Collective Effects in Multi-field Inflation

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Plagiarism Declaration

I, Sultan Hassan, know the meaning of plagiarism and declare that all of the work in the thesis, save for that which is properly acknowledged, is my own.

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Abstract

We present a new model of multi-field inflation in the limit when, N, the number of fields is very large. To implement this limit, we reformulate the problem in terms of a colourless bilocal field $\sigma(x, y)$ which encodes the collective degrees of freedom of the N scalar fields. As a concrete example, we apply the collective field theory formalism to the bosonic O(N) vector model with quartic self interaction minimally coupled to gravity and show how this may be used to model the quantum to classical transition out of inflation.

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Contents

To truth I always seek

Chapter 1

Introduction

The standard model of cosmology presents a historical scenario of the universe at different epochs and the evolution of the universe through to the present day. Despite the success of the standard model in explaining the observed universe, there still remain several problems that require a precise fine tuning of cosmological parameters. These problems include the flatness, horizon and monopole problems [1].

According to the standard model of cosmology, the Hubble parameter, H, is proportional to the dominant form of energy density, ρ, in the universe via Friedmann equation, $H^2 \propto \rho$, which comes from Einstein field Equations (EFEs). Because H decreases as the universe expands, ρ evolves away from the critical energy density, ρ_c, which is the energy density required for a flat universe. Current observations [1–4] find that the present universe possesses a nearly critical energy density $\rho \simeq \rho_c$. For the universe to be close to flat today, it is necessary that the universe must have been incredibly close to flat at early times. This is the so-called flatness problem.

Observations of the cosmic microwave background (CMB) [2–6] find that the universe is almost homogeneous, to an accuracy of one part in ten-thousand. For the opposite sides of the sky to be in thermal equilibrium, they must have been in causal contact at some time in the past. Nevertheless, the standard model of cosmology suggests that the initial universe consisted of many causally disconnected regions. This is known as the horizon problem.

The particle spectra of Grand Unified Theories (GUTs), in which a gauge symmetry breaks during a phase transition, predict the existence of heavy monopoles [1, 2]. These monopoles should come to dominate the energy density of the universe. The lack of observed monopoles is known as the monopole problem.

Any solution of these three problems that makes the dynamics of the universe seem more natural without excessive fine tuning of initial conditions would clearly be very desirable.

To date, inflation has been the most elegant resolution of these problems. Inflation is a period of superluminal growth in the early universe. Superluminal inflation is not in conflict with the special theory of relativity, since one may not send signals via space-time expansion [7]. Inflation implies a positive acceleration of the scale factor ($\ddot{a}(t) > 0$), which in turn implies an equation of state with a negative pressure ($p < 0$). No type of matter possesses this characteristic, so often a scalar field is chosen to drive inflation. The scalar field responsible for inflation is called the inflaton.

Inflation may take two causally connected regions and rapidly separate them so that they are removed from each others past light cone. Some time after inflation ends, the light cones of these causally disconnected regions intersect again, taking care of the horizon problem. In addition, inflation resolves the monopole problem by diluting unwanted relic particles to observably small levels. Finally, the resolution of the flatness problem comes from the fact that, during inflation, H increases rapidly so that ρ approaches ρ_c. When inflation ends, ρ begins to move away from ρ_c. As long as inflation lasts long enough, ρ will be close to ρ_c in the present universe [2, 7].

Inflation explains the origin of the large scale structure of the universe as the result of quantum fluctuations in the inflaton field. These perturbations in the inflaton field produce the density perturbations required for structure formation in the present universe [1]. In other words, inflation fills the evolutionary gap between the initial quantum universe and the current classical universe.

Modelling inflation is a subtlety that has puzzled cosmologists for several decades. Presently, there are several competing models which successfully explain the dynamics of inflation [2, 8]. The most commonly studied models consider a single inflaton driving inflation. Single Field Inflationary Models (SFIMs) are convenient in the simplicity of their form. SFIMs predict a nearly scale invariant Gaussian spectrum for the CMB, which is in good agreement with current observations [2–6]. The detection of any non-Gaussianity in the CMB would severely constrain SFIMs. Also, SFIMs produce only adiabatic perturbation modes and no isocurvature (entropy) modes. Isocurvature modes are a natural consequence of the presence of multiple interacting fields which lead to large non-Gaussianities in the CMB. To date, only a single scalar particle has been detected, which has properties similar to those of the Higgs boson such as zero spin and positive parity [9].

Most theories of high energy physics, including GUTs and string theories, deal with a large number of fields, moduli and dimensions [10]. Because inflation occurred at a relatively high energy scale, multiple fields may have collectively participated in driving inflation. Another motivation to study multi-field inflation is the possible detection of non-Gaussianities in the CMB [2, 11, 12], which, in single field inflation, is deemed below detectable threshold. Theoretically, when the number of fields becomes very large, the potential energy becomes much greater than the kinetic energy and the slow roll condition is spontaneously satisfied [1, 2, 11]. In this context, multi-field inflation may be more natural than single field models.

Several models of multi-field inflation have been studied in the literature. We pay attention to those models in which the main motivation is to achieve the desired number of e-foldings (~ 70). Doing so solves the problems of the standard model of cosmology without the need for fine tuning.

One early multi-field inflation model is *assisted inflation*, in which inflation is driven by a cooperative dynamic of multiple scalar fields with similar exponential potentials [13]. In [13], the slow roll approximation is circumvented by obtaining a ratio between the scalar fields that turns out to be itself a slow roll parameter. In this case, the scalar fields are redefined and rescaled according to their ratio in order to absorb the slow roll approximation into new definitions of the scalar fields. Another interesting model is *N-flation* [8]. N-flation assumes a collection of axions with periodic potentials which, in a leading order expansion, gives rise to the potential $m^2\phi^2$ of chaotic inflation. This model [8] applies the slow roll conditions spontaneously by expressing them in terms of the number of fields N which is a very large number. In this string theory motivated model [8], N is very large due to string compactifications. *Large N cosmology* [11] is another model in which the slow roll condition is applied automatically according to a generic form of the inflatons potential energy that dominates over the kinetic energy in the large N limit. In addition, in [11], the set of N differential equations is significantly reduced to two solvable coupled differential equations of certain gauge invariant variables, which are sufficient to obtain the adiabatic and entropy perturbation.

In 2008, S. Weinberg introduced the Effective Field Theory of Inflation (EFTI) [14]. EFTI attempts to establish a generic theory of inflation, which is one of the main aims of the present work. EFTI begins with the Einstein-Hilbert lagrangian which takes the following form

$$\mathcal{L} = \sqrt{g}\left[-\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_c\partial_\nu\phi_c - V(\phi_c)\right] , \tag{1.1}$$

where $\sqrt{g}$, R, M_P, ϕ_c, $V(\phi_c)$ represent the determinant of the metric tensor $g^{\mu\nu}$, the Ricci scalar, the reduced Planck mass, the scalar field and the potential energy respectively.

Note that the subscript ($_c$) runs from 1 to the total number of scalar fields N. EFTI considers Equation (1.1) as the approximate form of the lagrangian of the generic theory of the inflation. To correct the theory, higher derivative terms must be added, which cannot be neglected at high energy scales [14–16]. Once these corrections have been considered, cosmological perturbation methods [12] may be applied, at the level of the action.

Similar to other models of multi-field inflation, EFTI does not allow large non-Gaussianities. Furthermore, EFTI requires a careful constraint of various unknown functions that are combined with the correction terms [14–16]. Given such problems, it is worthwhile to think of different models of multi-field inflation to build a generic theory of inflation.

In this work, we revisit and study, with different techniques and more powerful tools, the dynamics of inflation which offers a clear picture of how the quantum-to-classical transition of early cosmological fluctuations occurred [17]. Such a transition is important to consider because the fluctuations in the CMB are observed as classical fluctuations. At present, no quantum nature of the CMB fluctuations has been detected [2–6]. This implies that inflation transformed these fluctuations from a quantum origin to a classical measure. This idea was investigated by Kiefer and Polarski [17], whose work suggested two reasons for the occurrence of such a transition. Firstly, the rapid expansion of the early universe hugely squeezed the quantum state of these fluctuations. Secondly, these quantum states interacted with their environments, which is known as quantum decoherence.

In 1973, G. 't Hooft considered a $1/N$ expansion of a large N quantum field thoery to determine the classical limit of the quantum systems [18]. Above, we argued that inflation can be defined as "the quantum-to-classical transition". This statement could be physically equivalent to "the classical limit of the quantum systems". Given such an equivalence, the large $1/N$ expansion becomes a robust approach to drive multi-field inflation from the initial quantum fluctuations up to the classical fluctuations present in the CMB.

Multi-field inflation is a system of many interacting scalar fields that can be solved as a problem of many-body physics. Investigating the collective effects of a many-body system is an efficient approach to address the complex problems that appear as a result of the presence and interactions between the constituents of the system.

We propose a very different and interesting model of multi-field inflation which, as far as we aware, has never been considered for either inflation or cosmology as a whole. We consider inflation driven by multiple fields, that collectively behave as an effective single field. We claim that inflation is collectively driven by a bilocal field $\sigma(x, y)$ where

$x \equiv (t_x, \vec{x})$ and $y \equiv (t_y, \vec{y})$. The bilocal field is a collective combination of scalar fields $\phi(x)$ defined as

$$\sigma(x,y) = \sum_{a=1}^{N} \phi^a(x)\phi^a(y)\,. \tag{1.2}$$

Historically, the idea of bilocality was inspired by the idea of non-locality, introduced in 1949 by H. Yukawa [19, 20]. Yukawa introduced non-locality to solve the divergence and infinity problems of the quantum theory of elementary particles, particularly for his theory of mesons. Yukawa found that, taking into account the radius of the particle and defining the local fields relative to the centre of mass coordinates to be non-local ones, he could write down a consistent effective quantum theory [19, 20].

Since then, several works [21, 22] have studied bilocal fields in details and have obtained an equation of motion and conservation laws [23] as a generalization of those of the local field theory. Interestingly, Tomonaga [24] used the collective picture of many body physics to obtain a linear field equation which is exactly solvable. This method easily solves complicated problems that arise from considering each field separately. Such methods are used, for example, in the treatment of a dense electron gas as a plasma [25–27]. The works of [25–27] were the first to propose the idea of studying the collective dynamics of quantum systems. The generalizations of [25–27] led to the *collective field theory* [28]. Collective field theory describes the dynamics of quantum systems in terms of their gauge invariant variables after applying a set of canonical transformations. As a consequence of these transformations, an effective Hamiltonian emerges, the large N limit of which is the classical approximation of the quantum theory [28].

The outline of this thesis is as follows: Chapter 2 presents a review of multi-field inflation in which the main features of "Large N Cosmology" [11] and "N-flation" [8] models are summarized as the main motivations for the present work. Chapter 3 contains a full description of the collective field theory and computations of the Schwinger Dyson equations, which are the equations of motion for the bilocal field $\sigma(x,y)$ of the O(N) vector model with quartic self interaction. This vector model will be minimally coupled to gravity. Cosmological perturbations are computed order by order to investigate the dynamics of bilocal field $\sigma(x,y)$ in chapter 4. The last chapter is devoted to a summary of the conclusion and a discussion of potential future research.

Chapter 2

Multi-field inflation

Multi-field inflation possesses several theoretical and observational advantages over single field inflation as a natural generalization and modification of single field models. In this chapter, we briefly review two of multi-field inflation models. Two such multi-field models, namely the "Large N Cosmology" [11] and "N-flation" [8] were the motivating ideas behind the present work. Section 2.1 describes "Large N Cosmology" [11] in which N scalar fields drive inflation and Section 2.2 describes "N-flation" [8] in which N pseudo-scalars drive inflation.

2.1 N scalar fields driving inflation

Single field inflation models are well studied due to the relative simplicity of the calculations involved. As the number of fields N increases, the probability of interaction between inflaton fields becomes very large and the computations become more complicated. In "Large N cosmology" [11], it was demonstrated that, under certain assumptions, a system of N coupled equations can be easily reduced to two solvable coupled equations of specific variables. These variables are gauge invariant variables, such that they are independent of the choice of coordinate system, as well as being observables [12]. The solution of the reduced system is sufficient to obtain the adiabatic and the entropy perturbations.

To begin, consider an N scalar field model whose Lagrangian, with a general potential $V(\phi)$, is given by

$$\mathcal{L} = \sum_I \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_I \partial_\nu \phi_I - V(\phi)\,. \tag{2.1}$$

The background evolution is completely specified by Einstein field equations (EFEs) of a homogeneous and isotropic universe, in addition to the Klein-Gordon equations

(KGEs) which yield the equation of motion of each scalar field. In such universe, the geometry of space-time can be represented by Robertson-Walker metric in the flat case (zero curvature, i.e. $k = 0$)

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j = a^2(d\eta^2 - \delta_{ij}dx^i dx^j), \tag{2.2}$$

where t represents the physical time and η stands for the conformal time, the latter is linked to the former via the scale factor $a(t)$ as $dt = ad\eta$. Matter can be described by the energy momentum tensor of a perfect fluid as follows

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}\,, \tag{2.3}$$

where ρ, p and u_μ are the energy density, the pressure and velocity of the fluid respectively. From the action of the theory, the energy momentum tensor of N scalar fields can be written as

$$T^\mu_\nu = \sum_I g^{\mu\nu}\partial_\mu\phi_I\partial_\nu\phi_I - \delta^\mu_\nu\mathcal{L}\,. \tag{2.4}$$

The EFEs and KGEs for the background are

$$\rho = \sum_I \frac{1}{2}\dot{\phi}_I^2 + V = 3H^2, \tag{2.5}$$

$$p = \sum_I \frac{1}{2}\dot{\phi}_I^2 - V = -2\dot{H} - 3H^2, \tag{2.6}$$

$$\ddot{\phi}_I + 3H\dot{\phi}_I + V_I = 0\,, \tag{2.7}$$

where the time derivative is represented by an over-dot, the Hubble parameter is defined as $H \equiv \frac{\dot{a}}{a}$, and $V_I \equiv \frac{\partial V}{\partial \phi_I}$.

To inflate the universe, according to Equation (2.5) and Equation (2.6), one requires a negative pressure ($p < 0$) that causes the acceleration of the scalar factor $\ddot{a}(t)$ to be positive. This condition is satisfied when the potential energy $V(\phi)$ dominates over the kinetic energy $\sum_I \frac{1}{2}\dot{\phi}_I^2$. This is known as the slow roll condition: $H^2 \gg \dot{H}$. In the large N limit, this condition is implicitly satisfied since a generic form of $V(\phi)$ may contain N sums over N fields while the kinetic energy term involves only one sum over N fields, therefore $V(\phi)$ is proportional to N^N and would definitely be dominant over N which is the order of the kinetic energy term.

2.1.1 The inflationary perturbations

Inflation enhances the generation of the early cosmological perturbations to evolve rapidly towards a classical picture. These perturbations can be split into two parts:

adiabatic perturbations and entropy perturbations.

