

17 Jan

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

FUNCTORIAL QUASI-UNIFORMITIES  
OVER PARTIALLY ORDERED SPACES

by

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A thesis prepared under the supervision of  
Professor G.C.L. Brümmer, in fulfilment of  
the requirements for the degree of Master  
of Science in Mathematics.

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1988

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ACKNOWLEDGEMENTS

I would like to thank my supervisor, Professor G.C.L. Brümmer, very much for all the help he gave me while I was writing this thesis; also, for his continuing interest in and encouragement of my work, and for the inspiring example he has been for me.

I wish also to express my gratitude to the Topology Research Group, through its chairman, Professor K.A. Hardie. This group, which is funded by the Foundation for Research Development, allowed me the use of its facilities, and provided a stimulating research environment in which I was able to learn much from both local and visiting mathematicians.

I further thank the Department of Mathematics, through its head, Professor R. I. Becker, for the excellent opportunities given me since I came here in 1986, and for a teaching assistantship from 1986 to 1988.

I am grateful to Mrs W.M. Fouquet for her fast and very efficient typing of this thesis.

I acknowledge a scholarship from the Foundation for Research Development for 1986 and 1987.

INTRODUCTION AND SUMMARY

Ordered spaces were introduced by Leopoldo Nachbin [1948 a, b, c, 1950, 1965]. We will be primarily concerned with completely regular ordered spaces, because they are precisely those ordered spaces which admit quasi-uniform structures. A recent and convenient study of these spaces is in the book by P. Fletcher and W.F. Lindgren [1982].

In this thesis we consider functorial quasi-uniformities over (partially) ordered spaces. The functorial methods which we use were developed by Brümmer [1971, 1977, 1979, 1982] and Brümmer and Hager [1984, 1987] in the context of functorial uniformities over completely regular topological spaces, and of functorial quasi-uniformities over pairwise, completely regular bitopological spaces. We obtain results which are to a large extent analogous to results in those papers. We also introduce some functors which relate our functorial quasi-uniformities to the structures studied by Brümmer and others (e.g. Salbany [1984]).

In more detail :

Definitions and terminology that will be assumed in the body of the thesis are given in Chapter 0.

Chapter 1 contains basic results about ordered topological spaces and the property of complete regularity in this context.

In Chapter 2 we give Brümmer's method for constructing right inverses (called "sections") of given forgetful functors. Applying this to a forgetful functor  $V: \underline{Quu} \rightarrow \underline{CTopOrd}$  (where Quu is the category of quasi-uniform spaces and CTopOrd the category of completely regular ordered spaces), we obtain functorial quasi-uniformities over ordered spaces.

Using functors first introduced by Salbany [1984], we obtain an embedding of the category of completely regular ordered spaces into the category of completely regular bispaces. We prove canonical properties of this embedding which are crucial in the sequel.

We establish functor diagrams similar to those used by Brümmer [1977, 1982], to investigate extensions and restrictions of  $V$ -sections to sections giving functorial quasi-uniformities over bispaces and sections giving functorial uniformities over topological spaces.

Chapter 3 concerns the quasi-uniform completion of functorial quasi-uniformities. In analogy with Brümmer and Hager's work [1984 and 1987], we define and characterize the

notion of a completion-true  $V$ -section. This is a section which, in a sense, commutes with the completion. Some special morphisms in the category of separated completely regular ordered spaces are characterized, in order to enable us to obtain epireflections induced by  $V$ -sections, in this category. We compare the resulting epireflective subcategories with the corresponding original ones in the completely regular topological spaces and bispaces. Extensions and restrictions of  $V$ -sections are investigated again, in connection with uniqueness and preservation of the completion-true property.

At the end of each chapter a section of notes giving acknowledgements and additional references will appear. Any results not acknowledged there, are, to the best of my knowledge, my own.

CHAPTER 0

PRELIMINARIES

We use mainly the terminology of [Herrlich and Strecker 1973] for categorical notions. The class of objects of a category  $X$  is written  $Ob X$ , and that of morphisms,  $Mor X$  (though we may write  $X \in X$  for  $X \in Ob X$  if no confusion can occur). If  $X$  and  $Y$  are objects of  $X$ , the set of morphisms from  $X$  to  $Y$  is denoted by  $X(X,Y)$ .

Let  $X$  be a category, and  $W: X \rightarrow \underline{Set}$  be a forgetful (i.e. faithful and amnesic) functor.

If  $f \in Mor X$ , we call  $Wf$  the underlying set map of  $f$ . Any functor between two categories which acts on morphisms by preserving the underlying set maps will be termed concrete.

Let  $W$  be faithful, and  $X_1, X_2 \in Ob X$ . We call  $X_1$  W-coarser than  $X_2$ , or  $X_2$  W-finer than  $X_1$ , (written  $X_1 \leq_W X_2$ ) iff there is a morphism  $h: X_2 \rightarrow X_1$  in  $X$  satisfying  $Wh = 1$  (where  $1$  is the appropriate identity morphism in Set). If  $F, G: Y \rightarrow X$  are functors, we write  $F \leq_W G$  iff  $FY \leq_W GY$  for all  $Y \in Y$ , and  $F <_W G$  iff  $F \leq_W G$  but  $FY \neq GY$  for some  $Y \in Y$ .

A source  $(f_\alpha: X \rightarrow X_\alpha)_{\alpha \in J}$  in  $X$  is called W-initial iff, given any source  $(g_\alpha: Y \rightarrow X_\alpha)_{\alpha \in J}$  in  $X$  and any morphism  $h$  in Set satisfying  $Wf_\alpha \cdot h = Wg_\alpha$  for all  $\alpha \in J$ , there exists a unique morphism  $k$  in  $X$  such that  $f_\alpha \cdot k = g_\alpha$  for all  $\alpha \in J$ , and  $Wk = h$ . A W-final sink may be defined dually.

Where it is clear what the functor  $W$  is, we may use the words "coarser" and "initial" instead of "W-coarser" and "W-initial".

A subcategory  $\mathcal{Y}$  of  $X$  is called initially dense in  $X$  iff for each  $X \in \text{Ob } X$ , the source  $X(X, \mathcal{Y})$ , where  $\mathcal{Y}$  ranges through the objects of  $\mathcal{Y}$ , is initial. The initial hull of a subcategory  $\mathcal{Y}$  of  $X$  consists of those objects  $Z$  of  $X$  for which the source  $X(Z, \mathcal{Y})$ , where  $\mathcal{Y}$  ranges through the objects of  $\mathcal{Y}$ , is initial. A finally dense subcategory and a final hull may be defined dually.

The functor  $W: X \rightarrow \text{Set}$  is called topological (resp. monotopological) iff, given any indexed class  $(X_\alpha: \alpha \in J)$  of  $X$ -objects and any source (resp. any monosource)  $(f_\alpha: S \rightarrow WX_\alpha)_{\alpha \in J}$  in Set, there exists a W-initial source  $(g_\alpha: X \rightarrow X_\alpha)_{\alpha \in J}$  in  $X$  satisfying  $Wg_\alpha = f_\alpha$  for all  $\alpha \in J$ . We call the pair  $(X, W)$ , or simply the category  $X$ , topological (resp. monotopological) iff, in addition to  $W$

being topological (resp. monotopological), we have that constant functions lift to morphisms in  $X$  and that the fibre  $W^{-1}S$  for any set  $S$  is small (i.e. it is a set, not a proper class).

The above terminology on initial sources and topological functors agrees with [Brümmer 1984a].

The table below lists some of the categories we will use. Some definitions and comments follow.

<u>Category</u>	<u>Objects</u>	<u>Morphisms</u>
<u>Set</u>	sets	functions
<u>Ord</u>	ordered sets	increasing functions
<u>Poset</u>	partially ordered sets	increasing functions
<u>Top</u>	topological spaces	continuous functions
<u>Creg</u>	completely regular topological spaces	continuous functions
<u>Tych</u>	Tychonoff spaces, i.e completely regular Hausdorff spaces	continuous functions
<u>Bitop</u>	bitopological spaces ("bispace")	bicontinuous functions
<u>Pcreg</u>	pairwise completely regular bispace	bicontinuous functions
<u>PTych</u>	pairwise Tychonoff bispace	bicontinuous functions

<u>Category</u>	<u>Objects</u>	<u>Morphisms</u>
<u>Unif</u>	uniform spaces	uniformly continuous functions
<u>Unif</u> <sub>0</sub>	separated uniform spaces	uniformly continuous functions
<u>CplUnif</u>	complete uniform spaces	uniformly continuous functions
<u>Quu</u>	quasi-uniform (quu) spaces	uniformly continuous functions
<u>Quu</u> <sub>0</sub>	separated quu spaces	uniformly continuous functions
<u>CplQuu</u>	complete quu spaces	uniformly continuous functions

By an ordered set  $(X, \leq)$  we mean a set  $X$  with an order (i.e. a reflexive and transitive relation) on it. A partially ordered set is an ordered set with an antisymmetric order relation. A function  $f: (X, \leq) \rightarrow (X', \leq')$  between ordered or partially ordered sets is called increasing (resp. decreasing) iff, for all  $x, y \in X$ ,  $x \leq y \Rightarrow f(x) \leq' f(y)$  (resp.  $f(y) \leq' f(x)$ ). A subset  $U$  of an ordered set is called increasing (resp. decreasing) iff  $x \in U, x \leq y \Rightarrow y \in U$  (resp.  $y \in U, x \leq y \Rightarrow x \in U$ ). The graph  $G_{\leq}$  of an order  $\leq$  on  $X$  is given by :

$$G_{\leq} = \{(x, y) \in X \times X \mid x \leq y\} .$$

We emphasize that by a completely regular topological space we do not necessarily mean one that is Hausdorff, but by a compact space, we do mean one with Hausdorff separation.

If  $\mathcal{P}$  and  $\mathcal{Q}$  are topologies on the same set,  $\mathcal{P} \vee \mathcal{Q}$  denotes the coarsest topology finer than both, i.e. the one with base  $\{P \cap Q: P \in \mathcal{P}, Q \in \mathcal{Q}\}$ .

We denote by  $\mathbb{R}$  the real numbers either with the usual topology, or simply regarded as a set (depending on the context). Similarly,  $\mathbb{I}$  denotes the closed unit interval  $[0,1]$  regarded as set or space.

For terminology on qu and bitopological spaces, we refer the reader to [Fletcher and Lindgren 1982], [Brümmer 1977, 1979 and 1982] and [Salbany 1970].

A bitopological space (or bispace) is a triple  $(X, \mathcal{P}, \mathcal{Q})$  where  $X$  is a set and  $\mathcal{P}$  and  $\mathcal{Q}$  are topologies on  $X$ . A bicontinuous function  $f: (X, \mathcal{P}, \mathcal{Q}) \rightarrow (X', \mathcal{P}', \mathcal{Q}')$  is a function between the underlying sets such that  $f: (X, \mathcal{P}) \rightarrow (X', \mathcal{P}')$  and  $f: (X, \mathcal{Q}) \rightarrow (X', \mathcal{Q}')$  are both continuous.

Let  $u$  be the upper topology on  $\mathbb{R}$  having as base  $\{U_a \mid a \in \mathbb{R}\}$ , where  $U_a = \{x \in \mathbb{R} \mid x < a\}$ . Similarly,

$l$  is the lower topology having base  $\{V_a \mid a \in \mathbb{R}\}$ , where  $V_a = \{x \in \mathbb{R} \mid x > a\}$ . We denote  $(\mathbb{R}, l, u)$  by  $\mathbb{R}_b$  and  $(\mathbb{I}, l, u)$  by  $\mathbb{I}_b$ . We note that our definition differs slightly from Brümmer's [1977, 1979 and 1982] in this respect: his  $\mathbb{R}_b$  and  $\mathbb{I}_b$  are  $(\mathbb{R}, u, l)$  and  $(\mathbb{I}, u, l)$  respectively; Salbany [1970 and 1984] uses the same spaces as Brümmer.

A bitopological space  $(X, P, Q)$  is called pairwise completely regular iff the source Bitop  $((X, P, Q), \mathbb{I}_b)$  is initial. This is equivalent to the following condition (see [Salbany 1970, Chp. 1, Prop. 2.2.2]): For each  $x_0 \in X$  and disjoint  $P$ -closed set  $P$ , there exists a bicontinuous  $f: (X, P, Q) \rightarrow \mathbb{I}_b$  such that  $f(x_0) = 1$  and  $f(x) = 0$  on  $P$ ; and for each  $x_0 \in X$  and disjoint  $Q$ -closed set  $Q$ , there exists a bicontinuous  $g: (X, P, Q) \rightarrow \mathbb{I}_b$  such that  $g(x_0) = 0$  and  $g(x) = 1$  on  $Q$ .

A bispace  $(X, P, Q)$  is pairwise Tychonoff iff it is pairwise completely regular and  $(X, P \vee Q)$  is a  $T_0$  space. (This then implies that  $(X, P, Q)$  is pairwise Hausdorff - see [Salbany 1970].)

A bispace  $(X, P, Q)$  is compact iff  $(X, P \vee Q)$  is compact. (Salbany [1970] calls these spaces "pairwise compact", and Brümmer [1977] calls them "S-compact".)

If  $(X, H)$  is a quu space, we denote by  $\tau(H)$  the topology on  $X$  having nbd system  $\{H[x] \mid H \in H\}$  for each  $x \in X$  (where  $H[x] = \{y \in X \mid (x, y) \in H\}$ ).  $H \vee H^{-1}$  is the coarsest uniformity finer than  $H$  and  $H^{-1}$ ; it has base  $\{H \cap H^{-1} \mid H \in H\}$ .

We define the concrete functors  $T, T_1$  and  $\bar{T}$  as follows:

- (i)  $T: \underline{\text{Unif}} \rightarrow \underline{\text{Creg}}$  is given by  $T(X, U) = (X, \tau(U))$ ,
- (ii)  $T_1: \underline{\text{Quu}} \rightarrow \underline{\text{Creg}}$  is given by  $T_1(X, H) = (X, \tau(H))$ ,
- (iii)  $\bar{T}: \underline{\text{Quu}} \rightarrow \underline{\text{Pcreg}}$  is given by  $\bar{T}(X, H) = (X, \tau(H), \tau(H^{-1}))$ .

A quu space  $(X, H)$  is separated iff  $\tau(H)$  is  $T_0$ , equivalently, iff  $\tau(H \vee H^{-1})$  is Hausdorff.

We call a quu space  $(X, H)$  complete iff  $(X, H \vee H^{-1})$  is a complete uniform space. (Fletcher and Lindgren [1982] call these spaces "bicomplete".) Note that completeness for uniform spaces is taken to imply Hausdorff separation. We denote the reflector to the complete spaces in  $\underline{\text{Quu}}$  by  $\bar{\gamma}$  (and the unit of the adjunction by  $\eta$ ). The reflector of  $\underline{\text{Unif}}$  to the complete uniform spaces will be called  $\gamma$ .

Let  $\mathbb{R}_q$  be the quu space with base  $\{H_\epsilon \mid \epsilon > 0\}$ , where  $H_\epsilon = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > x - \epsilon\}$ .  $\Pi_q$  is the quu subspace of  $\mathbb{R}_q$  with underlying set  $[0, 1]$ . Again, our definition is slightly different from the usual: Brümmer [1977] uses  $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y < x + \epsilon\}$ ,  $\epsilon > 0$ , as basic

entourages. We make these changes so that the spaces mentioned will fit naturally with the ordered spaces we will define later; the results concerning them are not significantly affected.

$\mathbb{R}_u$  and  $\mathbb{I}_u$  are  $\mathbb{R}$  and  $\mathbb{I}$  equipped with their usual uniformities.

The set-theoretic assumptions, and terminology about sets and classes, in [Herrlich and Strecker 1973, pp. 328-331] will suffice for all our needs.

At the end of the thesis is a diagram (Figure 1) illustrating the functors we will most often use.

CHAPTER 1

ORDERED TOPOLOGICAL SPACES

1. The topological category  $\text{TopOrd}$

Our reason for introducing the category  $\text{TopOrd}$  is to have a topological category as setting for investigating ordered spaces; and we wish to restrict this setting to those objects where the topology and the order are linked by the closed graph property. This property is especially attractive because any quasi-uniformity induces an order with a closed graph (as we shall see in 3.10).

1.1 Definition (1) An ordered topological space is a triple  $(X, \mathcal{T}, \leq)$  where  $X$  is a set,  $\mathcal{T}$  is a topology on  $X$ ,  $\leq$  is an order (i.e. a reflexive and transitive relation) on  $X$ , and the graph  $G_{\leq}$  of  $\leq$  is closed in  $(X, \mathcal{T}) \times (X, \mathcal{T})$ . (We may write  $X$  for  $(X, \mathcal{T}, \leq)$  if no confusion can occur.)

(2) The category  $\text{TopOrd}$  has all ordered topological spaces as objects and all continuous, increasing functions between them as morphisms.

(3) A partially ordered topological space is an ordered topological space with an antisymmetric order relation.

(4) The category TopPoset consists of all partially ordered topological spaces and continuous, increasing functions.

1.2 Remark Our "ordered topological space" is called a "topological space equipped with a closed preorder" by [Nachbin 1965]. What we call a "partially ordered topological space" is also called so by some other authors, e.g. [Salbany 1984] and [Künzi and Brümmer 1987]; but it is called a "topological space equipped with a closed order" by [Nachbin 1965], a "topological ordered space" by [Fletcher and Lindgren 1982], an "ordered topological space" by [Blatter 1975] and a "pospace" by [Gierz et al 1980]. Moreover, [Burgess and Fitzpatrick 1977], [Choe and Park 1979] and [Matolcsy 1984] use the term "ordered topological space" to denote a set with a topology and an unrelated partial order; and these objects are called "partially ordered topological spaces" by [Choe 1979] and [Choe and Garcia 1973].

1.3 Examples Let  $\mathbb{I}_0 = (\mathbb{I}, \text{usual topology, usual order})$  and  $\mathbb{R}_0 = (\mathbb{R}, \text{usual topology, usual order})$ .  $\mathbb{I}_0$  and  $\mathbb{R}_0$  are both clearly objects of TopPoset.

1.4 Lemma (1)  $(X, T, \leq) \in \text{TopPoset} \Rightarrow (X, T) \text{ Hausdorff}$ .  
(2)  $(X, T, \leq) \in \text{TopOrd}$  and  $(X, T) \text{ Hausdorff} \nRightarrow (X, T, \leq) \in \text{TopPoset}$ .

Proof: (1) [Nachbin 1965, Prop. 2].

(2) The two-point set  $X$  with the discrete topology and the indiscrete order (i.e.  $x \leq y$  for all  $x, y \in X$ ) provides a counter-example. □

1.5 Proposition TopOrd is a topological category.

Proof: The forgetful functor  $U: \text{TopOrd} \rightarrow \text{Set}$  is given by  $U(X, T, \leq) = X$ . For some index class  $J$ , let  $(X_\alpha, T_\alpha, \leq_\alpha)_{\alpha \in J} \in \text{TopOrd}$  and  $(f_\alpha: X \rightarrow X_\alpha)_{\alpha \in J}$  be a source in Set. We must find  $(X, T, \leq) \in \text{TopOrd}$  such that the source  $(f_\alpha: (X, T, \leq) \rightarrow (X_\alpha, T_\alpha, \leq_\alpha))_{\alpha \in J}$  is initial.

