

Inflation Modelling for Long-term Liability Driven Investments

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

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Abstract

A regime-switching model allows a process to switch randomly between different regimes which have different parameter estimates. This study investigates the use of a two regime-switching model for inflation in South Africa as a means of determining a hedging strategy for inflation linked liabilities of a financial institution. Each regime is modeled using an autoregressive process with different parameters and the change in regimes is governed by a two state Markov chain. Once the parameters have been estimated, the predictive validity of the regime-switching process as a model for inflation in South Africa is tested and a hedging strategy is outlined for a set of inflation linked cash flows. The hedging strategy is to invest in inflation linked bonds, the number of which is determined through the use of a Rand-per-point methodology that is applied to the inflation linked cash flows and inflation linked bonds. Over the period from January 2008 to June 2013 this hedging strategy was shown to be profitable.

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Chapter 1

Introduction

There are certain financial institutions that have long-term liabilities, such as pension funds and life insurance. These institutions will attempt to invest in assets in order to ensure that cash is available when the liabilities are required to be paid. The asset classes invested in are chosen with the aim of hedging the liabilities of the institution, hence this form of investment is known as liability-driven investment (LDI). The financial institutions will engage in asset-liability modelling in order to effectively hedge their liabilities exposure.

There are many sources of uncertainty in asset-liability models as the stochastic asset models used are not necessarily an indication of what will happen in reality. This may result in a divergence between the model outputs and the real-world economic developments.

One of the sources of uncertainty in the modelling of liabilities is inflation over the period. As a result, it is very important to have models of future inflation that follow real-world inflation growth as closely as possible when determining the financial institution's exposure to its liabilities, particularly in the long-term. An effective model for inflation will assist in more effective liability-driven investing.

Inflation is defined as a “sustained increase in the general price level of goods across the economy” (Malvaez, 2005). The South African Reserve Bank (SARB) aims to maintain the level of inflation within a target band of three to six percent. This inflation targeting policy was implemented in 2000. The inflation rate in South Africa is estimated using the Consumer Price Index (CPI) (Leung and Wu, 2011). The CPI measure consists of a basket of weighted goods and services. The CPI in South Africa is compiled by Stats SA.

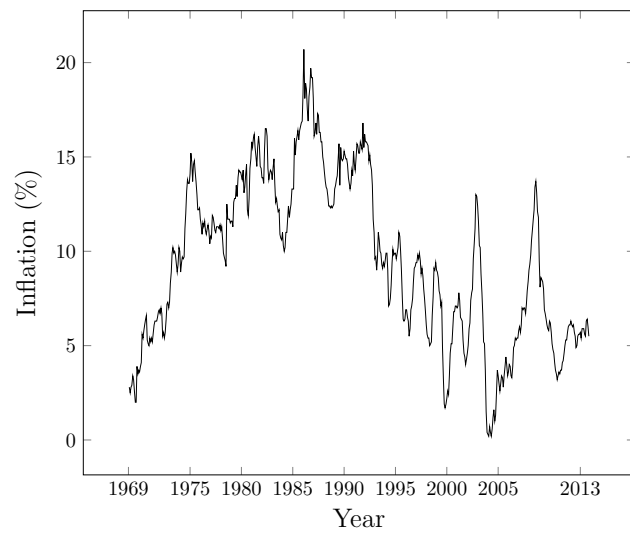


Fig. 1.1: Year-on-year inflation for January 1969 to October 2013

Chapter 2

Regime-Switching Process

This paper considers a regime-switching process as a model for inflation in South Africa. A regime-switching lognormal process is a method used to incorporate stochastic volatility into a model (Hardy, 2001). This approach is able to capture more extreme values than a general independent lognormal approach. It allows the process to switch randomly between N regimes, each of which has different model parameters. At any time the process is assumed to be Markov. The Markov property in the regime-switching model ensures that the probability of changing the regime is dependent only on the current regime. Maitland (2011) suggests that the regime-switching model is a reasonable model for inflation in South Africa, particularly when modelling long-term inflation projections.

The main attraction of the regime-switching process is that it is more representative of the market than the usual stochastic process models, which assume lognormality and constant parameters over the time period of the model. This is due to regime-switching models allowing the model to switch between different states of volatility at different times. In the case of inflation that is modeled using a two-regime Markov switching process, the two regimes could be chosen as a high-inflation regime and a low-inflation regime (Maitland, 2011). It should be noted that multiple regimes can be used.

Switching between regimes allows the model to account for changes in monetary policy, such as a change in the repo rate by the Monetary Policy Committee (MPC), as well as accounting for changes in variables during economic downturns (Hamilton, 2008).

However, there are disadvantages to using a Markov regime-switching process. Since the regime-switching model assumes that the inflation process is Markov, the price process is memoryless. Silvestrov and Stenberg (2004) state that this is a major disadvantage of the model since, empirically, inflation is not memoryless. Monetary policy decisions, which affect inflation, are made based on economic developments of the recent past. Hence, the use of a memoryless process is certainly a disadvantage

when modelling inflation as it does not reflect the actual economic conditions.

Chapter 3

Methodology and Data

3.1 Data

South African inflation data from January 1969 to October 2013 and real yield curves and nominal swap curves from 30 January 2008 to 7 January 2014 were provided by Old Mutual Specialised Finance. The nominal swap curve only has annual rates after two years and therefore linear interpolation is used to obtain rates on a monthly basis. The real yield curves and the CPI index are used to calculate the prices of inflation linked bonds over this period.

Scaled real and nominal cash flows from a pension fund were provided by Old Mutual Liability Driven Investments. The nominal cash flows are transformed from the real cash flows through the use of the inflation projections from the regime-switching inflation model.

3.2 Regime-switching methodology

Maitland (2011) proves that the model with two regimes fits South African inflation data the best. This is proved through the use of a likelihood ratio test.

Consider the interval $[t, t + 1)$, where each time point is in months. Let ρ_t denote the regime applying in this interval, where $\rho_t = 1, 2$. Let y_t be the annual CPI reading in the month $t + 1$.

Hamilton (2008) describes a process for a change in regimes as an autoregressive process of order one (AR(1)), which is written as

$$y_t = c_{\rho_t} + \phi_{\rho_t} y_{t-1} + \varepsilon_{\rho_t}, \quad (3.1)$$

where $\varepsilon_{\rho_t} \sim \mathcal{N}(0, \sigma_{\rho_t}^2)$, c_{ρ_t} is the intercept term of the equation and ϕ_j is the parameter that indicates the stability of the model. If $|\phi_j| = 1$ then the model has infinite variance and is not stationary (Hamilton, 1994). If $|\phi_j| < 1$ then the model is stationary. Under this model, each parameter will take on a different value under a different regime.

Maitland (2011) determined that the best model for inflation under the Markov switching regime is the AR(1) model with switching only in the intercept term. In this case Equation (3.1) is written as

$$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_t, \quad (3.2)$$

where $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$. Under this model, only the intercept term will switch values when there is a regime change, whereas the values ϕ and σ remain constant through regime changes.

Another model described by Simon (1996) is the AR(1) model with switching in the intercept term and in the variance term,

$$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_{\rho_t}, \quad (3.3)$$

where $\varepsilon_{\rho_t} \sim \mathcal{N}(0, \sigma_{\rho_t}^2)$. Under this model, the values of the intercept term and the variance term switch values when there is a regime change, whereas the value of ϕ remains constant through regime changes.

In order to forecast the above autoregressive models we need to have an estimate of the probability of moving from one regime to another. The model that gives the probability of changing from $\rho_t = 1$ to $\rho_t = 2$ and vice versa is a two-regime Markov chain. These probabilities are represented within a transition matrix P , where the matrix is of size $N \times N$ for an N -regime process. The elements of this matrix for each node are given by

$$P_{ij} = \mathbb{P}[\rho_t = j | \rho_{t-1} = i], \quad (3.4)$$

for $i, j = 1, 2$. The dependence on only the previous observation is a result of the Markov property.

The probability that inflation has transitioned to the alternative regime can be estimated by maximum likelihood estimation (Simon, 1996). The maximum likelihood estimate (MLE) of the probability that the process is in one of the regimes is calculated by dividing the number of transitions from regime i to regime j by the total number of transitions from regime i . This is given by Lee *et al.* (1968) as

$$\hat{P}_{ij} = \frac{n_{ij}}{\sum_{j=1}^2 n_{ij}}, \quad (3.5)$$

where n_{ij} is the number of times that regime i is followed by regime j . In the context of the inflation model, let the high inflation regime be regime i and let the low inflation regime be regime j . To determine n_{ij} a value that separates the high inflation and the low inflation regime must be chosen. This value can be chosen as six percent when considering the inflation targeting band of three to six percent in South Africa. An inflation reading greater than or equal to six percent in a

particular month would then be in the high inflation regime and an inflation reading below six percent in a particular month would then be in the low inflation regime. A movement in inflation from greater than or equal to six percent to below six percent would then add a count of one to the value n_{ij} .

The sum of these probabilities will then equal one. Simon (1996) defines a prevailing state as the state that has a probability estimate of greater than 50 percent.

The parameters of the autoregressive equations (3.1) to (3.3) can be estimated once the probabilities of moving from one regime to another are known. For the two-regime conditionally independent model with switching in all terms there are six parameters to be estimated, $\Theta = \{c_1, c_2, \phi_1, \phi_2, \sigma_1, \sigma_2\}$. For the two-regime autoregressive model with switching only in the intercept term there are four parameters to be estimated, $\Theta = \{c_1, c_2, \phi, \sigma\}$. For the two-regime autoregressive model with switching in the intercept term and the variance term there are five parameters to be estimated, $\Theta = \{c_1, c_2, \phi, \sigma_1, \sigma_2\}$. These parameters are estimated by maximising the likelihood function over the parameters.

