On the Topological Entropy of Nilpotent Groups of Finite Rank

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Contents

ABSTRACT

The present PhD thesis deals with some finiteness conditions on the topological entropy of continuous endomorphims of periodic locally compact Heisenberg p-groups (p prime). These are relevant models of locally compact nilpotent groups and their structure may be described very well by semidirect products. Firstly, we consider the class of locally compact abelian groups that are compactly generated; we recognize a relevant subclass among these, namely the slender groups, and observe that slender groups have very small values of topological entropy for their endomorphisms. Then we investigate a more general case of nilpotent periodic locally compact p-groups, reducing the computations to maximal p-subgroups. The role of p-groups becomes somehow fundamental, so we focus on locally compact Heisenberg p-groups which are studied especially on the field $\mathbb{Q}_p$ of p-adic rationals and on the ring $\mathbb{Z}_p$ of p-adic integers.

KEYWORDS AND PHRASES

- Topological Entropy
- Compactly Generated Locally Compact Abelian Groups
- Dynamical Systems
- Slender Groups
- Heisenberg p-Groups

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DECLARATION

I, Olwethu Waka, hereby declare that the work on which this thesis is based is my original work (except where acknowledgments indicate otherwise) and that neither the whole work nor any part of it has been, is being, or is to be submitted for another degree in this or any other university. I authorize the University to reproduce for the purpose of research either the whole or any portion of the contents in any manner whatsoever.

INTRODUCTION

The present PhD thesis discusses the notion of topological entropy in periodic locally compact groups and in particular on certain locally compact groups which can be constructed as groups of matrices. Entropy has been a known phenomenon in many areas of applied sciences, most notably in thermodynamics and dynamical systems. In the 1930s, Shannon defined the entropy in information theory. In the 1950s Kolmogorov, Sinai and Shannon [21, 32, 33] introduced a measure theoretic entropy in ergodic theory. For the first time in 1965, Alder, Konheim and McAndrew [1] introduced the notion of topological entropy for continuous self-maps of compact spaces. In 1973, Bowen [3] extended this notion of topological entropy and considered uniform continuous self-maps of noncompact spaces, and then Hood [20] in 1973 considered the entropy of uniform continuous self-maps on uniform spaces.

In the case of compact metric spaces, the entropy of Alder, Konheim and McAndrew [1] was proved to coincide with the uniform entropy of Bowen [3] and Hood [20], in fact much more can be said on this, see [12]. In [11], locally compact abelian groups of finite topological entropy were considered and it was seen that groups that admit structural decomposition may have zero entropy, or small enough topological entropy, and from this consideration it is interesting to focus on the topological entropy of compactly generated locally compact abelian groups.

Locally compact abelian groups, which are compactly generated, may be decomposed into the direct sum of the groups $\mathbb{R}^n$, $\mathbb{Z}^m$ and K, where K is compact abelian and n, m are nonnegative integers. The present work is based on [30], where we imposed conditions on these groups to arrive at zero entropy or at finite entropy. Moreover the topological entropy of compactly generated locally compact abelian groups depends on the compact summand in the decomposition. Our work follows some previous works done also by Peters [25], who considered the topological entropy of topological automorphisms of locally compact abelian groups.

With the aid of upper triangular matrices, abstract nilpotent groups of large enough nilpotency class can be constructed, and we considered specifically some families of nilpotent groups of nilpotency class two. We found among these that the Heisenberg groups play an interesting role: they can be constructed over several topological rings, and also on the p-adic rationals. The additive group of the p-adic rationals is very interesting because it is a totally disconnected p-group of finite topological p-rank. The corresponding construction of the Heisenberg groups on the p-adic rationals offers examples of a locally compact group which is neither abelian nor compact and it has nonzero finite topological entropy.

The background for all this thesis is the following: Section 1 contains some classical results in abstract abelian group theory. The material in this section is based on the well known textbooks of Robinson [27, Chapter 4] and of Fuchs [13], but also on that of Hofmann and Morris [19, Appendix 1]. On abstract abelian groups, we focus strongly on the theorems of structure, namely the structure theorem of finitely generated abstract abelian groups in Proposition 1.12. We recall the structure theorem precisely for our purposes in the class of compactly generated locally compact abelian groups in Theorem 2.26 of Section 2, and we view Theorem 2.26 as a topological generalization of Proposition 1.12 of the abstract case. The notions of divisibility and injectivity are also recalled in Section 1 for our use in Section 6 when we discuss the notion of abstract abelian slender groups which will be characterized as those countable torsion-free abstract abelian groups with no nontrivial divisible subgroups. We switch between divisibility and injectivity, since this is convenient in situations such as Proposition 1.5. The notion of the Prüfer rank is introduced also in Section 1 for abstract abelian group, we take this approach as will see in Definition 2.18 of Section 2 that the notion of dimension of compact abelian groups depends strongly on the that of Prüfer rank of abstract groups. Lastly we introduce the notion of abstract nilpotent group: we view this as a natural generalization of abstract abelian group. We construct abstract nilpotent groups of large enough nilpotency class via the use of upper triangular matrices, and when we restrict the nilpotency class to two, we arrive at the Heisenberg groups of (3.28) in Section 3; these groups are at the heart of the present PhD thesis.

In Section 3, we recall the basic notions of functional analysis. We start with the notions of quadratic and bilinear forms, and it is a known fact that if we have one, then we can always recover the other. The material of this section is based mainly on the well known textbooks of Rudin [28, 29]. We recall the notions of the norm and the norm induced by the inner product, with the metrics they induce, we arrive the Banach and Hilbert spaces respectively when these spaces are complete with respect to the induced metric.

A Hilbert space $\mathcal{H}$ over some field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ can be used to produce examples of nonabelian topological groups, for instance, $\mathrm{Gl}(\mathcal{H})$ can be used instead of $\mathrm{Gl}(n, \mathbb{R})$ in Example 2.2 of Section 2, and in Section 4, we will exhibit an example of a Banach space as the space of Lebesgue integrable, the so called L^2-space for some measure μ on a locally compact group. The bilinear forms that we introduced at the beginning of Section 3 are largely used later on; for instance, we use bilinear forms in Example 3.6 where we construct a binary operation on $\mathbb{Z}_p^3$ in order to define Heisenberg groups over $\mathbb{Z}_p$. The final Section 8 contains the proofs of the main results which are prepared in Sections 5 and 7.

NOTATION

Symbols	Description
$\mathbb{Z},\ \mathbb{P},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{C}$	Sets of integers, primes, rationals, reals, and complex numbers
$\mathbb{Z}(n) := \mathbb{Z}/n\mathbb{Z}$	Integers modulo n
$\mathbb{T} := \mathbb{R}/\mathbb{Z}$	Torus group
$\mathbb{Z}_p, \mathbb{Q}_p$	p-adic integers, p-adic rationals
$\underset{i\in I}{\mathrm{Cr}}\ G_i,\ \underset{i\in I}{\mathrm{Dr}}\ G_i$	Cartesian product and the direct product of the groups $\{G_i : i \in I\}$
$\mathbb{Z}^{\mathbb{N}},\ \mathbb{Z}^{(\mathbb{N})}$	The Specker group and the free abelian group of countable rank
$A \oplus B$	Direct sum of two abelian groups A and B
$\widehat{A}$	Character group of a locally compact abelian group A
$\dim(A)$	Dimension of the compact abelian group A
$r_p(A),\ r_0(A),\ r(A)$	p-rank, torsion-free rank, Prüfer rank of an abstract abelian group A (p prime)
$\Omega^n(A)$	n-power of the abstract abelian group A
$\Omega_n(A)$	n-socle of the abstract abelian group A
$d(G)$	Minimum number of generators of the abstract finitely generated group G
$Z(G)$	Centre of a group G, i.e.: $\{z \in G : [z, g] = 1, \forall g \in G\}$
$[x, y]$	Commutator of two elements in a group; i.e.: $x^{-1}y^{-1}xy$
$G' = [G, G]$	Derived subgroup of the group G, i.e.: $\langle [g, h] : g, h \in G \rangle$
$\mathrm{Gl}(n, F)$	Group of $n \times n$ nonsingular matrices over the field F
$S_n,\ A_n$	Symmetric, alternating groups of degree n
G_0	Connected component of the identity in the locally compact group G
$\mathrm{rank}_p(G)$	p-rank of the periodic locally compact p-group G (p prime)
$\mathbb{H}_n(R)$	$(n+2) \times (n+2)$-matrix over the topological ring R (Heisenberg group)

1. SOME CLASSICAL RESULTS ON ABELIAN AND NILPOTENT GROUPS

We begin to introduce the notion of topological groups, following [16, 17, 19]. The main idea is that a topological group is a group which is a topological space at the same time. Moreover, the group operation and the topology should be compatible between themselves.

Definition 1.1 ([19], Definition 1.1)**.** A *topological group* is a group G together with a Hausdorff topology $\mathcal{T}_G$ on G such that the maps

$$\begin{cases} (x,y) \mapsto xy : G \times G \to G, \\ \\ x \mapsto x^{-1} : G \to G, \end{cases}$$

are continuous, where $G \times G$ has the product topology $\mathcal{T}_{G\times G}$.

When we have a topological group G as per Definition 1, one usually refers to $\mathcal{T}_G$ as *group topology* and we say that G is *compact* (or *locally compact*) if the group topology is compact (or locally compact). Likewise, a topological group G is abelian, if the group operation is abelian. To fix terminology, we will refer to topological groups as *groups* and to groups without topology as *abstract groups*. Note that Definition 1.1 applies always to topological groups with the discrete topology, so abstract groups can be also interpreted at the level of topological groups with the discrete topology.

In this section we discuss the notion of abstract abelian groups and record some very elementary facts to which we shall refer very often in later on. Abelian groups are written additively (unless clearly stated otherwise). Specifically we discuss the notions of divisibility and injectivity, torsion and torsion-freeness, and the notion of a rank.

Let A be an abstract abelian group, if $a \in A$ is an element of finite order, then a is said to be *torsion*, or otherwise *torsion-free*. We denote the subset of all torsion elements of A by $\mathrm{tor}(A)$, and this is in fact an abstract subgroup of A with the property that $A/\mathrm{tor}(A)$ is torsion-free, see [19, Definition A1.16, Remark A1.17]. Given $n \in \mathbb{N}$, we may define the following two important abstract subgroups of A:

$$\begin{cases} \Omega_n(A) = \{a \in A : na = 0\} \\ \\ \Omega^n(A) = \{na : a \in A\}, \end{cases} \tag{1.1}$$

called *n-socle* and *n-power subgroup*, respectively. See also [16, 22, 26] for possible generalizations of (1.1) in the context of topological groups which are not necessarily abstract abelian groups.

In particular, if p is a prime, we call an element $a \in A$ a *p-element* if a has order a multiple of p, and $\Omega_{p^m}(A) = \{a \in A : p^m a = 0\}$, where m runs through $\mathbb{N}$, and this is again an abstract subgroup of A, called a *primary p-component* of A . Of course, the word "multiple" becomes "power" when we use the multiplicative notation, instead of the additive notation. We record the following result which relates the primary p-components of A with $\mathrm{tor}(A)$.

Proposition 1.2 ([27], Theorem 4.1.1)**.** *Let A be an abstract abelian group. Then*

$$\mathrm{tor}(A) = \bigoplus_{p \in \mathbb{P}} \Omega_{p^m}(A), \tag{1.2}$$

that is, $\mathrm{tor}(A)$ *may be always decomposed in the direct sum of primary p-components of A.*

There are more details on Proposition 1.2 in the discussions in [19, Definition A1.18, Theorem A1.19]. A simple observation should be done here: by controlling primary p-components of an abstract abelian group we can control the structure of the whole abstract group, since results of decompositions are possible. This logic applies also to more general situations where locally compact groups are present: later on we are going to illustrate certain generalizations of Proposition 1.2 as done in [16, 22, 26].

Example 1.3 ([19], Torsion and Torsion-free Abstract Abelian Groups)**.** All finite abelian groups are torsion abstract abelian groups. The Prüfer group $\mathbb{Z}(p^\infty)$ will be explicitly constructed in

(2.25), following the standard arguments in [13, 19, 27]. This is an example of an infinite torsion abstract abelian group. In fact, we have

$$\mathbb{Z}(p^\infty) = \bigoplus_{p\in\mathbb{P}} \Omega_{p^m}(\mathbb{Q}/\mathbb{Z}), \tag{1.3}$$

where $\mathbb{Q}$ denotes the usual additive group of the rational numbers and $\mathbb{Z}$ the usual additive group of the integers. Note also that $\mathbb{Z}$, $\mathbb{Q}$ and the usual additive group $\mathbb{R}$ of the real numbers are all examples of torsion-free abstract abelian groups.

Similarly, an element $a \in A$ is said to be *divisible* if $a \in \Omega^n(A)$ for all $n \in \mathbb{N}$. Of course, A is divisible if and only if $A = \Omega^n(A)$, this means that $\Omega^n(A)$ can be used to measure how far A is from being divisible. The opposite situation applies when this subgroup is very small. An abstract abelian group A is called *reduced* if the only divisible subgroup is the trivial subgroup. In other words, A is reduced if

$$\mathrm{Div}(A) := \bigcap_{n=1}^{\infty} \Omega^n(A) = 0. \tag{1.4}$$

We do not go further into details here and mention only that [19, Definition A1.29, Proposition A1.31, Theorem A1.32, Proposition A1.35, Theorem A1.37, Proposition A1.39, Theorem A1.42] collect a series of fundamental results on the role of the divisible abstract abelian groups. For our purposes later in this thesis, more specifically in Lemmas 6.4 and 6.8, it is appropriate to recall just one more notion that is a closely related to the divisibility.

Definition 1.4 ([19], Definition A1.34)**.** Let A, H_1 and H_2 be abstract abelian groups. We call A *injective*, if for every injective homomorphism $\iota : H_1 \to H_2$ and every homomorphism $\alpha : H_1 \to A$ there exists a homomorphism $\beta : H_2 \to A$ with the property that $\alpha = \beta \circ \iota$.

We may visualize Definition 1.4 in terms of the following two commutative diagrams: the first is a triangular commutative diagram, illustrating the condition $\alpha = \beta \circ \iota$, but the second is a rectangular diagram, illustrating the equivalent condition $\alpha \circ \mathrm{id}_A = \beta \circ \iota$ where id_A is the identical homomorphism of A.

(1.5)

$$\begin{tikzcd}
H_1 \arrow[rr, "\iota"] \arrow[dr, "\alpha"'] & & H_2 \arrow[dl, dashed, "\beta"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1 \arrow[r, "\iota"] \arrow[d, "\alpha"'] & H_2 \arrow[d, dashed, "\beta"] \\
A \arrow[r, "\mathrm{id}_A"'] & A
\end{tikzcd}$$

Removing the identity map id_A of A from the above rectangular diagram we get the triangular one. This is a classical technique, which can be applied when we have large abstract abelian groups and want to determine whether they are injective, or not. We now observe that divisible abstract abelian groups are precisely the injective abstract abelian groups.

Proposition 1.5 ([27], Theorem 4.1.2)**.** *An abstract abelian group is injective if and only if it is divisible.*

Let us now look at some examples of divisible abstract abelian groups.

Example 1.6 ([13, 16, 19, 27], Examples of Divisible Abstract Abelian Groups)**.** All finite abelian groups are not divisible. The group of integers $\mathbb{Z}$ is an example of an infinite torsion-free abstract abelian group, which is not divisible, while $\mathbb{Q}$ and $\mathbb{R}$ are examples of infinite torsion-free divisible abstract abelian groups. The Prüfer group $\mathbb{Z}(p^\infty)$ is an example of a torsion abstract abelian group which is divisible.

Now we proceed to the discussion of the notion of rank in an abstract abelian group.

Definition 1.7 ([13], §1.1)**.** If A is an abstract abelian group and $X \subseteq A$ is a nonempty subset of A, the subgroup of A *generated by* X is denoted by $\langle X \rangle$.

The elements of $\langle X\rangle$ turn out to be $\mathbb{Z}$-linear combinations of elements of X, that is,

$$a \in \langle X\rangle \Longleftrightarrow a = l_1x_1 + l_2x_2 + \cdots + l_nx_n, \tag{1.6}$$

where $l_i \in \mathbb{Z}$ and $x_i \in X$ with $i = 1, 2, \ldots, n$.

If $X = \{a\}$ for $a \in A$, we write $\langle a\rangle$ instead of $\langle\{a\}\rangle$, and call this *the cyclic* subgroup of A generated by a. It is a well known result in abelian group theory that if $\langle a\rangle$ is infinite, then $\langle a\rangle \cong \mathbb{Z}$; and if $\langle a\rangle$ is finite, then $\langle a\rangle \cong \mathbb{Z}(n)$, for some natural number n. Somehow cyclic subgroups are important because they are the smallest objects which we need to have, in order to describe the structure of large abstract abelian groups. We will see more details later on.

Following [27, §4.2] and [13, §3.4], we introduce the notion of linear independence on subsets of abstract abelian groups and the notion of a rank of an abstract abelian group which turns out be the cardinality of a maximal linearly independent subset.

Definition 1.8 ([13], §1.1)**.** Let X be a nonempty subset of the abstract abelian group A such that $0 \notin X$. Then we call X *linearly independent* (or simply *independent*) if

$$l_1x_1 + l_2x_2 + \cdots + l_nx_n = 0 \Longrightarrow l_1x_1 = l_2x_2 = \cdots = l_nx_n = 0, \tag{1.7}$$

with $x_i \in X$, $l_i \in \mathbb{Z}$ and $i = 1, 2, \ldots, n$. The set X is said to be *linearly dependent* (or simply *dependent*) if it is not independent.

An independent subset is called a *basis* (of an abstract abelian group A) if it generates A, that is, if $\langle X\rangle = A$. An independent subset X is called *maximal* if it is not contained in any proper independent subset of A. If instead we consider now a maximal independent subset $Y \subseteq A$ that contains all the elements of order some power of p (p prime), then we call the cardinality $r_p(A)$ of Y the *p-rank* of A. Similarly, if we consider a maximal independent subset $Z \subseteq A$ that contains all the torsion-free elements, then we call the cardinality $r_0(A)$ of Z the *torsion-free rank* of A.

Definition 1.9 ([27], Prüfer Rank)**.** The *Prüfer rank* (or simply the *rank*) of an abstract abelian group A is defined to be

$$r(A) \coloneqq r_0(A) + \max_{p\in\mathbb{P}} r_p(A). \tag{1.8}$$

For instance, since $\mathbb{Z}$ is infinite cyclic and torsion-free, it is a rank 1 group, in fact its rank equals its torsion-free rank, that is, $\max\limits_{p\in\mathbb{P}} r_p(\mathbb{Z}) = 0$ and $r(\mathbb{Z}) = r_0(\mathbb{Z}) = 1$. This notion of rank as it appears in (1.8), will be used in Section 2 in Definition 2.18 when we introduce the notion of dimension of compact abelian groups.