Let  $T$  be the topology making the source  $(f_\alpha: (X, T) \rightarrow (X_\alpha, T_\alpha))_{\alpha \in J}$  initial, and let  $\leq$  be the order defined by  $x \leq y \iff f_\alpha(x) \leq_\alpha f_\alpha(y) \forall \alpha \in J$ .  $\leq$  is clearly reflexive and transitive.  $G_\leq = \bigcap_{\alpha \in J} (f_\alpha \times f_\alpha)^{-1} G_{\leq_\alpha}$ , thus  $G_\leq$  is closed in  $(X, T) \times (X, T)$ . So  $(X, T, \leq) \in \text{TopOrd}$ .

Let  $(g_\alpha: (X', T', \leq') \rightarrow (X_\alpha, T_\alpha, \leq_\alpha))_{\alpha \in J}$  be a source in TopOrd, and  $h: X' \rightarrow X$  be a set map satisfying  $U f_\alpha \cdot h = U g_\alpha \forall \alpha \in J$ . We show that  $h: (X', T', \leq') \rightarrow (X, T, \leq)$  is increasing (that it is continuous is clear).

## 2.1 Construction of final sinks in TopOrd

Suppose that, for some index class  $J$ , we are given  $(X_\alpha, T_\alpha, \leq_\alpha) \in \text{TopOrd} \quad \forall \alpha \in J$  and a sink  $(f_\alpha: X_\alpha \rightarrow X)_{\alpha \in J}$  in Set. We must find  $(X, T, \leq) \in \text{TopOrd}$  such that  $(f_\alpha: (X_\alpha, T_\alpha, \leq_\alpha) \rightarrow (X, T, \leq))_{\alpha \in J}$  is final.

Let  $T$  be the topology making the sink  $(f_\alpha: (X_\alpha, T_\alpha) \rightarrow (X, T))_{\alpha \in J}$  final, i.e.  $U \subseteq X$  is open in  $T$  iff  $f_\alpha^{-1}(U)$  is open in  $T_\alpha$  for each  $\alpha \in J$ . Let  $G_0 = \{(x, y) \in X \times X \mid \exists \alpha \in J \text{ and } \exists a, b \in X \text{ such that } a \leq_\alpha b, f_\alpha(a) = x \text{ and } f_\alpha(b) = y\}$ . Then the graph of  $\leq$  is given by :

$G_\leq = \cap \{V \subseteq X \times X \mid G_0 \cup \Delta \subseteq V \text{ and } V \text{ is closed and transitive}\}$ , where  $\Delta$  is the diagonal of  $X \times X$  and by "closed" we mean closed in  $(X, T) \times (X, T)$ .  $G_\leq$  is clearly reflexive, transitive and closed.

We check that  $(f_\alpha: (X_\alpha, T_\alpha, \leq_\alpha) \rightarrow (X, T, \leq))_{\alpha \in J}$  is a final sink. Let  $(g_\alpha: (X_\alpha, T_\alpha, \leq_\alpha) \rightarrow (X', T', \leq'))_{\alpha \in J}$  be a sink in TopOrd for which there exists a set map  $h: X \rightarrow X'$  satisfying  $h \cdot U f_\alpha = U g_\alpha \quad \forall \alpha \in J$ . We show that  $h: (X, T, \leq) \rightarrow (X', T', \leq')$  is increasing, i.e. that  $G_\leq \subseteq (h \times h)^{-1} G_{\leq'}$ . (That  $h$  is continuous is clear, since  $T$  is the final topology.)

Since  $(h \times h)^{-1} G_{\leq'}$  is closed and transitive and contains  $\Delta$ , the result will follow if it contains  $G_0$ . Take  $(x, y) \in G_0$ .

Then  $\exists \alpha \in J$  and  $a, b \in X$  such that  $a \leq_{\alpha} b$ ,  $f_{\alpha}(a) = x$  and  $f_{\alpha}(b) = y$ .

$$\begin{aligned} \text{Now } a \leq_{\alpha} b &\Rightarrow g_{\alpha}(a) \leq' g_{\alpha}(b) \\ &\Rightarrow h(f_{\alpha}(a)) \leq' h(f_{\alpha}(b)) \\ &\Rightarrow (f_{\alpha}(a), f_{\alpha}(b)) \in (h \times h)^{-1} G_{\leq'} \\ &\Rightarrow (x, y) \in (h \times h)^{-1} G_{\leq'}. \quad \square \end{aligned}$$

2.2 Proposition TopPoset is finally dense in TopOrd.

Proof: Let  $(X, T, \leq) \in \text{TopOrd}$ . We must prove that the total sink  $S = (f: (\tilde{X}, \tilde{T}, \tilde{\leq}) \rightarrow (X, T, \leq) \mid (\tilde{X}, \tilde{T}, \tilde{\leq}) \in \text{TopPoset}, f \in \text{TopOrd}((\tilde{X}, \tilde{T}, \tilde{\leq}), (X, T, \leq)))$  is final.

First we show that  $T$  is the final topology for the given sink. We use the following result in [Bourbaki 1966, pp. 135-136]: Any topological space is a quotient of a Hausdorff space. So there exists a Hausdorff space  $(X', T')$  and a final surjection  $q: (X', T') \rightarrow (X, T)$ . Then  $(X', T', =) \in \text{TopPoset}$  and  $q: (X', T', =) \rightarrow (X, T, \leq)$  is continuous and increasing. Thus  $(X, T, \leq)$  has the final topology for one of the maps in the sink  $S$ , and so for the entire sink as well.

Next we show that  $\leq$  is the final order for the given sink. Let  $H = \{V \subseteq X \times X \mid G_0 \cup \Delta \subseteq V \text{ and } V \text{ is closed and transitive}\}$ , where  $G_0 = \{(x, y) \in X \times X \mid \exists \text{ a map } f: (\tilde{X}, \tilde{T}, \tilde{\leq}) \rightarrow (X, T, \leq) \text{ in the sink } S \text{ and } \exists a, b \in \tilde{X} \text{ such}$

that  $a \approx b$ ,  $f(a) = x$  and  $f(b) = y$ . We show that  $G_0$  is the graph of  $\leq$ , and then that  $G_0 = H$ .

Suppose  $x \leq y$ . Let  $(Y, T', \leq')$  be the two-point set  $\{a, b\}$  with the discrete topology and graph  $G_{\leq'} = \{(a, a), (b, b), (a, b)\}$ . Then  $(Y, T', \leq') \in \text{TopPoset}$ . The function  $f: (Y, T', \leq') \rightarrow (X, T, \leq)$  defined by  $f(a) = x$  and  $f(b) = y$  is in the sink  $S$ , so  $(x, y) \in G_0$ . Conversely,  $(x, y) \in G_0 \Rightarrow x \leq y$  (since each  $f$  is increasing). Thus  $x \leq y \iff (x, y) \in G_0$ .

Now  $G_0$  is reflexive and transitive (since  $\leq$  is) and  $G_0$  is closed (since it is the graph of  $\leq$ ). Thus  $G_0 = H$ , and the result now follows from 2.1. □

It is known that Poset is initially and finally dense in Ord ([Hoffmann 1979], [Alderton 1984, Prop. 2.2.2. and Prop. 2.2.4.]). However, we do not know whether TopPoset is initially dense in TopOrd or not.

### 3. Complete regularity in TopOrd.

For the remainder of this thesis we will be primarily concerned with the completely regular ordered topological spaces. This concept was introduced in [Nachbin 1965].

3.1 Definition (1) An ordered topological space  $X$  is called completely regular iff the source  $\text{TopOrd}(X, \mathbb{I}_0)$  is initial.

(2) The category CTopOrd is the full subcategory of TopOrd containing the completely regular ordered topological spaces.

(3) The category UTopOrd is the full subcategory of TopPoset containing the completely regular partially ordered topological spaces.

3.2 Remark Our "completely regular ordered topological spaces" are called "uniformizable preordered spaces" by [Nachbin 1965]. Our objects of UTopOrd are called "completely regular ordered spaces" by [Nachbin 1965], [Blatter 1975], [Choe 1979] and [Vrancken-Mawet 1985], "uniform ordered spaces" by [Fletcher and Lindgren 1982] and [Salbany 1984], "N-spaces" by [Lindgren and Fletcher 1979] and " $T_3$ -ordered spaces" by [Burgess and Fitzpatrick 1977]. (The latter authors use the term "completely regular ordered space" for a slightly weaker structure.)

In [Nachbin 1965] the completely regular ordered spaces are defined by the property in the following proposition.

3.3 Proposition  $(X, \tau, \leq) \in \text{CTopOrd}$  iff the following two conditions are satisfied :

(1) If  $a \in X$  and  $U$  is a nbd of  $a$  there exist continuous, real-valued functions  $f$  and  $g$  on  $X$ ,  $f$  increasing and  $g$  decreasing, such that (i)  $0 \leq f \leq 1$ ,  $0 \leq g \leq 1$ , (ii)  $f(a) = 1$ ,  $g(a) = 1$  and (iii)  $\inf \{f(x), g(x)\} = 0$  if  $x \in X - U$ .

(2) If  $a, b \in X$  and  $a \leq b$  is false, there exists a continuous, increasing, real-valued function  $f$  on  $X$  such that  $f(a) > f(b)$ .

Proof: This is implicit in [Salbany 1984]; the proof is analogous to the well-known proof that a topological space is completely regular iff it has the initial topology for its bounded, real-valued, continuous functions (see e.g. [Willard 1970, Thm. 14.12]). □

3.4 Definition We say  $(X, T, \leq) \in \text{TopOrd}$  admits (or is determined by) the quasi-uniformity  $H$  iff  $\tau(H \vee H^{-1}) = T$  and  $G_{\leq} = \bigcap H (= \bigcap \{H \mid H \in H\})$ .

The following characterization of CTopOrd is also well-known.

3.5 Proposition  $(X, T, \leq) \in \text{CTopOrd}$  iff  $(X, T, \leq)$  admits a quasi-uniformity.

Proof: [Nachbin 1965, Thm.9] □

3.6 Examples Clearly  $\mathbb{I}_q$  and  $\mathbb{R}_q$  determine  $\mathbb{I}_0$  and  $\mathbb{R}_0$  respectively. Thus  $\mathbb{I}_0$  and  $\mathbb{R}_0$  are members of CTopOrd.

3.7 Lemma Let  $(X, \mathcal{T}, \leq) \in \text{CTopOrd}$  with  $\mathcal{T}$  compact. Then  $(X, \mathcal{T}, \leq)$  admits a unique quasi-uniformity.

Proof: [Fletcher and Lindgren 1982, Thm. 4.21]. □

3.8 Lemma Let  $(X, \mathcal{T}, \leq) \in \text{CTopOrd}$ .

Then  $\leq$  is antisymmetric iff  $\mathcal{T}$  is Hausdorff, equivalently, iff  $\mathcal{T}$  is  $T_0$ .

Proof: See [Nachbin 1965, p.54]. □

3.9 Proposition (1) CTopOrd is a topological category.

(2) UTopOrd is a monotopological category.

Proof: (1) CTopOrd is defined as the initial hull of  $\{\mathbb{I}_0\}$  in TopOrd, so CTopOrd is a bireflective subcategory of TopOrd (since, over Set, initial hulls are bireflective hulls). Thus CTopOrd is topological, because any bireflective subcategory of a topological category over Set is topological.

(2) Similar to (1). □

We now define a functor  $V: \underline{Quu} \rightarrow \underline{CTopOrd}$  which will play an analogous role to  $\bar{T}: \underline{Quu} \rightarrow \underline{Pcreg}$ . We prove its most important property: its preservation of initial sources.

3.10 Definition of  $V$  Let  $V: \underline{Quu} \rightarrow \underline{CTopOrd}$  be the concrete functor given by  $V(X, H) = (X, \tau(H \vee H^{-1}), \cap H)$  for all  $(X, H) \in \underline{Quu}$ .

We check that  $V$  sends objects and morphisms of  $Quu$  to  $CTopOrd$ .

Take  $(X, H) \in \underline{Quu}$ .  $\cap H$  is easily seen to be reflexive and transitive. [Nachbin 1965, Prop.8] states that it is closed in  $(X, \tau(H \vee H^{-1})) \times (X, \tau(H \vee H^{-1}))$ . Thus  $V(X, H) \in \underline{TopOrd}$ ;  $V(X, H) \in \underline{CTopOrd}$  by Proposition 3.5.

Now take  $f: (X, H) \rightarrow (Y, K)$  a morphism in  $Quu$ . Then  $f: (X, H \vee H^{-1}) \rightarrow (Y, K \vee K^{-1})$  is uniformly continuous, so  $f: (X, \tau(H \vee H^{-1})) \rightarrow (Y, \tau(K \vee K^{-1}))$  is continuous. Also  $(f \times f)^{-1} \cap K = \cap (f \times f)^{-1} K$ , and since each  $(f \times f)^{-1} K$ ,  $K \in K$ , is an entourage of  $(X, H)$ , we have  $\cap H \subseteq (f \times f)^{-1} \cap K$ , i.e.  $f$  is increasing; and so  $V$  is a functor as claimed. □

3.11 Proposition  $V$  preserves initial sources.

Proof: Take an initial source  $(f_\alpha: (X, H) \rightarrow (X_\alpha, H_\alpha))_{\alpha \in J}$  in  $Quu$ . We must prove  $(f_\alpha: (X, \tau(H \vee H^{-1}), \cap H) \rightarrow (X_\alpha, \tau(H_\alpha \vee H_\alpha^{-1}), \cap H_\alpha))_{\alpha \in J}$  initial in  $CTopOrd$ .

It is easily seen that  $(f_\alpha: (X, H \vee H^{-1}) \rightarrow (X_\alpha, H_\alpha \vee H_\alpha^{-1}))_{\alpha \in J}$  is initial in Unif, and it is well-known that  $(f_\alpha: (X, \tau(H \vee H^{-1})) \rightarrow (X_\alpha, \tau(H_\alpha \vee H_\alpha^{-1})))_{\alpha \in J}$  is then initial in Creg (cf. [Kelley 1955, pp. 182-183]).

It remains to show that the order is initial, i.e. that  $\bigcap_{\alpha \in J} (f_\alpha \times f_\alpha)^{-1}(\bigcap H_\alpha) = \bigcap_{\alpha \in J} (f_\alpha \times f_\alpha)^{-1}(\bigcap \{K \mid K \in H_\alpha\})$ . Since the source  $(f_\alpha: (X, H) \rightarrow (X_\alpha, H_\alpha))_{\alpha \in J}$  is initial,  $H$  has subbase  $S = \{(f_\alpha \times f_\alpha)^{-1}K \mid K \in H_\alpha, \alpha \in J\}$ .

Thus  $\bigcap H = \bigcap S$

$$\begin{aligned} &= \bigcap_{\alpha \in J} (\bigcap \{(f_\alpha \times f_\alpha)^{-1}K \mid K \in H_\alpha\}) \\ &= \bigcap_{\alpha \in J} (f_\alpha \times f_\alpha)^{-1}(\bigcap \{K \mid K \in H_\alpha\}). \end{aligned}$$

□

#### 4. Notes

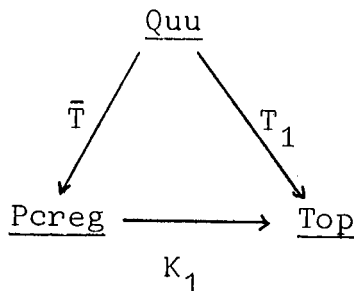
The definitions of TopOrd , (1.1), and the functor  $V$ , (3.10), were suggested by my supervisor; he also helped with the proof of Proposition 2.2, and with suggestions concerning the development of this chapter. By a happy coincidence Professor D. Pumplün proved in a lecture at the University of Cape Town, the result of Bourbaki's which we use in Proposition 2.2 (in fact he showed the stronger result that any topological space is an open quotient of a Hausdorff space).

CHAPTER 2

CONNECTIONS BETWEEN SPACES, BISPACES AND  
ORDERED SPACES

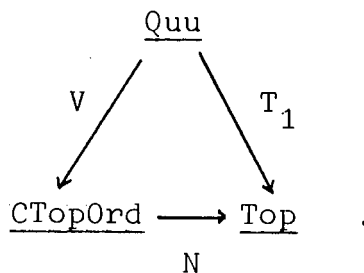
5. The first topology

We shall investigate connections between CTopOrd and Pcreg. To understand the possible settings for our work, it is important to note the following result, which contrasts with the simple fact that the triangle



commutes, where  $K_1$  is the functor which takes the first topology. (The functors  $\bar{T}$  and  $T_1$  are defined in Chapter 0.)

5.1 Proposition There is no functor  $N$  making this diagram commute:



Proof We exhibit two different topological spaces admitting two (obviously different) quasi-uniformities which are sent to the same ordered space by  $V$ .

For any  $(X, T) \in \underline{\text{Top}}$  the Pervin quu  $P$  on  $(X, T)$  (see [Fletcher and Lindgren 1982, Prop.2.1]) has subbase  $\{S_U \mid U \in T\}$ , where  $S_U = (U \times U) \cup ((X - U) \times X)$ . If  $x \in U$ ,  $(S_U \cap S_U^{-1})[x] = U$ , and if  $x \in X - U$ ,  $(S_U \cap S_U^{-1})[x] = X - U$ . Now  $\tau(P \vee P^{-1})$  is easily seen to be the Skula topology (see [Skula 1969], [Brümmer 1969, Prop. 4.1] and [Salbany 1970, Chp. 1, Prop. 3.5.2.4]) with base  $\{U_1 - U_2 \mid U_1, U_2 \in T\}$ .

Let  $\leq$  be the order with graph  $\cap P$ .

Then  $x \leq y \iff (x, y) \in (U \times U) \cup ((X - U) \times X) \quad \forall U \in T$   
 $\iff (x \in U \Rightarrow y \in U) \quad \forall U \in T$   
 $\iff y$  is in every  $T$ -nbd of  $x$   
 $\iff x \in \text{cl}_T\{y\}$ .

Hence  $\leq$  is the specialization order of  $T$ .

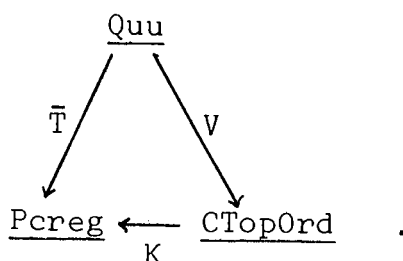
Consider now the space  $(\mathbb{R}, \text{usual topology})$ .

$T_1(\mathbb{R}, P) = (\mathbb{R}, \text{usual topology})$  and  $V(\mathbb{R}, P) = (\mathbb{R}, \tau(P \vee P^{-1}), \cap P)$ .

The Skula topology determined by the usual topology on  $\mathbb{R}$  is discrete (since every singleton is the intersection of an open and a closed set); and the specialization order of a Hausdorff space is discrete. Thus  $V(\mathbb{R}, P) = (\mathbb{R}, \text{discrete topology, discrete order})$ .

