A

Stochastic Asset-Liability Model

Using

Stable Distributions

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Notwithstanding the above acknowledgements, the work contained herein is my own. Where it has been necessary to use the work of others, proper citations have been made. Any errors or omissions contained herein are due to me.

Gary Finkelstein
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Chapter One: Introduction, Aims and Motivation

1.1 The MGWP Stochastic Investment Model

1.1.1 In 1977, the Institute and Faculty of Actuaries established a Joint Working Party to investigate suitable reserving bases for investment guarantees under investment-linked life assurance business. The Maturity Guarantees Working Party (MGWP) reported in 1980 (Ford *et. al.*, 1980).

1.1.2 The working party recommended the use of the following stochastic investment model to represent the movements of the unit price R_t, of a UK equity unit. Let

P_t = The unit price at time t (excluding reinvestment of income)
D_t = The gross dividend per unit received during period t
Yp_t = The prospective dividend yield at the end of period t
τ = The rate of tax (assumed constant)

Then

$$R_t = R_{t-1}.(P_t + (1-\tau)D_t)/P_{t-1} \tag{1.1}$$

where

$$P_t = D_{t+1}/Yp_t$$
$$\log D_t = \log D_{t-1} + \mu_D + Z_D(t)$$
$$\log Yp_t = A(\log Yp_{t-1} - \log\mu_Y) + \log\mu_Y + Z_Y(t)$$

and where

$Z_D(t)$ is an error term assumed to be Normally distributed with mean 0 and variance σ_D^2

$Z_Y(t)$ is an error term assumed to be Normally distributed with mean 0 and variance σ_Y^2 and is independent of $Z_D(t)$.

1.1.3 The complete parameterisation of the model is summarised below:

Dividend Yields	Dividend Growth	Tax
$A = 0.6$ $\mu_Y = 5\%$ $\sigma_Y^2 = 0.20^2$	$\mu_D = 4\%$ $\sigma_D^2 = 0.13^2$	$\tau = 0.375$

To run the model, initial values for $R_0, P_0,$ and Yp_0 need to be chosen. R_0 and P_0 can arbitrarily be set to 1.0, while Yp_0 needs to be set to an initial value reflecting the circumstances actually prevailing at time $t = 0$.

The above parameter values are the original values that were put forward by the MGWP. They were derived from an analysis of experience over the period 1919 to 1979. Some of them, notably the tax rate, would need to be revised for use today.

1.2 Aims and Motivation

1.2.1 The salient feature under examination in this thesis is the assumption that the error terms, $Z_D(t)$ and $Z_Y(t)$, are Normally distributed. This assumption is common to most of the stochastic asset models that are in widespread use within the actuarial profession. An example is the well known Wilkie model (Wilkie (1984, 1995)).

1.2.2 In order to examine the effect of changing the assumption of Normally distributed error terms in isolation, it will be useful to fix the parameters proposed by the MGWP. A comparison can then be made of the results obtained from using the original MGWP model, and those that would have been obtained had an alternative probability distribution been used. For completeness, updated parameter estimates will also be provided, but this is very much of peripheral importance to the thesis.

1.2.3 The thesis is not concerned with alternative formulae to those given in section 1.1 (except to the extent that the definitions of $Z_D(t)$ and $Z_Y(t)$ will change).

1.2.4 There is strong empirical evidence (see paragraph D2.13 of Ford *et. al.*(1980) and section 3.3 below), that the Normal distribution may be inappropriate. The main objections to the use of the Normal distribution stem from two main problems which often result when a stochastic model is fitted to financial data:

- the sample residuals tend to be much more skewed than those consistent with a Normal distribution,
- the sample residuals tend to have a much higher kurtosis (tail thickness) than the Normal

1.2.5 The problem of insufficient kurtosis with the Normal distribution could be significant. This is because the weight of the tails of the distribution will determine the extent of stochastic fluctuations in the model. Some authors, including Ford *et. al.* (1980), Wilkie(1984, 1995) and Carter(1991) attempt to overcome this problem by somewhat arbitrarily increasing the standard deviation parameter of the distribution. This however, is only a partial solution; since while it may introduce larger stochastic fluctuations more frequently, it also results in an inappropriate frequency of smaller variations. This is illustrated in Figure 1 below which shows a Cauchy distribution, a Normal distribution and an adjusted Normal distribution. In addition, the specific adjustment to the standard deviation parameter cannot be supported by the data.

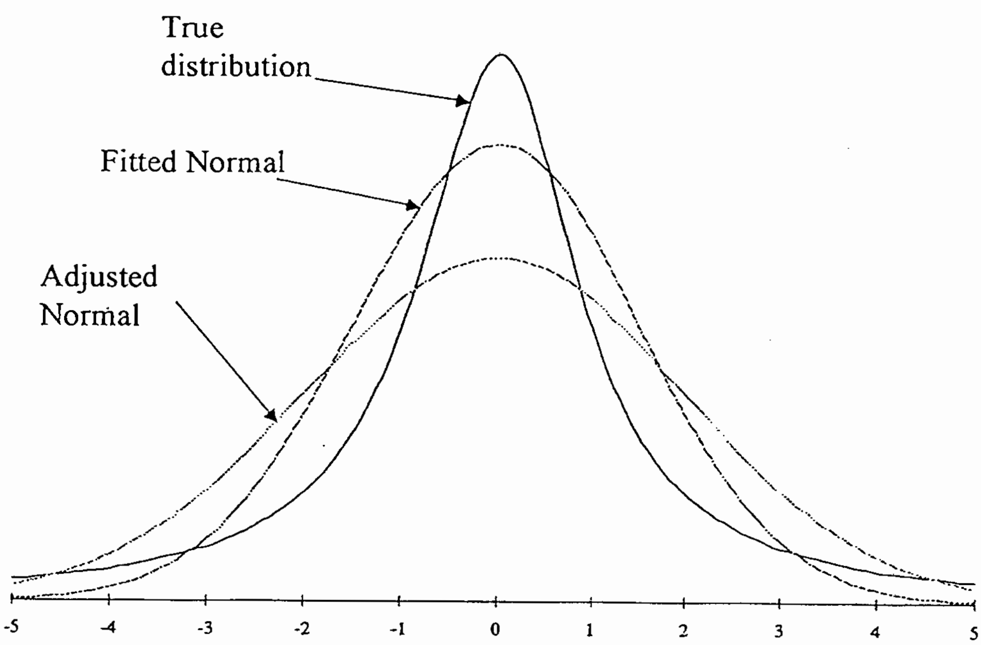

Figure 1

1.2.6 Tenenbein(1988) and Becker(1991) investigated the instability over time of sample standard deviations in financial data. They concluded that the volatility of a finite variance probability distribution, such as the Normal distribution, may not be sufficient to model the fluctuations of the actual investments.

1.2.7 A number of alternative (non-Normal) distributions have been proposed for use with financial data. These include the stable family (Mandelbrot(1963), Fama and Roll(1968, 1971), Tenenbein(1988), Becker(1991), Walter(1991) and Wilkie(1995)). Student's *t*-distribution has also been put forward as an alternative to the Normal distribution (Praetz(1972), and Blattberg and Gonedes(1974) and Taylor(1986)). Thomson (1994) used a number of different distributions in his model of various South African assets. In addition to the Normal distribution, Thomson uses a translated beta distribution for share dividend growth, and Student's *t*-distribution for long term interest rates.

1.2.8 One approach that has received widespread attention in econometric and statistical literature is the use of mixed distributions. Apparent non-stationarity due to sharp changes in time series is accounted for through an assumption of "unobserved heterogeneity". Here a large proportion of the stochastic disturbances might be assumed to have one distribution (e.g. a Normal distribution), while the remaining proportion would have a different (possibly also Normal) distribution. The theory of unobserved heterogeneity through mixed distributions is covered in Vaupel and Yashin(1985). Other proponents in favour of mixed distributions include Teichmoeller(1971), Barnea and Downes(1973) and Brenner(1974).

1.2.9 There are however a number of problems with mixed distributions. These include fundamental things such as identification of the distributions, parameter estimation and the appropriate magnitude and time period between random shocks. Blattberg and Gondedes(1974) cover some of the problems with the earlier attempts at using mixed distributions. Simkowitz and Beedles (1980) assert that the skewness feature is so pervasive that even well behaved mixed distributions are inadequate to explain it. They do however suggest that stock returns could be a mixture of two stable distributions with one or both being skewed. The resultant mixture would not strongly demonstrate stability since it would be a mixture of stable distributions with different index parameters. It would however be in the domain of attraction[1] of a stable distribution. Geoghegan et.al(1992) report that over the long term, the difference between mixed distribution models (with finite variance) and the simpler linear model is limited to small differences in the mean square prediction errors, and perhaps in the predicted values themselves if the random shock distribution is asymmetrical. They concluded that mixed models are more likely to be of concern for short-term models.

1.2.10 Another approach that has attracted some attention is the use of heteroscedastic models. These are models in which the parameters, (usually the standard deviation parameter) at a point in time are themselves obtained by sampling from a certain probability distribution. Examples of this approach can be found in Engle(1982), Clarkson(1991), Harris(1994) and Hua(1994). Engle modelled the variance of the current stochastic disturbance as being dependent on the magnitudes of the previous residuals, thereby capturing the notion that there is more variability at times of higher levels of inflation. His model is known as an auto-regressive conditional heteroscedastic (ARCH) model since the model had non-constant variances conditional on the past, although the unconditional variances were constant. Geoghegan *et. al.* (1992) report that the ARCH approach may well appear more realistic in the short term, but in the medium and long term the advantages are less,

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[1] See paragraph 2.10 for an explanation of the domain of attraction property of stable distributions

[2] See paragraph 2.10 for a statement of, and reference for the Generalised Central Limit Theorem

with very little difference in the prediction errors and values (compared with more traditional linear models).

1.2.11 With the exception of Walter(1991) and Klein(1993), the stable distributions appear to have not yet found their way into widely used actuarial models (Geoghegan *et. al.* (1992)). Investigation into the use of the stable distributions with actuarial models has been put forward as an area of further research by a number of authors including Ford *et. al.* (1980), Wilkie(1984, 1995), Tenenbein(1988) and Geoghegan *et. al.*(1992).

1.2.12 Previous attempts to model share prices with stable distributions appear to have concentrated on the simple univariate (log) price model. In addition, they concentrated on short term (daily, weekly, monthly) price changes. The attempts include:

- Fama and Roll (1968) (US data)
- Teichmoeller (1971) (US data)
- Praetz (1972) (Australian data)
- Blattberg and Gonedes (1974) (US data)
- Koutrouvelis (1980) (US data)
- Walter (1991) (French data)

1.2.13 Most of the attempts (all of the above except for Koutrouvelis), have restricted their attention to symmetric stable distributions. Fielitz and Smith (1972), Leitch and Paulson (1975) and Simkowitz and Beedles (1980) investigated US share price and log price changes, and argued that the symmetry assumption is incorrect. Klein (1993) applied the symmetric stable distributions to modelling US Treasury bond rates.

1.2.14 This thesis will concentrate on the bivariate (dividend yield, dividend growth) model proposed by Ford *et. al.*(1980), and the UK data on which it was based. No assumption about the symmetry of the probability distributions will be made.

1.3 Additional Literature - The Case Against Using Stable Distributions

1.3.1 The validity of the stable distribution assumption (for the univariate price and log-price model) appears to be controversial. For example, Hagerman(1978), Perry(1983), Akgiray and Booth(1988) and Lau et.al.(1990) found evidence against the stable model. Officer(1972) tested for both longitudinal stability (i.e. stability over time) and cross-sectional stability (i.e. stability across stocks in a portfolio). He rejected the stable model on the grounds of longitudinal stability over short time periods (daily data), but failed to reject the model over longer time periods (monthly data). He also rejected the stable model on the grounds of cross-sectional stability, but noted that the stocks could be *non-homogeneous* stable (meaning that different stocks are stable but with different index parameters).

1.3.2 Some authors (e.g. Praetz(1972) and Blattberg and Gonedes(1974)) compared the goodness of fit of Student's *t*-distribution with that of the *symmetric* stable distributions, and found in favour of the *t*-distribution. Praetz(1972) used a minimum χ^2 procedure on the univariate log price model of Australian shares. Although Blattberg and Gonedes(1974) found problems with this approach, they arrived at the same conclusion by examining log-likelihood ratios. The apparently superior fit of the *t*-distribution was taken as evidence in favour of it.

1.3.3 Feller(1966) comments (pages 52-53) that methods based on optimising the goodness of fit can be misleading, if there is no good reason for using a particular distribution. As an example, he describes how population theory in the past relied heavily on the goodness of fit of the logistic distribution function, despite there being strong theoretical reasons why it could not always be suitable. The problem with goodness of fit optimisation is that it is often possible to find alternative distributions which can improve the goodness of fit. This can lead to many contradictory theoretical models being supported by the same observational material.

1.3.4 The evidence against the stable model is discussed in further detail in Chapter 5, together with other problem areas.

1.4 Theoretical Support for the Stable Distributions

1.4.1 Fama(1963) has put forward a possible explanation and reason for the appropriateness of the stable distributions for modelling financial data. Central to Fama's reasoning is that the price changes in a speculative series can be regarded as a result of the influx of new information into the market and of the re-evaluation of existing information. At any point in time, there will be many items of information available. Thus price changes between transactions reflect the effects of many different bits of information.

1.4.2 In the simplest case the price changes implied by the different bits of information may themselves follow stable distributions with constant index and skewness parameters, but possibly differing values for the location and scale parameters. If the effects of the different pieces of information combine in an additive manner, then the price changes from transaction to transaction will also follow a stable distribution with the same index and skewness parameters. Since the price changes for intervals such as a day, week, month or year are the simple sums of changes from transaction to transaction, the changes for these intervals will also be stable, with the same index and skewness parameters.

1.4.3 Even if the price changes implied by the different bits of information are not stable, if their sum has a limiting distribution, then by the generalised central limit theorem, their limiting sum will be stable[2]. If there are many bits of information involved in the transaction, the asymptotic distribution may be a satisfactory approximation. It may be however that there are not enough bits of information involved in individual transactions to ensure that the limiting distribution is sufficiently closely achieved. In this case, as long as there are many transactions per day, week, month or year, the distributions of price changes for these longer period distributions should more closely follow a stable distribution.

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[2]See paragraph 2.10 for a statement of, and reference for, the Generalised Central Limit Theorem

1.4.4 There has been some work carried out into identifying those distributions whose sum have a limiting distribution. Gnedenko and Doeblin(1954) have shown that a necessary and sufficient condition is

$$\frac{F(-x)}{1-F(x)} \to \frac{C_1}{C_2} \quad \text{as} \quad x \to \infty$$

and for every constant $k > 0$,

$$\frac{1-F(x)+F(-x)}{1-F(kx)+F(-kx)} \to k^{\alpha} \quad \text{as} \quad x \to \infty$$

where F is the cumulative distribution function of a random variable X and C_1 and C_2 are constants.

In particular, any variable that is asymptotically Paretian (regardless of whether it is also stable) will satisfy these conditions. This is because if X is asymptotically Paretian then

$$\frac{F(-x)}{1-F(x)} \to \left[\frac{|-x|/V_2}{x/V_1}\right]^{-\alpha} = \left(\frac{V_2}{V_1}\right)^{\alpha}, \text{ and}$$

$$\frac{1-F(x)+F(-x)}{1-F(kx)+F(-kx)} \to \frac{(x/V_1)^{-\alpha} + (|-x|/V_2)^{-\alpha}}{(kx/V_1)^{-\alpha} + (|-kx|/V_2)^{-\alpha}} = k^{\alpha}$$

1.4.5 Central to the theoretical support of the stable distributions is the idea that the effects of items of information combine in an additive manner. Mandelbrot (1963b) has shown however that the above reasoning can be generalised further. He showed that as long as the effects of individual bits of information are asymptotically Paretian, various types of complicated combinations of these effects will also be asymptotically Paretian. For example, although there are many bits of information in the market at a given time, the price change for individual transactions may depend solely on what the transactors regard as the most important piece of information. Mandelbrot has shown that, if the effects of individual items of information are asymptotic Paretian with index α, the distribution of the largest effect will also be asymptotically Paretian with the same index α.

1.4.6 The above discussion has concentrated on the theoretical reasons why the stable distribution could be suitable for a univariate model of share prices. The arguments are however quite general, and can be applied to any speculative series. It is clear that the reasoning can be extended to dividend yields since, if there is an immediate change in share prices, without any change in the company's dividends, then the dividend yield by definition must change. Of course a change in dividends could also be amongst those items of information that can cause a change in prices and dividend yields. To apply the arguments to the dividends for an individual company, one must extend the arguments to the factors affecting the dividends, rather than 'bits of information' in the market affecting prices.

1.4.7 The above arguments may not seem sufficiently strong or compelling to use a stable distribution. However, to my knowledge, no other wholly satisfactory reason or justification has been given for using any other alternative distribution. The arguments do at least seem to justify further research in the subject.

1.5 Summary of the Aims, Scope and Structure of the Thesis

1.5.1 The thesis has two main aims:

1. to determine how and whether a stable (non-Gaussian) distribution can be suitably fitted to UK share price data and incorporated into the MGWP model, and
2. to examine the implications (mainly on reserving for investment guarantees) of replacing the assumption of Normally distributed error terms by a more general stable distribution.

1.5.2 This thesis will concentrate on the bivariate (dividend yield, dividend growth) model proposed by Ford *et. al.*(1980), and the UK data on which it was based. No assumption about the symmetry of the probability distributions will be made.

1.5.3 The thesis will focus on the stochastic component (i.e. the underlying probability distributions) of the MGWP model. Although the parameter values given in section 1.1.3 will be updated to allow for experience since 1980, possible modifications to the deterministic component of the model (i.e. the formulae in section 1.1.2) are strictly outside the scope of this thesis.

1.5.4 A review of the stable distributions is provided in Chapter 2. The fitting of the stable distributions to the MGWP model and data is presented in Chapter 3 and the implications of changing the assumption of Normally distributed error terms to a more general stable distribution is covered in Chapter 4. Chapter 5 deals with potential problem areas, and this is followed by suggestions for further research, a summary and conclusion.

Chapter Two : A Review of the Stable Distributions

2.1 Definition:

A random variable Z is said to be stable, or to have a stable distribution, if:

i) for every positive integer n, and

ii) for every set of independent and identically distributed random variables $\{Z_1, Z_2, \ldots, Z_n\}$ with the same distribution as Z,

there exist constants $a_n > 0$ and b_n such that the sum $Z_1 + Z_2 + \ldots + Z_n$ has the same distribution as $a_n Z + b_n$.

Loosely speaking, the stable distributions are those that are closed under addition after change of scale and location.

2.2 Stability is the only assumption which we shall make for the error terms in the stochastic investment model. Clearly this is a much weaker assumption than normality, since the Normal distribution is in fact a special case of a stable distribution. This follows from the well known fact that if $Z \sim N(\mu, \sigma^2)$ then $S_n = Z_1 + Z_2 + \ldots + Z_n$ is also Normal, with mean $n\mu$ and variance $n\sigma^2$. Thus, with $a_n = \sqrt{n}$ and $b_n = n$, the Normal distribution satisfies the definition of a stable distribution. It follows that rejection of a stable model implies rejection of the Normal model too.

2.3 The definition given in paragraph 2.1 is very broad, and in fact there are infinitely many distributions which fall into the class. With a few exceptions, (given in the next paragraph), the probability density function of the stable distributions cannot be written in closed form. Series expansions (Bergstrom(1952)) and approximations (Cramer(1962)) are however available. In addition, Zolotarev(1954, 1956, 1961) has derived representations of several stable densities in terms of higher transcendental functions (specifically the modified Bessel function and the Whittaker function). Numerical tabulations of a wide range of stable densities can be found in Holt and Crowe(1973).

2.4 The complete list of stable distributions whose density functions can be written in closed form is given below.

(I) The Normal distribution with mean δ and standard deviation $\gamma\sqrt{2}$. Its probability density function is

$$f(z) = \frac{1}{2\gamma\sqrt{\pi}} \exp\left[-\frac{1}{4}\left(\frac{z-\delta}{\gamma}\right)^2\right]$$

and its characteristic function is

$$\Psi(t) = E[e^{itz}] = \exp(i\delta t - \gamma^2 t^2)$$

where $i = \sqrt{-1}$.

(II) The Cauchy distribution with location δ and dispersion γ has probability density function

$$f(z) = \frac{\gamma}{\pi\left[\gamma^2 + (z-\delta)^2\right]}$$

and a characteristic function given by

$$\Psi(t) = E[e^{itz}] = \exp(i\delta t - |\gamma t|)$$

The sum of n independent and identically distributed random Cauchy variables is also Cauchy, with location parameter $n\delta$ and dispersion parameter $n\gamma$.

(III) The Brownian exit distribution with parameter γ. The probability density function is

$$f(z) = \sqrt{\frac{\gamma}{2\pi z^3}} \exp\left(\frac{-\gamma}{2z}\right)$$

and its characteristic function is

$$\Psi(t) = E[e^{itz}] = \exp\left(-(1-i)\sqrt{|\gamma t|}\right).$$

The sum of n independent and identically distributed Brownian exit random variables with parameter γ is also Brownian exit, with parameter $n^2\gamma$.

A Brownian exit random variable Z is the first time that a particle under Brownian motion moves above some level $\sqrt{\gamma}$. If $\gamma = 1$, then Z is also sometimes referred to as a Levy distribution. It is also the reciprocal of a chi-squared variable with one degree of freedom and is of type V in the Pearson system of frequency curves.

(IV) The degenerate variable with distribution function $F(z) = 1$ if $z \geq \delta$ and 0 otherwise can also be argued to be a member of the stable class. (Choose $a_n = n,\ b_n = 0.$) Its characteristic function is given by

$$\Psi(t) = E[e^{itz}] = \exp(i\delta t)$$

The sum of n independent and identically distributed degenerate variables is also degenerate, with parameter $n\delta$.

2.5 A closed form expression for the characteristic function of a stable distribution is always available. Levy (1924, 1925) and Gnedenko and Kolmogorov (1954) have shown that the characteristic function of any stable distribution is given by

$$\Psi(t) = E\left[e^{itZ}\right] = \begin{cases} \exp(i\delta t - |\gamma t|^{\alpha}(1 + i\beta \operatorname{sgn}(t)\tan(\pi\alpha/2)) & \text{if } \alpha \neq 1 \\ \exp(i\delta t - |\gamma t|^{\alpha}(1 + i\beta \frac{2}{\pi}\operatorname{sgn}(t)\ln|t|)) & \text{if } \alpha = 1 \end{cases} \quad (2.1)$$

where

sgn(t) is +1 if t is positive and -1 if t is negative, and $i = \sqrt{-1}$.

2.6 It follows that the entire class of stable distributions can be parameterised with the four parameters α, β, γ and δ.

The parameter δ is the location parameter. It simply shifts the distribution to the left or right. In the case of the symmetric distributions (e.g. the Gaussian), δ will be the mean of the distribution.

The parameter γ is the scale parameter. It compresses or expands the distribution about δ and in proportion to γ. The standard deviation parameter σ, and the scale parameter γ of the Normal distributions are related through $\sigma = \sqrt{2}\gamma$.

The parameter β determines the skewness of the distribution. This parameter is restricted to the range [-1, 1]. If $\beta = 0$, the distribution is symmetric about δ. In this case the median, the mode, and when finite the mean, coincide. If $\beta > 0$, then the distribution is negatively skewed. (This means that it has a long left hand tail and a heavy mass on the right hand side. In this case the median is less than the mode.) If $\beta < 0$, then the distribution is positively skewed, and the median is greater than the mode.

The parameter α is known as the characteristic exponent. It lies in the range (0,2] and identifies a particular member of the family. If $\alpha = 2$ a Normal distribution results. If $\alpha = 1$, a Cauchy distribution results and if α = 1/2 a Brownian exit distribution results. The smaller α, the thicker the tails of the distribution. When $\alpha < 2$, moments of order α and greater are infinite. Thus if $\alpha < 2$, then the variance of the distribution is infinite. This property is important since it implies that a stable distribution with $\alpha < 2$ will introduce more extreme stochastic fluctuations in the investment model.

The first aim of this thesis involves determining the appropriate values for the parameters α, β, γ, and δ (and in particular whether the appropriate stable distribution is non-Normal).

In this thesis, the abbreviation $S(\alpha, \beta, \gamma, \delta)$ will be used to denote a stable variable with parameters α, β, γ and δ.

2.7 If $Z_1, Z_2, \ldots Z_n$ are independent and identically distributed stable random variables with parameters α, β, γ and δ, then $S_n = \sum_{i=1}^{n} Z_i$ is stable (by definition), and has parameters $\alpha, \beta,\ n^{1/\alpha}\gamma$ and $n\delta$.

2.8 The stable distribution can be standardised. If $X \sim S(\alpha, \beta, \gamma, \delta)$, then the variable $Z = \dfrac{X - \delta}{\gamma}$ is $S(\alpha, \beta,\ 1,\ 0)$.

In this thesis, the abbreviation $S(\alpha, \beta)$ will be used to denote a standardised stable variable with parameters α and β.

2.9 The standardization property is important since it reduces the problem of simulating from a general $S(\alpha, \beta, \gamma, \delta)$ to simulating from the less general $S(\alpha, \beta)$ distribution. If z is a sample point from an $S(\alpha, \beta)$ distribution, then $z' = \gamma z + \delta$ is a sample point from an $S(\alpha, \beta, \gamma, \delta)$ distribution.

Chambers et.al.(1976) deduced the following representation for the standardized random variable $S(\alpha, \beta)$:

$$S(\alpha,\beta) = \begin{cases} \dfrac{\sin\{\alpha(\phi - \phi_0)\}}{\{\cos\phi\}^{1/\alpha}} \left[\dfrac{\cos(\phi - \alpha(\phi - \phi_0))}{W} \right]^{(1-\alpha)/\alpha} & \text{if } \alpha \neq 1 \\ \dfrac{2}{\pi}\left(\left(\dfrac{\pi}{2} + \phi\beta\right) \tan\phi - \beta \ln\left(\dfrac{\pi/2\, W \cos\phi}{\pi/2 + \phi\beta} \right) \right) & \text{if } \alpha = 1 \end{cases} \qquad (2.2)$$

where

$$\phi \sim U(-\pi/2, \pi/2)$$
$$W \sim \text{exponential}(1)$$
ϕ and W are mutually independent
$$\phi_0 = -\frac{k(\alpha)}{2\alpha}\pi\beta$$
$$k(\alpha) = 1 - |1-\alpha|$$

This remarkable result is used to simulate from a $S(\alpha,\beta)$ distribution. The result is in fact an extension of an observation of Kanter (1975) which in turn was derived from the integral representations of Ibragimov and Chernin (1959) and Zolotarev (1966).

A number of alternative methods are available for simulating stable random variables. These include the methods of DuMouchel(1971, pp121-24) and Bartels(1978).

The method of Chambers et.al will be used in this thesis. The computer program implementing their method is given in the appendix.

2.10 If $X_1, X_2, \ldots$ are independent random variables having a distribution function $F(x)$, and if there exist numbers A_n, B_n, such that $B_n^{-1}\left(\sum_{i=1}^{n} X_i - A_n\right)$ has a distribution which converges to the distribution function $G(x)$, then $F(x)$ is said to belong to the domain of attraction of $G(x)$.

It is often believed that an important factor affecting the widespread use of the Normal distribution, is the fact that this distribution has a domain of attraction stemming from the Central Limit Theorem (CLT). In fact all the stable distributions have a domain of attraction. According to the Generalised Central Limit Theorem (GCLT) (Feller(1966, 1971)), if a sum of independent and identically distributed random variables (from any distribution) has a limiting distribution, then the limiting distribution will be a member of the stable class.

The stable distributions are the only distributions that have a domain of attraction (as defined above).

If the summands have finite variance, then the CLT applies, and the limiting distribution is the Normal distribution. If the summands have infinite variance, then the GCLT applies, and the limiting distribution is stable (non-Normal).

From this one can prove that a stable distribution cannot have finite variance if it is not the Normal distribution.

The Central Limit Theorem also tells us that all finite variance distributions are in the domain of attraction of the Normal distribution, and we can take

$$A_n = nE[X] \text{ and } B_n = \sqrt{nVar[X]}.$$

Necessary and sufficient conditions for an infinite variance distribution to belong to the domain of attraction of a particular stable non-Normal distribution are given in Gnedenko and Kolmogorov (1954). (See also section 1.4.4 of this thesis.)

2.11 Levy(1925) showed that both tails of the non-Normal stable distributions follow an asymptotic form of the law of Pareto, except when $|\beta| = 1$.

For the Cauchy distribution $\alpha = 1$, $\beta = 0$ and $z\Pr(Z > z) = z\Pr(Z < -z) = 1/\pi$.

More generally, if $\alpha \neq 1$, there exist two constants C_1 and C_2, with $\beta = \dfrac{C_2 - C_1}{C_1 + C_2}$,

$$z^{\alpha}\Pr(Z > z) \to C_1 \text{ as } z \to \infty, \text{ and}$$
$$z^{\alpha}\Pr(Z < -z) \to C_2 \text{ as } z \to -\infty.$$

If $\alpha \neq 2$ then $C_1 = pC$ and $C_2 = qC = (1-p)C$ where $p = \dfrac{1+\beta}{2}$ and

$$C = \begin{cases} \dfrac{\gamma^{\alpha}}{\Gamma(1-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1 \\ \dfrac{2\gamma^{\alpha}}{\pi} & \text{if } \alpha = 1 \end{cases}$$

The constants C_1 and C_2 are non-zero if and only if $|\beta| \neq 1$.

It is this property which explains how the parameters α and β affect the tail thickness of the distribution. It is also this property which explains why a number of authors (e.g. Mandelbrot(1963), Blattberg and Sargent(1971) and Wiener(1975)) refer to the distributions as "Stable Paretian".

2.12 The behaviour of the stable distribution when $\beta = \pm 1$ is covered in Skorokhod(1954, 1961).

If $\beta = +1$, then the distribution is J-shaped. If in addition $\alpha < 1$, then there is no mass above δ. The corners of the distribution are smoothed slightly when $1 < \alpha < 2$.
If $\beta = -1$, then the cumulative distribution function is the reflection about the vertical axis of the cumulative distribution function for the case $\beta = +1$. If $\beta = -1$ and $\alpha < 1$, then there is no mass below δ.

If $\alpha < 1$ and $\beta = +1$, the interval over which the density is non-zero is $(-\infty,\ 0)$.

If $\alpha < 1$ and $\beta = -1$, the interval is the positive half of the real line.

In all other cases the interval is the entire real line.

2.13 Suppose X and Y are random variables with $X = \log Y$. If X is stable, then Y is said to be log-stable. Apart from the log-Normal distribution, the log-degenerate distribution, and the cases where $\beta = 1$, an important property of the log-stable class of distributions is that they have infinite means as well as infinite variances. Thus even if the mean of the stable distribution is finite, if its variance is infinite and $\beta \neq 1$, then the *mean* of the log-stable distribution is also infinite. This result ought not to be surprising, since if $Y = e^X$, one might intuitively reason that

$$E[Y] = E\left[e^X\right] > E\left[X^2\right] = \infty \text{ when } \alpha < 2.$$

To prove the result when $|\beta| \neq 1$, one can use the Paretian tail property of paragraph 2.11 to show that $e^x f(x) \to e^x C_1 \alpha x^{-(\alpha+1)}$ as $x \to \infty$. Now since $e^x C_1 \alpha x^{-(\alpha+1)} \to \infty$ as $x \to \infty$, it follows that the integral $E\left[e^X\right] = \int_{-\infty}^{\infty} e^x f(x)dx$ diverges.

When $\beta = +1$, the Paretian tail property does not hold. In this case the distribution of X has a heavy left hand tail (i.e. significant probability mass when x is negative and e^x is small), and insufficient probability mass when x and e^x are large for the integral $\int_{-\infty}^{\infty} e^x f(x)dx$ to diverge.

If X is Normal (i.e. $\alpha = 2$), then it is well known that

$$E[Y] = \exp\left(E[X] + \frac{VAR[X]}{2}\right), \text{ and}$$

$$VAR[Y] = \exp(2E[X] + VAR[X])(\exp(VAR[X] - 1).$$

Chapter Three : Fitting the Model

The model fitting exercise involves two stages:

i. Determining the parameters μ_D, μ_Y and A for the series $D(t)$ and $Y(t)$.

ii. Determining the parameters α, β, γ and δ of the stable distributions for the error terms $Z_D(t)$ and $Z_Y(t)$.