We shall also recall here the following result, which asserts that both the p-rank and torsion-free rank do not depend on the chosen maximal independent subset, and hence we may speak without ambiguity of *the* rank $r(A)$.

Proposition 1.10 ([27], Theorem 4.1.2)**.** *Let A be an abstract abelian group. Then any two maximal linearly independent subsets which consist of elements of order some power of p have the same cardinality. Moreover any two maximal linearly independent subsets that consist of torsion-free elements have the same cardinality. In particular, $r_0(A)$, $r_p(A)$ and $r(A)$ are well-defined and depend only on A.*

Before we consider some examples of the rank for some standard groups we know, it is useful to introduce here the notion of free abstract abelian group, or briefly *free abelian group*.

Definition 1.11 ([13, 19], Specker Group)**.** Consider the infinite cyclic nondivisible torsion-free abstract abelian group $\mathbb{Z}$ of rank 1. The Cartesian sum of $\mathbb{Z}$

$$\mathbb{Z}^{\mathbb{N}} \coloneqq \operatorname*{Cr}_{n\in\mathbb{N}} \mathbb{Z}, \tag{1.9}$$

(on a set of indices which is $\mathbb{N}$) is called *Specker group*.

This group has several interesting properties: for instance, each of its countable abstract abelian subgroups is a free abstract abelian group, even if itself is not a free abstract abelian group. In particular, the direct sum of $\mathbb{Z}$

$$\mathbb{Z}^{(\mathbb{N})} \coloneqq \operatorname*{Dr}_{n\in\mathbb{N}} \mathbb{Z} \tag{1.10}$$

is a countable abstract abelian subgroup, called *free abstract abelian group of infinite rank* $|\mathbb{N}|$.

If A is an abstract abelian group, of course $\{a_i : i \in I\} \subseteq A$ is a basis for $\mathbb{Z}^{(\mathbb{N})}$ if $\langle a_i \rangle \cong \mathbb{Z}$ for each $i \in I$, in fact, in this case we must have $I = \mathbb{N}$. When $|I| = n < \infty$, we write

$$\mathbb{Z}^n, \tag{1.11}$$

and call this *free abstract abelian group of finite rank* n. Then we have

Proposition 1.12 ([27], Finitely Generated Abstract Abelian Groups)**.** *An abstract abelian group A is finitely generated if and only if it is a direct sum of finitely many cyclic groups of infinite or prime-power orders.*

A well-known corollary of this result is that an abstract abelian group is finite if and only if it is a direct sum of finitely many cyclic groups of prime-power orders. Now consider F to be any finite abelian group and $\mathbb{Z}^n$ the free abstract abelian group of finite rank n. Proposition 1.12 asserts that an abstract abelian group A is finitely generated if and only if

$$A \cong \mathbb{Z}^n \oplus F, \tag{1.12}$$

that is, if and only if it is a direct sum of a free abelian group of finite rank and finite abelian group. Note that (1.12) gives a precise description of A via a result of decomposition.

Remark 1.13. In case of topological groups, there is a generalization of Proposition 1.12 in terms of compactly generated abelian groups: instead of looking for a finite subset, we look for a compact subset that generates the group, and this turns out to be a very interesting class of topological abelian groups. We will see more details on this aspect in Theorem 2.26 in Section 2.

We have seen already that the free abstract abelian groups are countable subgroups of the Specker group, even if the Specker group is not big enough. We may find even larger abstract abelian groups, called $\aleph_1$-*free groups*. Recall from [27, §4.3] that a subgroup H of an abstract abelian group A is *pure* if all the elements of H that are divisible by $n \in \mathbb{N}$ in A, are also divisible by n in H. More details can be found in [19, Appendix 1] on pure abelian subgroups.

Definition 1.14 ([19], Definition A1.63)**.** An abstract abelian group is called $\aleph_1$-*free* if every subgroup, which is pure and of finite rank, is a free abstract abelian group.

Furthermore, [19, Proposition A1.64] asserts that the $\aleph_1$-free groups are precisely those abstract abelian groups whose countable subgroups are free. Therefore, an obvious example of $\aleph_1$-free groups is the Specker group $\mathbb{Z}^{\mathbb{N}}$.

Example 1.15 ([27], Rank of Cyclic Groups)**.** Let A be a finitely generated abelian group of the form $A \cong \mathbb{Z}^n \oplus \mathbb{Z}(p) \oplus \mathbb{Z}(p)$ for a prime p and $d(A)$ be the minimal number of generators of A. By Proposition 1.12, we have that $d(A) = n + 2$ and clearly $r_0(A) = r(\mathbb{Z}^n) = n$ and $r_p(A) = 2$. Of course, if we have $m \geq 2$ copies of $\mathbb{Z}(p)$, instead of two copies only, we get $A \cong \mathbb{Z}^n \oplus \mathbb{Z}(p)^m$ with $r_0(A) = r(\mathbb{Z}^n) = n$ and $r_p(A) = m$.

We conclude this section by introducing the class of *nilpotent groups*. These groups are generally nonabelian, but any abelian group is nilpotent. In other words, the class of nilpotent groups include abstract abelian groups.

If G is an abstract group and $x, y \in G$, we call the element

$$[x, y] := x^{-1}y^{-1}xy, \tag{1.13}$$

the commutator of x and y. The set

$$Z(G) := \{z \in G \ : \ [z, g] = 1, \ \forall g \in G\} \tag{1.14}$$

is called *centre* of G and is a normal subgroup of G. Note that H is a normal subgroup of an abstract group G, if $H^a := a^{-1}Ha$ is contained in H for all elements $a \in G$, see [27]. Note that if H is a subgroup of an abstract group G contained in $Z(G)$, then automatically H is normal in G. If $X \subseteq G$ is a nonempty subset, the intersection of all normal subgroups N of G with $X \subseteq N$ is a normal subgroup called *normal closure* of X in G (usually denoted by X^G) and for a nonnormal subgroup H we always have $H \subseteq H^G$ so that $H = H^G$ in case H is contained in $Z(G)$. All these observations become somehow trivial when G is an abstract abelian group; in fact in this situation we would have $Z(G) = G$ and every subgroup of G automatically is normal and abelian.

Dually to $Z(G)$, we introduce

$$G' = [G, G] \coloneqq \langle [x, y] : x, y \in G \rangle \tag{1.15}$$

which is called *commutator subgroup* of an abstract group G. This is also a normal abstract subgroup of G with the property that if H is normal in G and G/H is abelian, then $[G, G] \subseteq H$.

In an abstract group G we call the *finite* chain

$$1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G \tag{1.16}$$

of abstract subgroups of G a *subgroup series* (or simply a *series*) of G. If $G_i \neq G_{i+1}$ for all $i = 0, 1, 2, \ldots, n-1$, the nonnegative integer n denotes the *length* of the series of G.

Definition 1.16 ([27], Upper Central Series)**.** An abstract group G is *nilpotent* if there is a series of length c (1.16) of G such that for all $i = 0, 1, 2, \ldots, c-1$ the following conditions are satisfied

(a). G_i is normal in G;
(b). G_{i+1}/G_i is a subgroup of $Z(G/G_i)$.

The integer c is called *the nilpotency class* of G.

A series in (1.16) satisfying the conditions of Definition 1.16 is called *upper central series.* Note also that a group G whose centre $Z(G)$ is trivial cannot be nilpotent since for $i = 0$, $G_0 = 1$ and $G/G_0 = G$ so that $Z(G) = 1$, so the condition (b) of Definition 1.16 will force the situation where $G_{i+1}/G_i = 1$ which implies that $G_i = G_{i+1}$ for all $i = 0, 1, \ldots, n$, which in the end implies that $G = 1$. As an evidence, S_3 is an example of a finite group that is *not* nilpotent since $Z(S_3) = 1$.

Dually we may reformulate the notion of nilpotency via a series commutator subgroups.

Definition 1.17 ([27], Lower Central Series)**.** An abstract group G is *nilpotent* if there is a series of length c of G, defined inductively by

$$G = \gamma_1(G) \geq \gamma_2(G) = G' \geq \gamma_3(G) = [\gamma_2(G), G] \geq \gamma_4(G) = [\gamma_3(G), G] \geq \ldots \geq \gamma_{c+1}(G) = 1 \tag{1.17}$$

of abstract subgroups $\gamma_i(G)$ of G where $i = 1, 2, \ldots, c$.

A series in (1.16) satisfying the conditions of Definition 1.17 is called *lower central series* and it turns out that the minimal number of steps to realize a chain in Definition 1.17 is equal to the minimal number of steps to realize a chain in Definition 1.16. Therefore we may speak without ambiguity of *the* nilpotency class of a nilpotent group, independently on the upper or lower central series which we consider. Naturally, all nontrivial abstract abelian groups are nilpotent of nilpotency class 1, and trivial groups are nilpotent of nilpotency class 0.

We want now to construct nilpotent groups of large enough nilpotency class which we will use when we work with the Heisenberg group of Section 3.

Let R be a commutative ring with identity and then set

$$UT(n, R) \coloneqq \left\{ \begin{pmatrix} 0 & r_{12} & \cdots & r_{1n} \\ 0 & 0 & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} : r_{ij} \in R \right\}. \tag{1.18}$$

Then we may consider the group of upper triangular matrices

$$U(n, R) \coloneqq \{ I_n + M : M \in UT(n, R) \}, \tag{1.19}$$

where I_n denote an n-by-n identity matrix. For $1 \leq m \leq n$, let $U_m(n, R)$ be a set of all n-by-n matrices whose $m-1$ superdiagonals have zeros, for instance, if $n = 4$ and $m = 3$, then

$$U_3(4, R) = \left\{ \begin{pmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : r \in R \right\}.$$

It turns out that $U_m(n, R)$ is a normal subgroup of $U(n, R)$, for each $m = 1, 2, \ldots, n$. Note that

$$U_1(n, R) = U(n, R) \quad \text{and} \quad U_n(n, R) = I_n. \tag{1.20}$$

It can also be shown that

$$[U_i(n, R), U(n, R)] \leq U_{i+1}(n, R), \quad \text{for all} \;\; i = 1, 2, \ldots, m-1, \tag{1.21}$$

which shows that

$$I_n = U_n(n,R) \ \leq\ U_{n-1}(n,R) \leq\ \dots\ \leq U_1(n,R) = U(n,R) \tag{1.22}$$

is a central series of $U(n,R)$, so that $U(n,R)$ is nilpotent of class at most $n-1$. In particular,

$$U_m(n,R)/U_{m+1}(n,R) \cong R^{n-m} \tag{1.23}$$

and if R is any field F of characteristic p, then $U(n,F)$ is a finite p-group as can be seen in [27, Exercise 5.1.11].

Example 1.18. Consider $R = \mathbb{Z}$ and $n = 3$. Then we consider $U(3,\mathbb{Z})$. For

$$\begin{pmatrix} 1 & r_1 & r_2 \\ 0 & 1 & r_3 \\ 0 & 0 & 1 \end{pmatrix} \in U(3,\mathbb{Z}),$$

consider

$$\begin{pmatrix} 1 & r_1 & r_2 \\ 0 & 1 & r_3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & r_1 & r_2 \\ 0 & 1 & r_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2r_1 & r_2 + r_1r_3 + r_2 \\ 0 & 1 & 2r_3 \\ 0 & 0 & 1 \end{pmatrix},$$

now that

$$\begin{pmatrix} 1 & 2r_1 & r_2 + r_1r_3 + r_2 \\ 0 & 1 & 2r_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \iff \begin{cases} 2r_1 = 0 \\ r_2 + r_1r_3 + r_2 = 0 \\ 2r_3 = 0, \end{cases}$$

which would imply that $r_1 = r_2 = r_3 = 0$. This means no nontrivial element of $U(3,\mathbb{Z})$ has finite order, and this can be shown for all $l \in \mathbb{N}$, so $U(n,\mathbb{Z})$ is a torsion-free nilpotent group.

We end this section by recalling the notion of Frattini subgroup for an abstract group.

Definition 1.19 ([27], Frattini Subgroup)**.** If G is an abstract group, the intersection of all maximal subgroups M of G is a subgroup of G, called *the Frattini subgroup* of G and denoted by

$$\mathrm{Frat}(G). \tag{1.24}$$

By [27, Theorem 5.2.12] the Frattini subgroup equals to the set of all nongenerators of G, where an element $g \in G$ is called a *nongenerator* if

$$G = \langle g, Y\rangle \Longrightarrow G = \langle Y \rangle, \tag{1.25}$$

for some $Y \subseteq G$. In the event that G possesses no maximal subgroup, $\mathrm{Frat}(G) = G$. This happens in the following situations: $\mathrm{Frat}(\mathbb{Q}) = \mathbb{Q}$ and $\mathrm{Frat}(\mathbb{Z}(p^\infty) = \mathbb{Z}(p^\infty)$, while $\mathrm{Frat}(\mathbb{Z}) = 0$ and $\mathrm{Frat}(S_n) = 1$. We extend this notion to some classes of topological groups in Remark 3.7 later on.

2. SOME NOTIONS ON PERIODIC LOCALLY COMPACT GROUPS

In the present section we recall some further concepts of topological group theory in [16, 19, 26]. Having in mind Definition 1.1, for a topological group G and an abstract subgroup H of G, we say that H is a *topological subgroup* of G (or briefly a *subgroup* of G, when it is clear the topology which is assigned on G) if H is an abstract subgroup of G, endowed with the induced topology of the group topology of G. We list a few examples arising from what we have seen so far.

Example 2.1 ([19], Examples of Abelian Topological Groups)**.**

1. Any abstract group is a topological group when equipped with the discrete topology. In particular, $\mathbb{Q}$ and $\mathbb{Z}$ are locally compact abelian groups with their discrete topologies.
2. The additive group of the real numbers, $\mathbb{R}$, is a topological group when given with the *standard topology*. It is a locally compact abelian group. Also, the multiplicative group of nonzero real number, $\mathbb{R}^\times$, is a topological group with the induced topology from $\mathbb{R}$.
3. The additive group of complex numbers, $\mathbb{C}$, is a topological group when equipped with the *standard topology*. The multiplicative group of nonzero complex numbers, $\mathbb{C}^\times$, is a topological group with the induced topology from $\mathbb{C}$. Both $\mathbb{C}$ and $\mathbb{C}^\times$ are locally compact abelian groups, but are not compact groups.
4. The subgroup
$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$
of $\mathbb{C}^\times$ is a compact abelian group with the induced topology. Note that $\mathbb{T}$ may be written additively as the quotient group $\mathbb{R}/\mathbb{Z}$, that is, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$; in fact $\mathbb{T}$ is a compact abelian group with the quotient topology of $\mathbb{R}$ modulo $\mathbb{Z}$.

Now we list the examples that are not necessarily abelian:

Example 2.2 ([19], Examples of Nonabelian Topological Groups)**.**

(a). The multiplicative group of the quaternions
$$\mathbb{H} = \{q = r \cdot 1 + x \cdot i + y \cdot j + z \cdot k \ : \ r, x, y, z \in \mathbb{R}\} \subseteq \mathbb{R}^4$$
which are nonzero is a topological group with respect to the induced topology from the standard topology of $\mathbb{R}^4$. We denote the nonzero quaternions by $\mathbb{H}^\times$.

(b). The unit quaternions
$$Q = \{q \in \mathbb{H} : |q| = 1\} \subseteq \mathbb{H}^\times$$
form a compact nonabelian subgroup with respect to the induced topology from $\mathbb{H}^\times$. Moreover, one can see from [19, Exercise E1.2(i) and (ii)] that the usual 3-dimensional sphere
$$\mathbb{S}^3 = \{(x, y, z, t) \in \mathbb{R}^4 \ : \ x^2 + y^2 + z^2 + t^2 = 1\}$$
may be endowed with the same structure of compact nonabelian group of Q.

(c). Given the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the abstract group of nonsingular n-by-n matrices
$$\mathrm{Gl}(n, \mathbb{K}) := \{A \in \mathrm{M}(n, \mathbb{K}) : \det(A) \neq 0\} \subseteq \mathbb{K}^{n^2}$$
is a topological group with respect to the induced topology and it is locally compact.

(d). The abstract subgroup
$$\mathrm{Sl}(n, \mathbb{K}) := \{A \in \mathrm{Gl}(n, \mathbb{K}) : \det(A) = 1\}$$
of $\mathrm{Gl}(n, \mathbb{K})$ is a topological group with respect to the induced topology from $\mathrm{Gl}(n, \mathbb{K})$.

(e). In a similar way, we may consider in $\mathrm{Gl}(n, \mathbb{R})$ the abstract subgroup
$$\mathrm{O}(n, \mathbb{R}) := \left\{A \in \mathrm{Gl}(n, \mathbb{R}) \ : \ AA^T = I_n\right\}$$
of all nonsingular n-by-n matrices with real coefficients which are self-adjoint, that is, satisfying the condition that the transpose equals the inverse, namely $A^T = A^{-1}$. This abstract subgroup can be equipped with the induced topology from $\mathrm{Gl}(n, \mathbb{R})$, considered as topological group, and it turns out to be a compact nonabelian group for all $n \geq 2$. Similarly, we may check that also
$$\mathrm{SO}(n, \mathbb{R}) := \mathrm{O}(n, \mathbb{R}) \cap \mathrm{Sl}(n, \mathbb{R})$$
is a compact nonabelian group for all $n \geq 2$, see [19, Examples 1.6].

At this point we want to introduce the notions of *direct* and *inverse systems*, producing more examples of topological groups that are of interest for us. We are going to recall these notions again from [16, 19, 26], but before to proceed, we mention the following definition.

Definition 2.3 ([19], Ponntryagin Dual)**.** For a topological group G, we denote by $\widehat{G}$ the *Pontryagin dual* (or the *character* group) of G, that is, the group of all continuous homomorphisms from G to the circle group $\mathbb{T}$.

The following result is very useful in describing the main features of the Pontryagin dual.

Proposition 2.4 ([19], Proposition 1.16 and Proposition 7.5(i))**.**

(i). *The character group of any abelian group is a compact abelian group.*
(ii). *The character group of a discrete abelian group is a compact abelian group, and a character group of a compact abelian group is a discrete abelian group.*

Recall also the following relevant concepts:

Definition 2.5 ([26], §1.1)**.** A nonempty set I together with a binary operation $\preceq$, that is, a function $\preceq: I \times I \to I$, (usually written as $I = (I, \preceq)$) is called a *directed partially ordered* set if the following conditions are satisfied:

(i). $i \preceq i$, for all $i \in I$;
(ii). if $i \preceq j$ and $j \preceq k$, then $i \preceq k$, for all $i, j \in I$;
(iii). if $i \preceq j$ and $j \preceq i$, then $i = j$, for all $i, j \in I$;
(iv). if $i, j \in I$, then there is some $k \in I$ such that $i, j \preceq k$, for all $i, j \in I$.