Consider also the space  $(\mathbb{R}, \text{discrete topology})$ . Let  $H$  be the discrete quu on  $\mathbb{R}$ . Then  $T_1(\mathbb{R}, H) = (\mathbb{R}, \text{discrete topology})$  and  $V(\mathbb{R}, H) = (\mathbb{R}, \text{discrete topology, discrete order})$ .  $\square$

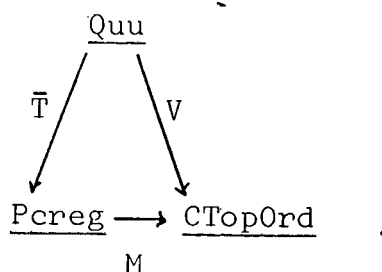
5.2 Corollary There is no functor  $K$  making this diagram commute:



Proof Suppose  $K: \text{CTopOrd} \rightarrow \text{Pcreg}$  satisfies  $KV = \bar{T}$ . Then  $K_1K$  satisfies  $K_1KV = K_1\bar{T} = T_1$ , contradicting Prop.5.1.  $\square$

### 6. Ordered spaces from bispaces

In [Charlton 1973] different ways of obtaining orders from bitopological spaces are given. Salbany [1984] uses one of these to define a functor:  $\text{Pcreg} \rightarrow \text{CTopOrd}$  having a right inverse. We change Salbany's definition slightly to obtain a unique functor  $M: \text{Pcreg} \rightarrow \text{CTopOrd}$  making the following diagram commute:



The further fact that  $M$  has a unique right inverse is very useful in the sequel.

6.1 Definition of  $M$  For any  $X \in \underline{\text{Pcreg}}$ , the source  $\underline{\text{Pcreg}}(X, \Pi_b)$  is initial. Define  $MX$  by giving  $MX$  the initial structure in  $\underline{\text{CTopOrd}}$  for the source  $\underline{\text{Pcreg}}(X, \Pi_b)$  to the object  $\Pi_0$  in  $\underline{\text{CTopOrd}}$ . Letting  $M$  act on morphisms by preserving set maps, one immediately has a functor  $M: \underline{\text{Pcreg}} \rightarrow \underline{\text{CTopOrd}}$ . □

We now describe  $M(X, P, Q)$  explicitly in terms of  $P$  and  $Q$ . We give a direct proof of the next lemma, which occurs in a different form in [Salbany 1970, Chp.1, Thm. 2.2.11].

6.2 Lemma Let  $(X, P, Q) \in \underline{\text{Pcreg}}$ . Then  $x \in \text{cl}_P\{y\} \iff y \in \text{cl}_Q\{x\}$ .

Proof " $\Rightarrow$ " Suppose  $x \in \text{cl}_P\{y\}$  but  $y \notin \text{cl}_Q\{x\}$ . By pairwise complete regularity there exists a bicontinuous map  $g: (X, P, Q) \rightarrow (\Pi, l, u)$  such that  $y \in g^{-1}(\{0\})$  and  $\text{cl}_Q\{x\} \subseteq g^{-1}(\{1\})$ . Now  $g(\text{cl}_P\{y\}) \subseteq \text{cl}_l\{g(y)\} = \text{cl}_l\{0\} = \{0\}$ . However,  $x \in \text{cl}_P\{y\} \Rightarrow g(x) = 0$ , which contradicts  $g(\text{cl}_Q\{x\}) = \{1\}$ .

" $\Leftarrow$ " Similar. □

6.3 Proposition For any  $(X, P, Q) \in \underline{\text{Pcreg}}$ ,  
 $M(X, P, Q) = (X, P \vee Q, \leq_0)$  where  $x \leq_0 y$  iff  $x \in \text{cl}_P\{y\}$ .

Proof Fix  $(X, P, Q) \in \underline{\text{Pcreg}}$ . Since the source  $S = \underline{\text{Pcreg}}$   
 $((X, P, Q), \Pi_b)$  is initial, so is  $(f: (X, P \vee Q) \rightarrow (\Pi, 1 \vee u) \mid$   
 $f \in \underline{\text{Pcreg}}((X, P, Q), \Pi_b))$ . It remains to check that  $\leq_0$  is  
the initial order for  $S$ , i.e. that  $x \leq_0 y \iff f(x) \leq f(y)$   
 $\forall f \in S$  (where  $\leq$  is the usual order on  $\Pi$ ).

" $\Rightarrow$ ": We show that if  $g: (X, P, Q) \rightarrow (X', P', Q')$  is  
bicontinuous, then  $g: (X, \leq_0) \rightarrow (X', \leq'_0)$  is increasing. Our  
result then follows, since  $\leq_0$  is clearly the usual order on  
 $\Pi$ . So, take  $g: (X, P, Q) \rightarrow (X', P', Q')$  bicontinuous and let  
 $x \leq_0 y$ . Suppose  $g(x) \not\leq'_0 g(y)$ , i.e.  $g(x) \notin \text{cl}_{P'}\{g(y)\}$ .  
Then there exists a  $P'$ -nbd  $U$  of  $g(x)$  such that  $g(y) \notin U$ ;  
and so  $g^{-1}(g(y)) \cap g^{-1}(U) = \emptyset$ . But then  $g^{-1}(U)$  is a  
 $P$ -nbd of  $x$  missing  $y$ , contradicting  $x \leq_0 y$ .

" $\Leftarrow$ ": Suppose  $x \not\leq_0 y$ . Then there exists  $U \in P$  such  
that  $x \in U$  but  $y \notin U$ . Since the source  $(f: (X, P) \rightarrow (\Pi, 1) \mid$   
 $f \in \underline{\text{Pcreg}}((X, P, Q), \Pi_b))$  is initial, there exist finitely  
many functions  $f_1, \dots, f_n$  of this source and finitely many  
sets  $U_1, \dots, U_n \in 1$  so that  $x \in \bigcap_{i=1}^n f_i^{-1}(U_i) \subseteq U$ . Now  
 $f_i(x) \in U_i$  for all  $i \in \{1, \dots, n\}$ , but there exists  
 $i_0 \in \{1, \dots, n\}$  such that  $f_{i_0}(y) \notin U_{i_0}$ . Then  
 $f_{i_0}(x) \leq f_{i_0}(y)$  since  $U_{i_0}$  is an increasing set w.r.t.  
the usual order on  $\Pi$ . □

6.4 Examples It is easily seen that  $M\Pi_b = \Pi_0$  and  $MIR_b = IR_0$ .

The following lemma was proved in [Charlton 1973, Cor. 3.2].

6.5 Lemma Let  $(X, P, Q) \in \text{Pcreg}$ . Then

- (1)  $U \in P \Rightarrow U$  is an increasing set w.r.t.  $\leq_0$ , and
- (2)  $U \in Q \Rightarrow U$  is a decreasing set w.r.t.  $\leq_0$ .

Proof (1) Suppose  $x \in U \in P$  and  $x \leq_0 y$ . Then  $x \in \text{cl}_P\{y\}$ , i.e. every  $P$ -open set containing  $x$  contains  $y$ . Thus  $y \in U$ .

(2) Similar. □

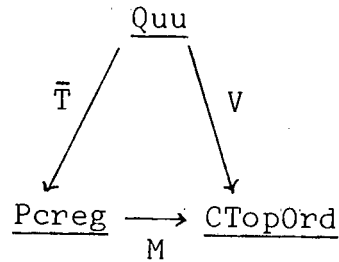
6.6 Proposition  $M$  preserves initial sources.

Proof Let the source  $(f_\alpha: (X, P, Q) \rightarrow (X_\alpha, P_\alpha, Q_\alpha))_{\alpha \in J}$  be initial. Then  $(f_\alpha: (X, P \vee Q) \rightarrow (X_\alpha, P_\alpha \vee Q_\alpha))_{\alpha \in J}$  is initial, and it remains to check that  $x \leq_0 y \iff f_\alpha(x) \leq_{0\alpha} f_\alpha(y) \forall \alpha \in J$ .

" $\Rightarrow$ ": Clear, since each  $f_\alpha$  is increasing.

" $\Leftarrow$ ": Similar to the proof of the initiality of the order in Proposition 6.3. □

6.7 Proposition The following diagram commutes :



Proof Fix  $(X, H) \in \text{Quu}$ . Then  $M\bar{T}(X, H) = M(X, \tau(H), \tau(H^{-1})) = (X, \tau(H \vee H^{-1}), \leq_0)$ , where  $x \leq_0 y$  iff  $x \in \text{cl}_{\tau(H)}\{y\}$ ; and  $V(X, H) = (X, \tau(H \vee H^{-1}), \cap H)$ . We must show that  $\leq_0 = \cap H$ .

" $\cap H \subset \leq_0$ " : Suppose  $x \not\leq_0 y$ . Then there exists a  $\tau(H)$ -open set containing  $x$  but not  $y$ . The collection  $\{H[x] \mid H \in H\}$  forms a nbd base at  $x$  in  $\tau(H)$ , so there exists  $H_0 \in H$  with  $x \in H_0[x]$  but  $y \notin H_0[x]$ . Thus  $(x, y) \notin H_0$  and so  $(x, y) \notin \cap H$ .

" $\leq_0 \subset \cap H$ " : Suppose  $(x, y) \notin \cap H$ . Then there exists  $H_0 \in H$  such that  $(x, y) \notin H_0$ , i.e.  $y \notin H_0[x]$ .  $H_0[x]$  is then a  $\tau(H)$ -nbd of  $x$  not containing  $y$ , so  $x \not\leq_0 y$ . □

Later (in Proposition 8.1) we will show that  $M$  is the only functor satisfying  $M\bar{T} = V$ .

## 7. Constructing sections of a functor

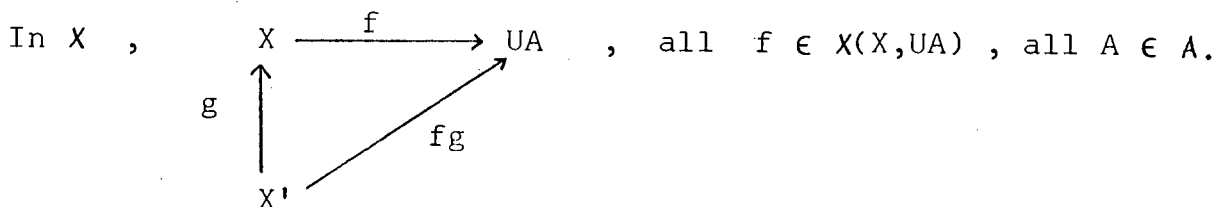
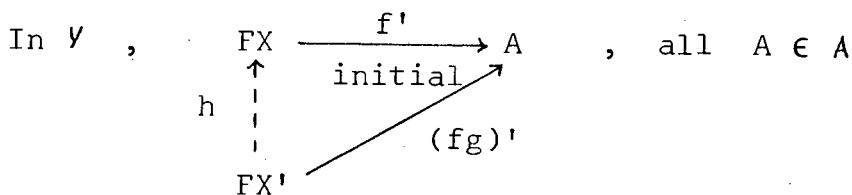
We introduce a spanning construction for creating right inverses to given forgetful functors, called "sections" of

these functors. The construction, which goes back to [Hušek 1967], has been developed in [Brümmer 1971] and [Brümmer 1976]. The term "spanning" was first used in [Brümmer 1979]. We give a fairly general description and indicate afterwards to which functors we will apply it.

7.1 Construction of a section of a given functor. Let  $X$  and  $Y$  be categories,  $U: Y \rightarrow X$  and  $E: X \rightarrow \underline{\text{Set}}$  be functors so that  $W = EU$  is topological and  $U$  preserves initial sources. Then we can define a functor  $F: X \rightarrow Y$  as follows:

Let  $A$  be a class of objects of  $Y$ . For any  $X \in \text{Ob } X$  there exists a unique  $W$ -initial source  $(f': B \rightarrow A)$  satisfying  $Wf' = Ef$ , where  $f$  ranges through  $X(X, UA)$  and  $A$  through  $A$ . Let  $FX = B$ . (We call  $(f': FX \rightarrow A)$  the source lifted from  $(f: X \rightarrow UA)$ .)

For any morphism  $g: X' \rightarrow X$  in  $X$ , consider the following diagrams :



The lower diagram commutes for all the given  $f$ . The source  $(f': FX \rightarrow A)$  is the one given by the definition of  $FX$ . The source  $((fg)': FX' \rightarrow A)$  consists of the maps  $fg: X' \rightarrow UA$  regarded as maps from  $FX'$  to  $A$ . They are morphisms in  $\mathcal{Y}$  by definition of  $FX'$ . Now the initiality of  $(f': FX \rightarrow A)$  gives us a unique morphism  $h: FX' \rightarrow FX$  satisfying  $Wh = Eg$ . We let  $Fg = h$ .

It is easily seen that  $F$  is a functor; we call it the functor spanned by  $A$  and write  $F = \langle A \rangle_U$ . A necessary and sufficient condition for  $\langle A \rangle_U$  to be a right inverse of  $U$ , will follow.

We will use the above construction in the following situations :

- (1)  $\mathcal{Y} = \underline{\text{Unif}}$  ,  $X = \underline{\text{Creg}}$  ,  $U = T$
- (2)  $\mathcal{Y} = \underline{\text{Quu}}$  ,  $X = \underline{\text{Pcreg}}$  ,  $U = \bar{T}$
- (3)  $\mathcal{Y} = \underline{\text{Quu}}$  ,  $X = \underline{\text{CTopOrd}}$  ,  $U = V$
- (4)  $\mathcal{Y} = \underline{\text{Pcreg}}$  ,  $X = \underline{\text{CTopOrd}}$  ,  $U = M$  .

The remaining propositions in this section are in the main special cases of results in [Brümmer 1976], so no proofs will be given here. It is of crucial importance that the functors  $T$ ,  $\bar{T}$ ,  $V$  and  $M$  preserve initial sources (see [Brümmer 1976, Thm. 2.3 and Prop. 2.5]).

7.2 Definition ([Brümmer 1976]) An object  $X_0$  of  $X$  is called a U-pivot iff  $\{X_0\}$  is initially dense in  $X$  and there is exactly one object  $Y_0$  in  $Y$  satisfying  $UY_0 = X_0$ .

7.3 Examples (1) That the compact space  $\mathbb{I}$  admits a unique uniformity is well-known (see e.g. [Bourbaki 1966, p.199]); thus  $\mathbb{I}$  is a T-pivot.

(2) By Lemma 3.7  $\mathbb{I}_0$  admits a unique  $quu$ ; so  $\mathbb{I}_0$  is a V-pivot.

(3) Salbany [1970, §4.5] shows that a compact, pairwise regular bispaces admits a unique  $quu$ ; thus  $\mathbb{I}_D$  is a  $\bar{T}$ -pivot.

(4) Since  $M\bar{T} = V$  and  $\mathbb{I}_0$  admits a unique  $quu$ , it also admits a unique pairwise completely regular space; so  $\mathbb{I}_0$  is an M-pivot.

7.4 Proposition ([Brümmer 1976, Thm. 3.5] and [Brümmer and Hager 1984, Prop.4]) Let  $A$  be a class of objects of  $Y$  and  $F = \langle A \rangle_U$ . The following are equivalent:

(1)  $F$  is a U-section, i.e.  $UF = 1$

(2)  $UA$  is initially dense in  $X$

In case a U-pivot  $X_0$  exists, these are also equivalent to:

(3)  $UFX_0 = X_0$ .

□

7.5 Proposition ([Brümmer and Hager 1984, Prop.5]) Any U-section  $F$  is spanned by its range  $R_F$ , i.e.  $F = \langle R_F \rangle_U$  where  $R_F = \{FX \mid X \in X\}$ . □

7.6 Proposition ([Brümmer 1976, Thm. 2.3]) Let  $F$  be any U-section and let  $M_F = \{Y \in Y \mid Y \leqslant FUY\}$ . Then  $M_F$  is the largest class spanning  $F$ . □

The following are examples of T-, V- and  $\bar{T}$ -sections.

7.7 Definition

$$\begin{array}{lll} C^* = \langle \{\Pi_u\} \rangle_T & C^{*\dagger} = \langle \{\Pi_q\} \rangle_V & \bar{C}^* = \langle \{\Pi_q\} \rangle_{\bar{T}} \\ C = \langle \{\mathbb{R}_u\} \rangle_T & C^\dagger = \langle \{\mathbb{R}_q\} \rangle_V & \bar{C} = \langle \{\mathbb{R}_q\} \rangle_{\bar{T}} \\ \phi = \langle \text{Ob } \underline{\text{Unif}} \rangle_T & \phi^\dagger = \langle \text{Ob } \underline{\text{Quu}} \rangle_V & \bar{\phi} = \langle \text{Ob } \underline{\text{Quu}} \rangle_{\bar{T}} \end{array}$$

7.8 Lemma ([Brümmer and Hager 1984, Example 9(3)])

(1)  $\phi = \langle \text{Ob } \underline{\text{CplUnif}} \rangle_T$

(2)  $\phi^\dagger = \langle \text{Ob } \underline{\text{CplQuu}} \rangle_V$

(3)  $\bar{\phi} = \langle \text{Ob } \underline{\text{CplQuu}} \rangle_{\bar{T}}$ . □

7.9 Proposition (1) ([Brümmer 1976, Prop. 3.12]) If  $UY_0$  is a U-pivot, then  $\langle \{Y_0\} \rangle_U$  is the coarsest U-section.

(2) ([Brümmer 1976, Cor. 2.7])  $\langle \text{Ob } Y \rangle_U$  is the finest U-section and is left adjoint to  $U$ . □

7.10 Corollary (1)  $c^*$ ,  $c^{*\dagger}$ ,  $\bar{c}^*$  and  $\langle \{\Pi_b\} \rangle_M$  are the coarsest T-, V-,  $\bar{T}$ - and M-sections, respectively.

(2)  $\phi$ ,  $\phi^\dagger$ ,  $\bar{\phi}$  and  $\langle \text{Ob } \underline{\text{Pcreg}} \rangle_M$  are the finest T-, V-,  $\bar{T}$ - and M-sections, and are left adjoint to T, V,  $\bar{T}$  and M respectively. □

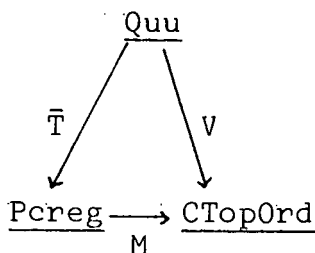
The next definition and proposition may be found in [Brümmer and Hager 1987, Defn. and Prop. 3.2].

7.11 Definition Let  $A \subseteq \text{Ob } \mathcal{Y}$  and let  $\text{init}(A)$  denote the initial hull (= bireflective hull) of  $A$ . Then the bireflector  $a: \mathcal{Y} \rightarrow \text{init}(A)$  determined by  $A$  is given as follows: Make the source  $(f: aY \rightarrow A)$ , where  $f$  ranges through  $\mathcal{Y}(Y, A)$  and  $A$  through  $A$ , W-initial. In other words,  $a$  is the reflector to the bireflective hull of  $A$  in  $\mathcal{Y}$ .

