Knowledge used for teaching counting: A case study of the treatment of counting by two Grade 3 teachers situated in schools serving working class communities in the Western Cape Province of South Africa

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PLAGIARISM DECLARATION

This work has not been previously submitted in whole, or in part, for the award of any degree. It is my own work. Each significant contribution to, and quotation in, this dissertation from the work, or works, of other people has been attributed, and has been cited and referenced.

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DEDICATION

This dissertation is dedicated to the ALMIGHTY GOD, The LORD JESUS CHRIST, and to HIS HOLY SPIRIT.

ABSTRACT

Knowing how to correctly count, is fundamental to the future mathematics success of young children. Earlier studies show that many South African primary school students underperform in mathematics even when evaluated with task below grade level. Reports suggest that this is a problem stemming from the poor pedagogic, and or content knowledge of classroom mathematics teachers.

Shulman (1986; 1987) refers to this area of knowledge as Pedagogic Content Knowledge (PCK). In the field of mathematics teaching and learning, Ball, Thames and Phelps (2008) refer to it as Mathematics Knowledge for Teaching (MKfT). Teachers' mathematics PCK, comprises of three core knowledge domain: (i) Teacher's Knowledge of Content and Teaching (KCT); (ii) Teacher's Knowledge of Content and Student (KCS); and (iii) teacher's Knowledge of Content and Curriculum (KCC). Teachers' KCS was considered in this study as it concerns what teachers know about what learners know and how they learn.

The general interest of this project was to study the construction of experience of mathematics (non-core domain knowledge) by genetic endowment on the basis of contextual data. More specifically, the particular interest of the study is on the construction of the experience of counting in the pedagogic situations of Grade 3 schooling. For that purpose, video records of mathematics teaching in two schools situated in working-class communities were analysed.

The study adopted an Integrated Causal Model approach which drew on resources from different disciplines such as mathematics education, cognitive science, evolutionary psychology and mathematics. The study was partly framed by Bernstein's pedagogic device, particularly with respect to his notion of evaluation, as well as the inter-related constructs of PCK, MKfT and KCS. The theoretical resources used to describe computations were drawn largely from Davis (2001, 2010b, 2011a, 2012, 2013a, 2015, 2018) and related work on the use of morphisms as elaborated in Baker *et al*. (1971), Gallistel & King, (2010), Krause (1969) and Open University (1970). These resources were used to produce the analytic framework for the production of and analysis of data. The analysis describes the computational activities of teachers and learners during the recorded lessons, specifically the computational domains made available pedagogically. In so doing, I was able to provide more illumination on what is described as teacher's KCS for teaching counting at the Grade 3 level.

From the generated data, the study finds that counting proper was restricted to the constitution and identification of very small ordered discrete aggregates which can be handled by human core domain object tracking system and approximate number system, and that an implicit reliance on numerical order derived from computations on aggregates was central to the teaching and learning of counting.

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LIST OF ABBREVIATIONS

ANS	Approximate Number System
CAG	Continuous Aggregate
CAP	Curriculum and Assessment Policy
CCK	Common Content knowledge
DAG	Discrete Aggregate
DBE	Department of Basic Education
FINSET	Finite Set
FP	Foundation Phase
HCK	Horizon Content Knowledge
ICM	Integrated Causal Model
KCC	Knowledge of Content and Curriculum
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KQ	Knowledge Quartet
MKfT	Mathematics Knowledge for Teaching
MKiT	Mathematics Knowledge in Teaching
NUM	Numbers
OTS	Object Tracking System
PCK	Pedagogic Content Knowledge
SCK	Specialised Content Knowledge
SMK	Subject Matter Knowledge
SPADE	Schools Performing Above Demographic Expectation
SSSM	Standard Social Science Model
SYM	Number Symbols
WRD	Number Words

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CHAPTER 1

Introduction

1.1 Introduction: Background to the study

From the research and professional literature, we know that intuitive mental structures shaping the work mathematics teachers do in the classroom is informed by teachers' experiences with content, the pedagogic setting, curriculum guidelines and the field of mathematics. Within mathematics teacher education research, this area of study is situated in research concerned with teacher's *mathematics knowledge for teaching* (MKiT/MKfT) (see Adler & Davis, 2006; Ball & Bass, 2000; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Turner & Rowland, 2011) [1]. The literature indicates that teachers who teach effectively hold specific forms of mathematics teaching knowledge, one of which is teacher's *knowledge of content and students* represented (KCS) (Ball et al., 2008).

MKfT broadly refers to knowledge recruited by teachers who undertake the work of classroom mathematics teaching (Ball, 1999, 2000). Many present-day researchers interested in this field of investigation do so by observing the pedagogic activities of teachers and learners in the classroom. That is, for example, by observing teacher's choice of mathematical representations, examples, treatment of learners' errors and mathematical ideas. In this field, and for this study, the work of Ball and her colleagues stands out as it is used by many researchers, nationally and internationally, to discuss the mathematics teaching knowledge of classroom teachers. From the work of Ball and her colleagues, teachers' MKfT is split into two broad knowledge categories which are: *subject matter knowledge* (SMK) and *pedagogic content knowledge* (PCK). As stated earlier, this study is interested in teachers' *knowledge of content and students* (KCS) which is a vital aspect of PCK.

PCK is a construct traceable to the seminal work of Shulman (1986; 1987). Pedagogic content knowledge refers to knowledge that integrates content and pedagogy in peculiar ways which enable students' understanding of the actual content. Ball et al. (2008) specify PCK as consisting of three sub-domains, which are: teachers' mathematical knowledge of students' thinking of content (KCS); knowledge of approaches for teaching a specific content (KCT); and knowledge about curricular specifications for teaching content (KCC). Researchers indicate that the KCS of mathematics teachers is an aspect of mathematical knowledge distinguishing mathematics teachers from mathematicians (Hill, Ball, & Schilling, 2008). Teachers' KCS relates to the mathematics teacher knowing content in ways a child at the grade level understands the content and the teacher being able to elaborate the mathematics content. In Chapter 2, we see that not much is provided in literature specific to how young children construct their learning of school mathematics in the classroom.

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[1] In some literature, mathematics knowledge for teaching is represented as either MKT or MKfT. For this study I will refer to it as MKfT following the use of Ball *et al*. (2008) for consistency.

However, a complementary area of research (e.g., Davis, 2013a, 2013b, 2016, 2018; Jaffer, 2018) shows that analytic descriptions of the mathematical computations[2] of teachers and learners can be used as resources in detailing the operational structures explicitly and implicitly used by teachers and learners. I argue that this approach to classroom mathematics teaching research adds depth to analysing teachers' KCS since it focuses on the computational details of observable pedagogic practices. Thus, this study hopes to contribute to current ideas on KCS through an examination of the computational activities of teachers and learners. In this way, it is possible to extend the literature by providing mathematical description of the teaching work of classroom mathematics teachers. The aim of this study is to use mathematical computational referents (see Davis, 2012; 2016; 2018) in analysing the mathematics lessons of two Grade 3 teachers relative to how the terms *count* and *counting* are used to refer by teachers and their learners. In order to achieve this, I refer to related fields (e.g., mathematics and cognitive science) in the formulation of analytic resources.

At this point, I refer to research in the field of cognitive science which supplies empirical evidence on the numerical cognition of young children. Research in the field of cognitive science interested in numerical cognition of infants and young children (e.g., Gallistel & Gelman, 2005; Gallistel & King 2011; Gelman 2009; Spelke & Kinzler, 2007; Xu & Spelke, 2000) report that humans are born with genetically endowed specialised knowledge systems from which formal or school mathematical knowledge can be developed. They refer to this rudimentary form of knowledge as *core domain* knowledge. Core domains enable learning in specific domains, like, for example, language and the quantification of experience. Gelman (2009) describes a domain of knowledge as "a set of coherent principles that form a structure and contains domain-specific entities […] that can combine to form other entities within the domain" (p. 247). She describes school knowledge such as classroom mathematics as *non-core domain* knowledge. That is, part of systems of "organized knowledge that are acquired later" (ibid. pp. 248–249; italics in the original). Literature in this area of research informs us that the numerical knowledge of young children in a domain holds a broad set of skills and if continually developed, it matures to an expected cognitive level. Research shows that preverbal infants can subitise small collections of objects based on their reaction to the cardinal values of sets presented in different formats (Izard, Sann, Spelke, & Streri, 2009; Wynn, 1995). These studies show that the ability to subitise small collection of objects is a skill shared by both infants and adults (Brannon, Jordan, & Jones, 2010; Halberda & Feigenson, 2008). This core domain skill is said to be associated with an *object tracking system* (OTS) and an *approximate number system* (ANS) (Dehaene, 2011), which allows for meaning making of the number system because of its mapping to numeric symbols and language (Mazzocco, Feigenson, & Halberda, 2011).

[2] This is to be understood as referring to the proposition that all thought as well as language is constituted by operations over domains of objects that serve as collections of argument and values for operational activity. Different scholars examined discourse following this approach (e.g., Chomsky, 2006, 2007; Davis, 2012, 2018; Gallistel & King, 2010; Pinker, 2007). In support, Gallistel and King (2010) informs that "brains are powerful organs of computation" and that computations, which are central to thought and to the communication of thought, are compositions of functions.

Cognitive scientists interested in child development (e.g., Markman, 1973; Markman, Horton, & McLanahan, 1980), describe the normal sequence of cognitive development in humans using ideas on aggregates which are precursors to mathematical conception of aggregates (Davis, 2016). Here, aggregation is described in terms of a hierarchical sequence of conceptions occurring in children which begins with an initial recognition to an entity. Such that the various components of an entity are given coherence by their association to the entity: *aggregate-as-object*. There is also a more conceptual recognition of aggregation where one begins to recognize similar entities that are closely bound together, as a *collection*. Members of a *collection* must have close similarities. For example, trees having close spatial proximity to each other to form a forest: *aggregate-as-collection*. Aggregate-as-collection is superseded by the notion of *aggregate-as-class*, where the components of an aggregate are conceived of as sharing common properties. Here, members of an aggregate do not have to be in close physical proximity. Davis (2016) points out that this early hierarchical conception of aggregates is in place long before the mathematical conception of collection in terms of a set.

One could hardly doubt this as even in societies with considerably basic number systems, such as the Amazonian hunter-gatherer groups, research (Everett, Berlin, Gonalves, *et al*. 2005) finds that they recognise, compare and order aggregates without explicit references to number (Butterworth & Reeve, 2012). This same ability is present even in preverbal children and young children (Halberda & Feigenson, 2008; Jordan & Brannon, 2006; Siegler, 2016). On this basis, Davis (2016) argues that sensitivity to aggregates is part of core domain knowledge and follows a hierarchical conceptual sequence (*aggregate-as-object*, *aggregate-as-collection*, *aggregate-as-class*) that emerges during early human development. In addition, one ought to pay attention to the implicit presence of a determinate reason for belonging to an aggregate, as it gives coherence to intuitive aggregates. Davis (2016) points to the common pedagogic use of operation on aggregates as precursors in elementary arithmetic operations, arguing that when one considers empirical realizations of elementary school arithmetic, one must consider the effects of core domain conceptions of aggregates used in what comes to be constituted as mathematics content in schooling which goes beyond mere observable pedagogic practices.

1.2 Contextualising the problem

Many studies (e.g., Graven, 2016; Spaull, 2013; Venkat & Spaull, 2015) point out that South African students perform poorly in mathematics as they move on to higher grades from Grade R. This is an ongoing research interest which some argue to be a problem resulting from the poor content knowledge of mathematics teachers (Venkat *et al*., 2015). Along these lines, there are some (e.g., Ally & Christiansen, 2013; Sorto & Sapire, 2011) who argue that this is a problem resulting from a prevalence of a *procedural* approach to mathematics teaching. It is believed that learners are driven towards mere production of answers to mathematical problem and suggests poor content knowledge of classroom teachers. By *procedural,* Hiebert and Lefevre's (1986) popularising of the notion, following Scheffler (1965) and colleague, identifies *conceptual* and *procedural*[3] knowledge as two kinds of teaching knowledge which are similar to Skemp's (1976) notions of *relational* and *instrumental* understanding. According to Skemp (1976), relational understanding refers to one's ability to deduce specific rules and procedures from more general mathematical relations, while instrumental understanding refers to the ability to apply a rule to the solution of a problem without necessarily understanding how it works. From the literature, teachers who show procedural knowledge are considered to have a disconnected and poor content knowledge structures because they are a presumed to have poor conceptual grasp of the knowledge. However, on the issue of knowledge, research in cognitive science points out that no procedure can be concept free (Chomsky, 2006; Pinker, 1997). Some researchers in mathematics education make a similar point (Davis, 2010; Vergnaud 1998). Therefore, there is a need to re-examine that which is described as procedural and conceptual when discussing teachers' MKfT. As we know, people, even in in pedagogic contexts, form mental connections differently and a teacher may formulate constructions that are considered legitimate in local pedagogic contexts but not necessarily outside of such contexts.

Some scholars (e.g., Reddy, Van der Berg, Lebani, & Berkowitz, 2006; Spaull, 2013) point to the impact of issues relating to socio-economic class membership on learning as primary explanatory reasons for the poor mathematics performances of South African learners. Along this line, learners who attend schools with fewer resources, categorised as working-class schools, are predisposed to perform poorly in mathematics as opposed to better performing counterparts attending more affluent schools, described as middle-class schools. Previous studies (see Adler, 2005; Adler & Davis, 2006). detail how that the legacy of apartheid has led to the persistence of this divide in South African schools and universities.

In this study, the two schools selected for study are situated in working-class communities in the Western Cape. Performance results of these schools have shown that learners perform slightly higher in numeracy and literacy in comparison to schools situated in similar contexts. Thus, it will be interesting to examine the mathematics teaching knowledge of the teachers in this schools.

1.3 Rationale

Reports pertaining to the early mathematics skill of young children in South African schools indicate that limitations apply mostly to the mathematical topic of *number, operations, and relationships.* On the one hand, this is not unusual since the bulk of the Foundation Phase curriculum is reasonably focused on those topics. However, given the amount of attention apparently paid to number, operations and relationships, it is a curious finding. From the research literature there is an indication that numeracy excellence in the early childhood years predicts later mathematics performance, particularly with respect to counting (Jordan, Glutting, &

[3] Hiebert and Lefevre, describe procedural knowledge as knowledge of the "formal language, or symbol representation systems of mathematics" and "the algorithms, or rules, for completing mathematical tasks"; and conceptual knowledge as "knowledge that is rich in relationships and forms" that forms "a connected web of knowledge" (1986, p.6).

Ramineni, 2010). In the field of mathematics education research, many agree with claims made by Gelman *et al*. (1986, pp. vi - vii) who points out principles guiding children's operations with numbers. According to these scholars, children's work with numbers follows two principles. The first principle concerns the use of counting when representing the size of a finite aggregate. The second concerns the definition and inter-relation of addition, subtraction, numerical order and equivalence. From their discussion of the growth of numerical knowledge and arithmetic competence in the young child, Gelman *et al*. (1986) argue that counting is central because it is how the numerosity of aggregates are decided, as well as a means by which arithmetic reasoning is initially formed.

As children begin to engage with their environments, they begin to recognise the meaning associated with phrases such as 'two sweets', 'four marbles' and so on. Alongside this, young children begin with learning by rote the sequencing of number names, numerals, and combination of numerals as a precursor to counting. When one observes pedagogic settings constructing learning on early grade referencing to the use of 'count' or 'counting', we see the use of variety of ways used by teachers and learners in constructing computational devices enabling them to arrive at desired outcomes. Yet, many such pedagogically fashioned computational resources often display features not found in mathematically principled elaborations of mathematics outside of primary schooling. In this study, I examine the resources used by Grade 3 mathematics teachers and learners when referencing *count* or *counting.*

1.4 Research Question

Given the centrality of *counting* to the growth of mathematical knowledge in young children, clarity on the treatment and references to *counting* at the Foundation Phase level contributes to our understanding of more complex mathematical ideas in pedagogic situations.

The focal research question framing this study seeks to understand

> *how two Grade 3 teachers, teaching in similar social-class contexts, generate structures relating to counting, during their lessons; and what the implications for the number knowledge of students might be.*

1.5 Overview of the research framework

This study adopts an *Integrated Causal Model* (ICM) approach by drawing on theoretical resources from different disciplines such as cognitive science, evolutionary psychology and mathematics. Tooby & Cosmides (1992) offer compelling arguments calling for an ICM approach to social science research. An ICM approach constructs descriptions and analyses by harnessing the causal connections between related components of a phenomenon, requiring resources from other disciplines for descriptive and explanatory purposes. In this way, one can connect social science research to the rest of science. Tooby and Cosmides argue that the current alternative, that is, the *Standard Social Science Model* (SSSM), "mischaracterizes" phenomena because of the failure to "causally locate their objects of study in the larger network of scientific knowledge" (p.23).

One of the propositions informing an ICM is that the human mind comes biologically fitted by natural selection with specialized domains of knowledge often referred to as core domain in cognitive science literature (Carey & Spelke, 1996; Spelke & Kinzler, 2007). Following an ICM approach shows how research in mathematics teaching and learning can productively draw on scientific work that has something to say on the object of investigation. In this instance, it requires a consideration of the structuring effects of human cognitive functioning on the computational activities of teachers and learners with respect to counting.

From an ICM perspective, the genesis of culture is considered as an internal product of the human mind in individuals who live in communities, as opposed to viewing culture exclusively as an external structure that shapes the human mind. Chomsky, who uses an ICM in his study of language, argues that human cognition involves the interplay of three factors which concerns: (a) genetic endowment, which affords human beings the basic resources needed to enable the growth of mathematics and language in individual (core domain); (b) the structuring of experience by genetic endowment on the basis of contextual data; (c) general properties of the world, such as biological and physical laws and principles of data processing and computational efficiency. The assumption of this study is primarily concerned at the second factor issues. Thus, the pedagogic activity of the two teachers studied are assumed to link with core domain knowledge, as would be the case with teachers working in any other social class setting. Also, since contextual data in different contexts vary, it would be interesting to examine variations that arise in the lessons examined.

1.6 A summary of the dissertation

Chapter 2 is a review of pertinent literature discussing mathematics teaching knowledge. The goal is to highlight the views of leading scholars in this area in terms of the approach used to examine teachers' knowledge of content and student in practice.

Chapter 3 details the general method of the study. From the methods, a series of propositions are derived to serve as a basis for the production of an analytic framework.

Chapter 4 is a description of the framework for the production of data and the protocols for analysis. It outlines the research design and considers issues relating to reliability, validity and generalisability of the study.

Chapter 5 and Chapter 6 provide descriptions of the data produced from the analyses.

Chapter 7 is a discussion on the analyses data and the conclusion of the study.

CHAPTER 2

A review of the literature

2.1 Introduction

Literature on the mathematics performances of South African students often argues that the poor performance of students stems largely from the poor content knowledge of schoolteachers (Taylor, 2011; Venkat & Spaull, 2015). The focus of this chapter is to review what is alluded to as the content knowledge for teaching number-related ideas to young learners. This chapter begins with a brief review on teacher knowledge and pedagogical content knowledge (PCK). Next is a review of teachers' knowledge of content and student (KCS) using the MKfT framework of Ball, Thames and Phelps (2008) as it resonates with my investigation on knowledge for teaching number-related ideas to young children.