We will write *poset* for a partially ordered set so that we write *directed poset* for a directed partially ordered set. Clearly $\mathcal{S} = (\mathcal{S}, \subseteq)$ is a directed poset where $\mathcal{S}$ is a collection of all the subsets of a nonempty set S and $\subseteq$ is the set inclusion. Similarly, $\mathbb{N} = (\mathbb{N}, \leq)$ is a directed poset where $\mathbb{N}$ is the set of natural numbers and $\leq$ is the usual inequality on $\mathbb{N}$.

Definition 2.6 ([26], §1.2)**.** Let $\{G_i : i \in I\}$ be a family of groups indexed by a directed poset I and $\alpha_{ij} : G_i \to G_j$ homomorphisms where $i \preceq j$ and $i, j \in I$. If the following requirements hold

(a). α_{ii} is the identity map on G_i;
(b). $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ whenever $i \preceq j \preceq k$ in I;

the collection

$$\mathcal{D} = \{(G_i, \alpha_{ij}) \ : \ i \preceq j \text{ and } i, j \in I\} \tag{2.1}$$

is called *direct system* of the groups $\{G_i : i \in I\}$.

In other words, if we have a direct system as in Definition 2.6, then the following diagram

$$\begin{tikzcd} G_i \arrow[rr, "\alpha_{ik}"] \arrow[dr, "\alpha_{ij}"'] & & G_k \\ & G_j \arrow[ur, "\alpha_{jk}"'] & \end{tikzcd} \tag{2.2}$$

is commutative whenever $i \preceq j \preceq k$ in I.

When Definition 2.6 is satisfied, we may introduce a relation

$$g_i \sim g_j \iff \alpha_{ik}(g_i) = \alpha_{jk}(g_j) \tag{2.3}$$

for some $k \in I$ with $i, j \preceq k$ on the set

$$\bigcup_{i \in I} G_i, \tag{2.4}$$

arranging i and j in I such that $G_i \cap G_j = \emptyset$, $g_i \in G_i$ and $g_j \in G_j$ for all $i \neq j$. It can be shown that $\sim$ is an equivalence relation on $\bigcup_{i \in I} G_i$ and the set

$$G := \left\{ [g_i] \in \bigcup_{i \in I} G_i / \sim : g_i \in [g_i] \right\} \tag{2.5}$$

of all the equivalence classes $[g_i]$ containing g_i endow G of the structure of abstract group. In fact the function

$$([g_i],[g_j]) \in G \times G \longmapsto [g_i][g_j] := [\alpha_{ik}(g_i)\alpha_{jk}(g_j)] \in G \tag{2.6}$$

makes G into an abstract group, where $[g_i]^{-1} = [g_i^{-1}]$ for each $[g_i] \in G$ and $1_G = [1_{G_i}]$.

Summarizing, if we have Definition 2.6 and construct (2.5), then by default there are *compatible homomorphisms*

$$\theta_i : g_i \in G_i \mapsto [g_i] \in G \tag{2.7}$$

for all $i \in I$, that is, homomorphisms θ_i such that $\theta_j \circ \alpha_{ij} = \theta_i$ for all $i, j \in I$. In this situation, $\{(G, \theta_i) \ : \ i \in I\}$ is called *direct limit* of the direct system $\mathcal{D}$ in (2.1) and we write briefly

$$G := \varinjlim_{i \in I} G_i. \tag{2.8}$$

In the preceding discussion, for any direct system of the form (2.1) we get to a direct limit (2.8). Let's see better the existence and uniqueness of this construction. More details are presented below and can be found also in [27, Theorem 1.4.9].

Proposition 2.7 ([26], Proposition 1.2.1)**.**
Let $\mathcal{D}$ be a direct system of abstract groups G_i on a directed poset of indices I. Then

(i). *A direct limit G of $\mathcal{D}$ exists.*

(ii). *If $\mathcal{D}' = \{(G'_i, \theta'_i) \ : \ i \in I\}$ is another direct system on I of abstract groups G'_i, then there exists a unique isomorphism of abstract groups $\varphi : G \to G'$ such that $\theta'_i = \varphi \circ \theta_i$ for all $i \in I$. In particular, G is unique up to isomorphisms of abstract groups.*

(iii). *$G = \bigcup\limits_{i \in I} \theta_i(G_i)$, where θ_i are in (2.7).*

(iv). *If all the homomorphisms $\alpha_{ij} : G_i \to G_j$ are injective, so are θ_i for all $i \in I$.*

Using Proposition 2.7, we find some well known abelian groups.

Example 2.8 ([19, 26, 27], Construction of Prüfer Groups)**.** Consider the direct poset $\mathbb{N}$ and fix a prime p, and denote by $\mathbb{Z}(p^n)$ a cyclic group of order p^n with $n \in \mathbb{N}$. Let $\alpha_n : \mathbb{Z}(p^n) \to \mathbb{Z}(p^{n+1})$ be a natural embedding, then

$$\{(\mathbb{Z}(p^n), \alpha_n) \ : \ n \in \mathbb{N}\} \tag{2.9}$$

is a direct system. By Proposition 2.7, we have that

$$\varinjlim_{n \in \mathbb{N}} \mathbb{Z}(p^n) = \bigcup_{n=1}^{\infty} \mathbb{Z}(p^n). \tag{2.10}$$

Now by [27, Exercises 4.1.1 and 4.1.2], we have that

$$\mathbb{Z}(p^\infty) = \varinjlim_{n \in \mathbb{N}} \mathbb{Z}(p^n) = \bigcup_{n=1}^{\infty} \mathbb{Z}(p^n), \tag{2.11}$$

that is, the direct limit is the Prüfer group.

Similarly, we may construct the abstract abelian group of the rationals.

Example 2.9 ([27], Additive Abstract Group of the Rationals)**.** Let $\{A_n \ : \ n \in \mathbb{N}\}$ be a collection of infinite cyclic groups each of which is isomorphic with $\mathbb{Z}$, and let α_n be the function which sends an integer m to a multiple of n, that is, $\alpha_n : m \in A_n \mapsto \alpha_n(m) = nm \in A_{n+1}$. Then

$$\{(A_n, \alpha_n) \ : \ n \in \mathbb{N}\} \tag{2.12}$$

is a direct system of infinite cyclic groups and by [27, Exercise 1.4.11] we have

$$\mathbb{Q} = \varinjlim_{n \in \mathbb{N}} A_n, \tag{2.13}$$

that is, the additive group of the rational numbers can be realized as a direct limit of infinite cyclic groups. In fact, if $\theta_n : A_n \to \mathbb{Q}$ denotes the embedding of $\mathbb{Z}$ into $\mathbb{Q}$, then Proposition 2.7 implies

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \theta_n(A_n). \tag{2.14}$$

Dually to the notion of direct limit of abstract groups, we have the notion of *inverse limit*, also called *projective limit*, of abstract groups.

Definition 2.10 ([19], Definition 1.25)**.** Let I be a directed poset and $\varphi_{ij} : G_j \to G_i$ homomorphisms of abstract groups for all $i \preceq j \in I$, where $\{G_i : i \in I\}$ is a collection of abstract groups. Then we call $\mathcal{I} = \{(G_i, \varphi_{ij}) \ : i \preceq j, \text{and}, i, j \in I\}$ an *inverse system* if

(a). φ_{ii} is the identity on G_i;
(b). $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ whenever $k \preceq j \preceq i$ in I.

In other words, we consider a diagram, which is the same of (2.2) but reversing the order on the indices i, j, k in I

$$\begin{tikzcd} G_i \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & G_k \\ & G_j \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd} \tag{2.15}$$

Note that the above diagram is commutative whenever $k \preceq j \preceq i$ in I.

Proposition 2.11 below is the counterpart for inverse limits of what we have seen in Proposition 2.7 for direct limits. Furthermore, we note that the inverse limit may be embedded into the Cartesian product and this allows us to have a better behaviour if we begin with topological groups and want to endow the inverse limit of a topological structure.

Proposition 2.11. *Let $\mathcal{I}$ be an inverse system of topological groups G_i and $P = \operatorname*{Cr}_{i \in I} G_i$ the Cartesian product of G_i with the product topology. If*

$$G \coloneqq \{(g_i)_{i \in I} \in P \ : \ \varphi_{ij}(g_j) = g_i \text{ whenever } i \preceq j \ \text{ and } \ i, j \in I\}, \tag{2.16}$$

then

(a). *G is a closed subgroup of P;*
(b). *if $\mathrm{inc} : G \to P$ is the embedding of G into P and $\mathrm{pr}_i : P \to G_i$ be the projection of P onto G_i, then $\varphi_i = \mathrm{pr}_i \circ \mathrm{inc} : G \to G_i$ is a continuous homomorphism and $\varphi_i = \varphi_{ij} \circ \varphi_j$ is satisfied for all $i \preceq j$ with $i, j \in I$;*
(c). *if all G_is are compact groups, then both P and G are compact groups.*

Motivated by Proposition 2.11, we have the following notion:

Definition 2.12 ([19], Definition 1.27, Inverse Limit)**.** Let $\mathcal{I}$ be an inverse system of topological groups as in Definition 2.10. Then the group G in Proposition 2.11 is called *inverse limit* and we write briefly

$$G = \varprojlim_{i \in I} G_i. \tag{2.17}$$

Again a construction with $I = \mathbb{N}$ should be recalled here, due to its relevance in our present context of investigation.

Example 2.13 ([19], Construction of p-adic Integers)**.** Define

$$\varphi_n \ : \ u + p^{n+1}\mathbb{Z} \in \mathbb{Z}(p^{n+1}) \longmapsto \varphi_n(u + p^{n+1}\mathbb{Z}) \coloneqq u + p^n\mathbb{Z} \in \mathbb{Z}(p^n), \tag{2.18}$$

which is a surjective homomorphism of finite p-groups realizing $\mathbb{Z}_p$ as the inverse limit of the inverse system

$$\{(\mathbb{Z}(p^n), \varphi_n) \ : \ n \in \mathbb{N}\} \tag{2.19}$$

and write

$$\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}(p^n). \tag{2.20}$$

This group $\mathbb{Z}_p$ is abelian and compact by Proposition 2.11 (b) and totally disconnected, in fact, this is the description of the so called *profinite groups*, that is, compact groups which are inverse limits of finite groups, see [19, 26].

Another relevant example is the following locally compact (noncompact) abelian group.

Example 2.14 ([19], Construction of p-adic Rationals)**.** Define

$$\Phi_n \ : \ v+p^{n+1}\mathbb{Z} \in \frac{1}{p^\infty}\mathbb{Z}/p^{n+1}\mathbb{Z} \longmapsto \Phi_n(v+p^{n+1}\mathbb{Z}) := v+p^n\mathbb{Z} \in \frac{1}{p^\infty}\mathbb{Z}/p^n\mathbb{Z} \tag{2.21}$$

which is also a surjective homomorphism of abstract abelian groups realizing $\mathbb{Q}_p$ as the inverse limit of the inverse system

$$\left\{ \left(\frac{1}{p^\infty}\mathbb{Z}/p^n\mathbb{Z}, \Phi_n \right) \ : \ n \in \mathbb{N} \right\}. \tag{2.22}$$

We write

$$\mathbb{Q}_p = \varprojlim_{n\in\mathbb{N}} \left(\frac{1}{p^\infty}\mathbb{Z}/p^n\mathbb{Z} \right) \tag{2.23}$$

and note that

$$\frac{1}{p^\infty}\mathbb{Z} = \bigcup_{n=1}^{\infty} \frac{1}{p^n}\mathbb{Z} \subseteq \mathbb{Q}, \qquad \frac{\frac{1}{p^\infty}\mathbb{Z}}{\mathbb{Z}} = \mathbb{Z}(p^\infty), \qquad \frac{\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}}}{\frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}}} \cong \mathbb{Z}(p^\infty), \tag{2.24}$$

$$\mathrm{incl} \ : \ a+p^{n+1}\mathbb{Z} \in \frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}} \mapsto a+p^{n+1}\mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}}, \qquad \mathrm{quot} \ : \ b+p^{n+1}\mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} \mapsto b+\mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{\mathbb{Z}}.$$

We may observe the Prüfer group

$$\mathbb{Z}(p^\infty) = \varinjlim_{n\in\mathbb{N}} \ \mathbb{Z}(p^n) = \varinjlim_{n\in\mathbb{N}} \ \langle g_n \rangle, \tag{2.25}$$

in (2.11), where $\langle g_n \rangle = \mathbb{Z}(p^n)$, but $\mathbb{Z}(p^\infty)$ is discrete so $\langle g_n \rangle = \overline{\langle g_n \rangle}$ for all $n \in \mathbb{N}$. If we consider the Pontryagin dual $\widehat{\mathbb{Z}}(p^\infty)$ of $\mathbb{Z}(p^\infty)$, then

$$\widehat{\mathbb{Z}}(p^\infty) = \mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}(p^n) = \varprojlim_{n\in\mathbb{N}} \langle g_n \rangle \tag{2.26}$$

is monothetic, that is, topologically generated by a single element, that is, $\mathbb{Z}_p = \overline{\langle x \rangle}$. In fact we have that in general $\overline{\langle x \rangle}$ is different from $\langle x \rangle = \mathbb{Z}$. See details in [16].

We summarise the information of direct and inverse limits, which we have found in Example 2.14, in the form of the following diagram:

$$\begin{tikzcd}
\ldots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \ldots \arrow[l] & 0 \arrow[l] \arrow[d] \\
\ldots & \frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}} \arrow[l] \arrow[d, "\mathrm{incl}"] & \frac{\mathbb{Z}}{p^{n}\mathbb{Z}} \arrow[l, "\varphi_{n-1}"'] \arrow[d, "\mathrm{incl}"] & \frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}} \arrow[l, "\varphi_{n}"'] \arrow[d, "\mathrm{incl}"] & \ldots \arrow[l] & \mathbb{Z}_p \arrow[l] \arrow[d, "\mathrm{incl}"] \\
\ldots & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n-1}\mathbb{Z}} \arrow[l] \arrow[d, "\mathrm{quot}"] & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n}\mathbb{Z}} \arrow[l, "\Phi_{n-1}"'] \arrow[d, "\mathrm{quot}"] & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} \arrow[l, "\Phi_{n}"'] \arrow[d, "\mathrm{quot}"] & \ldots \arrow[l] & \mathbb{Q}_p \arrow[l] \arrow[d, "\mathrm{quot}"] \\
\ldots & \mathbb{Z}(p^\infty) \arrow[l] \arrow[d] & \mathbb{Z}(p^\infty) \arrow[l, equal] \arrow[d] & \mathbb{Z}(p^\infty) \arrow[l, equal] \arrow[d] & \ldots \arrow[l, equal] & \mathbb{Z}(p^\infty) \arrow[l, equal] \arrow[d] \\
\ldots & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \ldots \arrow[l] & 0 \arrow[l]
\end{tikzcd} \tag{2.27}$$

We have expanded the list of our examples of topological groups:

Example 2.15 ([19], Examples of Totally Disconnected Locally Compact Abelian p-groups)**.**

1. The Prüfer group $\mathbb{Z}(p^\infty)$ is a discrete abelian group.
2. The group of p-adic integers, $\mathbb{Z}_p$, is a compact abelian group (see Proposition 2.4). Moreover, this group has a structure of a ring.

3. The group of p-adic rationals, $\mathbb{Q}_p$, is a locally compact abelian group. Moveover, this group has also the structure of a field.

The following result is fundamental for compact abelian groups; we report it for a later use.

Lemma 2.16 ([19], Corollary 8.5)**.** *The following two conditions are equivalent for a compact abelian group A:*

(i). *A totally disconnected;*
(ii). $\mathrm{Div}(A) = 0$.

It is also appropriate to recall here that we may formulate for a compact abelian group the notion of *dimension*, which is intimately related to the notion of rank in the case of abstract abelian groups in Section 1.

Definition 2.17 ([19], Dimension of Compact Abelian Groups)**.** Let G be a compact abelian group and $\widehat{G}$ be its character group. Since $\widehat{G}$ is discrete by Proposition 2.4, it can be viewed as an abstract abelian group; a nonempty subset $X \subseteq \widehat{G}$ is called *free* if $\langle X \rangle$ is an abstract abelian free group, that is, if $\langle X \rangle = \mathbb{Z}^{(X)}$.

Arguing with the Zorn's Lemma, there is always a maximal free subset $X \subseteq \widehat{G}$, see [13, 19, 27].

Definition 2.18 ([19], Lemma 8.13)**.** Let G be a compact abelian group and $X \subseteq \widehat{G}$ be a maximal free subset. Then the *dimension* $\dim(G)$ of G is defined by

$$\dim(G) \coloneqq |X| = r(\widehat{G}). \tag{2.28}$$

Note that $r(\widehat{G})$ in Definition 2.18 agrees with Definition 1.9. By [19, Lemma 8.13], the maximality of X in Definition 2.18 is equivalent to say that $\mathrm{tor}(\widehat{G}/\langle X \rangle) = \widehat{G}/\langle X \rangle$.

Example 2.19. Consider the compact abelian group $\mathbb{T}$. Since $\widehat{\mathbb{T}} = \mathbb{Z}$, we know that $\mathbb{Z}$ is free and so are all of its subgroups and these subgroups take the form $n\mathbb{Z}$ for some $n \in \mathbb{N}$. Since $\mathbb{Z}/n\mathbb{Z}$ is finite, it is torsion and according to Definition 2.18 we have $\dim(\mathbb{T}) = r(\mathbb{Z}) = 1$.

We have just seen how the rank of an abstract abelian group is intimately related to the dimension of a compact abelian group; we have seen that the dimension of $\mathbb{T}$ is exactly the torsion-free rank in Definition 1.9 of $\mathbb{Z}$. This is of course a consequence of Proposition 2.4. We now introduce the notion of rank which is suited for locally compact groups, that is, for groups which look like those in Example 2.15.

Definition 2.20 ([16], Primary Components in Locally Compact Groups)**.** For an arbitrary locally compact group G (not necessarily abelian), an element $g \in G$ is called *p-element*, if the sequence g^{p^k} with $k \in \mathbb{N}$ tends to the identity element in G (with p prime). A locally compact group G is called *p-group*, if G coincides with

$$G_p \coloneqq \{g \in G : g \text{ is a } p\text{-element}\}\,. \tag{2.29}$$

A maximal p-subgroup of a locally compact group G is called *Sylow p-subgroup* of G, or primary p-component of G.

Note that if $G = A$ is an abstract abelian group in Definition 2.20, that is, a topological abelian group with the discrete topology, we find exactly $\Omega_{p^m}(A) = A$, which has been discussed in Section 1. Note also that in a locally compact group G we have that G_p is a closed subgroup by [16, Lemma 2.6], when G is totally disconnected.

