Furthermore, in contrast to the HJM case, it is not possible under the LIBOR model to ensure that the discretised system is arbitrage-free by attempting to derive an appropriate drift that might resemble the HJM drift conditions.

1.3 Dissertation Outline

The outline of this dissertation is as follows. Chapter 2 provides the model specification and how the simulation of the LIBOR forward rates is undertaken. Chapter 3 details how the LIBOR forward rates are applied to swaptions and explains how the pricing and hedging of swaptions is conducted under the Black-76 and LIBOR methods, respectively. Chapter 4 provides the results and Chapter 5 concludes.

Chapter 2

LIBOR MARKET MODEL

This chapter provides some background about swaps and swaption pricing in general.

2.1 Model Specification

We now discuss the notation and framework in which we work. Consider an equispaced tenor structure defined by

$$T_i = T_0 + i\delta \text{ for } i = 1, 2, \ldots, n,$$

where $\delta := \delta_i = T_i - T_{i-1}$ is the accrual period (or year fraction) and $T_0 = 0$. The time-t price of a zero-coupon bond paying a unit of currency at maturity T_i is denoted by $P_i(t) := P(t, T_i)$.

The LIBOR forward rate $F_i(t) := F(t; T_{i-1}, T_i)$ is defined as the simple forward interest rate between tenor dates T_{i-1} and T_i, and is related to the zero coupon bonds by

$$F_i(t) = \frac{1}{\delta_i}\left(\frac{P_{i-1}(t) - P_i(t)}{P_i(t)}\right). \tag{2.1}$$

Following Robbertze (2021), in Figure 2.1, we illustrate these forward LIBOR rates as follows, at time $t = 0$, all other rates are "alive", except $F_1(0)$, which is fixed. This is depicted by Figure 2.1.

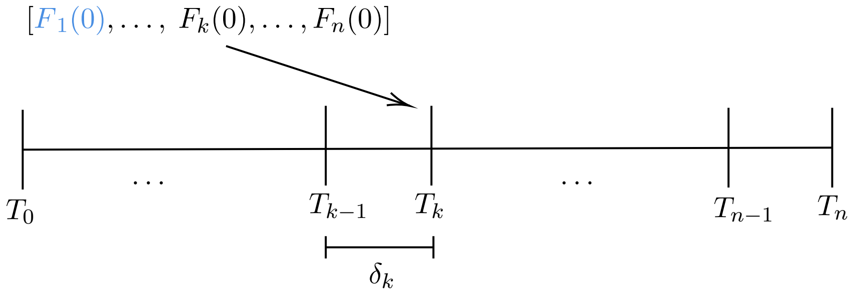

Fig. 2.1: Forward rates, $F_i(0)$, visualisation.

At any other time-t between two tenor dates, T_{k-1} and T_k, only rates after and excluding F_k are "alive", with $F_k(t)$ and those before being fixed. This is depicted in Figure 2.2.

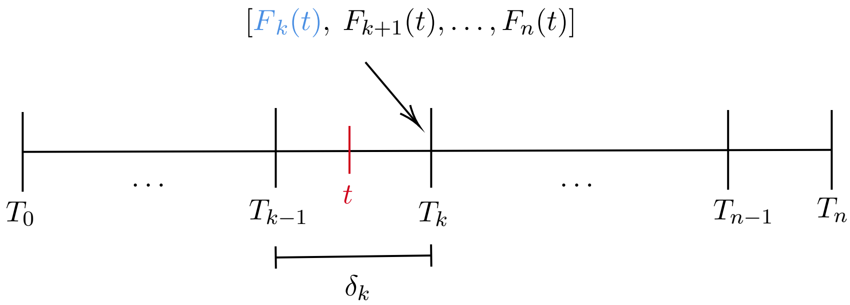

Fig. 2.2: Forward rates, $F_i(t)$, visualisation.

Now consider the asset $H_i(t) = \frac{1}{\delta}(P_{i-1}(t) - P_i(t))$ and $T > t$ being one of the tenors T_j. Using $P_i(t)$ as the numeraire, we have that

$$F_i(t) = \frac{H_i(t)}{P_i(t)} = \mathbb{E}^i \left(\frac{H_i(T)}{P_i(T)} \middle| \mathcal{F}_t \right) = \mathbb{E}^i \left(F_i(T_{i-1}) | \mathcal{F}_t \right), \quad \text{for } i > k,$$

where the ELMM (equivalent local martingale measure) associated with the numeraire $P_i(t)$ is $\mathbb{P}^i$ with Brownian motion W_i. Thus, under $\mathbb{P}^i$, the LIBOR forward rate process F_i is a martingale.

Define $\eta(t) := \min\{k : T_k \geq t\}$ to be the index of the most recently reset rate. From this, in this dissertation, we shall model the evolution of the LIBOR forward rates using a multi-dimensional process whose SDE takes the following form:

$$dF_i(t) = \mu_i(t)F_i(t)dt + \sigma_i(t)F_i(t)dW_i(t), \tag{2.2}$$

for $i = \eta(t) + 1, \ldots, n$, where σ_i represents the instantaneous volatility of the i^{th} forward rate independently of the other $n-1$ forward rates and $d\langle W_i, W_j \rangle := \rho_{ij}$ is the instantaneous correlation between these Brownian motions. Hence, the model form under consideration is Geometric Brownian Motion (GBM).

REMARK 2.1.1. In the Monte Carlo simulations, it is imperative that all LIBOR forward rates[1] $F_2, F_3, \ldots, F_n$ are specified under one pricing measure. In the next section, we show how this can be accomplished.

2.2 LIBOR Model under the spot measure

In this section, we demonstrate how all the LIBOR forward rates $F_2, F_3, \ldots, F_n$ can be specified under one pricing measure, as hinted by Remark 2.1.1. The content in

[1] Recall that F_1 is fixed.

this section follows largely from McWalter (2021) lecture 10.

The relationship between the LIBOR forward rates and the bond prices in (2.1) can be inverted to give the bond prices at the tenor dates T_i in terms of the LIBOR forward rates as follows:

$$P_k(T_i) = \prod_{j=i+1}^{k} \frac{1}{1+\delta_j F_j(T_i)},$$

for $k = i+1, \dots, n$.

Notice that at any other time-t between two tenor dates, T_{i-1} and T_i, the index $\eta(t)+1$ is that of the first forward rate that has not yet expired, thus we have

$$P_k(t) = P_{\eta(t)}(t) \prod_{j=\eta(t)+1}^{k} \frac{1}{1+\delta_j F_j(t)}, \tag{2.3}$$

for $k = \eta(t), \dots, n$ (where we define the empty product $\prod_{j=\eta(t)+1}^{\eta(t)} = 1$).

We now show how we can model all the LIBOR forward rates in (2.2) under the spot measure. Notice that this is a lognormal model, thus for our model to be arbitrage-free, we require the deflated (discounted) bond prices to be martingales under the risk-neutral measure. Thus, we need to derive the appropriate $\mu_i(t)$ to satisfy this requirement. To achieve this, we require a simply compounded numeraire asset since all the forward rates are specified using simple compounding.

