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Well-posedness And Long-time Dynamics Of  $\beta$ -plane  
Ageostrophic Flows

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## Abstract

# Well-posedness And Long-time Dynamics Of $\beta$ -plane Ageostrophic Flows

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We elucidate in a concrete way dynamical challenges concerning nonlinear analysis of dissipative  $\beta$ -plane ageostrophic flows. We employ the simplifying assumptions of the Boussinesq and hydrostatic approximations to the governing equations of non-barotropic viscous compressible fluid flows. As a result we obtain the dissipative ageostrophic equations which describe the motion of a viscous incompressible stratified fluid with Coriolis force. The problem of ageostrophic flows is encountered in many applications including modelling and forecasting of mesoscale and synoptic scale eddies. Furthermore, geostrophic and quasigeostrophic equations are derived from the dissipative ageostrophic equations where use is made of a perturbation analysis of the flow fields using the Rossby number. We prove the existence and uniqueness of solutions to the initial-boundary value problems corresponding to the dissipative  $\beta$ -plane ageostrophic flows and then establish attractors of solutions. We solve the problems utilizing energy stability, Gronwall's inequality, results about the Stokes problem and the technique of Faedo-Galerkin approximations. Another important aspect of our results is in the proof of existence of Lipschitz invariant manifolds for ageostrophic flows given by a Lipschitz function determined as a fixed point of a continuous mapping in Banach spaces. Our approach has been to renorm the Banach spaces and obtain stronger a priori estimates followed by an application of the contraction mapping principle of Banach-Cacciopoli. The invariant manifold yield long-time approximations for ageostrophic flows and this approach is taken because the ideas are applicable to problems other than viscous incompressible stratified fluid with Coriolis force. A major task ahead is to bring these techniques to the resolution of realistic atmosphere-ocean models including the extension of the theory to transient and aperiodic atmosphere-ocean phenomena. We conclude with a derivation and discussion of Melnikov integrals and other separation functions for aperiodic singularly perturbed dynamical system that plays a significant role in an analysis of  $\beta$ -plane ageostrophic and quasigeostrophic flows. Under nondegeneracy assumptions, we are able to obtain transversality of intersection of stable and unstable manifolds of invariant sets associated with orbits for ageostrophic and quasigeostrophic flows in terms of Melnikov functions.

## The Vita of Maleafisha Stephen Tladi

Maleafisha Stephen Tladi attended Clark University, Worcester where he earned his Bachelor of Arts, summa cum laude, with highest honors in Mathematics in 1992. He has been awarded teaching assistantships at Brown University, Providence where he obtained his Master of Science in Mechanical Engineering in 1996 and received his Master of Science in Applied Mathematics in 1993. During his studies at Brown University, Maleafisha had a privilege and opportunity to accept appointment at the University of Cape Town, first as a Junior Research Fellow and subsequently as an Assistant Lecturer in the Department of Mathematics and Applied Mathematics. At the University of Cape Town, Maleafisha designed and developed the curricula for the module Dynamical Systems and Bifurcations for students in the Department of Mathematics and Applied Mathematics which he successfully implemented and intimately taught from 2000 to 2003. In addition, he worked with Professor Daya Reddy on a project designated as "Energy theory in the stability of non-Newtonian fluid flows". Among professional memberships Maleafisha belongs to the Dynamical Systems Group of the Society for Industrial and Applied Mathematics fostering interaction between researchers and teachers with interests in dynamical systems theory and applications. Duly elected to SIGMA XI, The Scientific Research Society in 1994 and to The PHI BETA KAPPA Society in 1991, he actively participates and presents research work at the annual congress of the South African Mathematical Society and the annual meeting of the South African Society for Numerical and Applied Mathematics. Maleafisha traveled extensively in Newport, Rhode Island including Saratoga Springs, New York as well as Tanglewood, Massachusetts in pursuit of jazz concerts in celebration of the liberating power of culture. In 1994 he obtained a Teaching Certificate from Brown University's Center for the Advancement of College Teaching and he was admitted as a member of the American Institute of Aeronautics and Astronautics in 1998. He received the Mathematics Award in 1992 that serves to recognize excellence in the Department of Mathematics and Computer Science at Clark University during commencement. Maleafisha met his wife Raesibe Mojapelo in the Bible study conducted by the Deacon Daniel Mashao and they courted for about a year, and then they got married on 09 September 2000. They live with their son Makebe Thapelo Tladi and their daughter Seageng Natalia Tladi in Cape Town.

First, I dedicate the thesis with respect to the Cape Peninsula—the domain and its people—my shelter during my instruction from Junior Research Fellow to Assistant Lecturer in Department of Mathematics and Applied Mathematics at the University of Cape Town. In recognition of her generosity all along the unmarked way, the manuscript is dedicated to the memory of my mother, Lestrina Seageng Tladi of Chuenespoort, Pietersburg.

University of Cape Town

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# Contents

<b>1</b>	<b>Introduction</b>	<b>7</b>
1.1	Observations of mesoscale eddies . . . . .	7
1.2	Modelling of mesoscale eddies . . . . .	8
1.3	Prototype $\beta$ -plane ageostrophic models . . . . .	12
1.4	Limitations of standard techniques . . . . .	17
1.5	The goals of this thesis . . . . .	19
<b>2</b>	<b>Models for ageostrophic flows</b>	<b>28</b>
2.1	Description of ageostrophic equations . . . . .	28
2.1.1	$\beta$ -plane ageostrophic equations . . . . .	28
2.1.2	$\beta$ -plane ageostrophic equations with Reynolds stress . . . . .	34
2.2	The initial-boundary value problems . . . . .	38
2.2.1	$\beta$ -plane ageostrophic equations . . . . .	38
2.2.2	$\beta$ -plane ageostrophic equations with Reynolds stress . . . . .	41
<b>3</b>	<b>Nonlinear analysis preliminaries</b>	<b>45</b>
3.1	Elements of Hilbert spaces and Sobolev spaces . . . . .	45
3.2	Properties of some functionals and operators . . . . .	50
<b>4</b>	<b>Well-posedness of solutions</b>	<b>60</b>

4.1	Existence, uniqueness and differentiability of solutions . . . . .	60
4.1.1	$\beta$ -plane ageostrophic equations . . . . .	60
4.1.2	$\beta$ -plane ageostrophic equations with Reynolds stress . . . . .	73
<b>5</b>	<b>Nonlinear stability of solutions</b>	<b>86</b>
5.1	Aspects of stability and attraction for ageostrophic flows . . . . .	86
5.1.1	$\beta$ -plane ageostrophic equations . . . . .	87
5.1.2	$\beta$ -plane ageostrophic equations with Reynolds stress . . . . .	94
<b>6</b>	<b>Lipschitz invariant manifolds</b>	<b>98</b>
6.1	Lipschitz invariant manifolds for ageostrophic flows . . . . .	98
6.1.1	$\beta$ -plane ageostrophic equations . . . . .	99
6.1.2	$\beta$ -plane ageostrophic equations with Reynolds stress . . . . .	99
<b>7</b>	<b>Conclusions</b>	<b>110</b>
7.1	Discussion of results . . . . .	110
7.2	New research directions . . . . .	113
<b>A</b>	<b>Quasigeostrophic flows</b>	<b>116</b>
A.1	$\beta$ -plane quasigeostrophic equations . . . . .	116
A.2	$\beta$ -plane quasigeostrophic equations with Reynolds stress . . . . .	119
<b>B</b>	<b>Melnikov function for singularly perturbed dynamical systems</b>	<b>122</b>
B.1	Geometry of the phase space . . . . .	123
B.2	Splitting of the heteroclinic manifold . . . . .	124
B.3	Adiabatic transport and lobe area . . . . .	128

# Chapter 1

## Introduction

### 1.1 Observations of mesoscale eddies

Mesoscale eddies in marine science are of particular interest to meteorologists and oceanographers because satellite-tracked surface drifters and analysis of observed motion of floats suggest the structures are an important mechanism for transport of salinity, kinetic energy, available potential energy, and enstrophy, the latter being the integrated squared vorticity. Prototype problems include dynamics of oceanic vortices, atmospheric vortex blocking, rings of the Agulhas Current, and the Gulf Stream ring systems [21, 35, 34, 39, 31, 23, 53]. These studies suggest the predictability of the meandering eddying currents and of the system of mesoscale vortices and gyres accompanying them, on time scales which are known to have well-defined evolution, structure, and spillover effects ending with an overall spin-down of coherent structures.

To cite crucial examples of coherent structures, also called the internal weather of the ocean or the atmosphere, in Miller *et al.* [55, 56], a perturbed current evolves nonlinearly into a large-amplitude meandering configuration that propagates zonally. Although there are two dominant time scales, the flow is aperiodic and dissipative, characteristics that challenge standard techniques in numerically and experimentally generated vector fields. The meticulously detailed observations of Lutjeharms *et al.* [35, 34] reveal that heat and salinity exchange around the Agulhas Current Retroflection takes place through mesoscale ring detachment with an associated volume transport of approximately  $0.5 - 1.5 Sv$  ( $1 Sv = 10^6 m^3 s^{-1}$ ). The resulting transport within these currents is undisputedly complex as validated by observed motion of convoluted spaghetti floats called *SOFAR* and *RAFOS*.

In particular, it is evident from the investigations tackled in [39, 31, 23, 54] that heat and salinity exchange around the Middle Atlantic Bight occurs through the mesoscale ring detachment of the Gulf Stream. Investigations of the flow patterns of such upwelling fronts are therefore of concern for biological studies of the highly productive ecosystems where nutrient budgets play a significant role. Furthermore, understanding the dynamics of exchange processes by mesoscale eddies is essential so that their effects in shelf-water transport can be accounted for, as observed by Joyce *et al.* [36, 37, 38] in their utilization of hydrographic data and acoustic doppler current profiles to estimate total volume transport for a streamer of the Middle Atlantic Bight shelf water.

Similarly, satellite-tracked dipole structure images of Hooker *et al.* [31, 32, 33] provide additional details in the observation of mesoscale eddies. Specifically, advanced mesoscale observation techniques in [31, 32, 33] have shown long-period fluctuations in the ring eccentricity as well as vorticity and rotation rate of WCR82B, the Gulf Stream warm core ring that detached in February 1982. This important discovery serves as a paradigm that the ocean is full of mesoscale eddies that should be considered as part of a dynamically linked ring system. Nevertheless, the information available on the observations of mesoscale eddies using moored instruments and remote sensors made it possible to reveal various discrepancies with mathematical modelling of mesoscale eddies utilizing system of nonlinear differential equations from geophysical fluid dynamics. For example, as observed in [36, 37, 38] in their utilization of moored instruments as well as Hooker *et al.* [31, 32, 33] in the remote sensing of warm-core rings, the ring system has bimodal distribution of spin-down with short-lived mean of 54 days and a long-lived mean of 229 days. This theory is far along in its development, however, it is not yet clear if such results qualitatively and quantitatively match those generated by dynamically consistent translating modons of Flierl *et al.* [24, 22, 23, 58] and dynamically consistent rotating modons of Mied *et al.* [40, 41, 50, 39].

## 1.2 Modelling of mesoscale eddies

We proceed to note that the modelling of the dynamics of ring systems or mesoscale and synoptic-scale eddies is imperative due to limitations of current oceanographic observations. Explorations using satellite-tracking of vorticity structures is usually masked by heating of the surface layer, and these may not always be detected by satellite measurements. In the description of strategies utilized in observing geophysical phenomena, unsatisfactory results from the viewpoint of oceanography and meteorology are due to the

fact that deployment of floats such as SOFAR and RAFOS is expensive. In this regard, the development and utilization of geophysical fluid dynamics models is imperative in supplementing costly observations and experiments such as those described above in an effort at understanding the dynamics of mesoscale eddying coherent structures.

Over the past decades, there have emerged several elegant complementary approaches driving current research in vortex dynamics work, using models from geophysical fluid dynamics. The first approach involved construction of closed-form solutions which delivered a dynamically consistent theory of geophysical vortical coherent structures. For example, there are two distinct and robust developments of exact closed-form vortex solutions called modons or Rossby solitons. The building blocks of modons are sinusoidal and Bessel functions. Researchers developing these solutions proliferated in the past decades and include the translating modons group of Flierl *et al.* [24, 22, 23, 58] and the rotating modons group of Mied *et al.* [40, 41, 50, 39]. These exact closed-form vortex solutions have been used to simulate oceanic ring systems. Regarding numerical simulations in the comparative survey [39, 23, 54, 53], Limpfhart *et al.* use rotating modon solutions to investigate WCR82B which continue to be a source of support and inspiration on the dynamics of dipole-structures in the Middle Atlantic Bight.

Although closed-form results compare well with oceanographic observations, this approach has fallen short of elucidating the problem due to challenges associated with nonlinearities of the system of partial differential equations. Moreover, most models for mesoscale eddies are quasigeostrophic and, cannot account for ageostrophic motions which from observational evidence are important in the calculation of ring systems quantities such as ring energies, enstrophy, and Lagrangian transport. The baroclinic dissipative quasigeostrophic equations which we derive in the Appendix govern the evolution of the streamfunction, denoted by  $\psi$ , whenever the Rossby number is asymptotically small, and are given by the system of equations

$$\frac{\partial q}{\partial t} + J(\psi, q) = \frac{Ek}{Ro} \Delta q,$$

$$\frac{\partial \rho}{\partial t} + J(\psi, \rho) + w = \frac{1}{Ed} \Delta \rho,$$

$$q = \Delta \psi + \frac{\partial^2 \psi}{\partial z^2} + \beta y,$$

$$u = -\frac{\partial \psi}{\partial y}; \quad v = \frac{\partial \psi}{\partial x}; \quad \rho = -\frac{\partial \psi}{\partial z};$$

where the nondimensional fields  $\underline{u}$ ,  $q$ ,  $\rho$  and  $\psi$  are fluid velocity, potential vorticity, density

and streamfunction, respectively. The geophysically relevant parameter  $Ro$  is the Rossby number,  $Ek$ , the Ekman number,  $\beta$ , the reference reciprocal Coriolis parameter, and  $Ed$ , the eddy diffusivity. In the above system of nonlinear differential equations,  $\Delta$  is the Laplacian operator, and  $J(.,.)$  is the Jacobian operator. Indeed, the translating modons in [24, 22, 23, 58] and the rotating modons in [40, 41, 50, 39] represent closed form vortex solutions of the above quasigeostrophic potential vorticity equation when for example dissipation terms are neglected.

Other useful simplifying assumptions include the replacement of vertical coordinate by density so that the quasigeostrophic equations may lend themselves to discretization in the vertical coordinate, resulting in layered models. We make a noteworthy remark that the introduction of two-layer quasigeostrophic models leads to the independence of horizontal velocity with respect to height. The phenomenon is called *barotropic* and complements *baroclinic*, the change of horizontal velocity with height for incompressible fluid flows.

Consideration of  $\beta$ -plane ageostrophic model in two space dimensions, which is derived in Chapter 2, and the assumption that fluid flow is confined parallel to the  $xz$ -plane, gives the set of differential equations

$$\begin{aligned}\frac{\partial}{\partial t} \Delta \psi + J(\psi, \Delta \psi) + \frac{1}{Ro} \frac{\partial \rho}{\partial x} &= \frac{Ek}{Ro} \Delta^2 \psi, \\ \frac{\partial \rho}{\partial t} + J(\psi, \rho) + \frac{Ro}{Fr^2} \frac{\partial \psi}{\partial x} &= \frac{1}{Ed} \Delta \rho, \\ w = \frac{\partial \psi}{\partial x}, \quad u = -\frac{\partial \psi}{\partial z},\end{aligned}$$

governing the evolution of vorticity  $\Delta \psi$  and density  $\rho$ . Here,  $Fr$  is the so-called Froude number, related to the Burger number through the formula  $Bu = (\frac{Ro}{Fr})^2$  which is a useful measure of stratification. In the above system of partial differential equations,  $\Delta$  is the Laplacian operator, and  $J(.,.)$  is the Jacobian operator. Further, we note that the reduction of the two-dimensional space variables governing ageostrophic equations with imposed spatial periodicity, no normal flow at the material boundary and fixed density at the bounding planes imply the existence of a sequence of eigenvalues and a corresponding sequence of eigensolutions for the Laplacian operator. Alternatively, by seeking solutions with ansatz of the form

$$\begin{aligned}\psi(x, z, t) &= \psi_1(t) \sin(kx) \cos(\pi z), \\ \rho(x, z, t) &= \rho_1(t) \cos(kx) \cos(\pi z) + \rho_2(t) \sin(2\pi z),\end{aligned}$$

and plugging into the partial differential equations we obtain the Lorenz system with phase portrait depicted in Figure 1.1 obtained with the DsTool package program [3]. Hence,

this two-mode Galerkin approximation yield the standard Lorenz ordinary differential equations

$$\begin{aligned}\frac{dX}{dt} &= \frac{1}{Ro}(Y - X), \\ \frac{dY}{dt} &= rX - Y - XZ, \\ \frac{dZ}{dt} &= -bZ + XY,\end{aligned}$$

where the trigonometric terms other than those appearing in the ansatz has been neglected. Time,  $\psi_1$ ,  $\rho_1$ , and  $\rho_2$  have been rescaled using

$$\begin{aligned}\tau &= (\pi^2 + k^2)t, \quad \psi_1 = \frac{\sqrt{2}}{\pi k}(\pi^2 + k^2)X, \\ \rho_1 &= \frac{\sqrt{2}}{\pi^2 k}(\pi^2 + k^2)Y, \quad \rho_2 = \frac{\sqrt{2}}{\pi^2 k}(\pi^2 + k^2)Z,\end{aligned}$$

and the parameters are given by

$$r = \frac{k^2 Ro(Ed)}{(\pi^2 + k^2)^3 Fr^2}, \quad b = \frac{4\pi^2}{\pi^2 + k^2}.$$

Treatment of transversality of inclination-flip homoclinic orbits for the Lorenz system is due to Dumortier *et al.* [85] and was further developed in [4, 5, 73] for an analysis of viscous perturbations of potential vorticity conserving flows.

Deeper understanding of mesoscale and synoptic coherent structures including dynamics of oceanic eddies and atmospheric vortex blocking has been based on Lagrangian transport investigated utilizing the system of non-autonomous ordinary differential equations

$$\begin{aligned}\frac{dx}{dt} &= -\frac{\partial}{\partial x}\psi(x, y, z, t), \\ \frac{dy}{dt} &= \frac{\partial}{\partial x}\psi(x, y, z, t), \\ \underline{x}(\alpha, t)_{t=0} &= \alpha,\end{aligned}$$

where  $\psi(x, y, z, t)$  is a streamfunction. Lagrangian transport, which is the lodestar in the analysis of transient and aperiodic geophysical phenomena. As an illustration consider the phase portrait for the benchmark streamfunction

$$\psi(x, z, t) = \gamma_1 \sin kx \sin \pi z + \varepsilon \gamma_2 \cos kx \sin \pi z \cos \omega t$$

when  $\varepsilon = 0$  satisfying the two-dimensional space variables  $\beta$ -plane ageostrophic equations with free boundary conditions. Using DsTool [3] we found that the geometry of the phase space of this model when  $\varepsilon = 0$  has hyperbolic saddle equilibria connected by heteroclinic

orbits. Under regular perturbation, that is,  $\varepsilon \neq 0$  the problem becomes that of studying chaotic transport in phase space between regions of stable and unstable manifolds of invariant sets.

Aside from their oceanographic and meteorological interest, the models here are chosen to illustrate necessary conditions for stability and instability. This system with the given streamfunction also exhibits chaotic dynamics as demonstrated in Wiggins [79], which is geometrical and entails showing the Melnikov function has simple zeros for small  $\varepsilon \neq 0$  which implies the existence of transverse heteroclinic orbits. In fact, the simple zeros of the Melnikov function have an elegant geometric interpretation. The Melnikov function represent the distance between the stable and unstable manifolds for invariant sets such as hyperbolic saddle equilibria. A derivation of a Melnikov function for singularly perturbed streamfunctions is given in the Appendix.

Fluid exchange between regimes bounded by pieces of stable and unstable manifolds of distinguished parcel trajectories in the flow is tackled in Pierrehumbert [59] and Balasuriya *et al.* [5] for solutions of the barotropic quasigeostrophic equations with flow fields giving a large-amplitude meandering configuration and vortex motions. For instance, numerically generated velocity vector fields of the barotropic quasigeostrophic equations using a pseudospectral scheme were developed by Flierl *et al.* [25, 60]. Similarly, the flow fields of their numerical results yield a meandering jet with vortical coherent structures. In the splendid research [55, 56], a perturbed current evolves nonlinearly into a large-amplitude meandering configuration that propagates zonally. Although there are two dominant time scales, the flow is aperiodic and dissipative, characteristics that challenge techniques in the theoretical investigation of geophysical phenomena.

### 1.3 Prototype $\beta$ -plane ageostrophic models

It is useful to recall that in this section we briefly review the system of partial differential equations governing geophysical fluid flows. We employ the Boussinesq approximation and other key constraints to the fundamental equations expressing the conservation of momentum, constitutive laws, conservation of mass, the balance of energy and equation of state which provided an viscous incompressible fluid with density heterogeneity and rotary terms. What is here suggestive will be in subsequent chapters exhaustive. The problem of  $\beta$ -plane ageostrophic flows is encountered in many applications including modelling and forecasting of mesoscale and synoptic scale eddies. In order to understand the behavior

of mesoscale and synoptic scale coherent structures, one has to model their impact using  $\beta$ -plane ageostrophic equations. The equations for the conservation of mass, energy, salt and momentum budget in a rotating framework of reference are simplified by the Boussinesq assumption which states that the effect of compressibility is negligible in the balance equations, with the exception of the buoyancy term and equation of state.

Scale analysis ensures consideration of the meridional change in the Coriolis parameter and the reciprocal Coriolis parameter. According to the investigations [20, 57], nonlinear Rossby waves and vortex coherent structures such as alternating cyclones and anticyclones of the Gulf stream and Agulhas current span numerous degrees of latitude; and for them, it is imperative to consider the meridional change in the Coriolis parameter  $f$  and the reciprocal Coriolis parameter  $f_*$ . The  $\beta$ -plane approximation to the Coriolis  $f$  and the reciprocal Coriolis  $f_*$  is obtained by considering  $\frac{y}{r_0}$  as sufficiently small and invoking the two-term Taylor series

$$f = f_0 + \beta_0 y, \quad f_* = r_0 \beta_0 - \frac{f_0}{r_0} y,$$

where  $f_0 = 2\omega \sin \varphi_0$  is the Coriolis parameter at reference latitude  $\varphi_0$ ;  $y = (\varphi - \varphi_0)r_0$  is the coordinate oriented southward,  $\omega$  is the angular rate for rotating framework of reference,  $r_0$  is the earth's radius, and  $\beta_0 = 2\omega r_0^{-1} \cos \varphi_0$  is the beta parameter. We note that the  $\beta$ -plane approximation is valid whenever the term  $\beta_0 y$  is small compared to the leading term  $f_0$ . The approximation in which the beta term  $\beta_0 y$  is not retained is called the  $f$ -plane approximation, and in this case we have

$$f = 2\omega \sin \varphi_0, \quad f_* = 2\omega \cos \varphi_0.$$

The inclusion of the distinguishing geophysical rotary term in the balance of momentum asserts that the inertial acceleration of geophysical fluids is the decomposition of the relative acceleration and the Coriolis acceleration due to the rotating framework of reference. The significance of the Coriolis rotary term is in the generation of planetary or Rossby waves that support geophysical jets and mesoscale vortices. And the validity of the  $\beta$ -plane approximation is a consequence of restriction to geophysical phenomena with length scales substantially smaller than the radius of the earth [12, 20, 57]. Consequently, no appeal will be made in this investigation to the spherical geometry of the earth which in a curvilinear coordinate system contains extraneous curvature terms.

Combining these heuristic arguments, we get the following novelty system that governs a prototypical viscous incompressible fluid with ambient rotation and density heterogeneity:

$$\frac{Ro}{Ek} \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) + \frac{1}{Ek} (\underline{u} \times \underline{k}) + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) + \frac{1}{Ek} \rho \underline{k} = -\frac{1}{Ek} \nabla p + \Delta \underline{u},$$

$$\nabla \cdot \underline{u} = 0,$$

$$Ed\left(\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho\right) + \frac{Ro(Ed)}{Fr^2} \underline{u} \cdot \underline{k} = \Delta \rho.$$

Here  $\underline{u} = (u, v, w)$ ,  $p$  and  $\rho$ , respectively, are fluid velocity, pressure and density. As in the preceding sections,  $Ro = \frac{U}{L f_0}$  is the Rossby number which compares the inertial term to the Coriolis force;  $Fr = \frac{U}{NH}$  is the Froude number which measures the importance of stratification;  $N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz}$  and we consider the case of stably stratified fluids for which the stratification or Brunt-Vaisala frequency  $N$  is take to be real;  $Ek = \frac{\nu}{H^2 f_0}$  measures the relative importance of frictional dissipation; and  $Ed = \frac{\nu}{f_0 N^2}$  is the nondimensional eddy diffusion coefficient. The geophysically relevant ratio  $\frac{Fr^2}{Ro}$  measures the significant effects of combined rotation and stratification to the dynamics of the flow. And the Reynolds number is given by the relation  $Re = \frac{Ro}{Ek} \gamma^2$  where the parameter  $\gamma$  represents the scale ratio  $\frac{L}{H}$ .

It is demonstrated in the Appendix that other geophysical fluid dynamics models can be derived from the equations of a viscous incompressible stratified fluid with the Coriolis force if use is made of a perturbation analysis of the flow fields with respect to the Rossby number. At the zero-th order we obtain the geostrophic equations (A.2) and in this limit the equations reduce to a balance between the Coriolis force and the pressure gradient. First-order terms in the Rossby number yield the quasigeostrophic equations (A.4) that govern the evolution of geostrophic pressure and potential vorticity.

In meteorological and oceanographic discourses, the above system is referred to as *primitive* partial differential equations. These noteworthy remarks serve as a motivation to name the flow of a viscous incompressible stratified fluid with Coriolis force, ageostrophic. This derives from the view that the ageostrophic model is a natural extension of the geostrophic and quasigeostrophic models. We consider the term ageostrophic partial differential equations a better description.

We remark that the  $\beta$ -plane ageostrophic equations may be considered as the Navier-Stokes equations in the presence of combined density stratification and the Coriolis rotary term. Alternatively, the  $\beta$ -plane ageostrophic equations are the Rayleigh-Benard flow problem modified to accomodate the effects of the Coriolis rotary term.

The problem of  $\beta$ -plane ageostrophic equations with Reynolds stress, to be ecountered in Chapter 2, can be described with sufficient accuracy with regard to asymptotic stability

and well-posedness of solutions, existence of inertial and approximate inertial manifolds. The concept of Reynolds stress is adapted from Chapter 4 of Pedlosky [57] and the work of Sirovich [63]. We include  $\beta$ -plane ageostrophic equations with Reynolds stress for completeness and clarity of exposition in mathematical modelling in geophysical fluid dynamics. Thus, we consider the following nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force with Reynolds stress:

$$\frac{Ro}{Ek} \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) + \frac{1}{Ek} (\underline{u} \times \underline{k}) + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) + \frac{1}{Ek} \rho \underline{k} + \Delta^6 \underline{u} = -\frac{1}{Ek} \nabla p + \Delta \underline{u},$$

$$\nabla \cdot \underline{u} = 0,$$

$$Ed \left( \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho \right) + \frac{Ro(Ed)}{Fr^2} \underline{u} \cdot \underline{k} + \Delta^6 \rho = \Delta \rho,$$

where  $\underline{u} = (u, v, w)$ ,  $p$  and  $\rho$ , respectively, are fluid velocity, pressure and density.

The following simplified considerations will provide some of the principal ideas to be encountered in the sequel on the theory of stability for the problem of  $\beta$ -plane ageostrophic equations. One of the most interesting stability criterion occurs when the geophysically relevant ratio  $\frac{Ro}{Ek} \rightarrow \infty$ . In this limit, we describe what happens to a basic state consisting of a stratified shear flow  $\underline{u} = (\hat{u}(z), 0)$  when the viscosity and diffusivity are neglected entirely in the problem of  $\beta$ -plane ageostrophic equations. The choice of the shear problem for  $\beta$ -plane ageostrophic equations when  $\frac{Ro}{Ek} \rightarrow \infty$  is justified in the spirit of a highly simplified analysis with the purpose of showcasing the fundamental ideas on the theory of stability to be addressed. Under these assumptions, the above problem of  $\beta$ -plane ageostrophic equations with or without Reynolds stress lead to the following dimensional equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} + \frac{g}{\rho_0} \rho &= -\frac{1}{\rho_0} \frac{\partial p}{\partial z}, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} &= 0. \end{aligned}$$

It should be noted at the outset that these equations are precisely the non-homogenous Euler equations where  $\underline{u} = (u, w)$  is the velocity,  $p$  the pressure,  $\rho$  the density, and  $g$  the gravity field. The non-homogenous Euler equation has been considered for well-posedness

from a variety of conventional viewpoints, for instance, the method of characteristics and successive approximations exploited by Valli and Zajackowski [82] and cited works therein. As stated, there are several approaches to solve this problem, Klainerman and Majda [51, 52] reformulate the equation using the vorticity in order to understand the behavior of solutions as key parameters vanish. Analyses of various exact closed-form solutions together with the description of related moored instruments and remote sensors observations have been carried out in [80, 81] with the work of Wiggins [79, 102] representing two key thrusts: first, construction of stable and unstable invariant manifolds of hyperbolic invariant sets in phase space that separate qualitatively different type of motions, and the dynamical evolution of the so-called turnstiles and lobes to describe transport between the different regions of motion. A numerical simulation of the problem with representative streamfunction  $\psi(x, z, t) = \tanh[z - \sin(x - ct)]$  is depicted in Figure 1.2 obtained using DsTool package program [3].

As preparation for the demonstration of the aforementioned procedures on weighted and weighted energy methods, consider substitution of small perturbations to the stratified shear flow  $\underline{u} = (\hat{u}(z), 0)$  into the above non-homogenous Euler equation and subsequent application of the curl operator to eliminate pressure and linearization of the resulting equation gives the vorticity equation [20, 57]

$$\frac{\partial \Delta \psi}{\partial t} + \hat{u} \frac{\partial \Delta \psi}{\partial x} + \left( \frac{N^2}{\hat{u}} - \frac{g}{\rho_0 \hat{u}} - \frac{d^2 \hat{u}}{dz^2} \right) \frac{\partial \psi}{\partial x} = 0,$$

$$u = -\frac{\partial \psi}{\partial z}; \quad w = \frac{\partial \psi}{\partial x};$$

where  $\Delta$  is the Laplacian operator,  $\Delta \psi$  is the vorticity, and  $\psi$  is the streamfunction. In this linearized problem,  $N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz}$  and we consider the case of stably stratified fluids for which the stratification or Brunt-Vaisala frequency  $N$  is real. Due to the dependence of the coefficients on  $z$  alone in the above evolution equation for the streamfunction, we seek a wave solution with appropriate form and plugging into the partial differential equation yield the following ordinary differential equation for the amplitude  $\varphi(z)$  supplemented with the choice of Dirichlet boundary conditions [20]:

$$\frac{d}{dz} \left[ (\hat{u} - c) \frac{d\varphi}{dz} \right] - \left[ k^2 (\hat{u} - c) + 1/2 \frac{d^2 \hat{u}}{dz^2} + \frac{1}{\hat{u} - c} \left( 1/4 \left( \frac{d\hat{u}}{dz} \right)^2 - N^2 \right) \right] \varphi = 0,$$

$$\varphi(a) = \varphi(b) = 0.$$

The stability of the stratified shear flow is asserted by analyzing the eigenvalues and eigensolutions of this Sturm-Liouville type problem. The utility of the algorithm lie in the fact that the decision on stability is achieved without explicitly solving the differential

equation. Heuristically, we find that if the Richardson number  $Ri$ , a geophysically relevant parameter, satisfies

$$Ri > \frac{1}{4}$$

then the stratified shear flow is stable in the sense that no anomalous phenomena arise. The Richardson number  $Ri$  is defined by the relation  $Ri^{-1} = N^{-2}(\frac{d\bar{u}}{dz})^2$ . In this research work we shall be concerned with the stability of the rest state of problem of  $\beta$ -plane ageostrophic equations with or without Reynolds stress for arbitrary ratio  $\frac{Ro}{Ek}$  and it is proved rigorously that the flow is stable to small perturbations if the spectrum of the linear operator satisfies hypotheses analogous to the stability criterion

$$Ri > \frac{1}{4}$$

of the stratified shear problem for the non-homogenous Euler equations. Aside from their oceanographic and meteorological interest, the models here are chosen to illustrate necessary and sufficient conditions for Lyapunov stability through generalized energy and Poincaré-Friedrichs inequalities. It should be added that no effort is made in this research work to make a comprehensive analysis of other theories of stability. As such references are only to stability theories which are relevant to the argument being advanced in this manuscript. Thus to accomplish an extensive investigation of the theory of Lyapunov stability, in addition to bifurcation theory, we give an account of energy methods that will yield conditions for overstability of rest states or basic flows satisfying  $\beta$ -plane ageostrophic equations supplemented with initial-boundary conditions and formulated in suitable function spaces. Energy stability criteria of solutions have certain general and convenient properties that make it possible to prove existence results without specifying which norm is utilized. Thus, this approach is taken because these splendid and elegant ideas are applicable to problems other than to the motion of a viscous incompressible stratified fluid with Coriolis force.

#### 1.4 Limitations of standard techniques

In the description of strategies utilized in modelling and observing geophysical phenomena, results are unsatisfactory due to challenges associated with nonlinearities of the systems from the framework of mathematical analysis. For example, since the quasigeostrophic equation is quadratic in the streamfunction, there is no general solution for this system of nonlinear partial differential equations. Additionally, the translating modons of Flierl *et al.* [24, 22, 23, 58] and the rotating modons of Mied *et al.* [40, 41, 50, 39] represent

closed-form vortex solutions of the quasigeostrophic model without dissipation, and as such the models have not accounted for the spin-down of mesoscale eddies.

These drawbacks suggest the need for alternative perspectives for examining solutions of the models. One such perspective is that of Lagrangian transport in oceanic flows, which has benefited from progress in dynamical systems theory. By courtesy of the dynamical systems approach, the orbits or trajectories of particles of fluid corresponding to ageostrophic and quasigeostrophic flows are determined by the system

$$\frac{dX(t)}{dt} = U(X(t), t), \quad X(0) = X_0,$$

where  $U$  is the velocity field satisfying the ageostrophic and quasigeostrophic system of partial differential equations. The application, see for example, [59, 5, 55, 56, 25, 60] of the methods of dynamical systems theory to the problem of Lagrangian transport in time-dependent vortex dynamics offers a renewal of potentially valuable approach in elucidating geophysical phenomena of interest.

Other techniques anchored on the dynamical systems approach entail the well-posedness and existence of Lipschitz invariant manifolds for the resulting system of partial differential equations from geophysical fluid dynamics. With this approach many properties of solutions to the feature models can be deduced without resorting to the cumbersome project of solving the ageostrophic equations numerically. Moreover, well-posedness results anticipate that numerical approximations such as finite-difference schemes to the derivatives in the equations are convergent in the sense of the Lax equivalence theorem. That is, a given finite-difference scheme to a well-posed initial-boundary value problem converges to the solution of the problem with the rate of convergence specified by the order of accuracy of the finite-difference scheme.

Existence and uniqueness of solutions for the inviscid primitive equations is tackled by Bourgeois and Beale [6], who also demonstrate that when the Rossby number is asymptotically small, the inviscid quasigeostrophic solutions are accurate approximations of solutions of the inviscid primitive equations for small initial data. The ageostrophic equations treated in [6] are a refinement of the quasigeostrophic equations and are not considered in this thesis. Their motivation for their refinement is to establish necessary and sufficient criterion for the suppression of fast-scale motions in the inviscid primitive equations.

Most feature models currently rely on quasigeostrophic dynamics which fortunately is amenable to slowly-modulated Hamiltonian systems theory. Thus, the work of Bourgeois and Beale [6] on the inviscid ageostrophic dynamics serves as an improvement in those

models that emphasize quasigeostrophic dynamics. However, the role of dissipation in the transport of mesoscale eddies and gyres which surely are viscous non-quasigeostrophic or dissipative ageostrophic processes represent some of the primary challenges in the investigation of geophysical phenomena. For example, as observed by Joyce *et al.* [36, 37, 38] in their utilization of moored instruments as well as by Hooker *et al.* [31, 32, 33] in the remote sensing of warm-core rings, the ring system has bimodal distribution of spin-down with short-lived mean of 54 days and a long-lived mean of 229 days. It is conjectural and still in dispute as to whether such results qualitatively and quantitatively match those generated by dynamically consistent translating modons of Flierl *et al.* [24, 22, 23, 58] and dynamically consistent rotating modons of Mied *et al.* [40, 41, 50, 39]. In a comparative survey of Limpfhart *et al.* [39, 23, 54, 53], it is argued that there is a need for dissipative eddy-permitting models for the examination of mesoscale eddies that spin-down under the action of viscosity.

## 1.5 The goals of this thesis

Concrete analysis of the equations for ocean dynamics can assist in the long range to build a synthesis of techniques in the framework of the theory of dynamical systems. In order to elucidate ageostrophic dynamics, it is desirable for the theory of dynamical systems to recognize and be able to supplement, or in some cases even supplant, the expensive observational and experimental framework of oceanography developed.

We address in a coherent way dynamical challenges for the dissipative  $\beta$ -plane ageostrophic equations. The problem of  $\beta$ -plane ageostrophic flows continue to be of interest for various purposes and the thesis aims are therefore as follows:

- A synthesis of stability, attractors, and invariant manifolds anticipated in ageostrophy;
- The problem of ageostrophic flows is among noteworthy and accessible mathematical problems and continue to have oceanographic and meteorological interest especially for the modelling and forecasting of mesoscale and synoptic scale eddies;
- The improvement beyond quasigeostrophy is imperative since the Rossby number need not be asymptotically small especially in situations such as the Gulf Stream ring systems and the Agulhas Current Retroflexion;
- The rationale of the Boussinesq and the hydrostatic approximations as well as the assumption of  $\beta$ -plane approximation serve as paradigmatic models that validate and guide dynamical systems theory and applications.

The preceding examples have illustrated the significance of ageostrophic equations with or without Reynolds stress. The different types of equations makes it necessary to incorporate these into a class of differential equations that is mathematically tractable in the sense of well-posedness, stability, attraction, and invariant manifolds properties of solutions corresponding to the system of differential equations. In this thesis we emphasize the dynamics and the resulting flow generated by the system of differential equations consisting of ageostrophic equations with or without Reynolds stress. To accomplish the search for some of the dynamics and geometry of the problems formulated as initial-boundary value problems, we first give an analysis of  $\beta$ -plane ageostrophic equations and in a subsequent analysis consider the treatment of  $f$ -plane ageostrophic equations.