The following process for determining the maximum likelihood estimates for Θ is outlined by Hamilton (2008).

For Equations (3.1), (3.2) and (3.3), y_t can be observed directly in the market, however, the value of ρ_t is determined based on the history of y_t and takes the form of two probabilities

$$\xi_{jt} = \mathbb{P}(\rho_t = j | \omega_t; \Theta), \quad (3.6)$$

for $j = 1, 2$, where these two probabilities must sum to one and the set of observations obtained at time t is given by

$$\omega_t = \{y_t, y_{t-1}, \dots, y_1, y_0\},$$

and Θ is a vector of population parameters. The value of the probabilities of being in a particular regime for each point in time are calculated iteratively for $t = 1, 2, \dots, T$, where for time t the probability is

$$\xi_{i,t-1} = \mathbb{P}(\rho_{t-1} = i | \omega_{t-1}; \Theta), \quad (3.7)$$

for $i = 1, 2$.

The next step is to calculate the densities of the two regimes at time t . This is then multiplied by the probability of transitioning into each of these regimes and by the probability of being in each regime at the previous time point. This gives the density of the observation at time t (Hamilton, 2008).

Consider Equation (3.2). The distribution of the first observation, y_1 , is required. The mean and variance of the first observation were obtained by Hamilton (1994)

by expanding Equation (3.2) as follows

$$\begin{aligned} y_t &= (c_{\rho_t} + \varepsilon_t) + \phi(c_{\rho_t} + \varepsilon_{t-1}) + \phi^2(c_{\rho_t} + \varepsilon_{t-2}) + \phi^3(c_{\rho_t} + \varepsilon_{t-3}) + \dots \\ &= \frac{c_{\rho_t}}{1 - \phi} + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \phi^3\varepsilon_{t-3} + \dots, \end{aligned} \quad (3.8)$$

where the sum of an infinite geometric series is used and $\rho_t = 1, 2$. Since $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$, taking expectations of Equation (3.8) gives

$$\mathbb{E}[y_t] = \frac{c_{\rho_t}}{1 - \phi}. \quad (3.9)$$

Therefore,

$$\mathbb{E}[y_1] = \mu = \frac{c_{\rho_1}}{1 - \phi}.$$

The variance of the first observation is then given by Hamilton (1994) as

$$\begin{aligned} \mathbb{E}[y_1 - \mu]^2 &= \mathbb{E}[\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \phi^3\varepsilon_{t-3} + \dots]^2 \\ &= (1 + \phi^2 + \phi^4 + \phi^6 + \dots)\sigma^2 \\ &= \frac{\sigma^2}{1 - \phi^2}. \end{aligned}$$

Since ε_t is normally distributed, y_1 is normally distributed. The probability density function of y_1 for the two regimes is then given by

$$f(y_1 | \rho_1 = j; \Theta) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2/(1 - \phi^2)}} \exp\left[-\frac{\{y_1 - [c_j/(1 - \phi)]\}^2}{2\sigma^2/(1 - \phi^2)}\right], \quad (3.10)$$

for $j = 1, 2$ (Hamilton, 1994). From Equation (3.2)

$$y_2 = c_j + \phi y_1 + \varepsilon_2. \quad (3.11)$$

Conditional on y_1 , $y_2 \sim \mathcal{N}(c_j + \phi y_1, \sigma^2)$. Hence, the density function of y_2 given y_1 for the two regimes is given by

$$f(y_2 | \rho_1 = j, y_1; \Theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y_2 - c_j - \phi y_1)^2}{2\sigma^2}\right], \quad (3.12)$$

for $j = 1, 2$ (Hamilton, 1994).

The above procedure from Equation (3.10) to Equation (3.12) can be repeated iteratively up until time t , giving the densities under the two regimes as

$$\eta_{jt} = f(y_t | \rho_t = j, \omega_{t-1}; \Theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y_t - c_j - \phi y_{t-1})^2}{2\sigma^2}\right], \quad (3.13)$$

for $j = 1, 2$ (Hamilton, 2008).

The conditional density of the observation at time point t is calculated as

$$f(y_t | \omega_{t-1}; \Theta) = \sum_{i=1}^2 \sum_{j=1}^2 p_{ij} \xi_{i,t-1} \eta_{jt}. \quad (3.14)$$

The above equation takes the conditional density of a particular regime and multiplies it by the probability of being in that regime at the previous time step and the probability of transitioning to that regime. This process weights the conditional densities of each regime before summing them to determine the conditional density of the observation at time point t .

The probability of being in regime j , for $j = 1, 2$, is then determined by calculating

$$\xi_{jt} = \frac{\sum_{i=1}^2 p_{ij} \xi_{i,t-1} \eta_{jt}}{f(y_t | \omega_{t-1}; \Theta)}. \quad (3.15)$$

After executing this iteration, the sample conditional log likelihood of the observed data is calculated as

$$\log f(y_1, y_2, \dots, y_T | y_0; \Theta) = \sum_{t=1}^T \log f(y_t | \omega_{t-1}; \Theta), \quad (3.16)$$

for the given value of Θ . The value of Θ is estimated by maximising the sample conditional log likelihood for the observed data by numerical optimisation, in which different guesses for the value of Θ are made. From these guesses the corresponding value for Equation (3.16) can be determined and are then used to determine the value $\hat{\Theta}$ for which Equation (3.16) is largest.

The numerical maximisation methods described by Hamilton (1994) include a grid search method, a steepest-ascent method, the Newton-Raphson method and the Davidon-Fletcher-Powell method.

The grid search method is only effective when a single parameter needs to be estimated. In this case there are multiple parameters so the grid search method is not appropriate. The steepest-ascent method is a better alternative as it can handle the estimation of multiple parameters, however it may require a large number of iterations. The Newton-Raphson method converges quicker than the steepest-ascent method provided that the second derivative of the log likelihood function exists and that the log likelihood function is concave.

The Newton-Raphson method that will be used here is described in Hamilton (1994). The first and second derivatives of the log likelihood function for each model with respect to each of the parameters is required and will be estimated using finite central differences.

Hamilton (2008) provides several options for the value ξ_{i0} that will be used to start the iterations. If the Markov chain is presumed to be ergodic, one can use the unconditional probabilities

$$\xi_{i0} = \mathbb{P}(\rho_0 = i) = \frac{1 - P_{jj}}{2 - P_{ii} - P_{jj}}. \quad (3.17)$$

A Markov chain that is ergodic allows for transitions from each state to every other state. In other words, there are no absorbing states.

The Markov chain is ergodic if $P_{11} < 1$, $P_{22} < 1$ and $P_{11} + P_{22} > 0$ (Hamilton, 1994). The above equation represents the unconditional probability that the process is in state j . Hamilton (2008) also suggests setting $\xi_{i0} = \frac{1}{2}$ or estimating ξ_{i0} using maximum likelihood estimation.

3.2.1 Model Selection

Makridakis *et al.* (1979) made use of three methods to compare the accuracy of multiple forecasting methods on 111 time series. The methods used are the mean average percentage error (MAPE), the mean square error (MSE) and Theil's U -coefficient. All of these methods require calculation of the error of the model. The error of the model at time t is given by Makridakis *et al.* (1979) as

$$e_t = y_t - \hat{y}_t, \quad (3.18)$$

where y_t is the actual value at time t and \hat{y}_t is a single-period-ahead forecast.

The MSE methodology was developed by Allen (1971) as an alternative to the use of the residual sum of squares to choose variables. The residual is the same as the error as calculated above.

The MSE methodology consists of taking the expectation of the square of Equation (3.18). The calculation of MSE is

$$MSE = \frac{1}{n} \sum_{i=1}^n e_i^2, \quad (3.19)$$

where n is the number of observations (Makridakis *et al.*, 1979). The disadvantage of using the MSE is that it cannot be used to compare across different time series.

The MAPE is calculated by Makridakis *et al.* (1979) as

$$MAPE = \frac{1}{n} \sum_{i=1}^n \left(\frac{|e_i|}{y_t} \right) (100). \quad (3.20)$$

This method can be used to determine which forecasting model is the best fit for the data without taking into account the size of the errors of the forecasting models.

Theil's U -coefficient is a method that was developed by Theil *et al.* (1966) that measures the extent to which two time series differ from each other. This statistic is given as

$$U = \sqrt{\frac{\sum_{i=1}^{n-1} \frac{e_{i+1}^2}{y_i}}{\sum_{i=1}^{n-1} \frac{(y_{i+1} - y_i)^2}{y_i}}}. \quad (3.21)$$

The two time series used will be the observed sample and the forecast sample. A value above one means the forecast differs significantly from the observed time

series, while a value below one means the forecast is a fairly good fit for the data (Makridakis *et al.*, 1979).

There are models that use the MLE to determine the model that best fits the data. Using the MLE will invariably result in choosing the model with the highest number of dimensions (Schwarz, 1978). In order to avoid this issue, an adjustment must be made to the MLE in order to accurately account for the effect of dimensionality in the estimate. There are several methods that can be used for model selection. These include the Schwarz Information Criterion (SIC), the corrected Akaike Information Criterion (AICc) and the Hannan and Quinn criterion (HQ).

For the SIC, Schwarz (1978) states that the model that should be chosen is the model for which

$$-2 \log \mathcal{L}_j(\Theta) + (k_j + 1) \log(n), \quad (3.22)$$

is the largest, where $\mathcal{L}_j(\Theta)$ is the likelihood function for model j with parameters Θ , k_j is the dimensionality of the model, and n is the number of observations.

The model with the smallest AIC is the model that fits the data best. The AIC does not consistently choose the model that fits the data best (Hurvich and Tsai, 1989). The use of the AICc method instead of the AIC method solves this issue. The model that fits the data best is the model for which

$$-2 \log \mathcal{L}_j(\Theta) + \frac{2(k_j + 1)n}{n - k_j - 2}, \quad (3.23)$$

is minimised (Hurvich and Tsai, 1989).