We create such an asset as follows: let its initial value be 1 at time $t = T_0 = 0$, which is then invested in $P(t, T_1)$. At T_1, the payoff of these bonds is reinvested in $P(t, T_2)$ and at T_2, the payoff is reinvested in $P(t, T_3)$, etc. Thus at time-t, the asset will have a value of

$$\beta(t) = P_{\eta(t)}(t) \prod_{j=1}^{\eta(t)} (1+\delta_j F_j(T_{j-1})). \tag{2.4}$$

Note that $\delta_j F_j(T_j)$ is the simple interest earned over the accrual period $[T_{j-1}, T_j]$. The product represents interest earned up to time $T_{\eta(t)} > t$, and must therefore be deflated (discounted) to time-t by multiplying it by the current value of the shortest dated bond $P_{\eta(t)}(t)$.

To achieve the arbitrage-free model requirement, the deflated bond prices given by

$$D_k(t) = \frac{P_k(t)}{\beta(t)},$$

must be martingales (under the ELMM associated with numeraire $\beta(t)$). The *spot LIBOR measure* is obtained by taking $\beta(t)$ as the numeraire.

From (2.3) and (2.4), we immediately get that

$$D_k(t) = \left(\prod_{j=\eta(t)+1}^{k} \frac{1}{1+\delta_j F_j(t)} \right) \prod_{j=1}^{\eta(t)} \frac{1}{1+\delta_j F_j(T_{j-1})}, \tag{2.5}$$

which appropriately cancels the $P_{\eta(t)}(t)$ factor, thus allowing $D_k(t)$ to be exclusively specified by the LIBOR forward rates.

Since under the spot LIBOR measure $D_k(t)$ is a (positive) martingale, we may assume that it has the following dynamics

$$dD_k(t) = D_k(t)\nu_k(t)dW_k(t), \tag{2.6}$$

for $k = \eta(t), \ldots, n$ and some process ν_k.

Since $F_1(T_0)$ is fixed (see Figure 2.1), then $D_1(t) = (1+\delta_1 F_1(T_0))^{-1}$ is constant. Also, from (2.5), we have the recursive relation

$$D_{k+1}(t) = D_k(t)\frac{1}{1+\delta_{k+1}F_{k+1}(t)}.$$

Thus, $\delta_{k+1}F_{k+1}(t)D_{k+1}(t) = D_k(t) - D_{k+1}(t)$.

Now, since $D_1(t)$ is a (constant) martingale, we proceed inductively as follows: given that $D_k(t)$ is a martingale, for $D_{k+1}(t)$ to be a martingale, we must have that $F_{k+1}(t)D_{k+1}(t)$ is a martingale. By the (multi-dimensional) Ito formula,

$$\begin{aligned} d(F_{k+1}(t)D_{k+1}(t)) = F_{k+1}(t)D_{k+1}(t)\big[\, &(\mu_{k+1}(t) + \nu_{k+1}\sigma_{k+1}(t))\,dt \\ &+ (\sigma_{k+1} + \nu_{k+1})\,dW_{k+1}(t)\big]. \end{aligned}$$

Thus by the martingale property, we immediately see that

$$\mu_{k+1}(t) = -\nu_{k+1}\sigma_{k+1}. \tag{2.7}$$

We thus need to find an explicit form for ν_k. To do this, we apply the Ito formula to $\log(D_k(t))$ (see (2.6)) to get

$$d\log(D_k(t)) = -\frac{1}{2}\left(\nu_k(t)\right)^2 dt + \nu_k dW_k(t). \tag{2.8}$$

So by taking logs of both sides of (2.5), we have

$$\log(D_k(t)) = -\sum_{j=\eta(t)+1}^{k} \log(1+\delta_j F_j(t)) - \sum_{j=1}^{\eta(t)} \log(1+\delta_j F_j(T_{j-1})). \tag{2.9}$$

Notice that the term on the right of the RHS of (2.9) is a constant at t since the forwards have reset and therefore are constants, thus

$$\begin{aligned} d\log(D_k(t)) &= - \sum_{j=\eta(t)+1}^{k} d\log(1+\delta_j F_j(t)) \\ &= - \sum_{j=\eta(t)+1}^{k} \left[\left(\frac{\delta_j F_j(t)\mu_j(t)}{1+\delta_j F_j(t)} - \frac{1}{2}\left(\frac{\delta_j F_j(t)\sigma_j(t)}{1+\delta_j F_j(t)}\right)^2 \right) dt \right. \\ &\quad \left. + \frac{\delta_j F_j(t)\sigma_j(t)}{1+\delta_j F_j(t)} dW_j(t) \right], \end{aligned} \tag{2.10}$$

where the last equality follows from using (2.2) and the Ito formula.

By comparing (2.8) and (2.10), we immediately see that

$$\nu_k(t)dW_k(t) = - \sum_{j=\eta(t)+1}^{k} \frac{\delta_j F_j(t)\sigma_j(t)}{1+\delta_j F_j(t)} dW_j(t). \tag{2.11}$$

By comparing the quadratic variation/covariation processes of the LHS and each term in the sum on the RHS of (2.11) with $dW_k(t)$ and using the fact that $\rho_{kk} = 1$, and $d\langle W_i, W_j\rangle = \rho_{ij}dt$, we have

$$\begin{aligned} \nu_k(t)dt &= \nu_k(t)d\langle W_k, W_k\rangle \\ &= - \sum_{j=\eta(t)+1}^{k} \frac{\delta_j F_j(t)\sigma_j(t)}{1+\delta_j F_j(t)} d\langle W_k, W_j\rangle \\ &= - \sum_{j=\eta(t)+1}^{k} \frac{\delta_j F_j(t)\sigma_j(t)}{1+\delta_j F_j(t)} \rho_{kj} dt. \end{aligned}$$

Thus

$$\nu_k(t) = - \sum_{j=\eta(t)+1}^{k} \frac{\delta_j F_j(t)\sigma_j(t)}{1+\delta_j F_j(t)} \rho_{kj}. \tag{2.12}$$

Using (2.12) and (2.7) we get that the SDE for the LIBOR forward rates in (2.2) may be written as

$$dF_i(t) = F_i(t) \sum_{j=\eta(t)+1}^{i} \frac{\delta_j F_j(t)}{1+\delta_j F_j(t)} \sigma_i(t)\sigma_j(t)\rho_{ij} dt + F_i(t)\sigma_i(t)dW_i(t), \tag{2.13}$$

for $i = \eta(t)+1, \ldots, n$.

2.2.1 Simulation

We now look at how to generate the LIBOR forward rates in (2.13) using a discrete time grid given by $0 = T_0 = t_0 < t_1 < \cdots < t_m = T_n$, which includes the tenors (T_i)

and times (t_i) that we are interested in. Thus, we do not necessarily assume that $t_i = T_i$ for all the update points t_i, thus we define $\gamma := \gamma_i = t_i - t_{i-1}$.

We consider a sufficiently smooth volatility function that represents mild market conditions. This volatility function resembles the first volatility structure in Dun *et al.* (2001), which they constructed from UK market data from July 1994 to September 1997 following the work by Dun (2000) and Brace (1997). To achieve this, we use the popular instantaneous volatility parameterisation given by

$$\sigma_i(t) = [a + b(T_i - t)]e^{-c(T_i - t)} + d, \tag{2.14}$$

for some constants a, b, c, and d which influence the shape of the volatility function.