2.2 Teacher knowledge

In the past, some have interpreted teacher's knowledge as predominantly 'craft knowledge'. Of course, we know that this is merely a non-theoretical platitude. However, there are some who have made efforts to show that productive teaching requires that the teacher draws from specific knowledge bases to develop their professional expertise and improve learning. Such studies (e.g., Adler & Davis, 2006; Ma, 2010; Perressini, Borko, Romagnano, Knuth, & Willis, 2004) draw on the seminal work of Shulman (1986; 1987) on *pedagogic content knowledge* (PCK) when interrogating subject specific knowledge required for teaching. PCK is described as a complex nature of knowledge-in-use for teaching because it integrates subject matter knowledge and knowledge about teaching and learning (Shulman, 1986).

Many researchers began to consider Shulman's conception of teachers' pedagogical content knowledge because PCK allowed the researcher to question *how teachers decided on what to teach, how to present the content, and questions to ask students*, which were the kinds of question Shulman began to ask in his work on teacher knowledge. Shulman argued for the existence of a "missing paradigm" in this regard. He and his colleagues described PCK as an overlapping of general pedagogic knowledge and subject matter knowledge. Thus, PCK presents a fusion of subject matter expertise with pedagogical strategies, and knowledge of the learner that is thought to produce high quality classroom practice and learning.

According to Shulman and his colleagues, PCK demands an ability to develop and implement learning instruction in specific ways that leads to enhanced understanding for the learner. Shulman described PCK as the professional knowledge which goes past merely knowing the subject matter and knowing generic teaching methods to the dimension of 'knowledge for teaching' the required subject matter. He specified pedagogic content knowledge PCK as

> the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations — in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others. … Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (Shulman, 1986, p. 7).

Following Shulman's specification of what PCK might refer to in practice, scholars began to review teacher knowledge using ideas on PCK. Mathematics education researchers, and teacher educators, who were keen to specify the nature of PCK in practice began to explore the concept in varied ways.

In this field, many agree with work suggesting that Mathematics Knowledge for Teaching (MKT) is a distinctive form of mathematical knowledge produced in, and used for, the practice of teaching. However, because Shulman did not specify the inner details of PCK, various elaborations of the construct emerged. Some researchers sought to examine the concept of PCK (Grossman, 1990), while others made efforts to specify the composition of PCK (e.g., Ball, et al., 2008; Fennema & Franke, 1992; Rowland, et al., 2003). The latter is where the interest of this study is partly situated, and it draws largely from the work of Ball and her colleagues. These authors modified PCK by identifying three sub-knowledge domains within the construct, using the framework referred to as *mathematics knowledge for teaching* (MKfT) (Ball, et al., 2008).

2.3 Mathematics knowledge for teaching

In the past several years we have seen research focused on how school children solve mathematical problems using various theoretical perspectives (e.g., behavioural theories, philosophical and pedagogic constructivism, and forth). In all of these, it is generally agreed that students bring a great deal of knowledge to almost any learning situation, which significantly influences how they learn from instruction (Carpenter & Fennema, 1992). So, the need to explore knowledge to do with the teaching of content to children is of necessity. To show the knowledge domains typically deployed by teachers in the work of mathematics teaching, Ball *et al.* (2008) introduced the mathematics knowledge for teaching (MKfT) framework (see Figure 2.1), which is an elaboration of Shulman's concept of PCK.

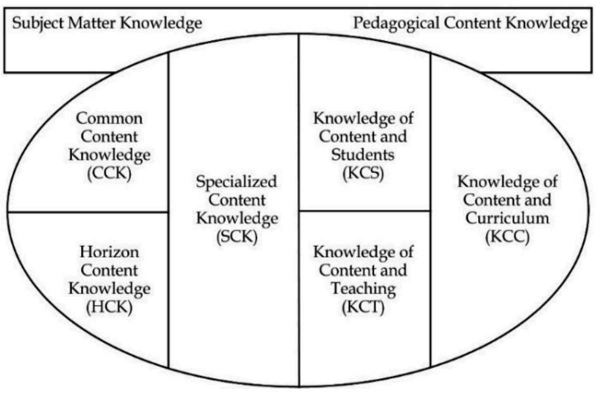

Figure 2.1 Framework of Mathematical Knowledge for Teaching (*Source*: Ball *et al.*, 2008)

The right part of Figure 2.1 specifies the components of PCK for mathematics teaching from the framework. It argues that teachers recruit knowledge from three unique sub-domains in their work of teaching. In all of these, Ball *et al*. (2008) indicate that this knowledge requires the blending of subject knowledge with other kinds of knowledge, such as: (a) knowledge of how learners think in different content domains; (b) knowledge for teaching; and (c) knowledge of the curriculum.

Using job analysis, Ball & Bass (2002; 2003) showed practice-based pedagogic activities, detailing evidence of the knowledge domains evident in mathematics lessons. By observing the *work* mathematics teachers do in the classroom, they found that as teachers *unpack* mathematics content to learners, they draw from specific knowledge domains. From an analysis of Grade 3 teaching, they argued that sixteen "mathematical tasks of teaching"[4] may occur daily in the classroom as teachers do the work of unpacking content. These tasks provide the context from which teachers must draw their mathematical knowledge in teaching.

Figure 2.1 shows that the knowledge domains are grouped in two broad categories. The first part contains the subject matter knowledge (SMK) which mainly concerns the subject or content knowledge. The second parts consider pedagogic content knowledge (PCK), which blends content with distinct aspects of pedagogy. Briefly, the first category of the framework, subject matter knowledge (SMK) is divided into three sub-domains, which is made up of common content knowledge (CCK), *specialised* content knowledge (SCK) and *horizon* content knowledge (HCK). CCK refers to the mathematical knowledge and skills, expected of any well-educated adult. For instance, teachers are expected to be able to do the work assigned to students. They need to be able to recognize wrong answers. Such as knowing that the product of 35 and 25 is 875 which, in a way, is general knowledge. SCK, refers to knowledge unique to teaching and is the mathematical knowledge and skills needed by teachers in their work; and (HCK), concerns knowing how mathematics topics connect across the curriculum.

Likewise, PCK is a composition of three other sub-domains, identified as follows:

(1) *knowledge of content and students* (KCS), which is a combination of knowing about learners and knowing about mathematics. From the framework, KCS is described as teachers' ability to design stimulating lessons that help the student formulate their own connections in understanding the content taught. This domain of knowledge speaks of what teachers know of the content and of how student know the content. According to Hill, et al. (2008), KCS can be divided into four distinct categories:

[4] The list includes tasks such as: responding to students 'why' questions, finding an example to make a specific mathematical point, evaluating the plausibility of students, choosing, and developing useful definitions etc. (Hill, Ball, & Schilling, 2008). These mathematical works of teaching is arranged around what they call "mathematical objects" because they are the mathematical instructional objects encountered by teachers while they teach. Examples includes explanations, representations, mathematical errors, and definitions.

1. Common student errors: That is, being able to identify errors that arise and being able to provide an explanation for such errors.
2. Students' understanding of content: This concerns teachers' ability to interpret student's production. Here, teacher is expected to be able to decide which student's production indicate good understanding.
3. Student developmental sequence: This deals with the teacher knowing problem types, topics, and activities which student find easier or more difficult to perform.
4. Common student computational strategies: This concerns knowledge of landmark numbers, facts and so on. (Hill, et al., 2008 p.380)

(2) *knowledge of content and teaching* (KCT) is described as a combination of knowing about teaching and knowing about mathematics. It informs activities such as the sequencing of examples and the ability to immediately spot algorithmic errors.

(3) *knowledge of content and curriculum* (KCC), which concerns curricular conceptions of content (Ball *et al*., 2008).

2.3.1 Knowledge of content and students (KCS)

According to Ball *et al*. (2008), teachers' KCS tells one what a teacher knows about the content relative to how learners typically learn the given content. That is, in such a way that they can anticipate what learners think mathematically and what they would likely find confusing. According to Hill *et al*. (2008, p. 375), KCS is described as knowledge "intertwined with knowledge of how students think about, know, or learn" content. So, a teacher who demonstrates KCS is believed to structure mathematics lessons in ways that addresses the common mathematics difficulties that learners encounter. To elaborate on what teachers' KCS look like in the practice of teaching, let us consider literature from the Cognitive Guided Instruction (CGI) study of Fennema, Carpenter, Franke, *et al*. (1996), which examined teacher's understanding of children's mathematical thinking.

The approach used by the authors was to monitor teachers' beliefs of their students to understand teacher's knowledge of students. For this, they used a 'problem identification' technique which classified basic arithmetic word problem operations (addition, subtraction, multiplication, division) into classes that were distinguished by the type of 'action' implied. They grouped arithmetic operation into four categories, as *change*, *combine*, *compare,* or *equalize*, and then, categorized them from easiest to most difficult classes of *join, separate, part-part whole* and *compare*. In this way, they argued, one could access the difficulty level of tasks and then connect that to teachers' knowledge of their students. In other words, the more complex a task is, the more knowledgeable a teacher is presumed to be and *vice versa*. Indeed, monitoring teachers' KCS this way tells one something about a teacher's mathematics skill but does not capture the aspect of KCS concerning *how* students know what they learn.

Another popular area of work used in mathematics teacher education research is that of Rowland and his colleagues on the Knowledge Quartet (KQ). The KQ examined the mathematics knowledge of pre-service teachers from the use of a framework that classifies situations in which teachers' SMK and PCK are observed in practice. The framework describes the interactions of four categories of knowledge seen during classroom mathematics teaching, namely, *foundation*, *transformation, connection,* and *contingency.*

Foundation category concerns the theoretical background relative to teachers' mathematics knowledge, beliefs and understanding that they acquire during teacher training. According to Rowland et al (2003), the indicators of *foundation* knowledge in the context of primary mathematics presumes moving from what is thought of as more concrete to more abstract notions of number, concentration on developing leaners' understanding, correct writing of mathematical expressions, demonstrating knowledge of common errors, and recognition of misconceptions in the planning of a lesson and avoiding them (Rowland et al., 2003).

The second category, *transformation*, relates to the mathematical knowledge-in-action which is the core aspect of the KQ framework. Drawing on Shulman's notion of PCK, Turner & Rowland (2011) specify *transformation* as comprising of the teacher's choices and use of examples, representations, use of instructional materials and demonstrations in teaching mathematics and explanation of mathematical ideas.

The *connection* category pertains to the coherence of the planning of the lesson or mathematics teaching displayed across a lesson or series of lessons. *Connection* examines how teacher makes decisions about sequencing and connectivity during teaching (Rowland et al., 2003).

The fourth category, *contingency*, focusses on the teacher's responses to classroom events that were not anticipated in the planning of the lesson (Turner & Rowland, 2011).

From the list of the knowledge dimensions listed by the authors, *transformation* resonates most with Ball and her colleague's description of KCS. For Rowland *et al*. (2003; 2005), *transformation* is the core part of the Knowledge Quartet (KQ) as it concerns teaching knowledge employed in practice, described as mathematics knowledge in teaching (MKiT). Transformation is in evidence when a teacher demonstrates the ability to transform the content of a lesson plan effectively for students (Rowland *et al*., 2003). This dimension is registered using observation of teachers' choice and use of examples, representations, instructional materials, and demonstrations in practice. By probing the foundational knowledge of the teacher from teachers' choice of examples. For example, using the lesson of a pre-service teacher (whom they referred to as Laura), they describe aspects of a lesson to inform the reader of the teacher's knowledge of content and her learner.

Laura's lesson began with her reminding students of the 'grid' method for multiplication that partitions numbers into tens and units. Then she invited a learner to demonstrate the method and went on to show a 'new way' of solving multiplication problems.

$$\begin{array}{lr} & 37 \\ & \times\, 9 \\ \hline 30 \times 9 & 270 \\ 7 \times 9 & 63 \\ \hline & 333 \end{array}$$

From this scenario, the KQ approach to examining Laura's teaching knowledge begins by questioning her *foundation* content knowledge. As such, Laura's *foundation* knowledge is judged based on her emphasis on the 'new way' of solving multiplication problems.

From the analysis by these authors of Laura's lesson, Laura recognised more than one possible written algorithm for whole number multiplication and curricular awareness. However, she did not make explicit the reasons for the 'new method' nor the connections between the grid method and the new method, given that her learners already appeared to be familiar with the grid method (Rowland, 2014).

I find that, just as with Shulman's conception of PCK, the understanding offered by Rowland and his colleagues on what teachers' knowledge of content and learners refer to, provides reasonable ideas on what is expected in the profession of mathematics teaching. However, the framework appears to be commonsensical in that its constituents are obvious things that a mathematics teacher needs to know and do as a teacher.

2.4 South African views on MKfT

Within the South African context, significant work has been done in this regard by scholars (e.g., Adler, 2005; Adler & Davis, 2006; Adler & Pillay, 2007) who draw on aspects of the MKfT work of Ball and colleagues in expounding on PCK. For example, Adler and Davis (2006) examined the kinds of mathematical and pedagogical or teaching competences that teacher educators are expected to display, and so, the kind of mathematical knowledge that is privileged. They argue for the dual nature of mathematics teaching knowledge in teacher education. That is, learning to *teach* mathematics, and learning mathematics for teaching–the 'subject-method' tension (Adler, Davis, Kazima, *et al*., 2005). They indicate that while mathematics teachers generally do the work of *interpreting*, *analysing* and *judging* students' mathematical thinking, the way in which it emerges and is approached differs in terms of its mathematical entailment (Adler *et al*., 2005). Thus, pointing out that Ball *et al*.'s (2004) description of *unpacking* as a distinctive feature of "knowing mathematics for teaching" and the learner is of a general notion. Employing Davis's (2005) treatment of Hegel's theory of judgements, Adler *et al*. (2005) show how pedagogic judgments legitimate appeals teachers use in trying to fix meaning in the classrooms.

Adler & Davis (2006) question what the primary and secondary objects (i.e., mathematics and/or teaching) of instruction are as they examine assessment tasks. They argue that there is an "absence, rather than presence of unpacked or elaborated mathematics for teaching" in mathematics teacher education courses (p. 291). Thus, they find that the kinds of mathematical work teachers do is not yet well understood. Along similar lines, Kazima, Pillay and Adler (2008) carried out two case studies in South African schools and conclude that teachers need more than knowledge of topics taught in mathematics. These reports suggest a need for some deeper description of mathematics teaching knowledge.

2.5 Number sense

Research on the numerical cognition of young children shows that young children are able to recognise small collections of objects quantitatively before they learn how to count with understanding (Gelman, *et al*., 1978). Then, as they begin to acquire language and learn numerical words and rhymes involving numbers, learning of counting ideas begin to develop. Thus, number sense is a natural phenomenon that develops as children engage with the environment in different contexts. Teachers are therefore tasked with the responsibility to know the rudimentary aspects of children's number-related knowledge to show that they deploy KCS effectively in the classroom. As previously mentioned, the early numerical skills of young children have a strong impact on their mathematics success in later schooling. Also, we know that PCK is supposedly that vital teaching knowledge base associated with the effective transformation of content in ways that is easy to understand in a given pedagogic context (Ball *et al*., 2008; Hill *et al*., 2004; Shulman, 1986; 1987). More so, a central component of the PCK of mathematics teachers is recognised as teachers' KCS, which is what the teacher knows of the specific content, and of the way students learn the given content (Ball *et al*., 2008).

In Chapter 1, we pointed out that numerical cognition has an innate component (core domain) that children are born with and which develops as they engage with culturally acquired noncore knowledge domains, such as school mathematics (Gelman *et al*., 1978). From literature it is clear that the innate features of number sense enable an early sensitivity to quantity in human infants by means of an Approximate Number System, and the OTS, which enables humans to track small numbers of objects, both of which remain operative over the lifespan of humans.

According to Dehaene (1997, p. 21) "every human being is endowed with a primal number sense, an intuition about numerical relations. Whatever is different in adult brains is the result of successful education, strategies, and memorization". Thus, well-developed number sense is premised on an understanding of quantity and relationships with numbers; exhibits of strategies indicating increased facility, flexibility, and fluency in operating with numbers in connected and sophisticated ways.

That said, a proposed framework (Mcintosh, Reys & Reys 1992), often used by mathematics teacher educators, specifies the features of number sense in three components: *numbers*, *operations* and *computational settings* (see Table 2.1).

From the description of McIntosh *et al*. (1992), we see an ordered arrangement of school level (noncore) progression of student learning on number-related knowledge. If one is, however, interested in examining teachers' KCS relative to what they know about young children's knowledge of number related ideas, there is a need to include the core domain features of number sense in such examination otherwise, we find ourselves implicitly adopting a blank slate conception of mind (cf. Pinker, 2002).

Table 2.1: *Numbers-Operations-Computational Settings* NS framework (*Source*: McIntosh *et al*. (1992, 4)).

Knowledge of and facility with NUMBERS	Sense of orderliness of numbers	Place value
		Relationship between number types
		Ordering numbers within and among number types
	Multiple representations of numbers	Graphical/symbolic
		Equivalent numerical forms (including decomposition/recomposition)
		Comparison to benchmarks
	Sense of relative and absolute magnitude of numbers	Comparing to physical referent
		Comparing to mathematical referent
	System of benchmarks	Mathematical
		Personal
Knowledge of and facility with OPERATIONS	Understanding the effect of operations	Operating on whole numbers
		Operating on fractions/decimals
	Understanding mathematical properties	Commutativity
		Associativity
		Distributivity
		Identities
		Inverses
	Understanding the relationships between operations	Addition/Multiplication
		Subtraction/Division
		Addition/Subtraction
		Multiplication/Division
Applying knowledge of facility with numbers and operations to COMPUTATIONAL SETTINGS	Understanding the relationship between problem context and the necessary computation	Recognise data as exact or approximate
		Awareness that solutions may be exact or approximate
	Awareness that multiple strategies exist	Ability to create and/or invent strategies
		Ability to apply different strategies
		Ability to select an efficient strategy
	Inclination to utilize an efficient representation and/or method	Facility with various methods (mental, calculator, paper/pencil)
		Facility with choosing efficient number(s)
	Inclination to review data and result for sensibility	Recognise reasonableness of data
		Recognise reasonableness of calculation

2.6 Summary

This chapter discussed aspects of teacher knowledge identified as PCK. Within mathematics teacher education research, we found that many researchers follow the work of Ball and colleagues on MKfT to examine teachers' PCK in the classroom and so, we reviewed PCK using the MKfT framework. From the MKfT framework, the literature reveals that teachers' KCS is a vital part of PCK. However, for one to attribute KCS to a mathematics teacher, Ball *et al*. (2008) suggest that a researcher needs to show that they know what students know of the topic or content. For this, they suggest that the researcher focus on key indicators such as teacher knowledge of students' common errors or misconceptions, teacher's awareness of what student find interesting, teachers' awareness of what students find difficult or easy to do. The positions of Ball and of Rowland and others are strongly empiricist in their reliance on the statistical prevalence of certain educational behaviours and traditions rather than asking questions of how things come to be. This study adopts a more rationalist approach to the problems of teaching and learning (cf. Chomsky, 2016b) and is descriptive rather than prescriptive, the latter being a feature of much of the work reviewed in this chapter.