The HQ model was used by Hannan and Quinn (1979) specifically to determine the autoregressive model that best fits the data. The model for which

$$-2 \log \mathcal{L}_j(\Theta) + 2(k_j + 1) \log \log n \quad (3.24)$$

is minimised is the model that best fits the data (Hannan and Quinn, 1979).

The results of the above model selection methods will be compared in order to determine whether a particular model is consistently recommended by the model selection methods.

3.2.2 Forecasting

The value of y_{t+1} of the AR(1) process can be forecast conditional on ω_t , ρ_{t+1} and Θ . This value is given by Hamilton (1994) as

$$\mathbb{E}[y_{t+1} | \rho_{t+1} = j, \omega_t; \Theta] = c_j + \phi_j y_t, \quad (3.25)$$

for $j = 1, 2$. Since we are in a two-state regime there are only two different forecasts. The forecast variable for the next time period is the sum of the forecasts for the two

regimes multiplied by $\xi_{j,t+1}$ for $j = 1, 2$, the probability of being in the j^{th} regime at the next time step (Hamilton, 1994). This gives the single-period forecast as

$$\mathbb{E}[y_{t+1}|\omega_t; \Theta] = \sum_{j=1}^2 (c_j + \phi_j y_t) \xi_{j,t+1}. \quad (3.26)$$

The above method of forecasting is only used for single period forecasting. The main use of this in the context of long-term inflation modelling is to determine which model fits the data best when performing model selection tests.

Consider an m -period forecast. In order to calculate this multi-period forecast, the law of iterated projections must be used. Let Ω_t and Ω_{t-1} be two information sets, with $\Omega_{t-1} \subset \Omega_t$. The law of iterated projections is then given by Sargent (1979) as

$$\mathbb{E}[\mathbb{E}(y_t|\Omega_t)|\Omega_{t-1}] = \mathbb{E}[y_t|\Omega_{t-1}]. \quad (3.27)$$

Consider Equation (3.1) with current time t for regime j , where $j = 1, 2$. This equation can be rewritten as

$$y_t = \mu + \phi_j(y_t - \mu), \quad (3.28)$$

where $\mu = c_j/(1 - \phi_j)$.

This relationship is shown below.

$$\begin{aligned} y_t &= \mu + \phi_j(y_t - \mu) \\ &= \frac{c_j}{1 - \phi_j} + \phi_j \left(y_t - \frac{c_j}{1 - \phi_j} \right) \\ &= \frac{c_j}{1 - \phi_j} + \phi_j \left(\frac{y_t(1 - \phi_j) - c_j}{1 - \phi_j} \right) \\ &= \frac{c_j + \phi_j y_t(1 - \phi_j) - \phi_j c_j}{1 - \phi_j} \\ &= \frac{c_j(1 - \phi_j) + \phi_j y_t(1 - \phi_j)}{1 - \phi_j} \\ &= c_j + \phi_j y_t. \end{aligned} \quad (3.29)$$

Iterating Equation (3.28) for m periods results in

$$y_{t+m} = c_j + \phi_j^m(y_t - \mu) + \varepsilon_{t+m} + \phi_j \varepsilon_{t+m-1} + \dots + \phi_j^{m-1} \varepsilon_{t+1}.$$

Using the law of iterated expectations produces

$$\begin{aligned}
\mathbb{E}[y_{t+1}|\rho_{t+1} = j, \omega_t; \Theta] &= \mu + \phi_j(y_t - \mu) \\
\mathbb{E}[y_{t+2}|\rho_{t+2} = j, \omega_t; \Theta] &= \mu + \phi_j(\mathbb{E}[y_{t+1}|\rho_{t+1} = j, \omega_t; \Theta] - \mu) \\
&= \mu + \phi_j(\phi_j(y_t - \mu)) \\
&= \mu + \phi_j^2(y_t - \mu) \\
&\vdots \\
\mathbb{E}[y_{t+m}|\rho_{t+m} = j, \omega_t; \Theta] &= \mu + \phi_j(\mathbb{E}[y_{t+m-1}|\rho_{t+m-1} = j, \omega_t; \Theta] - \mu) \\
&= \mu + \phi_j^m(y_t - \mu) \\
&= \frac{c_j}{1 - \phi_j} + \phi_j^m \left(y_t - \frac{c_j}{1 - \phi_j} \right)
\end{aligned} \tag{3.30}$$

To forecast m -periods ahead in the regime-switching model the probability of being in each regime at that time must be forecast. If the current state is given as $\rho_t = i$, then the probability of transitioning to state $\rho_{t+m} = j$ is determined using the matrix P^m , where the appropriate probability is given by the row j , column i element of the matrix (Hamilton, 1993). The probability of being in each regime can then be calculated as

$$\mathbb{E}[\xi_{t+m}|\xi_t, \xi_{t-1}, \dots, \xi_1, \omega_t] = P^m \xi_t, \tag{3.31}$$

where ξ_t is a vector containing the probability of being in each of the regimes at time t and P is the transition matrix. The answer will be a vector which holds the probability of being in each regime at time $t + m$.

In order to determine the optimal forecast given only the observed data up to date t , Hamilton (1993) applied the law of iterated expectations to Equation (3.31) to get

$$\mathbb{E}[\xi_{t+m}|\omega_t] = P^m \hat{\xi}_{t|t}, \tag{3.32}$$

where $\hat{\xi}_{t|t}$ is the optimal inference of the current probability of being in each regime.

The following process must be followed in order to calculate the optimal inference $\hat{\xi}_{t|t}$. Let $\hat{\xi}_{t|t-1}$ be an $N \times 1$ vector, where N is the number of regimes. The i^{th} element of the vector represents $\mathbb{P}[\rho_t = i|\omega_{t-1}]$, which is the probability of being in regime i at time t conditional on the observations up to time $t - 1$. Hamilton (1993) uses a filter to calculate $\hat{\xi}_{t|t-1}$ for every t . This filter requires iterating on

$$\hat{\xi}_{t+1|t} = \frac{P \cdot (\hat{\xi}_{t|t-1} \odot \eta_t)}{1'(\hat{\xi}_{t|t-1} \odot \eta_t)} \tag{3.33}$$

where \odot denotes element-by-element multiplication and 1 is an $N \times 1$ vector of ones. The starting point that Hamilton (1993) suggests for the iteration is

$$\hat{\xi}_{1|0} = \frac{1 - P_{jj}}{2 - P_{ii} - P_{jj}}. \tag{3.34}$$

Hamilton (1993) then calculates the required value $\hat{\xi}_{t|t}$ for Equation (3.32) as

$$\frac{(\hat{\xi}_{t|t-1} \odot \eta_t)}{1'(\hat{\xi}_{t|t-1} \odot \eta_t)}.$$

The optimal m -period ahead forecast is given by Hamilton (1994) as the expected value of each regime at time point $t + m$, where t is the current time, multiplied by the expected probability of being in each regime at time point m , given that at time t the prevailing regime is regime j . This is written as

$$\begin{aligned} \mathbb{E}[y_{t+m}|\omega_t; \Theta] &= \sum_{j=1}^2 \mathbb{E}[y_{t+m}|\rho_{t+m} = j, \omega_t; \Theta] \mathbb{E}[\xi_{t+1}|\omega_t] \\ &= \left[\sum_{j=1}^2 \frac{c_j}{1 - \phi_j} + \phi^m \left(y_t - \frac{c_j}{1 - \phi_j} \right) \right] P^m \hat{\xi}_{t|t}. \end{aligned} \quad (3.35)$$

A major disadvantage of using this method to forecast long-term inflation is that there is no random term in the forecast. Simulation can be used as an alternative forecasting mechanism to remedy this issue.

At the current time, the information known is the probability of being in each regime, the probability of transitioning from one regime to another, the conditional density of the observation and the value of inflation. The value of inflation one month from the current time can be calculated by using any of Equation (3.1) to Equation (3.3) and the probability of being in each regime at that time. Consider using Equation (3.3) as the autoregressive model. Then

$$y_{t+1} = (c_1 + \phi y_t + \varepsilon_1)\xi_{1t} + (c_2 + \phi y_t + \varepsilon_2)\xi_{2t}, \quad (3.36)$$

where $\varepsilon_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $\varepsilon_2 \sim \mathcal{N}(0, \sigma_2^2)$ are randomly generated from their respective Normal distributions.

The probability of being in each regime at $t + 1$ must now be calculated in order to simulate the value of inflation at time $t + 2$. This requires calculating $\eta_{j,t+1}$, the density of each regime at time $t + 1$, using Equation (3.13). Following this the conditional density of the observation at time point $t + 1$ must be calculated using Equation (3.14). Now the probability of being in each regime can be calculated using Equation (3.15) and the value of inflation at time $t + 2$ can be calculated by updating Equation (3.36). This process can be repeated up to $t + n$, where n can be any positive integer.

To determine the predictive validity of the simulated data a sample of data will be split into a testing sample and a validation sample. The testing sample will be used to estimate parameters and the validation sample will be the actual

observations that occur after the testing sample data period. The validation sample will be tested against the simulated sample, where the simulation uses the parameter estimates from the testing sample.

The two-sample Kolmogorov-Smirnov test can be used to test if the validation sample and the simulated sample come from the same continuous distribution (Massey, 1951). The continuous cumulative distribution function (CDF) of the simulated sample will be compared against the CDF of the observed sample for the same time period. This test tests the null hypothesis that the two samples come from the same continuous CDF at a chosen significance level.

