# Simple rule, hidden meaning: the scalar product in engineering mathematics

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## Abstract

Engineering is a highly mathematical field of study with different university courses requiring proficiency at different types of mathematics. Engineering dynamics requires the skilful use of vectors in various ways and proficiency at vector arithmetic, algebra and geometry is of vital importance to incoming students. This paper reports on findings from the administering of a vector proficiency assessment instrument across two semesters of a dynamics course. Findings suggest that problems requiring use of the scalar product embedded within a context are of the highest difficulty level. We argue that the geometric role of the scalar product is weakly understood by the majority of students, leading to poor performance at any problem requiring more than a basic calculation. We suggest that lecturers of engineering mathematics foreground the geometric role and that lecturers of engineering courses be aware of the level of challenge manifest in these problems.

Keywords: engineering dynamics; vectors; Rasch analysis; scalar product

## Introduction

In modern engineering practice, and especially in computational mechanics, vector algebra is an indispensable tool for the solution of challenging problems, and, hence, proficiency in the algebraic manipulation of vectors is an absolute necessity for all engineering students. For this reason it is given great emphasis in first year mathematics courses taken by engineering students.

By contrast, vector geometry, that is the graphical representation of vectors using arrow headed line segments, while far from ignored, receives less emphasis. In particular, vector geometry is typically used for the description of problems, rather than the solution of problems, where algebraic manipulation is the preferred method. It could be argued that, within the context of typical first-year mathematics problems, vector diagrams, if used at all, are a halfway point between the ubiquitous vector arithmetic/algebra and the accurate and detailed vector diagrams used elsewhere, such as in dynamics. It is therefore imperative that the students grasp what vector geometry they encounter in first year in order to prepare them for the more advanced uses of vector geometry later. In this paper we discuss how even that minimal geometric understanding is worryingly absent, with serious implications for the teaching and learning of advanced vector use.

The Dynamics Education Research Group (DERG) was established at the University of Cape Town (UCT) to investigate a range of educational issues related to the teaching and learning of dynamics, of which several are mathematical in nature. One of the avenues under investigation is the degree to which students entering a second-year dynamics course retain the vector mechanics proficiency which they acquire in first-year mathematics. In particular, we are interested in the difference between the contexts in which students find a given vector manipulation to be evident or obscured.

In preparing for writing this paper, we searched the education literature using several databases and search engines and could find nothing other than textbooks and teaching guides looking at the teaching and learning of the scalar product. While we stumbled across quite a few scholarly treatments of the teaching and learning of the vector (cross) product, the literature has no work (or nothing easily found) on the scalar (dot) product [1,2,3]. We contend that this gap in the literature exists because the academic teacher's view of the topic is that it is so easily understood and so devoid of complexity that there is little to say. In this paper we show empirically that a view of the scalar product as unproblematic is not correct and that students do struggle with using the scalar product in processes considered straightforward by the teacher or lecturer. We conclude with some ideas about why the students might find the scalar product unexpectedly challenging and offer suggestions for teachers of vector algebra and geometry.

## Research methodology

### *Instrument and cohort*

A test instrument was designed to assess proficiency at the vector algebra and geometry topics covered in the first-year mathematics course. This test was written in two consecutive semesters by the students registered for the Mechanical Engineering dynamics course for students in their second academic year of study at the University of Cape Town. Ethics approval was sought and obtained for the running of this study. The first and second semester cohorts consisted of 71 students and 129 students respectively, of whom 63 and 107 respectively gave consent for their data to be used in the analysis reported in this paper (Tables 1 and 2).

**Table 1. Student numbers by Engineering programme**

| | Mech Eng | Mechatronics | Elec-Mech | Electrical | Totals |
|---|---|---|---|---|---|
| Semester 1 | 65 | 6 | 0 | 0 | 71 |
| Semester 2 | 98 | 5 | 25 | 1 | 129 |
| Totals | 163 | 11 | 25 | 1 | 200 |

**Table 2. Students stating consent for data use**

| | Gave consent | Withheld consent | |
|---|---|---|---|
| Semester 1 | 63 | 8 | 71 |
| Semester 2 | 107 | 22 | 129 |
| | 170 | 30 | 200 |

Two of the students (both Semester 2 students) who gave consent for their data to be included in the analysis performed so well on the test that their results were excluded by the statistical software as the test was a poor fit for students of their level of proficiency. Therefore the data set finally used in the analysis discussed in this paper consisted of 168 students.

The first semester cohort saw an assessment instrument of 29 items. Item 13 in that instrument was not well constructed and has been removed from the data analysis. In the second semester, the students saw an instrument of 31 items, where Items 1-12 and 14-29 were identical to the first instrument, Item 13 was a better constructed version of the original Item 13 and Items 30 and 31 were new.

The items were chosen to represent a variety of typical vector topics encountered in first-year mathematics, including using the scalar product and the vector product, doing basic vector arithmetic, reading vector information off diagrams, calculating moduli, working with unit vectors, working with lines and planes and other geometric objects, such as spheres, and solving for parameters in vector parametric expressions. In this paper we discuss in detail the student responses to two of the scalar product items. We begin by briefly defining and contextualising the scalar product as encountered in a typical first-year mathematics course.