These two stages have influenced the choice of structure for this chapter, as follows:

- 3.1 : Description of the Data on which the Model is Based
- 3.2 : Determination of the Parameters μ_D, μ_Y and A
- 3.3 : Non-Normality of the Residuals
- 3.4 : Methods of Estimating the Parameters of a Stable Distribution
- 3.5 : Determination of the Parameters of the Stable Distributions for the Error Terms
- 3.6 : Choice of the Alternative Model

3.1 Description of the Data

3.1.1 The data used for the fitting exercise will be primarily the UK data used by the MGWP. This comprised:

- The DeZoete and Bevan Income Index (1919 to 1978), and a prospective dividend yield time series, which was derived from the DeZoete and Bevan Income and Price indices.
- A dividend and prospective dividend yield index which was derived from price and historic yield indices published by the Financial Times and the Institute of Actuaries.

The data is described and tabulated in Appendix D of Ford *et.al.*(1980) and reproduced in Appendix I of this thesis.

3.1.2 The FT-Actuaries data commenced in 1928 with the Actuaries Investment Index (Ordinary Industrials). This was an unweighted geometric mean of share prices published by the Institute and Faculty of Actuaries. The data was superseded in 1962 by the FT-Actuaries Indices, published by the Financial Times, in conjunction with the Institute and Faculty of Actuaries. The post 1962 data is based on the 500 Share Index (UK Ordinary Shares, Industrials and Oils), which was a weighted arithmetic index. The data was combined by separately chain-linking the price and dividend indices, and adjusting dividend yields pre-1962 accordingly.

3.1.3 One of the questions left open by the MGWP (see paragraph D1.3 of their report) was the relative significance of basing the model on geometric and arithmetic time series. This question is addressed in sections 3.5.6 and 3.5.7 of the thesis. This is done by constructing data which is a hybrid of the FT-Actuaries and DeZoete and Bevan indices. The hybrid time series are based on the annual DeZoete and Bevan data until 1962, and the FT-Actuaries (December) data thereafter. This is useful since it gives rise to an arithmetic index covering the whole time period (1919 to 1993). It also enables the effect of replacing the geometric Actuaries Investment index by the arithmetic FT-Actuaries index to be estimated.

The hybrid time series were constructed by separately chain-linking the dividend and price indices at December 1962. The hybrid dividend yield series was then constructed by using the relation $Y_{p_t} = D_{t+1} / P_t$ for all t.

3.1.4 In addition to the above, consideration is given to data relating to experience after the MGWP report was published. For this purpose, the hybrid indices described above were extended to include the FT-Actuaries data ending December 1993.

3.1.5 Appendix I also shows the hybrid time series, and the data needed for its construction. Table I.1 in the Appendix includes the DeZoete and Bevan data as used by the MGWP. The FT-Actuaries data ending December 1993 appears in Table I.2(a), and the hybrid indices for the periods ending December 1978 and December 1993 in Tables I.3(a) and I.3(b) respectively. In each table, the actual data for the price, dividend and dividend yield series are shown in columns (1), (2) and (8) respectively. Columns (3) and (9) show the transformed data $log(D_t/D_{t-1})$ and $log(Y_t)$ respectively. The other columns are explained in section 3.3.

3.1.6 The following notation will be used:

Abbreviation	Description
Dividend Yields	
DEZY	DeZoete and Bevan dividend Yields Annual Data 1.1.1919 to 1.1.1978 (Series D of MGWP(1980)
FTAJUNY	FT-Actuaries dividend Yields Annual Data end June 1930 to end June 1977 (Series E of MGWP(1980))
FTADECY	FT-Actuaries Dividend Yields Annual data; end Dec 1930 to end Dec 1977
FTAQTRY	FT-Actuaries Dividend Yields Quarterly Data; 1930 to 1977
HYBRIDY	The Dividend Yield Index based on Annual DEZY 1.1.1919 to 31.12.1961 December FT-Actuaries thereafter
Dividend Growth	
DEZD	De-Zoete and Bevan Income Index Annual Data; 1919 to 1978 (Series A of MGWP(1980)
FTAJUND	FT-Actuaries Derived Dividend Index Annual Data; End June 1930 to End June 1978 (Series B of MGWP(1980)
FTADECD	FT-Actuaries Derived Dividend Index Annual Data; End Dec 1930 to End Dec 1978
FTAQTRD	FT-Actuaries Derived Dividend Index Quarterly Data; 1930 to 1978
HYBRIDD	Dividend Index based on Annual DEZD 1919 to 1961 December FTA thereafter

3.2 Determination of the Parameters μ_D, μ_Y and A

3.2.1 The parameters μ_D, μ_Y and A were estimated by the MGWP using the method of least squares. This is the method traditionally used when the error terms are assumed to be Normally distributed. This is because the least squares estimators are also the maximum likelihood estimators when the Normal distribution applies.

3.2.2 However, when a stable non-Normal distribution applies, it is not immediately obvious that the least squares estimators are consistent[1] in probability. The problem was foreshadowed by Mandelbrot(1963) when he recounted Bienayme's argument in the discussion following Cauchy's 1853 paper (Bienayme(1853)). Bienayme argued that a method based on a minimization of the sum of squares of sample deviations could not reasonably be used if the expected value of this sum was known to be infinite.

The problem was also foreshadowed in section 4.4 of Geoghegan et.al(1992), when they pointed out that it *may* be necessary to re-fit the parameters of the Wilkie model, if stable non-Normal residuals were assumed.

3.2.3 Method of moments estimators are also impractical since moments of order greater than α don't exist, except when $\alpha = 2$. Davis and Resnick (1989) discuss a variation of the method of moments whereby the parameters of the time series model are obtained by equating the sample and theoretical correlation functions.

3.2.4 Fortunately, the use of the least squares estimates *are* indeed consistent - at least for the relatively simple MGWP model. The theory supporting this can be found in a series of papers including Kanter and Steiger (1974), Yohai and Maronna (1977), Hannan and Kanter (1977), Davis and Resnick (1989) and Cline(1989).

[1] See Section 3.3 of Lloyd(1984) (Volume 6) for an introduction to and definition of consistency

Kanter and Steiger (1974) proved that the least squared estimators for the parameters of a finite order auto-regressive process with symmetric error terms in the domain of attraction of a stable law process are consistent in probability. Hannan and Kanter (1977) improved on the consistency result of Kanter and Steiger by relaxing the requirement that the innovation sequences must be symmetric, and put a lower bound on the rate of convergence of the least squares estimators. Davis and Resnick (1989) went even further and showed that the sample correlation function of any Moving Average process converges in distribution to the ratio of two independent stable random variables with indices α and $\alpha/2$ respectively. By the Wold-Decomposition Theorem (see Chatfield (1975) and Miamee and Pourahmadi(1988)) this result immediately gives the limit distribution for the least squares estimators of the parameters in any finite order auto-regressive process.

3.2.5 In addition, for the data at hand, the least squares estimates turn out to be the same (to 1 significant figure) as the estimates derived by minimizing

$$\sum_t \|z_t\|^p \ (p = 1.5 \text{ for dividends and } 1.65 \text{ for dividend yields}).$$

where z_t is a sample residual. (The value for p was deliberately chosen to be slightly less than a conservative estimate for the relevant α, so that there is little doubt about the convergence of the summation.)

3.2.6 Paragraphs 3.2.4 and 3.2.5 are particularly fortunate, since it means that we can accept the values for μ_D, μ_Y and A given in MGWP(1980) as parameters for the model based on pre-1980 experience.

3.2.7 If experience since the MGWP published their report is taken into account, then the parameters μ_D, μ_Y and A *do* need to be re-estimated, and the method of least squares can once again be used.

The least squares estimates derived from the hybrid data set over the period December 1918 to December 1993 are:

$\mu_D = 5.77\%$

$\mu_Y = 5.11\%$

$A = 0.60$

The estimates for μ_D and μ_Y are rounded downwards to 5.5% and 5.0% respectively. This avoids introducing spurious accuracy. It is noted that downward rounding adjustments result in implicit margins for prudence. This could be argued to be appropriate, due to the possibility of model mis-specification error.

The reasons for using the hybrid data set are given in paragraphs 3.5.6 and 3.5.7.

Strictly speaking, introducing new data (i.e. post 1980 experience) means that the appropriate shape of the model (i.e. in Box and Jenkins(1970) notation, a (1,0,0) model for the log of dividend yields and a (0,1,0) model for the log of dividends) ought to be re-examined. This however is beyond the scope of this dissertation.

3.3 Non-Normality of the Residuals

3.3.1 In each of the tables in Appendix I, columns (5) and (11) show the residuals that result when the model is compared with the data. Column (5) shows the residuals for the dividend growth model, and is equal to column (3) - column (4). Column (11) shows the residuals for the dividend yield model, and is derived by subtracting column (10) from column (9).

3.3.2 If the error terms $Z_D(t)$ and $Z_Y(t)$ of the MGWP model are assumed to be Normal, then the residuals should constitute a random sample from the relevant Normal distribution.

3.3.3 The comments in section 1.2 suggest that in assessing the suitability of the Normal distribution, the skewness and kurtosis of the data should be examined. Kendall and Stuart (1977) describe the following widely used measures of skewness and kurtosis:

- Skewness $\sqrt{B}_1 = \frac{\mu_3}{\mu_2^{3/2}}$
- Kurtosis $B_2 = \frac{\mu_4}{\mu_2^2} - 3$

where $\mu_k = E[(X - \mu)^k]$

Under the assumption that the data (in our case the residuals) constitute a random sample of size n from a Normal distribution, then $\sqrt{B}_1$ and B_2 are also approximately Normal, with zero means and variances equal to $6/n$ and $24/n$ respectively.

3.3.4 The table below shows the skewness and kurtosis measures of the residuals that result when the MGWP model is fitted to the various data sets. The standard errors of the sample skewness and kurtosis are also shown along with the probability of the observed values.

	Skewness	Std	Prob Value	Comment	Kurtosis	Std	Prob Value	Comment
Dividend Growth								
DEZD 1919 to 1978	-0.274	0.3189	0.195114	don't reject	1.633	0.6378	0.005228	reject
DEZD 1930 to 1978	-0.657	0.3536	0.03158	?	1.249	0.7071	0.03867	?
FTADECD	-1.985	0.3536	9.93E-09	reject	5.957	0.7071	0	reject
FTAJUND	-2.045	0.3536	3.67E-09	reject	5.463	0.7071	5.55E-15	reject
FTAQTRD	-1.774	0.1759	0	reject	6.519	0.3517	0	reject
HYBRIDD to 1978	-0.360	0.3189	0.12881	don't reject	2.159	0.6378	0.000355	reject
HYBRIDD to 1993	-0.318	0.2847	0.13201	don't reject	1.727	0.5695	0.001213	reject
Dividend Yields								
DEZY 1919 to 1977	1.651	0.3162	8.9E-08	reject	7.961	0.6325	0	reject
DEZY 1930 to 1977	1.726	0.3536	5.28E-07	reject	7.336	0.7071	0	reject
FTADECY	2.097	0.3536	1.52E-09	reject	9.623	0.7071	0	reject
FTAJUNY	1.199	0.3536	0.000348	reject	3.362	0.7071	9.95E-07	reject
FTAQTRY	0.569	0.1759	0.000609	reject	4.821	0.3517	0	reject
HYBRIDY to 1977	1.665	0.3162	7.0E-08	reject	9.745	0.6325	0	reject
HYBRIDY to 1993	1.881	0.2828	1.5E-11	reject	9.177	0.5657	0	reject

It is noted that both the dividend growth and the dividend yield model can be rejected on the grounds of Kurtosis. Regardless of which data set is used, the probability of Type-I error will be very low.

The dividend yield model can also be safely rejected on the grounds of skewness, regardless of the data set used. However the dividend growth model can only be rejected on the grounds of skewness for the FT-Actuaries' data, but not for the DeZoete & Bevan nor the Hybrid time series. This last anomaly is discussed further in paragraph 3.5.6.

3.3.5 Further justification for replacing the Normal distribution by a more general stable one can be found by examining the residuals directly. For each of the tables in Appendix I, columns (6) and (12) show the probability values of the observed residuals of the dividend growth and dividend yield models respectively. The probability values are calculated if the assumption of Normally distributed error terms were true. In columns (7) and (13), observations whose probability values are less than 0.05 and 0.005 are flagged as being unlikely and extreme outliers respectively.

For example, the movements in the DeZoete & Bevan Dividend Index during 1921, and again in 1959, are considered to be unlikely with the least squares Normal model, and the movements during 1920 and 1930 are extremely unlikely. Three of the observations of the DeZoete & Bevan Dividend Yield series are considered to be unlikely, which is the expected number from 60 observations. However, an additional one observation, 1974, is extremely unlikely. With all the data sets, the frequency of unlikely, and extremely unlikely observations is higher than is consistent with the least squares Normal model.

3.4 Methods for Estimating the Parameters of a Stable Distribution

3.4.1 A number of methods are available for fitting a stable distribution to the sample residuals. The relative merits of these methods are discussed in Akgiray and Lamoureux (1989). The methods fall into three main classes:

- Fractile Methods
- Maximum Likelihood Methods
- Methods Involving the Sample Characteristic Function

3.4.2 The Fractile Methods

Included in this class are the methods of Fama and Roll (1968 and 1971) and McCulloch (1986). The idea is to base the parameter estimates on sample fractiles. For example, the location parameter δ could be based on the median, and the scale parameter γ could be based on the inter-quartile range. Allowing for the other parameters of the stable distribution involves more complicated functions of the ordered statistics, the distributions of which have been found and tabulated. The methods then entail comparing the observed fractiles (or functions thereof) with the tabulated values, and hence deducing the parameter values.

Akgiray and Lamoureux (1989) report that until recently, the most popular method has been that due to Fama and Roll (1968 and 1971). This is probably due to its main advantage which is that it is relatively simple and easy to implement.

McCulloch's (1986) method is a significant improvement over that of Fama and Roll. Without losing the relative simplicity of Fama and Roll's method, McCulloch relaxed the assumption that the distribution be symmetric (ie. $\beta = 0$). McCulloch also relaxed the assumption that $\alpha \geq 1$. In addition, the problem of the discontinuity of the location parameter δ with asymmetrical cases as the index parameter α passes unity was resolved. McCulloch's estimators for α and γ are similar to those of Fama and Roll, except that the asymptotic bias was eliminated.

3.4.3 Maximum Likelihood Estimators

Maximum Likelihood estimates are theoretically attainable, (ref. DuMouchel (1971, 1973 and 1975), but tend to be most impractical due to the lack of knowledge about the form of the density function in the general case. DuMouchel's strategy was to use the series expansions of Bergstom(1952) for symmetric densities, and to use a multinomial approximation to the likelihood function in the general case.

It follows that the maximum likelihood estimators can only be approximated. Simulation experiments suggest a downward bias in the estimation of the index parameter α and that this bias increases when α is close to 2. In addition, the method suffers from the disadvantage of requiring a relatively large amount of computational effort.

3.4.4 Methods Involving the Sample Characteristic Function

Included in this class are the methods of Press (1972), Paulson *et. al.*(1975), Wiener (1975) and Koutrouvelis (1980, 1981). The method that has been recommended by Akgiray and Lamoureux(1989), on the grounds of practicality, relative precision for the index parameter α and lack of bias, is that due to Koutrouvelis. It is noted that Koutrouvelis's method is in fact an extension of Wiener's method to allow for both skewed and symmetric distributions. The method is also consistent in probability.

3.4.5 Koutrouvelis's Method

Koutrouvelis's method involves evaluating the modulus of the sample characteristic function at a number of points close to zero. The following describes how the method works. Computer code which implements the method is provided in Appendix III.

Given a random sample $z_1, z_2, \ldots, z_n$ from a random variable Z, the sample characteristic function is defined to be

$$\psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itZ_j}$$

where $i = \sqrt{-1}$.

The modulus of the sample characteristic function at a particular point t is evaluated using

$$|\psi_n(t)| = \sqrt{\frac{1}{n}\left(\sum_j \cos(z_j t)\right)^2 + \left(\sum_j \sin(z_j t)\right)^2}\ .$$

In addition, Koutrouvelis's method makes use of the following relations which are derived from the characteristic function (equation (2.1)):

- $\log(-\log|\psi(t)|^2) = \log(2\gamma^\alpha) + \alpha \log|t|$ (3.1)
- $\dfrac{\mathrm{Im}(\psi(t))}{\mathrm{Re}(\psi(t))} = \tan\left(\delta t - \beta\gamma^\alpha \tan\left(\dfrac{\pi\alpha}{2}\right)\mathrm{sgn}(t)|t|^\alpha\right) \qquad \alpha \neq 1$ (3.2)

where $Re(\psi(t))$ and $Im(\psi(t))$ are the real and imaginary parts of $\psi(t)$ respectively.

Having evaluated $|\psi_n(t_k)|$ at a number of points ($k = 1, 2, ..., K$), the method involves using (3.1) by estimating α and γ by regressing $y = \log(-\log|\psi_n(t)|^2)$ on $w = \log(|t|)$ in the model

$$y_k = c + \alpha w_k + \varepsilon_k$$

where $c = \log(2\gamma^\alpha)$, and ε_k is an error term.

The slope of the line of best fit gives the estimate for α. This and the y-intercept is then used to estimate γ using $\gamma = \left(e^c/2\right)^{1/\alpha}$.

A second regression is used, together with the estimates for α and γ, to estimate β and δ. The second regression uses (3.2) by regressing

$$z = g(u) = \tan^{-1}\left(\mathrm{Im}(\psi_n(u))/\mathrm{Re}(\psi_n(u))\right) + \pi k_n(u)$$

on u and $\mathrm{sgn}(u)|u|^\alpha$ in the model

$$z_l = \delta u_l - \beta\gamma^\alpha \tan\frac{\pi\alpha}{2}\mathrm{sgn}(u)|u_l|^\alpha + \eta_l$$

where

u_l are the points at which $|\gamma_n(u)|$ are evaluated,

η_l are error terms, and

$k_n(u_l)$ is an integer needed only for non-principal branches of the arctan function.

Note that if non-principal branches of the arctan function are introduced, then for any given u_l, the term $\pi k_n(u_l)$ simply shifts the result of the non-principal arctan function to the principal branch. The regression simplifies if δ is assumed *a priori* to be zero.

Koutrouvelis(1980) explains that it is best to evaluate $|\gamma_n(u)|$ at points close to zero, and recommends using $t_k = \pi k/25$ (k = 1, 2, ..., K) for the first regression, and $u_l = \pi l/50$ (l = 1, 2, ..., L) for the second regression. The optimal numbers K and L of points for various sample sizes and index parameters were found by simulation and tabulated in the 1980 paper.

Koutrouvelis also points out that some of the inter-dependencies of the estimators can be reduced by using standardized data. This is effected by transforming each residual z_t to $z_t' = (z_t - \delta)/\gamma_0 = z_t/\gamma_0$ where γ_0 is an initial estimate for γ and when $\delta = 0$. If γ_1 is the estimate for the scale parameter after the data has been standardized by dividing by γ_0, then $\tilde{\gamma} = \gamma_1\gamma_0$ will be the estimate for the scale parameter for the unstandardized data. The initial estimate can be found using one of the fractile methods.

The above results in an iterative process whereby the *i'th* iteration will use the results of the *(i-1)'th* iteration to standardize the data before performing the regressions to update the parameter estimates. The process halts when the difference between the results of the successive iterations are satisfactorily small.

3.5 Determination of the Parameters of the Stable Distributions for the Error Terms

3.5.1 The results of applying Koutrouvelis's method to the sample residuals derived from comparing the MGWP model and data, are given below:

	Alpha	Beta	Gamma	Delta
Dividend Yields				
DEZY 1919-1978	1.77	(0.27)	0.099	0.004
DEZY 1930-1977	1.78	(0.46)	0.106	(0.001)
FTADECY	1.72	(0.85)	0.101	0.004
FTAJUNY	1.75	(0.72)	0.089	0.002
FTAQTRY	1.71	(0.47)	0.092	0.000
HYBRIDY 1919-1978	1.75	(0.39)	0.098	0.005
HYBRIDY 1919-1993	1.79	(0.71)	0.099	0.005
Dividend Growth				
DEZD 1919-1978	1.77	0.61	0.052	0.002
DEZD 1930-1978	1.88	1.00	0.049	0.005
FTADECD	1.60	0.92	0.045	0.002
FTAJUND	1.52	0.75	0.044	(0.001)
FTAQTRD	1.52	0.44	0.045	0.000
HYBRIDD 1919-1978	1.74	0.58	0.048	0.005
HYBRIDD 1919-1993	1.76	0.56	0.050	0.003

3.5.2 The relatively small size of the δ estimates is taken as evidence in favour of the choices for μ_D, μ_Y and A. There is no sizable difference in the estimates for α, β or γ if the location parameter δ is set *a priori* to zero. For both the dividend yield and the dividend growth models, the parameter δ will henceforth be assumed to be zero.

3.5.3 With the dividend yield model, a natural choice for $\tilde{\gamma}$ seems to be 0.10. There seems to be a slight discrepancy between the estimate for α based on the DeZoete & Bevan data, and that based on the FT-Actuaries data. If $\tilde{\alpha}$ is based on the DeZoete & Bevan data, then 1.77 seems to be an appropriate choice. However, if $\tilde{\alpha}$ is based on the FT-Actuaries data, then the estimate could be slightly lower (between 1.70 and 1.78). This discrepancy is explained in paragraphs 3.5.5 to 3.5.7. In addition, there appears to be a wide range of possible estimates for β. This is examined further in paragraph 3.5.9, and the choice of this parameter is deferred to section 3.6.

3.5.4 With the dividend growth model, the natural choice for $\tilde{\gamma}$ seems to be 0.05. There is again a significant difference between the estimate for α based on the DeZoete & Bevan data, and that based on the FT-Actuaries data, and this too is explained in the following three paragraphs. The choice for this β, is also deferred to section 3.6.

3.5.5 Since the parameter α determines the tail thickness of the distribution, the estimate for α is very much influenced by the outliers in the data. An inspection of the data will reveal that the more extreme variations on the FT-Actuaries Investment Indices occurred in 1931 and 1932 when the derived dividend index fell by 26.2% and 27.4% respectively. By contrast, the DeZoete and Bevan Income Index fell by 19.6% in 1931 and only 10.0% the following year.

3.5.6 There are two possible reasons for the observations in the preceding paragraph. The first reason will be that the two indices have different constituents. The second reason is that the two indices are constructed differently.

The DeZoete and Bevan Index is an arithmetically weighted index, while the Actuaries Investment Index (published until 1962) is a geometric index. (The effect of this difference was another question left open by the MGWP.)[2] It is also noted that in the early 1930's DeZoete and Bevan Index comprised 30 shares, while the Actuaries Investment Index comprised 123 shares. The precise constituents of the two indices is not known.

It is believed that the different construction of the two indices is the main reason for the discrepancies in 3.5.3 and 3.5.4, as well as the anomaly referred to in paragraph 3.3.4. This is because a geometric index is more sensitive to large falls in prices (or dividends) of its constituents. In an extreme case, if any one of the constituents falls in value by 100%, the whole geometric index collapses. By contrast, the impact on an arithmetic index depends on the weight given to the relevant constituent in the index, and the impact is normally much less severe.

The effect of the two indices having different constituents could be relatively small if at least some of the thirty constituents in the DeZoete and Bevan Index are also in the Actuaries Investment Index. In order to be able to quantify accurately the relative effects of the different constitution and construction of the two indices, details of the actual constituents and their weightings in the early 1930's are needed. The amount of work involved does not seem justified, and is outside the scope of this thesis.

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[2]Paragraph D1-3 of MGWP(1980) refers.

3.5.7 Instead, a new hybrid index was constructed. The hybrid index is based on the annual DeZoete and Bevan data until 1962, and the FT-Actuaries data thereafter. It was constructed by separately chain-linking the dividend and price indices at December 1962. The hybrid dividend yield series was then constructed by using the relation $Yp_t = D_{t+1}/P_t$ for all t.

The results of fitting the stable distribution to the residuals that result from comparing the MGWP model to the hybrid data, are also shown in paragraph 3.5.1. It should be noted that the hybrid data set behaves much more like the DeZoete & Bevan data, than like the FT-Actuaries data. This is what one would expect considering that the Hybrid consists of 40 years of DeZoete & Bevan data, and approximately twenty years of FT-Actuaries data. However, it also helps confirm that the main reason for the large difference between the DeZoete & Bevan results and the FT-Actuaries results is the data before 1962, and in particular the outliers in 1930 and 1931.

3.5.8 The impact that the outliers have on the parameter estimates can be illustrated by investigating what the parameter values would have been had the outliers not been present. This is most conveniently done with the FTADECD, FTAJUND and DEZD time series since these time series do not involve any auto-correlation and the outliers are amongst the first one or two observations. The table below shows the results of applying Koutrouvelis's method with the first one or two observations absent.

	Alpha	Beta	Gamma
DEZD 1919-1978	1.77	0.61	0.052
DEZD 1920-1978 (ie. exclude first year)	1.85	0.95	0.051
DEZD 1930-1978 (ie. same period as FTADECD)	1.88	1.00	0.049
FTADECD	1.60	0.92	0.045
FTADECD excluding first year	1.74	0.95	0.045
FTAJUND	1.52	0.75	0.044
FTAJUND excluding first year	1.65	0.95	0.043
FTAJUND excluding first two years	1.88	0.95	0.043

The results confirm that much of the difference between the FTADECD and DEZD parameter values is due to the more extreme observations in the FTADECD series in 1932/3.

The results also illustrate how sensitive the parameter estimates can be to outliers in the data. This can be argued to be an advantage, since it is the extreme stochastic fluctuations associated with these outliers which we wish to be able to model accurately. It is therefore advantageous for the parameter estimation method to make maximum use of the information content of the extreme observations in the data. However the lack of robustness of the estimators reduces one's confidence in the reliability of the parameter values especially since the data set is invariably incomplete. A potential problem also arises when the extreme observations in the data are the most distant observations. Is it sensible to give a high weighting to observations made a long time ago? These ideas are discussed further in section 5.1.

It is not known how sensitive the parameter estimators will be to any one outlier when much larger data sets are available.

3.5.9 Another important point to be made is that the parameters of the stable distribution cannot be reliably estimated with only 60 data points. This applies especially to the skewness parameter β. This fact can be established with the following simulation experiment.

SIMULATION EXPERIMENT
1. Simulate a random sample of size n (e.g. 60) from a population with known α, β, γ and δ
2. Use Koutrouvelis's Method to estimate the parameters from this sample.
3. Repeat the simulation and estimation procedures a number of times. Hence derive estimates for the standard errors of the Koutrouvelis estimators.

The standard errors depend on the true values of n, α, β, γ and δ. The results of this experiment for a particular set of values for n, α, β, γ and δ are shown below.

ACTUAL VALUES					AVERAGE OF ESTIMATES				STD ERRORS			
α	β	γ	c	n	α	β	γ	c	α	β	γ	c
1.75	-0.5	0.10	0	2000	1.750	- 0.499	0.100	0.0003	0.005	0.024	0.001	0.002
1.75	-0.5	0.10	0	240	1.753	- 0.505	0.100	0.0004	0.008	0.031	0.001	0.003
1.75	-0.5	0.10	0	60	1.745	-0.441	0.097	0.0015	0.065	0.284	0.008	0.016

More detailed sampling statistics and simulation experiments of this kind are given in Koutrouvelis(1980). It is also worth noting that the above experiment provides a useful means of checking the simulation and parameter estimation software against each other.

3.5.10 One may attempt to deal with the problem of unreliable estimators associated with short time series by using more frequent observations and the additive property of stable distributions. Using quarterly (non-overlapping) data as an example, the analysis in paragraph D2-12 of the MGWP report can be generalised with the following relationships:

- If $\log Y_t = \log \mu_Y + A.(\log Y_{t-1} - \log \mu_Y) + Z_t$ with $Z_t \sim S(\alpha,\beta,\gamma,0)$ then
  $\log Y_{t+4} = \log \mu_Y + A^4(\log Y_{t-1} - \log \mu_Y) + Z_t'$ with $Z_t' \sim S(\alpha,\beta,(1+A^\alpha + A^{2\alpha} + A^{3\alpha})^{1/\alpha}\gamma,0)$

- If $\log D_t = \log D_{t-1} + \mu_D + Z_t$ with $Z_t \sim S(\alpha,\beta,\gamma,0)$
  then $\log D_{t+4} = \log D_t + 4\mu_D + Z_t'$ with $Z_t' \sim S(\alpha,\beta,4^{1/\alpha}\gamma,0)$.

The equivalent annual models which results when the FT-Actuaries quarterly time-series (FTAQTRD and FTAQTRY) are used, are shown in the table in paragraph 3.5.1. The reasonably close agreement between the annualised quarterly estimates, and the estimates derived from annual data, can be taken as evidence that the data come from a stable distribution (or at least are within its domain of attraction).

It is not clear however, whether or not more frequent sampling reduces the uncertainty in the Koutrouvelis estimates for the various stable distribution parameters. In the case of the Normal distribution, sampling more frequently reduces the uncertainty in the maximum likelihood variance estimates, but does nothing to reduce the uncertainty with the mean. More frequent sampling also fails to reduce the uncertainty with the maximum likelihood estimates for the location parameter δ of the stable distributions.

In any case, the DeZoete and Bevan time series is available only on an annual basis. This is unfortunate since we noted (in paragraph 3.5.6) that the FT-Actuaries data is based on a geometric index until 1962, and this makes it less suitable.

3.6 A possible alternative model

3.6.1 Model for Pre-1980 Experience

A possible refinement to the MGWP model is one which uses the following stable distributions instead of the relevant Normal distributions:

- For Dividend Yields $\alpha_Y = 1.75$ $\beta_Y = -0.5$ $\gamma_Y = 0.10$ $\delta_Y = 0$
- For Dividend Growth $\alpha_D = 1.75$ $\beta_D = 0.5$ $\gamma_D = 0.05$ $\delta_D = 0$

The final choice of parameters is ultimately a matter of judgment. I have based my choice primarily on the DeZoete & Bevan data set, which was the longest arithmetic index available at the time the model was published. I have also applied some rounding to the parameter estimates, to avoid introducing spurious accuracy. This is justified in view of the fact that the parameters cannot be regarded as reliable, due to the limited data available. The rounded parameter estimates corroborate well with parameter estimates based on the hybrid time series, especially for pre-1980 experience.

3.6.2 Updated Model to Allow for Post-1980 Experience

If experience since the MGWP published their report is taken into account, then the parameters need to be updated. The following values were derived from the least squares estimates applicable for the hybrid time series over the period December 1918 to December 1993.

$\mu_D = 5.5\%$

$\mu_Y = 5.0\%$

$A = 0.60$

The least squares estimates have again been rounded downwards to avoid introducing spurious accuracy. Downward adjustments are conservative. The hybrid time series were used as they are based on the longest arithmetic indices available.

The same stable distributions can be used for the error terms in the model for post-1980 experience as were used in the model for pre-1980 experience. There is a case for increasing the value chosen for α_Y to 1.80, but this seems unwise in view of the relatively large uncertainties about parameter estimates derived from small data sets, and the fact that low index parameters are conservative.

Strictly speaking, introducing new data (ie. post 1980 experience) means that the appropriate shape of the model (i.e. in Box and Jenkins(1970) notation, a (1,0,0) model for the log of dividend yields and a (0,1,0) model for the log of dividends) ought to be re-examined. This however is beyond the scope of this dissertation.

3.6.3 Goodness of Fit

The Kolmogorov-Smirnov test (see Chapter 14 of Lloyd (1984)) was used to test the goodness of fit of the proposed models to the relevant data sets. This involves comparing the empirical cumulative distribution function with the theoretical distribution function specified in the null hypothesis, which was found by simulation. In all cases the test failed to reject the model at the 5% significance level.

The advantage of the Kolmogorov-Smirnov (KS) test over the Chi-squared test is that the KS test uses the individual observed values, without grouping of data (and subsequent loss of information). It is also believed to be more powerful than the Chi-squared test (section 14.2.1 of Lloyd (1984)).