Following [16, 19], we denote by G_0 the connected component of the identity of a locally compact group G and say that G is *compactly covered*, if for an arbitrary $x \in G$ we can always find a compact subgroup C of G such that $x \in C$. From [16, p.5], a *compact element* of G is an element $g \in G$ such that $\overline{\langle g \rangle}$ is compact and the set

$$\mathrm{comp}(G) = \{g \in G \mid g \text{ is a compact element}\} \tag{2.30}$$

is described in [16, Proposition 1.3, Lemma 1.6]. For instance, $G = \mathrm{comp}(G)$ when G is locally compact abelian, but in general $\mathrm{comp}(G)$ is just a subset of G, not necessarily a subgroup. Note that $\mathrm{comp}(G)$ is denoted by $B(G)$ in [2, 10, 11]; similarly, G_0 by $c(G)$.

Definition 2.21. Following [16, Proposition 1.3], we call *periodic* those locally compact groups G such that $G_0 = 1$ and $\overline{\langle g \rangle}$ is compact for all $g \in G$.

Of course, periodic locally compact groups are totally disconnected, so their Sylow p-subgroups are closed and $\mathrm{comp}(G) = G$ by [16, Lemma 1.6]. Among periodic locally compact group, there are locally compact abelian groups which are totally disconnected. These are largely studied in [16]. In particular we have the groups of Example 2.14.

Remark 2.22. According to [19, Definition 8.7], we may specialize Definition 2.20 in the following way: a compact abelian group A is called a *compact p-group* if its character group, $\widehat{A}$ agrees with its primary p-component. A locally compact abelian group turns out to be a *p-group* if it is a union of compact abelian p-groups.

If A is a compact abelian group, then $\widehat{A}$ is an abstract abelian group by Proposition 2.4, hence subject to the results which we have illustrated in Section 1. Therefore, for a compact abelian group, a p-group may be defined by the considerations which we have presented in Section 1 up to its Pontryagin dual. The situation is the same even in the more general case of locally compact abelian groups.

Example 2.23. Exploring Definition 2.20, let us take the p-adic integers $\mathbb{Z}_p$. This is a compact abelian torsion-free group, as mentioned earlier. We have that

$$\widehat{\mathbb{Z}_p} \simeq \bigcup_{n=1}^{\infty} \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z} = \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}(p^\infty), \tag{2.31}$$

and $\mathbb{Z}(p^\infty)$ (as an abstract group) is an infinite torsion abelian p-group, or, to say better, it is a discrete infinite torsion abelian p-group. Therefore it coincides with its p-primary component.

In terms of tensor products, we know by [19, Proposition A1.45] that for any abstract abelian group A, we have the following isomorphism of abstract groups:

$$\mathbb{Q} \otimes (A/\mathrm{tor}(A)) \cong \mathbb{Q} \otimes A. \tag{2.32}$$

By [19, Definition A1.59], we have that

$$r(A) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes A), \tag{2.33}$$

where $(\mathbb{Q} \otimes A)$ is viewed as a $\mathbb{Q}$-vector space. Then Definition 2.18 also describes the Prüfer rank of an abstract group A as the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes A$. Consider now (2.32) and (2.33), we get that

$$r(A) = r\left(A/\mathrm{tor}(A)\right), \tag{2.34}$$

in particular, if $A = \mathbb{Z}(p^\infty)$, and taking into account that $\mathbb{Z}(p^\infty)/\mathrm{tor}(\mathbb{Z}(p^\infty) = 0$, we get

$$r(\mathbb{Z}(p^\infty) = r(0) = 0. \tag{2.35}$$

Therefore we find that

$$\dim(\mathbb{Z}_p) = r(\mathbb{Z}(p^\infty)) = r_0(\mathbb{Z}(p^\infty)) + \max_{p \in \mathbb{P}} r_p(\mathbb{Z}(p^\infty)) = 0. \tag{2.36}$$

This illustrates Definition 2.20.

The previous example shows the relevance of what is called a 0-dimensional topological abelian group. Somehow one can think about these groups, at least in the context of compact abelian groups, as topological groups with no closed subgroups of the form $\mathbb{T}^n$. In fact $\mathbb{T}^n$ is a connected compact abelian group, so it cannot be 0-dimensional and totally disconnected, see for instance [19, Proposition 2.42].

Example 2.24. Considering Example 2.14, or also [19, Example 1.38, Exercise E1.16], we have

$$\mathbb{Q}_p = \mathbb{Z}_p \cup \frac{1}{p}\mathbb{Z}_p \cup \frac{1}{p^2}\mathbb{Z}_p \cup \dots, \tag{2.37}$$

where $\frac{1}{p^i}\mathbb{Z}_p \cong \mathbb{Z}_p$ for all $i = 0, 1, 2, \dots$. Here $\mathbb{Q}_p$ is the countable union of compact abelian p-groups, and therefore by Definition 2.20, it is a locally compact abelian p-group. Here $\widehat{\mathbb{Q}_p} \simeq \mathbb{Q}_p$.

At this point, we shall mention one of the most important notions which will be used in the present thesis.

Definition 2.25 ([16], Topologically Finitely Generated Groups)**.** A locally compact group G is *topologically finitely generated*, if there exists a finite subset X of G such that $G = \overline{\langle X \rangle}$. In particular, a locally compact p-group G has *finite p-rank*, if

$$\operatorname{rank}_p(G) = \max\{\operatorname{rank}_p(H) \mid H \text{ closed subgroup of } G\} \tag{2.38}$$

is a positive integer, where also the following quantities are positive integers

$$\operatorname{rank}_p(H) = \min\{|Y| \mid Y \subseteq H \text{ and } \overline{\langle Y \rangle} = H\}. \tag{2.39}$$

For compact p-groups, see also [26, §2.4] for an equivalent formulation of Definition 2.25. We shall mention here that in case G is a discrete abelian group, Definition 2.25 is in agreement with the notion of Prüfer rank, given in Definition 1.9.

We shall also note that the notion of rank of locally compact nonabelian p-group will be crucial in successive results of the form of Lemma 3.8. In fact it is possible to calculate the p-rank for Heisenberg groups later on in Section 3.

Finally, we mention that [16, 19] introduce the notion of a *compactly generated* locally compact group G, if there exists a compact set C in G such that $G = \langle C \rangle$. It is possible to provide examples of periodic locally compact groups, which are not compactly generated. It is also possible to provide examples which show that "topologically finitely generated groups" and "compactly generated groups" are two different notions, but we want just to describe here the locally compact abelian groups which are compactly generated:

Theorem 2.26 ([19], Theorem 7.57)**.** *Every compactly generated locally compact abelian group is isomorphic to the direct sum $\mathbb{R}^d \oplus \mathbb{Z}^m \oplus K$ for a compact abelian group K and two nonnegative integers d, m.*

3. BASIC FACTS ON QUADRATIC FORMS AND HEISENBERG GROUPS

We have already encountered abstract groups of matrices in Example 1.18, and also in (1.18) and (1.19). Moreover we have seen also topological groups of matrices in Example 2.2. We give some more details of construction here, concerning these important examples.

Let $\mathrm{M}(n,\mathbb{R})$ be the vector space of n-by-n square matrices with real coefficients. Of course, we may think of $\mathbb{R}^{n^2}$ instead of $\mathrm{M}(n,\mathbb{R})$, writing an appropriate isomorphism of finite dimensional real vector spaces. If $A \in \mathrm{M}(n,\mathbb{R})$, we denote by a_{ij} the (real) coefficients of A where $i,j = 1,2,\dots,n$. The function

$$F \; : \; \mathrm{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \longmapsto F(\mathrm{x}) := \sum_{i,j=1}^{n} a_{ij} x_i x_j \in \mathbb{R} \tag{3.1}$$

is called *quadratic form* associated with A. From the usual product of row-by-column of matrices, we note that the product of the matrix $\mathrm{x} \cdot A$ by the transpose matrix x^T of x is exactly

$$(x_1 \; x_2 \; x_3 \; \dots \; x_n) \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{pmatrix} = \sum_{i,j=1}^{n} a_{ij} x_i x_j = F(\mathrm{x}). \tag{3.2}$$

Therefore (3.1) can be rewritten equivalently in terms of the matrix equation (3.2). A quadratic form F such that

$$\begin{cases} F(\mathrm{x}) \geq 0 \quad \forall \mathrm{x} \in \mathbb{R}^n \\ \\ F(\mathrm{x}) = 0 \Longleftrightarrow \mathrm{x} = 0 \end{cases} \tag{3.3}$$

is called *positive definite*. Let's see a few more details on the quadratic forms.

Definition 3.1. Consider V as an $\mathbb{R}$-vector space of finite dimension. The map

$$f \; : \; (u,v) \in V \times V \longmapsto f(u,v) \in \mathbb{R} \tag{3.4}$$

is said to be *bilinear* if

$$\begin{cases} f(\lambda_1 u_1 + \lambda_2 u_2, v) = f(\lambda_1 u_1, v) + f(\lambda_2 u_2, v) = \lambda_1 f(u_1, v) + \lambda_2 f(u_1, v) \\ \\ f(u, \lambda_1 v_1 + \lambda_2 v_2) = f(u, \lambda_1 v_1) + f(u, \lambda_2 v_2) = \lambda_1 f(u, v_1) + \lambda_2 f(u, v_2), \end{cases} \tag{3.5}$$

for all $u_1, u_2, u, v_1, v_2, v \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

Of course, bilinear maps are linear maps with respect to two n-tuples of variables in $V \times V$, but they have codomain is the field where V is defined. If in addition to (3.5), the condition

$$f(u,v) = f(v,u), \tag{3.6}$$

is satisfied for all $u, v \in V$, we say that f in (3.4) is *symmetric*.

Following the classical terminology in [17, 28, 29, 31], we say that f in (3.4) is *alternating* if

$$f(u,u) = 0 \Longleftrightarrow u = 0, \tag{3.7}$$

for all $u \in V$, or that f is *nondegenerate*. Also, f is called *skew-symmetric* if

$$f(u,v) = -f(v,u), \tag{3.8}$$

for all $u, v \in V$. Let's be more formal, concerning these relevant notions:

Definition 3.2 ([29], Scalar Products)**.** An alternating positive definite real quadratic form is called a *scalar* product or a *Euclidean* product.

Let's see a typical example, where the above notions have application.

Example 3.3. Consider the special case where $V = \mathbb{R}^n$ and

$$f \ : \ (\mathrm{x},\mathrm{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto f(\mathrm{x},\mathrm{y}) = \mathrm{x} \cdot I_n \cdot \mathrm{y}^T = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}. \tag{3.9}$$

In other words, consider the product of matrices

$$(x_1 \ x_2 \ x_3 \ \dots \ x_n) \cdot \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \dots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i. \tag{3.10}$$

One can check that f is bilinear and symmetric. It is called *usual scalar product* on $\mathbb{R}^n$.

It is then clear that if we assign a symmetric f in (3.4), also called *bilinear form*, we may associate F in (3.1) to f just specializing the second variable $\mathrm{y} = \mathrm{x}$ in (3.4); we refer to F as the *quadratic form associated to f.* The main idea is the following: we consider the expression for f

$$f(\mathrm{x},\mathrm{y}) = \mathrm{x} \cdot A \cdot \mathrm{y}^T, \tag{3.11}$$

where $A \in \mathrm{M}(n,\mathbb{R})$ and $\mathrm{x},\mathrm{y} \in \mathbb{R}^n$ are assigned. Then we check that such an f is bilinear, that is, it satisfies (3.5). Moreover, (3.11) is symmetric if and only if $A = A^T$ if and only if A is a real symmetric matrix. In particular, $A = I_n$ or any diagonal n-by-n matrix is symmetric. Then we consider (3.1) associated to (3.11), that is,

$$F(\mathrm{x}) = f(\mathrm{x},\mathrm{x}) = \mathrm{x} \cdot A \cdot \mathrm{x}^T \coloneqq \sum_{i,j=1}^{n} a_{ij} x_i x_j. \tag{3.12}$$

Vice-versa, we may start with F in (3.1) and construct f in (3.4), as follows:

Proposition 3.4 ([18], Polar Decomposition Theorem)**.** *Given $A \in \mathrm{M}(n,\mathbb{R})$ and a quadratic form F in (3.1), if A is symmetric, then the bilinear form f in (3.4) can be obtained by F via the formula*

$$f(u,v) = \frac{1}{2}\left(F(u+v) - F(u) - F(v)\right), \tag{3.13}$$

for all $u, v \in V = \mathbb{R}^n$.

We can say also more, that is, we may describe better f when F is a scalar product, imposing appropriate conditions on the matrix A which is associated to F. We shall also mention that the construction of Proposition 3.4 shows that a quadratic form can always be associated with a bilinear form. It can be useful to observe that (3.1) can be rewritten in the form

$$F(\mathrm{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j = P(\mathrm{x}) + Q(\mathrm{x}), \tag{3.14}$$

where $P(\mathrm{x}) = a_{11}x_1^2 + \ldots + a_{nn}x_n^2$ is a polynomial of degree 2 and $Q(\mathrm{x}) = a_{12}x_1x_2 + \ldots + a_{1n}x_1x_n + a_{23}x_2x_3 + \ldots + a_{2n}x_2x_n + \ldots + a_{n-1n}x_{n-1}x_n$ as well. In particular, if $Q(\mathrm{x}) \equiv 0$ and $a_{11} = a_{22} = \ldots = a_{nn} = 1$, then we have what we usually know from the Euclidean vector spaces

$$F(\mathrm{x}) = f(\mathrm{x},\mathrm{x}) = \mathrm{x} \cdot I_n \cdot \mathrm{x}^T = (x_1, x_2, \ldots, x_n) \cdot (x_1, x_2, \ldots, x_n)^T = \sum_{i=1}^{n} x_i^2, \tag{3.15}$$

which gives the usual norm in $\mathbb{R}^n$

$$\|\mathrm{x}\| \coloneqq \sqrt{F(\mathrm{x})} = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{3.16}$$

There are no significant changes apply if we replace in the above discussions $A \in \mathrm{M}(n,\mathbb{R})$ with $B \in \mathrm{M}(n,\mathbb{C})$ and $\mathbb{R}^n$ with $\mathbb{C}^n$. In fact Definition 3.1 continues to be valid and (3.9) becomes

$$f \ : \ (\mathrm{x},\mathrm{y}) \in \mathbb{C}^n \times \mathbb{C}^n \mapsto f(\mathrm{x},\mathrm{y}) = \mathrm{x} \cdot I_n \cdot \overline{\mathrm{y}}^T = (x_1, x_2, \ldots, x_n) \cdot (\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)^T = \sum_{i=1}^{n} x_i \overline{y}_i \in \mathbb{C}. \tag{3.17}$$

Now consider a bilinear form f on $\mathbb{C}$. Then we call f *Hermitian* if it is both alternating and *sesquilinear,* that is,

$$f(u, v) = \overline{f(v, u)}, \tag{3.18}$$

for all $u, v \in V$. We do not get into the details, because the logic is very similar, with [18, 29].

We focus instead of some applications of previous definitions, on considering the following structures which are well known in Lie theory and functional analysis.

Definition 3.5 ([29], Inner Product)**.** Let H be a $\mathbb{C}$-vector space, the map

$$(\cdot \mid \cdot) : (\mathrm{x}, \mathrm{y}) \mapsto (\mathrm{x} \mid \mathrm{y}) : H \times H \to \mathbb{C} \tag{3.19}$$

is called an *inner product* on H if the following conditions hold:

(a). $(\mathrm{x} \mid \mathrm{x}) = 0 \iff \mathrm{x} = 0$;
(b). $(\mathrm{x} \mid \mathrm{x}) \geq 0$;
(c). $(\lambda \mathrm{x} \mid \mathrm{y}) = \lambda(\mathrm{x} \mid \mathrm{y})$;
(d). $(\mathrm{x} + \mathrm{y} \mid \mathrm{z}) = (\mathrm{x} \mid \mathrm{z}) + (\mathrm{y} \mid \mathrm{z})$;
(e). $(\mathrm{x} \mid \mathrm{y}) = \overline{(\mathrm{y} \mid \mathrm{x})}$,

for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in H$ and $\lambda \in \mathbb{C}$.

In this situation, a vector space H with $(\cdot \mid \cdot)$, written $H = (H, (\cdot \mid \cdot))$, is called a *unitary* or an *inner product* space. Moreover, it can be shown that the conditions (a) to (d) imply the inequality

$$\|(\mathrm{x} \mid \mathrm{y})\| \leq \|\mathrm{x}\| \|\mathrm{y}\|, \tag{3.20}$$

for all $\mathrm{x}, \mathrm{y} \in H$. This inequality (3.20) is known as the *Schwarz Inequality.*

Consider now the norm on H induced by (b) of Definition 3.5, that is,

$$\|\mathrm{x}\| := \sqrt{(\mathrm{x} \mid \mathrm{x})}. \tag{3.21}$$

It is easy to check that (3.21) satisfies the following conditions:

(N1). $\|\mathrm{x}\| \geq 0$; (nonnegative)
(N2). $\|\mathrm{x}\| = 0 \iff \mathrm{x} = 0$; (nondegenerate)
(N3). $\|\mathrm{x} + \mathrm{y}\| \leq \|\mathrm{x}\| + \|\mathrm{y}\|$, (triangular inequality)

for all $\mathrm{x}, \mathrm{y} \in H$. Consider the topology $\mathcal{T}_H$ where each open subset is defined to be the union of open balls with respect to the metric induced by Eq. (3.21), that is,

$$d(\mathrm{x}, \mathrm{y}) := \|\mathrm{x} - \mathrm{y}\|, \tag{3.22}$$

for all $\mathrm{x}, \mathrm{y} \in H$. If $H = (H, \mathcal{T}_H)$ is a *complete* metric space (that is, all the Cauchy sequences in H converge in H), then H is called a *Hilbert Space*, and we denote a Hilbert space by $\mathcal{H}$.

Hilbert spaces are a special kind of much more general spaces called *Banach spaces*, that is, those vector spaces that are complete with respect to the metric induced by the norm where the norm is *not* necessarily derived from the inner product in Definition 3.5. In other words, a Banach space need not be an inner product space. In Section 4 we will give an example of a Banach space which is also a Hilbert space.

Scalar products on Hilbert spaces are fundamental for the main results in the representation theory of topological groups, see [17, 18, 19]. For instance, one can see, as application of the representation theory of topological groups, that any compact group can be always embedded in the cartesian product of orthogonal groups of matrices, see [19, Chapter 2]. The proof of this result relies on the construction of an appropriate Hilbert space where it is possible to embed arbitrary models of compact groups.

We do not touch this delicate aspects, but go ahead to illustrate some interesting examples of bilinear forms which can be constructed for certain groups of upper triangular matrices.