Furthermore, we make the piecewise constant assumption that at time-t_i in the period $[T_{j-1}, T_j]$ the volatility is constant and is defined as

$$\sigma_{ij}^2 := \frac{1}{\delta_j} \int_{T_{j-1}}^{T_j} \sigma_i^2(t)dt, \tag{2.15}$$

for $j = \eta(t_i) + 1, \eta(t_i) + 2, \ldots, n$ and zero otherwise. Before we proceed, let us visualise the specification of (2.15) in Figure 2.3.

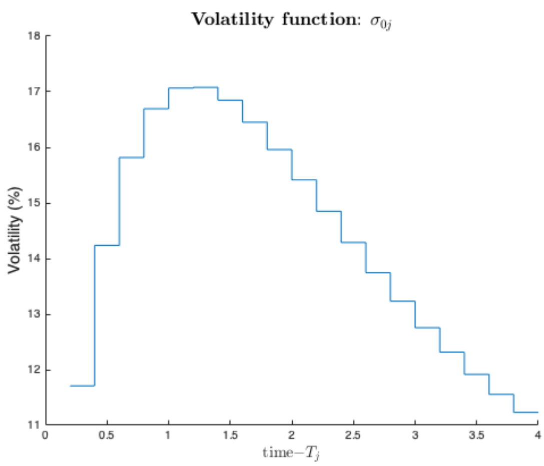

Fig. 2.3: Piecewise constant volatility.

Here we have considered the tenor structure where $\delta = 0.2$, $T_0 = 0$ and $T_n = 4$ and thus we only have non-zero values for the interval $[\delta, T_n]$ at time $t = 0$. Furthermore, we let $a = 0.01, b = 0.2, c = 0.95$, and $d = 0.09$.

Also, one must decide on how to generate correlated random variables. Thus, we now discuss the structure of the correlation matrix which follows from Brigo and Mercurio (2006) Section 6.9.

The correlation matrix $\rho^{(r)}$ has been chosen to be the rank-r reduced version of the full rank Schoenmakers and Coffey (2003) formulation, ρ^{SC}, whose elements are given by

$$\rho^{\text{SC}} = \exp\left[-\frac{|i-j|}{n-2}\left(-\log(\rho_\infty) + \zeta\frac{n-i-j}{n-3}\right)\right],$$

where r represents the number of factors for the model, $0 \leq \zeta \leq -\log(\rho_\infty)$ and $2 \leq i, j \leq n$ are the tenor indices[2]. Notice that the number of factors can be as large as there are tenors (i.e., $r \leq n-1$). Thus r represents the *rank* of the correlation matrix $\rho^{(r)}$.

Since ρ^{SC} is a positive definite symmetric matrix, then we can express its spectral decomposition as follows,

$$\rho^{\text{SC}} = VDV',$$

where V is a real orthogonal matrix (i.e., $V'V = VV' = I$) whose columns are the eigenvectors of ρ^{SC}, and D is a diagonal matrix of the associated positive eigenvalues of ρ^{SC} located in the corresponding position to the columns of V.

Now let $\Gamma := V\sqrt{D}$ so that we have both

$$\rho^{\text{SC}} = \Gamma\Gamma', \text{ and } \Gamma'\Gamma = D.$$

We achieve the rank-r reduced version of ρ^{SC} by using the approach of keeping the largest r eigenvalues of D and shrinking the matrix V by keeping the corresponding eigenvectors. That is, we define the diagonal matrix $D^{(r)}$ of size $(r \times r)$ by keeping the largest r diagonal terms of D and define $V^{(r)}$ by keeping the corresponding eigenvectors.

Now if we define $\Gamma^{(r)} := V^{(r)}\sqrt{D^{(r)}}$, then we would have that the *candidate* correlation matrix should be

$$\bar{\rho}^{(r)} = \Gamma^{(r)}(\Gamma^{(r)})'.$$

The problem here is that, in general, even though $\bar{\rho}^{(r)}$ is positive semi-definite, it does not necessarily have ones in the diagonal. Thus throwing away some eigenvalues from D alters the diagonal. We therefore interpret $\bar{\rho}^{(r)}$ as a *covariance* matrix, say $\Lambda^{(r)}$, and hence the solution is to set

$$\Lambda^{(r)} = \left(\lambda_{i,j}^{(r)}\right) := \Gamma^{(r)}(\Gamma^{(r)})'. \tag{2.16}$$

It immediately follows that the associated rank-r *correlation* matrix can be defined as follows,

$$\rho^{(r)} = \left(\rho_{i,j}^{(r)}\right) := \frac{\lambda_{i,j}^{(r)}}{\sqrt{\lambda_{i,i}^{(r)}\lambda_{j,j}^{(r)}}},$$

[2] Recall again that F_1 is fixed.

for $2 \leq i, j \leq n$. Since $i, j \neq 1$, this means that the element $\rho_{i,j}^{(r)}$ sits in position $\{i-1, j-1\}$ in the matrix $\rho^{(r)}$. The same is true for $\Lambda^{(r)}$.

From the resultant correlation $\rho^{(r)}$, we construct the matrix $\hat{\Gamma}^{(r)} = \hat{V}^{(r)}\sqrt{\hat{D}^{(r)}}$ by performing an eigendecomposition of $\rho^{(r)}$, to get the matrix of eigenvectors $\hat{V}^{(r)}$ and the diagonal matrix $\hat{D}^{(r)}$ of corresponding eigenvalues. Notice now that

$$\rho^{(r)} = \hat{\Gamma}^{(r)}(\hat{\Gamma}^{(r)})'.$$

So, if Z, of size $(r \times 1)$ is $N(0, I)$ where I is a $(r \times r)$ identity matrix then $Y = \hat{\Gamma}^{(r)} Z \sim N(0, \hat{\Gamma}^{(r)} I (\hat{\Gamma}^{(r)})')$. Thus, $Y \sim N(0, \rho^{(r)})$ as required[3].

Once the decision on how to generate correlated random variables has been made, one can proceed as follows, noting that $\hat{F}_i$ represents the approximated LIBOR forward rates. Initialise the LIBOR rates as $\hat{F}_i(t_0) = F_i(t_0)$ for $i = 1, 2, \ldots, n$. Here $F_i(t_0)$ is the initial term structure, which can be calibrated using the bond prices in the market or specified.

Some approximation schemes that can be considered for generating the Ito process in (2.13) could be the Euler-Maruyama method or the Milstein method. The Euler-Maruyama method would proceed as follows,

$$\hat{F}_i(t_k) = \hat{F}_i(t_{k-1}) + \hat{\mu}_i(t_{k-1})\hat{F}_i(t_{k-1})\gamma_k + \sqrt{\gamma_k}\sigma_i(t_k)\hat{F}_i(t_{k-1})Y_i,$$

for $i = \eta(t_k) + 1, \ldots, n$ and we would only need to try and construct $\hat{\mu}_i$ to ensure that the discounted bonds are martingales, just like the HJM case. Unfortunately, this is not possible.

We instead proceed as follows; working with the log of the process, we use the more accurate discretisation

$$\hat{F}_i(t_k) = \hat{F}_i(t_{k-1})\exp\left(\left(\hat{\mu}_i(t_{k-1}) - \frac{1}{2}\sigma_i^2(t_k)\right)\gamma_k + \sqrt{\gamma_k}\sigma_i(t_k)Y_i\right),$$

where

$$\hat{\mu}_i(t_{k-1}) = \sum_{j=\eta(t_k)+1}^{i} \frac{\delta_j \hat{F}_j(t_{k-1})}{1 + \delta_j \hat{F}_j(t_{k-1})} \sigma_i(t_{k-1})\sigma_j(t_{k-1})\rho_{ij}, \tag{2.17}$$

for $i = \eta(t_k) + 1, \ldots, n$, where $Y_i \sim N(0, \rho)$. Notice that this is the exact solution for GBM, where we have fixed the drift over the period $[t_{k-1}, t_k]$ to be the initial drift.