For both the dividend yield and dividend growth models, we have added two extra parameters (α, β) to those originally proposed by Ford et.al. (1980). We would therefore expect that our model should fit the data more closely than that proposed by Ford et.al. (1980). The real question is whether the model is sufficiently improved by the extra parameters.

The main problem with the KS test is that it is not entirely distribution free when the parameters in the null hypothesis are not completely specified and need to be estimated from the data (i.e. a composite test). Hogg and Klugman (1984) explain that if the KS approach is used, then we must realise that the actual significance level will be somewhat smaller than nominal significance level. In other words, if the KS test rejects the null hypotheses, at the 5% level, then the probability of erroneously rejecting the model will be *less than* 5%. However, if, as has happened in our case, the KS test fails to reject the model, then we should not feel too confident about accepting it.

The chi-squared test is more widely used when the parameters are not completely specified. The approach involves using *k - 1 - m* degrees of freedom, where *k* is the number of cells (or groupings) of the data and *m* is the number of parameters estimated. It should be noted however that this will lead to the real significance level being somewhat greater than the quoted significance level when the parameters have not been estimated by using a minimum chi-squared procedure (section 3.6 of Hogg and Klugman(1984).) This applies in our situation where Koutrouvelis's method has been used. This means that the probability of erroneously rejecting the model could be *greater than* 5%. However if the test fails to reject the model, then we can feel more confident about its acceptance since it would have passed a more stringent test. (The probability of Type II error is less than it would be had the nominal significance level been equal to the actual significance level.)

	Chi-square value		Chi-square value
Dividend Yields		**Dividend Growth**	
DEZY 1919-1978	7.99	**DEZD 1919-1978**	1.52
FTADECY	8.94	**FTADECD**	2.95
FTAQTRY	7.80	**FTAQTRD**	2.84
HYBRIDY 1919-1978	7.67	**HYBRIDD 1919-1978**	1.50
HYBRIDY 1919-1993	7.50	**HYBRIDD 1919-1993**	1.48

The results of applying the chi-squared test with the various data sets are shown above. In each case, eight cells were used and the probability values for each cell were found by simulation. The critical values of the Chi-Squared distribution with *d* degrees of freedom and significance probability *p* are shown below. (Source: Institute and Faculty of Actuaries (1980))

d.o.f. *d* / *p*	7	6	5	4	3	2
5%	14.07	12.59	11.07	9.49	7.82	5.99
2.5%	16.01	14.45	12.83	11.14	9.35	7.38

The dividend growth model is not rejected at the 5% significance level with all data sets and with as little as 2 degrees of freedom.

The dividend yield model for the hybrid data can be accepted[3] at the 5% level with 3 or more degrees of freedom. The model is acceptable provided we do not subtract a degree of freedom for one of the parameters, say the skewness parameter. This seems reasonable since it has already been noted (paragraph 3.5.9) that the skewness parameter cannot be reliably estimated from the data and it was chosen by judgment (and arguably somewhat arbitrarily).

The fit of the dividend yield model for the other data sets is not quite as good. These models can be accepted at the 5% significance level with 4 or more degrees of freedom or at the 2.5% significance level with 3 or more degrees of freedom. This does not seem wholly unacceptable if we allow for the fact that:

- these data sets (especially the geometric data sets) had less influence on the choice of parameter values and can therefore be expected to require more degrees of freedom
- the parameter values, especially the location parameters, have been subject to adjustment
- the real significance level of the chi-squared test is greater than quoted

Notwithstanding this, it could be argued that the goodness of fit of the dividend yield model is not ideal, and there is some evidence of over-parameterisation.

3.6.4 Sensitivity to Choice of Parameters and Rounding Adjustments

The sensitivity of the model to the choice of parameters is investigated in section 4.3.
The effect of the particular rounding adjustments that have been made is examined in section 4.4.

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[3]or rather 'not rejected'

3.6.5 An Improvement to the Existing Model

Despite the relatively large uncertainties surrounding the appropriate stable law parameters, the proposed refinement should still improve the MGWP model. This is because the Normal distribution is in fact a special case of a stable distribution, and rejection of a stable distribution by definition implies rejection of the Normal distribution too.

Chapter 4: Impact of Using Stable Distributions

This rather long chapter is structured by introducing (in sub-section 4.1) the notation, definitions and liability models that were used in the MGWP report, and which have been extended in this paper.

Sub-section 4.2 gives some theoretical implications for replacing the Normal distributions by stable non-Normal distributions. Probability distributions for forecasts of the dividend and dividend yield variables will be specified. In addition, it will be proved that the annual investment return has infinite mean (as well as infinite variance). This leads to the failure of the sample means and variances as measures of risk and reward and to the inapplicability of a large part of financial economics based on mean-variance optimisation and power law utility functions.

The impact of changing the parameters of the MGWP investment model (including the parameters of the stable distributions) are investigated with sensitivity tests in section 4.3. Separate sensitivity tests are used for:

- The Location, Scale and Auto-Correlation Parameters (section 4.3.1)
- The Index (or Tail Thickness) Parameters of the Stable Distributions (section 4.3.2)
- The Skewness Parameters (section 4.3.3)
- Initial Yields (4.3.4)

The possible alternative models (including the original MGWP model) that could be derived from the various data sets are compared in section 4.4. In this section, the differences between the MGWP model and the proposed alternative are quantified in detail.

Finally the behaviour of the proposed alternative investment model with various different liability portfolios is investigated in section 4.5.

4.1 Notations and Definitions

4.1.1 Let *I(t)* denote the rate of return in year *t* of the projection.

Let *G(t)* denote the geometric return over the *t* year period.

Thus if *R(t)* is the share price or equity unit price (allowing for reinvestment of dividend income) at time *t*, then

$$1 + I(t) = \frac{R(t)}{R(t-1)}, \text{ and}$$

$$1 + G(t) = \left(\frac{R(t)}{R(0)}\right)^{1/t} = \left(\prod_{s=1}^{t}(1 + I(s))\right)^{1/t}$$

The model for *R(t)* is given in section 1.1.

4.1.2 In view of the fact (see sections 4.2.2 and 4.2.3) that the annual investment returns have infinite means and variances, the effect of using a stable distribution will be investigated by examining the sample fractiles.

If X is a random variable, define $Q_X(p)$ such that

$$\Pr(X \geq Q_X(p)) = p.$$

Equivalently,

$$Q_X(p) = F_X^{-1}(1-p)$$

where $F_X^{-1}(t)$ is the inverse of the cumulative distribution function of X.

The median of X is $MED[X] = Q_X(0.5)$. Exactly half the population values are greater than the median, and exactly half the observations are less than it.

The inter-quartile range of X is $IQR[X] = Q_X(0.25) - Q_X(0.75)$. The inter-quartile range is the interval in which 50% of the population values lie.

The effects of using the stable distributions on investment performance will be investigated by examining:

- *MED[I(t)];* the median of the annual investment return in the *t*'th year of projection
- *MED[G(t)];* the median of the geometric investment return over a *t* year period
- *IQR[I(t)];* the inter-quartile range of the annual investment return
- *IQR[G(t)];* the inter-quartile range of the geometric investment return

The inter-quartile range is a widely used measure of the spread of values that the variable X can take. However it ignores observations in the tail of the distribution. Since these observations are critical to our investigation on extreme stochastic fluctuations, it will also be necessary to examine the some of the fractiles in more detail. The values $Q_{I(t)}(0.005)$ and $Q_{G(t)}(0.005)$ illustrate how large some (0.5%) of the observations are. The values $Q_{I(t)}(0.995)$ and $Q_{G(t)}(0.995)$ illustrate how small some of the observations are.

In all cases, the sample fractiles, derived from $n = 5000$ simulations, are used as the estimates for the population fractiles.

4.1.3 The MGWP investment model was developed primarily for the purpose of managing maturity guarantees under equity-linked insurance policies. It therefore seems sensible to investigate the effect of using stable non-Gaussian distributions within a model for equity-linked insurance policies with maturity guarantees. The same liability model that was used by the MGWP will be used here. A summary of this model is given below. Appendix E of Ford *et.al.*(1980) provides more details.

4.1.4 Consider an insurance policy with a premium of 1 p.a. invested for a term of n years. At maturity, the policyholder is promised that his benefit will be the accumulated amount of his investments, subject to a guaranteed minimum being a return of the premiums paid.

Let S_n = the accumulated amount of the investments of 1 p.a. after n years.

The maturity guarantee is then n and the maturity benefit is $B_n = max(n, S_n)$.

Define the guarantee claim amount $G_n = max(0, n-S_n)$.

It is noted that B_n, S_n and G_n are related through $B_n = G_n + S_n$.

4.1.5 Now consider a cohort of policies written simultaneously at time $t=0$.

Ignore mortality so that all policies reach maturity.

Let P_t denote the premium payable for policies maturing in year t $(t=m_1, \ldots, m_2)$. The quantities m_1 and m_2 can be defined to be the earliest and latest maturity dates of policies in the cohort.

The total sum assured is given by

$$T = \sum_{t=m_1}^{m_2} tP_t.$$

The total guarantee claim for a particular realization is

$$C = \sum_{t=m_1}^{m_2} P_t G_t.$$

The total discounted claim for a particular realization is

$$DC = \sum_{t=m_1}^{m_2} P_t G_t v^t$$

where $v = 1/(1+i)$ and i is the discount rate of interest. Ford et.al.(1980) fixed i at 4%.

The claim ratio, as a percentage of the total sum assured, is defined as

$$CR = 100 \times C/T,$$

and the discounted claim ratio (as a percentage of total sum assured) is defined as

$$DCR = 100 \times DC/T.$$

4.1.6 The statistics to be examined for each set of n (typically 5000) simulations are:

- *NZ*, the number of simulations in which *CR* and *DCR* are non-zero
- *MNZ,* the mean of the NZ non-zero observations of *CR* and *DCR*.
- Specific quantiles $Q_{CR}(p)$ and $Q_{DCR}(p)$ of the simulated distributions of *CR* and *DCR* respectively. $Q_{CR}(p)$ and $Q_{DCR}(p)$ are defined analogously to the variables $Q_{I(t)}(p)$ and $Q_{G(t)}(p)$ in section 4.1.2. The specific quantiles to be examined correspond to p = 0.001, 0.005, 1% and 5%.

The quantity *MNZ* provides a measure of the expected severity of a claim. The quantile $Q_{CR}(k/n)$ provides an estimate for the contingency reserve needed, as a percentage of the total sum assured, to ensure that the probability of ruin is limited to k/n. The quantile $Q_{DCR}(k/n)$ gives the reserve needed, as a percentage of total sum assured, if discounting is allowed. The ruin probability will be limited to k/n if it is assumed that the reserves can be invested in a risk-free investment and earn 4% p.a.

4.1.7 It can be argued that the MGWP's approach of calculating the reserves by discounting the claims at a certain rate of interest is questionable. If one chooses to discount at a fixed rate, then what value should this be? A widely used deterministic approach is to discount at a rate equal to an assumed risk-free rate of return, to which a risk margin could be added. This is problematic in a stochastic environment where there is no such thing as a 4% risk-free investment return and margins for risk are incorporated by examining stochastic ruin probabilities.

Possible alternative approaches include:

- discounting at 0% (i.e. the mis-matching reserves are kept under the mattress)
- discounting at the rate earned on the underlying unitized fund
- introducing *stochastic* risk free interest rates which are consistent with the remainder of the stochastic model

In order to compare like with like, I shall adopt the approach used by the MGWP, but concentrate on the undiscounted claim ratios.

4.1.8 The MGWP investigated the reserving requirements for various different insurance portfolios. The portfolios investigated include (amongst others):

- A cohort of identical policies all written for a term of 10 years
- A cohort of identical policies all written for a term of 20 years
- A standard portfolio being chosen as representative of a plausible typical mix of new business. The standard portfolio is reproduced in Appendix II. Except where otherwise indicated, all comparisons will be made by reference to this standard portfolio.

4.1.9 The MGWP assumed a rate of tax, τ, of 37.5% on dividend income. UK tax rates have subsequently changed a number of times. Except where otherwise indicated, a gross investor will be assumed.

4.1.10 In order to be able to run the model, it is necessary to assign a value to the initial yield Y_o. In practice, the dividend yields prevailing at the time of application would be used. Except where otherwise indicated, Y_o will be set to equal μ_Y. The sensitivity of the model to Y_o will be investigated in section 4.3.4.

4.2 Some Theoretical Implications of Using Stable Distributions

4.2.1 Forecasting more than one step ahead

The distribution of $\log D_{t+s}$, given the value of dividends D_t at time t, is

$$\log D_{t+s}\Big| D_t \sim S(\alpha_D, \beta_D, s^{1/\alpha}\gamma_D, \log D_t + s\mu_D)$$

and the distribution of $\log Yp_{t+s}$, given the value of dividend yields Yp_t at time t is

$$\log Yp_{t+s}\big| Yp_t \sim S\left(\alpha_Y, \beta_Y, \left[\frac{A^{\alpha s}-1}{A^{\alpha}-1}\right]^{1/\alpha}\gamma_Y, A^s \log Yp_t + \left(1-A^s\right)\log\mu_Y\right)$$

These facts can be established by re-writing the model in the above form (with $s = 1$) to establish an inductive basis. One can then use the techniques outlined in paragraph 3.5.10 and the principle of induction to show that the if the result is true an integer s, then the result holds true for $s+1$.

4.2.2 Moments of the Annual Investment Return

In this section I shall prove that if

a) either α_D or α_Y is less than 2

then the variance of the annual investment return of the model is infinite.

If in addition,

b) $|\beta_D| \neq 1$

then the mean of the annual investment return is also infinite.

One might intuitively expect the variance of the annual investment return to be infinite when the error terms $Z_D(t)$ and/or $Z_Y(t)$ have infinite variance (i.e. case (a)). Less intuitive is the result that the annual investment return has infinite mean, even when α_D and α_Y are greater than 1, so that the error terms have finite means (i.e. case (b)).

To prove the result, first introduce some notation and let $\left(\frac{P_t}{P_{t-1}}|Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right)$ denote the variable $\frac{P_t}{P_{t-1}}$ *given that* $Yp_{t-1} = y_{t-1}$ and $Yp_t = y_t$. Next assume that $\alpha_D < 2$ and $|\beta_D| \neq 1$, and note that

$$P_t = \frac{D_{t+1}}{Yp_t}$$
$$\Rightarrow \frac{P_t}{P_{t-1}} = \frac{D_{t+1}}{D_t}\frac{Yp_{t-1}}{Yp_t}$$
$$\Rightarrow \log\left(\frac{P_t}{P_{t-1}}|Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right) = \log\left(\frac{D_{t+1}}{D_t}\right) + \log\left(\frac{y_{t-1}}{y_t}\right).$$

Thus $\log\left(\frac{P_t}{P_{t-1}}|Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right)$ is a stable (non-Gaussian) random variable. Consequently, $\left(\frac{P_t}{P_{t-1}}|Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right)$ is *log*-stable (non-Gaussian) and therefore, by the result in paragraph 2.13 it has infinite mean and variance (when $\alpha_D < 2$ and $|\beta_D| \neq 1$.) Now since

$$E\left[\frac{P_t}{P_{t-1}}\right] = E\left[E\left[\frac{P_t}{P_{t-1}}|Yp_{t-1} = y_1;\ Yp_t = y_2\right]\right]$$
$$= \iint E\left[\frac{P_t}{P_{t-1}}|Yp_{t-1} = y_1;\ Yp_t = y_2\right] f_{Yp_{t-1},Yp_t}(y_1;\ y_2)dy_1dy_2,$$

it follows that the unconditional mean of $\frac{P_t}{P_{t-1}}$ is infinite. (The integrands in the above integration are either zero, when the joint probability density $f_{Yp_{t-1},Yp_t}(y_1;\ y_2) = 0$, or infinite, when the joint probability density $f_{Yp_{t-1},Yp_t}(y_1;\ y_2) > 0$.)

Using $VAR\left[\frac{P_t}{P_{t-1}}\right] = E\left[VAR\left[\frac{P_t}{P_{t-1}}|Yp_{t-1};Yp_t\right]\right] + VAR\left[E\left[\frac{P_t}{P_{t-1}}|Yp_{t-1};Yp_t\right]\right]$ establishes the fact that the unconditional variance of $\frac{P_t}{P_{t-1}}$ is infinite when α_D is less than 2.

A similar reasoning can be used for the case when α_Y is less than 2 and then to show that the annual investment return $\frac{R_t}{R_{t-1}} = \frac{P_t}{P_{t-1}} + \frac{D_t}{P_{t-1}}$ has infinite variance (when (a) holds) and also infinite mean (when (a) and (b) hold).

4.2.3 Failure of the Sample Mean and Standard Deviation as Measures of Risk and Reward

In the previous section it was shown that if a stable non-Gaussian distribution is used, then the population means and standard deviations for the investment returns will be infinite. It was also noted that for any finite number of simulations, the *sample* means and standard deviation *will* be finite. One might be tempted therefore, to compute the sample means and standard deviations and use them as measures of the amount of risk and reward inherent in the model under consideration.

The idea of using the sample moments as measures of risk and reward is flawed. Suppose after n simulations, E_n equals the average of *I(t)*, the annual rate of return in year *t*. Suppose also that after n simulations, S_n equals the sample standard deviation of *I(t)*. If the sample moments were useful measures of risk or reward, then the variables E_n and S_n should be able to fluctuate within some band for large enough n. This is necessary in order that some useful information can be inferred from a given simulation (e.g. to form comparisons with alternative models). The following proof shows that the sample moments do not fluctuate within any band, regardless of the number of simulations, and both E_n and S_n tend to infinity for large n with probability 1.

Let I_n denote the value of *I(t)* on the n'th simulation. The variables $I_1, I_2, \cdots I_n$ are independent and identically distributed with the same distribution as *I(t)*. Note also that $I_n \geq -1$, since asset prices cannot go below zero.[1]

[1]We are effectively restricting our attention to *limited liability* share companies and we ignore the impact of any wind-up costs for the shareholders or additional liabilities they may inherit when they acquire their shares.

Now let $M > 0$ be an arbitrarily large constant and consider the variable min{*I(t), k*} for large *k*. This variable is bounded in [-1, *k*] and therefore has finite mean. By the monotone convergence theorem (see chapter 1, section 3.5 of Durrett(1991) or section 5.3 of Williams(1991)), and the result in 4.2.2 above, we have

$$\lim_{k\to\infty} E[\min\{I(t),k\}] = E[I(t)] = \infty.$$

In particular, we can find K such that $E[\min\{I(t),K\}] > M$.

Now define:

$$E_n = \frac{I_1 + I_2 + \cdots + I_n}{n}, \text{ and}$$

$$E_n^* = \frac{\min\{I_1,K\} + \min\{I_2,K\} + \cdots + \min\{I_n,K\}}{n}$$

By Kolmogorov's strong law of large numbers (chapter 1, section 8.3 of Durrett(1991) or section 12.10 of Williams(1991)), and by hypothesis on K, we have with probability 1,

$$\lim_{n\to\infty} E_n^* = E[\min\{I(t),K\}] > M.$$

However, $E_n \geq E_n^*$, and therefore $\liminf_{n\to\infty} E_n \geq \liminf_{n\to\infty} E_n^* > M$.

Now since M was arbitrary, it follows that $\liminf_{n\to\infty} E_n = \infty$ with probability 1. This concludes the proof for E_n. A similar (but slightly longer) proof is possible for S_n.

An important implication of the above developments is that a large part of financial economics based on mean-variance optimisation or power-law utility functions will be inapplicable under a stable (or rather log-stable non-Gaussian) probability framework.

4.3 Sensitivity Tests

4.3.1 The Location, Scale and Auto-Correlation Parameters

In section E2.5 of their paper, the MGWP investigated the sensitivity of changing the basis. They did this by individually changing each of the parameters μ_D, μ_Y, σ_D, σ_Y and A both upwards and downwards by a small and large amount. This experiment is repeated with the error terms assumed to follow the proposed alternative model, and with the tax rate set to zero.

The results are shown in tables IV.1(a), IV.1(b), and IV.1(c) respectively. Note that the scale parameters σ_D and σ_Y of the Normal distribution are re-written using the general stable distribution notation γ_D and γ_Y, where $\gamma_D = \sigma_D / \sqrt{2}$ and $\gamma_Y = \sigma_Y / \sqrt{2}$.

The results show that the same general conclusions that were made by the MGWP can be made with the proposed alternative model. These are summarised as follows:

i. The effect of a change in basis is, in general, far from linear.
ii. The results tend to be most sensitive to the dividend location, followed by the yield location and then the auto-regressive parameter A.
iii. The parameters affect the different distributions in different ways, and the ranking order just quoted depends on which statistic one is considering. For example, the sample standard deviations of the investment returns seem to be most sensitive to the scale parameter σ_Y, while the claim frequency rate is most sensitive to changes in μ_D.

iv. Care is needed when choosing the basis. A too optimistic view of the dividend growth location parameter (the mean growth rate when Normal distributions are involved), may have a large effect in reducing the required contingency reserves, but may be unjustified in the light of experience. Given the difficulties and uncertainties surrounding the choice of parameter estimates, some implicit margins for caution could be built into the reserving basis. This could come in the form of conservative adjustments to the parameters of the investment model, and/or reserving at lower probabilities of ruin.

v. If adjustments to the parameter values are to be incorporated in the basis (e.g. for rounding or to cater for possible model mis-specification error) then:

   - for the location parameters, downwards adjustments are conservative
   - for the scale parameters, upwards adjustments will be conservative.

   Low values for the location parameters means that a low performing asset which is less capable of supporting the maturity guarantees would be modelled. High values for the scale parameters result in more volatile stochastic fluctuations, and increased frequency and severity of guarantee claims.

If the more general stable (non-Gaussian) distributions are used, then two additional types of parameter are introduced. These are the index and skewness parameters which are investigated in the following two sections.

4.3.2 Tail Thickness and the Index Parameters α_D and α_Y

In this experiment we shall fix the skewness, scale and location parameters at the values of the proposed alternative model, and investigate the effect of choosing different index parameters.

Tables IV.2(a) ,to (i) (in appendix IV) show the values of *MED[I(30)]*, *IQR[I(30)]*, $Q_{I(30)}(0.995)$, $Q_{I(30)}(0.0055)$ *NZ*, *MNZ*, $Q_{CR}(0.001)$, $Q_{CR}(0.01)$ and $Q_{CR}(0.05)$, with the skewness, scale and location parameters fixed at

$$\begin{array}{lll} \beta_D = +0.5 & \gamma_D = 0.05 & \delta_D = 0 \ (\mu_D = 0.04) \\ \beta_Y = -0.5 & \gamma_Y = 0.10 & \delta_Y = 0 \ (\mu_Y = 0.05) \end{array}$$

and the index parameters varying between 1 and 2 in 0.1 increments.

The results illustrate how the fluctuations within the model increases when either or both of the index parameters are reduced. For example, *IQR[I(30)]* increases from 25.5% when both index parameters are two (Normally distributed error terms) to 40.7% when both index parameters are one. The results are even more noticeable when tail fractiles are considered. For example, $Q_{I(30)}(0.995)$ reduces from -29.6% when both index parameters are two to -98.6% when the yield index parameter is one, and to -84% when the dividend growth index parameter is one. At the same time, $Q_{I(30)}(0.005)$ increases from 70% when both index parameters are two to 5.6×10^9% when they are both one. In the same sets of simulations, the reserving requirement (for a 1:1000 chance of ruin) increases from 3% of total sum assured to 86% of total sum assured (a 2800% increase in relative terms).

By reducing the index parameters, the kurtosis and tail thickness of the distributions are increased, according to the law of Pareto (paragraph 2.11). Very fat tails result in a greater proportion of random sample points having extreme values, and the stochastic fluctuations therefore become more extreme. This contributes to more frequent and more severe guarantee claims, with a consequential increase in the reserving requirement.

The reserving requirements at the less extreme ruin probabilities (e.g. 1% or 5%) are less sensitive to changes in the index parameters. For example, reducing the index parameters from 2.0 to 1.90 has no material impact on the reserving requirement corresponding to a 5% ruin probability. By contrast, the same reduction in index parameters causes the reserving requirement for a 1:1000 ruin probability to increase five times. A *very* small reduction in either index parameter (e.g. from 2 to 1.999) does not have a significant effect on the reserving requirement, except perhaps at extremely low ruin probabilities.

It is also interesting to note that such small changes in index parameters do not materially impact on the number or severity of claims. For example, when both index parameters are 1.999, the number and size of claims are $NZ = 82.5$ and $MNZ = 0.50$ respectively. These observations should be contrasted with the fact that even with a *very* small reduction in index parameters, the variance of the investment returns become infinite. This illustrates the point that volatility, as measured in terms of variance or standard deviation, is not a satisfactory measure of risk under a stable non-Gaussian probability framework.

Table IV.2(a) shows that while the median of the annual investment return increases slightly with reductions in α_D, the same is not necessarily true with reductions in α_Y. This suggests that it would be difficult to replace the mean-variance optimization theory in financial economics with optimizations involving say the median and the inter-quartile range.

4.3.3 Skewness

In this section we shall fix the index, scale and location parameters at the values of the proposed alternative model, and investigate the effect of choosing different skewness parameters.

Tables IV.3(a) to (i) show the values of *MED[I(30)]*, *IQR[I(30)]*, $Q_{I(30)}(0.005)$, $Q_{I(30)}(0.995)$, *NZ*, *MNZ*, $Q_{CR}(0.001)$, $Q_{CR}(0.01)$ and $Q_{CR}(0.001)$ with the index, scale and location parameters fixed at

$$\alpha_D = 1.75 \qquad \gamma_D = 0.05 \qquad \delta_D = 0 \ \ (\mu_D = 0.04)$$
$$\alpha_Y = 1.75 \qquad \gamma_Y = 0.10 \qquad \delta_Y = 0 \ \ (\mu_Y = 0.05)$$

and the skewness parameters β_D and β_Y varying between -1 and +1 in increments of 0.5.

It is important to state at outset that the relative significance of the skewness parameter β depends also on the parameters α, γ and δ of the stable distribution concerned. For example, as $\alpha \to 2$ the value of β becomes irrelevant. (This is because the term $\tan\left(\pi\alpha/2\right)$ in equation (2.1) approaches 0 as $\alpha \to 2$.)

The following observations were made with the particular parameter values associated with the proposed alternative model:

i. For a given β_Y, the median of the annual investment return, *MED[I(30)]*, increases as β_D increases. By contrast, for a given β_D, *MED[I(30)]* decreases as β_Y increases.
ii. The inter-quartile range of the annual investment return, *IQR[I(30)]* is not materially affected by changes in the skewness parameters.
iii. The above two observations again illustrate the difficulty of attempting to optimize any trade-offs between the median and the inter-quartile range of the investment return. The inter-quartile range can be relatively insensitive to changes in a particular parameter if the quantiles $Q_{I(30)}(0.25)$ and $Q_{I(30)}(0.75)$ both change by the same amount and in the same direction.

iv. The more extreme fractiles can also convey useful information, and should be examined in conjunction with the claim statistics. Tables IV.3(c) and (d) show that the large returns, $Q_{I(30)}(0.005)$, decrease as β_D increases as do the small (negative) returns $Q_{I(30)}(0.995)$. These observations are consistent with the dividend growth model having finite mean when $\beta_D = +1$. By contrast, $Q_{I(30)}(0.005)$ and $Q_{I(30)}(0.995)$ increase as β_Y increases. (The observations are explained below.)

v. The claim frequency statistic *NZ*, increases with increasing β_D. The behaviour of *NZ* with changes in β_Y depends on β_D. If $\beta_D > 0$, then *NZ* decreases with increasing β_Y.

vi. The mean claim severity *MNZ* reacts similarly. It increases with increasing β_D, and its behaviour with changes in β_Y depends on β_D. If $\beta_D > 0$, then *MNZ* increases with increasing β_Y.

vii. Overall the reserving requirement $Q_{CR}(0.001)$, which depends on both the number and severity of claims, appears to be more sensitive to β_D than to β_Y. The reserving requirement increases with increasing β_D, and its behaviour with changes in β_Y also depends on the value of β_D. If $\beta_D > 0$, then the reserving requirement tends to increase with increasing β_Y.

viii. When considering the contingency reserves required for maturity guarantees, we can say that positive β assumptions (negative skewnesses) are conservative. The data supported a positive assumption for β_D and a negative assumption for β_Y. It was noted however (ref. paragraph 3.5.9) that the limited data availability meant that the β parameters could not be reliably estimated

ix. Overall, the skewness parameters appear to have much less impact on the simulation results than do the index, scale and location parameters. The observation that the model is relatively insensitive to the choice of skewness parameters (especially β_Y when β_D is fixed) is an important factor to consider when settling on the final basis.

Recall that positive β*'s* imply negatively skewed distributions for the error terms. This means that the distributions have positive masses and long left hand tails. The opposite applies with negative β*'s*. The tail follows an asymptotic form of the law of Pareto when $|\beta \neq 1|$ (paragraph 2.11).

A long left hand tail for the dividend growth model means that the probability of large *falls* in dividends remains relatively large. This explains why a positive β_D assumption is conservative. If $\beta_D = +1$, the mean dividend growth is finite and one might intuitively expect higher ruin probabilities, larger claim severities and larger reserving requirements than a model whose mean dividend growth is infinite.

The above is consistent with the work of Leitch and Paulson (1975) who concluded that the skewness parameter can be used as a measure of "downside risk" in a univariate price model. Leitch and Paulson reached this conclusion by observing that the location parameter lies to the left of the median of distributions which are negatively skewed (ie. have a positive β). Since δ is interpreted as the rate of return (in a univariate price model), or the rate of dividend growth (in the MGWP dividend growth model), it follows that negative skewness for dividend growth is undesirable for reserving purposes. Put another way, a negative skewness (or positive β_D) assumption is conservative.

To understand the behaviour of the model to changes in β_Y, one must realise that the dividend yield model is a "reversion to trend" one. Thus if at a point in time the prevailing level of yields is above/(below) the long term trend μ_Y, then the model (ignoring the error terms) will project downward/upward moving yields respectively. This corresponds to upward/(downward) moving prices respectively.

If the model is projecting upward moving yields (and downward moving prices), then large positive error terms would increase the fall in prices. In this case, a distribution for which the probability of a large positive error term is relatively large (ie. a positively skewed distribution with a negative β_Y) is conservative. The converse applies if the model is projecting downward moving yields.

The initial conditions define whether dividend yields start off above, below, or equal to μ_Y, and therefore influence whether the model is likely to be projecting upward or downward moving yields.

4.3.4 The Initial Yield Parameter Y_o

In order to be able to run the model, it is necessary to assign a value to the initial yield Y_o. This is not really a basis assumption, since in a practical application the dividend yields prevailing at the time of application would be used. Despite this, the sensitivity of the model to this parameter needs to be examined, in order to understand the behaviour of the model in different conditions.

The results of running the proposed alternative model with values of $Y_0 = 2.5\%$, 5% and 7.5% are shown in Appendix IV, table IV.1(d). The results are similar to those observed by the MGWP. The main features are:

- Low values for Y_0 tend to result in lower investment returns in the early years of projection, and have less impact in the later years. Overall, the median geometric rate of return over the 30 year period is increased / (decreased) by approximately the same amount by which Y_0 exceeds / (is less than) μ_Y.
- The inter-quartile ranges of the investment returns are not very sensitive to the initial yield Y_0.
- Since low initial yields tend to depress the long term geometric investment returns, the ability of the assets to support the maturity guarantees reduces as the prevailing yields reduce. Consequently the claim frequencies and reserving requirements are higher if the initial yields are lower than average.

The behaviour of the model to changes in initial conditions is very much dependent on the fact that the model uses an auto-regressive process to project successive dividend yields. If the parameter A were zero, then the successive yields in the early years of the projection would be much less dependent on the initial yield, and a very different pattern of investment returns would result.

4.4 Simulation Investigations into the Differences between the Alternative Models

4.4.1 Initial Comparison of the Alternative Models

Let model A denote the proposed alternative model (paragraph 3.6.1). This model is based primarily on the DeZoete & Bevan time series, but with rounding of the parameter values. The adjusted parameter values could also be derived by fitting the model to the hybrid data series, which was based on the DeZoete & Bevan index until 1962 and the FT-Actuaries thereafter.