The two-sample Kolmogorov-Smirnov test requires two samples of variables, where with each sample each variable must have the same continuous CDF (Smirnov, 1948). Let the validation sample of variables be given by (y_1, \dots, y_m) with continuous CDF $F_m(y)$ and let the simulated sample of variables be given by $(\hat{y}_1, \dots, \hat{y}_n)$ with CDF $F_n(\hat{y})$.

The next step is to calculate step-functions for each sample (Smirnov, 1948). Let $M(z)$ be the number of observations from the sample (y_1, \dots, y_m) which have a value that is less than or equal to z . The step-function for this sample is then calculated as $F_m^*(z) = M(z)/m$. This is the probability of a value that is less than or equal to z occurring and hence represents the continuous CDF of the sample. Let $N(z)$ be the number of observations from $(\hat{y}_1, \dots, \hat{y}_n)$ which have a value less than or equal to z . The CDF of this sample is then represented by $F_n^{**}(z) = N(z)/n$. The test statistic $D_{m,n}$ is given by Smirnov (1948) as the maximum value of the difference $|F_m^*(z) - F_n^{**}(z)|$.

The critical value for the two-sample Kolmogorov-Smirnov test is dependent on the choice of significance level chosen. The critical values for a sample size greater than 35 observations are given by Massey (1951) in Table 3.1.

Tab. 3.1: Critical values of two-sample Kolmogorov-Smirnov test

	Significance level				
	0.20	0.15	0.10	0.05	0.01
Critical value	$\frac{1.07}{\sqrt{(N)}}$	$\frac{1.14}{\sqrt{(N)}}$	$\frac{1.22}{\sqrt{(N)}}$	$\frac{1.36}{\sqrt{(N)}}$	$\frac{1.63}{\sqrt{(N)}}$

For the two-sample Kolmogorov-Smirnov test, the value of N in Table 3.1 is $mn/(m+n)$ (Smirnov, 1948), where m is the size of the validation sample and n is the size of the simulated sample. If the test statistic $D_{m,n}$ is greater than the critical value at a significance level, then the null hypothesis that the two samples have the same continuous CDF is rejected at that significance level, else it is accepted. For

critical values for sample sizes lower than 35 see Massey (1951).

Chapter 4

Hedging real cash flows

Real cash flows of an institution that are to be paid out for the period 31 January 2008 to 30 June 2013 need to be hedged. It is assumed that the starting date is 1 January 2008.

The nominal value of the cash flows that the institution pays out will be dependent on the change in the level of the CPI index from the starting date to the date of each cash flow. For a cash flow at time t , the nominal value paid out on this date will be

$$Nominal = \frac{CPI_d}{CPI_{base}}, \quad (4.1)$$

where CPI_d is the level of the CPI at time d and CPI_{base} is the level of inflation at the base date, which in this case is 1 January 2008.

In order to hedge these cash flows, a simulation of inflation to each cash flow date is required. The regime-switching methodology provides parameter estimates that can be used to simulate year-on-year inflation, hence to obtain the simulated CPI value at time $d + 1$ we use the formula

$$CPI_{d+1} = CPI_{d-12+1}(1 + i), \quad (4.2)$$

where i is the regime-switching estimate of year-on-year inflation for that time period. This process is repeated iteratively to project CPI levels for the entire period from 31 January 2008 to 30 June 2013.

Once the nominal cash flows have been calculated using the simulation of CPI they are discounted using the nominal swap curve to calculate the value of the cash flows at the current time. The expected value of the cash flows can then be calculated as

$$\mathbb{E}[V] = \frac{1}{n} \sum_{i=1}^n V_i,$$

where n is the number of simulations and V_i is the value of simulation i .

The real cash flows are to be hedged by inflation-linked bonds. The pricing specifications for inflation-linked bonds in South Africa are provided in Bond Exchange

Tab. 4.1: Bond specifications

Bond	Issue date	Coupon	Maturity date
R189	20/03/2000	6.25%	31/03/2013
R197	30/05/2001	5.5%	07/12/2023
R202	15/08/2003	3.45%	07/12/2033

of South Africa (2006) and Bond Exchange of South Africa (2005).

There are certain CPI conventions used by Bond Exchange of South Africa (2005) that must be implemented in the South African market. It is assumed that the CPI figure for a certain month only applies to the first calendar day of that month. Linear interpolation between the CPI figure of that month and the next month is used in order to obtain CPI figures for any other day of that month. There is also a time lag in the publishing of the CPI figure for a particular month. The standard convention for determining the CPI figure for any date is to interpolate between the inflation figure four months ago and the inflation figure three months ago. For example, if we wish to value an inflation linked bond on 7 June 2013, we would linearly interpolate between the CPI figure for February 2013 and the CPI figure for March 2013.

The bonds that have been chosen to hedge the real cash flows are the R189, R197 and R202. The specifications of each bond are displayed in Table 4.1. The coupons are paid semi-annually and the bonds are priced using the bond pricing specifications in Bond Exchange of South Africa (2005). The only difference is the use of the real yield instead of the yield-to-maturity of a standard bond.

The Bond Exchange of South Africa (2006) price of the bonds is then adjusted for inflation using Equation (4.1). Similarly, the coupons are real cash flows, hence the actual nominal coupons must also be adjusted for inflation when paid out. In these cases the base CPI value will be the CPI value as calculated on the issue date of the bond. Using linear interpolation and the time lag convention, this base CPI is calculated as

$$CPI_{base} = CPI_j + \left[\frac{d-1}{D} \times (CPI_{j+1} - CPI_j) \right], \quad (4.3)$$

where CPI_j is the CPI figure of the month that precedes the issue date by four months, CPI_{j+1} is the CPI figure of the month that precedes the issue date by three months, d is the calendar date of the month of issue, for example $d = 7$ for a 7 June issue date, and D is the number of days in the month of issue.

The optimal hedge of the liability cash flows is found by calculating the Rand-per-point (RPP) values of the liability cash flows and each of the bonds. This value

is calculated by perturbing the nominal swap curve, which is used to discount the projected nominal cash flows, by one basis point (0.01%). However, inflation-linked bond prices are not determined using the nominal swap curve. In order to hedge the real cash flows with the inflation-linked bonds, the simulated inflation values will be used to determine the nominal value of the future coupons of each inflation-linked bond. This is done in the same manner as determining the nominal cash flows as described earlier. The value of the inflation-linked bond is then determined by discounting the simulated nominal coupons by the nominal swap curve. Now the RPP of the bonds can be calculated as well.

Risk buckets of length six months will be used. This means that all the interest rates in a six month period will be perturbed by one basis point. Once the nominal swap curve has been perturbed, the difference between the unperturbed value and the perturbed value of the liability cash flows and each bond is calculated. This change in value is the RPP. The above process is repeated after the RPP for the first bucketed period is calculated.

The RPP is used to determine the number of each bond to hold to hedge the liability cash flows. This is done by finding the number of the bonds that make the sum of the RPP of the liability cash flows and the RPP of the bonds as close to zero as possible and such that the absolute value of each separate RPP is minimised.

The RPP values are calculated at each time point during the hedging period. In order to do this the simulated CPI values need to be updated at each time point. In other words, after the January 2008 RPP values are calculated, the simulations are then run again from the actual February 2008 CPI value to the end point, which in our case is June 2013. This process is iterated repeatedly with each simulation being for one period less than the previous simulation. The last simulation will then be from May 2013 to June 2013.

The actual price of the inflation linked bonds at each time point is calculated using the actual recorded CPI values and real yields of each bond. The amount of each bond as calculated using the RPP values is then used to calculate the value of the bond holdings at each time point. The profit and loss at each point is calculated using the amount of each bond held at the earlier time point and the change in value of the bond. The actual nominal liabilities are subtracted from the profit and loss of the bonds to get the overall profit and loss at each time point. At each time point the profits and losses are invested at the prevailing risk-free rate, which is assumed to be the nominal swap curve rate. The total profit and loss from the time period is then calculated by adding all the profit and losses together.

Chapter 5

Results

Using Equation (3.5) and South African CPI data from 1969 to 2013, the probability of remaining in a high-inflation state is given by $p_{22} = 0.9811$ and the probability of remaining in a low-inflation state is given by $p_{11} = 0.9683$. These probabilities will remain the same for all variations of autoregressive regime-switching models used to model inflation for this particular data period.

The parameter estimates for Equation (3.1) to Equation (3.3) as determined by maximum likelihood estimation are shown in Table 5.1.

The intercept term of the high-inflation state is quite large in all of the models in Table 5.1 due to the particularly high-inflation that was observed in the period 1973 to 1993, in which inflation was well above 10%. These levels of inflation were as a result of the 1970's oil price shock and the sanctions placed on South Africa. The sanctions resulted in large scale disinvestment from South Africa, leading to depreciation of the Rand. This depreciation led to an increase in the price of imports and hence very high inflation levels. The introduction of inflation targeting in 2000 by the SARB means that those levels of inflation will be very unlikely to occur again and therefore adjustments must be made to the model for forecasting purposes. These adjustments may include reducing the probability of remaining in a high-

Tab. 5.1: Parameter estimates for AR(1) models

Model	High-inflation regime	Low-inflation regime
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_t$	$y_t = 0.0961 + 0.3451y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0162^2)$	$y_t = 0.0426 + 0.3451y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0162^2)$
$y_t = c_{\rho_t} + \phi_{\rho_t} y_{t-1} + \varepsilon_{\rho_t}$	$y_t = 0.0929 + 0.3458y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0128^2)$	$y_t = 0.0326 + 0.6099y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0133^2)$
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_{\rho_t}$	$y_t = 0.0848 + 0.4219y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0148^2)$	$y_t = 0.0367 + 0.4219y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0142^2)$

Tab. 5.2: Parameter estimates for AR(1) models for the data period from 1993 to 2013

Model	High-inflation regime	Low-inflation regime
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_t$	$y_t = 0.0545 + 0.4806y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0099^2)$	$y_t = 0.0269 + 0.4806y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0099^2)$
$y_t = c_{\rho_t} + \phi_{\rho_t} y_{t-1} + \varepsilon_{\rho_t}$	$y_t = 0.0540 + 0.4659y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0076^2)$	$y_t = 0.0177 + 0.6939y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0083^2)$
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_{\rho_t}$	$y_t = 0.0379 + 0.6618y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0075^2)$	$y_t = 0.0171 + 0.6618y_{t-1} + \varepsilon_t$ $\varepsilon_t \sim \mathcal{N}(0, 0.0069^2)$

Tab. 5.3: Model selection tests

Model	1963-2013			1993-2013		
	MSE	MAPE	Theil's U	MSE	MAPE	Theil's U
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_t$	0.00023762	35.8746%	4.3890	0.000079189	36.5463%	3.6382
$y_t = c_{\rho_t} + \phi_{\rho_t} y_{t-1} + \varepsilon_{\rho_t}$	0.00016065	30.0297%	4.0882	0.000054152	26.5859%	3.0174
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_{\rho_t}$	0.00018706	31.1166%	3.9875	0.000042998	24.9904%	2.8082

inflation regime and the probability of transitioning from the low-inflation regime to the high-inflation regime.