7.12 Proposition Let  $A \subseteq \text{Ob } \mathcal{Y}$ ,  $\langle A \rangle_U$  be a U-section and  $a$  be the bireflector determined by  $A$ . Then  $\langle A \rangle_U = a\langle \text{Ob } \mathcal{Y} \rangle_U$ . □

## 8. Bispaces from ordered spaces: A unique section

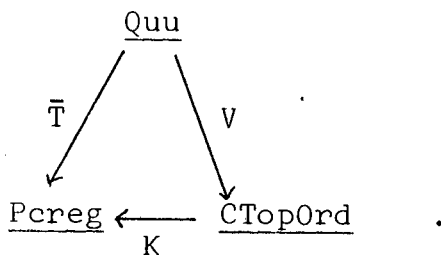
We return to the discussion of the diagram



Fix  $X \in \underline{\text{Pcreg}}$ . The source  $(f: \text{LMX} \rightarrow \mathbb{I}_b)$ , where  $f$  ranges through  $\underline{\text{CTopOrd}}(\text{MX}, \mathbb{I}_0)$ , is initial, as is the source  $\underline{\text{Pcreg}}(X, \mathbb{I}_b)$ . Since each function in the latter source is also in the former  $\text{LMX} \geq X$ ; thus  $\text{LM} \geq 1$ .

We give an example showing that  $\text{LM} \neq 1$ . Consider the closed unit interval  $\mathbb{I}$  with topology  $\mathcal{P}$  having basic open sets of the form  $(a, b]$ , and  $\mathcal{Q}$  having basic open sets of the form  $[c, d)$ . The quu space  $(\mathbb{I}, \mathcal{H})$  having base  $\{H_\epsilon \mid \epsilon > 0\}$ , where  $H_\epsilon = \{(x, y) \mid x - \epsilon < y \leq x\}$ , satisfies  $\bar{T}(\mathbb{I}, \mathcal{H}) = (\mathbb{I}, \mathcal{P}, \mathcal{Q})$ . Thus  $(\mathbb{I}, \mathcal{P}, \mathcal{Q}) \in \underline{\text{Pcreg}}$ . Now  $\text{LM}(\mathbb{I}, \mathcal{P}, \mathcal{Q}) = L(\mathbb{I}, \text{discrete topology}, \text{discrete order}) = (\mathbb{I}, \text{discrete topology}, \text{discrete topology}) \neq (\mathbb{I}, \mathcal{P}, \mathcal{Q})$ . □

8.5 Corollary  $\text{LV} > \bar{T}$ ; in fact there is no functor  $K$  making the following diagram commute:



Proof Since  $M\bar{T} = V$  and  $\text{LM} > 1$ ,  $\text{LV} = \text{LM}\bar{T} \geq \bar{T}$ . Note that the remainder of the statement was proved in Corollary 5.2; however, it is not necessary to invoke the functor  $T_1$ , as the following discussion shows. Let  $K: \underline{\text{CTopOrd}} \rightarrow \underline{\text{Pcreg}}$  be any functor. The space  $(\mathbb{I}, \mathcal{P}, \mathcal{Q})$  used in the previous proposition is different from the pcreg space  $(\mathbb{I}, \text{discrete topology}, \text{discrete topology})$ , so there are quu spaces

$(\mathbb{I}, H) \neq (\mathbb{I}, K)$  satisfying  $\bar{T}(\mathbb{I}, H) = (\mathbb{I}, P, Q)$  and  $\bar{T}(\mathbb{I}, K) = (\mathbb{I}, \text{discrete topology}, \text{discrete topology})$ . However  $V(\mathbb{I}, H) = V(\mathbb{I}, K) = (\mathbb{I}, \text{discrete topology}, \text{discrete order})$  (see proof of the previous proposition). Hence  $KV \neq \bar{T}$ .  $\square$

8.6 Proposition  $L$  is the unique right inverse of  $M$ , and  $L$  is left adjoint to  $M$ .

Proof By Corollary 7.10 (1),  $L$  is the coarsest  $M$ -section. It is also the finest: Let  $F$  be any  $M$ -section. Then  $L = L(MF) \geq F$  since  $LM \geq 1$ . Corollary 7.10 (2) shows that  $L$  is left adjoint to  $M$ .  $\square$

8.7 Proposition  $L$  is a full embedding of CTopOrd to a full, coreflective subcategory of Pcreg; the coreflector is  $LM$ .  $\square$

## 9. V-sections versus $\bar{T}$ -sections

We give a simple and useful relationship between  $V$ -sections and  $\bar{T}$ -sections, and then investigate extending  $V$ -sections to  $\bar{T}$ -sections.

9.1 Proposition Let  $A \subseteq \text{Ob } \underline{\text{Quu}}$ . Then  $\langle A \rangle_V$  is a  $V$ -section iff  $\langle A \rangle_{\bar{T}}$  is a  $\bar{T}$ -section, and in this case  $\langle A \rangle_{\bar{T}} \cdot L = \langle A \rangle_V$ .

Proof We show that  $\langle A \rangle_V$  is a V-section iff  $\Pi_q$  is a subspace of a product of objects of  $A$ . The proof of the first assertion is then complete, since the same characterization holds for  $\langle A \rangle_{\bar{T}}$  being a  $\bar{T}$ -section.

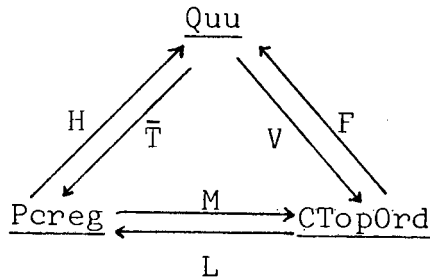
Suppose  $\langle A \rangle_V$  is a V-section. Then  $VA$  is initially dense in  $\underline{CTopOrd}$  and so the source  $\underline{CTopOrd}(\Pi_0, VA)$ , where  $A$  ranges through  $A$ , is initial.  $\Pi_0$  may therefore be embedded in a product of  $VA$ 's, say  $\Pi_0 \subseteq \prod_{\alpha \in J} VA_\alpha$ . Let the unit interval be given the subspace  $qu$  relative to  $\prod_{\alpha \in J} A_\alpha$ ; this  $qu$  has to be  $\Pi_q$ , since by Lemma 3.7  $\Pi_0$  admits a unique  $qu$ .

Conversely, suppose that  $\Pi_q \subseteq \prod_{\alpha \in J} A_\alpha$  for some spaces  $A_\alpha \in A$ . We show that  $VA$  is initially dense in  $\underline{CTopOrd}$ . Fix  $X \in \underline{CTopOrd}$ . Consider the source:  $X \xrightarrow{f} \Pi_0 \longleftrightarrow \prod_{\alpha \in J} VA_\alpha$   
 $\xrightarrow{\pi_\beta} VA_\beta$ , where  $f$  ranges through  $\underline{CTopOrd}(X, \Pi_0)$  and  $\pi_\beta$  through the usual projection maps. Since this source is initial, so is the entire source  $\underline{CTopOrd}(X, VA)$ , where  $A$  ranges through  $A$ .

Next we show that  $\langle A \rangle_{\bar{T}L} = \langle A \rangle_V$ . Since  $V = M\bar{T}$  and  $L$  and  $\bar{\phi}$  are left adjoints of  $M$  and  $\bar{T}$  respectively (see Corollary 7.10),  $\bar{\phi}L$  is left adjoint to  $V$  (see e.g. [Herrlich and Strecker 1973, Prop. 27.8]). Thus  $\bar{\phi}L = \phi^\dagger$ . By Proposition 7.12,  $\langle A \rangle_{\bar{T}} = a\bar{\phi}$  and  $\langle A \rangle_V = a\phi^\dagger$ , where  $a$  is the bireflector determined by  $A$ . Then  $\langle A \rangle_{\bar{T}L} = a\bar{\phi}L = a\phi^\dagger = \langle A \rangle_V$ .

□

We consider the diagram



where  $H$  and  $F$  are  $\bar{T}$ - and  $V$ -sections respectively.

- 9.2 Lemma (1) If  $H$  is a  $\bar{T}$ -section, then  $VH = M$ .  
 (2) If  $F$  is a  $V$ -section, then  $\bar{T}F = L$ .  
 (3) If  $F$  is a  $V$ -section, then  $FM$  is not a  $\bar{T}$ -section.

Proof (1)  $VH = (M\bar{T})H = M$  since  $\bar{T}H = 1$ .  
 (2)  $M(\bar{T}F) = VF = 1$ , so  $\bar{T}F$  is an  $M$ -section and uniqueness of  $L$  (Prop. 8.6) gives  $\bar{T}F = L$ .  
 (3) From Proposition 9.1 it is clear that there exists a  $\bar{T}$ -section  $H$  satisfying  $HL = F$ . Then  $\bar{T}(FM) = \bar{T}HLM = LM \neq 1$ . □

9.3 Lemma The  $V$ -section  $F$  factors through the  $\bar{T}$ -section  $H$  iff  $HL = F$ .

Proof If  $HK = F$  for some functor  $K$ , then  $K = (\bar{T}H)K = \bar{T}F = L$ . □

The lemma justifies the following terminology :

9.4 Definition We say the V-section  $F$  extends to the  $\bar{T}$ -section  $H$ , or  $H$  restricts to  $F$ , iff  $HL = F$ .

9.5 Examples  $\bar{C}^*$ ,  $\bar{C}$  and  $\bar{\phi}$  restrict to  $C^{*\uparrow}$ ,  $C^\uparrow$  and  $\phi^\uparrow$  respectively (by Prop. 9.1). (These functors were defined in Defn. 7.7.)

9.6 Proposition Let  $F = \langle A \rangle_V$  be a V-section.

(1) If  $H$  is a  $\bar{T}$ -section extension of  $F$ , then  $R_F \subseteq M_H \subseteq M_F$ . (These classes were defined in Props. 7.5 and 7.6.)

(2) Among the  $\bar{T}$ -sections to which  $F$  extends there is a finest, viz.  $\hat{F} = \langle M_F \rangle_{\bar{T}}$ ; and a coarsest, viz.  $\check{F} = \langle R_F \rangle_{\bar{T}}$ .  $\check{F}$  may also be expressed as  $\langle FV(A) \rangle_{\bar{T}}$ .

(3) If  $H: \text{Pcreg} \rightarrow \text{Quu}$  is a functor satisfying  $\check{F} \leq H \leq \hat{F}$ , then  $HL = F$ .

(4)  $M_{\hat{F}} = M_F$ .

(5) The class of all  $\bar{T}$ -sections to which  $F$  extends is a large-complete lattice under the ordering "coarser than" (though we make no claim that this class has more than one member).

Proof (1) By Proposition 9.1,  $\langle M_H \rangle_V = \langle M_H \rangle_{\bar{T}L}$ . However,  $M_H$  spans  $H$ , so  $\langle M_H \rangle_V = HL = F$ . Since  $M_F$  is the largest class spanning  $F$ ,  $M_H \subseteq M_F$ . To see that  $R_F \subseteq M_H$ , take any  $B \in \text{CTopOrd}$ , i.e.  $FB \in R_F$ .  $H\bar{T}(FB) = HLB = FB$ , so  $FB \leq H\bar{T}(FB)$ , i.e.  $FB \in M_H$ .

(2) Proposition 9.1 shows that  $\hat{F}$  and  $\underset{\vee}{F}$  are  $\bar{T}$ -sections. That  $\hat{F}$  is the finest, and  $\underset{\vee}{F}$  the coarsest,  $\bar{T}$ -section extending  $F$  is clear from (1). We show that  $\underset{\vee}{F} = \langle FV(A) \rangle_{\bar{T}}$ .  $\langle FV(A) \rangle_{\vee}$  is a  $V$ -section since  $V(FV(A)) = VA$  is initially dense in  $\underline{CTopOrd}$ .  $\langle FV(A) \rangle_{\vee} \geq \langle A \rangle_{\vee}$  since  $FVA \geq A$  and  $V(FVA) = VA$  for each  $A \in A$ . However  $\langle A \rangle_{\vee} = \langle R_F \rangle_{\vee}$ , and  $FV(A) \subset R_F$  implies that  $\langle FV(A) \rangle_{\vee} \leq \langle R_F \rangle_{\vee}$ . So  $\langle FV(A) \rangle_{\vee} = \langle A \rangle_{\vee}$ . Thus  $\langle FV(A) \rangle_{\bar{T}}$  is a  $\bar{T}$ -section extension of  $F$ , and is coarser than  $\langle R_F \rangle_{\bar{T}}$ ; hence it must equal  $\underset{\vee}{F}$ .

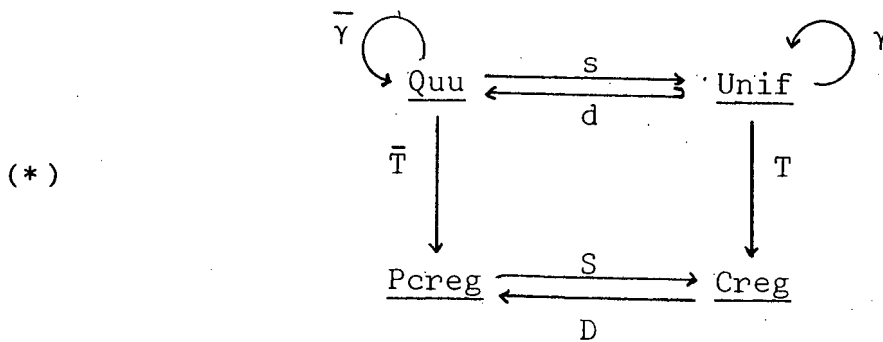
$$\begin{aligned} (3) \quad \underset{\vee}{F} \leq H \leq \hat{F} &\Rightarrow \underset{\vee}{FL} \leq HL \leq \hat{FL} \\ &\Rightarrow F \leq HL \leq F \\ &\Rightarrow HL = F. \end{aligned}$$

(4) By (1),  $M_{\hat{F}} \subseteq M_F$ . However,  $\hat{F} = \langle M_F \rangle_{\bar{T}}$  and  $M_{\hat{F}}$  is the largest class spanning  $\hat{F}$ , so  $M_F \subseteq M_{\hat{F}}$ .

(5) By (3), we may write the class of all  $\bar{T}$ -section extensions of  $F$  as  $C = \{H \mid \bar{T}H = 1 \text{ and } \underset{\vee}{F} \leq H \leq \hat{F}\}$ . It suffices to show that this class is closed under  $\leq$ -suprema. Let  $\{H_{\alpha} \mid \alpha \in J\}$  be a class of functors in  $C$ , for some index class  $J$ . Then it is easily verified that  $\langle \bigcup_{\alpha \in J} M_{H_{\alpha}} \rangle_{\bar{T}}$  is the supremum of the given  $H_{\alpha}$ 's. □

## 10. Comparing spaces and ordered spaces.

In [Brümmer 1977] and [Brümmer 1982] spaces and bitopological spaces are compared with the aid of the following functor diagram:



where  $T$  and  $\bar{T}$  are the usual forgetful functors,  
 $d$  is the natural inclusion functor,  
 $s$  is the concrete symmetrization functor given by  
 $s(X, H) = (X, H \vee H^{-1})$ ,  
 $D$  is the concrete functor given by  $D(X, T) = (X, T, T)$ ,  
 $S$  is the concrete functor given by  $S(X, P, Q) = (X, P \vee Q)$ ,  
 $\gamma$  and  $\bar{\gamma}$  are the reflectors onto the complete spaces.

10.1 Proposition [Brümmer 1977] (1)  $sd = 1$

(2)  $\bar{T}d = DT$

(3)  $Ts = S\bar{T}$

(4)  $d\gamma = \bar{\gamma}d$

(5)  $\gamma s = s\bar{\gamma}$  .

□

We now set up a similar square, replacing the category Pcreg by CTopOrd, and  $\bar{T}$  by  $V$ . We will obtain several results analogous to those of Brümmer, particularly in connection with extending T-sections to V-sections.

10.2 Definition of B Let  $B: \underline{CTopOrd} \rightarrow \underline{Creg}$  be the concrete forgetful functor given by  $B(X, T, \leq) = (X, T)$  for all  $(X, T, \leq) \in \underline{CTopOrd}$ .

We check that  $B$  sends objects of  $\underline{CTopOrd}$  to  $\underline{Creg}$ : Take  $(X, T, \leq) \in \underline{CTopOrd}$ . There exists  $(X, H) \in \underline{Quu}$  such that  $V(X, H) = (X, H \vee H^{-1}, \cap H) = (X, T, \leq)$ . Then  $(X, H \vee H^{-1}) \in \underline{Unif}$  and  $T(X, H \vee H^{-1}) = (X, T)$ . Thus  $(X, T) \in \underline{Creg}$ . That morphisms of  $\underline{CTopOrd}$  are mapped to morphisms of  $\underline{Creg}$  is clear.  $\square$

10.3 Definition of J Let  $J: \underline{Creg} \rightarrow \underline{CTopOrd}$  be the concrete functor given by  $J(X, T) = (X, T, cl\Delta)$ , where the relation  $cl\Delta$  is the closure of the diagonal in  $(X, T) \times (X, T)$ . We note that for  $(X, T) \in \underline{Tych}$ ,  $J(X, T) = (X, T, \Delta)$ .

We check that  $J$  sends objects and morphisms of  $\underline{Creg}$  to  $\underline{CTopOrd}$ : Fix  $(X, T) \in \underline{Creg}$ . There exists  $(X, H) \in \underline{Unif}$  such that  $T(X, H) = (X, T)$ . Regard  $(X, H)$  now as a quu space. Then  $V(X, H) = (X, \tau(H), \cap H) = (X, T, cl\Delta)$  (see [Bourbaki 1966, p.172]). Thus  $(X, T, cl\Delta) \in \underline{CTopOrd}$ .

Now let  $f: (X, T) \rightarrow (X', T')$  be a morphism in  $\underline{Creg}$ . We show that  $f: (X, T, cl\Delta) \rightarrow (X', T', cl\Delta')$  is increasing. Since  $f$  is continuous,  $(f \times f)(cl\Delta) \subseteq cl((f \times f)(\Delta)) \subseteq cl\Delta'$ . Thus  $(x, y) \in cl\Delta \Rightarrow (f(x), f(y)) \in cl\Delta'$ , as required.  $\square$

10.4 Proposition  $J$  is the finest right inverse of  $B$ , and is left adjoint to  $B$ .

Proof Clearly  $BJ(X,T) = (X,T)$ . Let  $F$  be any right inverse of  $B$ , and  $F(X,T) = (X,T, \leq_F)$ . Then  $G_{\leq_F} \supseteq \text{cl}\Delta$ , since  $(X,T, \leq_F) \in \text{TopOrd}$ .  $(X,T, \leq_F)$  is thus coarser than  $(X,T, \text{cl}\Delta)$ .

