NOVEL FITTED SCHEMES BASED ON MIMETIC FINITE DIFFERENCE METHOD FOR OPTIONS PRICING

by

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Declaration

I hereby declare that this thesis submitted is my original work at the University of Cape Town, and that any work done by others or by myself previously has been duly acknowledged and referenced, and has not been submitted for any other degree.

DAVID SENA ATTIPOE, 15 MARCH 2021

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Abstract

Numerical methods have been increasingly important for finding approximate solutions of partial differential equations (PDEs) describing financial models since only a few of them have analytical solutions. Indeed, in the pricing of derivative securities such as European options, the underlying PDE, the so called Black-Scholes equation, is known to have a closed-form solution when the coefficients are constant. In the case of an American put option, however, there is no analytical solution, even for constant coefficients. In this thesis, we propose alternative schemes based on mimetic finite difference to overcome the known limitations of the finite difference method while pricing options. The standard mimetic finite difference method is known in fluid dynamics to preserve important properties of the continuous problem in the discrete case thereby resulting in more accurate approximations. The underlying Black-Scholes differential operator is known to be degenerate at the boundary when the stock price equals zero. At this singularity, important properties of the PDE are lost. A negative consequence here is that the classical finite difference scheme applied to such problems is no longer monotone and hence fails to give an accurate approximation when the stock price is small. Therefore, more sophisticated techniques that are adapted to handle the degeneracy must be sought. Our proposed scheme, a fitted local approximation method, is able to handle the degeneracy of the Black-Scholes differential operator near the boundary at zero. The novel combined schemes are called fitted mimetic finite difference methods and are used for spatial discretization of the Black-Scholes PDE in one and two dimensional domains. Furthermore, rigorous mathematical convergence proofs of the methods for the one dimensional case are provided where the standard Euler method is used for temporal discretization. Numerical simulations show that the proposed numerical methods (in one and two dimensional domains) applied to both European and American options are more accurate compared to the standard finite difference method and the standard fitted finite volume methods.

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Introduction

In this thesis, we consider the valuation of derivative securities, which are financial instruments that promise a future value and payoff based primarily on the value of the underlying [13, 19]. Hence finding the fair value of a derivative has always been extremely crucial in the financial industry. One such derivative security is an option, a contract that gives to its holder the non-obligatory rights to buy (call) or sell (put) a specified quantity of assets at a fixed price (strike price) on (European option) or before (American option) a given date (maturity/expiry date) [13, 19]. European and American options have a rather simple financial structure and are both known as vanilla options. There are however several other kinds of options contracts with a more complex financial structure called exotic options. Some examples include Asian options (these are contracts with averaged asset price over some time to determine payoff) [16] and Binary options (these are "all or nothing" contracts which pay in full the amount) [16], amongst others.

Over the years, options have been valued primarily based on two fundamental models, i.e. the well-known Black-Scholes model (a model based on the concept of risk-neutral valuation) [9] and the Heston model (a model based on the observation of implied volatilities) [17]. We consider the valuation based on the Black-Scholes model. Fischer Black and Myron Scholes (1972) showed that the value of the European call option is governed by a second-order parabolic differential equation with respect to time and the price of the underlying assets [9]. Indeed, further in [9], the closed form solution for the value was obtained with constant parameters. In higher dimensions, several authors either resort to or develop numerical techniques [12, 13], even though closed-form solutions may exist or have been found to exist [16, 37]. In general, these analytical solutions have helped to understand the financial behaviour of the derivatives.

However, unlike European options, the value of an American option can be quite challenging to find, in that, at each time step we have to determine both the value of the option and whether or not it is optimal to exercise at that value (this is known as the early exercise constraint) [13, 19, 32, 38]. This makes the valuation of an American option a free boundary problem. This means there are two regions marked by the value of an asset at each time step and this informs whether an investor should hold or exercise the option. In [32, 33, 38] however, it is shown that American option pricing is governed by a linear complementarity problem (LCP) involving the Black-Scholes differential operator and a constraint on the value of the option. The LCP fully encapsulates the finance reasoning behind American options to a system of partial differential inequalities (PDIs) under appropriate constraints [32, 33, 38]. There is however a challenge when solving PDIs since we have to deal with the free and moving boundary. Also, the complementarity problem is, in general, analytically not solvable. Hence numerical methods are sought in both practice and research, of which several have been proposed in the literature. Some of the popular methods considered are the lattice method, *Cox et al.* (1979) [21], Monte-Carlo method, *Boyle et al.* (1997), explicit method, *Hull* (2005) [19], and the power penalty methods [32, 38].

In 1982, the first linear penalty approach for the complimentarity problem was proposed by *A. Bensoussan et.al.* [7] and has since been considered by many others in the literature. Indeed with a rational choice of parameter, a desirable accuracy in the approximate solution is obtained.

The penalty approach adds a small and continuous penalty term which converts the PDIs to a single partial differential equation under appropriate final and boundary conditions [32, 38]. The solution of the resulting penalized PDE is known to converge to that of the original problem (LCP) for both single and multi-assets American option [18, 32, 33, 38]. It is well known in the literature, that the convergence of a linear penalty method (l_1) is of $\mathcal{O}(\lambda^{\frac{1}{2}})$, where $\lambda > 1$ is the penalty parameter [14]. It was further established in [14, 32, 33, 38] that a penalty method (l_k) is expected to have an order $(\mathcal{O}(\lambda^{\frac{1}{k}}))$, for $\lambda > 1, k > 1$ parameters. It is important to remark that for multidimensional problems [33], the resulting PDE is fairly easy to discretize on general and unstructured meshes. Also, unlike PDIs, there are several tools for solving PDEs, and these tools have strong theoretical backgrounds [25]. Therefore it is preferable to consider converting the PDI to a PDE.

In the continuous case, many boundary value problems satisfy the maximum principle. Because of numerical stability, that is, when there are no spurious oscillations, it is imperative that the corresponding discrete problems also satisfy certain maximum principles. Such numerical schemes are called monotone schemes. These schemes if stable and consistent, are normally convergent as shown by [10]. The classical finite difference method is a natural technique for building monotone schemes. The standard finite difference method (SFDM) is amongst the commonly used standard techniques for spatial discretization of differential equations [13]. They have been the simplest way to efficiently find the solution of differential equations [13, 41, 45]. However, the Black-Scholes differential operator is known to be a convection-diffusion operator which becomes a convection-dominated operator for small volatility of asset price [13, 32, 38]. This causes numerical oscillations when standard finite difference methods are used to find the solution and this also significantly affects the accuracy of sensitivity (hedging) parameters [32, 38]. This presents a huge drawback to handling problems of our kind.

Generally, a standard upwind finite difference scheme is used to overcome this challenge and its stability depends on the associated *Courant-Friedrichs-Lewy* (CFL) condition being satisfied [13, 42]. This alternative method is heavily influenced by the biased directional flow of the convection-diffusion problem and overcomes the instability of the central differencing scheme of the standard finite difference scheme by giving numerically stable results. Moreover, the initial condition of the PDE has a discontinuity in its first derivative when the stock price is equal to the strike price E. This discontinuity has an adverse impact on the accuracy of the solution when the standard finite difference method or standard upwind finite difference is used (see [13, chapter 26]). Furthermore, the Black-Scholes differential operator is known to be degenerate towards the boundary close to zero $(S = 0)$ [13]. At this point, important characteristic properties of the PDE are lost. The standard finite difference method applied to the problem is no longer monotone and hence fails to give accurate approximations as S becomes small. A common and widely used approach by many authors dealing with finite difference/volume/element methods for the Black-Scholes equation is to apply an Euler transformation to remove the singularity of the differential operator. The transformed interval becomes $(-\infty, \infty)$ due to the Euler transformation. When solving the transformed problem, however, the domain would have to be truncated on both sides and this introduces two additional parameters in the numerical method. It is normal in computational finance [13] to truncate the domain on the right since the Black-Scholes operator is regular for $S > 0$. However, the truncation on the left hand of the domain artificially removes the degeneracy and this causes computational errors. Therefore, for the spatial discretization of the Black-Scholes PDE, it is important to build methods that are suitable to handle the degeneracy near the boundary at $S = 0$, and the discontinuity at $S = E$, where in general $S \in \Omega \subset \mathbb{R}^d, d \geq 1$. A fitted finite volume formulation coupled with a fitted local approximation to the solution was proposed for the one dimensional Black-Scholes PDE in [43], and a rigorous convergence proof provided in [5]. They showed that the system matrix of the discretization scheme is monotonic and in this case, the discrete maximum principle is satisfied and thus the discrete solution always remains non-negative. Although this fitted finite volume method is stable, it is only first order with respect to asset price variables.

In this thesis, we propose the novel mimetic finite difference methods (MFDM) to handle the discontinuity and the degeneracy of the Black-Scholes PDE. For non-degenerate PDEs, the standard MFDM is a high-quality (second order and above [15, 23, 28, 29, 45]) spatial discretization technique which follows from the well-known support operator method (SOM) [23, 45]. The SOMs are known to guarantee stability on general grids [15, 23, 28, 29, 45] in high dimension. Also, the standard MFDM tends to preserve important properties of the underlying continuum problem such as conservation laws, solution symmetries, and fundamental properties of vector and tensor calculus (such as divs, grads, and curls). This makes the standard MFDMs far superior to the standard finite difference methods. However, the standard MFDM can not handle the degeneracy at $S = 0$. To overcome this further drawback, we propose novel schemes by combining the standard fitted method [5, 43] with the standard MFDM. Indeed the standard fitted method is used to handle the degeneracy of the PDE in the region where the stock price approaches zero (degeneracy region). In the region where the PDE is non-degenerate, we apply the standard MFDM method. The novel numerical techniques from this combination are called fitted MFDMs. This combination will obviously improve the accuracy of the current fitted finite volume method in the literature since more of the approximations involved here are second-order in space. These novel spatial discretization techniques will be applied to the Black-Scholes PDE in one and two spatial dimensions.

The chapters of this thesis are as follows: In Chapter 1, our main contribution is to extend the existence and uniqueness proof for the option problem in three spatial dimensions (see for example [32] for one dimensional proof). To do this, we consider important function spaces and theorems which will be used throughout this work. We further introduce basic notions in relation to option pricing and the Black-Scholes equation for European options. Further in the chapter, we introduce the linear complementarity problem that governs the multidimensional American option problem. A variational formulation of the complementarity problem is considered. The theory of variational forms and complementarity problems is well established in the literature and hence there are several tools to establish the unique solvability of variational problems [7, 18]. The chapter further introduces the power penalty method [32, 38] that converts the partial differential inequalities of the LCP to a PDE and we demonstrate the continuity of the problem. The formulations and proofs in this chapter follow primarily from [32, 33]. It is important to mention here that the main contributions in this thesis are in Chapters 2, 3 and 4.