Example 3.6. For any prime p, consider a bilinear form

$$\omega : (x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p \to \omega(x, y) \in \mathbb{Z}_p \tag{3.23}$$

and the set $\mathbb{Z}_p^3$ endowed with the binary operation

$$\begin{aligned} \square \ : \ ((x_1, y_1, z_2), (x_2, y_2, z_2)) \in \mathbb{Z}_p^3 \times \mathbb{Z}_p^3 \mapsto (x_1, y_1, z_1) \,\square\, (x_2, y_2, z_2) \\ = \Big(x_1 + x_2, \quad y_1 + y_2, \quad z_1 + z_2 + \omega(x_1, y_2)\Big) \in \mathbb{Z}_p^3. \end{aligned} \tag{3.24}$$

See terminology in [2, Definitions 2.1 and 2.2]. In particular, $\Box$ is not a commutative operation and $(\mathbb{Z}_p^3, \Box)$ satisfies the algebraic axioms of group. Of course, the construction depends on ω and $(\mathbb{Z}_p^3, \Box)$ is topologically isomorphic to $\mathbb{H}(\mathbb{Z}_p)$ with the matrix product. The center, the Frattini subgroup and the derived subgroup of $\mathbb{H}(\mathbb{Z}_p)$ satisfy

$$\mathrm{Frat}(\mathbb{H}(\mathbb{Z}_p)) \supseteq Z(\mathbb{H}(\mathbb{Z}_p)) = \{M(0,0;c) \mid c \in \mathbb{Z}_p\} = \overline{[\mathbb{H}(\mathbb{Z}_p), \mathbb{H}(\mathbb{Z}_p)]} \simeq \mathbb{Z}_p. \tag{3.25}$$

We can now look at topological generators and relations for $\mathbb{H}(\mathbb{Z}_p)$, finding that

$$\begin{aligned}\mathbb{H}(\mathbb{Z}_p) = \overline{\langle M(1,0;0), M(0,1;0), M(0,0;1) \mid [M(1,0;0), M(0,1;0)] = M(0,0;1),} \\ \overline{[M(1,0;0), M(0,0;1)] = [M(0,1;0), M(0,0;1)] = I_3\rangle}.\end{aligned} \tag{3.26}$$

Later in Remark 3.7, we will see that the p-rank of $\mathbb{H}(\mathbb{Z}_p)$ can be reduced to the Frattini quotient, i.e.:

$$\mathrm{rank}_p(\mathbb{H}(\mathbb{Z}_p)) = \mathrm{rank}_p(\mathbb{H}(\mathbb{Z}_p)/\mathrm{Frat}(\mathbb{H}(\mathbb{Z}_p))) = 2. \tag{3.27}$$

In particular, we see that (3.12) induces (3.15) and this is a very special situation if we look at the general context of quadratic and bilinear forms. The usefulness of the bilinear forms are obvious in the following construction

Given a commutative unitary topological ring R, the *Heisenberg group* on R is the group of all $(n+2)\times(n+2)$-matrices of the following form

$$M(A,B;c) = \left(\begin{array}{c|ccccc|c} 1 & a_1 & a_2 & \dots & a_n & c \\ \hline 0 & 1 & 0 & \dots & 0 & b_1 \\ 0 & 0 & 1 & \dots & 0 & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & b_n \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} 1 & A & c \\ O & I_n & B \\ 0 & O & 1 \end{pmatrix}, \tag{3.28}$$

where the block O is of all zeros, I_n denotes an identity matrix $n \times n$, A the n-tuple row $(a_1, \dots, a_n)$, B the n-tuple column $(b_1, \dots, b_n)$. Of course, for $n = 1$ we get the usual representation of the Heisenberg group as group of matrices 3×3.

In particular, the type of matrices in (3.28) has coefficients m_{ij} such that

$$m_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i > j, \text{ or } 1 < i < j < n-1. \end{cases} \tag{3.29}$$

Note that $\mathrm{Gl}(R^{n+2})$ is the general linear group of dimension $n+2$ of all invertible matrices with coefficients in R, and the set of all the type of matrices (3.28) is denoted by $\mathbb{H}_n(R)$ and equipped with the product topology induced by the product topology on $R^{(n+2)^2}$. In particular, one can check that $\mathbb{H}_n(R)$ is nilpotent of class 2, since the center

$$Z(\mathbb{H}_n(R)) = \overline{[\mathbb{H}_n(R), \mathbb{H}_n(R)]} = \left\{ \begin{pmatrix} 1 & O & c \\ O & I_n & O \\ 0 & O & 1 \end{pmatrix} \mid c \in R \right\} \tag{3.30}$$

is topologically isomorphic to $(R,+)$ and the central quotient

$$\mathbb{H}_n(R)/Z(\mathbb{H}_n(R)) \cong \underbrace{(R,+) \times (R,+) \times \ldots \times (R,+)}_{2n-\text{times}}, \tag{3.31}$$

is topologically isomorphic to $2n$ copies of $(R,+)$. Note that for $R = \mathbb{Z}$, or $\mathbb{Z}_p$, or $\mathbb{Z}(p)$,(3.31) is topologically generated by the matrices of the following form

(3.32)

$$\left(\begin{array}{c|ccccc|c} 1 & 1 & 0 & \dots & 0 & 1 \\ \hline 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{c|ccccc|c} 1 & 0 & 1 & \dots & 0 & 1 \\ \hline 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \dots \left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & \dots & 1 & 1 \\ \hline 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

along with the corresponding ones where the role of A is played by B in the last column

(3.33)

$$\left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \dots \left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

In particular, we describe the nonabelian compact p-group $\mathbb{H}(\mathbb{Z}_p)$ below for $n = 1$.

We now extend the notion of the Frattini subgroup in (1.24) to totally disconnected compact abelian groups $G = \mathbb{Z}_p$. Note from [26, Chapter 2] that the *Frattini subgroup* $\mathrm{Frat}(G)$ of a profinite group G is defined as the intersection of all its maximal open subgroups. Moreover it is a characteristic subgroup of G. An element g of a profinite group G is a *nongenerator* if it can be omitted from every generating set of G, that is, $G = \overline{\langle X \rangle}$ whenever $G = \overline{\langle X, g \rangle}$. In particular,

Remark 3.7. We have the fact that the set of all nongenerators of a profinite group G coincides with $\mathrm{Frat}(G)$, see [26, Lemma 2.8.1]. Moreover [26, Lemma 2.8.6] shows that the minimal number of generators of a topologically finitely generated profinite group G agrees with the minimal number of generators of $G/\mathrm{Frat}(G)$. For compact p-group G, this means that $\mathrm{rank}_p(G) = \mathrm{rank}_p(G/\mathrm{Frat}(G))$.

Indeed, $\mathrm{Frat}(G) = \overline{\Omega^1(G)[G,G]}$ for any compact p-group G by [26, Lemma 2.8.7 (c)]. Since this is true of course when $G = \mathbb{H}(\mathbb{Z}_p)$, in (3.25) one can compute $\mathrm{Frat}(\mathbb{H}(\mathbb{Z}_p))$ as follows. Now $[\mathbb{H}(\mathbb{Z}_p), \mathbb{H}(\mathbb{Z}_p)] = Z(\mathbb{H}(\mathbb{Z}_p))$ and the groups

$$\Omega^1(\mathbb{H}(\mathbb{Z}_p)) = \begin{pmatrix} 1 & p\mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 1 & p\mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix} \tag{3.34}$$

are compact for all primes p, so $\mathrm{Frat}(\mathbb{H}(\mathbb{Z}_p)) = \Omega^1(\mathbb{H}(\mathbb{Z}_p))Z(\mathbb{H}(\mathbb{Z}_p))$, since $\Omega^1(\mathbb{H}(\mathbb{Z}_p))Z(\mathbb{H}(\mathbb{Z}_p))$ is compact. Therefore, $\mathbb{H}(\mathbb{Z}_p)/\mathrm{Frat}(\mathbb{H}(\mathbb{Z}_p)) \cong \mathbb{Z}(p) \times \mathbb{Z}(p)$. This proves the second equality in (3.27). Example 3.6 holds more generally than $R = \mathbb{Z}_p$, see [15, Theorem 2.5 and Lemma 5.5] and [2, §4]. Now we look at $\mathbb{H}_n(\mathbb{Z}_p)$ and $\mathbb{H}_n(\mathbb{Q}_p)$ for n large enough.

Lemma 3.8. *The Heisenberg group $\mathbb{H}_n(\mathbb{Q}_p)$ is a locally compact nonabelian p-group of nilpotency class two and*

$$\mathrm{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)) = \mathrm{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))) = 2n.$$

Proof. Looking at (3.28), (3.29), (3.30) and (3.31) with $R = \mathbb{Q}_p$, it is clear that $\mathbb{H}_n(\mathbb{Q}_p)$ is a locally compact nonabelian p-group of nilpotency class two . Now consider the structure of the Heisenberg group and observe that there are two abelian subgroups K_1 and K_2 intersecting trivially and a closed normal subgroup H_1 such that we have

$$H_1 = Z(\mathbb{H}_n(\mathbb{Q}_p)) \times K_2 \simeq \mathbb{Q}_p \times \mathbb{Q}_p^n \quad \text{and} \quad K_1 \simeq \mathbb{Q}_p^n, \tag{3.35}$$

where

$$\mathbb{H}_n(\mathbb{Q}_p) = H_1 \rtimes K_1 = \{h_1 k_1 \mid h_1 \in H_1 \text{ and } k_1 \in K_1\} \tag{3.36}$$

$$= \{z k_2 k_1 \mid z \in Z(\mathbb{H}_n(\mathbb{Q}_p))\ k_2 \in K_2, k_1 \in K_1\} = \{[u_2, u_1] k_2 k_1 \mid k_2, u_2 \in K_2 \text{ and } k_1, u_1 \in K_1\}.$$

Note also that

$$Z(\mathbb{H}_n(\mathbb{Q}_p)) = \overline{[\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]} = \overline{[K_2, K_1]}. \tag{3.37}$$

Therefore the p-rank of $\mathbb{H}_n(\mathbb{Q}_p)$ is reduced to that of K_1 plus that of K_2, i.e., $2n$. □

Lemma 3.8 can be proved with the idea of Example 3.6, that is, noting that

$$\mathrm{Frat}(\mathbb{H}_n(\mathbb{Q}_p)) \supseteq Z(\mathbb{H}_n(\mathbb{Q}_p)), \tag{3.38}$$

and that the quotient $\mathbb{H}_n(\mathbb{Q}_p)/\mathrm{Frat}(\mathbb{H}_n(\mathbb{Q}_p))$ has p-rank $2n$, but we gave an argument based on the structure of semidirect product for Heisenberg groups. Moreover also here one could argue that $\mathrm{Frat}(\mathbb{H}_n(\mathbb{Q}_p)) = \overline{\Omega^1(\mathbb{H}_n(\mathbb{Q}_p))[\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]}$, even if in this situation we don't have a compact p-group but a periodic locally compact p-group.

Remark 3.9. Looking at [2, Lemma 2.4 and Theorem 2.5], one can show that $\mathbb{H}(\mathbb{Z}_p)$ possesses abelian maximal subgroups of the following form

$$H_1 = Z(\mathbb{H}(\mathbb{Z}_p)) \oplus \overline{\langle M(1,0;0)\rangle} \simeq {\mathbb{Z}_p}^2 \quad \text{and} \quad H_2 = Z(\mathbb{H}(\mathbb{Z}_p)) \oplus \overline{\langle M(0,1;0)\rangle} \simeq {\mathbb{Z}_p}^2 \tag{3.39}$$

satisfying the next conditions:

$$H_1 \cap H_2 = Z(\mathbb{H}(\mathbb{Z}_p)), \quad H_1 \cap \overline{\langle M(0,1;0)\rangle} = 1, \quad H_2 \cap \overline{\langle M(1,0;0)\rangle} = 1, \tag{3.40}$$

$$\mathbb{H}(\mathbb{Z}_p) = H_1 \rtimes \overline{\langle M(0,1;0)\rangle} = H_2 \rtimes \overline{\langle M(1,0;0)\rangle} \simeq {\mathbb{Z}_p}^2 \rtimes \mathbb{Z}_p. \tag{3.41}$$

In particular (3.41) shows that any element of $\mathbb{H}(\mathbb{Z}_p)$ can be written uniquely as product of an element of H_1 and one of the elements of $\overline{\langle M(0,1;0)\rangle} = K_1$, but any element of H_1 can be also written uniquely as product of an element of $Z(\mathbb{H}(\mathbb{Z}_p))$ and one of the elements of $\overline{\langle M(1,0;0)\rangle} = K_2$ by (3.39). In Fig. 1 we identify the aforementioned subgroups in the lattice of closed subgroups of $\mathbb{H}(\mathbb{Z}_p)$, that is, in

$$\mathcal{SUB}(\mathbb{H}(\mathbb{Z}_p)) = \{C \mid C \text{ is a closed subgroup of } \mathbb{H}(\mathbb{Z}_p)\}. \tag{3.42}$$

At the first level (beginning from the bottom of Fig. 1) we find the trivial subgroup.
At the second level there are three subgroups isomorphic to $\mathbb{Z}_p$.
At the third level there are two subgroups isomorphic to the additive group ${\mathbb{Z}_p}^2$.
At the fourth level we find the entire group.
Fig. 1 shows only the subgroups that can be directly deduced from (3.41), so not all $\mathcal{SUB}(\mathbb{H}(\mathbb{Z}_p))$.

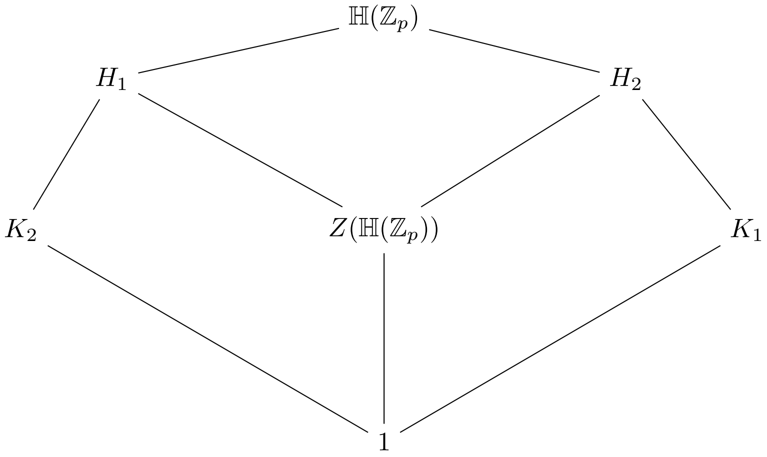

Figure 1: Some relevant subgroups in $\mathcal{SUB}(\mathbb{H}(\mathbb{Z}_p))$.

In fact one can see that, given the cardinality of the continuum $\mathfrak{c}$ and fixed $\xi \in \mathbb{Z}_p$, the subset M_ξ of all matrices $M(a, \xi a; t) \in \mathbb{H}(\mathbb{Z}_p)$ is a maximal abelian subgroup of $\mathbb{H}(\mathbb{Z}_p)$, and of course there are $\mathfrak{c}$ maximal abelian subgroups of the type M_ξ.

4. MEASURES ON LOCALLY COMPACT GROUPS

In this section we recall some notions of the theory of integration of Lebesgue with the intention to introduce the *Haar measure* in Section 5 later on. We also look at examples of Banach spaces, namely the so called L^2-spaces. We are going to recall some well known concepts from the functional analysis, available in [6, 9, 17, 18, 19, 28, 29].

Definition 4.1 ([6], §1.1)**.** Let X be a nonempty set and $\mathcal{M} \subseteq \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the powerset of X. Then $\mathcal{M}$ is called a *σ-algebra* if the following conditions are satisfied:

1. $\emptyset, X \in \mathcal{M}$;
2. if $A \in \mathcal{M}$, then $X \setminus A := A^c \in \mathcal{M}$;
3. if $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{M}$, then $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{M}$;
4. if $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{M}$, then $\bigcap\limits_{i=1}^{\infty} A_i \in \mathcal{M}$.

According with Definition 4.1, we call the pair $(X, \mathcal{M})$ a *measurable space*. It is a well result that given a subcollection $\mathcal{S} \subseteq \mathcal{P}(X)$, there is the smallest σ-algebra containing $\mathcal{S}$, and such a σ-algebra is denoted by $\sigma(\mathcal{S})$ and called *the σ-algebra generated by $\mathcal{S}$*. Now suppose that $X = (X, \mathcal{T})$ is a topological space. The σ-algebra $\sigma(\mathcal{T})$ is called *Borel* σ-algebra on X, and it is denoted also by $\mathcal{B}(X)$. This is the smallest σ-algebra containing all the open sets.

Since we are interested in topological groups, from now on we will consider only Borel σ-algebras $\mathcal{B}(G)$ where G is a topological group and the group topology on G to which we refer is clear from the context and always Hausdorff from the definition of a topological group, see Definition 1.1.

Definition 4.2 ([6], §1.2)**.** Let G be a topological group and $\mathcal{B}(G)$ be the Borel σ-algebra on G. We a call a function $\mu : \mathcal{B}(G) \to [0, +\infty]$ a *Borel measure* on G if the following conditions are satisfied:

1. $\mu(\emptyset) = 0$;
2. if $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{B}(G)$ is a subfamily such that $A_i \cap A_{i+1} = \emptyset$ for all $i \in \mathbb{N}$, then $\mu\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} \mu(A_i)$.

Formally, the pair $(G, \mathcal{B}(G))$ is called *Borel measurable space*, and if μ is a Borel measure on G, the triple $(G, \mathcal{B}(G), \mu)$ a *Borel measure space.* Less formally, we may just refer to a topological group G as a measure space with Borel measure μ, having in mind $(G, \mathcal{B}(G), \mu)$.

Definition 4.3 ([6], Characteristic and Simple Functions)**.** Let $E \subseteq G$ be a subset of a topological group G. The function $\chi_E : E \to \{0, 1\}$, defined by

$$x \in E \longmapsto \chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E, \end{cases} \tag{4.1}$$

is called *characteristic function* of E. A function $s : g \in G \mapsto s(g) \in \mathbb{R}$ is a *simple function* if the image $s(G)$ of s is given by

$$s(G) = \sum_{i=1}^{n} r_i \chi_{E_i}(g), \quad \text{where} \quad E_i = \{g \in G \ : \ s(g) = r_i\} \subseteq G \tag{4.2}$$

and $r_1, r_2, \ldots, r_n \in \mathbb{R}$.

Simple functions are $\mathbb{R}$-linear combinations of characteristic functions on a prescribed choice of sets $E_1, E_2, \ldots, E_n$, which are selected among appropriate families of subsets of the whole measure space. Most of the times one is looking for the sets $E_1, E_2, \ldots, E_n$ in such a way that they form a partition of a bigger set which we want to measure. The idea is to approximate larger sets with small sets for which we have a clear control of their measure. This is a classical technique in geometric measure theory. Now we introduce the notion of measurable functions.