Let model Z be a model whose stable distribution parameters would be based on the DeZoete & Bevan time series, without adjustment. Referring to paragraph 3.5.1, the parameters for this model might be

$$\alpha_D = 1.77 \quad \beta_D = 0.6 \quad \gamma_D = 0.05 \quad \delta_D = 0$$
$$\alpha_Y = 1.77 \quad \beta_Y = -0.3 \quad \gamma_Y = 0.10 \quad \delta_Y = 0$$

A comparison of the simulation results for models A and Z reveal the extent of any margins that have been incorporated as a result of adjusting the parameter values.

Let model F be a model whose stable distribution parameters would be based on the FT-Actuaries (quarterly) data. The annualised parameters for this model might be

$$\alpha_D = 1.52 \quad \beta_D = 0.4 \quad \gamma_D = 0.045 \quad \delta_D = 0$$
$$\alpha_Y = 1.70 \quad \beta_Y = -0.5 \quad \gamma_Y = 0.095 \quad \delta_Y = 0$$

Let M denote the model proposed by the MGWP (see paragraph 1.1.2).

The models are compared in Table IV.4(a). The standard portfolio with no tax was used. The following observations are made:

i. Model F appears to have the most volatile stochastic fluctuations. Although the inter-quartile (and also inter-decile) ranges of the annual and geometric investment returns are lower for model F than with model M, the tail quantiles $Q_{I(30)}(0.005)$, $Q_{I(30)}(0.995)$, $Q_{G(30)}(0.005)$ and $Q_{G(30)}(0.995)$, are more extreme under model F than any of the other models. In addition, the claim frequencies, claim severities, and reserving requirements are highest under model F. These observations are a result of model F having the lowest index parameters.

ii. In all cases, the geometric investment statistics tend to be smaller than the corresponding year on year statistics. The observation that $MED[G(t)] < MED[I(t)]$ follows from the fact that the geometric mean of a set of non-equal positive numbers $\{(1+I(1)), (1+I(2)), ..., (1+I(t))\}$ is less than the arithmetic mean.

iii. Models A and Z are very similar, with model A being slightly more volatile. Model A was derived from model Z after rounding the parameters. The differences shown in table IV.4(a) reflect the effects of these adjustments. Overall, the effect of rounding the parameters on the reserving requirements was almost negligible. The slightly higher volatility in Model A will be due to the fact that it has slightly lower index parameters, although this is confused by the slightly different skewnesses.

iv. When comparing model M (the MGWP model) with model A (the stable non-Gaussian alternative), we observe that under model A:

   - The inter-quartile range of the investment returns are smaller but the tail quantiles of the investment returns are more extreme
   - The expected severity of a guarantee claim is greater
   - The guarantee claim frequency rate is lower. (The number NZ of non-zero claims, per 5000 simulations, is smaller.)
   - The reserving requirements at the extreme tail probabilities are larger, but at the less extreme ruin probabilities (e.g. 5%), the reserving requirements are similar.

	Model M	Model A
α_D	2.00	1.75
α_Y	2.00	1.75
β_D	0.00	+0.5
β_Y	0.00	-0.5
γ_D	$0.0919 = 0.13/\sqrt{2}$	0.05
γ_Y	$0.1414 = 0.20/\sqrt{2}$	0.10

The different simulation results for models A and M are entirely attributable to their different parameterisations. The fact that there are as many as six different parameters makes accounting for the different models a little complicated.

On the one hand, the fact that model A has lower index parameters suggests that this model should be more volatile. The population variances of the investment returns are infinite, and the associated extreme stochastic fluctuations should result in higher claim frequencies, claim severities, and reserving requirements.

On the other hand, the fact that model A has smaller scale parameters will tend to make it less volatile. (In the extreme case of the scale parameters approaching zero, the model would degenerate into a deterministic one, and there would be no stochastic fluctuations at all.) Note that a reduction in the scale parameter is a natural consequence of fitting a model with more parameters to a given data set.

The different simulation results will also be partly due to the different skewnesses in the two models. The relative impact of changing the individual parameters on the difference between models A and M is explored in more detail in the next sub-section.

4.4.2 Quantifying the Effect of Individual Parameter Changes on the Differences between Model A and Model M

Let $Q(\alpha_D,\beta_D,\gamma_D,\delta_D,\alpha_Y,\beta_Y,\gamma_Y,\delta_Y)=Q_{CR}(0.001)$ using a stochastic model whose parameters are $\alpha_D,\beta_D,\gamma_D,\delta_D,\alpha_Y,\beta_Y,\gamma_Y$ and δ_Y. Let superscripts A and M be used to denote models A and M respectively.

We know from section 4.4.1 (and table IV.4(a)) that

- Q(model A) = Q(1.75, 0.5, 0.05, 0, 1.75, -0.5, 0.10, 0) = 28%
- Q(model M) = Q(2.0, 0.0, 0.0919, 0, 2.0, 0.0, 0.1414, 0) = 12%

Let Δ = Q(model A) - Q(Model M).

Then (using an analysis of surplus approach to quantify the difference between the models),

$$
\begin{aligned}
\Delta &= Q(\alpha_D^A,\beta_D^A,\gamma_D^A,\delta_D^A,\alpha_Y^A,\beta_Y^A,\gamma_Y^A,\delta_Y^A) - Q(\alpha_D^M,\beta_D^M,\gamma_D^M,\delta_D^M,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,\delta_Y^M) \\
&= Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^A,\beta_Y^A,\gamma_Y^A,0) - Q(\alpha_D^M,\beta_D^M,\gamma_D^M,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0) \\
&= Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^A,\beta_Y^A,\gamma_Y^A,0) - Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^A,\beta_Y^A,\gamma_Y^M,0) \\
&\quad + Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^A,\beta_Y^A,\gamma_Y^M,0) - Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^A,\beta_Y^M,\gamma_Y^M,0) \\
&\quad + Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^A,\beta_Y^M,\gamma_Y^M,0) - Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0) \\
&\quad + Q(\alpha_D^A,\beta_D^A,\gamma_D^A,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0) - Q(\alpha_D^A,\beta_D^A,\gamma_D^M,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0) \\
&\quad + Q(\alpha_D^A,\beta_D^A,\gamma_D^M,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0) - Q(\alpha_D^A,\beta_D^M,\gamma_D^M,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0) \\
&\quad + Q(\alpha_D^A,\beta_D^M,\gamma_D^M,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0) - Q(\alpha_D^M,\beta_D^M,\gamma_D^M,0,\alpha_Y^M,\beta_Y^M,\gamma_Y^M,0)
\end{aligned}
$$

A similar analysis can be done to quantify the source of change in variables other than $Q_{CR}(0.001)$. Table IV.4(b) shows the results of 5,000 computer simulations of the variables *NZ, MNZ,* $Q_{CR}(0.001)$ $Q_{CR}(1\%)$, $Q_{CR}(5\%)$, *MED[I(30)], IQR[I(30)],* $Q_{I(30)}(0.005)$, $Q_{I(30)}(0.995)$, *MED[G(30)], IQR[G(30)],* $Q_{G(30)}(0.005)$ and $Q_{G(30)}(0.995)$ for the 7 relevant values for $\alpha_D, \beta_D, \gamma_D, \alpha_Y, \beta_Y$, and γ_Y.. From these results we deduce the effect of changing each individual stable law parameter from the MGWP value to the new alternative value.

The effects of the individual parameter changes are shown in the table below. Positive entries in this table record increases in the relevant variable as a result of the change; negative values record decreases.

Parameter Change	Impact on Variable Q(0.001)	Q(0.01)	Q(0.05)	NZ	MNZ
γ_Y	-3	-2	-1	-427	0.3
β_Y	-2	+1	0	+147	0.0
α_Y	+8	+2	+1	+466	-0.3
γ_D	-19	-12	-3	-631	-1.0
β_D	+7	+3	+1	-43	+0.6
α_D	+25	+12	+2	+368	+1.9
TOTAL	+16	+4	0	-120	+1.5

Parameter Change	MED[I(30)]	IQR[I(30)]	$Q_{I(30)}(.995)$	$Q_{I(30)}(.005)$
γ_Y	-0.3	-10.1	+12.3	-83.4
β_Y	+0.8	+0.1	-2.4	-43.2
α_Y	+0.5	+2.7	-22.0	+166.0
γ_D	-0.6	-4.0	+6.7	-17.1
β_D	+0.7	+0.1	-1.9	-16.9
α_D	+0.2	+1.3	-6.6	+28.6
TOTAL	+1.3	-9.9	-13.9	+34

Parameter Change	MED[G(30)]	IQR[G(30)]	$Q_{G(30)}(.995)$	$Q_{G(30)}(.005)$
γ_Y	-0.1	-0.3	-0.1	-6.5
β_Y	0.0	0.0	-0.1	+4.3
α_Y	+0.1	+0.3	-0.5	+7.7
γ_D	-0.2	-1.8	+3.5	-4.4
β_D	+0.2	0.0	-0.9	-5.7
α_D	+0.1	0.9	-3.7	+9.3
TOTAL	+0.1	-0.9	-1.8	+4.7

The above tables illustrate the fact that changes in the parameters affect the distributions of the different variables in different ways, and the relative significance of each variable depends on which statistic one is considering. Similar observations were made by the MGWP in their experiments on sensitivity to changes in basis.

For example, we find that the expected claim severities (MNZ) are most affected by the reduction in the index parameter α_D from 2 to 1.75. The second most influential parameter change on MNZ is the reduction in the dividend scale parameter γ_D from $0.13/\sqrt{2}$ to 0.05, and this acted in the opposite direction to the change in the index parameter. By contrast, the claim frequency variable NZ is most affected by the change in γ_D, followed by the reduction in the yield index parameter α_Y.

Overall, the reserve requirement for a 1:1000 chance of ruin is higher for the alternative model, and this is mainly due to the reduction in α_D. The second most influential factor was the reduction in γ_D, which only partially offsets the change in α_D.

It is also relevant to note that the reserving requirements for less extreme ruin probabilities (e.g. $Q_{CR}(1\%)$, or $Q_{CR}(5\%)$), are much less sensitive to the index parameters. For these statistics, the reductions in the index and corresponding scale parameters were about the same. This is because of the fact that the index parameter controls the stochastic fluctuations from the extreme tail of a distribution (see paragraph 2.11), and the scale parameter affects the fluctuations elsewhere.

The consequences of paragraph 2.11 are also evident in the investment statistics. The inter-quartile range, which is a measure of less extreme stochastic fluctuations, is more sensitive to a change in the scale parameter of either the dividend yield or dividend growth model, than it is to a change in the corresponding index parameter. By contrast, the statistics $Q_{I(30)}(0.995)$, $Q_{I(30)}(0.005)$, $Q_{G(30)}(0.995)$ and $Q_{G(30)}(0.005)$, which measure more extreme stochastic fluctuations, are more sensitive to changes in the index parameters than scale parameters.

Introducing a skewness parameter into either the dividend yield or the dividend growth model generally has the smallest impact on the claim and reserving statistics. This point could be taken into account when deciding on a final basis (i.e. whether or not to include any allowance for skewness).

4.5 Different Liability Portfolios

The same set of simulations was used to record the usual statistics for a variety of 'simple' portfolios, other than the standard portfolio. The simulation results are shown in Tables IV.5(a) and (b) for models M and A respectively.

The portfolios chosen included :

- A single policy of 1 p.a. for a term of 5, 10, 20 and 30 years (columns 1 to 4 respectively).
- A group of five policies, each one with a premium of 1 p.a., and maturing in each of years 10, 15, 20, 25, 30 respectively (column 5).
- A group of twenty-one policies, each one with a premium of 1 p.a., and maturing in each of years 10, 11, 12, ..., 30 respectively (column 6).
- A group of ten policies, each one with a premium of 1 p.a., and maturing in each of years 1 to 10 (column 7)
- A group of twenty policies, each one with a premium of 1 p.a., and maturing in each of years 1 to 20 (column 8)

The simulations are done on the basis of zero tax. A comparison of the figures in Table IV.5(a) with those in Table E2.6 of Ford *et.al.* (1980) shows the effect of changing the tax basis from 37.5% to zero. It can be seen that the main impact of operating in a gross tax environment is to reduce the expected claim frequencies, claim severities and consequential reserving requirements. Eliminating the imposition of tax improves the performance of the asset and increases its ability to support the maturity guarantees. The benefit of a gross tax environment increases with increasing term.

Consider now the single policy portfolios (columns 1 to 4 of Tables IV.5(a) and (b)). The MGWP found that the number NZ of non-zero claims reduces substantially as the term increases. The MGWP also observed that the mean of the non-zero claims, MNZ, was relatively stable over time. They found that the most extreme claims were large and similar for all policy durations, although the inner quantiles reduced with term. Overall, the rate of decline of MNZ with policy term was very slow.

By comparison, we find that the number of non-zero claims with the alternative model also reduces substantially with term, although the rate of reduction is noticeably less. The mean of the non-zero claims is less stable with the alternative model, and in fact increases with policy term. The overall effect is that the reserving requirement for very low ruin probabilities decreases substantially more slowly with the alternative model. The reserving requirements for less extreme ruin probabilities behave similarly to the those given by model M. These results reflect the fact that the main impact of using the alternative model is to generate more extreme stochastic fluctuations, due to the smaller index parameters, at the expense of fewer less extreme stochastic fluctuations.

As pointed out by the MGWP, the discounted claim ratio for a single policy is simply the claim ratio for that policy, discounted by its term. No further information is derived from it.

The MGWP also investigated into the effect of a spread of maturity dates, using the portfolios in columns 5 and 6 of Tables IV.5(a) and (b). The portfolio in column five has only five policies, while that in column six has twenty-one. Both portfolios have approximately the same mean term -- twenty years. It is therefore interesting to compare the results of these two portfolios with each other, and also with the results of the single policy (term 20) portfolio in column 3.

We find that the claim frequency rates are higher for the larger portfolios, but the mean claim severities (as a percentage of total sum assured) decrease as the spread in the portfolios increase. Overall the reserving requirements (for low ruin probabilities) decrease as the size of the portfolio increases. These results apply to both model A and model M. These results can be deduced by general reasoning. The greater the number of policies in the portfolio, the greater the probability that at least some of them will become a claim. Also, the greater the spread in the portfolio, the larger the total sum assured relative to the size of any of the claims that do occur.

The portfolios in columns 7 and 8 suffer relatively high claim frequency rates. This illustrates the fact that most of the claims in such a portfolio would occur in the early years (in respect of the short term policies in the portfolio). The expected claim severities are also relatively high, decreasing as the number of policies and size of total sum assured is increased. The discounted claim ratios increase significantly as the term of the portfolio decreases. These results apply to the alternative model in the same way as they do to the MGWP model.

It is unlikely that any office would in practice have a liability portfolio like any of the ones which have been considered. The portfolios do however provide some insight as to the implications of providing maturity guarantees with portfolios of different terms and spreads. In addition one gains some insight as to the position for an on-going portfolio in respect of its future premiums, although the accumulated past premiums and asset shares also need to be taken into account. Investigating into the on-going experience of an existing portfolio, and allowing for its past experience, is beyond the scope of this paper.

Chapter Five: Potential Problems

It is apparent from the literature survey in section 1.3 that the suitability of stable distributions to model financial variables, especially share prices, is not widely accepted. Empirical evidence against their use is presented in Officer(1972), Blattberg and Gonedes(1974), Akgiray and Booth(1988) and Lau and Lau(1990). Their research has concentrated on the univeriate price and log price models for US share prices, and with the exception of Akgiray and Booth(1988), on the symmetric stable distributions.

This chapter considers the evidence against the univariate stable model in more detail, and with the potential repercussions on the bivariate (dividend yield, dividend growth) model considered in this thesis. The chapter also deals with other problem areas related to the both the univariate and bivariate models.

5.1 Sensitivity to Outliers in the Data

5.1.1 An important problem area with the stable model stems from the fact that the parameter estimates of the stable distribution are sensitive to outliers in the data. The sensitivity of the Koutrouvelis estimators to outliers in the UK data was demonstrated in section 3.5.8 of this thesis. The sensitivity of the maximum likelihood estimators to outliers in US share price data was investigated by DuMouchel(1971, 1975, 1983). DuMouchel explained the sensitivity by showing that the Fisher information[1] per observation is relatively large.

[1]See section 3.3.3 of Lloyd (1984) (Volume 6) for an explanation of Fisher information.

5.1.2 There are both positive and negative aspects to this sensitivity. On the positive side, the model makes good use of all the information in the data. Provided the stable law assumption is valid, the high Fisher information per observation should improve the extent to which stochastic fluctuations of appropriate magnitude and frequency are include, and, according to DuMouchel(1983), the problem of distinguishing the case $\alpha = 2$ from the case $\alpha \neq 2$ becomes relatively easy.

5.1.3 However, the sensitivity to the outliers in the data reduces the robustness of the model. If the data had excluded just one or two observations, e.g. if the series had started one or two years later, a very different set of parameter values could result. This was demonstrated in section 3.5.8 with the UK FT Actuaries Dividend data. It means that the model parameters can become very unstable when measured over different time periods. It also results in a potential for making misleading conclusions if the data set was not large enough, and puts into doubt the reliability of the projections resulting from the model.

5.1.4 The sensitivity of the parameter estimates to the outliers in the data can give rise to a potential problem when the outliers are also the most distant observations. The question arises whether it is appropriate to give the same high weighting to observations that were made a long time ago.

5.1.5 While the high Fisher information per observation reduces problem of distinguishing the case $\alpha = 2$ from the case $\alpha \neq 2$, if the stable law assumption is *not* valid, then the outliers in the data could lead to a biased measure of tail behaviour (DuMouchel(1983)). For example, a distribution with exponential tails could easily be diagnosed as having infinite variance.

5.2 Sensitivity to the Stability Assumption

5.2.1 DuMouchel(1983) demonstrated the sensitivity of the parameter estimates to the stable law assumption by generating a random sample from a distribution which is relatively hard to distinguish from the stable distribution. The alternative distribution used by DuMouchel had Pareto tails, but matched the Normal distribution at the centre. The density of this alternative distribution is given by

$$f(x)=\begin{cases}\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} & \text{for } |x|<1\\[2ex] \dfrac{\Phi(-1)}{\sigma}\left[\dfrac{1+\gamma(|x|-1)}{\sigma}\right]^{-\left(\frac{1}{\gamma}+1\right)} & \text{for } |x|\geq 1\end{cases}$$

where $\gamma \geq 0$.

5.2.2 The tail behaviour of these distributions is so different from the Normal that the parameter estimation methods are likely to result in stable distributions with index parameters $\alpha < 2$. Consequently a biased measure of tail behaviour results, and in particular, an erroneous diagnosis of infinite variance can be made.

5.2.3 Thus there is a potential problem, not only due to the sensitivity of the parameter estimation methods to outliers in the data, but also to the lack of robustness to the stability assumption.

5.3 Tests of the Stability Assumption

The validity of the stable distribution assumption appears to be controversial. Explicit tests of the stability assumption have been carried out by Teichmoeller(1971), Officer(1972), Blattberg and Gonedes(1974), Akgiray and Booth(1988) and Walter(1991). These articles were introduced in the literature survey (section 1.3). and are discussed in more detail below.

5.3.1 Teichmoeller

Teichmoeller(1971) accepted the stable law hypothesis, although he also came out in favour of the mixed distribution approach.

5.3.2 Officer

Officer(1972) pointed out that Teichmoeller did not allow for fact that the Fama and Roll estimates for the index parameter (which were used by Teichmoeller) are biased, and argued that had he done so, Teichmoeller may have reached a different conclusion.

Officer tested for both longitudinal and cross-sectional stability. Longitudinal stability refers to sums of stock returns over time while cross-sectional stability refers to sums of stock returns across stocks (i.e. portfolio returns). He used monthly returns over the period January 1926 to June 1968 and daily returns over the shorter period February 1962 to November 1969. Officer failed to reject the stable model for monthly and longer period (at least up to five months) returns. On the other hand, the results of summing daily returns over the shorter period showed very slight increases in the index parameter (α values) and although the increases are slight, this was not consistent with the fact that the Fama and Roll estimators for α have a downward bias. Increasing α values (towards two) are more consistent with a finite variance model, since it suggests that the Central Limit Theorem is operating.

The idea of Officer's test of cross-sectional stability was to use the fact that if stock returns were generated by a finite variance process, then one would expect, through the Central Limit Theorem, that the α values of sums of stocks (in a portfolio) would approach two as the number of stocks increased. One problem with this idea is that stock-returns are not independently distributed (cross-sectionally). Officer attempted to remove the dependence by fitting a model of the market and operating on the residuals that resulted. His results showed that while the α values for direct sums of individual stocks were good approximations for the average α values of the component stocks, the same was not true with the residuals.

The main problems with Officer's rejection of the stable model are:

- A relatively short time period was used for the daily returns. The results from the daily returns do not agree with those from the monthly returns (which used a longer time period) and it has been noted (section 5.1 above) that misleading conclusions can be made if the data time period is too short (since rare stochastic fluctuations may not be present).
- There is the possibility that the data used by Officer was non-stationary. Although the use of a relatively short time period reduces the possibility of non-stationarity, Officer did not explicitly test for non-stationarity and the results of Fielitz(1980) suggest that the data was non-stationary. The problems associated with non-stationary data are discussed further in section 5.4 below.
- The results could be consistent with a non-homogenous stable model (meaning that the stock returns of different companies follow different stable distributions). It is noted that there was evidence that the different stock returns had different α values.
- Officer considered only symmetric stable distributions. It is not known whether he would have reached the same conclusions had he allowed for skewness. More recent studies by Simkowitz and Beedles(1980) and Fielitz and Rozelle(1983) have shown that the empirical return distributions are significantly skewed, and the only stable laws that have any possibility of fitting empirical distributions are the asymmetric ones.

- Officer may not have removed all the dependence between stocks when testing for cross-sectional stability. King(1966) has shown that in addition to stock returns being dependent by being listed on the same stockmarket, there is also some correlation due to stocks being in the same sector or industry. Officer made no allowance for intra-industry dependence.

5.3.3 Blattberg and Gonedes

Blattberg and Gonedes(1974) did similar experiments into longitudinal stability. They also used a relatively short period (1957 to 1962), analysed only daily data and considered only symmetric stable distributions. Their results for the daily data were consistent with Officer's. They also illustrated how the index parameter α varies between different companies, suggesting that the stock returns could be non-homogenous stable.

5.3.4 Akgiray and Booth

Akgiray and Booth(1988) investigated longitudinal stability and tail behaviour using Koutrouvelis's method applied to US data over the period 1975 to 1980. This analysis is an improvement over that of Officer and Blattberg and Gonedes since Koutrouvelis's method results in unbiased estimates for α and can allow for the more general asymmetric distributions. One of their main observations was that the empirical tail shapes are significantly different (thinner) than those of the stable distributions with parameters estimated from the data. They concluded that although the distribution of stock returns appeared to be homogeneous and similar in overall shape to stable distributions with index $\alpha < 2$, the stable law assumption may still be invalid or misleading because parameter estimation methods are not sufficiently robust to differences between empirical and theoretical tail behaviour.

5.3.5 Lau, Lau and Wingender

Lau et.al.(1990) asserted that US daily share price changes are unlikely to follow a stable distribution since the empirical values of kurtosis are significantly smaller than one might expect from an infinite variance (and also infinite kurtosis) stable distribution.

Recall that one of the main incentives for replacing the Normal distribution by a more general stable distribution is because the data appear to have higher kurtosis than is consistent with the Normal distribution (see sections 1.2.4 and 3.4). It therefore follows that the assertion of Lau et.al. is particularly relevant. If it is shown that their assertion is applicable to the annual bivariate dividend yield - dividend growth model for UK equities, then it could be difficult to justify replacing the Normal distribution by a stable non-Normal distribution. One might consider whether the underlying distribution belongs to the domain of attraction of either the Normal or stable non-Normal distribution, and this could influence which (if either) were appropriate.

Lau et.al. simulated 5,000 observations from various stable distributions. For each set of 5,000 observations, they were able to compute one b_2 and one b_4 where

$$b_2 = \frac{m_4}{(m_2)^2} \quad \text{and}$$

$$b_4 = \frac{m_6}{(m_2)^3}$$

where m_k is the k'th central moment.

This step is repeated 100 times to obtain 100 b_2 and b_4's. From these, "expected" sampling distributions of b_2 and b_4 were derived. Various percentiles, the mean and standard deviation of these expected sampling distributions were presented.

The sample b_2 and b_4 values derived from the stock return data were compared with the "expected" values derived from the above simulation procedure using a stable distribution with index parameter derived using Koutrouvelis's method. In a large proportion of cases, the sample b_2 and b_4 values were lower than expected.

The main weakness with Lau et.al's test (by their own admission) is that it is heuristic in nature, and not formal in the Neyman-Pearson sense. The problem arises because the relevant stable distributions has infinite moments, while any finite sample must have finite moments. No matter how large a sample is used to calculate expected b_2 and b_4 values, it is not reasonable to expect these values to fluctuate within some band so that meaningful comparisons can be made (see also section 4.2.3) although observed values can be compared with the distribution percentiles.

Lau et.al.'s primary analysis made use of 10 years' of daily US share price changes (covering the period July 1962 to June 1982). They also considered log share price changes with similar conclusions. They also considered the period February 1968 to December 1987 (so as to include the events of October 1987) and while the evidence against the stable distribution was weaker with this data set, they still asserted that the stable model was not satisfactory.

Lau et.al did not consider share price changes over a longer time period (e.g. week, month or year), and they did not consider the question of whether the returns could come from an infinite variance distribution which although is not stable, is in the domain of attraction of a stable distribution.

Walter

Walter(1991) tested share price changes on the Paris stockmarket. He used Koutrouvelis's method applied to daily share prices of shares comprising the CAC40 index over the period 1 April 1986 to 30 June 1989. He concluded that the stability assumption was reasonable.

5.4 Stationarity

5.4.1 Loosely speaking, a stationary time series is one whose statistical properties do not change over time. A rigorous definition of stationarity can be found in Kendall and Stuart (1966) (pp 477 to 480) and Chatfield (1980) (section 3.2). A time series is said to be ***strictly*** stationary if the joint distributions of $X(t_1)$, $X(t_2)$, ..., $X(t_n)$ is the same as the joint distribution of $X(t_1+\tau)$, $X(t_2+\tau)$, ..., $X(t_n+\tau)$ for all t_1, t_2, ..., t_n, τ. In other words shifting the time origin by an amount τ has no effect on the joint distributions, which must therefore depend only on the intervals between t_1, t_2, ..., t_n. This definition holds for all n. In particular, $n = 1$ implies that the distribution of $X(t)$ must be the same for all t, so that all *possible* moments must be time invariant.

5.4.2 Verifying strict stationarity involves checking that *all* statistical properties are invariant over time. Since an infinite number of possible statistics exist, such verifications are not usually feasible in practical terms. Consequently, the concept of ***second-order*** stationarity is usually used. A process is said to be second-order stationary (or weakly stationary) if its mean is constant and its auto-covariance function depends only on the lag τ, so that $Cov[X(t), X(t+\tau)] = h(\tau)$.

5.4.3 Fielitz(1980) investigated the stationarity of US log share price changes over the period December 1963 to November 1968. Both daily and weekly data were investigated. The daily data coincides very closely with the data used by Officer(1972). Both the weekly and daily data showed no statistically significant variation over time in mean values. However, both the weekly and daily data showed significant variation over time in mean square values. Thus a finite variance homoscedastic model is inappropriate since the data exhibit second-order non-stationarity.

5.4.4 Fielitz does however make the point that the results *could* be consistent with a stable non-Normal distribution with index parameter $1 < \alpha < 2$, since these distributions have finite mean and infinite variance. In order to investigate the stationarity of infinite variance processes (such as those under stable non-Normal distributions) alternative measures of dispersion must be found (instead of the variance). Possible alternative measures are given in Stuck(1978) and Cline and Brockwell(1985).

5.4.5 Another possible explanation of the behaviour of the data is that it could be generated by a finite variance heteroscedastic process (see section 1.2.10).

5.4.6 In section 5.1 it was noted that the parameter values of the stable distribution are very sensitive to outliers in the data. If the time series is too short, there may not be sufficient outliers in the data to support the stable law hypothesis. One of the key differences between the data used by Akgiray and Booth(1988) compared with the data used by Walter(1991) is that the latter's data set included the 1987 stockmarket crash. Akgiray and Booth stated that they deliberately kept their time series short due to the uncertainty about the stationarity of the time series if pre-1975 data were included. I question how their results would have been affected had the investment experience during the 1974 oil-crisis and previous years been included. I also submit that the relatively short time period used by Officer and Blattberg and Gonedes for the daily returns compared with the monthly returns could explain why they rejected the model for daily returns but not for monthly returns.

5.4.7 Potential non-stationarity in the data is not only a key issue that could resolve the controversy about the validity of the stability assumption. Non-stationarity in the data is a potential problem with the MGWP model (and my refinement to it) in its own right. I have assumed that the MGWP were justifiably satisfied with the stationarity of their data and have not conducted any further tests in this area. I have also not tested whether the updated time series that I have used are still stationary. Conducting tests of stationarity under the assumption of infinite variance stable distributions is an area left for further research.

5.5 Limited Volume of Data

In section 3.5.9 the uncertainties in the stable distribution parameter values due to the limited volume of data were estimated. In particular, the β values could not be reliably estimated. The values chosen were rather arbitrary, but arguably no less arbitrary than assuming a symmetric or Normal distribution. This could be a potentially serious problem for certain applications. However, in the context of reserving for maturity guarantees, Chapter four of the thesis showed that the skewness parameter is the least significant of the stable distribution parameters.

5.6 Goodness of Fit

5.6.1 The problems with testing the goodness of fit of the stable distributions to the sample residuals were discussed in Section 3.6.3.

5.6.2 The problem with the Kolmogorov Smirnov test is that it is not truly distribution free when the parameters of the distribution are not completely specified and are estimated from the data (i.e. for composite tests). This means that the real significance level was somewhat *less* than the level quoted for this test.

5.6.3 The problem with the χ^2 test was that it was not clear how many degrees of freedom to use. The method of Cramer(1946) which involves subtracting one degree of freedom for each parameter estimated from the data is strictly speaking only accurate when minimum χ^2 estimators are used. In addition, not all the data sets had the same influence on the final choice of parameters. It was also the case that some of the parameter values used were subject to adjustment and were not exactly equal to their sample estimates. This is particularly the case for the skewness parameter, for which it has already been noted (paragraphs 3.5.9 and 5.5) that it could not be reliably estimated from the limited data available and was chosen by judgement (and arguably somewhat arbitrarily).

5.6.4 Overall the dividend growth model was accepted at the 5% significance level with all data sets and with as little as 2 degrees of freedom. On the other hand, the dividend yield model could be accepted at the 5% level with 3 or more degrees of freedom with the hybrid data and with 4 or more degrees of freedom with the other data. The results for the dividend yield model are not 'ideal' and it is noted that there could be a problem with over-parameterisation.

5.6.5 Some authors have proposed alternative distributions by optimising a goodness of fit statistic. The work of Praetz(1972) and Blattberg and Gonedes(1974) was introduced in section 1.3.

Praetz(1972) used a minimum χ^2 procedure on the univariate log price model of Australian shares. Blattberg and Gonedes(1974) pointed out that Praetz's work was flawed because he standardised his data by dividing by a standard deviation instead of a stable distribution scale parameter. They also pointed out that the minimum χ^2 procedure is not suitable for distinguishing the Normal from the stable non-Normal distributions since the χ^2 statistic for the Normal distribution assumes α = 2, and therefore is biased in favour of the Normal distribution.

Instead Blattberg and Gonedes estimated the ratio of the log-likelihood function of the students *t*-distribution divided by the log-likelihood function of the symmetric stable distribution. Log-likelihood ratios greater than one were taken as evidence in favour of the *t*-distribution (since it apparently provided a superior fit).

Thomson(1994) fitted a translated beta distribution to South African share dividend growth using the minimum χ^2 procedure. He does not however state whether this provided a better fit than the stable distribution, or whether he attempted to fit a stable distribution at all.

5.6.6 Notwithstanding the comments of Feller(1966) (see also section 1.3.3 of this thesis) about the pitfalls of choosing a probability distribution by optimising a goodness of fit statistic, the fact that better fit models can be found is a potential problem area for the stable distribution model. If reasons can be found to explain why these better fitting distributions might be more appropriate, then the stable model may need to be discarded in favour of those alternatives.