In Chapter 2, we present the beginning of our main work. The contribution here is building the novel spatial discretization scheme by combining the mimetic finite difference method [23, 45] with the fitted finite volume scheme [43] in one dimension. The resulting scheme is applied to the one-dimensional case of the Black-Scholes PDE. We establish the existence and consistency results of the semi-discrete solution in appropriate function spaces. The novel spatial discretization method is coupled with the standard implicit time-stepping method. We present a rigorous convergence proof for the full discrete solution coupled with the standard implicit method. We perform numerical simulations to show the accuracy of the proposed method compared to other standard discretization techniques (finite difference and finite volume methods) and the standard mimetic finite difference method. It is important to mention here that, to the best of our knowledge, the standard mimetic finite difference method has not previously been applied to problems in finance.

In Chapter 3, we present an extension of the novel fitted mimetic finite difference method to the two dimensional option problem. The special contribution in this chapter is that we split the diffusion matrix corresponding to the second order operator into a diagonal matrix tensor [23, 28, 29] and mixed terms matrix tensor. We then apply a standard mimetic finite difference method to the diagonal matrix tensor term. The mixed terms matrix tensor is discretized using a standard central finite difference method. An upwind scheme [42] is adopted for the advection terms. As in Chapter 2, we apply the fitted finite volume method [12, 43] to handle the degeneracy of the Black-Scholes differential operator near the boundary at $x = 0$ and $y = 0$. We perform numerical simulations to show the accuracy of the method compared to other standard discretization techniques, the standard mimetic finite difference, and the fitted finite volume methods [43]. The development of the technique in this thesis remains a contribution in this chapter.

As in Chapter 3, we present in Chapter 4 an extension of the fitted mimetic finite difference method to the two dimensional problem. Unlike in Chapter 3 where we split the diffusion matrix term corresponding to the second order term, here in this chapter, we keep the full diffusion matrix term. We construct a standard mimetic finite difference method to discretize this two dimensional diffusion term [28, 29]. We further adopt the upwind scheme to discretize the advection term. As before, we use the fitted finite volume method [12, 43] to handle the degeneracy of the differential operator. We present, in appropriate function spaces, the existence and consistency results of all semi-discrete solutions (mimetic and fitted mimetic methods). We perform numerical simulations to demonstrate the accuracy of the fitted mimetic (full matrix tensor) method to the standard mimetic (full matrix tensor), fitted mimetic (diagonal matrix tensor, chapter 3), standard mimetic (diagonal matrix tensor), and the fitted finite volume methods. As expected, the family of mimetic methods (in Chapters 3 and 4) outperformed the fitted finite volume method in terms of their accuracy when compared to the exact solution of the European option.

Finally, in Chapter 5, we summarise our findings and provide our conclusion of the thesis. We also give some ideas for further work.

1. Basic Notions

The main aim of this first chapter is to recall important basic notions in finance related to option pricing. As a preliminary, we recall important definitions of function spaces and theorems which we will adopt later in this thesis. Further in the chapter, we give an introduction to option pricing with a focus on the so-called Black-Scholes model. The model is primarily based on the no-arbitrage pricing theory under important assumptions. We present the multi-dimensional Black-Scholes equation, the underlying PDE that governs the value of a European call option. Furthermore in this chapter, we recall the formulation of the Linear Complementarity Problem (LCP) with associated constraints that govern the value of an American option. The function spaces and associated theorems considered earlier in the chapter will assist with establishing continuity, existence, and convergence proofs. We remark here that, detailed information to the above can be found in [1, 6, 9, 13, 19, 32] and the appropriate references therein.

1.1 Preliminaries

This section lays the preliminaries of the chapter. We will present here the function spaces and theorems that guarantee the solution to partial differential equations. The function spaces stated here will be used throughout this document. They are however all for continuous problems. Where appropriate, we shall state the associated discrete spaces to help establish the existence and convergence of the discrete solutions.

1.1.1 Function spaces and theorems

We introduce in this subsection important function spaces, notations, and major theorems that will be used throughout this document. We remark that most of the definitions are standard and follow from [1, 7, 8, 25] and appropriate references therein.

For any open set $\Omega \subset \mathbb{R}^d$ with $d \geq 1$, we define the following concepts:

Definition 1.1.2. [Continuous and Bounded Functions [8, 22]]

We denote by $C(\Omega)$ the space of all continuous real-valued functions on Ω, by $C^l(\Omega)$ (respectively, $C^l(\overline{\Omega})$) the set of functions whose derivatives up to order l are continuous on Ω (respectively $\overline{\Omega}$, where $\overline{\Omega}$ is the closure of Ω). It is important to note that, $C(\overline{\Omega})$ necessarily contains bounded and uniformly continuous functions on Ω. This then implies, $C^l(\overline{\Omega})$ on Ω is the space of bounded and uniformly continuous derivatives up to order l [25].

Definition 1.1.3. [Measure Space [8, 25]]

We define $(\Omega, \mathcal{M}, \mu)$ as a measure space, that is, Ω is a set and

(i) $\mathcal{M}$ is a σ-algebra in Ω, that is, $\mathcal{M}$ is a collection of subsets of Ω such that:

   (a) $\phi \in \mathcal{M}$,
   (b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$,
   (c) $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$ whenever $A_n \in \mathcal{M}$ for all n,

(ii) μ is a measure, that is, $\mu : \mathcal{M} \to [0, \infty]$ satisfies

   (a) $\mu(\phi) = 0$
   (b) $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \cup_{n=1}^{\infty} \mu\left(A_n\right)$ whenever (A_n) is a disjoint countable family of members of $\mathcal{M}$. The members of $\mathcal{M}$ are measurable sets.

Definition 1.1.4. [Normed Vector Space [8, 25]]

We assume here that Ω is a vector space [25] over $\mathbb{R}$.

(i) A function $\|\cdot\| : \Omega \to \mathbb{R}^+$ is called a norm if it satisfies the following conditions

   (a) $\|x + y\| \leq \|x\| + \|y\|$,
   (b) $\|\lambda x\| = |\lambda|\|x\|$, for any $\lambda \in \mathbb{R}$,
   (c) $\|x\| = 0, \text{if and only if } x = 0.$

In the case where only (a) and (b) hold but (c) is not necessarily imposed, then $\|\cdot\|$ is called a semi-norm.

(ii) Ω together with $\|\cdot\|$ is called a normed vector space (nvs).

(iii) $(\Omega, \|\cdot\|)$ is called a Banach space if it is a complete normed vector space [1].

(iv) A function $\langle \cdot, \cdot \rangle : \Omega \times \Omega \to \mathbb{R}$ is called an inner product if it satisfies

   (a) $\langle x, y \rangle = \langle y, x \rangle$,
   (b) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for any $\alpha, \beta \in \mathbb{R}$,
   (c) $\langle x, x \rangle \geq 0$ for all x and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(v) A complete inner product space is known as a Hilbert Space.

Definition 1.1.5. [L^p Space [8, 25]]

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. The space of Lebesgue measurable functions, $L^p(\Omega)$, consist of equivalence classes of measurable functions $v : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} |v|^p d\Omega < \infty, \tag{1.1.1}$$

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[1] That is if $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence ($\|x_m - x_n\| \to 0,$ as $n, m \to \infty$) then $\|x_j - x\| \to 0$ for some $x \in \Omega$.

with norm

$$\|v\|_{L^p(\Omega)} = \|v\|_p = \left(\int_\Omega |v|^p d\Omega \right)^{1/p}, \tag{1.1.2}$$

where two measurable functions are equivalent if they are equal μ-$a.e$, that is

$$v \sim u \Leftrightarrow \|v-u\|_p = 0 \Leftrightarrow \int_\Omega |v-u|^p d\Omega = 0. \tag{1.1.3}$$

For $p = \infty$, $L^\infty(\Omega)$ is defined as a space consisting of pointwise $a.e$-equivalence classes of essentially bounded measurable functions $v : \Omega \to \mathbb{R}$ with norm

$$\|v\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |v| < \infty, \tag{1.1.4}$$

where

$$\operatorname*{ess\,sup}_\Omega v = \inf \left\{ a \in \mathbb{R} : \mu\{x \in \Omega : v(x) > a\} = 0 \right\}, \tag{1.1.5}$$

and equivalently

$$\operatorname*{ess\,sup}_\Omega v = \inf \left\{ \sup_\Omega u : u = v \text{ pointwise } a.e \right\}. \tag{1.1.6}$$

We define the special case for $p = 2$. We define $L^2(\Omega)$ as the vector space of equivalence classes of functions $v : \Omega \to \mathbb{R}$ such that

$$L^2(\Omega) := \left\{ v : \left(\int_\Omega v^2 d\Omega \right)^{1/2} < \infty \right\}. \tag{1.1.7}$$

This space is equipped with the norm

$$\|v\|_0^2 := \int_\Omega v^2 d\Omega, \tag{1.1.8}$$

and further with the inner-product

$$(u, v) = \int_\Omega uv d\Omega, \ u, v \in L^2(\Omega). \tag{1.1.9}$$

As a vector space, $L^2(\Omega)$ is equipped with the following operations

$$[v] + [u] = [v + u], \quad \lambda[v] = [\lambda v], \tag{1.1.10}$$

where $[v]$ denotes the equivalence classes of v induced by $\sim$.

We introduce a weighted inner product on $L^2_\varpi(\Omega)$, mainly to handle the degeneracy of differential operators, e.g. Black-Scholes differential operator in our case, by

$$(u, v)_\varpi := \int_\Omega w^2 uv d\Omega \tag{1.1.11}$$

where w is the weight. This induces the corresponding weighted norm

$$\|v\|_{0,\varpi} := \sqrt{(v,v)_{\varpi}} = \left(\int_{\Omega} w^2 v^2 d\Omega\right)^{1/2}. \tag{1.1.12}$$

Hence the space of equivalence class of all weighted square-integrable functions is defined as

$$L^2_{\varpi}(\Omega) := \{v : \|v\|_{0,\varpi} < \infty\}.$$

It is very clear with the use of standard arguments that the pair $(L^2_{\varpi}(\Omega), (\cdot,\cdot)_{\varpi})$ is a Hilbert space (cf., for example [1]). To demonstrate this, we consider the following theorem below.