***The scalar product (or dot product)***

Given two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, the projection of $\vec{a}$ onto $\vec{b}$ (or, alternately, the component of $\vec{a}$ in the direction of $\vec{b}$ ) can be calculated using trigonometry and vector scaling to give:

$$\frac{|\vec{a}|\cos\theta}{|\vec{b}|}\vec{b} = \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{b}|^2}\vec{b}$$

where $\theta$ is the angle between the two vectors [4,5]. It is geometrically valuable to understand the factor $|\vec{a}|\cos\theta$ as "the amount" of $\vec{a}$ in the direction of $\vec{b}$ and hence the product $|\vec{a}|\cos\theta \cdot \frac{\vec{b}}{|\vec{b}|}$ turns that "amount" of $\vec{a}$ into a vector quantity. By defining the scalar product as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

the projection expression above becomes

$$\frac{|\vec{a}|\cos\theta}{|\vec{b}|}\vec{b} = \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{b}|^2}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}.$$

Using the law of cosines we can extrapolate from the definition that

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

a calculation which is easy to remember and to perform. In fact, this form is so convenient that it is often remembered as the definition of the scalar product. Indeed, it is provided as the definition of the scalar product in many textbooks [4].

Use of the scalar product as part of a larger process, or embedded in a context, is usually related to angles, either determining an angle between two vectors, or imposing a condition of orthogonality on a system of vectors since the scalar product of two orthogonal vectors is zero.

There are eight items in the assessment requiring use of the scalar product (see Appendix A). Two of those items simply provide two vectors and ask for their product (Items 1 and 14) and the other six require use of the scalar product as part of a larger process (Items 6, 7, 13, 21, 22 and 23). Items 6 and 23 ask for the component of a vector in the direction of another, both given. Item 21 asks for the distance between a point and a plane, requiring a self-chosen vector to be broken into components. Items 13 and 22 ask for the angle between two vectors, in both cases vectors which are not given directly and have to be determined from provided information. Item 7 requires use of the scalar triple product. We employed the Rasch measurement model to engage statistically with the items, to determine which items were found to be the most difficult and how students at different levels of proficiency responded to the items.

***The Rasch measurement method***

The Rasch measurement model is based on the requirement that measurement in the social sciences, including education, should aspire to the rigour of measurement instruments in the sciences, such as thermometers or rulers. The model originates with Georg Rasch [6] and has been discussed in detail elsewhere. [7, 8]

The first step in our process was to construct an instrument measuring a single construct of interest, in our case vector mechanics. The second was to administer the instrument to the study group, in our case second-year engineering students. The data were analysed using RUMM2030 software. [9] The items on the test were all multiple choice and were all on the single topic of vector mechanics. An important feature of this measurement model is that the "construct of interest" be unidimensional, that is that the instrument is asking questions all centred around a single topic.

***Results and discussion***

The Rasch analysis software (RUMM2030) allows the data analyst to understand an assessment instrument in many different ways. Four outputs of the software will be included in this paper, namely (1) the fit statistics, (2) the item map, (3) the item characteristic curves for two of the items and (4) the multiple choice distractor curves for the same two items. To augment the statistical analysis, we include data from the students' rough working of the multiple choice items under scrutiny in a bid to understand where and how things are going wrong.

The fit statistics provide measures of the unidimensionality of the instrument under scrutiny and the fit of the instrument to the group responding to the instrument. The requirement behind the model is that a unidimensional property is being measured and that the difficulty level of the items will remain invariant across different cohorts and sub-cohorts. Our study is young, having been run in only two cohorts to date, one of them small, so the data are still being collected to measure robust invariance; however the fit statistics at this point in our journey suggest that the instrument is measuring a unidimensional property and that the instrument is an adequate fit to the cohorts. For those accustomed to reading such statistics we include some in Table 3. For those not familiar with such fit statistics, simply note the "acceptable" rating of unidimensionality, meaning that this test instrument was measuring a single construct of interest (vector mechanics) at an "acceptable" level.

**Table 3. Fit statistics for the current analysis**

| Item fit residual | | Person fit residual | | Chi Square interaction | | PSI (w/o extr.) | Unidimensionality t-tests (95%CI) |
|---|---|---|---|---|---|---|---|
| Mean | SD | Mean | SD | Value (df) | p | | |
| -0.167 | 1.184 | -0.188 | 0.765 | 145.07 (96) | 0.00092 | 0.68736 | 2.7% (acceptable) |

One output of the Rasch analysis process is an axis (the item map) which locates on the left individuals along a continuum of proficiency with the construct of interest and also on the right the items along a continuum of difficulty (see Figure 1). An individual (marked on Figure 1 with the × symbol) at the same point on the axis as an item indicates that a student of that level of proficiency has a 50% chance of answering an item of that level of difficulty correctly. If the item is located lower than the student on the axis, then it is easier for the student, and if the item is located higher than the student then the item is more challenging. For illustration, Figure 1 indicates that items 14 and 1 (marked as I0001.1 and I0014.1) were very easy for all the students. Items 10 and 15 were of moderate difficulty (50% chance of getting them correct) for the student with the lowest level of proficiency, that student identified as the lowest × on the left hand side of the axis.