A more appropriate inflation model under these conditions would be to use the observations only from the period 1993 to 2013 when performing maximum likelihood estimation. The reason for using this data period is that the apartheid sanctions placed on South Africa were lifted at this time, fundamentally changing the structure of the economy. The parameter estimates for the models using the observations from this time period are displayed in Table 5.2.

The MSE, MAPE and Theil's U -coefficient for each model for each different time period is displayed in Table 5.3.

The model with the lowest MSE and MAPE for the period from 1963 to 2013 is the autoregressive model with switching in all terms, as shown in Equation (3.1). The Theil's U -coefficient for each model in this time period is well above one. The reason for this is the poor modelling of the extreme levels of inflation by the models. This can be seen in Figure 5.2. The model with the lowest MSE, MAPE and U -coefficient for the period from 1993 to 2013 is the autoregressive model with switching in the intercept term and in the variance term, as shown in Equation (3.3). All of these values are significantly lower than those from the period 1963 to 2013, suggesting

Tab. 5.4: Model selection values for MLE tests

Model	1963-2013			1993-2013		
	SIC	AICc	HQ	SIC	AICc	HQ
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_t$	-2,823.9	-2,832.5	-2,829.1	-1,515	-1,522	-1,519.2
$y_t = c_{\rho_t} + \phi_{\rho_t} y_{t-1} + \varepsilon_{\rho_t}$	-3,059	-3,067.6	-3,064.2	-1,633.5	-1,640.5	-1,637.7
$y_t = c_{\rho_t} + \phi y_{t-1} + \varepsilon_{\rho_t}$	-2,945.9	-2,954.4	-2,951.1	-1,653.6	-1,660.5	-1,657.8

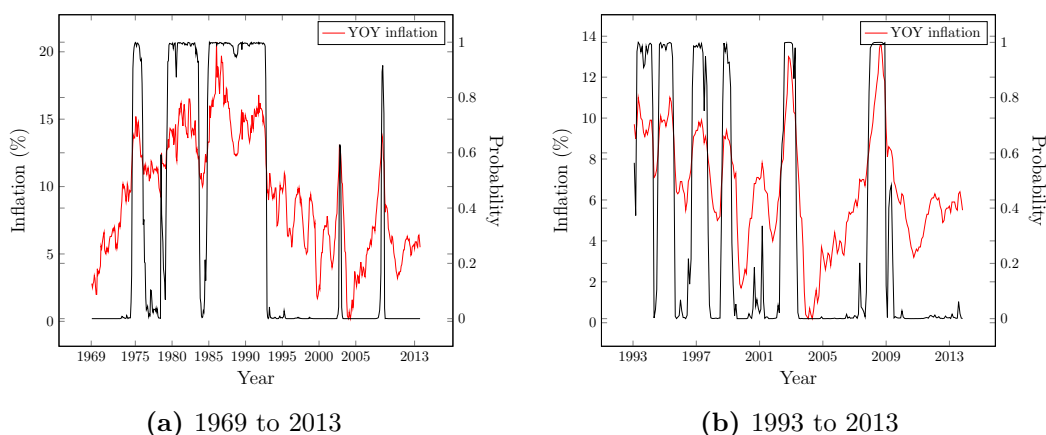


Fig. 5.1: Probability of being in a high-inflation state

that the use of a smaller data set may provide better results in the South African economy.

The values for the SIC, AICc and the HQ model selection methods for each of the three models for the period from 1963 to 2013 is shown in Table 5.4.

From these results it can be seen that the model that best fits the data from the period January 1969 to October 2013 is the autoregressive model with switching in all terms. This is due to this model displaying a minimum value for each model selection method.

The model that best fits the data from the period January 1993 to October 2013 is the autoregressive model with switching in the intercept and variance term. This indicates that the tests that use the MLE to determine the model that fits the data best provide the same results as the MSE, MAPE and Theil's U-coefficient test.

The probability of being in a high-inflation regime during the period January 1969 to August 2013 is compared to the actual inflation recorded during that period in Figure 5.1a for the model based on Equation (3.2). The probability of being in a high-inflation regime when using the data period from 1993 to 2013 to determine the estimates for the model given by Equation (3.3) is shown in Figure 5.1b.

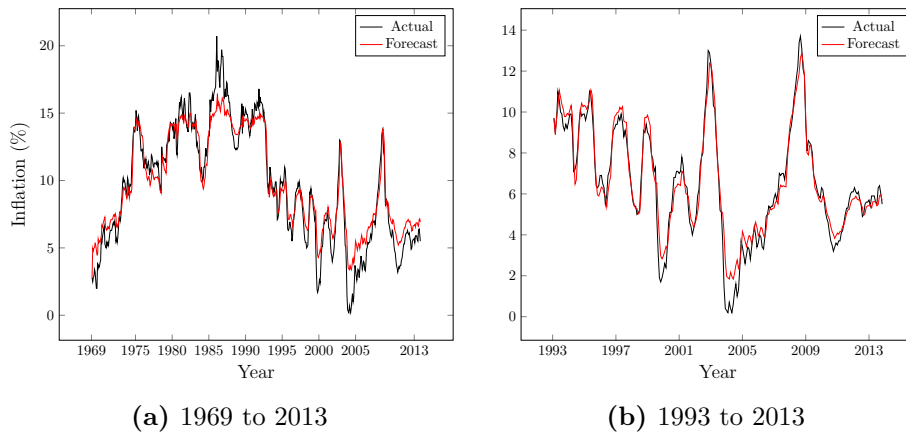


Fig. 5.2: Forecasted values of inflation

When considering the longer data period in Figure 5.1a it can be seen that generally when inflation is high, the probability of being in the high-inflation regime is high, which is desirable. However, when inflation fell in 1976 and 1984 the probability of being in the high-inflation regime fell to approximately zero even though inflation was well above 10%. This indicates that under this model, a fall in inflation may lead to a large understating of the probability of being in the high-inflation regime. The probabilities of being in a high-inflation regime for Equation (3.1) and Equation (3.3) are similar to Figure 5.1a.

When considering the shorter data period, it can be seen that Figure 5.1b provides an accurate representation of the probability of being in the high-inflation regime. There are no noticeable deviations from what would be expected, indicating that the model is a better fit for the data than the model determined using observations from the period 1969 to 2013.

The single-period forecasts for the period from 1969 to 2013 determined using Equation (3.26) are shown in Figure 5.2a, while the single-period forecasts for the period from 1993 to 2013 are displayed in Figure 5.2b.

When considering Figure 5.2a, the only periods where the actual data and the forecasted data differ significantly are in 1986, when inflation spiked to over 20 percent, and when inflation dropped below six percent in the early 2000's. This is due to the level of the mean force of inflation for the low inflation regime (Maitland, 2011). The mean force of inflation is given by Equation (3.9). The forecasted value of inflation is pulled towards this value, which in this case is 8.357%. Any values of inflation below this value will result in forecasts being pulled back towards this mean force of inflation.

When considering Figure 5.2b, the only deviations from the observed inflation

is at the levels of inflation below two percent due to the mean force of inflation for the low inflation regime, which is 5.61%.

Multi-period forecasts of length one-, two- and five-years are displayed in Figure 5.3. The forecasts shown in this figure are not rolling forecasts. This means that for the one-year forecast, for example, that from the current date a forecast is made for one-year using Equation (3.35). Then the next one-year forecast is made with the initial value as the actual inflation figure in one year's time. This process is repeated until the end of the data period. Similar processes can be followed for the five- and ten-year forecasts.

The multi-period forecasts also require the forecast of the probability of being in each regime for each future time point. These probabilities are calculated using Equation (3.31). Using a one-year forecast as an example, the forecast of the probability of being in each regime in one year's time will be given by $P^{12}\hat{\xi}_{t|t}$, where t is the current time and 12 is the number of months in the forecast. The above process accounts for the sharp increases and decreases in the multi-period forecasts as the initial value of each forecast gets set to the actual value of inflation at the time the forecast is made. This indicates that the forecasted value of inflation is very quickly pulled to the mean force of inflation value for the current regime, which is a significant disadvantage to using this form of multi-period forecasting.

A noticeable problem with the longer-term multi-period forecasts in Figure 5.3b and Figure 5.3c is that if inflation is in a particular regime when the forecast is made, the forecast values will likely remain in that regime for the duration of the forecasted period.

Long-term multi-period forecasts can also be simulated to allow for randomness in the forecast, unlike the previous method of long-term forecasting. Figure 5.4 displays four simulations compared to the actual CPI observed for the period January 2004 to October 2013. These four simulations were collected from a larger sample of simulations.

