5

PRICING INFLATION-LINKED DERIVATIVES USING THE JARROW-YILDIRIM MODEL

A dissertation submitted to the department of
Mathematics and Applied Mathematics
Faculty of Science
at the University of Cape Town
in partial fulfillment of the requirements
for the degree
Master of Science.

by

Selebelo I. Selamolela

Supervised by

Prof. Ronald Becker

May 2009

Contents

Abstract

Since the late 1990's, there's been a steady increase of government inflation linked bonds. As a result there's a slow introduction of inflation linked swaps and derivatives in the market. Inflation linked derivatives are a convenient way to get exposure to inflation. Institutions like pension funds, which have liabilities linked to inflation, may need to buy protection against rising inflation. On the other hand, institutions which have income linked to inflation maybe well positioned to offer such protection.

In this thesis we price inflation linked swaps, Caplet, Floorlet and Option on real zero coupon bond on foreign-currency analogy, as Hughston (1998) [20]. The nominal assets are thought of as domestic assets, real assets as foreign assets and the consumer price index is interpreted as the exchange rate between the nominal and real assets.

We price the inflation linked derivatives using Jarrow and Yildirim (2003) [23] three factor HJM model. We assume that volatilities of all asset price, including consumer price index, are deterministic.

We use the Hull-White short rate model for the forward volatility. This enables us to derive explicit pricing formula for inflation linked swaps, Caplet, Floorlet and Option on real zero coupon bond. We estimate volatility parameters for the volatility structure from the market prices of nominal and real bonds.

Lastly, we take an example of pricing a path dependent inflation derivative which can only be priced by simulation.

Acknowledgement

I would like to take this opportunity to acknowledge God for the opportunity to study for this degree and being able to endure till the end. I would like to send a special word of thanks to my supervisors and lecturers especially : Prof. Becker, Dr Peter Ouwehand, Dr Diane Wilcox , Prof. Troskie, Prof. Barr, Dr West (Financial Modelling Agency), Prof. David Taylor(University of Witwatersrand), Fernado Durrel. I would like to thank the National Research Fellowship and Deutscher Akademischer Austausch Dienst who sponsored my studies. I would like to thank everyone who contributed positively to the success of this thesis, namely Gill Raine (Rand Merchant Bank), Mark Greenwood (Rand Merchant Bank)Merlene Pillay (Bond Exchange of South Africa), University of Cape Town Interlibrary loans and Chanta (Standard Corporate Merchant Bank). I would like to thank all of my classmates, because without them I would not have been able to endure the race, especially Edward Nthite (Arbitrageur), Mphendulo Mkwanazi, Anita, Asanda Qina, Faeez Safedien, Mark de Arjou and Petrus Bosman. I would also like to thank my family for their constant support through the years, especially : Lydia , Tertia, Neo, Mosa, Lopang,Brian, Mathlodi, Mamane Kedi, Jake, Maureen, Bra Jeff, Katlego, Koketso, Ausie Morwesi and Uncle Seladi. I also like to thank the council of men who made sure that I run the whole race, namely : Evidence Buthelezi,Ntala Lelimo,Radichaba Malapane, Mpho Mokoena, Sandile Mchunu, Saul Rakwena and Ezekiel and Viwe Vilakazi. I would also like to thank the His People church for everything they have done for me. I would also like to thank Mamello for her encouragement. I would like to thank Zareer and Sobahle (Consultant), Allan and Lydia (Cape Community FM).

- guarantees the value of fixed income

He proposed that indexation of debt to be compulsory in private contracts [24]. The challenge in his proposal would be to define the index.

In 1886, Alfred Marshall, further expantiated on work which has been written on indexation. He proposed that a law be passed which stipulated the use of indexation in contracts for deferred payments [26]. This indexation protected the debtors and creditors from the effects of inflation. Irving Fisher 1922 said."The ideal is that neither debtor nor creditor should be worse off from having been deceived by unforseen changes" [12].

John Maynard Keynes supported indexation of debt. In 1924 he proposed that the Royal Commission on National Debt and Taxation that the British gorvenment should issue index-linked bonds [25]. He believed that the gorvenment would save on interest cost given that investors might be prepared to pay a premium on the bonds.

Index debt became popular in the second half of the 21st century after the second world war. High inflation caused many countries to issue indexed debt as a form of stabilizing prices e.g. Finland 1945. The issue of indexed bonds was discontinued because of devaluation of respective currencies. Devaluation of these currencies would cause high import price. A combination of high import prices and the supply of indexed linked debt caused fears of high pressure on inflation.

In the 1950 and 1960 a few countries issued indexed debt i.e. Austria, France, Sweden and Iceland. During this period numerous countries which suffered from hyperinflation issued indexed debt as a means of maintaining long-term debt contract e.g. Argentina, Brazil and Mexico. Hyperinflation can be defined as astronomical rise in prices in which the currency purchasing power diminishes. The same trend was evident in other countries which experienced increasing inflation. In the 1970 moderately high inflation was a permanent feature of the economic system. This fact motivated subsequent issue of indexed bonds e.g. in the United Kingdom, Australia, Denmark and New Zealand.

In 1981 the British government took a significant step in issuing indexed-linked debt. These bonds had a high demand because investors were familiar with high inflation since the 1970's and there was poor expectation of the government's ability to control inflation given a secret oil price shock in 1979. The first Australian indexed securities were issued in 1983 by State Electricity Commission Victoria.

The latest issuers of indexed bonds are the United States of America (US) and Turkey in 1997. The motive for the US Treasury to issue these bonds is twofold: firstly, the purchasing power of the investor would be protected and secondly, the government will save costs in servicing debt.

See table 1 for a summary of countries which issue indexed bonds. Some of the tabulated instruments may not be on issue.

Table 1.1 : Summary of countries which have issued indexed bonds [1]

COUNTRY	ISSUE DATE	INDEX USED
Argentina	1972-89	Non-agricultural wholesale prices
Australia	1983-	Consumer prices
	1991	Average weekly earnings
Austria	1953	Electricity prices
Brazil	1964-90	Wholesale prices
	1991-	General prices
Canada	1991-	Consumer prices
Chile	1966-	Consumer prices
Colombia	1967	Wholesale prices
	1995-	Consumer prices
Czech Republic	1997-	Consumer prices
Denmark	1982-	Consumer prices
Finland	1945-67	Wholesale prices
France	1952, 1973	Gold price
	1956	Level of industrial production
	1956	Average value of French security
	1957	Price of equities
Greece	1997-	Consumer prices
Hungary	1995-	Consumer prices
Iceland	1955	Consumer prices
	1964-80	Cost of Building Index
	1980-94	Credit Term Index
	1995-	Consumer prices
Ireland	1983-	Consumer prices
Israel	1955-	Consumer prices
Italy	1983	Deflator of GDP at factor cost
Mexico	1989-	Consumer prices
New Zealand	1977-84	Consumer prices
	1995-	Consumer prices
Norway	1982	Consumer prices
Poland	1992-	Consumer prices
Sweden	1952	Consumer prices
	1994-	Consumer prices
South Africa	2000-	Consumer prices
	1994-	Consumer prices
Turkey	1994-1997	Wholesale prices
	1997-	Consumer prices
United Kingdom	1975-	Consumer prices
	1981-	Consumer prices
United States	1742, 1780	Commodity prices
	1997-	Consumer prices

[1] Source: Inflation-indexed securities : Bonds, Swaps and Other Derivatives by Deacon M, Derry A, Mirfendereski D (Page 5)

1.2 Indexed Securities

We should now appreciate that fluctuations in inflation causes an increase or decrease in the value of wealth for either debtors or creditors. Index linked securities provide opportunities for investors and issuers to remove this risk and uncertainty. In the next paragraphs we will discuss the following topics: economic analysis and benefits for monetary policy, rationale of issuers and investors in issuing indexed securities.

1.2.1 Benefits for monetary policy and economic analysis

Inflation linked bonds can be used by economist to determine objective inflation expectations. The inflation expectation is determined by the market. This is very useful to monetary policy authorities, firstly they meaningfully forecast inflation. Secondly, they measure market perception on monetary policy. If inflation expectations decrease then the market perceive's current monetary policy to be robust.

This method is based on Fisher's equation. The equation simplistically states that nominal interest rates are equal to the sum of inflation expectation and real interest rates. If we take the difference between nominal yield of conventional bonds and real yield of index linked bonds with the same maturity will give a market expectation of average inflation for the life of the bonds.

There are flaws to this method, namely:

- Bonds with the same maturity are rare
- Yields of bonds with different coupons are not comparable
- Long-dated bonds may have risk and/or liquidity premiums

The difference between yields of conventional bonds and real yields of indexed bonds include inflation risk premiums. Tax effects might also be incorporated. The Bank of England believe that the change in inflation expectations is important rather than the actual inflation level. Changes in measured inflation will be unaffected if inflation premiums and tax effects are constant for a period of time. "The Bank of England found that changes in Britain inflation rate correlate fairly closely with changes in expected inflation" [33].

There are many methods of forecasting inflation. The method described above is reliable because the prediction is backed by investments.

In this paragraph we will elaborate on the benefits of inflation indexed bonds. Firstly, inflation indexed bonds give the investor long term assets with fixed long term real yields. Secondly, the Treasury will benefit from inflation protection from these inflation indexed bonds. Consequently, the Treasury will save on interest expense. Lastly, policymakers will benefit by gaining information about real interest rates and market inflation expectations both in the short and long term.

In this paragraph we will elaborate on the limitations of inflation indexed bonds. Firstly, there are many inflation indices and none of them meets the ideal condition. Secondly, there limitations due to the choice of inflation index. Thirdly, measurement biases of the inflation index e.g. Statistics South Africa are currently reviewing the CPI index because it was over-estimated. Fourthly, limitations caused by lag of indexation. Lastly, limitations due to taxation.

1.2.2 The Issuer's Perspective

One should always keep in mind that financial markets are a subset of international financial systems. "The financial system can be defined as a set of arrangements embracing the lending and borrowing of funds by non-financial institutions to facilitate the transfer of funds, to provide additional money when required, and to create markets in debt instruments so that the price and allocation of funds are determined efficiently" [13]. This definition acknowledges four sectors of a financial system, which are : borrowers and lenders, financial institutions, financial instruments and financial markets. Financial markets facilitate issues and trading (dealing) of financial instruments through institutional arrangements and conventions.

The South African financial sector has been extremely dynamic in the past two decades. This has resulted in the changes made by monetary authorities and the financial sector which brought about South Africa's re-entry into global markets. The ripple effects of these changes are a change in implementing monetary policy, developments of new financial instruments and products, and higher activity in financial markets. In the subsequent paragraph we will generate a discussion of different types of issuers of indexed linked securities.

Governments and Parastatals

Indexation of securities offers government the benefits in cost and management of public debt. Moreover, it gives government an opportunity to estimate the market long term inflation expectation. An issue of such securities can be viewed by the market as constructive efforts by government to keep inflation under control.

Public funded infrastructure projects

Capital indexed bonds are similar to infrastructure projects which has cashflows which are linked to the Consumer Price Index. Capital indexed bonds are bonds whose capital value of the bond is linked to an index. Refer to the next chapter for details of this structure. There are usually lower cashflows in the early stages of the loan which need to be paid to the investor because the project would have not yet generated considerable income. Moreover, these projects have a long duration, so there is less debt service pressure on the project in the early stages. It is important to note the real cost remain constant during the financing period of the project.

The South African inflation-linked security market emerged in the mid nineties. The evolution of financing South African toll roads initiated the creation of this market. The N1 Warmbath to Pietersburg toll road was the first project to be implemented. A series of similar projects were thus implemented. These projects set the platform for the development of CPI-bonds listed on the Bond Exchange of South Africa.

The are other issuers who participate in this market. Privately funded projects and corporations are some of these issuers. Privately funded projects function very similarly to public funded projects. They fund projects by issuing indexed securities. The cashflows of these projects are linked to the inflation index.

1.2.3 The investor's perspective

The primary benefit for investors to invest in index linked securities is to hedge their positions against inflation. Investors transfer inflation risk which is a constituent of long term debt to the issuer. Investors invest in these instrument to fulfill specific risk and return objectives which would not be offered by existing assets. In the subsequent paragraphs we will discuss two reasons why investors should include index linked securities in a portfolio.

Firstly, investors include index linked securities in their portfolio to match their liabilities. Liabilities for pension funds and insurance companies have a similar structure to index linked securities. Indexed securities broaden the investors selection of asset. These particular types of investment benefit investors who are risk averse i.e. pension funds.

Secondly, investors include indexed securities for the purpose of diversifying their portfolio. This point will be discussed in detail in the subsequent chapter

1.3 South African Inflation Bond Market

South Africa offered inflation-indexed bonds in 2000. The motive behind the issue was a response to the backdrop in indexed linked cooperate debt and CPI bonds which have been issued to finance toll roads. The South African government had two compelling reasons for issuing this bond. Firstly, the view of a fall in inflation in the medium term was prevalent. As a result the government issued inflation-indexed bonds as a form of reducing the real cost of servicing debt. Secondly, the government thought there would be sufficient demand of this product from the pension fund industry.

The government objective is to develop the 10 to 30 year real (inflation-indexed) yield curve. It also wanted to increase the inflation index bond proportion to 10 percent of the South African government debt. Table 1.1 shows international market capitalization of inflation bond markets. The United States of America ranks first with a total of 226 billion, whilst South Africa has 4.5 billion [2].

Table 1.1 : Inflation linked bond market capitalization

	US	UK	France	Sweden	Canada	Italy	South Africa
Market value ($ US billions)	224	103.1	76.5	23.1	20.6	19.2	4.5
Number of indexed bonds	13	9	2	5	4	2	5
Longest maturity	2032	2035	2032	2028	[illegible]	2014	[illegible]

The South African indexed bonds use the Consumer Price Index for all urban areas as their index. These securities are semi-annual capital-indexed bonds (refer to later chapters). The lag in the South African instruments is four months. This is caused by the delay in releasing CPI data. These bonds have a built in deflation floor which protects the stakeholders from deflation. Deflation is a decline in general prices of goods and services that increase purchasing power of money. These concepts are discussed in details in subsequent chapters. The tax treatment of these instruments will not be covered in this thesis.

On 15 March 2000 the department held its first inflation-indexed bond auction. The value of this auction was targeted at one billion Rands. The bid from the real yield ranged from 5 to 10 percent. A real yield was set at 6.5 percent. This resulted in R495 million issue of this bond. Subsequent bond auctions followed. The bond code is R189 and it pays a semi-annual coupon of 6.25 percent. It matures on 31st March 2013.

On the 25th May 2001 a new inflation-index bond was issued. The bond code is R197. This bond pays 5 percent semi-annual coupon and it matures on 7th December 2023.

In November 2001 the Trans-Caledon Tunnel Authority (TCTA) issued a government guaranteed bond. This instrument was used as part of the financing strategy of the

2

- Source : Barclays Capital. Last updated 31 March 2004
- Note : Bonds of less than 1 year to maturity or with a market value of less than 100 million dollars are excluded from the statistics

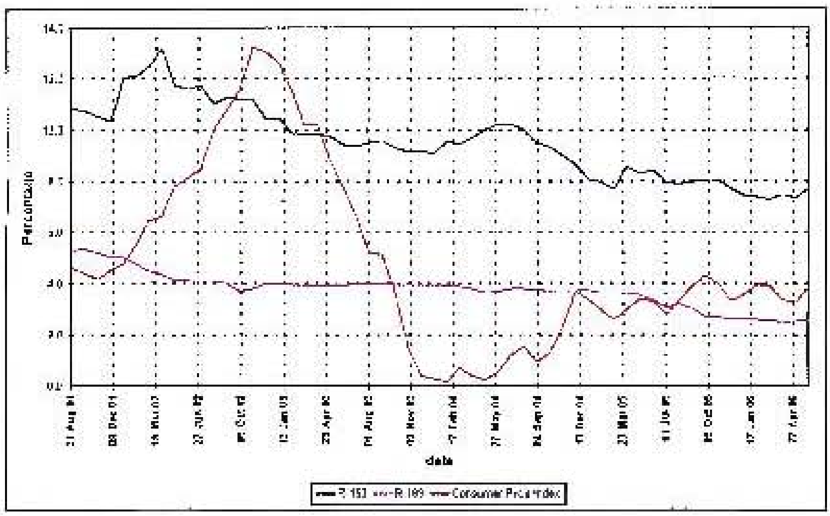
Lesotho Highlands Water Project. This bond bridged the gap between the two existing bonds. The bond code of this bond is WS05. It pays a 5 percent coupon and it matures on 1st August 2018. This bond along with others is listed on the Bond Exchange of South Africa.

In April 2002 another bond was launched under the bond code R198. This bond has reached maturity on the 31st March 2008. Embedded in the conditions of this issue is the intention of the South African Treasury to strip the bond. The coupons for the bond of this instrument are at 3.8 percent. The reason behind issuing the R198 is to meet the demand of the retail banks. These banks use this bond to hedge CPI linked structured products and synthetic products. This bond also appeals to investors who want to hedge short term inflation exposure.

On the 15th August 2003 the South African treasury issued another longer maturity bond. The bond matures on the 7th December 2033 and it trades under the bond code R202. The bond has a coupon rate of 3.45 percent.

The bond real yield curve has been declining. This may be caused by two factors. Firstly, the investors concern of a rise in inflation and secondly a demand in CPI bonds. See the figure below which shows the R189 and R153 historical yields. These bonds have similar maturity. The graph shows year to year inflation. The average year on year inflation is 4.8 percent over the period ranging from 31 March 2000 and 31 May 2004. This is moderate and it confirms the demand for CPI bonds and a view by investors who invest in these instruments that they expect inflation to rise.

Figure 1.1 : Historical graph of R153,189 and the year on year CPI



1.4 Derivatives

1.4.1 Background

The inflation-linked derivatives market has developed significantly. These developments have given rise to renewed pricing and hedging methodologies. Moreover, these instruments are investigated with the purpose of bringing value to issuers and investors. In the next paragraph we will discuss shortfalls of not issuing inflation-linked derivatives.

Firstly, the issuer may require small inflation exposure and this would not justify the costs associated with a bond issue. Secondly, the issuer may want to secure certain levels of rates prior to large issues. Lastly, investors may not be willing to take on credit risk associated with issuers. These are some of the problems experienced due to the lack of inflation-linked derivatives. These are some of the problems encountered in inflation linked bond markets. Issuing inflation linked derivatives will provide solutions to these problems.

Before we propose any solutions we will mention key variables which drive inflation linked derivatives. They are as follows:

- Maturity
- Timing
- Index

- Profile
- Size

In this paragraph we will discuss some of the advantages of this instrument. Firstly, investors and issuers can select a favorable index. Secondly, one can improve liquidity of inflation linked bonds. Thirdly, they provide market timing advantages. Fourthly, they provide customized cash flows, arbitrage trades and creation of hybrid structures. Lastly, they it gives efficient risk transfer.

In the subsequent paragraph we will discuss the stages of development in a derivative market.

- Stage 1: There are no tradable instruments in the market. Derivatives prices are determined by supply and demand. This stage is referred to as an incomplete market.
- Stage 2: A few tradable instruments are available in the market. However supply and demand governs prices.
- Stage 3: There are many tradable market instruments in the market. The market is complete. Supply and demand plays a secondary role in determining prices.
- Stage 4: The market has reached a level of maturity. The market is thus liquid and complete. Currently, there is no market which has reached this level.

1.4.2 Inflation derivatives

Since the late 1990's, there's been a steady increase of government inflation linked bonds. As a result there's a slow introduction of inflation linked swaps and derivatives in the market. Inflation linked derivatives are a convenient way to get exposure to inflation. Institutions like pension funds, which have liabilities linked to inflation, may need to buy protection against rising inflation. On the other hand, institutions which have income linked to inflation maybe well positioned to offer such protection.

The trend for inflation linked derivatives has continued to grow steady, especially in developed economies. The most popular being, inflation linked year on year swaps. An inflation linked swap is a swap where an institution is obliged to pay or receive the inflation rate in exchange of a fixed payment. Inflation linked options with non-linear payoff's are still illiquid, e.g. options.

Hughston (1998) [20] was one of the first to pioneer studies on pricing inflation linked derivatives using the foreign-currency analogy. The nominal assets are thought of as domestic assets, real assets as foreign assets and the consumer price index is interpreted as the exchange rate between the nominal and real assets.

For one to price inflation derivatives one has to specify a model to be valued. In most of contemporary literature on pricing inflation linked derivatives the one must prescribe the foreign currency analogy. This is the same methodology that has been used by Barone and Castagna (1997) [2] and Jarrow and Yildirim (2003).

This methodology is intuitive. There's no redundancy modeling the real rate, nominal rate and inflation index together. The true real rate will only be known when as soon as the Consumer Price Index is published.

Jarrow and Yildirim (2003) [23] use a three factor Heath Jarrow Morton (HJM) model to price TIPS (Treasury Inflation Protected Securities) and options written on inflation. They rely on the foreign currency analogy to develop their pricing methodology. They assume that volatilities of all asset prices, including consumer price index, are deterministic. They use the Hull-White short rate model for the forward volatility. This enables us to derive explicit pricing formula for options on inflation. We estimate volatility parameters for the volatility structure from the market prices of nominal and real bonds.

Mercurio (2005) [29] uses the Jarrow and Yildirim method to price zero coupon and year on year inflation linked swaps, caplets and floorlets.

Hinnerich (2006) [18] extended the Jarrow and Yildrim methodology to an HJM model which has more than 3 factors. He also allowed the possibilities of jumps in the economy. Standard multidimensional Wiener and point processes drive the random process that describes the consumer price index, nominal and real markets. They assume the volatilities of asset prices and consumer prices are deterministic. Moreover, they also assume the point process is deterministic. These two above mention facts ensures that there's a close form solution. They price year-on-year inflation indexed swaps, options written on TIPS, zero coupon inflation linked swaptions and year-on-year inflation linked swaptions.

Chapter 2

Inflation model for derivative pricing

2.1 Inflation linked derivatives

2.1.1 Inflation option

Inflation option is an option with a payoff were it will payout if the difference between relative changes of prices from reference and expiry date and this difference is compared to specified fixed rate.

2.1.2 Inflation swaps

There are two main swaps that dominate the inflation linked derivative market, namely: zero coupon inflation linked swap and year on year inflation linked swap. Zero coupon inflation linked swap is a contract which agrees to pay a fixed rate in an exchange of a rate which is linked to a consumer price index, over a specified term. The year on year inflation linked swap contract agrees to pay a fixed rate in exchange to the average rate of inflation over the current year.

2.1.3 Inflation caps

This pays out if the inflation, which is measured by relative increase of the inflation index, exceeds a specified threshold over a defined period.

2.1.4 Inflation swaption

Inflation swaption gives one a right to enter into an inflation swap ,zero coupon inflation linked swap and year on year inflation linked swap, at a specified future date with a given fixed rate.

2.2 Jarrow Yildirim model

The purpose of this section is to develop a pricing model for inflation linked derivatives. We assume a foreign currency analogy is used to implement this methodology. In our case, the nominal rands correspond to the domestic currency, real rands corresponds to the foreign currency and inflation index corresponds to the spot exchange rate [1]. Consequently, one expects fluctuations in the real, nominal and inflation rate to be correlated.

2.2.1 Notation and definitions

The following notations will be used in this paper :

- r for real and n for nominal
- $P_n(t,T)$: time t in Rands price of a nominal zero-coupon bond maturing at time T. Nominal zero coupon bonds are conventional bonds which pay a certain nominal return. However, inflation might erode the real value of the bond.
- I(t) : Consumer Price Index Urban (CPI) at time t. The units are Rand per CPI units
- $P_r(t,T)$: time t price of real zero-coupon bond maturing at time T in CPI-U units. A real zero coupon bond is a bond whose real value is certain. However, future cashflows(coupons and principal) has to be adjusted by a relative change in prices.
- $f_k(t,T)$: time t forward rates for date T where $k \epsilon \{r,n\}$

The discount bonds are represented by

$$P_k(t,T) = \exp(-\int_t^T f_k(t,u)du) \qquad (2.1)$$

- $r_k = f_k(t,t)$: the time t spot rate where $k \epsilon \{r,n\}$
- $B_k = \exp(\int_0^t r_k(s)ds)$:time t money market account value for $k \epsilon \{r,n\}$.
- $\mathbf{B}(0)$: time 0 price of conventional coupon bearing bond in Rands.

The nominal coupon bearing bond is represented by the following equation:

$$\mathbf{B}(0) = \sum_{t=1}^{T} CP_n(0,t) + FP_n(0,T) \tag{2.2}$$

where C is the coupon payments, T is time to maturity, F is the face value.

- $\mathbf{B}_{CPI}(0)$: time 0 price of inflation linked bond (real coupon bearing bond).

The real coupon bearing bond is represented by the following equation:

$$\mathbf{B}_{CPI}(0) = \{\sum_{t=1}^{T} C_j I(0) P_r(0,t) + F_j I(0) P_r(0,T)\} / I(t_0) \tag{2.3}$$

where C_j are the adjusted coupons , F_j is the adjusted principal, $I(0)$ is CPI-U at settlement date of security, $I(t_0)$ is CPI-U at the issue date of security and T_j is the maturity. The ratio $I(0)/I(t_0)$ in equation 2.3 is used to adjust for inflation and it is applied to the coupon and principal at maturity.

An instrument associated with cashflow structure of equation 2.3 is called a capital indexed bond. Let us look at a hypothetical example in order to illustrate the mechanics of this bond. Figure 2.1 and Table 2.1 illustrates a cashflow profile of a 5 percent annual coupon bond which is maturing in 10 years. The inflation profile has been assumed. Moreover we assumed a 1 million principal value. The table shows how coupons and the principal are adjusted for fluctuations in inflation.