Some commentary on the organization of the chapters of the thesis is in order. The chapters in the sequel introduce and treat, respectively, models for ageostrophic flows, elements of nonlinear analysis, well-posedness of solutions to ageostrophic equations, nonlinear stability of solutions for ageostrophic equations, Lipschitz invariant manifolds for ageostrophic equations, the Melnikov function for singularly perturbed dynamical systems, and quasigeostrophic equations. In Chapter 2 we give a derivation of the equations of dissipative  $\beta$ -plane ageostrophic flows. We employ the simplifying assumptions of the Boussinesq and hydrostatic approximations to the governing equations of non-barotropic viscous compressible fluid flows. As a result we obtain the dissipative  $\beta$ -plane ageostrophic equations which describe the motion of a viscous incompressible stratified fluid with Coriolis force. In the scaling analysis of the governing equations of geophysical fluid flows we introduce the Rossby number which compares the inertial term to the Coriolis force, the Froude number which measures the importance of stratification, the Ekman number which measures the relative importance of frictional dissipation, and the nondimensional eddy diffusion coefficient, that will be taken into account in the mathematical analysis of stability, attractors and invariant manifolds for ageostrophic equations. In order to solve the  $\beta$ -plane ageostrophic equations with or without Reynolds stress terms, one has to specify suitable boundary and initial conditions that lead to a formulation of initial-boundary value problems.

The relevant boundary conditions depend on the properties the fluid is supposed to have at a material boundary or internal boundary. For example, under natural conditions of fluid in motion, both tangential and normal components of density and velocity are continuous across a material boundary. This amounts to a statement about conservation property of momentum flux whose whose measure is provided by the gradient of velocity or rate-of-strain. Invoking the above heuristic approach to the current situation of  $\beta$ -

plane ageostrophic equations we assume the fields to be periodic along the horizontal coordinates  $x$  and  $y$  with periods  $L_1$  and  $L_2$ , respectively. The flow domain is assigned on an open rectangular periodic region  $\Omega = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0, h)\}$  with boundary  $\Gamma = \{z = 0, h\}$ . The domain  $\Omega$  represents the flow region and the surfaces  $z = 0$  and  $z = h$  represent the lower and upper boundaries of the ocean, respectively. To make this more precise, the system of  $\beta$ -plane ageostrophic equations

$$\frac{Ro}{Ek} \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) + \frac{1}{Ek} (\underline{u} \times \underline{k}) + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) + \frac{1}{Ek} \rho \underline{k} = -\frac{1}{Ek} \nabla p + \Delta \underline{u},$$

$$\nabla \cdot \underline{u} = 0,$$

$$Ed \left( \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho \right) + \frac{Ro(Ed)}{Fr^2} \underline{u} \cdot \underline{k} = \Delta \rho,$$

are subject to the mixed boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{on } \Gamma, \\ \rho(z = 0) = \rho_l, \\ \rho(z = h) = \rho_u, \end{aligned}$$

where  $\rho_l$  and  $\rho_u$  are constants. These boundary conditions are the stress-free boundary conditions for horizontal velocity components, no normal velocity and density is fixed at the bounding surfaces.

In Chapter 3 we review and develop relevant results from Hilbert spaces and Sobolev spaces. In the investigation of the initial-boundary value problems, it should be proved that the solution of the equations in a variety of function spaces is well-posed using auxiliary results available from this chapter. Inspired by the classical results on Stokes problem we develop the function spaces for  $U = (u, v, w, \rho)$  in Chapter 3 so that the continuity equation appears as a constraint for the velocity in the function spaces. The above system appended mixed Dirichlet and Neumann type boundary conditions admits a special class of basic flow solutions. Our viewpoint in Chapter 3 is to discuss the notions of stability, attraction, and invariant manifolds. It is these facts which will enable us to get some control on the spectra of the linear operators and thereby make applicable the principle of linear stability implies nonlinear stability using generalized energy and Poincaré-Friedrichs inequalities which express norms.

The existence, uniqueness and differentiability of solutions for the initial-value problems associated with  $\beta$ -plane ageostrophic flows is treated in Chapter 4. In addition, we illustrate that uniqueness and continuity with respect to initial conditions of the solution  $U(t)$

of the initial-value problem generate a dynamical system provided by continuous solution operators  $S(t)$ ,  $S(t)$ ,  $t \in \mathbb{R}_+$ . We further prove that the solution operators  $S(t)$ ,  $t \in \mathbb{R}_+$  are injective and as a result we obtain the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ . The essential properties of the solution operators which we establish are that

$$S(t)U_0 = U(t) \equiv S(U_0, t)$$

and that  $S$  satisfies the *group property*

$$S(t)U(s) = S(s)U(t)$$

$$S(0)U = U \quad \forall t, s \in \mathbb{R}.$$

We solve the problems utilizing more refined a priori energy type estimates, Gronwall's inequality, classical results about the Stokes problem, the best-known technique of Faedo-Galerkin approximation, the Lebesgue dominated convergence theorem and the mini-max principle.

The goal of Chapter 5 is to reconcile Lyapunov stability, asymptotic stability, attractors and the application of Lyapunov functionals and generalized energy which are equivalent to some norm induced by the inner product of solutions. It should be emphasized at this point, that the link between linear and nonlinear stability has been quite controversial. A more relaxed and perhaps more practical requirement to settle this question is accomplished as a particular problem of choosing Lyapunov functionals which furnish necessary and sufficient conditions for stability. A closer look at generalized energy and Poincaré-Friedrichs inequalities constructions that express norms show that the splitting of the operator of the problem into linear and nonlinear parts which are required to satisfy key constraints that yield necessary and sufficient conditions for stability. As regards the asymptotic stability of the solutions of the initial-boundary value problems for viscous  $\beta$ -plane ageostrophic equations, according to more refined a priori estimates that we develop, one of the stability of the rest state is investigated using the additive decomposition of the Lyapunov functional

$$E(t) = \mu_1 E_1(t) + \mu_2 E_2(t),$$

where

$$E_1(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 + \frac{Ro}{Ek} |\nabla \underline{u}|^2 + Ed |\nabla \rho|^2 \right) dx,$$

which is a specification of energy and entropy production in the  $H^1$ -Sobolev norm and

$$E_2(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 \right) dx + \nu_1 |\Pi(\underline{u} \times \underline{k})|^2 + \nu_2 \Pi(\underline{u} \times \underline{k}) \cdot \Pi(\rho \underline{k}) dx,$$

which is an suitable coupling functional. Consequently,  $f$ -plane ageostrophic system is stable if the criterion

$$\frac{1}{Ek} < \max \left\{ \sigma_L^2, \frac{\pi^2 Ro (Ed)^2}{Fr^2 (3 + 2Ed)} \right\},$$

is satisfied and on  $\beta$ -plane, energy stability manifests itself by the requirement

$$\frac{1}{Ek} < \max \left\{ \sigma_L^2, \frac{\beta Ro^2 (Ed)^2 \pi^2}{Fr^2 (3 + 2Ed)} \right\}$$

when the generalized energy functional  $E(t)$  is utilized. In the above energy stability criteria,  $\mu_1, \nu_1, \nu_2$ , and  $\mu_2$  are given positive constants which depend on the aforementioned geophysically relevant parameters. Additionally, the linear operators  $\Pi(\underline{u} \times \underline{k})$  and  $\Pi(\rho \underline{k})$  are defined by

$$\Pi(\underline{u} \times \underline{k}) = \underline{u} \times \underline{k} - \nabla \Psi,$$

and

$$\Pi(\rho \underline{k}) = \rho \underline{k} - \nabla \Phi,$$

where the scalar functions  $\Psi$  and  $\Phi$  satisfy the following inhomogenous elliptic equations with Neumann boundary conditions:

$$\begin{aligned} \Delta \Psi &= \nabla \cdot (\underline{u} \times \underline{k}) \text{ in } \Omega, \\ \frac{\partial \Psi}{\partial z} &= 0 \text{ on the boundary } \Gamma, \end{aligned}$$

$$\begin{aligned} \Delta \Phi &= \frac{\partial \rho}{\partial z} \text{ in } \Omega, \\ \frac{\partial \Phi}{\partial z} &= 0 \text{ on the boundary } \Gamma. \end{aligned}$$

One of the advantages of the result on energy theory in the nonlinear stability analysis of ageostrophic flows described in this exposition is that it may be considered as an extension and generalization of the one obtained using the Lyapunov functional

$$E_1(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed \rho^2 + \frac{Ro}{Ek} |\nabla \underline{u}|^2 + Ed |\nabla \rho|^2 \right) dx,$$

provided the condition

$$\frac{1}{Ek} < \sigma_L,$$

holds. And in this case,  $\sigma_L$  is given by the mini-max problem

$$\frac{1}{\sigma_L} = \sup_{U=(\underline{u}, \rho)} \left\{ \frac{2 \int_{\Omega} \rho \underline{u} \cdot \underline{k} dx}{\int_{\Omega} (|\nabla \underline{u}|^2 + |\nabla \rho|^2) dx} \right\}.$$

Therefore, based on Poincaré-Friedrichs type inequalities it follows that a minimum value for  $\sigma_L$  is given by

$$\frac{1}{Ek} < \sigma_L$$

below which instability cannot set in. Among the best quantitative estimate for the value of  $\sigma_L$ , given implicitly by the aforementioned inequality can be obtained by solving the eigenvalue problem

$$\begin{aligned} \frac{1}{Ek}(\underline{u} \times \underline{k}) + \frac{\beta Ro}{Ek}(y\underline{u} \times \underline{k}) + \frac{1}{Ek}\rho\underline{k} + \frac{1}{Ek}\nabla p - \Delta\underline{u} &= -\frac{Ro}{Ek}\lambda\underline{u}, \\ \nabla \cdot \underline{u} &= 0, \\ \frac{Ro(Ed)}{Fr^2}\underline{u} \cdot \underline{k} - \Delta\rho &= -Ed\lambda\rho, \\ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w &= 0 \text{ on } \Gamma, \\ \rho(z=0) &= \rho_l, \\ \rho(z=h) &= \rho_u. \end{aligned}$$

It should be emphasized at this point, that this eigenvalue problem may be formidable especially for other stress boundary conditions, just as for Sturm-Liouville problems. As we prove in the chapters that follow, one is indeed dealing with an eigenvalue problem with eigenfunctions satisfying orthonormality conditions and lead to the representation of the desired solution as an expansion in terms of a series of orthonormal eigensolutions. Consequently, all results about the eigenfunctions approach are valid, however the special form of the above novelty eigenvalue problem strengthens some of these significant results.

We further infer from stronger a priori estimates that the attractor  $A$  for both viscous  $\beta$ -plane ageostrophic without Reynolds stress in two space variables and viscous  $\beta$ -plane ageostrophic with Reynolds stress in three space variables is given by the  $\omega$ -limit set of  $Q = B_{2\rho_2}$ ,

$$A = \omega(Q) = \bigcap_{s \geq 0} Cl(\bigcup_{t \geq s} S(t)Q),$$

where  $B_{2\rho_2}$  denotes an open ball of radius  $2\rho_2$ , which depends on geophysically relevant parameters. The utility of the group property and continuity of the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ , gives the following invariance property of the above established attractor:

$$S(t)A = A, \quad \forall t \in \mathbb{R}.$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the initial-boundary value problems for  $\beta$ -plane ageostrophic equations is both positively and negatively invariant and consists of orbits or trajectories that are defined for all  $t \in \mathbb{R}$ .

In Chapter 6 we give a proof for the existence of Lipschitz invariant manifolds for the initial-value problems corresponding to  $\beta$ -plane ageostrophic flows which is formulated as a fixed point problem in Banach spaces. The key to the proof is the connection between the spectrum of  $\exp(Lt)$  and the dichotomic projections for  $\exp(Lt)$  as well as suitable bounds on the nonlinear operators. Using the spectral gap protocol together with exponential dichotomies of the variational equation and by virtue of a priori energy type estimates, we get the formula

$$\begin{aligned}\frac{dp}{dt} + Lp + (\Pi_m)N(U) &= 0, \\ \frac{dq}{dt} + Lq + (I - \Pi_m)N(U) &= 0,\end{aligned}$$

where  $\Pi_m$  is a spectral projection of rank  $m$ ,  $N$  is nonlinear,  $p(t) = \Pi_m U(t)$ ,  $q(t) = (I - \Pi_m)U(t)$ , and  $U(t) = p(t) + \Phi_{Lip}(p(t))$ . The existence of an inertial manifold for the problems is given by

$$\Lambda = \text{graph}(\Phi_{Lip}),$$

with a Lipschitz function  $\Phi_{Lip}$  determined as a fixed point of a continuous mapping defined by variation of constants map from the preceding functional differential equation. Furthermore, the inertial manifold  $\Lambda$  continues to be of interest for the following distinguished properties:

- it is a finite-dimensional normally hyperbolic invariant manifold,
- and it consists of trajectories with exponential decay or exponential growth. Certainly there are challenges in establishing existence and uniqueness of solution which yield a Lipschitz function whose graph is an invariant manifold in Banach spaces. Our approach has been to renorm the Banach spaces and the obtain stronger a priori estimates followed by an application of Gronwall's inequality and the contraction mapping principle of Banach-Cacciopoli. The invariant manifold yield long-time approximation for  $\beta$ -plane ageostrophic flows.

We conclude the description with Appendices devoted to the derivation of the geostrophic equations and quasigeostrophic potential vorticity equations as well as a Melnikov function for singularly perturbed dynamical systems. As already emphasized, deeper understanding of mesoscale and synoptic coherent structures including dynamics of oceanic eddies and atmospheric vortex blocking has been based on the quasigeostrophic potential vorticity equations whose derivation from the ageostrophic system is illustrated by a systematic use of scaling and asymptotic series in the Rossby number. Finally, we consider Melnikov integrals and other separation functions for aperiodic singularly perturbed dynamical system that plays a significant role in an analysis of  $\beta$ -plane ageostrophic and

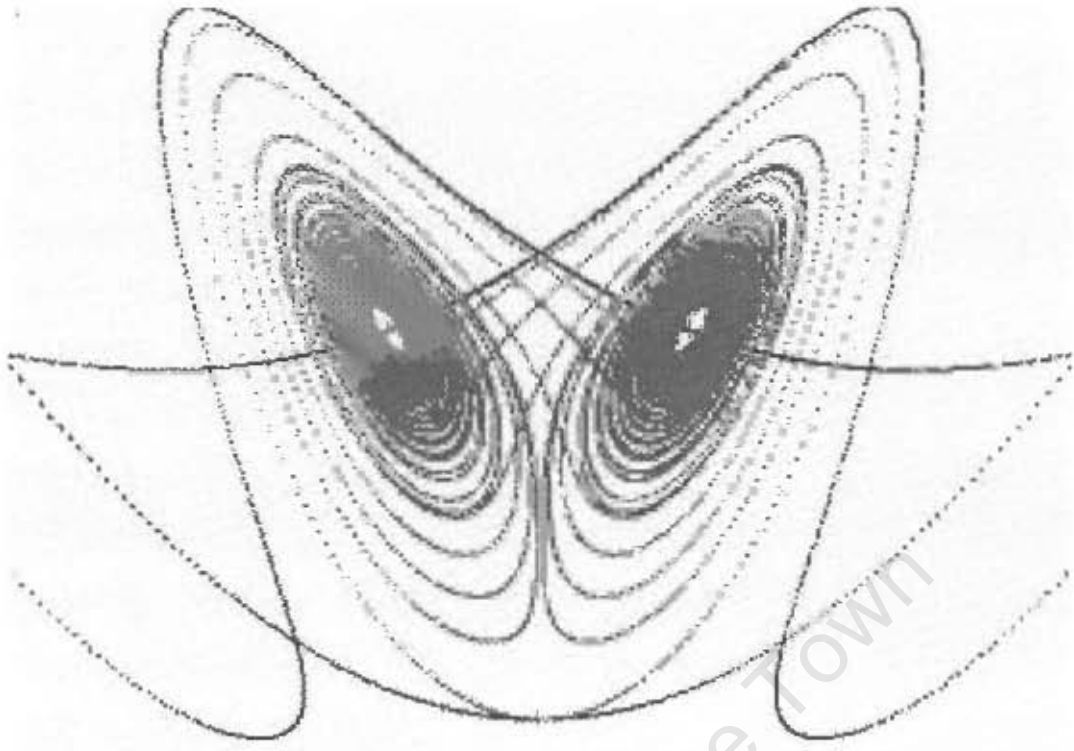


Figure 1.1: Lorenz strange attractor

quasigeostrophic flows.

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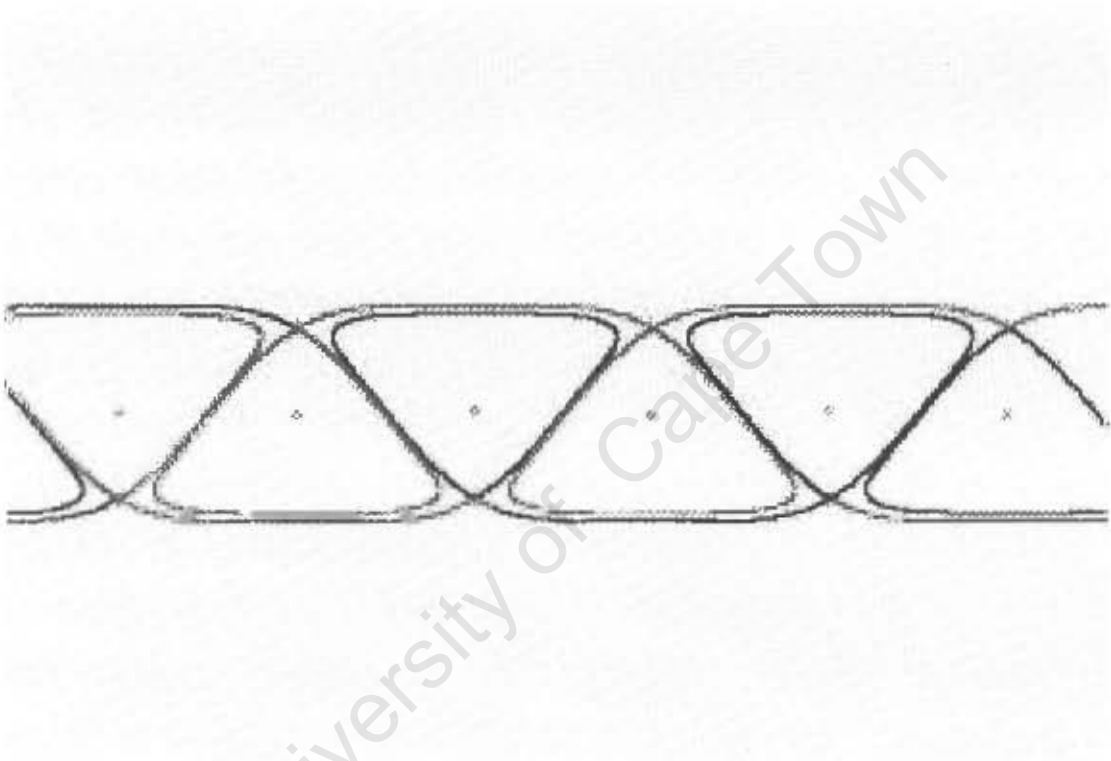


Figure 1.2: Centers, saddles and invariant manifolds

## Chapter 2

# Models for ageostrophic flows

### 2.1 Description of ageostrophic equations

#### 2.1.1 $\beta$ -plane ageostrophic equations

The objective of this chapter is to develop prototype geophysical fluid dynamical models as the first effort towards understanding the impact of ageostrophic flows in the ocean. First, we review the fundamental assumptions and techniques involved in the derivation of the system of partial differential equations governing geophysical fluid flows [12, 20, 57]. The evolution equations for the quantities of interest, which may be scalar-, or vector-, or tensor-valued, are derived with the assistance of Reynolds' transport theorem, conservation laws, and constitutive assumptions [11, 12, 30]. The basic fields in the description of the motion and states of geophysical phenomena are the velocity, the pressure, the density, the temperature, and the salinity. The equations governing these fields consists of the mass balance or continuity equation, the momentum equation, the energy equation, and the equation for salt budget.

Consider a geophysical fluid occupying a region  $B \in \mathbb{R}^3$  as depicted in figures 1.1 and 1.2. For convenience in all that follows, we will adopt a Cartesian coordinate system on a spherical rotating earth and consider the case of a flow with characteristic horizontal length scale shorter than the earth's radius  $r_0$  extending over a small range of latitudes centred on  $\varphi = \varphi_0$  such that the  $x$ -axis is directed westward, the  $y$ -axis is southward, the  $z$ -axis is oriented upward,  $y = (\varphi - \varphi_0)r_0$ ,  $\varpi$  is the angular rate for rotating framework of reference. And we denote the velocity components in these directions by  $u$ ,  $v$ , and  $w$ , respectively.

Heuristically, Reynolds' transport theorem asserts that the rate at which the total density for a fluid domain occupying  $B(t)$  is changing is equal to the sum of the rate of change within  $B(t)$  fixed in its current position and the rate at which density is transported out of the domain  $B(t)$  across its boundary. The use of the divergence theorem to transform the surface integral into a volume integral, and the arbitrariness of  $B(t)$  give the equation

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u} = 0, \quad (2.1)$$

for mass balance, where  $\underline{u} = (u, v, w)$  is the velocity of the fluid,  $\rho$  is density,  $t$  is time, and  $\underline{x} = (x, y, z)$  is the coordinate system. Equation (2.1) is the continuity equation for compressible fluids.

Similarly, in order to obtain the *equation of motion* we utilize Reynolds' transport theorem, and the principle of balance of linear momentum which states that the rate of change of total momentum is equal to the total body and surface forces acting on a domain. We decompose the velocity gradient  $\nabla \underline{u}$  into the symmetric rate-of-deformation tensor  $D$  and the skew-symmetric vorticity tensor  $\Omega$ , that is,

$$\begin{aligned} D &= \frac{1}{2}(\nabla \underline{u} + \nabla^T \underline{u}) \\ \Omega &= \frac{1}{2}(\nabla \underline{u} - \nabla^T \underline{u}), \end{aligned} \quad (2.2)$$

and write the constitutive equation of a compressible Newtonian fluid constitutive assumption

$$\sigma = -p + 2\mu(D - \frac{1}{3} \nabla \cdot \underline{u}), \quad (2.3)$$

where  $\sigma$  is the deviatoric stress tensor,  $\mu$  is the dynamic viscosity,  $p$  is the pressure,  $\nabla$  is the gradient operator, and  $D$  is the rate-of-strain tensor. In addition, consideration of the important ambient rotation of geophysical fluids and application of the simplifying effect of the continuity equation yield the following equation for the conservation of linear momentum, customarily called the *equation of motion*:

$$\rho \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times (\underline{f} \times \underline{x}) + \underline{f} \times \underline{u} + g \underline{k} \right) = -\nabla p + \mu \Delta \underline{u} + \frac{\mu}{3} (\nabla \cdot \underline{u}). \quad (2.4)$$

Here  $g$  is the gravitation acceleration,  $\underline{f} = (0, f, f_*)$  is the earth's rotation,  $f$  is Coriolis parameter, and  $f_*$  is reciprocal Coriolis parameter.

### The $\beta$ -plane approximation

Scale analysis ensures consideration of the meridional change in the Coriolis parameter and the reciprocal Coriolis parameter. According to the investigations [20, 57], nonlinear

Rossby waves and vortex coherent structures such as alternating cyclones and anticyclones of the Gulf stream and Agulhas current span numerous degrees of latitude; and for them, it is imperative to consider the meridional change in the Coriolis parameter  $f$  and the reciprocal Coriolis parameter  $f_*$ . The  $\beta$ -plane approximation to the Coriolis  $f$  and the reciprocal Coriolis  $f_*$  is obtained by considering  $\frac{y}{r_0}$  as sufficiently small and invoking the two-term Taylor series

$$\begin{aligned} f &= f_0 + \beta_0 y, \\ f_* &= r_0 \beta_0 - \frac{f_0}{r_0} y, \end{aligned} \quad (2.5)$$

where  $f_0 = 2\varpi \sin \varphi_0$  is the Coriolis parameter at reference latitude  $\varphi_0$ ;  $y = (\varphi - \varphi_0)r_0$  is the coordinate oriented southward,  $\varpi$  is the angular rate for rotating framework of reference,  $r_0$  is the earth's radius, and  $\beta_0 = 2\varpi r_0^{-1} \cos \varphi_0$  is the beta parameter. We note that the  $\beta$ -plane approximation is valid whenever the term  $\beta_0 y$  is small compared to the leading term  $f_0$ . The Cartesian coordinate system where the beta term  $\beta_0 y$  is not retained is called the  $f$ -plane approximation, and in this case we obtain

$$\begin{aligned} f &= 2\varpi \sin \varphi_0, \\ f_* &= 2\varpi \cos \varphi_0. \end{aligned} \quad (2.6)$$

The inclusion of the distinguishing geophysical rotary term in the balance of momentum equation asserts that the inertial acceleration of geophysical fluids can be decomposed into the relative acceleration, and the Coriolis acceleration due to the rotating framework of reference. The significance of the Coriolis rotary term is in the generation of planetary or Rossby waves that support geophysical jets and mesoscale vortices. And the validity of the approximation (2.5) is a consequence of restriction to geophysical phenomena with length scales substantially smaller than the radius of the earth [12, 20, 57]. Consequently, no appeal will be made in this investigation to the spherical geometry of the earth which in curvilinear coordinate system contain challenging extraneous curvature terms. Formal passage from the  $\beta$ -plane approximation to retain the extraneous curvature terms corresponding to the full geometry of the spherical rotating earth is treated in [12, 57]. The mathematical analysis of the resulting geophysical fluid equations with extraneous curvature terms is examined in Lions *et al.* [43, 44].

Equations (2.1) – (2.4) are supplemented by equations of state, energy and salt budgets. In order to obtain these equations we use equation of the balance of energy, which asserts that the rate of change of the internal energy supplied to a geophysical fluid parcel is balanced by the heat out of the fluid parcel and the power or work done by the system against external forces. Thus, by consideration of the power produced by the surface forces and Fourier's constitutive hypothesis for the rate of heat, and subsequent application of

the continuity equation, we obtain the temperature form of the energy equation,

$$\rho C_v \left( \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T \right) + p \nabla \cdot \underline{u} = \kappa_T \Delta T + \mu \nabla \cdot (\underline{u} \cdot \nabla \underline{u}) - \frac{2}{3} \mu (\nabla \cdot \underline{u})^2. \quad (2.7)$$

An alternative derivation of the energy equation may be achieved by exploiting the enthalpy relation [11].

The equations must be completed with the addition of the state equation

$$\rho = \rho_0 [1 - \alpha_T (T - T_0) + \alpha_S (S - S_0)], \quad (2.8)$$

and the salt budget

$$\frac{\partial S}{\partial t} + \underline{u} \cdot \nabla S = \kappa_S \Delta S, \quad (2.9)$$

which is a formula of conservation of salt content. In the evolution equations (2.7)–(2.9),  $T$  is absolute temperature,  $C_v$  is heat capacity,  $\kappa_T$  is thermal conductivity,  $S$  is salinity,  $T_0, S_0, \rho_0$  are reference values of temperature, salinity and density,  $\alpha_T$  is coefficient of thermal expansion,  $\alpha_S$  is coefficient of saline contraction, and  $\kappa_S$  is coefficient of saline diffusion. The coupled equations (2.1)–(2.9) yield the following system of partial differential equations governing non-barotropic viscous compressible  $\beta$ -plane oceanic flows:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u} &= 0, \\ \rho \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times (\underline{f} \times \underline{x}) + \underline{f} \times \underline{u} + g \underline{k} \right) &= -\nabla p + \mu \Delta \underline{u} + \frac{\mu}{3} (\nabla \cdot \underline{u}), \\ \rho C_v \left( \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T \right) + p \nabla \cdot \underline{u} &= \kappa_T \Delta T + \mu \nabla \cdot (\underline{u} \cdot \nabla \underline{u}) - \frac{2}{3} \mu (\nabla \cdot \underline{u})^2, \\ \rho &= \rho_0 [1 - \alpha_T (T - T_0) + \alpha_S (S - S_0)], \\ \frac{\partial S}{\partial t} + \underline{u} \cdot \nabla S &= \kappa_S \Delta S. \end{aligned} \quad (2.10)$$

Here  $\underline{f}$  is given by (2.5).

The above system of partial differential equations (2.10) consisting of the conservation of mass, energy, momentum, and salt budget simplifies to a class of problems governing viscous compressible  $\beta$ -plane oceanic flows. The next reformulation of viscous compressible  $\beta$ -plane oceanic flows is desirable from the framework of qualitative and quantitative properties of evolution equations. We make a noteworthy remark that our current purpose is in obtaining the dissipative  $\beta$ -plane ageostrophic equations which describe the motion

of a viscous incompressible stratified fluid with Coriolis force. First, we get

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u} = 0,$$

$$\rho \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times (\underline{f} \times \underline{x}) + \underline{f} \times \underline{u} + g \underline{k} \right) = -\alpha_T \nabla T + \alpha_S \nabla S + \mu \Delta \underline{u} + \frac{\mu}{3} (\nabla \cdot \underline{u}),$$

$$\rho C_v \left( \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T \right) + \rho \nabla \cdot \underline{u} = \kappa_T \Delta T + \mu \nabla \cdot (\underline{u} \cdot \nabla \underline{u}) - \frac{2}{3} \mu (\nabla \cdot \underline{u})^2,$$

$$\frac{\partial S}{\partial t} + \underline{u} \cdot \nabla S = \kappa_S \Delta S$$

where we employed the simplifying assumption

$$p = \rho = \rho_0 [1 - \alpha_T (T - T_0) + \alpha_S (S - S_0)] \equiv p(\rho).$$

Secondly, by repeating the same analysis with the pseudo-hydrostatic approximation

$$p(x, y, z, t) = p_s(x, y, t) + b \int_z^0 \rho dz \equiv p(\rho, T, S)$$

whose validity is a result of scale analysis, we obtain the non-barotropic viscous compressible  $\beta$ -plane oceanic equations

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u} = 0,$$

$$\begin{aligned} \rho \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times (\underline{f} \times \underline{x}) + \underline{f} \times \underline{u} + g \underline{k} \right) &= -\nabla p_s - b \nabla \int_z^0 [-\alpha_T (T - T_0) \\ &+ \alpha_S (S - S_0)] dz + \mu \Delta \underline{u} + \frac{\mu}{3} (\nabla \cdot \underline{u}), \end{aligned}$$

$$\begin{aligned} \rho C_v \left( \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T \right) + p_s \nabla \cdot \underline{u} &= \kappa_T \Delta T + \mu \nabla \cdot (\underline{u} \cdot \nabla \underline{u}) - \frac{2}{3} \mu (\nabla \cdot \underline{u})^2 \\ &- (\nabla \cdot \underline{u}) b \int_z^0 \rho dz, \end{aligned}$$

$$\frac{\partial S}{\partial t} + \underline{u} \cdot \nabla S = \kappa_S \Delta S.$$

Examination of the well-posedness of solutions to the initial-boundary value problems corresponding to the above two systems of viscous compressible  $\beta$ -plane oceanic flows with slight modification for the Coriolis terms has been tackled in [117, 118, 119, 120, 121] and further extended in the investigation [71] utilizing a strategy inspired in part

by Lions *et al.* [43, 44] for certain primitive equations of atmosphere and ocean with extraneous curvature terms. The conceptual strategy entails utility of results about the inhomogeneous Stokes problem [66] and the nonhomogeneous Euler equation [6, 82]. Aside from their oceanographic and meteorological merit, the models here are chosen to illustrate conditions for energy or Lyapunov stability, that is, the significant concept of continuous dependence of solutions on initial data on infinite intervals of time. It should be added that no effort is made in this work to make a comprehensive analysis of other theories of stability. As such references are only to techniques which are relevant to the argument being advanced in this research and development.

### The Boussinesq and hydrostatic approximations

Next, the equations for the conservation of mass, momentum, energy and salt budget are simplified by the Boussinesq assumption which states that the effect of compressibility is negligible in the balance equation except in the buoyancy term and the equation of state. Employing the additive decomposition

$$\rho = \rho_0 + \rho'(x, y, z, t), \quad \rho' \ll \rho_0 \quad (2.11)$$

in (2.1) and retaining terms multiplied by  $\rho_0$ , we obtain

$$\nabla \cdot \underline{u} = 0, \quad (2.12)$$

which is the continuity equation for an incompressible fluid. Substitution in (2.10) of the additive decompositions (2.11) and the hydrostatic approximation

$$p = p_0(z) + p'(x, y, z, t)$$

with  $p_0(z) = P_0 - \rho_0 g z$  being the hydrostatic pressure, taking into account the continuity equation (2.12), and retaining dominating terms, we obtain the set of equations

$$\begin{aligned} \nabla \cdot \underline{u} &= 0, \\ \rho_0 \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times \underline{u} \right) + g \rho \underline{k} &= -\nabla p + \mu \Delta \underline{u}, \\ \rho_0 C_v \left( \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T \right) &= \kappa_T \Delta T, \\ \rho &= \rho_0 [1 - \alpha_T (T - T_0) + \alpha_S (S - S_0)], \\ \frac{\partial S}{\partial t} + \underline{u} \cdot \nabla S &= \kappa_S \Delta S. \end{aligned} \quad (2.13)$$

We set  $\kappa_T = \frac{\kappa_T}{\rho_0 C_v}$ , and  $\nu = \frac{\mu}{\rho_0}$  is the kinematic viscosity.

In order to circumvent the challenges of directly accounting for molecular diffusion and oceanic salt finger pattern formations which result from the competitive effects of the diffusivities of heat and salt budgets, we examine the case where the salt and heat diffusivities are assumed to be equal to the eddy diffusivity, that is,  $\kappa = \kappa_T = \kappa_S$ . The choice of the eddy diffusivity [20, 57], which forms the basis of the model analyzed here, incorporates the ubiquitous geophysical phenomena that temperature, density and salinity structures of the ocean are influenced primarily by chaotic transport and mixing within jet streams and geophysical eddying currents on the scale of the Rossby deformation radius. The existence and persistence of jet streams and other geophysical coherent structures in the ocean make plausible the notion that molecular diffusion is weak to be directly significant in the evolution equations governing the motion of a viscous incompressible stratified fluid with the Coriolis force.

Application of (2.11) in the equation of state (2.8), use of the operators  $\Delta$  and  $\frac{\partial}{\partial t} + \underline{u} \cdot \nabla$ , and taking (2.13) in consideration give,

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho = \kappa \Delta \rho. \quad (2.14)$$

Thus, dropping the primes from  $\rho$  and  $p$ , the above Boussinesq approximation gives the novelty system

$$\begin{aligned} \rho_0 \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times \underline{u} \right) + g \rho \underline{k} &= -\nabla p + \mu \Delta \underline{u}, \\ \nabla \cdot \underline{u} &= 0, \\ \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho &= \kappa \Delta \rho. \end{aligned} \quad (2.15)$$

These equations govern the flow of a viscous incompressible stratified fluid with the Coriolis force and are called the  $\beta$ -plane ageostrophic equations. In the next section we turn to the conditions on the velocity and stress at a material boundary in order to formulate an initial-boundary value problem.

### 2.1.2 $\beta$ -plane ageostrophic equations with Reynolds stress

The fundamental ideas in the derivation of the primitive equations with Reynolds stress are due to Pedlosky [57] and Sirovich [63]. Our hypotheses are very similar to Chapter 4 of Pedlosky [57]. For simplicity of exposition, consider the following additive decomposition

of flow fields for the preceding ageostrophic equations (2.15):

$$\begin{aligned}\underline{u} &= \langle \underline{u} \rangle + \underline{u}', \\ p &= \langle p \rangle + p', \\ \rho &= \langle \rho \rangle + \rho'.\end{aligned}\tag{2.16}$$

Here the time-averaging operator  $\langle \cdot \rangle$  is defined by

$$\langle \underline{u} \rangle = \frac{1}{\tau} \int_{\tau} \underline{u}(\underline{x}) \chi(\underline{x}) d\underline{x},$$

with  $\tau$  being a characteristic evolution time-scale and  $\chi(\underline{x})$  a probability density function. The terms in the splitting (2.16) represent coherent and incoherent flow fields, respectively. Furthermore, we assume that the incoherent velocity field satisfies

$$\langle \underline{u}' \rangle = 0.$$

Employing the decomposition (2.16) in the ageostrophic equations (2.23) and invoking the time-averaging operator, we obtain the system

$$\begin{aligned}\rho_0 \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times \underline{u} \right) + g \rho \underline{k} &= -\nabla p + \mu \Delta \underline{u} + \nabla \cdot A, \\ \nabla \cdot \underline{u} &= 0,\end{aligned}\tag{2.17}$$

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + (\rho_u - \rho_l) \underline{u} \cdot \underline{k} = \kappa \Delta \rho + \nabla \cdot W,$$

for coherent flow fields in which  $A$  and  $W$  are Reynolds stress [57] fields. The presence of the effective dissipative terms  $\nabla \cdot A$  and  $\nabla \cdot W$  implies that although the incoherent flow fields have zero average, the momentum and density fluxes of the fluctuations, which are quadratic in the incoherent fluctuations, need not vanish when the time-averaging operator is employed. As far as the contributions of the incoherent flow fields are concerned, we observe that a prototypical fluid parcel with fluctuation velocity  $\underline{u}'$  in the  $x$ -direction will transport  $z$ -momentum  $\rho_0 \underline{u}'$  with a nonzero flux across arbitrary surface given by the quadratic relation  $-\rho_0 \langle \underline{u}' \underline{u}' \rangle = A_{xz}$ . What is here suggestive is in Pedlosky [57] exhaustive. It is similarly convenient to introduce the following quadratic protocols for

the other Reynolds stress fields:

$$\begin{aligned}
A_{xx} &= -\rho_0 \langle \acute{u}\acute{u} \rangle, \\
A_{yy} &= -\rho_0 \langle \acute{v}\acute{v} \rangle, \\
A_{zz} &= -\rho_0 \langle \acute{w}\acute{w} \rangle, \\
A_{yx} &= -\rho_0 \langle \acute{u}\acute{v} \rangle = A_{xy}, \\
A_{zx} &= -\rho_0 \langle \acute{u}\acute{w} \rangle = A_{xz}, \\
A_{zy} &= -\rho_0 \langle \acute{v}\acute{w} \rangle = A_{yz}, \\
W_x &= -\rho_0 \langle \acute{u}\acute{\rho} \rangle, \\
W_y &= -\rho_0 \langle \acute{v}\acute{\rho} \rangle, \\
W_z &= -\rho_0 \langle \acute{w}\acute{\rho} \rangle.
\end{aligned} \tag{2.18}$$

It is important to realize that the additive decomposition of the ageostrophic flow quantities into coherent and incoherent terms and consideration of the averaging operator to obtain the Reynolds stress fields or wind stress fields have resulted in a set of equations that is not closed. We adopt the following closure approximations for the Reynolds stress fields (2.18):

$$\begin{aligned}
A_{xx} &= -2\varepsilon\rho_0 \Delta^5 \frac{\partial u}{\partial x}, \\
A_{yy} &= -2\varepsilon\rho_0 \Delta^5 \frac{\partial v}{\partial y}, \\
A_{zz} &= -2\varepsilon\rho_0 \Delta^5 \frac{\partial w}{\partial z}, \\
A_{yx} &= -\varepsilon\rho_0 \Delta^5 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = A_{xy}, \\
A_{zx} &= -\varepsilon\rho_0 \Delta^5 \left( \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} \right) = A_{xz}, \\
A_{zy} &= -\varepsilon\rho_0 \Delta^5 \left( \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} \right) = A_{yz},
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
W_x &= -\delta\left(\frac{\partial^2 u}{\partial x^2} + \Delta^5 \frac{\partial \rho}{\partial x}\right), \\
W_y &= -\delta\left(\frac{\partial^2 v}{\partial y^2} + \Delta^5 \frac{\partial \rho}{\partial y}\right), \\
W_z &= -\delta\left(\frac{\partial^2 w}{\partial z^2} + \Delta^5 \frac{\partial \rho}{\partial z}\right),
\end{aligned} \tag{2.20}$$

where  $\varepsilon$  and  $\delta$  are parameters. The geophysical motivation of the choice of the closure protocols (2.19) – (2.20) will become clear in the sequel when we give a mathematical treatment of the resulting evolution equations. As a validation from a dynamical systems approach, the closure protocols lead to well-posedness in the sense of Hadamard and the existence of attractors for the solution of the system of partial differential equations. The closure protocols (2.19) – (2.20) are the simplest which allow existence of stability and attractors. Another rationale and validation desirable from the craft of the asymptotic behavior of trajectories such as homoclinic and heteroclinic orbits [17, 13, 79, 132] follows from the fact that a significant quantity in analyzing whether the solution of an evolution equation is bounded in some suitable norm is that of Lyapunov functionals of the system with interpretations such as fictitious energy, enstrophy and entropy production which are decreasing along solutions. Geophysical fluid dynamical processes are identified with trajectories of a dynamical system in a suitable phase space and the investigation of asymptotic behavior is reduced to the structure of  $\omega$ -limit sets of these orbits.