Of particular interest in this paper is the clustering of the items requiring use of the dot, or scalar, product. The two items simply asking for the calculation of a scalar product are the two easiest items on the test. The other six items requiring use of the scalar product are six of the eight most difficult items of the test, as determined by Rasch analysis. We can contrast this bimodal behaviour to that of the items involving the vector product. The items involving the vector product are spread throughout the central portion of the continuum, more challenging than the "basic" scalar product and less challenging than the process-oriented scalar product items. What's more, the "basic" vector product items (2 and 10) are themselves scattered among the spread of more process-oriented vector product items.

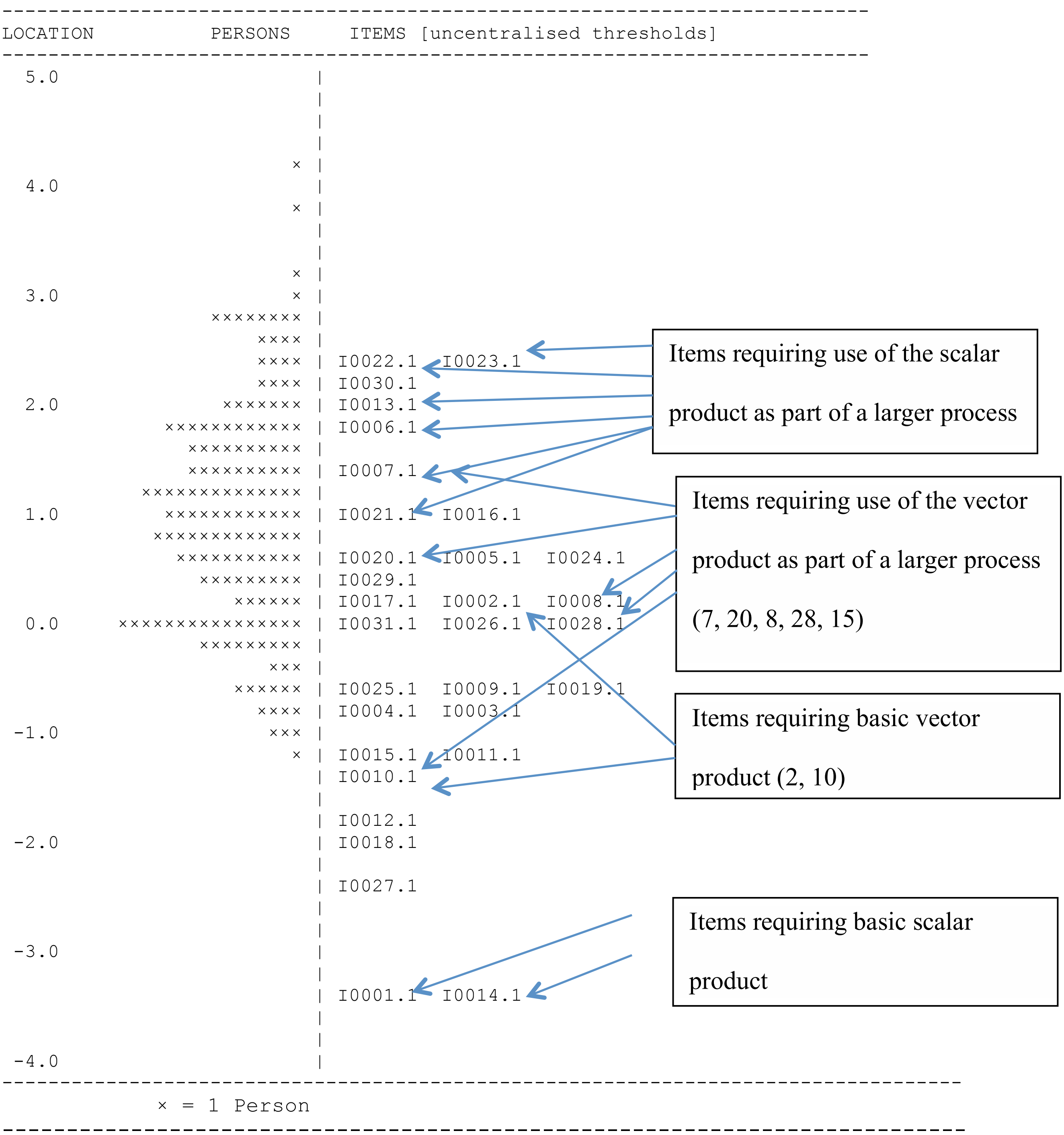



**Figure 1. The item map**

The fact that all of the items requiring use of the scalar product (other than simple calculations) are among the most difficult items on the assessment focusses attention on those items. Item 23, in particular, was found to be the most difficult item on the test by the students being assessed. Since Items 6 and 23 asked for the same process to be carried out, we looked more closely at these two items. In this paper we shall focus on items 6 and 23 (below, correct answers underlined). See Appendix B for solutions of these two items.

6. Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to $\vec{b} =< 1,3,4 >$. The parallel component is

(A) $< \frac{1}{2}, \frac{3}{2}, 2 >$ (B) $< 3, 9, 12 >$ <u>(C) $< -2,-6,-8 >$</u>

(D) $< -1,-3,-4 >$ (E) $< 1, 3, 4 >$

23. Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$. The parallel component is:

<u>(A) $8\hat{i} - 4\hat{j} + 12\hat{k}$</u> (B) $2\hat{i} - \hat{j} + 3\hat{k}$ (C) $-2\hat{i} + 1\hat{j} - 3\hat{k}$

(D) $4\hat{i} - 2\hat{j} + 6\hat{k}$ (E) $6\hat{i} - 3\hat{j} + 9\hat{k}$

For each of the item analyses below, we have included the item characteristic curve (the ICC) for each item as well as the MCQ distractor curves. The ICC positions a dot for each student proficiency quartile (located and indicated on the horizontal proficiency axis) at the probability level (vertical probability axis) at the collective probability (or expectation value) of students in that quartile answering the question correctly. In an ideal world where every student responds perfectly according to his or her proficiency level, those dots would lie along the indicated logistic curve. In an utterly random world where every student guessed the answer, the dots would lie horizontally at the probability level of 0.2 (if, as in our case, there are 5 distractors per question). The MCQ distractor curves show what expectation value each quartile indicated of answering each specific distractor (where A=1, B=2, and so on) to the questions.

***ICC and MCQ distractor graphs for Item 23***

"Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$. The parallel component is:" [Item 23]

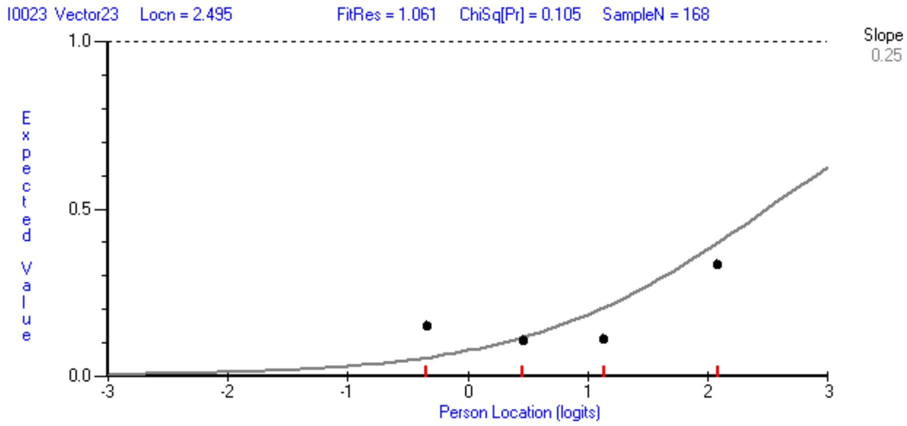



**Figure 2**. The ICC for Item 23.

The spread of the quartile responses (the quartiles' proficiencies are indicated by the small red bars on the horizontal axis) indicates some guessing, with the lowest quartile getting the answer correct more often than expected and highest quartiles less often (Figure 2).

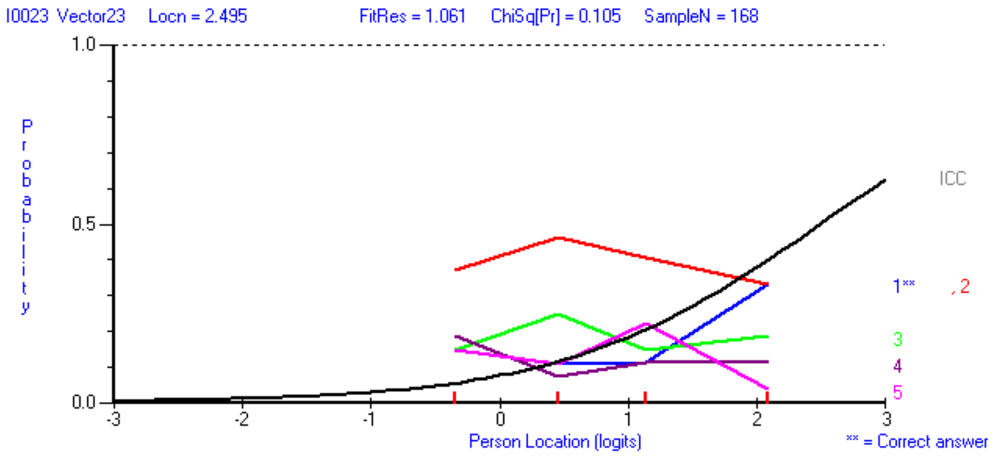



**Figure 3. The MCQ distractor graph for Item 23**

While A is the correct answer (shown as "1" and the blue line on the diagram above; also note the two little stars indicating 1 as the correct answer on the right hand side of the graph), option 2 (the red line) is more likely to be chosen by every quartile, with equal likelihood indicated for the top quartile (Figure 3). Option 2 (B) is the second vector ( $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$ ) given in the question, suggesting either that students chose it as a likely looking guess, or that they believe that the component of vector $\vec{v}$ in the direction of vector $\vec{u}$ is the entirety of vector $\vec{u}$ .