Figure 2.1 : Graph showing cashflow payments of a Capital Indexed Bond

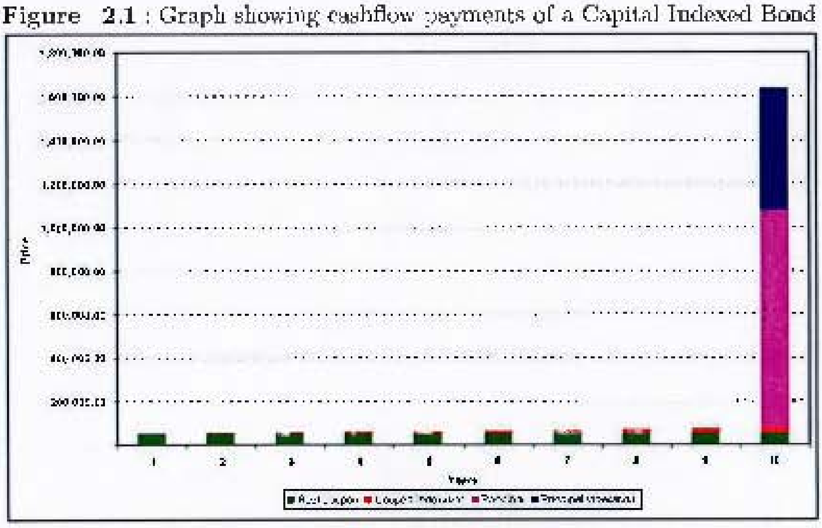


Table 2.1 : Example of cashflow payments of a Capital Indexed Bond

Year	Real coupon	Inflation rate (%)	Compounded Inflation (%)	Coupon payments	Coupon Indexation	Principal	Principal payments	Principal Indexation
1	50,000.00	6.0%	106.00%	53,000.00	3,000.00			
2	50,000.00	5.5%	111.83%	55,915.00	5,915.00			
3	50,000.00	3.9%	116.19%	58,096.69	8,096.69			
4	50,000.00	3.6%	120.37%	60,187.13	10,187.13			
5	50,000.00	3.6%	123.88%	61,892.74	11,892.74			
6	50,000.00	3.2%	127.96%	63,976.51	13,976.51			
7	50,000.00	4.1%	133.20%	66,599.66	16,599.66			
8	50,000.00	4.9%	139.73%	69,862.93	19,862.93			
9	50,000.00	5.5%	147.41%	73,705.39	23,705.39			
10	50,000.00	5.7%	155.81%	77,906.60	27,906.60	1000000	1558131.334	558131.334

- We define the price in Rands of a real zero-coupon bond without an issue date adjustment as:

$$P_{CPI}(t,T) = I(t)P_r(t,T) \tag{2.4}$$

We firstly define our probability space as $(\Omega, F, \mathbb{P}, \mathbf{F}_t)$. This probability space characterizes the uncertainty in the economy. Ω is our state space, F is a set of possible events (σ - algebra on Ω), $\mathbb{P}$ is statistical probability measure on (Ω, F) and $(\mathbf{F}_t t\epsilon\{0,T\})$ is the standard filtration generated by Brownian motion $(W_n(t), W_r(t), W_I(t) : t\epsilon[0,T])$. The brownian motions are initialized at at zero and they have the following correlations:

- $dW_n(t)dW_r(t) = \rho_{nr}dt$
- $dW_n(t)dW_I(t) = \rho_{nI}dt$
- $dW_r(t)dW_I(t) = \rho_{rI}dt$

Given the initial forward rate $f_n^*(0,T)$, we assume that the nominal forward rate evolves as:

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t) \tag{2.5}$$

where $\alpha_n(t,T)$ is random and $\sigma_n(t,T)$ is deterministic function of time subject to smoothness and boundedness conditions (See section 2.3). The deterministic volatility implies that that the nominal term structure of interest rates generates a Gaussian economy.

Similarly, given the initial forward rate $f_r^*(0,T)$, we assume that the real forward rate evolves as:

$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t) \tag{2.6}$$

where $\alpha_r(t,T)$ is random and $\sigma_r(t,T)$ satisfy the same conditions as expression 2.45. The inflation index evolution if given by:

$$dI(t) = I(t)\mu_I dt + \sigma_I(t)dW_I(t) \tag{2.7}$$

where $\mu_I(t)$ is random and $\sigma_I(t)$ is deterministic function of time subject technical smoothness and boundedness conditions [1]. The deterministic volatility in 2.47 implies that the inflation index follows a geometric brownian motion. Consequently, we know that the logarithm of the inflation index dynamics is normally distributed. This assumption complements the the imposed Gaussian HJM economy.

2.2.2 Arbitrage free foreign currency analogy

In this section we state restrictions that need to be imposed upon the stochastic process stated so that the foreign currency analogy constructed is both arbitrage free and complete. This foreign currency economy is arbitrage free and complete if there exist a unique equivalent probability measure $\mathbb{Q}$ such that [22]:

$$\frac{P_n(t,T)}{B_n(t)}, \quad \frac{I(t)P_r(t,T)}{B_n(t)} \quad and \quad \frac{I(t)B_r(t)}{B_n(t)} \quad are \quad \mathbb{Q}-martingales \tag{2.8}$$

By Girsanov's theorem then there exists market prices of risk $(\lambda_n(t), \lambda_r(t), \lambda_I(t) : t\epsilon[0,T])$ [2] such that:

$$\hat{W}_k(t) = W_k(t) - \int_0^t \lambda_k(s)ds \quad for \quad t\epsilon\{n,r,I\} \tag{2.9}$$

are $\mathbb{Q}$ brownian motions.

In the following proposition we provide the necessary conditions needed on various bond price evolutions so that the economy is arbitrage free.

Proposition 2.1 : Arbitrage free term structure

$$\alpha_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)ds - \lambda_n(t)\right) \tag{2.10}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)ds - \sigma_I(t)\rho_{rI} - \lambda_r(t)\right) \tag{2.11}$$

$$\mu_I(t) = r_n(t) - r_r(t) - \sigma_I(t)\lambda_I(t) \tag{2.12}$$

Expression 2.10 is the arbitrage free forward rate drift restriction as in the HJM model (see appendix 7). Expression 2.11 is the analogous arbitrage free forward rate drift restriction for the real forward rates. Expression 2.12 is the Fisher equation. It shows the relationship between nominal interest rates to real interest rates and expected inflation rate.

---

[1] $\mu_I(t)$ is $\mathbf{F}_t$ - adapted and jointly measurable with $\mathbf{E}[\int_0^T |\mu_I(t)|^2 dt] < \infty$ $\mathbb{P}$ - a.s. and $\sigma_I(t)$ is deterministic function of time with $\int_0^T \sigma_I^2(s)ds < \infty$ $\mathbb{P}$ - a.s

[2] The market price of risk are $\mathbf{F}_t$. Moreover, the Radnon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ at time T is: $\frac{d\mathbb{Q}}{d\mathbb{P}} = exp(-\frac{1}{2}\int_0^T < \sum_{k\epsilon\{n,r,I\}} \lambda_k(s)dW_k(s), \sum_{k\epsilon\{n,r,I\}} \lambda_k(s)dW_k(s) > + \sum_{k\epsilon\{n,r,I\}} \int_0^T \lambda_k(s)dW_k(s))$ where $< .,. >$ is the quadratic variation process

In the next proposition we apply Itô's lemma and the above proposition to generate price processes which hold under the martingale measure.

Proposition 2.2 : The term structure evolutions under the martingale measure

$$df_n(t,T) = \sigma_n(t,T)\int_t^T \sigma_n(t,s)ds + \sigma_n(t,T)d\hat{W}_n(t) \tag{2.13}$$

$$df_r(t,T) = \sigma_r(t,T)\left[\int_t^T \sigma_r(t,s)ds - \rho_{rI}\sigma_I(t)\right]dt + \sigma_r(t,T)d\hat{W}_r(t) \tag{2.14}$$

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r]dt + \sigma_I(t)d\hat{W}_I(t) \tag{2.15}$$

$$\frac{dP_n(t,T)}{P_n(t,T)} = r_n(t)dt - \int_t^T \sigma_n(t,s)dsd\hat{W}_n(t) \tag{2.16}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t) - \rho_{rI}\sigma_I(t)\int_t^T \sigma_r(t,s)ds\right]dt - \int_t^T \sigma_r(t,s)dsd\hat{W}_r(t) \tag{2.17}$$

$$\frac{dP_{CPI}(t,T)}{P_{CPI}(t,T)} = r_n(t)dt + \sigma_I(t)d\hat{W}_I(t) - \int_t^T \sigma_r(t,s)dsd\hat{W}_r(t) \tag{2.18}$$

The above stochastic differential equations are proved in appendix 2. These stochastic processes will prove useful when pricing derivatives. We are going to use the process $P_{CPI}(t,T)$ as a stochastic process for inflation linked zero coupon bond price.

2.3 Closed form solution

2.3.1 Inflation linked swaps

An inflation linked swap is a swap of inflation linked cash flow over predetermined fixed rate. The exchange of these cash flows are at given set dates $T_1, ..., T_m$. The inflation rate used to calculate inflation cash flows is the percentage return of the CPI index over the time it applies to.

There two main inflation linked swap, namely: zero coupon swap V_{ZCIIS} and year on year swap V_{YYIIS}.

Inflation linked zero coupon swap

In this section we price a zero coupon inflation linked swap. We use the same methodology used by Jarrow-Yildrim to price this instrument.

In a V_{ZCIIS}, the party pay the recipient party a fixed amount of:

$$N[(1+K)^M - 1] \tag{2.19}$$

at maturity, $T_M = M$ years
N= nominal value
K= fixed rate

This fixed payment is in exchange of a floating amount of:

$$N[\frac{I(T_M)}{I_0} - 1] \tag{2.20}$$

where $I(t)$ is the inflation index, CPI. The value of the inflation-indexed leg V_{ZCIIS} at time t, where $0 \leq T \leq T_M$, $\mathcal{F}_t$ is $\sigma-$ algebra.

This result is implied by no-arbitrage pricing theory. According to Jarrow-Yildrim foreign currency analogy, we know that the nominal price of the real zero coupon bond equals one unit of CPI index at maturity. The maturity correspond to the maturity of the bond. See the following equation:

$$V_{ZCIIS}(t, T_M, I_0, N) = N\mathbf{E}_n\left(e^{-\int_t^{T_M} n(u)du}\left[\frac{I(T_M)}{I_0} - 1\right]|\mathcal{F}_t\right) \tag{2.21}$$

where

$$I(t)P_r(t,T) = I(t)\mathbf{E}_r\left(e^{\int_t^T r(u)du}|\mathcal{F}_t\right) = \mathbf{E}_n\left(e^{\int_t^T n(u)du}I(T)|\mathcal{F}_t\right) \tag{2.22}$$

So if we substitute this result to equation for V_{ZCIIS}, then:

$$V_{ZCIIS}(t, T_M, I_0, N) = N\left[\frac{I(t)}{I_0}P_r(t, T_M) - P_n(t, T_M)\right] \tag{2.23}$$

Simplify equation at $t = 0$.

$$V_{ZCIIS}(0, T_M, N) = N\left[P_r(t, T_M) - P_n(t, T_M)\right] \tag{2.24}$$

So we are able to strip real zero-coupon bond prices from observed zero coupon inflation indexed swaps.

Year on year swap

In a year on year inflation linked swap, the one party pays the recipient a fixed amount at each time T_j

$$N\chi_j K \tag{2.25}$$

where χ_j is the fixed leg interval $[T_{j-1}, T_j]$. N and K are the nominal and fixed rate respectively. This fixed payment is in exchange of a floating amount of:

$$N\chi_j \left[\frac{I(T_j)}{I_{j-1}} - 1\right] \tag{2.26}$$

The value of the floating leg at time T_j, where $t < T_j$ is

$$V_{YYIIS}(t, T_{j-1}, T_j, \chi_j, N) = N\chi_j \mathbf{E}_n \left(e^{-\int_t^{T_j} n(u)du} \left[\frac{I(T_j)}{I_{j-1}} - 1\right] |\mathcal{F}_t\right) \tag{2.27}$$

If we assume that $t < T_{j-1}$ so that we can simplify equation V_{YYIIS}, then

$$N\chi_j \mathbf{E}_n \left(e^{-\int_t^{T_{j-1}} n(u)du} \mathbf{E}_n \left(e^{-\int_{T_{j-1}}^{T_j} n(u)du} \left[\frac{I(T_j)}{I_{j-1}} - 1\right] |\mathcal{F}_{T_{j-1}}\right) |\mathcal{F}_t\right) \tag{2.28}$$

Notice that the linear expectation is simply $V_{ZCIIC}(T_{j-1}, T_j, I(T_{j-1}, T_j), 1)$

This will enable us to use the explicit result of the zero coupon swap formula. The value of the floating leg of the year on year inflation linked swap becomes:

$$N\chi_j \mathbf{E}_n \left(e^{-\int_t^{T_{j-1}} n(s)du} [P_r(T_{j-1}, T_j) - P_n(T_{j-1}, T_j)] |\mathcal{F}_t\right) \tag{2.29}$$

$$= N\chi_j \mathbf{E}_n \left(e^{-\int_t^{T_{j-1}} n(s)du} P_r(T_{j-1}, T_j)|\mathcal{F}_t\right) - N\chi_j P_n(T_{j-1}, T_j)$$

The expectation in the equation above is model depend. Since we are using Jarrow Yildrim foreign currency analogy to price inflation linked swaps, we discover, for example, the real forward price depends on the volatilities of the real interest rates and inflation rates and their correlation.

Pricing with JY model

Consider Q_n^T as the $T-$ forward measure with maturity T and E_n^T is the corresponding expectation, so the above equation becomes:

$$V_{YYIIS}(t, T_{j-1}, T_j, \chi_j, N) = N\chi_j \mathbf{E}_n \left(e^{-\int_t^{T_{j-1}} n(s)du} P_r(T_{j-1}, T_j)|\mathcal{F}_t\right) - N\chi_j P_n(T_{j-1}, T_j) \tag{2.30}$$

We apply the change in numeraire techniques, developed by Geman [13](1995), to get the evolution of the real instantaneous rate under $Q_n^{T_{j-1}}$ forward measure. We get

$$dr(t) = [\rho_{n,r}\sigma_n\sigma_r B_n(t,T_{j-1})\vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]dt + \sigma_r dW_r^{T_{j-1}} \tag{2.31}$$

Note $W_r^{T_{j-1}}$ is a $Q_n^{T_{j-1}}-$ brownian motion. Under the $Q_n^{T_{j-1}}$ measure $P_r(T_{j-1},T_j)$ is lognormally distributed.

$$P_r(t,T) = A_r(t,T)e^{-B_r(t,T)r(t)} \tag{2.32}$$

Note that the zero coupon bond price formula in Hull and White is as follows:

$$B_r(t,T) = \frac{1}{a_r}\left[1 - e^{-a_r(T-t)}\right] \tag{2.33}$$

$$A_r(t,T) = \frac{P_r^M(0,T)}{P_r^M(0,t)}e^{B_r(t,T)f_r^M(0,t) - \frac{\sigma_r^2}{4a_r}(1-e^{-2a_r t})B_r(t,T)^2} \tag{2.34}$$

We use the above results and solve V_{YYIIS}. We get [30]

$$V_{YYIIS}(t,T_{j-1},T_j,\chi_j,N) = N\chi_j P_n(t,T_{j-1})\frac{P_r(t,T_j)}{P_r(t,T_{j-1})}e^{C(t,T_{j-1},T_j)} - N\chi_j P_n(t,T_j) \tag{2.35}$$

where

$$C(t,T_{j-1},T_j) = \sigma_r B_r(T_{j-1})\left[B_r(t,T_{j-1})\left(\rho_{r,I}\sigma_I - \frac{1}{1}\sigma_r B_r(t,T_{j-1}) + \frac{\rho_{n,r}\sigma_n}{a_n+a_r}(1+a_r B_n(t,T_{j-1}))\right)\right] \tag{2.36}$$

$$-\sigma_r B_r(T_{j-1})\left[\frac{\rho_{n,r}\sigma_n}{a_n+a_r}B_n(t,T_{j-1})\right]$$

We observe that the expectation of the real zero coupon under the $Q_n^{T_{j-1}}$ forward measure is a product of the forward price of the real bond and a correctional factor. This pricing framework is consistent with the Jarrow Yildrim approach. We note the correction term is a function of instantaneous volatilities of the real, nominal rates and CPI index; instantaneous correlation between the nominal and real rates, and correlation between the CPI index and real rates. We also note if the instantaneous volatility is zero then the correction factor disappears.

So the floating leg of the inflation linked swap is the sum of all the floating legs. So

$$V_{YYIIS}(\mathbf{t},\mathbf{T},\chi,\mathbf{N}) = N\chi_{\omega(t)}\left[\frac{I(t)}{I(T_{\omega(t)-1})}P_r(t,T_{\omega(t)}) - P_n(t,T_{\omega(t)})\right] \tag{2.37}$$

$$+N\Sigma_{j=\omega(t)+1}^{M}\chi_j\left[P_n(t,T_{j-1})\frac{P_r(t,T_j)}{P_r(t,T_{j-1})}e^{C(t,T_{j-1},T_j)}-P_n(t,T_j)\right]$$

we can simplify the above equation if we set

$\chi := \{\psi_1, ..., \psi_M\}$
$\omega(\mathbf{t}) = min\{j : T_j > t\}$ we defined $T_{\omega(t)-1} \leq t < T_{\omega(t)}$
$\mathbf{T} := \{T_1, ..., T_M\}$
and $t = 0$

$$V_{YYIIS}(\mathbf{0},\mathbf{T},\chi,\mathbf{N}) = N\chi_1[P_r(0,T_1) - P_n(0,T_1)) \tag{2.38}$$

$$+N\Sigma_{i=2}^{M}\chi_j\left[P_n(0,T_{j-1})\frac{P_r(0,T_j)}{P_r(0,T_{j-1})}e^{C(0,T_{j-1},T_j)}-P_n(0,T_j)\right]$$

$$= N\sum_{i=1}^{M}\chi_j P_n(0,T_j)\left[\frac{1+\tau_j F_n(0;T_{j-1},T_j)}{F_r(0;T_{j-1},T_j)}e^{C(0,T_{j-1},T_j)}-1\right] \tag{2.39}$$

The advantage of this pricing framework is that the model are gaussian so they are analytically tractable. The disadvantages is that there are possibilities of negative rates and there are difficulties in estimating historical real rates.

2.3.2 Caplet and Floorlet

In this section we consider pricing inflation indexed caplets and floorlets. The caplet is simply a call option on the inflation rate implied by the CPI index. Similarly a floorlet is a put option on the inflation rate.

$$N\chi_j\left[\nu\left(\frac{I(T_j)}{T_{j-1}}-1-\kappa\right)\right]^+ \tag{2.40}$$

where κ is the strike, χ_j is tenor for the interval $[T_{j-1}, T_j]$, N is the contract nominal value, and $\nu = 1$ for a caplet and $\nu = -1$ for floorlet.

If we set $K := 1 + \kappa$, and we apply standard no-arbitrage pricing theory then the value $t \leq T_{j-1}$ of the above equation at time T_j is:

$$V_{IICaplet}(t,T_{j-1},T_j,\chi_j,K,N,\nu) = N\chi_j\mathbf{E}_n\left(e^{-\int_t^{T_j}n(u)du}\left[\nu\left(\frac{I(T_j)}{T_{j-1}}-K\right)\right]^+|\mathcal{F}_t\right) \tag{2.41}$$

$$= N\chi_j P_n(t,T_j)\mathbf{E}_n^{T_j}\left(\left[\nu\left(\frac{I(T_j)}{T_{j-1}} - K\right)\right]^+ |\mathcal{F}_t\right) \tag{2.42}$$

We now derive explicit formulae under the Jarrow-Yildrim model. We have assumed that the nominal, real and CPI is gaussian. The price of inflation linked cap are similar to the price of a swaption, forward starting option. Moreover, this makes the CPI lognormally distributed under Q_n forward mearsure. Thus the distribution CPI is retained. If $\frac{I(T_j)}{I(T_{j-1})}$ is a random lognormal variable with $\mathbf{E}\left(\frac{I(T_j)}{I(T_{j-1})}\right) = b$ and the standard deviation is $std\left[ln\left(\frac{I(T_j)}{I(T_{j-1})}\right)\right] = \pi$. So,

$$\mathbf{E} = ([\nu(X-K)]^+) = \nu b\Phi\left(\nu\frac{ln\frac{n}{K} + \frac{1}{2}\pi^2}{\pi}\right) - \nu K\Phi\left(\nu\frac{ln\frac{n}{K} + \frac{1}{2}\pi^2}{\pi}\right) \tag{2.43}$$

where Φ is the standard normal distribution function.

Since $\frac{I(T_j)}{I(T_{j-1})}$ is conditional to $\mathcal{F}_t$ is lognormally distributed under the $Q_n^{T_j}$ - forward mearsure. We can determine this expectation from the price of the V_{YYIIS}, so

$$\mathbf{E}_n^{T_j}\left(\frac{I(T_j)}{I(T_{j-1})}|\mathcal{F}_t\right) = \frac{P_n(t,T_{j-1})}{P_n(t,T_j)}\frac{P_r(t,T_{j-1})}{P_r(t,T_j)}e^{C(t,T_{j-1},T_j)} \tag{2.44}$$

And the variance of the $\frac{I(T_j)}{I(T_{j-1})}$ is:

$$Var_n^{T_j}\left(ln\frac{I(T_j)}{I(T_{j-1})}|\mathcal{F}_t\right) = X^2(t,T_{j-1},T_j) \tag{2.45}$$

where

$$X^2(t,T_{j-1},T_j) = \frac{\sigma_n^2}{2a_n^3}(1-e^{-a_n(T_j-T_{j-1})})^2[1-e^{-2a_n(T_{j-1}-t)}] + \frac{\sigma_r^2}{2a_r^3}(1-e^{-a_r(T_j-T_{j-1})})^2[1-e^{-2a_r(T_{j-1}-t)}] \tag{2.46}$$

$$-2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_n a_r(a_n+a_r)}(1-e^{-a_n(T_j-T_{j-1})})(1-e^{-a_r(T_j-T_{j-1})})[1-e^{-(a_n+a_r)(T_{j-1}-t)}]$$
$$+\sigma_I^2(T_j-T_{j-1}) + \frac{\sigma_n^2}{a_n^2}\left[T_j - T_{j-1} + \frac{2}{a_n}e^{-a_n(T_j-T_{j-1})} - \frac{3}{2a_n}\right]$$
$$+\frac{\sigma_r^2}{a_r^2}\left[T_j - T_{j-1} + \frac{2}{a_r}e^{-a_r(T_j-T_{j-1})} - \frac{3}{2a_r}\right]$$
$$-2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_n a_r}\left[T_j - T_{j-1} - \frac{1-e^{a_n(T_jT_{j-1})}}{a_n} + \frac{1-e^{a_r(T_jT_{j-1})}}{a_r} + \frac{1-e^{(a_n+a_r)(T_jT_{j-1})}}{a_n+a_r}\right]$$
$$+2\rho_{n,I}\frac{\sigma_n\sigma_I}{a_n}\left[T_j - T_{j-1} - \frac{1-e^{a_n(T_jT_{j-1})}}{a_n}\right] - 2\rho_{r,I}\frac{\sigma_r\sigma_I}{a_r}\left[T_j - T_{j-1} - \frac{1-e^{a_r(T_jT_{j-1})}}{a_r}\right]$$

So the price of $V_{IICaplet}$ is:

$$V_{IICaplet}(t,T_{j-1},T_j,\chi_j,K,N,\nu) = \nu N\chi_j P_n(t,T_j)\left[\frac{P_n(t,T_{j-1})}{P_n(t,T_j)}\frac{P_r(t,T_{j-1})}{P_r(t,T_j)}e^{C(t,T_{j-1},T_j)}.\Phi(\varepsilon_+) - K\Phi(\varepsilon_-)\right] \tag{2.47}$$

where $\varepsilon_{\frac{+}{-}}$ is:

$$\varepsilon_{\frac{+}{-}} = \nu \frac{ln \frac{P_n(t,T_{j-1})P_r(t,T_j)}{KP_n(t,T_j)P_r(t,T_{j-1})} + C(t,T_{j-1},T_j) -/+ \frac{1}{2}X^2(t,T_{j-1},T_j)}{X(t,T_{j-1},T_j)}$$

A similar approach can be adopted to pricing floorlets.

2.3.3 Option on real zero coupon bond

Background

In this chapter we price inflation linked zero-coupon bond options. We assume the volatility function is exponentially decaying. This fact will enable us to derive explicit bond option formulaes.

Pricing model

We have developed a pricing model for valuing real zero-coupon bond derivatives. We consider pricing a plain European option on an asset $P_{CPI}(t,T)$. The pricing model is:

$$\frac{dP_{CPI}(t,T)}{P_{CPI}(t,T)} = r_n(t)dt + \sigma_I(t)d\hat{W}_I(t) - \int_t^T \sigma_r(t,s)dsd\hat{W}_r(t) \tag{2.48}$$

where

$$P_{CPI}(t,\tau) = I(t)P_r(t,\tau) \tag{2.49}$$

$P_r(t,T)$ is the real zero-coupon bond price. This value is stripped out of the coupon bearing bonds. The value of this real zero-coupon bond is adjusted for inflation by multiplying it by the index $I(t)$.

We observe that the pricing model has two sources of noise. It is thus called a two factor model. The first factor $\hat{W}_I(t)$ can be interpreted as a source of noise that last for a long time affecting all maturity equally. The second factor $\hat{W}_r(t)$ affects short maturity rates more than long term rates.