2.2.2 Implementation

We now provide pseudo-code for implementing the above simulation. We shall consider a single LIBOR forward path in an array called F, which contains evolved

[3] We may drop the superscript (r) on $\rho^{(r)}$ and simply write ρ when it is clear which rank we are referring to or when the rank chosen does not affect the results. The same is true for $\Lambda^{(r)}$.

rates. To simplify the implementation further, we assume that $t_i = T_i$, that is, we only evolve the forward rates on the tenor dates.

The algorithm also incorporates functionality for pricing a derivative H with maturity $T_\alpha \leq T_m$ which produces cash flows x_i at time-t_i. Note that T_α is a tenor date. Now for a single path, proceed as follows:

1. Initialise $F_i = F_i(t_0)$ for $i = 1, 2, \ldots, n$, either by using (2.1) and the observed bonds or specified some other way.

2. Set $\beta = 1$.

3. For $k = 1, \ldots, \alpha$ perform the following,

   (a) Update the numeraire by setting $\beta = \beta(1 + \gamma_k F_{k-1})$.

   (b) If $k < \alpha$ perform the following:

      i. Compute $\mu_i = \hat{\mu}_i(t_{k-1})$, for $i = k+1, \ldots, n$, using (2.17).

      ii. Generate $[Y_i] = Y \sim N(0, \rho)$ for $i = k+1, \ldots, n$.

      iii. Set $F_i = F_i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2(t_k)\right)\gamma_k + \sqrt{\gamma_k}\sigma_i(t_k)Y_i\right)$ for $i = k+1, \ldots, n$.

   (c) Discount cash flows c_k using $\frac{1}{\beta}$, (c_k may depend on updated forward rates).

4. Return H, as the sum of the discounted cash flows.

Notice that we are keeping a single version of the forward rates rather than storing the evolution through time. Thus, the values stored in F just before the execution of step 3(b)iii, are $F_i = \hat{F}_i(t_{k-1})$ for $k < i$.

Immediately after execution of step 3(b)iii, F stores the realised simple forward rates, $F_l = \hat{F}_l(t_{l-1})$ for $l \leq k$, and the newly evolved forward rates, $F_i = \hat{F}_i(t_k)$ for $i > k$, which are still alive at time-t_i.

2.2.3 Predictor-Corrector

In the log-discretisation proposed above, discretisation error arises because the drifts are state-dependent. In particular, the evolution is implemented assuming that the drift is constant and equal to the drift at the beginning of the period.

As indicated earlier, the drifts are sufficiently complicated such that a definitive solution to the LIBOR forward rate SDE cannot be obtained. The approach proposed by Hunter *et al.* (2001) allows us to produce a more accurate estimate of the drift required over an update period.

The idea is to evolve the rates to the end of the period and then compute the terminal drift using the evolved rates. Then, using the same random variates used to estimate the terminal drift, the initial rates are evolved using a drift computed as the average of the initial and terminal drift.

To implement this approach, one must replace step 3(b)i and 3(b)iii in the algorithm shown in Section 2.2.2 in the following manner: step 3(b)i gets replaced by

1. Compute

$$\tilde{F}_i(t_k) = F_i(t_{k-1})\exp\left(\left(\mu_i^{\text{init}}(t_{k-1}) - \frac{1}{2}\sigma_i^2(t_k)\right)\gamma_k + \sqrt{\gamma_k}\sigma_i(t_k)Y_i\right),$$

where

$$\mu_i^{\text{init}}(t_{k-1}) = \sum_{j=\eta(t_k)+1}^{i} \frac{\delta_j F_j(t_{k-1})}{1+\delta_j F_j(t_{k-1})}\sigma_i(t_{k-1})\sigma_j(t_{k-1})\rho_{ij},$$

for $i = \eta(t_k)+1, \ldots, n$, where $Y_i \sim N(0, \rho)$. Note that $\eta(t_k) = k$.

2. Using the intermediate values $\tilde{F}_i(t_k)$, compute

$$\mu_i^{\text{term}}(t_{k-1}) = \sum_{j=\eta(t_k)+1}^{i} \frac{\delta_j \tilde{F}_j(t_{k-1})}{1+\delta_j \tilde{F}_j(t_{k-1})}\sigma_i(t_{k-1})\sigma_j(t_{k-1})\rho_{ij}.$$

Step 3(b)iii gets replaced by

1. Compute the new rates

$$F_i(t_k) = F_i(t_{k-1})\exp\left(\frac{1}{2}\left(\mu_i^{\text{init}}(t_{k-1}) + \mu_i^{\text{term}}(t_{k-1}) - \sigma_i^2(t_k)\right)\gamma_k + \sqrt{\gamma_k}\sigma_i(t_k)Y_i\right),$$

where the Y_i's are the same normal random realizations used to compute $\tilde{F}_i(t_k)$.

To assess the effectiveness of the two algorithms, we show the recovery of the initial term structure by pricing zero-coupon bonds for various maturities. The simulated values are compared with the bond prices used to generate the initial LIBOR forward rates. Here, for simplicity, we have decide to keep the volatility flat across all tenors by simply setting $\sigma_i(t) = x\%$ $(x \in [0, 100])$. Everywhere else, we use the above parameterisation in (2.14) to conduct our analysis.

We use the closed form Vasicek short rate model to generate the zero coupon bond prices with the following parameters[4],

$$r_0 = 0.07, \qquad \alpha = 0.15, \qquad b = 9\%, \qquad \sigma_v = 2\%,$$

[4] Here we are not implementing the Vasicek short rate model but we are using it as input rates for the initial term structure. We could have used any term structure or taken rates directly from the market.

for maturities at two-year intervals out to 40 years.

The bond prices are simulated using the standard algorithm in Section 2.2.2 and the Predictor-Corrector method, where they are respectively computed as the mean of the corresponding deflator, $\frac{1}{\beta}$, at each time step, using $100\,000$ sample paths. We have kept the volatility for all time points fixed at $\sigma_i(t_k) = 20\%$.

Using these simulated bond prices, we compute the recovered forward rates using (2.1) and compare these with the input values.

Figure 2.4 shows that both algorithms can adequately recover the input bond prices. However, Figure 2.5 demonstrates that the original algorithm displays a long-term bias as a result of cumulative discretisation error, while the Predictor-Corrector method gives accurate results (within the Monte-Carlo sample error).

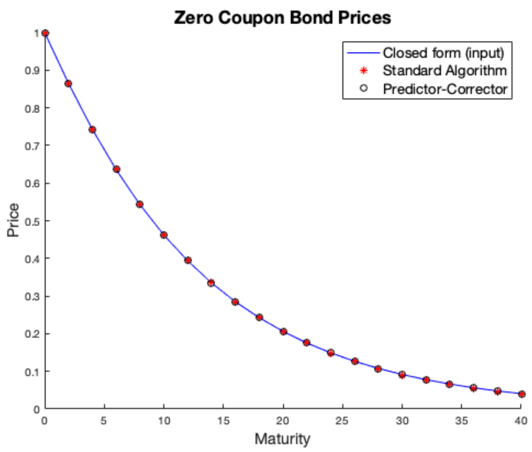

Fig. 2.4: Recovered ZCB prices.