This chapter also briefly discussed the views of some South African scholars (e.g., Adler & Davis, 2006) who draw attention to the silence on the dual nature of mathematics teaching and teacher education in the MKfT framework as used in many studies.

Finally, this chapter briefly discussed literature on number sense as this leads to the focus of this study which is to illuminate teachers' KCS by observing and examining their referential use and treatment of the terms 'count' or 'counting' in the classroom. The concluding aspect of this chapter found that mathematics education literature on the numbers and operations framework of young children excludes core domain features when examining number sense. Following this observation, we observe an incomplete assessment on the number related knowledge of classroom teachers and learners. Chapter 3 discusses the theoretical propositions used to frame the procedures for the production of data and the protocols for the analysis of data.

CHAPTER 3

Theoretical framework

3.1 Introduction

For one to generate data required for analysis there is a need to draw on appropriate theoretical referents to produce analytical procedures. This chapter sets out the general methodology applied in this study. From the methods, theoretical propositions that serve as a basis to produce data are produced to address the research question concerned with how FP teachers who teach at the Grade 3 level treat learning relating to counting. To start, a foundational proposition taken for the analytic procedure in this study is that pedagogy is fundamentally evaluative (Bernstein, 2000). That is, evaluation distinguishes the legitimate from non-legitimate utterances and statements and it reveals the criteria for the recognition and realisation of legitimate text in a pedagogic context. The structure of a mathematics lesson is visible through observation and description of the domains of objects and associated operations populated in a given *evaluative event* (Davis, 2003; 2005; 2011b). The approach underpinning this study draws on the methodological resources developed in Davis (2005; 2013; 2016; 2018), framed by a computational theory of mind, that posits that thought is computational in nature (Chomsky, 2006, 2007; Gallistel & King, 2010; Pinker, 1997, 2007).

3.2 The core system of numerical representation

In Chapter 2 we noted the tendency of research in mathematics teacher education to pay insufficient attention to the fundamental, biologically endowed basis for numerical cognition. Recall that in Chapter 1 the orientation used for research in this study is an integrated approach that attends to both the internal and external aspects of knowledge generation. This approach is in line with Tooby & Cosmides' (1992) call for an Integrated Causal Model (ICM) approach to social science research. Using an ICM approach necessitates the recognition that "the human mind consists of a set of evolved information-processing mechanisms instantiated in the human nervous system" (p.21).

Davis (2016; 2018) argues that the use of an ICM approach to teacher education research necessitates a recognition that humans come genetically endowed with specialized domains of knowledge, which research in cognitive science (e.g., Carey & Spelke, 1996; Spelke & Kinzler, 2007) refer to as *core domain* knowledge. This requires one to consider human, species-typical cognitive functioning, specifically the structuring effects of such on the computational activity of teachers and their learners (Davis, 2018). As stated in Chapter 1, core domain knowledge, refers to the genetically endowed knowledge system. That is, domains that benefit from biological underpinnings such as those for language and number (Gelman & Williams, 1998; Spelke, 2000). Literature in cognitive science and contemporary developmental psychology show that the presence of innate core systems of numerical representation in human infants are tuned to specific types of information which continues to function throughout the life span.

Experiments in this field of research show that infants demonstrate evidence of tracking distinct individual objects of up to a limit of three or four and show sensitivity to continuous variables (Feigenson, Dehaene, & Spelke, 2004; Jordan & Brannon, 2006; Xu, Spelke, & Goddard, 2005). Thus, it is believed that this innate mechanism starts out in its rudimentary form and, later, new organized mental structures are mounted, which are referred to as noncore domains (such as school mathematics learning of number) (Gelman, 2009). Note that Gelman (2000, p. 854) specifies a *domain* of knowledge as

> a set of interrelated principles […] a body of knowledge constitutes a domain of knowledge if we can show that a set of interrelated principles organizes its rules of operations and entities.

The idea on the presence of a core domain numerical knowledge is prefigured in Kant's (1781) argument on the necessary presence of cognitive schemas for number, space and time in humans from birth. That is, knowledge acquired independent of any experience as opposed to a posterior knowledge derived from experience. Thus, in cognitive science and contemporary developmental psychology, research suggests that numerical knowledge is *a priori*, present from birth in humans and animals (Siegler, 2016). Such a stance is central to rationalist approaches to the study of human cognition.

For Chomsky, the study of human cognition as a biological system involves engaging with the interplay of three factors.

(i) *Genetic endowment* (core domain) is required by and afforded to all humans. It enables humans interpret and interact with the environment.

(ii) The construction of *experience* on the basis of genetic endowment, which leads to variations in performance, like in, for example, school mathematics (Davis, 2016; 2018). According to Chomsky (2005), this subcategory is of significance in determining the nature of attainable languages (e.g., Afrikaans or Zulu). Davis concurs with this and points out that contextual data differs across pedagogic contexts because the classroom is a dynamic space with learners and teachers from different experiential backgrounds (ibid., 2016). According to Davis (2011b), *computational* activity entails a composition of the operational activities of teachers and learners relating to operations over domains of objects that serve as collections of arguments and values for the operational activity.

(iii) *General properties of the world* that have structuring effects on everything that emerges in the world, e.g., biological, and physical laws (Chomsky, 2005).

Chomsky (2006) proposes that, in empirical research, we ought to strive to realise three levels of adequacy, which are: (1) *observational adequacy*, (2) *descriptive adequacy*, and (3) *explanatory adequacy* (see also Boeckx (2006)). Paraphrasing Davis, adequacy is realized when descriptive and analytic resources can generate adequate description of the observed computational activity; that is, from the "data made available to students that capture the specifics and range of apparent operational resources that emerge in pedagogic situations" (Davis 2018, p. 3). Therefore, this level of adequacy requires one to set aside any preconceptions about teachers and learners during observation of the pedagogic computational activity. It requires the recognition of all computational specifics of teacher and learner activity. It must include the recognition of operational features that do not necessarily accord with the researcher's expectations of the content, nor what is routinely associated with a particular topic, nor what is considered as 'proper' mathematics (Davis, 2018).

On this basis, Davis (2013, p.35), following Chomsky, notes that satisfying adequacy requires the interrelation of (i) expression, (ii) syntax and (iii) semantics. The level of expression contains information relevant to the interpretation of expressive or lexical elements entailed in communicating mathematical thinking through speech, writing, gesture or any other semiotic medium. This level is directly observable from what teachers and learners say and do. The syntax comprises of compositions of operations and their associated domains and codomains. Relating the level of expression to the level of semantics is not directly observable but is inferred from an account of syntax. Thus, Davis (2010) argues, to ascertain what is constituted as mathematics in a pedagogic situation, there is need to describe what it is teachers and learners say and do as they go about doing mathematics, read off from the oral, written or gestural acts (scriptural practices) during pedagogic interaction. Therefore, this requires the researcher to make decisions regarding what teachers and learners refer to by conduction an examination of their computational activity.

3.3 Pedagogic communication and pedagogic discourse

In pedagogic situations, teachers are presumed to hold the desired knowledge required to be learnt and are tasked with the responsibility of simplifying knowledge in ways that can be easily understood by learners (i.e., PCK). Following our discussion on KCS in Chapter 2, the mathematics teacher is expected to know how students typically understand a mathematics topic, what they find difficult or easy, and how to structure learning in the pedagogic context, all of which contributes to what gets to be produced as legitimated knowledge in the classroom. Thus, the notion that the relationship between teachers and learners are essentially evaluative from the criteria made visible through the act of evaluation (Bernstein, 2000; Davis & Johnson, 2007).

In his theory of the *pedagogic device,* Bernstein (2000) provides a general structure for describing the process of knowledge transformation from domain-specific knowledge to school knowledge during pedagogic communication. Bernstein's discussion of the pedagogic device reveals that the entirety of the pedagogic device is condensed in the *evaluation*. According to Bernstein (2000, p.50)

> the key to pedagogic practice is continuous evaluation [...]. This is what the device is about. Evaluation condenses the meaning of the whole device. We are now in a position where we can derive the whole purpose of the device. The purpose of the device is to provide a symbolic ruler of consciousness.

Thus, the device can be used to describe classroom pedagogic practices. According to Bernstein, the "intrinsic grammar" of the pedagogic device is provided by three hierarchical interrelated rules, which are the *distributive* rule, the *recontextualising* rule, and the *evaluative* rule (Bernstein, 1996, p. 43). Briefly, the distributive rule concerns "who gets what", here the "what" refers to the knowledge chosen to be made available and the "who" refers to those to whom the selected knowledge is made available. In other words, the distributive rule operates within the field of production of knowledge and occurs through the process of knowledge production. Recontextualising rule derives from the distributive rule and it regulates the formation of specific pedagogic discourse (Bernstein, 1996) which is concerned with how knowledge is organised for distribution (e.g., school mathematics curricula and pedagogic prescriptions).

According to Bernstein, pedagogic discourse follows a principle which embeds two discourses, that is, an *instructional* discourse, and the *regulative* discourse (Bernstein, 1996). The instructional discourse is the discourse which creates specialized skills, and it is embedded in the regulative discourse. The regulative discourse is a moral discourse, which creates order, fixes pedagogic relations, and is identified as the dominant discourse. Singh (2002) explains that the moral order of the classroom is a necessary pre-requisite for the transmission of instructional discourses, hence it is the dominant discourse in pedagogy. So, while the distributive rule is concerned with power in terms of who gets what knowledge, the recontextualising rule operates within the field of recontextualisation of knowledge where the recontextualised content no longer resembles the original because it has been pedagogically transformed.

The evaluative rule constitutes specific pedagogic practices. It is here that the actual process of transmission and acquisition of knowledge in a pedagogic situation occur. That is, the level where the actual distributive effects of the device occur; and where what counts as valid acquisition of instructional (curricular content) and regulative (social conduct and character) requirements are recognised (Singh 2002). According to Bernstein, the contents of the distributive and recontextualising rules are condensed in the evaluative rule, thus the entire structure of the device is condensed at this level and visible from the *recognition* and *realisation* criteria during pedagogic discourse.

3.3.1 Evaluation, the recognition of pedagogic criteria and KCS

We know that teachers and students relates to knowledge in various ways.. For instance, learners commonly make judgements based on what they think their teacher expects them to say or do. Teachers, on the other hand, following curricular prescriptions and traditions, often using well-established analogies and metaphors to communicate what counts as legitimate knowledge to learners (Davis, 2010). These pedagogic practices are composites of evaluative criteria, and in Bernstein's (1990, p. 50) terms, "the key to pedagogic practice is continuous evaluation", where *evaluation* refers to student-teacher interactions, questions, problems, tests, projects, and examinations (Davis, 2010).

According to Davis (2013a), although evaluative action assists teachers and learners know when the desired outcome for a given mathematical problem is achieved, it is also possible to use completely different domains, codomains and operations or operation-like manipulations and still achieve the same apparent expressive outcome. Therefore, evaluative activities are referred to as mechanisms that mediate the field of knowledge encountered by learners. So, in order to describe the evaluative activities and criteria of teachers and learners, an examination of what it is teachers and learners "say and do" is required, which, according to Davis (2013a), is read off from their oral, written, or gestural acts. Hence Davis' (2011a, 2013a) argument that, as researchers, we accept as intended mathematics whatever emerges computationally in the pedagogic setting, whether or not it is questionable from the viewpoint of 'proper mathematics'.

Bernstein indicates that evaluation reveals *recognition* and *realisation rules (*Bernstein, 1990), arguing that the recognition rule, "regulates what meanings may be put together, what referential relations are privileged, whereas the realization rule establishes what counts as a legitimate text" (ibid. p.24). In other words, pedagogic evaluation makes available criteria that mark out what is to count as legitimate knowledge statements in a pedagogic context, and how such knowledge ought to be realised (Bernstein, 1996: 49-51).

In mathematics education, however, Davis (2013a) points out that one is able to produce the legitimate text without necessarily observing the recognition rule offered by teacher in realising the required output. An example of such is the common use of concatenation techniques by young children when doing place value addition. Davis argues that Bernstein's treatment of recognition rule is problematic on at least two counts:

(1) The first problem for Bernstein is that of the use of identical expressions by pedagogic agents in producing mathematical statements since the nature of recruited computational domains are not immediately obvious at the level of expression. Davis points out that school mathematics comprises of compositions of operations, and that operations are, by definition, functions. It is therefore possible to replace a rule for a function with a different, equivalent rule, to produce the same expressive output from a given expressive input, even though a different rule is used (ibid., 2013a).

(2) Compositions of operations are regulated by higher-level propositions and decision-making, the logic of which need not agree across individuals in pedagogic situations. The observation that pedagogic situations are communicative and given that communication depends on shared language use when referencing various aspects of the world, *reference* emerges as a problem for Bernstein's propositions on recognition and realisation. Natural languages, unlike formal languages, have no reference function. This simply means that there is no necessary association of any lexical element with a particular semantic content (Chomsky, 2016; Strawson, 1950).

From (1) and (2) it should be clear that the notions of *recognition* and *realisation* in the Bernsteinian sense are rather shaky and imprecise.

Recall that in Chapter 2 it was argued that many researchers in mathematics teacher education describe the mathematics knowledge of teachers. Specifically, it was pointed out that PCK is a construct that many in mathematics teacher education use when discussing classroom knowledge for teaching mathematics. On this account, the mathematics knowledge for teaching framework of Ball and colleagues (Ball *et al.*, 2008) stood out as one of the most popular approaches recruited by many international and South African-based researchers. The framework categorized mathematics teaching knowledge into six sub-domains of knowledge. Knowledge of content and student (KCS) stood out as it is concerned with the interrelations between curriculum-prescribed mathematics knowledge and learners.

Following this approach to classroom observation, the researcher focuses on items such as teachers' knowledge of different methods typically used by learners in solving mathematical problems and promptness in locating where errors occur. As mentioned in Chapter 2, the PCK framework is theoretically vague, largely commonsensical and empiricist. Consequently, this creates a methodological problem in that it does not move beyond the surface features of the knowledge that passes for mathematics in classrooms.

As previously indicated, FP teachers tend to elaborate mathematics in a manner that exploits children's intuitive (core domain) understandings of computations with small sets of objects. The kinds of pedagogic strategies used by teachers might be thought of as entailing the implicit and explicit construction of mappings from structures formed by taking operations over finite sets to structures formed by taking operations over numbers. By using a computational approach in the analysis of mathematics lessons, the object that translates into the basic arithmetic activity is generated from the scriptural practices from which operational activities are examined (Davis, 2010b). From here, the computational domains and logics of operation are used as resources for discussing the mathematics knowledge of teachers as it relates to teachers' KCS.

3.4 Computational analyses

Studying the computational features of scriptural practices of teachers and learners provides an opportunity for one to draw conclusions regarding the pedagogic practice within the context (Davis, 2010b). In order to describe computational features, one has to examine the activity of teachers and learners. That is, what is realized as mathematics in a pedagogic setting depends on the criteria made available by the mathematical objects and operations in use (Davis & Johnson, 2007). This includes the 'stuff' enunciated and written by teacher and students, which is visible from their scriptural activities (i.e., verbal, written, and gestural).

3.4.1 Operations and domains of objects

To recognise the objects and operations from operational activity, Davis considers the workings, or rules, of operations, which are functions. Following Cantor's definition of a set, "we are to understand [a set as] any collection into a whole […] of definite and separate objects of our intuition or our thought. These objects are called the 'elements'" (Cantor 1952 p. 85). It follows that the fundamental idea for the composition of set is that of membership. Members of a set are referred to as elements and sets are defined by their elements. Thus, the sets of entities making up the domains and codomains of functions can be comprised of any objects, such as physical objects, numbers, or even concepts such as freedom or love (Gallistel & King, 2010a).

So, an operation is a function defined over a set of objects, and takes in one or more objects as inputs, referred to as its argument(s), and generates an output, referred to as its value. In elementary arithmetic, the basic operations used are usually binary, with arguments and values of the same type. For instance, if the arguments are sets, then so are the values. Likewise, if the arguments are numbers, then the values are numbers.

However, in pedagogic settings, the computational activities of teachers and learners may involve operations not normally found in the body of mathematics knowledge or may entail what Davis (2010a) refers to as *operation-like manipulations*, which are very often common procedures used to obtain solutions in school settings. As such, there are instances where the processes in which the arguments and values of the processes are of different types. For example, much of what happens in elementary school mathematics requires teachers and their learners to use the cardinalities of finite sets in their computations. In such instances, the arguments are usually discrete finite sets, and the values are natural numbers. There are also instances where the arguments are numbers, but the values are sets, or other types of mathematical objects, like line segments. In order to accommodate such instances, I use the terms *map* or *mapping* when referring to all processes that take in some arguments and generate some value. In other words, the terms *map* and *mapping* will be used to refer to the familiar basic operations as well as to functions where the arguments and values are of different types. Where one is referring only to a specific basic operation, the term *operation* will be used. Describing the operational activity of teachers and learners in terms of their objects and operations in pedagogic context allows for a comparison of the mathematical activity in the classroom with the objects and operations found in the mathematics body of knowledge referred to as the *mathematics encyclopaedia* by Davis (2011a). The mathematics encyclopaedia refers to the network of "formal rules, concepts and systems" described by Mac Lane (1986: 409), which underpin the presentation and development of mathematics.

3.4.2 Structures, structure preservation and representations

Krause (1969) has argued that the concept of *representation* is fundamental to all branches of mathematics because it allows us to connect a newly encountered system to a known system in ways that allow us to construct new knowledge systems. A morphism can be thought of as a *representation* of the mappings or functions that connect two systems, one of which is a *representing* system and the other, a *represented* system (Gallistel & King, 2011).

A *structure* is defined as a set with an operation or mapping defined over a set. For example, the natural number ($\mathbb{N}$) defined over addition is a structure which has properties such as commutativity, associativity, and closure. A structure will be indicated by the symbol $(S,*)$, where S is some set and * is some operation or mapping. Addition defined over the natural numbers will, for example, be indicated as a structure by use of the symbol $(\mathbb{N}, +)$. More generally, an object of the type $(A,\circ)$, where $\circ$ is an operation and A is a collection of objects that serves as the domain for ○, is referred to as a structure. A representation requires the articulation of at least two structures. The structures are related by mappings that associate the objects of one structure with that of the other, as well as the mappings internal to one structure (e.g., the operations) with those of the other.

As a general example, consider the sets A and B together with the binary operations ○ and *, where $(A,\circ)$ and $(B,*)$ are two structures of interest. Suppose, further, that there exists a mapping, f, such that for all a_i, $a_j \in A$, we have $f(a_i), f(a_j) \in B$. If it is the case that $f(a_i \circ a_j) = f(a_i) * f(a_j)$ then f is said to be a morphism that preserves structure from $(A,\circ)$ to $(B,*)$. In other words, a morphism links both the sets, A and B, as well as the binary operations, ○ and *, defined over the A and B, respectively (Baker, Bruckheimer & Flegg, 1971). Since it is the case that $f(A) \equiv B$, we can write $f{:}(A,\circ) \to (f(A),*)$. A representation thus consists of some mapping, f, together with structures $(A,\circ)$ and $(f(A),*)$ such that $f{:}(A,\circ) \to (f(A),*)$.