Definition 4.4 ([28], Definition 11.13)**.** Let G be a topological group with $\mathcal{B}(G)$ Borel σ-algebra on G and $A \in \mathcal{B}(G)$. A function $f : \mathcal{B}(G) \to [-\infty, +\infty]$ is called *$\mathcal{B}(G)$-measurable* (or briefly *measurable*) if for all $r \in \mathbb{R}$ we have $\{x \in A : f(x) \leq r\} \in \mathcal{B}(G)$.

Of course, in the previous definition we think of $[-\infty, +\infty]$ as the extended real line with the usual topology. If f is $\mathcal{B}(G)$-measurable, then we may also split canonically f in two parts which are always nonnegative, namely

$$\begin{cases} f^+ := \max(f, 0), \\ \\ f^- := -\min(f, 0). \end{cases} \tag{4.3}$$

Clearly both f^+ and f^- are $\mathcal{B}(G)$-measurable whenever f is $\mathcal{B}(G)$-measurable. This allows us to have a theory of measurable functions, just focusing on those which are nonnegative.

We are then ready to introduce the integral in the sense of Lebesgue. Let G be a topological group and $(G, \mathcal{B}(G), \mu)$ be a Borel measure space. Assume that we have simple functions s as per Definition 4.3, and note that simple functions are always measurable, as long as $E_i \in \mathcal{B}(G)$ in (4.2). If $E \in \mathcal{B}(G)$, define

$$I_E(s) := \sum_{i=1}^{n} r_i \mu(E \cap E_i). \tag{4.4}$$

Then we have the following definition of a *Lebesgue integral* of a measurable function f;

Definition 4.5 ([28], Definition 11.21)**.** Let G be a topological group and f a measurable and nonnegative function of the measure space $(G, \mathcal{B}(G), \mu)$, and $E \in \mathcal{B}(G)$. Define

$$\int_E f \mathrm{d}\mu := \sup \{ I_E(s) : 0 \leq s \leq f \}. \tag{4.5}$$

We call $\int_E f \mathrm{d}\mu$ in (4.5) the *Lebesgue integral of f with respect to μ over E.*

Of course, we may formulate the above notion for any measure space, so the condition of working with topological groups is not necessary, but we designed the definition above for the purposes of the present study, where we basically use only measure spaces on topological groups. We shall also note that if $(G, \mathcal{B}(G))$ is a measurable space with $E \in \mathcal{B}(G)$ and f a $\mathcal{B}(G)$-measurable function which is not necessarily nonnegative, then we may consider f^+ and f^- as in (4.3) and consequently the integrals

$$\begin{cases} \int_E f^+ \mathrm{d}\mu, \\ \\ \int_E f^- \mathrm{d}\mu. \end{cases} \tag{4.6}$$

Assuming that at least one of the integrals in (4.6) is finite, we may extend Definition 4.5 to nonnegative functions, observing that

$$\int_E f \mathrm{d}\mu = \int_E f^+ \mathrm{d}\mu - \int_E f^- \mathrm{d}\mu. \tag{4.7}$$

Therefore we may give the notion of Lebesgue integrability for arbitrary functions.

Definition 4.6 ([28], Definition 11.22)**.** If both the integrals (4.6) are finite (and, consequently (4.7)), we say that an arbitrary measurable function $f : E \to -[\infty, +\infty]$ is *Lebesgue integrable* on E in the measure space $(G, \mathcal{B}(G), \mu)$, where G is a topological group with Borel σ-algebra $\mathcal{B}(G)$.

Now we move on to the notion of square integrable functions in the sense of Lebesgue. This will provide a set of functions which forms a vector space under the usual addition of functions and the usual scalar multiplication of functions. When an appropriate norm is defined, this set of functions turns out to be a Banach space. Let's see better the details.

Definition 4.7 ([28], Definition 11.34)**.** Let G be a topological group and $G = (G, \mathcal{B}(G), \mu)$ a measure space, where μ is a positive Borel measure and f a measurable function on G. Define

$$L^2(G, \mathbb{R}) := \left\{ f : G \to \mathbb{R} \ \mid \ \sqrt{\int_G |f|^2 \mathrm{d}\mu} < +\infty \right\}. \tag{4.8}$$

This is the set of *square Lebesgue integrable real functions* on G. We may also define

$$L^2(G,\mathbb{C}) := \left\{ f : G \to \mathbb{C} \ \mid \ \sqrt{\int_G |f|^2 \mathrm{d}\mu} < +\infty \right\}. \tag{4.9}$$

This is the set of *square Lebesgue integrable complex functions* on G.

Of course, we always consider the usual topologies on $\mathbb{R}$ and $\mathbb{C}$. We could also consider the real part of the complex functions in $L^2(G,\mathbb{C})$, obtaining in particular $L^2(G,\mathbb{R})$, but we report both $L^2(G,\mathbb{R})$ and $L^2(G,\mathbb{C})$ here, since it is good to have a clear distinction between these two functional spaces in the representation theory of compact groups, see [18, 19].

Consider $L^2(G,\mathbb{C})$ as in Definition 4.7. If $f, g \in L^2(G,\mathbb{C})$ and $f = f_1 + i f_2$ and $g = g_1 + i g_2$ is a decomposition of f and g in their real and imaginary parts, respectively, then

$$(f \mid g) = \int_G f\overline{g} \mathrm{d}\mu \tag{4.10}$$

satisfies the conditions of Definition 3.5 in Section 3, that is, (4.10) defines an inner product in $L^2(G,\mathbb{C})$. In particular,

$$\|f\|_2 = \sqrt{(f \mid f)} = \sqrt{\int_G |f| \mathrm{d}\mu} \tag{4.11}$$

is a norm on $L^2(\mu)$ which is induced by the inner product (4.10). This is the well known L^2-*norm* and it allows us to give the structure of Hilbert space on $L^2(G,\mathbb{C})$ (and similarly on $L^2(G,\mathbb{R})$).

In this situation, we have the L^2-*metric* induced by (4.11), that is,

$$d_2(f,g) = \|f - g\|_2. \tag{4.12}$$

Then d_2 in (4.12) is a metric in $L^2(G,\mathbb{C})$ and the inequality of functions $f, g \in L^2(G,\mathbb{C})$ should be taken *almost everywhere*, that is, $f(x) \neq g(x)$ for all $x \in A$ and for each $A \in \mathcal{B}(G)$ with $\mu(A) = 0$.

In the present situation we may consider $f(x), f_1(x), f_2(x), \ldots \in L^2(G,\mathbb{C})$, $x \in G$ and $f_n(x)$ is a sequence of functions in $L^2(G,\mathbb{C})$ converging to $f(x)$ if we have in $L^2(G,\mathbb{C})$ that

$$\lim_{n\to\infty} f_n(x) = f(x), \tag{4.13}$$

that is, if for all $\varepsilon > 0$ there is $M \in \mathbb{N}$ such that $\|f_n(x) - f(x)\|_2 < \varepsilon$ whenever $n \geq M$. Then if for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|f_n(x) - f_m(x)\|_2 < \varepsilon$, whenever $n, m \geq N$, we say that $f_n(x)$ is a *Cauchy sequence* in $L^2(G,\mathbb{C})$. There is of course a similar formulation if we replace $L^2(G,\mathbb{C})$ with $L^2(G,\mathbb{R})$ and consider real functions instead of complex functions.

Clearly, all the convergent sequences in a metric space are Cauchy, but not all Cauchy sequences are convergent in a metric space. If in a metric space the Cauchy sequences are convergent, then the metric space is said to be *complete*. Therefore we shall think of the Hilbert spaces $L^2(G,\mathbb{C})$ and $L^2(G,\mathbb{R})$ in terms of completeness as we have just mentioned, that is, with respect to the L^2-metric.

Proposition 4.8 ([29], Theorem 11.42)**.** *Let f_n be a Cauchy sequence in $L^2(G,\mathbb{C})$. Then there is a function $f \in L^2(G,\mathbb{C})$ such that f_n converges to f in $L^2(G,\mathbb{C})$.*

Of course, we may replace $L^2(G,\mathbb{C})$ with $L^2(G,\mathbb{R})$ in the above proposition and everything is still valid. We conclude this section by noting that:

Remark 4.9 ([6, 18, 29], Examples of Banach Spaces)**.** We may give a weaker formulation of Definition 4.7, beginning from a topological group G and a measure space $G = (G, \mathcal{B}(G), \mu)$, where μ is a positive Borel measure by defining

$$L^p(G,\mathbb{R}) = \left\{ f : G \to \mathbb{R} \ \mid f \text{ measurable on } G \text{ such that } \left(\int_G |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} < +\infty \right\}, \tag{4.14}$$

where $0 < p < \infty$. This is the set of L^p *integrable real functions* on G. It can be proved that $L^p(G,\mathbb{R})$ is a metric space, considering the L^p-*metric*

$$d_p(f,g) = \|f - g\|_p = \left(\int_G |f - g|^p \mathrm{d}\mu\right)^{\frac{1}{p}}. \tag{4.15}$$

Also here the inequality of functions $f, g \in L^p(G, \mathbb{R})$ should be taken *almost everywhere*. On the other hand, here we cannot introduce an inner product as in case of $p = 2$, so we have an interesting example of a Banach space which is not a Hilbert space.

According to the notions which we have seen in Definition 4.7, we can say something more on the measures in functional spaces, introducing a classical result of functional analysis due to Riesz, see [6, 28, 29]. Let G be a locally compact group and $G = (G, \mathcal{B}(G), \mu)$ a measure space where μ is a positive Borel measure. Consider the space of complex continuous functions with compact support

$$\text{supp}(f) = f^{-1}(\mathbb{C} \setminus \{0\}), \tag{4.16}$$

on G, that is,

$$C_c(G, \mathbb{C}) = \{f : G \to \mathbb{C} \ \mid \ f \ \text{ is continuous with compact } \text{supp}(f)\}. \tag{4.17}$$

This set forms a vector space with the respect to the pointwise sum of functions and to the scalar multiplication by elements of $\mathbb{C}$. By a *functional*, we mean a complex function defined on $C_c(G)$,

$$\Lambda : C_c(G, \mathbb{C}) \to \mathbb{C}. \tag{4.18}$$

Obviously, any functional Λ is linear in the sense of Section 3. There are different ways to endow the vector space $C_c(G, \mathbb{C})$ with a topology, which is compatible with the algebraic structure of vector space. One way is to consider all the functions $\mathbb{C}^{C_c(G,\mathbb{C})}$ from $C_c(G, \mathbb{C})$ to $\mathbb{C}$ and use the product topology of $\mathbb{C}$. On the other hand, we may give a better topology on this space. We may consider

$$\|f\|_\infty := \{|f(x)| : x \in G\}, \tag{4.19}$$

the so called supremum norm and define a norm by

$$\|\Lambda\|_\infty = \sup\{|\Lambda(f)| : f \in C_c(G, \mathbb{C}) \text{ and } \|f\|_\infty \le 1\} \tag{4.20}$$

on the set of all functionals from $C_c(G, \mathbb{C})$ to $\mathbb{C}$ which evaluate functions of bounded supremum norm. We say that $\Lambda : C_c(G, \mathbb{C}) \to \mathbb{C}$ is a *continuous functional* if there is a constant $M > 0$ such that

$$|\Lambda(f)| \le M\|f\|_\infty, \tag{4.21}$$

for all $f \in C_c(G, \mathbb{C})$. See also [9, 28, 29] for the classical results about functionals on Banach spaces.

The space of *continuous functionals* from $C_c(G, \mathbb{C})$ to $\mathbb{C}$ forms a Banach space with respect to (4.20); it is denoted by $\mathcal{C}_c(G, \mathbb{C})'$ and called *dual space* of $C_c(G, \mathbb{C})$. Note also that a functional Λ is said to be *positive* if $\Lambda(f) \ge 0$ whenever $f(x) \ge 0$ for all $x \in G$. The following result shows that the elements of $C_c(G, \mathbb{C})'$ are just integrals. We will use these integrals in Section 5 when we introduce the Haar measure on a locally compact group.

Proposition 4.10 ([29], Theorem 2.14)**.** *Let G be a locally compact group and Λ a positive linear functional on $C_c(G, \mathbb{C})$. Then there exists a σ-algebra $\mathcal{M}(G)$ such that $\mathcal{B}(G) \subseteq \mathcal{M}(G)$ and a unique positive measure μ on $\mathcal{M}(G)$ which represents Λ in the following sense:*

$$\Lambda(f) = \int_G f d\mu, \qquad \forall \ f \in C_c(G, \mathbb{C}). \tag{4.22}$$

Moreover

(i). *$\mu(K) < \infty$ for every compact subset $K \subseteq G$;*
(ii). *For all $E \in \mathcal{M}(G)$, we have $\mu(E) = \inf\{\mu(V) : E \subseteq V, V \text{ is open}\}$;*
(iii). *$\mu(E) = \sup\{\mu(K) : K \subseteq E, \ K \text{ is compact}\}$ for open E and for $E \in \mathcal{M}(G)$ of $\mu(E) < \infty$;*
(iv). *If $E \in \mathcal{M}(G)$, $A \subseteq E$ and $\mu(E) = 0$, then $A \in \mathcal{M}(G)$.*

Proposition 4.10 is known as the *Riesz Representation Theorem* for positive functionals in locally compact Hausdorff spaces. In fact it holds for any locally compact space X, so not specifically for locally compact groups. We will see in Section 5 that the conditions (ii) and (iii) in Proposition 4.10 are always satisfied when μ is the Haar measure, in fact, these are regularity conditions for a Borel measure.

5. THE NOTION OF TOPOLOGICAL ENTROPY FOR LOCALLY COMPACT GROUPS

In this section we introduce the notion of topological entropy in locally compact groups. We give a definition of the notion of a *left Haar* measure (see Section 4 for the discussion on general measures), and together with this we report a well known result on the existence and uniqueness of the left Haar measure on locally compact group. Then we will give the definition of the notion of topological entropy in locally compact groups. We will conclude this section with a few examples on the computation of topological entropy for some well known locally compact groups.

Let G be a locally compact group and $\mathcal{B}(G)$ be the Borel σ-algebra on G and μ be the Borel measure in the sense of Section 4. A Borel measure

$$\mu : \mathcal{B}(G) \to [0, +\infty] := \mathbb{R}_+ \cup \{+\infty\} \tag{5.1}$$

is called *μ-inner regular* if

$$\mu(B) = \sup\{\mu(C) : C \subseteq B, C \text{ is compact}\} \tag{5.2}$$

and *μ-outer regular* if

$$\mu(B) = \inf\{\mu(U) : B \subseteq U, U \text{ is open}\}, \tag{5.3}$$

for all $B \in \mathcal{B}(G)$. A Borel measure μ that is both μ-inner and μ-outer regular is called *μ-regular.*

Definition 5.1 ([17], Remark 15.8)**.** Let G be a locally compact group. A μ-regular Borel measure on G is called a *left Haar* measure on G if the following conditions are satisfied

(i). $\mu(B) > 0$ for all nonempty open subset $B \in \mathcal{B}(G)$;
(ii). $\mu(B) < \infty$ for all compact $B \in \mathcal{B}(G)$; and
(iii). $\mu(gB) = \mu(B)$ for all $g \in G$ and $B \in \mathcal{B}(G)$.

It is a well known result in Abstract Harmonic Analysis that any locally compact group possesses a left Haar measure. Now, we state this result below for the sake of completeness.

Theorem 5.2 ([17], Theorem 15.5 and [19], Theorem 2.8)**.** *Let G be a locally compact group. Then there exists a unique (up to a multiplicative constant $0 < c \in \mathbb{R}$) left Haar measure on G. If G is compact, then $c = 1$, that is, there is one and only one left Haar measure on a compact group G.*

Let G be a locally compact group and denote by $\mathcal{CT}(G)$ the collection of all compact neighbourhoods of the identity of G, and by μ a left Haar measure on G. Let φ on G be a continuous endomorphism, $V \in \mathcal{CT}(G)$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The set

$$C_n(\varphi, V) = V \cap \varphi^{-1}(V) \cap \ldots \cap \varphi^{-n+1}(V) \in \mathcal{CT}(G) \tag{5.4}$$

defines the *n-th φ-cotrajectory of V. The topological entropy* of φ (as in [20]) is

$$\mathsf{h}_{\mathsf{top}}(\varphi) = \sup\left\{ \limsup_{n\to\infty} \left(\frac{-\log \mu(C_n(\varphi, V))}{n} \right) \;\middle|\; V \in \mathcal{CT}(G) \right\}. \tag{5.5}$$

Adler and others [1, 3, 25] investigated the aforementioned notions, stressing on dynamical properties of topological structures in relations with ergodic theory and mathematical physics.

In view of [10, 11], we may introduce the *topological entropy of* a locally compact group G as

$$\mathsf{E}_{\mathsf{top}}(G) = \{\mathsf{h}_{\mathsf{top}}(\varphi) \mid \varphi \in \mathrm{End}(G)\}, \tag{5.6}$$

where $\mathrm{End}(G)$ denotes the ring of continuous endomorphisms of G and $\mathrm{Aut}(G)$ the group of continuous automorphisms of G. Here we investigate the cardinality of Eq. (5.6) and relations with structural properties, as made in [3, 5, 11, 24, 31, 34, 37].

Denoting with $\widehat{\mathbb{Q}}$ the topological dual (in the sense of Pontryagin) of the additive group $\mathbb{Q}$ of the rationals, we note that

$$\inf\{\mathsf{E}_{\mathsf{top}}(G) \setminus \{0\} \mid G \text{ is a compact group}\} = \inf(\{\mathsf{h}_{\mathsf{top}}(\varphi) \mid \varphi \in \mathrm{Aut}(\widehat{\mathbb{Q}}^n),\ n \in \mathbb{N}\} \setminus \{0\}) \tag{5.7}$$

and a formula of Yuzvinski [37] shows that $\mathsf{h}_{\mathsf{top}}(\varphi)$ can be calculated from the solutions of the characteristic polynomial of φ (see [23, 37]). Looking at locally compact groups, we also note that $\mathsf{h}_{\mathsf{top}}(\psi)$ is finite for any $\psi \in \mathrm{End}(\mathbb{R})$. Actually, we can do much more: given $t \in \mathsf{E}_{\mathsf{top}}(\mathbb{R}) \setminus \{+\infty\}$ we may construct $\psi \in \mathrm{Aut}(\mathbb{R})$ such that $\mathsf{h}_{\mathsf{top}}(\psi) = t$, see [3, 35].