We check the adjunction using a universal mapping. Let  $(X,T) \in \text{Creg}$  and  $i: (X,T) \rightarrow BJ(X,T)$  be the identity map. Now let  $f: (X,T) \rightarrow B(X',T', \leq')$  be continuous for some  $(X',T', \leq') \in \text{CTopOrd}$ . Then  $Jf: (X,T, \text{cl}\Delta) \rightarrow (X',T', \leq')$  satisfies  $BJf \circ i = f$ .  $Jf$  is clearly unique.  $\square$

10.5 Proposition  $J$  is a full embedding of  $\text{Creg}$  to a full, coreflective subcategory of  $\text{CTopOrd}$ ; the coreflector is  $JB$ .  $\square$

We thus have the following diagram :

$$\begin{array}{ccc}
 \text{Quu} & \xrightleftharpoons[\text{d}]{\text{s}} & \text{Unif} \\
 \uparrow \text{F} \quad \downarrow \text{V} & & \downarrow \text{T} \quad \uparrow \text{G} \\
 \text{CTopOrd} & \xrightleftharpoons[\text{J}]{\text{B}} & \text{Creg}
 \end{array}$$

(\*\*)

$F$  and  $G$  are  $V$ - and  $T$ -sections respectively;  $T, V, s, d, J$  and  $B$  are as defined previously.

10.6 Lemma (1)  $JT = Vd$

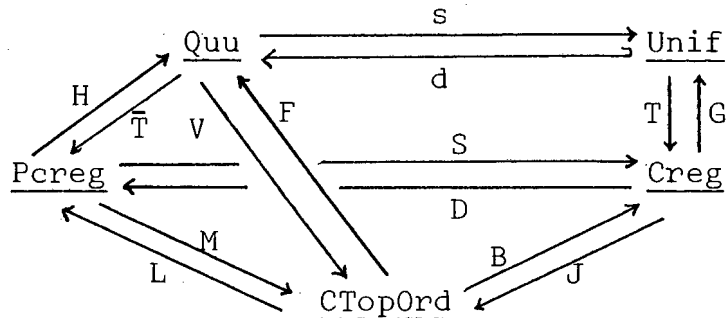
(2)  $Ts = BV$ .

Proof (1)  $JT(X,H) = (X, \tau(H), cl\Delta)$  and  $Vd(X,H) = (X, \tau(H), \cap H)$  but  $cl\Delta = \cap H$ .

(2)  $Ts(X,H) = T(X, H \vee H^{-1}) = (X, \tau(H \vee H^{-1}))$  and  $BV(X,H) = B(X, \tau(H \vee H^{-1}), \cap H) = (X, \tau(H \vee H^{-1}))$ . □

### 11. Relations between $\bar{T}$ -, V- and T-sections

We combine Brümmer's diagram (\*) involving Pcreg with ours (\*\*) involving CTopOrd via the functors M and L, to obtain :



H, F and G are  $\bar{T}$ -, V- and T-sections respectively. The other functors are as defined already.

11.1 Lemma  $LJ = D$ .

Proof Since L is left adjoint to M (Prop. 8.6), and J is left adjoint to B (Prop. 10.4), LJ is left adjoint

to  $BM = S$  ; however, this adjoint is  $D$  (see [Brümmer 1977]). □

11.2 Definition (1) ([Brümmer 1977]) The T-section  $G$  extends to the  $\bar{T}$ -section  $H$  , or  $H$  restricts to  $G$  , iff  $dG = HD$ .

(2) We say the T-section  $G$  extends to the V-section  $F$  , or  $F$  restricts to  $G$  , iff  $dG = FJ$  .

11.3 Remark (1) ([Brümmer 1977]) If the  $\bar{T}$ -section  $H$  restricts to the T-section  $G$  , then  $G = sHD$  , since  $sd = 1$ .

(2) If the V-section  $F$  restricts to the T-section  $G$  , then  $G = sFJ$  .

The next proposition will allow us to obtain several results concerning V-sections from known ones about  $\bar{T}$ -sections.

11.4 Proposition Let  $H, F$  and  $G$  be  $\bar{T}$ -, V- and T-sections respectively.

(1) If  $H$  restricts to  $F$  and  $F$  restricts to  $G$  , then  $H$  restricts to  $G$  .

(2) If  $H$  restricts to  $F$  and  $H$  restricts to  $G$  , then  $F$  restricts to  $G$  .

Proof (1) We have  $HL = F$  (see Defn. 9.4) and  $FJ = dG$ . Then  $HD = HLJ$  (since  $LJ = D$ ) and  $HLJ = FJ = dG$ .

(2) We have  $HL = F$  and  $HD = dG$ . Then  $FJ = HLJ = HD = dG$ .

□

We now investigate the problem of which V-sections restrict to T-sections, and which T-sections extend to V-sections. We model our discussion on [Brümmer 1977] and [Brümmer 1982]; the following three definitions are taken from [Brümmer 1977].

We define an involution functor in each of the categories Quu, Pcreg and CTopOrd. Each one will be denoted by  $c$  - the meaning will be clear from the context.

11.5 Definition The concrete conjugation functor  $c$  is defined as follows:

- (1) In Quu,  $c(X, H) = (X, H^{-1})$  where  $H^{-1} = \{H^{-1} \mid H \in H\}$ .
- (2) In Pcreg,  $c(X, P, Q) = (X, Q, P)$ .
- (3) In CTopOrd,  $c(X, T, \leq) = (X, T, \geq)$  where  $\geq$  is the opposite order to  $\leq$ .

11.6 Definition A  $\bar{T}$ -, M- or V-section  $F$  is called odd iff  $Fc = cF$ .

11.7 Definition Call an object  $X$  of Quu, Pcreg or CTopOrd symmetric iff  $cX = X$ , and a class  $A$  of objects symmetric iff  $A \in A \Rightarrow cA \in A$ .

11.8 Proposition Let  $F$  be a  $\bar{T}$ -,  $M$ - or  $V$ -section spanned by a class  $A$ .

(1)  $A$  symmetric  $\Rightarrow F$  odd.

(2)  $M_F$  symmetric  $\Leftrightarrow F$  odd.

Proof This statement is given for  $\bar{T}$ -sections in [Brümmer 1977, Prop. 4.2]. We give the proof for the case when  $F$  is a  $V$ -section. The other proofs are similar.

(1) Suppose  $A$  is symmetric. Fix  $X \in \underline{CTopOrd}$ . The following diagrams illustrate the constructions of  $FcX$  and  $cFX$ :

$$FcX \xrightarrow[\text{initial}]{f'} A, \text{ all } A \in A$$

$$cX \xrightarrow{f} VA, \text{ all } f \in \underline{CTopOrd}(cX, VA), \text{ all } A \in A$$

and  $FX \xrightarrow[\text{initial}]{g'} A, \text{ all } A \in A$

$$X \xrightarrow{g} VA, \text{ all } g \in \underline{CTopOrd}(X, VA), \text{ all } A \in A.$$

Since  $(g': FX \rightarrow A)$  is an initial source, so is  $(cg': cFX \rightarrow cA)$ . Now  $Vc = cV$ , since  $Vc(X, H) = (X, \tau(H \vee H^{-1}), \cap H^{-1}) = cV(X, H)$ . So the underlying set maps of  $(f': FcX \rightarrow A)$  and  $(cg': cFX \rightarrow cA)$  are the same, and since both sources are initial,  $cFX = FcX$ .

(2) Since  $F = \langle M_F \rangle_V$ ,  $M_F$  symmetric  $\Rightarrow F$  odd by (1).  
 Conversely, suppose  $F$  is odd, and fix  $X \in M_F$ . Then  
 $X \leq FVX$ , i.e. the identity map  $i: FVX \rightarrow X$  is uniformly  
 continuous. We must show that  $cX \in M_F$ , i.e. that the  
 identity map  $j: FVcX \rightarrow cX$  is uniformly continuous. However,  
 $ci: cFVX \rightarrow cX$  is uniformly continuous, and  $cFVX = FcVX =$   
 $FVcX$ , so the result follows.  $\square$

11.9 Proposition (1)  $L$  is odd.

(2) If a  $\bar{T}$ -section  $H$  restricts to a  $V$ -section  $F$ , then  
 $H$  odd  $\Rightarrow F$  odd.

(3) If  $F$  is an odd  $V$ -section then its  $\bar{T}$ -section extensions  
 $\hat{F} = \langle R_F \rangle_{\bar{T}}$  and  $\hat{F} = \langle M_F \rangle_{\bar{T}}$  are odd.

Proof (1) Since  $L$  is the unique right inverse of  $M$ ,  
 $L = \langle Ob \text{ Pcreg} \rangle_M$ .  $Ob \text{ Pcreg}$  is certainly a symmetric class,  
 so  $L$  is odd.

(2) We have that  $HL = F$  and that  $Lc = cL$  (since  $L$  is  
 odd, by (1)). So  $cF = c(HL) = HcL = HLc = Fc$ .

(3)  $R_F$  is easily seen to be symmetric if  $F$  is odd; so  
 $\langle R_F \rangle_{\bar{T}}$  is odd. Also,  $F$  odd  $\Rightarrow M_F$  symmetric  $\Rightarrow \langle M_F \rangle_{\bar{T}}$  odd.  $\square$

11.10 Proposition All odd  $\bar{T}$ -sections restrict to  $T$ -sections.

Proof [Brümmer 1977, Prop. 4.1].  $\square$

11.11 Proposition All odd V-sections restrict to T-sections.

Proof Let  $F$  be an odd V-section. Then  $\hat{F}$  is a  $\bar{T}$ -section extension of  $F$  that is odd, by Proposition 11.9. By the previous proposition,  $\hat{F}$  restricts to some T-section, say  $G$ . But Proposition 11.4 (2) then shows that  $F$  restricts to  $G$ . □

11.12 Remark If  $F$  is an odd V-section, we do not know whether all the  $\bar{T}$ -section extensions of  $F$  are odd, but we do know that they all restrict to T-sections (by Proposition 11.4 (1)).

11.13 Proposition [Brümmer 1982, Thm. 5.3] Let  $G = \langle A \rangle_T$  be a T-section.

(1) If  $d(R_G) \cup \{\Pi_q\} \subseteq B \subseteq s^{-1}(M_G)$  then  $\langle B \rangle_{\bar{T}}$  is a  $\bar{T}$ -section extension of  $G$ .

(2) If  $H$  is a  $\bar{T}$ -section extension of  $G$ , then  $d(R_G) \cup \{\Pi_q\} \subseteq M_H \subseteq s^{-1}(M_G)$ .

(3)  $\langle d(R_G) \rangle_{\bar{T}}$  is an extension of  $G$ , but it is not a  $\bar{T}$ -section.

(4) The class of  $\bar{T}$ -section extensions of  $G$  has a coarsest member,  $\underset{\sim}{G} = \langle d(R_G) \cup \{\Pi_q\} \rangle_{\bar{T}} = \langle dGT(A) \cup \{\Pi_q\} \rangle_{\bar{T}}$ , and a finest,  $\underset{\approx}{G} = \langle s^{-1}(M_G) \rangle_{\bar{T}}$ . Both  $\underset{\sim}{G}$  and  $\underset{\approx}{G}$  are odd. □

11.14 Proposition Let  $G = \langle A \rangle_T$  be a T-section.

(1) If  $d(R_G) \cup \{\Pi_q\} \subseteq B \subseteq s^{-1}(M_G)$  then  $\langle B \rangle_V$  is a V-section extension of  $G$ .

(2) If  $F$  is a V-section extension of  $G$ , then  $d(R_G) \cup \{\Pi_q\} \subseteq M_F \subseteq s^{-1}(M_G)$ .

(3)  $\langle d(R_G) \rangle_V$  is an extension of  $G$ , but it is not a V-section.

(4) The class of V-section extensions of  $G$  has a coarsest member,  $\underset{\sim}{G} = \langle d(R_G) \cup \{\Pi_q\} \rangle_V = \langle dGT(A) \cup \{\Pi_q\} \rangle_V$ , and a finest,  $\tilde{G} = \langle s^{-1}(M_G) \rangle_V$ . Both  $\underset{\sim}{G}$  and  $\tilde{G}$  are odd.

(5) If  $F: \underline{CTopOrd} \rightarrow \underline{Quu}$  is a functor satisfying  $\underset{\sim}{G} \leq F \leq \tilde{G}$ , then  $F$  restricts to  $G$ .

(6)  $M_{\tilde{G}} = s^{-1}(M_G)$ .

(7) The class of V-section extensions of  $G$  is a large-complete lattice under the ordering "coarser than" (though we make no claim that it has more than one member).

Proof (1) Follows from the previous proposition, using Proposition 9.1 and Proposition 11.4 (2).

(2) By Proposition 9.6 the finest  $\bar{T}$ -section extension of  $F$  is  $\hat{F} = \langle M_F \rangle_{\bar{T}}$ , and  $M_{\hat{F}} = M_F$ . Now  $\hat{F}$  restricts to  $G$ , so, by the previous proposition,  $d(R_G) \cup \{\Pi_q\} \subseteq M_{\hat{F}} \subseteq s^{-1}(M_G)$ .

(3) Follows from the previous proposition, Proposition 9.1 and Proposition 11.4 (2).

(4) Since  $c\Pi_q \cong \Pi_q$ , we may express  $\underset{\sim}{G}$  as  $\langle d(R_G) \cup \{\Pi_q, c\Pi_q\} \rangle_V$ . It is then clear that  $\underset{\sim}{G}$  and  $\tilde{G}$  are spanned by symmetric classes, and so are odd.

Let  $F$  be any  $V$ -section restricting to  $G$ , and let  $H$  be any  $\bar{T}$ -section restricting to  $F$ . Then  $H$  restricts to  $G$ . By the previous proposition,  $\langle d(R_G) \cup \{\Pi_q\} \rangle_{\bar{T}} \leq H \leq \langle s^{-1}(M_G) \rangle_{\bar{T}}$ . Applying  $L$  on the right gives  $\langle d(R_G) \cup \{\Pi_q\} \rangle_V \leq F \leq \langle s^{-1}(M_G) \rangle_V$ , since  $HL = F$ . Applying  $L$  to  $\langle d(R_G) \cup \{\Pi_q\} \rangle_{\bar{T}} = \langle dGT(A) \cup \{\Pi_q\} \rangle_{\bar{T}}$  also gives  $\tilde{G} = \langle dGT(A) \cup \{\Pi_q\} \rangle_V$ .

$$(5) \quad \tilde{G} \leq F \leq \tilde{G} \Rightarrow \tilde{G}J \leq FJ \leq \tilde{G}J \\ \Rightarrow dG \leq FJ \leq dG \\ \Rightarrow FJ = dG.$$

(6)  $M_{\tilde{G}} \supseteq s^{-1}(M_G)$  since  $M_{\tilde{G}}$  is the largest class spanning  $\tilde{G}$ , and  $s^{-1}(M_G) \supseteq M_{\tilde{G}}$  by (2).

(7) Similar to Proposition 9.6 (5). □

11.15 Definition Let  $[0, \infty)_q$  be the quu subspace of  $R_q$  having underlying set  $[0, \infty)$ .

11.16 Proposition The  $\bar{T}$ -section spanned by  $[0, \infty)_q$  does not restrict to a  $T$ -section.

Proof As for [Brümmer 1982, Prop. 5.1]. □

11.17 Proposition The  $V$ -section spanned by  $[0, \infty)_q$  does not restrict to a  $T$ -section (and is thus not odd).

Proof If  $\langle \{[0, \infty)_q\} \rangle_V$  restricted to a T-section, so would  $\langle \{[0, \infty)_q\} \rangle_{\bar{T}}$  (since  $\langle \{[0, \infty)\} \rangle_{\bar{T}}$  restricts to  $\langle \{[0, \infty)\} \rangle_V$  ). □

11.18 Proposition [Brümmer 1977, Thm. 4.3]. The  $\bar{T}$ -sections  $\bar{C}^*$ ,  $\bar{C}$  and  $\bar{\phi}$  are odd, and restrict to the T-sections  $C^*$ ,  $C$  and  $\phi$  respectively. □

11.19 Proposition The V-sections  $C^{*\dagger}$ ,  $C^\dagger$  and  $\phi^\dagger$  are odd, and restrict to the T-sections  $C^*$ ,  $C$  and  $\phi$  respectively.

Proof By Proposition 9.1,  $\langle \{\Pi_q\} \rangle_{\bar{T}.L} = \langle \{\Pi_q\} \rangle_V$ , i.e.  $\bar{C}^*$  restrict to  $C^{*\dagger}$ . Proposition 11.9 (2) and Proposition 11.18 then show that  $C^{*\dagger}$  is odd. Since  $\bar{C}^*$  restricts to  $C^{*\dagger}$  and  $\bar{C}^*$  restricts to  $C^*$ ,  $C^{*\dagger}$  restricts to  $C^*$ .

The other proofs are similar. □

11.20 Proposition [Brümmer 1977, Thm. 4.5].  $\bar{C}^*$  is the unique  $\bar{T}$ -section extension of the T-section  $C^*$ . □

11.21 Proposition  $C^{*\dagger}$  is the unique V-section extension of the T-section  $C^*$ .

Proof Suppose that  $F = \langle A \rangle_V$  also restricts to  $C^*$ . Then  $\langle A \rangle_{\bar{T}}$  and  $\bar{C}^*$  both restrict to  $C^*$ , and the uniqueness result for  $\bar{T}$ -section extensions of  $C^*$  (see the previous proposition), gives  $\langle A \rangle_{\bar{T}} = \bar{C}^*$ . Then  $\langle A \rangle_{\bar{T}.L} = \bar{C}^*.L$ , i.e.  $\langle A \rangle_V = C^{*\dagger}$ . □

In the next chapter we will see that  $C$  and any  $T$ -section finer than  $C$  have at least two distinct  $V$ -section extensions.

11.22 Proposition  $\bar{C}^*$  is the unique  $\bar{T}$ -section extension of the  $V$ -section  $C^{*\dagger}$ .

Proof If  $H_1$  and  $H_2$  are  $\bar{T}$ -sections restricting to  $C^{*\dagger}$ , then they both restrict to  $C^*$ , contradicting the uniqueness result of  $\bar{T}$ -section extensions of  $C^*$ . □

## 12. Notes

Definitions 6.1 and 8.2, as well as Proposition 8.4, occur in [Salbany 1984]. The statement of Proposition 6.3 also occurs in that paper, without proof. Our proof uses a result of Charlton's [1973, Prop. 5.4]. We note that our functor  $M$  gives an order opposite to that given by Salbany's functor; also the bispaces given by  $L$  are conjugates of those given by the corresponding functor of Salbany's. These changes make no significant difference to the results.

Our items 9.3, 9.4 and 9.6 are closely modelled on [Brümmer 1977, Prop. 3.1 and Thm. 3.2].

I am indebted to my supervisor for assistance with the proofs of Propositions 5.1 and 9.1, for suggesting the definition (10.3) of the functor  $J$  and for observations on 8.3, 8.6, 8.7, 10.4 and 11.1.