The benefit of using a two-regime process can be observed as the changes in inflation for the simulation appear to have a similar basic structure to the observed CPI. The structure referred to here is the movements between the low and high inflation levels and the volatility of inflation. The parameters that are used in Equation (3.36) to calculate the simulations are estimated over the period January 1994 to December 2003.

The predictive validity of the simulations for the period is determined through the use of a two-sample Kolmogorov-Smirnov test with a 99% significance levels. The results of this test for the four simulations are displayed in Figure 5.5. In order for the null hypothesis that the simulated sample and the observed sample come

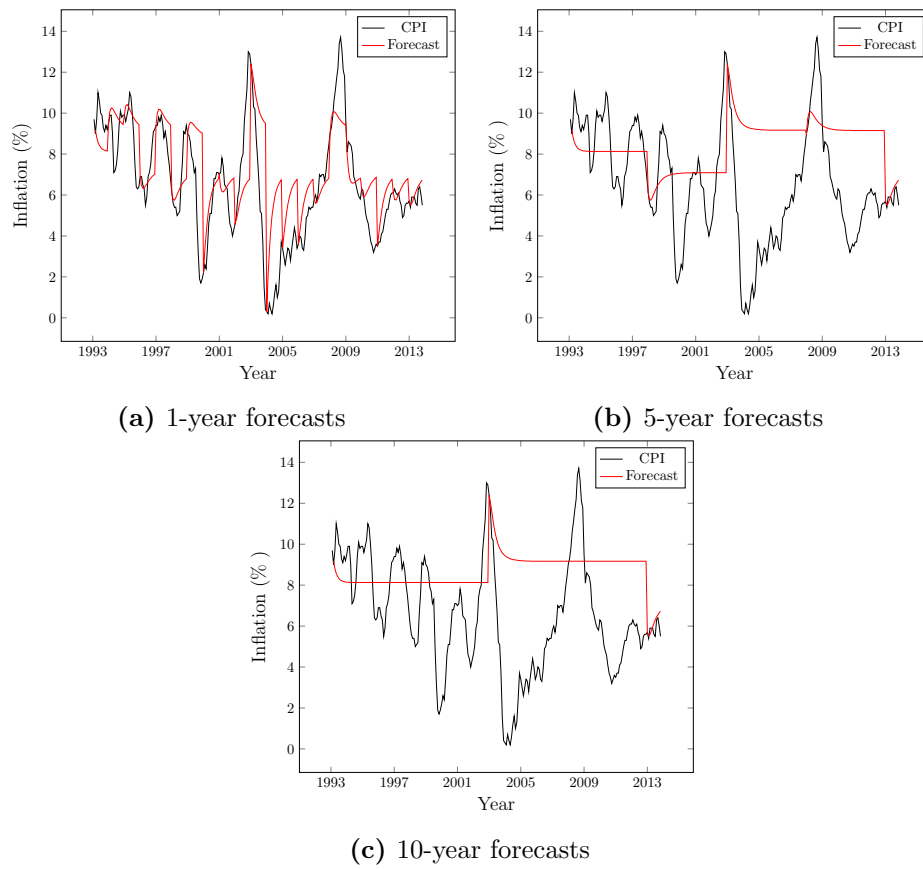


Fig. 5.3: Multi-period forecasts for the data period 1993 to 2013

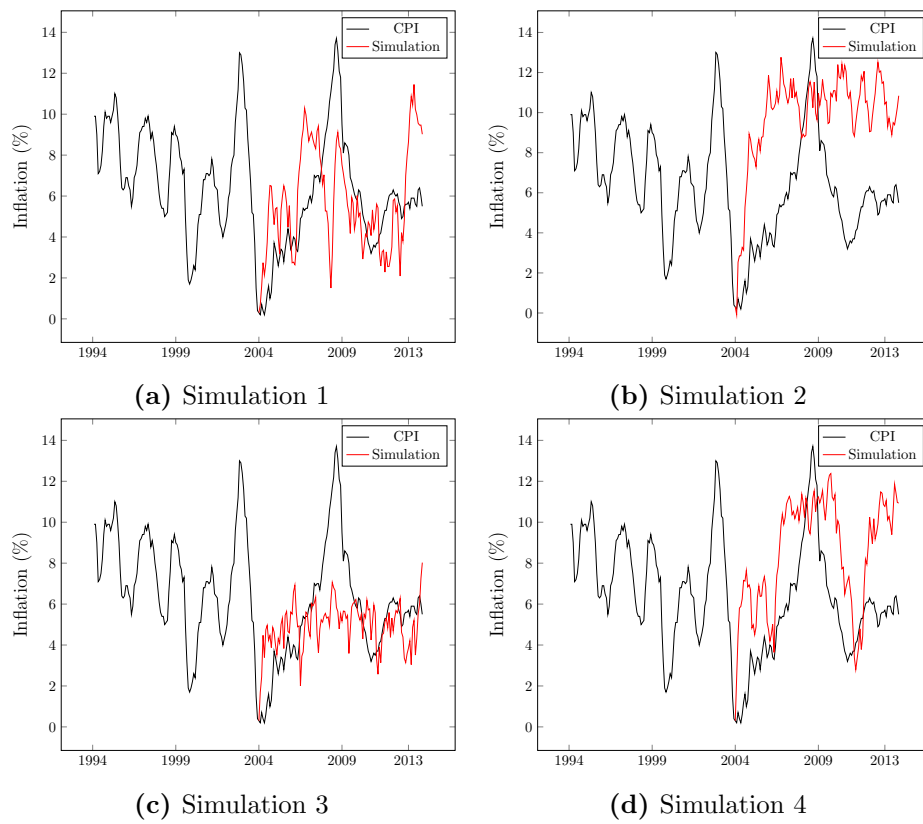


Fig. 5.4: Four simulations for the period 2004 to 2013

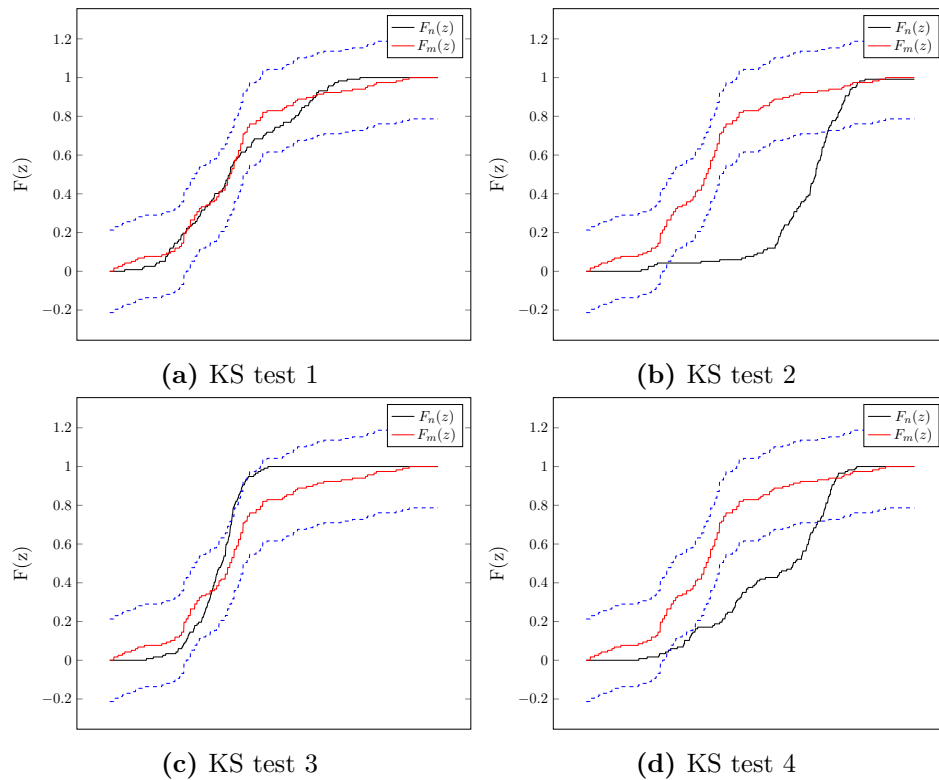


Fig. 5.5: Sample of Kolmogorov-Smirnov Tests

from the same continuous CDF to be accepted, the two samples must fall within the blue dashed line, which represents the critical value.

The simulations Figure 5.4b and Figure 5.4d correspond to the two tests in which the null hypothesis is rejected in Figure 5.5b and Figure 5.5d. Both of these simulations have inflation levels which remain in the high-inflation regime for longer periods of time than the observed CPI. As was earlier suggested, in the time of inflation targeting in South Africa it may be more applicable to have a lower probability of transitioning from the low inflation regime to the high inflation regime and a lower probability of remaining in the high inflation regime. The estimates of the probabilities in the transition matrix were $p_{22} = 0.9225$, which is the probability of remaining in the high inflation regime, and $p_{11} = 0.9083$, which is the probability of remaining in the low inflation regime. These values were adjusted to $p_{22} = 0.8$ and $p_{11} = 0.95$. The simulations and the two-sample Kolmogorov-Smirnov test were then rerun and the results are displayed in Figure 5.6 and Figure 5.7.