2.1.1.1 Adiabatic perturbations

Adiabatic perturbations arise when the equations of motion of the curvature perturbation have a vanishing source term. This indicates that the baryonic number density contains no fluctuations [12]. Furthermore, the main source of the adiabatic perturbation is the energy density perturbation, which is responsible for inhomogeneities in the spatial curvature along the direction of the background evolution [29].

For an N scalar field theory, the perturbed metric (in Newtonian gauge with equal fluctuations in space and time) takes the following form

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 - 2\Phi)\delta_{ij}dx^i dx^j \,. \tag{2.8}$$

From this, one can determine the perturbed EFEs and KGEs to obtain a system of three equations, similar to the background system (Equation (2.5),(2.6),(2.7)), that governs the dynamics of the perturbations. For the sake of simplicity, we may reformulate the system of equations in terms of the Sasaki-Mukhanov variables Q_I

$$Q_I \equiv \delta\phi_I + \frac{\dot{\phi}_I}{H}\Phi \,, \tag{2.9}$$

where the Q_I are gauge invariant variables that can be canonically quantized [11]. Interestingly, all the important information and the expressions can be written down in terms of two specific gauge invariant variables, defined as

$$\kappa = \sum_I \dot{\phi}_I Q_I, \quad \mu = \sum_I V_I Q_I. \tag{2.10}$$

From this definition, the comoving curvature ζ and the metric fluctuation Φ can be expressed in terms of κ and μ as

$$\zeta = -\frac{H\kappa}{2\dot{H}}, \quad \Phi = -\frac{a^2}{2k^2}\left\{\dot{\kappa} + \left(6H + \frac{\dot{H}}{H}\right)\kappa + 2\mu\right\}. \tag{2.11}$$

2.1.1.2 The entropy perturbations

In contrast to adiabatic perturbations, entropy perturbations concern the source term in the equation of motion of the curvature fluctuations. This implies fluctuations in the baryonic number density. As a result, the matter exhibits a non-uniform spatial distribution. Entropy perturbation emerges as a phenomenon of a system of multiple fields, because such systems make interactions between the fields possible. Single field models produce small amounts of entropy perturbation and therefore we may not be able to measure them. In multi-field models, however, the entropy perturbations grow in a direction perpendicular to the background evolution [29], and play a crucial role in seeding the primordial perturbations that lead to structure formation.

The total entropy perturbation is given by

$$\mathcal{S} = H\left(\frac{\delta p}{\dot{p}} - \frac{\delta \rho}{\dot{\rho}}\right), \tag{2.12}$$

and can be written in terms of κ and μ as

$$\mathcal{S} = -\frac{2(\ddot{H} + 3H\dot{H})\mu + (\ddot{H} + 6H\dot{H})\left[\dot{\kappa} + \frac{\dot{H}+3H^2}{H}\kappa\right]}{6\dot{H}(\ddot{H} + 3H\dot{H})}. \tag{2.13}$$

Equation (2.11) and Equation (2.13) represent the adiabatic and the entropy perturbation in terms of κ and μ respectively. Moreover, from the canonically quantized formalism that Q_I has, we can compute the correlation functions of these perturbations in terms of κ and μ as well. Now, the complicated computations of N inflatons in terms of the Sasaki-Mukhanov variables Q_I can be simplified and written as two coupled differential equations. In particular, the system simplifies quite significantly with the imposition of additional symmetries.

2.1.2 N-fields inflation with $SO(N)$ symmetry

Imposing an $SO(N)$ symmetry restricts the potential energy to the following form

$$V = V(B), \; B \equiv \sum_I \phi_I^2 . \tag{2.14}$$

According to Equation (2.7), we can write the equation of motion of the background in terms of B as the following

$$\ddot{B} + 3H\dot{B} + 4B\frac{dV(B)}{dB} + 4\dot{H} = 0. \tag{2.15}$$

Equation (2.5) and Equation (2.6) may be solved to find an expression of V in terms of H, such that the background dynamics can completely be specified by only B and H for a given set of initial conditions. Since the summation over N is inherent in B by definition, the background evolution does not depend on N. As we have mentioned regarding the adiabatic perturbation, we can obtain a system of three equation that fully describes the dynamics of the perturbation from the perturbed EFEs and KGEs. After applying the change of variables from Φ and ϕ to κ and μ via Q_I, we can write the perturbation equations as two coupled differential equations for κ and μ as the following [11]

$$\ddot{\kappa} + 9H\dot{\kappa} + \left(\frac{k^2}{a^2} + 3\dot{H} + 18H^2 - 2\frac{\dot{H}^2}{H^2} + \frac{\ddot{H}}{H}\right)\kappa + 2\dot{\mu} + \left(6H - 2\frac{\dot{H}}{H}\right)\mu = 0, \tag{2.16}$$

$$\ddot{\mu} + (3H - 2u)\dot{\mu} + \left(\frac{k^2}{a^2} + 6\dot{H} + \frac{\ddot{H}}{H} + u^2 - 3Hu - \dot{u} + 4B\frac{d^2V}{dB^2}\right)\mu - 4\frac{dV}{dB}\dot{\kappa} - \left\{12H\frac{dV}{dB} - \frac{4}{H}B\left(\frac{dV}{dB}\right)^2 - \frac{1}{H^2}(\dot{H} + 3H^2)(\ddot{H} + 6H\dot{H})\right\}\kappa = 0. \tag{2.17}$$

where,

$$u = \frac{d}{dt}\log\left(\frac{dV}{dB}\right) = \frac{\frac{d^2V}{dB^2}(\ddot{H} + 6H\dot{H})}{\left(\frac{dV}{dB}\right)^2}. \tag{2.18}$$

To solve these coupled equations for κ and μ, we need to consider a particular form of the potential energy V, which obeys the $SO(N)$ symmetry such as

$$V(B) = \frac{1}{2}m^2B\,, \tag{2.19}$$

for the massive fields and

$$V(B) = \frac{\lambda}{4}B^2. \tag{2.20}$$

as in ϕ^4 theory. Having defined the potential energy, we may now determine the exact form of κ and μ, which allows us to compute the adiabatic and entropy perturbation through Equation (2.11) and Equation (2.13).

To this end, we realized that applying the large N limit simplifies the complicated system of equations and, in the next section, we will see how inflation occurs naturally and slow roll condition is a natural consequence of such a limit.

2.2 N-axions driving inflation

The amount of inflation, which is the number of e-foldings of the exponential expansion, is a very important quantity to test the ability of inflationary models to fill the gaps and resolve the problems of the standard model of Cosmology. A suitable range from 50 to 70 e-foldings [8] is known to be sufficient to fill these gaps. Nevertheless, the special reason behind this particular range remains unknown in theories of inflation. A single inflaton is less likely to be capable for inflation to last this long ($\sim$ 60 e-folds), and the probability to dwell in such a range increases directly as the number of inflaton N increases too. To achieve this range, one imposes slow roll conditions which require a flat potential that allows inflaton to slowly roll down the potential to oscillate about its minimum. This leads to a rigorous constraint on the inflaton's nature as to possess a very small mass [30, 31]. Moreover, the flatness of the potential gives rise to primordial perturbations which are almost identical to the observed perturbations in the CMB. However, the main feature of N-flation is to extract a sufficient amount of inflation by a collection of many axions without fine tuning problems.

N-flation [8] is a string motivated model of inflation. String theory is a theory at high energy scale ($\sim$ Planck mass M_p) which attempts to unify the four fundamental forces of nature into a single framework as an extension of standard model of elementary particle physics. The high energy limit of string theory is constructed in ten dimensions while the low energy limit is achieved by reducing these ten dimensions down to four via string compactifications [10].

There are many worries one has to face at the low energy limit of string theory. The main problems are radiative corrections and moduli stabilization. Radiative corrections concern interactions between inflatons which might spoil the flatness of the potential [32], nevertheless, N-flation [8] shows that radiative corrections could scale the amount of inflation and fix the limits of the obtained e-foldings without fine tuning which makes inflation more natural. Moduli stabilization tries to fix the resultant moduli due to compactifications, one example of such a subtle is that decompactifications of the four dimensions may lead to infinity rather than ten dimensions [33], which is still an open problem in the low energy limit of string theory.

N-flation [8] presents an inflationary universe that is driven by a collection of N axions, where axion is a pseudo-scalar that changes the sign under reflection symmetry and appears with symmetry breaking as a result [32]. Having combined axions to play collectively the role of the inflaton, Dimopoulos [8] shows that the predictions and results of $m^2\phi^2$ chaotic inflation can be reproduced to some level of accuracy.

However, each axion possesses a periodic potential V_n due to the breaking of its shift symmetry ($\phi_n \rightarrow \phi_n + constant$) which secures the flatness of its potential V_n. A general potential V is established due to the breaking of a global shift symmetry that includes the combination of all independent shift symmetries. In this case, the general potential is given by

$$V = \sum_{n=1}^{N} V_n \,, \tag{2.21}$$

and each axion has a periodic potential V_n which can be written as

$$V_n = \Lambda_n^4 \left[1 - \cos\left(\frac{2\pi\phi_n}{f_n} \right) \right] \,, \tag{2.22}$$

where f_n represents the axion decay constant and Λ_n denotes the dynamically generated scale of V_n. The form of the general potential V, Equation (2.21), shows no cross couplings between ϕ_n. Taylor expansion of the periodic potential V_n can be written as

$$V_n = \frac{1}{2}\left(\frac{2\pi\Lambda_n^2}{f_n}\right)^2 \phi_n^2 + \cdots = \frac{1}{2} m_n^2 \phi_n^2 + \cdots \,. \tag{2.23}$$

The higher order terms can be neglected, for small field values, and V_n becomes similar to that of $m^2\phi^2$ chaotic inflation, when all masses are equal $m_n = m$.

The polar transformation of ϕ fields, Φ, is very useful to obtain the collective behaviour of N-flation, which has the following form

$$\Phi = r\, e^{i\theta} \,, \tag{2.24}$$

where r stands for the radial field and θ represents the angular field. The radial field r can be expressed in terms of ϕ fields as

$$r^2 = \sum_{n=1}^{N} \phi_n^2 \,. \tag{2.25}$$

As an initial arrangement of axion fields ϕ, consider each axion field ϕ has a sub-Planckian displacement from the minimum, which is determined by its vacuum expectation value (vev) as $\langle \phi_{n0} \rangle = \alpha_n M_p$, where the maximum displacement is fixed by each axion decay constant

$$\alpha_n^2 \lesssim \frac{f_n^2}{M_p^2} \,. \tag{2.26}$$

It is worthwhile to mention that this scenario is equivalent to that of the radial field r with a super-Planckian vev as $\langle r_0 \rangle = \sqrt{N}\alpha M_p$, and the angular field θ does not contribute to N-flation dynamics, because it has a very big kinetic term $r^2(\partial\theta)^2$, which breaks slow roll conditions at the angular direction of N-flation evolution.

In N-flation, for a general potential Equation (2.21), Friedmann equation can be written as

$$3H^2 = \frac{1}{M_p^2} \sum_n \left[\frac{1}{2}\dot{\phi}_n^2 + V_n \right] = \rho \,, \tag{2.27}$$

where ρ is the energy density and the equation of motion of each field is

$$\ddot{\phi}_n + 3H\dot{\phi}_n + \frac{\partial V_n}{\partial \phi_n} = 0\,. \tag{2.28}$$

Slow roll conditions imply

$$3H^2 = \frac{1}{M_p^2}\sum_n V_n = V/M_p^2\,, \tag{2.29}$$

and

$$3H\dot{\phi}_n = -\frac{\partial V_n}{\partial \phi_n}\,. \tag{2.30}$$

For the initial conditions that mentioned earlier about the radial field r, $\langle r_0 \rangle = \sqrt{N}\alpha M_p$, and considering $V = \frac{1}{2}\sum m^2\phi_n^2 = \frac{1}{2}m^2r^2$ with Equation (2.29), one finds

$$3H^2 = N\alpha^2 m^2\,, \tag{2.31}$$

which means the friction term in the equation of motion Equation (2.28), $3H\dot{\phi}_n$, is N dependent and all fields unite together to apply the friction force. The e-foldings number of slow roll inflation is given by [34]

$$N_e(\phi) = \frac{1}{M_p^2}\int_{\phi_{end}}^{\phi} \frac{V}{V'}\,d\phi\,, \tag{2.32}$$

where prime denotes the derivative with respect to ϕ. In context of N-flation, the required N_e is determined as function of N as

$$N_e = \frac{\alpha^2 N}{4}\,. \tag{2.33}$$

Another way to quantify the flatness of the potential is to consider the slow roll parameters, ϵ and η, which are defined as

$$\epsilon \equiv \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2, \quad \eta \equiv M_p^2\frac{V''}{V}\,, \tag{2.34}$$

where slow roll conditions implies that $\epsilon \ll 1$ and $\eta \ll 1$. These parameters can be estimated in terms of N as

$$\epsilon \sim \frac{1}{\alpha^2 N^2}, \quad \eta \sim \frac{1}{\alpha^2 N}\,. \tag{2.35}$$

Similarly, the energy density perturbation is a function of N as

$$\frac{\delta\rho}{\rho} \sim N\alpha^2\frac{m}{M_p}\,. \tag{2.36}$$

The observed fluctuations in the CMB require that $\frac{\delta\rho}{\rho} \sim 10^{-5}$, which sets a constraint on axion mass as to have $m \sim 10^{10}$TeV.

We realize that all the important quantities (N_e, ϵ, η & $\frac{\delta\rho}{\rho}$) are expressed in terms of N, which shows that N-flation does not require fine tuning issues, because N is very high due to string compactifications, and automatically implies $\epsilon \ll 1$ and $\eta \ll 1$.

In this chapter, we have discussed multi-field inflation by a brief review of two papers which can be considered as a starting point for the present thesis. The first model, "Large N cosmology" [11], is a generalization of a single field inflation in which we learned how a complicated set of N coupled equations, that encode the evolution of N inflatons, can be simplified and reduced to two coupled equations of gauge invariant variables, and the solutions of the simplified set are sufficient to obtain the adiabatic and entropy perturbations. The second model, "N-flation" [8], is string theory inspired, where inflation is driven by a collection of axions, and its predictions are identical to those of chaotic inflation. In this model, we learned how to express all the important quantities (e-foldings number, slow roll parameters, density perturbations, ... etc) in terms of the number of fields N, in order to avoid fine tuning subtleties using the fact that string theory predicts large number of N fields in the low energy limit due to compactifications, therefore the theory would apply the large N limit spontaneously which falls directly into slow roll regime to drive inflation.

Chapter 3

Collective Effects in Large N Cosmology

In the previous chapter, we have argued that a large number N of inflaton fields possesses many advantages over theories of single field inflation. These benefits include measurable isocurvature perturbations and large non-Gaussianities in the CMB as well as avoiding tuning requirements to ensure a natural inflationary dynamics. These advantages emerge as the number of fields N increases. Generally, the theory of inflation becomes simple, complicated, more complicated and efficiently simple as the number of fields N goes from one (single field inflation), to a small finite N, to large finite N and finally $N \to \infty$ respectively. One problem that commonly arises is the appearance of N as an upper limit of the summation as in the action of multi-field inflation. This problem provides subtleties to distinguishing and extracting all explicit dependence on N from the action. Mathematically, it is much easier to take $N \to \infty$ by having N as a parameter that may then be varied [35]. This problem may be solved by choosing a suitable change of variables. The method is illustrated as follows: We begin with a many body quantum theory and change from the original variables to gauge invariant *collective* variables using a point canonical transformation. An effective Hamiltonian emerges as a result of this transformation, whose large N limit ($N \to \infty$) is the classical limit of the quantum theory. The evaluation of the Jacobian of the point canonical transformations is the main subtlety to be determined in this collective method [36].