Chapter 6 : Summary, Further Research and Conclusion

6.1 Summary

6.1.1 Problems with the Existing MGWP Model

The MGWP model can be rejected on the grounds of kurtosis, and possibly on the grounds of skewness too. The residuals that result from fitting both the dividend yield and the dividend growth models to either the FT-Actuaries data or the DeZoete and Bevan data have more kurtosis than can be supported by the assumption of a Normal distribution.

The dividend yield model can also be safely rejected on the grounds of skewness, regardless of the data set used. However the dividend growth model can only be rejected on the grounds of skewness with the FT-Actuaries' data, but not with the DeZoete & Bevan nor the Hybrid time series. This last anomaly is due to the fact that the Actuaries Investment Index was a geometric index until 1962, while the DeZoete & Bevan and the Hybrid Indices were arithmetically weighted over the whole period of investigation.

6.1.2 Refining the MGWP Model

6.1.2.1 The model can however be refined by replacing the Normality assumptions by more general stable distributions, with index parameters less than two.

6.1.2.2 The results of Kanter and Steiger (1974), Yohai and Maronna (1977), Hannan and Kanter (1977) and Davis and Resnick (1989) establish that the least squares estimates for the parameters μ_D, μ_Y and A are consistent in probability, when the error terms have a stable distribution. We can thus accept the values for these parameters that were recommended by the MGWP for the pre-1980 experience.

6.1.2.3 If experience since the MGWP published their report is taken into account, then the parameters μ_D, μ_Y *and* A need to be updated, and the method of least squares can again be used. The following values were derived from the least squares estimates applicable for the hybrid time series over the period December 1918 to December 1993.

$\mu_D = 5.5\%$

$\mu_Y = 5.0\%$

$A = 0.60$

The least squares estimates have again been rounded downwards to avoid introducing spurious accuracy. Downward rounding adjustments are conservative. The hybrid time series were used as they are based on the longest arithmetically indices available.

6.1.2.4 Using the updated parameter values implies using a weaker reserving basis, as was demonstrated in section 4.3. This is because the updated parameter values imply a more optimistic assumption about the likely future performance of the underlying assets (in particular dividend growth). Whether or not it is appropriate to use the updated figures must also depend on one's assessment about the likely *future* performance of the assets over the long term.

6.1.2.5 In addition, it is noted that, strictly speaking, the introduction of new data (ie. post 1980 experience) requires the shape of the model to be re-examined. This means checking whether, a first order auto-regressive model is appropriate for the log of dividend yields, and, whether a once differenced model is appropriate for the log of dividends. This however is beyond the scope of this dissertation.

6.1.2.6 The method that has been recommended for fitting the appropriate stable distributions is due to Koutrouvelis (1980, 1981). Koutrouvelis's method is recommended on the grounds of ease of computation, lack of bias, and also relative precision of the estimate of the index parameter. The results of applying Koutrouvelis's method to fit the model to the data are given in paragraph 3.5.1.

6.1.2.7 The discrepancy in the estimates that result from fitting the model to the FT-Actuaries and the DeZoete and Bevan time series are primarily due to the fact that until 1962, the Actuaries' Investment Index was a geometric index. By contrast, the DeZoete and Bevan was an arithmetically weighted one. Geometric indices are more sensitive to sharp downward falls of the individual assets. This could result in larger downward fluctuations, lower index parameters, and negative skewness.

6.1.2.8 The model with stable distribution parameters

$\alpha_D = 1.75 \qquad \alpha_Y = 1.75$

$\beta_D = +0.5 \qquad \beta_Y = -0.5$

$\gamma_D = 0.05 \qquad \gamma_Y = 0.10$

has been proposed for consideration as an alternative to the MGWP model. These parameters can be used for the error terms in the model for pre-1980 experience, as well as in the updated model.

6.1.2.9 The parameters for the stable distributions have also been rounded to avoid introducing spurious accuracy. This is in recognition of the fact that the parameters (especially the skewness parameters) cannot be estimated reliably with small data sets. The significance of the rounding adjustments were examined in section 4.3 and were found to be small.

6.1.2.10 The estimates for the skewness parameters are particularly unreliable. The simulation results in sections 4.2.3 and 4.4 show however that the model is relatively insensitive to the skewness parameters (as compared to the other stable distribution parameters).

6.1.2.11 It is questionable whether the alternative stable distribution model provides a better fit to the data than the existing Normal distribution model. On the one hand, the fact that lower scale parameters result indicates that a better fit is achieved. However, this is to be expected since additional parameters have been introduced into the model. If allowance is made for the loss of degrees of freedom due to the estimation of the additional parameters, then, in the case of the dividend yield model, there is evidence of over-parameterisation. The goodness of fit is better for the arithmetic indices than the geometric indices. This is to be expected since the arithmetic indices had more influence on the final choice of parameter values.

6.1.2.12 The stable distribution is a generalisation of the Normal distribution. One can argue that rejection of the stable model implies rejection of the Normal model too.

6.1.2.13 Some authors have suggested alternative distributions which provide a better fit to financial data than both the stable (non-Normal) and the Normal distributions. There does not however appear to be a satisfactory explanation as to why these alternative distributions should be more appropriate than the stable distributions.

6.1.3 Impact of Using Stable (Non-Normal) Distributions

6.1.3.1 One consequence of using the more general stable (non-Normal) distributions is that the population variances of the error terms become infinite. This leads to the annual investment return also having infinite variance. In addition, in a large number of cases (see section 4.2.2), the annual investment return will also have infinite mean, due to the log-stable transformation.

6.1.3.2 Large parts of financial economics based on mean-variance optimisation are inapplicable under a stable (or log-stable) probability distribution framework. This follows from the population variances (and possibly also means) being infinite, which leads to the failure of sample variances (and possibly also means) as measures of risk and return respectively.

6.1.3.3 The lower the index parameters, the thicker the tails of the distributions, and the more extreme are the stochastic fluctuations. In the context of maturity (or other investment) guarantees, this will tend to increase guarantee claim frequencies, mean claim severities, and the reserving requirements for low probabilities of ruin.

6.1.3.4 Using stable (non-Normal) distributions also results in lower scale parameters. This means that a better fit to the data is achieved. This is not surprising, in view of the fact that more parameters are introduced into the model. Lower scale parameters tend to reduce the stochastic fluctuations in the model. (In the limit, as the scale parameters approach zero, the model degenerates into a deterministic one, and there are no stochastic fluctuations at all.)

6.1.3.5 The reductions in index and scale parameters act against each other. The relative impact of changing the stable distribution parameters is investigated using simulation techniques. The method of Chambers, Mallows and Stuck (1976) is used to simulate from the stable distributions.

6.1.3.6 The significance of the skewness parameters depends on the value of the other parameters. The introduction of skewness parameters into the model was found however to have generally less impact than the changes to the index and scale parameters. It was found that negative skewness (positive β) for dividend growth is conservative. If the initial conditions are neutral (i.e. $Y_0 = \mu_Y$), and dividend growth is negatively skewed, then negative skewness for dividend yields is also conservative.

6.1.3.7 The overall impact of replacing the MGWP model by the proposed alternative is to:

- decrease the frequency of guarantee claims (lower scale parameters dominating)
- increase the mean severity of guarantee claims (lower index parameters dominating)
- increase the reserving requirements for low ruin probabilities (lower index parameters dominating)

6.1.3.8 It was also noted that the reserving requirements for the less extreme ruin probabilities were similar with the two models. (Here the reductions in scale and index parameters offset each other almost exactly.)

6.1.3.9 When liability portfolios of different terms are used, we find that the claim frequency rates and reserving requirements (at low ruin probabilities) decrease with increasing term, although the rates of reduction are less with the alternative model than with the MGWP model. The behaviour of the reserving requirements at the less extreme ruin probabilities is once again similar in the two models. These results reflect the fact that the main impact of using the alternative model is to generate more extreme stochastic fluctuations, at the expense of fewer less extreme fluctuations.

6.1.3.10 If the spread of policies in a portfolio is increased, without changing the mean term of the whole portfolio, then the mean claim severities (as a percentage of total sum assured) and the required contingency reserves, decrease. This is despite the fact that the number of non-zero claims may increase, and applies to the alternative model in the same way as it does to the MGWP model.

6.1.4 Problem Areas

6.1.4.1 There are a number of potential problems associated with the proposed alternative model. The main problems relate to the sensitivity of the parameter estimates to the outliers in the data, to the assumption that the underlying distribution is stable, and to the assumption that the data is stationary.

6.1.4.2 The validity of the stable distribution assumption for financial data is controversial. Some evidence against the assumption has been found in the context of univariate price and log-price models. The earlier evidence focused on the symmetric stable distributions. The more recent evidence has been against the more general asymmetric stable distributions too. The more recent evidence is based on the observation that the empirical distributions have thinner tails and lower kurtosis than that consistent with the fitted stable distributions. Most of the evidence has focused on relatively short term models (daily price changes) and data collected over relatively short periods (between five and ten years).

6.1.4.3 The main reason for many practitioners using data collected over a relatively short time period is to ensure that the time series are stationary. However, if the data collection period is too short, then there may not be sufficient outliers (which by their nature are very rare) to support the stable distribution hypothesis. Methods for testing the stationarity of the data under an infinite variance stable distribution framework have not yet been developed. The fact that the stationarity of the MGWP data has not been tested is in itself a potential problem area.

6.1.4.4 The other problem areas relate to the difficulties of reliably estimating the parameters of the stable distributions with the limited data available, and the difficulties of testing the goodness of fit given the number of parameter estimates that have been made. The fact that better fitting distributions can be found is also a potential problem area.

6.2 Suggestions for Further Research

6.2.1 There is considerable scope for further research in the area of parameter estimation so that

- the trade-off between the sensitivity of the parameter estimates to outliers in the data and the extent to which all the information in the data is used is better understood
- the trade-off between the efficiency of the parameter estimation method and its robustness to the stability assumption is better understood. In particular it would be desirable to find a method that is more capable of detecting the cases of $\alpha < 2$ from $\alpha = 2$
- the relationship between the reliability of the parameter estimates and the volume of data is better understood. In particular it would be desirable to find an estimation method which can provide more reliable skewness parameter estimates with small data sets
- numerical methods can be found to make DuMouchel's maximum likelihood estimates easier to implement

6.2.2 Further research is needed in order to formally test the stationarity of the MGWP's data and the post-1980 experience update to it. Practical methods for investigating the stationarity of the data when the distribution is assumed to have infinite variance need to be developed.

6.2.3 The nature of the tails of the empirical distribution and the sample kurtosis should be investigated further. The methods of Akgiray and Booth (1988) and Lau et. al.(1990) can be used to test the suitability of the stable distribution assumption with the UK dividend yield and dividend growth data used in this thesis.

6.2.4 There is also scope for further research in the area of testing the goodness of fit of the stable distributions to the data, especially the dividend yield data. There was some doubt as to the appropriate number of degrees of freedom to use for the χ^2 test, while the extension of the Kolmogorov-Smirnov procedure to a composite test (i.e. where the distributions are not fully specified and parameter values need to be estimated from the data) is a well known problem that remains unresolved (Lloyd(1984)).

6.2.5 Some authors have found alternative distributions that provide a superior fit to financial data. Further research is needed to explain this phenomenon, and also to ascertain whether these alternative distributions provide a superior fit to the UK dividend yield and dividend growth data.

6.2.6 By definition of the word 'model', all models will be imperfect. It would therefore be desirable to have a list of criteria to enable the choice of one model in preference to another. It seems that a case can be made for doing further research to evaluate the relative merits of the stable model vs. alternatives such as the *t*-distribution, heteroscedastic and mixed distribution models.

6.2.7 This thesis has focused solely on refining the MGWP's bivariate model for UK equity price data. Ford et.al.(1980) also considered applying their model to US and property investments. Further research is needed to investigate the suitability of the model with stable distributions to other countries and asset classes.

6.2.8 The thesis has also focused solely on the stochastic element (i.e. the distribution of the error terms) of the MGWP model. The question of whether the deterministic element of the model can be improved (especially in the light of post 1980 experience) was kept outside the scope of the thesis. This question is left for further research.

6.2.9 Since Ford et.al.(1980) published the MGWP model, a number of other models have been suggested and are now in widespread use. These models also tend to use the Normal or other finite variance distributions. An example is the Wilkie model (Wilkie(1986, 1995)). The refinement of these models so as to incorporate infinite variance stable distributions is another possible area for further research.

6.2.10 In assessing the impact of the stable distributions the thesis focused primarily on reserving for maturity guarantees under unit-linked life assurance business. The refined model can also be used in other actuarial applications such as the funding of pension schemes, product pricing, resilience test reserving and dynamic solvency testing for life insurance companies, and product pricing and claims reserving for non-life insurance.

The application of the stable distributions to non-life insurance could differ from the other applications in that stable distributions could be used for the distribution of claims rather than for the distribution of investment returns. Here the stable distributions would be used on the liability side of the model, rather than (or in addition to) on the asset side. The use of stable distributions for modelling large claims in non-life insurance can be found in Beirlant and Teugels (1992).

6.2.11 Applications for the refined model also exist in the wider field of financial economics. Examples include the pricing of options and other derivative contracts and the adaptation of mean-variance optimisation theory to allow for more appropriate measures of risk and reward under a stable distribution framework. Finally, one could follow the approach of Mandelbrot (1982, 1989) and investigate the implications of the fractal and chaotic nature of the UK stockmarket under the stable distribution hypothesis.

6.3 Conclusion

6.3.1 The two primary aims of the thesis were given in paragraph 1.2.14. These were:

1. to attempt to determine whether a stable (non-Normal) distribution can be fitted to the UK data and incorporated in the MGWP model
2. to examine the implications (mainly on reserving for investment guarantees) of replacing the assumption of Normally distributed error terms by a more general stable distribution.

6.3.2 In pursuit of the first aim of the thesis, it was found that

- the least squares estimates for the parameters μ_D, μ_Y and A are consistent in probability when the error terms have a stable distribution. We can thus accept the values for these parameters that were recommended by the MGWP for the pre-1980 experience, and also use the method of least squares to derive updated values in the light of post-1980 experience. The suggested updated parameter values are given in paragraph 6.1.2.3 above.
- Various methods are available for fitting a stable distribution to the error terms. The method of Koutrouvelis (1980, 1981) is recommended on the grounds of ease of computation, lack of bias, and also relative precision of the index parameter. The results of applying Koutrouvelis's method to fit the model to the data are given in paragraph 3.5.1. The suggested stable distribution parameter values to use are given in paragraph 6.1.2.8 above. These apply to a model for both pre and post 1980 experience for arithmetic time series.

6.3.3 A number of potential problems have been encountered in the fitting of the stable distributions. The main problems include:

- the validity of the stability assumption is questionable in the light of the large body of evidence in the literature against this assumption for other financial data
- the reliability of the stable distribution parameter estimates (especially the skewness parameter) is questionable, given the limited volume of data available

- potential non-stationarity in the data, which was not tested
- possible over-parameterisation and uncertainties about the goodness of fit of the dividend yield model due to uncertainty about the appropriate number of degrees of freedom to use with the parameter estimation methods used
- the fact that other distributions can be found that provide a superior fit to financial data

6.3.4 By definition of the word 'model', all models must be imperfect. However models are needed to deal with problems that arise in real life situations. The problems with the model used must therefore be accepted (and understood), until a superior model can be found. The question of whether the stable distribution model is superior to some of the other models available remains unanswered.

6.3.5 In pursuit of the second aim of the thesis, the main implications of using the stable distributions were found to be:

- infinite means and variances for the annual investment return, due to the log-stable transformation
- the failure of sample means and variances as measures of return and risk and the inapplicability of large parts of financial economics dealing with mean-variance optimisation
- in the context of maturity (or other investment) guarantees, more extreme stochastic fluctuations (due to lower index parameters) but fewer less extreme fluctuations (due to lower scale parameters) result. The overall impact is to:
    - decrease the frequency of guarantee claims (lower scale parameters dominating)
    - increase the mean severity of guarantee claims (lower index parameters dominating)
    - increase the reserving requirements for low ruin probabilities (lower index parameters dominating), and possibly reduce the reserving requirement at more modest (i.e. higher) probabilities of ruin.

- For both the dividend growth and dividend yield models, the skewness parameters were found to be relatively less significant than the corresponding index and scale parameters.

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APPENDICES

I. Data

I.1 : De Zoete and Bevan Data
I.2 : FT Actuaries Data
I.3 : Hybrid Data (Chainlink of De Zoete and Bevan with FT Actuaries)

II. The Standard Liability Portfolio

III. Computer Programmes

III.1 : Koutrouvelis's Method for Stable Distribution Parameter Estimation
III.2 : Asset-Liability Simulation Using the Method of Chambers et.al.(1976)

IV. Tables of Simulation Results

IV.1 Sensitivity of the Alternative Stable Distribution Model to Changes in the Location, Scale, Auto-Correlation and Initial Yield Parameters

Table IV.1(a) : Sensitivity to Changes in the Location Parameters
Table IV.1(b) : Sensitivity to Changes in the Scale Parameters
Table IV.1(c) : Sensitivity to Changes in the Auto-Correlation Parameters
Table IV.1(d) : Sensitivity to Changes in Initial Yields

IV.2 Sensitivity of the Model to Changes in the Index Parameters

Table IV.2(a) : Effect of Changes in Index Parameters on $MED[I(30)]$
Table IV.2(b) : Effect of Changes in Index Parameters on $IQR[I(30)]$
Table IV.2(c) : Effect of Changes in Index Parameters on NZ
Table IV.2(d) : Effect of Changes in Index Parameters on MNZ
Table IV.2(e) : Effect of Changes in Index Parameters on $Q_{CR}(0.001)$
Table IV.2(f) : Effect of Changes in Index Parameters on $Q_{CR}(0.01)$

IV.3 Sensitivity of the Model to Changes in Skewness Parameters

Table IV.3(a) : Effect of Changes in Index Parameters on $MED[I(30)]$
Table IV.3(b) : Effect of Changes in Index Parameters on $IQR[I(30)]$
Table IV.3(c) : Effect of Changes in Index Parameters on NZ
Table IV.3(d) : Effect of Changes in Index Parameters on MNZ
Table IV.3(e) : Effect of Changes in Index Parameters on $Q_{CR}(0.001)$
Table IV.3(f) : Effect of Changes in Index Parameters on $Q_{CR}(0.01)$

Appendix I.1

Table I.1 : Analysis of the DeZoete Bevan Data

	Price (1)	D (2)	..ln(D/D) (3)	..model (4)	..residl (5)	Prob Value (6)	Remark (7)	Yp (8)	...ln (9)	..model (10)	..residl (11)	Prob Value (12)	Remark (13)
1918	100							6.5	-2.73337	-2.73337	0	0.5	
1919	135.5	6.5						6.58	-2.72114	-2.83831	0.117178	0.260148	
1920	93.73	8.91	0.315372	0.04	0.275372	0.000965	EXTREME OUTLIER	7.98	-2.52823	-2.83097	0.302742	0.048359	UNLIKELY
1921	84.74	7.48	-0.17494	0.04	-0.21494	0.007755	UNLIKELY	8.52	-2.46275	-2.71523	0.252478	0.082995	
1922	119.8	7.22	-0.03538	0.04	-0.07538	0.198007		6.79	-2.68972	-2.67595	-0.01377	0.46988	
1923	125.87	8.13	0.118706	0.04	0.078706	0.187745		6.3	-2.76462	-2.81212	0.047504	0.39719	
1924	145.61	7.93	-0.02491	0.04	-0.06491	0.232429		6.03	-2.80842	-2.85707	0.048642	0.394784	
1925	182.88	8.78	0.101823	0.04	0.061823	0.243171		5.3	-2.93746	-2.88335	-0.05412	0.383269	
1926	179.2	9.69	0.098618	0.04	0.058618	0.254613		5.48	-2.90407	-2.96077	0.056706	0.377858	
1927	200.6	9.82	0.013327	0.04	-0.02667	0.381957		5.28	-2.94124	-2.94073	-0.00051	0.498879	
1928	231.7	10.6	0.076433	0.04	0.036433	0.340816		5.05	-2.98578	-2.96304	-0.02274	0.450351	
1929	191.68	11.71	0.099589	0.04	0.059589	0.251117		5.53	-2.89498	-2.98976	0.09478	0.301531	
1930	158.9	10.6	-0.09959	0.04	-0.13959	0.058		5.36	-2.92621	-2.93528	0.009076	0.480143	
1931	125.7	8.52	-0.21844	0.04	-0.25844	0.001807	EXTREME OUTLIER	6.11	-2.79524	-2.95402	0.158773	0.191851	
1932	162.27	7.67	-0.1051	0.04	-0.1451	0.051147		4.93	-3.00983	-2.87544	-0.13439	0.23046	
1933	204.9	8	0.042125	0.04	0.002125	0.490455		4.44	-3.11452	-3.00419	-0.11032	0.272495	
1934	245.59	9.11	0.129931	0.04	0.089931	0.155618		4.08	-3.19907	-3.067	-0.13207	0.23435	
1935	269.9	10.02	0.09521	0.04	0.05521	0.267078		3.93	-3.23653	-3.11774	-0.11879	0.25728	
1936	310.85	10.6	0.056271	0.04	0.016271	0.427316		3.87	-3.25192	-3.14021	-0.1117	0.269985	
1937	259.1	12.03	0.12655	0.04	0.08655	0.16489		4.67	-3.06401	-3.14944	0.085431	0.319638	
1938	220.57	12.1	0.005802	0.04	-0.0342	0.350091		5.25	-2.94694	-3.0367	0.089757	0.311202	
1939	213.68	11.58	-0.04393	0.04	-0.08393	0.172326		5.63	-2.87706	-2.96646	0.089397	0.3119	
1940	191.97	12.03	0.038124	0.04	-0.00188	0.491574		6.1	-2.79688	-2.92453	0.127648	0.24186	
1941	224.25	11.71	-0.02696	0.04	-0.06696	0.225431		4.99	-2.99773	-2.87642	-0.12131	0.252842	
1942	252.64	11.19	-0.04542	0.04	-0.08542	0.168058		4.35	-3.13499	-2.99693	-0.13806	0.224387	
1943	270.94	10.99	-0.01803	0.04	-0.05803	0.256724		4.13	-3.18689	-3.07929	-0.1076	0.277476	
1944	293.3	11.19	0.018035	0.04	-0.02197	0.402326		3.88	-3.24934	-3.11043	-0.13891	0.223	
1945	299.25	11.38	0.016837	0.04	-0.02316	0.397116		4.02	-3.21389	-3.14789	-0.06599	0.358649	
1946	340.45	12.03	0.055546	0.04	0.015546	0.43052		4.03	-3.2114	-3.12663	-0.08478	0.32092	
1947	319.49	13.72	0.131451	0.04	0.091451	0.151564		3.95	-3.23145	-3.12514	-0.10632	0.279841	
1948	294.26	12.62	-0.08357	0.04	-0.12357	0.082048		4.53	-3.09445	-3.13717	0.042717	0.407351	
1949	264.64	13.33	0.054734	0.04	0.014734	0.434114		5.26	-2.94504	-3.05496	0.109923	0.273227	
1950	279.25	13.92	0.04331	0.04	0.00331	0.485137		5.59	-2.88419	-2.96532	0.081126	0.328128	
1951	287.3	15.61	0.114585	0.04	0.074585	0.2005		5.75	-2.85597	-2.92881	0.072837	0.34472	
1952	270.64	16.52	0.05666	0.04	0.01666	0.425598		6.37	-2.75357	-2.91188	0.158304	0.192553	
1953	318.91	17.24	0.04266	0.04	0.00266	0.48805		6.24	-2.77419	-2.85044	0.076245	0.337859	
1954	453.63	19.9	0.143487	0.04	0.103487	0.121953		5.09	-2.97789	-2.86281	-0.11509	0.263887	
1955	480.57	23.09	0.14868	0.04	0.10868	0.110524		4.94	-3.0078	-2.98503	-0.02278	0.450277	
1956	413.61	23.74	0.027762	0.04	-0.01224	0.445198		5.88	-2.83361	-3.00298	0.169362	0.176394	
1957	384.6	24.32	0.024138	0.04	-0.01586	0.429121		6.71	-2.70157	-2.89846	0.19689	0.140022	
1958	542.6	25.82	0.05985	0.04	0.01985	0.411567		5.38	-2.92248	-2.81924	-0.10325	0.285543	
1959	810.83	29.2	0.123019	0.04	0.083019	0.174944		4.34	-3.1373	-2.95178	-0.18551	0.154385	
1960	789.7	35.19	0.186593	0.04	0.146593	0.049405	UNLIKELY	4.62	-3.07478	-3.08067	0.005895	0.4871	
1961	766.4	36.49	0.036276	0.04	-0.00372	0.483277		4.8	-3.03655	-3.04316	0.006604	0.485549	
1962	732.5	36.81	0.008731	0.04	-0.03127	0.362386		5.2	-2.95651	-3.02023	0.063714	0.363334	
1963	927.8	38.11	0.034707	0.04	-0.00529	0.476238		4.75	-3.04703	-2.9722	-0.07483	0.34071	
1964	846.2	44.03	0.144395	0.04	0.104395	0.119899		5.86	-2.83702	-3.02651	0.189488	0.149261	
1965	863.3	49.63	0.119724	0.04	0.079724	0.184672		5.83	-2.84215	-2.90051	0.058352	0.37443	
1966	759.4	50.34	0.014205	0.04	-0.0258	0.385732		6.38	-2.752	-2.90358	0.151583	0.202804	
1967	979.67	48.45	-0.03827	0.04	-0.07827	0.189077		4.82	-3.0324	-2.84949	-0.1829	0.157815	
1968	1310.97	47.22	-0.02571	0.04	-0.06571	0.229664		3.83	-3.26231	-3.01773	-0.24457	0.089824	
1969	1071.7	50.21	0.061397	0.04	0.021397	0.404805		4.77	-3.04282	-3.15568	0.112852	0.267907	
1970	931.79	51.12	0.017962	0.04	-0.02204	0.402007		4.97	-3.00175	-3.02399	0.022237	0.451449	
1971	1281.5	46.31	-0.09882	0.04	-0.13882	0.059014		3.73	-3.28876	-2.99934	-0.28942	0.056157	
1972	1317.66	47.8	0.031668	0.04	-0.00833	0.462625		3.85	-3.2571	-3.17155	-0.08555	0.319411	
1973	935.04	50.73	0.059492	0.04	0.019492	0.413138		5.85	-2.83873	-3.15255	0.313823	0.042556	UNLIKELY
1974	437.19	54.7	0.075346	0.04	0.035346	0.345314		13.39	-2.01066	-2.90153	0.890868	5.11E-07	EXTREME OUTLIER
1975	1065.89	58.54	0.067847	0.04	0.027847	0.37693		5.98	-2.81675	-2.40469	-0.41206	0.011888	UNLIKELY
1976	1035.28	63.74	0.085102	0.04	0.045102	0.305778		7.03	-2.65498	-2.88834	0.233359	0.100219	
1977	1407.99	72.78	0.132629	0.04	0.092629	0.148472		6.01	-2.81175	-2.79128	-0.02046	0.455306	
1978	1451.7	84.62	0.150729	0.04	0.110729	0.106231							

Table I.2(b) : Analysis of the FT-Actuaries' Data (June Figures)

	P (1)	D (2)	..ln(D/D (3)	..model (4)	..residl (5).	Prob Value (6)	Remark (7)	Yp (8)	...ln (9)	...model (10)	...resid (11)	Prob Value (12)	Remark (13)
1930		4.552						4.486	-3.1042	-3.1042	0	0.5	
1931	74.79	3.355	-0.3051	0.04	-0.3451	0.00027	EXTREME OUTLIER	4.311	-3.144	-3.0608	-0.0832	0.30014	
1932	55.79	2.405	-0.3329	0.04	-0.3729	0.00009	EXTREME OUTLIER	4.903	-3.0153	-3.0847	0.06937	0.33106	
1933	44.79	2.196	-0.0909	0.04	-0.1309	0.09485		3.777	-3.2762	-3.0075	-0.2688	0.04523	UNLIKELY OBS
1934	62.91	2.376	0.07878	0.04	0.03878	0.34882	-	3.773	-3.2773	-3.164	-0.1133	0.23777	
1935	70.79	2.671	0.11703	0.04	0.07703	0.22014		3.657	-3.3085	-3.1647	-0.1439	0.18242	
1936	77.50	2.834	0.05924	0.04	0.01924	0.42359		3.958	-3.2294	-3.1834	-0.046	0.38594	
1937	83.10	3.289	0.14889	0.04	0.10889	0.13766		4.243	-3.1599	-3.136	-0.0239	0.44004	
1938	82.61	3.505	0.06361	0.04	0.02361	0.40653		4.753	-3.0464	-3.0942	0.04784	0.38157	
1939	66.40	3.156	-0.1049	0.04	-0.1449	0.07333		5.144	-2.9673	-3.0261	0.05879	0.35557	
1940	61.80	3.179	0.00726	0.04	-0.0327	0.37147		7.126	-2.6414	-2.9787	0.33728	0.01681	UNLIKELY OBS
1941	38.90	2.772	-0.137	0.04	-0.177	0.0381	UNLIKELY OBS	5.532	-2.8946	-2.7831	-0.1115	0.24127	
1942	50.60	2.799	0.00969	0.04	-0.0303	0.38071		4.998	-2.9961	-2.9351	-0.0611	0.35024	
1943	60.40	3.019	0.07566	0.04	0.03566	0.36044		4.254	-3.1573	-2.996	-0.1613	0.15474	
1944	75.32	3.204	0.05947	0.04	0.01947	0.42266		3.9	-3.2442	-3.0927	-0.1515	0.16993	
1945	84.51	3.296	0.02831	0.04	-0.0117	0.45339		4.25	-3.1583	-3.1448	-0.0134	0.46626	
1946	85.91	3.651	0.10229	0.04	0.06229	0.2663		4.364	-3.1318	-3.0932	-0.0385	0.4041	
1947	92.97	4.057	0.10544	0.04	0.06544	0.25604		3.957	-3.2297	-3.0774	-0.1523	0.16865	
1948	103.29	4.087	0.00737	0.04	-0.0326	0.37187		4.599	-3.0793	-3.1361	0.05677	0.36031	
1949	89.76	4.128	0.00998	0.04	-0.03	0.38181		5.355	-2.9271	-3.0459	0.11875	0.22721	
1950	77.42	4.146	0.00435	0.04	-0.0356	0.3605		5.503	-2.8999	-2.9546	0.0547	0.36521	
1951	83.88	4.616	0.10738	0.04	0.06738	0.24982		4.973	-3.0011	-2.9382	-0.0629	0.3459	
1952	98.61	4.904	0.06052	0.04	0.02052	0.41855		6.82	-2.6853	-2.999	0.31367	0.02408	UNLIKELY OBS
1953	74.28	5.066	0.0325	0.04	-0.0075	0.47006		6.767	-2.6931	-2.8095	0.11637	0.23177	
1954	86.33	5.842	0.14252	0.04	0.10252	0.1522		5.765	-2.8534	-2.8142	-0.0392	0.40247	
1955	115.85	6.679	0.1339	0.04	0.0939	0.17345		4.948	-3.0062	-2.9103	-0.0959	0.27294	
1956	144.40	7.145	0.06744	0.04	0.02744	0.39168		6.006	-2.8124	-3.002	0.18959	0.11618	
1957	126.02	7.569	0.05765	0.04	0.01765	0.42983		5.167	-2.9629	-2.8857	-0.0771	0.31351	
1958	142.66	7.371	-0.0265	0.04	-0.0665	0.25262		6.003	-2.8129	-2.976	0.16311	0.1521	
1959	129.42	7.769	0.05259	0.04	0.01259	0.44982		5.334	-2.9311	-2.886	-0.045	0.38834	
1960	172.33	9.192	0.16819	0.04	0.12819	0.09953		4.949	-3.006	-2.9569	-0.0491	0.37867	
1961	211.13	10.449	0.12817	0.04	0.08817	0.18854		4.533	-3.0938	-3.0019	-0.0919	0.28132	
1962	228.24	10.346	-0.0099	0.04	-0.0499	0.30855		5.473	-2.9053	-3.0546	0.14922	0.17361	
1963	188.56	10.32	-0.0025	0.04	-0.0425	0.33508		5.109	-2.9742	-2.9415	-0.0327	0.41848	
1964	228.61	11.69	0.12465	0.04	0.08465	0.19821		5.184	-2.9596	-2.9828	0.0232	0.4419	
1965	253.57	13.145	0.11731	0.04	0.07731	0.21933		6.052	-2.8048	-2.974	0.16927	0.14315	
1966	222.41	13.46	0.02368	0.04	-0.0163	0.43507		4.899	-3.0161	-2.8812	-0.135	0.19759	
1967	264.95	12.98	-0.0363	0.04	-0.0763	0.22229		5.163	-2.9637	-3.008	0.04432	0.39004	
1968	253.03	13.064	0.00645	0.04	-0.0335	0.3684		3.768	-3.2786	-2.9765	-0.3021	0.0285	UNLIKELY OBS
1969	367.99	13.866	0.05958	0.04	0.01958	0.42225		4.193	-3.1718	-3.1655	-0.0063	0.48421	
1970	339.80	14.248	0.02718	0.04	-0.0128	0.44889		4.963	-3.0032	-3.1013	0.09819	0.26812	
1971	294.36	14.609	0.02502	0.04	-0.015	0.44036		4.142	-3.184	-3.0002	-0.1838	0.12347	
1972	381.43	15.799	0.07831	0.04	0.03831	0.35057		3.635	-3.3146	-3.1087	-0.2059	0.09734	
1973	456.59	16.597	0.04928	0.04	0.00928	0.46298		4.221	-3.1651	-3.187	0.02193	0.44506	
1974	427.74	18.055	0.0842	0.04	0.0442	0.32896		8.219	-2.4987	-3.0974	0.59863	0.00008	EXTREME OUTLIER
1975	238.82	19.629	0.08359	0.04	0.04359	0.33119		7.556	-2.5828	-2.6975	0.1147	0.23499	
1976	290.37	21.94	0.1113	0.04	0.0713	0.23752		6.87	-2.678	-2.748	0.06998	0.32966	
1977	368.70	25.33	0.14368	0.04	0.10368	0.14949		6.071	-2.8016	-2.8051	0.00345	0.49133	
1978	464.82	26.219	0.10801	0.04	0.06801	0.24785							