It can be observed that the structure of the simulated sample is very similar to the structure of the actual CPI in all the simulations apart from Figure 5.6b. The null hypothesis of the two-sample Kolmogorov-Smirnov test is accepted in all the

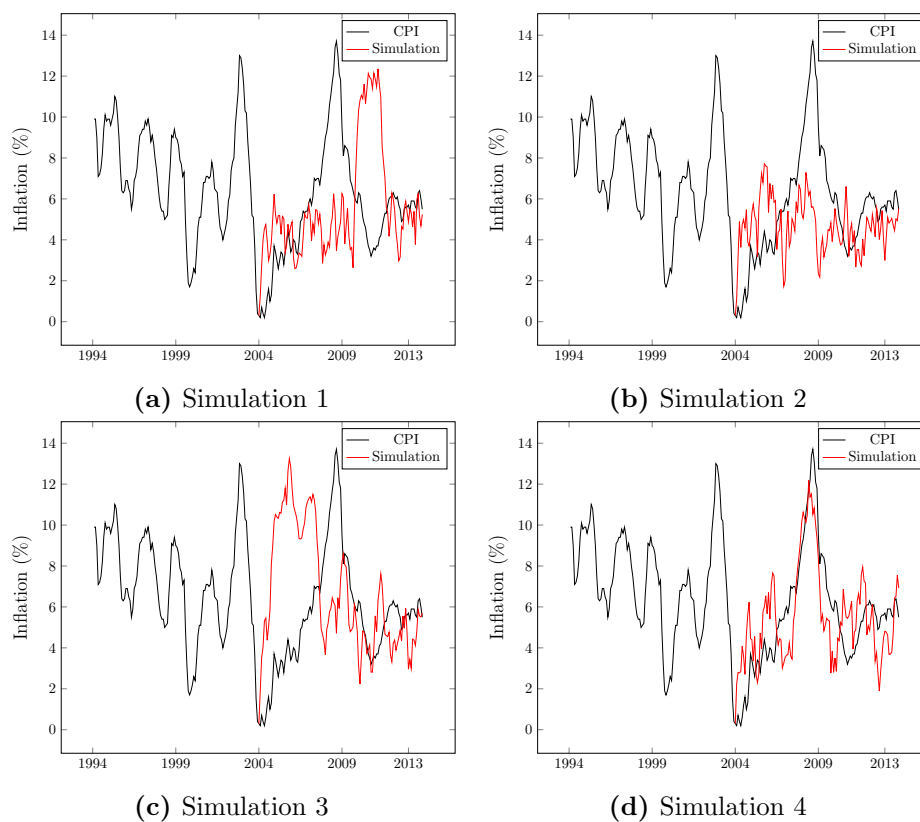


Fig. 5.6: Four simulations for the period 2004 to 2013 after adjusting transition probabilities

cases apart from that simulation, where there is no transition to the high-inflation regime. All the other simulations display a transition to the high inflation regime that followed soon after by a transition back to the low inflation regime. This is representative of a shock to inflation levels that is soon rectified. It can be argued that the transition probabilities be adjusted lower the further we go into the inflation targeting era, as inflation shocks will be less likely.

A qualitative test for the predictive validity of the model is to compare the actual monthly returns of the CPI index over the period January 2004 to October 2013 to the monthly returns of a sample of the simulated CPI indexes over the same time period. The simulation is run using parameter estimates derived using this same time period. This is done in order to compare whether the monthly CPI returns that are simulated share the same characteristics of the monthly returns of the data that is used to estimate the model parameters. This will be used to analyse whether the simulated CPI has stylistic facts that are similar to the actual CPI index. The actual monthly returns are displayed in Figure 5.8 and the simulated monthly returns are compared to the actual monthly returns in Figure 5.9.

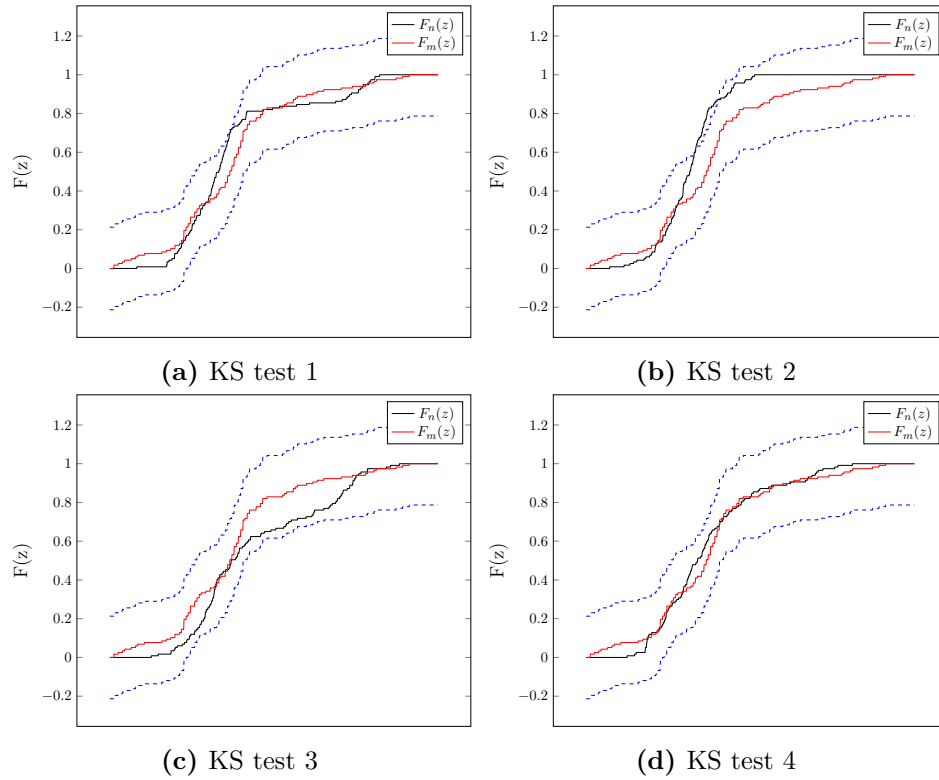


Fig. 5.7: Sample of Kolmogorov-Smirnov Tests

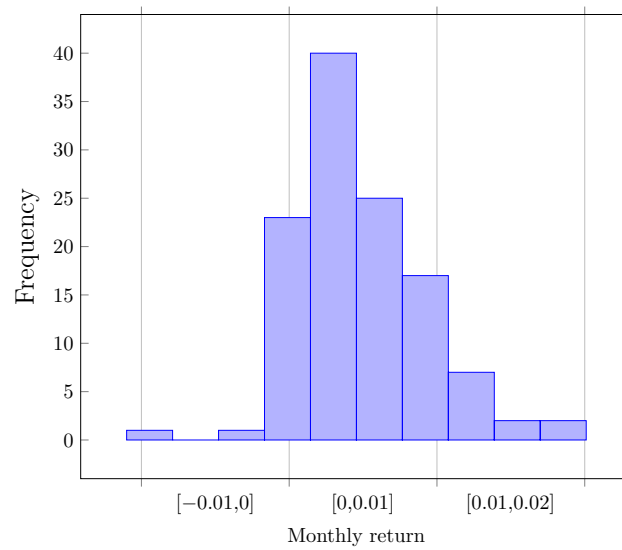


Fig. 5.8: Monthly return on actual CPI index

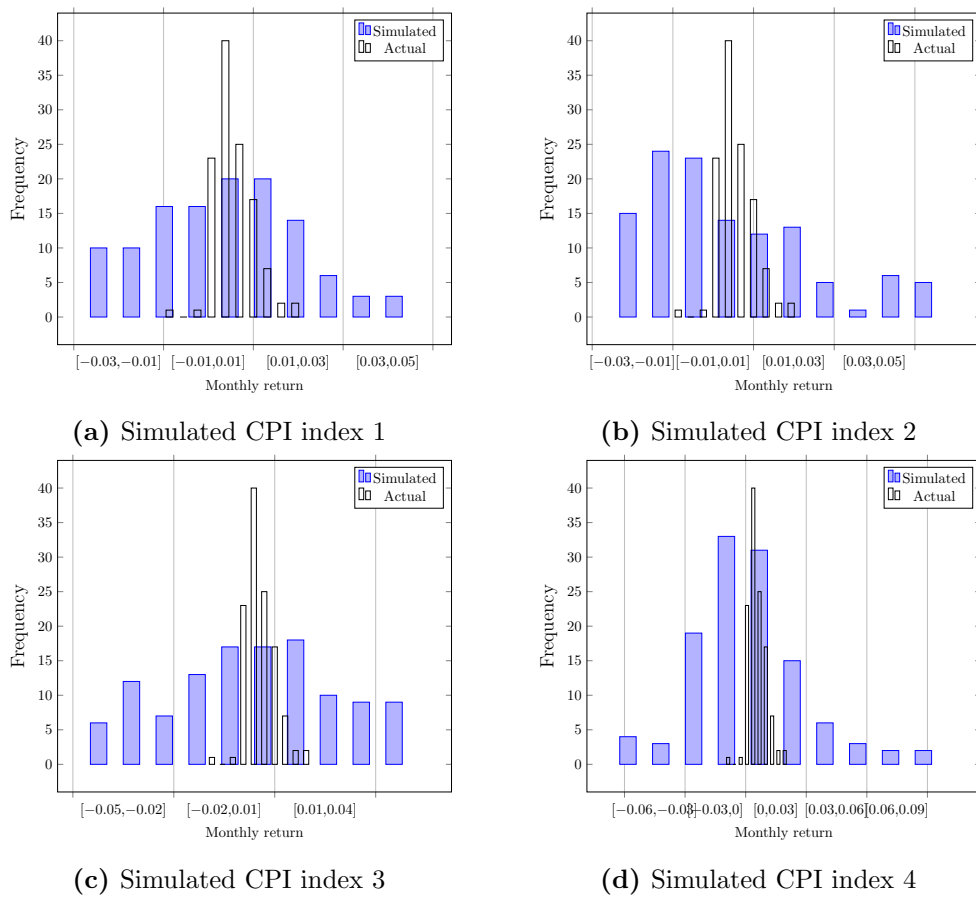


Fig. 5.9: Sample of monthly returns on simulated CPI index

The most noticeable difference between the actual monthly returns and the simulated monthly returns is that the simulated monthly returns are far more volatile than the actual monthly returns and they have a wider range of returns than the actual monthly returns. The majority of the actual monthly returns fall between -0.5% and 1% . While there is still a high frequency of the simulated samples in this interval, the frequency is almost half the number in the actual monthly return. This indicates that adjustments may need to be made to the volatilities in the regime-switching model. It also suggests that generating random variables from the normal distribution may be inappropriate for the model of inflation in South Africa.