Following [4, 10, 11, 14, 34], we introduce (for a locally compact group G)

$$\mathfrak{E}_0 = \{G \mid \mathsf{E}_{\mathsf{top}}(G) = \{0\}\} \quad \text{and} \quad \mathfrak{E}_{<\infty} = \{G \mid \mathsf{E}_{\mathsf{top}}(G) = [0, +\infty)\} \tag{5.8}$$

and note that there are results, which describe the abelian cases in $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$. The characterization of groups in $\mathfrak{E}_0$ can indicate the presence of structural theorems. For instance, finite abelian groups are in $\mathfrak{E}_0$ and have a decomposition in direct product. On the other hand, very little is known in the nonabelian case in $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$.

Now let $m \in \mathbb{N}$ and $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous automorphism where $\mathbb{R}^m$ has the product topology. Then $\varphi \in \mathrm{End}(\mathbb{R}^m) = \mathrm{GL}(m, \mathbb{R})$ and in [3, Corollary 16] it was shown that

$$\mathtt{h}_{top}(\varphi) = \sum_{|\lambda_i|>1} \log|\lambda_i|, \tag{5.9}$$

where $\{\lambda_i : i = 1, 2, \ldots, n\}$ is the family of all eigenvalues of $\varphi \in \mathrm{GL}(m, \mathbb{R})$. Therefore, we have that $\mathbb{R}^m \in \mathfrak{E}_{<\infty} \setminus \mathfrak{E}_0$, and in particular, $\mathbb{R} \in \mathfrak{E}_{<\infty} \setminus \mathfrak{E}_0$.

The following result describes locally compact abelian groups of finite rank and their topological entropy is known to be finite.

Theorem 5.3 ([16], Theorem 3.97)**.** *A locally compact abelian p-group A has finite p-rank if and only $A \simeq \mathbb{Z}_p^\alpha \times \mathbb{Q}_p^\beta \times \mathbb{Z}(p^\infty)^\gamma \times E_p$ for some nonnegative integers $\alpha, \beta, \gamma, \delta$ and a finite p-group E_p of $\mathrm{rank}_p(E_p) = \delta$. In particular, A belongs to $\mathfrak{E}_{<\infty}$ and $\mathrm{rank}_p(A) = \alpha + \beta + \gamma + \delta$. The case of G in $\mathfrak{E}_0$ is characterized by the condition $\beta = 0$.*

For further computations of topological entropy, we consider the class of topologically quasihamiltonian groups, investigated recently in [16].

Definition 5.4 ([36], Definition 2.11)**.** Let G be a topological group and then we say that:

(i). a closed subgroup H of G is *topologically quasinormal* if $\overline{HL} = \overline{LH}$ for all the closed subgroups L of G;

(ii). G is *topologically quasihamiltonian* if every closed subgroup is topologically quasinormal.

Recalling the notion of compactly covered group, given in Section 2, we have the following recent result:

Proposition 5.5 ([36], Proposition 4.10)**.** *Let G be a compactly covered locally compact topologically quasihamiltonian group, $\phi \in \mathrm{End}(G)$. Then*

$$\mathsf{h}_{\mathsf{alg}}(\phi) = \sup\left\{\log|:\phi^{-1}(A)| \ : \ A \leq G, \ open, \ \phi^{-1}(A) \leq A, |A : \phi^{-1}(A)| < \infty\right\}. \tag{5.10}$$

Now one can observe that among finite groups, topologically quasihamiltonian groups are just finite quasihamiltonian groups, and these turns out to be always finite nilpotent groups, see [16, 27]. Therefore the above proposition shows a first attempt to study nilpotent groups and their topological entropy. We should also observe that in general finite Heisenberg p-groups are nilpotent but are not quasihamiltonian. This motivates most of the results in the present thesis.

6. SOME RESULTS ON TOPOLOGICAL ENTROPY OF SLENDER GROUPS

In this section we are going to focus on the specific class of abstract torsion-free abelian groups and we then topologize them and check their topological characterizations for later use in this thesis. Recall the Cartesian sum of countably many copies of $\mathbb{Z}$ in Section 1 which we called the Specker group in Eq. (1.9), denoted by $\mathbb{Z}^{\mathbb{N}}$ and its countable subgroup in Eq. (1.10) denoted by $\mathbb{Z}^{(\mathbb{N})}$, which we called the free-abelian group of infinite rank $|\mathbb{N}|$.

First we give the abstract characterization of the slender groups. Lemma 6.4 says that slender groups are necessarily reduced, that is, they are those abelian groups without nontrivial divisible subgroups. Theorem 6.7 gives a topological characterization; it says that reduced torsion-free abelian groups are exactly those slender groups that do not accept some topologies, such topologies are explicitly described in this result. Lemma 6.8 is our first result in slender groups and it says that slender groups cannot be compact, and we calculate topological entropy of slender groups in our First Main result in Theorem 8.1.

Definition 6.1 (Slender Groups, See [13], p. 489)**.** A discrete torsion-free abelian group A is *slender* if, for every homomorphism

$$\alpha : (e_i)_{i\in\mathbb{N}} \mapsto \alpha\left((e_i)_{i\in\mathbb{N}}\right) : \mathbb{Z}^{\mathbb{N}} \to A,$$

we have $\alpha\left((e_i)_{i\in\mathbb{N}}\right) = 0$ for almost all i where $(e_i)_{i\in\mathbb{N}}$ is the sequence in $\mathbb{Z}$ with the i-th component equal 1 and 0 elsewhere.

The following basic properties of slender groups follow from the definition and we record them for our later frequent reference to them.

Lemma 6.2 ([13], Chapter 13, §2)**.**

(i) *Subgroups of slender groups are slender,*
(ii) *Slender groups are torsion-free,*
(iii) $\mathbb{Q}$, $\mathbb{Z}_p$, *and* $\mathbb{Z}^{\mathbb{N}}$ *are not slender,*
(iv) $\mathbb{Z}$ *is slender, and the direct sums of slender groups are slender. In particular,* $\mathbb{Z}^{(\mathbb{N})}$ *is slender*
(v) *A torsion-free abelian group* A *is slender if and only if for homomorphism* $\varphi : \mathbb{Z}^{\mathbb{N}} \to A$*, the image* $\varphi(\mathbb{Z}^{\mathbb{N}})$ *is an abstract finitely generated abelian group.*

Remark 6.3. To be begin with, by this Lemma 6.2 and Definition 1.14 of $\aleph_1$-free groups, we remark that an $\aleph_1$-free group is slender if and only if it does not contain a copy of the Specker group, $\mathbb{Z}^{\mathbb{N}}$.

E. Sasiada has (see [27, Exercise 4.4.8]) showed that all reduced countable torsion-free abelian groups are slender, also, R. Nunke noted that (see [13, Chapter 12, §3, Theorem 3.5]) all abelian groups that do not include $\mathbb{Q}$, $\mathbb{Z}^{\mathbb{N}}$, $\mathbb{Z}_p$ and $\mathbb{Z}(p)$ for any prime p are slender. We summarize these results in the following lemma.

Lemma 6.4 ([13], Chapter 13; §2, Lemma 2.3, Sasiada's Theorem and §3, Theorem 3.5)**.**

(i) *An abelian group which is slender must be reduced. In addition, if the group is countable, then the condition of being reduced is necessary and sufficient to conclude that the group is slender.*
(ii) *An abelian group is slender if and only if it does not contain a copy of* $\mathbb{Q}$, $\mathbb{Z}^{\mathbb{N}}$, $\mathbb{Z}_p$ *or* $\mathbb{Z}(p)$ *for any prime* p.

In line with the discussion in [13, Chapter 1, §7] and [8, §1], consider an abelian group A and a filter $\mathcal{F}$ in the subgroup lattice $\mathrm{L}(A)$ of A, that is, $\mathcal{F} \subseteq \mathrm{L}(A)$. Automatically $\mathcal{F}$ defines a topology on A if we declare $\mathcal{B} = \{U : U \in \mathcal{F}\}$ to be a basis of open neighbourhoods at the neutral element of A and if for every $a \in A$ the cosets $a + \mathcal{B} = \{a + U : U \in \mathcal{B}\}$ form a basis of open neighbourhoods at $a \in A$. We call this topology so obtained a *linear $\mathcal{F}$-topology* on A (or simply a *linear topology* on A). Abelian groups equipped with this topology are called *linear groups* (in the sense of Orsatti and De Marco in [8]). A linear group is *complete* if it is Hausdorff and every Cauchy net in A converges in A. Every abelian group can be topologized in this manner, Orsatti and De Marco characterized these groups in [8].

Definition 6.5 (See [8])**.** An abelian group A belongs to the class $L\Omega$ if it admits a *linear complete and nondiscrete, Hausdorff topology.* We say that A belongs to the class $L\Omega_1$, if it belongs to $L\Omega$ and in addition its topology is metrizable.

Note that just like as in other group topologies in general, the conditions in Definition 6.5 are not readily available in all linear groups, that is, not all linear groups are Hausdorff, complete or discrete and each of these conditions needs to be checked individually, but, of course, $A \in L\Omega_1$ implies $A \in L\Omega$, and the converse is generally false.

Proposition 6.6 ([8], Theorem 2.3)**.** *A torsion-free abelian group possesses a metrizable linear complete nondiscrete topology if and only if it contains a copy of* $\mathbb{Z}_p$*, or of* $\mathbb{Z}^{\mathbb{N}}$ *for a prime* p*.*

Clearly all groups of $L\Omega_1$ are characterized in Proposition 6.6. Furthermore, Lemma 6.2 shows that both $\mathbb{Z}_p$ and $\mathbb{Z}^{\mathbb{N}}$ are not slender, hence Proposition 6.6 implies that A cannot be slender, if it can be endowed with a metrizable linear complete nondiscrete topology. This is recorded below:

Theorem 6.7 ([8], De Marco and Orsatti)**.** *Let* A *be a reduced torsion-free abelian group. Then* A *is slender if and only if* A *does not belong to* $L\Omega_1$*.*

From all that we have discussed in this section, we have the following result.

Lemma 6.8. *There are no nontrivial compact abelian slender groups.*

Proof. Assume that A is a compact abelian slender group. Lemma 6.4 along with Lemma 2.16 (i) and (ii) imply that A is totally disconnected. Then A should be profinite by [19, Theorem 1.34], hence inverse limit of finite groups. Profinite abelian groups are not slender; $\mathbb{Z}_p$ is a counterexample. From this contradiction, there are no nontrivial compact abelian slender groups. ☐

7. MORE RESULTS ON THE TOPOLOGICAL ENTROPY

When we have a totally disconnected locally compact group G, van Dantzig [7] proved that

$$\mathcal{U}(G)=\{V\leq G \mid V \text{ compact and open}\}\tag{7.1}$$

is contained in $\mathcal{CT}(G)$ and is local basis. From [11, Proposition 3.4], we have that

$$\mathsf{h}_{\mathsf{top}}(\varphi)=\sup\left\{\lim_{n\to\infty}\left(\frac{\log|V:C_n(\varphi,V)|}{n}\right)\;\middle|\; V\in\mathcal{U}(G)\right\},\tag{7.2}$$

where $C_n(\varphi,V)\in\mathcal{U}(G)$ and the index $|V:C_n(\varphi,V)|$ is finite. In fact, the set $\mathsf{E}_{\mathsf{top}}(G)$ turns out to be a countable subset of the real half-line in this situation.

Some relevant facts are reported below. The first refers to discrete groups.

Remark 7.1 (See [11], Remark 2.4)**.** Discrete groups belong to $\mathfrak{E}_0$. □

The second refers to the additive group of p-adic integers.

Corollary 7.2 (See [11], Corollary 2.2)**.** *Let G be a locally compact group and $\varphi\in\mathrm{End}(G)$. If $\mathcal{S}\subseteq\mathcal{CT}(G)$ is a local basis of G and each element of $\mathcal{S}$ is a φ-invariant subgroup, then $\mathsf{h}_{\mathsf{top}}(\varphi)=0$. In particular, this applies to $\mathbb{Z}_p^n$, hence $\mathbb{Z}_p^n\in\mathfrak{E}_0$.*

The computation of the topological entropy of continuous endomorphisms is somehow harder than that of continuous automorphisms, but we have results for totally disconnected groups.

Corollary 7.3 (See [11], Lemma 2.3, Theorem 3.11)**.**
Let G be a locally compact group and $\varphi\in\mathrm{End}(G)$.

(a) *If H is a φ-invariant closed subgroup of G, then $\mathsf{h}_{\mathsf{top}}(\varphi_{|_H})\leq\mathsf{h}_{\mathsf{top}}(\varphi)$, and, if in addition H is normal, then $\mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/H})\leq\mathsf{h}_{\mathsf{top}}(\varphi)$, where $\bar{\varphi}_{G/H}:G/H\to G/H$ is induced by φ.*
(b) *If $\mathcal{S}\subseteq\mathcal{U}(G)$ is a local basis of G such that $\varphi^{-n}(V)$ is normal in G for all n and $V\in\mathcal{S}$, then $\mathsf{h}_{\mathsf{top}}(\varphi)=\mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/\ker\varphi})$.*
(c) *If G is totally disconnected and $\varphi\in\mathrm{Aut}(G)$, then $\mathsf{h}_{\mathsf{top}}(\varphi)=\mathsf{h}_{\mathsf{top}}(\varphi_{|N})+\mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/N})$, where N is a closed normal subgroup of G.*

The third is related to the p-adic rationals. Denoting the p-adic norm with $|-|_p$, Yuzvinski's Formula [23, 37] helps with the following computations:

Theorem 7.4 (See [23])**.** *For $n\in\mathbb{N}$ and $\varphi\in\mathrm{End}(\mathbb{Q}_p^n)$, we have*

$$\mathsf{h}_{\mathsf{top}}(\varphi)=\sum_{|\lambda_i|_p>1}\log|\lambda_i|_p,\tag{7.3}$$

where λ_i (with $1\leq i\leq n$) is eigenvalue of φ in a finite extension of $\mathbb{Q}_p$. In particular, $\mathbb{Q}_p^n\in\mathfrak{E}_{<\infty}$.

Further criteria of finiteness of topological entropy are related to the finiteness of p-rank as seen in Theorem 5.3. Furthermore, Theorem 5.3 shows that the p-rank is preserved under Pontryagin duality. Indeed,

$$\widehat{G}=(\mathbb{Z}_p^\alpha\times\mathbb{Q}_p^\beta\times\mathbb{Z}(p^\infty)^\gamma\times E_p)^\wedge\cong\mathbb{Z}_p^\gamma\times\mathbb{Q}_p^\beta\times\mathbb{Z}(p^\infty)^\alpha\times E_p,\tag{7.4}$$

and so $\mathrm{rank}_p(\widehat{G})=\mathrm{rank}_p(G)$. In particular, it can be seen that $\mathbb{Q}_p^\beta\in\mathfrak{E}_{<\infty}\setminus\mathfrak{E}_0$, $\mathbb{R}^d\in\mathfrak{E}_{<\infty}\setminus\mathfrak{E}_0$, $\mathbb{Z}_p^\gamma\in\mathfrak{E}_0$, $E_p\in\mathfrak{E}_0$ and $\mathbb{Z}(p^\infty)^\alpha\in\mathfrak{E}_0$. Further results are reported below in the Abelian case.

Theorem 7.5 (See [11], Theorems 1.1 and 1.2)**.** *Let A be a locally compact abelian group.*

(i) *If A belongs to $\mathfrak{E}_{<\infty}$, then its dimension should be finite;*
(ii) *The viceversa of (i) above is true when A is compact and A/A_0 belongs to $\mathfrak{E}_{<\infty}$;*
(iii) *If A belongs to $\mathfrak{E}_0$, then A is totally disconnected; moreover a profinite group belongs to $\mathfrak{E}_0$ if and only if it belongs to $\mathfrak{E}_{<\infty}$;*
(iv) *If A is periodic, then A is in $\mathfrak{E}_0$ if and only if all its Sylow p-subgroups A_p do the same.*

In the arguments which are used to prove Theorem 7.5, the main logic is to find decompositions of the endomorphisms in portions where we can control the finiteness of the topological entropy. In fact we say that *the Addition Theorem holds* for (G,φ,N) of a locally compact group G with $\varphi\in\mathrm{End}(G)$ and a φ-invariant closed normal subgroup N of G, if

$$\mathsf{h}_{\mathsf{top}}(\varphi)=\mathsf{h}_{\mathsf{top}}(\varphi_{|_N})+\mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/N}),\tag{7.5}$$

or briefly, we write that $AT(G,\varphi,N)$ *holds*. Of course, (7.5) is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\iota"] \arrow[d, "\varphi_{|N}"'] & G \arrow[r, "\pi"] \arrow[d, "\varphi"'] & G/N \arrow[r] \arrow[d, "\bar{\varphi}_{G/N}"'] & 0 \\
0 \arrow[r] & N \arrow[r, "\iota"] & G \arrow[r, "\pi"] & G/N \arrow[r] & 0
\end{tikzcd} \tag{7.6}$$

Similarly, $AT(G)$ *holds* if $AT(G,\varphi,N)$, which is depending on φ and N in general, is satisfied by all φ and N. From [11, Proposition 3.6], if N is a fully invariant open subgroup of G and $AT(N)$ holds, then also $AT(G)$ holds.

At this point it is important that we pause and look closely at the structure of compactly generated locally compact abelian groups of Theorem 2.26. First, we note that the groups that appear in the decomposition are either compact or totally disconnected, or isomorphic to $\mathbb{R}^d$ for some nonnegative integer d. Because of this observation, we record the following result:

Lemma 7.6 (See [11], Lemma 3.1)**.** *Let A, B be two locally compact groups that either are compact, or totally disconnected or isomorphic to $\mathbb{R}^d$ for some nonnegative integer d, and $f \in \mathrm{End}(A)$, $g \in \mathrm{End}(B)$. Consider $A \times B$ with the product topology and $f \times g \in \mathrm{End}(A \times B)$. Then*

$$\mathsf{h}_{\mathsf{top}}(f \times g) = \mathsf{h}_{\mathsf{top}}(f) + \mathsf{h}_{\mathsf{top}}(g). \tag{7.7}$$

Again the situation is very clear computationally for locally compact abelian groups.

Theorem 7.7 (See [11], Theorems 1.8 and 1.9)**.** *Let A be a totally disconnected locally compact abelian group. Then, for every $\varphi \in \mathrm{End}(A)$, we have*

$$\mathsf{h}_{\mathsf{top}}(\varphi) = \sum_{p \text{ prime}} \mathsf{h}_{\mathsf{top}}(\varphi_{|A_p}). \tag{7.8}$$

If A is also periodic, then $AT(A)$ holds if and only if $AT(A_p)$ holds for all Sylow p-subgroups A_p.

As application of Corollary 7.2, we have that a compact p-group G with local basis $\{\Omega^n(G) \mid n \in \mathbb{N}\} \subseteq \mathcal{U}(G)$ should belong to $\mathfrak{E}_0$. Note that this applies to $\mathbb{Z}_p^n \times E_p$, where E_p is finite p-group.