***ICC and MCQ distractor graphs for Item 6***

"Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to $\vec{b} =< 1,3,4 >$. The parallel component is:" [Item 6]

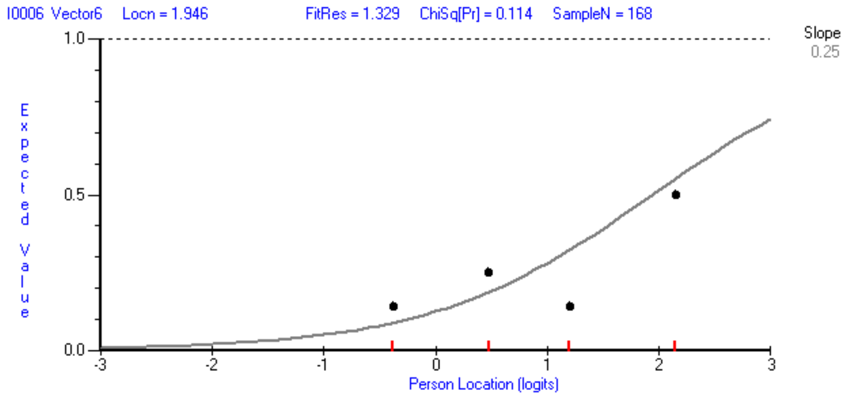



**Figure 4. The ICC for Item 6**

This item's ICC (Figure 4) shows a roughly similar guessing look to that of Item 23's ICC, with the lower two quartiles answering it correctly more often than expected and the top two quartiles less often. Nevertheless, this item was answered slightly better than Item 23.

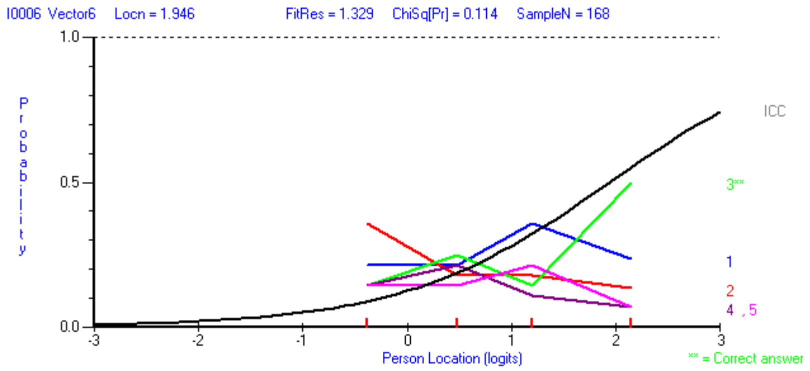



**Figure 5 .The MCQ distractor curves for Item 6**

In Figure 5, the correct answer of C (labelled 3 on the graph, the green line) is the most likely answer for the top quartile, but all of the distractors are contenders throughout, particularly A (1, blue) and B (2, red). The phenomenon noticed for Item 23, that of the second vector given in the question being the answer most popular with the students, is not observed here in Item 6; that option would be E (5, pink, on the graph above).

*Types of errors*

In order to understand what the students were doing when working on these items, we turned to looking through their rough work. Some students did their rough work in the margins of the question paper, others did it in a booklet provided and collected by the tutors. Since there was no stipulated requirement to make working apparent, in only a few cases (see Table 4) could working be identified for these two items.

**Table 4. Number of identifiable instances of rough work**

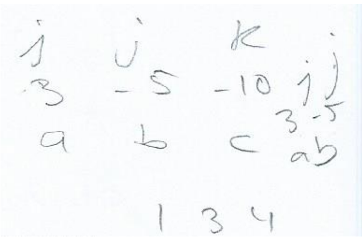
| | On question paper | In booklet | Totals |
|---|---|---|---|
| Item 6 | 31 | 3 | 34 |
| Item 23 | 2 | 8 | 10 |
| Totals | 33 | 11 | 44 |

One interesting feature is that far less working was done for Item 23 than Item 6. One hypothesis is that the students recognised Item 23 as being very similar to Item 6, encountered earlier in the paper. Having found Item 6 to be challenging possibly led them to not attempt Item 23 with as much diligence. To test this hypothesis we shall swap the order of these two items in the next iteration of this instrument.

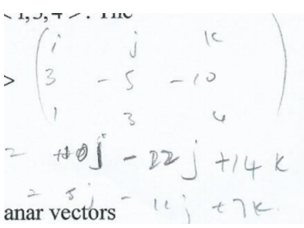
Some of the rough work was so minimal that it is impossible to see where the student made an error, for example simply writing a relevant formula and writing nothing further. In other cases, however, sufficient working was available to see patterns emerge across the cohort. Four types of error were identified, namely (1) using the vector product, (2) drawing the vectors in 3-dimensions and failing to find that helpful, (3) being confused as to how to use the scalar product and (4) weak skill at doing basic arithmetic.

Two constraints prohibit us from inferring frequency of error from availability of rough work evidence. Firstly, rough work is not available for the majority of the students, for a variety of reasons, and secondly, in several of the cases where rough work does provide interesting information the student has not given consent for us to publicise the data. Consequently, the examples given below are merely illustrative, not exhaustive and do not allow us to discern which errors have greater impact.

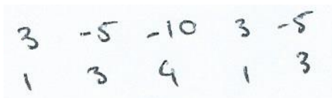
The students whose working is shown in Figure 6 are carrying out a vector product on the two given vectors, which indicates a lack of understanding of how to interpret the result of a vector product as well as a lack of knowledge of what process to use for the items in question.