Theorem 1.2. *The pair* $(L^2_{\varpi}(\Omega), (\cdot,\cdot)_{\varpi})$ *is a Hilbert space.*

Proof. We recall that a Hilbert space is defined as a complete inner product space. To show that the pair $(L^2_{\varpi}(\Omega), (\cdot,\cdot)_{\varpi})$ is a Hilbert space, we essentially have to prove that

(i) $(\cdot,\cdot)_{\varpi}$ is an inner product.

(ii) $L^2_{\varpi}(\Omega)$ is complete.

Let $u, v \in L^2_{\varpi}(\Omega)$, $w > 0$ and $\alpha, \beta \in \mathbb{R}$. Then we first show that $(\cdot,\cdot)_{\varpi}$ is an inner product, that is

(a) $(u,v)_{\varpi} = (v,u)_{\varpi}$. Then from the left hand side, we have

$$(u,v)_{\varpi} = \int_{\Omega} w^2 uv d\Omega = \int_{\Omega} w^2 vu d\Omega = (v,u)_{\varpi}.$$

Also, we show that

(b) $(\alpha u + \beta v, z)_{\varpi} = \alpha(u,z)_{\varpi} + \beta(v,z)_{\varpi}$. That is

$$\begin{aligned}(\alpha u + \beta v, z)_{\varpi} &= \int_{\Omega} w^2(\alpha u + \beta v) z d\Omega = \int_{\Omega} w^2 \alpha u z d\Omega + \int_{\Omega} w^2 \beta v z d\Omega \\ &= \alpha(u,z)_{\varpi} + \beta(v,z)_{\varpi}.\end{aligned}$$

Finally we show that

(c) $(u,u) \geq 0$ for all u. That is

$$(u,u) = \int_{\Omega} w^2 u^2 d\Omega \geq 0, \text{ for all } v. \tag{1.2.1}$$

Furthermore,

$$\begin{aligned}(u,u)_{\varpi} = 0 \Rightarrow \int_{\Omega} w^2 u^2 d\Omega = 0 \Rightarrow u = 0, \quad \text{and} \\ u = 0 \Rightarrow \int_{\Omega} w^2 u^2 d\Omega = 0 \Rightarrow (u,u)_{\varpi} = 0.\end{aligned}$$

Therefore, $(\cdot,\cdot)_\varpi$ is an inner product.

Next we show that $L^2_\varpi(\Omega)$ is complete. That is we show that every Cauchy sequence in $L^2_\varpi(\Omega)$ converges. This means if $\{v_n\}$ is a Cauchy sequence in $L^2_\varpi(\Omega)$, then there exists a function $v \in L^2_\varpi(\Omega)$ such that $\{v_n\}$ converges to v in $L^2_\varpi(\Omega)$.

Now let $\{v_n\}$ be a Cauchy sequence in $L^2_\varpi(\Omega)$. Then by definition, there exists a sequence $\{n_k\}$ $(k \in \mathbb{N})$ such that

$$\|v_{n_{k+1}} - v_{n_k}\|_{0,\varpi} < \frac{1}{2^k}. \tag{1.2.2}$$

We choose a function $u \in L^2_\varpi(\Omega)$. Then by the Cauchy-Schwartz inequality,

$$\int_\Omega |w^2 u(v_{n_{k+1}} - v_{n_k})|\, d\Omega \le \|v_{n_{k+1}} - v_{n_k}\|_{0,\varpi}\|u\|_{0,\varpi} \le \frac{\|u\|_{0,\varpi}}{2^k}. \tag{1.2.3}$$

Summing the above for each k, we have

$$\sum_{k=1}^{\infty} \int_\Omega |w^2 u(v_{n_{k+1}} - v_{n_k})|\, d\Omega \le \|u\|_{0,\varpi}, \tag{1.2.4}$$

then it follows that

$$|w^2 u(x)| \sum_{k=1}^{\infty} \Big| v_{n_{k+1}}(x) - v_{n_k}(x) \Big| < \infty \quad a.e. \text{ on } \Omega \text{ (since its integral is also finite)}. \tag{1.2.5}$$

Then we have

$$\sum_{k=1}^{\infty} \Big| v_{n_{k+1}}(x) - v_{n_k}(x) \Big| < \infty \quad a.e. \text{ on } \Omega. \tag{1.2.6}$$

If (1.2.6) diverges on a set E of positive measure, then we could take $u(x)$ to be non-zero on a subset of E of positive measure which contradicts (1.2.5).

Since the k-th partial sum of the telescoping series (1.2.6) which converges almost everywhere on Ω is

$$v_{n_{k+1}}(x) - v_{n_1}(x), \tag{1.2.7}$$

we see that

$$v(x) = \lim_{k\to\infty} v_{n_k}(x) \tag{1.2.8}$$

defines $v(x)$ for almost all $x \in \Omega$.

For a given $\epsilon > 0$, we choose an integer N such that $n \ge N,\ m \ge N$ implies $\|v_n - v_m\|_{0,\varpi} \le \epsilon$. Then by Fatou's Theorem [8], if $n_k > N$ then

$$\|v - v_{n_k}\|_{0,\varpi} \le \liminf_{k\to\infty} \|v_{n_i} - v_{n_k}(x)\|_{0,\varpi} \le \epsilon. \tag{1.2.9}$$

That is, $v - v_{n_k} \in L^2_\varpi(\Omega)$. Now since, $v = (v - v_{n_k}) + v_{n_k}$, we see that $v \in L^2_\varpi(\Omega)$. Since ϵ was arbitrary,

$$\lim_{k\to\infty} \|v - v_{n_k}\|_{0,\varpi} = 0. \tag{1.2.10}$$

Finally, the inequality

$$\|v - v_n\|_{0,\varpi} \leq \|v - v_{n_k}\|_{0,\varpi} + \|v_{n_k} - v_n\|_{0,\varpi} \tag{1.2.11}$$

shows that v_n converges to v in $L^2_\varpi(\Omega)$, since by taking n and n_k large enough each of the terms on the right can be made arbitrarily small. ☐

Now in the multi-dimensional case, in this dissertation $1 < d \leq 3$, we define a general weighted inner product on $\left(L^2_\varpi(\Omega)\right)^d$ by

$$(\mathbf{u}, \mathbf{v})_\varpi := \int_\Omega \left(\sum_{i=1}^{d} w_i^2 u_i v_i \right) d\Omega \tag{1.2.12}$$

for any $\mathbf{u} = (u_1, u_2, ..., u_d)^T$ and $\mathbf{v} = (v_1, v_2, ..., v_d)^T \in \left(L^2_\varpi(\Omega)\right)^d$, where w_i are weights. The induced weighted L^2-norm is

$$\|\mathbf{v}\|_{0,\varpi} := \sqrt{(\mathbf{v}, \mathbf{v})_\varpi} = \left(\int_\Omega \left(\sum_{i=1}^{d} w_i^2 v_i^2 \right) d\Omega \right)^{1/2}. \tag{1.2.13}$$

The space of all multi-dimensional weighted-square-integrable functions is defined as

$$\mathbf{L}^2_\varpi(\Omega) := \{\mathbf{v} \in \left(L^2_\varpi(\Omega)\right)^d : \|\mathbf{v}\|_{0,\varpi} < \infty\}. \tag{1.2.14}$$

Furthermore, the pair $(\mathbf{L}^2_\varpi(\Omega), (\cdot,\cdot)_w)$ is Hilbert space [1].

Definition 1.2.1. [Sobolev Spaces [8, 22]]

Let $l \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^l_p(\Omega)$ essentially consists of the functions in $L^p(\Omega)$ whose partial derivatives up to order l, in terms of distributions, are identified with functions in $L^p(\Omega)$. With the set $\alpha = (\alpha_1, ..., \alpha_d)$ and $|\alpha| = \sum_{i=1}^{d} \alpha_i$, the derivatives are denoted by

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_d} x_d}. \tag{1.2.15}$$

We define the Sobolev space $W^l_p(\Omega)$, the set of all functions $v : \Omega \to \mathbb{R}$, by

$$W^l_p(\Omega) := \{v : v \in L^p(\Omega), D^\alpha v \in L^p(\Omega), \text{ for all multi} - \text{indicies } \alpha \text{ with } |\alpha| \leq l\}, \tag{1.2.16}$$

where $D^\alpha v$ represents the weak derivatives of v. $W^l_p(\Omega)$ is equipped with norm

$$\|v\|_{l,p} := \left(\sum_{|\alpha| \leq l} \int_\Omega |D^\alpha v|^p \right)^{1/p}, \tag{1.2.17}$$

and semi-norm $|\cdot|_{l,p}$. $W_p^l(\Omega)$ is a Banach Space. When $p=2$, we have the special case denoted by $H^l(\Omega)$ and it is defined by

$$H^l(\Omega) := \left\{v : \Omega \to \mathbb{R} | \ v \in L^2(\Omega), \text{ the weak derivatives } D^\alpha v \text{ exists in } L^2(\Omega)\right\}. \quad (1.2.18)$$

$H^l(\Omega)$ is a Hilbert space and it is equipped with the norm $\|\cdot\|_l$.

Furthermore, for $0 < l < 1$ and $1 \leq p < \infty$, we define the Sobolev space with fractional order, by

$$W_p^l(\Omega) := \left\{ v \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{nl+p}} dxdy < \infty \right\}, \quad (1.2.19)$$

and

$$H^l(\Omega) = W_2^l(\Omega).$$

We define the following semi-norm in $W_p^l(\Omega)$

$$|v|_{l,p}^p := \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{nl+p}} dxdy, \quad (1.2.20)$$

and norm

$$\|v\|_{l,p}^p := \|v\|_p^p + |v|_{l,p}^p. \quad (1.2.21)$$

Furthermore, an important concept to consider is the inclusion between certain Sobolev spaces. These inclusion maps (which are continuous) allow for an equivalence class of functions f belonging to one Sobolev space to be contained in another Sobolev space, and thereby taking certain improved local behaviour within that space. These inclusions are important for some of the analysis in this work. The following is the Sobolev embedding theorem, where embedding here means the continuous inclusion of functions within the spaces. The embedding theorem is in two parts.

Theorem 1.3. [Sobolev Embedding Theorems [8, 22]]

Let $k, l \geq 0$ and $p \in [1,\infty)$, $\Omega \in \mathbb{R}^d$.