The equations governing the flow  $\beta$ -plane of ageostrophic equations with Reynolds stress (2.19) – (2.20) induce damping mechanisms in the nature of viscosity, diffusion, stratification and rotation effects that manifest themselves in the evolution equation through the existence of Lyapunov functionals which are equivalent to some norm induced by the inner product.

We now proceed with the derivation of the geophysical fluid dynamics model. Introducing the above closure hypothesis into the equations (2.17) for coherent flow fields, we obtain the set of partial differential equations

$$\begin{aligned}
\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times \underline{u} + \frac{g}{\rho_0} \rho \underline{k} + \varepsilon \Delta^6 \underline{u} &= -\frac{1}{\rho_0} \nabla p + \nu \Delta \underline{u}, \\
\nabla \cdot \underline{u} &= 0,
\end{aligned} \tag{2.21}$$

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + (\rho_u - \rho_l) \underline{u} \cdot \underline{k} + \delta \Delta^6 \rho = \kappa \Delta \rho,$$

governing  $\beta$ -plane ageostrophic flows with Reynolds stress. The presence of *effective* dissipative terms  $\Delta^6 \underline{u}$  and  $\Delta^6 \rho$  is equivalent to stress due to the impact of the incoherent

terms on the coherent flow. In the next section we turn to the conditions on the velocity and stress at a material boundary in order to formulate an initial-boundary value problem. The preceding examples have illustrated the significance of ageostrophic equations with or without Reynolds stress. The variety of equations makes it necessary to incorporate these into a class of differential equations that is mathematically tractable in the sense stability, attractors, and Lipschitz invariant manifolds properties of solutions. Therefore, in this research work we emphasize the dynamics and the resulting flow generated by the system of partial differential equations consisting of ageostrophic equations with or without Reynolds stress.

## 2.2 The initial-boundary value problems

### 2.2.1 $\beta$ -plane ageostrophic equations

Next, concerning the boundary conditions we assume the fields to be periodic along the horizontal coordinates  $x$  and  $y$  with periods  $L_1$  and  $L_2$ , respectively. The flow domain is assigned on an open rectangular periodic region  $B = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0, h)\}$  with boundary  $\Gamma = \{z = 0, h\}$ . The domain  $B$  represents the flow region and the surfaces  $z = 0$  and  $z = h$  represent the lower and upper boundaries of the ocean, respectively. To system (2.15) we prescribe mixed boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{on } \Gamma, \\ \rho(z = 0) = \rho_l, \end{aligned} \tag{2.22}$$

$$\rho(z = h) = \rho_u,$$

where  $\rho_l$  and  $\rho_u$  are constants. The natural boundary conditions (2.22) are the stress-free boundary conditions for horizontal velocity components, no normal velocity and density is fixed at the bounding surfaces. In order to eliminate rigid motions we impose the condition

$$\int_B u dx = \int_B v dx = 0.$$

We introduce the conducting solutions

$$\begin{aligned} \rho &= \bar{\rho} - \rho_l + \frac{z}{2}(\rho_u - \rho_l), \\ p &= \bar{p} + z\rho_l g - \frac{g}{4}z^2(\rho_u - \rho_l), \end{aligned}$$

for density and pressure, respectively. Substitution in (2.15) leads to the system of partial differential equations

$$\begin{aligned} \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \underline{f} \times \underline{u} + \frac{g}{\rho_0} \rho \underline{k} &= -\frac{1}{\rho_0} \nabla p + \nu \Delta \underline{u}, \\ \nabla \cdot \underline{u} &= 0, \end{aligned} \quad (2.23)$$

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + (\rho_u - \rho_l) \underline{u} \cdot \underline{k} = \kappa \Delta \rho,$$

governing  $\beta$ -plane ageostrophic flows.

We introduce the nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force without Reynolds stress. In order to achieve this objective, the system of partial differential equations and boundary conditions is made nondimensional with the length and velocity scales

$$x = L\bar{x}, \quad y = L\bar{y}, \quad z = H\bar{z}, \quad t = \frac{L}{U}\bar{t}$$

$$u = U\bar{u}, \quad v = U\bar{v}, \quad w = \frac{UH}{L}\bar{w}, \quad L_1 = L\bar{L}_1$$

$$L_2 = L\bar{L}_2, \quad \rho = \frac{1}{gH}f_0\rho_0UL\bar{\rho}, \quad p = f_0\rho_0UL\bar{p}$$

$$\rho_u - \rho_l = \frac{1}{g}N^2\rho_0, \quad \bar{\beta} = \frac{L \cos \varphi_0}{r_0 R_0 \sin \varphi_0}$$

in which rotation and stratification effects are important. Here  $T = \frac{L}{U}$  represents the time scale,  $U$  represents the horizontal velocity scale,  $W$  the vertical velocity scale,  $L$  the length scale,  $H$  the vertical length scale, etc. It is recognized at the outset that the above scaling need not be fit and proper for all scales of nonlinear waves and coherent structures of oceanographic and meteorological interest. To ensure that the measure of the effect of variations of density defined by the stratification or Brunt-Vaisala frequency  $N$ ,

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz}$$

is real (which implies static stability of the stratified fluid), we assume that  $\frac{d\rho}{dz} < 0$  for  $0 \leq z \leq h$ . It is known that an unstable density stratification leads to penetrative convective motions.

Employing the  $\beta$ -plane approximation, substituting the above scaling into the system (2.23), and omitting bars from nondimensional quantities, we obtain the following nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid with the Coriolis force:

$$Ro\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) + (1 + y Ro \beta_0) \underline{u} \times \underline{k} + \rho \underline{k} = -\nabla p + Ek \Delta \underline{u},$$

$$\nabla \cdot \underline{u} = 0,$$

$$\frac{Fr^2}{Ro} \left( \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho \right) + \underline{u} \cdot \underline{k} = \frac{Fr^2}{Ro(Ed)} \Delta \rho.$$

Here we set the aspect ratio  $\frac{H}{L}$  to unity. The evolution equations have been scaled so that the relative order of each term is measured by the dimensionless parameter multiplying it. As above, the parameter  $Ro = \frac{U}{Lf_0}$  is the Rossby number which compares the inertial term to the Coriolis force;  $Fr = \frac{U}{NH}$  is the Froude number which measures the importance of stratification;  $Ek = \frac{\nu}{H^2 f_0}$  measures the relative importance of frictional dissipation;  $Ed = \frac{\nu}{f_0 N^2}$  is the nondimensional eddy diffusion coefficient. And the Reynolds number is given by the relation  $Re = \frac{Ro}{Ek} \gamma^2$  where the parameter  $\gamma$  represents the scale ratio  $\frac{L}{H}$ . And the ratio  $\frac{Fr^2}{Ro}$  measures the significant influence of both rotation and stratification to the dynamics of the flow.

We make the vertical density boundary conditions homogeneous using

$$\rho = \bar{\rho} + \rho_l + \frac{1}{g} z \rho_0 N^2,$$

$$p = \bar{p} - z \rho_l - \frac{1}{2g} z^2 \rho_0 N^2.$$

Omitting bars the corresponding boundary conditions (2.22) become

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \rho = 0 \quad \text{at } \Gamma,$$

$$\int_B u dx = \int_B v dx = 0.$$

Additionally, the flow domain reduces to  $B = \{(x, y, z) \in (0, L) \times (0, L) \times (0, 1)\}$  with boundary  $\Gamma = \{z = 0, 1\}$ . Furthermore, velocity and density are given at initial time  $t = 0$

$$\underline{u}(\underline{x}, 0) = \underline{u}_0(\underline{x})$$

$$\rho(\underline{x}, 0) = \rho_0(\underline{x}).$$

We conclude this section with a recapitulation of the set of equations whose further description and mathematical analysis will be considered in subsequent chapters. The evolutionary equation for the  $\beta$ -plane ageostrophic equations governing geophysical fluid flows is the following initial-boundary value problem:

$$\begin{aligned} \frac{Ro}{Ek} \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) + \frac{1}{Ek} (\underline{u} \times \underline{k}) + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) + \frac{1}{Ek} \rho \underline{k} &= -\frac{1}{Ek} \nabla p + \Delta \underline{u}, \\ \nabla \cdot \underline{u} &= 0, \end{aligned} \quad (2.24)$$

$$Ed \left( \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho \right) + \frac{Ro(Ed)}{Fr^2} \underline{u} \cdot \underline{k} = \Delta \rho,$$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \rho = 0 \text{ at } \Gamma, \quad (2.25)$$

$$\int_B u dx = \int_B v dx = 0.$$

$$\begin{aligned} \underline{u}(\underline{x}, 0) &= \underline{u}_0(\underline{x}) \\ \rho(\underline{x}, 0) &= \rho_0(\underline{x}). \end{aligned} \quad (2.26)$$

In the next chapter we show how the definition of function spaces should be defined appropriately due to the above imposed boundary conditions. The initial-boundary value problem (2.24) – (2.26) represent the form of  $\beta$ -plane ageostrophic equations that we will investigate.

### 2.2.2 $\beta$ -plane ageostrophic equations with Reynolds stress

We embark on showing how to impose a set of natural boundary conditions to  $\beta$ -plane ageostrophic equations with Reynolds stress. Concerning the boundary conditions, we assume the fields to be periodic along the coordinates  $x, y$  and  $z$  with periods  $L_1, L_2$  and  $L_3$ , respectively. The flow domain is assigned on an open rectangular periodic region

$$B = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0, L_3)\}.$$

To system (2.21) we prescribe the space-periodic boundary conditions

$$\begin{aligned} \underline{u}(\underline{x} + L\mathbf{e}_i, t) &= \underline{u}(\underline{x}, t), \\ \rho(\underline{x} + L\mathbf{e}_i, t) &= \rho(\underline{x}, t), \\ p(\underline{x} + L\mathbf{e}_i, t) &= p(\underline{x}, t), \end{aligned} \quad (2.27)$$

where  $\{e_1, e_2, e_3\}$  is the standard basis of  $\mathbb{R}^3$  and with the simplifying assumption that the period  $L = L_1 = L_2 = L_3$ . We further assume the derivatives of  $\underline{u}$ ,  $\rho$  and  $p$  are also space-periodic. In order to eliminate rigid motions we consider the case where the average velocity and pressure vanish

$$\int_B \underline{u}(\underline{x}, t) d\underline{x} = 0,$$

$$\int_B p(\underline{x}, t) d\underline{x} = 0.$$

It is recognized at the outset that the above space-periodic boundary conditions may not be appropriate for the deliniation of closed-form solutions such as modons which decay to zero at infinity, and is well-defined at internal boundaries with patch conditions that ensure that the basic fields such as velocity and density are continuous. In the next chapter we show how the definition of function spaces should be defined appropriately due to the above imposed boundary conditions. We introduce the nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force with Reynolds stress. In order to achieve this objective, the system of partial differential equations and boundary conditions are made nondimensional with the following length and velocity scales

$$x = L\bar{x}, \quad y = L\bar{y}, \quad z = H\bar{z}, \quad t = \frac{L}{U}\bar{t}$$

$$u = U\bar{u}, \quad v = U\bar{v}, \quad w = \frac{UH}{L}\bar{w}, \quad L_1 = L\bar{L}_1$$

$$L_2 = L\bar{L}_2, \quad \rho = \frac{1}{gH}f_0\rho_0UL\bar{\rho}, \quad p = f_0\rho_0UL\bar{p}$$

$$\rho_u - \rho_t = \frac{1}{g}N^2\rho_0, \quad \bar{\beta} = \frac{L \cos \varphi_0}{r_0 Ro \sin \varphi_0}$$

in which rotation and stratification effects are essential.

Employing the  $\beta$ -plane approximation, substituting the above scaling into the system (2.23), and omitting bars from nondimensional quantities, we obtain the following nondimensional form of the equations governing the flow of a viscous incompressible stratified

fluid under the Coriolis force with Reynolds stress:

$$Ro\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) + (1 + y Ro \beta_0) \underline{u} \times \underline{k} + \rho \underline{k} + Ek \Delta^6 \underline{u} = -\nabla p + Ek \Delta \underline{u},$$

$$\nabla \cdot \underline{u} = 0,$$

$$\frac{Fr^2}{Ro} \left( \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho \right) + \underline{u} \cdot \underline{k} + \frac{Fr^2}{Ed(Ro)} \Delta^6 \rho = \frac{Fr^2}{Ro(Ed)} \Delta \rho.$$

Here we set the aspect ratio  $\frac{H}{L}$  to unity. The evolution equations have been scaled so that the relative order of each term is measured by the dimensionless parameter multiplying it. As above, the parameter  $Ro = \frac{U}{Lf_0}$  is the Rossby number which compares the inertial term to the Coriolis force;  $Fr = \frac{U}{NH}$  is the Froude number which measures the importance of stratification;  $Ek = \frac{\nu}{H^2 f_0}$  is the Ekman number measures the relative importance of frictional dissipation; and  $Ed = \frac{\nu}{f_0 N^2}$  is the nondimensional eddy diffusion coefficient.

We conclude the section with the following initial-boundary value problem for  $\beta$ -plane ageostrophic equations with Reynolds stress terms:

$$\frac{Ro}{Ek} \left( \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) + \frac{1}{Ek} (\underline{u} \times \underline{k}) + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) + \frac{1}{Ek} \rho \underline{k} + \Delta^6 \underline{u} = -\frac{1}{Ek} \nabla p + \Delta \underline{u},$$

$$\nabla \cdot \underline{u} = 0,$$

$$Ed \left( \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho \right) + \frac{Ro(Ed)}{Fr^2} \underline{u} \cdot \underline{k} + \Delta^6 \rho = \Delta \rho, \quad (2.28)$$

$$\begin{aligned} \underline{u}(\underline{x} + Le_i, t) &= \underline{u}(\underline{x}, t), \\ \rho(\underline{x} + Le_i, t) &= \rho(\underline{x}, t), \\ p(\underline{x} + Le_i, t) &= p(\underline{x}, t), \end{aligned} \quad (2.29)$$

$$\int_B \underline{u}(\underline{x}, t) dx = 0,$$

$$\int_B p(\underline{x}, t) dx = 0,$$

$$\begin{aligned} \underline{u}(\underline{x}, 0) &= u_0(\underline{x}), \\ \rho(\underline{x}, 0) &= \rho_0(\underline{x}). \end{aligned} \quad (2.30)$$

It is useful to recall that in this chapter we derived the system of partial differential equations governing geophysical fluid flows. We employed the Boussinesq approximation and other key restrictions to the fundamental equations expressing the conservation of momentum, constitutive laws, conservation of mass, the balance of energy and equation of state which provided an viscous incompressible fluid with density heterogeneity and rotary terms. In the scaling analysis of the governing equations of geophysical fluid flows we introduced the aforementioned parameters, that is, the Rossby number which compares the inertial term to the Coriolis force, the Froude number which measures the importance of stratification, the Ekman number which measures the relative importance of frictional dissipation, and the nondimensional eddy diffusion coefficient, that will be taken into account subsequently in the mathematical analysis of the initial-boundary value problems (2.28) – (2.30) and (2.24) – (2.26).

In the subsequent chapters we elucidate in a concrete way dynamical challenges concerning well-posedness, stability, attraction, and invariant manifolds properties of solutions associated with the initial-boundary value problems (2.28) – (2.30) and (2.24) – (2.26) reformulated in suitable function spaces. In addition, geostrophic and quasigeostrophic equations are derived from the  $\beta$ -plane ageostrophic equations with or without Reynolds stress where use is made of a perturbation analysis of the flow fields using the Rossby number.

## Chapter 3

# Nonlinear analysis preliminaries

### 3.1 Elements of Hilbert spaces and Sobolev spaces

The purpose of this introductory section is to develop the definitions of relevant function spaces and briefly examine propositions appropriate for analysis of the evolution equations established in the previous chapter. In the investigation of the initial-boundary value problems, it should be proved that the solution of the equations in a variety of function spaces is well-posed using the auxiliary results available from this chapter. The proof of existence and uniqueness of solution to initial-boundary problems utilizing Hilbert spaces and Sobolev spaces techniques will be of central importance in this work. Therefore, we provide some notation, preliminary concepts, definitions of some differential operators and function spaces.

We represent the flow region by a bounded domain denoted by the symbol  $B = \{(x, y, z) \in \mathbb{R}^3\}$  of three-dimensional Euclidean space. We assume  $B$  has a locally Lipschitz boundary  $\Gamma = \partial B$ , that is,  $\Gamma$  is locally the graph of a Lipschitz function which holds when  $B$  is of class at least  $C^1$ . Scalar-, vector-, and tensor-valued functions are assumed to be real and locally summable in the sense of Lebesgue, whereas their derivatives will be interpreted in the generalized sense of the theory of distributions of Schwartz. Thus, throughout this work we will use the standard Lebesgue spaces  $L^p(B)$ ,  $1 \leq p \leq \infty$ , which consist of  $p$ -integrable functions on  $B$  with norms [69, 70]

$$\|\underline{u}\|_{L^p}^p = \int_B |\underline{u}(x)|^p dx, \quad 1 \leq p < \infty,$$

and

$$\|\underline{u}\|_{L^\infty} = \operatorname{ess\,sup}_{x \in B} |\underline{u}(x)|.$$

Here  $dx$  is the Lebesgue measure on  $\mathbb{R}^3$  and the assumption of identification through the *a.e.* equivalence relation. The Lebesgue spaces  $L^p(B)$  are complete normed linear spaces, i.e., Banach spaces. We denote the inner product of the Hilbert space  $L^2(B)$  by

$$(\underline{u}, \underline{v}) = \int_B \underline{u}(x) \cdot \underline{v}(x) dx$$

and the associated norm is defined with

$$(\underline{u}, \underline{u}) = \|\underline{u}\|_{L^2}^2$$

for any  $\underline{u}, \underline{v} \in L^2(B)$ . The major point to note is that the Cauchy-Schwarz inequality

$$|(\underline{u}, \underline{v})| \leq \|\underline{u}\| \|\underline{v}\|$$

and the triangle inequality

$$\|\underline{u} \pm \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$

are a consequence of Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

$\frac{1}{p} + \frac{1}{q} = 1$ , and the convexity of the function  $\phi(t) = t^p$  on the interval  $[0, \infty)$  for  $1 \leq p < \infty$ , where  $a$  and  $b$  are nonnegative scalars.

The function space of real continuous functions on  $B$  is denoted by  $C(B)$ . The spaces of infinitely continuous differentiable functions on  $B$  with compact support in  $B$  is denoted by  $C_0^\infty(B)$ . We define a function to be of compact support in the domain  $B$  if it is nonzero only on a bounded subdomain  $B_s$  of the domain  $B$  with the subdomain lying at a positive distance from  $\Gamma$ .

We define the Sobolev spaces  $H^k(B)$ ,  $k = -1, 0, 1, \dots$ , with generalized derivatives up to order  $k$  belonging to  $L^2(B)$  and norms  $\|\cdot\|_k$ . These are Hilbert spaces endowed with the inner product

$$\left( (\underline{u}, \underline{v}) \right)_k = \sum_{|\alpha| \leq m} (D^\alpha \underline{u}, D^\alpha \underline{v}) = \sum_{|\alpha| \leq m} \int_B D^\alpha \underline{u}(x) \cdot D^\alpha \underline{v}(x) dx.$$

Here  $\alpha = (\alpha_1 \dots \alpha_n)$  is a multi-index with each  $\alpha_i$  a nonnegative integer;  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and  $D^\alpha u$  denotes the partial derivative  $\partial^{|\alpha|} u / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ . The Sobolev  $k$ -norm associated with the above inner product is defined by

$$\|\underline{u}\|_k^2 = \left( (\underline{u}, \underline{u}) \right)_k$$

whereas the  $H^k$ -pseudonorm is given by

$$\|\underline{u}\|_k^2 = \sum_{|\alpha|=k} \|D^\alpha \underline{u}\|_{L^2}^2.$$

The Sobolev space  $H_0^1(B)$  of functions in  $H^1(B)$  which vanish on  $\Gamma$  in the sense of traces, and its dual  $H^{-1}(B)$ , will be employed. Duality pairing of two elements  $l \in H^{-1}(B)$  and  $u \in H_0^1(B)$  is represented by the notation  $\langle l, u \rangle$ .

It follows that the concept of Sobolev spaces underlies the construction of a priori energy type estimates for suitable functions and their derivatives up to order  $k$ . Thus, in order to prove well-posedness and other properties of solutions to initial-value problems we resort to a priori energy type estimates which are derived from the bound of the  $L^2$ -norm of the solution in terms of the Sobolev  $k$ -norm of the initial condition.

In the sequel we will require the Sobolev space  $H^{k,per}(B_p)$  of space-periodic functions in the periodic domain

$$B_p = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0, L_3)\}.$$

Let  $\underline{u}(x)$  represent a space-periodic function or, equivalently, the space-periodic extension of a function. Define the Fourier series expansion

$$\underline{u}(x) = \sum_{k \in \mathbb{Z}^3} u_k \exp(2i\pi k \cdot \frac{x}{L})$$

with  $\bar{u}_k = u_{-k}$ , and

$$\sum_{k \in \mathbb{Z}^3} (1 + |k|^2) |u_k|^2 < \infty.$$

Furthermore, in the space-periodic case we assume that the flow average vanishes, that is,

$$\frac{1}{|B_p|} \int_{B_p} \underline{u}(x) dx = 0.$$

The rigorous characterization of the Sobolev spaces  $H_{even}^{k,per}(B_p)$  and  $H_{odd}^{k,per}(B_p)$  of functions belonging to  $H^{k,per}(B_p)$  that are even and odd, respectively, in  $z$  may be found in the work of Bourgeois and Beale [6]. We will have need of the following useful result from [6] that characterizes the spaces  $H_{even}^{k,per}(B_p)$  and  $H_{odd}^{k,per}(B_p)$ .

**Lemma 3.1.1** *Suppose that the flow domain is given by  $B = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0, 1)\}$  with the boundary  $\Gamma = \{z = 0, 1\}$ . Consider a function  $u \in H^k(B)$  for a given integer  $k$ . Then  $u$  can be extended to  $H_{even}^{k,per}(B_p)$  if and only if all odd  $z$  derivatives of  $u$  with index less than  $k$  are zero on the boundary  $\Gamma$  of  $B$ . Also,  $u$  may be extended to  $H_{odd}^{k,per}(B_p)$  if and only if all even  $z$  derivatives of  $u$  with index less than  $k$  including  $u$  itself are zero on the boundary  $\Gamma$  of  $B$ .*

Several a priori estimates and important properties have been developed for Sobolev spaces and are customarily referred to as the Sobolev embedding theorem and Rellich's compact injection theorem. These results are established in the expositions [42, 66, 67, 68].

Next, we exhibit a series of results for Sobolev spaces which will be required in establishing existence, uniqueness and differentiability properties of solution for the initial-boundary value problems. In the investigation of the differentiability properties of generalized solutions to the initial-boundary value equations reformulated in variational form, the following Banach spaces are employed. Given that  $I = (a, b) \subset \mathbb{R}$  is an open interval and  $Y$  is a complete normed space, then  $L^p(I; Y(B)), 1 \leq p \leq \infty$ , are spaces of functions from  $I$  into  $Y$  which are Banach spaces with corresponding norms

$$\|f\|_{L^p(I; Y(B))}^p = \int_a^b \|f(t)\|_Y^p dt, \quad 1 \leq p < \infty$$

$$\|f\|_{L^\infty(I; Y(B))} = \text{ess sup}_{t \in I} \|f(t)\|_Y.$$

Thus,  $L^p(I; H^k(B)), 1 \leq p \leq \infty$ , are function spaces which consist of  $p$ -integrable functions with values in  $H^k(B)$ . Similarly,  $C(\bar{I}; H^k(B))$  is the space of vector functions  $\underline{u}(x, t)$  such that  $\underline{u}(\cdot, t)$  is an element of  $H^k(B)$  for all  $t \in \bar{I}$  and the function  $t \rightarrow \underline{u}(\cdot, t)$  with values in  $H^k(B)$  is continuous on  $\bar{I}$ . Similarly,  $C_b(\bar{I}; H^k(B))$  denotes the space of vector functions  $\underline{u}(x, t)$  such that  $\underline{u}(\cdot, t)$  is an element of  $H^k(B)$  for all  $t \in \bar{I}$  and the function  $t \rightarrow \underline{u}(\cdot, t)$  with values in  $H^k(B)$  is a bounded continuous function on  $\bar{I}$ .

The following proposition adapted from Temam [66, 67, 68] considers the sense in which integration by parts is valid in the functional reformulation of initial-boundary value equations.

**Lemma 3.1.2** *Suppose that  $Y$  is a given complete normed linear space with dual denoted by  $Y'$  and assume that  $u, f \in L^1(I; Y)$ . Then the following results are equivalent:*

(1)  *$u$  is a.e. equal to the Lebesgue integral of  $f$ , that is, there is  $\xi \in Y$  such that for  $t \in I$  a.e.*

$$u(t) = \xi + \int_0^t f(s) ds.$$

(2) *Given a test function  $\phi \in C_0^\infty(a, b)$ ,*

$$\int_a^b \frac{d\phi}{dt} u(t) dt = - \int_a^b f(t) \phi(t) dt.$$

(3) *Given  $\eta \in Y'$ , the identity*

$$\frac{d}{dt} \langle u, \eta \rangle = \langle f, \eta \rangle \tag{3.1}$$

is valid in the scalar distributional sense on  $(a, b)$ . Whenever conditions (1) – (3) are satisfied,  $f = \frac{du}{dt}$  is considered the  $Y$ -valued distributional derivative of  $u$  and in this case  $u$  is a.e. equal to an element of  $C(I; Y)$ .

Next, we consider the following useful result from Temam [66, 67, 68] that will be utilized in the chapters that follow.

**Lemma 3.1.3** Consider  $V \subset H \subset V'$ , where the injections are continuous and each Hilbert space is dense in the following one. If a function  $u \in L^2((a, b); V)$  and its derivative  $\frac{du}{dt}$  are elements of  $L^2((a, b); V')$ , then  $u$  is a.e. equal to an element of  $C(I; H)$  as in the above lemma and the identity

$$\frac{d}{dt}|u|^2 = 2 \langle \frac{du}{dt}, u \rangle \quad (3.2)$$

is valid in the scalar distribution sense on  $(a, b)$  since the functions  $t \rightarrow |u(t)|^2$  and  $t \rightarrow \langle \frac{du}{dt}(t), u(t) \rangle$  are both elements of  $L^1(a, b)$ .

We proceed with other significant results from Sobolev spaces. A Poincaré-Friedrichs inequality states that there exists a constant  $P_F \in (0, \infty)$  such that

$$\int_B |\underline{u}|^2 dx \leq P_F \int_B |\nabla \underline{u}|^2 dx,$$

for all  $\underline{u} \in H_0^1(B)$ . It should be emphasized at this point, that we made simplifying assumptions that the flow region  $B$  is a bounded open set of  $\mathbb{R}^3$  smooth enough for integration by parts to hold. Assuming the functions satisfy the homogeneous Dirichlet or Neumann type boundary conditions and in the case of space periodicity subject to vanishing in mean,

$$\int_B \underline{u} dx = 0,$$

then an elementary upper bound for the existence of  $P_F \in (0, \infty)$  in the Poincaré-Friedrichs inequality which is particularly suited for later purposes is provided by

$$P_F \leq \pi^{-2}. \quad (3.3)$$

For a comprehensive and very elegant review of Poincaré-Friedrichs inequalities in the case of unbounded and exterior domains, see [28, 29, 42, 67, 1].

By the Poincaré-Friedrichs inequality, the  $H^1$ -pseudonorm  $|\underline{u}|_1$  is equivalent to the Dirichlet norm

$$\|\nabla \underline{u}\|_{L^2} \equiv \|\underline{u}\|_1$$

and is a norm on  $H_0^1(B)$ .

For simplicity of exposition, the function spaces of vector-, or tensor-valued functions which have components in one of the spaces defined above, will be denoted by the same symbol.

### 3.2 Properties of some functionals and operators

Next we turn to the decomposition of the space  $L^2(B)$  into two orthogonal spaces denoted by  $H$  and  $H^\perp$  with the fundamental ideas due to [42, 66, 67, 68]. The space  $H$  consists of all smooth divergence-free or solenoidal vector functions of compact support in  $B$  and the orthogonal complement  $H^\perp$  consists of the gradients of all functions which are single-valued in  $B$ . An analysis for the existence of the decomposition of the space  $L^2(B)$  which is particularly suited for later purposes is specified by

$$L^2(B) = \begin{cases} H \oplus H^\perp & \text{for the case of Lipschitz open bounded domain} \\ H \oplus G \oplus J & \text{for the case of open bounded domain of class } C^2 \end{cases}$$

where

$$H^\perp = \{\underline{u} \in L^2(B), \underline{u} = \nabla p, p \in H^1(B)\},$$

$$G = \{\underline{u} \in L^2(B), \underline{u} = \nabla p, p \in H^1(B), \Delta p = 0\},$$

and

$$J = \{\underline{u} \in L^2(B), \underline{u} = \nabla p, p \in H_0^1(B)\}.$$

Inspired by the results on the Stokes problem we develop the function spaces for  $U$  so that the continuity equation appears as a constraint in the function spaces. In what follows we set  $H_1 = L^2(B)$ . The function space  $V_1$  is the space of functions in  $H^1(B)$  satisfying condition (2.25). The function spaces  $H_1$  and  $V_1$  are Hilbert spaces endowed, respectively, with norms  $\|\cdot\|_{H_1} \equiv \|\cdot\|_{L^2}$  and  $\|\cdot\|_{V_1} \equiv \|\cdot\|_1$ . Additionally, the function spaces  $H_0$  and  $V_0$  are given by

$$H_0 = \{\underline{u} \in L^2(B) : \nabla \cdot \underline{u} = 0 \text{ in } B, \underline{u} \cdot \underline{n} = 0 \text{ at } \Gamma\},$$

with  $\underline{n}$  the unit outward normal on the boundary  $\Gamma$ , and

$$V_0 = \{\underline{u} \in V_1 : \nabla \cdot \underline{u} = 0\};$$

these are Hilbert spaces with norm denoted by  $\|\cdot\|_1$ .

Furthermore, we will find it useful to introduce the following function spaces

$$Q = \{\underline{u} \in L^2(B) : \nabla \cdot \underline{u} = 0, \underline{u} \text{ satisfies space-periodicity boundary conditions}\}.$$

Then we define

$$H = \begin{cases} H_1 \times \text{closure of } Q \text{ in } L^2(B) & \text{for the case of space-periodicity boundary conditions} \\ H_1 \times H_0 & \text{for the case of boundary conditions (2.25)} \end{cases}$$

$$V = \begin{cases} V_1 \times \text{closure of } Q \text{ in } H^1(B) & \text{for the case of space-periodicity boundary conditions} \\ V_1 \times V_0 & \text{for the case of boundary conditions (2.25)}. \end{cases}$$

The function spaces  $H$  and  $V$  are Hilbert spaces equipped, respectively, with norms  $\|\cdot\|_H \equiv \|\cdot\|_{L^2}$ ,  $\|\cdot\|_V \equiv \|\cdot\|_1$  and  $\|\cdot\|_X \equiv \|\cdot\|_{L^2}$  corresponding to the product Hilbert structure. By virtue of the Riesz-Fréchet representation theorem we identify the dual space  $H'$  of  $H$  with  $H$  and as such we get

$$V \subset H = H' \subset V' \quad (3.4)$$

where the two inclusions are compact continuous. For elegant characterization of the spaces  $H$  and  $V$ , see avid researchers [66, 67, 68].

As a consequence of the identifications (3.4), the inner product in  $H$  of  $\underline{u} \in H$  and  $\underline{v} \in V$  is the same as the inner product of  $\underline{u}$  and  $\underline{v}$  in the duality between  $V'$  and  $V$ . And for each  $\underline{v} \in V$  the form

$$\underline{v} \rightarrow ((\underline{u}, \underline{v})) \in \mathbb{R}$$

is a linear continuous on  $V$ . Hence there exists an element of  $V'$  denoted by  $A\underline{u}$  such that

$$\langle A\underline{u}, \underline{v} \rangle = ((\underline{u}, \underline{v})), \forall \underline{v} \in V.$$

The linear continuous mapping

$$\underline{u} \rightarrow A\underline{u}$$

is an isomorphism from  $V$  onto  $V'$ .

Based on integration by parts formula which is a direct consequence of the Stokes problem we are in a position to introduce the Stokes operator  $A_s : V \rightarrow V'$  and the Laplace operator  $A_l : V \rightarrow V'$  by setting

$$a_s(\underline{u}, \underline{v}) = (A_s \underline{u}, \underline{v}) = ((\underline{u}, \underline{v})) = \left( (\underline{u}, \underline{v}) \right)_1,$$

$$a_l(\underline{u}, \underline{v}) = (A_l \underline{u}, \underline{v}) = ((\underline{u}, \underline{v})) = \left( (\underline{u}, \underline{u}) \right)_1.$$

Here  $a_s$  and  $a_l$  are the corresponding bilinear forms

$$a_s : V \times V \rightarrow \mathbb{R}, \quad a_l : V \times V \rightarrow \mathbb{R}$$

which are coercive and continuous. Hence the isomorphisms

$$A_s : V \rightarrow V', \quad A_l : V \rightarrow V'$$

may be extended to unbounded self-adjoint positive linear operators, respectively, from  $D(A_s)$  into  $H_0$  and from  $D(A_l)$  into  $H_1$ . Furthermore,  $A_s^{-1}$  and  $A_l^{-1}$  are compact self-adjoint linear operators in  $H_1$  and  $H_0$ .

The domain  $D(A_l)$  is given by the Hilbert space

$$D(A_l) = V_1 \cap H^2(B).$$

The domain of the Stokes operator  $A_s$  is assigned by the Hilbert space

$$D(A_s) = V_0 \cap H^2(B),$$

and is endowed with the norm  $\|\underline{u}\|_{D(A_s)} = \|A_s \underline{u}\|_{L^2}$  which is equivalent to the natural norm  $\|\cdot\|_2$  of the Sobolev space  $H^2(B)$ .

Due to the presence of the boundary condition (2.25) and the flow region  $B = \{(x, y, z) \in (0, L) \times (0, L) \times (0, 1)\}$  with boundary  $\Gamma = \{z = 0, 1\}$ , we note that the above definition of Stokes operator is equivalent to

$$\Pi \Delta \underline{u} = \Delta \underline{u} - \nabla p,$$

where the scalar function  $p$  satisfies the following homogenous elliptic equation with Neumann boundary conditions:

$$\Delta p = 0 \quad \text{in } B,$$

$$\frac{\partial p}{\partial z} = \Delta w \quad \text{at } \Gamma.$$

Additionally, from well-known a priori estimates for the Stokes problem the inequality

$$\sup |\underline{u}|^2 \leq c_1 \int_B (|\Pi \Delta \underline{u}|^2 + |\nabla \underline{u}|^2) dx,$$

holds.

In order to ensure that the solution of an evolution equation is bounded in some suitable norm, we need to introduce the following Hilbert spaces

$$D(L) = \begin{cases} \{U \in \Upsilon : \int_B (|\Delta \underline{u}|^2 + |\Delta \rho|^2) dx < \infty, \rho, \underline{u} \text{ satisfy (2.25)}\}, \\ \{U \in \Upsilon : \int_B (|\Delta^6 \underline{u}|^2 + |\Delta^6 \rho|^2) dx < \infty, \rho, \underline{u} \text{ satisfy (2.29)}\}. \end{cases}$$

The spaces  $D(L^n)$  are Hilbert spaces with obvious inner products and corresponding norms  $\|U\|_{D(L^n)} = \|L^n U\|$ . Furthermore, the dual of  $D(L^{1/2})$  is denoted by  $D(L^{-1/2})$ . Under appropriate assumptions it is possible to establish existence of the injections

$$D(L^{1/2}) = V \subset D(L) \subset \Upsilon = H \subset D(L^{-1/2}). \quad (3.5)$$

The density and compactness of the injections will find applications in the sequel when we prove well-posedness of solutions to the initial-boundary value problems.

In addition to the Stokes operator and Laplace operator, with the assistance of the orthogonal projection  $\Pi$  we require the bilinear mapping  $\mathbf{b}(\cdot, \cdot)$  defined by

$$\mathbf{b}(\underline{u}, \underline{v}) = \begin{cases} \Pi(\underline{u} \cdot \nabla) \underline{v} = \mathbf{b}_s(\underline{u}, \underline{v}) & \text{for the case of boundary conditions (2.25)} \\ (\underline{u} \cdot \nabla) \underline{v} = \mathbf{b}_i(\underline{u}, \underline{v}) & \text{for the case of space-periodicity boundary conditions} \end{cases}$$

where  $\underline{u}, \underline{v} \in D(A_s)$  for the bilinear operator  $\mathbf{b}_s(\underline{u}, \underline{v})$  and  $\underline{u} \in D(A_s)$ ,  $\underline{v} \in D(A_l)$  for the bilinear operator  $\mathbf{b}_i(\underline{u}, \underline{v})$ .