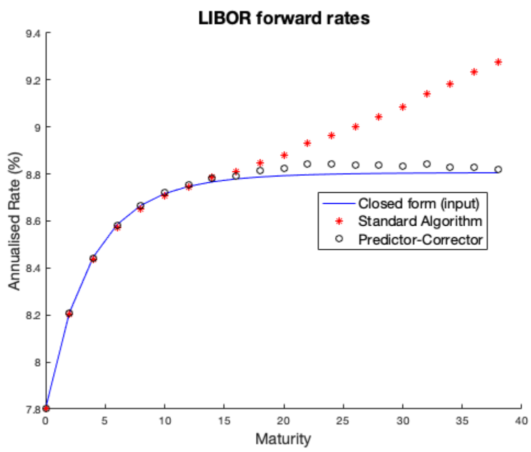

Fig. 2.5: Recovered forward rates.

Chapter 3

APPLICATION TO SWAPTION PRICING

Generally, swaptions (as well as caps and floors) are quoted using the Black-76 volatilities in financial markets. The Black-76 formula assumes, among other things, that the distribution of an asset at maturity is lognormal under a specific risk-neutral measure. As a result, a standardised formula can be used for pricing options.

Since a swaption is an option to enter into a swap, we begin by discussing a swap instrument. A swap is a contract whereby two parties agree to exchange different types of cash flows in the future. In particular, an interest rate swap exchanges a fixed interest rate for a floating rate of interest.

The holder of a *payer* swap with tenor $T_\alpha \times (T_\beta - T_\alpha)$ (equivalently, $T_\alpha \backslash T_\beta$) will receive the floating leg and pay the fixed leg at dates $T_{\alpha+1}, T_{\alpha+2}, \ldots, T_\beta$. In a *receiver* swap, payments move in the opposite direction.

3.1 Valuing Payer Swap

In a $T_\alpha \times (T_\beta - T_\alpha)$ payer swap, payments are made as follows:

- Payments will be made at times $T_{\alpha+1}, T_{\alpha+2}, \ldots, T_\beta$.
- For each period $[T_i, T_{i+1}]$, $i = \alpha, \ldots, \beta - 1$, the LIBOR forward rate $F_{i+1}(T_i)$ is fixed at time T_i and the floating leg $\delta_{i+1} F_{i+1}(T_i)$ is received at time T_{i+1}.
- For the same period the fixed leg $\delta_{i+1} K$ is paid at time T_{i+1}.
- Thus, the cash-flow at time T_{i+1} is $\delta_{i+1}(F_{i+1}(T_i) - K)$. This is the payoff of a FRN (floating rate note) and thus has a time-t value given by

$$\delta_{i+1} P_{i+1}(t)(F_{i+1}(t) - K).$$

Hence, the time-t value, $t \leq T_\alpha$, for the $T_\alpha \times (T_\beta - T_\alpha)$ payer swap is given by

$$\text{Pswap}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)(F_i(t) - K). \tag{3.1}$$

Alternatively, a payer swap is equivalent to a position that is long a FRN paying LIBOR, and short a fixed coupon bond with a coupon rate K. Hence the time-t value of a payer swap can also be expressed as

$$\text{Pswap}(t) = P_\alpha(t) - \left(K \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t) + P_\beta(t) \right). \tag{3.2}$$

Swap rate

The par or forward swap rate $S_\alpha^\beta(t)$ of a $T_\alpha \times (T_\beta - T_\alpha)$ payer swap, is the value of K for which $\text{Pswap}(t) = 0$. Thus, from (3.1)

$$S_\alpha^\beta(t) = \frac{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t) F_i(t)}{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)}. \tag{3.3}$$

Similarly from (3.2), we get that

$$S_\alpha^\beta(t) = \frac{P_\alpha(t) - P_\beta(t)}{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)}. \tag{3.4}$$

From (3.3), we note that the swap rate is just a weighted average of LIBOR forward rates, that is

$$S_\alpha^\beta(t) = \sum_{i=\alpha+1}^{\beta} w_i F_i(t) \text{ where } w_i = \frac{\delta_i P_i(t)}{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)}.$$

Now that we have covered swaps, in the following section we discuss swaption instruments.

3.2 Swaption pricing and hedging

Swaptions are options that give the holder the right but not the obligation to enter into a particular interest rate swap at a pre-specified time in the future. As discussed earlier, market practice is to compute swaption prices using the Black-76 formula. These prices are quoted as implied Black volatilities in the market, which are usually for par swap rates.

A $T_\alpha \times (T_\beta - T_\alpha)$ European payer swaption with strike K is a contract which at the exercise date T_α gives the holder the right but not the obligation to enter into a $T_\alpha \times (T_\beta - T_\alpha)$ swap with a fixed swap rate K. A payer swaption is a T_α contingent claim defined as

$$\text{Pswpn}(T_\alpha) = \max\{\text{Swap}(T_\alpha), 0\} = \sum_{i=\alpha+1}^{\beta} \delta_i P_i(T_\alpha) \max\{S_\alpha^\beta(T_\alpha) - K, 0\}.$$

Thus, when expressed under the numeraire $\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)$, a payer swaption can be considered as a call option on $S_\alpha^\beta(t)$ with strike K.

3.2.1 Black-76

The Black-76 payer swaption price for a $T_\alpha \times (T_\beta - T_\alpha)$ swaption is given by

$$\text{Pswpn}_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t) \Big[S_\alpha^\beta(t)\Phi(d_1) - K\Phi(d_2) \Big], \tag{3.5}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\hat{\sigma}_{\alpha,\beta}^{\text{swap}} \sqrt{T_\alpha - t}} \left[\log\left(\frac{S_\alpha^\beta(t)}{K} \right) + \frac{1}{2} (\hat{\sigma}_{\alpha,\beta}^{\text{swap}})^2 (T_\alpha - t) \right], \\ d_2 &= d_1 - \hat{\sigma}_{\alpha,\beta}^{\text{swap}} \sqrt{T_\alpha - t}, \end{aligned}$$

and the constant $\hat{\sigma}_{\alpha,\beta}^{\text{swap}}$ is the *implied* Black volatility.

However, this price formula requires the assumption that $S_\alpha^\beta(t)$, under the measure $\mathbb{P}_\alpha^\beta$ corresponding to the numeraire $\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)$, assumes the following log-normal dynamics

$$dS_\alpha^\beta(t) = \hat{\sigma}_{\alpha,\beta}^{\text{swap}} S_\alpha^\beta(t) dW_\alpha^\beta(t).$$

Unfortunately, this formulation is not consistent with our formulation of log-normal forward rates which are used to imply the associated swap rates, thus, our formulation does not allow for log-normality of swap rates. Nonetheless, prices in the market are usually quoted as implied volatilities, which will prove helpful in the subsequent volatility approximation. To date, several volatility approximations exist, such as:

- The Rebonato formula; see Brigo and Mercurio (2006) proposition 6.15.1.
- The Hull–White formula; see Brigo and Mercurio (2006) proposition 6.15.2.