Part 1: *If $k > l$, $p < q \in [1,\infty)$, and $\frac{1}{p} - \frac{k}{d} = \frac{1}{q} - \frac{l}{d}$, then there exists a continuous embedding*

$$W_p^k(\Omega) \hookrightarrow W_p^l(\Omega). \quad (1.3.1)$$

Part 2: *Let $C^{r,\alpha}(\Omega)$ be a Holder space. If $d < p$ and $\frac{1}{p} - \frac{k}{d} = \frac{r+\alpha}{d}$, $\alpha \in (0,1]$, then the following embedding exists*

$$W_p^k(\Omega) \hookrightarrow C^{r,\alpha}(\Omega). \quad (1.3.2)$$

Remark 1.3.1. As a special case, if $k = 1,\ l = 0$, for a Sobolov conjugate p^* with $\left(\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{d}\right)$, we have

$$W_p^1(\Omega) \hookrightarrow L^{p^*}(\Omega). \tag{1.3.3}$$

We are particularly interested in the space $H^1(\Omega)$ and its associated subspaces. Consider for example $v \in L^2(\Omega)$ with $\Omega \subset \mathbb{R}$. All functions in $L^2(\Omega)$ are defined except to a zero-measure set and hence a restriction of $v \in L^2(\Omega)$ to this set is not possible. However, functions in Sobolev spaces are more regular than functions in $L^2(\Omega)$. The restriction of functions is considered to the boundary of Ω ($\partial\Omega$) and is known as the trace of the function on the boundary of Ω [8].

Assume that Ω is a Lipschitz bounded domain, that is, the boundary $\partial\Omega$ can be locally represented by Lipschitz continuous function. We define

$$C^\infty(\mathbb{R}^d)|_\Omega := \left\{v : \Omega \to \mathbb{R} |\ v \text{ can be extended to } \overline{v} : \Omega \to \mathbb{R} \text{ and } \overline{v} \in C^\infty(\mathbb{R}^d)\right\}. \tag{1.3.4}$$

Then, $C^\infty(\mathbb{R}^d)|_\Omega$ is dense in $H^1(\Omega)$. Therefore, the mapping that restricts v to $\partial\Omega$

$$\begin{aligned} T : \left(C^\infty(\mathbb{R}^d)|_\Omega, \|\cdot\|_1\right) &\to \left(L^2(\partial\Omega), \|\cdot\|_0\right), \\ v &\mapsto v|_{\partial\Omega} \end{aligned}$$

is continuous. That is, there exists a unique, linear and continuous extension

$$T : \left(H^1(\Omega), \|\cdot\|_1\right) \to \left(L^2(\partial\Omega), \|\cdot\|_0\right). \tag{1.3.5}$$

Then, $T(v) \in L^2(\partial\Omega)$ and there exists some constant $C > 0$ such that

$$\|T(v)\|_0 \leq C\|v\|_1, \text{ for all } v \in H^1(\Omega) \tag{1.3.6}$$

where $T(v) \in L^2(\partial\Omega)$ is called the trace of $v \in H^1(\Omega)$.

We further define $H_0^l(\Omega)$ a subspace of $H^l(\Omega)$ by

$$H_0^l(\Omega) = \{v \in H^l(\Omega) : Tv = v|_{\partial\Omega} = 0\}, \tag{1.3.7}$$

where T is a trace operator defined by $T : H^l(\Omega) \to L^p(\partial\Omega)$. This space is important because differential equations (or PDEs) are often coupled with boundary conditions. This space is relevant in our analysis to capture the behaviour and value of v at the boundary $\partial\Omega$.

This then leads to the following:

$$H_0^1(\Omega) := \left\{v : v \in L^2(\Omega), \nabla v \in L^2(\Omega), \text{ and } v|_{\partial\Omega} = 0\right\}, \tag{1.3.8}$$

where the derivation in ∇v is understood in the weak sense. This space plays an important role in the analysis of a Dirichlet problem using the fact that for a bounded domain Ω, the injection from $H_0^1(\Omega)$ to $L^2(\Omega)$ is compact.

Now, to handle the degeneracy of the PDE, we define the weighted Sobolev space,

$$H^1_{0,\varpi}(\Omega) := \left\{ v : v \in L^2(\Omega), \nabla v \in L^2_{\varpi}(\Omega) \text{ and } v|_{\partial\Omega} = 0 \right\},$$

where $\partial\Omega$ denotes the boundary of Ω. We define the weighted inner product on $H^1_{0,\varpi}(\Omega)$ by $(\cdot,\cdot)_H := (\cdot,\cdot) + (\cdot,\cdot)_{\varpi}$, which induces the norm

$$\|v\|_{1,\varpi} = \left[\|v\|_0^2 + \|\nabla v\|_{0,\varpi}^2\right]^{1/2} = \left[(w \cdot \nabla v, w \cdot \nabla v) + (v, v)\right]^{1/2},$$

where '$\cdot$' is the standard inner product of $\mathbb{R}^d$.

These definitions further extend to the multi-dimensional space, where in our particular case, we define

$$H^1_{0,\varpi}(\Omega) := \left\{ v : v \in L^2(\Omega), \nabla v \in \mathbf{L}^2_{\varpi}(\Omega) \text{ and } v|_{\partial\Omega} = 0 \right\}.$$

Then we define the weighted inner product on $H^1_{0,w}(\Omega)$ by $(\cdot,\cdot)_H := (\cdot,\cdot) + (\cdot,\cdot)_{\varpi}$, which is equipped with the norm

$$\|v\|_{1,\varpi} = \left[\|v\|_0^2 + \|\nabla v\|_{0,\varpi}^2\right]^{1/2} = \left[(w \cdot \nabla v, w \cdot \nabla v) + (v, v)\right]^{1/2}.$$

Finally for any Hilbert space $H(\Omega)$ of classes of functions defined on Ω, we let $L^p((0,T); H(\Omega))$ denote the space defined by

$$L^p((0,T); H(\Omega)) = \{v(\cdot,t) : v(\cdot,t) \in H(\Omega) \ a.e. \text{ in } (0,T) : \|v(\cdot,t)\|_H \in L^p((0,T))\} \tag{1.3.9}$$

which induces the norm

$$\|v\|_{L^p((0,T);H(\Omega))} = \left(\int_0^T \|v(\cdot,t)\|_H^p dt\right)^{1/p},$$

where $\|\cdot\|_H$ is the natural norm on $H(\Omega)$. Furthermore, for $\Theta = \Omega \times (0,T)$, we have that

$$L^p((0,T); L^p(\Omega)) = L^p(\Omega \times (0,T)) = L^p(\Theta). \tag{1.3.10}$$

Further, we shall consider important theorems and results which we will use in this work.

Definition 1.3.2. [Eqiuvalence Norm [22, 25]]

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a space V. They are said to be equivalent if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|u\|_1 \leq \|u\|_2 \leq C_2\|u\|_1, \text{ for all } u \in V. \tag{1.3.11}$$

For a theory to govern the unique solvability of problems in the variational form, we consider the theorem below

Theorem 1.4. [8, 22]

For $t \in [0,T]$, let $A(\cdot,\cdot;t) : V \times V \to \mathbb{R}$, be a bilinear form, where V is a Hilbert space, which satisfies

$$A(u,u;t) \geq C\|u\|_V^2, \quad \text{for all } u \in V$$

$$A(u,v;t) \leq M\|u\|_V\|v\|_V, \quad \text{for all } u,v \in V$$

for some positive constants $M, C > 0$.

Let $F : V \to \mathbb{R}$ be a continuous linear functional which is

$$|F(v)| \leq N\|v\|_V, \text{ for all } v \in V,$$

for $t \in (0,T)$. Then, there exists a unique $u \in V$, with $u(0)$ given, satisfying

$$\left(\frac{du}{dt}, v\right) + A(u,v;t) = F(v), \quad \text{for all } v \in V \tag{1.4.1}$$

Proof. The proof of the Theorem 1.4 can be found in [22]. ☐

1.5 Option basics

An option is a tradable derivative contract, written by the option writer, that gives to its holder (option buyer) the full non-obligatory rights to buy (call option) or sell (put option) a specified amount of assets (e.g. stocks, shares, commodities, currencies, etc.) at an agreed fixed price (strike price) on (European option) or any time before (American option) a particular date (maturity or expiry date) [9, 13, 19, 42]. This definition of an option has a simple financial structure. Such options are known as vanilla options. The two vanilla options are European and American options. There are however other types with a more complex financial structure known as exotic options. Our main focus in this thesis is the vanilla options. For more information on the exotic options please see standard texts [13, 19] and some of the references therein.

1.5.1 Black Scholes pricing model and European options

The valuation of options has always been crucial to practitioners who have adopted several pricing models and valuation theories to find the fair value [44]. Amongst these theories was the no-arbitrage valuation, which in the spring of 1973 was the basis for the important seminal paper [9]. The authors, Fischer Black and Myron Scholes, showed that the value of the European call option was governed by a partial differential equation (a result of Itô's lemma [13, 19] for stochastic differential equations) and that of the underlying security price in time [9]. Their initial assumption was that the underlying asset price $S(t)$ followed the well known geometric Brownian motion process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ t \in [0,T];\ S(0) = S_0 > 0, \tag{1.5.1}$$

where μ is the total expected return on the asset, σ is the volatility and W is a Wiener (standard Brownian motion) process that captures the uncertainty of the underlying in the market [19, 44]. They assumed that the no-arbitrage principle holds and short selling was permissible, and non-integer quantities of the asset can be traded. They further assumed that trading in this asset was continuous and unlimited, that is, there were no transaction costs, no taxes, or any other form of market friction. Further, their model constants r, the risk-free interest rate, and the volatility σ were known by investors on the market. The so-called Black-Scholes partial differential equation that governs the price of the European call option $V(s,t)$ on $R^+ \times [0,T)$ is given by

$$\frac{\partial V(s,t)}{\partial t} + rs\frac{\partial V(s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} - rV(s,t) = 0 \tag{1.5.2}$$

with associated boundary and final conditions

$$\begin{aligned} V(s,T) &= V^*(s) \\ V(0,t) &= V^*(0)e^{-r(T-t)} \\ \lim_{s\to\infty} V(s,t) &= s - Ee^{-r(T-t)}. \end{aligned}$$

where $V^*(s) = \max\{s-E, 0\}$ is the payoff for the call option. Below is a diagram of the payoff function

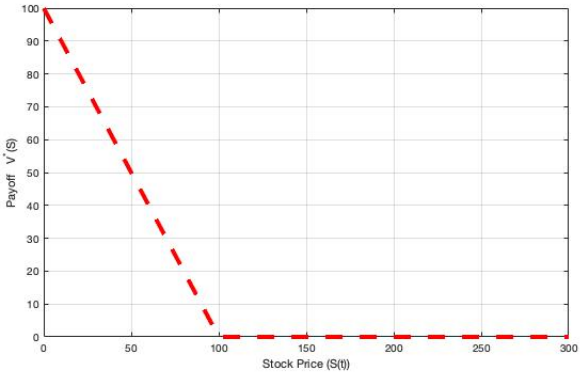

Figure 1.1: Payoff function for European call option with $E = 100$ and $S(T) = 300$.