3.1 Collective Field Theory

The idea of studying the collective behaviour of many-body systems originated in the 1950s when the collective motions of a dense electron gas was treated as oscillations in a plasma. This study was performed by applying a set of canonical transformations to the dense electron gas [25–27]. A generalization of this idea was formulated for the study of the large N limit of quantum systems [28]. This generalization led to the *collective field theory*. We present here a summary of collective transformations of quantum systems[1]. We emphasize that it is not necessary to determine an explicit form of the Jacobian of such transformations. The derivative of the logarithm of the Jacobian is sufficient to investigate the collective properties of quantum theories.

To begin we consider a Schrödinger equation with a wave function ψ. The Hamiltonian takes the form

$$\hat{H} = \left[-\frac{1}{2} \sum_{a=1}^{N} \frac{\partial^2}{\partial q^{a\,2}} + V(q^a) \right] , \tag{3.1}$$

where the first term represents the kinetic energy operators and the second term the potential operators. The original variables, q^a, are related to the collective variables, Q^a, via an arbitrary function f:[2]

$$Q^a = f^a(q) \, , \tag{3.2}$$

and its inverse F:

$$q^a = F^a(Q) \, . \tag{3.3}$$

The transformation of the potential operators and the wave function is straightforward, simply by replacing q with $F(Q)$. However, the transformation of the kinetic energy operators may be computed by the chain rule

$$\frac{\partial}{\partial q^a}\psi(q) = \sum_b \frac{\partial Q^b}{\partial q^a}\frac{\partial}{\partial Q^b}\psi(F(Q)) = \sum_b \frac{\partial f^b}{\partial q^a}\frac{\partial}{\partial Q^b}\psi(F(Q)) \tag{3.4}$$

$$\frac{\partial^2}{\partial q^{a\,2}}\psi(q) = \frac{\partial}{\partial q^a}\left[\frac{\partial}{\partial q^a}\psi(q)\right] = \left[\sum_b \frac{\partial^2 f^b}{\partial q^{a\,2}}\frac{\partial}{\partial Q^b} + \sum_{b\,c}\frac{\partial f^b}{\partial q^a}\frac{\partial f^c}{\partial q^a}\frac{\partial}{\partial Q^b}\frac{\partial}{\partial Q^c}\right]\psi(F(Q)) \, , \tag{3.5}$$

Equation (3.4) and Equation (3.5) imply that the kinetic energy operator takes the following form

$$-\frac{1}{2}\sum_a \frac{\partial^2}{\partial q^{a\,2}} = \frac{1}{2}\left[-i\sum_{a\,b}\frac{\partial^2 f^b}{\partial q^{a\,2}}\frac{\partial}{i\partial Q^b} + \sum_{a\,b\,c}\frac{\partial f^b}{\partial q^a}\frac{\partial f^c}{\partial q^a}\frac{\partial}{i\partial Q^b}\frac{\partial}{i\partial Q^c}\right] . \tag{3.6}$$

From Equation (3.6), we define two quantities, $\omega^a(Q)$ and $\Omega^{ab}(Q)$. We now demonstrate that $\omega^a(Q)$ and $\Omega^{ab}(Q)$ provide all the knowledge we need to know about the Jacobian of the transformation, J. This Jacobian will be used to impose the Hermiticity condition

[1]Based on "Quantum Theory of Many-Variable Systems and Fields" by Sakita [36] and "Collective field approach to the large-N limit" by Jevicki and Sakita [37]

[2]This transformation is so-called the point canonical transformation as demonstrated in [36].

on the effective Hamiltonian H_{eff}. $\omega^a(Q)$ and $\Omega^{ab}(Q)$ are defined by

$$\omega^a(Q) \equiv -\sum_b \frac{\partial^2 f^a}{\partial q^{b\,2}}\,, \tag{3.7}$$

$$\Omega^{ab}(Q) \equiv \sum_c \frac{\partial f^a}{\partial q^c}\frac{\partial f^b}{\partial q^c}\,. \tag{3.8}$$

The transformed Hamiltonian may be written as

$$\hat{H} = \left[\frac{1}{2}\left(i\sum_a \omega^a(Q)P_a + \sum_{a\,b}\Omega^{ab}(Q)P_aP_b\right) + \tilde{V}(Q)\right]\,, \tag{3.9}$$

where P_a is the momentum conjugate to Q^a ($P_a = \frac{\partial}{i\partial Q^a}$) and $\tilde{V}(Q) \equiv V(F(Q))$.

We notice that the Hamiltonian has lost its Hermiticity after the change of variables from q^a to Q^a. To restore it, we need to introduce the Jacobian of the transformation, J. The Hermitian Hamiltonian is the so-called effective Hamiltonian H_{eff}

$$H_{eff} = J^{1/2}HJ^{-1/2}\,. \tag{3.10}$$

To compute H_{eff} through Equation (3.9), we only need to consider P_a since the rest of the parameters in the expression of $\hat{H}$ operate trivially on the wave function ψ. The effective form of P_a is determined by considering the wave function ψ as follows

$$\begin{aligned} J^{1/2}P_aJ^{-1/2}\psi(F(Q)) &= J^{1/2}\frac{\partial}{i\partial Q^a}J^{-1/2}\psi(F(Q)) \\ &= \frac{\partial}{i\partial Q^a}\psi(F(Q)) + \frac{i}{2}J^{-1}\left(\frac{\partial J}{\partial Q^a}\right)\psi(F(Q)) \\ &= \left[P_a + \frac{i}{2}\frac{\partial \ln J}{\partial Q^a}\right]\psi(F(Q))\,, \end{aligned}$$

which means that

$$J^{1/2}P_aJ^{-1/2} = P_a + iC_a \;\text{ where }\; C_a = \frac{1}{2}\frac{\partial \ln J}{\partial Q^a}\,. \tag{3.11}$$

Combining Equation (3.9) and Equation (3.10) with Equation (3.11), the effective Hamiltonian can be written as

$$H_{eff} = \frac{1}{2}\left[i\sum_a \omega^a(Q)(P_a + iC_a) + \sum_{a\,b}\Omega^{ab}(Q)(P_a+iC_a)(P_b+iC_b)\right] + \tilde{V}(Q)\,. \tag{3.12}$$

We assume that P_a, Q_a and C_a are Hermitian so that

$$P_a = P_a^\dagger,\;\; Q_a = Q_a^\dagger,\;\; C_a = C_a^\dagger\,. \tag{3.13}$$

Since H_{eff} is Hermitian, H_{eff} must be equal to its adjoint that requires $H_{eff} - H^{\dagger}_{eff} = 0$. Applying this condition results in an equation that constrains the parameters ω^a, Ω^{ab} and C_a as

$$\omega^a + 2\sum_b \Omega^{ab} C_b + \sum_b \frac{\partial \Omega^{ab}}{\partial Q^b} = 0\,. \tag{3.14}$$

Equation (3.14) may be solved for C_a in terms of ω^a and Ω^{ab}. Having obtained a solution of C_a, means that we have found a solution of the derivative of the logarithm of the Jacobian ($\frac{\partial \ln J}{\partial Q^a}$) which is sufficient to compute the effective Hamiltonian. H_{eff} may now be written as

$$H_{eff} = \frac{1}{2}\sum_{ab}\Big(P_a \Omega^{ab} P_b + C_a \Omega^{ab} C_b\Big) + \tilde{V}\,. \tag{3.15}$$

3.2 The Bilocal Fields $\sigma(x, y)$ and The Schwinger-Dyson Equations

In this section, with the aim of investigating the collective dynamics of the inflationary universe, we consider the change of variable techniques of the *collective field theory*. We begin with a system written in terms of the scalar fields ϕ^a and rewrite this system in terms of the collective variables which are the bilocal fields $\sigma(x, y)$. We consider the $O(N)$ vector model [38, 39] whose action is given by:

$$S = \int d^4x \sum_{a=1}^{N}\left(\frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}\mu^2 \phi^a \phi^a - \frac{g}{8}(\phi^a\phi^a)^2\right)\,, \tag{3.16}$$

where μ is the mass of ϕ^a and g represents the coupling constant of the self interaction term. This action is invariant under the $O(N)$ symmetry which is the $N-$dimensional rotation group

$$\phi^a \to A^{ab}\phi^b\,, \tag{3.17}$$

where A^{ab} is an $N \times N$ orthogonal matrix. A^{ab} is taken to satisfy $AA^T = A^TA = \mathbb{I}$ with $\mathbb{I}$ is an $N \times N$ unit matrix. Here, superscript T denotes the transpose of A. The theory is also invariant under the reflection symmetry $\phi^a \to -\phi^a$ [38, 39].

The expectation values of the product of ϕ^a fields, which are known as correlation functions (or sometimes correlators), are very important quantities. These correlation functions reveal the dynamics of the quantum theory. We are interested in the bilocal field $\sigma(x, y)$ which is taken to be a collective field that transforms as a singlet of the symmetry. $\sigma(x, y)$ is taken as such because it is defined through the multiplication of two ϕ fields that are evaluated at different points in space-time

$$\sigma(x,y) = \sum_{a=1}^{N} \phi^a(x)\phi^a(y)\,, \tag{3.18}$$

where $x = (t_x, \bar{x})$ and $y = (t_y, \bar{y})$. Since the bilocal field $\sigma(x,y)$ is a gauge invariant operator, therefore it is a physically observable quantity [3]. We will consider the bilocal field $\sigma(x,y)$ as the inflaton that drives inflation, rather than the individual fields ϕ^a each contribute to driving inflation.

In order to obtain the effective action, we apply the change of variables at the level of the path integral Z as [38–40]

$$Z = \int \mathcal{D}\phi^a e^{iS} = \int \mathcal{D}\sigma J e^{iS} = \int \mathcal{D}\sigma e^{iS_{eff}}\,, \tag{3.19}$$

where the effective action S_{eff} is defined by

$$S_{eff} \equiv S - i \ln J\,. \tag{3.20}$$

We now compute the Jacobian in Equation (3.20), which will contribute to the equation of motion found from minimizing the effective action of $\sigma(x,y)$ fields. For any arbitrary test functional $F[\sigma]$ that is constructed from invariant correlators $\sigma(x,y)$, the derivative of the logarithm of the Jacobian can be obtained from an identity which essentially produces the Schwinger-Dyson Equations [38, 40]

$$\int \mathcal{D}\phi \frac{\delta}{\delta\phi^a(x)} \left[\phi^a(y) F[\sigma] e^{iS}\right] = 0\,. \tag{3.21}$$

In Equation (3.21), the superscript a is summed over the N fields. Taking the functional derivatives and functional average yields

$$\langle N\delta(x-y)F[\sigma]\rangle + \langle \phi^a(y)\frac{\delta F[\sigma]}{\delta\phi^a(x)}\rangle + i\langle \phi^a(y)F[\sigma]\frac{\delta S}{\delta\phi^a(x)}\rangle = 0\,, \tag{3.22}$$

where in first term the delta function is [39]

$$\frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x-y)\,. \tag{3.23}$$

N in Equation (3.22) appears as a parameter that comes from the contraction of the superscript a as opposed to being an upper limit of the summation. This allows the large N limit to be easily applied. The exponential that carries the action, e^{iS}, is absorbed by definition of expectation values, which are obtained by taking the functional average.

[3]Not only for this reason, in fact there are other reasons to consider $\sigma(x,y)$ as the inflaton which will be discussed later.

A similar identity to Equation (3.21) can be determined by a change of variables from $\phi(x)$ to $\sigma(x,y)$ via Equation (3.19) in terms of the effective action [40]

$$\int \mathcal{D}\sigma \int dz \frac{\delta}{\delta\sigma(x,z)} \left[\sigma(y,z)F[\sigma]e^{iS_{eff}}\right] = 0\,, \tag{3.24}$$

where $\mathcal{D}$ denotes a functional integration and d is a coordinate integration. Similarly, the average of the last identity can be written as

$$\langle K\delta(x-y)F[\sigma]\rangle + \left\langle \int dz\,\sigma(y,z)\frac{\delta F[\sigma]}{\delta\sigma(x,z)} \right\rangle + i\left\langle \int dz\,\sigma(y,z)F[\sigma]\frac{\delta S_{eff}}{\delta\sigma(x,z)} \right\rangle = 0\,, \tag{3.25}$$

where $K = \delta(0)\int dz$ is the volume expansion factor [38, 40]. In order to extract a differential expression of the Jacobian which is inherent in S_{eff} by definition, we apply the chain rule to change derivatives of ϕ^a in Equation (3.22) to derivatives of $\sigma(x,y)$ in Equation (3.25). The transformation is written as [38]

$$\begin{aligned}\frac{\delta}{\delta\phi^a(x)} = \int dz \int dy \frac{\delta\sigma(y,z)}{\delta\phi^a(x)}\frac{\delta}{\delta\sigma(y,z)} &= \int dz \int dy\delta(x-y)\phi^a(z)\frac{\delta}{\delta\sigma(y,z)} \\ &+ \int dz \int dy\delta(x-z)\phi^a(y)\frac{\delta}{\delta\sigma(y,z)}\,.\end{aligned} \tag{3.26}$$

Using the following property of delta function [39]

$$f(x) = \int dy\delta(x-y)f(y)\,, \tag{3.27}$$

and integrating over y in the first term and z in the second term yields

$$\frac{\delta}{\delta\phi^a(x)} = \int dz\phi^a(z)\frac{\delta}{\delta\sigma(x,z)} + \int dy\phi^a(y)\frac{\delta}{\delta\sigma(y,x)}\,, \tag{3.28}$$

Since we know that $\sigma(x,y)$ is totally symmetric ($\sigma(x,y) = \sigma(y,x)$) [38], we may change the dummy variable of integrations from y to z in the second term of Equation (3.28) to write

$$\frac{\delta}{\delta\phi^a(x)} = 2\int dz\phi^a(z)\frac{\delta}{\delta\sigma(x,z)}\,. \tag{3.29}$$

Inserting the transformation Equation (3.29) into Equation (3.22) gives

$$\langle N\delta(x-y)F[\sigma]\rangle + 2\langle\int dz\,\sigma(y,z)\frac{\delta F[\sigma]}{\delta\sigma(x,z)}\rangle + 2i\langle\int dz\,\sigma(y,z)F[\sigma]\frac{\delta S}{\delta\sigma(x,z)}\rangle = 0\,. \tag{3.30}$$

Combining Equation (3.30) with Equation (3.25) demonstrates that

$$i\left\langle \int dz\,\sigma(y,z)F[\sigma]\frac{\delta(S_{eff}-S)}{\delta\sigma(x,z)} \right\rangle = \frac{1}{2}\langle N\delta(x-y)F[\sigma]\rangle - \langle K\delta(x-y)F[\sigma]\rangle\,. \tag{3.31}$$

By using the definition of the effective action, Equation (3.20), we obtain

$$\left\langle \int dz\, \sigma(y,z) F[\sigma] \frac{\delta \ln J}{\delta \sigma(x,z)} \right\rangle = \frac{1}{2}\langle N\delta(x-y)F[\sigma]\rangle - \langle K\delta(x-y)F[\sigma]\rangle \,. \tag{3.32}$$

Since $F[\sigma]$ is an arbitrary function, the last equation requires that

$$\int dz\, \sigma(y,z) \frac{\delta \ln J}{\delta \sigma(x,z)} = \left(\frac{N}{2} - K\right)\delta(x-y)\,. \tag{3.33}$$

We may neglect the contribution of the volume expansion factor K to leading order in N and write

$$\int dz\, \sigma(y,z) \frac{\delta \ln J}{\delta \sigma(x,z)} = \frac{N}{2}\delta(x-y)\,. \tag{3.34}$$

The Jacobian then satisfies this last differential equation, Equation (3.34), which is sufficient to obtain the equation of motion of the bilocal fields $\sigma(x,y)$: the Schwinger-Dyson Equations.