Explicit bond option formula

Theorem 6.1: Explicit bond option formula

If we consider a European call option $C_{CPI}(t)$ with maturity T and strike K. Note that τ is the maturity of the bond and $T \leq \tau$.

$$C_{CPI}(0,T) = P_r(0,T)I(0)N\left(\frac{\log\frac{P_r(0,T)I(0)}{KP_n(0,T)} + \frac{1}{2}\theta^2}{\theta}\right) - KP_n(0,T)N\left(\frac{\log\frac{P_r(0,T)I(0)}{KP_n(0,T)} - \frac{1}{2}\theta^2}{\theta}\right) \tag{2.50}$$

Proof:

An expression for $\int_0^t r_n(u)du$

As in section 2.3 we obtained the equation of the short rate to be:

$$r_n(t) = f_n(0,t) - \int_0^t \sigma_n(s,t)S_n(s,t)ds + \int_0^t \sigma_n(s,t)d\hat{W}_n(s) \tag{2.51}$$

Integrating this equation, we obtain

$$\int_0^t r_n(u)du = \int_0^t f_n(0,u)du - \int_0^t \int_0^u \sigma_n(s,u)S_n(s,u)dsdu + \int_0^t \int_0^u \sigma_n(s,u)d\hat{W}_n(s)du \tag{2.52}$$

Interchange the order of integration gives

$$\int_0^t r_n(u)du = \int_0^t f_n(0,u)du - \int_0^t \int_0^u \sigma_n(s,u)S_n(s,u)duds + \int_0^t \int_0^u \sigma_n(s,u)dud\hat{W}_n(s) \tag{2.53}$$

and simplifying gives

$$\int_0^t r_n(u)du = \int_0^t f_n(0,u)du + \frac{1}{2}\int_0^t S_n^2(s,u)ds - \int_0^t S_n(s,t)d\hat{W}_n(s) \tag{2.54}$$

An expression for $B_n(t)$

Recall from section 2.2.1 that $B_n(t) = e^{\int_0^t r_n(u)du}$, and that $P_n(0,t) = e^{-\int_0^t f_n(0,u)du}$, we obtain

$$B_n(t) = \frac{1}{P_n(0,t)}\int_0^t e^{\frac{1}{2}S_n^2(s,t)ds - \int_0^t S_n(s,t)d\hat{W}_n(s)} \tag{2.55}$$

An expression for $Z_r(t,T)$

We have

$$Z_r(t,T) = \frac{P_r(t,T)I(t)}{B_n(t)} \tag{2.56}$$

So that

$$Z_r(T,T) = \frac{P_r(T,T)I(T)}{B_n(T)} = \frac{I(T)}{B_n(T)} \tag{2.57}$$

$$Z_r(0,T) = P_r(0,T)I(0) \tag{2.58}$$

Also

$$\frac{dZ_r(t,T)}{Z_r(t,T)} = \sigma_I(t)d\hat{W}_I(t) + S_r(t,T)d\hat{W}_r(t) \tag{2.59}$$

The solution of this equation is

$$Z_r(t,T) = Z_r(0,T)e^{-\frac{1}{2}\int_0^t(\sigma_I^2(u)+S_r^2(u,T))du + \int_0^t(\sigma_I(u)d\hat{W}(u)+S_r(u,T)d\hat{W}_r(u))} \tag{2.60}$$

$$= P_r(0,T)I(0)e^{-\frac{1}{2}\int_0^t(\sigma_I^2(u)+S_r^2(u,T))du + \int_0^t(\sigma_I(u)d\hat{W}(u)+S_r(u,T)d\hat{W}_r(u))} \tag{2.61}$$

European option on $P_{CPI}(t,T)$

We will denote a European call option on a real zero coupon bond by $C_{CPI}(0,T)$, where T is the date of option expiry. Let the bond maturity be τ where $T \leq \tau$. Note that $P_{CPI}(t,T) = P_r(T,\tau)I(T)$

$$C_{CPI}(0,T) = E^{\mathbf{Q}}\left(\left(\frac{P_r(T,\tau)I(T) - K}{B_n(t)}\right)^+\right) \tag{2.62}$$

$$= E^{\mathbf{Q}}\left(\left(\frac{P_r(T,\tau)I(T)}{B_n(t)} - \frac{K}{B_n(T)}\right)^+\right) \tag{2.63}$$

$$= E^{\mathbf{Q}}\left(\left(Z_r(T,\tau) - \frac{K}{B_n(T)}\right)^+\right) \tag{2.64}$$

Substitute

$$= E^{\mathbf{Q}}(P_r(0,\tau)I(0)e^{-\frac{1}{2}\int_0^T(\sigma_I^2(u)+S_r^2(u,\tau))du+\int_0^T(\sigma_I(u)d\hat{W}_I(u)+S_r(u,\tau)d\hat{W}_r(u))} \tag{2.65}$$

$$-KP_n(0,t)e^{-\frac{1}{2}\int_0^T S_n^2(s,T)ds-\int_0^T S_n(s,T)d\hat{W}_n(s)})^+ \tag{2.66}$$

We can write the formula as

$$C_{CPI}(0,T) = E^{\mathbf{Q}}(K_1 e^{X-\frac{1}{2}\sigma_X^2} - K_2 e^{Y-\frac{1}{2}\sigma_Y^2})^+ \tag{2.67}$$

where

$$X = \int_0^T (\sigma_I(u)d\hat{W}_I(u) + S_r(u,\tau)d\hat{W}_r(u)) \tag{2.68}$$

$$Y = \int_0^T S_n(s,t)d\hat{W}_n(s) \tag{2.69}$$

$$\sigma_X^2 = \int_0^T (\sigma_I^2(u) + S_r^2(u,\tau))du \tag{2.70}$$

$$\sigma_Y^2 = \int_0^T S_n^2(s,T)ds \tag{2.71}$$

Result of Amin and Jarrow

Let X, Y be normally distributed with mean 0. Then the expected value $E^{\mathbf{Q}}$ in the last expression of $C_{CPI}(0,T)$ is given by:

$$C_{CPI}(0,T) = K_1 N(\phi) - K_2 N(\phi - \theta) \tag{2.72}$$

where θ is the variance of $X - Y$ and

$$\phi = \frac{\log \frac{K_1}{K_2} + \frac{1}{2}\theta^2}{\theta} \tag{2.73}$$

The call price is

$$C_{CPI}(0,T) = P_r(0,T)I(0)N\left(\frac{\log\frac{P_r(0,T)I(0)}{KP_n(0,T)} + \frac{1}{2}\theta^2}{\theta}\right) - KP_n(0,T)N\left(\frac{\log\frac{P_r(0,T)I(0)}{KP_n(0,T)} - \frac{1}{2}\theta^2}{\theta}\right) \quad (2.74)$$

where

$$\theta^2 = var(X-Y) = var(\int_0^T (\sigma_I(u)d\hat{W}_I(u) + S_r(u,\tau)d\hat{W}_r(u) - S_n(s,t)d\hat{W}_n(s))) \quad (2.75)$$

$$d(X-Y) = \sigma_I(t)d\hat{W}_I(t) + S_r(u,\tau)d\hat{W}_r(t) - S_n(t,T)d\hat{W}_n(t) \quad (2.76)$$

$$d(X-Y)d(X-Y) = (\sigma_I^2(t)d\hat{W}_I(t) + S_r^2(u,\tau)d\hat{W}_r(t) - S_n^2(t,T)d\hat{W}_n(t) \quad (2.77)$$

$$+2(\rho_{rI}\sigma_I(t)S_r(t,\tau) - \rho_{nI}\sigma_I(t)S_n(t,T) - \rho_{nr}S_r(t,\tau)S_n(t,T)))dt \quad (2.78)$$

Hence

$$\theta^2 = var(X-Y) = \int_0^T (\sigma_I^2(u)d\hat{W}_I(u) + S_r^2(u,\tau)d\hat{W}_r(u) - S_n^2(u,T)d\hat{W}_n(u) \quad (2.79)$$

$$+2(\rho_{rI}\sigma_I(t)S_r(u,\tau) - \rho_{nI}\sigma_I(u)S_n(u,T) - \rho_{nr}S_r(u,\tau)S_n(u,T)))du \quad (2.80)$$

Parameters

We assume the following form of parameter

$$\sigma_n(t,T) = \sigma_n e^{-\lambda_n(T-t)}, \quad \sigma_r(t,T) = \sigma_r e^{-\lambda_r(T-t)} \quad (2.81)$$

$$S_n(t,T) = \int_t^T \sigma_n(t,s)ds = \frac{\sigma_n}{\lambda_n}(e^{-\lambda_n(T-t)-1}) \quad (2.82)$$

$$S_r(t,T) = \int_t^T \sigma_r(t,s)ds = \frac{\sigma_r}{\lambda_r}(e^{-\lambda_r(T-t)-1}) \quad (2.83)$$

$$\sigma_I(t) = \sigma_I \quad constant \quad (2.84)$$

We substitute the parameters into the variance equation. See appendix 3 for the final derivation of the variance equation. We have now developed everything to price options on the real zero coupon bond.

2.4 Pricing complicated derivatives by simulation

In this section we price complicated hybrid inflation derivatives. Firstly, we compare inflation instruments with other assets classes. We highlight differences and similarities of these asset classes. Secondly, we outline the two broad categories of derivatives on inflation. Thirdly, we discuss the different types of inflation derivatives. Lastly, we take an example of pricing a path dependent derivative which can only be priced by simulation.

2.4.1 Background

Inflation is a unique asset class because it displays similar characteristics as other major asset classes like equities, forex and interest rates. Inflation is an interest rate product because it is traded in the form of inflation-linked debt. However, the CPI can also be viewed as the forex rate between fixed interest (i.e. nominal) and inflation-linked (i.e. real) currencies. This is consistent with the Jarrow-Yildrim foreign currency analogy to pricing inflation linked derivatives. Inflation as an asset class may exhibit mean reversion and stochastic volatility - a characteristic it shares with equities.

2.4.2 Categories

There are two main categories of inflation linked derivatives, namely model dependent and independent derivatives.

Currently in South Africa there are various inflation linked products which vary in dimensions e.g. term, inflation indices and indexation lags. The building blocks of these products are permutations of long and short inflation linked zero coupon bonds.

Currently in South Africa, inflation swaps and zeros are model independent and do not require a model of random behaviour of inflation and real rates. However, these instruments do require a yield curve to interpolate zero rates. This yield curve is just a sophisticated interpolation model which will produce zero rates that is consistent with market rates.

To price derivatives that have payoffs which are nonlinear functions of CPI zeros such as caps, floors, swaptions and inflation-bond options we need a model of random variation of the CPI zero price at maturity. These are called model dependent derivatives. We have added innovation to pricing inflation linked swaps in South Africa. We have used Jarrow Yildrim foreign currency model to price zero coupon inflation linked swap and the year on year inflation linked swap.

2.4.3 Type

Most of inflation linked derivatives mimic the way the South African inflation-linked debt and pension increases. There are various ways in which these increases can vary which leads to distinctive types of CPI indexation.

Type1 : This type of indexation is unlimited. It is determined by calculating the relative difference between the last increase.

Type2 : This type takes the minimum of the unlimited indexation (type 1) and the compounded indexation caped at the specified level - in this case 5 percent.

Type3 : This type of indexation increases subject to a minimum floor increase rate and some a minimum of the highest level to date.

Type4 : It is determined by calculating the relative difference between the last increases. This rate would be subject to a maximum cap level or a minimum in the scenario when pricing floors.

Table 2.2 and **Figure 2.2** use the case of a cap at 5 percent p.a. to illustrate the difference between the cap types for a hypothetical CPI scenario.

Table 2.2 : Table of indexed annuity type with CPI cap of 5 percent

t	CPI (Type 1)	CPI t/t-1	5% cap (Index)	Type 2 cap at 5% payment	Type 2 cap at 5% increase	Type 3 cap at 5% payment	Type 3 cap at 5% increase	Type 4 cap at 5% payment	Type 4 cap at 5% increase
0	100.00		100.00	100.00		100.00		100.00	
1	102.90	2.9%	105.00	103.00	3.00%	103.00	3.0%	103.00	3.0%
2	110.51	7.4%	110.25	110.25	7.04%	108.15	5.0%	108.15	5.0%
3	113.93	3.1%	115.76	[illegible]	[illegible]	113.95	5.4%	111.72	3.3%
4	116.12	1.9%	121.55	[illegible]	1.90%	116.12	1.9%	113.84	1.9%
5	126.57	9.0%	127.63	126.57	9.00%	121.92	5.0%	119.55	5.0%

Figure 2.2 : Graph of indexed annuity type with CPI cap of 5 percent

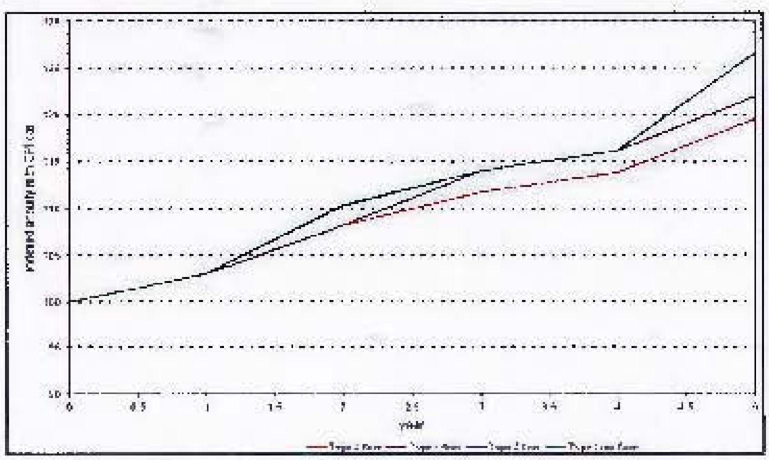
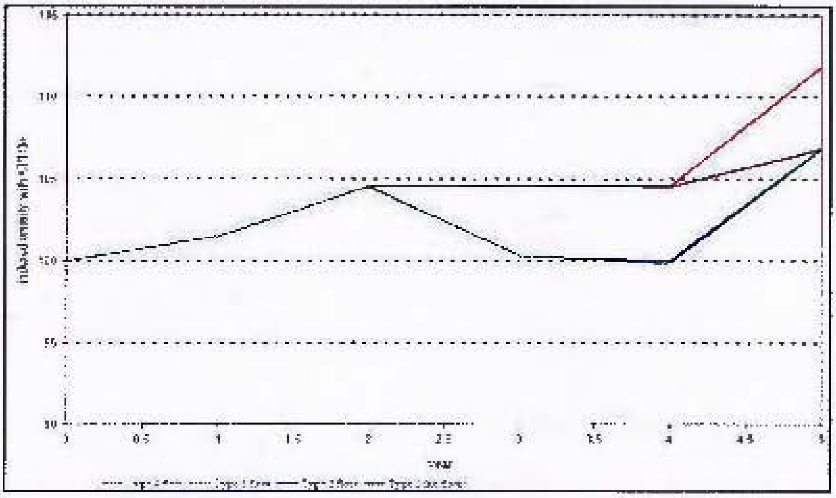


Table 2.3 and **Figure 2.3** use the case of a floor at 5 percent p.a. to illustrate the difference between the floor types for a hypothetical CPI scenario.

Table 2.3 : Table of indexed annuity type with CPI floor of 0 percent

t	CPI (Type 1)	CPI y/y%	0% floor index	Type 2 floor at 0% payment	Type 2 floor at 0% increase	Type 3 floor at 0% payment	Type 3 floor at 0% increase	Type 4 floor at 0% payment	Type 4 floor at 0% increase
0	100.0		100.0	100.0		100.0		100.0	
1	101.5	1.5%	100.0	101.5	1.50%	101.5	1.5%	101.5	1.5%
2	104.5	3.0%	100.0	104.5	3.00%	104.5	3.0%	104.5	3.0%
3	100.4	-4.0%	100.0	100.4	-4.00%	104.5	0.0%	104.5	0.0%
4	99.9	0.5%	100.0	100.0	-0.40%	104.5	0.0%	104.5	0.5%
5	106.9	7.0%	110.9	106.9	6.85%	106.9	2.2%	111.9	7.0%

Figure 2.3 : Graph of indexed annuity type with CPI floor of 0 percent



2.4.4 Complicated derivative

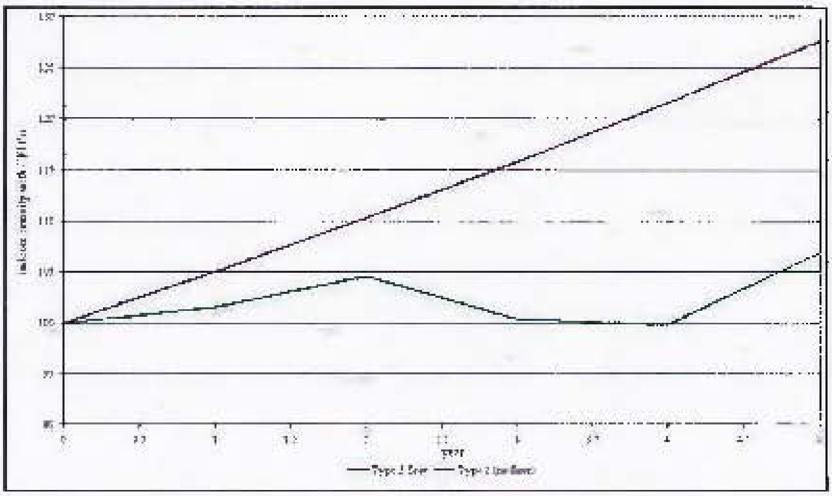
In this section we price a 5 year annuity which commenced on 31/05/2006. Payments are monthly and inflation increases are annually in arrears. Inflation indexation is lagged 4 months, consistent with South Africa CPI-bonds. We will value the annuity assuming it has various inflation derivative forms type 1 and type 3 floor at 5 percent.

Table 2.4 and Figure 2.4 use the case of a type 1 and type 3 floor at 5 percent p.a. to illustrate the difference between the floor types for a hypothetical CPI scenario.

Table 2.4 : Table of indexed annuity type with CPI floor of 5 percent

t	CPI (Type 1)	CPI y/y%	5% floor index	Type 2 floor at 5% payment	Type 2 floor at 5% increase	Type 3 floor at 5% payment	Type 3 floor at 5% increase
0	100.0		100.0	100.0		100.0	
1	101.5	1.5%	105.0	105.0	5.00%	105.0	5.0%
2	104.8	3.0%	110.3	110.3	5.00%	110.3	5.0%
3	100.4	-4.0%	115.6	115.6	5.00%	125.6	5.0%
4	99.9	-0.5%	121.6	121.6	5.00%	121.6	5.0%
5	106.9	7.0%	127.6	127.6	5.00%	127.6	5.0%

Figure 2.4 : Graph of indexed annuity type with CPI floor of 5 percent



Price annuity

Now that we have described the annuity which we wish to price, in this section we define the payoff of a type 1 and type 3 floor at 5 percent.

$$\textbf{Payoff} = \textbf{max}(C_T, CPI(t-1)) \tag{2.85}$$

$$= C_0 + \textbf{max}_{\tau=0,1,\dots,T} C_\tau - C_0 \tag{2.86}$$

This shows that a type 3 payoff is equivalent to the initial CPI, C_0, plus the payoff on a T-year discrete time lookback option on the maximum of the CPI with a fixed strike at C_0. We use the simulated inflation process in equation 2.15 as the driver of pricing this instrument. The inflation model is consistent with the foreign currency framework proposed by Jarrow Yildrim.

The path dependency of type 3 floors complicates the valuation. It is unlikely do derive a closed form valuation formulae for this derivative. The path dependency becomes complex and valuation by Monte Carlo simulation offers the path of least resistance. In chapter 6 we state the results of this instrument.

Chapter 3

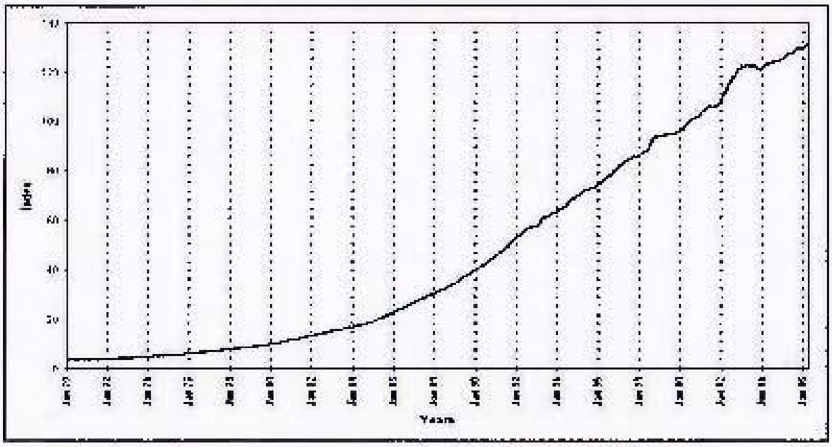
Data Description

3.1 Consumer Price Index

3.1.1 background

The value of the Consumer Price Index (CPI) is published on a monthly basis by Statistics South Africa. We are going to use the Consumer Price Index for all urban consumers (CPI-U) as our index for inflation. The Consumer Price Index calculates the cost of purchasing fixed quantities of goods and services for any particular month. The implication of the lag, delay in the release of data, in the CPI data is that inflation-linked bonds (CPI bonds) do not provide exact real returns but an approximation of the real return. Moreover, we must note that it is the best available approximation. We have collected data from Statistics South Africa. Figure 3.1 shows a graph of CPI-U data. The range of the data is from 01 January 1970 to 31 May 2006.

Figure 3.1 : Consumer Price Index According to Urban Consumers



We need to determine the daily reference CPI for the settlement dates ranging in the sample 2 May 2002 to 23 June 2003. The reference CPI is factored into the pricing procedure for inflation-linked bonds. Daily CPI values ,on any day t, are determined by interpolating as follows [5]:

$$CPI(t) = CPI_{M-4} + (\frac{d-1}{D})(CPI_{M-3} - CPI_{M-4}) \quad (3.1)$$

where :

- d=calendar day corresponding to t
- M=calendar month corresponding to t
- D=number of days in calendar month
- CPI_r=reference CPI for the calendar month r

3.1.2 Examples

In this paragraph we are going to give examples on how to determine reference CPI for various dates. Assume the following CPI index values:

Month	CPIValue
October 2003	100
November 2003	100.5
December 2003	101.2
January 2004	101.7
February 2004	102.4

If we had to determine the reference CPI on the 20 February 2004. The equation would be:

$$CPI_{October2003} + \frac{(20-1)}{30}(CPI_{November2003} - CPI_{October2003}) \tag{3.2}$$

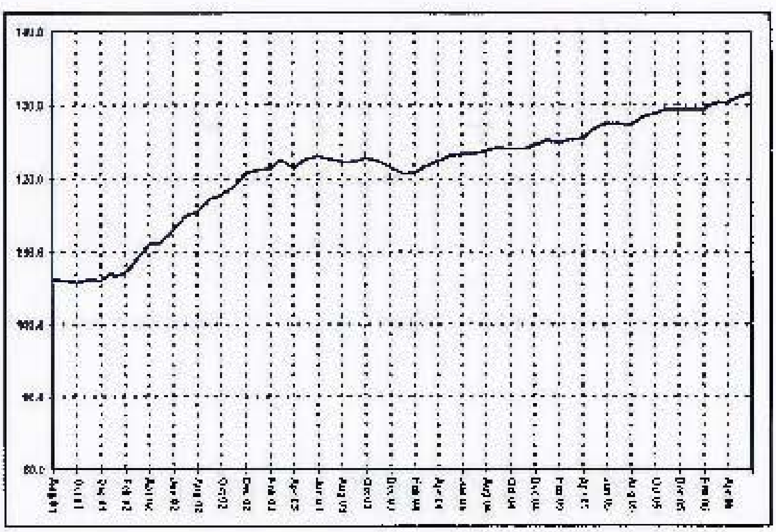
$$= 100 + \frac{19}{30}(100.5 - 100) = 100.4 \tag{3.3}$$

The reference CPI at the beginning of each month will correspond to a 4 month prior CPI value. For example the CPI value on the 1 of February 2004 will correspond to the October 2003 CPI value.

3.1.3 Reference CPI

In this graph we summarize the calculated reference CPI in a sample that we will use for our analysis. See Figure 3.2 for a graph of the Reference CPI.

Figure 3.2 : Reference CPI ranging from 02 Aug 2001 to 31 May 2006



3.2 Methodology of pricing inflation-linked products

Inflation-linked bonds are adjusted by factoring the CPI index on the coupon and principal at maturity. In the subsequent paragraph we will describe the methodology followed by Bond Exchage of South Africa (BESA) when pricing these instruments.

3.2.1 Determine settlement date

The conversion according to the South African bond market is t+3.

3.2.2 Determine the reference CPI for settlement i.e. CPI(t)

The reference CPI is determined as described in section 3.1.

3.2.3 Calculate the index ratio

The index ratio $\frac{CPI_s}{CPI_t}$ is a ratio of the reference CPI for the settlement date of the bond over the issue date of the bond. This ratio will only be applied to the pricing framework if the ratio is greater than one at the day of redemption of the bond. The reason being that the inflation linked instrument protects the investor from rising inflation. If there is a deflation in inflation from the issue date to the redemption of the bond, the apparent gain in purchase power of the investor will not be adjusted.

3.2.4 Determine bond prices using BESA bond pricing formula

See appendix 4 for the bond pricing specifications. The bond pricing formula is used to obtain the value of the all-in-price. The inputs into the pricing formula are the real yield and coupon. The value of the bond can be written as follows :

$$\mathbf{A} = \frac{CPI(s)}{CPI(t)} \sum_{i=1}^{n} e^{-R*\eta_i} * p_i \tag{3.4}$$

where :

- s=settlement date
- t=issue date
- R=term structure of real rates, were R=$r - X$, X=term structure of rates of inflation and r=term structure of nominal rates

- $CPI(s)$=reference CPI for settlement date of bond

- $CPI(t)$=reference CPI for issue date of bond

- η_i=period from settlement date to CD_i, were $CD_1, ..., CD_n$ are coupon dates that follows.

- for 1<i<n p_i becomes $p_1 = e * \frac{c}{2}$, $p_i = \frac{c}{2}, p_n = 1 + \frac{c}{2}$ and c is the coupon.

Inflation-linked bonds will have similar attributes to standard fixed coupon bonds issued by BESA. The book closing date and coupon dates will be published for each inflation indexed bond. See table below.

Table 3.1 : CPI bond data published by BESA on 31 August 2003

StockCode	**R198**	**R189**	**WS05**	**R197**
Issuer	RSA	RSA	TCTA	RSA
CouponRate	3.800	6.250	5.000	5.500
Redemption	31/Mar/08	31/Mar/13	01/Aug/18	07/Dec/23
InterestPayable	31Mar/30Sep	31Mar/30Sep	01Feb/01Aug	07Jun/07Dec
BookCloseDate	17Mar/16Sep	17Mar/16Sep	22Jan/22Jun	24May/23Nov
ListingDate	26/Apr/02	20/Mar/00	18/Nov/01	25/May/01
IssueAmount	1,095,000,000	13,408,435,174	1,350,000,000	7,132,517,477

3.2.5 Apply index ratio to all-in-price

The result obtained from BESA bond pricing formula is multiplied by the index ratio. This will give the unrounded all-in-price formula.

$$\mathbf{A}_{CPI} = \frac{CPI(s)}{CPI(t)} * \mathbf{A} \tag{3.5}$$

3.2.6 Rounding

All-in-Price

The all-in-price **A** is rounded and then it is multiplied by an unrounded index ratio. The result is rounded to 5 decimal places.

Accrued Interest

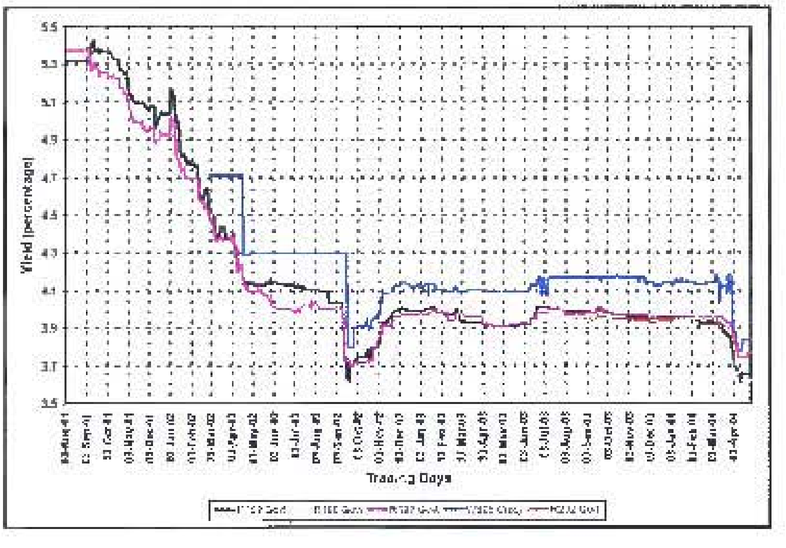
Accrued interest is determined by adjusting the accrued interest of a vanilla bond for the inflation index. Initially the accrued interest is rounded and it is multiplied with the unrounded index ratio. This will give the unrounded accrued interest on the inflation-linked bond which is then rounded.

Clean Price

The rounded clean price of inflation-linked bond is the difference between the rounded all-in-price and accrued interest on inflation-linked bonds.

The data of available inflation-linked securities were obtained from **BESA** (See disc). See Figure 3.3 for graphical representation of the CPI All in price.

Figure 3.3 : CPI yield for the period ranging from **02 Aug 2001** to 31 May 2006



Example of pricing

Suppose we want to determine the All in Price of an Inflation linked bond(R189) on the 1 June 2004. If we assume that the bond traded at a yield of 3.5 percent.

- The price of the conventional bond at 3.5 percent yield: R 1,108.300
- CPI at February 2004: 102.4
- CPI at bond issue: 100

- Index ratio: $\frac{102.4}{100} = 1.024$
- The price of R189 at 3.5 percent yield: R 1,108,300*1.024 = R 1,134,899

3.3 Yield curves

There are two main categories of term structure estimation method, namely equilibrium and empirical methodology. The equilibrium methodology assumes that certain state variables follow stochastic process which explain the evolution of the term structure in time. The resulting term structure of interest rates is a theoretical one consistent with arbitrage free condition in an efficient market but, unless allowing for multi-factor models it is hardly able to fit the actually observed market data Bin-Huei Lin [3].