CHAPTER 3

COMPLETION-TRUE SECTIONS

13. Definitions and characterizations

A T-section  $G$  is called completion-true if there is an endofunctor  $r$  of  $\underline{Creg}$  satisfying  $\gamma F = Fr$ . The well-known equations  $\gamma C^* = C^* \beta$  (where  $\beta$  is the Stone-Čech compactification functor) and  $\gamma C = C \nu$  (where  $\nu$  is the Hewitt realcompactification functor) (see [Gillman and Jerison 1960, Thm. 15.13]) are instances of this. The term "completion-true" was first introduced (for T- and  $\bar{T}$ -sections) in [Brümmer 1982], although the concept was implicit in [Brümmer 1979] and [Brümmer and Salbany 1977]. The theory has been developed in [Brümmer and Hager 1984, 1987, 1988]. The completion-true T-sections are exactly those which can be spanned by classes of complete spaces. The subcategory  $E(G)$  of Tych containing those spaces  $X$  for which  $GX$  is complete, is an epireflective subcategory of Tych lying between the compact and the topologically complete spaces. Moreover, any epireflective subcategory of Tych between these two extremes may be realized as  $E(G)$  for some completion-true  $G$ .

In this section we give analogous definitions and results for V-sections. In the Notes, §19, we shall list the items in [Brümmer and Hager 1984, 1987, 1988] which were the models for the present results.

13.1 Definition (1) ([Brümmer 1982]) A  $\bar{T}$ -section  $H$  is called completion-true (or  $\bar{\gamma}$ -true) iff there is an endofunctor  $r: \underline{Pcreg} \rightarrow \underline{Pcreg}$  satisfying  $\bar{\gamma}H = Hr$ .

(2) A V-section  $F$  is called completion-true (or  $\bar{\gamma}$ -true) iff there is an endofunctor  $r: \underline{CTopOrd} \rightarrow \underline{CTopOrd}$  satisfying  $\bar{\gamma}F = Fr$ .

(3) ([Brümmer 1982]) A T-section  $G$  is called completion-true (or  $\gamma$ -true) iff there is an endofunctor  $r: \underline{Creg} \rightarrow \underline{Creg}$  satisfying  $\gamma G = Gr$ .

13.2 Proposition A V-section  $F$  is completion-true iff  $\bar{\gamma}F = FV\bar{\gamma}F$ .

Proof " $\Rightarrow$ ": Suppose  $\bar{\gamma}F = Fr$  for some  $r$ . Applying  $V$  on the left gives  $V\bar{\gamma}F = r$ .

" $\Leftarrow$ ": Use  $r = V\bar{\gamma}F$ . □

13.3 Corollary If a V-section  $F$  is completion-true, then  $V\bar{\gamma}F$  is idempotent.

Proof  $(V\bar{\gamma}F)(V\bar{\gamma}F) = V\bar{\gamma}.FV\bar{\gamma}F = V\bar{\gamma}.\bar{\gamma}F = V\bar{\gamma}F$ . □

13.4 Definition and Proposition Let  $H, F$  and  $G$  be  $\bar{T}$ -,  $V$ - and  $T$ -sections respectively.

$$(1) \bar{E}(H) = \{X \in \underline{\text{Pcreg}} \mid HX \text{ is a complete quu space}\} \\ = \{X \in \underline{\text{Pcreg}} \mid \bar{T}\bar{Y}HX = X\} .$$

$$(2) E^\uparrow(F) = \{X \in \underline{\text{CTopOrd}} \mid FX \text{ is a complete quu space}\} \\ = \{X \in \underline{\text{CTopOrd}} \mid V\bar{Y}FX = X\} .$$

$$(3) E(G) = \{X \in \underline{\text{Creg}} \mid GX \text{ is a complete uniform space}\} \\ = \{X \in \underline{\text{Creg}} \mid T\gamma GX = X\} .$$

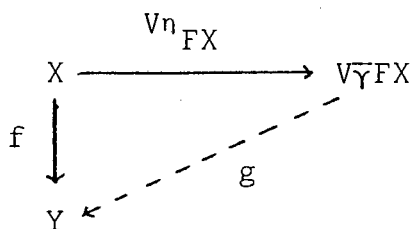
Proof (1) Fix  $X \in \underline{\text{Pcreg}}$ .  $HX$  complete  $\Rightarrow \bar{Y}HX = HX \Rightarrow \bar{T}\bar{Y}HX = X$ . Conversely, suppose  $\bar{T}\bar{Y}HX = X$ . The completion reflector  $\eta_{HX}: HX \rightarrow \bar{Y}HX$  is initial, and  $\bar{T}\eta_{HX} = 1$ ; thus  $\eta_{HX}$  is an isomorphism.

(2) and (3) are similar. □

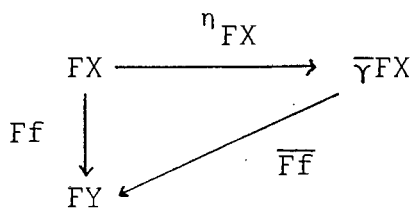
13.5 Proposition Let  $F$  be a  $V$ -section. Then  $V\bar{Y}F$  is idempotent iff  $V\bar{Y}F$  is the reflector of  $\underline{\text{CTopOrd}}$  onto  $E^\uparrow(F)$ .

Proof " $\Leftarrow$ ": All reflectors are idempotent.

" $\Rightarrow$ ": Fix  $X \in \underline{\text{CTopOrd}}$ .  $V\bar{Y}FX \in E^\uparrow(F)$ , since  $V\bar{Y}F(V\bar{Y}FX) = V\bar{Y}FX$ . The reflection map we require is  $V\eta_{FX}: X \rightarrow V\bar{Y}FX$ . To see this, take any  $Y \in E^\uparrow(F)$  and any morphism  $f: X \rightarrow Y$  in  $\underline{\text{CTopOrd}}$ . We must find a map  $g$  making the following diagram commute :



In the next diagram  $FY$  is complete, since  $Y \in E^\uparrow(F)$ .



A map  $\bar{F}f$  making the diagram commute exists because of the universal property of  $\bar{\gamma}FX$ . Put  $g = V(\bar{F}f)$ . □

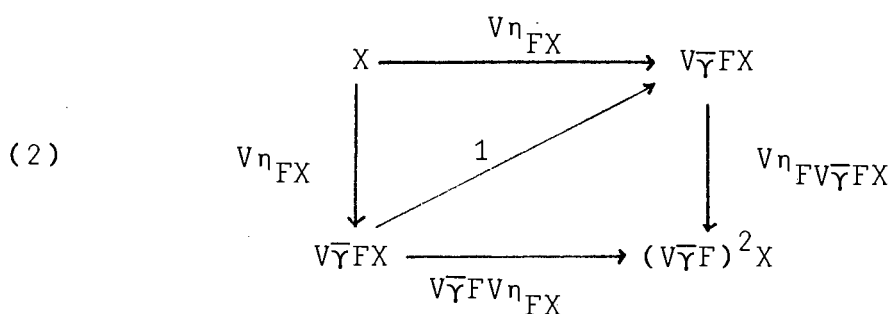
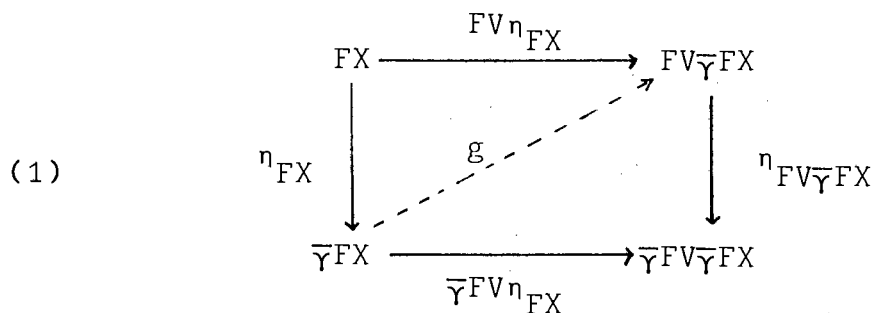
13.6 Remark (1) If  $F$  is a completion-true  $V$ -section, then  $V\bar{\gamma}F$  is the reflector of CTopOrd onto  $E^\uparrow(F)$ , by Corollary 13.3 and Proposition 13.5.

(2) Surprisingly,  $E^\uparrow(F)$  is epireflective in UTopOrd for every  $V$ -section  $F$  (see Prop. 15.1), though  $V\bar{\gamma}F$  need not be idempotent (see Example 17.8 (1)), i.e.  $V\bar{\gamma}F$  need not be the reflector onto  $E^\uparrow(F)$ .

The next result is useful for determining whether or not a given  $V$ -section is completion-true.

13.7 Proposition For any  $V$ -section  $F$ ,  $FV\bar{\gamma}F \leq \bar{\gamma}F$ .

Proof Consider the diagrams

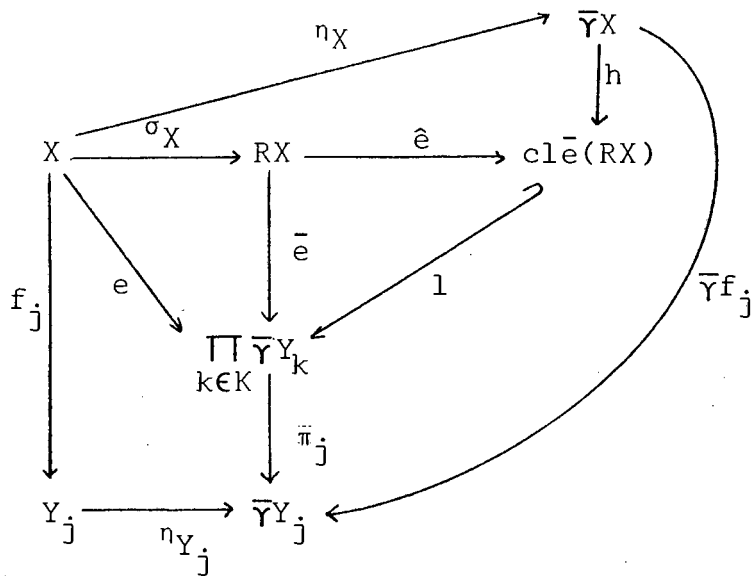


The outer square of (1) commutes by naturality of  $\eta$ . Since diagram (2) is simply  $V$  applied to diagram (1), the outer square of (2) commutes.  $(V\bar{\gamma}F)^2X$  has a Hausdorff topology and  $V\eta_{FX}$  is dense (regarded as a map between the underlying topological spaces), so  $V\eta_{FV\bar{\gamma}FX} = V\bar{\gamma}FV\eta_{FX}$ . Thus the lower triangle in (2) commutes. Consider the diagonal  $g$  in (1) with the identity as underlying set map. We have  $\eta_{FV\bar{\gamma}FX} \circ g = \bar{\gamma}FV\eta_{FX}$ , so initiality of  $\eta_{FV\bar{\gamma}FX}$  shows  $g$  uniformly continuous, as required.  $\square$

13.8 Proposition  $\bar{\gamma}$  preserves initial sources.

Proof Let  $(f_j: X \rightarrow Y_j)_{j \in J}$  be an initial source in Quu for some index class  $J$ . There is a set  $K \subseteq J$  such that  $(f_j: X \rightarrow Y_j)_{j \in K}$  is still initial. We show that  $(\bar{Y}f_j: \bar{Y}X \rightarrow \bar{Y}Y_j)_{j \in K}$  is initial, and it then follows trivially that  $(\bar{Y}f_j: \bar{Y}X \rightarrow \bar{Y}Y_j)_{j \in J}$  is.

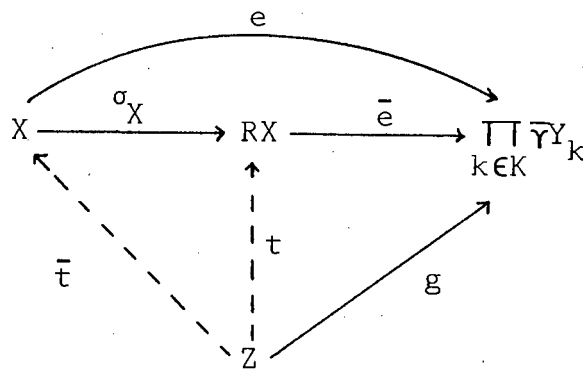
This diagram illustrates the maps we will be using :



The source  $(\pi_j: \prod_{k \in K} \bar{Y}Y_k \rightarrow \bar{Y}Y_j)_{j \in K}$  consists of the usual projection maps, and the maps  $n_{Y_j}: Y_j \rightarrow \bar{Y}Y_j$  are the completion reflectors. The map  $e$  satisfying  $\pi_j \cdot e = n_{Y_j} \cdot f_j \quad \forall j \in K$  exists by the definition of a product, and is initial since the source  $(f_j: X \rightarrow Y_j)_{j \in K}$  is, and each  $n_{Y_j}$  is.

Let  $\sigma_X: X \rightarrow RX$  be the separated quu reflection (i.e. the reflection of Quu to Quu<sub>0</sub>). An explicit construction of this reflection may be found in [Salbany 1970, Chp. 3, Prop. 3.11].

The map  $\bar{e}$  satisfying  $\bar{e} \cdot \sigma_X = e$  exists by the universal property of  $RX$  (since  $\prod_{k \in K} \bar{Y}_k$  is separated). We show that  $\bar{e}$  is an embedding. To show  $\bar{e}$  initial, let  $Z$  be a quu space,  $g: Z \rightarrow \prod_{k \in K} \bar{Y}_k$  be uniformly continuous and  $t$  be a set map satisfying  $\bar{e}t = g$ . We must show  $t$  uniformly continuous. We have the diagram :



$\sigma_X$  is a surjection (see [Salbany 1970, Chp.3, Prop.3.11]), so there exists a set map  $r$  such that  $\sigma_X \cdot r = 1$ . Let  $\bar{t} = rt$ . Then  $\sigma_X \cdot \bar{t} = \sigma_X \cdot rt = t$ . Now initiality of  $e$  shows  $\bar{t}$  uniformly continuous, so  $t$  is also uniformly continuous. Thus  $\bar{e}$  is initial, and since  $RX$  is separated,  $\bar{e}$  is an embedding.

Let  $\hat{e}$  be the natural embedding of  $RX$  into  $cl \bar{e}(RX)$ , and let  $l$  be the natural inclusion map. Note that  $cl \bar{e}(RX)$  refers to the  $T(H \vee H^{-1})$ -closure of  $\bar{e}(RX)$ , where  $H$  is the quu of  $\prod_{k \in K} \bar{Y}_k$ . By [Salbany 1970, Chp. 3, Prop.3.5],  $cl \bar{e}(RX)$  is a complete quu space. Since  $\hat{e} \cdot \sigma_X$  is initial and dense and  $cl \bar{e}(RX)$  is complete there is an isomorphism,  $h$ , between  $\bar{Y}X$  and  $cl \bar{e}(RX)$ .



$i: FVA \rightarrow A$  is uniformly continuous because  $A \in M_F$ , i.e.  $A \leq FVA$ . So we may set up the diagram :

$$\begin{array}{ccc}
 \bar{\gamma}FX & \xrightarrow{\bar{\gamma}f'} & \bar{\gamma}A = A \\
 \uparrow h & & \uparrow i \\
 FV\bar{\gamma}FX & \xrightarrow{FV\bar{\gamma}f'} & FV\bar{\gamma}A = FVA
 \end{array}$$

Since  $\bar{\gamma}f'$  and  $i.FV\bar{\gamma}FX$  have the same underlying set maps, the identity set map  $h: FV\bar{\gamma}FX \rightarrow \bar{\gamma}FX$  makes the diagram commute.  $h$  is uniformly continuous because  $(\bar{\gamma}f': \bar{\gamma}FX \rightarrow \bar{\gamma}A)$  is initial. This shows that  $\bar{\gamma}FX \leq FV\bar{\gamma}FX \quad \forall X \in \underline{CTopOrd}$ . The converse inequality is given by Proposition 13.7, so  $\bar{\gamma}F = FV\bar{\gamma}F$ . □

13.10 Remark Brümmer [1979] showed that a T-section (resp.  $\bar{T}$ -section) is completion-true iff it is spanned by a class of complete uniform (resp. quu) spaces.

13.11 Examples (1) ([Brümmer 1982])  $C^*$ ,  $\bar{C}^*$ ,  $C$ ,  $\bar{C}$ ,  $\phi$  and  $\bar{\phi}$  are completion-true.

(2)  $C^{\dagger}$ ,  $C^{\uparrow}$  and  $\phi^{\dagger}$  are completion-true (since  $\Pi_q$  and  $\mathbb{R}_q$  are complete, and  $\phi^{\dagger} = \langle \underline{CplQuu} \rangle_V$ , see Lemma 7.8).

13.12 Proposition For any V-section  $F$  the following are equivalent :

(1)  $F$  is completion-true

- (2)  $\bar{\gamma}R_F \subseteq R_F$
- (3)  $F = \langle \bar{\gamma}R_F \rangle_V$
- (4)  $F = \langle R_F \cap \underline{\text{CplQuu}} \rangle_V$
- (5)  $F = \langle M_F \cap \underline{\text{CplQuu}} \rangle_V$
- (6)  $\bar{\gamma}R_F = R_F \cap \underline{\text{CplQuu}}$
- (7)  $\bar{\gamma}R_F \subseteq M_F$  .

Proof (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) follows from the proof of Proposition 13.9.

(3)  $\Rightarrow$  (4): Suppose (3) holds. Then, by 13.9, (1) holds. Since we have shown that (1)  $\Rightarrow$  (2),  $\bar{\gamma}R_F \subseteq R_F$  . Hence  $\bar{\gamma}R_F \subseteq R_F \cap \underline{\text{CplQuu}} \subseteq M_F$  , and since  $\bar{\gamma}R_F$  and  $M_F$  both span  $F$  , so does  $R_F \cap \underline{\text{CplQuu}}$ .

(4)  $\Rightarrow$  (5): Follows since  $R_F \cap \underline{\text{CplQuu}} \subseteq M_F \cap \underline{\text{CplQuu}} \subseteq M_F$ .

(5)  $\Rightarrow$  (1): Follows from Proposition 13.9.

(2)  $\Rightarrow$  (6): If (2) holds, then  $\bar{\gamma}R_F \subseteq R_F \cap \underline{\text{CplQuu}}$  ; and since the other inclusion always holds,  $\bar{\gamma}R_F = R_F \cap \underline{\text{CplQuu}}$ .

(6)  $\Rightarrow$  (2): Trivial.

(1)  $\Rightarrow$  (7):  $F$  completion-true  $\Rightarrow \bar{\gamma}R_F \subseteq R_F \Rightarrow \bar{\gamma}R_F \subseteq M_F$  (since  $R_F \subseteq M_F$ ).

(7)  $\Rightarrow$  (1): Suppose  $\bar{\gamma}R_F \subseteq M_F$  . Fix  $X \in \underline{\text{CTopOrd}}$  . Then  $\bar{\gamma}FX \in M_F$  , i.e.  $\bar{\gamma}FX \leq FV\bar{\gamma}FX$  . Proposition 13.7 then gives  $\bar{\gamma}F = FV\bar{\gamma}F$  . □

13.13 Proposition Let  $F$  be a  $V$ -section. Then (1)  $\Rightarrow$  (2)

$\Rightarrow$  (3) where:

- (1)  $\bar{\gamma}M_F \subseteq M_F$

$X \in \underline{CTopOrd}$ . The maps  $g, h: X \rightarrow Y$  given by  $g(x) = y_1$  and  $h(x) = y_2$  are continuous and increasing and satisfy  $fg = fh$ . Thus  $f$  is not monic.