3.2.1 The Jacobian from the collective field theory

In section 3.1, we have defined two quantities $\omega^a(Q)$ and $\Omega^{ab}(Q)$, Equation (3.7) and Equation (3.8), which allow us to derive an expression of a differential equation of the Jacobian through Equation (3.14). This expression is sufficient to determine the effective Hamiltonian. Here, we will obtain the previous result of Equation (3.34) from Equation (3.14) to establish the connection with the collective field theory. In this case, the original variables q^a are the scalar fields $\phi^a(x)$ and the collective variables Q^q are the bilocal fields $\sigma(x,y)$. ω and Ω may be transformed into functional integrals as [40]

$$\omega(x,y) = -\int dz \frac{\delta^2}{\delta\phi^a(z)\delta\phi^a(z)}\sigma(x,y) \tag{3.35}$$

$$\Omega(x,y;x',y') = \int dz \left[\frac{\delta}{\delta\phi^a(z)}\sigma(x,y)\right]\left[\frac{\delta}{\delta\phi^a(z)}\sigma(x',y')\right]\,. \tag{3.36}$$

From Equation (3.35), $\omega(x,y)$ is

$$\begin{aligned}
\omega(x,y) &= -\int dz \frac{\delta^2}{\delta\phi^a(z)\delta\phi^a(z)} \sum_b \phi^b(x)\phi^b(y) \\
&= -\int dz \frac{\delta}{\delta\phi^a(z)} \sum_b \delta_{ab}\left[\delta(x-z)\phi^b(y) + \delta(y-z)\phi^b(x)\right] \\
&= -\int dz \sum_b \delta_{ab}\left[\delta(x-z)\delta(y-z) + \delta(y-z)\delta(x-z)\right] \\
&= -2N\delta(x-y)\,.
\end{aligned} \tag{3.37}$$

From Equation (3.36), $\Omega(x, y; x', y')$ reads

$$\begin{aligned}\Omega(x, y; x', y') = \int dz \sum_b \delta_{ab} \left[\delta(x - z)\phi^b(y) + \delta(y - z)\phi^b(x)\right] \\ \sum_c \delta_{ac} \left[\delta(x' - z)\phi^c(y') + \delta(y' - z)\phi^c(x')\right] \\ = \delta(x - x')\sigma(y, y') + \delta(y' - x)\sigma(x', y) \\ + \delta(y - x')\sigma(x, y') + \delta(y - y')\sigma(x, x') \,. \end{aligned} \tag{3.38}$$

In terms of functional integrals, Equation (3.14) becomes

$$\omega(x, y) + \int dx'dy'\Omega(x, y; x', y')\frac{\delta \ln J}{\delta\sigma(x', y')} + \int dx'dy'\frac{\delta\Omega(x, y; x', y')}{\delta\sigma(x', y')} = 0 \,. \tag{3.39}$$

The second and third terms of Equation (3.39) may be computed separately. Using Equation (3.38), the second term is given by

$$\begin{aligned}\int dx'dy'\Omega(x, y; x', y')\frac{\delta \ln J}{\delta\sigma(x', y')} = \int dx'dy'\delta(x - x')\sigma(y, y')\frac{\delta \ln J}{\delta\sigma(x', y')} \\ + \int dx'dy'\delta(y' - x)\sigma(x', y)\frac{\delta \ln J}{\delta\sigma(x', y')} + \int dx'dy'\delta(y - x')\sigma(x, y')\frac{\delta \ln J}{\delta\sigma(x', y')} \\ + \int dx'dy'\delta(y - y')\sigma(x, x')\frac{\delta \ln J}{\delta\sigma(x', y')} \,. \end{aligned} \tag{3.40}$$

Integrating over x' in the first term and third term, and integrating over y' in the second and fourth term yields

$$\begin{aligned}\int dx'dy'\Omega(x, y; x', y')\frac{\delta \ln J}{\delta\sigma(x', y')} = \int dy'\sigma(y, y')\frac{\delta \ln J}{\delta\sigma(x, y')} \\ + \int dx'\sigma(x', y)\frac{\delta \ln J}{\delta\sigma(x', x)} + \int dy'\sigma(x, y')\frac{\delta \ln J}{\delta\sigma(y, y')} \\ + \int dx'\sigma(x, x')\frac{\delta \ln J}{\delta\sigma(x', y)} \,. \end{aligned} \tag{3.41}$$

Changing the dummy indices x' and y' to z gives

$$\int dx'dy'\Omega(x, y; x', y')\frac{\delta \ln J}{\delta\sigma(x', y')} = 4\int dz\sigma(y, z)\frac{\delta \ln J}{\delta\sigma(x, z)} \,. \tag{3.42}$$

Using Equation (3.38), the third term of Equation (3.39) becomes

$$\begin{aligned}\int dx'dy'\frac{\delta\Omega(x,y;x',y')}{\delta\sigma(x',y')} &= \int dx'dy'\delta(x-x')\frac{\delta\sigma(y,y')}{\delta\sigma(x',y')} + \int dx'dy'\delta(y'-x)\frac{\delta\sigma(x',y)}{\delta\sigma(x',y')} \\ &+ \int dx'dy'\delta(y-x')\frac{\delta\sigma(x,y')}{\delta\sigma(x',y')} + \int dx'dy'\delta(y-y')\frac{\delta\sigma(x,x')}{\delta\sigma(x',y')} \\ &= \int dx'dy'\delta(0)[\delta(x-x')\delta(y-x') + \delta(y-x')\delta(x-x')] \\ &+ \int dx'dy'\delta(0)[\delta(y'-x)\delta(y-y') + \delta(y-y')\delta(x-y')] \\ &= 4K\delta(x-y)\,, \end{aligned} \tag{3.43}$$

where in the last line we used

$$\frac{\delta\sigma(z,w)}{\delta\sigma(x,y)} = \delta(z-x)\delta(w-y)\,. \tag{3.44}$$

Substituting Equation (3.43), Equation (3.42) and Equation (3.37) into Equation (3.39) gives

$$-2N\delta(x-y) + 4\int dz\sigma(y,z)\frac{\delta \ln J}{\delta\sigma(x,z)} + 4K\delta(x-y) = 0\,. \tag{3.45}$$

To leading order in N, we obtain

$$\int dz\sigma(y,z)\frac{\delta \ln J}{\delta\sigma(x,z)} = \frac{N}{2}\delta(x-y)\,, \tag{3.46}$$

which is of the same from as Equation (3.34).

3.2.2 The Schwinger-Dyson Equations

In terms of ϕ^a, the Lagrangian is obtained from the action Equation (3.16) via $S = \int \mathcal{L} d^4x$ as

$$\mathcal{L} = \sum_{a=1}^{N}\left(\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu^2\phi^a\phi^a - \frac{g}{8}(\phi^a\phi^a)^2\right)\,. \tag{3.47}$$

Using Equation (3.18), we may write the Lagrangian in terms of the bilocal fields $\sigma(x,y)$ as

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - \frac{1}{2}\mu^2\sigma(x,x) - \frac{g}{8}\sigma^2(x,x)\,, \tag{3.48}$$

where in the first term one takes the derivatives first and then sets the equality between x and y. Here $\eta^{\mu\nu}$ is the Minkowski metric for flat space. The equation of motion of $\sigma(x,y)$ describes the classical dynamics of the quantum theory which can be derived, as in classical field theory, from applying Euler-Lagrange equation. The action of the

$\sigma(x,y)$ fields is the effective action which can be minimized as [38]

$$\frac{\delta S_{eff}}{\delta\sigma(x,y)} = \frac{\delta S}{\delta\sigma(x,y)} - i\frac{\delta \ln J}{\delta\sigma(x,y)} = 0\,, \tag{3.49}$$

where in the last line we used the definition of the effective action Equation (3.20). To make use of Equation (3.34), we multiply Equation (3.49) by $\sigma(y,z)$ and integrate over coordinates y to obtain

$$\int dy\,\sigma(y,z)\frac{\delta S}{\delta\sigma(x,y)} - i\int dy\,\sigma(y,z)\frac{\delta \ln J}{\delta\sigma(x,y)} = 0\,. \tag{3.50}$$

Combining Equation (3.34) and Equation (3.50) yields

$$\int dy\,\sigma(y,z)\frac{\delta S}{\delta\sigma(x,y)} = i\frac{N}{2}\delta(x-z)\,. \tag{3.51}$$

We manipulate the left hand side of Equation (3.51) by first deriving the functional derivative of the action with respect to the bilocal field $\sigma(x,y)$. Next, we multiply the result by $\sigma(y,z)$ and integrate over y. The action consists of two terms, the kinetic term and the potential term, which can be computed separately as follows. For the potential term, we find [38]

$$\begin{aligned}\frac{\delta}{\delta\sigma(x,y)}&\left(\int dp\left[-\frac{1}{2}\mu^2\sigma(p,p) - \frac{g}{8}\sigma^2(p,p)\right]\right)\\ &= \int dp\left[-\frac{1}{2}\mu^2\delta(p-x)\delta(p-y) - \frac{g}{4}\sigma(p,p)\delta(p-x)\delta(p-y)\right]\\ &= \left[-\frac{1}{2}\mu^2\delta(x-y) - \frac{g}{4}\sigma(y,y)\delta(x-y)\right]\,.\end{aligned} \tag{3.52}$$

Multiply Equation (3.52) with $\sigma(y,z)$ and integrate over y leads to

$$\begin{aligned}\int dy\,\sigma(y,z)&\left[-\frac{1}{2}\mu^2\delta(x-y) - \frac{g}{4}\sigma(y,y)\delta(x-y)\right]\\ &= -\frac{1}{2}\mu^2\sigma(x,z) - \frac{g}{4}\sigma(x,z)\sigma(x,x)\,.\end{aligned} \tag{3.53}$$

Similarly, for the kinetic term, we find

$$\begin{aligned}\frac{\delta}{\delta\sigma(x,y)}&\int dp\left(\frac{1}{2}\eta^{\mu\nu}\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_2^\nu}\sigma(p_1,p_2)\right)\Big|_{p_1=p_2=p}\\ &= \frac{1}{2}\eta^{\mu\nu}\int dp\left(\frac{\partial}{\partial p_1^\mu}\frac{\partial}{\partial p_2^\nu}\delta(p_1-x)\delta(p_2-y)\right)\Big|_{p_1=p_2=p}\\ &= \frac{1}{2}\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\delta(x-y)\,.\end{aligned} \tag{3.54}$$

Multiplying by $\sigma(z,y)$ and integrating over y leads to

$$\begin{aligned}
&\frac{1}{2}\eta^{\mu\nu}\int dy\sigma(y,z)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\delta(x-y) \\
&\qquad = \frac{1}{2}\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\int dy\left(\sigma(y,z)\frac{\partial}{\partial y^\nu}\delta(x-y)\right) \\
&\qquad = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\int dy\left(\delta(x-y)\frac{\partial}{\partial y^\nu}\sigma(y,z)\right) \\
&\qquad = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\sigma(x,z)\,.
\end{aligned} \tag{3.55}$$

In the second line we pulled out the derivative with respect to x because $\sigma(y,z)$ is independent of x. In the third line, we integrate by parts to take the derivative of $\sigma(y,z)$ where the first term of integration by parts $\delta(x-y)\sigma(y,z)$ vanishes at the boundary. The equation of motion may be constructed by substituting Equation (3.55) and Equation (3.53) into Equation (3.51) to give [38]

$$\left(-\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}-\mu^2-\frac{g}{2}\sigma(x,x)\right)\sigma(x,z)=iN\delta(x-z)\,. \tag{3.56}$$

This is the Schwinger-Dyson Equation which represents the equation of motion for the bilocal field $\sigma(x,y)$. In order to solve this equation, we Fourier transform it by considering the translationally invariant ansatz of the bilocal field [38, 40]

$$\sigma(x,y)=\int\frac{d^4p}{(2\pi)^4}\,\exp\left(\,ip.(x-y)\,\right)\sigma(p)\;. \tag{3.57}$$

Substituting this ansatz into the Schwinger-Dyson Equation, we obtain

$$\sigma(p)=\frac{iN}{p^2\,-\mu^2\,-\,\frac{g}{2}\int\frac{d^4k}{(2\pi)^4}\sigma(k)}\,, \tag{3.58}$$

where p is the four-momentum such that $p\equiv(E,\vec{p})$, E is the energy that corresponds to the time dimension, and $\vec{p}$ are momenta in the spatial directions of the bilocal field $\sigma(x,y)$.

To this end, we have obtained the Schwinger-Dyson Equations for the bilocal fields $\sigma(x,y)$. In the next chapter, we will couple the $O(N)$ vector model to gravity in order to compute the cosmological perturbations of an inflationary universe which is collectively driven by the bilocal field $\sigma(x,y)$.

Chapter 4

Towards a collective dynamics of inflationary universe

In this chapter, we adopt a consistent approach to compute the dynamics of the bilocal field $\sigma(x, y)$. We compute the Einstein Field Equations (EFEs) of the bilocal field $\sigma(x, y)$ that is coupled to gravity. We begin with the Lagrangian of the bilocal field $\sigma(x, y)$ of Equation (3.48) and re-write it as

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \sigma(x, y)(x, y) \bigg|_{x=y} - \frac{1}{2} \mu^2 \sigma(x, x) - \frac{g}{8} \sigma^2(x, x) \,, \tag{4.1}$$

where the metric tensor is represented by $g^{\mu\nu}$, a general space-time metric opposed to the Minkowski metric $\eta^{\mu\nu}$ used in the previous section.[1]

4.1 Cosmological Perturbations

We adopt the standard approach from the theory of cosmological perturbations [12]. This theory is very successful in that it efficiently explains the dynamics of an expanding universe in a gauge invariant approach. Furthermore, the theory provides a simple mathematical framework to determine the evolution of perturbations from initial perturbations in the inflaton fields to the present perturbations lying on the observational background, such as those in the CMB. In this method, the background is considered homogeneous and isotropic with a perturbation found by taking a small deviation away from homogeneity in the background. This method agrees with the fact that observations show that the universe is approximately homogeneous and isotropic on large scales.