### Stability, attraction, and Lipschitz invariant manifolds

We introduce ideas which are required in the formulation of the stability of invariant sets to be customized for use in the problem of  $\beta$ -plane ageostrophic equations with or without Reynolds stress terms. The technique used in the stability problem of  $\beta$ -plane ageostrophic equations is energy and weighted energy theory and the framework for its utility in this context has been provided, for example, in [28, 29]. The existence, uniqueness and differentiability properties of solutions to the initial-boundary value problems can be solved in a variety of function spaces introduced in the preceding section. The appropriate one for the purpose of stability analysis will be Hilbert spaces which follows from the fact that a significant quantity in analyzing whether the solution of an initial-value problem is bounded in some suitable norm is that of Lyapunov functionals of the system with interpretations such as energy and weighted energy which are decreasing along solutions of the initial-boundary value problems formulated in function spaces. The nonlinearity and the spectrum of the linear operator must be taken into account when defining stability using the energy and weighted energy methods. As preparation for the illustration of these techniques, consider an initial-value problem

$$\frac{du}{dt} = Lu, \quad u(0) = u_0,$$

in an appropriate Hilbert space then a self-adjoint operator  $L$  is defined to be essentially dissipative if the following hold:

$$(LU, U) \leq 0 \quad \forall \quad U \in D(L)$$

$$(LU, U) = 0 \Rightarrow U = 0.$$

The complement of essentially dissipative is called essentially non-dissipative. By the classical spectral theorem [28, 29, 48, 129], essential dissipativity is equivalent to the spectrum of  $L$  being nonnegative, with zero not an eigenvalue. For every essentially dissipative operator  $L$ , the bilinear form

$$(U, W)_L = -(LU, W) \quad \forall U, W \in D(L)$$

defines a scalar product in  $D(L)$ . We denote by  $H_L$  the completion of  $D(L)$  in the norm  $\|\cdot\|_L$  given by

$$\|U\|_L^2 = -(LU, U) \quad \forall U \in D(L).$$

It is these definitions which will enable us to get some control on the spectrum of the linear operator and thereby make applicable the principle of linear stability implies nonlinear stability using energy and weighted arguments. Hence, energy and weighted energy theory for stability is achieved as a particular case of the problem of choosing Lyapunov functionals which gives necessary and sufficient conditions for stability, just as for bifurcation theory, chaos, and overstability.

Thus, the case of essential dissipativity of linear operators in Hilbert spaces developed here may be considered as an extension of matrix theory in  $\mathbb{R}^n$ . Just as in Jones and Dafermos [73], the most crucial utility of essential dissipativity is to obtain exponential decay estimates for solutions of the initial-boundary value problems that yield Lyapunov stability of solutions. The ideas can be less abstractly formalized in the finite dimensional case. To make this precise, we observe that each basis of  $\mathbb{R}^n$  renders a scalar product denoted by  $(\cdot, \cdot)$ . It is known that if  $L$  is an  $n \times n$  matrix for which

$$\alpha_1 < \operatorname{Re}(\lambda) < \alpha_2 \tag{3.6}$$

for all  $\lambda$  in the spectrum  $\sigma(L)$  of  $L$ , then from [73] we get

$$\alpha_1(u, u) \leq (Lu, u) \leq \alpha_2(u, u) \tag{3.7}$$

where  $\alpha_1$  and  $\alpha_2$  are nonzero constants. Thus based on the inequalities (3.6) – (3.7), it follows that if we set  $u(t)$  to be a solution of a system of linear ordinary differential equations

$$\frac{du}{dt} = Lu, \quad u(0) = u_0, \tag{3.8}$$

then we get

$$|u_0| \exp(\alpha_1 t) \leq |u(t)| \leq |u_0| \exp(\alpha_2 t),$$

where  $|\cdot|$  is the Euclidean norm. Consequently, if  $Re(\lambda) < 0 \quad \forall \lambda \in \sigma(L)$  we obtain stability of the rest state and if there exists  $\lambda \in \sigma(L)$  with  $Re(\lambda) > 0$  then the rest state is unstable.

We emphasize that the results on linear stability belong to finite-dimensional space  $\mathbb{R}^n$ . Formal passage from the finite dimensional space  $\mathbb{R}^n$  to the infinite-dimensional case of Hilbert spaces will be established in the sequel utilizing essential dissipativity of the linear operators and suitable Lyapunov functionals which are equivalent to some norm induced by the inner product of solution.

We have observed that for the system of linear ordinary differential equations (3.8), the structure near the rest state is described by  $\sigma(L)$ , the spectrum of  $L$ . Another related question to be addressed is whether this structure holds for a nonlinear evolution equation if the linearization at a given rest state has this structure in the infinite-dimensional phase space. The general question as to what extent the linearized equations determine the structure of the nonlinear case is fundamental in the investigation of dynamical systems and will recur throughout this work. In the subsequent chapters we embark on the challenge of determining a necessary and sufficient criteria for asymptotic stability of the initial-boundary value problems (2.28) – (2.30) and (2.24) – (2.26).

We define the *orbit* or *trajectory* [73, 14, 15] of motion for an initial-value problem starting at the initial condition  $u_0 \in H$  to be the set

$$\Xi = \{(u, t) : u \in H, t \in \mathbb{R}_+\} = \cup_{t \in \mathbb{R}_+} S(t)u_0.$$

Uniqueness and continuity with respect to initial conditions of the solution  $u(t)$  generate a dynamical system prescribed by continuous solution operators  $S(t)$ ,  $t \in \mathbb{R}_+$  which is a mapping of the phase space  $H$  into itself. Whenever the solution operators  $S(t)$ ,  $t \in \mathbb{R}_+$  are injective then we denote by  $S(-t)$  the inverse mapping of  $S(t)H$  onto  $H$ . As a result we obtain the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ . The crucial property of the solution operators which will be employed is that

$$S(t)u_0 = u(t) \equiv S(u_0, t)$$

satisfies the *group property*

$$S(t)u(s) = S(s)u(t),$$

$$S(0)u = u \quad \forall t, s \in \mathbb{R}.$$

That the solution operators  $S(t)$  are continuous is a consequence of the continuity of  $u(t)$  in time and the initial conditions. The group property is a consequence of the injectivity

of the solution operators, and is equivalent to the backward uniqueness of solution for the initial-value problem.

In this work we shall adopt the definition that a set  $Q \subset H$  is *positively invariant* [92, 14, 15, 73] relative to  $H$  for the dynamical system if for each  $u \in Q$ ,

$$\cup_{t \in [0, s]} S(t)u \subset H \Rightarrow \cup_{t \in [0, s]} S(t)u \subset Q \quad \forall s > 0.$$

The set  $Q \subset H$  is called *negatively invariant* relative to  $H$  for the dynamical system if for each  $u \in Q$ ,

$$\cup_{t \in [0, s]} S(-t)u \subset H \Rightarrow \cup_{t \in [0, s]} S(-t)u \subset Q \quad \forall s \geq 0.$$

If the following useful technique for the absorbing or trapping of trajectories

$$S(t)Q = Q, \quad \forall t \in \mathbb{R},$$

holds, then the set  $Q \subset H$  is called *invariant*. We remark that an invariant set is both positively and negatively invariant and consists of orbits or trajectories that are defined for all  $t \in \mathbb{R}$ .

A definition of stability and instability of invariant sets such as critical points, and periodic orbits using  $\varepsilon - \delta$  arguments is given in Hale [18] and the context which is particularly suited for later purposes is provided in [28, 29]. Following [28, 29], a rest state of an initial value problem is *stable* in the  $H^1$ -Sobolev norm if and only if for all  $\varepsilon > 0$ , there is  $\delta > 0$  such that

$$\|u_0\|_{H^1} < \delta \implies \sup_{t \in [0, \infty)} \|u\|_{H^1} < \varepsilon.$$

Furthermore, a rest state of an initial value problem is *asymptotically stable* in the  $H^1$ -Sobolev norm if and only if it is stable and there is  $\gamma \in (0, \infty]$  such that

$$\|u_0\|_{H^1} < \delta \implies \lim_{t \rightarrow \infty} \|u\|_{H^1} = 0.$$

We have defined energy or Lyapunov stability and energy or Lyapunov asymptotic stability using the  $H^1$ -Sobolev norm. Analogous definitions using a similar argument is possible. A rest state is said to be *unstable* if it is not stable. Energy and weighted energy stability has certain general and convenient properties that make it possible to prove existence results without mentioning which norm is utilized. Thus, this approach is taken because these splendid and elegant ideas are applicable to problems other than to the motion of a viscous incompressible stratified fluid with Coriolis force.

We introduce some results from [92, 74, 99, 132] concerning properties of the spectrum  $\sigma(L)$  that will be employed in the investigation of Lipschitz invariant manifolds. Suppose

the spectrum  $\sigma(L)$ , of the linear operator  $L$  is specified by  $\sigma(L) = \sigma^s \cup \sigma^c \cup \sigma^u$  with

$$\sigma^s = \{\lambda \in \sigma(L) : \operatorname{Re}(\lambda) < 0\},$$

$$\sigma^c = \{\lambda \in \sigma(L) : \operatorname{Re}(\lambda) = 0\},$$

$$\sigma^u = \{\lambda \in \sigma(L) : \operatorname{Re}(\lambda) > 0\},$$

where  $\sigma^s$ ,  $\sigma^c$ , and  $\sigma^u$  are spectral subsets of  $\sigma(L)$ , two of which are bounded. The assumption implies there are invariant subspaces corresponding to  $\sigma^s$ ,  $\sigma^c$ , and  $\sigma^u$  denoted by the subspaces  $H^s$ ,  $H^c$ , and  $H^u$ , respectively, so that the decomposition of the vector space  $H$

$$H = H^s \oplus H^c \oplus H^u,$$

is satisfied. The correspondence is that

$$\sigma^s = \sigma(L|_{H^s}), \quad \sigma^c = \sigma(L|_{H^c}), \quad \sigma^u = \sigma(L|_{H^u}).$$

The investigation of a dynamical system in the neighborhood of an invariant set lead to the challenge of constructing the stable, unstable and center manifolds established, for example, in [13, 18, 92, 73] and cited works therein. The stable and unstable invariant manifolds consist of orbits or trajectories which decay to the invariant set in either positive or negative time and correspond to the eigenspaces in the linearized version. Their status as Lipschitz manifolds follows from the fact that they are constructed as graphs of Lipschitz functions in the neighborhood of the invariant set. Furthermore, the manifolds are invariant under the dynamical system relative to some neighborhood of the invariant set. For example, given a neighborhood,  $Q$ , of zero, we get the following definition of the local stable manifold and the local unstable manifold:

$$W_{loc}^s = \{u \in Q : S(t)u \in Q \forall t \geq 0 \text{ and } S(t)u \rightarrow 0 \text{ exponentially as } t \rightarrow \infty\},$$

$$W_{loc}^u = \{u \in Q : S(t)u \in Q \forall t \leq 0 \text{ and } S(t)u \rightarrow 0 \text{ exponentially as } t \rightarrow -\infty\}.$$

Given a neighborhood,  $Q$ , of the rest state zero, a local centre manifold is a Lipschitz manifold  $W_{loc}^c \subset Q$  such that

- $W_{loc}^c$  is invariant relative to  $Q$ ,
- The natural continuous projection  $P^c : H \rightarrow H^c$  implies  $P^c(W_{loc}^c)$  contains a neighborhood of 0 in  $H^c$ ,
- $W_{loc}^c \cap W_{loc}^s = \{0\}$  and  $W_{loc}^c \cap W_{loc}^u = \{0\}$ . In particular, there are Lipschitz functions

$\Phi^u$ ,  $\Phi^s$ , and  $\Phi^c$  so that

$$\Phi^u : P^u(Q) \rightarrow H^s \oplus H^c : W_{loc}^u = \text{graph}\Phi^u, \quad \Phi^u(0) = 0, \quad \text{and} \quad d\Phi^u(0) = 0;$$

$$\Phi^s : P^s(Q) \rightarrow H^u \oplus H^c : W_{loc}^s = \text{graph}\Phi^s, \quad \Phi^s(0) = 0, \quad \text{and} \quad d\Phi^s(0) = 0; \quad (3.9)$$

$$\Phi^c : P^c(Q) \rightarrow H^s \oplus H^u : W_{loc}^c = \text{graph}\Phi^c, \quad \Phi^c(0) = 0, \quad \text{and} \quad d\Phi^c(0) = 0.$$

Here,  $d\Phi(0)$  denotes the Fréchet derivative of  $\Phi$  at zero.

It is known that of central significance in the analysis of dynamical systems is the asymptotic behavior of trajectories such as homoclinic and heteroclinic orbits [17, 13, 79, 132] as depicted in figure 1.1 and 1.2. Geophysical fluid dynamical processes are identified with trajectories of a dynamical system in a suitable phase space and the investigation of asymptotic behavior is reduced to the structure of  $\omega$ -limit sets of these orbits. Invoking these ideas, we define the  $\omega$ -limit set of  $Q \subset H$  with

$$\omega(Q) = \bigcap_{s \geq 0} \text{Cl}(\bigcup_{t \geq s} S(t)Q),$$

where the closure is taken in the Hilbert space  $H$ . From the fact that the arbitrary intersection of closed sets is closed, we deduce that  $\omega$ -limit set of  $Q$  is closed. An invariant set  $\Omega \subset H$  is defined as an *attractor* if there is a neighborhood  $\Xi$  of  $Q$  such that

$$\omega(\Xi) = Q.$$

The definition of attraction using  $\epsilon - \delta$  arguments is given in Temam [68].

In addition to stability and attraction, it is important to note that other properties of the solutions of initial-boundary value problems are also possible. From the modern geometric theory of dynamical systems, differentiability of solution lies in the fact that trajectories nearby a given trajectory yield a variation on the base trajectory which is approximated, to first order, by a linear nonautonomous differential equation. Thus, the equation of variation is obtained by linearizing the nonlinear initial-value problem about a solution and govern the evolution of the Fréchet derivative. The technique of proving differentiability of the solution operators is of great importance as it uses the linearized flow or the variational equation.

A given dynamical system  $u_0 \rightarrow S(t)u_0$  generated by an initial-value problem is Fréchet differentiable in a Hilbert space  $\Upsilon$  if there exists  $L(t, u_0) : \xi \rightarrow \Psi(t)$  given by the solution of the corresponding variational equation such that

$$\frac{\|S(t)v_0 - S(t)u_0 - L(t, u_0).(v_0 - u_0)\|^2}{\|v_0 - u_0\|^2} = o(\|v_0 - u_0\|)$$

as  $v_0 \rightarrow u_0$ .  $L(t, u_0)$  is known as the Fréchet derivative, and has properties similar to the derivative in finite dimensional space: the chain rule holds and the mean value theorem is valid [13].

The following Gronwall's lemma [73, 18, 66, 67, 68] will obtain use in establishing a priori energy type estimates, for example, on the difference between two solutions of the nonlinear initial-value problems to be examined in the sequel.

**Lemma 3.2.1** *Suppose that  $f, g$  and  $u$  are nonnegative locally integrable functions on an interval  $I = (t_0, \infty)$  with the derivative of  $u$ ,  $\frac{du}{dt}$ , locally integrable on  $I = (t_0, \infty)$  and satisfying the differential inequality*

$$\frac{du}{dt} \leq gu + f \quad \text{for } t \geq t_0. \quad (3.10)$$

*If in addition the estimates*

$$\begin{aligned} \int_t^{t+\alpha} g(s) ds &\leq \alpha_1, \\ \int_t^{t+\alpha} f(s) ds &\leq \alpha_2, \\ \int_t^{t+\alpha} u(s) ds &\leq \alpha_3, \end{aligned}$$

*hold for  $t \geq t_0$ , where  $\alpha, \alpha_1, \alpha_2$  and  $\alpha_3$  are nonnegative constants, then the following Gronwall's inequality is valid:*

$$u(t + \alpha) \leq \left( \frac{\alpha_3}{\alpha} + \alpha_2 \right) \exp(\alpha_1) \quad \forall t \geq t_0.$$

The strategy of the Gronwall's lemma is particularly remarkable and interesting in that it can be utilized in the analysis of stability and attractors for solutions of initial-value problems. Obviously this lemma is an extension of the variation of constants formula. Below, we describe analytically how to obtain Lipschitz invariant manifolds for general solution of three-dimensional space variables initial-boundary value problem for viscous  $\beta$ -plane ageostrophic flows with Reynolds stress utilizing the Gronwall's inequality.

## Chapter 4

# Well-posedness of solutions

### 4.1 Existence, uniqueness and differentiability of solutions

#### 4.1.1 $\beta$ -plane ageostrophic equations

The purpose of this section is to develop results of existence, uniqueness and differentiability of solutions to the initial-boundary value problem (2.24) – (2.26) for  $\beta$ -plane ageostrophic flows. We prove well-posedness of solutions utilizing the machinery introduced earlier, and additional techniques developed in this section.

The evolution system of  $\beta$ -plane ageostrophic equations (2.24) – (2.26) is equivalent to the following initial-value problem: find  $U(., t) \in \Upsilon$  such that for a.a.  $t \in (0, T)$ ,

$$\begin{aligned} M \frac{dU}{dt} + LU + N(U) &= 0, \\ U(0) &= U_0, \end{aligned} \tag{4.1}$$

given a suitable choice of function  $U_0$ .

We prove that the initial-value problem admits a unique differentiable solution, provided that time  $T$  and the initial data of the problem are sufficiently small. Without loss of generality, and for simplicity of exposition we assume

$$R = \frac{Ro(Ed)}{Fr^2} = \frac{1}{Ek}.$$

In the above initial-value problem, we have rearranged the operators so that

$$MU = \begin{pmatrix} \frac{Ro}{Ek} u \\ Ed \rho \end{pmatrix};$$

$$\begin{aligned}
N(U) &= B(U, U) - FU; \\
B(U, U) &= B(U) = \begin{pmatrix} \frac{Ro}{Ek} \Pi \underline{u} \cdot \nabla \underline{u} \\ Ed \underline{u} \cdot \nabla \rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_s(\underline{u}, \underline{u}) \\ \mathbf{b}_l(\underline{u}, \rho) \end{pmatrix}; \\
FU &= - \begin{pmatrix} \frac{1}{Ek} \Pi(\rho \underline{k}) \\ \frac{1}{Ek} \underline{u} \cdot \underline{k} \end{pmatrix}; \\
LU &= TU + SU; \\
TU &= \begin{pmatrix} -\Pi \Delta \underline{u} + \frac{\beta Ro}{Ek} \Pi(y \underline{u} \times \underline{k}) \\ -\Delta \rho \end{pmatrix} = \begin{pmatrix} A_s \underline{u} + \frac{\beta Ro}{Ek} \Pi(y \underline{u} \times \underline{k}) \\ A_l \rho \end{pmatrix}; \\
SU &= \begin{pmatrix} \frac{1}{Ek} \Pi(\underline{u} \times \underline{k}) \\ 0 \end{pmatrix}.
\end{aligned}$$

Well-posedness of this problem as the Navier-Stokes equations in the presence of stratification, has been examined utilizing functional-analytic techniques by Temam *et al.* ([66, 67, 26, 27] and cited works therein), but without the Coriolis terms.

In order to develop basic results concerning the existence of a unique solution in the large for the general three-dimensional space variables for the initial-value problem (4.1), we derive a priori energy type estimates useful in proving that (4.1) generate a dynamical system,  $U(t) = S(t)U_0$ , where  $U(t)$  is the unique solution uniformly bounded in finite-time and  $S(t)$  is a group of continuous nonlinear solution operators. The principal result concerning the existence of such a unique solution will be proven using the Faedo-Galerkin technique, and a priori energy type estimates. Concerning the linear operator  $L$  of the initial-value problem (4.1), the inequality

$$(LU, U) = (TU, U) = \frac{Ek}{Ro} \|\nabla \underline{u}\|^2 + \frac{1}{Ed} \|\nabla \rho\|^2 \quad (4.2)$$

holds. It follows from standard results that the operator  $T$  is self-adjoint and its spectrum  $\sigma(T)$  satisfies  $\sigma(T) \subseteq [0, \infty)$ . The following result on the spectrum of  $L$  developed in [28, 29, 48, 42] will be utilized.

**Lemma 4.1.1**  $\sigma(L) \subseteq [0, \infty) \cup \Pi$ , where  $\Pi$  is either empty or an at most denumerable set consisting of isolated, positive eigenvalues  $\lambda_n = \lambda_n(R)$  with finite multiplicity such that

$$\infty > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \dots,$$

clustering at zero. Furthermore, whenever

$$R \leq P_F$$

then  $\Pi = \emptyset$ , whereas if

$$R \geq P_F,$$

then  $\Pi \neq \emptyset$ . Also, zero is not an eigenvalue if

$$R \leq P_F,$$

with  $P_F \in (0, \infty)$  obtained from the Poincaré-Friedrichs type result (3.3).

Before giving the proof of the lemma, we make some remarks and investigate the steady linear version of the nonlinear initial-value problem. This examination demonstrates that the steady linear system has a unique solution, and provides useful properties which are employed in an analysis of the nonlinear initial-value problem. We begin by recalling that when  $\sigma(L) \subseteq (0, \infty)$ , which holds if

$$R \leq P_F,$$

then

$$(LU, U) = (TU, U) \geq \lambda \|U\|^2 > 0, \quad (4.3)$$

which in turn implies that the bilinear form induced by  $L$  is coercive and  $L^{-1}$  is compact and self-adjoint. This establishes existence and uniqueness of a solution to the steady linear problem, a consequence of the Riesz-Fréchet representation theorem or the Lax-Milgram lemma. From mini-max problems,  $\lambda$  is the smallest eigenvalue of the Laplace and Stokes operators. Also, the symmetry of the bilinear form  $a(U, W)$  together with the coerciveness of  $a(U, U)$  imply the existence of a sequence of positive eigenvalues and a corresponding sequence of eigensolutions which yield an orthonormal basis  $\{W_i\}$  of  $\Upsilon$  such that

$$LW_i = \lambda_i W_i, \quad (4.4)$$

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots,$$

and  $\lambda_i \rightarrow \infty$  as  $i \rightarrow \infty$ . Furthermore, we have the identities

$$mc_0 \lambda_1 \leq \lambda_m \leq mc_1 \lambda_1 \quad \text{for } m = 1, 2, \dots \quad (4.5)$$

which holds for positive constants  $c_0$  and  $c_1$ .

Moreover, the inequalities

$$\|L^{\frac{1}{2}}U\| \geq \lambda_1^{\frac{1}{2}} \|U\|, \quad (4.6)$$

$$\|LU\| \geq \lambda_1^{\frac{1}{2}} \|L^{\frac{1}{2}}U\|,$$

are satisfied.

Next, we focus our attention to establishing the above lemma.

**Proof:** According to a spectral perturbation theorem, if  $TU$  is closed and  $SU$  is *relatively*

compact with respect to  $TU$ , then  $\sigma(L = T + S)$  and  $\sigma(T)$  differ by at most a denumerable number of isolated, positive eigenvalues of finite multiplicity clustering at zero. Thus it suffices to show that  $S$  is relatively compact with respect to  $T$ , since from [28, 29, 48, 42] we have  $\sigma(T) \subseteq [0, \infty)$ . By definition,  $S$  is relatively compact with respect to  $T$  if for any sequence  $\{U_n\} \subset \Upsilon$ ,

$$\|U_n\| + \|TU_n\| \leq \alpha, \quad \text{uniformly in } n, \quad (4.7)$$

then there is a subsequence  $\{U_{n_k}\} \subset \{U_n\}$  such that  $SU_{n_k}$  is strongly convergent. With strong convergence, we can make strong statements. By virtue of Ascoli-Arzelà theorem  $\{SU_{n_k}\}$  contains a uniformly convergent subsequence which is a Cauchy sequence in a complete normed space  $\Upsilon$ .

We proceed to put together a series of results that will be employed in establishing properties of solutions to the initial-value problem (4.1). The following mini-max principle employed in [66, 26] will be utilized:

$$\lim_{t \rightarrow \infty} \sup \|\rho(t)\|^2 \leq \|\Omega\|. \quad (4.8)$$

Here  $\|\Omega\|$  is the measure of  $\Omega \subset \mathbb{R}^3$  and represents the flow domain.

We define geophysically relevant parameters  $\alpha$  and  $\alpha_1$  by  $\alpha = \max\{\frac{Ek}{Ro}, \frac{1}{Ed}\}$  and  $\alpha_1 = \max\{\frac{1}{Ro}, \frac{Ro}{Fr^2}\}$ . By the orthogonal property of the nonlinear operator, we get

$$(B(U), U) = \int_{\Omega} [(\underline{u} \cdot \nabla) \underline{u}] \cdot \underline{u} + ((\underline{u} \cdot \nabla) \rho) \rho dx = 0.$$

Next, we exhibit a series of a priori estimates which will be required in establishing the existence, uniqueness and differentiability properties of solutions to the problem. Taking the inner product of (4.1) with  $U$ , utilizing inequality (4.6), the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} \|U\|^2 + \alpha \lambda_1 \|U\|^2 \\ & \leq \frac{d}{dt} \|U\|^2 + \alpha \|L^{\frac{1}{2}} U\|^2 \\ & \leq \frac{\lambda_1}{\alpha} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \|\rho\|^2. \end{aligned} \quad (4.9)$$

The use of the Gronwall's inequality (3.10) in (4.9) and the mini-max principle (4.8) provides the a priori estimate

$$\|U(t)\|^2 \leq \|U_0\|^2 \exp(-\alpha t) + \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4 \frac{\lambda_1^2}{\alpha^2} \|\Omega\| (1 - \exp(-\alpha t)). \quad (4.10)$$

Therefore, we get energy stability with flow energy and entropy production specified by

$$E(t) = \frac{1}{2} \|U(t)\|^2 = \frac{1}{2} \int_{\Omega} (\underline{u}^2 + \rho^2) dx.$$

Furthermore, in the limit we obtain

$$\lim_{t \rightarrow \infty} \sup \|U(t)\|^2 \leq \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^4 \frac{\lambda_1^2}{\alpha^2} \|\Omega\| \equiv \rho_1^2, \quad (4.11)$$

which shows that  $U(t)$  is uniformly bounded for all time in  $\Upsilon$ .

Next, by the mini-max principle (4.8) and uniform bound (4.11), it follows that for any  $U_0 \in \Upsilon$  and  $\varepsilon$  there exists  $T_1 = T_1(U_0, \varepsilon)$  such that

$$\|\rho(t)\| \leq \|\Omega\|^{\frac{1}{2}} + \varepsilon \quad \text{for } t \geq T_1. \quad (4.12)$$

Integration of the second inequality in (4.9) and by virtue of (4.12) we obtain the a priori energy type estimate

$$\int_0^{T_1} \|L^{\frac{1}{2}}U(t)\|^2 dt \leq \frac{1}{\alpha} \left\{ \|U_0\|^2 + \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \frac{\lambda_1}{\alpha} (\|\Omega\|^{\frac{1}{2}} + \varepsilon)^2 \right\} \equiv \rho_{11}^2.$$

To dominate the nonlinearity, we proceed by taking the inner product of (4.1) with  $LU$  and making use of the inequality (4.6), the Cauchy-Schwarz inequality, Young's inequality and Sobolev inequality, to obtain

$$\begin{aligned} & \frac{d}{dt} \|L^{\frac{1}{2}}U\|^2 + \lambda_1 \alpha \|L^{\frac{1}{2}}U\|^2 \\ & \leq \frac{d}{dt} \|L^{\frac{1}{2}}U\|^2 + \alpha \|LU\|^2 \end{aligned} \quad (4.13)$$

$$\leq 2\alpha \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \|\rho\|^2 + \frac{c_0^2}{4\alpha} \|L^{\frac{1}{2}}U\|^6,$$

containing spatial derivatives of higher order. For simplicity of exposition, by setting

$$c = \max\left\{ \frac{c_0^2}{4\alpha}, 2\alpha \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 (\|\Omega\|^{\frac{1}{2}} + \varepsilon)^2 \right\},$$

with the assistance of (4.13) and (4.12) we get the differential inequality

$$\frac{d}{dt} (1 + \|L^{\frac{1}{2}}U\|^2) \leq c(1 + \|L^{\frac{1}{2}}U\|^2)^3. \quad (4.14)$$

Invoking Gronwall's inequality (3.10) in the differential inequality (4.14), we have

$$\|L^{\frac{1}{2}}U\|^2 \leq 1 + \|L^{\frac{1}{2}}U_0\|^2 \quad (4.15)$$

which is valid for finite-time

$$t \leq T_2(\|L^{\frac{1}{2}}U_0\|) = \frac{3}{8c(1 + \|L^{\frac{1}{2}}U_0\|^2)}. \quad (4.16)$$

Consequently, putting the pieces together we obtain

$$\sup_{t \in [0, T_2]} \|L^{\frac{1}{2}}U\|^2 \leq 1 + \|L^{\frac{1}{2}}U_0\|^2 \equiv \rho_{20}^2, \quad (4.17)$$

which shows that  $U(t)$  is bounded for finite-time  $T_2$  in  $D(L)$ . Integrating the differential inequality (4.14) and utilizing (4.17) provide the a priori energy type estimate

$$\int_0^{T_2} \|LU(t)\|^2 dt \leq \frac{1}{\alpha} \{ \|L^{\frac{1}{2}}U_0\|^2 + 2\alpha \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 (\|\Omega\|^{\frac{1}{2}} + \varepsilon)^2 + \frac{c_0^2}{4\alpha} \rho_{20}^6 \} \equiv \rho_{21}^2.$$

We proceed to state existence, uniqueness, continuity and differentiability with respect to initial conditions of solution to the initial-value problem (4.1) and utilize the above a priori estimates to prove the results. Set  $I = (0, T)$  and  $\bar{I} = [0, T]$  where the finite time  $T > 0$  satisfies  $T = \min\{T_1, T_2\}$  with  $T_1$  and  $T_2$  specified in (4.12) and (4.16).

**Proposition 4.1.2** *Under the above hypotheses for  $U_0 \in \Upsilon$  given, (4.1) generate the dynamical system  $U(t) = S(t)U_0$  satisfying*

$$U \in C(\bar{I}; \Upsilon) \cap L^2(I; D(L^{1/2})).$$

Furthermore, if  $U_0 \in D(L^{1/2})$  then

$$U \in C(\bar{I}; D(L^{1/2})) \cap L^2(I; D(L)).$$

**Proof:** For fixed  $m$ , the Faedo-Galerkin approximation

$$U_m = \sum_{i=1}^m g_{im}(t) W_i$$

of the solution  $U$  of (4.1) is defined by the finite-dimensional system of nonlinear ordinary differential equations for  $g_{im}(t)$  given by

$$\frac{dU_m}{dt} + LU_m + \Pi_m N(U_m) = 0, \quad (4.18)$$

$$U_m(0) = \Pi_m U_0,$$

where  $\Pi = \Pi_m$  is the spectral projection of rank  $m$  in  $\Upsilon, D(L^{1/2}), D(L^{-1/2}),$  or  $D(L)$  onto the space spanned by the  $m$  eigenfunctions of  $L$  given in (4.4). Let an eigenspace  $\mathbf{X}_0 = \Pi_m D(L)$  be the finite-dimensional Hilbert space spanned by the  $m$  eigenfunctions of  $L$  given in (4.4) with inner product  $((\cdot, \cdot))$  induced by  $D(L)$ . Set  $\mathbf{X}_1 = \Pi_m D(L^{1/2})$  and  $\mathbf{X}_2 = \Pi_m D(L^{-1/2})$ . By the a priori estimates on  $L$  and  $N$  we get existence of solutions to the finite-dimensional system of nonlinear ordinary differential equations (4.18). The technique of the proof is fundamental in an analysis of dynamical systems and will recur

throughout this work. We illustrate this proof used in Temam [66, 67, 68], customized for  $\beta$ -plane ageostrophic equations without Reynolds stress. The result consists of examining the linear initial-value problem

$$\begin{aligned}\frac{dU_m}{dt} + LU_m &= 0, \\ U_m(0) &= \Pi_m U_0.\end{aligned}\tag{4.19}$$

For any  $U_m(0) \in \mathbf{X}_0$ , we get existence and uniqueness of solution to the linear initial-value problem (4.19) given by

$$U_m \in C([0, \infty); \mathbf{X}_0) \cap C((0, \infty); \mathbf{X}_1).$$

Moreover, we have a mapping given by the linear solution operators

$$S(t) = \exp(-tL) : U_m(0) \rightarrow U_m(t),$$

which are continuous from the function space  $\mathbf{X}_2$  into  $\mathbf{X}_0$ ,  $\forall t > 0$ , and from  $\mathbf{X}_0$  into itself,  $\forall t \geq 0$ . Invoking the same inequalities that led to the linear analogue of the differential inequality (4.13) and by utilization of Gronwall's inequality (3.10), we find that

$$\|LU_m(t)\|^2 \leq \|LU_m(0)\|^2 \exp(-\lambda_1 \alpha t).$$

Consequently, we obtain the estimate

$$\|\exp(-tL)\|_{\mathbf{L}(\mathbf{X}_0)} \leq \exp(-\lambda_1 \alpha t) \quad \forall t \geq 0,$$

which implies that the norm of  $\exp(-tL)$  in  $\mathbf{L}(\mathbf{X}_0)$  is bounded and therefore continuous. Here,

$$\mathbf{L}(\mathbf{X}_0) = \mathbf{L}(\mathbf{X}_0, \mathbf{X}_0)$$

denotes the Banach space of bounded operators from  $\mathbf{X}_0$  into itself. Similarly,

$$\|\exp(-tL)\|_{\mathbf{L}(\mathbf{X}_2, \mathbf{X}_0)} \leq \left(\frac{1}{2t} + \lambda_1 \alpha\right)^{\frac{1}{2}} \exp(-\lambda_1 \alpha t), \quad \forall t > 0,$$

implies that the norm of  $\exp(-tL)$  in  $\mathbf{L}(\mathbf{X}_2, \mathbf{X}_0)$  is bounded. Uniqueness of a solution to the linear initial-value problem (4.19) follows by considering the difference of two solutions  $U_m(t) = U_m^1(t) - U_m^2(t)$ , with  $U_m^1(t)$  and  $U_m^2(t)$  satisfying (4.19). By courtesy of the Gronwall's inequality, we get

$$\|LU_m(t)\|^2 \leq \|LU_m(t_0)\|^2 \exp(-\lambda_1 \alpha (t - t_0)).$$

Hence, we obtain uniqueness through  $\|LU_m(t)\| = 0$  as  $t_0 \rightarrow -\infty$  due to the fact that  $\|LU_m(t_0)\|$  is bounded.

Integration of the nonlinear initial-value problem (4.18) using lemma (3.1) and the above a priori energy type estimates provide a continuous mapping defined by the integral of (4.18), given by the formula

$$U_m(t) = S(t - t_0)U_m(t_0) - \int_{t_0}^t S(t - \tau)\Pi_m N(U_m)d\tau,$$

where the continuous solution operators  $S(t) = \exp(-tL)$  satisfy the linear initial-value problem (4.19). This integral equation defines a continuous mapping in  $\mathbf{X}_0$  from a closed ball of radius  $\rho_{21}^2$  into itself. Thus, by virtue of a fixed point argument, the finite-dimensional system of nonlinear ordinary differential equations (4.18) has at least one solution  $U_m$  inside the ball of radius  $\rho_{21}^2$  contained in  $\mathbf{X}_0$ .

The passage to the limit

$$m \rightarrow \infty \text{ and } T_m \rightarrow T$$

follows from the subsequent arguments and a priori estimates. Utilizing the differential inequality (4.9) yields

$$\begin{aligned} \frac{d}{dt}\|U_m\|^2 + \alpha\lambda_1\|U_m\|^2 \\ \leq \frac{\lambda_1}{\alpha}\left(\frac{R_0}{Fr^2} + \frac{1}{R_0}\right)^2\|\rho_m\|^2 \end{aligned} \tag{4.20}$$

which together with the a priori estimate (4.10) and the next one implies that  $U_m$  remains bounded in

$$L^\infty(I; \Upsilon) \cap L^2(I; D(L^{1/2})).$$

This useful result combined with the weak compactness implies there exists a subsequence, also denoted by  $U_m$ , and

$$U \in L^\infty(I; \Upsilon) \cap L^2(I; D(L^{1/2}))$$

such that

$$U_m \rightarrow \begin{cases} U \in L^2(I; D(L^{1/2})) & \text{weakly} \\ U \in L^\infty(I; \Upsilon) & \text{weak-star.} \end{cases}$$

By invoking the same inequalities that led to the differential inequality (4.13), we obtain the boundness of  $N(U_m)$  and  $\Pi_m N(U_m)$  in  $L^2(I; D(L^{-1/2}))$ . Furthermore, from (4.18), it follows that  $\frac{dU_m}{dt}$  is also bounded in  $L^2(I; D(L^{-1/2}))$ . Employing this result and weak compactness, we get

$$\frac{dU_m}{dt} \rightarrow \frac{dU}{dt} \in L^2(I; D(L^{-1/2})) \text{ weakly.}$$

By virtue of the Lebesgue dominated convergence theorem as well as the above weak convergence results,

$$U_m \rightarrow U \in L^2(I; \Upsilon) \text{ strongly.}$$

Therefore, by application of the above results (3.1) – (3.2) we pass to the limit in (4.18) and obtain the required result (4.1). Additionally, using the properties listed in the above results (3.1) – (3.2) and the compact injections (3.5) we obtain

$$U \in C(\bar{I}; \Upsilon) \cap L^2(I; D(L^{1/2})).$$

A similar argument is employed in illustrating

$$U \in C(\bar{I}; D(L^{1/2})) \cap L^2(I; D(L))$$

and due to the estimates (4.17), we conclude that if  $U_0 \in D(L^{1/2})$  and by the Faedo-Galerkin technique

$$U \in C(\bar{I}; D(L^{1/2})) \cap L^2(I; D(L)) \quad \forall T > 0.$$

According to Gronwall's inequality (3.10), uniqueness and continuity with respect to initial conditions of solution  $U(t)$  follows from considering the difference between two solutions of the initial-value problem (4.1) and employing a priori estimates (4.10) – (4.17) including the inequalities (3.1) – (3.2) provided the hypotheses of the lemmas hold.

Uniqueness and continuity with respect to initial conditions of solution  $U(t)$  generate a dynamical system which is described by continuous solution operators  $S(t)$ ,  $t \in \mathbb{R}$  defined by

$$S(t)U_0 = U(t) \equiv S(U_0, t)$$

satisfying the group property

$$S(t)U(s) = S(s)U(t), \tag{4.21}$$

$$S(0)U = U \quad \forall t, s \in \mathbb{R}$$

such that  $t+s \leq T$ . That the solution operators  $S(t)$  are continuous is a consequence of the continuity of  $U(t)$  in time and the initial conditions. The group property is a consequence of the injectivity of the solution operators which is equivalent to the backward uniqueness of the solution to the initial-value problem (4.1).

We proceed with an analysis of Fréchet differentiability with respect to initial conditions of the solution  $U(t)$  to the initial-value problem (4.1) with the work in Temam [66, 67, 68] as a source of inspiration. The technique of the proof uses the linearization of the evolution equation (4.1) about the difference of two given solutions. Let  $u$  and  $v$  be solutions of the initial value problem (4.1) corresponding to the initial conditions  $u(0) = u_0$  and  $v(0) = v_0$ ,

repectively. Then the difference  $w = v - u$  satisfies the semilinear problem

$$\begin{aligned} \frac{dw}{dt} + Lw + N(v) - N(u) &= 0, \\ w(0) &= v_0 - u_0, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} N(v) - N(u) &= B(v, v) - B(u, u) - Fw \\ &= B(v, w) + B(w, u) - Fw \\ &= B(u, w) + B(w, u) + B(w, w) - Fw. \end{aligned} \tag{4.23}$$

The linearization of the nonlinear equation (4.22) about a given solution reduces to the equation

$$\begin{aligned} \frac{d\Psi}{dt} + L\Psi + l_0(t)\Psi &= 0, \\ \Psi(0) &= v_0 - u_0 = \xi, \end{aligned} \tag{4.24}$$

where  $l_0(t)w = B(u(t), w) + B(w, u(t))$  and  $l_0(t) \in \mathbf{L}(D(L^{1/2}), D(L^{-1/2}))$ . As above,

$$\mathbf{L}(D(L^{1/2}), D(L^{-1/2}))$$

denotes a Banach space of bounded linear operators from  $D(L^{1/2})$  to  $D(L^{-1/2})$ . Consequently, the above equation is a linear nonautonomous ordinary differential equation.