Moving to the level of the particular, we note that numbers and arithmetic operations used by teachers in pedagogic situations are very often collections of familiar objects and processes for manipulating such collections. In the foundation phase, counting is often used to map operations defined over the class of finite sets, like *disjoint union* and *relative complement*, respectively, to the operations of addition and subtraction defined over the natural numbers. In classroom situations, foundation phase teachers often make use of small collections of objects as resources to help learners grasp basic arithmetic operations (addition, subtraction, multiplication, division).[5]

Figure 3.1 shows a typical example that uses counting of collections of dolls to elaborate whole number addition. Here we have two structures that are to be related, one of which concerns disjoint union defined over finite sets, (FINSET,$\uplus$)[6], while the other concerns addition defined over ℕ, (ℕ,+). The learner is required to combine the two sets and then count the elements of the resultant set to arrive at a value for the computation 3 + 2.

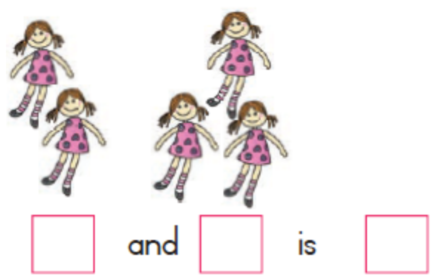

Figure 3.1: An implied mapping from (FINSET, $\uplus$) to (ℕ, +) (*Source*: DBE, 2020, p.54).

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[5] Note that the basic arithmetic operations are functions in two variables defined over suitable domains. For example, addition over the natural numbers is of the form $+{:}(x,y) \to z$, where $x,y,z \in \mathbb{N}$ (Davis, 2011a). ℕ will be taken to include 0 for convenience.

[6] The symbol FINSET will be used to refer to the class of finite sets; $\uplus$ is the symbol used to indicate the disjoint union of sets.

The mapping that associates (FINSET, ⊎) with (ℕ, +) is referred to as COUNT, which associates disjoint union with addition. The computation involving the sets in Figure 3.1 are to be represented by the computations involving the addition of natural numbers. The mapping from (FINSET, ⊎) to (ℕ,+) is shown diagrammatically in Figure 3.2.

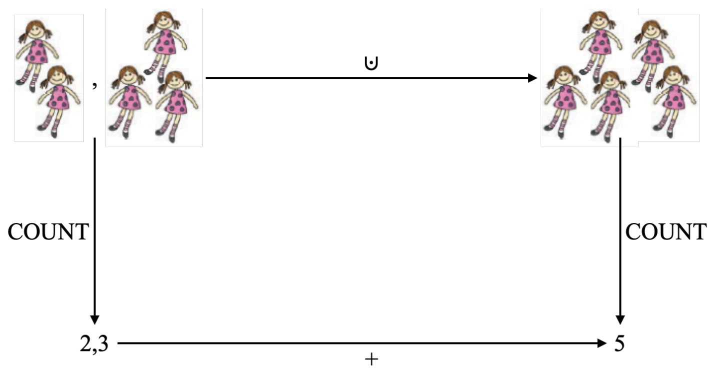

Figure 3.2: COUNT maps (FINSET, ⊎) to (ℕ, +).

We can write ⊎: FINSET×FINSET → FINSET and +: ℕ×ℕ →ℕ to indicate the general mappings associated with the structures (FINSET, ⊎) to (ℕ,+), respectively, and we write COUNT:(FINSET,⊎) → (ℕ,+) to indicate the general mapping from (FINSET,⊎) to (ℕ,+). The structure (FINSET, ⊎) is the *represented system*, while (ℕ, +) is the *representing system*.

3.5 The differences between *tuples*, *sequences* and *counts*

In foundation phase teaching, teachers often rely on morphisms as a way to recontextualise prescribed mathematics content for their learners. The general approach used for the teaching of basic arithmetic, for example, entails the use of counting to map the operations implicitly defined over finite sets to operations defined over natural numbers. From the early grades we often witness certain precursors of counting, such as rote recitation of number names and listing of numerals. These kinds of pedagogic activities, along with actual counting, entail the production and use of implicit and explicit mappings from various collection of objects to the natural numbers and vice versa. However, there are three very similar types of functions that are constituted through such processes, each of which can be described as a mapping from a subset of the natural numbers to some or other set:

(i) *n-tuples,* which are finite ordered lists, like, for example, a tuple of cartesian coordinates, or the initial list of number words.

For example, let A = {"One", "Two", "Three", …, "Twenty"} and $T_{20} = \{x \in \mathbb{N} \mid 1 \leq x \leq 20\}$, then the function $t{:}T_{20} \rightarrow A$ defines a 20-tuple on A. More generally, $t{:}T_n \rightarrow A$, where A is a set of n elements, defines an n-tuple on A.[7]

(ii) *Rule-bound sequences*, finite or infinite, which are lists of things having a rule enabling the production an element of a list.

For example, let $A = \{x \in \mathbb{N} \mid 1 \leq x \leq 20, x \text{ even}\}$, then the function $s{:}\mathbb{N} \rightarrow A$ defines a rule-bound sequence on A, specifically, 2, 4, 6, 8, …, 18, 20. More generally, $s{:}\mathbb{N} \rightarrow A$, where A is some set having a membership rule, defines a rule-bound sequence on A.

(iii) *Counts*, which describe sets in terms of their cardinality.

Let A be some set and $\mathbb{N}(n) = \{x \in \mathbb{N} \mid 1 \leq x \leq n\}$ with $\mathbb{N}(0) = \varnothing$ (the empty set), then the bijective function $c{:}\mathbb{N}(n) \rightarrow A$ defines a count on A, and n is the cardinality of A. For example, in Figure 3.2 we see the count function used three times. If we refer to the sets in Figure 3.2 as A_1, A_2 and A_3, reading left to right, then we see that $c{:}\mathbb{N}(2) \rightarrow A_1$, $c{:}\mathbb{N}(3) \rightarrow A_2$ and $c{:}\mathbb{N}(5) \rightarrow A_3$ have been used in the computation, as demonstrated in Figure 3.3.

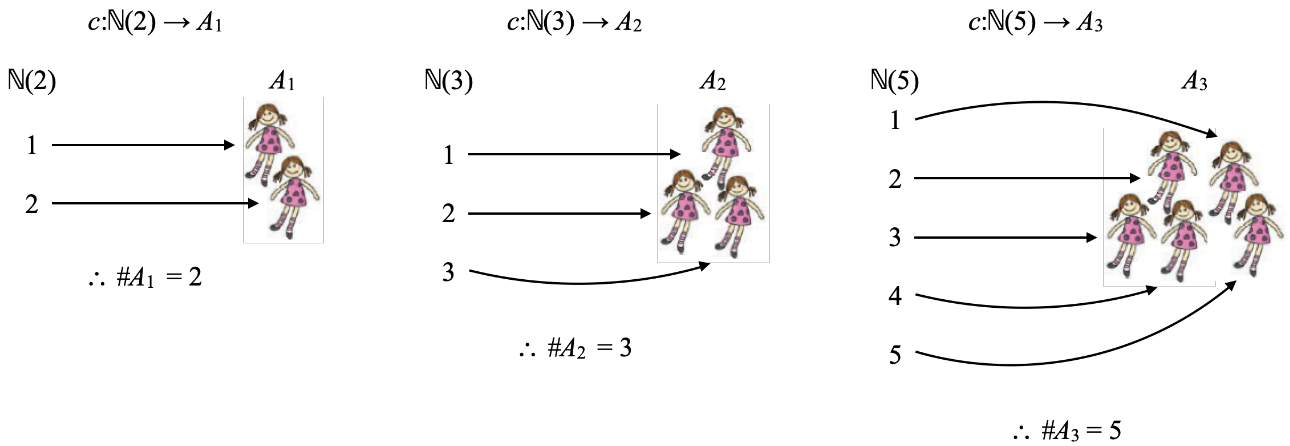

Figure 3.3: Counts on the sets in Figure 3.2.

Gellman & Gallistel (1978) present a cognitive scientific description of the process of learning to count, which they refer to as *counting principles*. For them, the attention of children is drawn to assigning only one counting word to each element in a particular collection (one-one principle) following a stable order (e.g., one, two, there and so on), where there is an implicit partitioning of objects counted from those that are yet to be counted. That is, knowledge of an adequate list of the counting order of number word (stable order principle). Next, young children recognise that the final word called out from the list registers the number of objects in a

[7] Counting-out rhymes, like *One, Two, Buckle My Shoe*, are often used to establish an initial list of number words as an ordered list. Published around 1805 and numerous times since, the rhyme circulated in oral cultures well before publication (Hahn & Morpurgo, 2015; Opie & Opie, 1952).

collection (cardinal principle). This activity is referred to as *counting.* Then, as children become more exposed to non-concrete objects, they recognise that the first three principles can be applied to any collection of things (abstraction principle) in any order (order-irrelevance principle) to generate a count on a collection of things. However, in foundation phase classrooms, there are situations where teachers and learners use the terms *count* and *counting* when performing number related activities, yet actual counting may not have occurred. However, through such pedagogic activity, children acquire knowledge of the sequence of the counting and natural numbers. Later they recognise that by adding 1 successively the order for counting number is formed mathematically by what is referred to as the *successor function*, which has the natural numbers as domain and codomain.

The distinctions drawn here between tuples, sequences and counts are crucial, and will be used as part of the formulation of an analytic framework in Chapter 4.

3.6 Summary: cognitive science, computational analyses and KCS

From cognitive science we see that the human brain is biologically endowed with core, skeletal computational resources that need to be taken into account when considering learning. Further, a central cognitive learning resource consists in the extension of computational resources through the use of structure-preserving mappings. It follows that the idea of KCS needs to be concerned not only with prescribed curriculum content and local computational traditions, but also with the implications of biological endowment if it is to be used in observationally and descriptively adequate ways. Failing that, only weak explanatory adequacy can be achieved.

The notion of computational structure-preservation can be used to relate core domain systems/knowledge, curriculum content and learner/teacher computational activity in order to enhance descriptions and analyses of the teaching and learning of foundation phase mathematics, and so of KCS. An analysis of pedagogic evaluation as it unfolds in the teaching and learning of mathematics provides insights into the semantic bases of the syntactical resources used and prioritised in pedagogic settings, thereby illuminating additional features of KCS in practice.

In this work, an important methodological difference from that of Ball and her colleagues is that KCS and related ideas are used anthropologically. That is, the position taken here is that KCS *is always present*, while for Ball and others, teachers possess KCS to lessor or greater degrees. Ball's position is, essentially, prescriptive, based as it is on an *a priori* model of good teaching practice. In contrast, the position elaborated in this work is descriptive and is concerned with how the computational features of pedagogic evaluation interact with core domain knowledge in the teaching and learning of school mathematics.

CHAPTER 4

Analytic framework

4.1 Introduction

The interest of this study is to describe the number related knowledge of Grade 3 mathematics teachers specific to the use of counting. This chapter presents the research design and procedures used to generate and analyse data. In Chapter 5 and Chapter 6, the propositions and procedures followed to examine the mathematics teaching knowledge of the two teachers examined in this study is put to work in order to address the research question.

4.2 Research design

To follow a systematic approach, there is need for a research design that affords descriptively adequate readings of the lessons examined in this study. McMillan & Schumacher (2010) describe a research design as the articulation of procedures for conducting a study such that it includes when, from whom, and under what conditions the data is obtainable. The design of a study functions as a blueprint detailing what needs to be done and how it will be conducted. It involves the application of a chosen method to address the research question(s).

Given (i) that the computational practices of pedagogically communication between teachers and their learners serve as information for data production, and (ii) analyses of computational practices require detailed descriptions, one effect of which is the generation of a great deal of data, a case study research approach was deemed suitable for this study given the word count restrictions. As Isaac & Michael (1995, p.46) point out, the purpose of case and field studies is to "study intensively the background, current status, and environmental interactions of a given social unit: an individual, group, institution, or community". According to Yin (2009), the *case* is identified as a unit, entity, or phenomenon with defined boundaries that the researcher can demarcate or "fence in" thereby determining the boundary of the study (ibid., p.27). Similarly, for Merriam (1998, p.27), the "single most defining characteristic of case study research lies in delimiting the object of study: the case".

An advantage of using a case study approach to research is that different methods are exploitable, and it is dependent on the methodological resources recruited (Yin, 2009). However, using a case study method is subject to the objection that events that are prevalent in one pedagogic situation may be rare in a different pedagogic situation. As such, the findings are not immediately generalizable to the practices of other teachers in similar contexts. What case study research does, however, enable one to do is to generate non-superficial hypotheses that can be investigated across larger, statistically significant, samples using other methods. One of the most significant difficulties encountered by the researcher is the construction of a productive research problem, and case study research with its detailed focus on the features of phenomena is helpful in that regard.

4.2.1 The cases

Two primary schools situated in working class communities in Western Cape were selected as the cases for this study. The schools participated in the *Schools Performing Above Demographic Expectation* (SPADE) project of the University of Cape Town's School of Education. The SPADE project selected fourteen schools situated in low-income communities in Western Cape (Hoadley, 2017; Hoadley & Galant, 2014). The two cases examined in this study were randomly selected from the SPADE project archive. The schools are situated in Paarl and Oudtshoorn, populated by learners known to come from homes with few economic resources, with parents mainly in low-status employment according to census data collected by the City of Cape Town. The schools were formerly administered by the apartheid era House of Representatives for so-called "coloured" children.

Literature on South African education informs us that the effects of the apartheid system are still very much in evidence in South African schools situated in working-class communities. As a group, students in such schools are known to perform poorer in tests of literacy and mathematics than do their counterparts in schools serving middle-class families. Some argue that teachers in such schools hold limited mathematics content knowledge for teaching and that classrooms are, typically, poorly resourced (Schollar, 2008; Spall, 2013; Spaull & Kotze, 2015; Taylor & Taylor, 2013). However, the classrooms of teachers selected for this study were colourful, displaying a variety of educational posters and resources. The classroom spaces were also well kept, with sufficient texts, desks, chairs, materials and mats to accommodate all learners.

4.3 Procedures for the production and analysis of data

This study hopes to illuminate the computational activities of teachers and learners. The purpose is to describe and analyse the computations used by Grade 3 teachers for teaching young children number-related content, specifically, the uses of counting. A computational approach was chosen for this investigation because it supplies a means for one to use mathematical resources as tools for examining the mathematics knowledge used by classroom teachers.

4.3.1 Information gathering

The observed lessons were video recorded. Field notes were taking and interviews with teachers were audio recorded by a SPADE researcher. The camera was located at the back of the classroom and captured the activities of teacher and learners but was focussed on the teacher for most of the time. From the video records, teachers' and learners' verbal and non-verbal communication were transcribed and annotated to include information on gestures and written work. Where necessary, communications were translated from Afrikaans to English and an Afrikaans-English bilingual mathematics educator reviewed the transcripts and the video records. From this, the transcripts shown in Appendices I and II were revised and finalised.

Chapter 3 stated that the computational features of the scriptural practices of teachers and learners provides an opportunity for one to draw conclusions regarding the semantic bases of computation in pedagogic situations

(Davis, 2010b). To describe computational features, the stages listed below guided the systematic examination of the activities of teachers and learners from scriptural activities.

Stage 1: Segment the lesson into *evaluative events* by specifying the topic of each event and the time spent addressing the topic.
Stage 2: Describe the syntactical features of computational activity within an event.
Stage 3: Deduce the semantic bases of computational activity from the analysis of computational syntax.

4.3.2 Segmenting lessons into evaluative events

Davis (2003) developed the notion of an *evaluative event* from his engagement with the work of Badiou, Bernstein, Freud, Hegel and Lacan, and in no way from the work of Shulman and Ball (as might be mistakenly concluded from some presentations of the work by Adler). The procedure for segmenting pedagogic activity into evaluative events begins with recognising the presentation of a specific content in some initial form and concluding with the presentation of the realisation of the content in a (possibly provisional) final form. The evaluative event is used to partition records of pedagogic situations into sections that are homogeneous with respect to the mathematical topic and the particular type of activity that participants in the pedagogic situation are engaged in (Davis, 2011a).

Evaluative events may be partitioned into sub-events, which are identified when teachers digress from the topic because of misunderstanding or for some other reason—an interruption in the pedagogic encounter—or dialogue entered into to assist learners in acquiring content related to the evaluative event (Davis, 2011a). In addition, sub-events may be used to indicate a change in the pedagogic activity of the teacher and/or learners. Lessons are segmented into series of evaluative events by marking out segments of pedagogic activity, starting from the introduction of particular mathematical content to the introduction of a new content, or the end of activity. A new evaluative event or sub-event is marked by the introduction of a different content.

Once the lessons have been segmented into evaluate events, the next step in the analysis is a description of the computations employed as well as the computational domains used (Davis, 2011a). The framework of Davis (2018) provides a useful protocol for specifying the domains (input and output sets) used.

4.3.3 Describing operations and morphisms of computational activity in mathematical terms

The protocol for the mathematical description of computational activity is as follows:

(i) One starts with an initial identification of the action(s) performed and of the material (objects, expressions) that are subjected to the action(s). Specifically, one considers the input of an action as well as its output.
(ii) One then describes the type of material acted upon; that is, the type of its inputs and outputs.

(iii) A description of the type of material acted upon, along with an analysis of the inputs and outputs of action, enables an initial description of the syntax of an action.

(iv) The syntax of an action is then redescribed in mathematical terms as an operation or mapping. All computationally distinct actions are described in mathematical terms that indicate the type of input (domain), the type of output (codomain) and a name for the action qua operation/mapping. Where necessary, the composition of operations/mappings is described. The term *composition* refers to the chaining of operations/morphisms, where the codomain of one operation is taken as the domain of the next operation/morphism in a chain of computations.

(v) The analysis of computational activity is summarized in mathematical notation as well as diagrammatically, in the form of diagrams that use arrows to indicate operations/morphisms and illustrations and/or symbols to indicate inputs and outputs. See Figure 3.2 and Figure 4.1 for examples.

(vi) Computational structures are identified by taking an operation/morphism together with its domain(s). The properties of computational structures are listed when it is productive to do so.

(vii) In instances where a computational link between one structure and another is employed, the mapping effecting the link is described. Mappings between structures are examined for structure preservation.

4.3.3.1 Fundamental mappings for number work and counting

Computations referred to in the use of *count* and *counting* by teachers exploit references to finite collections of things believed to be familiar to young children, so that finite sets are implicitly being referenced (Davis 2018, p.6). Figure 4.1 shows a series of interconnected computational relations believed to occur in FP classrooms when treating concepts on numbers. Relations between aggregates, natural numbers, number words and number symbol are kinds of computational objects typically used by FP teachers when teaching number related concepts.