Student 20152043



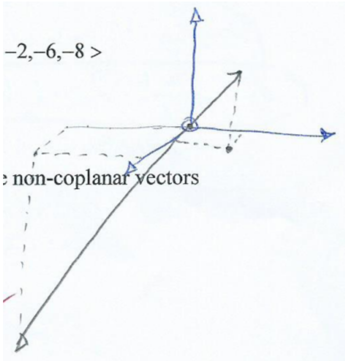
Student 20152026



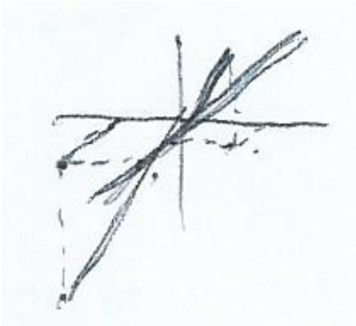
Student 20152023

**Figure 6. Vector product error**

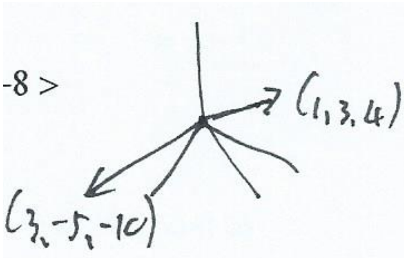
For exercises such as represented in Items 6 and 23, a two dimensional diagram serving as an aid to structuring the calculation correctly is as much as is needed. Students proficient with this type of problem might no longer need to draw any diagram as they can already “see” in their mind’s eye what the geometric implications of the problem are. Some students in their rough work drew 3-dimensional diagrams. Certain students did so and went on to correctly answer the questions (falling back on the necessary algebra after drawing the diagram). Three dimensional vector sketches are not inherently bad; however they are not directly useful either in this context. Any diagram, 3-d or 2-d, might have helped some students conceptualise the problem and gain clarity in what algebra to use. Figure 7 shows the diagrams of students who got no further in their working than a 3-d diagram and ultimately answered the question incorrectly.




Student 20152089



Student 20152030

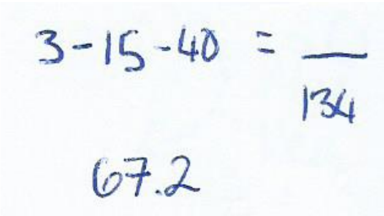



Student 20152111

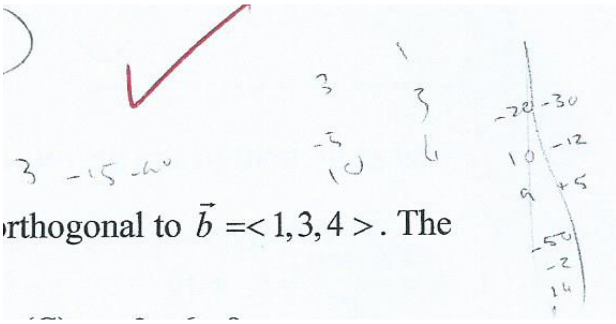
**Figure 7. 3-d diagrams with no further progress**

A third pattern in the types of error is failure to use the scalar product correctly. In Figure 8 below, Student 20152012 has carried out a scalar product and is using the modulus of one of the vectors, but has not put these otherwise useful pieces of information together in a useful way. (S/he seems to have calculated an angle in degrees as well, although the relevance of the 67.2 is unclear). Student 20152100 has carried out a scalar product and thereafter there is no apparent progress although the student seems to be trying out other calculations as well. Student 20152045 has rather worryingly carried out a version of the scalar product, yet has written the result in vector form, that is s/he has written

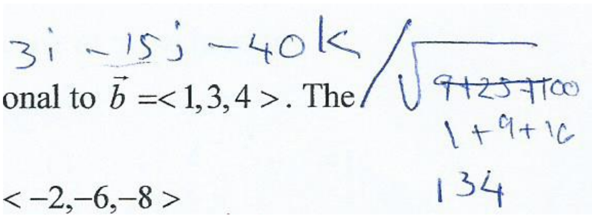
$(3)(1)+(-5)(3)+(-10)(4)$ as $(3)(1)\hat{i}+(-5)(3)\hat{j}+(-10)(4)\hat{k}$ . That student has then calculated the individual vector moduli but does not know what to do with them.