The empirical methodology attempts to find a close representation of the term structure at any point in time, given some observed interest rate data (See Patrick Hagan and Graeme West [15]).

This method uses curve fitting techniques with observed coupon bond prices. The empirical methodology generates realistic yield curve patterns. There are many approximation functions for curve fitting techniques. In the next paragraph we will briefly describe some of the models.

McCulloch [28] uses polynomial splines to fit the discount function. This model will thus produce estimates of continuous time discount functions. This model is linear so ordinary least squares regression are used. Once the model is built it can be used to estimate the whole term structure.

Vasicek and Fong [36] have pioneered the method of exponential spline fitting. The output of this method are forward rates that are smooth continuous function in time.

The resulting term structure of interest rates obtained from the empirical methodology can be used to price contingent claims.

We are going to use the empirical methodology to construct the yield curve. We will thus determine the real and nominal zero coupon bond which we will use for estimating the volatility structure. The volatility structure is an important input in the HJM model.

This term structure will replicate the input data. For example, if we price one input bond as a series of cash flows from the curve we should get the accurate All-in-Price.

We will use the bootstrap and interpolation method to construct the yield curve. The bootstrap method is a procedure for calculating the real zero coupon yield using market data [15]. There is incomplete information to completely describe all points on the yield curve. The purpose of the interpolation methods is to complete the yield curve using the bootstrapped output.

We provide in the following section data for the nominal, real and swap curve. This data is constructed using the empirical methodology [31]. The nominal curve is consistent with the Bond Exchange of South Africa.

3.4 Nominal zero curve

3.4.1 Short-term instrument

We need to have a history of the South African nominal yield curve. We will use the nominal yield curve to strip nominal zero coupon bonds. Once we determine the nominal zero coupon bond prices we will calculate their respective returns over the sample period. We will then be able to determine historical volatilities of various zero coupon bond maturities. This data will be used in the following chapter for estimating the volatility function. The volatility function is an input in the HJM model.

In South Africa the approach to constructing the yield curve is to use JIBAR in the short end, FRA's in the medium part and Bonds in the long end. There are different levels of credit worthiness used to construct the yield curve. The short and medium end of the curve is not default free, because of the lack of liquid and credit in the short end term.

The following instruments are used to derive the South African Rand (ZAR) Government curve:

Data	Type
SAFEX Call	Over-Night
1 Month	JIBAR
3 Month	JIBAR
6 Month	JIBAR
12 Month	JIBAR

3.4.2 Long-term instrument

Data	Type
R153	Bond
R157	Bond
R186	Bond
R204	Bond
R201	Bond
R203	Bond
R207	Bond

Figure 3.4 : Nominal yield curve surface ranging from 02 Aug 2001 to 31 May 06

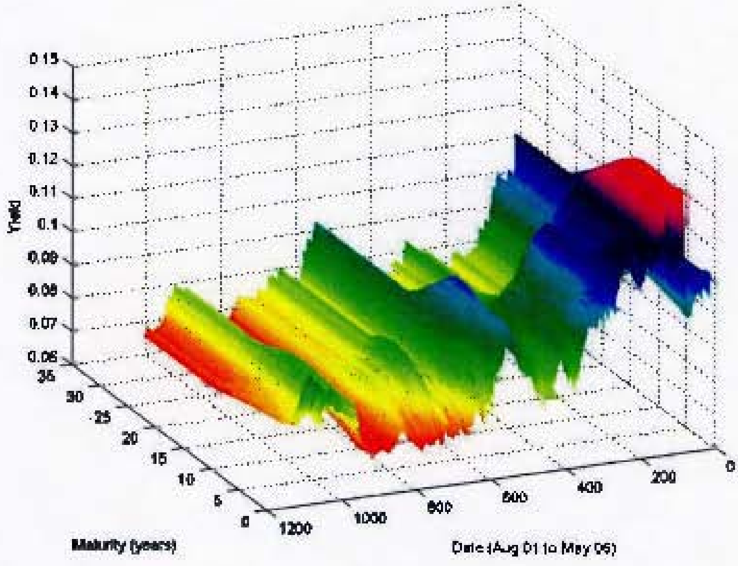
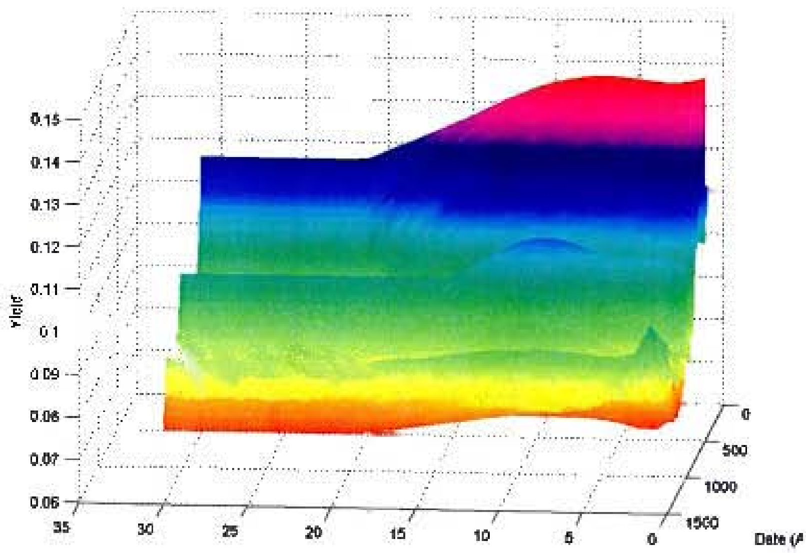


Figure 3.5 : Nominal yield curve surface ranging from 02 Aug 2001 to 31 May 06



3.5 Real zero curve

Similarly, we need to have a history of the South African real yield curve. We will use the real yield curve to strip real zero coupon bonds. Once we determine the real zero coupon bond prices we will calculate their respective returns over the sample period. We will then be able to determine historical volatilities of various zero coupon bond maturities. This data will be used in the following chapter for estimating the volatility function. The volatility function is an input in the HJM model.

The following instruments are used to derive the South African Rand (ZAR) Real Zero curve:

Data
Swap Zero Curve
Published and forecast CPI (see section 3.7)

3.5.1 Long-term instrument

Data	Type
R198	Bond
R189	Bond
R197	Bond
R202	Bond

Figure 3.6 : Real yield curve surface ranging from 02 Aug 2001 to 31 May 06

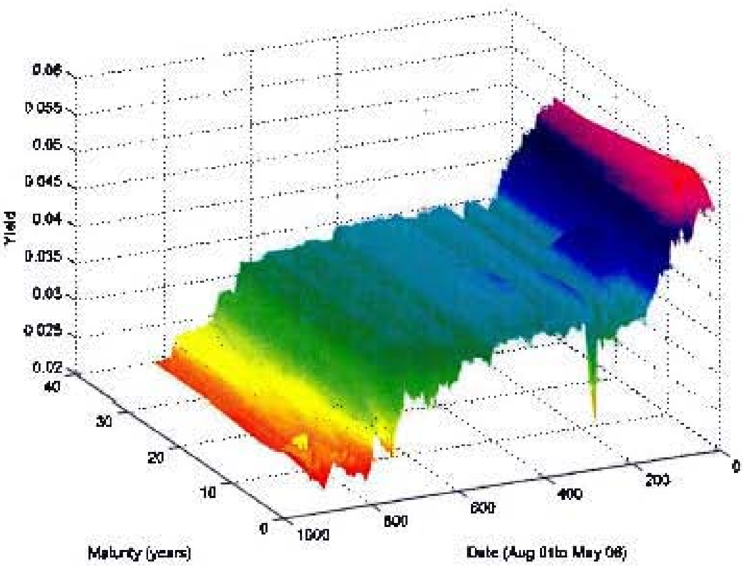
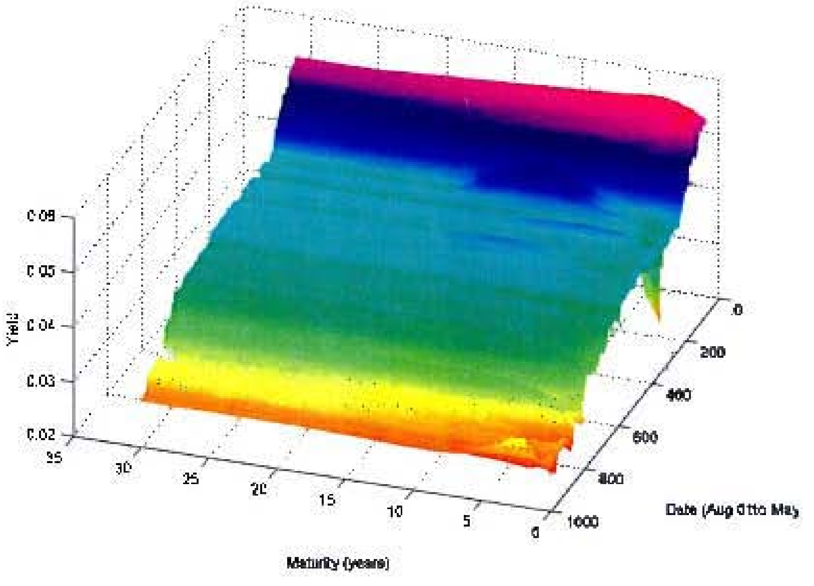


Figure 3.7 : Real yield curve surface ranging from 02 Aug 2001 to 31 May 06



3.6 Swap zero curve

A swap is an exchange of a fixed payments for floating payment on the same notional. A swap curve is a graph of fixed payments relative to their maturity. We can then strip swap rates from the curve.

We use the swap curve as a discount curve to price derivative e.g. zero coupon swaps and year on year swaps. We also use the swap curve as an input to building the real zero curve.

The following instruments are used to derive the South African Rand (ZAR) SwapZero curve:

Data	Type
SAFEX Call	Over-Night
1 Month	JIBAR
3 Month	JIBAR

3.6.1 Long-term instrument

Data	Type
1X4	FRA
2X5	FRA
3X6	FRA
4X7	FRA
5X8	FRA
6X9	FRA
7X10	FRA
8X11	FRA
9X12	FRA
12X15	FRA
15X18	FRA
18X21	FRA
21X24	FRA
1-10 Year	Swap rate
12 Year	Swap rate
15 Year	Swap rate
20 Year	Swap rate
25 Year	Swap rate
30 Year	Swap rate

Figure 3.8 : Swap yield curve surface ranging from 02 Aug 2001 to 31 May 06

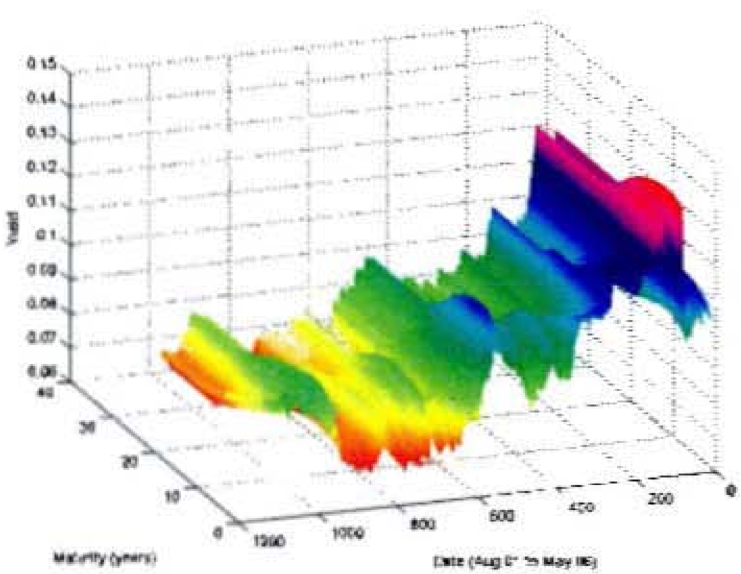
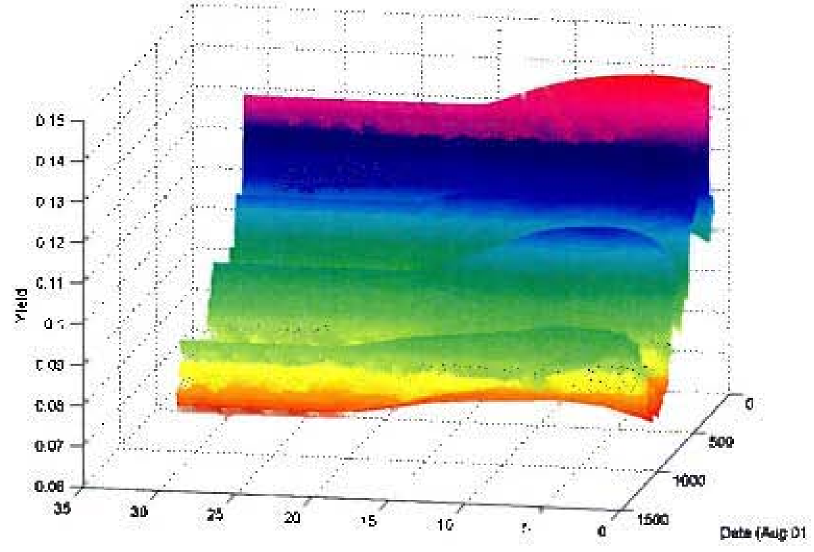


Figure 3.9 : Swap yield curve surface ranging from 02 Aug 2001 to 31 May 06



3.7 Deriving a forward CPI curve

In this section we describe the forward CPI curve. We propose Jarrow and Yildrim forward CPI curve model, namely:

$$CPI(T-t) = CPI(t) * \left(\frac{1+n}{1+r}\right)^{(T-t)} \tag{3.6}$$

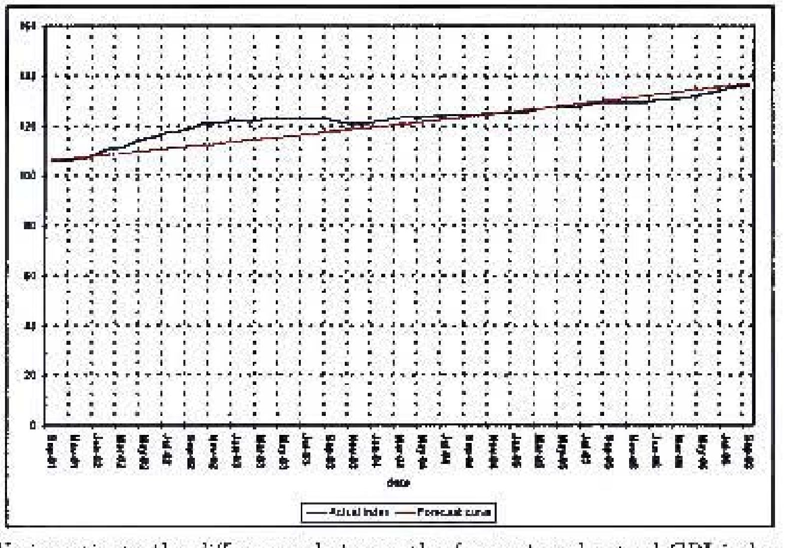
where

- n is the nominal rate
- r real rate
- t and T are starts and end dates respectively

They use the foreign currency analogy to pricing exchange rates. The real rates correspond to foreign prices, nominal yield corresponds to domestic prices and inflation correspond to the spot exchange rates. This analogy provides a consistent frameworks to develop pricing models for foreign exchange rates, interest rate derivatives and inflation linked bond and derivatives. This model is consistent with general theoretical mordern option pricing theory.

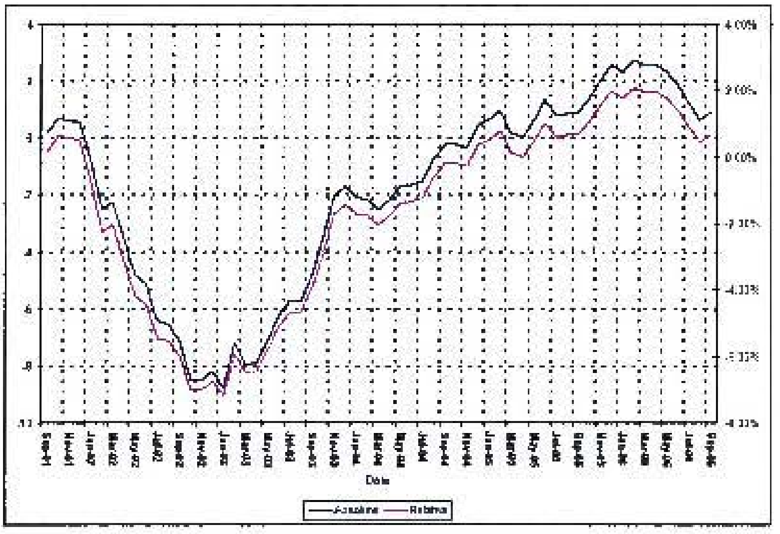
In the following graph we plot the actual CPI inflation index. We also plot the forecast CPI index using the foward CPI. See figure 3.10.

Figure 3.10 : Plot of CPI index versus Forecasted CPIX



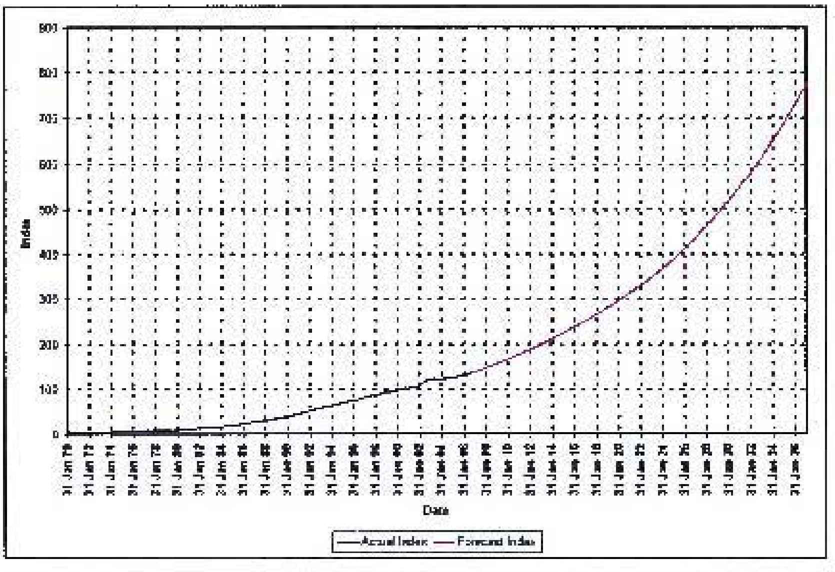
We investigate the differences between the forecast and actual CPI index. We notice that the absolute and relative difference between the actual and forecast CPI index increase from Jan 2002 to November 03; it peaks on the month of January 2003. This discrepancy is a resultant of incorrect repoting of South Africa's CPI data. After intervention the CPI index normalised.

Figure 3.11 : Absolute and Relative Graph



In the following graph we plot the actual CPI index with the forecasted CPI curve. We constructed a 30 year forecast curve.

Figure 3.12 : Plot of Actual and Forecast CPI Index



Chapter 4

Estimation of Parameters

4.1 Background

Volatility functions lie at the core of the HJM approach to pricing interest rate derivatives. The initial term structure and volatility functions are the only inputs needed to price and hedge derivatives. The volatility function describes the form of the term structures motion through time. We are going to describe some of the HJM volatility functions in the next paragraph.

There are a number of approaches that can be used to select a volatility structure. Market data can be used to infer volatility functions. There are two market driven approaches namely, historical and implied volatility functions. They can be estimated using historical term structure movements. The implied method chooses volatility functions that best fit derivatives prices.

The simplest model to choose for the volatility function is to keep it constant. All forward rates will have the same volatility and the forward curve will move in upward and downward parallel shifts. This model is known as the Ho and Lee model. The only disadvantage with this model is that interest rates can become negative if rates are low.

One can force the HJM model to conform to the Ho and Lee or Vasicek models. See table 4.1 for a summary of possible volatility functions.

Table 4.1 : HJM volatility functions, σ a constant, is a deterministic function

Model	Volatility function
Historical	Historical volatility function
	Implied volatility function
Continuous Ho-Lee	σ
Vasicek (Hull-White)	$\sigma e^{-\lambda(s-t)}$

4.2 Estimate parameters for real forward rate

We are going to estimate the volatility function of a one factor model using an exponentially declining volatility. This is called the generalized Vasicek model. There are two main reasons for selecting this volatility function. Firstly, longer maturity forward rates fluctuate less than shorter maturity rates. Secondly, when a bond approaches maturity it converges to par value. So a bond which is close to maturity will be relatively less volatile. So we assumed a volatility function of the following form:

$$\sigma_r(t,T) = \sigma_r e^{-\lambda_r(T-t)} \tag{4.1}$$

where $\sigma_r > 0$ and $\lambda_r > 0$ are constant parameters.

4.2.1 Establish variance equation

We state the bond dynamics under the martingale measure as follows (See equation 2.57):

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t) - \rho_{rI}\sigma_I(t)\int_t^T \sigma_r(t,s)ds\right]dt - \int_t^T \sigma_r(t,s)dsd\hat{W}_r(t) \tag{4.2}$$

We also know that using expression 4.2 given the volatility function that bond return evolves according to the following distribution :

$$\frac{\Delta P_r(t,T)}{P_r(t,T)} - \left[r_r(t) - \rho_{rI}\sigma_I(t)\int_t^T \sigma_r(t,s)ds\right]\Delta t \sim N\left(0, \left(\int_t^T \sigma_r(t,s)ds\right)^2 \Delta t\right) \tag{4.3}$$

We take the variances of the bond return (equation 4.2) in order to estimate the parameters of volatility function, so:

$$var\left(\frac{\Delta P_r(t,T)}{P_r(t,T)}\right) = \mathbf{E}\left[\frac{\Delta P_r(t,T)}{P_r(t,T)} - \left[r_r(t) - \rho_{rI}\sigma_I(t)\int_t^T \sigma_r(t,s)ds\right]\Delta t\right]^2 = \left(\int_t^T \sigma_r(t,s)ds\right)^2 \Delta t \tag{4.4}$$

Where Δt are daily observations (1/365). The expected return on the bond $\left(r_r(t) - \rho_{rI}\sigma_I(t)\int_t^T \sigma_r(t,s)ds\right)\Delta t$ is small relative to its standard deviation $(\int_t^T \sigma_r(t,s)ds)\sqrt{\Delta t}$, so:

$$var\left[\frac{\Delta P_r(t,T)}{P_r(t,T)}\right] = \left(\int_t^T \sigma_r(t,s)ds\right)^2 \Delta t \tag{4.5}$$

Substitute exponential declining volatility, so:

$$var\left[\frac{\Delta P_r(t,T)}{P_r(t,T)}\right] = \left(\int_t^T \sigma_r e^{-\lambda_r(s-t)}ds\right)^2 \Delta t \tag{4.6}$$

$$var\left[\frac{\Delta P_r(t,T)}{P_r(t,T)}\right] = \sigma_r^2 \left(\frac{e^{-\lambda_r(T-t)}-1}{\lambda_r}\right)^2 \Delta t \tag{4.7}$$

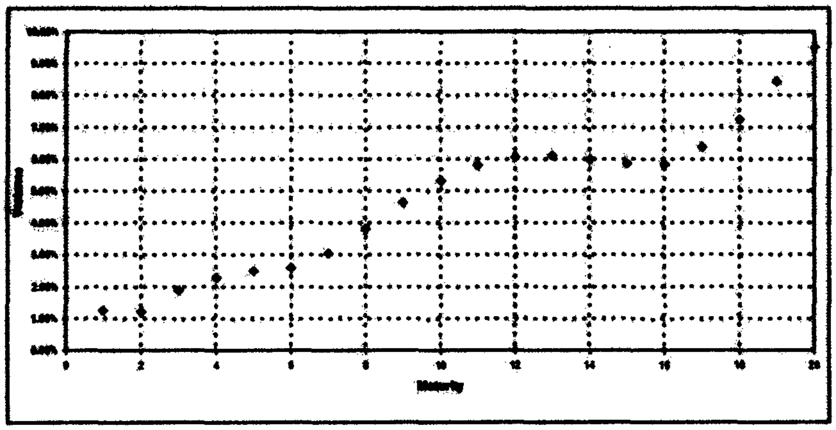
4.2.2 Variance of real zero coupon bond

To estimate the unknown parameters (σ_r and λ_r) in equation 4.7 we need the variance of the real zero coupon bond returns. We obtained these variances from the previous chapter. The following table and graph will summarize the results of the variances of real zero coupon bond returns.

Table 4.2 : Estimate of the real zero coupon bond sample variance from 01 Aug 2001 to 31 May 2006

Bond Ticker	Annualized volatility percentages
P(0,1)	1.243
P(0,2)	1.219
P(0,3)	1.881
P(0,4)	2.293
P(0,5)	2.492
P(0,6)	2.606
P(0,7)	3.066
P(0,8)	3.834
P(0,9)	4.644
P(0,10)	5.330
P(0,11)	5.816
P(0,12)	6.072
P(0,13)	6.110
P(0,14)	5.996
P(0,15)	5.860
P(0,16)	5.835
P(0,17)	6.383
P(0,18)	7.252
P(0,19)	8.442
P(0,20)	9.507

Figure 4.1 : Sample variance of real zero coupon bond returns from 02 Aug 2001 to 31 May 06



The variances of the real zero coupon bond are relatives low compared with the same maturity nominal zero coupon bonds. This is so because inflation linked bonds are less liquid than conventional nominal bonds. We note the medium and long term real zero coupon seem to deviate from the apparent linearity of real zero coupon bond variances and maturity relationship. This implies that the medium and long term real zero coupon bond are more volatile. This observation is intuitive given that the majority of inflation linked bond issuance sit in the medium and longer term region of the curve.

4.2.3 Estimate parameters by downhill simplex method

In this section we will explain the method used to solve for the parameters in equation (4.7). This is a multidimensional minimization problem because we have to find the minimum of a function which has more than one independent variable. We are going to use the downhill simplex method which was introduced by Neder and Mead [30].

The downhill simplex method (its also called the amoeba) does not depend on the existence of derivatives. This makes this method suitable for graphs which might have places which are flat, discontinuous or where there is a sharp change in their gradient. The method requires functional evaluations

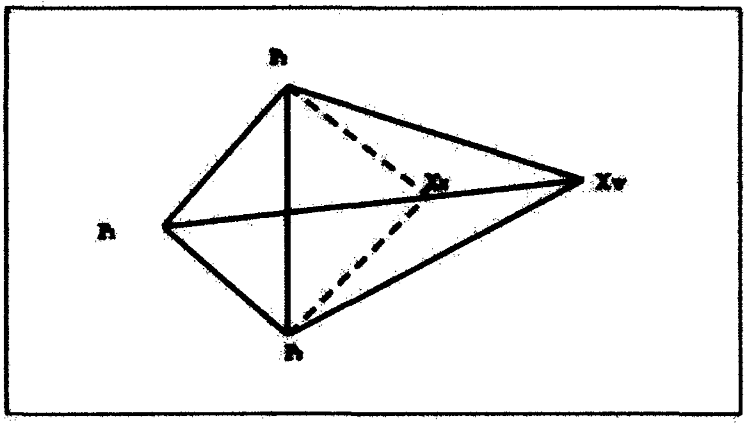
Rather than choosing a single point guess at what the parameters are, this method chooses points which form the simplest shape possible in a dimensional space called a simplex. A precise definition of a simplex is the geometric figure consisting, in N dimensions, of $N+1$ points (or vertices) and all their interconnecting line segments, polygonal faces [35]. In two dimensional space a simplex is a triangle and in three dimension its a tetrahedron. We only focus on simplexes that enclose a finite N dimensional space i.e.non-degenerate.