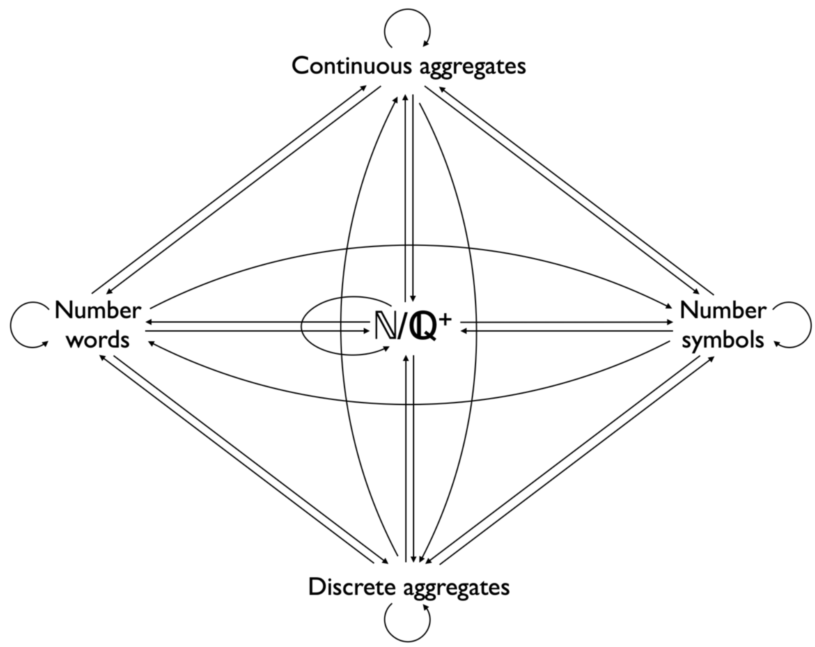

Figure 4.1: Relations between aggregates, numbers, number words and symbols (*Source*: Davis, 2018).

Table 4.1: Descriptions of the mappings indicated in Figure 4.1 (*Source*: Davis, 2018)

IDENTIFIER	DOMAIN	CODOMAIN	MAPPING
CAGWRD	continuous aggregates	number words	CAGWRD: CAG → WRD
CAGSYM	continuous aggregates	number symbols	CAGSYM: CAG → SYM
CAGNUM	continuous aggregates	numbers	CAGNUM: CAG → NUM
CAGDAG	continuous aggregates	discrete aggregates	CAGDAG: CAG → DAG
CAG*	continuous aggregates	continuous aggregates	CAG*: CAG → CAG
DAGWRD	discrete aggregates	number words	DAGWRD: DAG → WRD
DAGSYM	discrete aggregates	number symbols	DAGSYM: DAG → SYM
DAGNUM	discrete aggregates	numbers	DAGNUM: DAG → NUM
DAGCAG	discrete aggregates	continuous aggregates	DAGCAG: DAG → CAG
DAG*	discrete aggregates	discrete aggregates	DAG*: DAG → DAG
WRDCAG	number words	continuous aggregates	WRDCAG: WRD → CAG
WRDNUM	number words	numbers	WRDNUM: WRD → NUM
WRDDAG	number words	discrete aggregates	WRDDAG: WRD → DAG
WRDSYM	number words	number symbols	WRDSYM: WRD → SYM
WRD*	number words	number words	WRD*: WRD → WRD
SYMCAG	number symbols	continuous aggregates	SYMCAG: SYM → CAG
SYMNUM	number symbols	numbers	SYMNUM: SYM → NUM
SYMDAG	number symbols	discrete aggregates	SYMDAG: SYM → DAG
SYMWRD	number symbols	number words	SYMWRD: SYM → WRD
SYM*	number symbols	number symbols	SYM*: SYM → SYM
NUMCAG	numbers	continuous aggregates	NUMCAG: NUM → CAG
NUMSYM	numbers	number symbols	NUMSYM: NUM → SYM
NUMDAG	numbers	discrete aggregates	NUMDAG: NUM → DAG
NUMWRD	numbers	number words	NUMWRD: NUM → WRD
NUM*	numbers	numbers	NUM*: NUM → NUM

The series of relations presented in the Figure 4.1 is shown by means of arrows, where each arrow indicates a mapping from some domain (set of inputs) to some codomain (set of possible outputs). The computational relations between the collections of entities can be thought of as mappings that take as input one or more elements of a particular aggregate which are associated by means of some process to one or more entities in the same aggregate or to entities in some other aggregate. In other words, the aggregate from which input entities are taken is called the *domain* of a mapping and the aggregate from which output entities are taken is called the *codomain*. Table 4.1 lists all the mappings. The term *range* can also be used in analyses to refer to the specific elements of the codomain that result from the use of a mapping. The small, discrete sets that are routinely used can be thought of as elements of the class of finite sets, which will be indicated by the symbol FINSET.

Two kinds of aggregates are identified, *discrete* and *continuous*. Discrete aggregates are indexed by DAG; continuous aggregates, by CAG. Numbers are indexed by NUM, number words by WRD, and number symbols by SYM. The reference symbols (DAG, CAG, NUM, WRD, SYM) allow one to distinguish the elements of one collection from another by referring to their types. When referring to number words and number symbols, forward slashes are used, as in /two hundred/ and /200/. When referring to numbers, the usual number word or symbol is used (i.e., without forward slashes). For example, *3* and *three* refer to the concept three; /3/ refers to the number symbol for 3; and /three/ refers to the number word for 3. More generally, whenever a symbol or word is placed between forward slashes it is the symbol or word that is indicated rather than its meaning. This awkward set of distinctions is necessary because of the problem of reference discussed in Chapter 3.

The data generated by application of (i) - (vii) are used to produce an account of the semantic bases of computational activity by relating that activity to what we know of biologically endowed cognitive resources in humans. Further, the outcome of the process described above affords us some insight into the specific KCS informing the practice of a teacher, thus enabling a move beyond commonsensical and prescriptive uses of that construct.

4.3.3.2 Currying (Schönfinkelisation)

Any operation, function or mapping has some input (the argument(s)) and output (the value). For example, the basic arithmetic operations (addition, multiplication, subtraction and division) are binary, which means that each has two arguments as input. Using the notation employed here, a binary computation like $2 + 3 = 5$ will be written as $+: (2,3) \rightarrow 5$. The operation can be rendered as unary by taking the second argument, 3, as part of the operator, +, thus producing a new operator, (+3). The computation now becomes $(+3): 2 \rightarrow 5$. The process of effecting a transformation from $+: (2,3) \rightarrow 5$ to $(+3): 2 \rightarrow 5$ is referred to as *currying* or *schönfnkelisation*, named for Haskell Curry (1900 - 1981) and Moses Schönfinkel (1888 - 1942), respectively. The term *currying* is the more common of the two and would be familiar to those who know of, or use, functional programming languages, like *Haskell*, or are familiar with the lambda calculus or typed formal languages.

Currying is very often used implicitly by FP teachers in the teaching of elementary arithmetic (Davis, 2018), but the phenomenon generally goes unrecognized by mathematics education researchers. It is for this reason that the brief account of currying presented here is included. We can expect to see numerous instances of currying when analysing FP mathematics lessons.

4.4 Reliability

Reliability refers to the extent to which a test, method or tool yields consistent results across a range of settings if used by researchers in different contexts and at different times (Lincoln & Guba, 1985; Merriam, 1998; Opie, 2004), specifically the extent to which a piece of research can be replicated to yield similar results in a different context of the same type, conducted by different researchers. The reliability of the method used in this study rests on the mathematical description of computational activity, which attempts to observe the demands of observational and descriptive adequacy. One of the foundational premises of this study is that the mind/brain is computational, a well-founded result of contemporary cognitive science which directs us to construct a computational description of the teaching and learning of school mathematics. The descriptive principles are very explicit and unambiguous and so can be assessed by others relatively easily.

A difficulty with reliability that one is obliged to confront is generated by the fact that the rules used to associate the input with the output of any operation, function or mapping can, in principle, vary infinitely. An elementary example of such variation is immediately obvious from the possibility of substituting an operation by a curried version of the operation. A slightly more complex example of rule variation can be seen in a transformation like $[f(x) = x^2 + 2x + 1] \rightarrow [f(x) = (x + 1)^2]$ where, for any given number x, the rules $x^2 + 2x + 1$ and $(x + 1)^2$, while different, nevertheless produce the same output value (Lawvere & Schanuel, 1997, 22–23).

The ineradicable potential for rule variation therefore requires of the researcher a sensitivity to such variations when attempting to describe the computational activity of research subjects. It follows from this observation that it is often necessary for the researcher to produce more than one analytic description of computations in order to meet the demand for observational and descriptive adequacy.

4.5 Validity

In qualitative research, there are various types of validity of interest; namely, descriptive and interpretive validity, and theoretical and explanatory validity (Maxwell, 1992). Descriptive validity requires accuracy in reporting facts, while interpretive validity requires accuracy in the interpretation of the facts found (Maxwell, 1992). Both descriptive and interpretive validity is achievable through careful, principled transcription and interpretation. The analysed lessons were video recorded, and the speech and learners and teachers were transcribed. As indicated earlier, where translations from Afrikaans to English were used, they were checked for accuracy by an established bilingual mathematics educator.

Theoretical and explanatory validity is achieved by establishing a systematic link between the theoretical and analytic frameworks of a study, as required by Maxwell (1992). The use of an ICM enables one to draw on resources from whichever fields of knowledge are pertinent to construct explanatory accounts of phenomena. That said, one has to acknowledge the potential of needing to revise explanations in future as knowledge of mind/brain activity becomes more robust, but that is an ineradicable component of all scientific endeavour.

4.6 Research ethics

The main ethical consideration in case study research is protecting the confidentiality and anonymity of the participants. Where images are shown of classroom activity, the faces of research subjects have been pixilated or blurred to ensure anonymity. This study adheres to the guidelines as set out by The University of Cape Town code of ethics for research. In addition, all proposal material for this study submitted for ethics clearance was approved by the School of Education Ethics Committee. The ethics clearance document is presented in the Appendices.

CHAPTER 5

Analysis: Lesson 1

5.1 Introduction

This chapter presents the production of data for the Lesson 1, using the protocol outlined in Chapter 4. Stages 1 and 2 of the analytic protocol will be carried out, with Stage 3 left to Chapter 7.

5.2 Production and analysis of data

Recall that the production of data and the analysis of lessons proceeds in three stages:

Stage 1: Segment the lesson into *evaluative events* by specifying the topic of each event and the time spent addressing the topic.
Stage 2: Describe the *syntactical features* of computational activity within an event.
Stage 3: Deduce the *semantic bases* of computational activity from the analysis of computational syntax.

5.2.1 Stage 1: Lesson 1 evaluative events (Teacher 1)

Lesson 1 began with the teacher (referred to as Teacher 1 from now on) instructing her learners to locate a specific number symbol on their individual number charts by pointing at it with a finger. Each learner had a booklet of number charts, spanning the number symbols from /1/ to /1000/.

Figure 5.1: Learners using a booklet of number charts listing number symbols from /1/ to /1000/.

Sitting on a mat in front of the classroom, each learner was required to point at a number symbol on the charts as instructed by Teacher1 (see Figure 5.1). The sequence of instructions was for learners to count in twos, then threes, twenties, fifties, and hundreds, starting from a number symbol announced by the teacher. Learners performed the tasks by paging to the appropriate chart, locating the number symbol as specified by Teacher 1, and then calling out the number words associated with the number symbols as they pointed at the number symbols. Table 5.1 details the different activities referred to as /counting/. The evaluative event concluded at 07:16.

Table 5.1: Evaluative events and sub-events referring to /counting/ in Lesson 1.

EE	SE	Content	Location	Duration
1	1.1	Everyone /counts/ in twos from /202/ to /220/	00.08-01:14	1' 06"
	1.2	Boys /count/ backwards in twos from /220/ to /202/	01:14-01:50	36"
	1.3	Girls /count/ in threes from /303/ to /330/	01:50-02:36	46"
	1.4	Boys /count/ backwards in threes from /330/ to /303/	02.36-03.10	34"
	1.5	Everyone /counts/ in twenties from /400/ to /500/	03.10-04.00	50"
	1.6	Everyone /counts/ backwards in twenties from /500/ to /400/	04:00-04:19	19"
	1.7	Everyone /counts/ in fifties from /300/ to /800/	04:19-06:35	2' 16"
	1.8	Everyone /counts/ in hundreds from /100/ to /1000/	06:35-07:16	41"
			Total	7' 08"

5.2.2 Stage 2: Computational syntax analysis

Teacher 1 began SE 1.1 by instructing learners to place a finger on /200/ and /count/ in twos up to /220/. In response, a number of the learners indicated that the number symbols on the number chart listing /220/ did not start with /200/, but rather with /201/. Consequently, Teacher 1 instructed her learners to start from /202/. (See Extract 5.1)

1. Teacher: Put your finger for me on /two hundred/. /Count/ for me in twos. Go from /two hundred/ up to /two hundred and twenty/. Everyone, together.
2. Learners: /Two hundred/.
3. Teacher: Put your finger on /two hundred/!
4. Learner: We don't have /two hundred/.
5. Teacher: /Two hundred and two/.
6. Learners: /Two hundred and two/. (Teacher 1 and the learners call out the number words in unison.) /Two hundred and four/, /two hundred and six/, /two hundred and eight/, /two hundred and ten/, /two hundred and twelve/, /two hundred and fourteen/, /two hundred and sixteen/, /two hundred and eighteen/, /two hundred and twenty/.

Extract 5.1: Teacher-learner interaction in SE 1.1.

By using the protocol detailed in Section 4.3.3, the syntactical aspects of the computations required by the task can now be described.

(i) Setting aside the absence of the number symbol /200/ from the number charts, the initial action demanded by Teacher 1 entails a mapping from the number word /two hundred and two/ (the input) to the number symbol /202/ (the output).

(ii) The input and output types of the initial mapping are, respectively, number words and number symbols. The number words and symbols listed in the number charts index the natural numbers—indicated by the usual symbol, ℕ—so that the number words might be indicated by the symbol /$\mathbb{N}^w$/ and the number symbols by /ℕ/.

(iii) Syntactically, we have a mapping of the form /$\mathbb{N}^w$/ → /ℕ/, where /$\mathbb{N}^w$/ is its domain and /ℕ/ its codomain.

(iv) Referring to Table 4.1, the initial mapping in use is WRDSYM: /two hundred and two/ → /200/, which is an instance of the general type WRDSYM: WRD → SYM. To fix the type more specifically, we can indicate that we are dealing with a mapping of the type WRDSYM: /$\mathbb{N}^w$/ → /ℕ/, which indicates the specific domain and codomain types. Moving beyond the start of the computational activity as initiated by Teacher 1, the subsequent general mapping employed recursively is SYMWRD: SYM → WRD. Specifying the type more precisely, we have SYMWRD: /ℕ/ → /$\mathbb{N}^w$/, with specific domain {/202/, /204/, …, /220/} and range {/two hundred and two/, …, /two hundred and twenty/}.

One further detail is required to describe the series of computations and is concerned with the generation of the specific domain and range of the series. With respect to the domain, the question is one of explaining how the learner is to get from /202/ to /204/ to /206/ and so forth. The instruction given by Teacher 1 is for learners to /count/ in twos by referring to their number charts.

The number charts are *n*-tuples, showing ordered lists of number symbols that serve as base material for the series of computations to be performed. It is here that counting proper enters. After locating a particular number symbol like /202/, for example, the learner implicitly constitutes a discrete aggregate of cardinality 2 by using the two elements of the *n*-tuple immediately following on from /202/, viz., {/203/, /204/}, thereby employing the mapping NUMDAG: NUM → DAG (refer to Table 4.1). The mapping can be made contextually precise by rewriting it as NUMDAG: 2 → {/203/, /204/}, the internal machinery of which entails counting to constitute an aggregate of the necessary cardinality. Of course, implicit checks of the correctness of the constituted aggregate may well use DAGNUM: DAG → NUM as well.

To complete this part of the analysis we need to fashion a mapping that captures the necessary computational details more concisely. Such a mapping would need two bits of information as input, one of which is an initial element of the *n*-tuple of interest, while the other is the number of subsequent elements of the *n*-tuple. More precisely, the mapping requires an initial number symbol, a strict observance of *n*-tuple order, and the cardinality of an ordered aggregate. The last element of the ordered aggregate is its penultimate output. The final output is produced by the use of SYMWRD. Since the effect of the mapping is one of *location* of a number symbol which is then mapped to a corresponding number word, it is helpful to refer to it as LOCATE. In general terms, we define LOCATE: (/$\mathbb{N}$/,$\mathbb{N}$) → /$\mathbb{N}^w$/ to do the core computational work required.

In the example under discussion, LOCATE: (/202/,2) → /two hundred and four/ is used, followed by WRDSYM: (/two hundred and four/) → /204/. The number symbol /204/ then becomes an input value to the next iteration of LOCATE and so on. In practice, it is more appropriate to use a curried form of LOCATE in which the second argument of the input—that is, the required cardinality of the target aggregate—is used as though it were part of the mapping, with the first argument—the number symbol—serving as the only input. In other words, we rewrite LOCATE: (/$\mathbb{N}$/,$\mathbb{N}$) → /$\mathbb{N}^w$/ as LOCATE(n): /$\mathbb{N}$/ → /$\mathbb{N}^w$/, where is a natural number. Specifically, LOCATE: (/202/,2) → /two hundred and four/ is replaced by LOCATE(2): (/202/) → /two hundred and four/.

(v) The chain of computations entailed in the task can be depicted by the arrow diagram in Figure 5.

Figure 5.2: Arrow diagram for the series of computations used in SE 1.1.

Having described the computational activity, we do need to ask whether the computational data might be explained in a different way. The answer to that question is: “Yes”. It is very possible that the arrow marked as LOCATE entails addition rather than counting. Figure 5.3 shows an arrow diagram that uses addition rather than LOCATE and counting.