[1] The replacement of $\eta^{\mu\nu}$ with $g^{\mu\nu}$ is, in general, non-trivial but possible, in this case, for reasons which will be discussed later.

The background can be described by Friedmann-Robertson-Walker metric as [12]

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\gamma_{ij} dx^i dx^j \,, \tag{4.2}$$

where $\bar{g}_{\mu\nu}$ is the background metric and $a(t)$ denotes the scalar factor which is only a function of time. Here, Latin letters $(i, j, ...)$ run from 1 to 3 whereas the Greek letters $(\mu, \nu, ...)$ run from 0 to 3. The zeorth component will always denote the time component and the spatial components represent the first, second and third components. The parameter γ_{ij} is given by

$$\gamma_{ij} = \delta_{ij} \left[1 + \frac{k}{4}(x_l x^l)\right]^{-2} \,, \tag{4.3}$$

where δ_{ij} is the Kronecker delta and k stands for the curvature which possesses three possible values as the following:

- $k = 0$ for a flat universe.
- $k = 1$ for a closed universe.
- $k = -1$ for an open universe.

For the reason that observations suggest that the current universe exhibits the flatness properties with a density close to the critical density, we will consider the flat case with $k = 0$. The metric is now re-written as

$$ds^2 = a^2(\eta)(d\eta^2 - \delta_{ij} dx^i dx^j) \,, \tag{4.4}$$

where the physical time (dt) is replaced by the conformal time $(d\eta)$ via the transformation $dt = a d\eta$.

4.1.1 The perturbed Einstein Field Equations

The Einstein Field Equations (EFEs) describe gravity as space-time curvature that is influenced by the presence of matter and energy. EFEs are given by[2] [41]

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = T^\mu_\nu \,, \tag{4.5}$$

where T^μ_ν represents the energy momentum tensor, R^μ_ν and R are Ricci tensor and scalar respectively. The Ricci scalar is related to the Ricci tensor by [41]

$$R = R^\mu_\mu \,. \tag{4.6}$$

[2] We use the convention $8\pi G = 1$, where G is the gravitational constant.

The Ricci tensor is obtained through a special contraction of the Riemann tensor $R^{\mu}_{\nu\alpha\beta}$ as [41]

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\nu\,\mu\,,\alpha} - \Gamma^{\alpha}_{\alpha\,\mu\,,\nu} + \Gamma^{\alpha}_{\alpha\,\beta}\Gamma^{\beta}_{\nu\,\mu} - \Gamma^{\alpha}_{\nu\,\beta}\Gamma^{\beta}_{\alpha\,\mu}\,, \tag{4.7}$$

where the connection coefficients (christoffel symbols, $\Gamma^{\alpha}_{\mu\nu}$) are defined in terms of the metric tensor as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta})\,. \tag{4.8}$$

Due to the fact that the existence of matter and energy affect the geometry of space-time through Equation (4.5), it follows that fluctuations of the metric tensor are induced by the fluctuations of scalar fields (in this case, inflatons). We shall assume a first order perturbation of a special case where inflaton perturbations produce an equal fluctuation in the space-time metric [41]

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = a^2(\eta)[(1+2\Phi)d\eta^2 - (1-2\Phi)\delta_{i\,j}dx^i dx^j]\,, \tag{4.9}$$

where Φ denotes the metric fluctuation which is a function in both space and time, and $g_{\mu\nu}$ is the perturbed metric tensor that contains the background $\bar{g}_{\mu\nu}$ and a perturbation contribution $\delta g_{\mu\nu}$. $g_{\mu\nu}(t,x)$ is given by [12, 41]

$$g_{\mu\nu}(t,x) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,x)\,. \tag{4.10}$$

We shall neglect higher order perturbations and assume $O((\delta g)^2) << 1$. This metric is obtained by choosing the longitudinal gauge. In the longitudinal gauge, all fluctuations in mixed directions vanish, that is, the metric has no off-diagonal components. This choice offers a fixed coordinate system, and sets Φ to be the Newtonian gravitational potential. We re-write the perturbed metric as covariant and contravariant tensors:

$$g_{\mu\nu} = a^2 \begin{pmatrix} (1+2\Phi) & \\ & -(1-2\Phi)\delta_{i\,j} \end{pmatrix}\,, \tag{4.11}$$

and

$$g^{\mu\nu} = a^{-2} \begin{pmatrix} (1-2\Phi) & \\ & -(1+2\Phi)\delta^{i\,j} \end{pmatrix}\,. \tag{4.12}$$

The metric may be split into background $\bar{g}$ and perturbation δg via Equation (4.10) as covariant tensors as

$$\bar{g}_{\mu\nu} = a^2 \begin{pmatrix} 1 & \\ & -\delta_{i\,j} \end{pmatrix}\,, \tag{4.13}$$

and

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 2\Phi & \\ & 2\Phi\delta_{i\,j} \end{pmatrix}\,, \tag{4.14}$$

and similarly contravariant forms can be written as

$$\bar{g}^{\mu\nu} = a^{-2} \begin{pmatrix} 1 & \\ & -\delta^{i\,j} \end{pmatrix}, \tag{4.15}$$

and

$$\delta g^{\mu\nu} = a^{-2} \begin{pmatrix} -2\Phi & \\ & -2\Phi\delta^{i\,j} \end{pmatrix}. \tag{4.16}$$

Our aim is to compute the EFEs which are a set of 10 differential equations that describe the evolution of the background and perturbations. To do this, we must first compute the perturbed form of Einstein tensor and the perturbed energy-momentum tensor. The Einstein tensor is expressed in terms of the Ricci tensor, which is determined by the connection coefficients (Christoffel symbols). We start by calculating the perturbed expressions of all christoffel symbols from the metric Equation (4.9). We then split the perturbed christoffel symbols into a background and perturbation, making use of the fact that the background contribution is always a function of time only and the perturbed contribution is a function of both space and time. Having found the perturbed christoffel symbols, we proceed to compute the perturbed Ricci tensor and finally the perturbed Einstein tensor.

We calculate the first perturbed christoffel symbol $\Gamma^0_{0\,0}$ from Equation (4.8) as follows:

$$\Gamma^0_{0\,0} = \frac{1}{2} g^{00} g_{00,0} = \frac{1}{2}(\bar{g}^{00} + \delta g^{00})\partial_0(\bar{g}_{00} + \delta g_{00})$$

$$\Gamma^0_{0\,0} = \frac{1}{2}(\bar{g}^{00}\partial_0\bar{g}_{00} + \bar{g}^{00}\partial_0\delta g_{00} + \delta g^{00}\partial_0\bar{g}_{00} + O(\delta g^2_{00}))\,.$$

By using covariant and contravariant forms of the background and the perturbed metric, we obtain

$$\Gamma^0_{0\,0} = \frac{1}{2}(a^{-2})\partial_0(a^2) + \frac{1}{2}(a^{-2})\partial_0(2a^2\Phi) + \frac{1}{2}(-2\Phi a^{-2})\partial_0(a^2) = \mathcal{H} + \Phi^{'}\,,$$

where $\mathcal{H} = \frac{a^{'}}{a}$, with a prime representing the derivative with respect to the conformal time η. Similarly, we find the perturbed christoffel symbols as

$$\Gamma^0_{0\,0} = \mathcal{H} + \Phi^{'}\,, \qquad \Gamma^0_{0\,k} = \Phi_{,k}\,, \qquad \Gamma^0_{i\,j} = \mathcal{H}\delta_{i\,j} - [4\mathcal{H}\Phi + \Phi^{'}]\delta_{i\,j}\,, \tag{4.17}$$

$$\Gamma^i_{0\,0} = \Phi_{,i}\,, \qquad \Gamma^i_{0\,j} = \mathcal{H}\delta^i_j - \Phi^{'}\delta^i_j\,, \qquad \Gamma^i_{k\,l} = -[\Phi_{,l}\delta^i_k + \Phi_{,k}\delta^i_l] + \Phi_{,i}\delta_{k\,l}\,. \tag{4.18}$$

These results can be split into a background and perturbation component as

$$\Gamma^{\alpha}_{\mu\,\nu}(t,x) = \bar{\Gamma}^{\alpha}_{\mu\,\nu}(t) + \delta\Gamma^{\alpha}_{\mu\,\nu}(t,x)\,. \tag{4.19}$$

From Equation (4.19), the background components are given by

$$\bar{\Gamma}^{0}_{0\,0} = \mathcal{H}\,, \qquad \bar{\Gamma}^{0}_{0\,k} = 0\,, \qquad \bar{\Gamma}^{0}_{i\,j} = \mathcal{H}\delta_{i\,j}\,, \tag{4.20}$$

$$\bar{\Gamma}^{i}_{0\,0} = 0\,, \qquad \bar{\Gamma}^{i}_{0\,j} = \mathcal{H}\delta^{i}_{j}\,, \qquad \bar{\Gamma}^{i}_{k\,l} = 0\,. \tag{4.21}$$

The perturbation components are given by

$$\delta\Gamma^{0}_{0\,0} = \Phi^{'}\,, \qquad \delta\Gamma^{0}_{0\,k} = \Phi_{,k}\,, \qquad \delta\Gamma^{0}_{i\,j} = -[4\mathcal{H}\Phi + \Phi^{'}]\delta_{i\,j}\,, \tag{4.22}$$

$$\delta\Gamma^{i}_{0\,0} = \Phi_{,i}\,, \qquad \delta\Gamma^{i}_{0\,j} = -\Phi^{'}\delta^{i}_{j}\,, \qquad \delta\Gamma^{i}_{k\,l} = -[\Phi_{,l}\delta^{i}_{k} + \Phi_{,k}\delta^{i}_{l}] + \Phi_{,i}\delta_{k\,l}\,. \tag{4.23}$$

Using these results, and substituting Equation (4.19) into Equation (4.7), we obtain

$$\begin{aligned} R_{\mu\nu} =& (\bar{\Gamma}^{\alpha}_{\nu\,\mu\,,\alpha} + \delta\Gamma^{\alpha}_{\nu\,\mu\,,\alpha}) - (\bar{\Gamma}^{\alpha}_{\alpha\,\mu\,,\nu} + \delta\Gamma^{\alpha}_{\alpha\,\mu\,,\nu}) + (\bar{\Gamma}^{\alpha}_{\alpha\,\beta} + \delta\Gamma^{\alpha}_{\alpha\,\beta})(\bar{\Gamma}^{\beta}_{\nu\,\mu} + \delta\Gamma^{\beta}_{\nu\,\mu}) \\ &- (\bar{\Gamma}^{\alpha}_{\nu\,\beta} + \delta\Gamma^{\alpha}_{\nu\,\beta})(\bar{\Gamma}^{\beta}_{\alpha\,\mu} + \delta\Gamma^{\beta}_{\alpha\,\mu}) \end{aligned}$$

$$\begin{aligned} R_{\mu\nu} =& \bar{\Gamma}^{\alpha}_{\nu\,\mu\,,\alpha} - \bar{\Gamma}^{\alpha}_{\alpha\,\mu\,,\nu} + \bar{\Gamma}^{\alpha}_{\alpha\,\beta}\bar{\Gamma}^{\beta}_{\nu\,\mu} - \bar{\Gamma}^{\alpha}_{\nu\,\beta}\bar{\Gamma}^{\beta}_{\alpha\,\mu} + \delta\Gamma^{\alpha}_{\nu\,\mu\,,\alpha} - \delta\Gamma^{\alpha}_{\alpha\,\mu\,,\nu} + \bar{\Gamma}^{\alpha}_{\alpha\,\beta}\delta\Gamma^{\beta}_{\nu\,\mu} \\ &+ \bar{\Gamma}^{\beta}_{\nu\,\mu}\delta\Gamma^{\alpha}_{\alpha\,\beta} - \bar{\Gamma}^{\alpha}_{\nu\,\beta}\delta\Gamma^{\beta}_{\alpha\,\mu} - \bar{\Gamma}^{\beta}_{\alpha\,\mu}\delta\Gamma^{\alpha}_{\nu\,\beta}\,. \end{aligned} \tag{4.24}$$

Here, we have neglected higher order perturbation terms, since $O(\delta\Gamma^2) << 1$ in a first order perturbation. Separating the perturbed Ricci tensor $R_{\mu\nu}$ into a background and perturbation components gives

$$R_{\mu\nu}(t,x) = \bar{R}_{\mu\nu}(t) + \delta R_{\mu\nu}(t,x)\,. \tag{4.25}$$

Comparing Equation (4.24) with Equation (4.25), the background part of Ricci tensor $\bar{R}_{\mu\nu}$ is defined as

$$\bar{R}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\nu\,\mu\,,\alpha} - \bar{\Gamma}^{\alpha}_{\alpha\,\mu\,,\nu} + \bar{\Gamma}^{\alpha}_{\alpha\,\beta}\bar{\Gamma}^{\beta}_{\nu\,\mu} - \bar{\Gamma}^{\alpha}_{\nu\,\beta}\bar{\Gamma}^{\beta}_{\alpha\,\mu}\,, \tag{4.26}$$

while the perturbed part reads

$$\delta R_{\mu\nu} = \delta\Gamma^{\alpha}_{\nu\,\mu\,,\alpha} - \delta\Gamma^{\alpha}_{\alpha\,\mu\,,\nu} + \bar{\Gamma}^{\alpha}_{\alpha\,\beta}\delta\Gamma^{\beta}_{\nu\,\mu} + \bar{\Gamma}^{\beta}_{\nu\,\mu}\delta\Gamma^{\alpha}_{\alpha\,\beta} - \bar{\Gamma}^{\alpha}_{\nu\,\beta}\delta\Gamma^{\beta}_{\alpha\,\mu} - \bar{\Gamma}^{\beta}_{\alpha\,\mu}\delta\Gamma^{\alpha}_{\nu\,\beta}\,. \tag{4.27}$$

From Equation (4.26) and Equation (4.27), we may obtain exact expressions of the background and perturbed components of Ricci tensor by substituting Equation (4.20), (4.21), (4.22) and (4.23) into Equation (4.26) and (4.27). Doing so produces

$$\bar{R}_{00} = -3\mathcal{H}^{'}\,, \quad \bar{R}_{0i} = 0\,, \quad \bar{R}_{ij} = [\mathcal{H}^{'} + 2\mathcal{H}^2]\delta_{i\,j}\,, \tag{4.28}$$

$$\delta R_{00} = \nabla^2\Phi + 3\Phi^{''} + 6\mathcal{H}\Phi^{'}\,, \delta R_{0i} = 2(\Phi^{'} + \mathcal{H}\Phi)_{,i}\,, \tag{4.29}$$

$$\delta R_{ij} = [-\Phi^{''} + \nabla^2\Phi - 4\mathcal{H}^{'}\Phi - 6\mathcal{H}\Phi^{'} - 8\mathcal{H}^2\Phi]\delta_{i\,j}\,. \tag{4.30}$$