The algorithm which is applied to minimize a function with N variables, starts with a non-degenerate simplex. We proceed to evaluate the function at each $N+1$ vertices. The vertex, X_W, which has the worst functional value is reflected about the hyperplane generated by the remaining N vertices to form a new vertex X_N.

We then proceed to evaluate the function at this newly created vertex. If X_N becomes the worst vertex then the simplex will contract along the direction $\overrightarrow{X_W X_N}$. If the newly generated vertex has the best functional value in the simplex then there will be an expansion of the simplex along the direction of $\overrightarrow{X_W X_N}$. See figure 4.2.

The simplex generation process will continue until the maximum iteration or the tolerance is reached.

Figure 4.2 : Simplex expansion



4.2.4 Results and discussion

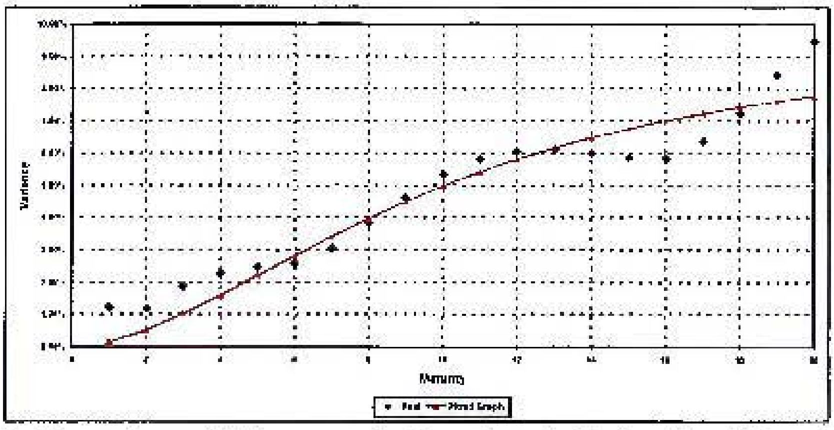
The downhill simplex method was implemented to determine σ_r and λ_r. The original amoeba algorithm was modified to solve the problem. The code is written in true basic and it is documented in appendix 5 [35]. The estimated parameters are:

Table 4.3 :Estimated parameters

λ_r	σ_r
0.13927	0.04139

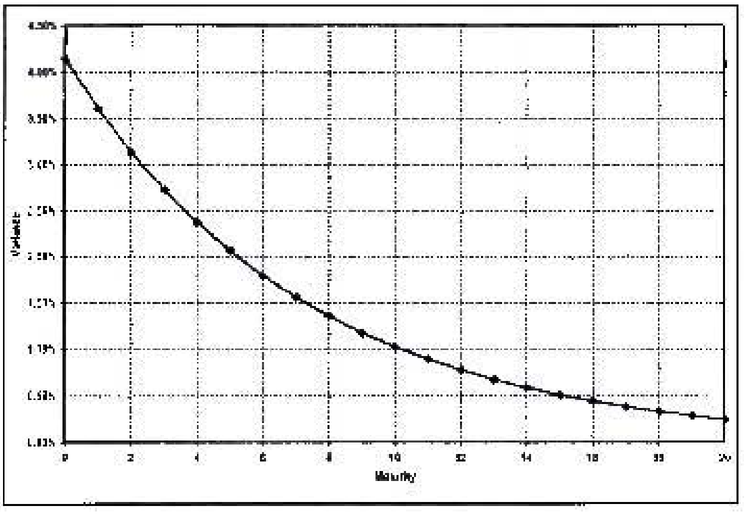
The parameters are positive. This will ensure that the volatility function is decaying in time. In the following graph we will plot the variance of bond returns and the fitted function. Graphically the fitted function fits the data well.

Figure 4.3 : Graph of fitted function and the variance of the real zero coupon bond returns



In this graph we plot the exponential decaying volatility function. We used the estimated parameters to plot this function. The volatility function in equation 4.1 and the estimated parameters (σ_r and λ_r) will be the input to price derivative.

Figure 4.4 : Graph of volatility function with parameter $\lambda_r = 0.13927$ and $\sigma_r = 0.04139$



4.3 Estimate parameters for nominal forward rate

4.3.1 Establish nominal variance equation

In this section we establish the nominal variance equation. We follow the same method as in section 4.2.1. We assume the volatility function is exponentially decaying, as follow:

$$\sigma_n(t,T) = \sigma_n e^{-\lambda_n(T-t)} \tag{4.8}$$

where $\sigma_n > 0$ and $\lambda_n > 0$ are constant parameters.
We use the evolution of the nominal bond under the martingale measure as a base to establish the nominal variance equation i.e.

$$\frac{dP_n(t,T)}{P_n(t,T)} = r_n(t)dt - \int_t^T \sigma_n(t,s)dsd\hat{W}_n(t) \tag{4.9}$$

We follow a similar outline as in 4.2.1 to find the expression of the nominal variance equation, i,e.

$$var\left[\frac{\Delta P_n(t,T)}{P_n(t,T)}\right] = \sigma_n^2\left(\frac{e^{-\lambda_n(T-t)} - 1}{\lambda_n}\right)^2 \Delta t \tag{4.10}$$

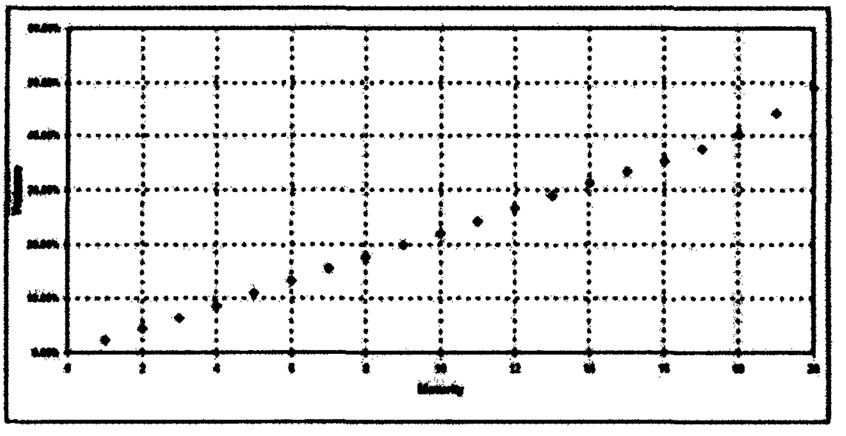
4.3.2 Variance of nominal zero coupon bond

To estimate the unknown parameters (σ_n and λ_n) in equation 4.10 we need the variance of the nominal zero coupon bond returns. We obtained these variances from the nominal zero curve from the previous chapter. The following table and graph will summarize the results of the variances of real zero coupon bond returns.

Table 4.4 : Estimate of the nominal zero coupon bond sample variance from 01 Aug 2001 to 31 May 2006

Bond	Annualized volatility
Ticker	percentages
P(0,1)	2.361
P(0,2)	4.482
P(0,3)	6.387
P(0,4)	8.690
P(0,5)	11.095
P(0,6)	13.394
P(0,7)	15.611
P(0,8)	17.771
P(0,9)	19.923
P(0,10)	22.072
P(0,11)	24.260
P(0,12)	26.650
P(0,13)	28.954
P(0,14)	31.298
P(0,15)	33.439
P(0,16)	35.454
P(0,17)	37.659
P(0,18)	40.398
P(0,19)	44.199
P(0,20)	48.991

Figure 4.5 : Sample variance of real zero coupon bond returns from 02 Aug 2001 to 31 May 06



Nominal zero coupon bond are more volatile than inflation linked bonds. The reason for the volatility is that the nominal bonds are more liquid. The relationship between the nominal zero coupon bonds and maturity is linear.

4.3.3 Estimate parameters by downhill simplex method

In this section we will explain the method used to solve for the parameters in equation (4.10). This is a multidimensional minimization problem because we have to find the minimum of a function which has more than one independent variable. We are going to use the downhill simplex method which was introduced by Neder and Mead [35].

4.3.4 Results and discussion

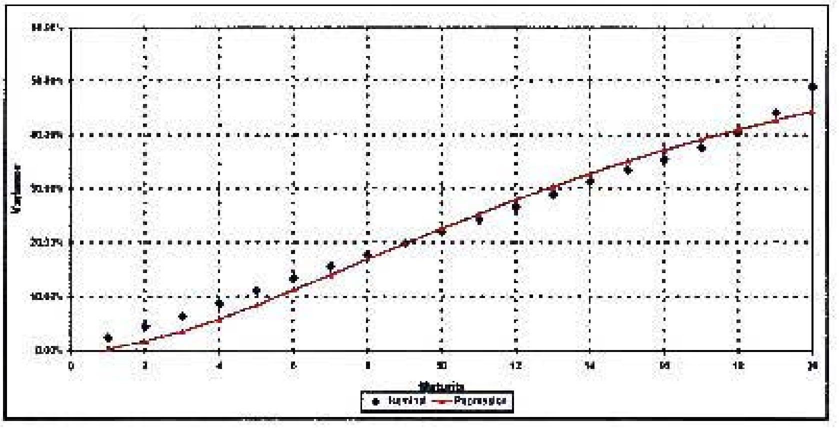
The downhill simplex method was implemented to determine σ_n and λ_n. The original amoeba algorithm was modified to solve the problem. The code is written in true basic and it is documented in appendix 5 [35]. The estimated parameters are:

Table 4.5 :Estimated parameters

λ_n	σ_n
0.09150	0.072620

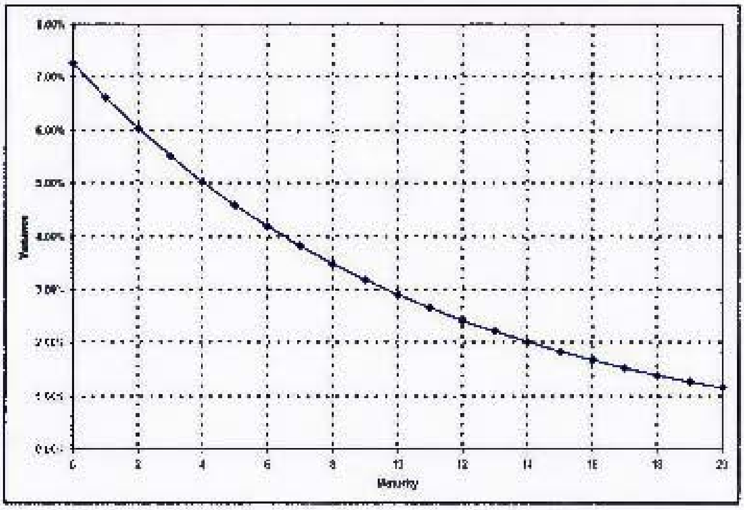
The parameters are positive. This will ensure that the volatility function is decaying in time. In the following graph we will plot the variance of bond returns and the fitted function.

Figure 4.6 : Graph of fitted function and the variance of the nominal zero coupon bond returns



In this graph we plot the exponential decaying volatility function. We used the estimated parameters to plot this function. The volatility function in equation 4.8 and the estimated parameters (σ_n and λ_n) will be the input to price derivative.

Figure 4.7 : Graph of volatility function with parameter $\lambda_n = 0.09150$ and $\sigma_n = 0.07262$



4.4 Estimate inflation index and correlations

The model we have adopted to model real zero-coupon bonds is $P_{CPI}(t,T)$(see equation 2.58) which depends on inflation rate σ_I, correlations between the real and nominal spot interest rate (ρ_{rn}), inflation rate and nominal spot interest rate (ρ_{In}), and inflation index and real spot rate of interest (ρ_{Ir}). We assume that the inflation index is constant. We compute estimate of these parameters as:

$$\sigma_I = \left[\frac{1}{\Delta} var(\frac{\Delta I(t)}{I(t)})\right]^{1/2} \tag{4.11}$$

$$\rho_{Ir} = cor\left(\frac{\Delta r_r(t)}{r_r(t)}, \frac{\Delta I(t)}{I(t)}\right) \tag{4.12}$$

$$\rho_{In} = cor\left(\frac{\Delta r_n(t)}{r_n(t)}, \frac{\Delta I(t)}{I(t)}\right) \tag{4.13}$$

$$\rho_{rn} = cor\left(\frac{\Delta r_r(t)}{r_r(t)}, \Delta r_n(t)\right) \tag{4.14}$$

where Δ =1/12. We have used historical monthly CPI-U data and monthly zero coupon bond returns. We cannot linearly interpolate daily CPI-U values with the hope of increasing the data observation. The reason is that linear interpolation procedure for creating daily index values is deterministic and it give spurious results for the inflation rate volatility. We found that the annualized volatility of the inflation index is 2.04 percent. We list in the following table the correlations stated in equations 4.12 to 4.14.

Table 4.6: Result of computed correlations

Ticker	ρ_{Ir}	ρ_{In}	ρ_{rn}
	percent	percent	percent
P(0,1)	23.80	15.87	39.48
P(0,2)	33.06	20.06	50.17
P(0,3)	38.98	24.23	35.26
P(0,4)	40.85	24.97	32.60
P(0,5)	40.34	25.23	41.73
P(0,6)	35.05	26.25	44.90
P(0,7)	42.51	27.06	44.01
P(0,8)	46.95	27.02	38.28
P(0,9)	45.82	26.36	31.68
P(0,10)	44.12	25.47	26.34
P(0,11)	42.88	25.61	22.83
P(0,12)	42.20	26.25	21.19
P(0,13)	41.87	27.04	21.60
P(0,14)	41.38	27.04	22.30
P(0,15)	39.81	27.18	23.64
P(0,16)	36.39	26.65	25.51
P(0,17)	36.55	25.40	25.24
P(0,18)	40.81	24.89	26.09
P(0,19)	44.40	26.35	26.75
P(0,20)	46.24	29.27	27.98

Chapter 5

Numerical examples

5.1 Price option

5.1.1 Background

In this section we implement the model by providing an example.

- $P_r(t,T)$: real zero-coupon bond maturing at τ
- $P_n(t,T)$: nominal zero-coupon bond maturing at T
- K: strike price
- τ: maturity of the zero coupon bond
- T: maturity of the option
- t: time today
- σ_I: volatility of inflation
- σ_r: real volatility function parameter
- λ_r: real volatility function parameter
- λ_n: nominal volatility function parameter
- σ_n: nominal volatility function parameter
- $I(t)$: inflation adjustment
- ρ_{Ir}: correlation between inflation index and real returns
- ρ_{In}: correlation between inflation index and nominal returns
- ρ_{nr}: correlation between nominal and real returns

Table 6.1 : Summary of inputs

Input

Ticker	Value
$P_r(t,T)$	0.9746
$P_n(t,T)$	0.9307
$I(t)$	1.0200
K	0.9746
σ_I	2.04%
σ_r	4.14%
σ_n	7.26%
λ_r	0.14
λ_n	0.09
ρ_{Ir}	23.8%
ρ_{In}	15.9%
ρ_{nr}	39.5%
T	1
τ	1
t	0

We price a one year European call option on a real zero coupon bond. We assume the above inputs which have been estimated from the previous chapters. Note that the standard unit for a South African bond is R100. We have changed the standard to be unitless i.e (R1 or R1 million). The inflation adjustment factor can be forecasted or one can take an average of yearly inflation rates. We assumed an adjustment of 2 percent.

Table 6.2 : Price of option

Output

Call Price	0.0822
% spot	8.43%

We computed the option price. The calculations are attached in appendix 6 and the disc. We have calculated the percentage of spot statistic. This is the ratio between the option premium and the spot price (real zero coupon bond). This gives us an intuitive understanding of the price. The price of the option is reasonable.

5.2 Historical option prices

We have calculated option prices for our sample period (2 August 2001 to 31 May 2006). We have used the estimated volatility structure to price these option. Firstly, we calculate 1 year, 2 year and 5 year maturity option on a respective 1,2 and 5 year real zero-coupon bond. We struck these options at varying strike level, i.e 70 percent, 80 percent, 90 percent and 100 percent of $P_r(t,T)$.

5.2.1 Results and discussion

In the subsequent charts we show results of historical option prices given the scenarios described.

Figure 6.1 : Call prices with strike $K = 0.7 * P_r(t, T)$ ranging from 2 August 2001 to 31 May 2006

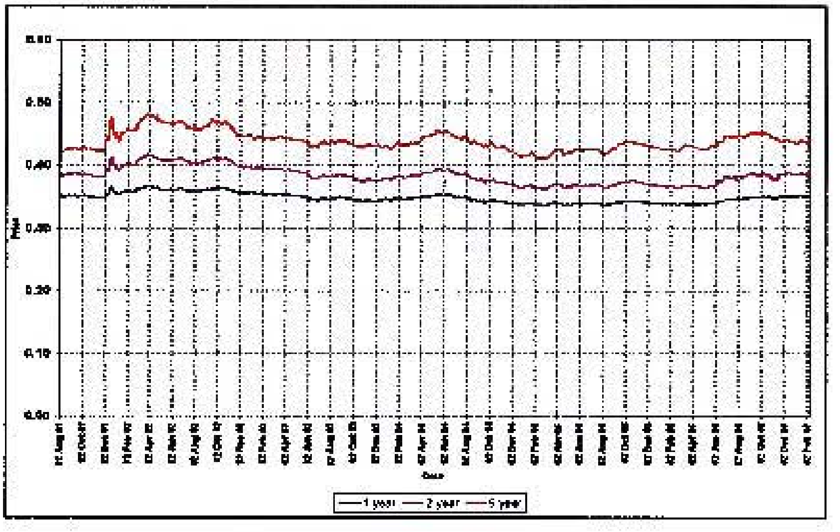


Figure 6.2 : Call prices with strike $K = 0.8 * P_r(t,T)$ ranging from 2 August 2001 to 31 May 2006

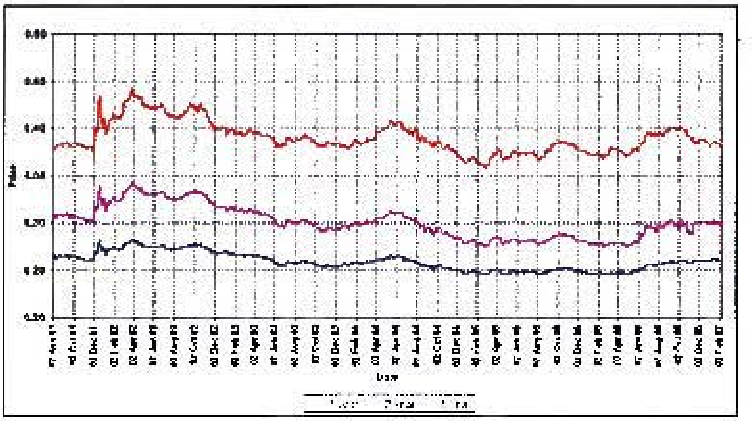


Figure 6.3 : Call prices with strike $K = 0.9 * P_r(t,T)$ ranging from 2 August 2001 to 31 May 2006

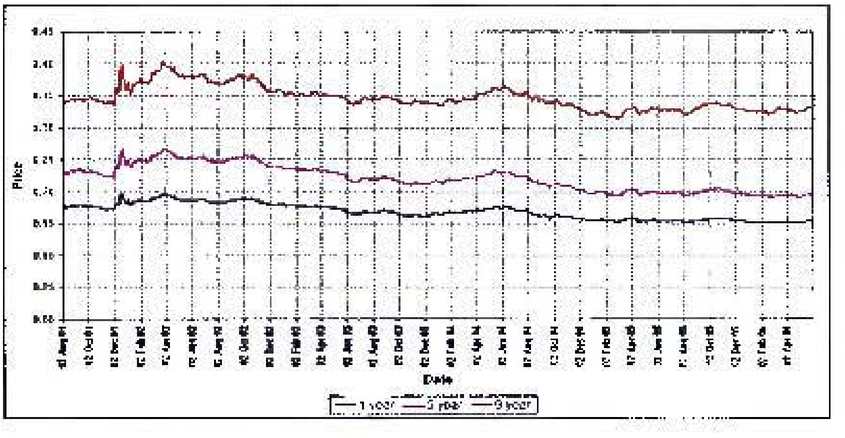
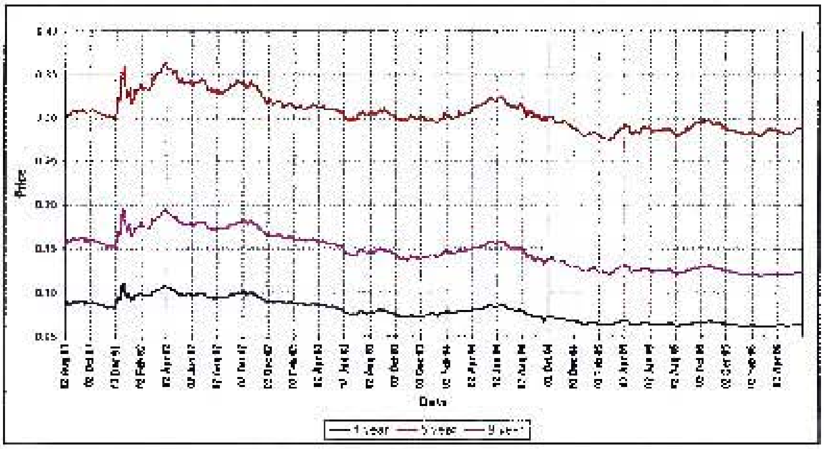


Figure 6.4 : Call prices with strike $K = 1 * P_r(t, T)$ ranging from 2 August 2001 to 31 May 2006



We make two logical conclusions. Firstly, the price of the options increases with decreasing strike. Secondly, the longer maturity options are much more expensive than short maturity options. In the next table we show option price matrix on the 23 June 2003. This table can be updated daily and it gives an indication of were fair option prices.

Table 6.3 : Price matrix on the 31 May 2006

Strike	1 year	2 years	5 years
0.7	0.3456	0.3781	0.4329
0.8	0.2574	0.2957	0.3859
0.9	0.1670	0.2156	0.3428
1	0.0768	0.1437	0.3034

5.3 Year on year swap

In this section we price the year on year inflation swap using the explicit formula that we derived in chapter 2. We have also calibrated this model using market data. See listed below a description of input variables:

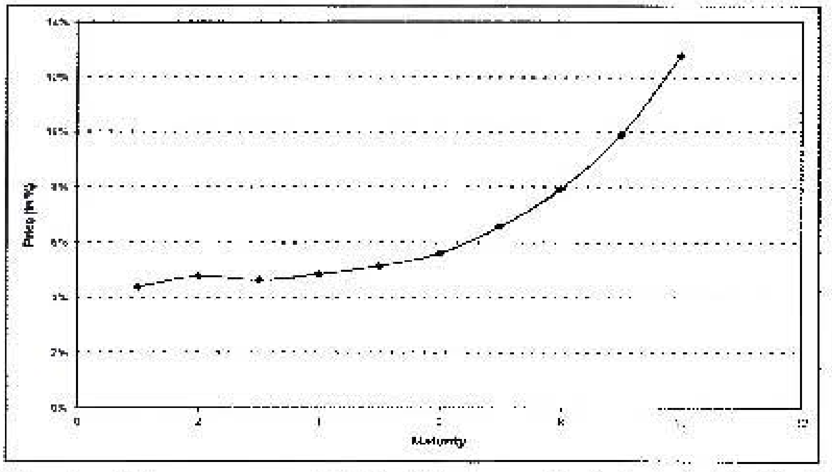
- $a_r = \lambda_r$: real volatility function parameter
- $a_n = \lambda_n$: nominal volatility function parameter
- σ_n: nominal volatility function parameter
- σ_r: real volatility function parameter
- σ_I: volatility of inflation

- T_j: time at end of fixed leg interval
- T_{j-1}: time at the beginning of fixed leg interval
- t: time today
- ρ_{rI}: correlation between inflation index and real returns
- ρ_{rn}: correlation between nominal and real returns
- ρ_{nI}: correlation between inflation index and nominal returns
- ψ_j: fixed leg interval
- N: Nominal

We price an inflation year on year swap which matures in one year and has a nominal of R 1 million. We price this swap as of the 31 May 2006. So we use the all market data that was quoted on this date, for example, swap, real and inflation data. Obviously we use the estimated parameter from the previous chapters, for example, $\lambda_r, \lambda_n, \sigma_n, \sigma_r, \sigma_I, \rho_{rI}, \rho_{rn}, \rho_{nI}$. See listed below a list of populated input variables and values. Please see a copy of the calculation on the disc attached at the back of this thesis.

Ticker	Value
$a_r = \lambda_r$	0.139
$a_n = \lambda_n$	0.092
σ_n	0.073
σ_r	0.041
σ_I	0.020
T_j	1
T_{j-1}	0
t	0
ρ_{rI}	0.238
ρ_{rn}	0.395
ρ_{nI}	0.250
ψ_j	1
N	100

Figure 6.5 : Swap prices (in percentages) on the 31 May 2006



The price of the year on year inflation linked swaps is relatively flat for the short term. We notice that there's slight drop of the 3 year quoted price. This could be explained by the drop in the breakeven inflation indicating that inflation should drop in 3 years. The price of this instrument thereafter increases exponential.

5.4 Numerical example: pricing derivative by simulation

5.4.1 Background

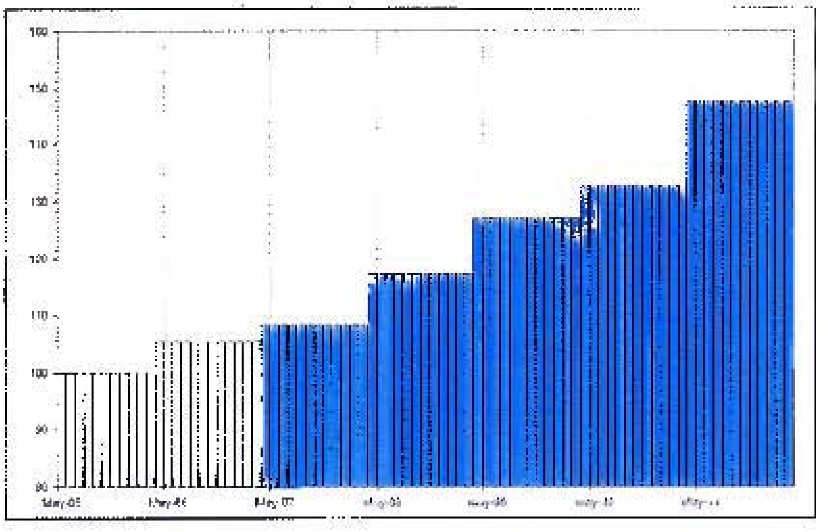
In this section we provide an application of the model outlined in section 2.3.4. We price complicated inflation derivatives, where explicit formulae are not achievable.