The difference  $\varphi = w - \Psi$  satisfies the semilinear problem

$$\begin{aligned} \frac{d\varphi}{dt} + L\varphi + l_0(t)\varphi + l_1(t; w(t)) &= 0, \\ \varphi(0) &= 0, \end{aligned} \tag{4.25}$$

where  $l_1(t; w(t)) = B(w, w) - Fw$ . From (4.23), we have  $N(v) - N(u) = l_0(t)w + l_1(t; w)$ . The linear operator  $L + l_0 + l_1$  may be viewed as a compact perturbation of  $L$  whose spectrum can be estimated directly [48, 42]. In fact,  $D(L + l_0 + l_1) = D(L)$  and  $\|\cdot\|_{D(L + l_0 + l_1)}$  is a norm equivalent to  $\|\cdot\|_{D(L)}$  on that function space.

**Proposition 4.1.3** *If  $u_0 \in \Upsilon$  and  $u$  is the unique solution of the nonlinear initial-value problem (4.1) then the corresponding variational equation (4.24) has a unique solution  $\Psi$  satisfying*

$$\Psi \in C(\bar{I}; \Upsilon) \cap L^2(I; D(L^{\frac{1}{2}})).$$

*Furthermore, the dynamical system  $u_0 \rightarrow S(t)u_0$  is Fréchet differentiable in  $\Upsilon$  with differential*

$$L(t, u_0) : \xi \rightarrow \Psi(t)$$

given by the solution of the variational equation.

**Proof:** Existence and uniqueness of the solution to the variational equation (4.24) is proved using Gronwall's lemma and the Faedo-Galerkin technique and the above a priori energy type estimates. Taking the inner product of the equation (4.22) with  $w$  and applying the Cauchy-Schwarz inequality, Young's inequality and the Sobolev inequalities, we get

$$\frac{d}{dt}\|w\|^2 + \|L^{\frac{1}{2}}w\|^2 \leq k^2\|w\|^2 \quad (4.26)$$

where  $k = c_1\rho_{21}$  and  $\rho_{21}$  is given in the estimate after (4.17). By virtue of Gronwall's inequality (3.10) we obtain

$$\|v(t) - u(t)\|^2 \leq \exp(k^2T)\|v_0 - u_0\|^2 \quad \forall t \in (0, T). \quad (4.27)$$

Let  $t_0 = \frac{1}{2k^2} \log 2$ ; then (4.27) reduces to

$$\|S(t)v_0 - S(t)u_0\|^2 \leq \frac{1}{2}\|v_0 - u_0\|^2 \quad (4.28)$$

for  $t_0 \leq t \leq 2t_0$  which implies the dynamical system is Lipschitz continuous with respect to the initial conditions. The a priori estimate (4.27) illustrates forward uniqueness. Furthermore, invoking the estimates (4.26), we find that

$$\int_0^t \|L^{\frac{1}{2}}w(s)\|^2 ds \leq \exp(k^2T)\|v_0 - u_0\|^2.$$

Additionally, taking the inner product of equation (4.25) with  $\varphi$  we get

$$\frac{d}{dt}\|\varphi\|^2 + \frac{1}{2}\|L^{\frac{1}{2}}\varphi\|^2 \leq k^2\|\varphi\|^2 + \frac{c_1^2}{\lambda_1}\|L^{\frac{1}{2}}w(t)\|^3. \quad (4.29)$$

By utility of Gronwall's inequality (3.10) we obtain the estimate

$$\|\varphi(t)\|^2 \leq \frac{c_1^2}{\lambda_1} \int_0^T \|L^{\frac{1}{2}}w(s)\|^3 ds$$

for all  $t \in \bar{I}$ . Consequently, it follows that

$$\|\varphi(t)\|^2 \leq \frac{c_1^2}{\lambda_1} \exp(3k^2T/2)\|v_0 - u_0\| \quad (4.30)$$

which gives the required result

$$\frac{\|S(t)v_0 - S(t)u_0 - L(t, u_0) \cdot (v_0 - u_0)\|^2}{\|v_0 - u_0\|^2} = o(\|v_0 - u_0\|)$$

as  $v_0 \rightarrow u_0$ . Thus, the dynamical system

$$S(t) : u_0 \rightarrow u(t)$$

is Fréchet differentiable at  $u_0$  in the Hilbert space  $\Upsilon$ .

Next, we investigate the problem of proving backward uniqueness of the solution to the initial-value equation (4.1) that establishes the injectivity properties and group properties of the solution operators  $S(t)$ . The backward uniqueness of solution is proved in approximately the same way as Fréchet differentiability result using the log-convexity method that has been employed in [2, 67]. In order to motivate this objective of the injectivity of the solution operators, consider two solutions  $u$  and  $v$  of the initial-value problem elaborated above such that at time  $t = t_*$  both  $u$  and  $v$  satisfy the equation (4.1) for  $t \in (t_* - \epsilon, t_*)$  and  $u(t_*) = v(t_*)$ . Given that this hypothesis is valid, backward uniqueness of the solution operators is accomplished if  $u(t) = v(t)$  for all time  $t < t_*$  whenever the solutions are well-defined. Alternatively, suppose that the solution operators satisfy

$$S(t) = u(t + s) \quad \forall t > 0 \quad s \in \mathbb{R},$$

then for  $\tau \in (0, \epsilon)$ , the problem of backward uniqueness is equivalent to satisfaction of the inequalities

$$S(\tau)u(t_* - \tau) = S(\tau)v(t_* - \tau) \Rightarrow u(t_* - \tau) = v(t_* - \tau), \quad (4.31)$$

which implies the injectivity of the solution operators  $S(\tau)$ .

**Proposition 4.1.4** *Suppose that*

$$u, v \in L^\infty(I; D(L^{\frac{1}{2}})) \cap L^2(I; D(L));$$

*then the difference  $w = v - u$  satisfies the differential equation (4.22) for  $t \in (0, T)$  and the solution operators  $S(t)$  are injective.*

**Proof:** We will use the notation  $\varphi = \varphi(t)$  for the quotient

$$\begin{aligned} \varphi &= \frac{\|w(t)\|_V^2}{\|w(t)\|_H^2} \\ &= \frac{(Lw(t), w(t))}{(w(t), w(t))}. \end{aligned}$$

Putting the quotient  $\varphi = \varphi(t)$  into inequalities (3.1) – (3.2) and invoking the Cauchy-Schwarz inequality, Young's inequality, Sobolev inequalities, continuity properties of the nonlinear operator established above and the fact that  $(Lw - \varphi w, w) = 0$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \varphi = \frac{((\frac{dw}{dt}, w))}{\|w\|^2} - \frac{\|w\|_V^2}{\|w\|^4} (\frac{dw}{dt}, w)$$

$$\begin{aligned}
&= \frac{1}{\|w\|^2} \left( \frac{dw}{dt}, Lw - \varphi w \right) \\
&= \frac{1}{\|w\|_V^2} (N(v) - N(u) - Lw, Lw - \varphi w) \\
&= -\frac{\|Lw - \varphi w\|^2}{\|w\|^2} + \frac{1}{\|w\|^2} (Lw, N(v) - N(u)) \\
&\leq -\frac{\|Lw - \varphi w\|^2}{2\|w\|^2} + \frac{\|N(v) - N(u)\|^2}{2\|w\|^2} \\
&\leq -\frac{\|Lw - \varphi w\|^2}{2\|w\|^2} + k^2 \varphi
\end{aligned}$$

where  $N(v) - N(u)$  is given in (4.23). The function  $k(t) = k_1(t) + \alpha_1$  is defined by the mapping

$$k_1(t) : t \rightarrow c(\|u(t)\|_V^{\frac{1}{2}} \|Lu(t)\|^{\frac{1}{2}} + \|v(t)\|_V^{\frac{1}{2}} \|Lv(t)\|^{\frac{1}{2}}) \in L^2(0, T),$$

and the geophysically relevant parameter  $\alpha_1$  denotes  $\alpha_1 = \max\{\frac{1}{Ro}, \frac{Ro}{Fr^2}\}$ .

The use of Gronwall's inequality (3.10) in the preceding differential inequality gives

$$\varphi(t) \leq \varphi(0) \exp(2 \int_0^t k^2(s) ds) \quad t \in I = (0, T).$$

Thus, if the difference  $w = v - u$  satisfies the above conditions, then

$$w(\tau) = 0 \quad \Rightarrow \quad w(t) = 0, \quad t \in \bar{I} = [0, T].$$

To illustrate this, we proceed by contradiction. Hence, suppose that  $\|w(t_0)\| \neq 0$  for  $t_0 \in [0, T)$ . As a result by continuity we have  $\|w(t)\| \neq 0$  on some open interval  $(t_0, t_0 + \epsilon)$  and we employ the notation  $t_* \leq T$  for the largest time for which

$$w(t) \neq 0 \quad \forall t \in [t_0, t_*).$$

It follows that  $w(t_*) = 0$ . In the interval  $[t_0, t_*)$  the mapping given by  $t \rightarrow \log \|w(t)\|$  is well-defined and by courtesy of inequalities (3.1) – (3.2) and the fact that  $w(t)$  satisfies the differential equation (4.22) we obtain

$$\frac{d}{dt} \log \frac{1}{\|w\|} \leq 2\varphi + k^2.$$

Consequently, by virtue of inequalities (3.1) – (3.2) for the time  $t \in [t_0, t_*)$

$$\log \frac{1}{\|w\|} \leq \log \frac{1}{\|w(t_0)\|} + \int_{t_0}^t 2\varphi(s) + k^2(s) ds$$

which illustrates the boundedness of  $\frac{1}{w(t)}$  from above as the time  $t \rightarrow t_*$  from below and this is a contradiction. This proves the desired result.

#### 4.1.2 $\beta$ -plane ageostrophic equations with Reynolds stress

The evolution system of  $\beta$ -plane ageostrophic equations with Reynolds stress (2.28) – (2.30) is equivalent to the following initial-value problem: to find  $U(\cdot, t) \in \Upsilon$  such that for a.a.  $t \in (0, T)$

$$\begin{aligned} M \frac{dU}{dt} + LU + N(U) &= 0, \\ U(0) &= U_0, \end{aligned} \quad (4.32)$$

given suitable choice of functions  $U_0$ . Without loss of generality, and for simplicity of exposition we assume

$$R = \frac{Ro(Ed)}{Fr^2} = \frac{1}{Ek}.$$

The goal of this section is to prove that the initial-value problem admits a unique differentiable solution. Concerning the operators in the above initial-value problem, we have

$$\begin{aligned} MU &= \begin{pmatrix} \frac{Ro}{Ek} \underline{u} \\ Ed \quad \rho \end{pmatrix}; \\ N(U) &= B(U, U) + CU - FU; \\ B(U, U) &= B(U) = \begin{pmatrix} \frac{Ro}{Ek} \underline{u} \cdot \nabla \underline{u} \\ Ed \quad \underline{u} \cdot \nabla \rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_s(\underline{u}, \underline{u}) \\ \mathbf{b}_l(\underline{u}, \rho) \end{pmatrix}; \\ CU &= - \begin{pmatrix} \Delta \underline{u} \\ \Delta \rho \end{pmatrix} = \begin{pmatrix} A_l \underline{u} \\ A_l \rho \end{pmatrix}; \\ FU &= - \begin{pmatrix} \frac{1}{Ek} (\rho \underline{k}) \\ \underline{u} \cdot \underline{k} \end{pmatrix}; \\ LU &= TU + SU; \\ TU &= \begin{pmatrix} \Delta^6 \underline{u} + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) \\ \Delta^6 \rho \end{pmatrix} = \begin{pmatrix} A_s^6 \underline{u} + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) \\ A_l^6 \rho \end{pmatrix}; \\ SU &= \begin{pmatrix} \frac{1}{Ek} (\underline{u} \times \underline{k}) \\ 0 \end{pmatrix}. \end{aligned}$$

In order to develop fundamental results concerning whether there exists a unique solution in the large for the general three-dimensional space variables for initial-value problem (4.32), we derive a priori energy type estimates useful in proving that (4.32) generate a dynamical system, denoted,  $U(t) = S(t)U_0$ , where  $U(t)$  is the unique solution uniformly bounded in time and  $S(t)$  is a group of continuous nonlinear solution operators. The principal result concerning the existence of such a unique solution will be proven by the utility of the Faedo-Galerkin technique, and a priori energy type estimates. The linear operator  $L$  of the initial-value problem (4.32) satisfies the inequality

$$(LU, U) = (TU, U) \equiv a(U, U) \quad (4.33)$$

where  $a(., .)$  is symmetric bilinear form. Therefore,  $L$  is self-adjoint. Additionally,

$$(LU, U) = (TU, U) \geq \lambda \|U\|^2 > 0 \quad (4.34)$$

which implies the bilinear form induced by  $L$  is coercive and  $L^{-1}$  is compact and self-adjoint. It is well-known that  $\lambda$  is the smallest eigenvalue of Laplace's operator and Stokes operator. Also, the symmetry of the bilinear form  $a(U, W)$ , together with the coerciveness of  $a(U, U)$  imply the existence of a sequence of positive eigenvalues and a corresponding sequence of eigensolutions which yield an orthonormal basis  $\{W_i\}$  of  $\Upsilon$  such that

$$LW_i = \lambda_i W_i \quad (4.35)$$

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots,$$

and  $\lambda_i \rightarrow \infty$  as  $i \rightarrow \infty$ . Furthermore, the eigenvalues satisfies the identities

$$\lambda_m \equiv \zeta \lambda_1 m^4 \text{ as } m \rightarrow \infty \quad (4.36)$$

for some positive constant  $\zeta$ .

Moreover, the inequalities

$$\|L^{\frac{1}{2}}U\| \geq \lambda_1^{\frac{1}{2}} \|U\|, \quad (4.37)$$

$$\|LU\| \geq \lambda_1^{\frac{1}{2}} \|L^{\frac{1}{2}}U\|,$$

hold. The following useful inequalities are utilized in the derivation of a priori energy type estimates:

$$(B(U), U) = \int_{\Omega} [((\underline{u} \cdot \nabla)\underline{u}) \cdot \underline{u}] + ((\underline{u} \cdot \nabla)\rho)\rho dx = 0,$$

$$\|CU\|^2 \leq \lambda_1 \|L^{\frac{1}{2}}U\|^2 \leq \lambda_1^{\frac{1}{2}} \|LU\| \|L^{\frac{1}{2}}U\|. \quad (4.38)$$

From the fact that the operators  $L$  and  $C$  commutes, then the inequality

$$\|L^{\frac{1}{2}}CU\|^2 \leq \lambda_1 \|LU\|^2 \quad (4.39)$$

is valid. By courtesy of the Sobolev inequality and and continuity properties of the nonlinearity  $B$ , the inequalities

$$\begin{aligned} \|B(U)\|^2 &\leq c_0^2 (CU, U) [\|CU\|^2 + \|U\|^2] \\ &\leq c_0^2 (\lambda_1 + 1/\lambda_1) \|U\| \|L^{1/2}U\|^2 \|LU\|, \end{aligned} \quad (4.40)$$

are satisfied. By virtue of Cauchy-Schwarz inequality, Young's inequality and (4.40) the estimates

$$|(B(U) + CU, LU)| \leq 54(c_1^4 \|U\|^2 \|L^{1/2}U\|^2 + \lambda_1) \|L^{1/2}U\|^2 + \frac{1}{4} \|LU\|^2, \quad (4.41)$$

are also satisfied, where  $c_1 = c_0^2(\lambda_1 + 1/\lambda_1)$ . Furthermore, the a priori estimates

$$\|L^{1/2}B(U)\|^2 \leq c \|LU\|^2 \quad (4.42)$$

holds. Concerning the solution to the steady linear case we have

$$\begin{aligned} (LU + CU, U) &= (TU, U) + (CU, U) \\ &\geq (\lambda + \lambda_1^{\frac{1}{6}}) \|U\|^2 > 0, \end{aligned} \quad (4.43)$$

which implies  $L + C$  is positive and hence the bilinear form it induces is coercive. This establishes existence and uniqueness of solution to the steady linear problem, the Riesz-Fréchet representation theorem or the Lax-Milgram lemma.

The following mini-max principle [66, 26] will be employed:

$$\lim_{t \rightarrow \infty} \sup \|F\|^2 \leq \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \|\Omega\|^2 \quad (4.44)$$

where  $\|\Omega\|$  is the measure of  $\Omega$ .

Next, we derive a series of a priori estimates which will be required in establishing properties of solutions. Taking the inner product of (4.32) with  $U$ , and invoking the inequalities (4.43), (4.37) and Young's inequality we obtain the differential inequalities

$$\begin{aligned} &\frac{d}{dt} \|U\|^2 + \alpha \lambda_1 \|U\|^2 \\ &\leq \frac{d}{dt} \|U\|^2 + \alpha \|L^{1/2}U\|^2 \\ &\leq \frac{1}{\alpha} \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \|\Omega\|^2 \end{aligned} \quad (4.45)$$

where  $\alpha = \lambda + \lambda_1^{\frac{1}{6}}$ . The use of Gronwall's inequality (3.10) in (4.45) provides the a priori estimate

$$\|U(t)\|^2 \leq \|U_0\|^2 \exp(-\alpha t) + \frac{1}{\alpha^2} \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \|\Omega\|^2 (1 - \exp(-\alpha t)) \quad (4.46)$$

which yields energy stability with flow energy and entropy production specified by

$$E(t) = \frac{1}{2} \|U(t)\|^2 = \frac{1}{2} \int_{\Omega} (\underline{u}^2 + \rho^2) dx.$$

Thus, we are able to assert that the asymptotic estimate,

$$\lim_{t \rightarrow \infty} \sup \|U(t)\|^2 \leq \frac{1}{\alpha^2} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \|\Omega\|^2, \quad (4.47)$$

is valid and this shows  $U(t)$  is uniformly bounded in time in the Hilbert space  $\Upsilon$ . To dominate the nonlinearity we again proceed as in the previous section by taking the inner product of (4.1) with  $LU$  and using inequalities (4.37) and (4.41), we obtain

$$\begin{aligned} & \frac{d}{dt} \|L^{1/2}U\|^2 + \lambda_1 \|L^{1/2}U\|^2 \\ & \leq \frac{d}{dt} \|LU\|^2 + \|LU\|^2 \end{aligned} \quad (4.48)$$

$$\leq 2 \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \|\Omega\|^2 + 108(c_1^4 \|U\|^2 \|L^{1/2}U\|^4 + \lambda_1 \|L^{1/2}U\|^2),$$

containing spatial derivatives of higher order. We estimate (4.45) using the inequality

$$\limsup_{t \rightarrow \infty} \int_t^{t+1} \|L^{1/2}U(s)\|^2 ds \leq (1/\alpha^2 + 1/\alpha^3) \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \|\Omega\|^2$$

and by virtue of Gronwall's inequality (3.10) in (4.48) we obtain the a priori estimate

$$\|L^{1/2}U(t)\|^2 \leq \rho_1^2, \quad (4.49)$$

where  $\rho_1^2 \equiv \nu_1 \nu_2$ , and the parameters  $\nu_1$  and  $\nu_2$  are given by

$$\nu_1 = (2 + 107\lambda_1(1/\alpha^2 + 1/\alpha^3) + 1/\alpha^2 + 1/\alpha^3) \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \|\Omega\|^2$$

and

$$\nu_2 = \exp(108c_0^2(1/\alpha^2 + 1/\alpha^3) \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \|\Omega\|^2).$$

Consequently, from the last inequality we have

$$\lim_{t \rightarrow \infty} \sup \|L^{1/2}U(t)\|^2 \leq \rho_1^2, \quad (4.50)$$

so that  $U(t)$  is uniformly bounded in time in  $D(L^{1/2})$ . Similarly, taking the inner product of (4.32) with  $L^2U$  and making use of the inequalities (4.37) and (4.42), we get

$$\begin{aligned} & \frac{d}{dt} \|LU\|^2 + \lambda_1 \|LU\|^2 \\ & \leq \frac{d}{dt} \|LU\|^2 + \|L^{3/2}U\|^2 \end{aligned} \quad (4.51)$$

$$\leq 3c^2 \|LU\|^4 + 3\lambda_1 \|LU\|^2 + 3 \left( \frac{1}{Ro^2} + \frac{Ro^2}{Fr^4} \right) \rho_1^2.$$

Using the differential inequality (4.48) and invoking Hölder inequality we obtain the a priori energy type estimate

$$\int_t^{t+1} \|LU(s)\|^2 ds \leq \nu_3^2$$

where  $\nu_3^2 = a_1 + a_2$ , and

$$a_1 = \lambda_1(1/\alpha^2 + 1/\alpha^3)\left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \|\Omega\|^2 + \rho_1^2,$$

and

$$a_2 = 108c_0^2(1/\alpha^2 + 1/\alpha^3)^2\left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^4 \|\Omega\|^4.$$

Substitution of this a priori energy type estimate into (4.51) yields

$$\|LU(t)\|^2 \leq \rho_2^2, \quad (4.52)$$

with  $\rho_2^2$  denoting

$$\rho_2^2 = 3c^2\nu_3^4 + 2\lambda_1\nu_3^2 + 3\left(\frac{1}{Ro^2} + \frac{Ro^2}{Fr^4}\right)\rho_1^2.$$

Consequently, putting the estimates together we obtain

$$\lim_{t \rightarrow \infty} \sup \|LU(t)\|^2 \leq \rho_2^2 \quad (4.53)$$

which shows  $U(t)$  is uniformly bounded in time in  $D(L)$ .

The following proposition on the existence, uniqueness, continuity and differentiability with respect to initial conditions of the solution to the initial-value problem (4.32) holds:

**Proposition 4.1.5** *Under the above hypotheses for  $U_0 \in \Upsilon$ , (4.32) generate dynamical systems  $U(t) = S(t)U_0$  satisfying*

$$U \in C([0, \infty); \Upsilon) \cap L^2((0, \infty); D(L^{1/2})).$$

*Additionally, if  $U_0 \in D(L^{1/2})$  then*

$$U \in C([0, \infty); D(L^{1/2})) \cap L^2((0, \infty); D(L)).$$

**Proof:** For fixed  $m$ , the Faedo-Galerkin approximation

$$U_m = \sum_{i=1}^m g_{im}(t)W_i$$

of the solution  $U$  of (4.32) is defined by the finite-dimensional system of nonlinear ordinary differential equations for  $g_{im}(t)$  given by

$$\frac{dU_m}{dt} + LU_m + \Pi_m B(U_m) + CU_m = \Pi_m F, \quad (4.54)$$

$$U_m(0) = \Pi_m U_0,$$

where  $\Pi = \Pi_m$  is the spectral projection of rank  $m$  in  $\Upsilon, D(L^{1/2}), D(L^{-1/2})$ , or  $D(L)$  onto the space spanned by the  $m$  eigenfunctions of  $L$  given in (4.4). Let an eigenspace

$\mathbf{X}_0 = \Pi_m D(L)$  be the finite-dimensional Hilbert space spanned by the  $m$  eigenfunctions of  $L$  given in (4.35) with inner product  $((\cdot, \cdot))$  induced by  $D(L)$ . Set  $\mathbf{X}_1 = \Pi_m D(L^{1/2})$  and  $\mathbf{X}_2 = \Pi_m D(L^{-1/2})$ . By a priori estimates on  $L$ ,  $C$  and  $N$  we get existence of solutions to the finite-dimensional system of nonlinear ordinary differential equations (4.54). The result consists of analyzing the linear initial-value problem

$$\begin{aligned} \frac{dU_m}{dt} + LU_m &= 0, \\ U_m(0) &= \Pi_m U_0. \end{aligned} \tag{4.55}$$

For any  $U_m(0) \in \mathbf{X}_0$ , we get existence and uniqueness of the solution to the linear initial-value problem (4.55) given by

$$U_m \in C([0, \infty); \mathbf{X}_0) \cap C((0, \infty); \mathbf{X}_1).$$

Moreover, we have a mapping given by the linear solution operators

$$S(t) = \exp(-tL) : U_m(0) \rightarrow U_m(t),$$

which are continuous from the function space  $\mathbf{X}_2$  into  $\mathbf{X}_0$ ,  $\forall t > 0$ , and from  $\mathbf{X}_0$  into itself,  $\forall t \geq 0$ . Invoking the same inequalities that led to the linear analog of the differential inequality (4.48) and by utilization of Gronwall's inequality (3.10) we find

$$\|LU_m(t)\|^2 \leq \|LU_m(0)\|^2 \exp(-\lambda_1 t).$$

Consequently, we obtain the estimate

$$\|\exp(-tL)\|_{\mathbf{L}(\mathbf{X}_0)} \leq \exp(-\lambda_1 t) \quad \forall t \geq 0,$$

which implies the norm of  $\exp(-tL)$  in  $\mathbf{L}(\mathbf{X}_0)$  is bounded and therefore continuous. Here,  $\mathbf{L}(\mathbf{X}_0) = \mathbf{L}(\mathbf{X}_0, \mathbf{X}_0)$  denotes the Banach space of bounded linear operators from  $\mathbf{X}_0$  into itself. Similarly,

$$\|\exp(-tL)\|_{\mathbf{L}(\mathbf{X}_2, \mathbf{X}_0)} \leq \left(\frac{1}{2t} + \lambda_1\right)^{\frac{1}{2}} \exp(-\lambda_1 t), \quad \forall t > 0,$$

implies the norm of  $\exp(-tL)$  in  $\mathbf{L}(\mathbf{X}_2, \mathbf{X}_0)$  is bounded. Uniqueness of solution to the linear initial-value problem (4.55) follows by considering the difference of two solutions  $U_m(t) = U_m^1(t) - U_m^2(t)$ , with  $U_m^1(t)$  and  $U_m^2(t)$  satisfying (4.55). Using the Gronwall's inequality, we get

$$\|LU_m(t)\|^2 \leq \|LU_m(t_0)\|^2 \exp(-\lambda_1 \alpha(t - t_0)).$$

Hence we obtain uniqueness through  $\|LU_m(t)\| = 0$  as  $t_0 \rightarrow -\infty$  due to the fact that  $\|LU_m(t_0)\|$  is bounded.

Consider a closed ball of radius  $\rho_2^2$  given by (4.53) contained in  $\mathbf{X}_0$ . Integration of the nonlinear initial-value problem (4.54) using lemma (3.1) and the above a priori energy type estimates provide a continuous mapping defined by the integral of (4.54), given by the formula

$$U_m(t) = S(t - t_0)U_m(t_0) - \int_{t_0}^t S(t - \tau)[\Pi_m N(U_m) + CU_m - \Pi_m F]d\tau,$$

where the continuous solution operators  $S(t) = \exp(-tL)$  satisfy the linear initial-value problem (4.19). This integral equation defines a continuous mapping in  $\mathbf{X}_0$  from a closed ball of radius  $\rho_2^2$  into itself. Thus, by virtue of a fixed point argument the finite-dimensional system of nonlinear ordinary differential equations (4.18) has at least one solution  $U_m$  inside the ball of radius  $\rho_2^2$  contained in  $\mathbf{X}_0$ .

The passage to the limit

$$m \rightarrow \infty \text{ and } T_m \rightarrow T = \infty$$

follows from the next arguments and a priori estimates. Use of the differential inequality (4.45) provides

$$\begin{aligned} \frac{d}{dt} \|U_m\|^2 + \alpha \lambda_1 \|U_m\|^2 \\ \leq \frac{1}{\alpha} \left( \frac{R_0}{Fr^2} + \frac{1}{R_0} \right)^2 \end{aligned} \tag{4.56}$$

which implies that  $U_m$  remains bounded in

$$L^\infty(I; \Upsilon) \cap L^2(I; D(L^{\frac{1}{2}})).$$

This useful result combined with the weak compactness implies there is a subsequence also denoted by  $U_m$  and

$$U \in L^\infty(I; \Upsilon) \cap L^2(I; D(L^{1/2}))$$

such that

$$U_m \rightarrow \begin{cases} U \in L^2(I; D(L^{1/2})) & \text{weakly} \\ U \in L^\infty(I; \Upsilon) & \text{weak-star.} \end{cases}$$

Using (4.40), we obtain the boundness of  $B(U_m)$  and  $\Pi_m B(U_m)$  in  $L^2(I; D(L^{-1/2}))$ . Moreover, from (4.54),  $\frac{dU_m}{dt}$  is also bounded in  $L^2(I; D(L^{-1/2}))$ . Utility of this result and weak compactness yield

$$\frac{dU_m}{dt} \rightarrow \frac{dU}{dt} \in L^2(I; D(L^{-1/2})) \text{ weakly.}$$

By virtue of the Lebesgue dominated convergence theorem and the preceding weak convergence results, we get

$$U_m \rightarrow U \in L^2(I; \Upsilon) \text{ strongly.}$$

Therefore, by the applying the above results (3.1) – (3.2) we pass to the limit in (4.54) and obtain the required result (4.32). Additionally, using the properties listed in the above results (3.1) – (3.2) and the compact injections (3.5), we obtain

$$U \in C(\bar{I}; \Upsilon) \cap L^2(I; D(L^{1/2})).$$

Similarly, from (4.48) and (4.51) we conclude that if  $U_0 \in D(L^{1/2})$  then by employing the Faedo-Galerkin technique we obtain

$$U \in C(\bar{I}; D(L^{1/2})) \cap L^2(I; D(L)) \quad \forall T > 0.$$

Due to Gronwall's lemma (3.10), uniqueness and continuity with respect to initial conditions of the solution  $U(t)$  follow from considering the difference between two solutions of the initial-value problem (4.32) and employing a priori estimates (4.47) – (4.53) including the inequalities (3.1) – (3.2) provided the hypotheses of the lemmas are satisfied. Uniqueness and continuity with respect to initial conditions of the solution  $U(t)$  generate the dynamical system which is prescribed by continuous solution operators  $S(t)$ ,  $t \in \mathbb{R}$  defined by

$$S(t)U_0 = U(t) \equiv S(U_0, t)$$

satisfying the group property

$$S(t)U(s) = S(s)U(t), \tag{4.57}$$

$$S(0)U = U \quad \forall t, s \in \mathbb{R}.$$

That the solution operators  $S(t)$  are continuous is a consequence of the continuity of  $U(t)$  in time and the initial conditions. The group property is a consequence of the injectivity of the solution operators which is equivalent to the backward uniqueness of the solution to the initial-value problem (4.32).

Next, we turn our attention to establish uniqueness, continuity and Fréchet differentiability with respect to initial conditions of solution  $U(t)$  for the initial-value problem (4.32). As in the previous section, the technique of the proof uses the linearization of the evolution equation (4.32) about the difference of two given solutions. For the sake of completeness, we repeat the analysis here. Consider  $u$  and  $v$  satisfying the evolution equation (4.32) associated with the initial conditions  $u(0) = u_0$  and  $v(0) = v_0$ , respectively. The difference  $w = v - u$  satisfies the semilinear problem

$$\frac{dw}{dt} + Lw + N(v) - N(u) = 0, \tag{4.58}$$

$$w(0) = v_0 - u_0,$$

where

$$\begin{aligned}
N(v) - N(u) &= B(v, v) - B(u, u) + Cw - Fw \\
&= B(v, w) + B(w, u) + Cw - Fw \\
&= B(u, w) + B(w, u) + B(w, w) + Cw - Fw.
\end{aligned} \tag{4.59}$$

The linearization of (4.58) along the dynamical system trajectory satisfies the equation

$$\begin{aligned}
\frac{d\Psi}{dt} + L\Psi + l_0(t)\Psi &= 0, \\
\Psi(0) &= v_0 - u_0 = \xi,
\end{aligned} \tag{4.60}$$

where  $l_0(t)w = B(u(t), w) + B(w, u(t)) + Cw + Fw$  and  $l_0(t) \in \mathbf{L}(D(L^{1/2}), D(L^{-1/2}))$ . As in the preceding section,

$$\mathbf{L}(D(L^{1/2}), D(L^{-1/2}))$$

is a notation for the Banach space of bounded linear operators from  $D(L^{1/2})$  to  $D(L^{-1/2})$ . We shall be concerned with the difference  $\varphi = w - \Psi$ , satisfying the problem

$$\begin{aligned}
\frac{d\varphi}{dt} + L\varphi + l_0(t)\varphi + l_1(t; w(t)) &= 0, \\
\varphi(0) &= 0,
\end{aligned} \tag{4.61}$$

where  $l_1(t; w(t)) = B(w, w) - Fw$ . From (4.59), we have  $N(v) - N(u) = l_0(t)w + l_1(t; w)$ . The linear operator  $L + l_0 + l_1$  in (4.61) may be viewed as a compact perturbation of  $L$  whose spectrum can be estimated directly [48, 42]. In fact,  $D(L + l_0 + l_1) = D(L)$  and  $\|\cdot\|_{D(L+l_0+l_1)}$  is a norm equivalent to  $\|\cdot\|_{D(L)}$ .

The following proposition is an assertion of Fréchet differentiability of solutions with respect to initial data:

**Proposition 4.1.6** *If  $u_0 \in \Upsilon$  and  $u$  is the unique solution of (4.32) then the equation (4.60) has a unique solution  $\Psi$  satisfying*

$$\Psi \in C([0, \infty); \Upsilon) \cap L^2((0, \infty); D(L^{\frac{1}{2}})).$$

*Furthermore, the dynamical system  $u_0 \rightarrow S(t)u_0$  is Fréchet differentiable in  $\Upsilon$  with differential*

$$L(t, u_0) : \xi \rightarrow \Psi(t)$$

*given by the solution of the variational equation.*

**Proof:** Existence and uniqueness of the solution to (4.60) is a consequence of the Faedo-Galerkin technique and the use of a priori estimates together with Gronwall's inequality (3.10). Next, we exhibit a series of a priori estimates which will be required in establishing the above proposition. Taking the inner product of (4.58) with  $w$  and applying the Cauchy-Schwarz inequality, Young's inequality, and the Sobolev inequalities, we have

$$\frac{d}{dt} \|w\|^2 + \|L^{\frac{1}{2}} w\|^2 \leq k^2 \|w\|^2, \quad (4.62)$$

where  $k = c_1 \rho_2$  and  $\rho_2$  is given in the estimate (4.52). By virtue of Gronwall's inequality (3.10), we get

$$\|v(t) - u(t)\|^2 \leq \exp(k^2 T) \|v_0 - u_0\|^2 \quad \forall t \in (0, T). \quad (4.63)$$

Let  $t_0 = \frac{1}{2k^2} \log 2$ ; then (4.63) reduces to

$$\|S(t)v_0 - S(t)u_0\|^2 \leq \frac{1}{2} \|v_0 - u_0\|^2 \quad (4.64)$$

for  $t_0 \leq t \leq 2t_0$  which implies the dynamical system is Lipschitz continuous with respect to the initial conditions.

Furthermore, from (4.62) we obtain a priori energy type estimate

$$\int_0^t \|L^{\frac{1}{2}} w(s)\|^2 ds \leq \exp(k^2 T) \|v_0 - u_0\|^2.$$

Taking the inner product of (4.61) with  $\varphi$  we get the differential inequality

$$\frac{d}{dt} \|\varphi\|^2 + \frac{1}{2} \|L^{\frac{1}{2}} \varphi\|^2 \leq k^2 \|\varphi\|^2 + \frac{c_1^2}{\lambda_1} \|L^{\frac{1}{2}} w(t)\|^3. \quad (4.65)$$

We repeat the construction using Gronwall's inequality and find a priori estimate

$$\|\varphi(t)\|^2 \leq \int_0^T \|L^{\frac{1}{2}} w(s)\|^3 ds \quad \forall t \in (0, \infty).$$

Consequently we obtain the bound

$$\|\varphi(t)\|^2 \leq \frac{c_1^2}{\lambda_1} \exp(3k^2 T/2) \|v_0 - u_0\| \quad (4.66)$$

which implies the needed result

$$\frac{\|S(t)v_0 - S(t)u_0 - L(t, u_0) \cdot (v_0 - u_0)\|^2}{\|v_0 - u_0\|^2} = o(\|v_0 - u_0\|)$$

as  $v_0 \rightarrow u_0$ . Hence, the dynamical system

$$S(t) : u_0 \rightarrow u(t)$$

is Fréchet differentiable at  $u_0 \in \mathcal{Y}$ .

Next, we examine the problem of proving backward uniqueness of the solution to the initial-value equation (4.32) that establishes the injectivity and group properties of the solution operators  $S(t)$ . The backward uniqueness of solution is proved in approximately the same way as Fréchet differentiability result using the standard log-convexity method as in the previous section. For the sake of completeness of presentation we cover the proof of the injectivity of solution operators here. In order to motivate this objective, consider two solutions  $u$  and  $v$  of the initial-value problem such that at time  $t = t_*$  both  $u$  and  $v$  satisfy (4.32) for  $t \in (t_* - \epsilon, t_*)$  and  $u(t_*) = v(t_*)$ . Given that this hypothesis is valid, backward uniqueness of the solution operators is accomplished if  $u(t) = v(t)$  for all time  $t < t_*$  whenever the solutions are well-defined. Alternatively, suppose that the solution operators satisfy

$$S(t) = u(t + s) \quad \forall t > 0 \quad s \in \mathbb{R};$$

then for  $\tau \in (0, \epsilon)$ , the problem of backward uniqueness is equivalent to satisfaction of the identities

$$S(\tau)u(t_* - \tau) = S(\tau)v(t_* - \tau) \Rightarrow u(t_* - \tau) = v(t_* - \tau) \quad (4.67)$$

which implies the injectivity of the solution operators  $S(\tau)$ .

**Proposition 4.1.7** *Suppose that*

$$u, v \in L^\infty(I; D(L^{1/2})) \cap L^2(I; D(L));$$

*then the difference  $w = v - u$  satisfies the differential equation (4.58) for  $t \in (0, T)$  and the solution operators  $S(t)$  are injective.*

**Proof:** We will use the notation  $\varphi = \varphi(t)$  for the quotient

$$\begin{aligned} \varphi &= \frac{\|w(t)\|_V^2}{\|w(t)\|_H^2} \\ &= \frac{(Lw(t), w(t))}{(w(t), w(t))}. \end{aligned}$$

Putting the quotient  $\varphi = \varphi(t)$  into inequalities (3.1) – (3.2) and invoking the Cauchy-Schwarz inequality, Young's inequality, Sobolev inequalities, continuity properties of the nonlinear operator established above and the fact that  $(Lw - \varphi w, w) = 0$ , we get

$$\frac{1}{2} \frac{d}{dt} \varphi = \frac{((\frac{dw}{dt}, w))}{\|w\|^2} - \frac{\|w\|_V^2}{\|w\|^4} (\frac{dw}{dt}, w)$$

$$\begin{aligned}
&= \frac{1}{\|w\|^2} \left( \frac{dw}{dt}, Lw - \varphi w \right) \\
&= \frac{1}{\|w\|_V^2} (N(v) - N(u) - Lw, Lw - \varphi w) \\
&= -\frac{\|Lw - \varphi w\|^2}{\|w\|^2} + \frac{1}{\|w\|^2} (Lw, N(v) - N(u)) \\
&\leq -\frac{\|Lw - \varphi w\|^2}{2\|w\|^2} + \frac{\|N(v) - N(u)\|^2}{2\|w\|^2} \\
&\leq -\frac{\|Lw - \varphi w\|^2}{2\|w\|^2} + k^2 \varphi
\end{aligned}$$

where  $N(v) - N(u)$  is given in (4.59). The function  $k(t) = k_1(t) + \alpha + \alpha_1$  is defined by the mapping

$$k_1(t) : t \rightarrow c(\|u(t)\|_V^{\frac{1}{2}} \|Lu(t)\|^{\frac{1}{2}} + \|v(t)\|_V^{\frac{1}{2}} \|Lv(t)\|^{\frac{1}{2}}) \in L^2(0, T),$$

and the geophysically relevant parameters  $\alpha_1$  and  $\alpha$  denote  $\alpha_1 = \max\{\frac{1}{Ro}, \frac{Ro}{Fr^2}\}$ , and  $\alpha = \max\{\frac{Ek}{Ro}, \frac{1}{Ed}\}$ , respectively.