To calculate the Einstein tensor, we still must determine the Ricci scalar R. The Ricci scalar is obtained by raising one index of $R_{\mu\nu}$ to construct the mixed Ricci tensor

$$R^{\mu}_{\nu} = g^{\mu\alpha}R_{\alpha\nu} = \bar{g}^{\mu\alpha}\bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta R_{\alpha\nu} + g^{\mu\alpha}\bar{R}_{\alpha\nu} \tag{4.31}$$

Without splitting into background and perturbation contributions, we evaluate R^{μ}_{ν} in our time and spatial directions as

$$R^0_0 = a^{-2}[-3\mathcal{H}^{'} + \nabla^2\Phi + 3\Phi^{''} + 6\mathcal{H}\Phi^{'} + 6\mathcal{H}^{'}\Phi]\,,$$

$$R^0_i = 2a^{-2}[\Phi^{'} + \mathcal{H}\Phi]_{,i} = -R^i_0\,,$$

$$R^i_j = -a^{-2}[\mathcal{H}^{'} + 2\mathcal{H}^2 - \Phi^{''} + \nabla^2\Phi - 2\mathcal{H}^{'}\Phi - 6\mathcal{H}\Phi^{'} - 4\mathcal{H}^2\Phi]\delta^i_j\,.$$

The Ricci scalar is found to be

$$R = R^{\mu}_{\mu} = R^0_0 + R^i_i = a^{-2}[-6(\mathcal{H}^{'} + \mathcal{H}^2) + 6\Phi^{''} - 2\nabla^2\Phi + 12(\mathcal{H}^{'} + \mathcal{H}^2)\Phi + 24\mathcal{H}\Phi^{'}]\,.$$

Using Equation (4.5), the perturbed Einstein tensor is given by

$$G^0_0 = a^{-2}[3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi^{'} - 6\mathcal{H}^2\Phi]\,,$$

$$G^0_i = 2a^{-2}[\Phi^{'} + \mathcal{H}\Phi]_{,i}\,,$$

$$G^i_j = a^{-2}[2\mathcal{H}^{'} + \mathcal{H}^2 - 2\Phi^{''} - 4\mathcal{H}^{'}\Phi - 2\mathcal{H}^2\Phi - 6\mathcal{H}\Phi^{'}]\delta^i_j\,.$$

We re-write the perturbed Einstein tensor in terms of background and perturbation contributions via $G^{\mu}_{\nu}(t,x) = \bar{G}^{\mu}_{\nu}(t) + \delta G^{\mu}_{\nu}(t,x)$ as

$$\bar{G}^0_0 = 3a^{-2}\mathcal{H}^2\,, \qquad \delta G^0_0 = a^{-2}[2\nabla^2\Phi - 6\mathcal{H}\Phi^{'} - 6\mathcal{H}^2\Phi]\,, \tag{4.32}$$

$$\bar{G}^0_i = 0\,, \qquad \delta G^0_i = 2a^{-2}[\Phi^{'} + \mathcal{H}\Phi]_{,i}\,, \tag{4.33}$$

$$\bar{G}^i_j = a^{-2}[2\mathcal{H}^{'} + \mathcal{H}^2]\delta^i_j\,, \qquad \delta G^i_j = a^{-2}[-2\Phi^{''} - 4\mathcal{H}^{'}\Phi - 2\mathcal{H}^2\Phi - 6\mathcal{H}\Phi^{'}]\delta^i_j\,. \tag{4.34}$$

This is the exact expression of the perturbed Einstein tensor.We now compute the right hand side (RHS) of Equation (4.5), which contains the energy-momentum tensor $T_{\mu\nu}$.

The energy-momentum tensor is defined as [41]

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\mathcal{L}\sqrt{-g})}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L} - 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\,, \tag{4.35}$$

where g is the determinant of the metric tensor. An alternate method to construct the energy-momentum tensor of the bilocal field is by analogy with the usual energy-momentum tensor of multi-field inflation of ϕ^4 theory. The energy momentum tensor is written as

$$T^{\mu}_{\nu} = \sum_{a=1}^{N}\left[\partial^{\mu}\phi^{a}\partial_{\nu}\phi^{a} - \left(\frac{1}{2}\partial_{\tau}\phi^{a}\partial^{\tau}\phi^{a} - \frac{1}{2}\mu^{2}\phi^{a}\phi^{a} - \frac{g}{8}(\phi^{a}\phi^{a})^{2}\right)\delta^{\mu}_{\nu}\right], \tag{4.36}$$

which in terms of the bilocal field $\sigma(x,y)$ becomes

$$\begin{aligned} T^{\mu}_{\nu} = g^{\mu\tau}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\nu}}\sigma(x,y)\bigg|_{x=y} - \delta^{\mu}_{\nu}\left[\frac{1}{2}\,g^{\tau\rho}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\rho}}\sigma(x,y)\bigg|_{x=y}\right] \\ - \left[\frac{1}{2}\mu^{2}\sigma(x,x) + \frac{g}{8}\sigma^{2}(x,x)\right]. \end{aligned} \tag{4.37}$$

To perturb the energy-momentum tensor, we must first determine the perturbation of the bilocal field $\sigma(x,y)$ which comes from the perturbation of the scalar fields $\phi(x)$ by definition of Equation (3.18). Perturbations in the scalar field $\phi(x)$ and the bilocal field $\sigma(x,y)$ are written as

$$\phi(t,x) = \phi(t) + \delta\phi(t,x) \tag{4.38}$$

$$\sigma(x,y) = \sigma(t_x,t_y) + \delta\sigma(x,y) \tag{4.39}$$

where

$$\sigma(t_x,t_y) = \sum \phi^{a}(t_x)\phi^{a}(t_y) \tag{4.40}$$

$$\delta\sigma(x,y) = \sum\left(\phi^{a}(t_x)\delta\phi^{a}(y) + \phi^{a}(t_y)\delta\phi^{a}(x)\right). \tag{4.41}$$

Substituting Equation (4.39) and Equation (4.10) into Equation (4.37) gives the perturbed energy momentum tensor:

$$\begin{aligned} T^{\mu}_{\nu} = (\bar{g}^{\mu\tau} + \delta g^{\mu\tau})\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\nu}}(\sigma(t_x,t_y) + \delta\sigma(x,y))\bigg|_{x=y} \\ - \delta^{\mu}_{\nu}\left[\frac{1}{2}\,(\bar{g}^{\tau\rho} + \delta g^{\tau\rho})\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\rho}}(\sigma(t_x,t_y) + \delta\sigma(x,y))\bigg|_{x=y}\right] \\ + \delta^{\mu}_{\nu}\left[\frac{1}{2}\mu^{2}(\sigma(t_x,t_x) + \delta\sigma(x,x)) + \frac{g}{8}(\sigma(t_x,t_x) + \delta\sigma(x,x))^{2}\right]. \end{aligned} \tag{4.42}$$

Neglecting higher order terms yields

$$\begin{aligned}
T^{\mu}_{\nu} = &\bar{g}^{\mu\tau}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\nu}}\sigma(t_x,t_y)\bigg|_{x=y} + \delta g^{\mu\tau}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\nu}}\sigma(t_x,t_y)\bigg|_{x=y} \\
&+ \bar{g}^{\mu\tau}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\nu}}\delta\sigma(x,y)\bigg|_{x=y} - \delta^{\mu}_{\nu}\left[\frac{1}{2}\,\bar{g}^{\tau\rho}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\rho}}\sigma(t_x,t_y)\bigg|_{x=y}\right] \\
&+ \delta^{\mu}_{\nu}\left[\frac{1}{2}\,\delta g^{\tau\rho}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\rho}}\sigma(t_x,t_y)\bigg|_{x=y} + \frac{1}{2}\,\bar{g}^{\tau\rho}\frac{\partial}{\partial x^{\tau}}\frac{\partial}{\partial y^{\rho}}\delta\sigma(x,y)\bigg|_{x=y}\right] \\
&+ \delta^{\mu}_{\nu}\left[\frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{8}\sigma^2(t_x,t_x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right]\,. \qquad (4.43)
\end{aligned}$$

The time-time component of the energy momentum tensor is given by

$$\begin{aligned}
T^{0}_{0} = &\bar{g}^{00}\frac{\partial}{\partial x^{0}}\frac{\partial}{\partial y^{0}}\sigma(t_x,t_y)\bigg|_{x=y} + \delta g^{00}\frac{\partial}{\partial x^{0}}\frac{\partial}{\partial y^{0}}\sigma(t_x,t_y)\bigg|_{x=y} \\
&+ \bar{g}^{00}\frac{\partial}{\partial x^{0}}\frac{\partial}{\partial y^{0}}\delta\sigma(x,y)\bigg|_{x=y} - \frac{1}{2}\,\bar{g}^{00}\frac{\partial}{\partial x^{0}}\frac{\partial}{\partial y^{0}}\sigma(t_x,t_y)\bigg|_{x=y} \\
&- \frac{1}{2}\,\delta g^{00}\frac{\partial}{\partial x^{0}}\frac{\partial}{\partial y^{0}}\sigma(t_x,t_y)\bigg|_{x=y} - \frac{1}{2}\,\bar{g}^{00}\frac{\partial}{\partial x^{0}}\frac{\partial}{\partial y^{0}}\delta\sigma(x,y)\bigg|_{x=y} \\
&+ \frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{8}\sigma^2(t_x,t_x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\,. \qquad (4.44)
\end{aligned}$$

In the evaluation above, we dropped spatial derivatives of the perturbed bilocal field $\delta\sigma(x,y)$. Despite the dependence of $\delta\sigma(x,y)$ on spatial coordinates x and y, spacial derivatives of $\delta\sigma(x,y)$ vanish from Equation (4.41), which easily can be verified. Simplifying the equation above yields

$$\begin{aligned}
T^{0}_{0} =&\frac{1}{2}\,\bar{g}^{00}\frac{\partial}{\partial \eta_x}\frac{\partial}{\partial \eta_y}\sigma(t_x,t_y)\bigg|_{x=y} + \frac{1}{2}\,\delta g^{00}\frac{\partial}{\partial \eta_x}\frac{\partial}{\partial \eta_y}\sigma(t_x,t_y)\bigg|_{x=y} \\
&+ \frac{1}{2}\,\bar{g}^{00}\frac{\partial}{\partial \eta_x}\frac{\partial}{\partial \eta_y}\delta\sigma(x,y)\bigg|_{x=y} + \frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{1}{2}\mu^2\delta\sigma(x,x) \\
&+ \frac{g}{8}\sigma^2(t_x,t_x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\,, \qquad (4.45)
\end{aligned}$$

where η_x and η_y are the conformal times coordinates x and y respectively. Substituting the metric elements using Equation (4.9) leads to

$$\begin{aligned}
T^{0}_{0} =&\frac{1}{2}\,a^{-2}\frac{\partial}{\partial \eta_x}\frac{\partial}{\partial \eta_y}\sigma(t_x,t_y)\bigg|_{x=y} - \frac{\Phi}{a^2}\frac{\partial}{\partial \eta_x}\frac{\partial}{\partial \eta_y}\sigma(t_x,t_y)\bigg|_{x=y} \\
&+ \frac{1}{2}\,a^{-2}\frac{\partial}{\partial \eta_x}\frac{\partial}{\partial \eta_y}\delta\sigma(x,y)\bigg|_{x=y} + \frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{1}{2}\mu^2\delta\sigma(x,x) \\
&+ \frac{g}{8}\sigma^2(t_x,t_x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\,. \qquad (4.46)
\end{aligned}$$

By a similar method, we may obtain the remaining perturbed energy-momentum tensor components. For the time-space components, one finds

$$T_i^0 = a^{-2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial y^i}\delta\sigma(x,y)\bigg|_{x=y} . \tag{4.47}$$

In arriving at Equation (4.47), we used the fact that the second term of the energy-momentum tensor, Equation (4.37), contains a Kronecker delta, which is zero for δ_i^0. The space-space components may be written as

$$\begin{aligned} T_j^i =& \delta_j^i\left[-\frac{1}{2}\,a^{-2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\bigg|_{x=y} + \frac{\Phi}{a^2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\bigg|_{x=y}\right] \\ &+ \delta_j^i\left[-\frac{1}{2}\,a^{-2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\delta\sigma(x,y)\bigg|_{x=y} + \frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{1}{2}\mu^2\delta\sigma(x,x)\right] \\ &+ \delta_j^i\left[\frac{g}{8}\sigma^2(t_x,t_x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right] . \end{aligned} \tag{4.48}$$

We split the results of the perturbed energy-momentum components into background and perturbation terms: $T_{\mu\nu}(t,x) = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(t,x)$.
The time-time components are

$$\bar{T}_0^0 = a^{-2}\left[\frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\bigg|_{x=y} + a^2\left(\frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{g}{8}\sigma^2(t_x,t_x)\right)\right] , \tag{4.49}$$

$$\begin{aligned} \delta T_0^0 =& a^{-2}\left[\frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\delta\sigma(x,y)\bigg|_{x=y} - \Phi\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\bigg|_{x=y}\right] \\ &+ \left(\frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right) , \end{aligned} \tag{4.50}$$

the time-space components are

$$\bar{T}_i^0 = 0 , \tag{4.51}$$

$$\delta T_i^0 = a^{-2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial y^i}\delta\sigma(x,y)\bigg|_{x=y} . \tag{4.52}$$

and the space-space components are

$$\bar{T}_j^i = a^{-2}\delta_j^i\left[-\frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\bigg|_{x=y} + a^2\left(\frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{g}{8}\sigma^2(t_x,t_x)\right)\right] , \tag{4.53}$$

$$\delta T^i_j = a^{-2}\delta^i_j \left[\Phi \frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{x=y} - \frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\delta\sigma(x,y)\Big|_{x=y}\right]$$

$$+\,\delta^i_j\left(\frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right). \tag{4.54}$$

With these results, we have all the perturbed quantities of EFEs, Equation (4.5). The EFEs represent a system of 10 differential Equations each for the background and the perturbation. The EFEs of the background and perturbations are written as

$$\bar{G}^\mu_\nu = \bar{T}^\mu_\nu \tag{4.55}$$

$$\delta G^\mu_\nu = \delta T^\mu_\nu\,. \tag{4.56}$$

We equate the background quantities of the EFEs, Equation (4.55), to obtain a dynamical system that completely describes the behaviour of the background evolution of the bilocal field $\sigma(x,y)$.

The time-time equation of the background, $\bar{G}^0_0 = \bar{T}^0_0$, reads

$$3\mathcal{H}^2 = \frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{x=y} + a^2\left(\frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{g}{8}\sigma^2(t_x,t_x)\right). \tag{4.57}$$

The time-space equations of the background vanishes, because the time-space quantities of EFEs vanish as $\bar{T}^0_i = \bar{G}^0_i = 0$. The space-space equations of the background can be written as

$$2\mathcal{H}' + \mathcal{H}^2 = -\frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{x=y} + a^2\left(\frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{g}{8}\sigma^2(t_x,t_x)\right). \tag{4.58}$$

From Equation (4.56), we obtain a dynamical set of equations that describes the perturbation evolution of the bilocal field $\sigma(x,y)$.