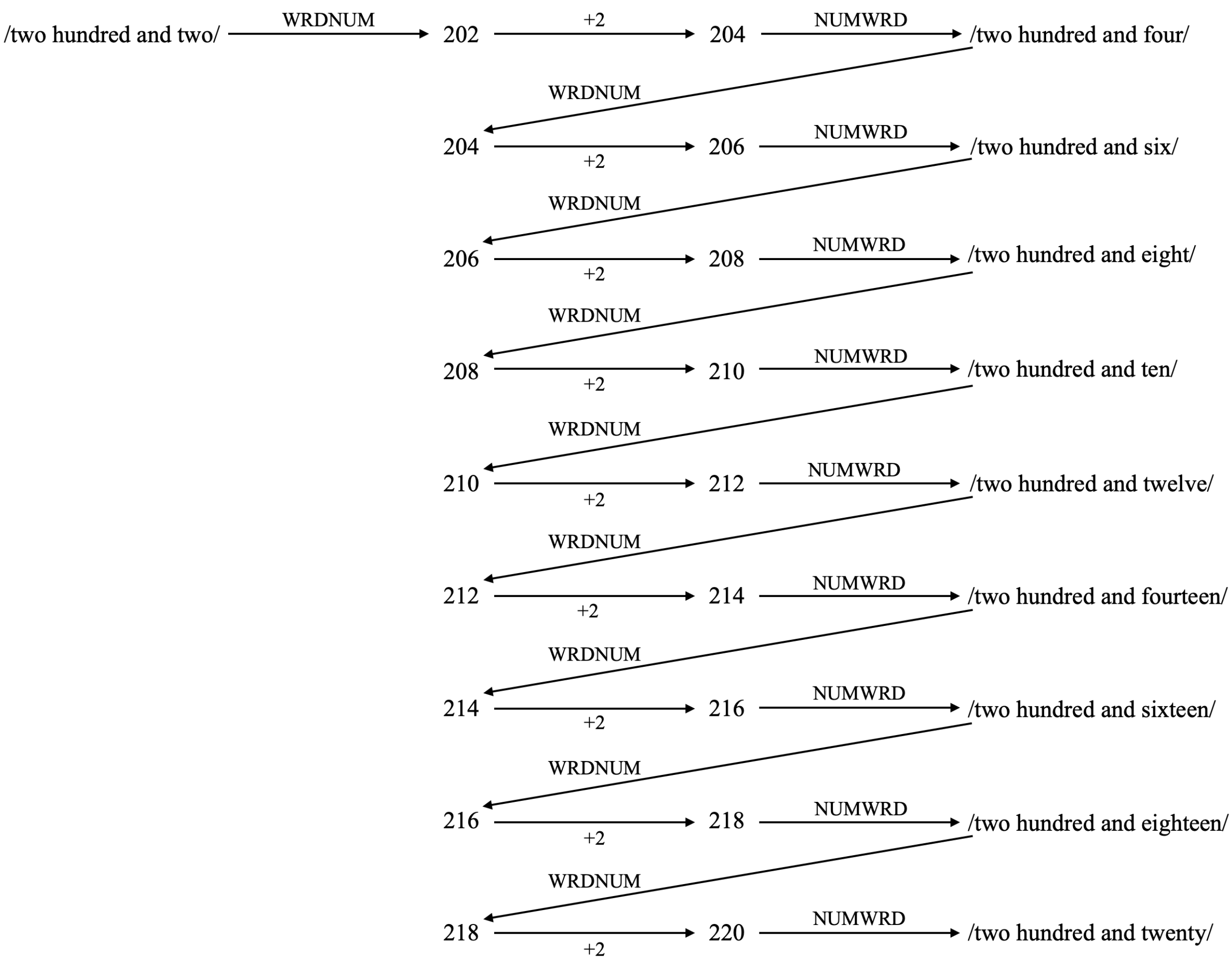

Figure 5.3: Alternate arrow diagram for the series of computations used in SE 1.1.

LOCATE, WRDSYM and SYMWRD might be retained to deal with the requirement that the learners point at the number symbols listed in the charts. If that is done, the arrow diagram shown in Figure 5.3 would have to be altered. However, from the video record it appears to be the case that very many of the learners do not point at the number symbols as they call out number words. Unfortunately, the SPADE data archive did not include interviews with learners, so one cannot know with absolute certainty that they were counting rather than adding or vice versa, or even using a combination of both strategies.

What is clear is that Teacher 1 required explicit references to number symbols and number words as part of the task, so that Figure 5.2 captures the intent of the task. For many of the learners it may well have been the case that Figure 5.3 is a more accurate depiction of actual learner activity.
In summary, the mappings of the computational procedure using LOCATE are as follows:

A.

WRDSYM: $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$

NUMDAG: $\mathbb{N} \rightarrow$ FINSET (implicit to LOCATE)

DAGNUM: FINSET $\rightarrow \mathbb{N}$ (implicit to LOCATE)

LOCATE: $(/\mathbb{N}/,\mathbb{N}) \rightarrow /\mathbb{N}^w/$ or LOCATE(n): $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$, where $n \in \mathbb{N}$ (curried version).

SYMWRD: $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$

The following mappings cover the procedure that uses addition:

B.

WRDNUM: $/\mathbb{N}^w/ \rightarrow \mathbb{N}$

+: $(\mathbb{N},\mathbb{N}) \rightarrow \mathbb{N}$ or $(+n)$: $\mathbb{N} \rightarrow \mathbb{N}$, where $n \in \mathbb{N}$ (curried version).

NUMWRD: $\mathbb{N} \rightarrow /\mathbb{N}^w/$

(vi) Referring to the task as intended by Teacher 1, much of the computational activity of SE 1.1 is concerned with computations on aggregates of number words and symbols. The aggregates that are subjected to counting proper are small, with each containing two elements. The mappings DAGNUM and NUMDAG include counting and are implicitly incorporated into LOCATE.

The entire computational procedure rests on 1-1 correspondences between number words and number symbols, number words and numbers, and number symbols and numbers, in a manner that preserves *order*. Structure preservation with respect to order is crucial to the success of the computations, hence the importance of the use of n-tuples of number symbols (number charts) and number words. Here we should recall that n-tuples are ordered lists, and that the particular n-tuples used are, in part, set by linguistic and writing conventions, but ultimately derive from and are over-determined by numerical order. Numerical order is encoded into the n-tuples of number words and number symbols through the use of bijective mappings (1-1 correspondences).

Three *computational structures* (refer to Section 3.4.2) of central importance to the task are ($\mathbb{N}$,Order), ($/\mathbb{N}/$,Order) and ($/\mathbb{N}^w/$,Order), even though they remain implicit since order is never explicitly addressed in the lesson.

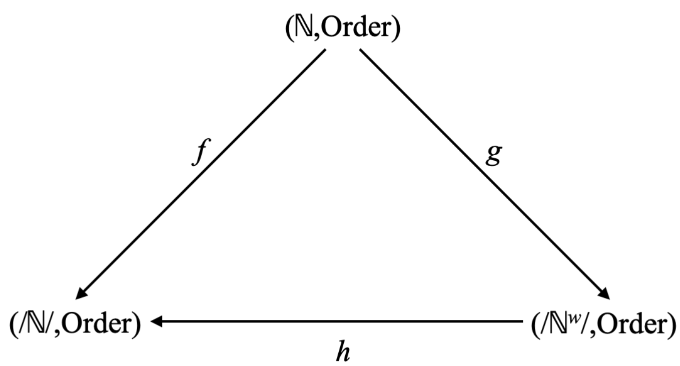

Figure 5.4: *f*, *g* and *h* are bijective mappings ensuring structure preservation with respect to order.

Figure 5.3 depicts the structure preservation relations between the three structures by way of the bijective mappings *f*, *g* and *h*. Since *f*, *g* and *h* are bijective, there exist three inverse mappings f^{-1}, g^{-1} and h^{-1}, each of which can be depicted by reversing the arrows in Figure 5.3.

Each of the sub-events of EE 1 can be described in a manner similar to that of SE 1.1 since the procedure remains the same for each task. SE 1.2, SE 1.4 and SE 1.6 require the learners to /count/ backwards, which means that subtraction may have been used in those tasks by some learners. The latter possibility can be covered by a procedure consisting of the following series of mappings:

C.

WRDNUM: $/\mathbb{N}^w/ \rightarrow \mathbb{N}$

-: $(\mathbb{N},\mathbb{N}) \rightarrow \mathbb{N}$ or (-n): $\mathbb{N} \rightarrow \mathbb{N}$, where $n \in \mathbb{N}$ (curried version).

NUMWRD: $\mathbb{N} \rightarrow /\mathbb{N}^w/$

Taking SE 1.2 as an example, the following arrow diagram describes the computational activity:

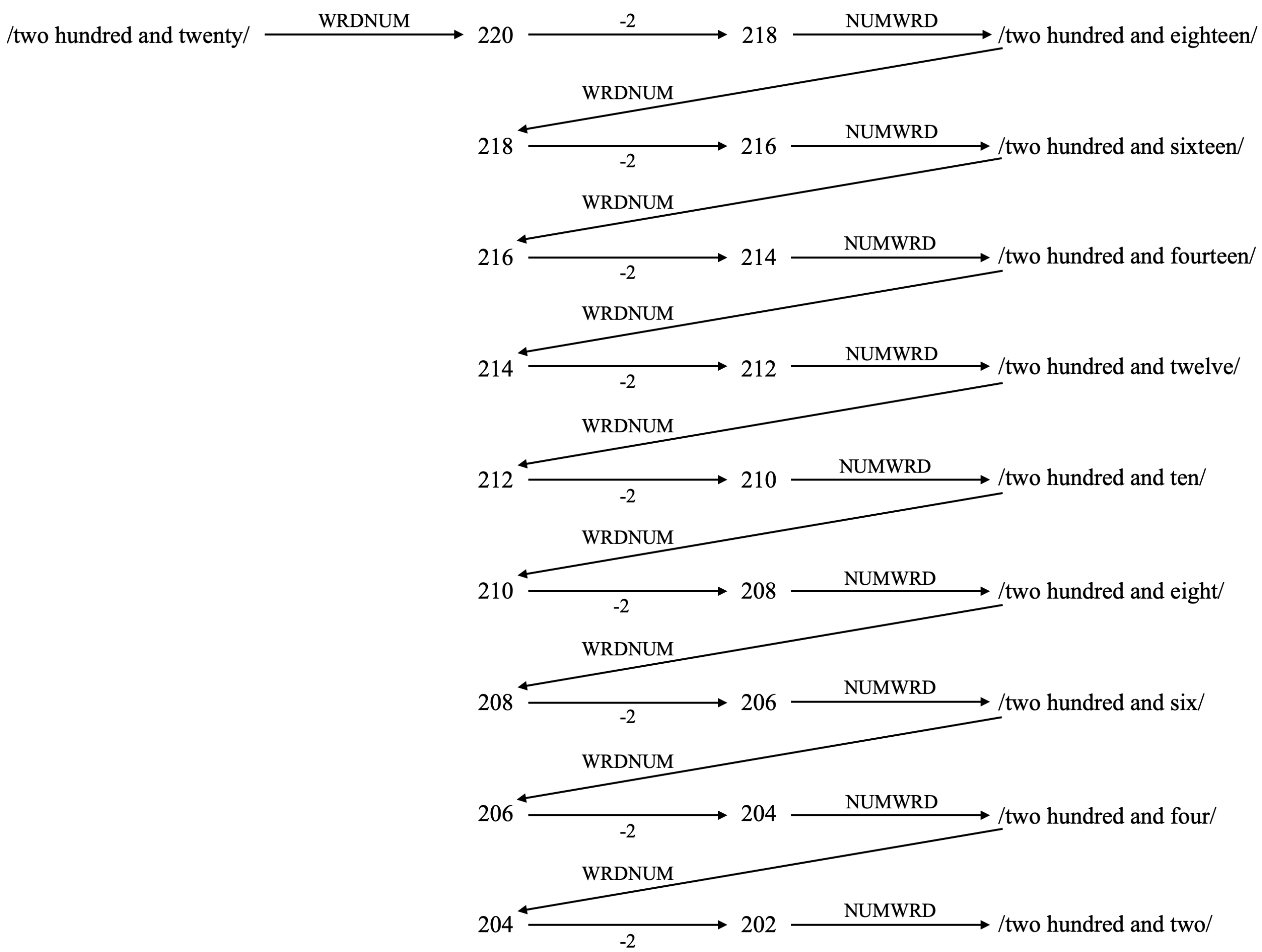

Figure 5.5: Arrow diagram for the series of computations employing subtraction in SE 1.2.

(vii) The issue of structure preservation, while touched on briefly under (vi), will be discussed in detail in Chapter 7, as part of Stage 3 of the analysis.

5.3 Summary comments

The arrow diagrams using LOCATE remain the same for /counting/ forwards and backwards. In each of the tasks making up EE 2, one or more main procedure is used recursively to perform the computations. The main procedures are shown in Figure 5.5.

/Counting/ forwards or backwards: $/\mathbb{N}^{w}/ \xrightarrow{\text{WRDSYM}} /\mathbb{N}/ \xrightarrow{\text{LOCATE}(n)} /\mathbb{N}/ \xrightarrow{\text{SYMWRD}} /\mathbb{N}^{w}/$

/Counting/ forwards (addition): $/\mathbb{N}^{w}/ \xrightarrow{\text{WRDNUM}} \mathbb{N} \xrightarrow{+n} \mathbb{N} \xrightarrow{\text{NUMWRD}} /\mathbb{N}^{w}/$

/Counting/ backwards (subtraction): $/\mathbb{N}^{w}/ \xrightarrow{\text{WRDNUM}} \mathbb{N} \xrightarrow{-n} \mathbb{N} \xrightarrow{\text{NUMWRD}} /\mathbb{N}^{w}/$

Figure 5.6: Arrow diagrams for the core procedures used recursively in EE 1.

LOCATE is the only mapping that uses counting directly, though implicitly, and we have to conclude that what is referred to as /counting/ is an indirect form of counting. This is not necessarily a problem. In fact, it is an intelligent move away from direct counting, but a few critical comments do need to be made about the pedagogy in that regard and will be discussed in Chapter 7.

CHAPTER 6

Analysis: Lesson 2

6.1 Introduction

This chapter presents the production of data for evaluative events concerned with /counting/ in Lesson 2, using the protocol outlined in Chapter 4. Stages 1 and 2 of the analytic protocol will be carried out, with Stage 3 left to Chapter 7.

6.2 Production and analysis of data

Recall once more that the production of data and the analysis of lessons proceeds in three stages:

Stage 1: Segment the lesson into *evaluative events* by specifying the topic of each event and the time spent addressing the topic.
Stage 2: Describe the *syntactical features* of computational activity within an event.
Stage 3: Deduce the *semantic bases* of computational activity from the analysis of computational syntax.

6.2.1 Stage 1: Lesson 2 evaluative events (Teacher 2)

Activity capturing the use of /count/ and /counting/ began with EE2, which started at 14:20 into the lesson of Teacher 2 and lasted for a duration of 5' 14". Seated on the mat at the front of the classroom, learners were instructed to /count/ by referring to a number symbol on a number chart.

Figure 6.1: Instructed by Teacher 2, a learner locates a number symbol on a number chart and calls out the associated number word.

Another /counting/ activity began at 27:01 and continued for a duration of 5' 27", which I shall refer to as EE5. Learners were instructed to count in threes, fours, and tens, referring to the number charts shown in Figure 5.2. The number charts used during the lesson consisted of ten charts, each showing one hundred number symbols (/1/ to /100/, /101/ to /200/, /201/ to /300/, /301/ to /400/, /401/ to /500/, /501/ to /600/, /601/ to /700/, /701/ to /800/, /801/ to /900/ and /901/ to /1000/). See Figure 5.3 for a close-up view of a few of the charts.

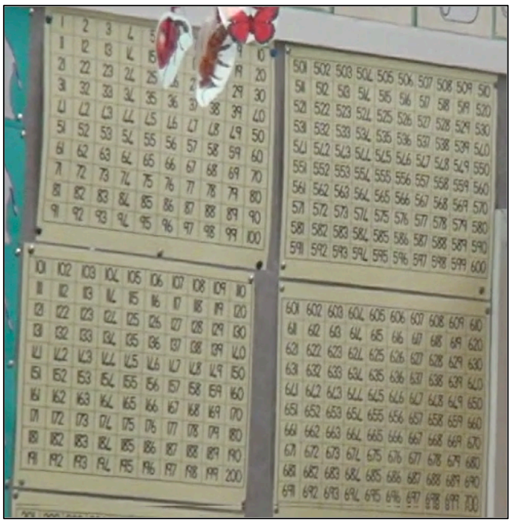

Figure 6.2: An example of the wall-mounted number charts used by Teacher 2.

Table 6.1 lists the tasks constituting the evaluative events and sub-events concerned with /counting/ activity in Lesson 2. The other evaluative events that constitute the lesson did not have /counting/ as a topic.

Table 6.1: Evaluative events and sub-events referring to /counting/ in Lesson 2

EE	SE	Content	Location	Duration
2	2.1	/Count/ in ones from /550/ to /588/	14:20-18:00	3' 40"
	2.2	/Count/ backwards in ones from /775/ to /766/	18:02-19:00	58"
	2.3	/Count/ in tens from /3/ to /103/	19:47-20:23	36"
5	5.1	/Count/ in threes from /3/ to /102/	27:01-29:30	2' 29"
	5.2	/Count/ in fours from /4/ to /96/	29:30-31:25	1' 55"
	5.3	/Count/ in tens from /9/ to /209/	31:31-32:34	1' 03"
			Total	10' 41"

6.2.2 Stage 2: Computational syntax analysis

Consider the following interaction between Teacher 2 and her learners during SE 2.1:

86. Teacher: So, over now to oral maths. Some counting, and... and ... (turning to learners) now turn to the counting chart please (Learners all turn to face the counting charts on the left wall of the classroom). Where's teacher's long stick? Where's my pointer? (A learner gets up and runs to fetch the teacher's pointer at the back of the classroom and hands it to Teacher 2)**.** There we go. Count for me in .. in ones and start

by /five hundred … and/... We will start with a nice big number today, /five hundred and .. fifty/. Come show me, where's /five hundred and fifty/? /Five … hundred … and … fifty/. Which number is that one? (Teacher 2 points at a number symbol on the chart at the bottom left of the collection of charts).

87. Learner: /Five hundred/.
88. Teacher: Yes! (Looking at the whole class) Now, will /five hundred and fifty/ be after /five hundred/ or before?
89. Learner: After.
90. Teacher: Beautiful! ... It will be after. Lylen come and help her. She said after /five hundred/. Take the pointer and help her. (Lylen points at /550 / on the next chart, which is situated at the top right of the collection of charts.)
91. Teacher: There we go. /Five hundred/. ... Thank you, my angel. Thank you. Where ... I can't see. (Stretching to point at /550/ on the chart) ... That is how you write /five hundred and fifty/.
92. Teacher: So, we are going to count in ones from /five hundred and fifty/ till ... /five hundred and eighty-eight/. (Teacher 2 points at /588/.) What number is this?
93. Learners: (Chorus) /Five hundred and eighty-eight/.
94. Teacher: So, you must remember now, from /five hundred and fifty/ .. till /five hundred and eighty-eight/. Let's start.
95. Learners: (Chorus): /Five hundred and fifty/, /five hundred and fifty-one/, /five hundred and fifty-two/, […], /five hundred and seventy-five/, (Teacher 2 motions a learner to come and hold the pointer.) /five hundred and seventy-six/, […], /five hundred and eighty-eight/.
96. Good! Thank you. /Five hundred and eighty-eight/.

Extract 6.1: Teacher-learner interaction in SE 2.1.

Once again, the syntactical aspects of the computations required by the task can now be described by using the protocol detailed in Section 4.3.3.

(i) Teacher 2 oriented her learners to the requirements of the task concerned with the identification of number symbols corresponding to number words by way of an example of mapping from the number word /five hundred and fifty/ (the input) to the number symbol /550/ (the output). Interestingly, she also drew the attention of the learners to order, though very briefly, in utterances 87 to 90.

(ii) The input and output types of the initial mapping are, respectively, number words and number symbols. As was the case in Lesson 1, the number words and the symbols listed in the number charts index the natural numbers, $\mathbb{N}$, so the number words will be indicated by the symbol /$\mathbb{N}^w$/ and the number symbols by /$\mathbb{N}$/ once again.

(iii) Syntactically, we have mappings of the type /$\mathbb{N}^w$/ $\rightarrow$ /$\mathbb{N}$/ and /$\mathbb{N}$/ $\rightarrow$ /$\mathbb{N}^w$/ in use, corresponding to WORDSYM and SYMWORD.