In this case we are going to look at the type 1 and 3 floor at 5 percent. We will be forced to use simulating methods to price these derivatives due to the path dependency of the derivative. Obviously we can structure any instrument to any level of sophistication; depending on the market and/or the need there off.

In this section we price a 5 year annuity which commenced on 31/05/2006. Payments are monthly and inflation increases are annually in arrears. Inflation indexation is lagged 4 months, consistent with South Africa CPI-bonds. We will value the annuity assuming it has various inflation derivative forms type 1 and type 3 floor at 5 percent.

See below a timeline of the derivative to be valued on the 31/05/2006. The bar that are shaded in blue means the inflation has not been published.

Figure 6.6 : See below a graph of a timeline of the annuity



5.4.2 Using simulation spreadsheet

To recap, to price this derivative we will simulate the path of the inflation process under the equivalent martingale measure, Q, using equation (2.15). The value of the derivative is then the present value of the average payoff of the derivative for sufficient simulations.

We have attached in appendix 7 a copy of the VBA code used to price the derivative simulation. We have also included as part of the thesis the spreadsheet outlining the calculation. The next section outlines the methodology that needs to be followed when using the simulation spreadsheet.

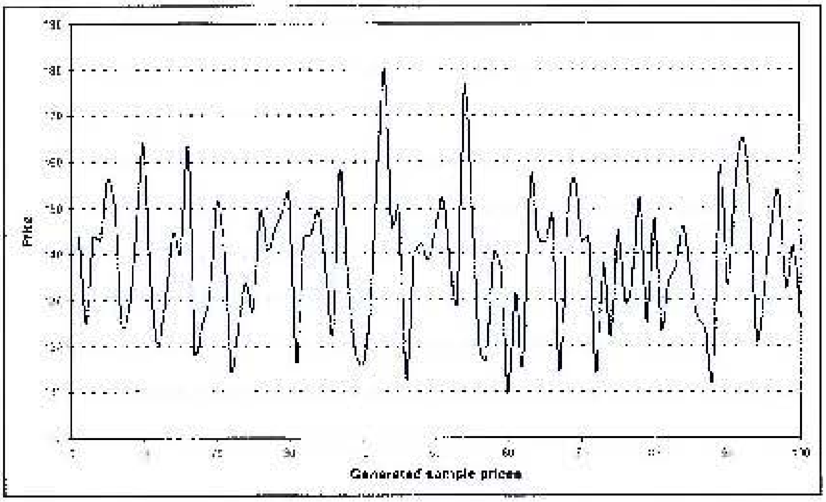
It is recommended that the valuation follow these steps: Firstly, draw a timeline of the derivative to be valued. Secondly, ensure REAL and SWAP recover the published CPI. Thirdly, specify the number of simulations in cell E99. Lastly, to run the simulation press the 'simulate' tab and observe the discounted simulated prices in cell I95.

Seeing that we are implementing the simulation calculation in excel, we refrain from using Excel random number generator because it only produces 32767 distinct random numbers and the numbers generated repeat too soon. Consequently, it is unsuitable for valuations of out-of-the-money caps and floors. The simulation spreadsheet uses the Mersenne Twister random number generator algorithm [27] for uniform random variables and the Box Muller transform [6] to transform these to normal random variables. The uniform random numbers are generated by the POPTOOLS.XLA VBA add-in for Excel (see http://www.cse.csiro.au/poptools/).

5.4.3 Results

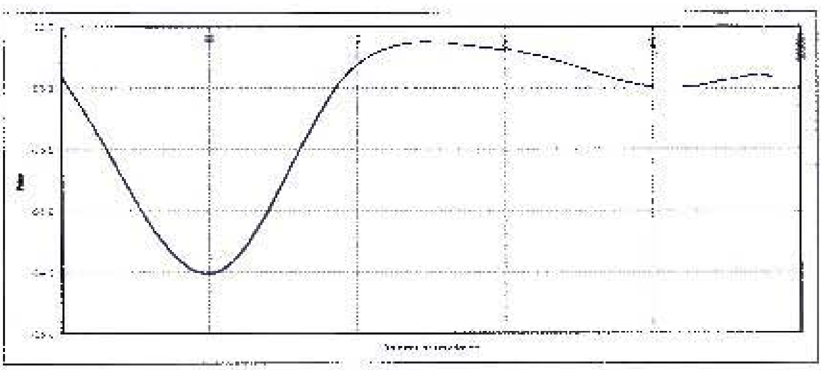
In this section we provide a typical price path given a specific number of simulation. See below a price sample paths given 100 simulations. Once we have simulated our price process we can take the average and discount it given the corresponding discount factor.

Figure 6.7 : Generated sample prices



Now we can visualize a typical price process. How do we know how many price processes do we need to simulated before we can accept the valuation. The purpose of this section is to test the stability of the valuation before we can accept it. We increase the number of valuation until we see a convergence in our valuation. We ran the calculation using 10, 100, 1000, 10000 and 50000 simulations. See figure 6.8 below.

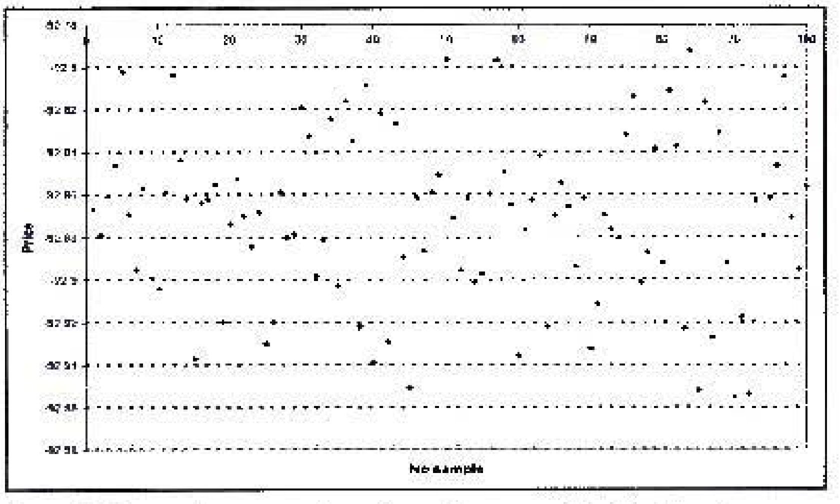
Figure 6.8 : Convergence of price



We observe that the price converges with increasing number of simulations. We have also observed that the calculation becomes slower with an increase in simulation. This can be impractical when pricing these derivatives in a work context. We could not run 100000 simulations due to the constrains in the number of rows available in excel - 65536. It's difficult to ascertain if the valuation is acceptable and stable.

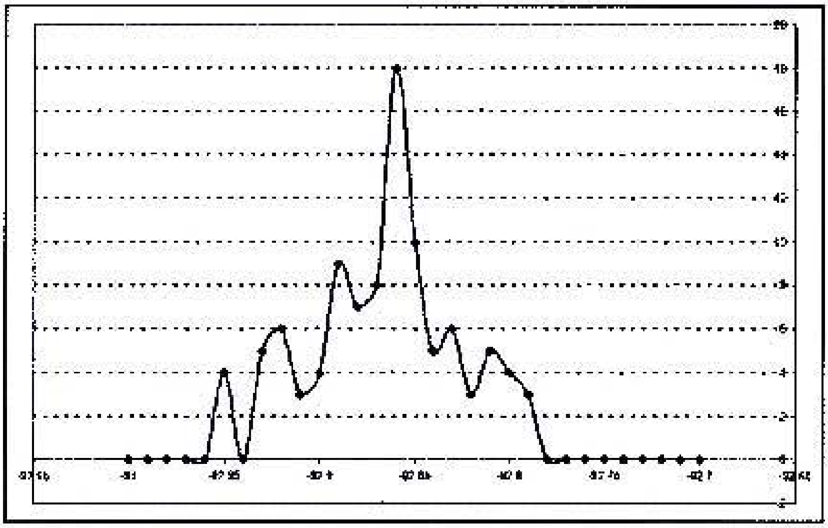
In the next results we simulate 100 valuations using 50000 simulations. See below a scatter plot of 100 valuations.

Figure 6.9 : Simulated prices at 50000



This calculation takes a very long time. On average it takes 29 minutes to run a valuation with 50000. The total time is takes to generate 100 valuations is 48 hours. We used this data to perform some calculate basic statistics. The average price is 92.87. See the histogram below.

Figure 6.10 : distribution the price of annuity



The histogram is a normally distributed curve distributed about the average. We thus propose that we use a simulation of 50000 provides acceptable and time valuation of this annuity.

Chapter 6

Conclusion

6.1 Inflation model for derivative pricing

- The purpose of this section is to develop a pricing model for inflation linked derivatives.
- Inflation linked swaps, Caplet and Floorlet , Option on real zero coupon bond
- We have added innovation to pricing inflation linked swaps in South Africa. We have used Jarrow Yildrim foreign currency model to price zero coupon inflation linked swap and the year on year inflation linked swap.

  The lag in the inflation data makes computations of current option prices approximation
- The choice of inflation index affects the final option price

6.2 Real, Nominal and swap curves

- We needed to have a history of the South African nominal yield curve. We will use the nominal yield curve to strip nominal zero coupon bonds. Once we determine the nominal zero coupon bond prices we will calculate their respective returns over the sample period. We will then be able to determine historical volatilities of various zero coupon bond maturities. This data will be used in the following chapter for estimating the volatility function. The volatility function is an input in the HJM model.
- The discount curve to price derivative e.g. zero coupon swaps and year on year swaps. We also use the swap curve as an input to building the real zero curve.

6.3 Estimated parameters

6.3.1 Real parameters

- The variances of the real zero coupon bond are relatives low compared with the same maturity nominal zero coupon bonds. This is so because inflation linked bonds are less liquid than conventional nominal bonds.
- We note the medium and long term real zero coupon seem to deviate from the apparent linearity of real zero coupon bond variances and maturity relationship. This implies that the medium and long term real zero coupon bond are more volatile. This observation is intuitive given that the majority of inflation linked bond issuance sit in the medium and longer term region of the curve.
- The parameters (σ_r and λ_r) are positive. This will ensure that the volatility function is decaying in time.
- The volatility function in equation 4.1 and the estimated parameters (σ_r and λ_r) will be the input to determining derivative prices.

6.3.2 Nominal parameters

- Nominal zero coupon bond are more volatile than inflation linked bonds. The reason for the volatility is that the nominal bonds are more liquid. The relationship between the nominal zero coupon bonds and maturity is linear.
- The parameters(σ_n and λ_n) are positive. This will ensure that the volatility function is decaying in time.
- The volatility function in equation 4.10 and the estimated parameters (σ_n and λ_n) will be the input to determining derivative prices.

6.3.3 Estimate parameters by downhill simplex method

- We discovered that equation 4.7 and 4.10 as a multidimensional minimization problem because we have to find the minimum of a function which has more than one independent variable. We use the downhill simplex method which was introduced by Neder and Mead to solve this problem.

6.3.4 Estimate inflation index and correlations

- We cannot linearly interpolate daily CPI-U values with the hope of increasing the data observation. The reason is that linear interpolation procedure for creating daily index values is deterministic and it give spurious results for the inflation rate volatility. So we only have 13 monthly observation for our sample period. We need to increase the sample period so that we can improve these estimates.

6.4 Pricing derivatives

6.4.1 Option on real zero coupon bonds

- We computed the option price. We have calculated the percentage of spot statistic. This is the ratio between the option premium and the spot price (real zero coupon bond). This gives us an intuitive understanding of the price. The price of the option is reasonable.
- We have calculated option prices for our sample period (01 Aug 2001 to 31 May 2006). We make two logical conclusions. Firstly, the price of the options increases with decreasing strike. Secondly, the longer maturity options are much more expensive than short maturity options.

6.4.2 Year on year swap

- We have succeffuly priced a year on year swap using the Jarrow Yildrim framework. This has never been done before with South African data.
- The price of the year on year inflation linked swaps is relatively flat for the short term. We notice that there's slight drop of the 3 year quoted price. This could be explained by the drop in the breakeven inflation indicating that inflation should drop in 3 years.

6.4.3 Pricing annuity by simulation

- we price a 5 year annuity which commenced on 31/05/2006. Payments are monthly and inflation increases are annually in arrears. Inflation indexation is lagged 4 months, consistent with South Africa CPI-bonds. We will value the annuity assuming it has various inflation derivative forms type 1 and type 3 floor at 5 percent.
- We observe that the price converges with increasing number of simulations. We have also observed that the calculation becomes slower with an increase in simulation. This can be impractical when pricing these derivatives in a work context. We could not run 100000 simulations due to the constrains in the number of rows available in excel - 65536.
- We thus propose that we use a simulation of 50000 provides acceptable and time valuation of this annuity.

Chapter 7

Appendix 1

7.1 Heath Jarrow Morton model

7.1.1 Background

The Heath-Jarrow-Morton (HJM) offers a new approach to interest rate option pricing. The input to the model is the term structure and the volatility of the term structure. HJM models the dynamics of the forward interest rate. Forward interest rate are possible rate which one can contract to borrow and lend at a future date. The initial forward rate curve is an input to the HJM model. This is taken from market data. The volatility of each forward rate must be specified in order to model the evolution of the entire term structure.

7.1.2 Assumptions

We assume that the economy is frictionless, competitive and continuous. By frictionless we mean that there are no arbitrage opportunities, no transaction costs in buying and selling financial securities, no restriction on trade and no differential taxes versus capital gains income. The frictionless assumption is reasonable and justifiable on two grounds. Firstly, large institutional traders approximate frictionless markets because their transactions costs are minimal. Secondly, we need to understand frictionless markets before we can comprehend the dynamics of markets with friction.

The markets are assumed to be competitive. This implies that the market for any financial security is perfectly liquid i.e. a trader can buy or sell shares of a traded security without influencing its price.

We assume a continuous trading economy with trading interval $\{0, \tau\}$.

7.1.3 Forward rate

We firstly define our probability space as $(\Omega, F, \mathbb{P}, \mathbf{F}_t)$. This probability space characterizes the uncertainty in the economy. Ω is our state space, F is a set of possible events (σ - algebra on Ω), $\mathbb{P}$ is statistical probability measure on (Ω,F) and ($\mathbf{F}_t t\epsilon\{0,T\}$) is the standard filtration generated by Brownian motion W_t.

The Heath, Jarrow and Morton (1992) approach is based on specification of the dynamics of the instantaneous and continously compounded forward rates $f(t,T)$. For any fixed maturity $T \leq \tau$ we assume that the forward rate evolves as :

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{7.1}$$

where $\alpha(t,T)$ is random and $\sigma(t,T)$ is deterministic function. These parameters satisfy smoothness and boundedness conditions and $\alpha(t,T)$ and $\sigma(t,T)$ are adapted process [1]. Volatility in expression 2.1, which is deterministic, implies that the real term structure of interest rates generates a Gaussian Economy.

Equation 2.1 is an infinite dimensional stochastic system i.e. one equation for each fixed T. We will use the observed forward curve as boundary value at t=0 i.e

$$f(0,T) = f^{*}(0,T), \forall T \geq 0 \tag{7.2}$$

In other words the initial condition $f^{*}(0,T)$, for any fixed maturity, is determined by the current value of the continuously compounded forward rate for any future date T. Consequently, we will obtain a perfect fit to the observed term structure. Therefore, the problem of inverting the yield curve is avoided. This is possible because of an inversely proportional relationship between the bond price and forward rates, given by :

$$P(t,T) = \exp(-\int_t^T f(t,s)ds) \tag{7.3}$$

It is evident from equation 2.3 that a specification of the forward rate is equivalent to a specification of the entire family of bond prices. The first task is to investigate when the bond market, which is induced by the forward rate, is arbitrage free. This condition can be satisfied if there exist a suitably defined martingale measure. Under this martingale probability the drift coefficient $\alpha(t,T)$ in the dynamics 2.1 of the forward rate is uniquely determined by the volatility and a stochastic process which can be interpreted as the risk premium.

Bond Market

In the following lemma we define the dynamics od the bond price equation under the actual probability $\mathbb{P}$. We observe that the drift and volatility coeficients in the bond

[1] $\alpha(s,T)$ is $\mathbf{F}_t$ - adapted and jointly measurable with $\int_0^T \mid \alpha(s,T) \mid ds<\infty$ $\mathbb{P}$ - a.s. and $\sigma(s,T)$ satisfies $\int_0^T \sigma^2(s,T)ds<\infty$ $\mathbb{P}$ - a.s

price can be expressed in terms of α and σ coefficients of the forward rate dynamics, and the short term interest rate $r(t) = f(t,t)$

Lemma 2.1 : The dynamics of the bond price $P(t,T)$ are determined by the expression:

$$dP(t,T) = P(t,T)m(t,T)dt + P(t,T)v(t,T)dW_t \tag{7.4}$$

where m and v are given by the following fomulae

$$m(t,T) = f(t,t) + A(t,T) - \frac{1}{2}|S(t,T)|^2, \quad v(t,T) = S(t,T) \tag{7.5}$$

and for any $t\epsilon[0,T]$ we thus have

$$A(t,T) = -\int_t^T \alpha(t,s)ds, \quad S(t,T) = -\int_t^T \sigma(t,s)ds \tag{7.6}$$

Proof :

For every fixed $T \leq \tau$, the dynamics of the instantaneous forward rate is given by the integrated formula of the 2.1

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dW_u \tag{7.7}$$

Let $Y(t,T)$=ln $P(t,T)$ then it will follow from 2.1 and 2.3 that

$$Y(t,T) = -\int_t^T f(t,s)ds \tag{7.8}$$

$$Y(t,T) = -\int_t^T f(0,s)ds - \int_t^T\int_0^t \alpha(u,s)duds - \int_t^T\int_0^t \sigma(u,s)dW_u ds \tag{7.9}$$

Apply Fubini's standard theorem

$$Y(t,T) = -\int_t^T f(0,s)ds - \int_0^t\int_t^T \alpha(u,s)duds - \int_0^t\int_t^T \sigma(u,s)dW_u ds \tag{7.10}$$

or equivalently

$$= -\int_0^T f(0,s)ds - \int_0^t\int_u^T \alpha(u,s)dsdu - \int_0^t\int_u^T \sigma(u,s)dsdW_u \tag{7.11}$$

$+\int_0^t f(0,s)ds + \int_0^t\int_u^t \alpha(u,s)dsdu + \int_0^t\int_u^t \sigma(u,s)dsdW_u$ As a result

$$Y(t,T) = Y(0,T) + \int_0^t r(s)ds - \int_0^t\int_u^T \alpha(u,s)dsdu - \int_0^t\int_u^T \sigma(u,s)dsdW_u \tag{7.12}$$

where we used the fact that $r(s) = f(s,s)$ and that the integral form of the forward rate over the interval $[0, s]$ is:

$$r(s) = f(s,s) = f(0,s) + \int_0^s \alpha(u,s)du + \int_0^s \sigma(u,s)dW_u \tag{7.13}$$

We substitute A and S as defined in the statement of the lemma 1.

$$dY(t,T) = \{r(t) + A(t,T)\}dt + S(t,T)dW_t \tag{7.14}$$

To conclude the proof we must apply the Itô formula to the process $P(t,T) = exp(Y(t,T))$.Now

$$dP(t,T) = de^{Y(t,T)} \tag{7.15}$$

$$dP(t,T) = e^{Y(t,T)}[dY - \frac{1}{2}d < Y >] \tag{7.16}$$

$$dP(t,T) = p(t,T)\{r(t) + A(t,T) + \frac{1}{2}|S(t,T)|^2\}dt + S(t,T)p(t,T)dW_t \tag{7.17}$$

Absence of Arbitrage

In this section we looking at a condition which will ensure that the absence of arbitrage opportunities across all bonds with varying maturity. This condition will also excludes arbitrage between bonds and the saving account. In order to achieve absence of arbitrage we assume that a martingale measure $\mathbb{Q}$ can be chosen simultaneously for all bonds and also between the savings account.

Spot martingale probability measure

There exist an adapted $\mathbb{R}^d$ value process γ_t such that:

$$\mathbb{E}_{\mathbb{P}}\{\varepsilon_\tau * (\int_0^{\cdot} \gamma_u \cdot dW_u)\} = 1 \tag{7.18}$$

and, for any maturity $T \leq \tau$, we have

$$A(t,T) = \frac{1}{2}|S(t,T)|^2 - S(t,T) \cdot \gamma_t \tag{7.19}$$

substitute for $A(t,T)$ and $S(t,T)$

$$\int_T^T \alpha(t,u)du + \frac{1}{2}|\int_T^T \sigma(t,u)du|^2 + \gamma_t \cdot \int_T^T \sigma(t,u)du = 0 \tag{7.20}$$

Take the partial derivative with respect to T

$$\alpha(t,T) + \sigma(t,T) \cdot (\gamma_t + \int_T^T \sigma(t,u)du) = 0 \tag{7.21}$$

for every $0 \leq t \leq T \leq \tau$.

For $\mathbb{Q}$ probability measure to be equivalent to $\mathbb{P}$ on $(\Omega, \mathbf{F}_t)$ it must satisfy:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \varepsilon_\tau * (\int_0^\cdot \gamma_u \cdot dW_u) := \exp(\int_0^\tau \gamma_u \cdot dW_u - \frac{1}{2}\int_0^\tau |\gamma_u|^2 dW_u), \quad \mathbb{P} - a.s. \quad (7.22)$$

where γ_t satisfy equation 2.18. Moreover, the γ_t process is associated with the risk premium. $\mathbb{Q}$ can be seen as the spot martingale measure for the HJM model. We define $\mathbb{Q}$ Brownian motion $\hat{W}$ by setting the Girsanov transformation to be:

$$\hat{W}_t = W_t - \int_0^t \gamma_u du, \quad \forall t \epsilon [0,T] \quad (7.23)$$

Corollary 2.1 : For any fixed maturity $T \leq \tau$, the dynamics of the bond price $P(t,T)$ under the spot martingale measure $\mathbb{Q}$ are:

$$dP(t,T) = P(t,T)r(t)dt + S(t,T)P(t,T)d\hat{W}_t \quad (7.24)$$

and the forward rate $f(t,T)$ is

$$df(t,T) = -\sigma(t,T)S(t,T)dt + \sigma(t,T)d\hat{W}_t \quad (7.25)$$

Lastly, short-term rate $r(t) = f(t,t)$ is given by the expression

$$r(t) = f(t,0) - \int_0^t \sigma(u,t)du + \int_0^t \sigma(u,t)d\hat{W}_u \quad (7.26)$$

Proof :

Firstly we solve $m(t,T)$ and $v(t,T)$ and then we substitute into the bond dynamics described equation 2.4 of lemma 2.1

$$m(t,T) = f(t,t) + A(t,T) - \frac{1}{2}|S(t,T)|^2 \quad (7.27)$$

substitute for $A(t,T)$ from equation 2.19

$$m(t,T) = f(t,t) + \frac{1}{2}|S(t,T)|^2 - S(t,T) \cdot \gamma_t - \frac{1}{2}|S(t,T)|^2 \quad (7.28)$$

consequently

$$m(t,T) = f(t,t) - S(t,T) \cdot \gamma_t \quad (7.29)$$

We also know that

$$v(t,T) = S(t,T) \quad (7.30)$$

Now we can substitute the above results in the bond dynamics

$$dP(t,T) = P(t,T)(f(t,t) - S(t,T) \cdot \gamma_t)dt + P(t,T)S(t,T)dW_t \quad (7.31)$$

Substitute the differential of the Girsanov transformation i.e. equation 2.23

$$dP(t,T) = P(t,T)(f(t,t) - S(t,T) \cdot \gamma_t)dt + P(t,T)S(t,T)(d\hat{W}_t + \gamma_t dt) \quad (7.32)$$

So

$$dP(t,T) = P(t,T)f(t,t)dt + P(t,T)S(t,T)d\hat{W}_t \quad (7.33)$$

We also know that $r(t) = f(t,t)$, so

$$dP(t,T) = P(t,T)r(t)dt + P(t,T)S(t,T)d\hat{W}_t \quad (7.34)$$

The proof for the forward rate and the short-term interest rate is similar.

Gaussian HJM model

Gaussian HJM model are cases where the volatility of the forward rate is deterministic. We are going to describe two Gaussian HJM model namely: Ho and Lee and the Generalized Vasicek model

Ho and Lee model :

We assume that the volatility is constant for each forward rate. Moreover, it is independent of maturity date and level of the forward rate. So $\sigma(t,T) = \sigma$ for positive values of σ. We substitute this expression 2.25 to determine the dynamics of the forward rate process $f(t,T)$ under the probability measure $\mathbb{Q}$ are:

$$df(t,T) = \sigma^2(T-t)dt + \sigma d\hat{W}_t \quad (7.35)$$

so that the dynamics of the bond price are:

$$dP(t,T) = P(t,T)r(t)dt + \sigma(t,T)P(t,T)d\hat{W}_t \quad (7.36)$$

Generalized Vasicek model :

Forward rates of longer maturity fluctuate less than forward rates of shorter maturity. To account for this fact in the HJM framework, we assume that the volatility of the forward rate is exponentially decaying:

$$\sigma(t,T) = \sigma e^{-\lambda(T-t)}, \qquad \forall t \epsilon [0,T] \quad (7.37)$$

for positive real numbers σ,$\lambda > 0$

$$S(t,T) = \int_t^T \sigma e^{-\lambda(u-t)} du = -\frac{\sigma}{\lambda}(e^{-\lambda(T-t)} - 1) \quad (7.38)$$

and consequently the forward rate is

$$df(t,T) = -\frac{\sigma^2}{\lambda} e^{-\lambda(T-t)}(e^{-\lambda(T-t)} - 1)dt + \sigma e^{-\lambda(T-t)} d\hat{W}_t \quad (7.39)$$

So the bond price satisfies

$$dP(t,T) = P(t,T)r(t)dt + \frac{\sigma}{\lambda}(e^{-\lambda(T-t)} - 1)P(t,T)d\hat{W}_t \quad (7.40)$$

Chapter 8

Appendix 2

8.1 System of SDE in the HJM framework

We will illustrate the use of systems by modelling foreign currency market using the approach of Amin and Jarrow (1991) [1].

Notations and definitions stated in section 2.4.2 (Proposition 2.1) will apply for the proof of the subsequent system of stochastic differential equations. Moreover, we assume the following:

$$\frac{dB_n(t)}{B_n(t)} = r_n(t)dt \tag{8.1}$$

$$B_n(0) = 1 \tag{8.2}$$

$$\frac{dB_r(t)}{B_r(t)} = r_r(t)dt \tag{8.3}$$

$$B_r(0) = 1 \tag{8.4}$$

We define

$$W(t) = \begin{pmatrix} W_n(t) \\ W_I(t) \\ W_r(t) \end{pmatrix} \tag{8.5}$$

$$\rho = \begin{pmatrix} 1 & \rho_{nI} & \rho_{nr} \\ \rho_{nI} & 1 & \rho_{Ir} \\ \rho_{nr} & \rho_{Ir} & 1 \end{pmatrix} \tag{8.6}$$

and

$$Z_n(t,T) = \frac{P_n(t,T)}{B_n(t)}, \quad Z_{rn}(t,T) = \frac{B_r(t,T)I(t)}{B_n(t)}, \quad Z_r(t,T) = \frac{P_r(t,T)I(t)}{B_n(t)} \tag{8.7}$$

We will write

$$A_n(t,T)=\int_t^T \alpha_n(t,u)du,\quad S_n(t,T)=-\int_t^T \sigma(t,u)du,\quad \nu_n(t,T)=\frac{1}{2}(S_n^2(t,T)-A_n(t,T)) \tag{8.8}$$

and similarly for $A_r(t,T), S_r(t,T), \nu_r(t,T)$.