The time-time equation is

$$2\nabla^2\Phi - 6\mathcal{H}\Phi' - 6\mathcal{H}^2\Phi = \frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\delta\sigma(x,y)\Big|_{x=y} - \Phi\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{x=y}$$

$$+\,a^2\left(\frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right), \tag{4.59}$$

the time-space equations are

$$2\partial_i[\Phi' + \mathcal{H}\Phi] = \frac{\partial}{\partial\eta_x}\frac{\partial}{\partial y^i}\delta\sigma(x,y)\Big|_{x=y}, \tag{4.60}$$

and the space-space equations read

$$-2\Phi'' - 4\mathcal{H}'\Phi - 2\mathcal{H}^2\Phi - 6\mathcal{H}\Phi' = \Phi\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{x=y} - \frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\delta\sigma(x,y)\Big|_{x=y} + a^2\left(\frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right). \quad (4.61)$$

In terms of the physical time t, the background contributions of EFEs may be written as

$$3H^2 = \frac{1}{2}\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_y}\sigma(t_x,t_y)\Big|_{x=y} + \frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{g}{8}\sigma^2(t_x,t_x) \quad (4.62)$$

$$2\dot{H} + 3H^2 = -\frac{1}{2}\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_y}\sigma(t_x,t_y)\Big|_{x=y} + \frac{1}{2}\mu^2\sigma(t_x,t_x) + \frac{g}{8}\sigma^2(t_x,t_x). \quad (4.63)$$

Similarly, the perturbation contributions of EFEs may be written as

$$2\frac{\nabla^2}{a^2}\Phi - 6H\dot{\Phi} - 6H^2\Phi = \frac{1}{2}\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_y}\delta\sigma(x,y)\Big|_{x=y} - \Phi\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_y}\sigma(t_x,t_y)\Big|_{x=y} + \frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x)$$

$$2\frac{\partial}{\partial x^i}[\dot{\Phi} + H\Phi] = \frac{\partial}{\partial t_x}\frac{\partial}{\partial y^i}\delta\sigma(x,y)\Big|_{x=y} \quad (4.64)$$

$$-2\ddot{\Phi} - 4\dot{H}\Phi - 6H^2\Phi - 6H\dot{\Phi} = \Phi\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_y}\sigma(t_x,t_y)\Big|_{x=y} - \frac{1}{2}\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_y}\delta\sigma(x,y)\Big|_{x=y} + \frac{1}{2}\mu^2\delta\sigma(x,x) + \frac{g}{4}\sigma(t_x,t_x)\delta\sigma(x,x). \quad (4.65)$$

We have computed EFEs of the background and the perturbation for the bilocal field $\sigma(x,y)$ inflation. This allows us to investigate the dynamical evolution of the background and the fluctuations, as well as demonstrating how the bilocal field $\sigma(x,y)$ induces fluctuations Φ in space-time.

4.1.2 The energy momentum tensor of $\sigma(x,y)$ in curved space

In this section, we will prove that the replacement of the metric tensor $\eta_{\mu\nu}$ with $g_{\mu\nu}$ in Equation (4.1) is valid. To generalize, we start with an action written as [42]

$$S = \int d^4x\sqrt{-g}\sum_{a=1}^{N}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a - \frac{1}{2}(\mu^2 + \xi R)\phi^a\phi^a - \frac{\lambda}{8}(\phi^a\phi^a)^2\right), [3] \quad (4.66)$$

[3] Here, the coupling constant g in Equation (3.16) is replaced with λ to avoid the confusion with the metric determinant g.

where the scalar field ϕ^a is coupled to the gravitational field by the term $\xi R\,\phi^a\phi^a$, R is the Ricci scalar and ξ is a numerical factor. This action may be re-written in terms of the bilocal field $\sigma(x,y)$ as

$$S = \int \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - (\,m^2+\xi R\,)\sigma(x,x) - \frac{\lambda}{4}\sigma^2(x,x)\right]dx^4\,. \quad (4.67)$$

The energy momentum tensor is defined as

$$T_{\mu\,\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\,, \quad (4.68)$$

where the functional derivative of the action is given by

$$\delta S = \int \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu}d^4x\,. \quad (4.69)$$

We apply a first order variation at level of the action as $S \to S+\delta S$, $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$,$\sqrt{-g} \to \sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $R \to R+\delta R$. Using this gives

$$\begin{aligned}\delta S = &\int \frac{1}{2}\sqrt{-g}\left[\delta g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - \xi\,\delta R\,\sigma(x,x)\right]dx^4 \\ &- \int \frac{1}{4}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\left[g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - (\,m^2+\xi R\,)\sigma(x,x) - \frac{\lambda}{4}\sigma^2(x,x)\right]dx^4\,.\end{aligned} \quad (4.70)$$

Using the identity

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + g^{\rho\sigma}g^{\mu\nu}(\delta g_{\rho\sigma;\mu\nu} + \delta g_{\rho\mu;\sigma\nu})\,, \quad (4.71)$$

and integrating by parts yields

$$\begin{aligned}\delta S = &\int \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}\left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - \xi\,\left[R_{\mu\nu} - g^{\beta\gamma}g_{\mu\nu}\sigma(x,x)_{;\gamma\beta} - \sigma(x,x)_{;\mu\nu}\right]\right]dx^4 \\ &- \int \frac{1}{4}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\left[g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - (\,m^2+\xi R\,)\sigma(x,x) - \frac{\lambda}{4}\sigma^2(x,x)\right]dx^4\,.\end{aligned} \quad (4.72)$$

Combining Equation (4.68), Equation (4.69) and Equation (4.72) gives

$$\begin{aligned}T_{\mu\,\nu} = &\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - \xi\,\left[R_{\mu\nu} - g^{\beta\gamma}g_{\mu\nu}\sigma(x,x)_{;\gamma\beta} - \sigma(x,x)_{;\mu\nu}\right] \\ &- \frac{1}{2}g_{\mu\nu}\left[g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - (\,m^2+\xi R\,)\sigma(x,x) - \frac{\lambda}{4}\sigma^2(x,x)\right]\,.\end{aligned}$$

Arranging the last line gives

$$T_{\mu\nu} = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\sigma(x,y)\bigg|_{x=y} + \xi\left[g_{\mu\nu}g^{\alpha\beta}\sigma(x,x)_{;\alpha\beta} + \sigma(x,x)_{;\mu\nu}\right]$$
$$- \xi G_{\mu\nu}\sigma(x,x) + \frac{1}{2}g_{\mu\nu}m^2\sigma(x,x) + \frac{\lambda}{8}g_{\mu\nu}\sigma^2(x,x)\,. \tag{4.73}$$

It is very interesting to notice the appearance of the Einstein tensor $G_{\mu\nu}$ in the obtained expression of the energy momentum tensor Equation (4.73). The mixed energy momentum tensor is defined as

$$T^\mu_\nu = g^{\mu\tau}T_{\tau\nu}\,, \tag{4.74}$$

and from Equation (4.73), we find

$$T^\mu_\nu = g^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial y^\nu}\sigma(x,y)\bigg|_{x=y} + \xi\left[g^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial x^\nu} - G^\mu_\nu\right]\sigma(x,x)$$
$$- \delta^\mu_\nu\left[\frac{1}{2}g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\sigma(x,y)\bigg|_{x=y} - \xi g^{\alpha\beta}\sigma(x,x)_{;\alpha\beta} - \frac{1}{2}m^2\sigma(x,x) - \frac{\lambda}{8}\sigma^2(x,x)\right]. \tag{4.75}$$

We may now consider a first order perturbation of $g^{\mu\nu}$ and $\sigma(x,y)$ to compute the perturbed energy momentum tensor. We find that

$$T^\mu_\nu = \bar{g}^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial y^\nu}\sigma(t_x,t_y)\bigg|_{x=y} + \bar{g}^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial y^\nu}\delta\sigma(x,y)\bigg|_{x=y} + \delta g^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial y^\nu}\sigma(t_x,t_y)\bigg|_{x=y}$$
$$+ \xi\left[\bar{g}^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial x^\nu}\sigma(t_x,t_x) + \bar{g}^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial x^\nu}\delta\sigma(x,x) + \delta g^{\mu\tau}\frac{\partial}{\partial x^\tau}\frac{\partial}{\partial x^\nu}\sigma(t_x,t_x)\right]$$
$$- \xi\left[\bar{G}^\mu_\nu\sigma(t_x,t_x) + \bar{G}^\mu_\nu\delta\sigma(x,x) + \delta G^\mu_\nu\sigma(t_x,t_x)\right]$$
$$- \delta^\mu_\nu\left[\frac{1}{2}\bar{g}^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\sigma(t_x,ty)\bigg|_{x=y} + \frac{1}{2}\bar{g}^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\delta\sigma(x,y)\bigg|_{x=y} + \frac{1}{2}\delta g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\sigma(t_x,ty)\bigg|_{x=y}\right]$$
$$+ \delta^\mu_\nu\xi\left[\bar{g}^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\sigma(t_x,t_x) + \bar{g}^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\delta\sigma(x,x) + \delta g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\sigma(t_x,t_x)\right]$$
$$+ \delta^\mu_\nu\left[\frac{1}{2}m^2\sigma(t_x,t_x) + \frac{1}{2}m^2\delta\sigma(x,x) + \frac{\lambda}{8}\sigma^2(t_x,t_x) + \frac{\lambda}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right]. \tag{4.76}$$

We consider the perturbed metric in the Newton gauge, Equation (4.9), and the perturbed bilcoal field, Equation (4.39), to find the perturbed energy momentum tensor which can be split into background and perturbation. The background is given by the time-time component

$$\bar{T}^0_0 = a^{-2}\left[\frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\bigg|_{t_x=t_y} + 2\xi\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_x}\sigma(t_x,t_x) - 3\xi\mathcal{H}^2\sigma(t_x,t_x)\right]$$
$$+ \frac{1}{2}m^2\sigma(t_x,t_x) + \frac{\lambda}{8}\sigma^2(t_x,t_x)\,, \tag{4.77}$$

while the perturbation contributes

$$\begin{aligned}\delta T_0^0 &= a^{-2}\left[\frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\delta\sigma(x,y)\Big|_{t_x=t_y} - \Phi\,\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{t_x=t_y}\right]\\ &+ 2\xi a^{-2}\left[\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_x}\delta\sigma(x,x) - \delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\delta\sigma(x,x) - 2\Phi\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_x}\sigma(t_x,t_x)\right]\\ &- \xi a^{-2}\left[3\mathcal{H}^2\delta\sigma(x,x) + [2\nabla^2\Phi - 6\mathcal{H}\Phi^{'} - 6\mathcal{H}^2\Phi]\sigma(t_x,t_x)\right]\\ &+ \frac{1}{2}m^2\delta\sigma(x,x) + \frac{\lambda}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\,. \end{aligned} \tag{4.78}$$

The time-space components have a vanishing background, $\bar{T}_i^0 = 0$, and the perturbed part equals

$$\delta T_i^0 = a^{-2}\left[\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial y^i}\delta\sigma(x,y)\Big|_{t_x=t_y} + \xi\left(\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial x^i}\delta\sigma(x,x) - 2[\Phi^{'} + \mathcal{H}\Phi]_{,i}\sigma(t_x,t_x)\right)\right]. \tag{4.79}$$

Similarly, for the space-space components we find

$$\begin{aligned}\bar{T}_j^i &= a^{-2}\delta_j^i\left[-\frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{x=y} + a^2\left(\frac{1}{2}m^2\sigma(t_x,t_x) + \frac{\lambda}{8}\sigma^2(t_x,t_x)\right)\right]\\ &+ \xi a^{-2}\delta_j^i\left[\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_x} - [2\mathcal{H}^{'} + \mathcal{H}^2]\right]\sigma(t_x,t_x)\,, \end{aligned} \tag{4.80}$$

and

$$\begin{aligned}\delta T_j^i &= a^{-2}\delta_j^i\left[\Phi\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\sigma(t_x,t_y)\Big|_{x=y} - \frac{1}{2}\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_y}\delta\sigma(x,y)\Big|_{x=y}\right]\\ &+ \delta_j^i\left(\frac{1}{2}m^2\delta\sigma(x,x) + \frac{\lambda}{4}\sigma(t_x,t_x)\delta\sigma(x,x)\right)\\ &+ a^{-2}\xi\delta_j^i\left[\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_x}\delta\sigma(x,x) + \delta^{\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\delta\sigma(x,x) - 2\Phi\frac{\partial}{\partial\eta_x}\frac{\partial}{\partial\eta_x}\sigma(t_x,t_x)\right]\\ &- a^{-2}\xi\left[\delta^{ik}\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j}\delta\sigma(x,x)\right] - a^{-2}\xi\delta_j^i[2\mathcal{H}^{'} + \mathcal{H}^2]\delta\sigma(x,x)\\ &- a^{-2}\xi\delta_j^i[-2\Phi^{''} - 4\mathcal{H}^{'}\Phi - 2\mathcal{H}^2\Phi - 6\mathcal{H}\Phi^{'}]\sigma(t_x,t_x)\,. \end{aligned} \tag{4.81}$$

It is worthwhile to mention that these results of the components of the energy momentum tensor (both for the background and for perturbations) are exactly the same as those previously computed (Equation (4.49) - (4.54)) but in the minimally coupled case when $\xi = 0$. This shows that we can restore the previous set of Einstein Field Equations only by setting $\xi = 0$ which means that a direct replacement of the metric tensor from a flat case to curved one is consistent without performing the computations starting from curved space.