- Mean-path-updated formula; see Van Appel and McWalter (2020).
- The Kawai expansion; see Kawai (2003).

For more information on the Rebonato formuala, see Rebonato and Coakley (1998) and Rebonato (1999), on the Hull-White formula, see Hull and White (2000).

Fortunately, Van Appel and McWalter (2020) investigate the accuracy of these four volatility approximations and show that the Kawai expansion is the most accurate yet computationally inefficient. This approximation will be used in this dissertation to calibrate for $\hat{\sigma}^{\text{swap}}_{\alpha,\beta}$, however, we do not explore how it is derived. The mean-path-updated formula is offered to strike a good balance between accuracy and efficiency.

The Kawai expansion

Under this approximation, the squared Black swaption volatility is

$$(\hat{\sigma}^{\text{swap}}_{\alpha,\beta})^2 = \frac{\Sigma}{\left(S^{\beta}_{\alpha}(0)\right)^2 T_{\alpha}}\Bigg(1 + \left(\frac{1}{S^{\beta}_{\alpha}(0)} - 2c_1\right) g_0 + \left(c_2 + \frac{11}{12\left(S^{\beta}_{\alpha}(0)\right)^2} - \frac{2c_1}{S^{\beta}_{\alpha}(0)}\right) g_0^2 + \left(c_3 + \frac{1}{12\left(S^{\beta}_{\alpha}(0)\right)^2}\right)\Sigma\Bigg).$$

This expression is used in 3.5 for calculating the swaption price. For more information on the constants and derivatives, see Van Appel and McWalter (2018, 2020).

Hedging method

We use "The Underlying Swap Method" as our hedging strategy as described in Dun *et al.* (2001). These authors also provide two additional hedging strategies. However, all these strategies are *equivalent*[1]. Thus, the specific method used is immaterial.

Define the *present value of a basis point* (*PVBP*) of a swap to be the sum

$$\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t).$$

[1] Static portfolio arguments can be used to transform one hedge portfolio into another.

Now, note that (3.5) can be rewritten as follows:

$$\begin{aligned}\text{Pswpn}_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)\left[S_\alpha^\beta(t)\Phi(d_1) - K\Phi(d_2)\right] \\ &= \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)\left(S_\alpha^\beta(t) - K\right)\Phi(d_1) - K\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)(\Phi(d_2) - \Phi(d_1)) \\ &= \Phi(d_1)\text{Pswap}(t) - K(\Phi(d_2) - \Phi(d_1))\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t). \qquad (3.6)\end{aligned}$$

Thus from (3.6) we obtain the hedge to replicate the swaption, which entails going long $\Delta = \Phi(d_1)$ (*option delta*) units of the underlying swap, and shorting $K(\Phi(d_2)-\Phi(d_1))$ units in the *PVBP*. We further impose the self-financing condition on the hedge and hence all profits and losses are accumulated in the spot numeraire $\beta(t)$.

3.2.2 LIBOR

The LIBOR payer swaption price for a $T_\alpha \times (T_\beta - T_\alpha)$ swaption based on Dun *et al.* (2001) is given by

$$\text{Pswpn}_{\alpha,\beta}(t) = \sum_{j=\alpha+1}^{\beta} \delta_j P_j(t)\left[F_j(t)\Phi\left(h_j^{(r)}\right) - K\Phi\left(\bar{h}_j^{(r)}\right)\right],$$

where r is the rank of the forward LIBOR covariance matrix[2] $\Lambda^{(r)}$ in (2.16) and

$$\begin{aligned}h_j^{(r)} &= -\frac{s_1 + d_1^{(j)} - \Gamma_{j,1} - \sum_{l=2}^{r} s_l\left(d_l^{(j)} - \Gamma_{j,l}\right)}{\sqrt{1 + \sum_{l=2}^{r} s_l^2}}, \\ \bar{h}_j^{(r)} &= -\frac{s_1 + d_1^{(j)} - \sum_{l=2}^{r} s_l d_l^{(j)}}{\sqrt{1 + \sum_{l=2}^{r} s_l^2}}.\end{aligned}$$

Here we define the empty sum $\sum_{l=2}^{1} := 0$. Additionally,

$$d_l^{(j)} = \sum_{i=1}^{j} \frac{\delta_i F_i(t)}{1 + \delta_i F_i(t)}\Gamma_{i,l},$$

and the r-dimensional vector s is obtained by solving the highly nonlinear fixed point problem such that

$$\delta\sum_{j=1}^{n} \frac{F_j(t)\exp\left((\Gamma_{j,:})\cdot(s + d^{(j)}) - \frac{1}{2}||\Gamma_{j,:}||^2\right) - K}{\prod_{i=1}^{j}(1 + \delta F_i(t))\exp\left((\Gamma_{i,:})\cdot(s + d^{(i)}) - \frac{1}{2}||\Gamma_{i,:}||^2\right)} = 0, \qquad (3.7)$$

[2] Note that we have dropped the superscript (r) on $\Gamma^{(r)}$ and wrote Γ in the definition of $h_j^{(r)}$, $\bar{h}_j^{(r)}$, $d_l^{(j)}$ and in (3.7).

where $\cdot$ represents a dot product, $d^{(j)} = [d_1^{(j)}, \dots, d_r^{(j)}]$, and $\Gamma_{i,:}$ represents the i^{th} row of Γ.

In Brace *et al.* (1997), it is shown, based on real-world observations, that the largest eigenvalue of the covariance matrix dominates all subsequent eigenvalues, and thus a rank one approximation is adequate. This assertion, therefore, reduces the fixed point problem in (3.7) to one dimension. However, for extreme volatility scenarios, a rank one approximation may not be sufficient, and thus a rank two or more approximation should be used.

Theoretically, for the one-dimensional case, there exists a unique solution for s as shown in Brace *et al.* (1997). However, depending on the optimization scheme used and the starting point for s, one may arrive at different solutions for s and swaption prices. As such, and before any further analysis, we conjecture that this pricing method will not be favored by market practitioners over the Black-76 pricing method since the latter will always produce the same price as the one quoted in the market by using the market quoted implied Black volatilities, ceteris paribus.

A possible extension to this dissertation would be to investigate if a unique solution for s exists for dimensions of two or more and explore which optimisation scheme is best suited to solve (3.7) and determine if there are optimal starting points for s.

Hedging Method

Dun *et al.* (2001) provide two hedging methods, one being an approximate hedge proposed by Brace *et al.* (2001) and the other using numerical evaluation of the required partial derivatives by difference quotient. Even though the latter provides more accurate results[3], it is quite demanding both computationally and in terms of its practical implementation. Thus, we consider the former hedging method, which provides an approximate hedge. We shall therefore assess how good the resultant hedge is, compared to the Black hedge discussed in Section 3.2.1.

The nonlinear fixed point problem implies that the LIBOR hedge will not be consistent upon multiple runs (where we have fixed all inputs and have set the seed for the random number generator) in the sense that different s values can be found as solutions to (3.7) at each time point. Furthermore, the time 0 prices found using the Black-76 and LIBOR method will not necessarily be the same. However, they should be very close to each other. This implies that the initial values for the hedges will not be the same, and hence the two hedging strategies cannot be compared.

[3] Numerical evaluation of partial derivatives ensures that one does not have to worry about the nonlinear fixed point problem.