They further obtained an explicit form for the value of the European call option $C(s,t)$ at time t given by [16, 44]

$$C(s,t) = sN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{1.5.3}$$

where $N(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{y^2/2}dy$

$$d_1 = \frac{\ln(s/E) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(s/E) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

with d_1 and d_2 related by

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

We can then rewrite the price of the call as

$$C(s,t) = e^{-r(T-t)}[SN(d_1)e^{r(T-t)} - EN(d_2)].$$

We can use the risk neutral valuation to explain the meaning, that is, $N(d_2)$ is the probability that an investor will exercise the option in a risk-neutral world. $EN(d_2)$ is the probability that the strike price will be paid and $se^{-r(T-t)}N(d_1)$ is the expected value of a variable which equals $S(T)$ when the option is in the money and zero otherwise in a risk-neutral setting. The interested reader may consult the references [13, 16, 19, 44] for more information on the Black-Scholes model.

Now, even though the single asset model (1.5.1) works well in both theory and practice, it is not sufficient when the payoffs of models are dependent on multiple correlated assets [13, 19, 44]. Pricing multiple assets option starts with the premise of the ith asset price follows the geometric Brownian motion process

$$ds_i(t) = \mu_i s_i(t)dt + \sigma_i s_i(t)dW_i(t), \quad t \in [0,T], \tag{1.5.4}$$

where μ_i denotes the drift rate for the ith asset, σ_i denotes the volatility of the ith asset, and W_i denotes the standard Brownian motion of the ith asset. Now, ρ_{ij} describes the correlation of W_i and W_j for a pair i, j. Then the value $V(s_1, \cdots, s_n, t)$ of an European multi-asset call option at time t is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}s_i s_j\frac{\partial^2 V}{\partial s_i \partial s_j} + r\sum_{i=1}^{n}s_i\frac{\partial V}{\partial s_i} - rV = 0, \tag{1.5.5}$$

$\Omega = \{(s_1, \cdots, s_n), s_i > 0, \;\; i \in \{1, 2, .., n\}\}$, ρ_{ij} is the correlation coefficient associated to the ith and jth assets and r is the interest rate, with appropriate final condition

$$V(s_1, \cdots, s_n, T) \;\; = \;\; V^*(s_1, \cdots, s_n).$$

The boundary conditions result from finding the solution to the $(n-1)$ dimensional problem, details can be found in [13, 19, 31, 37]. That is, to solve the n dimensional problem, the solution to the $(n-1)$ dimensional problem is the recommended boundary condition used. The closed-form solution for the two dimensional European option problem is well documented in [31, 37].

Definition 1.5.2. [Degenerate PDEs [8]]

Let us consider the following second order PDE:

$$-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)+\sum_{j=1}^{N}b_j\frac{\partial u}{\partial x_j}+cu=f, \qquad \text{in } \Omega, \tag{1.5.6}$$

with boundary conditions on $\partial\Omega$ or on part of $\partial\Omega$, and the functions $a_{ij}(x)\in C^1(\overline{\Omega}),\ 1\leq i,j\leq N$. The coefficients a_{ij} of the PDE in (1.5.6) are said to satisfy the ellipticity condition (see [8]) if the following inequality holds

$$\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\geq\alpha|\xi|^2, \quad \text{for all } \ x\in\Omega, \quad \text{for all } \ \xi\in\mathbb{R}^d \ \ \textit{with } \alpha>0. \tag{1.5.7}$$

The PDE is said to be degenerate if the coefficients a_{ij} do not satisfy the ellipticity condition (1.5.7) but only

$$\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\geq 0, \quad \text{for all } x\in\Omega, \quad \text{and for all } \xi\in\mathbb{R}^d. \tag{1.5.8}$$

Then from (1.5.5), the multi-dimensional Black-Scholes equation is given by

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}s_is_j\frac{\partial^2 V}{\partial s_i\partial s_j}+r\sum_{i=1}^{n}s_i\frac{\partial V}{\partial s_i}-rV=0, \tag{1.5.9}$$

is said to be degenerate (or degenerate elliptic) at the point $s=(0,0,...,0,0)$, n-times, since at the point $(0,0,...,0,0)$, n-times, we have

$$\sum_{i,j=1}^{N}s_is_j\xi_i\xi_j=0, \qquad \text{for all } \xi\in\mathbb{R}^d. \tag{1.5.10}$$

As we mentioned earlier in the introduction, the standard finite difference method is applied to the PDE is no longer monotone and hence fails to give accurate approximations as s becomes small. This will therefore have a negative impact on the accuracy of the solution of the PDE. We will consider a fitted local approximation technique that was proposed by [43] to handle the degeneracy close to the boundary at zero.

1.6 American options

Unlike the European options, which can only be exercised at maturity, investors holding American options get the liberty to exercise any time until (and including) maturity. This makes pricing American options quite challenging since at each time the investor would have to determine an optimal value of the option, this is called early exercise constraint which makes the valuation of an American option a free boundary problem [13, 32, 38]. That is, at each time there is a particular value of the asset, which marks the boundary between two regions: to one side one should hold the option and to another side, one should exercise the option. So, if the investors act optimally, the value of an American option cannot fall below the value that would be obtained if it were exercised early [19, 21]. In this section, we consider the so-called linear complementarity problem (LCP) that encapsulates the American option pricing problem. We will consider the two-dimensional case of the problem. The one dimensional case was considered in [32, 38].

1.6.1 Linear Complementarity Problem

In this subsection, we describe the linear complementarity problem which is well known to govern a multi-asset American put option [32, 38]. Now let the market prices for individual non-dividend paying assets be $\{s_i\}_{i\in\mathbb{N}}$ which follows (1.5.4).

Then it follows from similar arguments, as with the one dimensional case [32, 38], that $V(s_1,\cdots,s_n,t)$ (the value of an American put option with expiry date $T>0$) is the solution of the following system of partial differential inequalities called Linear Complementarity Problem (LCP) [6, 19, 24]

$$\begin{cases} \mathcal{L}V\left(V(s_1,\cdots,s_n,t)-V^*(s_1,\cdots,s_n)\right)=0, \quad a.e.\ in\ \ \Theta=\Omega\times(0,T) \\ V(s_1,\cdots,s_n,t)-V^*(s_1,\cdots,s_n)\geq 0, \quad a.e.\ in\ \Theta=\Omega\times(0,T) \\ \mathcal{L}V\geq 0, \quad a.e.\ in\ \Theta=\Omega\times(0,T) \end{cases} \tag{1.6.1}$$

where V^* is the payoff function which is defined as

$$V^*(s_1,\cdots,s_n)=\max\left(E-\sum_{i=1}^{n}\alpha_i s_i, 0\right),$$

and E, being the agreed exercise price at expiry date T. Note that we are considering the transformation $t=T-t$ here, so we have

$$\mathcal{L}V=\frac{\partial V}{\partial t}-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}s_is_j\frac{\partial^2 V}{\partial s_i\partial s_j}-r\sum_{i=1}^{n}s_i\frac{\partial V}{\partial s_i}+rV, \tag{1.6.2}$$

$\Omega=\{(s_1,\cdots,s_n), s_i>0,\ \ i\in\{1,2,..,n\}\}$, α_i is the weight associated to the ith asset and r is the interest rate.

The boundary conditions and the final condition are given by

$$\begin{aligned} V(\mathbf{S},t) &= g_i(\mathbf{S},t),\ i=1,....,n,\ \mathbf{S}\in\Omega_i\times[0,T], & (1.6.3)\\ \lim_{s_i\to\infty} V(\mathbf{S},t) &= G_i(s_1,\cdots,s_{i-1},0,s_{i+1},\cdots,s_n,t),\ i=1,...,n,\ \ (\mathbf{S},t)\in\Omega\times[0,T], & (1.6.4)\\ V(\mathbf{S},0) &= V^*(\mathbf{S}) = \mathbf{S}\in\Omega, & (1.6.5)\end{aligned}$$

where $\Omega_i = \{(s_1,\cdots,s_{i-1},0,s_{i+1},\cdots,s_n);\ s_j\geq 0,\ \text{for all}\ j\in\{1,2,..,n\}\}$ and $\mathbf{S}=(s_1,\cdots,s_n)$.

The functions $g_i(\cdot,\cdot)$ and $G_i(\cdot,\cdot)$ are given and provide suitable boundary conditions. Typically, we determine $g_i(\cdot,\cdot)$ by solving the $(n-1)$-dimensional American put option problem and $G_i(\cdot,\cdot)$ is identical to zero.

1.6.2 Well-posedness of three assets model problem

The well-posedness for one asset and two assets are well-known [6, 18, 32, 33, 38]. Here we provide the well-posedness for three assets options pricing.

Reformation of LCP

In [32, 38] the LCP was reformulated for convenience during theoretical analysis. We write the LCP problem in conservative form to facilitate the theoretical analysis into a variational form. The variational forms have been extensively studied in [6, 7, 22]. The well-posedness here is studied in a truncated domain $\Omega=[0,X]\times[0,Y]\times[0,Z]$, where we have assumed that $X\gg E$, $Y\gg E$ and $Z\gg E$ [6, 24, 33, 38]. For the sake of simplicity, we set $x=s_1, y=s_2$ and $z=s_3$. Let V_0 be a thrice differentiable function satisfying the boundary conditions in (1.6.3)-(1.6.5) [32, 33]. Let us consider the transformation

$$u(x,y,z,t)=e^{\beta t}\left(V_0-V\right), \tag{1.6.6}$$

with $\beta=\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_2^3+\rho_{12}\sigma_1\sigma_2+\rho_{13}\sigma_1\sigma_3+\rho_{23}\sigma_2\sigma_3)$ [33].