Appendix I.3

Table I.3(a) : Analysis of the Hybrid Data
(Period ending December 1978)

	P (1)	D (2)	Ln(D/D) (3)	mdl-78 (4)	resdl-78 (5)	Prob Value (6)	Remark (7)	Yp (8)	Ln(Yp) (9)	mdl (10	residl (11)	Prob Value (12)	Remark (13)
1918	100							6.5	-2.73337	-2.73337	0	0.5	
1919	135.5	6.5						6.5	-2.73337	-2.83831	0.10495	0.28717	
1920	93.73	8.91	0.31537	0.04	0.27537	0.00069	EXTREME OUTLIER	6.58	-2.72114	-2.83831	0.11718	0.26528	
1921	84.74	7.48	-0.17494	0.04	-0.21494	0.00626	UNLIKELY OBS	7.98	-2.52823	-2.83097	0.30274	0.05258	
1922	119.8	7.22	-0.03538	0.04	-0.07538	0.19061		8.52	-2.46275	-2.71523	0.25248	0.0883	
1923	125.87	8.13	0.11871	0.04	0.07871	0.18028		6.79	-2.68972	-2.67595	-0.01377	0.47062	
1924	145.61	7.93	-0.02491	0.04	-0.06491	0.22542		6.3	-2.76462	-2.81212	0.0475	0.39965	
1925	182.88	8.78	0.10182	0.04	0.06182	0.23632		6.03	-2.80842	-2.85707	0.04864	0.3973	
1926	179.2	9.69	0.09862	0.04	0.05862	0.24795		5.3	-2.93746	-2.88335	-0.05412	0.38605	
1927	200.6	9.82	0.01333	0.04	-0.02667	0.37834		5.48	-2.90407	-2.96077	0.05671	0.38076	
1928	231.7	10.6	0.07643	0.04	0.03643	0.33606		5.28	-2.94124	-2.94073	-0.00051	0.49891	
1929	191.68	11.71	0.09959	0.04	0.05959	0.2444		5.05	-2.98578	-2.96304	-0.02274	0.45156	
1930	158.9	10.6	-0.09959	0.04	-0.13959	0.05245		5.53	-2.89498	-2.98976	0.09478	0.30598	
1931	125.7	8.52	-0.21844	0.04	-0.25844	0.00134	EXTREME OUTLIER	5.36	-2.92621	-2.93528	0.00908	0.48063	
1932	162.27	7.67	-0.1051	0.04	-0.1451	0.04594	UNLIKELY OBS	6.11	-2.79524	-2.95402	0.15877	0.19772	
1933	204.9	8	0.04212	0.04	0.00212	0.49015		4.93	-3.00983	-2.87544	-0.13439	0.23598	
1934	245.59	9.11	0.12993	0.04	0.08993	0.14808		4.44	-3.11452	-3.00419	-0.11032	0.27744	
1935	269.9	10.02	0.09521	0.04	0.05521	0.26064		4.08	-3.19907	-3.067	-0.13207	0.23983	
1936	310.85	10.6	0.05627	0.04	0.01627	0.42504		3.93	-3.23653	-3.11774	-0.11879	0.26245	
1937	259.1	12.03	0.12655	0.04	0.08655	0.15735		3.87	-3.25192	-3.14021	-0.1117	0.27497	
1938	220.57	12.1	0.0058	0.04	-0.0342	0.34558		4.67	-3.06401	-3.14944	0.08543	0.32375	
1939	213.68	11.58	-0.04393	0.04	-0.08393	0.16479		5.25	-2.94694	-3.0367	0.08976	0.31547	
1940	191.97	12.03	0.03812	0.04	-0.00188	0.49131		5.63	-2.87706	-2.96646	0.0894	0.31616	
1941	224.25	11.71	-0.02696	0.04	-0.06696	0.21833		6.1	-2.79688	-2.92453	0.12765	0.24724	
1942	252.64	11.19	-0.04542	0.04	-0.08542	0.16052		4.99	-2.99773	-2.87642	-0.12131	0.25808	
1943	270.94	10.99	-0.01803	0.04	-0.05803	0.2501		4.35	-3.13499	-2.99693	-0.13806	0.22998	
1944	293.3	11.19	0.01803	0.04	-0.02197	0.3993		4.13	-3.18689	-3.07929	-0.1076	0.28234	
1945	299.25	11.38	0.01684	0.04	-0.02316	0.39393		3.88	-3.24934	-3.11043	-0.13891	0.22861	
1946	340.45	12.03	0.05555	0.04	0.01555	0.42834		4.02	-3.21389	-3.14789	-0.06599	0.36197	
1947	319.49	13.72	0.13145	0.04	0.09145	0.14404		4.03	-3.2114	-3.12663	-0.08478	0.32501	
1948	294.26	12.62	-0.08357	0.04	-0.12357	0.07557		3.95	-3.23145	-3.12514	-0.10632	0.28467	
1949	264.64	13.33	0.05473	0.04	0.01473	0.43205		4.53	-3.09445	-3.13717	0.04272	0.40958	
1950	279.25	13.92	0.04331	0.04	0.00331	0.48467		5.26	-2.94504	-3.05496	0.10992	0.27816	
1951	287.3	15.61	0.11459	0.04	0.07459	0.19313		5.59	-2.88419	-2.96532	0.08113	0.33207	
1952	270.64	16.52	0.05666	0.04	0.01666	0.42327		5.75	-2.85597	-2.92881	0.07284	0.34833	
1953	318.91	17.24	0.04266	0.04	0.00266	0.48767		6.37	-2.75357	-2.91188	0.1583	0.19842	
1954	453.63	19.9	0.14349	0.04	0.10349	0.11465		6.24	-2.77419	-2.85044	0.07625	0.34161	
1955	480.57	23.09	0.14868	0.04	0.10868	0.10338		5.09	-2.97789	-2.86281	-0.11509	0.26896	
1956	413.61	23.74	0.02776	0.04	-0.01224	0.44347		4.94	-3.0078	-2.98503	-0.02278	0.45149	
1957	384.6	24.32	0.02414	0.04	-0.01586	0.4269		5.88	-2.83361	-3.00298	0.16936	0.18235	
1958	542.6	25.82	0.05985	0.04	0.01985	0.40881		6.71	-2.70157	-2.89846	0.19689	0.14599	
1959	810.83	29.2	0.12302	0.04	0.08302	0.16742		5.38	-2.92248	-2.81924	-0.10325	0.29027	
1960	789.7	35.19	0.18659	0.04	0.14659	0.04429	UNLIKELY OBS	4.34	-3.1373	-2.95178	-0.18551	0.16038	
1961	766.4	36.49	0.03628	0.04	-0.00372	0.48275		4.62	-3.07478	-3.08067	0.00589	0.48742	
1962	732.5	36.81	0.00873	0.04	-0.03127	0.35821		4.8	-3.03655	-3.04316	0.0066	0.4859	
1963	867.542	38.941	0.05628	0.04	0.01628	0.42501		5.316	-2.93445	-3.02023	0.08578	0.32309	
1964	781.66	44.648	0.13676	0.04	0.09676	0.1305		5.146	-2.96695	-2.95896	-0.00799	0.48295	
1965	821.207	47.883	0.06995	0.04	0.02995	0.36395		6.126	-2.79263	-2.97846	0.18584	0.15996	
1966	761.739	47.757	-0.00263	0.04	-0.04263	0.3102		5.815	-2.84473	-2.87387	0.02914	0.43803	
1967	957.393	47.39	-0.00771	0.04	-0.04771	0.28969		6.221	-2.77724	-2.90513	0.12789	0.24683	
1968	1379.63	48.141	0.01572	0.04	-0.02428	0.38896		5.028	-2.99015	-2.86464	-0.12551	0.25087	
1969	1166.91	49.241	0.02259	0.04	-0.01741	0.41987		3.569	-3.33288	-2.99238	-0.3405	0.0342	UNLIKELY OBS
1970	1050.99	51.685	0.04844	0.04	0.00844	0.46094		4.429	-3.117	-3.19802	0.08103	0.33226	
1971	1455.82	53.615	0.03666	0.04	-0.00334	0.48453		5.101	-2.97573	-3.06849	0.09276	0.30979	
1972	1608.27	58.197	0.082	0.04	0.042	0.31279		3.998	-3.21938	-2.98373	-0.23564	0.10362	
1973	1112.27	60.907	0.04551	0.04	0.00551	0.47446		3.787	-3.2736	-3.12992	-0.14368	0.22095	
1974	507.242	66.284	0.0846	0.04	0.0446	0.30219		5.959	-2.82027	-3.16245	0.34218	0.03352	UNLIKELY OBS
1975	1224.18	74.007	0.11021	0.04	0.07021	0.20736		14.59	-1.92483	-2.89045	0.96562	1.2E-07	EXTREME OUTLIER
1976	1211.8	83.797	0.12424	0.04	0.08424	0.1639		6.845	-2.68165	-2.35319	-0.32846	0.03938	UNLIKELY OBS
1977	1714.2	97.731	0.15382	0.04	0.11382	0.09305		8.065	-2.51764	-2.80728	0.28965	0.06054	
1978	1781.28	109.02	0.10931	0.04	0.06931	0.21036		6.36	-2.75514	-2.70887	-0.04627	0.40221	

Appendix I.3

Table I.3(b) : Analysis of the Hybrid Data
(Updated to December 1993)

HYBRID AND UPDATED

	P	D	Ln(D/D)	mdl-93	resdl-93	Prob Value	Remark	Yp	Ln(Yp)	mdl	resdl	Prob Value	Remark
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
918	100							6.5	-2.7334	-2.7334	0	0.5	
919	135.5	6.5						6.58	-2.7211	-2.8383	0.11718	0.25605	
920	93.73	8.91	0.31537	0.055	0.26037	0.00135	EXTREME OUTLIER	7.98	-2.5282	-2.831	0.30274	0.04516	UNLIKELY OBS
1921	84.74	7.48	-0.1749	0.055	-0.2299	0.00402	EXTREME OUTLIER	8.52	-2.4628	-2.7152	0.25248	0.0789	
1922	119.8	7.22	-0.0354	0.055	-0.0904	0.14877		6.79	-2.6897	-2.6759	-0.0138	0.46929	
1923	125.87	8.13	0.11871	0.055	0.06371	0.23139		6.3	-2.7646	-2.8121	0.0475	0.39521	
1924	145.61	7.93	-0.0249	0.055	-0.0799	0.17852		6.03	-2.8084	-2.8571	0.04864	0.39276	
1925	182.88	8.78	0.10182	0.055	0.04682	0.2947		5.3	-2.9375	-2.8833	-0.0541	0.38103	
1926	179.2	9.69	0.09862	0.055	0.04362	0.30757		5.48	-2.9041	-2.9608	0.05671	0.37553	
1927	200.6	9.82	0.01333	0.055	-0.0417	0.31549		5.28	-2.9412	-2.9407	-0.0005	0.49886	
1928	231.7	10.6	0.07643	0.055	0.02143	0.40244		5.05	-2.9858	-2.963	-0.0227	0.44938	
1929	191.68	11.71	0.09959	0.055	0.04459	0.30365		5.53	-2.895	-2.9898	0.09478	0.29797	
1930	158.9	10.6	-0.0996	0.055	-0.1546	0.03739	UNLIKELY OBS	5.36	-2.9262	-2.9353	0.00908	0.47975	
1931	125.7	8.52	-0.2184	0.055	-0.2734	0.00081	EXTREME OUTLIER	6.11	-2.7952	-2.954	0.15877	0.18719	
1932	162.27	7.67	-0.1051	0.055	-0.1601	0.03249	UNLIKELY OBS	4.93	-3.0098	-2.8754	-0.1344	0.22606	
1933	204.9	8	0.04212	0.055	-0.0129	0.44101		4.44	-3.1145	-3.0042	-0.1103	0.26854	
1934	245.59	9.11	0.12993	0.055	0.07493	0.19388		4.08	-3.1991	-3.067	-0.1321	0.22999	
1935	269.9	10.02	0.09521	0.055	0.04021	0.32151		3.93	-3.2365	-3.1177	-0.1188	0.25315	
1936	310.85	10.6	0.05627	0.055	0.00127	0.49416		3.87	-3.2519	-3.1402	-0.1117	0.266	
1937	259.1	12.03	0.12655	0.055	0.07155	0.20477		4.67	-3.064	-3.1494	0.08543	0.31634	
1938	220.57	12.1	0.0058	0.055	-0.0492	0.28533		5.25	-2.9469	-3.0367	0.08976	0.30778	
1939	213.68	11.58	-0.0439	0.055	-0.0989	0.12709		5.63	-2.8771	-2.9665	0.0894	0.30849	
1940	191.97	12.03	0.03812	0.055	-0.0169	0.42289		6.1	-2.7969	-2.9245	0.12765	0.23757	
1941	224.25	11.71	-0.027	0.055	-0.082	0.17241		4.99	-2.9977	-2.8764	-0.1213	0.24866	
1942	252.64	11.19	-0.0454	0.055	-0.1004	0.12354		4.35	-3.135	-2.9969	-0.1381	0.21994	
1943	270.94	10.99	-0.018	0.055	-0.073	0.19995		4.13	-3.1869	-3.0793	-0.1076	0.27359	
1944	293.3	11.19	0.01803	0.055	-0.037	0.33503		3.88	-3.2493	-3.1104	-0.1389	0.21854	
1945	299.25	11.38	0.01684	0.055	-0.0382	0.33001		4.02	-3.2139	-3.1479	-0.066	0.35598	
1946	340.45	12.03	0.05555	0.055	0.00055	0.49749		4.03	-3.2114	-3.1266	-0.0848	0.31764	
1947	319.49	13.72	0.13145	0.055	0.07645	0.18911		3.95	-3.2315	-3.1251	-0.1063	0.27598	
1948	294.26	12.62	-0.0836	0.055	-0.1386	0.05511		4.53	-3.0944	-3.1372	0.04272	0.40556	
1949	264.64	13.33	0.05473	0.055	-0.0003	0.49878		5.26	-2.945	-3.055	0.10992	0.26928	
1950	279.25	13.92	0.04331	0.055	-0.0117	0.44641		5.59	-2.8842	-2.9653	0.08113	0.32496	
1951	287.3	15.61	0.11459	0.055	0.05959	0.24611		5.75	-2.856	-2.9288	0.07284	0.34182	
1952	270.64	16.52	0.05666	0.055	0.00166	0.49237		6.37	-2.7536	-2.9119	0.1583	0.1879	
1953	318.91	17.24	0.04266	0.055	-0.0123	0.44345		6.24	-2.7742	-2.8504	0.07625	0.33485	
1954	453.63	19.9	0.14349	0.055	0.08849	0.15388		5.09	-2.9779	-2.8628	-0.1151	0.25983	
1955	480.57	23.09	0.14868	0.055	0.09368	0.14012		4.94	-3.0078	-2.985	-0.0228	0.4493	
1956	413.61	23.74	0.02776	0.055	-0.0272	0.37678		5.88	-2.8336	-3.003	0.16936	0.17169	
1957	384.6	24.32	0.02414	0.055	-0.0309	0.36102		6.71	-2.7016	-2.8985	0.19689	0.13533	
1958	542.6	25.82	0.05985	0.055	0.00485	0.47771		5.38	-2.9225	-2.8192	-0.1032	0.28176	
1959	810.83	29.2	0.12302	0.055	0.06802	0.21652		4.34	-3.1373	-2.9518	-0.1855	0.14966	
1960	789.7	35.19	0.18659	0.055	0.13159	0.06466		4.62	-3.0748	-3.0807	0.00589	0.48685	
1961	766.4	36.49	0.03628	0.055	-0.0187	0.41457		4.8	-3.0366	-3.0432	0.0066	0.48526	
1962	732.5	36.81	0.00873	0.055	-0.0463	0.29691		5.316	-2.9344	-3.0202	0.08578	0.31565	
1963	867.542	38.941	0.05628	0.055	0.00128	0.49412		5.146	-2.967	-2.959	-0.008	0.48218	
1964	781.66	44.648	0.13676	0.055	0.08176	0.17299		6.126	-2.7926	-2.9785	0.18584	0.14924	
1965	821.207	47.883	0.06995	0.055	0.01495	0.43159		5.815	-2.8447	-2.8739	0.02914	0.43525	
1966	761.739	47.757	-0.0026	0.055	-0.0576	0.25324		6.221	-2.7772	-2.9051	0.12789	0.23715	
1967	957.393	47.39	-0.0077	0.055	-0.0627	0.23488		5.028	-2.9901	-2.8646	-0.1255	0.24128	
1968	1379.63	48.141	0.01572	0.055	-0.0393	0.32538		3.569	-3.3329	-2.9924	-0.3405	0.02839	UNLIKELY OBS
1969	1166.91	49.241	0.02259	0.055	-0.0324	0.35437		4.429	-3.117	-3.198	0.08103	0.32516	
1970	1050.99	51.685	0.04844	0.055	-0.0066	0.46987		5.101	-2.9757	-3.0685	0.09276	0.3019	
1971	1455.82	53.615	0.03666	0.055	-0.0183	0.4163		3.998	-3.2194	-2.9837	-0.2356	0.09369	
1972	1608.27	58.197	0.082	0.055	0.027	0.3778		3.787	-3.2736	-3.1299	-0.1437	0.21075	
1973	1112.27	60.907	0.04551	0.055	-0.0095	0.45647		5.959	-2.8203	-3.1625	0.34218	0.02778	UNLIKELY OBS
1974	507.242	66.284	0.0846	0.055	0.0296	0.36648		14.59	-1.9248	-2.8905	0.96562	3.3E-08	EXTREME OUTLIER
1975	1224.18	74.007	0.11021	0.055	0.05521	0.26227		6.845	-2.6817	-2.3532	-0.3285	0.03306	UNLIKELY OBS
1976	1211.8	83.797	0.12424	0.055	0.06924	0.21242		8.065	-2.5176	-2.8073	0.28965	0.05256	
1977	1714.2	97.731	0.15382	0.055	0.09882	0.12734		6.36	-2.7551	-2.7089	-0.0463	0.39788	
1978	1781.28	109.02	0.10931	0.055	0.05431	0.26565		7.861	-2.5433	-2.8514	0.30812	0.04237	UNLIKELY OBS
1979	1826.52	140.03	0.25033	0.055	0.19533	0.01218	UNLIKELY OBS	8.382	-2.4791	-2.7242	0.24516	0.08509	
1980	2277.77	153.103	0.08925	0.055	0.03425	0.34649		6.757	-2.6946	-2.6857	-0.0088	0.48026	
1981	2454.56	153.917	0.0053	0.055	-0.0497	0.28338		6.787	-2.6902	-2.815	0.12489	0.24237	
1982	3128.22	166.601	0.07919	0.055	0.02419	0.3902		5.75	-2.856	-2.8124	-0.0436	0.40369	
1983	3730.9	179.87	0.07663	0.055	0.02163	0.40155		5.997	-2.8139	-2.9119	0.09796	0.29182	
1984	4823.77	223.753	0.21831	0.055	0.16331	0.0299	UNLIKELY OBS	5.27	-2.9431	-2.8866	-0.0565	0.37596	
1985	5557.2	254.193	0.12755	0.055	0.07255	0.20151		5.208	-2.955	-2.9642	0.0092	0.47947	
1986	6790	289.44	0.12985	0.055	0.07485	0.19413		4.762	-3.0445	-2.9713	-0.0732	0.34102	
1987	7101.84	323.322	0.1107	0.055	0.0557	0.26043		5.294	-2.9386	-3.025	0.0864	0.31442	
1988	7481.6	375.958	0.15083	0.055	0.09583	0.13468		5.831	-2.842	-2.9615	0.11947	0.25194	
1989	9719.33	436.226	0.14868	0.055	0.09368	0.14011		4.922	-3.0115	-2.9035	-0.108	0.2729	
1990	8437.35	478.343	0.09217	0.055	0.03717	0.33418		6.121	-2.7934	-3.0052	0.21172	0.11811	
1991	9896.64	516.444	0.07664	0.055	0.02164	0.40152		5.225	-2.9517	-2.8744	-0.0774	0.33259	
1992	11252.6	517.126	0.00132	0.055	-0.0537	0.26805		4.366	-3.1313	-2.9693	-0.162	0.18238	
1993	13222.6	491.236	-0.0514	0.055	-0.1064	0.11011							

APPENDIX II : THE STANDARD LIABILITY PORTFOLIO

Term t	Premium P_t	Sum Assured tP_t
10	30	300
11	6	66
12	7	84
13	8	104
14	9	126
15	50	750
16	10	160
17	10	170
18	10	180
19	10	190
20	50	1000
21	9	189
22	8	176
23	7	161
24	6	144
25	30	750
26	5	130
27	5	135
28	5	140
29	5	145
30	10	300