4.1.3 The perturbed Schwinger-Dyson Equation

In the previous chapter, we obtained the equation of motion of the bilocal field $\sigma(x, y)$, which is the Schwinger-Dyson Equation. In this section, we will apply the perturbation methods to the Schwinger-Dyson Equation to obtain the equation of motion of the background $\sigma(t_x, t_y)$ and the fluctuation $\delta\sigma(x, y)$. The Schwinger-Dyson Equation can be written as[4]

$$\left(-g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \mu^2 - \frac{g}{2}\sigma(x,x)\right)\sigma(y,x) = iN\delta(y-x)\,. \tag{4.82}$$

We consider the perturbed bilocal field and the perturbed metric tensor, Equation (4.39) and Equation (4.10), to perturb the Schwinger-Dyson Equation as follows

$$(-(\bar{g}^{\mu\nu}+\delta g^{\mu\nu})\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \mu^2 - \frac{g}{2}(\sigma(t_x,t_x)+\delta\sigma(x,x)))(\sigma(t_y,t_x)+\delta\sigma(y,x)) = iN\delta(y-x)\,. \tag{4.83}$$

The first term of the left hand side (LHS) of Equation (4.83) may be perturbed as

$$\begin{aligned} &-(\bar{g}^{\mu\nu}+\delta g^{\mu\nu})\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}(\sigma(t_y,t_x)+\delta\sigma(y,x)) = \\ &-\bar{g}^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\sigma(t_y,t_x) - \bar{g}^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\delta\sigma(y,x) - \delta g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\sigma(t_y,t_x)\,. \end{aligned} \tag{4.84}$$

Equation (4.84) may be expanded as

$$\begin{aligned} = &-\bar{g}^{00}\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0}\sigma(t_y,t_x) - \bar{g}^{00}\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0}\delta\sigma(y,x) - \bar{g}^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\delta\sigma(y,x) \\ &-\delta g^{00}\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\sigma(t_y,t_x)\,. \end{aligned}$$

In the last equation, we keep the spatial derivatives of $\delta\sigma(x, y)$ because the derivative is taken with respect to x twice which possesses a value through Equation (4.41). This is in contrast to the earlier computation of Equation (4.44) where the spatial derivative of $\delta\sigma(x, y)$ vanished because the spatial derivatives were with respect to x and y. Evaluating the metric components leads to

$$= -\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\sigma(t_y,t_x) - \frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\delta\sigma(y,x) + a^{-2}\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\delta\sigma(y,x) + 2\Phi\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\sigma(t_y,t_x) \tag{4.85}$$

In Equation (4.85), we changed from the conformal time η to the physical time t. Substituting Equation (4.85) into Equation (4.83) gives the perturbed form of the equation

[4] Here, we made another replacement of $\eta^{\mu\nu}$ with $g^{\mu\nu}$ which is, in general, non-trivial; compare with Equation (3.56). In this case, this replacement is valid and will be discussed in the next chapter.

of motion:

$$iN\delta(y-x) = -\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\sigma(t_y,t_x) - \frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\delta\sigma(y,x) + a^{-2}\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\delta\sigma(y,x)$$
$$+ 2\Phi\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\sigma(t_y,t_x) - \mu^2\sigma(t_y,t_x) - \mu^2\delta\sigma(y,x) - \frac{g}{2}\sigma(t_x,t_x)\sigma(t_y,t_x)$$
$$- \frac{g}{2}\delta\sigma(x,x)\sigma(t_y,t_x) - \frac{g}{2}\sigma(t_x,t_x)\delta\sigma(y,x)\,. \tag{4.86}$$

The perturbed Schwinger-Dyson Equation may be split into two equations for the background and perturbation with Dirac delta functions as a source for both

$$-\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\sigma(t_y,t_x) - \mu^2\sigma(t_y,t_x) - \frac{g}{2}\sigma(t_x,t_x)\sigma(t_y,t_x) = iN\delta(t_y-t_x)\,, \tag{4.87}$$

$$-\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\delta\sigma(y,x) + a^{-2}\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\delta\sigma(y,x) + 2\Phi\frac{\partial}{\partial t_x}\frac{\partial}{\partial t_x}\sigma(t_y,t_x) - \mu^2\delta\sigma(y,x)$$
$$-\frac{g}{2}\delta\sigma(x,x)\sigma(t_y,t_x) - \frac{g}{2}\sigma(t_x,t_x)\delta\sigma(y,x) = iN\delta(y-x)\,. \tag{4.88}$$

Notice that we sourced the differential equation of $\sigma(t_y,t_x)$ by a delta function in one dimension and that of $\delta\sigma(y,x)$ by a delta function in four dimension in order to ensure the symmetry of the system. This is because the background is a function of time only, while the perturbation is a function of both space and time. We seek solutions for $\delta\sigma(x,y)$ and $\sigma(t_x,t_y)$. These equations may be solved by taking their Fourier transform. The Fourier transform of $\sigma(x,y)$ is given by

$$\sigma(x,y) = \int \frac{d^4p}{(2\pi)^4}\,\exp\left(\,ip.(x-y)\,\right)\sigma(p)\,, \tag{4.89}$$

where $p \equiv (E,\vec{p})$. From this ansatz, we write a similar ansatz for $\sigma(t_x,t_y)$ as

$$\sigma(t_x,ty) = \int \frac{dE}{2\pi}\,\exp\left(\,ip.(t_x-ty)\,\right)\sigma(E)\,. \tag{4.90}$$

Similarly, the Fourier transform of $\delta\sigma(x,y)$ is

$$\delta\sigma(x,y) = \int \frac{d^4p}{(2\pi)^4}\,\exp\left(\,ip.(x-y)\,\right)\delta\sigma(p)\,. \tag{4.91}$$

Applying the Fourier transform of $\sigma(t_x,t_y)$ to Equation (4.87) yields

$$\sigma(E) = \frac{iN}{E^2\,-\mu^2\,-\,\frac{g}{2}\int\frac{d\omega}{2\pi}\sigma(\omega)}\,, \tag{4.92}$$

which is of the same form as Equation (3.58) but in one dimension. The Fourier transform of the differential equation of $\delta\sigma(x,y)$ is non-trivial, despite using the constructed ansatz of $\delta\sigma(x, y)$. In this chapter, we have studied the $O(N)$ vector model that is coupled to gravity. We have obtained the Schwinger-Dyson Equations and investigated the dynamics of the bilocal field $\sigma(x, y)$.

Chapter 5

Conclusion and discussion

Due to its growing observational evidence, inflation has become a fundamental part of early universe cosmology. There is theoretical controversy that concerns modelling inflation based on the presence of various successful inflationary models from single field to multi-field inflation. Although the recent Planck results [6] favours single field inflation models, multi-field inflation models have interesting observational predictions including large non-Gaussianties in the CMB. Multi-field inflation has been studied in several models very broadly [2, 13, 15, 29, 41]. Nevertheless, little research has been focused on the large N limit of multi-field inflation [8, 11]. In the large N limit, N inflatons effectively drive inflation as a single collective field. The large N limit of multi-field inflation is a remarkably interesting subject with enormous observational and theoretical advantages over single field inflation. Observational motivations for its study include the persistence of the adiabatic perturbations and Gaussianties in the CMB. There are also purely theoretical motivations such as dramatic computational simplifications and the natural occurrence of the observed dynamics without the imposition of the slow roll condition [8, 11].

We have reviewed two models of multi-field inflation which stand as a driving force behind the present work. These models are "Large N cosmology" [11] and "N-flation" [8]. The first model, "Large N cosmology" [11], shows that the adiabatic and the entropy perturbations of large N of inflatons, in fact, are easily obtained by solving two coupled differential equations for two gauge invariant variables (κ and μ). These two coupled differential equations are sufficient to explain inflationary dynamics without the need to solve a large set of N differential equations for N inflatons. This system of equations is reduced by applying a change of variables, from the original inflatons ϕ_I to Sasaki-Mukanov variables Q_I and then to particular observables (κ and μ). The second model, "N-flation" [8], implies that inflation occurs naturally without imposing conditions in multi-field models. This model avoids the slow roll condition by yielding expressions for the slow roll parameters (η and ϵ) in terms of the number of fields N. Evidently, η and ϵ are inversely proportional to N. Because N-flation is string theory inspired, a large number N of axions emerge due to requirements from string compactifications. Hence η and ϵ become very small when N is large, and the slow roll conditions are satisfied.

We have proposed a new model of inflation which, as far as we are aware, has not been studied before in the literature. This model is a special type of multi-field inflation in which the dynamics of multiple inflatons mimics those of single field inflation. In this model, an individual scalar field is not capable of giving rise to inflation by itself. Moreover, although single scalar field might be responsible alone for the inflation of the universe, the amount of inflation required to solve the problems of the standard model of cosmology may not be achieved by a single scalar field. In the large N limit, however, the modes of individual scalar fields die off and the collective mode of large numbers of these scalar fields becomes dominant giving rise to inflation. This collective mode can be determined within the framework of *collective field theory* [28, 36, 37].

With the hope of studying the collective dynamics of multi-field inflation, we have introduced collective field theory. Collective field theory is a successful approach to studying the dynamics of theories by their gauge invariant variables (observables). This theory suggests a point canonical transformation through which the original variables are transformed into observables. In turn, an effective action emerges whose classical limit is determined by taking the large N limit. Although, in general, the evaluation of the Jacobian of this transformation is subtle, nevertheless, we have shown that the Jacobian can be derived. Interestingly, no explicit expression for the Jacobian is required - a differential equation for the logarithm of the Jacobian is sufficient to obtain the equations of motion for the gauge invariant observables.

In our multi-field model, inflation is effectively driven by a single collective field, which is the bilocal inflaton field $\sigma(x, y)$. This bilocal inflaton field $\sigma(x, y)$ is a collective combination of the individual scalar fields ϕ_I. We have considered the $O(N)$ vector model, with quartic self-interactions, and applied the techniques of the *collective field theory*. A point canonical transformation has been applied to change the variables from the individual scalar fields ϕ_I to the bilocal inflaton field $\sigma(x, y)$. The equation of motion for the bilocal inflaton field $\sigma(x, y)$ is the Schwinger-Dyson equation, which can be determined by a direct minimization of the $\sigma(x, y)$ action, being the effective action S_{eff}.

We have successfully obtained the Schwinger-Dyson equation for the bilocal inflaton field $\sigma(x, y)$ (but in flat space-time). For this bilocal field $\sigma(x, y)$ to drive inflation, however, we had to minimally couple the $O(N)$ vector model to gravity, which is written in terms of $\sigma(x, y)$. This we do in order to compute the Einstein field equations. Coupling this $\sigma(x, y)$ model to gravity, a subtlety arises due to the presence of the Minkowski metric $\eta_{\mu\nu}$ in the action and Schwinger-Dyson equation for $\sigma(x, y)$. We replaced the Minkowski metric $\eta_{\mu\nu}$ with a general metric tensor $g_{\mu\nu}$ in both the action and Schwinger-Dyson equation for $\sigma(x, y)$. With this replacement, we have chosen the metric tensor $g_{\mu\nu}$ in the Newtonian gauge, with equal fluctuations, Φ, in both the time and spatial directions. We have computed the perturbed Einstein field equations and the perturbed Schwinger-Dyson equation to investigate the collective dynamics of multi-field inflation.

To check that replacing $\eta_{\mu\nu}$ with $g_{\mu\nu}$ is consistent, we have non-minimally coupled the $O(N)$ vector model to gravity, using curved space with metric $g_{\mu\nu}$ as a starting point and computed the energy momentum tensor and the Einstein field equations. As a result, we have found that the Einstein field equations obtained by the replacement of the metric tensor, and the equations obtained in the non-minimally coupled case, agree in the minimally coupled case which requires no coupling constant ($\xi = 0$). In addition, the effective energy momentum tensor, which may be determined from the effective action S_{eff}, does not exist because $\frac{\delta \ln J}{\delta g^{\mu\nu}} = 0$, and the previous set of Einstein field equations remains unaffected. If one replaces $\eta_{\mu\nu}$ with $g_{\mu\nu}$ in the $\sigma(x, y)$ action, the Schwinger-Dyson equation is unchanged and identical to the Schwinger-Dyson equation in the flat space [38, 40], since $\frac{\delta g^{\mu\nu}}{\delta\sigma(x,y)} = 0$. We therefore emphasize that the replacement of $\eta_{\mu\nu}$ with $g_{\mu\nu}$ is a consistent replacement.

Having obtained the perturbed expressions for the Einstein field equations and Schwinger-Dyson equation, one may proceed to find their solutions. As an attempt to solve these equations, we have used a translatially invariant ansatz for the bilocal field $\sigma(x, y)$, with the Fourier transform $\sigma(p)$, and obtained the Fourier transform of the background contribution to Schwinger-Dyson equation, denoted $\sigma(E)$. We did not solve the perturbed contribution to the Schwinger-Dyson equation, which is the equation of motion for $\delta\sigma(x, y)$. This equation of motion for $\delta\sigma(x, y)$ is coupled to the fluctuation in the metric tensor (Newtonian potential), $\Phi(x)$, demonstrating how $\delta\sigma(x, y)$ may induce $\Phi(x)$. Similarly, the Fourier transform $\delta\sigma(x, y)$ equation of motion may reveal a solution. If we construct, by analogy with $\sigma(p)$, a Fourier transform $\delta\sigma(p)$, we must have a similar Fourier transform of the Newtonian potential, $\Phi(x)$, in order to Fourier transform the whole equation consistently. Since $\sigma(x, y)$ is defined at two points in space-time and $\Phi(x)$ is defined at one, their Fourier transforms dot not have the same form. It is possible that we may not even need to solve the perturbed Schwinger-Dyson equation. The Einstein field equations may be sufficient to investigate the collective dynamics of inflation. With the lack of a consistent expression of the Fourier transform of the bilocal field $\sigma(x, y)$ relative to that of the Newtonian potential, $\Phi(x)$, the Einstein field equations may not be solved.

Besides the results of the Cosmic Background Explorer (COBE)[3] and Wilkinson Microwave Anisotropy Probe (WMAP)[4, 5], the recent Planck data[6] also favours single field models to drive inflation. This is due to the agreement of predictions with the adiabatic, Gaussian, homogeneous, isotropic and scale invariant spectrum of the CMB. Although, we have studied a new model of multi-field inflation, nevertheless, the COBE, WMAP and recently Planck results do not exclude our model. As we have mentioned, in our model, inflation is driven by the single collective bilocal field $\sigma(x, y)$ which becomes dominant in the large N limit.

There is much research left to be done in future work. The future developments of our new model include:

- The analytical and numerical solutions of the perturbed Einstein field equations and Schwinger-Dyson equation. The analytical solutions may be achieved through better understanding of how to couple the bilocal field $\sigma(x, y)$ to gravity which allows us to construct a consistent form of the Fourier transform of $\sigma(x, y)$ relative to the metric fluctuations $\Phi(x)$. The numerical solutions may be obtained by building and implementing efficient numerical algorithms for our perturbed system of equations.

- Applying the large N limit to these solutions will yield predictions which may be compared with observation, possibly confirming our suggested model of inflation driven by the single collective bilocal field $\sigma(x, y)$.

- Verifying that the single collective bilocal field $\sigma(x, y)$ may be responsible for the quantum-to-classical transition of the initial cosmological fluctuations required for structure formation. The essential property of the large N limit of this new model is *factorization* [38, 43]. Factorization of the correlation functions of our observables $\sigma(x, y)$ of this new model may be written as

$$\langle\sigma(x_1, y_1)\sigma(x_2, y_2)\rangle = \langle\sigma(x_1, y_1)\rangle\langle\sigma(x_2, y_2)\rangle \text{ at } N = \infty\,.$$

  This factorization relation, for any observables $\sigma(x, y)$, implies that

$$\langle\sigma^2(x, y)\rangle = \langle\sigma(x, y)\rangle^2 \text{ at } N = \infty\,.$$

  Therefore, at the large N limit, the fluctuations vanish suggesting that the large N limit is a classical limit of our new model [38, 43]. For this reason, the collective mode $\sigma(x, y)$ may be used to describe the quantum-to-classical transition process.

- Studying and constraining the non-Gaussianities in the CMB map. The non-Gaussianties may be studied by the computation of the three point correlation functions which is, in general, a very non-trivial problem [44]. The factorization property of our new model reduces the complexity of studying the non-Gaussianities dramatically by computing only one point correlation function.

This new model of inflation presents a novel application of collective field theory and large N dynamics to the early universe, as well as potentially providing an attractive resolution of many of the problems encountered in the standard model of cosmology.

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