" $\Leftarrow$ ": As for the category Set.

The same proof applies to UTopOrd since the one point space has antisymmetric order.  $\square$

14.2 Proposition In CTopOrd a morphism is epic iff it is surjective.

Proof " $\Rightarrow$ ": Suppose  $f: X \rightarrow Y$  is not surjective. Then there exists  $y_0 \in Y - f(X)$ . Let  $Z$  be the two point space  $\{a, b\}$  with the indiscrete topology and order. The quu with a single entourage determines  $Z$ , thus  $Z \in \underline{CTopOrd}$ . Define maps  $g, h: Y \rightarrow Z$  by:  $g(y_0) = a$  and  $g(y) = b$  if  $y \neq y_0$ ;  $h(y) = b$  for all  $y \in Y$ .  $g$  and  $h$  are continuous and increasing, and satisfy  $gf = hf$ . Hence  $f$  is not epic.

" $\Leftarrow$ ": As for the category Set.  $\square$

The next result was proved in [Vrancken-Mawet 1985]. We provide another proof, which uses the known characterization of the epimorphisms in the separated quu spaces.

14.3 Proposition In UTopOrd a morphism is epic iff it is dense. (By  $f$  "dense" we mean  $Bf$  is dense, where  $B: \underline{CTopOrd} \rightarrow \underline{Creg}$  is the forgetful functor defined earlier.)

Proof " $\Rightarrow$ ": Suppose  $f: X \rightarrow Y$  is an epi in UTopOrd. Let  $F$  be any  $V$ -section. Then  $Ff: FX \rightarrow FY$  is easily seen to be an epi in Quu<sub>0</sub>. By [Brümmer 1971, Thm. 2.5.3],  $TsFf: TsFX \rightarrow TsFY$  is dense, i.e.  $Bf: BX \rightarrow BY$  is dense (since  $Ts = BV$ : see Lemma 10.6).

" $\Leftarrow$ ": As for the category of Hausdorff topological spaces. □

14.4 Proposition In UTopOrd a morphism is an extremal mono iff it is a closed embedding.

Proof Note that by a "closed" morphism  $f$  in UTopOrd we mean one for which  $Bf$  is closed; the proof then goes through as in the category of Hausdorff spaces, using Proposition 14.3. □

14.5 Proposition (1) UTopOrd is complete.  
(2) UTopOrd is wellpowered and cowellpowered.  
(3) UTopOrd is an (epi, extremal mono)-category.

Proof (1) [Herrlich 1974, Cor. 5.2].

(3) The proofs go through as for the category of Hausdorff topological spaces.

(3) [Herrlich and Strecker 1973, Thm. 34.5]. □

15. Epireflections in  $UTopOrd$  induced by functorial quasi-uniformities.

By [Herrlich and Strecker 1973, Thm. 37.1], if  $A$  is a full and isomorphism-closed subcategory of an  $E$ -cowellpowered  $(E, M)$ -category  $B$  that has products, then  $A$  is  $E$ -reflective in  $B$  iff  $A$  is closed under formation of products and  $M$ -subobjects in  $B$ . Recall from 13.4 that for any  $V$ -section  $F$ ,  $E^\uparrow(F) = \{X \in CTopOrd \mid FX \text{ is a complete quu space}\}$ . We may thus show  $E^\uparrow(F)$  epireflective in  $UTopOrd$  by showing closure under products and closed subspaces.

15.1 Proposition Let  $F$  be any  $V$ -section.  $E^\uparrow(F)$  is an epireflective subcategory of  $UTopOrd$  lying between  $CptTopOrd$  (those spaces in  $UTopOrd$  having compact underlying topologies) and  $TopOrdCpl$  (those admitting complete quasi-uniformities).

Proof Take  $X \in E^\uparrow(F)$  and  $A$  a closed subspace of  $X$ .  $FX$  is complete; we show  $FA$  is. Let  $i: A \rightarrow X$  be the inclusion map.  $F_i: FA \rightarrow FX$  is uniformly continuous, so  $FA$  is finer than the quu subspace  $Y$  of  $FX$  with point set  $A$ . Now  $sY$  is a closed subspace of the complete uniform space  $sFX$ , so  $sY$  is complete. However,  $sFA \geq sY$ , and their underlying topologies are the same, so  $sFA$  is a complete uniform space, i.e.  $FA$  is a complete quu space.

Proof  $E^\uparrow(F) \supseteq VA$  since  $A \in A \Rightarrow A \in M_F \Rightarrow FVA \geq A \Rightarrow sFVA \geq sA$ , and since  $sFVA$  and  $sA$  have the same underlying topologies,  $sFVA$  is complete, i.e.  $FVA$  is complete. Since  $E^\uparrow(F)$  is an epireflective subcategory of UTopOrd, it remains only to show that  $E^\uparrow(F)$  is contained in the epireflective hull of  $VA$ .

Fix  $X \in E^\uparrow(F)$ . We show that  $X$  is in the epireflective hull of  $VA$  by showing that  $X$  is a closed subspace of a product of members of  $VA$  (see [Herrlich and Strecker 1973, Cor. 37.6]). Let  $(f': FX \rightarrow A)$  be the initial source lifted from  $(f: X \rightarrow VA \mid f \in \text{CTopOrd}(X, VA), A \in A)$ . Choose a subset  $(f_i': FX \rightarrow A_i)_{i \in I}$  (for some index set  $I$ ) of maps from the source  $(f': FX \rightarrow A)$  so that the source  $(f_i': FX \rightarrow A_i)_{i \in I}$  remains initial. Then there is an embedding  $e$  making the following diagram commute for all  $i \in I$ :

$$\begin{array}{ccc}
 FX & \overset{e}{\dashrightarrow} & \prod_{j \in I} A_j \\
 & \searrow f_i' & \downarrow \pi_i \\
 & & A_i
 \end{array}$$

Now  $se: sFX \rightarrow s \prod_{i \in I} A_i = \prod_{i \in I} sA_i$  is an embedding of a complete space into a complete (and hence separated) space;  $se$  is therefore closed. Thus  $\forall e: X \rightarrow \prod_{i \in I} VA_i$  is the required closed embedding.

$\overline{V}F$  being the reflector of CTopOrd onto  $E^\dagger(F)$  may be seen from Remark 13.6, since  $F$  here is completion-true.  $\square$

15.3 Remark The corresponding original results for  $T$ - and  $\bar{T}$ -sections in [Brümmer 1979 and 1982] and [Brümmer and Hager 1984] were:

(1) If  $G$  is a  $T$ -section spanned by a class  $A$  of complete uniform spaces, then  $E(G)$  is the epireflective hull of  $TA$  in Tych, and  $T\gamma G$  reflects Creg onto  $E(G)$ .  $T\gamma C^* = \beta$ , the Stone-Čech compactification,  $T\gamma C = v$ , the Hewitt realcompactification, and  $T\gamma\phi = \delta$ , the Dieudonné topological completion. Moreover,  $E(C^*)$  consists of the compact spaces,  $E(C)$  of the realcompact ones and  $E(\phi)$  of the topologically complete ones (i.e. those admitting complete uniformities).

(2) If  $H$  is a  $\bar{T}$ -section spanned by a class  $A$  of complete quu spaces, then  $\bar{E}(H)$  is the epireflective hull of  $\bar{TA}$  in PTych, and  $\bar{T}\bar{\gamma}H$  reflects Pcreg onto  $\bar{E}(H)$ .  $\bar{T}\bar{\gamma}C^*$  is  $\bar{\beta}$ , the reflector to the compact bispaces in Pcreg.

15.4 Definition (1)  $\beta^\dagger = \overline{V\gamma C^*}^\dagger$ , i.e.  $\beta^\dagger$  is the reflector of CTopOrd onto the epireflective hull of  $\Pi_0$  in UTopOrd.

(2)  $v^\dagger = \overline{V\gamma C}^\dagger$ , i.e.  $v^\dagger$  is the reflector of CTopOrd onto the epireflective hull of  $\mathbb{R}_0$  in UTopOrd.

The next result shows that every epireflective subcategory of UTopOrd between CptTopOrd and TopOrdCpl arises as  $E^\uparrow(F)$  for some V-section  $F$ .

15.5 Proposition For each epireflective subcategory  $S$  of UTopOrd between CptTopOrd and TopOrdCpl there is a completion-true V-section  $F$  satisfying  $E^\uparrow(F) = S$ . In general there is more than one such  $F$ ; there is a finest such  $F$ .

Proof Let  $F_0 = \langle \phi^\uparrow S \rangle_V$  where  $\phi^\uparrow S = \{ \phi^\uparrow S \mid S \in S \}$ . Since  $\phi^\uparrow S$  consists of complete quu spaces and contains  $\Pi_q$ , it spans a completion-true V-section (see Prop. 7.4 and Prop. 13.9). By Proposition 15.2  $E^\uparrow(F_0)$  is the epireflective hull of  $V\phi^\uparrow S = S$ , thus  $E^\uparrow(F_0) = S$ , since  $S$  is epireflective in UTopOrd.

Suppose  $F = \langle A \rangle_V$  is another completion-true V-section satisfying  $E^\uparrow(F) = S$ . We show that  $F \leq F_0$ . Now  $VA \subseteq S$ , so  $\phi^\uparrow VA \subseteq \phi^\uparrow S$ ; and then  $F = \langle A \rangle_V \leq \langle \phi^\uparrow VA \rangle_V \leq \langle \phi^\uparrow S \rangle_V = F_0$ .

Next we give an example showing that  $F$  is in general not unique. Let  $S = E^\uparrow(C^\uparrow)$  and let  $F = \langle \phi^\uparrow S \rangle_V$ . Assume that  $F = C^\uparrow$ . Let  $S = (\mathbb{R}, \text{discrete topology})$  and  $S_0 = (\mathbb{R}, \text{discrete topology, discrete order})$ ; then  $S_0 = JS$  (where  $J$  is the functor defined in 10.3).  $S_0 \in S$  since :

$$\begin{aligned}
 \overline{V\Upsilon}C^\uparrow S_0 &= \overline{V\Upsilon}C^\uparrow JS \\
 &= \overline{V\Upsilon}dCS \quad (\text{since } C^\uparrow J = dC \text{ by Prop. 11.19}) \\
 &= Vd\Upsilon CS \quad (\text{since } d\Upsilon = \overline{V}d \text{ by Prop. 10.1}) \\
 &= JT\Upsilon CS \quad (\text{since } Vd = JT \text{ by Lemma 10.6}) \\
 &= JS \quad (\text{since } S \text{ topologically complete} \Rightarrow S \\
 &\text{realcompact, by Shirota's Theorem; see e.g. [Gillman and} \\
 &\text{Jerison 1960, Thm. 15.20]}). \\
 &= S_0 .
 \end{aligned}$$

We assumed that  $F = C^\uparrow$ , so  $M_F = M_{C^\uparrow}$ . Since  $S_0 \in S$ ,  $\phi^\uparrow S_0 \in M_F$ ; thus  $\phi^\uparrow S_0 \leq C^\uparrow V\phi^\uparrow S_0 = C^\uparrow S_0$ , i.e.  $\phi^\uparrow S_0 = C^\uparrow S_0$ . Then  $\phi^\uparrow JS = C^\uparrow JS$  and so  $\phi S = CS$  (since  $\phi^\uparrow J = d\phi$  and  $C^\uparrow J = dC$ , see Prop. 11.19). This however, is well known to be false: If  $CS = \phi S = (\mathbb{R}, \text{discrete uniformity})$ , there exist finitely many set maps  $f_1, \dots, f_n$  between the set  $\mathbb{R}$  and itself, satisfying  $\Delta = \bigcap_{i=1}^n (f_i \times f_i)^{-1} H$  where  $H = \{(x,y) \in \mathbb{R} \times \mathbb{R} : |x-y| < 1\}$ . Then one obtains an uncountable set of points in  $\mathbb{R}^n$  that are at distances of more than one unit from each other, which is impossible.  $\square$

16. Epireflections in UTopOrd induced by epireflections in PTych.

The result in this section provides a relationship between certain epireflectors in PTych and in UTopOrd. We need some categorical facts about PTych.

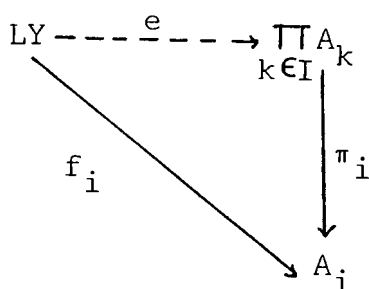
By a "closed" (resp. "dense") morphism  $f$  in  $\underline{PTych}$  we mean one for which  $Sf$  is closed (resp. dense); by a "closed" subspace  $X$  of  $Y \in \underline{PTych}$  we mean one for which  $SX$  is a closed subspace of  $SY$ , and by "closure" in  $X \in \underline{PTych}$  we mean that in  $SX$ . The epimorphisms in  $\underline{PTych}$  are the dense maps ([Salbany 1970, Chp.1, Prop. 3.4.1]) and the extremal monomorphisms are the closed embeddings ([Brümmer 1971, Thm. 2.5.7]).  $\underline{PTych}$  is complete, wellpowered and cowellpowered ([Brümmer 1971, Prop. 2.5.9]), and so is an (epi, extremal mono)-category ([Herrlich and Strecker 1973, Thm. 34.5]). Then, by [Herrlich and Strecker 1973, Thm. 37.4] the epi-reflective hull of any  $A \subset \underline{PTych}$  consists of closed subspaces of products of members of  $A$ .

16.1 Proposition Let  $A$  be a class of objects of  $\underline{PTych}$  that is initially dense in  $\underline{Pcreg}$ . Let  $\bar{\alpha}$  denote the reflector of  $\underline{Pcreg}$  onto the epi-reflective hull of  $A$  in  $\underline{PTych}$ , and let  $\alpha^\dagger$  denote the reflector of  $\underline{CTopOrd}$  onto the epi-reflective hull of  $MA$  in  $\underline{UTopOrd}$ . Then  $\bar{\alpha}L = L\alpha^\dagger$ .

Proof First we show that if  $Y$  is in the epi-reflective hull of  $MA$  in  $\underline{UTopOrd}$ , then  $LY$  is in the epi-reflective hull of  $A$  in  $\underline{PTych}$ .

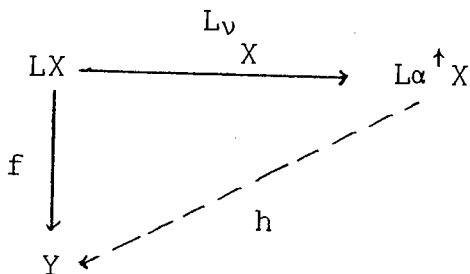
The source  $\underline{Pcreg}(LY, A)$ , where  $A$  ranges through  $A$ , is initial, since  $A$  is initially dense in  $\underline{Pcreg}$ . To any  $f \in \underline{Pcreg}(LY, A)$  assign the codomain restriction

$\tilde{f}: LY \rightarrow cl_A f(LY)$ . Two maps  $f, g \in \underline{P}reg(LY, A)$  will be called "equivalent" iff there is a homeomorphism  $h$  satisfying  $h\tilde{f} = \tilde{g}$ . Since the codomain restricted maps are all epis the cowellpoweredness of  $\underline{P}reg$  ([Brümmer 1971, Prop. 2.1.15]) shows that  $LY$  determines only a set of equivalence classes. Thus there is an initial set-indexed source  $(f_i: LY \rightarrow A_i)_{i \in I}$  representing the given one. An embedding  $e$  then exists that makes the following diagram commute for all  $i \in I$ :

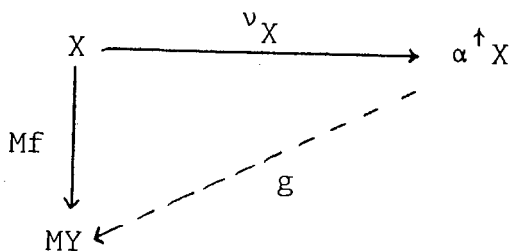


$Me: Y \rightarrow \prod_{k \in I} MA_k$  is an embedding; it is closed because this is a canonical construction for the reflection onto the epireflective hull of  $MA$  (see [Herrlich and Strecker 1973, Thms. 37.1 and 37.4]).  $LY$  is thus a closed subspace of a product of members of  $A$ , and so is in the epireflective hull of  $A$ .

Now we show that  $La^\uparrow$  has the universal property of  $\bar{\alpha}L$ . Fix  $X \in \underline{C}TopOrd$ . Then  $La^\uparrow X$  is in the epireflective hull of  $A$  in  $\underline{P}Tych$ , by the above discussion. Let  $v_X: X \rightarrow \alpha^\uparrow X$  be the reflection map associated with  $\alpha^\uparrow$ . Then the reflection map required here is  $Lv_X$ . To see this, let  $f: LX \rightarrow Y$  be any morphism in  $\underline{P}reg$  with  $\bar{\alpha}Y = Y$ . We seek a map  $h: La^\uparrow X \rightarrow Y$  making the following diagram commute:



By applying  $M$  to the diagram we get :



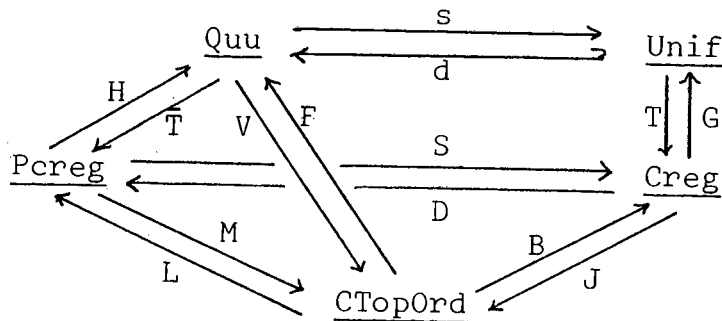
Now  $\bar{\alpha}Y = Y \Rightarrow \alpha^\uparrow MY = MY$  (since, if  $e: Y \rightarrow \Pi A$  is a closed embedding into some product of members of  $A$ , then  $Me: MY \rightarrow \Pi MA$  is a closed embedding into a product of members of  $MA$ ). Thus the universal property of  $v_X$  gives a morphism  $g: \alpha^\uparrow X \rightarrow MY$  satisfying  $g.v_X = Mf$ . The following composite then satisfies the requirements for  $h$  (where  $i$  is the (bicontinuous) identity map):  $L\alpha^\uparrow X \xrightarrow{Lg} LMY \xrightarrow{i} Y$ .  $\square$

16.2 Remark (1) Salbany's result ([1984]) that  $L\beta^\uparrow = \bar{\beta}L$  is a special case of Proposition 16.1.

(2) In general,  $M\bar{\alpha} \neq \alpha^\uparrow M$  since  $M\bar{\beta} \neq \beta^\uparrow M$  (see [Salbany 1984]).

17. Extensions and restrictions of completion-true sections

We return to the diagram (see §11)



where  $H$ ,  $F$  and  $G$  are  $\bar{T}$ -,  $V$ - and  $T$ -sections respectively. We now investigate various completion-true extensions and restrictions.