The use of Gronwall's inequality (3.10) in the above differential inequality gives

$$\varphi(t) \leq \varphi(0) \exp\left(2 \int_0^t k^2(s) ds\right) \quad t \in (0, T).$$

Thus, if the difference  $w = v - u$  satisfies the above conditions and

$$w(\tau) = 0 \quad \Rightarrow \quad w(t) = 0, \quad t \in [0, T].$$

To demonstrate this, we proceed by contradiction. Suppose that  $\|w(t_0)\| \neq 0$  for  $t_0 \in [0, T)$ . As a result by continuity we have  $\|w(t)\| \neq 0$  on some open interval  $(t_0, t_0 + \epsilon)$  and we employ the notation  $t_* \leq T$  for the largest time for which

$$w(t) \neq 0 \quad \forall t \in [t_0, t_*).$$

It follows that  $w(t_*) = 0$ . In the interval  $[t_0, t_*)$  the mapping given by  $t \rightarrow \log \|w(t)\|$  is well-defined and by courtesy of inequalities (3.1) – (3.2) and the fact that  $w(t)$  satisfies the differential equation (4.22) we obtain

$$\frac{d}{dt} \log \frac{1}{\|w\|} \leq 2\varphi + k^2.$$

Consequently, by the virtue of inequalities (3.1) – (3.2) for the time  $t \in [t_0, t_*)$  the following is valid:

$$\log \frac{1}{\|w\|} \leq \log \frac{1}{\|w(t_0)\|} + \int_{t_0}^t 2\varphi(s) + k^2(s) ds$$

which illustrates the boundedness of  $\frac{1}{w(t)}$  from above as the time  $t \rightarrow t_*$  from below and this is a contradiction. Therefore the result is established.

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## Chapter 5

# Nonlinear stability of solutions

### 5.1 Aspects of stability and attraction for ageostrophic flows

In this chapter we establish results on energy stability criteria and attractors of solutions in a class of  $\beta$ -plane ageostrophic equations by adapting a priori estimates employed in establishing well-posedness and differentiability. Hence, the results of this section on nonlinear stability differ in form rather than in essence from their counterparts of well-posedness and differentiability of solutions. The technique used in the stability problem of  $\beta$ -plane ageostrophic equations is energy theory and the framework for its utility in this context is due to Galdi and Rionero [29] and was further extended in Galdi and Padula [29]. Specifically, the techniques used are spectral properties of the linear operator and a priori estimates for the nonlinear operator.

Furthermore, the following proposition [14, 15, 67, 68, 26, 27] on the characterization of attractors and determining the compactness of an  $\omega$ -limit set will be utilized:

**Proposition 5.1.1** *Suppose that for  $Q \subset H$ ,  $Q \neq \emptyset$ , and that for some  $s > 0$  the set  $\cup_{t \geq s} S(t)Q$  is relatively compact in  $H$ . Then the  $\omega$ -limit set of  $Q$ , denoted by  $\omega(Q)$ , is nonempty, compact and invariant. In particular, if there exists an open set  $K \subset H$  and a bounded set  $A$  of  $K$  such that for all  $A_0 \subset K$ ,  $A_0$  bounded, there is  $t_0 > 0$  with  $S(t)A_0 \subset A$  for all  $t \geq t_0$  then  $\omega(A)$  is an attractor.*

The proposition is of central importance in the construction of attractors. In order to prove the major hypothesis that  $\cup_{t \geq s} S(t)Q$  is relatively compact in  $H$ , it suffices to illustrate that this set is bounded in a space  $V$  compactly imbedded in  $H$  by employing

the existence of the injections

$$D(L^{1/2}) = V \subset D(L) \subset \Upsilon = H \subset D(L^{-1/2}).$$

The density and compactness of the injections will find applications in the sequel when we prove existence of attractors for the initial-value problems.

### 5.1.1 $\beta$ -plane ageostrophic equations

The goal of this section is to develop energy stability criteria results for  $\beta$ -plane ageostrophic flows (2.24) – (2.26). For this purpose, the functional formulation for the evolution equation of  $\beta$ -plane ageostrophic equations is restated equivalently in the form

$$\begin{aligned} M \frac{dU}{dt} + LU + N(U) &= 0, \\ U(0) &= U_0, \end{aligned} \tag{5.1}$$

where  $U = (u, v, w, \rho)$ .

Without loss of generality, and for simplicity of exposition we suppose

$$R = \frac{Ro(Ed)}{Fr^2} = \frac{1}{Ek}.$$

In the above initial-value problem, we have rearranged the operators so that

$$N(U) = B(U, U);$$

$$MU = \begin{pmatrix} \frac{Ro}{Ek} \underline{u} \\ Ed \quad \rho \end{pmatrix};$$

$$LU = TU + \frac{1}{Ek}(SU - FU);$$

$$TU = - \begin{pmatrix} \Pi \Delta \underline{u} + \frac{\beta Ro}{Ek} \Pi(\underline{y} \underline{u} \times \underline{k}) \\ \Delta \rho \end{pmatrix} = \begin{pmatrix} A_s \underline{u} + \frac{\beta Ro}{Ek} (\underline{y} \underline{u} \times \underline{k}) \\ A_l \rho \end{pmatrix};$$

$$SU = \begin{pmatrix} \Pi(\underline{u} \times \underline{k}) \\ 0 \end{pmatrix};$$

$$FU = - \begin{pmatrix} \Pi(\rho \underline{k}) \\ \underline{u} \cdot \underline{k} \end{pmatrix};$$

$$B(U, U) = B(U) = \begin{pmatrix} \frac{Rc}{Ek} \Pi \underline{u} \cdot \nabla \underline{u} \\ Ed \quad \underline{u} \cdot \nabla \rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_s(\underline{u}, \underline{u}) \\ \mathbf{b}_l(\underline{u}, \rho) \end{pmatrix}.$$

In nonlinear stability analysis we take the number in (3.3) and we note that we have reformulated the initial-boundary value problem for  $\beta$ -plane ageostrophic flows in order to reflect the influence of the results in [28, 29] for Lyapunov and energy stability theory.

The first step is to define the functionals  $G(\underline{u}, \rho)$  using

$$\frac{1}{P_F} = \sup_{U=(\underline{u}, \rho)} \left\{ \frac{2 \int_{\Omega} \rho \underline{u} \cdot \underline{k} dx}{\int_{\Omega} (|\nabla \underline{u}|^2 + |\nabla \rho|^2) dx} \right\} = \sup G(\underline{u}, \rho),$$

where the supremum is taken for  $U \in D(L)$ . We shall put together various inequalities about the equation (5.1).

Taking the inner product of  $LU$  with  $U$  we get

$$(LU, U) = (1 + R)J(U) \quad (5.2)$$

with the functional  $J(U)$  given by

$$J(U) = \int_{\Omega} (|\nabla \underline{u}|^2 + |\nabla \rho|^2) dx. \quad (5.3)$$

It is appropriate at this point to note that if  $R > P_F$  then the identity

$$-(LU, U) \leq -(1 + \frac{R}{P_F})J(U) < 0 \quad (5.4)$$

is valid, and in this case  $-L$  is essentially dissipative. Additionally, the inequalities

$$(\frac{P_F - R}{P_F})J(U) \leq \|U\|_L^2 \leq (\frac{P_F + R}{P_F})J(U) \quad (5.5)$$

hold, which shows that the norm  $\|U\|_L^2$  induced by the linear operator  $L$  is equivalent to the norm  $J(U)$  whenever  $-L$  is essentially dissipative.

The following result on the spectrum of  $-L$  will be utilized:

**Lemma 5.1.2**  $\sigma(-L) \subseteq (-\infty, 0] \cup \Pi$ , where  $\Pi$  is either empty or an at most denumerable set consisting of isolated, positive eigenvalues  $\lambda_n = \lambda_n(R)$  with finite multiplicity such that

$$\infty > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \dots,$$

clustering at zero.

**Proof:** According to a perturbation theorem [28, 48, 42], if  $TU$  is closed and  $MU = SU - FU$  is relatively compact with respect to  $T$  then  $\sigma(-L = -(T + S - F))$  and  $\sigma(T)$  differ by at most a denumerable number of isolated, positive eigenvalues of finite multiplicity clustering at zero. Thus it suffices to show that  $M = S - F$  is relatively compact with respect to  $T$  since from [66, 28, 48, 42] we have  $\sigma(-T) \subseteq (-\infty, 0]$ . By definition,  $M = S - F$  is relatively compact with respect to  $T$  if for any sequence  $\{U_n\} \subset \Upsilon$ ,

$$\|U_n\| + \|TU_n\| \leq \alpha, \quad \text{uniformly in } n, \quad (5.6)$$

then there is a subsequence  $\{U_{n_k}\} \subset \{U_n\}$  such that  $MU_{n_k}$  is strongly convergent. By virtue of Ascoli-Arzelà theorem,  $\{MU_{n_k}\}$  contains a uniformly convergent subsequence which is a Cauchy sequence in a complete normed space  $\Upsilon$ . Thus  $\{MU_{n_k}\}$  is strongly convergent, which illustrates that  $M = S - F$  is relatively compact with respect to  $T$ .

It remains to estimate the nonlinearity  $N$ . First, from well-known a priori estimates for the Stokes problem the inequality

$$\sup |\underline{u}|^2 \leq c_1 \int_B (|\Pi \Delta \underline{u}|^2 + |\nabla \underline{u}|^2) dx,$$

holds. Next, we employ this inequality to exhibit a series of a priori estimates on the nonlinearity which will be needed in establishing necessary and sufficient nonlinear stability of ageostrophic flows (5.1). Consideration of the Sobolev inequality and continuity properties of the nonlinearity  $B(U)$  yield the following uniform bounds

$$\begin{aligned} \|NU\|^2 &\leq (TU, U)c_0^2(\|TU\|^2 + \|U\|^2) \\ &\leq c_0^2\|T^{\frac{1}{2}}U\|(\lambda_1\|L^{\frac{1}{2}}U\|^2 + \|U\|^2) \\ &\leq c_0^2\lambda^{-\frac{1}{2}}\|TU\|(\lambda_1\|L^{\frac{1}{2}}U\|^2 + \|U\|^2) \\ &\leq c_0^2\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\|L^{\frac{1}{2}}U\|(\lambda_1^{\frac{1}{2}}\|LU\|^2 + \|U\|^2) \\ &\leq c_0^2c\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\|L^{\frac{1}{2}}U\|(\|LU\|^2 + \|U\|^2) \\ &= c_0^2c\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}J(U)(\|LU\|^2 + \|U\|^2) \end{aligned} \quad (5.7)$$

for the nonlinearity  $N(U) = B(U)$ , where  $c = \max\{\lambda_1^{\frac{1}{2}}, 1\}$ . Putting the estimates (4.37) and (5.7) together and invoking the equivalence relation (5.5) gives the required a priori

estimate for the nonlinearity

$$\begin{aligned}\|NU\|^2 &\leq c_0^2 c \lambda^{-\frac{1}{2}} \lambda_1^{\frac{1}{2}} J(U) (\|LU\|^2 + \|U\|^2) \\ &\leq c_1 c_0^2 c \lambda^{-\frac{1}{2}} \lambda_1^{\frac{1}{2}} \|U\|_L^2 (\|LU\|^2 + \lambda_1^{-1} \|U\|_L^2)\end{aligned}\tag{5.8}$$

which is necessary for nonlinear stability.

The fundamental ideas in the derivation of nonlinear stability are due to Galdi and Padula [28] and for simplicity of exposition we employ their techniques. Taking the inner product of the equation (5.1) with  $U$  and employing inequality (3.2) yields

$$\frac{1}{2} \frac{d}{dt} (U, U) = (LU, U) + (NU, U).$$

Similarly, taking the inner product of the equation (5.1) with  $LU$  and invoking inequality (3.2) gives

$$\frac{1}{2} \frac{d}{dt} (LU, U) = (LU, LU) + (LU, NU).$$

Defining generalized energy with

$$E(t) = \frac{1}{2} \{ \|U\|^2 + \|U\|_L^2 \}\tag{5.9}$$

then we get

$$\frac{dE}{dt} = (LU, U) + (NU, U) - (LU, LU) - (LU, NU).\tag{5.10}$$

Applying the Cauchy-Schwarz inequality and Young's inequality in (5.10), we obtain

$$\begin{aligned}\frac{dE}{dt} &\leq (LU, U) + \frac{3}{2} \|LU\|^2 + \|U\| \|NU\| + \frac{1}{2} \|NU\|^2 \\ &\leq -\mu_0 (\|LU\|^2 + \|U\|_L^2) + \|U\| \|NU\| + \frac{1}{2} \|NU\|^2.\end{aligned}\tag{5.11}$$

Since  $L$  is essentially dissipative and the nonlinear operator satisfies the preceding a priori estimate (5.8), then plugging in (5.11) and use of the generalized energy functional (5.9) lead to

$$\begin{aligned}\frac{dE}{dt} &\leq -\mu_0 (\|LU\|^2 + \|U\|_L^2) [1 - c^{\frac{1}{2}} \|U\|_L \|U\| (\|LU\|^2 + \|U\|_L^2)^{-\frac{1}{2}} - \frac{c}{2} \|U\|_L^2] \\ &\leq -\mu_0 (\|LU\|^2 + \|U\|_L^2) [1 - \frac{c}{2} \|U\|_L^2] \\ &\leq -\mu_0 (\|LU\|^2 + \|U\|_L^2) [1 - cE].\end{aligned}\tag{5.12}$$

According to the inequality (5.12) and whenever

$$\gamma = (1 - cE(0)) > 0,\tag{5.13}$$

we obtain the a priori energy estimate

$$\mu_0(\|LU\|^2 + \|U\|_L^2)(1 - cE(0)) \leq -\frac{dE}{dt}$$

which through (3.1) implies

$$\int_0^\infty \mu_0(\|LU\|^2 + \|U\|_L^2)(1 - \frac{c}{\mu_1}E(0))dt \leq 2E(0).$$

We proceed by recalling the following useful property

$$-\frac{1}{2} \frac{d}{dt}(LU, U) = -(LU, LU) + (LU, NU).$$

Invoking the Cauchy-Schwarz inequality, Young's inequality and the a priori estimate (5.8) we get the differential inequality

$$\frac{d}{dt}\|U\|_L^2 \leq 2c\|U\|_L^2(\|LU\|^2 + \|U\|_L^2). \quad (5.14)$$

In order to establish the hypotheses of Gronwall's inequality (3.10) we note that (5.14) may be restated equivalently using

$$\frac{d\Phi}{dt} \leq h_1(t) + \frac{2c}{\mu_1}h_2(t)\Phi, \quad (5.15)$$

where

$$\Phi(t) = \|U\|_L^2(t+1),$$

$$h_1(t) = \|U\|_L^2,$$

and

$$h_2(t) = \|LU\|^2 + \|U\|_L^2.$$

Consequently, by an assertion of Gronwall's lemma (3.10) we get

$$\Phi(t) \leq \exp[2c \int_0^t h_2(s)ds] \{ \Phi(0) + \int_0^t \exp[-2c \int_0^s h_2(\alpha)d\alpha] h_1(s)ds \}.$$

In addition, the following finite energy solution

$$\Phi(t) \leq 2E(0)(1 + \frac{1}{\gamma\mu_0}) \exp[\frac{4(1-\gamma)}{\mu_0\gamma}] \quad (5.16)$$

holds. Therefore, substitution of  $\Phi(t) = \|U\|_1^2(t+1)$  into (5.16) gives the desired asymptotic stability with Lyapunov functional  $\|U\|_1^2$ . More particularly, we get asymptotic stability with energy functional

$$E_1(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 + \frac{Ro}{Ek} |\nabla \underline{u}|^2 + Ed|\nabla \rho|^2 \right) dx.$$

Using similar arguments we obtain asymptotic stability with generalized energy functional

$$E(t) = \mu_1 E_1(t) + \mu_2 E_2(t),$$

where

$$E_2(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 \right) dx + \nu_1 |\Pi(\underline{u} \times \underline{k})|^2 + \nu_2 \Pi(\underline{u} \times \underline{k}) \cdot \Pi(\rho \underline{k}) dx.$$

The generalized energy stability for this problem is therefore given by the criterion

$$\frac{1}{Ek} < \max \left\{ P_F^2, \frac{\pi^2 Ro (Ed)^2}{Fr^2 (3 + 2Ed)} \right\},$$

on  $f$ -plane and the condition

$$\frac{1}{Ek} < \max \left\{ P_F^2, \frac{\beta Ro^2 (Ed)^2 \pi^2}{Fr^2 (3 + 2Ed)} \right\},$$

on  $\beta$ -plane with respect to Lyapunov functional  $E(t)$ .

Next, we turn our attention to develop results on the attractor for  $\beta$ -plane ageostrophic flows (2.24) – (2.26) in the case of two-dimensional space variables. We show existence of the attractor for the dynamical system by proceeding essentially using the techniques employed in [14, 15, 67, 68, 26, 27] which have been summarized in the above proposition, though the calculations that we present in this examination are tedious and involved.

Putting the estimate (4.10) and (4.9) together gives the bound

$$\limsup_{t \rightarrow \infty} \int_t^{t+1} \|L^{\frac{1}{2}} U(s)\|^2 ds \leq \frac{\|\Omega\|}{\alpha} \left[ \frac{\lambda_1^2}{\alpha^2} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4 + \frac{\lambda_1}{\alpha} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \right].$$

Taking the inner product of the in the initial-value problem (4.1) with  $LUi$ , we obtain the differential inequalities

$$\begin{aligned} & \frac{d}{dt} \|L^{\frac{1}{2}} U\|^2 + \lambda_1 \|L^{\frac{1}{2}} U\|^2 \\ & \leq \frac{d}{dt} \|L^{\frac{1}{2}} U\|^2 + \|LU\|^2 \leq \rho_{22}^2 \end{aligned} \tag{5.17}$$

such that  $\rho_{22}^2 = \nu_0 + \nu_1 \nu_2$  follows from successive application of Gronwall's inequality (3.10). The parameters  $\nu_0$ ,  $\nu_1$  and  $\nu_2$  are given by

$$\begin{aligned} \nu_0 &= \frac{3\|\Omega\|}{\nu} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2, \\ \nu_1 &= \frac{27c}{4\nu^3} + \frac{2}{\kappa^3} (a_{23} + a_{22}) \exp(a_{21}), \end{aligned}$$

$$\begin{aligned}
\nu_2 &= \frac{\|\Omega\|}{\nu^2} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4 (a_{13} + a_{12})^2 \exp(2a_{11}), \\
a_{11} &= \frac{27c}{4\nu^3} \left( \left( \frac{1}{\nu} \right)^2 \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 + \frac{1}{\nu^3} \right) \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 \frac{\|\Omega\|^2}{\nu^2} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4, \\
a_{12} &= 3\|\Omega\|^2 \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2, \\
a_{13} &= \left( \left( \frac{1}{\nu} \right)^2 \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 + \frac{1}{\nu^3} \right) \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4 \|\Omega\|, \\
a_{21} &= \frac{9c\|\Omega\|^3}{2\kappa^5} \left( \frac{1}{\kappa} + \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^7 \kappa^2 \nu^2 \right) \left( \frac{1}{\nu^2} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 + \frac{1}{\nu^3} \right)^2 \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4, \\
a_{22} &= \frac{\|\Omega\|^3}{2\nu^4 \kappa} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4 \left( \frac{1}{\nu^2} \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^2 + \frac{1}{\nu^3} \right)^2 \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^4, \\
a_{23} &= \left( \frac{1}{\kappa} + \frac{1}{\kappa^2 \nu^2} \right) \left( \frac{Ro}{Fr^2} + \frac{1}{Ro} \right)^6 \|\Omega\|.
\end{aligned}$$

The use of Gronwall's inequality in the differential inequality (4.9) provides the estimate

$$\lim_{t \rightarrow \infty} \sup \|L^{\frac{1}{2}} U(t)\|^2 \leq \frac{\rho_{22}^2}{\alpha} \equiv \rho_2^2 \quad (5.18)$$

which shows  $U(t)$  is uniformly bounded in time in  $D(L^{\frac{1}{2}})$ . Putting the estimate (5.18) and (5.17) together we obtain

$$\lim_{t \rightarrow \infty} \sup \int_t^{t+1} \|LU(s)\|^2 ds \leq \left(1 + \frac{1}{\alpha}\right) \rho_2^2. \quad (5.19)$$

We infer from the more refined a priori estimates (5.18) that the attractor for (4.1) is given by the  $\omega$ -limit set of  $Q = B_{2\rho_2}$ ,

$$A = \omega(Q) = \bigcap_{s \geq 0} Cl(\cup_{t \geq s} S(t)Q), \quad (5.20)$$

where  $B_{2\rho_2}$  denotes an open ball of radius  $2\rho_2$ , which depends on the geophysically relevant parameters. The closure is taken in the Hilbert space  $D(L^{\frac{1}{2}})$ . Utility of the group property and continuity of the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ , gives the following invariance property of the above established attractor:

$$S(t)A = A, \quad \forall t \in \mathbb{R}. \quad (5.21)$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the two-dimensional space variables initial-value problem for viscous  $\beta$ -plane ageostrophic equations without Reynolds stress is both positively and negatively invariant and consists of orbits or trajectories that are defined for all  $t \in \mathbb{R}$ .

### 5.1.2 $\beta$ -plane ageostrophic equations with Reynolds stress

The aim of this section is to establish nonlinear stability results for the rest state of ageostrophic equations with Reynolds stress (2.28) – (2.30). In order to give nonlinear stability bounds that agree approximately with established paradigmatic example of initial-value problem (5.1), the functional formulation for the evolution equation of  $\beta$ -plane ageostrophic equations with Reynolds stress is restated equivalently in the form

$$\begin{aligned} M \frac{dU}{dt} + LU + N(U) &= 0, \\ U(0) &= U_0, \end{aligned} \quad (5.22)$$

where  $U = (u, v, w, \rho)$ . Without loss of generality, and for simplicity of exposition we assume

$$R = \frac{Ro(Ed)}{Fr^2} = \frac{1}{Ek}.$$

In the above initial-value problem, we have rearranged the operators so that

$$\begin{aligned} MU &= \begin{pmatrix} \frac{Ro}{Ek} \underline{u} \\ Ed \ \rho \end{pmatrix}; \\ N(U) &= B(U, U); \\ LU &= TU + \frac{1}{Ek}(SU - FU); \\ TU &= \begin{pmatrix} \Delta^6 \underline{u} + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) \\ \Delta^6 \rho \end{pmatrix} = \begin{pmatrix} A_s^6 \underline{u} + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) \\ A_l^6 \rho \end{pmatrix}; \\ SU &= \begin{pmatrix} (\underline{u} \times \underline{k}) \\ 0 \end{pmatrix}; \\ FU &= - \begin{pmatrix} (\rho \underline{k}) \\ \underline{u} \cdot \underline{k} \end{pmatrix}; \\ B(U, U) &= B(U) = \begin{pmatrix} \frac{Ro}{Ek} \underline{u} \cdot \nabla \underline{u} - \Delta \underline{u} \\ Ed \ \underline{u} \cdot \nabla \rho - \Delta \rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_s(\underline{u}, \underline{u}) - A_l \underline{u} \\ \mathbf{b}_l(\underline{u}, \rho) - A_l \rho \end{pmatrix}. \end{aligned}$$

From the above definition of the linear operator  $L$ , we obtain

$$(LU, U) = (TU, U) + \frac{2}{Ek} \int_{\Omega} \rho \underline{u} \cdot \underline{k} dx, \quad (5.23)$$

which is a modification of the bilinear form (4.33) that was employed in proving well-posedness of solution for the problem of ageostrophic equations with Reynolds stress. The first step in nonlinear stability analysis to define the functionals  $G(\underline{u}, \rho)$  using

$$\frac{1}{P_F} = \sup_{U=(\underline{u}, \rho)} \left\{ \frac{2 \int_{\Omega} \rho \underline{u} \cdot \underline{k} dx}{(TU, U)} \right\} = \sup G(\underline{u}, \rho),$$

where the supremum is taken for  $U \in D(L)$ . We proceed to derive the necessary inequalities about the equation (5.22) that will be need in the illustration of energy stability.

First, taking an inner product of  $LU$  with  $U$  we obtain

$$(LU, U) = (1 + R)J(U) \quad (5.24)$$

with the functional  $J(U)$  given by

$$J(U) = (TU, U). \quad (5.25)$$

Note that if  $R > P_F$  then the relation

$$-(LU, U) \leq -(1 + \frac{R}{P_F})J(U) < 0 \quad (5.26)$$

hold, and in this case  $-L$  is essentially dissipative. Additionally, we have

$$(\frac{P_F - R}{P_F})J(U) \leq \|U\|_L^2 \leq (\frac{P_F + R}{P_F})J(U), \quad (5.27)$$

which shows that the norm  $\|U\|_L^2$  induced by the linear operator  $L$  is equivalent to the norm  $J(U)$  whenever  $-L$  is essentially dissipative.

We require the following lemma on the spectrum of  $-L$  whose validity may be proved by using similar arguments employed in the case of  $\beta$ -plane ageostrophic equations:

**Lemma 5.1.3**  $\sigma(-L) \subseteq (-\infty, 0] \cup \Pi$ , where  $\Pi$  is either empty or an at most denumerable set consisting of isolated, positive eigenvalues  $\lambda_n = \lambda_n(R)$  with finite multiplicity such that

$$\infty > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \dots,$$

*clustering at zero.*

Next, we proceed to illustrate that suitable a priori estimates for the nonlinear operator  $N$  are available. Putting estimates (4.37) and (4.38) together into the inequality (4.40)

give the desired bounds

$$\begin{aligned}
\|NU\|^2 &\leq (CU, U)c_0^2(\|CU\|^2 + \|U\|^2) \\
&\leq c_0^2\|C^{\frac{1}{2}}U\|(\lambda_1\|L^{\frac{1}{2}}U\|^2 + \|U\|^2) \\
&\leq c_0^2\lambda^{-\frac{1}{2}}\|CU\|(\lambda_1\|L^{\frac{1}{2}}U\|^2 + \|U\|^2) \\
&\leq c_0^2\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\|L^{\frac{1}{2}}U\|(\lambda_1^{\frac{1}{2}}\|LU\|^2 + \|U\|^2) \\
&\leq c_0^2c\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\|L^{\frac{1}{2}}U\|(\|LU\|^2 + \|U\|^2) \\
&= c_0^2c\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}J(U)(\|LU\|^2 + \|U\|^2),
\end{aligned} \tag{5.28}$$

for the nonlinearity  $N(U) = B(U)$ , where  $c = \max\{\lambda_1^{\frac{1}{2}}, 1\}$ . Consequently, a priori estimates (5.28) on the nonlinearity means that the nonlinear operator is suitably dominated by the linear operator  $L$ . Putting the above estimates together and invoking the equivalence relation (5.27) gives

$$\begin{aligned}
\|NU\|^2 &\leq c_0^2c\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}J(U)(\|LU\|^2 + \|U\|^2) \\
&\leq c_1c_0^2c\lambda^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\|U\|_L^2(\|LU\|^2 + \lambda_1^{-1}\|U\|_L^2),
\end{aligned} \tag{5.29}$$

which is the needed a priori estimate for the nonlinearity.

Hence, by virtue of Gronwall's inequality (3.10), we get asymptotic stability of the rest state of  $\beta$ -plane ageostrophic equations with Reynolds stress (5.22) with respect to generalized energy functional  $\|U\|_1^2$ . Furthermore, we get asymptotic stability with generalized energy functional

$$E_1(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 \right) dx + \frac{Ro}{Ek} (TU, U).$$

Using similar arguments we obtain asymptotic stability with generalized energy functional

$$E(t) = \mu_1 E_1(t) + \mu_2 E_2(t),$$

where

$$E_2(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 \right) dx + \nu_1 |\underline{u} \times \underline{k}|^2 + \nu_2 (\underline{u} \times \underline{k}) \cdot (\rho \underline{k}) dx.$$

The generalized energy stability criteria for this problem is therefore given by the requirements

$$\frac{1}{Ek} < \max \left\{ P_F^2, \frac{\pi^2 Ro (Ed)^2}{Fr^2 (3 + 2Ed)} \right\},$$

and

$$\frac{1}{Ek} < \max \left\{ P_F^2, \frac{\beta Ro^2 (Ed)^2 \pi^2}{Fr^2 (3 + 2Ed)} \right\},$$

on  $f$ -plane and  $\beta$ -plane, respectively. In this case,

$$\frac{1}{P_F} = \sup_{U=(\underline{u}, \rho)} \left\{ \frac{2 \int_{\Omega} \rho \underline{u} \cdot \underline{k} dx}{(TU, U)} \right\} = \sup G(\underline{u}, \rho),$$

where the supremum is taken for  $U \in D(L)$  and the linear operator  $T$  is

$$TU = \begin{pmatrix} \Delta^6 \underline{u} + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) \\ \Delta^6 \rho \end{pmatrix} = \begin{pmatrix} A_s^6 \underline{u} + \frac{\beta Ro}{Ek} (y \underline{u} \times \underline{k}) \\ A_l^6 \rho \end{pmatrix},$$

as above in the functional reformulation of the initial-value problem (5.22).

We further the analysis by focussing our attention to the development of results for the attractor of the  $\beta$ -plane ageostrophic equations with Reynolds stress (2.28) – (2.30). We show existence of attractors for the dynamical system by adapting the splendid techniques employed in [14, 15, 67, 68, 26, 27] which have been summarized in the aforementioned proposition. Concerning the results for the attractor of the general three-dimensional space variables initial-value problem for viscous  $\beta$ -plane ageostrophic equations with Reynolds stress we infer from more refined a priori estimates (4.53) that the attractor for (4.32) is given by the  $\omega$ -limit set of  $Q = B_{2\rho_2}$ ,

$$A = \omega(Q) = \bigcap_{s \geq 0} Cl(\bigcup_{t \geq s} S(t)Q), \quad (5.30)$$

where  $B_{2\rho_2}$  denotes an open ball of radius  $2\rho_2$ , which depends on the geophysically relevant parameters. The closure is taken in the Hilbert space  $\Upsilon$ . Utility of the group property and continuity of the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ , gives the following invariance property of the above established attractor:

$$S(t)A = A, \quad \forall t \in \mathbb{R}. \quad (5.31)$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the three-dimensional space variables initial-value problem for viscous  $\beta$ -plane ageostrophic equations with Reynolds stress is both positively and negatively invariant and consists of orbits or trajectories that are defined for all  $t \in \mathbb{R}$ .

## Chapter 6

# Lipschitz invariant manifolds

### 6.1 Lipschitz invariant manifolds for ageostrophic flows

The investigation of a dynamical system in the neighborhood of an invariant set leads to the challenge of constructing the stable, unstable and center manifolds defined by the identities (3.9) and established in [73, 134, 86, 92] and cited works therein. The stable and unstable invariant manifolds consist of orbits or trajectories which decay to the invariant set in either positive or negative time and correspond to the eigenspaces in the linearized version. Their status as Lipschitz manifolds follows from the fact that they are built as graphs of Lipschitz functions over an eigenspace spanned by the eigensolutions of the linear operator, and are tangent at the invariant set. Furthermore, the manifolds are invariant under the dynamical system relative to some neighborhood of the invariant set.

In this chapter we consider the problem of constructing Lipschitz invariant manifolds where the invariant sets are given by the attractors (5.20) and (5.30) for the nonlinear initial-value problems (4.1) and (4.32). These attracting invariant manifolds called *inertial manifolds* are finite-dimensional Lipschitz invariant manifolds and make possible the reduction of the dynamics of the infinite-dimensional system of initial-value problems to a finite-dimensional system of ordinary differential equations. Another aim of this work is to extend the notion of inertial manifolds established in various investigations and obtain new inertial manifolds for ageostrophic flows.

**Definition:** A given subset  $\Lambda \subseteq \Upsilon$  is called an inertial manifold for the initial-value problems (4.1) and (4.32) whenever the following properties are satisfied:

- it is a finite-dimensional normally hyperbolic invariant manifold,

- and it consists of trajectories with exponential decay or exponential growth. Hence, inertial manifolds are graphs of Lipschitz functions, and are characterized by the exponential rate at which trajectories on them approach the attractor as  $t \rightarrow +\infty$  or repeller as  $t \rightarrow -\infty$ . Here, the repeller is a time-reversed object of the attractor which is well-defined due to the validity of the group property of the dynamical system.

### 6.1.1 $\beta$ -plane ageostrophic equations

In the construction of invariant manifolds we note that we have reformulated the initial-boundary value problem for ageostrophic flows in order to reflect the influence of the investigation [86] on new classes of inertial manifolds for the the Navier-Stokes equations. The sophisticated work gives more refined results on approximate inertial manifolds and as such validates this exposition from a benchmark point of view and offers the opportunity of invoking established results as a guide for Lipschitz invariant manifolds. The inertial manifolds of [86] are improvements of the results in [65, 27, 67]. The work in [86, 65, 27, 67] plays a major role in this study of ageostrophic equations that govern the flow of a viscous incompressible stratified fluid under the the Coriolis force.

We make a noteworthy remark that Titi's criterion [65] for inertial manifolds warrants that the eigenspace  $\mathbf{X}_0$  utilized in the Faedo-Galerkin technique

$$U_m = \sum_{i=1}^m g_{im}(t)W_i$$

of the solution  $U$  of (4.1) and (4.32) is considered as an approximate inertial manifold with the dynamics of the infinite-dimensional functional differential equations governed by the finite-dimensional system of nonlinear ordinary differential equations (4.18) and (4.54). Heuristically, this follows from the strong result (5.30) on attractors and the sharp a priori energy type estimate (4.53) in the case of ageostrophic equations with Reynolds stress. In the case of  $\beta$ -plane ageostrophic equations the result on approximate inertial manifolds is only valid for the two-dimensional space variables, as indicated by the strong result (5.20) on attractors. The details for the proof of the existence of inertial manifolds are given in the next section when we examine  $\beta$ -plane ageostrophic equations with Reynolds stress.

### 6.1.2 $\beta$ -plane ageostrophic equations with Reynolds stress

In this section we construct Lipschitz invariant manifolds where the invariant sets are given by the attractors (5.30) for the nonlinear initial-value problem (4.32). Similar to

the construction of center-stable invariant manifolds in geometric singular perturbation theory for dynamical systems [73, 74, 92, 96], we consider the modified differential equation

$$\begin{aligned} \frac{dU}{dt} + LU + \eta_\gamma(\|L^{\frac{1}{2}}U\|)N(U) &= 0, \\ U(0) &= U_0, \end{aligned} \tag{6.1}$$

where  $\eta : \mathbb{R}_+ \rightarrow [0, 1]$  is a  $C^\infty$ -bump function and  $\eta_\gamma(s) = \eta(\frac{s}{\gamma})$  for  $s \geq 0$ . In particular,

$$\eta_\gamma(\|L^{1/2}U\|) = \begin{cases} 1 & \text{for } \|L^{1/2}U\| \leq \gamma \\ 0 & \text{for } \|L^{1/2}U\| \geq 2\gamma. \end{cases}$$

The hypothesis  $\eta_\gamma(\|L^{1/2}U\|) = 1$  which holds for  $\|L^{1/2}U\| \leq \gamma = \rho_2$  implies the initial-value problems (4.32) and (6.1) are identical in the  $D(L^{\frac{1}{2}})$ -neighborhood of the attractor. Furthermore, a priori estimates (4.47) – (4.53) are satisfied by the solutions of the modified initial-value problem (6.1). The advantage of (6.1) compared to the initial-value problem (4.32) is that (6.1) has an inertial manifold given by a graph Lipschitz function above the root space of the linear operator  $L$ .

Hence, existence, uniqueness and differentiability properties of solution for the problem (6.1) hold. The result of the existence and uniqueness of solutions for the initial-value problem (6.1) yields

**Proposition 6.1.1** *Under the above hypotheses for  $U_0 \in \Upsilon$ , (6.1) generate dynamical systems  $U(t) = S(t)U_0$  satisfying*

$$U \in C([0, \infty); \Upsilon) \cap L^2((0, \infty); D(L^{1/2})).$$

*Additionally, if  $U_0 \in D(L^{1/2})$  then*

$$U \in C([0, \infty); D(L^{1/2})) \cap L^2((0, \infty); D(L)).$$

Thus, if  $\|L^{1/2}U\| \geq 2\rho_2$  then employing the inequality (4.51) we obtain

$$\begin{aligned} &\frac{d}{dt}\|L^{1/2}U\|^2 + \lambda_1\|L^{1/2}U\|^2 \\ &\leq \frac{d}{dt}\|L^{1/2}U\|^2 + \|LU\|^2 \leq 0. \end{aligned} \tag{6.2}$$

By virtue of Gronwall's inequality (3.10) in the differential inequality (6.2) we get more refined a priori estimate

$$\|L^{1/2}U(t)\|^2 \leq \|L^{1/2}U_0\|^2 \exp(-\lambda_1 t). \tag{6.3}$$

Consequently, if  $\|L^{1/2}U_0\| > \rho_3$  where  $\rho_3 \geq 2\rho_2$  then  $U(t)$  decay exponentially in  $D(L^{1/2})$  to a ball of radius  $\rho_3$ . From (6.3), we obtain the estimate

$$\|\exp(-tL^{1/2})\|_{\mathbf{L}(D(L^{1/2}))} \leq \exp(-\lambda_1 t) \quad \forall t \geq 0,$$

which implies the norm of  $\exp(-tL^{1/2})$  in  $\mathbf{L}(D(L^{1/2}))$  is bounded and therefore continuous. Here,  $\mathbf{L}(D(L^{1/2})) = \mathbf{L}(D(L^{1/2}), D(L^{1/2}))$  denotes the Banach space of bounded linear operators from  $D(L^{1/2})$  into itself. Similarly, it is straightforward to demonstrate that

$$\|\exp(-tL)\|_{\mathbf{L}(D(L^{-1/2}), D(L))} \leq \left(\frac{1}{2t} + \lambda_1\right)^{\frac{1}{2}} \exp(-\lambda_1 t), \quad \forall t > 0,$$

implies the norm of  $\exp(-tL)$  is bounded in the Banach space  $\mathbf{L}(D(L^{-1/2}), D(L))$  of bounded linear operators from  $D(L^{-1/2})$  into  $D(L)$ .