Remark 7.8. Groups of the form $\mathbb{Z}_p \times E_p$ for the E_p finite nonabelian p-group are among the easiest examples of infinite nilpotent compact p-groups which can be produced in $\mathfrak{E}_0$. Looking at [22, Section 3.1], a finite p-group E_p is of maximal class if it has order p^n with $n > 3$ and nilpotency class is $c = n - 1$. Their construction can be found in [22, Examples 3.1.5]. Now $\mathbb{Z}_p \times E_p$ has nilpotency class exactly n by Fitting's Lemma [22, Lemma 1.1.21]. This means that we have already an example of an infinite nonabelian compact p-group of nilpotency class arbitrarily large in $\mathfrak{E}_0$. □

8. MAIN RESULTS AND THEIR PROOFS

Our first main result can be now formulated:

Theorem 8.1 (First Main Theorem)**.** *Let A be a compactly generated locally compact abelian group. With the notations of Theorem 2.26, the following statements are satisfied :*

(a). *If A is slender, then $A \in \mathfrak{E}_0$. Viceversa, if $A \in \mathfrak{E}_0$ and $K = 0$, then A is slender.*
(b). *Assume that K is connected. Then $A \in \mathfrak{E}_{<\infty}$ if and only if $A \simeq \mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$ for some nonnegative integers d, m, s.*

Proof of Theorem 8.1. (a). If A is a compactly generated locally compact abelian group, then Theorem 2.26 implies $A \cong \mathbb{R}^d \oplus \mathbb{Z}^m \oplus K$ for a compact abelian group K and nonnegative integers m, d. Assume in addition that A is slender. Lemma 6.2 shows that subgroups of slender groups are slender. Then Lemma 6.4 implies $d = 0$, that is, $A \cong \mathbb{Z}^m \oplus K$. Lemma 6.8 implies $K = 0$. Hence $A \cong \mathbb{Z}^m$, and since $\mathbb{Z}^m \in \mathfrak{E}_0$, the first part of the result follows. Assume now that $A \in \mathfrak{E}_0$ and that $A \cong \mathbb{R}^d \oplus \mathbb{Z}^m$. Since $\mathbb{R}^d \in \mathfrak{E}_\infty \setminus \mathfrak{E}_0$, A should be totally disconnected by Theorem 7.5 (iii) and so $A \cong \mathbb{Z}^m$ which is slender. The result is clear.

(b). From Theorem 2.26 and the assumption that K is a connected compact abelian group, we have that $A \cong \mathbb{R}^d \oplus \mathbb{Z}^m \oplus K$ with K of $\dim(K)$ eventually infinite. Then

$$\dim(A) = \dim(\mathbb{R}^d) + \dim(\mathbb{Z}^m) + \dim(K) = d + 0 + \dim(K) \tag{8.1}$$

and this shows that $\dim(A) < \infty$ if and only if $\dim(K) < \infty$ if and only if $K = \mathbb{T}^s$ for some nonnegative integer s, see [19, Corollary 8.22 (5)]. From Theorem 7.5 (i), this means that if $A \in \mathfrak{E}_{<\infty}$, then $\dim(A) < \infty$ hence $\dim(K) < \infty$, and so $A \cong \mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$. Conversely, assume that $A \cong \mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$. We may apply Lemma 7.6 with summands $\mathbb{R}^d \in \mathfrak{E}_{<\infty}$, $\mathbb{Z}^m \in \mathfrak{E}_0$ and $\mathbb{T}^s \in \mathfrak{E}_{<\infty}$, concluding $A \in \mathfrak{E}_{<\infty}$. Note that the computations of topological entropy, which allows us to have $\mathbb{R}^d \in \mathfrak{E}_{<\infty}$, $\mathbb{Z}^m \in \mathfrak{E}_0$ and $\mathbb{T}^s \in \mathfrak{E}_{<\infty}$, are well known, see [3, 23, 34]. The result follows. □

Note that the computations of the topological entropy of continuous automorphisms (not endomorphisms) of $\mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$ are available in [25, pp. 475–476]. Also [5, 24] contain the computations of the topological entropy of continuous endomorphims, but mostly of Lie groups. We go ahead and describe the finiteness of the topological entropy for some nonabelian locally compact groups, looking at the behaviour of the Sylow p-subgroups.

Theorem 8.2 (Second Main Theorem)**.** *The continuous automorphisms of a nilpotent periodic locally compact p-group G have finite topological entropy whenever $\mathrm{rank}_p(G)$ is finite.*

Proof of Theorem 8.2. First assume that G has $\mathrm{rank}_p(G) < \infty$. We note that closed subgroups and quotients of G are again periodic locally compact p-groups. The topological lower central series of G of length c has closed characteristic p-subgroups $\overline{\gamma_i(G)}$ (with $i = 1, 2, \ldots, c$) such that

$$G = \overline{\gamma_1(G)} \geq \overline{\gamma_2(G)} = \overline{[G,G]} \geq \overline{\gamma_3(G)} = \overline{[[G,G],G]} \geq \ldots \geq \overline{\gamma_c(G)} \geq \overline{\gamma_{c+1}(G)} = 1 \tag{8.2}$$

and $\overline{\gamma_i(G)}/\overline{\gamma_{i+1}(G)}$ are locally compact abelian p-groups for all i. Note also that closed subgroups and quotients of a periodic locally compact p-group of finite p-rank have finite p-rank. This means that if G has finite p-rank, then $\overline{\gamma_i(G)}/\overline{\gamma_{i+1}(G)}$ are of the form of those in Theorem 5.3, and in particular continuous automorphisms of $\overline{\gamma_i(G)}/\overline{\gamma_{i+1}(G)}$ have finite topological entropy. Now we do induction on c. Assume $c = 1$. Then G is a locally compact abelian group of finite p-rank and the result is true by Theorem 5.3, because in this situation the continuous automorphisms of G should have finite topological entropy. Assume $c > 1$ and that the result is true for all periodic nilpotent locally compact p-groups of derived length at most $c - 1$. Then the continuous automorphisms of $N = \overline{\gamma_c(G)}$ have finite topological entropy, since N is abelian, but also those of G/N have finite topological entropy, since G/N is a locally compact abelian p-group of finite p-rank. From Addition Theorem for continuous automorphisms of totally disconnected locally compact abelian groups (see [25, Addition Theorem 10], or Corollary 7.3 (b)) we conclude that $AT(G, \varphi, N)$ holds for every continuous automorphism φ of G. The result follows. □

Theorem 7.5 (iv) (as well as Theorem 7.7) show that Addition Theorems may be reduced to Addition Theorems on p-Sylow subgroups. This means that the presence of a decomposition helps to determine groups in $\mathfrak{E}_0$ or in $\mathfrak{E}_{<\infty}$, just looking at Sylow p-subgroups in $\mathfrak{E}_0$ or in $\mathfrak{E}_{<\infty}$.

Remark 8.3. For compactly generated locally compact abelian groups, Theorem 2.26 shows that Lemma 7.6 can be applied and so we have an Addition Theorem. This helps to reduce the computation of the topological entropy of continuous endomorphisms to the topological entropy of continuous endomorphisms arising from factors.

We can always find periodic locally compact p-groups G of $\mathrm{rank}_p(G) = r$ in $\mathfrak{E}_{<\infty}$, looking at the direct sum $G = \mathbb{Z}_p^r$ of r copies of the additive group of p-adic integers $\mathbb{Z}_p$. On the other hand, it is possible to find periodic locally compact p-groups of nilpotency class two and of finite p-rank, looking at Heisenberg p-groups $\mathbb{H}_n(\mathbb{Q}_p)$ constructed with upper triangular $(n+2)\times(n+2)$ matrices with coefficients in the field of p-adic rationals $\mathbb{Q}_p$. These are neither abelian nor compact groups, and have finite topological entropy and finite p-rank large enough.

Theorem 8.4 (Third Main Theorem)**.** *The Heisenberg group $\mathbb{H}_n(\mathbb{Q}_p)$ is a periodic locally compact nonabelian p-group of nilpotency class 2 with $\mathrm{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)) = 2n$, where n is an arbitrary positive integer. Moreover $\mathbb{H}_n(\mathbb{Q}_p)$ belongs to $\mathfrak{E}_{<\infty}$, but not to $\mathfrak{E}_0$.*

Proof of Theorem 8.4. From Lemma 3.8, the Heisenberg group $\mathbb{H}_n(\mathbb{Q}_p)$ is a periodic locally compact nonabelian p-group of nilpotency class two and $\mathrm{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)) = 2n$. Then we shall only prove that $\mathbb{H}_n(\mathbb{Q}_p)$ belongs to $\mathfrak{E}_{<\infty}$, but not to $\mathfrak{E}_0$.

Assume that $n = 1$. From [11, Theorem 6.8] we know that $\mathbb{H}(\mathbb{Q}_p)$ belongs to $\mathfrak{E}_{<\infty}$, but not to $\mathfrak{E}_0$. Then there exists a subgroup S of $\mathbb{H}_n(\mathbb{Q}_p)$ which is isomorphic to $\mathbb{H}(\mathbb{Q}_p)$ as topological group, for instance S can be realized putting in (3.28) the condition $a_i = b_i = 0$ for all $i = 2, 3, \ldots, n$. This is sufficient to show that $\mathbb{H}_n(\mathbb{Q}_p)$ cannot be in $\mathfrak{E}_0$, since it contains a subgroup S which is not in $\mathfrak{E}_0$. It remains to check that $\mathbb{H}_n(\mathbb{Q}_p)$ belongs to $\mathfrak{E}_{<\infty}$ and we adapt to the argument of [11, Proof of Theorem 6.8] for this scope.

Consider $\varphi \in \mathrm{End}(\mathbb{H}_n(\mathbb{Q}_p))$ and $N = \ker\varphi$; we claim that $\mathsf{h}_{\mathsf{top}}(\varphi) < \infty$.

Assume that $N = 1$. We claim that $\varphi \in \mathrm{Aut}(\mathbb{H}_n(\mathbb{Q}_p))$. Since $Z(\mathbb{H}_n(\mathbb{Q}_p))$ is fully invariant, $\varphi_{|Z(\mathbb{H}_n(\mathbb{Q}_p))}$ is injective, hence $\varphi_{|Z(\mathbb{H}_n(\mathbb{Q}_p))}$ is a continuous automorphism of $Z(\mathbb{H}_n(\mathbb{Q}_p))$. In particular, $\varphi^{-1}(Z(\mathbb{H}_n(\mathbb{Q}_p))) = Z(\mathbb{H}_n(\mathbb{Q}_p))$ and so $\bar{\varphi}_{|\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))}$ on $\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))$ is injective. In fact it is a continuous automorphism of $\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p)) \simeq \mathbb{Q}_p^{2n}$. Now $\mathbb{H}_n(\mathbb{Q}_p)$ is a totally disconnected locally compact group, which can be also realized as union of countably many compact sets, and so φ is a continuous automorphism by the Open Mapping Theorem [19, Appendix 1, Exercise EA1.21]. We may apply Addition Theorems on closed normal subgroups for continuous automorphisms of locally compact groups as per Corollary 7.3 (c), concluding $\mathsf{h}_{\mathsf{top}}(\varphi) < \infty$ from the fact that both $\mathsf{h}_{\mathsf{top}}(\varphi_{|Z(\mathbb{H}_n(\mathbb{Q}_p))}) < \infty$ and $\mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))}) < \infty$ by Theorem 5.3.

Now assume that $N = \ker\varphi \neq 1$. First we show that $N \cap Z(\mathbb{H}_n(\mathbb{Q}_p))$ is nontrivial and then that $Z(\mathbb{H}_n(\mathbb{Q}_p)) \subseteq N$. If there exists some $y \in N \setminus Z(\mathbb{H}_n(\mathbb{Q}_p))$, then there exists $x \in \mathbb{H}_n(\mathbb{Q}_p)$ such that $[x, y]$ is nontrivial. This implies that $N \cap [\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]$ is nontrivial, because $[x, y] \in N$. The claim follows and $N \cap Z(\mathbb{H}_n(\mathbb{Q}_p))$ is a nontrivial closed subgroup of $Z(\mathbb{H}_n(\mathbb{Q}_p))$, hence $Z(\mathbb{H}_n(\mathbb{Q}_p))/(N \cap Z(\mathbb{H}_n(\mathbb{Q}_p)))$ is torsion because it is a nontrivial quotient of $\mathbb{Q}_p$. On the other hand, $Z(\mathbb{H}_n(\mathbb{Q}_p))/(N \cap Z(\mathbb{H}_n(\mathbb{Q}_p))) \cong \varphi(Z(\mathbb{H}_n(\mathbb{Q}_p)))$ is a subgroup of $\mathbb{H}_n(\mathbb{Q}_p)$ (up to continuous isomorphisms), hence torsion-free. Consequently $Z(\mathbb{H}_n(\mathbb{Q}_p))/(N \cap Z(\mathbb{H}_n(\mathbb{Q}_p)))$ is trivial, and the other claim $Z(\mathbb{H}_n(\mathbb{Q}_p)) \subseteq N$ follows. Since N contains $Z(\mathbb{H}_n(\mathbb{Q}_p)) = [\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]$, we may apply Addition Theorems as per Corollary 7.3 (b), hence $\mathsf{h}_{\mathsf{top}}(\varphi) = \mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{\mathbb{H}_n(\mathbb{Q}_p)/N})$ is finite by Theorem 5.3. Therefore the result follows.

□

References

[1] R.L. Adler, A.G. Konheim and M.H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965) 309–319. 5, 32

[2] M. Bonatto and D. Dikranjan, *Generalized Heisenberg groups and self-duality*, Quest. Answ. Gen. Topol. **37** (2019) 89–108. 19, 25, 26, 27

[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971) 401–414. 5, 32, 33, 38

[4] A. G. Bruno and S. Virili, *Algebraic Yuzinvinski formula*, J. Algebra. **423** (2015) 114—147. 32

[5] A. Caldas and M. Patrão, *Entropy and its variational principle for locally compact metrizable systems*, Ergodic Theory Dyn. Syst. **38** (2018) 540–565. 32, 38

[6] D. L. Cohn, *Measure Theory*, Birkhäuser, 1980, Boston. 28, 30, 31

[7] D. van Dantzig, *Studien over topologische Algebra*, Dissertation, 1931, Amsterdam. 36

[8] G.De Marco and A.Orsatti, *Complete linear topologies on abelian groups*, Sympos. Math. **13** (1974) 153–161 34, 35

[9] A. Deitmar, *Principle of Harmonic Analysis*, Springer, 2009, Berlin. 28, 31

[10] D. Dikranjan and M. Sanchis, *Dimension and entropy in compact topological groups*, J. Math. Anal. Appl. **476** (2019) 337—366. 19, 32

[11] D. Dikranjan, A. Giordano Bruno and F. G. Russo, *Finiteness of topological entropy for locally compact abelian groups*, Glasgow Math. J. **361** (2020) 403—442. 5, 19, 32, 36, 37, 39

[12] D. Dikranjan, M. Sanchis and S. Virili, *New and old facts about entropy in uniform spaces and topological groups*, Topology Appl. **159** (2012) 1916-–1942. 5

[13] L. Fuchs, *Abelian Groups*, Springer Cham, Switzerland, 2015. 5, 9, 10, 19, 34

[14] A. Giordano Bruno and S. Virili, *Topological entropy in totally disconnected locally compact groups*, Ergod. Theory Dyn. Syst. **37** (2017) 2163–2186. 32

[15] T. Grundhöfer and M. Stroppel, *Automorphisms of Verardi groups: small upper triangular matrices over rings*, Beitr. Algebra Geom. **49** (2008) 1–31. 26

[16] W. Herfort, K. H. Hofmann and F. G. Russo, *Periodic Locally Compact Groups*, de Gruyter, 2019, Berlin. 8, 9, 14, 15, 18, 19, 20, 21, 33

[17] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis: Structure of Topological Groups, Integration Theory and Group Representation*, Vol. 1, Springer, 1979, New York. 8, 22, 24, 28, 32

[18] J. Hilgert and K.H. Neeb, *Structure and Geometry of Lie Groups*, Springer, 2012, Berlin. 23, 24, 28, 30

[19] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, de Gruyter, 2020, Berlin. 5, 8, 9, 10, 11, 14, 15, 16, 17, 18, 19, 20, 21, 24, 28, 30, 32, 35, 38, 39

[20] B. M. Hood, *Topological entropy and uniform spaces*, J. London Math. Soc. **8** (1974) 633—641. 5, 32

[21] A. N. Kolmogorov, *New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces*, Doklady Akad. Nauk. SSSR **119** (1958) 861—864. 5

[22] C. R. Leedham-Green, S. McKay, *The structure of groups of prime power order*, Oxford University Press, 2002, Oxford. 8, 37

[23] D. A. Lind and T. Ward, *Automorphisms of solenoids and p-adic entropy*, Ergod. Theory Dyn. Syst. **8** (1988) 411–419. 32, 36, 38

[24] M. Patrão, *The topological entropy of endomorphisms of Lie groups*, Isr. J. Math. **234** (2019) 55–80. 32, 38

[25] J. Peters, *Entropy of automorphisms on locally compact abelian groups*, Pacific J. Math. **96** (1981) 475—488. 5, 32, 38

[26] L. Ribes and P. Zalesskii, *Profinite Groups*, Springer, 2000, Berlin. 8, 14, 15, 16, 17, 21, 26

[27] D. J. S. Robinson, *A Course in the Theory of Groups.* Springer, 1996, New York. 5, 8, 9, 10, 11, 12, 13, 16, 19, 33, 34

[28] W. Rudin, *Principles of Mathematical Analysis.* McGraw-Hill, 1976, Singapore. 5, 22, 28, 29, 31

[29] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1986, Singapore. 5, 22, 24, 28, 30, 31

[30] F.G. Russo and O. Waka, *On locally compact groups of small topological entropy*, arXiv:2304.08156 5

[31] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser, 1995, Basel. 22, 32

[32] Y. G. Sinai, *On the concept of entropy of a dynamical system*, Doklady Akad. Nauk. SSSR **124** (1959) 786—781. 5

[33] C. E. Shannon, *The Mathematical Theory of Communication*, Physics Today. **3** (1950) 31–32. 5

[34] S. Virili, *Entropy for endomorphisms of locally compact abelian groups*, Topology Appl. **159** (2012) 2546—2556. 32, 38

[35] P. Walters, *An Introduction to Ergodic Theory*, Springer, 1969, Berlin. 32

[36] W. Xi, M. Shlossberg, and D. Toller, *Algebraic entropy on topologically quasihamiltonian groups*, Topology Appl. **272** (2020) 107093. 33

[37] S. Yuzvinski, *Metric properties of endomorphisms of compact groups*, Izv. Acad. Nauk SSSR, Ser. Mat. **29** (1965) 1295–1328 (in Russian). English Translation: Amer. Math. Soc. Transl. (2) **66** (1968) 63–98. 32, 36

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