Student 20152012



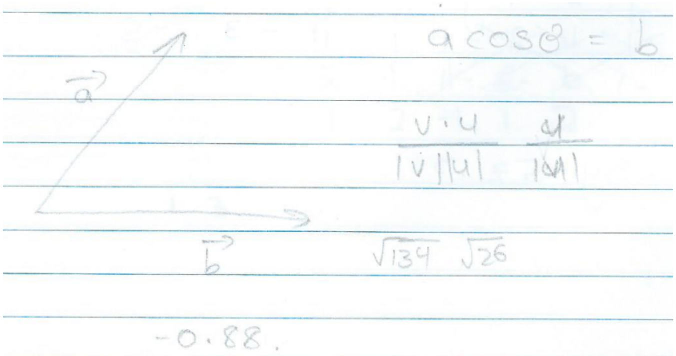

Student 20152100

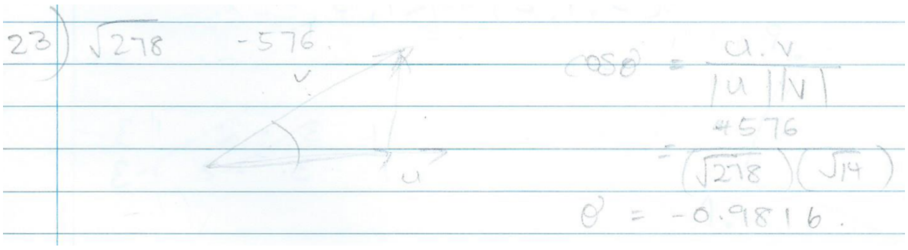



Student 20152045

**Figure 8. Problematic use of the scalar product**

Student 20152046 (Figure 9) was one of the few students for whom hand-written working was found for both of the items 6 and 23. S/he has used the scalar product correctly, but in each case has used it to calculate an angle, not the vector required by the problem. It is as if the student recognises the problem format as requiring the scalar product, but then uses the scalar product in the only form with which s/he is comfortable, that of solving for an angle.

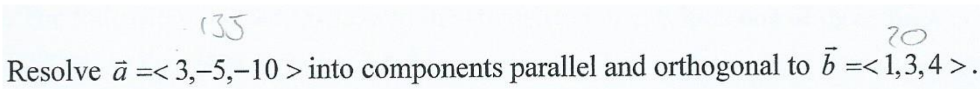


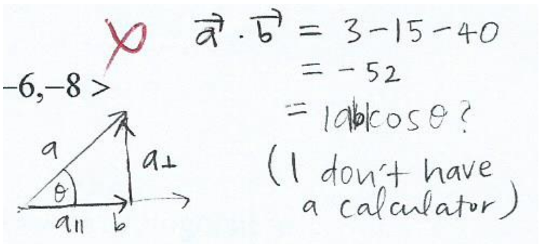

Student 20152046

**Figure 9. Problematic use of the scalar product, continued**

The final (and sadly commonplace) source of error is an inability to do error free mental arithmetic. In Figure 10 this finding is illustrated with one case of a student calculating moduli incorrectly (written over the vectors given) and another giving up since s/he did not have a calculator handy (for a calculation which did not strictly speaking demand one).


Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to $\vec{b} =< 1,3,4 >$.

Student 20152016




Student 20152037

**Figure 10. Inability to do arithmetic**

**Conclusion**

We ran an assessment based on first-year mathematics vector mechanics. The students being assessed were students registered for the second-year engineering course in dynamics. We analysed the data using the Rasch measurement method. Analysis of the results drew our attention to a number of interesting phenomena, one of which was the surprising difficulty of several items involving the use of the scalar (dot) product. Of those six challenging items, two were identical in the process demanded, which was the finding of the component of a vector in the direction of a second vector, or alternatively, finding the projection of a vector onto a second vector.

Recognising the importance of this process in engineering contexts, such as dynamics, we looked more closely at the data available and discerned several facts about these two items.

The item encountered first (6) was analysed as being easier than the second (23). One possibility is that Item 23 was avoided (and the answer was either omitted or guessed) as a result of the difficulty of Item 6 and hence Item 23's inherent difficulty was skewed, a hypothesis backed up by less rough working being found for the later item. A second possibility resides in the fact that the notation used in Item 23 (the unit vector notation – and the preferred notation in the dynamics course) was less familiar to the students than the parentheses notation in Item 6, backed up by a similar pattern being observed throughout the test on other item pairs. Future uses of versions of this assessment instrument will swap the order and we shall observe any consequences of that change. Our ongoing research will investigate the apparent lack of familiarity with the

The errors we observed in student rough work included several which suggest that the geometric interpretation of the scalar product is poorly grasped, for example using the vector product instead of the scalar product, drawing 3-dimensional diagrams when none are needed and using the scalar product to determine an angle when this result was not required.

As a result of the concerted efforts of the first-year mathematics lecturers, students in second-year engineering courses generally show great algebraic proficiency. However, when students encounter situations where physical problems are required to be represented using vectors they often encounter unexpected difficulties. Furthermore, in certain courses, such as second-year dynamics, students generally do not struggle to follow vector based solutions presented in class, but subsequently struggle to solve similar problem by themselves. It is a common refrain that students prefer to follow an algebraic approach because they struggle to 'see' the geometric interpretation.

We argue that the geometric role of the scalar product is understood weakly, if at all, by the majority of the students. This weak understanding is in spite of the geometric role of the scalar product being taught and demonstrated in the first-year mathematics course. The item map (Figure 1) suggests that the computational challenge of the vector product is greater than that of the scalar product, while using the vector product in a context is less challenging than using the scalar product in a context. Perhaps the very simplicity of the final form scalar product used for calculations results in an under appreciation of the geometric significance, whereas the greater complexity of the vector product provides a cognitive spur for engaging with the vector product's geometric interpretation.