APPENDIX III.1 : Asset-Liability Simulation Using the Method of Chambers et.al

```
program AssLiab (input, output);

(* Program written by Gary Steele Finkelstein
 * Dept of Business Science
 * University of Cape Town
 * Rhodes Gift,  South Africa

 * Program developed as part of the MBusSc degree

 * Program implements an Asset-Liability Model.

 * The assets are modelled using a refinement of the
 * MGWP model (ref JIA, 1980). The refinement involves
 * replacing the assumption that the error terms of the
 * model are Normally distributed, by the assumption that
 * they have a more general Stable distribution.

 * The liability model is the same as that used by the MGWP.
 * It consists of regular premium unit linked insurance policies
 * with maturity guarantees; the maturity guarantees being a
 * return of the premiums paid.

 * Various insurance portfolios with different mean terms
 * are implemented (procedure UpdatePrems).

 * The program has the ability to compute the reserving requirements
 * and ruin probabilities associated with the liabilities.
 * This done using simulation techniques. Several thousand pseudo-
 * random scenarios are generated by sampling from the assumed
 * distribution for the error terms.

 * The method of Chambers, Mallows and Stuck (JASSA, 1976)
 * is used to sample from the Stable distributions
 * (See functions Stable() and StdSP())
 *)


uses Dos;

const
        NSIMS = 5000;
        TERMT = 30; TERMTP1 = 31;
        MAXRAT = 100;
        MAXCDF = 1000;
    TAX = 0.0; (* 0.375 *)
        (* YMU = 0.05;*)  (* A = 0.6; *) g = 0.0; c = 1.0; (* Ypo = YMU;*)
type
        RatVector = array [0..MAXRAT] of double;
        time_series = array [0..TERMTP1] of double;
        (* CdfVector = array[1..MAXCDF] of double; *)
    ObsVector = array[1..NSIMS] of double;
```

```
var  (* global variables *)

  DMU, YMU, varINC, Ypo, A : double;
  SZy, SZd, NZy, NZd : time_series;
  dummy : time_series;
  Y, D, P, Premium, Assets, I, J, Liabs, GI, GJ : time_series;
  CR, DCR : double;
  IObs (* GObs *)(* DErrObs *)  : ObsVector; (* Use one for CHI-SQUARE TESTS *)

  MSZy, MSZd, MY, MD, MP, MAssets, MI, MJ, MLiabs, MGI, MGJ : time_series;
  M2SZy, M2SZd, M2Y, M2D, M2I, M2GI : time_series;
  VSZy, VSZd, VY, VD, VI, VGI : time_series;

  alphaY, betaY, gammaY, alphaD, betaD, gammaD : double;
  alphaDLL, alphaYLL, alphaDUL, alphaYUL, alphaINC,
  betaDLL, betaYLL, betaDUL, betaYUL, betaINC : double;
  gammaY_MGWP, gammaD_MGWP, betaY_MGWP, betaD_MGWP, alphaY_MGWP,
alphaD_MGWP,
  gammaY_FTA, gammaD_FTA, alphaY_FTA, alphaD_FTA, betaY_FTA, betaD_FTA : double;
  alphaY_alt, alphaD_alt, betaY_alt, betaD_alt, gammaY_alt, gammaD_alt : double;
  alphaY_Z, alphaD_Z, betaY_Z, betaD_Z, gammaY_Z, gammaD_Z : double;
  SimK, NZ, Seed, StartSeed : longint;
  Mean_N, Mean_NZ : double;

  RatCounts, RuinProbs, DCRcounts, DCRprobs : RatVector;
  RatDensity, DCRdensity : RatVector;
  ratindex : longint;
  ruin_5ile, ruin_2ile, ruin_1ile, ruin_5mile, ruin_2mile, ruin_1mile : double;
 (*
  CauchyCdf, S1_2Cdf, S1_5Cdf, S1_8Cdf, S2Cdf : CdfVector;
 *)
  Hour, Minute, Second, Sec100 : word;
  REG_PREM : boolean;

function square(x:double):double;
begin
   square := x*x;
end;

function pow(a.b:double) : double;
begin
   if (abs(a) < 0.00001) then pow := 0.0
   else
     pow := exp(b*ln(a));
end;
```

```
function Random (var Seed : longint) : double;
const
        { MULTIPLIER = 25173 or * 69069 or * 16807 or 314159221 }
        MULTIPLIER = 25173 ;
        {MODULUS = 65536 or * 4294967296 or maxint or 1000000000 }
        MODULUS = 65536 ;
        { INCREMENT = 13849 or * 1 or * 0 or * 211324863 }
        INCREMENT = 13849 ;
var     r : double;
begin
    if (Seed > MODULUS) then
    begin
        writeln(' ### Error : Seed = ', Seed : 15);
        writeln('           : SimK = ', SimK : 15);
    end;

        Seed := (MULTIPLIER * Seed + INCREMENT) mod MODULUS;
        r := Seed / Modulus;

        Random := r;
end;

function Expo (var Seed : longint; lambda : double) : double;
begin
        Expo := -ln(1-Random(Seed))/lambda;
end;

function Tan2 (x:double) : double;
begin
        Tan2 := (sin(x)/cos(x))/x;
end;

function D2(x:double):double;
begin
        D2 := (exp(x)-1.0)/x;
end;

function Cauchy (var Seed : longint) : double;
const PI = 3.1415927;
var u1, u2, v : double;
begin
  (*
   repeat
         u1 := Random(Seed);
         u2 := Random(Seed);
         v  := 2*u2 - 1;
   until (u1*u1 + v*v < 1);
   Cauchy := v/u1;
  *)
  u1 := Random(Seed);
  Cauchy := sin(PI*(u1-0.5))/cos(PI*(u1-0.5));
end;
```

```
function SymmSP (var Seed : longint; alpha, gamma, delta : double) : double;
const
   PI = 3.1415927;
   PIBY2 = 1.570796327;
var
  V,W : double;
begin
     (* write('DEBUG : SEED = '); write (Seed :5); *)

         V := (Random(Seed) - 0.5)*PI;
         W := Expo(Seed,1);

     (*
      write('DEBUG : Seed, V, W = ');
      writeln(Seed : 5, V:7:2 ,W : 7:2);
     *)

         SymmSP := delta + gamma*sin(alpha*V)/(pow(cos(V),1/alpha))
                       * (pow(cos((1-alpha)*V)/W,(1-alpha)/alpha));
end;

function Stable(var Seed : longint; alpha, beta, gamma, delta : double) : double;
(* this algorithm works only for alpha <> 1 *)
const
   PI = 3.1415927;
   PIBY2 = 1.570796327;
var
  V,W : double;
  PHIo : double;
begin
     (* write('DEBUG : SEED = '); write (Seed :5); *)

      if (abs(alpha-1) < 0.001) then
      writeln('### error : Call to function Stable when alpha = 1');

         V := (Random(Seed) - 0.5)*PI;  (* V = PHI *)
         W := Expo(Seed,1);
      PHIo := -0.5*PI*beta*(1-abs(1-alpha))/alpha;

     (*
      write('DEBUG : Seed, V, W = ');
      writeln(Seed : 5, V:7:2 ,W : 7:2);
     *)

         Stable := delta + gamma*sin(alpha*(V-PHIo))/(pow(cos(V),1/alpha))
                       * (pow(cos(V-alpha*(V-PHIo))/W,(1-alpha)/alpha));
end;
```

```
function StdSP(var Seed : longint; alpha, beta : double) : double;
(* function for more general STANDARDIZED stable variate,
  where alpha is not restricted to be <> 1. Algorithm with a
  maths coprocessor is less numerically sensitive          *)
const
  PI = 3.141592654;
  PIBY2 = 1.570796327;
var
  Uphi, StdW, eps, tou, a, b, BigB, zeta, z, d : double;
begin
  if (abs(alpha - 1.0) <= 0.0001) then StdSP := Cauchy(Seed)
  else begin (* alpha <> 1 *)
  eps := 1.0 - alpha;
  if (eps > -0.99) then
        tou := beta / ( PIBY2*Tan2(PIBY2*eps) )
  else
        tou := PIBY2 * beta * eps * (1.0-eps) * Tan2(PIBY2*(1-eps));
  tou := tou/(-eps);
  (* tou = tan (alpha * phiO *)
  Uphi := (Random(Seed) - 0.5)*PI;
  StdW := Expo(Seed, 1);
  zeta := ((cos(eps*Uphi) - tou*sin(eps*Uphi))/(StdW*cos(Uphi)));
  StdSP := (sin(alpha*Uphi)/cos(Uphi) - tou*(cos(alpha*Uphi)/cos(Uphi)-1))
                 * pow(zeta,eps/(1-eps))
           + tou * (1 - pow(zeta,eps/(1-eps)));
  end; (* else alpha <> 1*)
end;


function StdZ (var Seed : longint) : double;
const
  PI = 3.1415927;
var
  u1, u2 : double;
begin
  u1 := Random(Seed);
  u2 := Random(Seed);
  StdZ := sqrt(-2*ln(u2))*cos(2*PI*u1);
end;
```

```
function ProbValue (var Obs : ObsVector; N : integer; num : double) : double;
(* function assumes Obs has been sorted *)
var i : integer;
   found : boolean;
   answer : double;
begin
   found := false;
   i:= 0;
   answer := 6.023E+23;
   while (not found) and (i < N) do
   begin
     if (Obs[i+1] > num) and (Obs[i] <= num) then
     begin
       found := true;
       answer := 0.5*(i+(i+1))/N;
      end
      else
        i := i+1;
   end;
   if not found then writeln ('### Warning : not found in Prob Value');
   ProbValue := answer;
end;

procedure BubbleSort (var Observations : ObsVector; N : integer);
var
  changed : boolean;
  here, end_of_scan : 1..NSIMS;
  temp : double;
begin
   end_of_scan := N;
   repeat
      end_of_scan := end_of_scan - 1;
      changed := false;
      for here := 1 to end_of_scan do
         if Observations[here] > Observations[here+1] then
         begin
            temp := Observations[here];
            Observations[here] := Observations[here+1];
            Observations[here+1] := temp;
            changed := true;
         end;
   until ((end_of_scan = 1) or ( not changed));
end;
```

```
(*
procedure UpdateCdfs (var FC, F1_2, F1_5, F1_8, F2 : CdfVector;
                         var Seed : longint );
var Cz, S1_2z, S1_5z, S1_8z, S2z : double;
   cdf_index : longint;
begin
   Cz := Cauchy(Seed);
   S1_2z := SymmSP(Seed, 1.5, 1.0, 0);
   S1_5z := SymmSP(Seed, 1.8, 1.0, 0);
   S1_8z := Stable(Seed, 1.8, 0, 1.0, 0);
   S2z := SymmSP(Seed, 2, 1.0, 0);

   for cdf_index := 1 to MAXCDF do
   begin
         if Cz <= cdf_index/100 then CauchyCdf[cdf_index] :=
                                        CauchyCdf[cdf_index] + 1.0;
         if S1_2z <= cdf_index/100 then F1_2[cdf_index] :=
                                        F1_2[cdf_index] + 1.0;
         if S1_5z <= cdf_index/100 then F1_5[cdf_index] :=
                                        F1_5[cdf_index] + 1.0;
         if S1_8z <= cdf_index/100 then F1_8[cdf_index] :=
                                        F1_8[cdf_index] + 1.0;
         if S2z <= cdf_index/100 then F2[cdf_index] := F2[cdf_index] + 1.0;
      end;
end;
*)
```

```
procedure Initialise (var SZy, SZd, Y, D, P, Assets, I, GI,
                          Liabs, J, GJ, Premium,
                          MSZy, MSZd, MY, MD, MP, MAssets, MI, MGI,
                          MLiabs, MJ, MGJ,
                M2SZy, M2SZd, M2Y, M2D, M2I, M2GI,
                VSZy, VSZd, VY, VD, VI, VGI
                              : time_series;
                      var RatCounts, RatDensity, RuinProbs,
                 DCRcounts, DCRdensity, DCRprobs : RatVector;
                var NZ : longint; var Mean_N, Mean_NZ : double
                          (* var FC, F1_2, F1_5, F1_8, F2 : CdfVector   *)
                );
var
  t, index : longint;

begin
   NZ := 0;
   Mean_N := 0.0; Mean_NZ := 0.0;

   for t := 0 to TERMTP1 do
   begin
         SZy[t] := 0.0;    SZd[t] := 0.0;    Y[t] := 0.0;      D[t] := 0.0;      P[t] := 0.0;
         Assets[t] := 0.0; I[t] := 0.0;      GI[t] := 0.0;     Liabs[t] := 0.0;  J[t] := 0.0;
         GJ[t] := 0.0;     dummy[t] := 0.0;     Premium[t] := 0.0;
     (*
         MSZy[t] := 0.0; MSZd[t] := 0.0;   MY[t] := 0.0; MD[t] := 0.0; MP[t] := 0.0;
         MAssets[t]:= 0.0; MI[t] := 0.0; MGI[t] := 0.0; MLiabs[t] := 0.0; MJ[t] := 0.0; MGJ[t] := 0.0;
     *)
   end;
  (*
   M2SZy := MY; M2SZd := MY; M2Y := MY; M2D := MY; M2I := MY; M2GI := MY;
   VSZy := MY; VSZd := MY; VY := MY; VD := MY; VI := MY; VGI := MY;
  *)

   for index := 0 to MAXRAT do
         RatCounts[index] := 0.0;

   RatDensity := RatCounts;   RuinProbs := RatCounts;   DCRcounts := RatCounts;
   DCRdensity := RatCounts;   DCRprobs := RuinProbs;

   (*
   for index := 1 to MAXCDF do
   begin
         FC[index] :=0; F1_2[index] := 0.0;
         F1_5[index] := 0.0; F1_8[index] := 0.0; F2[index] := 0.0;
   end;
   *)
end;
```

```
procedure UpdatePrems(var P : time_series);
var t : 0..TERMTP1;
begin
   (*
    P[TERMT] := 1.0
   *)

     P[10] := 30;  P[11] := 6;  P[12] := 7;  P[13] := 8;  P[14] := 9;  P[15] := 50;
     P[16] := 10; P[17] := 10; P[18] := 10; P[19] := 10; P[20] := 50;     P[21] := 9;  P[22] := 8;
     P[23] := 7;  P[24] := 6;  P[25] := 30;  P[26] := 5;  P[27] := 5;  P[28] := 5;  P[29] := 5;
     P[30] := 10;

   (*
    for t := 1 to TERMTP1 do
    begin
       if ( t >= TERMT ) then P[t] := 0.0
       else begin
           if REG_PREM then P[t] := 1.0
           else {SING_PREM} P[t] := 0.0;
       end;
    end;
   *)
end;

procedure UpdateSeries(alphaY, betaY, gammaY, alphaD, betaD, gammaD : double;
          var SZy, SZd, Y, D, P, Assets, I, GI, Liabs, J, GJ, Premium: time_series;
      var CR, DCR : double;
          var seed : longint);

var
   t, TM1, TM2 : 0..TERMTP1;
   logY, logD, Rprice, GUprice, GClaim : time_series;
   TSA, TotClaim, TotDiscClaim : double;
begin
   Y[0] := Ypo; logY[0] := ln(Y[0]); D[0] := 1.0; logD[0] := ln(D[0]);
   Assets[0] := 0.0; Liabs[0] := 0.0; { immediately BEFORE prem now due }
   I[0] := 0.0; J[0] := 0.0; GI[0] := 0.0; GJ[0] := 0.0;
  TSA := 0.0; TotClaim := 0.0; TotDiscClaim := 0.0;

   for t := 1 to TERMTP1 do
   begin
          (* SZy[t] := SymmSP(seed, alphaY, 1.0, 0.0);
           SZd[t] := SymmSP(seed, alphaD, 1.0, 0.0); *)
            SZy[t] := Stable(seed, alphaY, betaY, 1.0, 0.0);
            SZd[t] := Stable(seed, alphaD, betaD, 1.0, 0.0);
          logY[t] := A*logY[t-1] + (1-A)*ln(YMU) + gammaY*SZy[t];
          logD[t] := logD[t-1] + DMU + gammaD*SZd[t];
          Y[t] := exp(logY[t]);
          D[t] := exp(logD[t]);
   end;

   P[0] := D[1]/Y[0];
   Rprice[0] := 1.0;
   GUprice[0] := 1.0;
```

```
for t := 1 to TERMT do
  begin

        P[t] := D[t+1]/Y[t];
    I[t] := (P[t] + (1-TAX)*D[t])/P[t-1] - 1;
    Rprice[t] := Rprice[t-1]*(1+I[t]);
   (*
        if (I[t] > 3.00) then
        begin
           writeln('I[t] of ',I[t]:7:4, 'detected on SimK',SimK :6);
           write('Y[t] = ',Y[t] : 8:5, 'Y[t-1] = ',Y[t-1]:8:5);
           write('D[t+1] = ',D[t+1]:8:3, 'D[t] = ',D[t]:8:3);
           writeln;
        end;
   *)

   (* Assets and Liabilities immediately BEFORE premium due *)

    Assets[t] := (Assets[t-1]+1)*(1+I[t]);
        if (I[t] < g) then J[t] := g else J[t] := g + (1-c)*(I[t]-g);
    GUprice[t] := GUprice[t-1]*(1+J[t]);
        (* Liabs[t] := (Liabs[t-1]+Premium[t-1])*(1+J[t]); *)
    if t  > Assets[t] then
      GClaim[t] := t - Assets[t]
    else
       GClaim[t] := 0.0;
        GI[t] := pow(Rprice[t]/Rprice[0],1/t) - 1.0;
        GJ[t] := pow(GUprice[t]/GUprice[0],1/t) - 1.0;

        { Ratio[t] := Liabs[t] / Assets[t]; }
    TSA := TSA + t*Premium[t];
    TotClaim := TotClaim + Premium[t]*GClaim[t];
    TotDiscClaim := TotDiscClaim + Premium[t]*GClaim[t]*pow(1/1.04,t);
  end;

  CR := TotClaim / TSA;
  DCR := TotDiscClaim / TSA;

  logY[TERMTP1] := 0.0;
  Y[TERMTP1] := 0.0;
  P[TERMTP1] := 0.0;

end;
```

```
procedure UpdateStats
       (CR , DCR : double;
        var RatCounts, DCRcounts, RatDensity, DCRdensity : RatVector;
        var NZ: longint; SimK : longint; var Mean_NZ, Mean_N : double;
            var SZy, SZd, Y, D, P, Assets, I, GI, Liabs, J, GJ,
        MSZy, MSZd, MY, MD, MP, MAssets, MI, MGI, MLiabs, MJ, MGJ,
        M2SZy, M2SZd, M2Y, M2D, M2I, M2GI,
        VSZy, VSZd, VY, VD, VI, VGI
        : time_series );
var
   t : 0..TERMTP1;
   ratindex : 0..MAXRAT;
   bigratio : double;
begin
   bigratio := 0.0;
   for t := 1 to TERMTP1 do
   begin
         MSZy[t] := ((SimK-1)*MSZy[t] + SZy[t])/SimK;
         MSZd[t] := ((SimK-1)*MSZd[t] + SZd[t])/SimK;
         MI[t] := ((SimK-1)*MI[t] + I[t])/SimK;
         MJ[t] := ((SimK-1)*MJ[t] + J[t])/SimK;
         MGI[t] := ((SimK-1)*MGI[t] + GI[t])/SimK;
         MGJ[t] := ((SimK-1)*MGJ[t] + GJ[t])/SimK;
         MD[t] := ((SimK-1)*MD[t] + D[t])/SimK;
         MY[t] := ((SimK-1)*MY[t] + Y[t])/SimK;
         MP[t] := ((SimK-1)*MP[t] + P[t])/SimK;
         MAssets[t] := ((SimK-1)*MAssets[t] + Assets[t])/SimK;
         MLiabs[t] := ((SimK-1)*MLiabs[t] + Liabs[t])/SimK;

      M2SZy[t] := ((SimK-1)*M2SZy[t] + SZy[t]*SZy[t])/SimK;
      M2SZd[t] := ((SimK-1)*M2SZd[t] + SZd[t]*SZd[t])/SimK;
         M2I[t] := ((SimK-1)*M2I[t] + I[t]*I[t])/SimK;
         M2GI[t] := ((SimK-1)*M2GI[t] + GI[t]*GI[t])/SimK;
         M2D[t] := ((SimK-1)*M2D[t] + D[t]*D[t])/SimK;
         M2Y[t] := ((SimK-1)*M2Y[t] + Y[t]*Y[t])/SimK;

      VSZy[t] := { (SimK/(SimK-1))* } (M2SZy[t] - MSZy[t]*MSZy[t]);
      VSZd[t] := { (SimK/(SimK-1))* } (M2SZd[t] - MSZd[t]*MSZd[t]);
      VY[t] := { (SimK/(SimK-1))* } (M2Y[t] - MY[t]*MY[t]);
      VD[t] := { (SimK/(SimK-1))* } (M2D[t] - MD[t]*MD[t]);
      VI[t] := { (SimK/(SimK-1))* } (M2I[t] - MI[t]*MI[t]);
      VGI[t] := { (SimK/(SimK-1))* } (M2GI[t] - MGI[t]*MGI[t]);
   end; {for t}

   Mean_N := (( SimK-1)*Mean_N + CR)/SimK;
   if CR > 0.0 then
   begin
      NZ := NZ + 1;
      Mean_NZ := ((NZ-1)*Mean_NZ + CR)/NZ;
   end;
```

```
  bigratio := CR;                 {want distribn of all N ratios}
                              {eg ratio = 0.333 (% of SA),pr=.01}
  if ( 100*bigratio > MAXRAT) then  {eg 100*bigratio = 33.3 }
        bigratio := MAXRAT/100; {note Ratio := Claim Ratio = max(0,Ratio) }
  if CR <= 0 then
    RatDensity[0] := RatDensity[0] + 1
  else
     RatDensity[round(100*CR+0.5)] := RatDensity[round(100*CR+0.5)]+1;
  for ratindex := 0 to round(100*bigratio) do
        RatCounts[ratindex] := RatCounts[ratindex] + 1.0;

  bigratio := DCR;                {want distribn of all N discounted ratios}
                              {eg ratio = 0.333 (% of SA),pr=.01}
  if ( 100*bigratio > MAXRAT) then  {eg 100*bigratio = 33.3 }
        bigratio := MAXRAT/100; {note DCR = Discounted Claim Ratio }
  if DCR <= 0 then
    DCRdensity[0] := DCRdensity[0] + 1
  else
     DCRdensity[round(100*DCR+0.5)] := DCRdensity[round(100*DCR+0.5)]+1;
  for ratindex := 0 to round(100*bigratio) do
        DCRcounts[ratindex] := DCRcounts[ratindex] + 1.0;

  {
  if Ratio[TERMT] > bigratio then bigratio := Ratio[TERMT];
  if (100*(bigratio - 1) > MAXRAT) then
        bigratio := (MAXRAT + 100)/100;
  for ratindex := 0 to trunc(100*bigratio - 100) do
        RatCounts[ratindex] := RatCounts[ratindex] + 1.0;
  }

end; {UpdateStats}


procedure FindPCile (var RuinProbs : RatVector;
                percentage : double; var pcile : double);

var ratindex : longint;
begin
  ratindex := 0;
  pcile := MAXRAT;
  while ((ratindex < MAXRAT) and (pcile = MAXRAT)) do
  begin
        if RuinProbs[ratindex] < percentage then
          pcile := ratindex
        else
          ratindex := ratindex + 1;
  end;
end;
```

```
procedure ShowFractiles (var Observations : ObsVector; N : integer);
var fract001, fract005, fract01, fract025, fract05, fract10, fract25, fract50,
   fract999, fract995, fract99, fract975, fract95, fract90, fract75,
   fract125, fract375, fract625, fract875,
   range50, range80, range90, range95, range99 : double;
begin
   fract001 := 100*0.5*(Observations[round(0.001*N)]+Observations[round(0.001*N)+1]);
   fract005 := 100*0.5*(Observations[round(0.005*N)]+Observations[round(0.005*N)+1]);
   fract01 := 100*0.5*(Observations[round(0.01*N)]+Observations[round(0.01*N)+1]);
   fract025 :=100*0.5*(Observations[round(0.025*N)]+Observations[round(0.025*N)+1]);
   fract05 := 100*0.5*(Observations[round(0.05*N)]+Observations[round(0.05*N)+1]);
   fract10 := 100*0.5*(Observations[round(0.10*N)]+Observations[round(0.10*N)+1]);
   fract125 := 100*0.5*(Observations[round(0.125*N)]+Observations[round(0.125*N)+1]);
   fract25 := 100*0.5*(Observations[round(0.25*N)]+Observations[round(0.25*N)+1]);
   fract50 := 100*0.5*(Observations[round(0.50*N)]+Observations[round(0.50*N)+1]);
   fract375 := 100*0.5*(Observations[round(0.375*N)]+Observations[round(0.375*N)+1]);
   fract625 := 100*0.5*(Observations[round(0.625*N)]+Observations[round(0.625*N)+1]);
   fract75 := 100*0.5*(Observations[round(0.75*N)]+Observations[round(0.75*N)+1]);
   fract875 := 100*0.5*(Observations[round(0.875*N)]+Observations[round(0.875*N)+1]);
   fract90 := 100*0.5*(Observations[round(0.90*N)]+Observations[round(0.90*N)+1]);
   fract95 := 100*0.5*(Observations[round(0.95*N)]+Observations[round(0.95*N)+1]);
   fract975 := 100*0.5*(Observations[round(0.975*N)]+Observations[round(0.975*N)+1]);
   fract99 := 100*0.5*(Observations[round(0.99*N)]+Observations[round(0.99*N)+1]);
   fract995 := 100*0.5*(Observations[round(0.995*N)]+Observations[round(0.995*N)+1]);
   fract999 := 100*0.5*(Observations[round(0.999*N)]+Observations[round(0.999*N)+1]);

   writeln;
   write('The smallest are :');
   writeln(100*Observations[1]:6:2, ' ',100*Observations[2]:6:2, ' ',100*Observations[3]:6:2);
   writeln;
   write('Fract001 = '); writeln(fract001:7:2);   write('Fract005 = '); writeln(fract005:7:2);
   write('Fract01 = '); writeln(fract01:7:2);   write('Fract025 = '); writeln(fract025:7:2);
   write('Fract05 = '); writeln(fract05:7:2);   write('Fract10 = '); writeln(fract10:7:2);
   write('Fract25 = '); writeln(fract25:7:2);   write('Fract50 = '); writeln(fract50:7:2);
   write('Fract75 = '); writeln(fract75:7:2);   write('Fract90 = '); writeln(fract90:7:2);
   write('Fract95 = '); writeln(fract95:7:2);   write('Fract975 = '); writeln(fract975:7:2);
   write('Fract99 = '); writeln(fract99:7:2);   write('Fract995 = '); writeln(fract995:7:2);
   write('Fract999 = '); writeln(fract999:7:2);
   writeln;
   writeln('The biggest are : ');
   writeln(Observations[NSIMS-5]:7, ' ',Observations[NSIMS-3]:7, ' ',Observations[NSIMS-2]:7, ' ',
        Observations[NSIMS-2]:7, ' ',Observations[NSIMS-1]:7, ' ',Observations[NSIMS]:7);
   writeln;

  (*
   writeln;
   write('Fract001 = '); writeln(fract001:7:2);   write('Fract005 = '); writeln(fract005:7:2);
   write('Fract125 = '); writeln(fract125:8:2);   write('Fract25 = '); writeln(fract25:8:2);
   write('Fract375 = '); writeln(fract375:8:2);   write('Fract50 = '); writeln(fract50:8:2);
   write('Fract625 = '); writeln(fract625:8:2);   write('Fract75 = '); writeln(fract75:8:2);
   write('Fract875 = '); writeln(fract875:8:2);
   writeln;
  *)

   range50 := fract75-fract25;   range80 := fract90-fract10;   range90 := fract95-fract05;
   range95 := fract975-fract025;   range99 := fract995-fract005;
```

```
   write('Range50 = '); writeln(range50:9:3);
   write('Range80 = '); writeln(range80:9:3);
   write('Range90 = '); writeln(range90:9:3);
   write('Range95 = '); writeln(range95:9:3);
   write('Range99 = '); writeln(range99:9:3);

end;

(*
procedure ShowDensities (var D1, D2 : RatVector);
var i : integer;
begin
    writeln('Range      D1       D2');
    writeln('----------------------');
    for i := 0 to MAXRAT do
    begin
        write(i:5);
        write(D1[i]:8:1);
        write(D2[i]:8:1);
        writeln;
    end;
end;
*)

(*
procedure ShowCdfs (var FC, F1_2, F1_5, F1_8, F2 : CdfVector);
var ci : longint;
begin
   for ci := 1 to MAXCDF do
   begin
         if ( (ci <= 100) and (ci mod 5 = 0)) then
         begin
            write(ci:5);
            writeln(FC[ci]/NSIMS :7:4, F1_2[ci]/NSIMS :7:4,
                  F1_5[ci]/NSIMS :7:4, F1_8[ci]/NSIMS :7:4, F2[ci]/NSIMS :7:4);
         end
      else if ( (ci <= 200) and (ci mod 20 = 0)) then
         begin
            write(ci:5);
            writeln(FC[ci]/NSIMS :7:4, F1_2[ci]/NSIMS :7:4,
                  F1_5[ci]/NSIMS :7:4, F1_8[ci]/NSIMS :7:4. F2[ci]/NSIMS :7:4);
         end
         else if ( (ci <= 600) and (ci mod 40 = 0)) then
         begin
            write(ci:5);
            writeln(FC[ci]/NSIMS :7:4, F1_2[ci]/NSIMS :7:4,
                  F1_5[ci]/NSIMS :7:4, F1_8[ci]/NSIMS :7:4, F2[ci]/NSIMS :7:4);
         end
         else if (ci mod 100 = 0) then
         begin
            write(ci:5);
            writeln(FC[ci]/NSIMS :7:4, F1_2[ci]/NSIMS :7:4,
                  F1_5[ci]/NSIMS :7:4, F1_8[ci]/NSIMS :7:4, F2[ci]/NSIMS :7:4);
         end;
   end; {for}
end; {proc}
*)
```

```
procedure Show6Series (var v1, v2, v3, v4, v5, v6 : time_series;
                                m, n : longint);
var t : longint;
begin
   for t := 0 to TERMTP1 do
   begin
         write(t:4);
         write(v1[t]:m:n, v2[t]:m:n, v3[t]:m:n,
              v4[t]:m:n, v5[t]:m:n, v6[t]:m:n);
         writeln;
   end;
end;


begin (* main *)

     REG_PREM := true;

     DMU := 0.04; YMU := 0.05; Ypo := YMU; A:= 0.6;

     gammaY_MGWP := 0.20/sqrt(2); betaY_MGWP := 0.0; alphaY_MGWP := 2.0;
     gammaD_MGWP := 0.13/sqrt(2); betaD_MGWP := 0.0; alphaD_MGWP := 2.0;
     gammaY_FTA := 0.11; betaY_FTA := -0.5; alphaY_FTA := 1.70;
     gammaD_FTA := 0.033; betaD_FTA := +0.4; alphaD_FTA := 1.52;
     gammaY_Z := 0.10; betaY_Z := -0.3; alphaY_Z := 1.77;
     gammaD_Z := 0.05; betaD_Z := +0.6; alphaD_Z := 1.77;

     gammaY_alt := 0.10; gammaD_alt := 0.05;
     betaY_alt := -0.50; betaD_alt := +0.50;
     alphaY_alt := 1.75; alphaD_alt := 1.75;

       alphaYLL := alphaY_alt; alphaINC := 0.100; alphaYUL := alphaYLL;
       alphaDLL := alphaD_alt; alphaDUL := alphaDLL;
       betaYLL := betaY_alt; betaINC := 0.50; betaYUL := betaYLL;
       betaDLL := betaD_alt; betaDUL := betaDLL;
       gammaY := gammaY_alt;
       gammaD := gammaD_alt;

  varINC := 0.025;
  while (Ypo <= YMU) do
  begin

  alphaY := alphaYLL;
  while (alphaY <= alphaYUL) do
  begin

   alphaD := alphaDLL;
   while (alphaD <= alphaDUL) do
   begin
    (*
     if (abs(alphaY-1.0) < 0.01) then alphaY := 1.01;
    *)

    betaY := betaYLL;
    while (betaY <=   betaYUL) do
    begin
```

```
betaD := betaDLL;
while (betaD <= betaDUL) do
begin

GetTime(Hour, Minute, Second, Sec100);
writeln; writeln; writeln;
writeln('NEW SIMULATION');
writeln('--------------');
writeln;
writeln('The starting time is', Hour:3,':',Minute:3,':',Second:3);
writeln;

StartSeed := 3; Seed := StartSeed;

Initialise(SZy, SZd, Y, D, P, Assets, I, GI, Liabs, J, GJ, Premium,
            MSZy, MSZd, MY, MD, MP, MAssets, MI, MGI, MLiabs, MJ, MGJ,
        M2SZy, M2SZd, M2Y, M2D, M2I, M2GI,
        VSZy, VSZd, VY, VD, VI, VGI,
        RatCounts, RatDensity, RuinProbs,
        DCRcounts, DCRdensity, DCRprobs,
      NZ, Mean_N, Mean_NZ
   (*   CauchyCdf, S1_2cdf, S1_5cdf, S1_8cdf, S2cdf              *)
   );

UpdatePrems(Premium);

for SimK := 1 to NSIMS do
begin

   if (SimK mod 100 = 0) then
   begin
      (* write (SimK : 6); *)
      if (SimK mod 500 = 0) then
      begin
         GetTime(Hour, Minute, Second, Sec100);
         write('  Time is ',Hour:3,':',Minute:3,':',Second:3);
         writeln;
      end;
   end;

  (*
   if ( alphaY = alphaLL) then
          UpdateCdfs(CauchyCdf, S1_2cdf, S1_5cdf, S1_8cdf, S2cdf, Seed);
   *)

       UpdateSeries(alphaY, betaY, gammaY, alphaD, betaD, gammaD,
                 SZy, SZd, Y, D, P, Assets, I, GI, Liabs, J, GJ, Premium,
         CR, DCR, Seed );


     IObs[SimK] := I[30];
   (* GObs[SimK] := GI[30]; *)
   (* YErrObs[SimK] := SZy[30]; *)   (* NB vars FOR CHI-SQUARED TESTS *)
   (* DErrObs[SimK] := SZd[30]; *)
```

```
        UpdateStats(CR, DCR, RatCounts, DCRcounts, RatDensity, DCRdensity,
          NZ, SimK, Mean_NZ, Mean_N,
                  SZy, SZd, Y, D, P, Assets, I, GI, Liabs, J, GJ,
                  MSZy, MSZd, MY, MD, MP, MAssets, MI, MGI,
          MLiabs, MJ, MGJ,
          M2SZy, M2SZd, M2Y, M2D, M2I, M2GI,
          VSZy, VSZd, VY, VD, VI, VGI );

end; {for SimK}


writeln; writeln('Sorting IObs ...');   (* eg sort also DErrObs *)
GetTime(Hour, Minute, Second, Sec100);
writeln('  Time is ',Hour:3,':',Minute:3,':',Second:3);
 BubbleSort(IObs, NSIMS);
GetTime(Hour, Minute, Second, Sec100);
writeln('Done !');
writeln('  Time is ',Hour:3,':',Minute:3,':',Second:3);
writeln;


for ratindex := 0 to MAXRAT do
begin
        RuinProbs[ratindex] := RatCounts[ratindex] / NSIMS;
   DCRprobs[ratindex] := DCRcounts[ratindex] / NSIMS;
end;

FindPCile(RuinProbs, 0.05, ruin_5ile);  FindPCile(RuinProbs, 0.02, ruin_2ile);
FindPCile(RuinProbs, 0.01, ruin_1ile);  FindPCile(RuinProbs, 0.005, ruin_5mile);
FindPCile(RuinProbs, 0.002, ruin_2mile);  FindPCile(RuinProbs, 0.001, ruin_1mile);

(* Show Output *)
writeln;  write('NSIMS = '); writeln(NSIMS:9);  writeln;
write('StartSeed = '); writeln(StartSeed:4);  writeln;

writeln('The Liab parameters g, c ...');
writeln(g:5:2,c:5:2);
if REG_PREM then writeln('REGULAR PREMIUM')
  else writeln('SINGLE PREMIUM');
write('DMU = ', DMU:6:2, '  YMU = ', YMU:7:4);
writeln('  A = ', A:6:3, ' Ypo = ', Ypo:7:4);
writeln('THE STABLE LAW PARAMETERS ...:');
write(' alphaY, betaY, gammaY ...');
writeln(alphaY:5:2, betaY:4:1, gammaY:6:3);
write(' alphaD, betaD, gammaD ...');
writeln(alphaD:5:2, betaD:4:1, gammaD:6:3);
writeln;
write('TAX '); writeln(TAX:8:3);
writeln;
```

```
(*
 if (alphaY = alphaLL) then
 begin
   writeln; writeln;
   writeln('The CDFs  FC, FS1.2, FS1.5, FS1.8, FS2 ... ');
   writeln;
   ShowCdfs(CauchyCdf, S1_2cdf, S1_5cdf, S1_8cdf, S2cdf);
   writeln;
 end;
*)
(*
writeln('The Densities of CR and DCR ...');
ShowDensities(RatDensity, DCRdensity);
*)

writeln; writeln('The fractiles of IObs ...');
ShowFractiles(IObs, NSIMS);

(*
writeln; writeln('The fractiles of DErrObs ...');
ShowFractiles(DErrObs, NSIMS);
writeln; writeln('The Prob Values of DErrObs ...'); writeln;
write('The P Value of -1.7 is '); writeln(ProbValue(DErrObs,NSIMS,-1.7):8:4);
write('The P Value of -1.0 is '); writeln(ProbValue(DErrObs,NSIMS,-1.0):8:4);
write('The P Value of -0.5 is '); writeln(ProbValue(DErrObs,NSIMS,-0.5):8:4);
write('The P Value of  0.0 is '); writeln(ProbValue(DErrObs,NSIMS,0.0):8:4);
write('The P Value of +0.5 is '); writeln(ProbValue(DErrObs,NSIMS,+0.5):8:4);
write('The P Value of +1.0 is '); writeln(ProbValue(DErrObs,NSIMS,+1.0):8:4);
write('The P Value of +1.7 is '); writeln(ProbValue(DErrObs,NSIMS,+1.7):8:4);
writeln;
*)
(*
 writeln;
 writeln('The Vectors MY, MD, MP, MAssets, MLiabs, MRatio...');
 Show6Series(MY, MD, MP, MAssets, MLiabs, MRatio, 10, 4);
 writeln;
*)
(*
 writeln;
 writeln('The Vectors MSZy, MSZd, MI, MGI, MY, MD, ...');
 Show6Series(MSZy, MSZd, MI, MGI, MY, MD, 10,4);
 writeln;
*)
(*
 writeln;
 writeln('The Vectors VSZy, VSZd, VI, VGI, VRatio, dummy ...');
 Show6Series(VSZy, VSZd, VI, VGI, VRatio, dummy, 10,4);
 writeln;
*)
```

```
writeln;
writeln('Summary ...');
write('MI[T] = '); writeln(100.0*MI[TERMT]:8:2);
write('MGI[T] = '); writeln(100.0*MGI[TERMT]:8:2);
write('MJ[T] = '); writeln(100.0*MJ[TERMT]:8:2);
write('MGJ[T] = '); writeln(100.0*MGJ[TERMT]:8:2);
write('SI[T] = '); writeln(100.0*sqrt(VI[TERMT]):8:2);
write('SGI[T] = '); writeln(100.0*sqrt(VGI[TERMT]):8:2);
write('NZ = '); writeln(NZ:7);
write('Mean_NZ = '); writeln(Mean_NZ:9:5);
write('Mean_N = '); writeln(Mean_N:9:5);
write('1st ruin m%ile = '); writeln(ruin_1mile:6:1);
write('5th ruin m%ile = '); writeln(ruin_5mile:6:1);
(* write('2nd ruin m%ile = '); writeln(ruin_2mile:6:1); *)
write('1st ruin %ile = '); writeln(ruin_1ile:6:1);
write('5th ruin %ile = '); writeln(ruin_5ile:6:1);
(* write('2nd ruin %ile = '); writeln(ruin_2ile:6:1); *)

write('Ruin[00%] = '); writeln(RuinProbs[0]:8:4);
write('Ruin[05%] = '); writeln(RuinProbs[5]:8:4);
write('Ruin[20%] = '); writeln(RuinProbs[20]:8:4);

writeln('DISCOUNTED STATISTICS : ');

FindPCile(DCRprobs, 0.05, ruin_5ile);  FindPCile(DCRprobs, 0.02, ruin_2ile);
FindPCile(DCRprobs, 0.01, ruin_1ile);  FindPCile(DCRprobs, 0.005, ruin_5mile);
FindPCile(DCRprobs, 0.002, ruin_2mile);  FindPCile(DCRprobs, 0.001, ruin_1mile);
write('1st ruin m%ile = '); writeln(ruin_1mile:6:1);
write('5th ruin m%ile = '); writeln(ruin_5mile:6:1);
(* write('2nd ruin m%ile = '); writeln(ruin_2mile:6:1);*)
write('1st ruin %ile = '); writeln(ruin_1ile:6:1);
write('5th ruin %ile = '); writeln(ruin_5ile:6:1);
(* write('2nd ruin %ile = '); writeln(ruin_2ile:6:1); *)
write('RuinDisc[00%] = '); writeln(DCRprobs[0]:8:4);
write('RuinDisc[05%] = '); writeln(DCRprobs[5]:8:4);
write('RuinDisc[20%] = '); writeln(DCRprobs[20]:8:4);

 betaD := betaD + betaINC;
end; {big while betaD}

 betaY := betaY + betaINC;
end; {big while betaY}

 alphaD := alphaD + alphaINC;
end; {big while alpha}

 alphaY := alphaY + alphaINC;
end; {big alpha}
Ypo := Ypo + varINC;
end; {MU}
end.
```