(iv) Here we have a variation on the LOCATE mapping discussed in Chapter 5 since Teacher 2 initially requires the learners to identify number symbols corresponding to number words, and also because the learners often call out the number words *before* Teacher 2 points at the corresponding number symbol. We can define the mapping LOCATE: $(/\mathbb{N}^w/,\mathbb{N}) \rightarrow /\mathbb{N}/$ along with LOCATE(n): $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$, which is its curried form, to describe the computational activity from utterance 86 to utterance 93.

Note that LOCATE(n): $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$ and LOCATE(n): $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$ (also used by Teacher 1) are different because the respective domains and codomains of the mappings are different. In order to avoid ambiguity, it would be helpful to mark the two varieties as different. Going forward, subscripts will be used to distinguish between the different types of LOCATE. LOCATE(n): $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$ will be indicated by $\text{LOCATE}_1(n)$: $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$, and LOCATE($n$): $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$ will be indicated by $\text{LOCATE}_2(n)$: $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$. In all other respects LOCATE_1 and LOCATE_2 are identical. Both entail the implicit use of NUMDAG and DAGNUM.

As was the case with the lesson of Teacher 1, the possibility of the use of addition rather than the constitution of aggregates (consisting of a single element in SE 2.1) needs to be taken into account.

SE 2.2 requires learners to /count/ backwards, in response to which learners may well have used subtraction. The rest of EE2 and all of EE 5 are concerned with /counting/ forward.

(v) The mappings of the series of procedures that can be used to describe the computational activity in EE 2 and EE 5 are as follows:

A.
WRDSYM: $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$
NUMDAG: $\mathbb{N} \rightarrow$ FINSET (implicit to LOCATE)
DAGNUM: FINSET $\rightarrow \mathbb{N}$ (implicit to LOCATE)
LOCATE_1: $(/\mathbb{N}/,\mathbb{N}) \rightarrow /\mathbb{N}^w/$, or, $\text{LOCATE}_1(n)$: $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$, where $n \in \mathbb{N}$ (curried version).
SYMWRD: $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$

B.
SYMWRD: $/\mathbb{N}/ \rightarrow /\mathbb{N}^w/$
NUMDAG: $\mathbb{N} \rightarrow$ FINSET (implicit to LOCATE)
DAGNUM: FINSET $\rightarrow \mathbb{N}$ (implicit to LOCATE)
LOCATE_2: $(/\mathbb{N}^w/,\mathbb{N}) \rightarrow /\mathbb{N}/$, or, $\text{LOCATE}_2(n)$: $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$, where $n \in \mathbb{N}$ (curried version).
WRDSYM: $/\mathbb{N}^w/ \rightarrow /\mathbb{N}/$

C.
WRDNUM: $/\mathbb{N}^w/ \rightarrow \mathbb{N}$
+: $(\mathbb{N},\mathbb{N}) \rightarrow \mathbb{N}$ or $(+n)$: $\mathbb{N} \rightarrow \mathbb{N}$, where $n \in \mathbb{N}$ (curried version).
NUMWRD: $\mathbb{N} \rightarrow /\mathbb{N}^w/$

D.

WRDNUM: $/\mathbb{N}^w/ \to \mathbb{N}$

-: $(\mathbb{N},\mathbb{N}) \to \mathbb{N}$ or ($-n$): $\mathbb{N} \to \mathbb{N}$, where $n \in \mathbb{N}$ (curried version).

NUMWRD: $\mathbb{N} \to /\mathbb{N}^w/$

(vi) It is highly probable that C was used rather than A and B in SE 2.1 because adding 1 is such a well-practised computation and, in fact, corresponds to the successor function for the natural numbers. Similarly, D rather A or B might have been used by learners in SE 2.2.

Arrow diagrams for A to D, are, *mutatis mutandis*, as those produced for Teacher 1 and will therefore not be presented here. The reader can refer to Figure 5.2 and Figure 5.3, making the necessary alterations.

(vii) As was the case with Teacher 1, much of the computational activity of SE 2.1 of Teacher 2 is concerned with computations on aggregates of number words and symbols. The aggregates that might be subjected to counting proper are trivial, each containing a single element. The mappings DAGNUM and NUMDAG include counting and are implicitly incorporated into LOCATE_i.

An interesting difference from the tasks used by Teacher 1 is that, in SE 2.3 and SE 5.3, Teacher 2 starts the tasks with reference to a number that is not a multiple of the 'skip' which is to be used. In SE 2.3, the task is to /count/ in tens from /3/ to /103/; in SE 5.3 it is to /count/ in tens from /9/ to /209/. Three and nine are not multiples of ten. Since such tasks are, in general, often referred to as 'skip counting' by FP teachers, we can refer to ten as the 'skip'.

(viii) The issue of structure preservation will be discussed in detail in Chapter 7, as part of Stage 3 of the analysis.

6.3 Summary comments

Unsurprisingly, order is once again central to the tasks. While Teacher 2 does draw some attention to order initially to assist her learners to focus on the correct chart, numerical order is not dealt with in any significant depth, as was the case with Teacher 1 as well. The reader can refer to Figure 5.3 once again to be reminded of the structure preservation mappings with regard to order. The core mappings shown in Figure 5.5 can be extended as in Figure 6.3 to account for the additional possibilities that emerge from the analysis of Lesson 2.

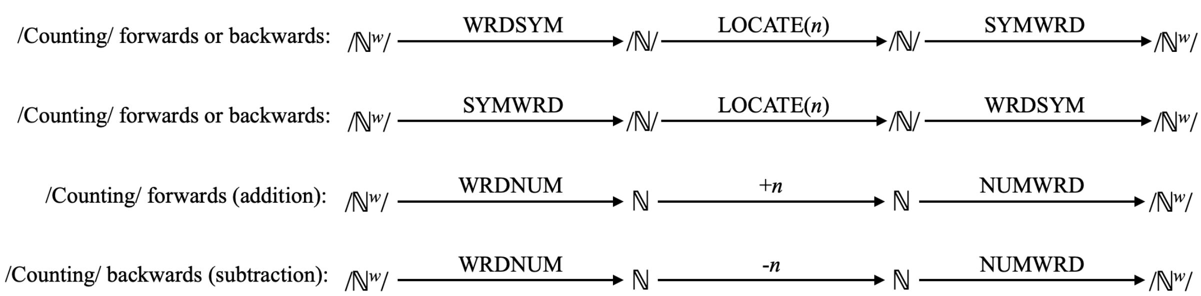

Figure 6.3: Arrow diagrams for the core procedures used recursively in EE 2 and EE 5 of Lesson 2.

The arrow diagrams showing the use of LOCATE remain the same for /counting/ forwards and backwards. In each of the tasks making up EE 2 and EE 5, one or more core procedure is used recursively to perform the computations. The core procedures are shown in Figure 6.3.

LOCATE is, once again, the only mapping that uses counting directly, though implicitly, and we have to conclude that what Teacher 2 referred to as /counting/ is an indirect form of counting, as was the case for Teacher 1. The semantic bases for the activity will be discussed in Chapter 7.

CHAPTER 7

Concluding discussion

7.1 Introduction

In this study, the general interest is in the construction of the experience of mathematics (non-core domain knowledge) by genetic endowment on the basis of contextual data. More specifically, the particular interest of the study is on the construction of the experience of counting in the pedagogic situations of Grade 3 schooling. This interest is linked with the field of mathematics education through the construct of PCK and its sub-constructs, specifically, KCS. In order to draw the discussion towards a conclusion, the analyses of computational activity conducted in Chapters 5 and 6 need to be brought into relation with curriculum prescriptions for the teaching of counting and what we know of genetic endowment as it pertains to the construction of quantified contextual data.

7.2 Stage 3: Semantic bases

In Chapters 5 and 6, a small series of related computational procedures were generated to account for the data. In those analyses it appeared that counting proper was restricted to the constitution and counting of very small discrete aggregates internal to the mapping LOCATE. However, LOCATE may well have been displaced by addition and subtraction since such computations required by the tasks were rather trivial. To the extent that such was the case, no counting occurred. If the mathematical definition of counting discussed in Section 3.5 is recalled, the reader will recognise that how the terms /count/ and /counting/ are used to refer in Lessons 1 and 2 is wider than that afforded directly by the definition of counting. One source of contextual data that has not been discussed analytically yet is the official curriculum for school mathematics to which the teachers and learners were subject, which is the subject of the next sub-section.

7.2.1 Counting in the Curriculum and Assessment Policy Statement for Grade 3 Mathematics

Plates 7.1 and 7.2 show the broad South African Grade 3 curriculum prescriptions that refer to: (i) the counting of objects, (ii) /counting/ forwards and backwards, (iii) the relations between number symbols and number words for the natural numbers (including zero), and (iv) the comparison and ordering of the natural numbers. The terms /count/ and /counting/ are used in relation to (i) and (ii) only in the curriculum prescriptions shown in Plates 7.1 and 7.2. However, the content of (iii) and (iv) clearly bear on (i) and (ii) because both counting proper and the ways in which /counting/ is used to refer, imply some understanding of the lexical/symbolic representation of natural numbers as well as of the ordering of representational resources used to index numbers (like number words and number symbols). Hence the inclusion of Plate 7.2.

In Chapters 5 and 6, it was argued that numerical order had an over-determining role on the use of ordered lists of number words and corresponding number symbols, thereby implicitly encoding the use of order in pedagogic activity. What that resulted in was the extensive use of n-tuples in the form of ordered lists of number symbols organised as so-called "number charts", which was the general device through which numerical order was inserted. One aspect of the computational activity of Lessons 1 and 2 that was not discussed in Chapters 5 and 6 was the use of rule-bound sequences.

GRADE 3 OVERVIEW 1. NUMBERS, OPERATIONS AND RELATIONSHIPS				
TOPICS	**TERM 1**	**TERM 2**	**TERM 3**	**TERM 4**
NUMBER CONCEPT DEVELOPMENT: Count with whole numbers				
1.1 **Count objects**	Group to at least 200 objects to estimate and count reliably. Give a reasonable estimate of a number of objects that can be checked by counting. The strategy of grouping is encouraged.	Group to at least 500 objects to estimate and count reliably. Give a reasonable estimate of a number of objects that can be checked by counting. The strategy of grouping is encouraged	Group to at least 700 objects to estimate and count reliably. Give a reasonable estimate of a number of objects that can be checked by counting. The strategy of grouping is encouraged.	Group to at least 1 000 objects to estimate and count reliably. Give a reasonable estimate of a number of objects that can be checked by counting. The strategy of grouping is encouraged.
1.2 **Count forwards and backwards**	Count forwards and backwards in: • 1s, from any number between 0 and 200 • 10s from any multiple of 10 between 0 and 200 • 5s from any multiple of 5 between 0 and 200 • 2s from any multiple of 2 between 0 and 200 • 3s from any multiple of 3 between 0 and 200 • 4s from any multiple of 4 between 0 and 200 • 100s to at least 500	Count forwards and backwards in: • 1s, from any number between 0 and 500 • 10s from any multiple of 10 between 0 and 500 • 5s from any multiple of 5 between 0 and 500 • 2s from any multiple of 2 between 0 and 500 • 3s from any multiple of 3 between 0 and 500 • 4s from any multiple of 4 between 0 and 500 • 50s,100s to at least 1 000	**Count forwards** and **backwards** in: • 1s, from any number between 0 and 700 • 10s from any multiple of 10 between 0 and 700 • 5s from any multiple of 5 between 0 and 700 • 2s from any multiple of 2 between 0 and 700 • 3s from any multiple of 3 between 0 and 700 • 4s from any multiple of 4 between 0 and 700 • 20s, 25s, 50s,100s to at least 1 000	**Count forwards** and **backwards** in: • 1s, from any number between 0 and 1 000 • 10s from any multiple of 10 between 0 and 1 000 • 5s from any multiple of 5 between 0 and 1 000 • 2s from any multiple of 2 between 0 and 1 000 • 3s from any multiple of 3 between 0 and 1 000 • 4s from any multiple of 4 between 0 and 1 000 • 20s,25s, 50s,100s to at least 1 000

Plate 7.1: Curriculum and Assessment Policy Statement for Grades 1 to 3 Mathematics prescriptions for counting and /counting/ in Grade 3 (Source: Department of Basic Education (2011, p. 73)).

TOPICS	**TERM 1**	**TERM 2**	**TERM 3**	**TERM 4**
NUMBER CONCEPT DEVELOPMENT: Represent whole numbers				
1.3 **Number symbols and number names**	**Identify, recognise and read numbers** • Identify, recognise and read number symbols 0 to 500 • Write number symbols 0 to 500 • Identify, recognise and read number names 0 to 250 • Write number names 0 to100	**Identify, recognise and read numbers** • Identify, recognise and read number symbols 0 to 1 000 • Write number symbols 0 to1000 • Identify, recognise and read number names 0 to 250 • Write number names 0 to 250	**Identify, recognise and read numbers** • Identify, recognise and read number symbols 0 to 1 000 • Write number symbols 0 to1000 • Identify, recognise and read number names 0 to 500 • Write number names 0 to 500	**Identify, recognise and read numbers** • Identify, recognise and read number symbols 0 to 1 000 • Write number symbols 0 to1000 • Identify, recognise and read number names 0 to1 000 • Write number names 0 to1000
NUMBER CONCEPT DEVELOPMENT: Describe, compare and order whole numbers				
1.4 **Describe, compare and order numbers**	Describe, compare and order numbers to 99. • Compare whole numbers up to 99 using smaller than, greater than, more than, less than and is equal to • Order whole numbers up to 99 from smallest to greatest, and greatest to smallest	Describe, compare and order numbers to 500. • Compare whole numbers up to 500 using smaller than, greater than, more than, less than and is equal to • Order whole numbers up to 500 from smallest to greatest, and greatest to smallest	Describe, compare and order numbers to 700. • Compare whole numbers up to 700 using smaller than, greater than, more than, less than and is equal to • Order whole numbers up to 700 from smallest to greatest, and greatest to smallest **Use ordinal numbers to show order, place or position** • Use, read and write ordinal numbers, including abbreviated form up to 31st	Describe, compare and order numbers to 999. • Compare whole numbers up to 999 using smaller than, greater than, more than, less than and is equal to • Order whole numbers up to 999 from smallest to greatest, and greatest to smallest

Plate 7.2: Curriculum and Assessment Policy Statement for Grades 1 to 3 Mathematics prescriptions for the representation of whole numbers and for the comparison and order of whole numbers in Grade 3 (*Source*: Department of Basic Education (2011, p. 74)).

Consider Topic 1.2 shown in Plate 7.1 and Topic 1.3 in Plate 7.2. One of the purposes of the curriculum prescriptions listed there appears to be the training of learners to move beyond the memorisation of various *n*-tuples and on to the use of rule-bound sequences (refer to Section 3.5) that are enabling of the identification, representation and generation of number words and number symbols for the natural numbers. Such training is enabling of indirect counting, where the presence of the implied discrete aggregates is not necessary. The strategy is essential to the construction of precise quantification that goes beyond that of the very small aggregates (up to four objects) that can be handled by the OTS (object tracking system) and the approximate quantification of larger aggregates that are processed by the ANS (approximate number system). In Topic 1.1 we find a curriculum prescription requiring learners to estimate the cardinality of relatively large discrete aggregates (implying the use of the ANS) and relate such estimates to counting proper: "Give a reasonable estimate of a number of objects that can be checked by counting".

The OTS and ANS, which serve as the core domain bases for quantification, suffer limitations that are to be overcome by the use of *n*-tuples and rule-bound sequences of number words and number symbols, thus enabling of indirect, exact counting that assists us to move beyond the innate human and general vertebrate ability to recognise the cardinalities of only very small aggregates (subitising). We now know that ANS acuity improves with schooling (Piazza, Pica, Izard, Spelke & Dehaene, 2013), so that there is a recognisable better-than-chance, though weak, convergence of estimation and exact counting, which results from exposure to formal mathematics education.

The ANS enables humans and non-human animals to compare distinct, relatively large, aggregates without any reference to number. How this is possible is described in mathematical terms in Theorem 1 (Davis, 2015).

Theorem 1: Finite sets can be ordered in terms of size without using counting
Proof
Let FINSET be the class of finite sets. The symbol $|S|$, where S is some set, will be used to refer to the size of S. Suppose A and B are two disjoint elements of FINSET, and that f is a function mapping A to B. Further, suppose that f is restricted to the construction of an injection, in so far as that is possible, for the purposes of effecting a comparison of A and B with respect to set size. Specifically, f is always generated by an attempt to construct injections and, when that is not possible, as in those cases where $|A| > |B|$, f is necessarily surjective. Under such conditions, f is either (1) injective, (2) surjective, or (3) bijective. Each case is discussed in turn.

(1) $f{:}A \rightarrow B$, where f is injective (i.e., 1-to-1 or into).
Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Since f is injective, for all pairs $a_i, a_j \in A, f(a_i) \neq f(a_j)$. That is, there is no $b_k \in B$ such that $b_k = f(a_i)$ AND $b_k = f(a_j)$. If it is the case that for all $b_k \in B$ there exists an $a_i \in A$ such that $f(a_i) = b_k$, then f is also bijective, and A and B are equinumerous. [1]
However, if there exist one or more $b_k \in B$ for which there does not exist an $a_i \in A$ such that $f(a_i) = b_k$, then A and B are not equinumerous. A partition of B, viz., $(B_1|B_2) \vdash B$, can be constructed such that B_1 contains all the b_k for which there exists an $a_i \in A$ such that $f(a_i) = b_k$ and B_2 contains all the b_k for which no such a_i exists. It follows that, with respect to A, B_2 is in excess of B_1. [2]
Now consider the set $C = A \cup B_2$. Then C is an element of FINSET. By transitivity, B and C are equinumerous since A and B_1 are equinumerous and $B = B_1 \cup B_2$. $(A|B_2) \vdash C$ is a partition of C. This means that, with respect

to C, B_2 is in excess of A. We can thus conclude that $|B|$ is greater than $|A|$, by which is meant that there exist elements of B—viz., the set B_2—for which there are no associated elements in A when we construct an injective mapping from A to B. [3]

It follows from [1], [2] and [3] that, if f is injective, then $|A| \leq |B|$.

(2) $f{:}A \to B$, where f is surjective (i.e., onto).

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Since f is surjective, for all $b_k \in B$ there exists an a_k such that $b_k = f(a_k)$. If it is the case that for all $b_k \in B$ there exists an $a_i \in A$ such that $f(a_i) = b_k$, then f is also bijective, and A and B are equinumerous. [4]

However, if for any pair b_i, $b_j \in B$, there exists an $a_k \in A$ such that $b_i = f(a_k)$ AND $b_j = f(a_k)$, then A and B are not equinumerous. A partition of A, viz. $(A_1|A_2) \vdash A$, can be constructed such that, for some bijective function g, $g{:}A_1 \to B$. Now consider the set $C = B \cup A_2$ where C is disjoint with respect to A and B. C is then an element of FINSET. By transitivity, A and C are equinumerous since B and A_1 are equinumerous and $A = A_1 \cup A_2$. Now, $(B|A_2) \vdash C$ is a partition of C. This means that, with respect to C, A_2 is in excess of B. We can thus conclude that $|A|$ is greater than $|B|$, by which is meant that there exist elements of A—viz., the set A_2—for which there are no associated elements in B when we construct an injective mapping from B to A. [5]

It follows from [4] and [5] that, if f is surjective, then $|A| \geq |B|$.