In this case of three-dimensional space variables  $\beta$ -plane ageostrophic flows with Reynolds stress (2.28) – (2.30), since  $D(L) \subset \Upsilon$  is dense, then of a sequence of positive eigenvalues and a corresponding sequence of eigensolutions which give an orthonormal basis  $\{W_i\}$  of  $\Upsilon$  such that (4.36) holds and therefore there exists a spectral gap condition

$$\frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1}^{1/2} - \lambda_m^{1/2}} > \zeta, \quad (6.4)$$

for sufficiently large  $m$  and therefore the gap between  $\lambda_m = \max\{\sigma(L|_{\Pi D(L^{1/2})})\}$  and  $\lambda_{m+1} = \min\{\sigma(L|_{PD(L^{1/2})})\}$  is suitably large. Moreover, the following result on the existence of exponential dichotomies for the initial-value problem (6.1) hold with a constant  $\kappa$  independent of  $m$ :

**Lemma 6.1.2**

$$\begin{aligned} \|\exp(-Lt)\Pi\|_{\mathbf{L}(\Upsilon)} &\leq \kappa \exp(-\lambda t) \quad \forall t < 0, \\ \|L^{1/2} \exp(-Lt)\Pi\|_{\mathbf{L}(\Upsilon)} &\leq \kappa \lambda^{1/2} \exp(-\lambda t) \quad \forall t < 0, \\ \|\exp(-Lt)P\|_{\mathbf{L}(\Upsilon)} &\leq \kappa \exp(-\nu t) \quad \forall t > 0, \\ \|L^{1/2} \exp(-Lt)P\|_{\mathbf{L}(\Upsilon)} &\leq \kappa (t^{-1/2} + \lambda_{m+1}^{1/2}) \exp(-\nu t) \quad \forall t > 0. \end{aligned}$$

Here,  $\Pi = \Pi_m$  is the spectral projection of rank  $m$  in  $\Upsilon, D(L^{1/2}), D(L^{-1/2}),$  or  $D(L)$  onto the eigenspace  $\mathbf{X}_0$  spanned by the eigenfunctions of  $L$ ;  $\lambda = \lambda_m + 2c\lambda_m^{1/2}$  and  $\nu = \lambda_{m+1} - 2c\lambda_{m+1}^{1/2}$ . And we have

$$P = P_m = I - \Pi_m.$$

These assumptions imply there are invariant subspaces denoted by the subspaces  $\Pi D(L^{1/2}) = \Pi_m D(L^{1/2})$ , and  $PD(L^{1/2}) = P_m D(L^{1/2})$ , respectively, so that the splitting of the vector space  $D(L^{\frac{1}{2}})$

$$D(L^{1/2}) = \Pi D(L^{1/2}) \oplus PD(L^{1/2}),$$

is satisfied. In this case, the correspondence is such that

$$\Pi D(L^{1/2}) = \mathbf{X}_0.$$

Moreover, the properties of exponential dichotomies assert that  $\Pi D(L^{1/2})$  is a  $m$ -dimensional subspace of solutions tending to zero uniformly and exponentially as  $t \rightarrow \infty$ . And the complementary subspace  $PD(L^{1/2})$  of solutions tending to infinity uniformly and exponentially as  $t \rightarrow \infty$  with the angle between these subspaces remaining bounded away from zero.

Given that  $U(t)$  is the solution of the modified initial-value problem (6.1), we define  $p$  and  $q$  with

$$p(t) = \Pi_m U(t),$$

and

$$q(t) = (I - \Pi_m)U(t).$$

Then we rewrite the initial-value problem (6.1) as systems

$$\frac{dp}{dt} + Lp + (\Pi_m)G(U) = 0 \text{ on } \Pi D(L^{1/2}) = \mathbf{X}_0, \quad (6.5)$$

and

$$\frac{dq}{dt} + Lq + (I - \Pi_m)G(U) = 0 \text{ on } PD(L^{1/2}), \quad (6.6)$$

where  $G(U) = \eta_\gamma(\|L^{1/2}U\|)N(U)$  and  $U = p + q$ . Also, the solution of the initial-value problem (6.1) has been decomposed into  $U = p + q$ , utilizing the invariance of the subspaces  $\Pi D(L^{1/2})$  and  $PD(L^{1/2})$  and the continuity of the spectral projection  $\Pi = \Pi_m$  and  $P = P_m = I - \Pi_m$ , and the uniqueness of solution to obtain the solutions of the initial-value problems (6.5) and (6.6).

We construct inertial manifolds for system (6.1) utilizing the Lyapunov-Perron technique which is based on the contraction mapping principle of Banach-Cacciopoli. In order to motivate the objective and the strategy of the Lyapunov-Perron technique, we consider the decomposition  $U(t) = p(t) + \Phi(p(t))$  which is a solution of the initial-value problem (6.1) if and only if  $p(t)$  and  $q(t) = \Phi(p(t))$  satisfy (6.5) and (6.6) with  $\Phi$  required to be a Lipschitz function from  $\Pi D(L^{1/2})$  into  $PD(L^{1/2})$ .

An inertial manifold  $\Lambda$  will be constructed as a graph

$$\Lambda = \text{graph}(\Phi),$$

provided a Lipschitz function  $\Phi$  exists. Consequently, part of the proof is in establishing the existence of a Lipschitz function  $\Phi$ . We adopt the approach of determining the

Lipschitz function  $\Phi$  as a fixed point of an integral equation to be derived subsequently and by using the splitting between different subspaces to obtain a priori estimates on the spectral projections defined above.

Given a Lipschitz function  $\Phi$  in Banach space and  $p_0 \in \Pi D(L^{1/2})$  then by employing (3.1) and (3.2) which follows from the above a priori estimates, we integrate (6.5) with  $U = p + \Phi(p)$  and  $p(0) = p_0$  and find that

$$p(t) = - \int_{-\infty}^t \exp[-(t - \tau)L\Pi]\Pi G(p(\tau) + \Phi(p(\tau)))d\tau.$$

With the assistance of Gronwall's inequality the solution given by the above integral is unique for all  $t \in \mathbb{R}$  by virtue of the Lipschitz continuity assumption of  $\Phi$  and the uniform boundedness  $PD(L^{1/2})$ . Thus,  $p(t)$  exists and is unique in the space of all continuous functions  $p \in C(\mathbb{R}; \Pi Y)$ . The function space  $C(\mathbb{R}; \Pi Y)$  is a complete normed space when given the norm

$$\sup\{\|Lp(t)\| : t \in \mathbb{R}\} < \infty.$$

Integration of the nonlinear initial-value problem (6.6) using (3.1), (3.2) and the above a priori energy type estimates provide a continuous mapping defined by the integral of (6.6), given by the formula

$$q(t) = S(t - t_0)q(t_0) - \int_{t_0}^t S(t - \tau)PG(p(\tau) + \Phi(p(\tau)))d\tau,$$

where the continuous solution operators  $S(t) = \exp(-tL)$  satisfy the linear version of problem (6.6). Above, we established the boundedness of  $S(t) = \exp(-tL)$  and by taking the limit  $t_0 \rightarrow -\infty$ , we get

$$q(t) = - \int_{-\infty}^t \exp[-(t - \tau)L\Pi]PG(p(\tau) + \Phi(p(\tau)))d\tau.$$

Setting  $t = 0$  in the above integral equation, we obtain

$$q(0) = - \int_{-\infty}^0 \exp(\tau LP)PG(p(\tau) + \Phi(p(\tau)))d\tau,$$

where

$$p = p(\tau, \Phi; p_0).$$

From the fact that  $q(0)$  is specified by the above integral equation and the observation that  $q(0)$  depends on  $\Phi$  and  $p_0 \in \Pi D(L^{1/2})$ , we obtain a mapping

$$\Phi \rightarrow \Gamma^t \Phi,$$

where  $\Phi$  is a function

$$p_0 \rightarrow \Phi(p_0)$$

and  $\Gamma^t\Phi$  maps  $p_0$  into  $q(0)$  given by the integral equation

$$\Gamma^t q(0) = - \int_{-\infty}^0 \exp(\tau LP) PG(p(\tau) + \Phi(p(\tau))) d\tau.$$

We require  $p(0) + q(0)$  to belong to the inertial manifold  $\Lambda$  which holds whenever

$$q(0) = \Phi(p_0) = \Gamma^t\Phi(p_0) \quad \forall \quad p_0 \in \Pi D(L^{1/2}) = \mathbf{X}_0.$$

This property of  $\Phi$  will be utilized to determine this function as a fixed point problem.

Consider the function space

$$\mathbf{F} = \{ \Phi : \Pi D(L^{1/2}) \rightarrow PD(L^{1/2}), \sup_{p \in \Pi D(L^{1/2})} \|L^{1/2}\Phi(p)\| \leq 1, \text{ with Lipschitz constant } \text{Lip}\Phi \leq 1 \}.$$

The function space  $\mathbf{F}$  is a complete normed space when given the norm

$$\|\Phi_1 - \Phi_2\|_{\text{Lip}} = \sup_{p \in \Pi D(L^{1/2})} \|L^{1/2}(\Phi_1(p) - \Phi_2(p))\|,$$

since  $PD(L^{1/2})$  is a Hilbert space with inner product induced by  $D(L^{1/2})$ .

A calculation using exponential dichotomies and the spectral gap condition (4.36) shows that the mapping  $\Gamma^t$  specified above by an infinite-dimensional integral equation is well-defined. Furthermore, using properties of the modified initial-value problem (6.1) and by virtue of the  $C^\infty$ -bump function we observe that the range of  $\Gamma^t\Phi$  is in a compact subset of  $PD(L^{1/2})$  and indeed does not depend on  $\Phi$ . Hence, the fixed point problem is to find  $\Phi \in \mathbf{F}$  and  $p_0 \in \Pi D(L^{1/2})$  so that  $\Gamma^t$  is a contraction mapping on  $\mathbf{F}$  well-defined by the integral equation

$$\Gamma^t\Phi(p_0) = - \int_{-\infty}^0 \exp(\tau LP) PG(p(\tau) + \Phi(p(\tau))) d\tau.$$

where  $p$  is the solution of the finite-dimensional ordinary differential equation (6.5) with initial condition  $p(0) = p_0$ .

**Proposition 6.1.3** *We set*

$$\nu_1 = \sup_{U \in D(L^{1/2})} \|G(U)\|, \tag{6.7}$$

and

$$\nu_2 = \sup_{U \in D(L^{1/2})} \|dG(U)\|_{\mathbf{L}(D(L^{1/2}), \Upsilon)}, \tag{6.8}$$

where  $\mathbf{L}(D(L^{1/2}), \Upsilon)$  denotes the Banach space of bounded operators from  $D(L^{1/2})$  into  $\Upsilon$  and  $dG$  is the Fréchet derivative of the nonlinearity  $G$ . Under the above hypotheses for the existence of exponential dichotomies and the spectral gap condition (4.36); then  $\Gamma^t$  is a contraction mapping of  $\mathbf{F}$  into itself. And the inertial manifold  $\Lambda$  is given by the graph of a Lipschitz function  $\Phi$  which is the unique fixed point of  $\Gamma^t$  on  $\mathbf{F}$ .

Next, we exhibit a series of results adapted from the general results invariant manifolds which will be customized for establishing validity of this proposition. The basic technique for obtaining the unique fixed point of  $\Gamma^t$  on  $\mathbf{F}$  is taking a norm followed by an application of Gronwall's inequality (3.10) and the variation of constants formula to verify the invariance of the inertial manifold  $\Lambda$ . This strategy is particularly remarkable and interesting in that it can be utilized to obtain a priori estimates on the projected solution operators for small time as well as  $t \rightarrow \pm\infty$ .

The first part of the proof consists of showing that  $\Gamma^t$  maps  $\mathbf{F}$  into itself and lastly we illustrate that indeed  $\Gamma^t$  is a contraction mapping on  $\mathbf{F}$ . Let  $\Phi \in \mathbf{F}$  and for  $p_0 \in \Pi D(L^{1/2})$  consider the integral equation

$$\Gamma^t \Phi(p_0) = - \int_{-\infty}^0 \exp(\tau LP) P G(p(\tau) + \Phi(p(\tau))) d\tau.$$

We need to show that for sufficiently large  $m$ ,  $\Gamma^t$  is a mapping of  $\mathbf{F}$  into itself. Taking norms and using the above hypotheses on exponential dichotomies, (6.7), and (6.8), we obtain the estimates

$$\begin{aligned} \|L^{1/2} \Gamma^t \Phi(p_0)\| &\leq \int_{-\infty}^0 \|\exp(\tau LP) P\|_{L(\mathbb{R})} \|G(p(\tau) + \Phi(p(\tau)))\| d\tau \\ &\leq \nu_1 \kappa \int_{-\infty}^0 (\tau^{1/2} + \lambda_{m+1}^{1/2}) \exp(\nu \tau) d\tau \\ &\leq 4\nu_1 \kappa \left( \frac{\lambda_{m+1}^{1/2}}{\nu} \right), \end{aligned}$$

with  $\lambda = \lambda_m + 2c\lambda_m^{1/2}$  and  $\nu = \lambda_{m+1} - 2c\lambda_{m+1}^{1/2}$  given in the above lemma on the existence of exponential dichotomies and spectral gap condition. Hence, whenever  $m$  is sufficiently large then we get

$$\|L^{1/2} \Gamma^t \Phi(p_0)\| \leq 4\nu_1 \kappa \left( \frac{\lambda_{m+1}^{1/2}}{\nu} \right) \leq 1 \quad \forall p_0 \in \Pi D(L^{1/2}).$$

We can now consider the difference of two solutions  $p(t) = p_1(t) - p_2(t)$ , with  $p_1(t)$  and  $p_2(t)$  satisfying the finite-dimensional ordinary differential equation (6.5) with initial conditions  $p_{01}$  and  $p_{02}$ , respectively. The utility of the variation of constants formula with  $p(t) = p_1(t) - p_2(t)$ , we obtain

$$p(t) = S(t)p(0) - \int_0^t S(t-\tau) (\Pi G(p_1(\tau) + \Phi(p_1(\tau))) - \Pi G(p_2(\tau) + \Phi(p_2(\tau)))) d\tau \quad \forall t \in \mathbb{R},$$

where the continuous solution operators  $S(t) = \exp(-tL)$  satisfy the linear version of problem (6.5).

Similarly, taking norms and using the above exponential dichotomies, (6.7), and (6.8), we have

$$\|L^{1/2}p(t)\| \leq \kappa \exp(-\lambda t) \|L^{1/2}p(0)\| + 2\kappa\nu_2 \int_t^0 \lambda^{1/2} \exp[-\lambda(t-\tau)] \|L^{1/2}p(\tau)\| d\tau \quad \forall t < 0.$$

In order to invoke the Gronwall's inequality (3.10) we note that the above estimate may be restated equivalently using

$$-\frac{dh}{dt} \leq \kappa \|L^{1/2}p(0)\| + 2\kappa\nu_2 \lambda^{1/2} h,$$

where

$$h(t) = \int_t^0 \exp[\lambda\tau] \|L^{1/2}p(\tau)\| d\tau.$$

By virtue of the Gronwall's inequality, we have

$$\|h(t)\| = \frac{\|L^{1/2}p(0)\|}{2\nu_2 \lambda^{1/2}} \exp(-2\kappa\nu_2 \lambda^{1/2} t).$$

Consequently, we obtain the following a priori estimates that will be employed in demonstrating that  $\Gamma^t$  is a mapping of  $\mathbf{F}$  into itself:

$$\|L^{1/2}p(t)\| \leq \kappa \exp(-\lambda t) \|L^{1/2}p(0)\| + \kappa \|L^{1/2}p(0)\| \exp[-(\lambda + 2\kappa\nu_1 \lambda^{1/2})t]$$

and this implies

$$\|L^{1/2}p(t)\| \leq \kappa \|L^{1/2}p(0)\| \exp[-(\lambda + 2\kappa\nu_1 \lambda^{1/2})t] \quad \forall t < 0. \quad (6.9)$$

From these together with the definition of the mapping  $\Gamma^t$  we deduce a priori estimates

$$\begin{aligned} & \|L^{1/2}(\Gamma^t\Phi(p_{01}) - \Gamma^t\Phi(p_{02}))\| \\ & \leq \int_{-\infty}^0 \|L^{1/2} \exp(\tau L) P\| \|G(p_1(\tau) + \Phi(p_1(\tau))) - G(p_2(\tau) + \Phi(p_2(\tau)))\| d\tau \\ & \leq 2\kappa\nu_1 \int_{-\infty}^0 (\tau^{1/2} + \lambda_{m+1}^{1/2}) \exp(\nu\tau) \|L^{1/2}p(\tau)\| d\tau, \end{aligned}$$

with  $\lambda = \lambda_m + 2c\lambda_m^{1/2}$  and  $\nu = \lambda_{m+1} - 2c\lambda_{m+1}^{1/2}$ .

Using (6.9), we obtain

$$\begin{aligned} & \|L^{1/2}(\Gamma^t\Phi(p_{01}) - \Gamma^t\Phi(p_{02}))\| \\ & \leq 4\kappa^2\nu_2 \|L^{1/2}p(0)\| \int_{-\infty}^0 \lambda_{m+1}^{1/2} \exp[(\nu - \lambda - 2\kappa\nu_2 \lambda^{1/2})\tau] d\tau \\ & \leq 16\kappa^2\nu_2 \|L^{1/2}p(0)\| \left( \frac{\lambda_{m+1}^{1/2}}{\nu - \lambda - 2\kappa\nu_2 \lambda^{1/2}} \right). \end{aligned}$$

We remark that the above estimates may be restated equivalently using

$$\|L^{1/2}(\Gamma^t\Phi(p_{01}) - \Gamma^t\Phi(p_{02}))\|$$

$$\leq 16\kappa^2\nu_2\|L^{1/2}p(0)\|\left(\frac{\lambda_{m+1}^{1/2} + \lambda_m^{1/2}}{\lambda_{m+1} - \lambda_m - k(\lambda_{m+1}^{1/2} + \lambda_m^{1/2})}\right),$$

where  $k = 2c_1 + 2\kappa\nu_2(1 + 2c_1\lambda^{-1/2})^{1/2}$ . By the validity of the spectral gap condition (6.4) we conclude that

$$\|\Gamma^t\Phi(p_{01}) - \Gamma^t\Phi(p_{02})\|_{Lip} = \sup_{p_0 \in \Pi D(L^{1/2})} \|L^{1/2}(\Gamma^t\Phi(p_{01}) - \Gamma^t\Phi(p_{02}))\| \leq \|L^{1/2}(p_{01} - p_{02})\|,$$

holds with Lipschitz constant  $\text{Lip}\Gamma^t\Phi \leq 1$  if we choose  $\zeta$  in the spectral gap condition (6.4) to satisfy

$$\zeta \geq k + 16\kappa^2\nu_2.$$

The above arguments and a priori estimates prove the result that for sufficiently large  $m$ ,  $\Gamma^t$  yield a mapping of  $\mathbf{F}$  into itself.

Next we illustrate that  $\Gamma^t$  is a contraction mapping on  $\mathbf{F}$  for  $m$  sufficiently large. For that goal we consider two functions  $\Phi_1, \Phi_2 \in \mathbf{F}$  and  $p \in \Pi D(L^{1/2})$  and let  $p_1$  and  $p_2$  be two solutions to the initial-value problem (6.5) with the same initial condition  $p_{01} = p_0 = p_{02}$ . The utility of the variation of constants formula with the difference of two solutions  $p(t) = p_1(t) - p_2(t)$ , yield

$$p(t) = - \int_0^t S(t-\tau)(\Pi G(p_1(\tau) + \Phi(p_1(\tau))) - \Pi G(p_2(\tau) + \Phi(p_2(\tau))))d\tau,$$

where the continuous solution operators  $S(t) = \exp(-tL)$  satisfy the linear version of problem (6.5). If  $t < 0$ , taking norms and using the above a priori estimates and the Lipschitz constant  $\text{Lip}\Gamma^t\Phi \leq 1$ , we get

$$\begin{aligned} \|L^{1/2}p(t)\| &\leq \kappa\nu_2\lambda^{1/2} \int_t^0 \exp[-\lambda(t-\tau)](\|\Phi_1 - \Phi_2\|_{Lip} + 2\|L^{1/2}p(\tau)\|)d\tau \\ &\leq \kappa\nu_2\lambda^{-1/2}\|\Phi_1 - \Phi_2\|_{Lip} \exp(-\lambda t) + 2\kappa\nu_2\lambda^{1/2} \int_t^0 \exp[-\lambda(t-\tau)]\|L^{1/2}p(\tau)\|d\tau. \end{aligned}$$

Using Gronwall's inequality (3.10) leads to the estimate

$$\|L^{1/2}p(t)\| \leq 2\kappa\nu_2\lambda^{-1/2}\|\Phi_1 - \Phi_2\|_{Lip} \exp[-(\lambda + 2\kappa\nu_2\lambda^{1/2})t].$$

From the above a priori estimates utilized in demonstrating that  $\Gamma^t$  is a mapping on the Banach space  $\mathbf{F}$ , we also have

$$\begin{aligned} &\|L^{1/2}(\Gamma^t\Phi(p_0) - \Gamma^t\Phi(p_0))\| \\ &\leq \int_{-\infty}^0 \|L^{1/2} \exp(\tau L)P\| \|G(p_1(\tau) + \Phi(p_1(\tau))) - G(p_2(\tau) + \Phi(p_2(\tau)))\| d\tau \end{aligned}$$

We conclude therefore that

$$\|L^{1/2}(\Gamma^t\Phi(p_0) - \Gamma^t\Phi(p_0))\| \leq K\left(\frac{\lambda_{m+1}^{1/2}}{\nu - \lambda - 2\lambda^{1/2}\kappa\nu_2}\right)\|\Phi_1 - \Phi_2\|_{Lip},$$

with the constant  $K$  specified by

$$K = 4\kappa\nu_2(1 + 4\kappa\lambda^{-1/2}).$$

If we choose  $\zeta$  in the spectral gap condition (6.4) to satisfy

$$\zeta \geq 2K + k,$$

where  $k = 2c_1 + 2\kappa\nu_2(1 + 2c_1\lambda^{-1/2})^{1/2}$  then we get

$$\|\Gamma^t\Phi_1 - \Gamma^t\Phi_2\|_{Lip} \leq \frac{1}{2}\|\Phi_1 - \Phi_2\|_{Lip}.$$

This proves the result that  $\Gamma^t$  is a contraction mapping on  $\mathbf{F}$  with Lipschitz constant equal to half. Hence, by the contraction mapping principle of Banach-Cacciopoli, there exists a unique Lipschitz function  $\Phi \in \mathbf{F}$  from  $\Pi D(L^{1/2})$  into  $PD(L^{1/2})$ .

From the validity of the existence of a Lipschitz function  $\Phi$ , we prove that  $\Lambda = \text{graph}(\Phi)$  satisfies the hypotheses of an inertial manifold. The invariance of  $\Lambda = \text{graph}(\Phi)$  is a consequence of the fixed point equation  $\Gamma^t\Phi = \Phi$  which reduces to

$$\Phi(p_0) = \Gamma^t\Phi(p_0) = -\int_{-\infty}^0 \exp(\tau LP)PG(U(p_0, \tau))d\tau.$$

Substitution of  $p(t) = p(t; \Phi, p_0)$  in the above integral equation and by virtue of the group property of the solution operators we obtain

$$\begin{aligned} \Phi(p(t)) &= -\int_{-\infty}^0 \exp(\tau LP)PG(p(\tau) + \Phi(p(\tau)))d\tau \\ &= -\int_{-\infty}^t \exp(\tau LP)PG(U(p_0, \tau))d\tau \quad \forall t \in \mathbb{R}. \end{aligned}$$

Additionally, a calculation involving differentiation with respect to  $t$  shows that  $p(t)$ ,  $q(t)$ , and  $U(t) = p(t) + q(t)$  satisfy the initial-value problems (6.5), (6.6), and (6.5), respectively, where  $q(t) = \Phi(p(t))$ . This proves the invariance of the inertial manifold  $\Lambda = \text{graph}(\Phi)$  and hence  $S(t)\Lambda \subseteq \Lambda$ .

Define  $\text{dist}(u, \Lambda) = \inf\{\|u - v\| : v \in \Lambda\}$  then using the Lipschitz property (4.64) for the dynamical system and the fact that, if  $\|L^{1/2}U_0\| > \rho_3$  where  $\rho_3 \geq 2\rho_2$  then  $U(t)$  decay exponentially in  $D(L^{1/2})$  to a ball of radius  $\rho_3$  we observe that for  $t_0 = \frac{1}{2k^2} \log 2$  then

$$\text{dist}(S(t_1)u_0, \Lambda) \leq \frac{1}{2}\text{dist}(u_0, \Lambda)$$

is valid with  $t_0 \leq t_1 \leq 2t_0$ . In general we obtain

$$\begin{aligned} \text{dist}(S(mt_1)u_0, \Lambda) &\leq \frac{1}{2^m} \text{dist}(u_0, \Lambda) \\ &\leq \exp\left(-\frac{t}{2t_0} \log 2\right) \text{dist}(u_0, \Lambda) \quad \text{for } t > t_0, \end{aligned}$$

which yield exponential decay with exponential rate  $\frac{1}{2t_0} \log 2$ .

Hence, the inertial manifold  $\Lambda = \text{graph}(\Phi)$  consists of orbits or trajectories which decay exponentially. Therefore, inertial manifolds are graphs of Lipschitz functions, and are indeed characterized by the exponential rate at trajectories on them approach the invariant set as  $t \rightarrow +\infty$ . We have to make sure that the master invariant manifold  $\text{graph}(\Phi)$  contains the attractor whose existence, we proved using more refined a priori estimates that it is given by the  $\omega$ -limit set of  $Q = B_{2\rho_2}$ ,

$$A = \omega(Q) = \bigcap_{s \geq 0} Cl(\bigcup_{t \geq s} S(t)Q),$$

where  $B_{2\rho_2}$  denotes an open ball of radius  $2\rho_2$ , which depends on geophysically relevant parameters. Furthermore, the use of the group property and continuity of the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ , gave the following invariance property of the above established attractor:

$$S(t)A = A, \quad \forall t \in \mathbb{R}.$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the general three-dimensional space variables initial-boundary value problem for viscous  $\beta$ -plane ageostrophic equations with Reynolds stress is both positively and negatively invariant and consists of trajectories that are defined for all  $t \in \mathbb{R}$ .

A similar argument establishes that the inertial manifold specified by  $\Lambda = \text{graph}(\Phi)$ , contains the attractor. Note that for  $u \in \omega(Q)$  implies that the solution  $S(t)u$  is defined  $\forall t \in \mathbb{R}$  and we have

$$\text{dist}(S(t)u, \Lambda) \leq 2\rho_2.$$

Therefore, if we set  $v = S(-t)$  with  $t \geq t_0$  as in the previous estimates then we obtain

$$\text{dist}(S(t)u, \Lambda) \leq 2\rho_2 \exp\left(-\frac{t}{2t_0} \log 2\right).$$

This proves the result that inertial manifold given by  $\Lambda = \text{graph}(\Phi)$  contains the attractor.

# Chapter 7

## Conclusions

### 7.1 Discussion of results

In this investigation we reviewed the equations of geophysical fluid dynamics as the first effort towards understanding the impact of dissipative ageostrophic flows in the ocean. The ageostrophic model, as the equations have been called, governs the flow of a viscous incompressible stratified fluid under the Coriolis force.

It has been proved that the general three-dimensional space variables initial-boundary value problem for viscous  $\beta$ -plane ageostrophic equations with Reynolds stress has a unique solution in the large that enjoys useful properties such as differentiability with respect to initial conditions. Furthermore, we illustrated that uniqueness and continuity with respect to initial conditions of the solution  $U(t)$  of the initial-value problem generate a dynamical system provided by continuous solution operators  $S(t)$ ,  $S(t)$ ,  $t \in \mathbb{R}_+$ . We further proved that the solution operators  $S(t)$ ,  $t \in \mathbb{R}_+$  are injective and as a result we obtained the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ . The crucial property of the solution operators which we established is the group property

$$S(t)U(s) = S(s)U(t)$$

$$S(0)U = U \quad \forall t, s \in \mathbb{R}.$$

As regards the asymptotic stability of the solutions of the initial-boundary value problems for viscous  $\beta$ -plane ageostrophic equations, according to more refined a priori estimates that we develop, one of the stability of the rest state is investigated using the additive

decomposition of the Lyapunov functional

$$E(t) = \mu_1 E_1(t) + \mu_2 E_2(t),$$

where

$$E_1(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 + \frac{Ro}{Ek} |\nabla \underline{u}|^2 + Ed |\nabla \rho|^2 \right) dx,$$

which is a specification of energy and entropy production in the  $H^1$ -Sobolev norm and a suitable coupling functional

$$E_2(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 \right) dx + \nu_1 |\Pi(\underline{u} \times \underline{k})|^2 + \nu_2 \Pi(\underline{u} \times \underline{k}) \cdot \Pi(\rho \underline{k}) dx,$$

whenever the criterion

$$\frac{1}{Ek} < \max \left\{ \sigma_L^2, \frac{\pi^2 Ro (Ed)^2}{Fr^2 (3 + 2Ed)} \right\},$$

is valid in the case of  $f$ -plane ageostrophic equations and if the criterion

$$\frac{1}{Ek} < \max \left\{ \sigma_L^2, \frac{\beta Ro^2 (Ed)^2 \pi^2}{Fr^2 (3 + 2Ed)} \right\},$$

holds in the case of  $\beta$ -plane ageostrophic equations. In the above stability criterion, it is noteworthy to observe that  $\mu_1$ ,  $\nu_1, \nu_2$ , and  $\mu_2$  are given positive constants which depend on the above geophysically relevant parameters and the finite number  $\sigma_L$  is given by

$$\frac{1}{\sigma_L} = \sup_{\underline{u}=(\underline{u},\rho)} \left\{ \frac{2 \int_{\Omega} \rho \underline{u} \cdot \underline{k} dx}{\int_{\Omega} (|\nabla \underline{u}|^2 + |\nabla \rho|^2) dx} \right\}.$$

One of the advantages of the result on energy theory in the stability of ageostrophic flows described in this work is that it may be considered as an extension and generalization of the one obtained using the Lyapunov functional

$$E_1(t) = \frac{1}{2} \int_{\Omega} \left( \frac{Ro}{Ek} \underline{u}^2 + Ed\rho^2 + \frac{Ro}{Ek} |\nabla \underline{u}|^2 + Ed |\nabla \rho|^2 \right) dx,$$

which is the specification of energy and the entropy production in the  $H^1$ -Sobolev norm, provided the new criterion

$$\frac{1}{Ek} < \sigma_L,$$

holds. These guarantee that stability criteria may be expressed utilizing the following geophysically relevant parameters:  $Ro$ , the Rossby number which compares the inertial term to the Coriolis force;  $Fr$ , the Froude number which measures the importance of stratification;  $Ek$ , the Ekman number which measures the relative importance of frictional dissipation;  $Ed$ , the nondimensional eddy diffusion coefficient. From theoretical and applied aspects of geophysical fluid dynamics, the Burger number defined by  $Bu = \left(\frac{Ro}{Fr}\right)^2$  is a useful measure of stratification. And the finite number  $\sigma_L$  is given by

$$\frac{1}{\sigma_L} = \sup_{\underline{u}=(\underline{u},\rho)} \left\{ \frac{2 \int_{\Omega} \rho \underline{u} \cdot \underline{k} dx}{\int_{\Omega} (|\nabla \underline{u}|^2 + |\nabla \rho|^2) dx} \right\}.$$

We alluded to the fact that the techniques employed in developing stability criteria have been completed without carrying out closed-form wave solutions for mesoscale and synoptic eddies. The presence of stratification and Reynolds stress introduced some new considerations that needed special attention since in this more general setting, close-form solutions need not be easily accessible. Thus, most cases of oceanographic or meteorological interest such as vortex spin-down, the decay of energy in the the nonlinear stability criteria need not account for the dissipation of energy and enstrophy for mesoscale and synoptic wave motions. However, we consider the energetic stability criteria as a first effort towards elucidating such geophysical phenomena. Furthermore, we remark that the stability criteria may give the possibility of a more rigorous investigation of the numerical approximation of the solution in the sense of the Lax equivalence theorem. Well-posedness and stability criteria results anticipate that numerical approximations such as finite-difference schemes to the derivatives in the equations are convergent in the sense of the Lax equivalence theorem, that is, a designed finite-difference scheme to a well-posed initial-boundary value problem converges to the solution of the partial differential equation with the rate of convergence specified by the order of accuracy of the scheme.

Additionally, we proved using more refined a priori estimates that the attractor is given by the  $\omega$ -limit set of  $Q = B_{2\rho_2}$ ,

$$A = \omega(Q) = \bigcap_{s \geq 0} Cl(\bigcup_{t \geq s} S(t)Q),$$

where  $B_{2\rho_2}$  denotes an open ball of radius  $2\rho_2$ , which depends on geophysically relevant parameters. The utility of the group property and continuity of the solution operators  $S(t)$  defined for all time  $t \in \mathbb{R}$ , gave the following invariance property of the above established attractor:

$$S(t)A = A, \quad \forall t \in \mathbb{R}.$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the general three-dimensional space variables initial-boundary value problem for viscous  $\beta$ -plane ageostrophic equations with Reynolds stress is both positively and negatively invariant and consists of orbits or trajectories that are defined for all  $t \in \mathbb{R}$ .

Furthermore we proved the existence of Lipschitz invariant manifolds for the general three-dimensional space variables initial-boundary value problem for viscous  $\beta$ -plane ageostrophic equations with Reynolds stress. The Lipschitz invariant manifold that we established possesses the following distinguishing characteristics: it is a finite-dimensional normally hyperbolic invariant manifold, and it consists of trajectories with exponential decay or exponential growth.

## 7.2 New research directions

This investigation represents a first effort in the theoretical and applied aspects of  $\beta$ -plane ageostrophic equations with or without Reynolds stress. The results are indeed encouraging, and we emphasize at this point that based on the manifestation of well-posedness and stability, it is possible to design numerical algorithms such as finite-difference schemes. In the derivation of  $\beta$ -plane ageostrophic equations with Reynolds stress, it was essential to note that the additive decomposition of the ageostrophic flow quantities into coherent and incoherent terms and consideration of the averaging operator to obtain the Reynolds stress fields resulted in a set of evolution equations that were not closed. In order to close the set of equations derived in Chapter 2, we adopted the Pedlosky closure protocols.

Progress utilizing other advantageous protocols for the Reynolds stress fields are possible, here we consider viscoelastic type stress as a closure relation for the Reynolds stress fields of ageostrophic equations. A variety of reasons suggest the use ageostrophic equations with viscoelastic-type stress given by the new system of partial differential equations

$$\begin{aligned} Ro\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) + \rho \underline{k} + (1 + Ro\beta y)\underline{u} \times \underline{k} &= -\nabla p + Ek(\Delta \underline{u} + \nabla \cdot \tau) \\ \nabla \cdot \underline{u} &= 0 \\ \frac{Fr^2}{Ro}\left(\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho\right) + \underline{u} \cdot \underline{k} &= \frac{Fr^2}{Ro(Ed)} \Delta \rho \\ We\left(\frac{\partial \tau}{\partial t} + \underline{u} \cdot \nabla \tau + \tau \Omega - \Omega \tau\right) - We(\tau D + D\tau) + \tau &= 2\varepsilon D. \end{aligned}$$

These primitive equations govern the flow of a viscoelastic incompressible stratified fluid under the Coriolis force. Here,  $\underline{u}$  is velocity of the fluid,  $\tau$  is the Reynolds stress tensor,  $\Omega$  is the vorticity tensor,  $D$  is the rate-of-strain tensor,  $p$  is the pressure, and  $\rho$  is the density. In the system of partial differential equations,  $\Delta$  is the Laplacian operator,  $\nabla$  is the gradient operator,  $Ro$  is the Rossby number,  $\varepsilon$  is the temporal shear-rate viscosity,  $Ek$  is the Ekman number,  $Fr$  is the Froude number,  $We$  is the Weissenberg number,  $\beta$  is reference reciprocal Coriolis parameter and  $Ed$  is nondimensional eddy diffusion coefficient. And the Reynolds number is given by the relation  $Re = \frac{Ro}{Ek}\gamma^2$  where the parameter  $\gamma$  represents the scale ratio  $\frac{L}{H}$ .

In particular, other geophysical fluid dynamics models established above are derived from the  $\beta$ -plane ageostrophic equations with viscoelastic-type stress where use is made of a perturbation analysis of the flow fields with respect to the Rossby number. First-order terms in the Rossby number yield a system of equations which coincide with (A.4), the baroclinic  $\beta$ -plane quasigeostrophic equations that govern the evolution of potential

vorticity.

Furthermore, we note that the reduction of the two-dimensional space variables governing  $\beta$ -plane ageostrophic equations with viscoelastic-type stress with imposed spatial periodicity, no normal flow at the material boundary and fixed density at the bounding planes and by appealing to a two-mode Galerkin approximation yield a four-dimensional system of nonlinear ordinary differential equations. Specifically, by seeking solutions with ansatz

$$\begin{aligned}\psi(x, z, t) &= \psi_1(t) \sin(kx) \cos(\pi z), \\ \rho(x, z, t) &= \rho_1(t) \cos(kx) \cos(\pi z) + \rho_2(t) \sin(2\pi z), \\ \tau_{xx}(x, z, t) &= \xi(t) \cos(kx) \sin(\pi z), \\ \tau_{zz}(x, z, t) &= \eta(t) \cos(kx) \sin(\pi z), \\ \tau_{xz}(x, z, t) &= \zeta(t) \cos(kx) \sin(\pi z),\end{aligned}$$

and substitution in the partial differential equations with pressure having been eliminated using the curl operator gives the four-dimensional ordinary differential equations

$$\begin{aligned}\frac{dX}{dt} &= \frac{1}{Ro}(Y - EkX - \lambda(Ek)W), \\ \frac{dY}{dt} &= rX - Y - XZ, \\ \frac{dZ}{dt} &= -bZ + XY, \\ \frac{dW}{dt} &= \delta(\varepsilon(\lambda)X - W),\end{aligned}$$

where the trigonometric terms other than those appearing in the ansatz has been neglected. Time,  $\xi$ ,  $\eta$ ,  $\zeta$ ,  $\psi_1$ ,  $\rho_1$ , and  $\rho_2$  have been rescaled utilizing

$$\begin{aligned}s &= (\pi^2 + k^2)t, \quad \psi_1 = \frac{\sqrt{2}}{\pi k}(\pi^2 + k^2)X, \\ \rho_1 &= \frac{\sqrt{2}}{\pi^2 k}(\pi^2 + k^2)Y, \quad \rho_2 = \frac{\sqrt{2}}{\pi^2 k}(\pi^2 + k^2)Z, \\ W &= \frac{k}{\pi\sqrt{2}}(\pi^2 + k^2)^{-1} \left\{ \frac{k}{\pi}(\eta - \xi) + \left(\frac{k^2}{\pi^2} - 1\right)\zeta \right\}\end{aligned}$$

and the parameters are given by

$$\begin{aligned}\delta &= \frac{1}{(\pi^2 + k^2)We}, \quad \lambda = \frac{\pi^4}{(\pi^2 + k^2)^2}, \\ r &= \frac{k^2 Ro(Ed)}{(\pi^2 + k^2)^3 Fr^2}, \quad b = \frac{4\pi^2}{\pi^2 + k^2}.\end{aligned}$$

Examination of transversality of inclination-flip homoclinic orbits for the Lorenz system is due to Dumortier *et al.* [85] and was further extended in [126]. Aside from their oceanographic and meteorological interest, the models here are chosen to illustrate necessary

conditions for Lyapunov stability and instability. This system also exhibits chaotic dynamics as demonstrated in Khayat [116], which is largely numerical and entails elementary bifurcations: pitchfork bifurcation and Poincaré-Andronov-Hopf bifurcations. Abreast of these results, in [126], we illustrate the existence of inclination-flip homoclinic orbits, via a bifurcation with a special type of eigenvalue condition, taking account of degeneracies. In order to utilize geometric singular perturbation theory and Melnikov techniques, it is required to perturb the problem and carry the analysis further to the question of the persistence of inclination-flip homoclinic orbits on a slow manifold. There are other viscoelastic-type ageostrophic systems that are amenable to these methods including further explorations of primitive equations governing the flow of a viscoelastic incompressible stratified fluid under the Coriolis force.