APPENDIX III.2 : Computer Programme for Koutrouvelis's Estimation Method

```
program Koutrouvelis (input, output);
const
    PI = 3.1415927;
    MaxN = 2000;
    KoutK = 10;
    KoutL = 16;
    MaxKL = 32;

type
   data_type = array [1..MaxN] of double;
   Kout_type = array [1..MaxKL] of double;

var
  data, std_data, ss_data : data_type;
  Y,X : Kout_type;
  k, l, n, j, try_count : integer;
  tk, ul : double;
  gamma, gamma1, gamma_guess, gamma_try, alpha, beta : double;
  slope, cut, modPHI, RePHI, ImPHI : double;

procedure ReadInData (var gamma_guess : double; var n : integer;
                 var data: data_type );
var
  j : integer;
begin
   (* writeln('Enter data : gamma_guess, n, random sample'); *)
   (* Assign(Input, 'simulate.out'); reset(Input); *)
   readln(gamma_guess);
   readln (n);
   while not eof do
        for j := 1 to n do
        begin
            (* read(data[j]); *)
            (* if (j mod 5 = 0) then readln; *)
            readln(data[j]);
        end;
end;

function sgn(x:double):double;
begin
   if x > 0 then sgn := +1.0
   else if x < 0 then sgn := -1.0
   else begin
       writeln('### Potential error, call to sgn x with x zero');
       sgn := +1.0;
   end;
end;

function pow(a,b : double) : double;
begin
   pow := exp(b*ln(a));
end;
```

```
function tan(x:double) : double;
begin
   tan := sin(x)/cos(x);
end;

function RE_Characteristic (var sdata : data_type;
                       n : integer; tk : double) : double;
var
  xj : 1..MaxN;
  CSum : double;
begin
   CSum := 0.0;
   for xj := 1 to n do
       CSum := CSum + cos(tk*sdata[xj]);
   RE_Characteristic := 1/n * CSum;
end;

function IM_Characteristic (var sdata : data_type;
                       n : integer; tk : double) : double;
var
  xj : 1..MaxN;
  SSum : double;
begin
   SSum := 0.0;
   for xj := 1 to n do
       SSum := SSum + sin(tk*sdata[xj]);
   IM_Characteristic := 1/n * SSum;
end;

function ValueCharacteristic (var sdata : data_type;
                       n : integer; tk : double) : double;
var
  xj : 1..MaxN;
  CSum, SSum : double;
begin
   CSum := 0.0;
   SSum := 0.0;
   for xj := 1 to n do
   begin
       CSum := CSum + cos(tk*sdata[xj]);
       SSum := SSum + sin(tk*sdata[xj]);
   end;
   CSum := 1/n * CSum;
   SSum := 1/n * SSum;
   ValueCharacteristic := sqrt( CSum*CSum + SSum*SSum );
end;
```

```
procedure Regress (var X, Y : Kout_type;
                var slope, cut : double);
var
  k : 1..KoutK;
  xbar, ybar, Sxx, Sxy : double;
begin
    xbar := 0.0;
    ybar := 0.0;
    for k := 1 to KoutK do
    begin
        xbar := xbar + X[k];
        ybar := ybar + Y[k];
    end;
    xbar := xbar / KoutK;
    ybar := ybar / Koutk;
    Sxx := 0.0;
    Sxy := 0.0;
    for k := 1 to KoutK do
    begin
        Sxx := Sxx + (X[k] - xbar)*(X[k] - xbar);
        Sxy := Sxy + (X[k] - xbar)*(Y[k] - ybar);
    end;
    slope := Sxy/Sxx;
    cut := ybar - xbar*slope;
end;

procedure OORegress (var X, Y : Kout_type;
                var slope : double);
(*  Regression through the Origon (0,0)   *)
(*  Ref : Larson Section 9.1, pg 476      *)
var
  k : 1..KoutL;
  Sxx, Sxy : double;
begin
    Sxx := 0.0;
    Sxy := 0.0;
    for k := 1 to KoutK do
    begin
        Sxx := Sxx + X[k]*X[k];
        Sxy := Sxy + X[k]*Y[k];
    end;
    slope := Sxy/Sxx;
end;
```

APPENDIX IV : TABLES OF SIMULATION RESULTS

IV.1 Sensitivity of the Proposed Alternative Model to Changes in the Location, Scale, Auto-Correlation and Initial Yield Parameters

Table IV.1(a) : Sensitivity of the Model to Changes in the Location Parameters

	Dividend Means					Yield Means				
μ_D	**8%**	**6%**	**4%**	**2%**	**0%**	4%	4%	4%	4%	4%
γ_D	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
μ_Y	5%	5%	5%	5%	5%	**5.5%**	**5.25%**	**5%**	**4.75%**	**4.5%**
A	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
γ_Y	10%	10%	10%	10%	10%	10%	10%	10%	10%	10%
Y_o	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
MED[I(30)]	14.2	12.0	9.9	7.8	5.8	10.4	10.1	9.9	9.7	9.4
MED[G(30)]	13.7	11.5	9.4	7.3	5.3	9.5	9.5	9.4	9.4	9.3
IQR[I(30)]	29.0	28.5	27.9	27.4	26.9	28.0	28.0	27.9	27.9	27.8
IQR[G(30)]	2.7	2.7	2.6	2.6	2.5	2.6	2.6	2.8	2.7	2.8
NZ	264	437	679	1197	2133	600	634	679	732	786
MNZ	.029	.026	.026	.024	.025	.026	026	.026	.026	.0265
$Q_{CR}(0.001)$	15	22	28	36	45	27	27	28	28	29
$Q_{CR}(0.005)$	10	12	15	19	27	14	14	15	16	17
$Q_{CR}(0.01)$	6	7	9	12	18	9	9	9	10	10
$Q_{CR}(0.05)$	1	2	2	4	6	2	2	2	3	3
$Q_{DCR}(0.001)$	9	12	14	18	21	14	14	14	15	15
$Q_{DCR}(0.005)$	5	7	8	11	14	8	8	8	9	9
$Q_{DCR}(0.01)$	4	4	5	7	9	5	5	5	5	6
$Q_{DCR}(0.05)$	1	1	2	2	4	2	2	2	2	2

Table IV.1(b) : Sensitivity of the Model to Changes in the Scale Parameters

	Dividend Scale					Yield Scale				
μ_D	4%	4%	4%	4%	4%	4%	4%	4%	4%	4%
γ_D	**1%**	**3%**	**5%**	**7%**	**9%**	5%	5%	5%	5%	5%
μ_Y	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
A	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
γ_Y	10%	10%	10%	10%	10%	**5%**	**7.5%**	**10%**	**12.5%**	**15%**
Y_o	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
MED[I(30)]	9.6	9.7	9.9	10.2	10.4	9.8	9.9	9.9	10.0	10.2
MED[G(30)]	9.2	9.3	9.4	9.5	9.6	9.3	9.3	9.4	9.5	9.6
IQR[I(30)]	24.9	26.5	27.9	30.1	32.5	16.8	22.3	27.9	33.9	40.1
IQR[G(30)]	1.2	1.8	2.6	3.5	4.3	2.4	2.5	2.6	2.8	3.0
NZ	434	542	679	920	1198	308	463	679	679	1196
MNZ	.015	.019	.026	.030	.035	.036	030	.026	.026	.023
$Q_{CR}(0.001)$	13	17	28	42	47	23	26	28	30	32
$Q_{CR}(0.005)$	8	10	15	23	27	15	15	15	15	16
$Q_{CR}(0.01)$	4	6	9	14	19	7	9	9	10	12
$Q_{CR}(0.05)$	1	2	2	3	5	1	2	2	3	4
$Q_{DCR}(0.001)$	7	8	14	21	23	12	14	14	15	16
$Q_{DCR}(0.005)$	4	5	8	11	14	8	8	8	9	9
$Q_{DCR}(0.01)$	2	4	5	7	9	4	5	5	6	6
$Q_{DCR}(0.05)$	1	1	2	2	3	1	1	2	2	2

Table IV.1(c) : Sensitivity to Changes in Auto-Correlation

μ_D	4%	4%	4%	4%	4%	4%
γ_D	5%	5%	5%	5%	5%	5%
μ_Y	5%	5%	5%	5%	5%	5%
A	**0.0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
γ_Y	10%	10%	10%	10%	10%	10%
Y_o	5%	5%	5%	5%	5%	5%
MED[I(30)]	9.5	9.8	9.8	9.9	10.1	12.6
MED[G(30)]	9.4	9.4	9.4	9.4	9.5	10.4
IQR[I(30)]	34.1	23.1	30.1	27.9	26.7	26.1
IQR[G(30)]	2.5	2.5	2.5	2.6	2.8	3.7
NZ	669	659	672	679	745	760
MNZ	.023	.024	.025	.026	.027	.030
Q_{CR}(0.001)	24	25	27	28	26	34
Q_{CR}(0.005)	14	14	14	15	16	17
Q_{CR}(0.01)	9	9	9	9	10	11
Q_{CR}(0.05)	2	2	2	2	3	3
Q_{DCR}(0.001)	13	13	14	14	14	18
Q_{DCR}(0.005)	8	8	8	8	9	9
Q_{DCR}(0.01)	5	5	5	5	6	6
Q_{DCR}(0.05)	2	2	2	2	2	2

Table IV.1(d) : Sensitivity to Changes in Initial Yields

μ_D	4%	4%	4%
γ_D	5%	5%	5%
μ_Y	5%	5%	5%
A	0.6	0.6	0.6
γ_Y	10%	10%	10%
Y_o	**2.5%**	**5%**	**7.5%**
MED[I(30)]	9.9	9.9	9.9
MED[G(30)]	6.8	9.4	11.0
IQR[I(30)]	27.9	27.9	27.9
IQR[G(30)]	2.5	2.6	2.6
NZ	1291	679	489
MNZ	.020	.026	.029
Q_{CR}(0.001)	31	28	26
Q_{CR}(0.005)	17	15	13
Q_{CR}(0.01)	11	9	8
Q_{CR}(0.05)	3	2	2
Q_{DCR}(0.001)	16	14	13
Q_{DCR}(0.005)	10	8	8
Q_{DCR}(0.01)	6	5	4
Q_{DCR}(0.05)	2	2	2

Appendix IV.2 : Sensitivity of the Model to Changes in the Index Parameters

Table IV.2(a) : Effect of Changes in Index Parameters on *MED[I(30)]*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	11.81	11.01	10.40	9.99	9.55	9.31	9.00	8.88	8.60	8.44	8.25
1.1	12.92	12.03	11.29	10.84	10.40	10.14	9.85	9.59	9.49	9.24	9.09
1.2	13.55	12.74	12.15	11.56	11.01	10.66	10.28	10.24	10.10	9.93	9.82
1.3	13.94	13.11	12.34	11.80	11.33	10.94	10.66	10.48	10.24	10.08	10.07
1.4	14.20	13.16	12.46	11.95	11.45	11.06	10.82	10.65	10.41	10.25	10.11
1.5	14.10	13.15	12.46	11.94	11.40	10.98	10.64	10.38	10.27	10.16	10.10
1.6	14.01	13.19	12.41	11.76	11.39	10.99	10.59	10.37	10.24	10.07	9.99
1.7	13.70	12.79	12.10	11.51	11.08	10.80	10.37	10.15	10.01	9.82	9.82
1.8	13.29	12.47	11.78	11.22	10.82	10.51	10.13	9.88	9.67	9.62	9.47
1.9	12.67	12.07	11.38	10.82	10.43	10.14	9.84	9.55	9.35	9.17	9.14
2.0	12.20	11.64	11.04	10.55	10.12	9.73	9.47	9.18	8.99	8.89	8.79

Table IV.2(b) : Effect of Changes in Index Parameters on *IQR[I(30)]*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	40.70	40.10	39.29	38.42	37.30	36.69	36.40	36.11	35.80	35.23	34.93
1.1	38.80	38.34	37.95	37.07	36.34	36.16	35.61	35.06	34.66	34.18	34.06
1.2	37.78	37.04	36.36	35.69	34.77	34.29	33.92	33.55	33.18	32.78	32.57
1.3	36.11	35.67	34.94	34.40	33.78	33.15	32.68	32.31	31.85	31.40	31.00
1.4	34.81	34.52	34.03	33.37	32.69	32.39	31.89	31.44	30.62	30.37	30.09
1.5	33.50	33.22	32.36	31.94	31.45	31.34	31.04	30.48	29.89	29.62	29.26
1.6	32.40	31.93	31.26	30.92	30.63	30.09	29.89	29.33	28.98	28.55	28.35
1.7	31.62	31.12	30.49	30.18	29.77	29.36	28.94	28.35	28.02	27.74	27.45
1.8	31.10	30.52	29.91	29.51	29.15	28.73	28.21	27.95	27.39	26.97	26.85
1.9	30.20	29.93	29.45	28.73	28.35	27.82	27.43	27.06	26.77	26.47	26.26
2.0	29.69	28.99	28.55	28.02	27.42	26.99	26.63	26.14	25.88	25.66	25.51

Table IV.2(c) : Effect of Changes in Index Parameters on $Q_{I(30)}(0.995)$

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	-98.55	-98.55	-98.55	-98.55	-98.55	-98.55	-98.55	-98.55	-98.55	-98.55	-98.55
1.1	-96.25	-96.25	-96.21	-96.21	-96.14	-96.14	-96.14	-96.14	-96.14	-96.13	-96.10
1.2	-95.48	-95.48	-95.35	-95.14	-95.14	-95.12	-95.06	-95.06	-95.06	-95.06	-94.93
1.3	-94.90	-94.67	-94.29	-94.09	-93.66	-93.64	-92.89	-92.77	-92.78	-92.78	-92.14
1.4	-93.80	-93.02	-92.99	-91.77	-90.42	-89.05	-89.07	-88.11	-88.14	-88.16	-87.29
1.5	-92.40	-91.44	-89.53	-88.37	-86.51	-82.40	-80.01	-80.05	-79.58	-79.63	-79.20
1.6	-91.78	-89.58	-86.31	-83.84	-81.55	-78.65	-73.81	-70.91	-71.03	-69.97	-69.27
1.7	-90.22	-88.99	-83.79	-77.79	-73.76	-72.32	-67.82	-61.98	-61.16	-60.22	-59.05
1.8	-86.40	-84.60	-81.97	-75.93	-68.25	-62.53	-60.00	-57.06	-51.13	-49.68	-46.95
1.9	-85.33	-81.42	-76.55	-70.01	-64.37	-55.96	-50.50	-48.36	-41.53	-38.78	-36.99
2.0	-83.95	-80.35	-73.46	-65.20	-56.28	-49.33	-41.83	-35.85	-33.94	-31.37	-29.56

Table IV.2(d) : Effect of Changes in Index Parameters on $Q_{I(30)}(0.005)$

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	5.6E+9	3.9E+6	3.5E+6	3.4E+6	3.4E+6	3.4E+6	3.4E+6	3.4E+6	3.3E+6	3.3E+6	3.0E+6
1.1	2.5E+6	2.3E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6
1.2	2.9E+5	7.7E+4	4.5E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4
1.3	7.6E+4	6.1E+3	4.5E+3	4.0E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3
1.4	2.3E+4	3.0E+3	1.4E+3	1.3E+3	1.2E+3	1.1E+3	1.1E+3	1.1E+3	1.1E+3	1.1E+3	1.1E+3
1.5	7.2E+3	1.7E+3	8.6E+2	6.6E+2	5.1E+2	5.0E+2	4.9E+2	4.8E+2	4.8E+2	4.9E+2	5.0E+2
1.6	2.9E+3	9.6E+2	5.3E+2	4.2E+2	3.3E+2	2.7E+2	2.7E+2	2.6E+2	2.6E+2	2.7E+2	2.7E+2
1.7	2.2E+3	5.7E+2	4.0E+2	2.8E+2	2.3E+2	2.0E+2	1.8E+2	1.7E+2	1.7E+2	1.7E+2	1.7E+2
1.8	1.5E+3	4.8E+2	2.6E+2	2.0E+2	1.7E+2	1.4E+2	1.4E+2	1.2E+2	1.1E+2	1.1E+2	1.1E+2
1.9	8.8E+2	4.3E+2	2.2E+2	1.5E+2	1.1E+2	1.0E+2	9.2E+1	8.9E+1	8.7E+1	8.4E+1	8.3E+1
2.0	8.7E+2	4.3E+2	2.0E+2	1.2E+2	9.9E+1	8.6E+1	7.9E+1	7.4E+1	7.1E+1	7.0E+1	7.0E+1

Table IV.2(e) : Effect of Changes in Index Parameters on *NZ*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	1980	2109	2175	2198	2199	2185	2147	2100	2079	2049	1038
1.1	1854	2006	2062	2081	2064	2075	2036	1975	1932	1902	1885
1.2	1755	1897	1959	1961	1929	1893	1853	1797	1779	1746	1709
1.3	1660	1774	1814	1820	1783	1713	1655	1588	1555	1521	1475
1.4	1545	1680	1699	1668	1626	1551	1471	1396	1362	1316	1267
1.5	1421	1530	1558	1529	1451	1386	1321	1240	1186	1125	1076
1.6	1317	1392	1389	1346	1278	1185	1110	1003	941	885	846
1.7	1208	1267	1263	1206	1129	1007	915	812	740	667	623
1.8	1111	1179	1156	1082	980	847	745	630	552	493	422
1.9	1057	1088	1052	977	862	728	617	496	392	314	254
2.0	1030	1010	975	878	748	598	464	345	242	165	82

Table IV.2(f) : Effect of Changes in Index Paramecters on *MNZ*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	7.90	7.43	6.78	6.12	5.53	5.05	4.69	4.46	4.23	4.07	4.10
1.1	7.34	6.80	6.18	5.47	4.87	4.27	3.86	3.63	3.42	3.24	3.02
1.2	7.14	6.57	5.87	5.15	4.52	3.95	3.49	3.22	2.94	2.75	2.54
1.3	7.27	6.72	6.00	5.15	4.43	3.85	3.36	3.07	2.79	2.56	2.34
1.4	7.54	6.81	6.08	5.27	4.45	3.76	3.25	2.94	2.60	2.36	2.11
1.5	7.93	7.17	6.29	5.38	4.58	3.74	3.09	2.72	2.37	2.12	1.83
1.6	8.33	7.61	6.73	5.76	4.80	3.94	3.15	2.75	2.33	2.01	1.63
1.7	8.86	8.12	7.12	6.09	5.06	4.20	3.33	2.79	2.29	1.89	1.45
1.8	9.44	8.50	7.51	6.49	5.49	4.59	3.63	3.01	2.37	1.79	1.26
1.9	9.75	9.03	8.01	6.89	5.90	4.95	3.93	3.22	2.56	1.84	1.00
2.0	9.80	9.52	8.41	7.39	6.45	5.62	4.74	3.98	3.20	2.16	0.47

Table IV.2(g) : Effect of Changes in Index Parameters on Q_{CR} *(0.001)*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	86	86	86	86	86	86	86	86	86	86	86
1.1	80	79	79	79	80	80	80	80	80	80	80
1.2	57	57	57	57	56	53	49	43	33	23	18
1.3	56	56	55	54	53	51	47	41	31	21	17
1.4	53	53	52	52	50	50	47	39	29	18	17
1.5	53	53	52	52	49	49	45	38	28	18	16
1.6	52	52	52	49	49	47	43	35	26	18	14
1.7	51	51	50	49	47	44	42	34	25	17	13
1.8	50	50	51	48	47	44	40	32	23	17	12
1.9	50	50	51	48	46	42	37	30	21	15	8
2.0	49	49	50	48	46	42	36	27	19	10	3

Table IV.2(h) : Impact of Changes in Index Parameeters on Q_{CR}*(1%)*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	44	43	41	37	33	32	27	22	19	18	16
1.1	39	38	35	32	27	23	20	18	17	15	14
1.2	36	36	34	31	26	22	18	16	14	14	13
1.3	36	35	33	30	26	21	17	15	14	13	12
1.4	35	34	33	30	26	21	17	14	13	12	11
1.5	35	35	33	31	26	21	16	13	12	10	9
1.6	35	35	32	30	26	20	15	12	11	9	6
1.7	35	35	33	30	26	20	15	11	9	6	5
1.8	35	34	33	30	27	21	14	11	7	5	4
1.9	34	34	33	30	27	21	15	10	6	4	2
2.0	34	34	33	30	27	21	15	10	5	2	1

Table IV.2(i) : Impact of Changes in Index Parameeters on Q_{CR}*(5%)*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	20	19	17	15	14	13	11	10	10	9	8
1.1	18	18	16	14	13	11	9	8	7	6	6
1.2	18	17	15	13	11	9	8	7	6	6	5
1.3	18	17	14	12	10	8	7	6	5	5	4
1.4	18	17	14	12	10	7	6	5	5	4	4
1.5	17	16	14	11	9	7	5	4	4	4	3
1.6	17	17	13	11	8	6	4	4	3	3	3
1.7	17	16	13	11	8	5	4	3	3	3	2
1.8	17	16	13	10	7	5	3	2	2	2	1
1.9	17	16	13	10	7	4	3	2	1	1	1
2.0	16	16	13	10	6	3	2	1	1	1	1

Appendix IV.3 : Sensitivity of The Model to Changes in the Skewness Parameters

Table IV.3(a) : Values of *MED[I(30)]* with different skewness parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	9.59	9.81	10.18	10.59	10.83
-0.5	8.95	9.21	9.58	9.92	10.30
0.0	8.31	8.60	9.01	9.37	9.76
+.5	7.80	8.13	8.45	8.73	9.10
+1.0	7.11	7.45	7.79	8.11	8.43

Table IV.3(b) : Values of *IQR[I(30)]* with different skewness parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	27.00	27.28	27.47	27.41	27.17
-0.5	27.82	27.91	27.92	27.93	27.72
0.0	27.84	28.10	28.12	28.22	27.82
+.5	27.23	27.64	27.71	27.73	27.52
+1.0	26.51	26.61	26.86	26.64	26.35

Table IV.3(c) : Values of $Q_{I(30)}$ *(.005)* with different skewness parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	162.36	158.75	145.68	112.99	98.76
-0.5	171.41	164.40	151.93	143.37	131.26
0.0	190.90	180.16	165.56	160.06	160.00
+.5	185.46	177.15	166.03	159.74	160.97
+1.0	199.21	194.30	177.90	155.26	156.34

Table IV.3(d) : Values of $Q_{I(30)}$ (.995) with different skewness parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	-55.47	-58.95	-59.17	-60.16	-60.70
-0.5	-53.85	-55.58	-56.01	-55.95	-58.74
0.0	-51.24	-53.41	-53.60	-56.15	-58.18
+.5	-48.24	-48.76	-50.43	-53.25	-55.29
+1.0	-42.65	-44.96	-46.85	-47.87	-51.94

TableIV.3(e) : Claim Frequency Rates with different Skewness Parameters (Number NZ of Claims per 5000 Simulations)

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	691	737	747	753	751
-0.5	613	660	672	679	680
0.0	481	540	572	580	584
+.5	357	415	448	468	491
+1.0	161	243	283	311	337

Table IV.3(f) : Mean Claim Severities MNZ with different Skewness Parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	1.41	1.84	2.18	2.50	2.78
-0.5	1.29	1.79	2.23	2.59	2.89
0.0	1.22	1.85	2.32	2.74	3.08
+.5	0.98	1.85	2.46	2.95	3.23
+1.0	0.51	2.12	3.06	3.70	4.06

Table IV.3(g) : Reserving Requirements $Q_{CR}(0.001)$ with different Skewness Parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	13	17	19	23	29
-0.5	13	17	19	28	36
0.0	10	17	22	30	38
+.5	9	15	24	32	38
+1.0	4	11	25	33	39

Table IV.3(h) : Reserving Requirements $Q_{CR}(1\%)$ with different Skewness Parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	5	7	9	10	11
-0.5	4	6	8	9	11
0.0	4	5	7	9	11
+.5	3	5	6	8	10
+1.0	3	3	6	7	10

Table IV.3(i) : Reserving Requirements $Q_{CR}(5\%)$ with different Skewness Parameters

β_D / β_Y	-1.0	-0.5	0.0	+.5	+1.0
-1.0	2	2	3	3	3
-0.5	2	2	2	2	3
0.0	2	2	2	2	2
+.5	1	1	2	2	2
+1.0	1	1	1	1	1

Appendix IV.4 : Simulation Investigations into the Differences between the Alternative Models

Table IV.4(a) : Comparison of the Alternative Models

	M	A	F	Z
MED[I(30)]	8.6	9.9	10.1	9.6
IQR[I(30)]	37.8	27.9	30.2	27.8
$Q_{I(30)}(0.995)$	-42.1	-56.0	-65.7	-54.3
$Q_{I(30)}(0.005)$	109.4	143.4	202.0	140.3
MED[G(30)]	9.3	9.4	9.5	9.4
IQR[G(30)]	3.5	2.6	2.5	2.6
$Q_{G(30)}(0.995)$	2.9	1.1	-0.2	1.3
$Q_{G(30)}(0.005)$	16.2	20.9	30.7	18.2
NZ	799	679	895	601
MNZ	.012	.026	.029	.026
$Q_{CR}(.001)$	12	28	30	28
$Q_{CR}(.005)$	7	15	18	14
$Q_{CR}(0.01)$	5	9	12	8
$Q_{CR}(0.05)$	2	2	3	2
$Q_{DCR}(.001)$	7	14	16	15
$Q_{DCR}(.005)$	4	8	11	8
$Q_{DCR}(0.01)$	3	5	7	5
$Q_{DCR}(0.05)$	2	2	2	2

Table IV.4(b) : Simulation Investigation into the Relative Impact of Individual Parameter Changes

α_D	β_D	γ_D	α_Y	β_Y	γ_Y	NZ	MNZ	$Q_{CR}(.001)$	$Q_{CR}(1\%)$	$Q_{CR}(5\%)$
1.75	0.5	0.05	1.75	(0.5)	0.10	679	0.026	28	9	2
1.75	0.5	0.05	1.75	(0.5)	0.1414	1106	0.023	31	11	3
1.75	0.5	0.05	1.75	0	0.1414	959	0.023	33	10	3
1.75	0.5	0.05	2.00	0	0.1414	493	0.026	25	8	2
1.75	0.5	0.0919	2.00	0	0.1414	1124	0.036	44	20	5
1.75	0	0.0919	2.00	0	0.1414	1167	0.030	37	17	4
2.00	0	0.0919	2.00	0	0.1414	799	0.011	12	5	2

α_D	β_D	γ_D	α_Y	β_Y	γ_Y	MED[I(30)]	IQR[I(30)]	$Q_{I(30)}(.995)$	$Q_{I(30)}(.005)$
1.75	0.5	0.05	1.75	(0.5)	0.10	9.9	27.9	-56.0	143.4
1.75	0.5	0.05	1.75	(0.5)	0.1414	10.2	38.0	-68.3	226.8.
1.75	0.5	0.05	1.75	0	0.1414	9.4	37.9	-65.9	270.0
1.75	0.5	0.05	2.00	0	0.1414	8.9	35.2	-43.9	104.0
1.75	0.5	0.0919	2.00	0	0.1414	9.5	39.2	-50.6	121.1
1.75	0	0.0919	2.00	0	0.1414	8.8	39.1	-48.7	138.0
2.00	0	0.0919	2.00	0	0.1414	8.6	37.8	-42.1	109.4

α_D	β_D	γ_D	α_Y	β_Y	γ_Y	MED[G(30)]	IQR[G(30)]	$Q_{G(30)}(.995)$	$Q_{G(30)}(.005)$
1.75	0.5	0.05	1.75	(0.5)	0.10	9.4	2.6	1.1	20.9
1.75	0.5	0.05	1.75	(0.5)	0.1414	9.5	2.9	1.2	27.4
1.75	0.5	0.05	1.75	0	0.1414	9.5	2.9	1.3	23.1
1.75	0.5	0.05	2.00	0	0.1414	9.4	2.6	1.8	15.4
1.75	0.5	0.0919	2.00	0	0.1414	9.6	4.4	-1.7	19.8
1.75	0	0.0919	2.00	0	0.1414	9.4	4.4	-0.8	25.5
2.00	0	0.0919	2.00	0	0.1414	9.3	3.5	2.9	16.2

Table IV.4(c) :
Relative Impact of Changing Individual Parameters from Model M to Model A Values

Parameter Change	$Q_{CR}(.001)$	$Q_{CR}(0.01)$	$Q_{CR}(0.05)$	NZ	MNZ
γ_Y	-3	-2	-1	-427	0.3
β_Y	-2	+1	0	+147	0.0
α_Y	+8	+2	+1	+466	-0.3
γ_D	-19	-12	-3	-631	-1.0
β_D	+7	+3	+1	-43	+.6
α_D	+25	+12	+2	+368	+1.9
TOTAL	+16	+4	0	-120	+1.5

Parameter Change	MED[I(30)]	IQR[I(30)]	$Q_{I(30)}(.995)$	$Q_{I(30)}(.005)$
γ_Y	-0.3	-10.1	+12.3	-83.4
β_Y	+0.8	+0.1	-2.4	-43.2
α_Y	+0.5	+2.7	-22.0	+166.0
γ_D	-0.6	-4.0	+6.7	-17.1
β_D	+0.7	+0.1	-1.9	-16.9
α_D	+0.2	+1.3	-6.6	+28.6
TOTAL	+1.3	-9.9	-13.9	+34

Parameter Change	MED[G(30)]	IQR[G(30)]	$Q_{G(30)}(.995)$	$Q_{G(30)}(.005)$
γ_Y	-0.1	-0.3	-0.1	-6.5
β_Y	0.0	0.0	-0.1	+4.3
α_Y	+0.1	+0.3	-0.5	+7.7
γ_D	-0.2	-1.8	+3.5	-4.4
β_D	+0.2	0.0	-0.9	-5.7
α_D	+0.1	0.9	-3.7	+9.3
TOTAL	+0.1	-0.9	-1.8	+4.7

Appendix IV.5 : Simulation Investigations into Model Behaviour with Different Liability Portfolios

Table IV.5(a) : Simulation Results for the MGWP Model with Different Liability Portfolios (gross investor)

Portfolio	All at Term 5 (1)	All at Term 10 (2)	All at Term 20 (3)	All at Term 30 (4)	1 each at 10,15,...30 (5)	1 each at 10,11,...30 (6)	1 each at 1...10 (7)	1 each at 1...20 (8)
NZ	965	307	39	4	419	799	3807	3938
MNZ	13.4	13.1	11.5	7.5	1.8	0.9	2.4	0.9
$Q_{CR}(.001)$	47	48	27	1	10	11	25	15
$Q_{CR}(.005)$	38	32	5	1	6	5	18	10
$Q_{CR}(0.01)$	33	23	1	1	5	4	16	8
$Q_{CR}(0.05)$	18	5	1	1	2	2	9	4
$Q_{DCR}(.001)$	39	33	13	1	6	6	20	10
$Q_{DCR}(.005)$	31	22	3	1	4	3	14	7
$Q_{DCR}(0.01)$	28	16	1	1	3	3	12	6
$Q_{DCR}(0.05)$	15	4	1	1	2	1	7	3

Table IV.5(b) : Simulation Results for the Alternative model with Different Liability Portfolios (gross investor)

Portfolio	All at Term 5 (1)	All at Term 10 (2)	All at Term 20 (3)	All at Term 30 (4)	1 each at 10,15,...30 (5)	1 each at 10,11,...30 (6)	1 each at 1...10 (7)	1 each at 1...20 (8)
NZ	1620	164	47	27	306	679	3122	3332
MNZ	15.1	19.8	37.9	36.7	5.1	2.3	2.0	1.0
$Q_{CR}(.001)$	82	76	80	68	26	23	38	29
$Q_{CR}(.005)$	49	40	37	6	16	13	21	18
$Q_{CR}(0.01)$	38	23	1	1	11	7	17	9
$Q_{CR}(0.05)$	11	1	1	1	2	3	7	4
$Q_{DCR}(.001)$	68	52	37	22	13	11	30	17
$Q_{DCR}(.005)$	41	27	17	2	8	7	16	11
$Q_{DCR}(0.01)$	31	16	1	1	6	4	13	6
$Q_{DCR}(0.05)$	9	1	1	1	1	2	6	3