5.1 Hedging

The expected value of the real liability cash flows is calculated by Monte Carlo simulation. 10,000 simulations of CPI were used to calculate this expected value. The expected value of the real cash flows at the starting point of 1 January 2008 was calculated to be R590,350

The first step in hedging the real liability cash flows is to calculate the RPP of the bonds and the cash flows. The RPP of the R189 bond has different characteristics to the RPP of the R197 and R202 bonds. Since the R189 matures on 31 March 2013, the bond expires during the hedging period. As a result, the nominal amount of the bond will be paid back during the hedging period. This will result in the RPP value of the R189 due to the perturbation of the interest rate in the last six month bucket period being significantly larger than the RPP values from perturbations in earlier six month buckets. This occurs because at the time of valuing the bond the discounted nominal amount will be the largest proportion of the value of that bond. The RPP for 100 of each bond at 1 January 2008 for a single simulation is displayed in Table 5.5 to illustrate this effect.

The effect of the nominal amount being paid back at maturity can be seen for the R189 bond for the time period January 2013 to June 2013. As a result, the hedging portfolio will consist of only a small amount of R189 bonds compared to R197 and R202 bonds. Another noticeable effect is the difference in RPP between the R197 and R202 bond. The R202 bond has lower values than the R197 since the R202 has a later maturity date than the R197.

It can be noticed that the RPP of the bonds for each six month bucket is far lower than the RPP of the liability cash flows. This is due to the inflation linked bonds only having one cash flow in each six month bucket, whereas the liability cash flows having six cash flows in each six month bucket.

Next the amount to invest in each bond is determined by finding the combination

Tab. 5.5: Rand per point of inflation linked bonds at 1 January 2008

Time period	Real cashflows	R189	R197	R202
Jan 2008-July 2008	21.2918	-0.1397	-0.2238	-0.1188
July 2008-Jan 2009	55.8138	-0.4114	-0.4340	-0.2304
Jan 2009-July 2009	88.0060	-0.6640	-0.6405	-0.3401
July 2009-Jan 2010	119.1441	-0.9379	-0.8449	-0.4486
Jan 2010-July 2010	146.0393	-1.1452	-1.0462	-0.5555
July 2010-Jan 2011	177.5580	-1.4332	-1.2653	-0.6718
Jan 2011-July 2011	201.5356	-1.6527	-1.4444	-0.7669
July 2011-Jan 2012	229.0136	-1.9007	-1.6461	-0.8740
Jan 2012-July 2012	247.6967	-2.1225	-1.8006	-0.9560
July 2012-Jan 2013	273.8636	-2.3528	-2.0276	-1.0765
Jan 2013-June 2013	287.5953	-40.3850	-2.1606	-1.1472

of bonds that minimises the overall portfolio RPP. This is done for every time step in the simulation. Once this is calculated the profit of the hedging strategy can be determined. The profitability of the hedging strategy over the period 1 January 2008 to 30 June 2013 for the simulations of the CPI index is displayed in Figure 5.10.

This indicates that the profitability of the hedging strategy is consistently around R380,000, with the lowest value being approximately R340,000 and the highest value being approximately R440,000. The strength of using this hedging strategy is the low variance of the strategy. The standard deviation of this hedging strategy is approximately R17,550. This hedging strategy thus performs well in hedging the inflation risk of inflation linked liabilities. However, it would have been expected that the profits of the hedging strategy were centred around zero if the hedge was performing very well.

One reason for the high profits occurring is that the calculation of the RPP of the bonds is calculated by adjusting the future real coupons and nominal repayment by the simulated value of inflation and then discounting these cash flows by the nominal swap curve. This method is used to ensure consistency in the calculation of the RPP of the real liability cash flows and the inflation linked bonds. However, this is a fundamentally different way of calculating the value of the bonds compared to the standard practice of using the Bond Exchange of South Africa (2006) bond pricing specifications and therefore the value of the bonds as calculated by each method may differ, resulting in a RPP value that may not be representative of the actual change in the value of the bond due to a basis point change in the nominal

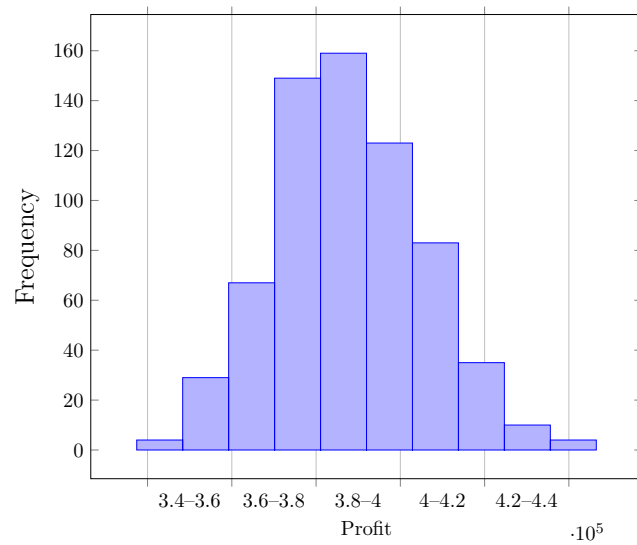


Fig. 5.10: Profit on hedging strategy

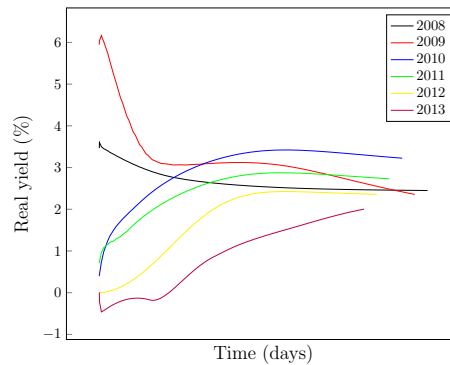


Fig. 5.11: Real yield curve from 2008 to 2013

swap curve. Unfortunately one cannot calculate the RPP of the inflation linked bonds using the Bond Exchange of South Africa (2006) specifications because the value of these bonds is only dependent on the real yield that a particular bond is trading at. Thus, the real yields may not be well hedged.

Another reason for the high profitability is that during the hedge period the value of the bonds used to hedge the real cash flows increased both as a result of an increase in inflation and a decrease in the level of the real yield curve. The real yield curve on the date 31 January for the years 2008 to 2013 is displayed in Figure 5.11.

This figure indicates that the shape of the real yield curve changed in the period between 2009 and 2010. The shorter term yields were initially greater than the longer term yields, but after 2009 this effect was reversed. This implies that demand for shorter term inflation linked products increased over this time. The subsequent

decreases in the level of the real yield curve indicate that the demand for inflation linked bonds increased from 2010 to 2013. The value of inflation linked bonds will therefore have increased over this period due to the inverse relationship between bond prices and yields. This would have been a significant driver in the hedging strategy being profitable.

It can be expected that the profit of the hedge would be centred around zero if the real yield curve remained stable over the period of the hedge. However, this is an unlikely scenario if the hedging period is short. If the real yield curve consistently increases over the hedging period then there may be a scenario where large losses occur over the hedging period. Hence, the effectiveness of this hedging strategy is dependent on the movement in the real yield curve over the time of the hedging period. As a result of these findings, care should be taken in the use of this method in hedging inflation linked liabilities. Further research should be carried out over longer periods to determine the effectiveness of the hedge when there is both an increase and a decrease in real yields during the longer time period.

An alternative hedging strategy that should be researched would be to use this inflation model to inform asset allocation of a portfolio that includes equities, bonds, inflation linked bonds and interest rate products. This portfolio would then be used to hedge the liability cash flows. This would require stochastic models of equities and interest rates that are linked to the regime-switching inflation model. From these stochastic models an efficient frontier can be constructed, where the construction is determined by the minimisation of a measure of asset allocation combinations. A possible measure is to use the investment surplus at a particular time, hence the efficient frontier would be determined by standard deviation and the investment surplus.

Chapter 6

Conclusion

Several tests were conducted to determine the appropriateness of using the regime-switching model to model inflation in South Africa over the period 1994 to 2013. These tests included the two-sample Kolmogorov-Smirnov test and comparing the monthly returns of the actual CPI index and the simulated CPI indices. These tests indicated that adjustments will need to be made to the estimated parameters of the model.

The two-sample Kolmogorov-Smirnov test showed that the probabilities of transitioning from one regime to another should be adjusted based on the current economic scenario. In South Africa's case, it can be expected that the probability of transitioning from the low inflation regime to the high inflation regime is going to be lower than the estimated probability derived from maximum likelihood estimation due to inflation targeting.

It could be seen when comparing the monthly returns of the actual CPI index and the simulated CPI indices over the period that the projected models had more volatile returns and a larger range of returns. Again this indicates that adjustments might have to be made to the volatility parameters.

The significant factor that arises out of the need to make adjustments is that it allows the user of the model to take a view on what inflation is going to look like over the period that they would like to model. They can adjust the transition probabilities based on whether they expect inflation to remain in a regime or to jump between regimes and they can adjust the volatility of the model based on what they expect the stability of inflation will be over the period.

The use of the regime-switching model as a means of determining a hedging strategy for inflation linked cash flows using inflation linked bonds showed positive results over the period January 2008 to June 2013. What should be noted during this period is that the real yield curve was steadily decreasing over the time period, leading to a steady increase in the value of inflation linked bonds, resulting in the hedging strategy being profitable.

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