The problem (1.6.1) becomes

$$\begin{cases} Lu(x,y,z,t)\leq f, \\ u(x,y,z,t)-u^*(x,y,z,t)\leq 0, \\ \left(Lu-f\right)\left(u(x,y,z,t)-u^*(x,y,z,t)\right)=0, \end{cases} \tag{1.6.7}$$

where L is the conservative differential operator given by

$$Lu=-u_t-\nabla\cdot\left(A\nabla u+\mathbf{b}u\right)+cu, \tag{1.6.8}$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sigma_1^2 x^2 & \frac{1}{2}\rho_{12}\sigma_1\sigma_2 xy & \frac{1}{2}\rho_{13}\sigma_1\sigma_3 xz \\ \frac{1}{2}\rho_{12}\sigma_1\sigma_2 xy & \frac{1}{2}\sigma_2^2 y^2 & \frac{1}{2}\rho_{23}\sigma_2\sigma_3 yz \\ \frac{1}{2}\rho_{13}\sigma_1\sigma_3 xz & \frac{1}{2}\rho_{23}\sigma_2\sigma_3 yz & \frac{1}{2}\sigma_3^2 z^2 \end{pmatrix}, \tag{1.6.9}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} rx - \sigma_1^2 x - \frac{1}{2}\rho_{12}\sigma_1\sigma_2 x - \frac{1}{2}\rho_{13}\sigma_1\sigma_3 x \\ ry - \sigma_2^2 y - \frac{1}{2}\rho_{12}\sigma_1\sigma_2 y - \frac{1}{2}\rho_{23}\sigma_2\sigma_3 y \\ rz - \sigma_3^2 z - \frac{1}{2}\rho_{12}\sigma_1\sigma_3 z - \frac{1}{2}\rho_{23}\sigma_2\sigma_3 z \end{pmatrix},$$

and

$$c = 4r - (\sigma_1^2 + \sigma_2^2 + \sigma_2^3 + \rho_{12}\sigma_1\sigma_2 + \rho_{13}\sigma_1\sigma_3 + \rho_{23}\sigma_2\sigma_3) + \beta, \tag{1.6.10}$$

$$\beta = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_2^3 + \rho_{12}\sigma_1\sigma_2 + \rho_{13}\sigma_1\sigma_3 + \rho_{23}\sigma_2\sigma_3). \tag{1.6.11}$$

The functions $f(x,y,z,t)$ and $u^*(x,y,z,t)$ are defined as

$$f(x,y,z,t) = e^{\beta t}\mathcal{L}V_0, \quad u^* = e^{\beta t}\left(V_0 - V^*\right).$$

The boundary and final conditions become,

$$\begin{aligned} u(0,y,z,t) &= 0 = u(X,y,z,t), \quad t \in [0,T], \quad y \in [0,Y], \quad z \in [0,Z] \\ u(x,0,z,t) &= 0 = u(x,Y,z,t), \quad t \in [0,T], \quad x \in [0,X], \quad z \in [0,Z] \\ u(x,y,0,t) &= 0 = u(x,y,Z,t), \quad t \in [0,T], \quad x \in [0,X], \quad y \in [0,Y], \end{aligned}$$

and

$$u(x,y,z,0) = u^*(x,y,z,0).$$

Well-posedness of variational form

From (1.2.12) and (1.2.13), we have that the weighted inner product on $(L^2(\Omega))^3$ is given by

$$(\mathbf{u},\mathbf{v})_\varpi := \int_\Omega (x^2 u_1 v_1 + y^2 u_2 v_2 + z^2 u_3 v_3) \mathrm{d}\Omega,$$

for any $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\mathbf{v} = (v_1, v_2, v_3)^T \in (L^2(\Omega))^3$. The corresponding weighted L^2-norm is

$$\|\mathbf{v}\|_{0,\varpi} := \sqrt{(\mathbf{v},\mathbf{v})_\varpi} = \left(\int_\Omega (x^2 v_1^2 + y^2 v_2^2 + z^2 v_3^2) \mathrm{d}\Omega\right)^{1/2}.$$

The energy norm of $H^1_{0,\varpi}(\Omega)$ is defined by

$$\|v\|_{1,\varpi}^2 := |v|_{1,\varpi}^2 + \|v\|_0^2, \quad |v|_{1,\varpi} := \|\nabla v\|_{0,\varpi}, \quad v \in H^1_{0,\varpi}(\Omega) \tag{1.6.12}$$

Theorem 1.7. *Assume that* $\rho_{ij} \in [0, 0.5]$. *Find* $u(t) \in H^1_{0,\varpi}(\Omega)$ *such that* for all $v \in H^1_{0,\varpi}(\Omega)$, *and* $t \in (0, T)$ *a.e.*

$$\left(-\frac{du}{dt}, v\right) + \mathbb{A}(u(t), v; t) = (f(t), v)$$
$$\text{with } u(T) \text{ given.} \tag{1.7.1}$$

Indeed, the proof Theorem 1.7 is done by using Theorem 1.4. That is, the proof depends on the existence and uniqueness requirements of Theorem 1.4 being fulfilled. This is therefore achieved by the following lemma.

Lemma 1.7.1. There exist positive constants C and M, such that

$$\mathbb{A}(v, v; t) \geq C\|v\|^2_{1,\varpi}, \tag{1.7.2}$$
$$|\mathbb{A}(v, w; t)| \leq M\|v\|_{1,\varpi}\|w\|_{1,\varpi}, \quad \forall\, v, w \in H^1_{0,\varpi}(\Omega), \tag{1.7.3}$$

where $\mathbb{A}$ is a bilinear form defined by

$$\mathbb{A}(v, w; t) = (A\nabla v + \mathbf{b}v, \nabla w) + (cv, w), \tag{1.7.4}$$

and the transformed LCP (1.6.7) has a unique solution.

Proof. We first show that $\mathbb{A}(\cdot, \cdot; t)$ is coercive. The proof is an extension of the work in [33]. Let $v \in H^1_{0,\varpi}(\Omega)$, then following integrating by parts line by line in [39], we have

$$\begin{aligned}\int_\Omega \mathbf{b}v \cdot \nabla v d\Omega &= \int_{\partial\Omega} v^2\mathbf{b} \cdot n ds - \int_\Omega v\nabla \cdot (\mathbf{b}v) d\Omega \\ &= -\int_\Omega v\mathbf{b} \cdot \nabla v d\Omega - \int_\Omega v^2\nabla \cdot \mathbf{b} d\Omega \\ &= -\frac{1}{2}\int_\Omega v^2\nabla \cdot \mathbf{b} d\Omega. \end{aligned} \tag{1.7.5}$$

Now by definition of the bilinear form, we have

$$\begin{aligned}\mathbb{A}(v, v; t) &:= (A\nabla v + \mathbf{b}v, \nabla v) + (cv, v) \\ &= (A\nabla v, \nabla v) + (\mathbf{b}v, \nabla v) + (cv, v).\end{aligned}$$

Using (1.7.5) allows to have

$$\begin{aligned}&\mathbb{A}(v, v; t) = \\ &\quad \frac{1}{2}\int_\Omega \left(\sigma_1^2 x^2 v_x^2 + \sigma_2^2 y^2 v_y^2 + \sigma_3^2 z^2 v_z^2 + 2\rho_{12}\sigma_1\sigma_2 xy v_x v_y + 2\rho_{13}\sigma_1\sigma_3 xz v_x v_z + 2\rho_{23}\sigma_2\sigma_3 yz v_y v_z\right) d\Omega \\ &\quad + \left(\left(c - \frac{1}{2}\nabla \cdot \mathbf{b}\right) v, v\right).\end{aligned}$$

Rearranging the terms and using the expression of the vector $\mathbf{b}$ yields

$$\begin{aligned}
\mathbb{A}(v,v;t) &= \frac{1}{2}\int_{\Omega}\left[(1-\rho_{12}-\rho_{13})\sigma_1^2x^2v_x^2+(1-\rho_{12}-\rho_{23})\sigma_2^2y^2v_y^2+(1-\rho_{13}-\rho_{23})\sigma_3^2z^2v_z^2\right]d\Omega\\
&+\frac{1}{2}\int_{\Omega}\left[\rho_{12}(\sigma_1xv_x+\sigma_2yv_y)^2+\rho_{13}(\sigma_1xv_x+\sigma_3zv_z)^2+\rho_{23}(\sigma_2yv_y+\sigma_3zv_z)^2\right]d\Omega\\
&+\left(\left(\beta+\frac{5}{2}r-\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\rho_{12}\sigma_1\sigma_2+\rho_{13}\sigma_1\sigma_3+\rho_{23}\sigma_2\sigma_3\right)\right)v,v\right)\\
&\geq C\left(\int_{\Omega}\left[\sigma_1^2x^2v_x^2+\sigma_2^2y^2v_y^2+\sigma_3^2z^2v_z^2\right]d\Omega+\|v\|_0^2\right)\\
&\geq C\left(|v|_{1,\varpi}^2+\|v\|_0^2\right).
\end{aligned}$$

Hence we have that,

$$\mathbb{A}(v,v;t) \geq C\|v\|_{1,\varpi}^2. \tag{1.7.6}$$

So the coerciveness of the bilinear form $\mathbb{A}(v,v;t)$ is now proved.

Now, we have that,

$$\begin{aligned}
|\mathbb{A}(v,w;t)| &= |\,(A\nabla v+\mathbf{b}v,\nabla w)+(cv,w)|\\
&\leq |\,(A\nabla v,\nabla w)\,|+|\,(\mathbf{b}v,\nabla w)+|(cv,w)|
\end{aligned}$$

Considering the first term

$$\begin{aligned}
&|\,(A\nabla v,\nabla w)\,|\\
&\leq \frac{1}{2}\left|\int_{\Omega}\left(\sigma_1^2x^2v_xw_x+\sigma_2^2y^2v_yw_y+\sigma_3^2z^2v_zw_z\right)d\Omega\right|\\
&+\left|\int_{\Omega}\left(\rho_{12}\sigma_1\sigma_2xy(v_xw_y+w_xv_y)+\rho_{13}\sigma_1\sigma_3xz(v_xw_z+w_xv_z)+\rho_{23}\sigma_2\sigma_3yz(v_yw_z+w_yv_z)\right)d\Omega\right|,
\end{aligned} \tag{1.7.7}$$

then considering the first term of (1.7.7) above and using the triangle inequality we have,

$$\begin{aligned}
&\frac{1}{2}\left|\int_{\Omega}\left(\sigma_1^2x^2v_xw_x+\sigma_2^2y^2v_yw_y+\sigma_3^2z^2v_zw_z\right)d\Omega\right|\\
&\leq \frac{1}{2}\left|\int_{\Omega}\left(\sigma_1^2x^2v_xw_x\right)d\Omega\right|+\frac{1}{2}\left|\int_{\Omega}\left(\sigma_2^2y^2v_yw_y\right)d\Omega\right|+\frac{1}{2}\left|\int_{\Omega}\left(\sigma_3^2z^2v_zw_z\right)d\Omega\right|\\
&=\frac{1}{2}\left|(\sigma_1xv_x,\sigma_1xw_x)\right|+\frac{1}{2}\left|(\sigma_2yv_y,\sigma_2yw_y)\right|+\frac{1}{2}\left|(\sigma_3zv_z,\sigma_3zw_z)\right|
\end{aligned}$$

and by using the Cauchy-Schwartz inequality for the standard L^2-inner product (1.1.9) with norm (1.1.8) we have,