(3) $f{:}A \to B$, where f is bijective (i.e., into and onto).

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Since f is bijective, for all pairs a_i, $a_j \in A$, $f(a_i) \neq f(a_j)$, and for all $b_k \in B$ there exists an a_k such that $b_k = f(a_k)$. Therefore, there exists an inverse function, $f^{1}{:}B \to A$ which is necessarily bijective. Since f and f^{1} are bijective, A and B are equinumerous. That is, $|A| = |B|$.

We could also argue more economically that since f is bijective, hence both injective and surjective, we have $|A| \leq |B|$ AND $|A| \geq |B|$, from which it follows that $|A| = |B|$.

In summary, given two finite sets A and B, and f is a function from A to B subject to the listed retractions, then we can say that

(1) if $f{:}A \to B$ is injective, then $|A| \leq |B|$;

(2) if $f{:}A \to B$ is surjective, then $|A| \geq |B|$;

(3) if $f{:}A \to B$ is bijective, then $|A| = |B|$.

Taking the elements of any sub-class of finite sets pairwise, we can thus generate a monotonically increasing ordering of those sets with respect to size by constructing a mapping, f, between each pair of sets subject to the conditions for the construction of f.

■

Returning to curriculum prescriptions concerned with the comparison and ordering of finite aggregates and its mapping to the comparison and ordering of natural numbers, we find such in Topic 1.4 of Grade 1 (see Plate 7.3). Theorem 1 assures us that such comparisons and orderings can always be made.

TOPICS	TERM 1	TERM 2	TERM 3	TERM 4
NUMBER CONCEPT DEVELOPMENT: Describe, compare and order whole numbers				
1.4 **Describe, compare and order numbers**	**Describe, compare and order up to 5 objects** • Compare collection of objects according to many, few; most, least; more than, less than; the same as, just as many as, different • Order collection of objects from most to least and least to most **Describe, compare and order numbers to 5** • Describe and compare whole numbers according to smaller than, greater than, more than, "less than, is equal to • Describe and order numbers: - from smallest to greatest and greatest to smallest - using the number line 1 - 5	**Describe, compare and order up to 10 objects** • Compare collection of objects according to many, few; most, least; more than, less than; the same as, just as many as, different • Order collection of objects from most to least and least to most **Describe, compare and order numbers to 10** • Describe and compare whole numbers according to smaller than, greater than, more than, "less than, is equal to • Describe and order numbers: - from smallest to greatest and greatest to smallest - before, after, in the middle/ between - using the number line 0 - 10	**Describe, compare and order up to 15 objects** • Compare collection of objects according to many, few; most, least; more than, less than; the same as, just as many as, different • Order collection of objects from most to least and least to most **Describe, compare and order numbers to 15** • Describe and compare whole numbers according to smaller than, greater than, more than, "less than, is equal to • Describe and order numbers: - from smallest to greatest and greatest to smallest - before, after, in the middle/ between - using the number line 0 - 15	**Describe, compare and order up to 20 objects** • Compare collection of objects according to many, few; most, least; more than, less than; the same as, just as many as, different • Order collection of objects from most to least and least to most **Describe, compare and order numbers to 20** • Describe and compare whole numbers according to smaller than, greater than, more than, "less than, is equal to • Describe and order numbers: - from smallest to greatest and greatest to smallest - before, after, in the middle/ between - using the number line 0 - 20 **Use ordinal numbers to show order, place or position** • Position objects in a line from first to tenth or first to last e.g. first, second, third … tenth, last • Ordinal numbers in the range first to tenth

Plate 7.3: Curriculum and Assessment Policy Statement for Grades 1 to 3 Mathematics prescriptions for the representation of whole numbers and for the comparison and order of whole numbers in Grade 1 (*Source*: Department of Basic Education (2011, p. 41)).

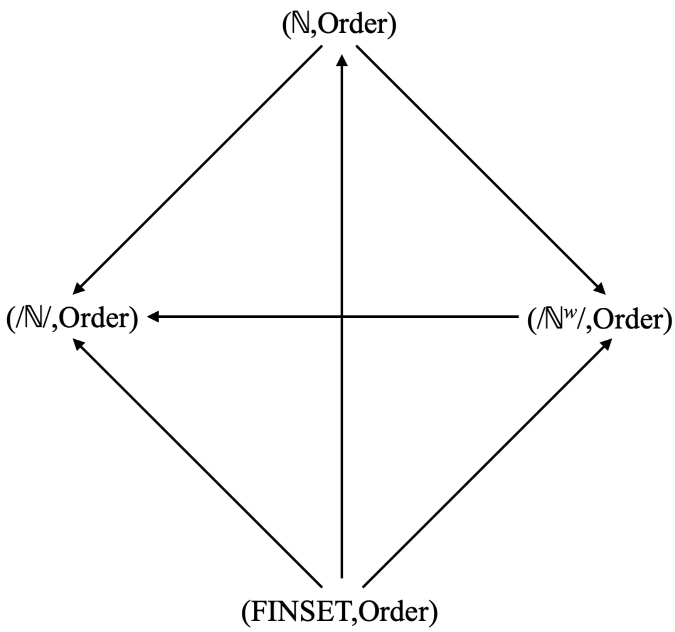

Figure 7.1: Mappings ensuring structure preservation with respect to order across discrete aggregates, natural numbers, number names and number symbols.

The content of Topic 1.4, shown in Plate 7.3, indicates an implied correspondence between the ordering of small discrete aggregates and natural numbers. The mapping from order over finite sets to order over the natural numbers is to be mediated by counting. We can now extend Figure 5.3 (see Chapter 5) to include the structure preserving mappings between the computational structure (FINSET,Order) and the other structures concerned with order (see Figure 7.1). A fuller elaboration of the structure preserving mappings shown in Figure 7.1 can be constructed, an example of which is shown in Figure 7.2.

Consider an order relation *bigger than* defined over finite sets and suppose that we have a pair of elements of FINSET, A and B. We can define *bigger than* as follows:

bigger than: $(A,B) \rightarrow A$ if $|A| > |B|$;

bigger than: $(A,B) \rightarrow B$ if $|A| < |B|$;

bigger than is undefined if $|A| = |B|$.

Also consider the order relation *greater than*, defined over the natural numbers. Suppose than m and n are a pair of natural numbers and that:

greater than: $(m,n) \rightarrow m$ if $m - n > 0$;

greater than: $(m,n) \rightarrow n$ if $n - m > 0$;

greater than is undefined if $m - n = 0$.

We thus have two structures constituted from order relations: (FINSET, *bigger than*) and ($\mathbb{N}$, *greater than*).

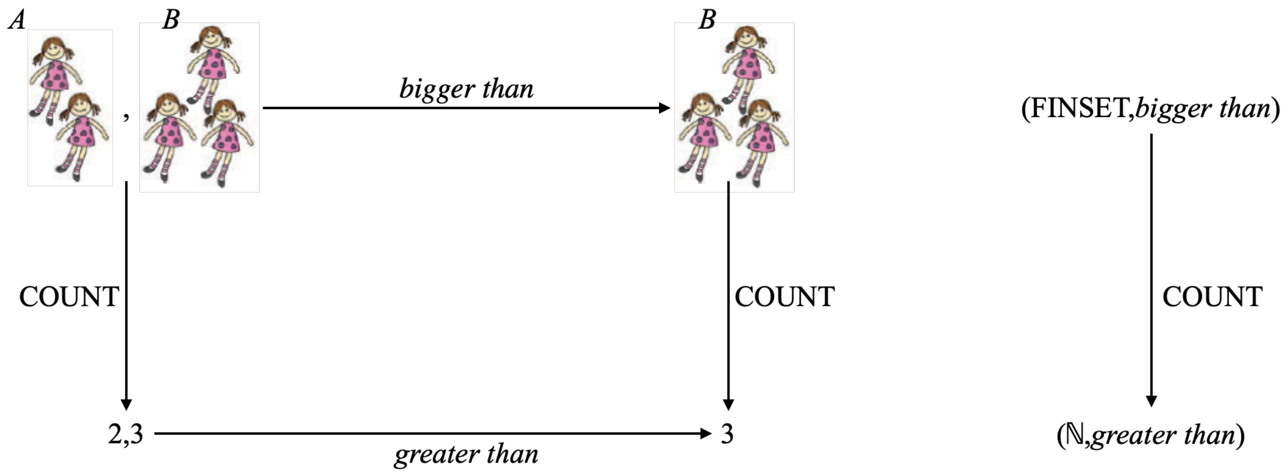

Figure 7.2: An example of the mapping COUNT ensuring the preservation of structure with respect to order from FINSET to the natural numbers.

From the analysis of the curriculum prescriptions, it is evident that the semantic basis for order across natural numbers, number words and number symbols derive from the innate recognition of and ability to order aggregates, which is due to the biologically endowed existence of the ANS in humans and non-human animals. Such knowledge appears to be taken for granted by Teacher 1 and Teacher 2, possibly because it was presumed to have been covered in Grade 1, and because such innate knowledge is familiar to the teachers and implicitly presumed to be a species property of humans. In less turgid language, we might say that such knowledge is taken as common sense by humans. From this it can be deduced that knowledge of aggregate order is part of the KCS of Teacher 1 and Teacher 2.

It is, therefore, interesting that a more mathematically precise study of order relations is generally not a substantial feature of the training of FP teachers and remains restricted to the rather superficial engagement with the so-called *ordinal numbers* in the guise of the use of comparative terms like “first”, “second” and so forth.

7.3 KCS and the teaching of counting

This study set out to examine the PCK of mathematics teachers (i.e., MKfT) at the Grade 3 level, specific to what is referred to as teachers’ knowledge of content and student (KCS). Of interest was teachers’ knowledge for teaching content relating to the use of the terms /count/ and /counting/ in the classroom, which was achieved by means of a computational analysis of the syntax made apparent by teachers and learners during lessons. The central purpose was to answer a research question concerned with how two Grade 3 teachers, from similar social class teaching contexts, generate structures referring to counting during their lessons; and what the implications of such practises might be for the number knowledge of learners.

The study took on an integrated causal model approach (ICM), which drew on resources from different disciplines such as mathematics education, cognitive science and mathematics. The study was framed by Bernstein’s pedagogic device, particularly with respect to the evaluative rule of the device, as well as the inter-related constructs of PCK, MKfT and KCS. The theoretical resources used to describe computations were drawn largely from Davis (2001, 2010b, 2011a, 2012, 2013a, 2015, 2018) and related work on the use of morphisms as elaborated in Baker *et al*. (1971), Gallistel & King, (2010), Krause (1969) and Open University (1970). These resources were used to produce the analytic framework for the analysis and production of data from the video records of two Grade 3 mathematics lessons. The analytic framework was used to describe the computational activities of teachers and learners during the recorded lessons, specifically the computational domains made available by teachers. In so doing, I was able to provide more illumination on what is described as teacher’s knowledge of content and student (KCS) for teaching counting at the Grade 3 level.

Knowledge of content and students (KCS) stands out as a core component to PCK as it is concerned with what teachers know about what learners know and how they learn. This knowledge informs the teacher on how to go about teaching particular curriculum content in order to meet the required learning needs. Indeed, one would hardly doubt that what a teacher knows about a given topic influences how they interact with and teach learners the particular content.

The position of Ball *et al*. (2008) on teachers’ KCS informs us that the mathematics teacher having KCS anticipates what learners are likely to think; predicts what they will find interesting or difficult; recognises and interprets students emerging and incomplete ideas and recognises learners’ misconceptions. From the video recording on the lessons of the two Grade 3 mathematics teachers examined in this study on their uses of

/count/ and /counting/, I find that following the KCS features as described by Ball to examine the mathematics PCK of teachers provides general prescriptions for the pedagogic strategies that are to be used by teachers. However, such an approach obscures the actual computational structures deployed when teaching mathematics content, such as that concerned with counting at the Grade 3 level.

As noted in previous chapters, counting is a fundamental resource used for the construction and elaboration of elementary arithmetic. From the data produced in the analyses of Lesson 1 and Lesson 2, we find that teachers do not show sensitivity to the implications of learners' use of specific orderings of finite list of *n* things (number chart) in performing all the /counting/ activities. In each sub-event, from both lessons, learners could complete the task by using *n*-tuples and the sequencing of number symbols and number words without counting proper. This finding raises the question of what /count/ and /counting/ refer to.

From the analyses, we see that although counting is intimately bound up with the counting of natural numbers, the terms /count/ and /counting/ are used to refer to various mappings between number symbols, number words and natural numbers, most of which is not counting proper, but which does begin to construct resources for the use of indirect counting of relatively large aggregates. Further, the primacy of the ordering of discrete aggregates as a semantic basis supporting the understanding of numerical order and the relations between aggregates, numbers, number words and number symbols remains obscure in the framework developed by Ball and her collaborators. That framework can be enhanced and made more productive by suspending its prescriptive thrust in favour of a more descriptive orientation that draws in explanatory resources from both cognitive science and mathematics.

7.4 Limitations and potential of the study

This study suffers the inevitable limitations of case study research, like the inability to generalise to other situations of FP teaching. The detailed computational analyses of counting in the lessons has, however, generated a strong suggestion that further study of quantitative and numerical order as it pertains to the teaching and learning of elementary arithmetic needs more extensive and explicit investigation because of the semantic importance of order to both direct and indirect counting, and hence to the growth of more sophisticated number sense in learners.

The SPADE data archive lacked records of computationally sensitive interviews with teachers and learners, so that important information for more in-depth analysis of teacher and learner understanding of counting was absent. Any future study would have to include such data.

The central achievement of the study is the production of a strong hypothesis that an understanding of order is central to the growth of knowledge of counting.

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Appendix A: Lesson 1

1. Teacher:	Put your finger for me on two hundred. Count for me in twos. Go from two hundred up to two hundred and twenty. Everyone, together.
2. Learners:	Two hundred.
3. Teacher:	Put your finger on two hundred!
4. Learners:	We don't have two hundred
5. Teacher:	Two hundred and two
6. Learners:	Two hundred and two, (Both teacher and learner count aloud). two hundred and four, two hundred and six, two hundred and eight, two hundred and ten, two hundred and twelve, two hundred and fourteen, two hundred and sixteen, two hundred and eighteen, two hundred and twenty
7. Teacher:	Right, boys, put your fingers on two hundred and twenty and count backward to two hundred and two. Two hundred and twenty, two hundred and eighteen, two hundred and sixteen, two hundred and fourteen, two hundred and twelve, two hundred and ten, two hundred and eight, two hundred and six, two hundred and four, two hundred two.
8. Teacher:	Right, girls, look, put your finger for me on three hundred and three. Count in threes up to three hundred and thirty.
9. Teacher:	(Teacher recites with the girls). Three hundred and three, three hundred and six, three hundred and nine, three hundred and twelve, three hundred and sixteen, three hundred and eighteen, three hundred and twenty-one, three hundred and twenty-four, three hundred and twenty-seven, three hundred and thirty.
10. Teacher:	Right, boys, put your finger on three hundred and thirty and count backwards.
11. Boys:	Three hundred and thirty, three hundred and twenty-seven, three hundred and twenty-four, three hundred and twenty-one, three hundred and eighteen, three hundred and sixteen, three hundred and twelve, three hundred and nine, three hundred and six, three hundred and three.
12. Teacher:	Right, put your finger; everyone put their fingers for me on four hundred. Count for me in twenties from four hundred up to five hundred.
13. Learners:	Four hundred, four hundred and twenty, four hundred and forty, four hundred and sixty, four hundred and eighty, four hundred and...,
14. Teacher:	Again! Four hundred and twenty.
15. Learners:	Four hundred and twenty, four hundred and forty, four hundred and sixty, (teacher counts along with a pitch higher than her learners) four hundred and eighty, five hundred

86.Teacher:	(Teacher speaking to interviewers) so over now to oral maths. Some counting, and... And ehn... (Turning to learners) now turn to the counting chart please. Learners all turn to face the counting charts on the left side of the classroom wall. Where is my pointer? A learner gets up and runs to fetch the teacher's pointer at the back of the Classroom. Count for me in 1's and start by 500 (five hundred) and enh.... 500 and we will start with a nice big number today 550 (five hundred and fifty). Come show me where is 550. Five...Hundred and fifty (repeats the number slowly). (A learner stands in front of the number chart. (Speaking to the learner) which number is that one?
87.Learner:	500 (five hundred)
88. Teacher:	Yes (Looking at the whole class) now will five hundred and fifty be after five hundred or before?
89.Learners:	(Learner standing in front responds) after
90.Teache:r	beautiful... it will be after. Lylen, come and help her. She said after five hundred. Take the pointer and help her. (Lylen points to 550 on the chart correctly)
91.Teacher:	There we go. 500 (five hundred) ... thank you my angel. Thank you. Umm wait.... I can't see (stretching pointer to top of the chart) ... that is how you write 550. So, we are going to count in ones from 550(five hundred and fifty) till... 588 (five hundred and eighty-eight). (Pointing to 588) what number is this?
92.Learners:	(chorus) 588 (five hundred and eighty- eight) (Teacher still pointing at the number charts)
93.Teacher:	so you must remember now from five hundred and fifty till five hundred and eighty-eight. Let's start (Pointing at the numbers on the chart and then motions a learner to come hold the pointer as the rest continues calling out the numbers in chorus)
94.Learners:	(Learners count in chorus.) Five hundred and fifty; five hundred and fifty-one, five hundred and fifty-two...five hundred and eighty-eight.
95.Teacher:	Good! Thank you. Five hundred and eighty-eight. Now let's count backward from... umm seven hundred and seventy-five till seven hundred and seventy... sixty-six. Backwards ...Abdul come show us where is ..., what is the number that teacher just said now? (A few learners respond unsurely)
96.Learners:	seven hundred and seventy-five
97.Teacher:	(repeats) seven hundred and seventy-five. Abdul come show us. (Abdul walks in front of the class and points to 775 on the number chart)
98.Teacher:	very good... until which number did I say?
99.Learners:	(a few chorus) seven hundred and sixty-six
100.Teacher:	sixty- six I said ... from seven hundred and seventy-five till seven hundred and sixty-six. Where is seven hundred and sixty-six?
101.Teacher:	There... now you must count backwards in ones. Come (Abdul points to 766 on the number chart)
102.Learners:	(counting backwards in ones in chorus) seven hundred and seventy- five, seven hundred and seventy-four, seven hundred and seventy-three...seven hundred and sixty-sixty.
103.Teacher:	we must...We are supposed to stop there.
104.Teacher:	Now we are going to count in tens. Tajrika come. We gonna count in tens down the umm ... you start by three. Can you reach? (Pointing to the chart and hands pointer to Tajrika).
105.Learners:	(Counting in chorus) three, thirteen, twenty- three, thirty-three...one hundred and three, one hundred and thirteen ...one hundred and ninety-three, two hundred and three.
106. Teacher:	Thank you, we stop there.

158. Teacher: Let’s count in fives and start by uhh ... what number (lesson interrupted for a few seconds by the P.A system) let’s start by eight hundred and seventy- five and then count till eight hundred and ninety-five.

159.Learners: (chorus) eight hundred and seventy- five, eight hundred and eighty, eight hundred and eighty-five, eight hundred and ninety, eight hundred and ninety- five.