To solve this problem, we use the Black-76 price to calibrate for the appropriate s vector, which ensures that the LIBOR swaption price is the same as the Black-76 price. Consequently, this will enforce that the initial values for the hedges are the same. Adding the self-financing condition to both hedges implies that the two hedges can be compared against each other. Remember that the LIBOR hedge we propose below is only an approximation, as discussed earlier.

Dun *et al.* (2001) proposed constructing the approximate hedge by partitioning the value of the swaption into distinct hedges for each forward rate, with all profits and losses invested in the *PVBP*. Implementing this, we get that

$$\text{Pswpn}_{\alpha,\beta}^{\text{Calib}^{\text{n}}}(t) = \sum_{j=\alpha+1}^{\beta} \delta_j P_j(t) F_j(t) \Phi\left(h_j^{(r)}\right) + \theta \sum_{j=\alpha+1}^{\beta} \delta_j P_j(t), \tag{3.8}$$

where θ is chosen so that equality holds. Here, $\text{Pswpn}_{\alpha,\beta}^{\text{Calib}^{\text{n}}}(t)$ is effectively the Black-76 price, which we obtain from the LIBOR pricing formulation after calibrating for the s vector which makes the Black-76 and LIBOR price equal.

It is important to note that the $h_j^{(r)}$ used in (3.8) is calculated using the calibrated s value. Thus, one cannot simply replace $\text{Pswpn}_{\alpha,\beta}^{\text{Calib}^{\text{n}}}(t)$ with the Black-76 price and leave $h_j^{(r)}$ unchanged as this will lead to inconsistent results.

Chapter 4

RESULTS

In this chapter, we provide our findings to the research specification as discussed in Section 1.1.

Since the forward LIBOR model is automatically fitted to the initial term structure as an initial condition, we compute the bond prices $P_i(t)$ under the Vasicek model for all the $T_0, T_1, \ldots, T_n$ tenor dates using the parameters[1]

$$r_0 = 0.07, \qquad \alpha = 0.15, \qquad b = 9\%, \qquad \sigma_v = 2\%.$$

We then calibrate the discrete forward curve at inception $F_i(T_0)$, for $i = 1, 2, \ldots, n$, to these prices.

As discussed earlier, the simulation is conducted under the spot measure within the LIBOR model framework. In the log-discretisation of the forward rates, we implement the method proposed by Hunter *et al.* (2001) called the predictor-corrector method, which provides a more accurate drift term.

4.1 Black-76 vs LIBOR prices

Recall that the volatility function in Figure 2.3 represents typical market conditions, and thus a rank-1 approximation should be adequate. However, we have used a rank-2 approximation to demonstrate the robustness of the algorithm presented here and its generality to handle any valid rank approximation. Even though we have used a rank-2 approximation here, the rank-1 prices are precisely the same.

In Table 4.1 below, we present the swaption Black-76 and LIBOR prices for various swaptions. For comparability, we have defined an in-the-money (ITM) swaption to be at $K := 0.8 \cdot S_\alpha^\beta(0)$, at-the-money (ATM) $K := S_\alpha^\beta(0)$, and out-the-money

[1] Recall again that here we are not implementing the Vasicek short rate model but we are using it as input rates for the initial term structure. We could have used any term structure or taken rates directly from the market.

(OTM) $K := 1.2 \cdot S_{\alpha}^{\beta}(0)$. We have also fixed the starting point for s when solving (3.7).

Swaption maturity(yrs)	Swaption Strike	Black (Rank-2) price (%)	LIBOR (Rank-2) price (%)	Difference (bp)
0.25/2	IN	2.35	2.35	0.02
	AT	0.34	0.14	20.14
	OUT	0.001	0.000	-0.16
1/2	IN	1.35	1.35	0.00
	AT	0.41	0.41	0.00
	OUT	0.06	0.06	0.00
2/4	IN	2.44	2.61	17.14
	AT	1.03	0.71	-30.89
	OUT	0.30	0.19	-11.08

Tab. 4.1: Swaption prices, Black-76 vs LIBOR.

As we would expect, Table 4.1 reveals that ITM swaptions are relatively expensive and OTM swaptions are the cheapest in general. Furthermore, the Black-76 and LIBOR prices are close to each other, and in some instances, they are the same.

From Table 4.1 we see that the starting point for s was good enough for the 1yr/2yr and 0.25yr/2yr LIBOR prices to be close to the Black-76 prices. However, the 2yr/4yr swaption price differences are relatively large. This is due to the starting point for s being fixed throughout instead of being chosen optimally. We have done this to stress the importance of the starting point for s when employing the optimisation scheme. Thus, if we were to choose a different starting point for s, we might see a lower difference in the 2yr/4yr swaption prices but a significant difference in the other swaptions. Therefore, each swaption requires an appropriate starting point determined by calibration or by modifying the optimisation scheme options to improve results.

4.2 Black-76 vs LIBOR PnL

We simulate 1000 paths of the forward LIBORs and use these to construct the hedging strategies as described in Section 3.2. The terminal swaption payoffs are subtracted from the respective portfolio values to produce the total profit or loss at maturity. This, in turn, provides us with a measure of the hedging error.

Since we adjust the portfolio at discrete time points, there will always be hedging errors even if all model assumptions are fulfilled. Theoretically, one must adjust the portfolio continuously, which will, in turn, guarantee perfect self-financing and replication of the option. However, this is impossible in practice. If there are no other sources of misspecification, this error can be made arbitrarily small by increasing the hedging frequency/hedging points (HP).

Below in Figure 4.1 and Table 4.2 for Black-76, and Figure 4.2 and Table 4.3 for LIBOR, we provide the estimated hedging PnL as a function of the hedging frequency for an ATM 2yr/4yr swaption. We observe that the PnL for both Black-76 and LIBOR hedging strategies exhibit Gaussian characteristics with means centred around zero and decreasing standard deviations as the number of hedge points increases. It is also evident that the 95% confidence intervals for the means and standard deviations become narrower as the number of hedge points increases.

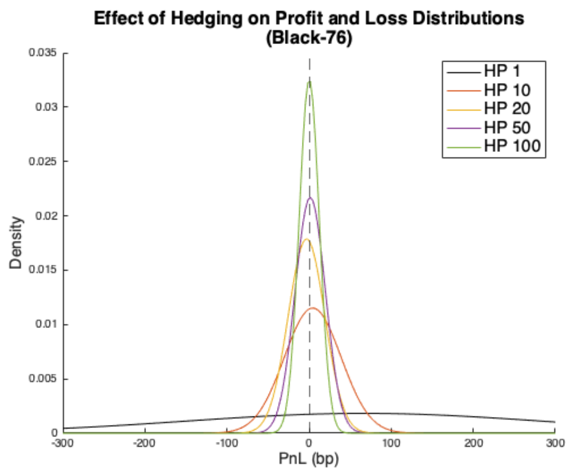

Fig. 4.1: Black PnL for an ATM 2yr/4yr swaption.

Rebalancing Frequency - Times until maturity	Hedging Profit and Loss (bp) Mean (95% Error)	Standard Deviation (95% Error)
1	42.7 (79.4)	212.6 (73.2)
10	4.4 (2.1)	34.6 (1.4)
20	-3.0 (1.3)	22.2 (0.9)
50	1.4 (1.1)	18.3 (0.7)
100	0.1 (0.7)	12.2 (0.5)

Tab. 4.2: Black ATM 2yr/4yr Swaption.