We suggest to lecturers of first-year engineering and science mathematics that they strongly emphasise the geometric role of the scalar product, including exercises whose solution require geometric interpretation, and do not allow the simplicity of the arithmetic to upstage it. We suggest to lecturers of engineering or science courses which utilise the scalar product to be aware of the challenge which scalar product contexts might embody to students.

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**Appendix A Scalar product items in increasing order of difficulty**

The items are ordered here from easiest to most difficult, as determined through the analysis.

The correct answers are underlined.

1. Evaluate the dot product $<5,1,1>\cdot<3,7,-1>$.

(A) 32 (B) 16 (C) 63 <u>(D) 21</u> (E) 50

14. Evaluate the dot product of $8\hat{i}-\hat{j}+4\hat{k}$ and $3\hat{i}+6\hat{j}-2\hat{k}$.

<u>(A) 10</u> (B) 24 (C) 11 (D) 25 (E) 38

21. Find the distance from the point $P(3,1,2)$ to the plane $x+3y-2z=4$.

(A) $\frac{1}{7}$ (B) $\sqrt{14}$ (C) $\frac{1}{7}\sqrt{14}$ (D) $\sqrt{7}$ (E) $\frac{1}{7}\sqrt{2}$

7. Determine the volume of the parallelepiped defined by the three non-coplanar vectors

$\vec{m}=<2,1,1>$, $\vec{n}=<4,1,2>$, $\vec{p}=<-1,3,1>$.

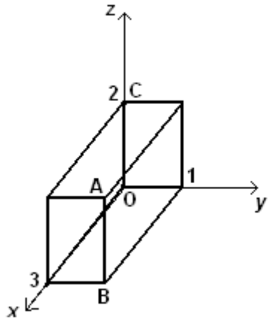
(A) $-3$ <u>(B) 3</u> (C) 8 (D) 2 (E) 5

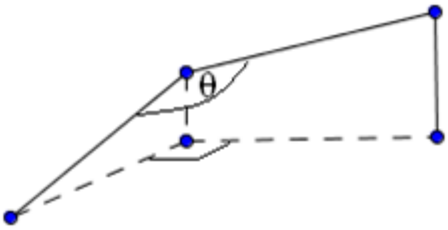
6. Resolve $\vec{a}=<3,-5,-10>$ into components parallel and orthogonal to

$\vec{b}=<1,3,4>$. The parallel component is

(A) $<\frac{1}{2},\ \frac{3}{2},2>$ (B) $<3,9,12>$ <u>(C) $<-2,-6,-8>$</u>

(D) $<-1,-3,-4>$ (E) $<1,3,4>$






13. Two water pipes are connected as shown in the diagram. The first pipe runs from south to north and rises up with a 20% grade. The second pipe runs from west to east and rises with a 10% grade. At the connection, determine the angle θ between the two pipes.

(A) 92.1° (B) 91.1° (C) 90° (D) 88.9° (E) 87.5°

22. What is the acute angle between the diagonals linking the corners A, B, C and O of the rectangular prism shown? (O is the origin.)

(A) 45° (B) 90° (C) 65° (D) 35° (E) 15°

23. Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$.

The parallel component is:

(A) $8\hat{i} - 4\hat{j} + 12\hat{k}$ (B) $2\hat{i} - \hat{j} + 3\hat{k}$ (C) $-2\hat{i} + 1\hat{j} - 3\hat{k}$

(D) $4\hat{i} - 2\hat{j} + 6\hat{k}$ (E) $6\hat{i} - 3\hat{j} + 9\hat{k}$

**Appendix B Solutions to Items 6 and 23**

6. Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to $\vec{b} =< 1,3,4 >$.

The parallel component is

(A) $< \frac{1}{2}, \frac{3}{2}, 2 >$ (B) $< 3, 9, 12 >$ (C) $< -2,-6,-8 >$

(D) $< -1,-3,-4 >$ (E) $< 1, 3, 4 >$

Let $\lambda\vec{b}$ be the parallel component and let $\vec{c}$ be the perpendicular component.

$$a = \lambda\vec{b} + \vec{c}$$

$$\vec{a}\cdot\vec{b} = \lambda\vec{b}\cdot\vec{b} = \lambda\left|\vec{b}\right|^2$$

$$\lambda\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\vec{b}$$

$$= \frac{3-15-40}{1+9+16} < 1, 3, 4 >$$

$$= -2 < 1, 3, 4 >$$

23. Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$. The parallel component is:

(A) $8\hat{i} - 4\hat{j} + 12\hat{k}$ (B) $2\hat{i} - \hat{j} + 3\hat{k}$ (C) $-2\hat{i} + 1\hat{j} - 3\hat{k}$

(D) $4\hat{i} - 2\hat{j} + 6\hat{k}$ (E) $6\hat{i} - 3\hat{j} + 9\hat{k}$

Let $\lambda\vec{u}$ be the parallel component and let $\vec{c}$ be the perpendicular component.

$$\vec{v} = \lambda\vec{u} + \vec{c}$$

$$\vec{v} \cdot \vec{u} = \lambda\vec{u} \cdot \vec{u} = \lambda|\vec{u}|^2$$

$$\lambda\vec{u} = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2}\vec{u}$$

$$= \frac{56}{14}\vec{u}$$

$$= 8\hat{i} - 4\hat{j} + 12\hat{k}$$