$$
\begin{aligned}
&\frac{1}{2}\left|\int_\Omega \left(\sigma_1^2x^2v_xw_x+\sigma_2^2y^2v_yw_y+\sigma_3^2z^2v_zw_z\right)d\Omega\right| \\
&\le \frac{1}{2}\left(\sigma_1xv_x,\sigma_1xw_x\right)^{1/2}\left(\sigma_1xv_x,\sigma_1xw_x\right)^{1/2}+\frac{1}{2}\left(\sigma_2yv_y,\sigma_2yw_y\right)^{1/2}\left(\sigma_2yv_y,\sigma_2yw_y\right)^{1/2} \\
&+\frac{1}{2}\left(\sigma_3zv_z,\sigma_3zw_z\right)^{1/2}\left(\sigma_3zv_z,\sigma_3zw_z\right)^{1/2} \\
&=\frac{1}{2}\left[\int_\Omega \sigma_1^2x^2v_x^2d\Omega\right]^{1/2}\left[\int_\Omega \sigma_1^2x^2w_x^2d\Omega\right]^{1/2}+\frac{1}{2}\left[\int_\Omega \sigma_2^2y^2v_y^2d\Omega\right]^{1/2}\left[\int_\Omega \sigma_2^2y^2w_y^2d\Omega\right]^{1/2} \\
&+\frac{1}{2}\left[\int_\Omega \sigma_3^2z^2v_z^2d\Omega\right]^{1/2}\left[\int_\Omega \sigma_3^2z^2w_z^2d\Omega\right]^{1/2} \\
&\quad\le \frac{1}{2}\left[\int_\Omega \left(\sigma_1^2x^2v_x^2+\sigma_2^2y^2v_y^2+\sigma_3^2z^2v_z^2\right)\right]^{1/2}\left[\int_\Omega \left(\sigma_1^2x^2w_x^2+\sigma_2^2y^2w_y^2+\sigma_3^2z^2w_z^2\right)\right]^{1/2} \\
&\quad\le M|v|_{1,\varpi}|w|_{1,\varpi}
\end{aligned}
$$

Now we consider the second term of (1.7.7),

$$
\begin{aligned}
&\left|\int_\Omega \left(\rho_{12}\sigma_1\sigma_2xy(v_xw_y+w_xv_y)+\rho_{13}\sigma_1\sigma_3xz(v_xw_z+w_xv_z)+\rho_{23}\sigma_2\sigma_3yz(v_yw_z+w_yv_z)\right)d\Omega\right| \\
\le\ &\left|\int_\Omega \rho_{12}\sigma_1\sigma_2xy(v_xw_y+w_xv_y)d\Omega\right|+\left|\int_\Omega \rho_{13}\sigma_1\sigma_3xz(v_xw_z+w_xv_z)d\Omega\right| \\
&+\left|\int_\Omega \rho_{23}\sigma_2\sigma_3yz(v_yw_z+w_yv_z)d\Omega\right| \\
\le\ &\rho_{12}\left(|(\sigma_1xv_x,\sigma_2yw_y)|+|(\sigma_1xw_x,\sigma_2yv_y)|\right)+\rho_{13}\left(|(\sigma_1xv_x,\sigma_3zw_z)|+|(\sigma_1xw_x,\sigma_3zv_z)|\right) \\
&+\rho_{23}\left(|(\sigma_2yv_y,\sigma_3zw_z)|+|(\sigma_2yw_y,\sigma_3zv_z)|\right)
\end{aligned}
$$

then by using the Cauchy-Schwartz inequality for the standard L^2-inner product (1.1.9) with norm (1.1.8) yields

$$
\begin{aligned}
&\left|\int_\Omega \left(\rho_{12}\sigma_1\sigma_2xy(v_xw_y+w_xv_y)+\rho_{13}\sigma_1\sigma_3xz(v_xw_z+w_xv_z)+\rho_{23}\sigma_2\sigma_3yz(v_yw_z+w_yv_z)\right)d\Omega\right| \\
\le\ &\rho_{12}\left[\left(\int_\Omega \sigma_1^2x^2v_x^2d\Omega\right)^{1/2}\left(\int_\Omega \sigma_2^2y^2w_y^2d\Omega\right)^{1/2}+\left(\int_\Omega \sigma_1^2x^2w_x^2d\Omega\right)^{1/2}\left(\int_\Omega \sigma_2^2y^2v_y^2d\Omega\right)^{1/2}\right] \\
&+\rho_{13}\left[\left(\int_\Omega \sigma_1^2x^2v_x^2d\Omega\right)^{1/2}\left(\int_\Omega \sigma_3^2z^2w_z^2d\Omega\right)^{1/2}+\left(\int_\Omega \sigma_1^2x^2w_x^2d\Omega\right)^{1/2}\left(\int_\Omega \sigma_3^2z^2v_z^2d\Omega\right)^{1/2}\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\rho_{23}\left[\left(\int_{\Omega}\sigma_2^2y^2v_y^2d\Omega\right)^{1/2}\left(\int_{\Omega}\sigma_3^2z^2w_z^2d\Omega\right)^{1/2}+\left(\int_{\Omega}\sigma_2^2y^2w_y^2d\Omega\right)^{1/2}\left(\int_{\Omega}\sigma_3^2z^2v_z^2d\Omega\right)^{1/2}\right] \\
\leq \;& \rho_{12}\left[\int_{\Omega}\left(\sigma_1^2x^2v_x^2+\sigma_2^2y^2v_y^2\right)d\Omega\right]^{1/2}\left[\int_{\Omega}\left(\sigma_1^2x^2w_x^2+\sigma_2^2y^2w_y^2\right)d\Omega\right]^{1/2} \\
& +\rho_{13}\left[\int_{\Omega}\left(\sigma_1^2x^2v_x^2+\sigma_3^2z^2v_z^2\right)d\Omega\right]^{1/2}\left[\int_{\Omega}\left(\sigma_1^2x^2w_x^2+\sigma_3^2z^2w_z^2\right)d\Omega\right]^{1/2} \\
& +\rho_{23}\left[\int_{\Omega}\left(\sigma_2^2y^2v_y^2+\sigma_3^2z^2v_z^2\right)d\Omega\right]^{1/2}\left[\int_{\Omega}\left(\sigma_2^2y^2w_y^2+\sigma_3^2z^2w_z^2\right)d\Omega\right]^{1/2}.
\end{aligned}
$$

Then clearly we have that,

$$
\begin{aligned}
& \left|\int_{\Omega}\left(\rho_{12}\sigma_1\sigma_2xy(v_xw_y+w_xv_y)+\rho_{13}\sigma_1\sigma_3xz(v_xw_z+w_xv_z)+\rho_{23}\sigma_2\sigma_3yz(v_yw_z+w_yv_z)\right)d\Omega\right| \\
\leq \;& \rho_{12}\left[\int_{\Omega}\left(\sigma_1^2x^2v_x^2+\sigma_2^2y^2v_y^2+\sigma_3^2z^2v_z^2\right)d\Omega\right]^{1/2}\left[\int_{\Omega}\left(\sigma_1^2x^2w_x^2+\sigma_2^2y^2w_y^2+\sigma_3^2z^2w_z^2\right)d\Omega\right]^{1/2} \\
& +\rho_{13}\left[\int_{\Omega}\left(\sigma_1^2x^2v_x^2+\sigma_2^2y^2v_y^2+\sigma_3^2z^2v_z^2\right)d\Omega\right]^{1/2}\left[\int_{\Omega}\left(\sigma_1^2x^2w_x^2+\sigma_2^2y^2w_y^2+\sigma_3^2z^2w_z^2\right)d\Omega\right]^{1/2} \\
& +\rho_{23}\left[\int_{\Omega}\left(\sigma_1^2x^2v_x^2+\sigma_2^2y^2v_y^2+\sigma_3^2z^2v_z^2\right)d\Omega\right]^{1/2}\left[\int_{\Omega}\left(\sigma_1^2x^2w_x^2+\sigma_2^2y^2w_y^2+\sigma_3^2z^2w_z^2\right)d\Omega\right]^{1/2} \\
\leq \;& M|v|_{1,\varpi}|w|_{1,\varpi}.
\end{aligned}
$$

Therefore from the first and second terms of (1.7.7), we have that,

$$
|\,(A\nabla v,\nabla w)\,|\leq M|v|_{1,\varpi}|w|_{1,\varpi}. \tag{1.7.8}
$$

Now we consider the case of $|(\mathbf{b}v,\nabla w|$ from (1.7.5), we have

$$
\begin{aligned}
|(\mathbf{b}v,\nabla w| &= \left|\int_{\Omega}\mathbf{b}v\cdot\nabla w d\Omega\right| \\
&= \left|\int_{\partial\Omega}wv\mathbf{b}\cdot nds-\int_{\Omega}w\nabla\cdot(\mathbf{b}v)d\Omega\right| \\
&= \left|-\int_{\Omega}w\mathbf{b}\cdot\nabla vd\Omega-\int_{\Omega}wv\nabla\cdot\mathbf{b}d\Omega\right| \\
&\leq \left|\int_{\Omega}w\mathbf{b}\cdot\nabla vd\Omega\right|+\left|\int_{\Omega}wv\nabla\cdot\mathbf{b}d\Omega\right| \\
&\leq \left|\int_{\Omega}w\mathbf{b}\cdot\nabla vd\Omega\right|+M\|v\|_0\|w\|_0.
\end{aligned}
$$

Now

$$\mathbf{b} = \begin{pmatrix} rx - \sigma_1^2 x - \frac{1}{2}\rho_{12}\sigma_1\sigma_2 x - \frac{1}{2}\rho_{13}\sigma_1\sigma_3 x \\ \\ ry - \sigma_2^2 y - \frac{1}{2}\rho_{12}\sigma_1\sigma_2 y - \frac{1}{2}\rho_{23}\sigma_2\sigma_3 y \\ \\ rz - \sigma_3^2 z - \frac{1}{2}\rho_{12}\sigma_1\sigma_3 z - \frac{1}{2}\rho_{23}\sigma_2\sigma_3 z \end{pmatrix},$$

and hence we have,

$$\left|\int_\Omega w\mathbf{b}\cdot\nabla v d\Omega\right| \leq \left|\int_\Omega w\Big[\left(r - \sigma_1^2 - \frac{\rho_{12}}{2}\sigma_1\sigma_2 - \frac{\rho_{13}}{2}\sigma_1\sigma_3\right)xv_x\right.$$

$$\left. + \left(r - \sigma_2^2 - \frac{\rho_{12}}{2}\sigma_1\sigma_2 - \frac{\rho_{23}}{2}\sigma_2\sigma_3\right)yv_y + \left(r - \sigma_2^2 - \frac{\rho_{13}}{2}\sigma_1\sigma_3 - \frac{\rho_{23}}{2}\sigma_2\sigma_3\right)zv_z\Big]d\Omega\right|$$

$$\left|\int_\Omega w\mathbf{b}\cdot\nabla v d\Omega\right| \leq M_1\left|\int_\Omega w\left(xv_x + yv_y + zv_z\right)d\Omega\right|.$$