Table 4.2 shows us the exact values of the PnL in Figure 4.1. The same is true for Figure 4.2 and Table 4.3.

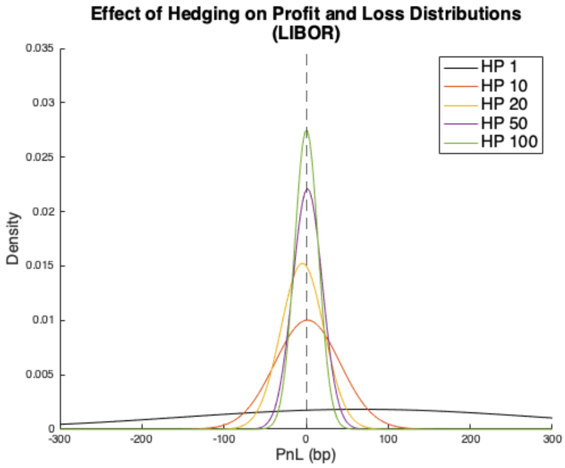

Fig. 4.2: LIBOR PnL for an ATM 2yr/4yr swaption.

Rebalancing Frequency (Times until Maturity)	Hedging Profit and Loss (bp) Mean (95% Error)	Standard Deviation (95% Error)
1	42.7 (79.4)	212.6 (73.2)
10	1.7 (2.4)	39.7 (1.6)
20	-4.5 (1.6)	26.1 (1.0)
50	1.7 (1.1)	18.0 (0.7)
100	0.4 (0.8)	14.4 (0.6)

Tab. 4.3: LIBOR ATM 2yr/4yr Swaption.

Our results agree with those in Dun *et al.* (2001), in the sense that the error bounds for the means and standard deviations decrease with an increasing number of hedge points and that the PnLs are centered around zero.

When comparing Figure 4.2 with Figure 4.1, it might seem as if the Black hedge is better than the LIBOR hedge; however, that is not the case. Recall that here we used an approximate LIBOR hedge, and the exact hedge would need to be constructed using numerical differentiation of the required partial derivatives. However, this approximate hedge is good enough for our analysis, and it is evident that both the Black and LIBOR hedges are equally effective in hedging swaption.

4.3 Hedging method comparison

In Figure 4.3 and Figure 4.4 we provide examples of the profiles produced by swap rates, swaption prices, and hedging portfolio values through time for a single realisation of the LIBOR forward rates.

We have considered the case where the swaption is re-balanced 20 times. A notable change in the swap rate is observed at each rehedging point, causing a shift in the swaption price. The value of the hedge portfolios follows that of the swaption price almost precisely.

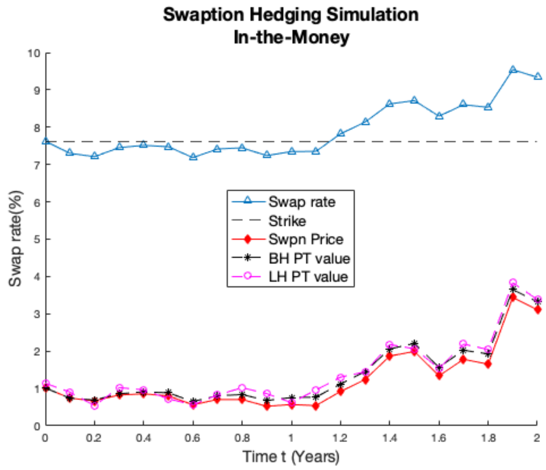

Fig. 4.3: Swaption hedging path for an ATM 2yr/4yr swaption which expires ITM.

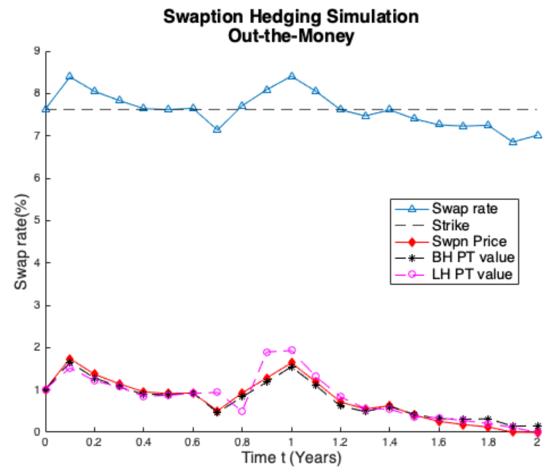

Fig. 4.4: Swaption hedging path for an ATM 2yr/4yr swaption which expires OTM.

We also notice that the two portfolio values lie almost perfectly on top of each other. A few points show more significant discrepancies because we have used an approximate LIBOR hedge here. However, this suggests that these hedging techniques are essentially equivalent. Fortunately, Dun *et al.* (2001) tests this assertion by considering 10000 paths and up to 640 rehedging points and shows that the Black hedge is not any "riskier" than the LIBOR hedge and hence the two are equivalent.

Chapter 5

Conclusion

We have implemented an algorithm for simulating the forward rates and have shown how one should update it to incorporate the predictor-corrector method, which provides a more accurate estimate of the drift term. To illustrate the improvement by the predictor-corrector method, we demonstrated the recovery of the initial term structure by pricing zero-coupon bonds for various maturities. This has ensured that the lognormal forward LIBOR framework is chosen to represent the ”true” model.

Our analysis reveals that the LIBOR and Black hedges are essentially equivalent. Hence, since the Black method is much more straightforward and well-understood by market practitioners, it should be used to price and hedge swaptions.

The pitfalls in the LIBOR price and hedge due to the nonlinear fixed point problem in (3.7) make the LIBOR approach unfavorable for use in the market as it heavily depends on the s parameter which is further dependent on the optimisation scheme used and the starting point for s. The computational overhead associated with the LIBOR implied swaption price and hedge further disadvantages it to the Black approach.

One may take the covariance matrix as given or calibrate it to cap/floor or swaption prices. Here we have chosen to use a rank-reduced version of the Schoenmakers and Coffey (2003) formulation, where we chose a rank-2 approximation which did not result in over-fitting as we realised stable swaption prices and hedging strategies. Dun *et al.* (2001) argue that using a higher rank for the covariance matrix may result in over-fitting and hence it is beneficial to keep the rank of the covariance matrix reasonably low.

A possible extension to this dissertation would be to look at the displaced lognormal forward LIBOR model (DLFM). Van Appel and McWalter (2020) present an algorithm to approximate moments for forward rates under a DLFM. Since the joint distribution of rates is unknown, they use a multi-dimensional full weak order-2

Ito-Taylor expansion in combination with a second-order Delta method. They then show that this accurately accounts for state dependence in the drift terms thus providing an improvement upon previous approaches.

Furthermore, as discussed briefly in the introduction, Lyashenko and Mercurio (2019) provide an extension to the LMM considered here which they call the FMM, and show that setting-in-arrears back-looking rates are a feasible replacement for IBORs from the analytics perspective. In addition to the martingale property under their corresponding forward measure, these rates have several other analytical properties that IBORs lack, such as a simple analytic formula for the drift under the risk-neural measure $\mathbb{Q}$.

Lastly, it is imperative to indicate that, using the FMM, an extension of the LMM, should have a fraction of the implementation cost of building the FMM from scratch since the general framework remains the same.

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