It is anticipated that the resulting inclination-flip homoclinic bifurcations will provide the impetus for complex upwelling pattern formations that will have important role in numerous applications, including the explanation of observed ageostrophic instabilities of geophysical fluid flows and deep understanding of mesoscale and synoptic coherent structures. Applied and theoretical aspects of inclination-flip homoclinic orbits and the related phenomena of orbit-flip bifurcations have been recently tackled in [124, 123, 74]. Therefore, the Lorenz type attractor has helped give focus to inclination-flip homoclinic orbits and orbit-flip bifurcations as objects of study.

## Appendix A

# Quasigeostrophic flows

### A.1 $\beta$ -plane quasigeostrophic equations

The baroclinic dissipative quasigeostrophic equations which we derive govern the evolution of the potential vorticity whenever the the Rossby number is considered asymptotically small. The quasigeostrophic system has been of oceanographic and meteorological interest especially for the modelling and forecasting of mesoscale and synoptic scale eddies. The standard derivation of the quasigeostrophic potential vorticity equations from the ageostrophic system, which is a set of primitive equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force with or without the Reynolds stress, is accomplished by the systematic use of scaling arguments and asymptotic series in the Rossby number. The presence of stratification and Reynolds stress introduces some new considerations that need special attention since in this more general setting, exact closed form solutions need not be easily accessible.

The quasigeostrophic limit is useful from the fact that the translating modons of Flierl *et al.* [24, 22, 23, 58] and the rotating modons of Mied *et al.* [40, 41, 50, 39] represent closed form vortex solutions of the quasigeostrophic potential vorticity equation when for example dissipation terms are neglected. Moreover, other simplifying assumptions include the replacement of the vertical coordinate by density so that the quasigeostrophic equations may lend themselves to discretization in the vertical, resulting in layered models. We note here that the introduction of two-layer quasigeostrophic models leads to the independence of horizontal velocity with height. The phenomena is called *barotropic* which complement *baroclinic*, the change of horizontal velocity with height.

The purpose of this section is to derive the quasigeostrophic equations from the  $\beta$ -plane ageostrophic flows (2.24). It is now possible to systematically utilize the presence of parameters in the evolution equations to embark on a perturbation analysis. The next aim is to derive geostrophic and quasigeostrophic models from the ageostrophic equations via a perturbation analysis of the flow fields at the Rossby number on the order of unity or less. In order to illustrate the asymptotic expansion, consider

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_0(\mathbf{x}, t, Ro, \beta, Fr, Ek, Ed, We, \epsilon) + \\
&\quad Ro\mathbf{u}_1(\mathbf{x}, t, Ro, \beta, Fr, Ek, Ed, We, \epsilon) \\
p &= p_0(\mathbf{x}, t, Ro, \beta, Fr, Ek, Ed, We, \epsilon) + \\
&\quad Ro p_1(\mathbf{x}, t, Ro, \beta, Fr, Ek, Ed, We, \epsilon) \\
\rho &= \rho_0(\mathbf{x}, t, Ro, \beta, Fr, Ek, Ed, We, \epsilon) + \\
&\quad Ro\rho_1(\mathbf{x}, t, Ro, \beta, Fr, Ek, Ed, We, \epsilon).
\end{aligned} \tag{A.1}$$

with assumption  $\frac{Fr^2}{Ro} = Ro$ . When the expansion (A.1) is inserted in the nondimensional system (2.24) and equating terms of the same order in the Rossby number,  $Ro$ , at zero-order we obtain *geostrophic* equations given by a set of equations

$$\begin{aligned}
v_0 &= \frac{\partial p_0}{\partial x}, \\
u_0 &= -\frac{\partial p_0}{\partial y}, \\
\rho_0 &= -\frac{\partial p_0}{\partial z}, \\
\nabla \cdot \mathbf{u}_0 &= 0, \\
w_0 &= 0.
\end{aligned} \tag{A.2}$$

In (A.2), horizontal velocity is divergence free and pressure plays the role of streamfunction. From consideration of the continuity equation and the realization that zero-order velocity is *isobaric*, that is, pressure of the system remain constant along streamlines, we obtain  $\frac{\partial w_0}{\partial z} = 0$  in addition of  $w_0 = 0$ . In order to derive equation for evolution of the geostrophic pressure,  $p_0 = \psi$ , we equate first-order terms in the Rossby number which

yield the system of equations

$$\begin{aligned}
\frac{\partial \underline{u}}{\partial t} + \underline{u}_0 \cdot \nabla u_0 - v_1 - y\beta_0 v_0 &= -\frac{\partial p_1}{\partial x} + \frac{Ek}{Ro} \Delta u_0, \\
\frac{\partial v_0}{\partial t} + \underline{u}_0 \cdot \nabla v_0 + u_1 + \beta_0 u_0 + \rho_1 &= -\frac{\partial p_0}{\partial y} + \frac{Ek}{Ro} \Delta v_0, \\
-\beta_0 u_0 + y u_0 + \rho_1 &= -\frac{\partial p_1}{\partial z}, \\
\nabla \cdot \underline{u}_0 &= 0, \\
\frac{\partial \rho_0}{\partial t} + \underline{u}_0 \cdot \nabla \rho_0 + w_1 &= \frac{1}{Ed} \Delta \rho_0.
\end{aligned} \tag{A.3}$$

The next aim is to rewrite the equations (A.3) in an equivalent setup with emphasis on geostrophic pressure which we denote by  $\psi = p_0$ , the streamfunction. We take the two-dimensional curl of horizontal momentum in (A.3) to eliminate the pressure gradient and utilize the fact that from (A.2) horizontal velocity is divergence free which together with suitable simplifying assumptions gives the following *quasigeostrophic* potential vorticity system

$$\begin{aligned}
\frac{\partial q}{\partial t} + J(\psi, q) &= \frac{Ek}{Ro} \Delta q, \\
\frac{\partial \rho}{\partial t} + J(\psi, \rho) + w &= \frac{1}{Ed} \Delta \rho, \\
q &= \Delta \psi + \frac{\partial^2 \psi}{\partial z^2} + \beta y, \\
u &= -\frac{\partial \psi}{\partial y}, \\
v &= \frac{\partial \psi}{\partial x}, \\
\rho &= -\frac{\partial \psi}{\partial z},
\end{aligned} \tag{A.4}$$

where the nondimensional fields  $\underline{u} = (u, v, w)$ ,  $q$ ,  $\rho$ , and  $\psi$  are fluid velocity, potential vorticity, density and streamfunction, respectively. The geophysically relevant parameter  $Ro$  is the Rossby number,  $Ek$  is the Ekman number,  $\beta$  is the reference reciprocal Coriolis parameter. Here  $J(.,.)$  is the Jacobian operator,  $\Delta$  is the Laplacian operator, and the operator  $\frac{\partial}{\partial t} + J(.,.)$  represents advection along fluid particle trajectories. Indeed, the skew-symmetric spin tensor  $\Omega$  defined in equation (2.2) is nonzero and consequently the flow is rotational.

## A.2 $\beta$ -plane quasigeostrophic equations with Reynolds stress

The aim of this section is to derive the quasigeostrophic equations from the ageostrophic flows with Reynolds stress (2.28). It is now possible to systematically utilize the presence of parameters in the evolution equations to embark on a perturbation analysis. The next aim is to derive geostrophic and quasigeostrophic models from the ageostrophic equations via a perturbation analysis of the flow fields at the Rossby number on the order of unity or less. In order to illustrate the asymptotic expansion, consider

$$\begin{aligned}\underline{u} &= \underline{u}_0(\underline{x}, t, Ro, \beta, Fr, Ek, Ed) + Ro\underline{u}_1(\underline{x}, t, Ro, \beta, Fr, Ek, Ed) \\ p &= p_0(\underline{x}, t, Ro, \beta, Fr, Ek, Ed) + Ro p_1(\underline{x}, t, Ro, \beta, Fr, Ek, Ed) \\ \rho &= \rho_0(\underline{x}, t, Ro, \beta, Fr, Ek, Ed) + Ro\rho_1(\underline{x}, t, Ro, \beta, Fr, Ek, Ed),\end{aligned}\tag{A.5}$$

with assumption  $\frac{Fr^2}{Ro} = Ro$ . When the expansion (A.5) is inserted in the nondimensional system (2.28) and equating terms of the same order in the Rossby number,  $Ro$ , at zero-order we obtain *geostrophic* equations given by a set of equations

$$\begin{aligned}v_0 &= \frac{\partial p_0}{\partial x}, \\ u_0 &= -\frac{\partial p_0}{\partial y}, \\ \rho_0 &= -\frac{\partial p_0}{\partial z}, \\ \nabla \cdot \underline{u}_0 &= 0, \\ w_0 &= 0.\end{aligned}\tag{A.6}$$

In (A.6), horizontal velocity is divergence free and pressure plays the role of streamfunction. From consideration of the continuity equation and the realization that zero-order velocity is *isobaric*, that is, pressure of the system remain constant along streamlines, we obtain  $\frac{\partial w_0}{\partial z} = 0$  in addition of  $w_0 = 0$ . In order to derive equation for evolution of the geostrophic pressure,  $p_0 = \psi$ , we equate first-order terms in the Rossby number which

yield the system of equations

$$\begin{aligned}
\frac{\partial u_0}{\partial t} + \underline{u}_0 \cdot \nabla u_0 - v_1 - y\beta_0 v_0 + \frac{Ek}{Ro} \Delta^6 u_0 &= -\frac{\partial p_1}{\partial x} + \frac{Ek}{Ro} \Delta u_0, \\
\frac{\partial v_0}{\partial t} + \underline{u}_0 \cdot \nabla v_0 + u_1 + \beta_0 u_0 + \rho_1 + \frac{Ek}{Ro} \Delta^6 v_0 &= -\frac{\partial p_0}{\partial y} + \frac{Ek}{Ro} \Delta v_0, \\
-\beta_0 u_0 + y u_0 + \rho_1 &= -\frac{\partial p_1}{\partial z} \\
\nabla \cdot \underline{u}_0 &= 0, \\
\frac{\partial \rho_0}{\partial t} + \underline{u}_0 \cdot \nabla \rho_0 + w_1 + \frac{1}{Ed} \Delta^6 \rho_0 &= \frac{1}{Ed} \Delta \rho_0.
\end{aligned} \tag{A.7}$$

The next goal is to rewrite the equations (A.7) in an equivalent setup with emphasis on geostrophic pressure which we denote by  $\psi = p_0$ , the streamfunction. We take the two-dimensional curl of horizontal momentum in (A.7) to eliminate the pressure gradient and utilize the fact that from (A.6) horizontal velocity is divergence free which together with suitable simplifying assumptions gives the following *quasigeostrophic* potential vorticity system

$$\begin{aligned}
\frac{\partial q}{\partial t} + J(\psi, q) &= \frac{Ek}{Ro} \Delta q + \left(\frac{Ek}{Ro} - \frac{1}{Ed}\right) \Delta \frac{\partial^2 \psi}{\partial z^2} + \frac{Ek}{Ro} \Delta^6 \Delta \psi, \\
\frac{\partial \rho}{\partial t} + J(\psi, \rho) + w &= \frac{1}{Ed} \Delta \rho - \frac{1}{Ed} \Delta^6 \rho, \\
q &= \Delta \psi + \frac{\partial^2 \psi}{\partial z^2} + \beta y, \\
u &= -\frac{\partial \psi}{\partial y}, \\
v &= \frac{\partial \psi}{\partial x}, \\
\rho &= -\frac{\partial \psi}{\partial z},
\end{aligned} \tag{A.8}$$

where the nondimensional fields  $\underline{u} = (u, v, w)$ ,  $q$ ,  $\rho$ , and  $\psi$  are fluid velocity, potential vorticity, density and streamfunction, respectively. As in the preceding section,  $J(\cdot, \cdot)$  is the Jacobian operator,  $\Delta$  is the Laplacian operator, and the operator  $\frac{\partial}{\partial t} + J(\cdot, \cdot)$  represents advection along fluid particle trajectories. Indeed, the skew-symmetric spin tensor  $\Omega$  defined in equation (2.2) is nonzero and consequently the flow is rotational.

Nonlinear analysis of the equations (A.4) is given in [6, 7] and further extended in [72] for both systems (A.4) and A.8. The well-posedness, stability, attractors, and Lipschitz invariant manifolds of the problem of  $\beta$ -plane quasi-geostrophic equations with or with-

out Reynolds stress provide further study of the work that have been carried out in [89, 20, 57, 75, 24, 76, 127] including barotropic and baroclinic instabilities. One of the advantages of the technique for the proof adapted for the system of non-dissipative partial differential equations associated with A.4 and A.8 is that it emphasizes an advection form for accounting for the characteristic propagation directions, and it leads in a natural way to the following non-autonomous ordinary differential equations

$$\begin{aligned}\frac{dx}{dt} &= -\frac{\partial}{\partial x}\psi(x, y, z, t), \\ \frac{dy}{dt} &= \frac{\partial}{\partial x}\psi(x, y, z, t), \\ \underline{x}(\alpha, t)_{t=0} &= \alpha.\end{aligned}$$

This formulation is desirable also from the framework of geometric singular perturbation theory and adiabatic dynamical systems. Since the established baroclinic non-dissipative quasigeostrophic system is a hybrid elliptic-hyperbolic equation the assertion of the proof follows from employing convergent iteration scheme to obtain velocity from the elliptic system and then solve the hyperbolic equations. Given that the operator  $\frac{\partial}{\partial t} + J(\psi, \cdot)$  denotes advection along the fluid particle trajectories, the superposition principle and the technique of characteristics yield well-posedness of solution through the initial value problem. Additionally, by virtue of hyperbolic saddle criterion and the Melnikov function, we give existence proof for homoclinic and heteroclinic bifurcations. In fact, the simple zeros of the Melnikov function have an elegant geometric interpretation. The Melnikov function represent the distance between the stable and unstable invariant manifolds for the perturbed system. In the final analysis, we prove the existence and uniqueness of solutions to the initial-boundary value problems corresponding to the dissipative  $\beta$ -plane quasigeostrophic flows and then establish attractors of solutions. We solve the problem utilizing more refined a priori energy type estimates, Gronwall's inequality, Poincaré-Friedrichs inequality, coerciveness of  $L = -(\Delta + \frac{\partial^2}{\partial z^2})$  supplemented with appropriate boundary conditions and the technique of Faedo-Galerkin approximations.

## Appendix B

# Melnikov function for singularly perturbed dynamical systems

We derive and discuss a Melnikov function for aperiodic singularly perturbed ordinary differential equations which is a generalization of the periodic results developed in [103, 106, 107]. The fundamental tools we adapt in the analysis of aperiodic singularly perturbed dynamical systems is a set of existence theorems due to Fenichel [96, 97, 98, 99] and extended to an aperiodic case in [73, 74]. For a comprehensive and very useful review of Melnikov functions for singularly and regularly perturbed dynamical systems, see avid researchers [133, 5, 130, 131, 124, 123]. It should be noted at the outset, that recent results innovated in [133] are conjectured to apply to mixing and transport due to the interaction of stable and unstable manifolds associated with complex invariant sets using Perron's method for the construction of invariant manifolds.

From the preceding chapters, we have observed that a deeper understanding of mesoscale and synoptic coherent structures including dynamics of oceanic eddies and atmospheric vortex blocking has been based on Lagrangian transport. The information as to how this occurs is contained in the following prototypical system of non-autonomous ordinary differential equations

$$\begin{aligned}\frac{dx}{dt} &= -\frac{\partial}{\partial x}\psi(x, y, z, t), \\ \frac{dy}{dt} &= \frac{\partial}{\partial x}\psi(x, y, z, t), \\ \underline{x}(\alpha, t)_{t=0} &= \alpha,\end{aligned}\tag{B.1}$$

where  $\psi(x, y, z, t)$  is a streamfunction. This formulation is desirable also from the framework of geometric singular perturbation theory and adiabatic dynamical systems. Lagrangian transport, which is the lodestar in the analysis of transient and aperiodic geo-

physical phenomena.

Specifically, we reformulate (B.1) using

$$\begin{aligned}\dot{x} &= f_1(x, y, z), \\ \dot{y} &= f_2(x, y, z), \\ \dot{z} &= \varepsilon,\end{aligned}\tag{B.2}$$

with  $(x, y) \in \mathbb{R}^2$  and  $z \in \mathbb{R}$ .

For convenience of notation we re-write equation (B.2) as

$$\begin{aligned}\dot{X} &= f(X, z), \\ \dot{z} &= \varepsilon,\end{aligned}\tag{B.3}$$

where  $X \in \mathbb{R}^2$ , and  $f = (f_1, f_2)$ . Furthermore,  $f \in C^r$  where  $r \geq 1$ . When  $\varepsilon = 0$  we will call system (B.3) the unperturbed equations.

## B.1 Geometry of the phase space

We make the following analytic and geometric assumptions about a class of ordinary differential equations given in (B.3):

**A1** *The unperturbed system has two hyperbolic saddle equilibria  $h^\pm(z, 0)$  connected by a heteroclinic orbit  $h_0(t, z)$  which varies smoothly with  $z \in \mathbb{R}$ .*

**A2** *A Melnikov function for equations (B.3), which we derive in the next section, has at least two distinct simple zeros.*

We emphasize at the outset that assumption A1 implies the unperturbed equation has  $C^r$  1-dimensional normally hyperbolic invariant manifolds denoted by  $M_0^\pm$ . Moreover,  $M_0^\pm$  have  $C^r$  2-dimensional stable and unstable manifolds denoted by  $W^s(M_0^\pm)$  and  $W^u(M_0^\pm)$ , respectively, which intersect in 2-dimensional heteroclinic manifolds. Without loss of generality, we focus on the following heteroclinic manifold:  $\Gamma = W^s(M_0^+) \cap W^u(M_0^-)$ .

By virtue of the implicit function theorem,  $M_0^\pm$  can be expressed as

$$\begin{aligned}M_0^\pm &= \{(X, z) = (h^\pm(z, 0), z) \in \mathbb{R}^2 \times I \mid f(h^\pm(z, 0), z) = 0 \text{ and} \\ &\det[D_X f(h^\pm(z, 0), z)] \neq 0 \forall z \in I = \mathbb{R}\}.\end{aligned}$$

The dynamics on  $M_0^\pm$  are given by  $\dot{x} = \dot{z} = 0$  which shows that  $M_0^\pm$  is invariant. We parametrize the unperturbed heteroclinic manifold by the heteroclinic orbit using

$$\Gamma = \{(h_0(t_0, z_0), z_0) \in \mathbb{R}^2 \times I \mid (t_0, z_0) \text{ are fixed.}\} \quad (B.4)$$

We apply geometric singular perturbation theory [74, 89] to establish the existence of nearby normally hyperbolic invariant manifolds  $M_\varepsilon^\pm$  for a class of ordinary differential equations (B.3) with  $\varepsilon > 0$  sufficiently small.

**Proposition B.1.1** *When  $\varepsilon > 0$  is sufficiently small, there exist manifolds  $M_\varepsilon^\pm$  that lie within  $O(\varepsilon)$  of  $M_0^\pm$ . Furthermore,  $M_\varepsilon^\pm$  have  $C^r$  stable and unstable manifolds  $W^s(M_\varepsilon^\pm)$  and  $W^u(M_\varepsilon^\pm)$  that lie within  $O(\varepsilon)$  of the manifolds  $W^s(M_0^\pm)$  and  $W^u(M_0^\pm)$ , respectively.*

An outline of the proof is given in Jones [74] which involves a sequence of transformations to bring the equation of aperiodic singularly perturbed system into a normal form called Fenichel normal form. In addition, the Wazewski's principle and the moving cone lemma [92, 73, 74] are utilized to prove the existence and smoothness of stable and unstable manifolds  $W^s(M_\varepsilon^\pm)$  and  $W^u(M_\varepsilon^\pm)$ .  $M_\varepsilon^\pm$  is obtained by taking the intersection of  $W^s(M_\varepsilon^\pm)$  and  $W^u(M_\varepsilon^\pm)$ . Locally, the implicit function theorem gives the intersection as graphs and consequently  $M_\varepsilon^\pm$  can be expressed using

$$M_\varepsilon^\pm = \{(x, z) = (h^\pm(z, \varepsilon), z) \in \mathbb{R}^2 \times I \mid \\ h^\pm(z, \varepsilon) = h^\pm(z, 0) + O(\varepsilon).\}$$

The dynamics on  $M_\varepsilon^\pm$  are given by  $\dot{z} = \varepsilon$  which shows that  $M_\varepsilon^\pm$  is invariant and the motions on  $M_\varepsilon^\pm$  are slow.

## B.2 Splitting of the heteroclinic manifold

An analytical technique which allows to predict the behavior of  $W^s(M_\varepsilon^\pm)$  and  $W^u(M_\varepsilon^\pm)$  for small  $\varepsilon$  is a separation function which under nondegeneracy assumption can be expressed as Melnikov integral and is interpreted as the distance function between  $W^s(M_\varepsilon^+)$  and  $W^u(M_\varepsilon^-)$  or between  $W^s(M_\varepsilon^-)$  and  $W^u(M_\varepsilon^+)$ . Also, under nondegeneracy conditions we get transversality of the intersection of  $W^s(M_\varepsilon^+)$  and  $W^u(M_\varepsilon^-)$  in terms of Melnikov function.

In order to compute the Melnikov function, we only need to know the parametrization of the unperturbed heteroclinic manifold (B.4) and the perturbed equation (B.3). Moreover,

in a special case where the perturbation is periodic then by virtue of assumption A2 we may appeal to Birkhoff-Smale theorem for normally hyperbolic invariant manifolds [102] and assert the existence of chaotic transport in system (B.3) or the best achievable fluid mixing in adiabatic perturbation of system (B.3).

The following simplified considerations will provide some of the principal ideas to be encountered in the derivation of a Melnikov function. For fixed  $z_0$ , recall that  $h^\pm(z_0, 0)$  are the hyperbolic saddle equilibria connected by the heteroclinic orbit  $h_0(t, z_0)$ . We adapt the approach due to Robinson [110]. Let  $\Pi$  be a transversal through  $h_0(0, z_0) = h_0^z(0)$  in the  $x$ -plane which is normal to  $f(h_0^z(0), z_0)$ . Consider solutions  $h_\varepsilon^s(t)$  and  $h_\varepsilon^u(t)$  in  $W^s(M_\varepsilon^+)$  and  $W^u(M_\varepsilon^-)$  respectively intersecting  $\Pi$  at  $t = 0$  with  $z = z_0$  at  $t = 0$ . Set  $h_0^z(t) = h_0(t, z_0)$  to the unperturbed heteroclinic orbit and let  $f(h_0^z(t)) = f(h_0^z(t), z_0)$ .

Then the time-dependent distance between  $W^s(M_\varepsilon^+)$  and  $W^u(M_\varepsilon^-)$  is given by

$$d(t, z, \varepsilon) = \frac{[h_\varepsilon^s(t) - h_\varepsilon^u(t)] \wedge f(h_0^z(t))}{\|f(h_0^z(t))\|} \quad (B.5)$$

where  $\wedge$  denotes the wedge product and  $\|\cdot\|$  denotes the Euclidean norm. By Taylor expanding (B.5) about  $\varepsilon = 0$  we obtain

$$\begin{aligned} d(t, z, \varepsilon) &= \varepsilon \frac{\partial d(t, z, 0)}{\partial \varepsilon} + O(\varepsilon^2) \\ &= \varepsilon M(t, z) + O(\varepsilon^2) \end{aligned} \quad (B.6)$$

where  $M(t, z)$  denote the time-dependent Melnikov function.

The time-independent measure of distance is obtained by setting  $t = 0$  in expression (B.6) which gives

$$\begin{aligned} d(z, \varepsilon) &= \varepsilon M(z) + O(\varepsilon^2) \\ &= \varepsilon \left( \frac{[\frac{\partial h_\varepsilon^s(0)}{\partial \varepsilon} - \frac{\partial h_\varepsilon^u(0)}{\partial \varepsilon}] \wedge f(h_0^z(0))}{\|f(h_0^z(0))\|} \right) + O(\varepsilon^2) \end{aligned} \quad (B.7)$$

An application of the implicit function theorem shows that if the Melnikov function  $M(z)$  has simple zeros and is independent of  $\varepsilon$ , then for  $\varepsilon > 0$  sufficiently small,  $W^s(M_\varepsilon^+)$  and  $W^u(M_\varepsilon^-)$  intersect transversely.

Using a similar argument, we can define a Melnikov technique to detect the persistence of periodic orbits, the so-called subharmonic Melnikov function denoted by  $M^{\frac{m}{n}}(z)$  where the period of the periodic orbit is given by  $\frac{mT}{n}$ ,  $m$  and  $n$  relatively prime [17, 13]. Furthermore, it can be shown that

$$\lim_{m \rightarrow \infty} M^{\frac{m}{n}}(z) = M(z),$$

which implies that heteroclinic bifurcation is the limit of countable sequence of subharmonic saddle-node bifurcations.

The above accounts show that the Melnikov function is a profound technique in dynamical systems and we proceed to give the expression for the Melnikov function for detecting the persistence of heteroclinic orbits (B.4) in a class of ordinary differential equations (B.3).

**Proposition B.2.1** *From equation (B.7), it is clear that the Melnikov function is the lowest order nonzero term in the Taylor expansion of the separation function and in the case of  $\text{Tr} D_x f = 0$  which is valid for Hamiltonian systems then we obtain*

$$M(z) = \int_{-\infty}^{\infty} \left[ \frac{\partial f(h_0^z(t))}{\partial z} \frac{\partial z}{\partial \varepsilon} \right] \wedge f(h_0^z(t)) dt \quad (\text{B.8})$$

with  $\frac{\partial z}{\partial \varepsilon}$  satisfying

$$\left( \frac{\partial z}{\partial \varepsilon} \right)' = 1,$$

and

$$\frac{\partial z}{\partial \varepsilon} = 0,$$

when  $t = 0$ .

Before giving proof to the proposition we emphasize that it follows by assumption A2 that the Melnikov function (B.8) must have at least two distinct simple zeros. The non-degeneracy assumption for existence of at least two distinct simple zeros for the Melnikov function will become clear in the sequel when we define and derive lobes.

**Proof:** From equation (B.7) we deduce

$$\begin{aligned} M(z) &= \frac{\left[ \frac{\partial h_\varepsilon^u(0)}{\partial \varepsilon} - \frac{\partial h_\varepsilon^s(0)}{\partial \varepsilon} \right] \wedge f(h_0^z(t))}{\|f(h_0^z(t))\|} \\ &= \frac{\frac{\partial h_\varepsilon^u(0)}{\partial \varepsilon} \wedge f(h_0^z(t)) - \frac{\partial h_\varepsilon^s(0)}{\partial \varepsilon} \wedge f(h_0^z(t))}{\|f(h_0^z(t))\|}. \end{aligned} \quad (\text{B.9})$$

For notation let  $h_\varepsilon^s(0) = h_\varepsilon^+$  and  $h_\varepsilon^u(0) = h_\varepsilon^-$ . Recall that  $(h_\varepsilon^\pm, z_0, \varepsilon)$  are points on  $\Pi_{z_0}$  which asymptote the hyperbolic saddles  $h^\pm(z_0, 0)$  in forward and backward time, respectively, under the flow generated by equation (B.3). Thus, the derivatives with respect to  $\varepsilon$  must be tangent to the flow. The derivatives are given by  $(\frac{\partial h_\varepsilon^\pm}{\partial \varepsilon}, 0, 1)$  evaluated at  $\varepsilon = 0$ . The tangent vectors to the flow are given by  $(\frac{\partial h_\varepsilon^\pm}{\partial \varepsilon}, 0, 1)$  and  $(f(h_0^z(t)), 1, 0)$ . We are interested in

$$\frac{\partial h_\varepsilon^\pm}{\partial \varepsilon} \wedge f(h_0^z(t))$$

and we shall think of the tangent vectors as the initial conditions for the equations of

variation or adjoint equation to system (B.3) evaluated at  $\varepsilon = 0$  given by

$$\begin{aligned}\delta\dot{X} &= D_X f(X, z)\delta X + \frac{\partial f(X, z)}{\partial z}\delta z \\ \delta\dot{z} &= \delta\varepsilon \\ \delta\dot{\varepsilon} &= 0\end{aligned}\tag{B.10}$$

We make a noteworthy remark that  $\delta X$  is a 2-vector and we shall be concerned with the wedge of two  $\delta X$  components of the initial condition for the equations (B.10). Let  $(\delta X, \delta z, \delta\varepsilon)$  and  $(\delta\hat{X}, \delta\hat{z}, \delta\hat{\varepsilon})$  be solutions of the equations (B.10). Furthermore, we set

$$w = \delta X \wedge \delta\hat{X}$$

with  $\delta\hat{X} = (\hat{\delta}x, \hat{\delta}y)$  and  $\delta X = (\delta x, \delta y)$ . The assertion

$$\begin{aligned}\dot{w} &= \delta x \hat{\delta}z + \delta\hat{x} \delta y - \delta y \hat{\delta}x - \delta y \delta\hat{x} \\ &= (\hat{\delta}y \delta z - \delta y \delta\hat{z}) \frac{\partial f_1}{\partial z} \\ &\quad + (\hat{\delta}z \delta x - \delta z \delta\hat{x}) \frac{\partial f_2}{\partial z} \\ &\quad + (\hat{\delta}y \delta x - \delta y \delta\hat{x}) \text{Tr} D_X f \\ &= (\hat{\delta}y \delta z - \delta y \delta\hat{z}) \frac{\partial f_1}{\partial z} \\ &\quad + (\hat{\delta}z \delta x - \delta z \delta\hat{x}) \frac{\partial f_2}{\partial z} \\ &= (\hat{\delta}y \frac{\partial z}{\partial \varepsilon} \delta\varepsilon - \delta y \frac{\partial z}{\partial \varepsilon} \delta\hat{\varepsilon}) \frac{\partial f_1}{\partial z} \\ &\quad + (\frac{\partial z}{\partial \varepsilon} \delta\hat{\varepsilon} \delta x - \frac{\partial z}{\partial \varepsilon} \delta\varepsilon \delta\hat{x}) \frac{\partial f_2}{\partial z}\end{aligned}\tag{B.11}$$

holds where the chain-rule and the assumption  $\text{Tr} D_X f = 0$  have been employed.

Let  $(\delta X, \delta z, \delta\varepsilon)$  be solutions of (B.10) that satisfy  $(\frac{\partial h_\pm}{\partial \varepsilon}, 0, 1)$  at  $t = 0$  and  $(\delta\hat{X}, \delta\hat{z}, \delta\hat{\varepsilon})$  be solutions satisfying  $(f(h_0^\pm(t)), 1, 0)$  at  $t = 0$ . By virtue of  $\delta\dot{\varepsilon} = 0$  we get  $\delta\varepsilon = 1$  and  $\delta\hat{\varepsilon} \equiv 0$ . Also, by utility of  $\delta\dot{z} = \delta\varepsilon$  we may choose  $\delta\hat{z} \equiv 1$  and  $\delta z = 0$  at  $t = 0$ . Consequently, substitution in (B.11) gives

$$\dot{w}_\pm = \frac{\partial f_1}{\partial z} \frac{\partial z}{\partial \varepsilon} \hat{\delta}y - \frac{\partial f_2}{\partial z} \frac{\partial z}{\partial \varepsilon} \hat{\delta}x.$$

Using  $(\hat{\delta}x, \hat{\delta}y) = f(h_0^z(t))$  we obtain

$$w_{\pm} = \left[ \frac{\partial f(h_0^z(t))}{\partial z} \frac{\partial z}{\partial \varepsilon} \right] \wedge f(h_0^z(t)).$$

In order to obtain an integral expression for the Melnikov function, we utilize the following lemma:

**Lemma B.2.2**

$$\lim_{T \rightarrow \infty} w_+(T) = \lim_{T \rightarrow \infty} w_-(-T) = 0.$$

Integrating  $w_{\pm}$  above and sending  $T \rightarrow \infty$  and applying the aforementioned lemma we get

$$M(z) = w_-(0) + w_+(0) = \int_{-\infty}^{\infty} \left[ \frac{\partial f(h_0^z(t))}{\partial z} \frac{\partial z}{\partial \varepsilon} \right] \wedge f(h_0^z(t)) dt$$

which is the desired result. When  $f$  in (B.3) takes the form  $f = (-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x})$  then the Melnikov function becomes

$$\begin{aligned} M(z) &= \int_{-\infty}^{\infty} \left[ \frac{\partial f(h_0^z(t))}{\partial z} \frac{\partial z}{\partial \varepsilon} \right] \wedge f(h_0^z(t)) dt, \\ &= \int_{-\infty}^{\infty} \frac{\partial z}{\partial \varepsilon} \{H, \frac{\partial H}{\partial z}\}(h_0^z(t)) dt \end{aligned} \tag{B.12}$$

where  $\{, \}$  is the Poisson bracket.

We conclude by rewriting the Melnikov function in a form useful in the derivation of adiabatic lobes using

$$M(z) = \int_{-\infty}^{\infty} \left[ \frac{\partial H}{\partial z}(h_0^z(t)) - \frac{\partial H}{\partial z}(h_0^+) \right] dt, \tag{B.13}$$

where  $(h_0^+ \in M_0^+)$  and  $h_0^z(t)$  is a parametrization of the heteroclinic manifold (B.4).

Hence, a method which enables an analysis of the behavior of  $W^s(M_{\varepsilon}^{\pm})$  and  $W^u(M_{\varepsilon}^{\pm})$  for small  $\varepsilon$  is the Melnikov integral which is interpreted as a distance function between  $W^s(M_{\varepsilon}^+)$  and  $W^u(M_{\varepsilon}^-)$ . Using a similar argument we get transversality of the intersection of  $W^s(M_{\varepsilon}^+)$  and  $W^u(M_{\varepsilon}^-)$  in terms of the Melnikov function.

### B.3 Adiabatic transport and lobe area

Analyses of various closed-form solutions together with the description of related moored instruments and remote sensors observations have been carried out in [80, 81] with the work in [79, 102] focussing on two key features: first, construction of stable and unstable

invariant manifolds of hyperbolic invariant sets in phase space that separate qualitatively different type of motions, and the dynamical evolution of turnstiles and lobes to describe transport between the different regions of motion. A numerical simulation of the problem (B.1) with representative streamfunction  $\psi(x, z, t) = \tanh[z - \sin(x - ct)]$  is depicted in Figure 1.2 obtained using the DsTool package program [3].

This section is devoted to a study of adiabatic transport in phase space of system (B.3) between qualitatively different regions bounded by pieces of the stable and unstable manifolds of invariant sets. We use the Melnikov function (B.13) to derive a computable expression for lobes, the sneaks of fluid transport. Local results about transport are investigated in Mackay *et al.* [108, 109] who studied regions separated by cantori using generating functions for area-preserving maps. Global results, which is a source of inspiration in this work, were developed in Rom-Kedar *et al.* [111, 112] employing regular perturbation theory and Kaper [103, 106] used theory of action. We apply the Kaper-Kovacic criterion [107] to adiabatic dynamical systems (B.3).

We begin by giving definitions of a primary intersection points and lobes.

**Definition B.3.1** *Let  $\Pi_z$  denote a slice (Poincaré section) through the heteroclinic manifold. Consider a heteroclinic point  $P \in W^s(M_\varepsilon^+) \cap W^u(M_\varepsilon^-) \cap \Pi_z$  and let  $X^s P$  denote the segment of  $W^s(M_\varepsilon^+) \cap \Pi_t$  from  $M_\varepsilon^+ \cap \Pi_z$  to  $P$  and let  $X^u P$  denote the segment of  $W^u(M_\varepsilon^-) \cap \Pi_t$  from  $M_\varepsilon^- \cap \Pi_z$  to  $P$ . Then  $P$  is called a primary intersection point.*

**Definition B.3.2** *Let  $P$  and  $Q$  be two adjacent primary intersection points. Then the region on the 2-dimensional slice  $\Pi_z$  bounded by the segments of  $W^s(M_\varepsilon^+) \cap \Pi_t$  and  $W^u(M_\varepsilon^-) \cap \Pi_t$  which connect  $P$  and  $Q$  is called a lobe.*

By virtue of the definition of primary intersection point, it follows that a zero of the Melnikov function (B.13) correspond to a primary intersection point. Additionally, from the definition of a lobe, it follows that we need two primary intersection points to define a lobe which means the Melnikov function should have at least two zeros. The importance of assumption A2 is now clear. It is useful to visualize how the motion of the lobes is manifested in terms of a set of equations (B.3). Kaper *et al.* [103, 106, 107] exploited this intuition and the results in [108, 109] to obtain the lobe in the form

$$A_L = \int_{z_0}^{z_1} M(z) dz + O(\varepsilon) \quad (\text{B.14})$$

where  $z_0$  and  $z_1$  are two consecutive simple zeros of the adiabatic Melnikov function (B.13). It follows from the research work in [124], that the lobe is analogous to the

Evans function, which is a topological invariant. In addition, the lobe (B.14) satisfies the identity

$$M(z) = \frac{dA_L(z)}{dz}, \quad (\text{B.15})$$

illustrating analogy of the Melnikov function  $M(z)$  with the derivative of the Evans function using the results due to Sandstede *et al.* [124].

By utility of (B.15), we deduce that adiabatic Melnikov function has simple zeros at those values of  $z$  for which the lobe area is local minimum or maximum. Thus, the simple zeros of a Melnikov function have an elegant geometric interpretation. Illustrating that a simple zero exists for a Melnikov integral provides a proof for chaos in the sense of Smale horseshoes which is expected to result in the randomization of fluid motion and typical fluid quantities such as heat transport.

These existence results demonstrate Lagrangian chaos for dynamically consistent velocity field. A treatment of Lagrangian transport in a system of non-autonomous ordinary differential equations (B.1) is available for periodic, quasiperiodic, almost periodic flows and new innovations for numerically and experimentally generated vector fields with general time-dependence is tackled by avid researchers [133, 80, 81, 79, 102]. In conclusion, we emphasize at this point that the recent results developed in [133] apply to mixing due to the interaction of stable and unstable manifolds corresponding to invariant sets employing a method due to Perron for the construction of invariant manifolds that complement techniques formulated and innovated in Gruendler [130, 131] using dichotomic projections, Melnikov integrals, and other separation functions.

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