8.2 The Market

We consider the market consisting of the three assets $P_n(t,T), P_r(t,T)I(t), B_r(t)I(t)$ with the numeraire $B_n(t)$. The numeraire-based assets are $Z_n(t,T), Z_r(t,T), Z_{rn}(t,T)$. We first obtain the equation of various assets in the actual measure. Then we construct the equivalent martingale measure that makes them martingales. Note that d is taken with respect to t in all cases.

lemma 1: The asset $Z_{rn}(t,T)$

$$\frac{dZ_{rn}(t,T)}{Z_{rn}(t,T)}=\frac{d\frac{B_r(t)I(t)}{B_n}}{\frac{B_r(t)I(t)}{B_n(t)}} \tag{8.9}$$

$$=\frac{de^{\int_0^t(r_r-r_n)}I(t)}{e^{\int_0^t(r_r-r_n)}I(t)} \tag{8.10}$$

$$=\frac{de^{\int_0^t(r_r-r_n)}I(t)}{e^{\int_0^t(r_r-r_n)}I(t)}+\frac{dI(t)}{I(t)} \tag{8.11}$$

$$=(r_r-r_n)dt+\frac{dI(t)}{I(t)} \tag{8.12}$$

$$=(r_r-r_n+\mu_I)dt+\sigma_I(t)dW_I(t) \tag{8.13}$$

lemma 2: The asset $Z_n(t,T)$

Firstly we need to prove:

$$\frac{dZ_n(t,T)}{Z_n(t,T)}=\nu_n(t,T)dt+S_n(t,T)dW_n(t) \tag{8.14}$$

Proof :

We know that from section 2.3 that the integral form of the forward rate is given as:

$$f(t,T)=f(0,T)+\int_0^t \alpha_n(s,T)ds+\int_0^t \sigma_n(s,T)dW_n(t) \tag{8.15}$$

while the equation of $r(t)=f(t,t)$, so

$$r(t)=f(t,T)=f(0,T)+\int_0^t \alpha_n(s,T)ds+\int_0^t \sigma_n(s,T)dW_n(t) \tag{8.16}$$

We now define the cash bond in terms of r

$$B_n(t) = exp(\int_0^t r_n(u)du) \tag{8.17}$$

$$= exp\left(\int_0^t f(0,u)du + \int_0^t du \int_0^u \alpha(s,u)ds + \int_0^t \int_0^u \sigma_n(s,u)dW_n(u)\right) \tag{8.18}$$

$$= exp\left(\int_0^t f(0,u)du + \int_0^t ds \int_s^t \alpha_n(s,u)du + \int_0^t dW_n(s) \int_s^t \sigma_n(s,u)du\right) \tag{8.19}$$

Now

$$Z_n(t,T) = \frac{P_n(t,T)}{B_n(t)} \tag{8.20}$$

$$= \frac{exp\left(-\left(\int_t^T f(0,u)du + \int_0^t ds \int_t^T \alpha_n(s,u)du + \int_0^T dW_n(s) \int_t^T \sigma_n(s,u)du\right)\right)}{exp\left(\int_0^t f(0,u)du + \int_0^t ds \int_s^t \alpha_n(s,u)du + \int_0^t dW_n(s) \int_s^t \sigma_n(s,u)du\right)} \tag{8.21}$$

$$= exp\left(-\left(\int_0^T f(0,u)du + \int_0^t ds \int_s^T \alpha_n(s,u)du + \int_0^t dW_n(s) \int_s^T \sigma_n(s,u)du\right)\right) \tag{8.22}$$

Let

$$A_n(t,T) = \int_t^T \alpha(t,u)du, \quad S_n(t,T) = -\int_t^T \sigma_n(t,u)du \tag{8.23}$$

So $Z_n(t,T)$ becomes

$$= exp\left(-\int_0^T f(0,u)du\right) exp\left(-\int_0^t A_n(s,T)ds + \int_0^t S_n(s,T)dW_n(s)\right) \tag{8.24}$$

$$= P_n(0,T)exp\left(-\int_0^t A_n(s,T)ds + \int_0^t S_n(s,T)dW_n(s)\right) \tag{8.25}$$

Apply the Itô lemma on $Z_n(t,T)$ to obtain

$$\frac{dZ_n(t,T)}{Z_n(t,T)} = \left(\frac{1}{2}S_n^2(t,T) - A_n(t,T)\right)dt + S_n(t,T)dW_n(t) \tag{8.26}$$

We conclude the proof by substituting the drift of the above stochastic equation to $\nu_n(t,T)$
We can also express $Z_n(t,T)$ as:

$$dZ_n(t,T) = d\left(\frac{P_n(t,T)}{B_n(t)}\right) = \frac{P_n(t,T)}{B_n(t)}\left(\frac{dP_n(t,T)}{P_n(t,T)} - \frac{dB_n(t)}{B_n(t)}\right) \tag{8.27}$$

When we equate the two values of $dZ_n(t,T)$ we obtain:

$$d\left(\frac{P_n(t,T)}{B_n(t)}\right) = (r_n + \nu_n)dt + S_n(t,T)dW_n(t) \tag{8.28}$$

A similar analysis for real bond market yields:

$$d\left(\frac{P_r(t,T)}{B_r(t)}\right) = (r_r + \nu_r)dt + S_r(t,T)dW_r(t) \tag{8.29}$$

lemma 3: The asset $Z_r(t,T)$

$$\frac{dZ_r(t,T)}{Z_r(t,T)} = \frac{d\left(\frac{P_r(t,T)I(t)}{B_n I(t)}\right)}{\frac{P_r(t,T)I(t)}{B_n I(t)}} \tag{8.30}$$

$$= \frac{d(P_r(t,T)I(t))}{P_r(t,T)I(t)} - \frac{dB_n(t)}{B_n(t)} \tag{8.31}$$

$$= \frac{d(P_r(t,T)I(t))}{P_r(t,T)} - r_n(t)dt \tag{8.32}$$

Further

$$\frac{d(P_r(t,T)I(t))}{P_r(t,T)I(t)} = \frac{dP_r(t,T)}{P_r(t,T)} + \frac{dI(t)}{I(t)} + \frac{dP_r(t,T)}{dP_r(t,T)}\frac{dP_r(t,T)}{P_r(t,T)} \tag{8.33}$$

$$= \frac{dP_r(t,T)}{P_r(t,T)} + \frac{dI(t)}{I(t)} + \sigma_I(t)S_r(t,T)\rho_{rI}dt \tag{8.34}$$

Using the last two equations and the expressions for $\frac{dP_r(t,T)}{P_r(t,T)}, \frac{dI(t)}{I(t)}$ we obtain:

$$\frac{dZ_r(t,T)}{Z_r(t,T)} = ((r_r(t) + \nu_r(t,T))dt + S_r(t,T)dW_r(t)) + (\mu_I dt + \sigma_I(t)dW_I(t)) + \sigma_I(t)\rho_{rI}dt - r_n dt \tag{8.35}$$

$$= ((r_r(t) - r_n(t) + \mu_I(t)) + \nu_r(t) + \sigma_I(t)S_r(t,T)\rho_{rI})dt + \sigma_I(t)dW_I(t) + S_r(t,T)dW_r(t) \tag{8.36}$$

The market equation and the equivalent martingale method(EMM)

The three market assets have the following equations:

$$\frac{dZ_n(t,T)}{Z_n(t,T)} = \nu_n(t,T)dt + S_n(t,T)dW_n(t) \tag{8.37}$$

$$\frac{dZ_{rn}(t,T)}{Z_{rn}(t,T)} = (r_r - r_n + \mu_I)dt + \sigma_I(t)dW_I(t) \tag{8.38}$$

$$\frac{dZ_r(t,T)}{Z_r(t,T)} = ((r_r(t) - r_n(t) + \mu_I(t)) + \nu_r(t) + \sigma_I(t)S_r(t,T)\rho_{rI})dt + \sigma_I(t)dW_I(t) + S_r(t,T)dW_r(t) \tag{8.39}$$

We represent the above system of equations in vector form, we firstly let:

$$\mu_1 = \nu_n(t,T) \tag{8.40}$$

$$\mu_2 = r_r - r_n + \mu_I \tag{8.41}$$

$$\mu_3 = ((r_r(t) - r_n(t) + \mu_I(t)) + \nu_r(t) + \sigma_I(t)S_r(t,T)\rho_{rI} \tag{8.42}$$

$$\bar{\mu} = (\mu_1, \mu_2, \mu_3)^T, \quad Z = (Z_n(t,T), Z_{rn}(t,T), Z_r(t,T))^T, \quad \mathbf{Z} = diag(Z_n(t,T), Z_{rn}(t,T), Z_r(t,T)) \tag{8.43}$$

$$\sum = \begin{pmatrix} S_n(t,T) & 0 & 0 \\ 0 & \sigma_I(t) & 0 \\ 0 & \sigma_I(t) & S_r(t,T) \end{pmatrix} \tag{8.44}$$

Thus the vector form of the system of market assets can be written as:

$$\mathbf{Z}^{-1}dZ = \bar{\mu} + \sum dW(t) = \sum d(W(t) + \int_0^t \sum{}^{-1}\bar{\mu}) = \sum d\hat{W}(t) \tag{8.45}$$

where Girsanov transformation is:

$$\hat{W}(t) = W(t) + \int_0^t \sum{}^{-1}\bar{\mu} \tag{8.46}$$

and

$$\sum{}^{-1} = \begin{pmatrix} \frac{1}{S_n(t,T)} & 0 & 0 \\ 0 & \frac{1}{\sigma_I(t)} & 0 \\ 0 & \frac{-1}{S_r(t,T)} & \frac{1}{S_r(t,T)} \end{pmatrix} \tag{8.47}$$

Let

$$\bar{q} = \sum{}^{-1}\bar{\mu} \tag{8.48}$$

Then

$$\bar{q} = \begin{pmatrix} \frac{\mu_1}{S_n(t,T)} \\ \frac{\mu_2}{\sigma_I(t)} \\ \frac{\mu_3-\mu_2}{S_r(t,T)} \end{pmatrix} \tag{8.49}$$

So

$$\hat{W} = \begin{pmatrix} \hat{W}_n(t) \\ \hat{W}_I(t) \\ \hat{W}_r(t) \end{pmatrix} = \begin{pmatrix} W_n(t) + \int_0^t \frac{\mu_1}{S_n(t,T)} \\ W_I(t) + \int_0^t \frac{\mu_2}{\sigma_I(t)} \\ W_r(t) + \int_0^t \frac{\mu_3-\mu_2}{S_r(t,T)} \end{pmatrix} \tag{8.50}$$

8.2.1 Stochastic differential equations of assets in martingale measure

The martingale form of the stochastic differential equation is:

$$\mathbf{Z}^{-1}dZ = \sum d\hat{W}(t) \tag{8.51}$$

and this yields:

$$\frac{dZ_n(t,T)}{Z_n(t,T)} = S_n(t,T)d\hat{W}_n(t) \tag{8.52}$$

$$\frac{dZ_{rn}(t,T)}{Z_{rn}(t,T)} = \sigma_I(t)d\hat{W}_I(t) \tag{8.53}$$

$$\frac{dZ_r(t,T)}{Z_r(t,T)} = \sigma_I(t)d\hat{W}_I(t) + S_r(t,T)d\hat{W}_r(t) \tag{8.54}$$

In the subsequent lemma's we prove bond price dynamics under the martingale measure.

lemma 4: The SDE for $\frac{dP_n(t,T)}{P_n(t,T)}$ in the martingale measure

$$dZ_n(t,T) = Z_n(t,T)S_n(t,T)d\hat{W}_n(t) = d\left(\frac{P_n(t,T)}{B_n(t)}\right) = \frac{P_n(t,T)}{B_n(t)}\left(\frac{dP_n(t,T)}{P_n(t,T)} - \frac{dB_n(t)}{B_n(t)}\right) \tag{8.55}$$

$$Z_n(t,T)S_n(t,T)d\hat{W}_n(t) = \frac{P_n(t,T)}{B_n(t)}\left(\frac{dP_n(t,T)}{P_n(t,T)} - \frac{dB_n(t)}{B_n(t)}\right) \tag{8.56}$$

$$S_n(t,T)d\hat{W}_n(t) = \frac{dP_n(t,T)}{P_n(t,T)} - r_n(t)dt \tag{8.57}$$

$$\frac{dP_n(t,T)}{P_n(t,T)} = r_n(t)dt + S_n(t,T)d\hat{W}_n(t) \tag{8.58}$$

lemma 5: The SDE for $\frac{dI(t)}{I(t)}$ in the martingale measure

$$\frac{dZ_{rn}(t,T)}{Z_{rn}(t,T)} = \sigma_I(t)d\hat{W}_I(t) = (r_r - r_n)dt + \frac{dI(t)}{I(t)} \tag{8.59}$$

$$\frac{dI(t)}{I(t)} = (r_n - r_r)dt + \sigma_I(t)d\hat{W}_I(t) \tag{8.60}$$

lemma 6: The SDE for $\frac{dP_r(t,T)}{P_r(t,T)}$ in the martingale measure

$$dZ_r(t,T) = Z_r(t,T)(\sigma_I(t)d\hat{W}_I(t) + S_r(t,T)d\hat{W}_r(t)) \tag{8.61}$$

Also

$$dZ_r(t,T) = d\left(\frac{P_r(t,T)I(t)}{B_n(t)}\right) = \frac{P_r(t,T)I(t)}{B_n(t)}\left(\frac{d(P_r(t,T)I(t))}{P_r(t,T)I(t)} - \frac{dB_n(t)}{B_n(t)}\right) \tag{8.62}$$

$$= \frac{d(P_r(t,T)I(t))}{B_n(t)} - \frac{d(P_r(t,T)I(t))}{B_n(t)}r_n(t)dt \tag{8.63}$$

Now equate the above expressions for $dZ_r(t,T)$

$$\frac{d(P_r(t,T)I(t))}{P_r(t,T)I(t)} = r_n(t)dt + \sigma_I(t)d\hat{W}(t) + S_r(t,T)d\hat{W}_r \tag{8.64}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} + \frac{dI(t)}{I(t)} + \sigma_I(t)S_r(t,T)\rho_{rI}dt = r_n(t)dt + \sigma_I(t)d\hat{W}(t) + S_r(t,T)d\hat{W}_r \tag{8.65}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = -\frac{dI(t)}{I(t)} - \sigma_I(t)S_r(t,T)\rho_{rI}dt + r_n(t)dt + \sigma_I(t)d\hat{W}(t) + S_r(t,T)d\hat{W}_r \tag{8.66}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = -((r_n-r_r)dt+\sigma_I(t)d\hat{W}_I(t))-\sigma_I(t)S_r(t,T)\rho_{rI}dt+r_n(t)dt+\sigma_I(t)d\hat{W}(t)+S_r(t,T)d\hat{W}_r \tag{8.67}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = (r_r - \sigma_I(t)S_r(t,T)\rho_{rI})dt + S_r(t,T)d\hat{W}_r \tag{8.68}$$

lemma 7: The SDE for $\frac{dP_r(t,T)I(t)}{P_r(t,T)}$ in the martingale measure

The asset $P_r(t,T)I(t)$ satisfies the SDE

$$\frac{d(P_r(t,T)I(t))}{P_r(t,T)I(t)} = r_n(t)dt + \sigma_I(t)d\hat{W}(t) + S_r(t,T)d\hat{W}_r \tag{8.69}$$

as was seen in lemma 8.

lemma 9: The SDE for $f_n(t,T), f_r(t,T)$ in the martingale measure

It follows as proved in section 2.3 that the dynamics of the nominal forward rate dynamics evolves as follows:

$$df_n(t,T) = -\sigma_n(t,T)S_n(t,T)dt + \sigma_n(t,T)d\hat{W}_n \tag{8.70}$$

For $f_r(t,T)$ we have:

$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r = \alpha_r(t,T)dt + \sigma_r(t,T)d(\hat{W}_r(t) - \int_0^t \frac{\mu_3 - \mu_2}{S_r(t,T)}) \tag{8.71}$$

$$= \alpha_r(t,T)dt + \sigma_r(t,T)d(\hat{W}_r(t) - \int_0^t \mu_3(u)du) \tag{8.72}$$

Note that:

$$q_3(t)S_r(t,T) = \mu_3 - \mu_2 = \nu_r(t,T) - \sigma_I S_r(t,T)\rho_{rI} \tag{8.73}$$

and $\sigma_I(t)$ is independent of T. Substitute for $\nu_r(t,T)$ to obtain:

$$(q_3(t) - \sigma_I(t)\rho_{rI})S_r(t,T) = \frac{1}{2}S_r^2(t,T) - A_r(t,T) \tag{8.74}$$

We differentiate the above equation with respect to T

$$(q_3(t) - \sigma_I(t)\rho_{rI})\sigma_r(t,T) = \sigma_r(t,T)S_r(t,T) - \alpha_r(t,T) \tag{8.75}$$

We substitute $q_3(t)$ into equation for $df_r(t,T)$

$$df_r(t,T) = -(\sigma_r(t,T)S_r(t,T) + \rho_{rI}\sigma_r(t,T)\int_t^T \sigma_I(u)du)dt + \sigma_r(t,T)d\hat{W}_r \tag{8.76}$$

Chapter 9

Appendix 3

$$\theta^2 = var(X-Y) = \int_0^T (\sigma_I^2(u)d\hat{W}_I(u) + S_r^2(u,\tau)d\hat{W}_r(u) - S_n^2(u,T)d\hat{W}_n(u) \tag{9.1}$$

$$+2(\rho_{rI}\sigma_I(t)S_r(u,\tau) - \rho_{nI}\sigma_I(u)S_n(u,T) - \rho_{nr}S_r(u,\tau)S_n(u,T)))du \tag{9.2}$$

Substitute parameter

$$\theta^2 = \int_0^T (\sigma_I^2(u)+\{\frac{\sigma_r}{\lambda_r}(e^{-\lambda_r(\tau-u)}-1)\}^2+\{\frac{\sigma_n}{\lambda_n}(e^{-\lambda_n(T-u)}-1)\}^2+2\rho_{rI}\sigma_I\frac{\sigma_r}{\lambda_r}(e^{-\lambda_r(\tau-u)}-1) \tag{9.3}$$

$$-\rho_{nI}\sigma_I\frac{\sigma_n}{\lambda_n}(e^{-\lambda_n(T-u)}-1) - \rho_{nr}\frac{\sigma_n}{\lambda_n}(e^{-\lambda_r(T-u)}-1)\frac{\sigma_r}{\lambda_r}(e^{-\lambda_r(\tau-u)}-1))du \tag{9.4}$$

Solve above equation

term 1

$$\int_0^T \sigma_I^2(u)du = \sigma_I^2(T) \tag{9.5}$$

term 2

$$\int_0^T \frac{\sigma_r^2}{\lambda_r^2}(e^{-\lambda_r(\tau-u)}-1)^2du = \frac{\sigma_r^2}{\lambda_r^2}\int_0^T (e^{-2\lambda_r(\tau-u)} - 2e^{-\lambda(\tau-u)}+1)^2du \tag{9.6}$$

$$= \frac{\sigma_r^2}{\lambda_r^2}\left[e^{-2\lambda_r\tau}\frac{1}{2\lambda_r}(e^{2\lambda_r T}-1) - 2e^{-\lambda_r\tau}\frac{1}{\lambda_r}(e^{\lambda_r T}-1)+T\right] \tag{9.7}$$

Similarly

term 3

$$\int_0^T \frac{\sigma_n^2}{\lambda_n^2}(e^{-\lambda_n(\tau-u)}-1)^2du = \frac{\sigma_n^2}{\lambda_n^2}\left[e^{-2\lambda_r\tau}\frac{1}{2\lambda_n}(e^{2\lambda_n T}-1) - 2e^{-\lambda_n\tau}\frac{1}{\lambda_n}(e^{\lambda_n T}-1)+T\right] \tag{9.8}$$

term 4

$$\int_0^T 2\rho_{rI}\sigma_I\frac{\sigma_r}{\lambda_r}(e^{-\lambda_r(\tau-u)}-1)du = 2\rho_{rI}\sigma_I\frac{\sigma_r}{\lambda_r}\left[\frac{e^{-\lambda_r\tau}}{\lambda_r}(e^{\lambda_r T}-1)-T\right] \tag{9.9}$$

Similarly

term 5

$$\int_0^T \rho_{nI}\sigma_I \frac{\sigma_n}{\lambda_n}(e^{-\lambda_n(\tau-u)}-1)du = 2\rho_{nI}\sigma_I\frac{\sigma_n}{\lambda_n}\left[\frac{e^{-\lambda_n\tau}}{\lambda_n}(e^{\lambda_n T}-1)-T\right] \tag{9.10}$$

term 6

$$\int_0^T \rho_{nr}\frac{\sigma_n}{\lambda_n}(e^{-\lambda_n(T-u)}-1)\frac{\sigma_r}{\lambda_r}(e^{-\lambda_r(\tau-u)}-1)du \tag{9.11}$$

$$= \int_0^T \rho_{nr}\frac{\sigma_n}{\lambda_n}\frac{\sigma_r}{\lambda_r}(e^{-\lambda_n(T-u)}-1)(e^{-\lambda_r(\tau-u)}-1)du \tag{9.12}$$

$$= \int_0^T \rho_{nr}\frac{\sigma_n}{\lambda_n}\frac{\sigma_r}{\lambda_r}(e^{-\lambda_n T-\lambda_r\tau+\lambda_n u+\lambda_r u} - e^{-\lambda_n(T-u)} - e^{-\lambda_r(\tau-u)})du \tag{9.13}$$

$$= \rho_{nr}\frac{\sigma_n}{\lambda_n}\frac{\sigma_r}{\lambda_r}(e^{-\lambda_n T-\lambda_r\tau}\frac{1}{\lambda_n+\lambda_r}(e^{(\lambda_n+\lambda_r)T}-1)-e^{-\lambda_n T}\frac{1}{\lambda_n}(e^{\lambda_n T}-1)-e^{-\lambda_r\tau}\frac{1}{\lambda_r}(e^{\lambda_r\tau}-1)+T) \tag{9.14}$$

We substitute the values of term 1 to term 6 into the variance equation θ^2

Chapter 10

Appendix 4

Bond pricing specification
This appendix will give a framework of the Johannesburg Stock Exchange (JSE) Gilt Clearing House (GCH) pricing formula) [5].

10.1 Bonds with more than 6 months to redemption

$$Unrounded - all - in - price = V_i^{\frac{d_1}{d_2}}\{\frac{1}{2}g(a_n^i + e) + 100V_i^n\} \qquad (10.1)$$

where :
d_1=number of days from settlement date to interest date
d_2=number of days from last to next interest date or from settlement date to next interest date if settlement falls on interest date
i=yield at which bond trades, as a percentage
$V_i = \frac{1}{1+\frac{i}{200}}$=present value of 1 payable in 6 months time
g=coupon as a percentage
n=number of complete six month periods from next interest date to redemption date
$a_n^i = (1 - V_i^n)/(\frac{i}{200})$
=present value of an annuity of 1 per six months, payable in arrears
e=1 if the bond is cum and 0 if ex

$$AccruedInterest = \frac{d_2e - d_1}{365}g \qquad (10.2)$$

Clean Price=(All-in-price)-(Accrued Interest)

Note :
1. Rounding convention:Clean price is rounded to 5 decimal places, accrued interest then rounded to 5 decimal places and added back to the clean price to arrive at the all-in-price.

2. Bonds are considered to be cum interest on coupon date

10.2 Bonds with less than 6 months to redemption

$$Unrounded - all - in - price = \frac{100 + e\frac{g}{2}}{1 + \frac{d_1}{365}\frac{i}{100}} \qquad (10.3)$$

the notation is the same as in section 1. The accrued interest and rounding is the same as for longer bonds.

10.3 Price to yield

The implied yield of a bond is the yield which produces an unrounded all-in-price equal to the target all-in-price, rounded to 5 decimal places.

10.4 Precision

All calculations and intermediate results should be carried to at least 11 significant digits; and preferably to full double precision (15-16 significant figures).

Chapter 11

Appendix 5.1

11.1 Process of running downhill simplex code

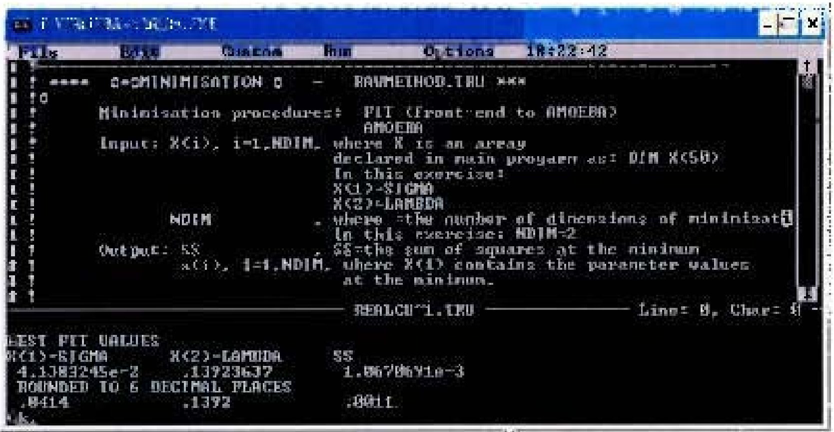
In this section of the appendix we describe the process of running the downhill simplex method. We estimate two parameter of the volatility function using this method. The code is documented and it is also provided in the disc attached to this thesis.

Open true basic. The file is located on $D : backslashTRUEBASICbackslashRUN.EXE$. The following window opens.

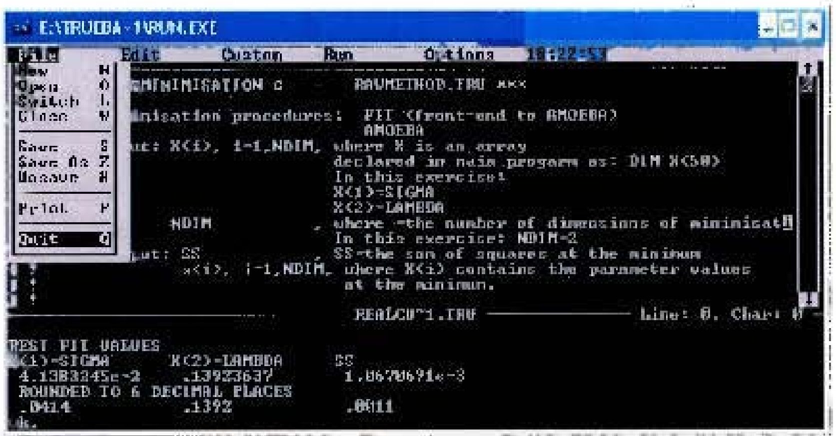
Load file by selecting *File* then *Open*.

A screen pops us. It has a list of files which can be loaded. Select file and click *ok*.