17.1 Proposition Let the  $\bar{T}$ -section  $H$  restrict to the  $V$ -section  $F$ . If  $H$  is  $\bar{Y}$ -true, so is  $F$ .

Proof  $H$   $\bar{Y}$ -true  $\Rightarrow H = \langle A \rangle_{\bar{T}}$  for some class  $A \subseteq Ob \ CplQuu$   
(by Remark 13.10)

$$\Rightarrow F = \langle A \rangle_{\bar{T}.L} = \langle A \rangle_V \quad (\text{by Prop. 9.1})$$

$$\Rightarrow F \text{ } \bar{Y}\text{-true} \quad (\text{by Prop. 13.9}). \quad \square$$

17.2 Proposition If the  $V$ -section  $F$  is  $\bar{Y}$ -true, then its coarsest  $\bar{T}$ -section extension,  $\bar{F}$ , is  $\bar{Y}$ -true.

Proof  $F = \langle R_F \rangle_V = \langle \bar{Y}R_F \rangle_V$  by Proposition 13.12.  $\bar{F} = \langle R_F \rangle_{\bar{T}}$  by Proposition 9.6, so  $\langle R_F \rangle_{\bar{T}} \leq \langle \bar{Y}R_F \rangle_{\bar{T}}$ . We show that

$\langle R_F \rangle_{\bar{T}} = \langle \bar{\gamma}R_F \rangle_{\bar{T}}$ . Fix  $X \in \text{Pcreg}$  and consider the construction of  $\langle \bar{\gamma}R_F \rangle_{\bar{T}} X$ :

$$\begin{array}{ccc}
 \langle \bar{\gamma}R_F \rangle_{\bar{T}} X & \xrightarrow[\text{initial}]{f'} & \bar{\gamma}FB = FV\bar{\gamma}FB, \text{ all } B \in \text{CTopOrd} \\
 \uparrow i & \nearrow \tilde{f} & \\
 \langle R_F \rangle_{\bar{T}} X & & \\
 \\ 
 X & \xrightarrow{f} & \bar{T}\bar{\gamma}FB = \bar{T}FV\bar{\gamma}FB,
 \end{array}$$

all  $f \in \text{Pcreg}(X, \bar{T}\bar{\gamma}FB)$ ,  
 all  $B \in \text{CTopOrd}$ .

(f') is the initial source lifted from (f) by the definition of  $\langle \bar{\gamma}R_F \rangle_{\bar{T}} X$ ; ( $\tilde{f}$ ) is lifted from (f) by the definition of  $\langle R_F \rangle_{\bar{T}} X$ . The identity set map  $i: \langle R_F \rangle_{\bar{T}} X \rightarrow \langle \bar{\gamma}R_F \rangle_{\bar{T}} X$  makes the diagram commute, and is uniformly continuous since (f') is initial. □

17.3 Proposition If the V-section  $F$  is strongly  $\bar{\gamma}$ -true, then its finest  $\bar{T}$ -section extension,  $\hat{F}$ , is strongly  $\bar{\gamma}$ -true.

Proof Clear, since  $M_{\hat{F}} = M_F$  (see Prop. 9.6). □

17.4 Remark Let the  $\bar{T}$ -section  $H$  restrict to the V-section  $F$ . We do not know whether  $H$  strongly  $\bar{\gamma}$ -true  $\Rightarrow F$  strongly  $\bar{\gamma}$ -true, nor whether  $\bar{T}\bar{\gamma}H$  idempotent  $\Rightarrow V\bar{\gamma}F$  idempotent.

17.7 Proposition (1) Let the V-section  $F$  restrict to the T-section  $G$ . Then:

(i)  $F$  strongly  $\bar{\gamma}$ -true  $\Rightarrow G$  strongly  $\gamma$ -true

(ii)  $F$   $\bar{\gamma}$ -true  $\Rightarrow G$   $\gamma$ -true

(iii)  $V\bar{\gamma}F$  idempotent  $\Rightarrow T\gamma G$  idempotent.

(2) The finest V-section extension,  $\tilde{G}$ , of a strongly  $\gamma$ -true T-section  $G$ , is strongly  $\bar{\gamma}$ -true.

(3) The coarsest V-section extension,  $\underset{\sim}{G}$ , of a  $\gamma$ -true T-section  $G$ , is  $\bar{\gamma}$ -true.

Proof (1) (i) We must show that  $\gamma M_G \subseteq M_G$ . Take  $X \in M_G$ , so that  $GTX \geq X$ .

$$\begin{aligned} \text{Then } dGT\gamma X &= FJT\gamma X \quad (\text{since } FJ = dG) \\ &= FVd\gamma X \quad (\text{since } JT = Vd, \text{ Lemma 10.6}) \\ &= FV\bar{\gamma}dX \quad (\text{since } d\gamma = \bar{\gamma}d, \text{ Prop. 10.1}). \end{aligned}$$

Now  $dX \in M_F$ , because  $FV(dX) = FJTX = dGTX \geq dX$ . Thus, since  $\bar{\gamma}M_F \subseteq M_F$ ,  $FV\bar{\gamma}dX \geq \bar{\gamma}dX$ . So  $dGT\gamma X \geq \bar{\gamma}dX = d\gamma X$ . Hence  $GT\gamma X \geq \gamma X$ .

(ii) Since  $F$  is  $\bar{\gamma}$ -true, its coarsest  $\bar{T}$ -section extension  $\underset{\sim}{F}$  is  $\bar{\gamma}$ -true (by Prop. 17.2). But  $\underset{\sim}{F}$  restricts to  $G$  (see Prop. 11.4(1)), so  $G$  is  $\gamma$ -true (by Prop. 17.5(1)).

$$\begin{aligned} \text{(iii) } T\gamma GT\gamma G &= T\gamma sFJT\gamma G \quad (\text{since } G = sFJ) \\ &= Ts\bar{\gamma}FVd\gamma G \quad (\text{since } \gamma s = s\bar{\gamma} \text{ and } JT = Vd) \\ &= BV\bar{\gamma}FV\bar{\gamma}dG \quad (\text{since } Ts = BV \text{ and } d\gamma = \bar{\gamma}d) \\ &= BV\bar{\gamma}FV\bar{\gamma}FJ \quad (\text{since } FJ = dG) \\ &= BV\bar{\gamma}FJ \quad (\text{since } V\bar{\gamma}F \text{ is idempotent}) \\ &= T\gamma G. \end{aligned}$$

- (2) Take  $Y \in M_G^\sim$ , i.e.  $Y \leq \tilde{G}VY$ . Then  $sY \leq s\tilde{G}VY \leq GBVY$  (by Lemma 17.6); but  $GBVY = GTsY$ , so  $sY \leq GTsY$ ; i.e.  $sY \in M_G$ . Then  $\gamma sY \in M_G$  by assumption, so  $s\bar{\gamma}Y \in M_G$ , i.e.  $\bar{\gamma}Y \in s^{-1}(M_G) = M_G^\sim$  (by Prop. 11.14). Hence  $\bar{\gamma}M_G^\sim \subseteq M_G^\sim$ .
- (3) If  $G$  is  $\gamma$ -true, then its coarsest  $\bar{T}$ -section extension  $G \approx \langle d(R_G) \cup \{\Pi_q\} \rangle_{\bar{T}}$  is  $\bar{\gamma}$ -true (by Prop. 17.5 (2)). But  $G \approx$  restricts to  $G \approx \langle d(R_G) \cup \{\Pi_q\} \rangle_V$ , so  $G \approx$  is  $\bar{\gamma}$ -true (by Prop. 17.1). □

17.8 Examples (1) By [Brümmer and Hager 1984, Example 18] there exists a  $T$ -section  $G$  for which  $T\gamma G$  is not idempotent. By Proposition 17.7 (1)(iii),  $V\bar{\gamma}F$  cannot be idempotent for any  $V$ -section extension  $F$  of  $G$ .

(2) (i)  $C^{*\dagger}$  is strongly  $\bar{\gamma}$ -true: By [Brümmer and Hager 1987, §7.6]  $C^*$  is strongly  $\gamma$ -true. Since  $C^{*\dagger}$  is the unique  $V$ -section extension of  $C^*$  (see Prop. 11.21), Proposition 17.7 (2) shows that  $C^{*\dagger}$  is strongly  $\bar{\gamma}$ -true. (Alternatively, one may see this by noting that  $M_{C^{*\dagger}}$  consists of the totally bounded quu spaces, since  $M_{C^{*\dagger}} = M_{\bar{C}^*}$  by Proposition 9.6(4), and  $M_{\bar{C}^*}$  consists of the totally bounded quu spaces [Brümmer 1977, Prop. 3.4].)

(ii)  $\phi^\dagger$  is strongly  $\bar{\gamma}$ -true.

(3)  $C^\dagger$  is  $\bar{\gamma}$ -true but not strongly  $\bar{\gamma}$ -true: We noted in 13.11 that  $C^\dagger$  is  $\bar{\gamma}$ -true. Brümmer and Hager have shown (see [Brümmer 1986]) that  $C$  is not strongly  $\gamma$ -true. Since  $C^\dagger$  restricts to  $C$ , Proposition 17.7 (1)(i) shows that  $C^\dagger$  cannot be strongly  $\bar{\gamma}$ -true.

17.9 Proposition Let the  $\bar{T}$ -section  $H$  restrict to the  $V$ -section  $F$ , and let  $F$  restrict to the  $T$ -section  $G$ .

Then :

- (1)  $E^\dagger(F) \subseteq B^{-1}(E(G))$
- (2)  $\bar{E}(H) \subseteq S^{-1}(E(G))$
- (3)  $\bar{E}(H) \subseteq M^{-1}(E^\dagger(F))$

and

- (a)  $E(G) = J^{-1}(E^\dagger(F))$
- (b)  $E(G) = D^{-1}(\bar{E}(H))$
- (c)  $E^\dagger(F) = L^{-1}(\bar{E}(H))$ .

Proof (1) Take  $X \in E^\dagger(F)$ . Then  $sFX$  is a complete uniform space.  $GBX = sFJBX \geq sFX$  (since  $G = sFJ$  and  $JB \geq 1$ ), and since  $GBX$  and  $sFX$  have the same underlying topologies,  $GBX$  must be complete.

(2) Similar to (1).

(3) Take  $X \in \bar{E}(H)$ . Then  $HX$  is a complete quu space.  $FMX = HLMX \geq HX$  (since  $F = HL$  and  $LM \geq 1$ ), and since  $V(FMX) = MX = V(HX)$  (see Lemma 9.2),  $FMX$  must be complete.

(a), (b) and (c) are easy to show. □

17.10 Proposition (1)(i)  $E^\dagger(C^*\dagger) = B^{-1}(E(C^*))$

(ii)  $\bar{E}(\bar{C}^*) = S^{-1}(E(C^*))$

(iii)  $\bar{E}(\bar{C}^*) = M^{-1}(E(C^*\dagger))$

(2) (i)  $E^\dagger(\underset{\sim}{C}) \neq B^{-1}(E(C))$ , where  $\underset{\sim}{C}$  is the coarsest  $V$ -section extension of  $C$ .

(ii)  $\bar{E}(\underset{\sim}{C}) \neq S^{-1}(E(C))$ , where  $\underset{\sim}{C}$  is the coarsest  $\bar{T}$ -section extension of  $C$ .

Proof (1) (i) Take  $X \in B^{-1}(E(C^*))$ . Then  $BX \in E(C^*)$ , i.e.  $C^*BX$  is complete. It is also totally bounded (since the range of  $C^*$  consists of totally bounded objects - see the proof of [Brümmer 1977, Thm. 4.3]), and thus is compact.  $C^{*\uparrow}X$  is therefore complete.

(ii) Similar to (i).

(iii) Take  $X \in M^{-1}(E(C^{*\uparrow}))$ . Then  $MX \in E(C^{*\uparrow})$ , i.e.  $C^{*\uparrow}MX$  is complete. Since  $C^{*\uparrow}MX = \overline{C^*}LMX$  and the range of  $\overline{C^*}$  consists of totally bounded spaces (see [Brümmer 1982, §3.4]),  $C^{*\uparrow}MX$  is totally bounded. Thus  $T(sC^{*\uparrow}MX)$  is compact. However  $TsC^{*\uparrow}MX = Ts\overline{C^*}X$ , so  $\overline{C^*}X$  is complete.

(2) (i) We show that  $\mathbb{R}_0 \in B^{-1}(E(C))$  but  $\mathbb{R}_0 \notin E^{\uparrow}(C)$ .  $T\gamma C(B\mathbb{R}_0) = T\gamma C\mathbb{R} = \nu\mathbb{R} = \mathbb{R}$ , since  $\mathbb{R}$  is realcompact; thus  $B\mathbb{R}_0 \in E(C)$ .

$$\begin{aligned} \text{Now } V\overline{\gamma} \underset{\approx}{C} \mathbb{R}_0 &= M\overline{T}\overline{\gamma} \underset{\approx}{C} L \mathbb{R}_0 \quad (\text{since } V = M\overline{T} \text{ and } \underset{\approx}{C} = \underset{\approx}{C}L) \\ &= M\overline{T}\overline{\gamma} \underset{\approx}{C} \mathbb{R}_b \quad (\text{since } L\mathbb{R}_0 = \mathbb{R}_b, \text{ see Example 8.3}) \\ &= M\overline{T}\overline{\gamma} \overline{C^*}\mathbb{R}_b \quad (\text{since } \overline{C^*}\mathbb{R}_b = \underset{\approx}{C}\mathbb{R}_b, \text{ see the} \\ &\text{proof of [Brümmer 1982, Cor. 5.8]}) \\ &= M\overline{\beta} \mathbb{R}_b . \end{aligned}$$

Applying  $B$  to the above equation yields  $BV\overline{\gamma} \underset{\approx}{C} \mathbb{R}_0 = BM\overline{\beta} \mathbb{R}_b = S\overline{\beta} \mathbb{R}_b$ . The bispace  $\overline{\beta} \mathbb{R}_b$  is compact, so  $S\overline{\beta} \mathbb{R}_b$  is compact. However  $\mathbb{R} (= B\mathbb{R}_0)$  is not compact, so  $V\overline{\gamma} \underset{\approx}{C} \mathbb{R}_0 \neq \mathbb{R}_0$ .

(ii) Similar to (i):  $\mathbb{R}_b \in S^{-1}(E(C))$  but  $\mathbb{R}_b \notin \overline{E}(\underset{\approx}{C})$ .  $\square$

17.11 Remarks (1) By Proposition 17.10 (1)(i) and Remark 15.3,  $E^\uparrow(C^{*\uparrow})$  consists of exactly those spaces in CTopOrd with compact underlying topologies.

(2) If  $F_1 \leq F_2$  are V-sections restricting to the T-section  $G$ , then  $E^\uparrow(F_1) = B^{-1}(E(G)) \Rightarrow E^\uparrow(F_2) = B^{-1}(E(G))$ .

We pose two questions:

17.12 Questions (1) Is  $E(H) = M^{-1}(E^\uparrow(F))$  for any  $\bar{T}$ -section restricting to some V-section  $F$ ?

(2) Given a T-section  $G$ , can one find a V-section extension  $F$  of  $G$  such that  $E^\uparrow(F) = B^{-1}(E(G))$ ?

Equivalently, does the finest V-section extension of  $G$  satisfy the equation?

## 18. Some non-unique extensions

We use techniques developed in the previous sections to show that certain T-sections do not have unique V-section extensions.

18.1 Proposition The T-section  $C$  has at least two distinct V-section extensions:  $\underline{C}$ , the coarsest, and  $C^\uparrow$ , the one spanned by  $\mathbb{R}_q$ .

Proof We show that  $V\bar{Y}\underline{C}R_0 \neq V\bar{Y}C^\uparrow R_0$ .  $V\bar{Y}C^\uparrow R_0 = \upsilon^\uparrow R_0 = R_0$  since  $\upsilon^\uparrow$  is the reflector onto the bireflective hull of  $R_0$  in UTopOrd. However,  $V\bar{Y}\underline{C}R_0 \neq R_0$  (see the proof of Prop. 17.10 (2)(i)). □

18.2 Proposition Let  $G$  be any T-section finer than  $C$ . Then the coarsest and finest V-section extensions of  $G$  are distinct.

Proof Recall that the coarsest and finest V-section extensions of  $G$  are  $\underline{G} = \langle d(R_G) \cup \{\Pi_q\} \rangle_V$  and  $\tilde{G} = \langle s^{-1}(M_G) \rangle_V$ , respectively.

Now  $\langle d(R_G) \cup \{\Pi_q\} \rangle_V R_0 = \langle d(R_G) \cup \{\Pi_q\} \rangle_{\bar{T}} L R_0$  (Prop. 9.1)  
 $= \langle d(R_G) \cup \{\Pi_q\} \rangle_{\bar{T}} R_D$  (since  $L R_0 = R_D$ , see Example 8.3)  
 $< \langle s^{-1}(M_G) \rangle_{\bar{T}} R_D$  (see proof of [Brümmer 1982, Cor. 5.9])  
 $= \langle s^{-1}(M_G) \rangle_V R_0$ . □

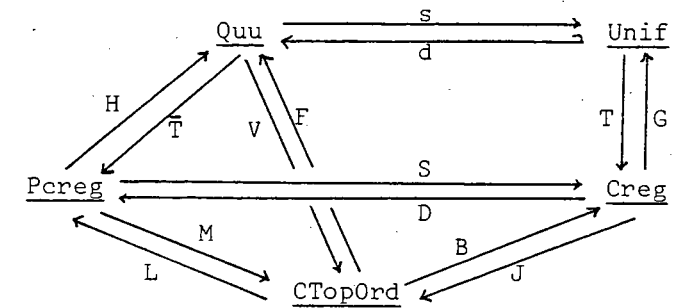
19. Notes

Many of our results in this section are analogous to results of Brümmer and Hager; in particular, our sources were :

- [Brümmer and Hager 1984, §10, 11, 8, 11, 12] for our §13.2, 13.9, 15.1, 15.2, 15.5,
- [Brümmer and Hager 1988, §12, 7, 10, 14, 15] for our §13.3, 13.4, 13.5, 13.12, 13.13,
- [Brümmer and Hager 1987, §7.4] for our §13.7, and
- [Brümmer 1982, §5.8, 5.9] for our §18.1 and 18.2.

I was provided with a manuscript of [Brümmer and Hager 1988]; consequently I was able to formulate 13.12 and 13.16 in analogy to corresponding results for T-sections. I wrote out the present proof for 16.1 because [Brümmer 1979, Prop. 3.3] provided no proof for the original result (which had D: Creg  $\rightarrow$  Pcreg in the place of the present L: CTopOrd  $\rightarrow$  Pcreg). I thank my supervisor for the suggestion to investigate relations between  $E(G)$ ,  $E^\dagger(F)$  and  $\bar{E}(H)$ , as in 17.9.

FIGURE 1



The functors in this diagram are defined in the following items of the text :

<u>Functor</u>	<u>Item</u>
T, $\bar{T}$	Chapter 0
V	3.10
s, d, S, D	§10
B	10.2
J	10.3
M	6.1
L	8.2
G (any T-section)	§7
F (any V-section)	§7
H (any $\bar{T}$ -section)	§7

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