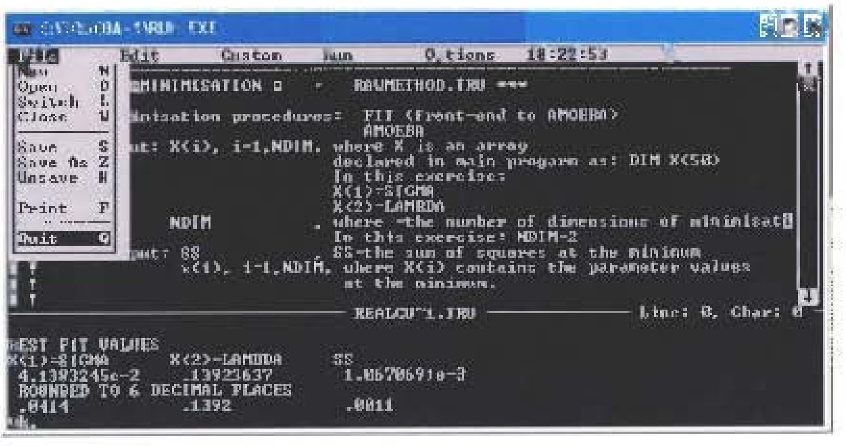
Once the file has been loaded. Go to the top menu and select *Run*.



The programs has run. It gives $SIGMA$ and $LAMBDA$ as output.



Once complete; go to the *File* tab and select *Quit*.


Edit
Custom
Run
Options
18:22:53
Open O
Switch L
Close W
Save S
Save As Z
Unsave N
Print P
Quit Q
MINIMISATION
RAWMETHOD.TRU
minisation procedures: FIT (front-end to AMOEBA)
AMOEBA
ut: X(i), i=1,NDIM, where X is an array
declared in main program as: DIM X(50)
In this exercise:
X(1)=SIGMA
X(2)=LAMBDA
NDIM
where =the number of dimensions of minimisat
In this exercise: NDIM=2
put: SS
SS=the sum of squares at the minimum
x(i), i=1,NDIM, where X(i) contains the parameter values
at the minimum.
REALCU~1.TRU
Line: 0, Char: 0
BEST FIT VALUES
X(1)=SIGMA
X(2)=LAMBDA
SS
4.1393245e-2
.13923637
1.0670691e-3
ROUNDED TO 6 DECIMAL PLACES
.0414
.1392
.0011

Chapter 12

Appendix 5.2

```
!*NON LINEAR ESTIMATION OF PARAMETERS RAWMETHOD.TRU *!
!Minimization procedures: FIT (front-end to AMOEBA) !
!AMOEBA !
!Input: X(i), i=1,NDIM, where X is an array declared in main program as:
!DIM X(50) !
!In this exercise:
! X(1)=SIGMA !
!X(2)=LAMBDA !
!NDIM =the number of dimensions of minimization problem !
!In this exercise: NDIM=2 !
!Output: SS, SS=the sum of squares at the minimum !
!Moreover x(i), i=1 and NDIM; where X(i) contains the parameter values at !the mini-
mum.
!
!In main program:
! DECLARE DEF FUNK !
!DIM X(50) !
!X(1)=SIGMA !
!X(2)=LAMBDA !
!NDIM=2 !
!CALL FIT(X,NDIM,SS) !
!END !
!
! DEF FUNK(X()) !
!END DEF !
!
!-----------------------------------------------------------

DECLARE DEF FUNK

DIM X(50)
```

```
LET X(1) = 1
LET X(2) = 1

LET NDIM = 2

!CLEAR
CALL FIT(X,NDIM,SS)
PRINT
PRINT "BEST FIT VALUES"
PRINT "X(1)=SIGMA","X(2)=LAMBDA","SS"
PRINT X(1),X(2),SS
! ** BETTER FORMATTING"
PRINT " ROUNDED TO 6 DECIMAL PLACES"
PRINT ROUND(X(1),6),ROUND(X(2),6),ROUND(SS,6)

SUB FIT(X(),NDIM,SS)
DIM P(51,50),Y(51)

LET TOL=0.000001
LET GRD=1.2

! SET UP P(NDIM+1,NDIM) MATRIX !
FOR I=1 TO NDIM+1
FOR J=1 TO NDIM
LET P(I,J)=X(J)
IF ((I-1) = J) THEN LET P(I,J)=GRD*P(I,J)
NEXT J
NEXT I

! SET UP Y(NDIM+1) VECTOR !
FOR I=1 TO NDIM+1
FOR J=1 TO NDIM
LET X(J)=P(I,J)
NEXT J
LET Y(I)=FUNK(X)
NEXT I

CALL AMOEBA(P,Y,51,50,NDIM,TOL,ITER)

FOR J=1 TO NDIM
LET X(J)=P(1,J)
NEXT J
LET SS=Y(1)
END SUB

!----------------------------------------
```

```
SUB AMOEBA(P(,),Y(),MP,NP,NDIM,FTOL,ITER)

DIM PR(50) !Dimentioned ndim+1
DIM PRR(50)
DIM PBAR(50)

LET ALPHA=1.0
LET BETA=0.5
LET GAMMA=2.0
LET ITMAX=500

LET MPTS=NDIM+1
LET ITER=0

LET return=0
DO WHILE return=0
LET ILO=1
IF Y(1) > Y(2) THEN
LET IHI=1
LET INHI=2
ELSE
LET IHI=2
LET INHI=1
END IF
FOR I=1 TO MPTS
IF Y(I)<Y(ILO) THEN LET ILO=I
IF Y(I)>Y(IHI) THEN
LET INHI=IHI
LET IHI=I
ELSE IF Y(I)>Y(INHI) THEN
IF I <> IHI THEN LET INHI=I
END IF
NEXT I

IF Y(IHI) = 0 THEN
PRINT "Y(IHI)=0"
END IF
LET RTOL=2*ABS(Y(IHI)-Y(ILO))/(ABS(Y(IHI))+ABS(Y(ILO)))
IF RTOL > FTOL THEN
IF ITER = ITMAX THEN
PRINT "AMOEBA EXCEEDING MAXIMUM ITERATIONS"
END IF
LET ITER=ITER+1

FOR J=1 TO NDIM
LET PBAR(J)=0
```

```
NEXT J

FOR I=1 TO MPTS
IF I <> IHI THEN
FOR J=1 TO NDIM
LET PBAR(J)=PBAR(J)+P(I,J)
NEXT J
END IF
NEXT I

FOR J=1 TO NDIM
LET PBAR(J)=PBAR(J)/NDIM
LET PR(J)= (1 + ALPHA)*PBAR(J)-ALPHA*P(IHI,J)
NEXT J

LET YPR=FUNK(PR)

IF YPR <= Y(ILO) THEN
FOR J=1 TO NDIM
LET PRR(J)=GAMMA*PR(J)+(1.-GAMMA)*PBAR(J)
NEXT J

LET YPRR=FUNK(PRR)
IF YPRR < Y(ILO) THEN
FOR J=1 TO NDIM
LET P(IHI,J)=PRR(J)
NEXT J
LET Y(IHI)=YPRR
ELSE

FOR J=1 TO NDIM
LET P(IHI,J)=PR(J)
NEXT J
LET Y(IHI)=YPR
END IF

ELSE IF YPR >= Y(INHI) THEN

IF YPR < Y(IHI) THEN
FOR J=1 TO NDIM
LET P(IHI,J)=PR(J)
NEXT J
LET Y(IHI)=YPR
END IF

FOR J=1 TO NDIM
```

```
LET PRR(J)=BETA*P(IHI,J)+(1.-BETA)*PBAR(J)
NEXT J
LET YPRR=FUNK(PRR)

IF YPRR < Y(IHI) THEN
FOR J=1 TO NDIM
LET P(IHI,J)=PRR(J)
NEXT J
LET Y(IHI)=YPRR
ELSE

FOR I=1 TO MPTS
IF I <> ILO THEN
FOR J=1 TO NDIM
LET PR(J)=0.5*(P(I,J)+P(ILO,J))
LET P(I,J)=PR(J)
NEXT J
LET Y(I)=FUNK(PR)
END IF
NEXT I
END IF

ELSE


FOR J=1 TO NDIM
LET P(IHI,J)=PR(J)
NEXT J
LET Y(IHI)=YPR
END IF
ELSE
LET return=1
END IF
LOOP

END SUB

END

DEF FUNK(X())

DIM ARRAY(21)

DIM MATURITY(21),SAMPLEVARIANCE(21)

   DATA 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20
```

```
DATA
0.007349,2.36070977631166,4.4821003707275,6.38721322964039,
8.6897921380025,11.0945859712152,13.3944617503504,15.6106495724248,
17.7707818144496,19.9227864010068,22.0717611953761,24.2596745438519,
26.6495553846224,28.9543752229589,31.2976003894423,33.4386187124921,
35.4543148615713,37.6586489559927,40.3984837005913,44.1990279575248,48.9913955850879

MAT READ MATURITY,SAMPLEVARIANCE

LET SIGMA = X(1) LET LAMBDA = X(2)

LET SUM =0.0 LET SS = 0.0

FOR a=1 TO 21
ARRAY(a) = SIGMA^2 * ((EXP(-LAMBDA * MATURITY(a)) - 1)/LAMBDA)^2
NEXT a

FOR h=1 TO 21
SS = SS + (SAMPLEVARIANCE(h) - ARRAY(h))^2
NEXT h

PRINT "SIGMA","LAMBDA","SS"
PRINT SIGMA,LAMBDA,SS
LET FUNK = SS
END DEF
```

Chapter 13

Appendix 6

```
' This code calculates a European call option on a real zero coupon bond

Option Explicit
Dim wsf As WorksheetFunction

' Pn :nominal zero-coupon bond maturing at T
' Pr :real zero-coupon bond maturing at T
' InfAdj : inflation adjustment
' K :strike price
' sigmaI : volatility of inflation
' sigmaN : nominal volatility function parameter
' sigmaR : real volatility function parameter
' lambdaR : real volatility function parameter
' lambdaN : nominal volatility function parameter
' CorrIR : correlation between inflation index and real
' CorrIN : Correlation between inflation index and nominal returns
' CorrNR : correlation between nominal and real returns
' Tk : maturity of the option
' Tau :maturity of the zero coupon bond
' t :time today

'-----------------------------------------
Function variance1(sigmaI, sigmaN, sigmaR, lambdaR,lambdaN,CorrIR,CorrIN, CorrNR, Tau, Tk, t)

Dim term1 As Double
Dim term2 As Double
Dim term3 As Double
Dim term4 As Double
Dim term5 As Double
Dim term6 As Double
```

$term1 = sigmaI^2 * Tk$

$term2 = (sigmaR/lambdaR)^2 * ((Exp(-2 * lambdaR * Tau)/(2 * lambdaR)) * (Exp(2 * lambdaR * Tk) - 1) - 2 * (Exp(-lambdaR * Tau)/lambdaR) * (Exp(lambdaR * Tk) - 1) + Tk)$

$term3 = (sigmaN/lambdaN)^2 * ((Exp(-2 * lambdaN * Tk)/(2 * lambdaN)) * (Exp(2 * lambdaN * Tk) - 1) - 2 * (Exp(-lambdaN * Tk)/lambdaN) * (Exp(lambdaN * Tk) - 1) + Tk)$

$term4 = (2*CorrIR*sigmaI*sigmaR/lambdaR)*((Exp(-lambdaR*Tau)/lambdaR)* (Exp(lambdaR * Tk) - 1) - Tk)$

$term5 = (CorrIN*sigmaI*sigmaN/lambdaN)*((Exp(-lambdaN*Tau)/lambdaN)* (Exp(lambdaN * Tk) - 1) - Tk)$

$term6 = ((CorrNR * sigmaN * sigmaR)/(lambdaN * lambdaR)) * ((Exp(-lambdaN * Tk - lambdaR * Tk)/(lambdaN + lambdaR)) * (Exp((lambdaN + lambdaR) * Tk) - 1) - (Exp(-lambdaN * Tk)/lambdaN) * (Exp(lambdaN * Tk) - 1) - (Exp(-lambdaR * Tau)/lambdaR) * (Exp(lambdaR * Tau) - 1) + Tk)$

Dim FVol As Double

$FVol = term1 + term2 + term3 + term4 - term5 - term6$

variance1 = FVol

End Function

!———————————————————

Function Fdede1(Pn,Pr,InfAdj,K,sigmaI,sigmaN,sigmaR,lambdaR,lambdaN,CorrIR,CorrIN,CorrNR,Tk,Tau,t)

Set wsf = Application.WorksheetFunction

Dim vol As Double

vol = variance1(sigmaI,sigmaN,sigmaR,lambdaR,lambdaN,CorrIR,CorrIN,CorrNR,Tk,Tau,t)

Fdede1 = (wsf.Log((Pr*InfAdj)/(K*Pn))+(0.5*vol))/Sqr(vol)

End Function

!———————————————————

Function Fdede2(Pn,Pr,InfAdj,K,sigmaI,sigmaN,sigmaR,lambdaR,lambdaN,CorrIR,CorrIN,CorrNR,Tk,Tau,t)

Set wsf = Application.WorksheetFunction

```
Dim vol As Double

vol =variance1(sigmaI,sigmaN,sigmaR,lambdaR,lambdaN,CorrIR,CorrIN,CorrNR,Tk,Tau,t)

Fdede2 = (wsf.Log((Pr*InfAdj)/(K*Pn))-(0.5*vol))/Sqr(vol)

End Function

'—————————————————————————————
Function EuroCall(Pn,Pr,InfAdj,K,sigmaI,sigmaN,sigmaR,lambdaR,lambdaN,CorrIR,CorrIN,CorrNR,Tk,T

Dim d1 As Double
Dim d2 As Double

d1 =Fdede1(Pn,Pr,InfAdj,K,sigmaI,sigmaN,sigmaR,lambdaR,lambdaN,CorrIR,CorrIN,CorrNR,Tk,Tau,t)

d2 =Fdede2(Pn,Pr,InfAdj,K,sigmaI,sigmaN,sigmaR,lambdaR,lambdaN,CorrIR,CorrIN,CorrNR,Tk,Tau,t)

EuroCall = (wsf.NormSDist(d1)*Pr*InfAdj)-(wsf.NormSDist(d2)*K*Pn)

End Function
```

Chapter 14

Appendix 7

```
Sub Simulate()
With Application
.Calculation = xlManual
.MaxChange = 0.001
End With

Application.ScreenUpdating = False

Range("W97:AE101").Select 'clear simulation range
Range(Selection, Selection.End(xlDown)).Select
Range(Selection, Selection.End(xlDown)).Select
Selection.ClearContents
Nosims = Sheets("simulation").Range("nosim").Value

For i = 1 To Nosims

Calculate
Sheets("simulation").Range("output").Offset(i - 1, -1).Value = I
Sheets("simulation").Range("output").Offset(i - 1, 0).Value = Sheets("simulation").Range("target").Offset(0,
0).Value
Sheets("simulation").Range("output").Offset(i - 1, 1).Value = Sheets("simulation").Range("target").Offset(0,
1).Value

If Round((i - 1) / 500, 0) = (i - 1) / 500 Then 'flash no. of simulations every 500
simulations
Application.ScreenUpdating = True
Application.ScreenUpdating = False
End If

Next I
```

```
With Application
.Calculation = xlAutomatic
.MaxChange = 0.001
End With

Application.ScreenUpdating = True

End Sub
'================================================================
'
' Box Muller Transformation to generate N(0,1) random variables from U(0,1) ones
'
' From Dupire - Monte Carlo Toolkit
' inputs = u and v, two uniform random numbers
' output = first N(0,1) variable (i=1) or second N(0,1) variable (i=2) independently of
the first

   Function NormSBM (u As Double, v As Double, i As Integer) As Double
Dim x, y As Double
If i = 1 Then NormSBM = ((-2 * Log(u))^0.5) * (Cos(2 * 3.14159265358979 * v))
Else
NormSBM = ((-2 * Log(u))^0.5) * (Sin(2 * 3.14159265358979 * v))
End If

End Function

'================================================================

' Cumulative Normal Subroutine
' Based on Chebyshev series expansions
' From W.J.Cody Math.Comp v23(1969)
Function normd(xa) As Double
Dim x, erf, c1(2, 4), c2(2, 8) As Double
Dim c3(2, 5), s1, S2, sqpi As Double
Dim i, ind As Integer

c1(1, 0) = 3209.37758913849: c1(2, 0) = 2844.23683343917
c1(1, 1) = 377.485237685302: c1(2, 1) = 1282.61652607737
c1(1, 2) = 113.86415415105: c1(2, 2) = 244.024637934444
c1(1, 3) = 3.16112374387057: c1(2, 3) = 23.6012909523441
c1(1, 4) = 0.185777706184603: c1(2, 4) = 1

c2(1, 0) = 1230.339354798: c2(2, 0) = 1230.33935480375
c2(1, 1) = 2051.07837782607: c2(2, 1) = 3439.36767414372
c2(1, 2) = 1712.04761263407: c2(2, 2) = 4362.61909014325
c2(1, 3) = 881.952221241769: c2(2, 3) = 3290.79923573346
```

```
c2(1, 4) = 298.6351381974: c2(2, 4) = 1621.38957456669
c2(1, 5) = 66.1191906371416: c2(2, 5) = 537.18110186201
c2(1, 6) = 8.88314979438838: c2(2, 6) = 117.693950891312
c2(1, 7) = 0.56418849698867: c2(2, 7) = 15.7449261107098
c2(1, 8) = 2.15311535474404E-08: c2(2, 8) = 1


c3(1, 0) = -6.58749161529838E-04: c3(2, 0) = 2.33520497626869E-03
c3(1, 1) = -1.60837851487423E-02: c3(2, 1) = 6.05183413124413E-03
c3(1, 2) = -0.125781726111229: c3(2, 2) = 5.27905102951428E-03
c3(1, 3) = -0.360344899949804: c3(2, 3) = 1.87295284992346
c3(1, 4) = -0.305326634961232: c3(2, 4) = 2.56852019228982
c3(1, 5) = -1.63153871373021E-02: c3(2, 5) = 1

erf = 0
If (xa = 0) Then GoTo 4
sqpi = 1.77245385090551
ind = 0
If (xa ¡ 0) Then ind = 1
x = Abs(xa / Sqr(2))
s1 = 0
S2 = 0
If (x ¿= 4) Then
For i = 0 To 5
s1 = s1 + c3(1, i)/x^(2 * i)
S2 = S2 + c3(2, i)/x^(2 * i)
Next I
erf = 1 - Exp(-x * x) / x * (1 / sqpi + s1 / S2 / x / x)
ElseIf (x ¿= 0.46875) Then
For i = 0 To 8
s1 = s1 + c2(1, i) * x^I
S2 = S2 + c2(2, i) * x^I
Next I
erf = 1 - Exp(-x * x) * s1 / S2
Else
For i = 0 To 4
s1 = s1 + c1(1, i) * x^(2 * i)
S2 = S2 + c1(2, i) * x^(2 * i)
Next I
erf = x * s1 / S2
End If
normd = (1 + erf) / 2
If (ind = 1) Then normd = 1 - normd

End Function
```

```
'==============================================================
' Inverse Cumulative Normal Subroutine
'
' Beasly and Springer,
' Applied Statistics 26, 1997, pp. 118-121
'

Function normi(u) As Double
Dim x, y, r As Double
Dim a0, a1, a2, a3, b0, b1, b2, b3, c0, c1, c2, c3, c4, c5, c6, c7, c8 As Double

a0 = 2.50662823884
a1 = -18.61500062529
a2 = 41.39119773534
a3 = -25.44106049637
b0 = -8.4735109309
b1 = 23.08336743743
b2 = -21.06224101826
b3 = 3.13082909833

If (u = 0) Or (u = 1) Then Exit Function

If Abs(u - 0.5) ¡ 0.42 Then GoTo 1

x = u - 0.5
y = x * x
normi = (x * (((a3 * y + a2) * y + a1) * y + a0)) / (((((b3 * y + b2) * y + b1) * y +
b0) * y + 1))
Exit Function


c0 = 0.337475482272615
c1 = 0.976169019091719
c2 = 0.160797971491821
c3 = 2.76438810333863E-02
c4 = 3.8405729373609E-03
c5 = 3.951896511919E-04
c6 = 3.21767881768E-05
c7 = 2.888167364E-07
c8 = 3.960315187E-07


r = Log(-Log(1 - u))
normi = c0 + r * (c1 + r * (c2 + r * (c3 + r * (c4 + r * (c5 + r * (c6 + r * (c7 + r *
c8)))))))
```

```
End Function

'=========================================================================
' Cumulative Normal Subroutine
'
' Andrew Smith
'


Function cumnorm(z As Double) As Double
'page 932 of Abramowitz + Stegun
Dim T As Double, temp As Double

Const p As Double = 0.2316419
Const b1 As Double = 0.31938153
Const b2 As Double = -0.356563782
Const b3 As Double = 1.781477937
Const b4 As Double = -1.821255978
Const b5 As Double = 1.330274429

If z ¿= 0 Then
T = 1 / (1 + p * z)
temp = b2 + T * (b3 + T * (b4 + T * b5))
cumnorm = 1 - incnorm(z) * T * (b1 + T * temp)
Else
T = 1 / (1 - p * z)
temp = b2 + T * (b3 + T * (b4 + T * b5))
cumnorm = incnorm(z) * T * (b1 + T * temp)

End If

End Function
Function incnorm(z As Double) 'called by the cumnorm subroutine
```

$incnorm = 0.398942280401433 * Exp(-0.5 * z^2)$

```
End Function
```

Bibliography

[1] **Amin K and Jarrow R** :*Pricing Foreign Currency Options under Stochastic Interest Rates*,Journal of International Money and Finance, 1991, Volume 10, Pages 310-329

[2] **Barone, E., and Castagna, A**, *The Information Content of Tips. Internal report*, 1997,SanPaolo IMI, Turin and Banca IMI, Milan

[3] **Bing-Huei Lin**, *Fitting term structure of interest rates using B-splines: the case of Taiwanese Government bonds*, Applied Financial Economics, 2002, Volume 12, Pages 57-75

[4] **Black F, Derman E, Toy W** (1990):*A One-Factor Model of Interest Rates and its Application to Treasury Bond Options*, Financial Analyst Journal, January-February 1990, Pages 33-39

[5] **Bond Exchange of South Africa: Bond Pricing Specification Formula**, Quant Financial Research, 20 August 1997, http://www.bondex.co.za/pricing/bondpricingformula.pdf

[6] **Box G.E.P and Muller M.E.**,*A note on the generation of random normal deviates*. Annals Math. Stat 29, 1958.

[7] **Boyle P, Wang T** (2001): *Pricing of New Securities in an Incomplete Market: the catch 22 of no-arbitrage pricing*, Mathematical Finance, Volume 11, Number 3, Pages 267-284

[8] **Brochaus O, Ferraris A, Gallus C, Long D, Martin R, Overhaus M**, *Monte Carlo Methods*,Chapter 7, Modeling and Hedging Equity Derivatives

[9] **Carverhill A and Pang K**, *Efficientand Flexible Bond Option Valuation in the Heath, Jarrow and Morton Framework*, Journal of Fixed Income, Volume 5, Page 70-77

[10] **Deacon M, Derry A, Mirfendereski D**: *Inflation-indexed securities:Bonds, Swaps and Other Derivatives*, Second Edition, Page 2

[11] **Evelyn G.S** (1798):*An Account of some endeavours to ascertain a standard of weight and measure*, Philosophical Transactions of the Royal Society of London, Volume 88, Pages 133-182

[12] **Fisher I** (1922):*The Purchasing Power of Money*, New York: Macmillan, Page 208

[13] **Fourie, Falkena, Kok**: *Student Guide To The South African Financial Systems*

[14] **Geman, H., El Karoui, N., and Rochet, J.-C.**,*Changes of Numeraire, Changes of Probability Measure and Option Pricing*,1995, Journal of Applied Probability 32, 443-458

[15] **Patrick Hagan and Graeme West**, *interpolation methods for yield curve construction*, Financial Modelling Agency, 19 First Ave East, Parktown, 2193, South Africa

[16] **Henrard M**, *TIPS Options in the Jarrow-Yildirim Model*, Risk, March 2006, Vol 19, No. 3, Pages 82 - 83

[17] **Hines W.W and Montgommery D.C**, *Probality and Statistics in Engineering and Management Science*, 1980, , 2nd edition, New York

[18] **Mia Hinnerich**, *Inflation Indexed Swaps and Swaptions*, 2006,Department of Finance, Stockholm School of Economics, [unpublished]

[19] **Ho and Lee** (1986):*Term Structure Movements and Pricing Interest Rate Contingent Claims*,The Journal of Finance, Volume 41, Number 5, Pages 1011-1028

[20] **Hughston, L.**, *Inflation Derivatives*, Department of Mathematics, King's College London, [unpublished]

[21] **Jamshidian F** :*Bond and Option Evaluation in the Gaussian Interest Rate Model*, Research in Finance, Volume 9, Pages 131-170

[22] **Jarrow R and Turnbull S** :*A Unified Approach for Pricing Contingent Claims on Multiple Term Structure*,Review of Quantitative Finance and Accounting, 10 January 1998, Pages 5-19

[23] **Jarrow, R. and Yildirim, Y.**, *Pricing Treasury Inflation Protected Securities and Related Derivatives using an HJM model*, 2003,Journal of Financial and Quantitative Analysis , Volume 38, 409-430

[24] **Jevons W.S** (1975):*Money and the Mechanism of Exchange*, New York: Appelton

[25] **Keynes J.M** (1927):*Statement before the committee on National Debt and Taxation*, In: Minutes of Evidence, Colwyn Committee. London: HMSO, Pages 278 and 287

[26] **Marshall A** (1868):*Remedies for fluctuations of general prices*, In: A.C. Pigou (edition), Memorials of Alfred Marshall, New York: Augustus M. Kelley

[27] **Matsumoto M and Nishimura T.**,*Mersenne twister - a 623-dimensionally equidistributed uniform pseudorandom number generator*. ACM Trans. on Modeling and Computer Simulation 8 I, 1998.

[28] **McCulloch J H**, *Mearsuring the Term Structure of Interest Rates*, Journal of Business, January 1971, Page 19-31

[29] **F. Mercurio**, *Pricing Inflation-Indexed Derivatives*, 2005,Quantitative Finance , Volume 5, 289

[30] **Nelder, J.A and Mead R**, *A Simplex method for function minimization*, Computer Journal,1965, Vol 7, Pages 308-313

[31] **Ivan Ruscic**,*RMB YIELD CURVES: TECHNICAL DETAILS*,Risk and Compliance, March 2006

[32] **Scrope P** (1968):*Principles of Political Economy*, New York: Augustus M. Kelly,Pages 413-419

[33] *South African CPI Security Manual*, Standard Coorperate and Merchant Bank(Treasury), Interest rate products

[34] **Shen P** :*Benefits and limitations of inflation indexed treasury bonds*

[35] **Press W.H, Teukolsky S. A, Vetterling W.T, Flannerg B.P**, *Downhill Sipmlex Method in Multidimensions*, Numerical Recipes in C: The Art of Scientific Computing

[36] **Vasicek A O and Fong G H**, *Term Structure Modelling Using Exponential Splines*, The Journal of Finance, Volume 37, Issue 2, Page 339-348

[37] **West G***South African Financial Instruments*, Pages 42-45

[38] **I-net**, *www.inet,co.za/corporate3/contactUS.htm*