Then by using the Cauchy-Schwartz inequality for the standard L^2-inner product (1.1.9) with norm (1.1.8), we have that,

$$\begin{aligned} \left|\int_\Omega w\mathbf{b}\cdot\nabla v d\Omega\right| &\leq M_1\left(\int_\Omega w^2 d\Omega\right)^{1/2}\cdot\left(\int_\Omega \left(xv_x + yv_y + zv_z\right)^2 d\Omega\right)^{1/2} \\ &\leq M\|w\|_0|v|_{1,\varpi} \end{aligned}$$

Hence we have that,

$$|(\mathbf{b}v, \nabla w)| \leq M\|w\|_0|v|_{1,\varpi} + \|v\|_0\|w\|_0. \tag{1.7.9}$$

Clearly,

$$|c(v, w)| \leq M\|v\|_0\|w\|_0. \tag{1.7.10}$$

Therefore from (1.7.8), (1.7.9) and (1.7.10), we have

$$\begin{aligned} |\mathbb{A}(u, v; t)| &\leq M\left(|v|_{1,\varpi}|w|_{1,\varpi} + \|w\|_0|v|_{1,\varpi} + \|v\|_0\|w\|_0\right) \\ \\ &\leq M||v||_{1,\varpi}||w||_{1,\varpi} \end{aligned}$$

Note that the map $F : V \to \mathbb{R}$ is defined by $v \to (f, v),$ for $v \in V$ and the application is continuous. ☐

Furthermore, the variational form of the LCP is given by

Definition 1.7.2. Let us define a convex and closed subset of $H^1_{0,\varpi}(\Omega)$

$$\mathcal{K} = \{v(t) : v(t) \in H^1_{0,\varpi}(\Omega),\ v(t) \leq u^*(t) \ \ a.e. \text{ in } (0, T)\}. \tag{1.7.11}$$

Find $u \in \mathcal{K}$ such that, for all $v \in \mathcal{K}$,

$$\left(-\frac{du}{dt}, v-u\right) + \mathbb{A}(u, v-u; t) \geq (f, v-u), \quad a.e. \text{ in } (0,T)$$
$$\text{with } \ u(T) \text{ given,} \tag{1.7.12}$$

where $\mathbb{A}(u, v; t)$ is a bilinear form defined in (1.7.4).

Theorem 1.8. *Equation (1.7.12) is the variational form for the complementarity problem (1.6.1).*

Proof. The proof of this theorem follows from [32, 39] without any modification. □

1.8.1 Power penalty formulation

The LCP (1.6.1), which is a system of partial differential inequalities (PDIs), is quite challenging to solve analytically. As a result, several numerical techniques have been used to find the solution of the discrete LCP. Indeed, one must ensure that the early exercise constraints of the LCP are satisfied before any discrete solution is found. A common practice to handle the early exercise condition is to advance the discrete solution over a time step while ignoring the constraint, and then explicitly apply the constraint [23]. This results in a solution which from the outset is flawed as the discrete form of the LCP is not approximately satisfied. Other methods like the relaxation method [13], linear programming technique and multigrid methods have been suggested. However, these methods have been found to be either particularly poor for pricing problems, have computational challenges or complex to implement. It is well known that the LCP can be converted to a single non-linear PDE by adding a non-linear term (or a penalty term) which satisfies all the constraints [32, 38]. This method of converting the LCP to a single non-linear PDE is known as the power penalty method. The advantage here is that, one can resort to standard methods to solve the resulting non-linear algebraic system. Furthermore, the power penalty method is a strong approximation tool to the solution of complementarity problems [32, 38]. The method is simple to implement as numerical methods for PDEs are well developed. It is also independent of the discretization scheme or meshing technique [32, 38]. Let V_λ denote the penalised solution of the LCP. Then by adding a power penalty term $\lambda[V^* - V_\lambda]_+^{1/k} = \lambda \max\{(V^* - V_\lambda)^{1/k}, 0\}$ to the LCP yields

$$LV_\lambda + \lambda\,[V^* - V_\lambda]_+^{1/k} = 0 \tag{1.8.1}$$

with (1.6.3)-(1.6.5), or

$$\mathcal{L}u_\lambda + \lambda\,[u^* - u_\lambda]_+^{1/k} = f(x,y,z,t), \qquad (x,y,z) \in \Omega \tag{1.8.2}$$

with boundary and final conditions

$$\begin{aligned}
u_\lambda(0,y,z,t) &= 0 = u_\lambda(X,y,z,t), \quad t \in [0,T], \quad y \in [0,Y], \quad z \in [0,Z],\\
u_\lambda(x,0,z,t) &= 0 = u_\lambda(x,Y,z,t), \quad t \in [0,T], \quad x \in [0,X], \quad z \in [0,Z]\\
u_\lambda(x,y,0,t) &= 0 = u_\lambda(x,y,Z,t), \quad t \in [0,T], \quad x \in [0,X], \quad y \in [0,Y]\\
u_\lambda(x,y,z,0) &= u^*(x,y,z,0),
\end{aligned}$$

In (1.8.2), u_λ is the penalised solution of the reformulated LCP, $k > 0$ is the power of the penalty term and $\lambda > 1$ is the penalty parameter. When $k = \frac{1}{2}$, the penalty approach corresponds to a quadratic penalty approach. The case for $k = 1$ is linear penalty. For $k > 1$, we have the so-called lower-order penalty approach [32, 38, 39]. Using the same arguments as in [32], (1.8.2) is well-posed.

Theorem 1.9. *Assume that* $\rho_{ij} \in [0, 0.5]$. *Find* $u_\lambda(t) \in H^1_{0,\varpi}(\Omega)$ *such that* for all $v \in H^1_{0,\varpi}(\Omega)$ *and* $t \in (0,T)$ *almost everywhere*

$$\left(-\frac{du_\lambda}{dt}, v\right) + \mathbb{A}(u_\lambda(t), v; t) + \lambda\left([u_\lambda(t) - u^*]_+^{1/k}, v\right) = (f(t), v)$$

$$\text{with} \ \ u(T) \ \text{given}. \tag{1.9.1}$$

Proof. As we did in the proof of Theorem 1.7, the proof here also follows the fact that the conditions of Theorem 1.4 are satisfied by Lemma 1.7.1, and hence the results thereafter are standard in [32] (where the one dimensional case was considered), following results in [22, page 37]. We will consider the linear penalty ($k = 1$) case here.

Now, since V_0 is linear and differentiable, we have that $f(s,t) = e^{\beta t}\mathcal{L}V_0$ is infinitely smooth in (x, y, z, t). To prove Theorem 1.9, we show that the operator on the left of (1.8.2) is strictly monotone and continuous. Then for any $v_1(t), v_2(t) \in H^1_{0,\varpi}(\Omega)$, almost everywhere in $(0,T)$, with final condition equal to $u^*(s,T)$ at $t = T$, it then follows from the integration by parts that,

$$(\mathcal{L}(v_1 - v_2), v_1 - v_2) + \lambda([v_1 - u^*]_+ - [v_2 - u^*]_+, v_1 - v_2) \tag{1.9.2}$$

$$= \left(-\frac{d(v_1 - v_2)}{dt}, v_1 - v_2\right) + \mathbb{A}(v_1 - v_2, v_1 - v_2; t) + \lambda([v_1 - u^*]_+ - [v_2 - u^*]_+, v_1 - v_2).$$

The difference however is the penalty term $\lambda[u_\lambda(t) - u^*]_+$. Now using the definition

$$[v]_+ = \max\{v, 0\}, \tag{1.9.3}$$

which is non-decreasing in v, and $[v]_- = \max\{0, -v\}$, we can write that

$$\begin{aligned} v_1 - v_2 &= v_1 - u^* - (v_2 - u^*) \\ &= [v_1 - u^*]_+ - [v_2 - u^*]_+ - \big([v_1 - u^*]_- - [v_2 - u^*]_-\big). \end{aligned} \tag{1.9.4}$$

Then we have that

$$\begin{aligned} \lambda([v_1 - u^*]_+ - [v_2 - u^*]_+, v_1 - v_2) &= \lambda\Big([v_1 - u^*]_+ - [v_2 - u^*]_+\Big)^2 \\ &\quad - \Big([v_1 - u^*]_+ - [v_2 - u^*]_+\Big)\Big([v_1 - u^*]_- - [v_2 - u^*]_-\Big) \\ &= \lambda\Big([v_1 - u^*]_+ - [v_2 - u^*]_+\Big)^2 - \underbrace{[v_1 - u^*]_+[v_1 - u^*]_-}_{=0} + \underbrace{[v_1 - u^*]_+[v_2 - u^*]_-}_{\geq 0} \\ &\quad + \underbrace{[v_2 - u^*]_+[v_1 - u^*]_-}_{\geq 0} - \underbrace{[v_2 - u^*]_+[v_2 - u^*]_-}_{=0} \\ &\geq \lambda\Big([v_1 - u^*]_+ - [v_2 - u^*]_+\Big)^2. \end{aligned} \tag{1.9.5}$$

Therefore

$$\lambda([v_1-u^*]_+-[v_2-u^*]_+,v_1-v_2)=\lambda\int_\Omega([v_1-u^*]_+-[v_2-u^*]_+)(v_1-v_2)d\Omega\geq 0 \quad (1.9.6)$$

Integrating both sides of (1.9.2) from 0 to T, using the coercive property of $\mathbb{A}$, (1.9.6) and with $e(\tau)=v_1(\tau)-v_2(\tau)$, we have

$$\begin{aligned}&\int_0^T[(\mathcal{L}e,e)+\lambda([v_1-u^*]_+-[v_2-u^*]_+,e)]d\tau\\ &=\int_0^T\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)d\tau+\int_0^T[\mathbb{A}(e,e)+\lambda([v_1-u^*]_+-[v_2-u^*]_+,e)]d\tau\\ &\geq\int_0^T\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)+C\int_0^T\|e\|^2_{1,\varpi}d\tau. \end{aligned}\quad (1.9.7)$$

We consider integration by part to get, for $t\in(0,T)$,

$$\begin{aligned}\int_0^T\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)d\tau &= \int_0^t\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)d\tau+\int_t^T\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)d\tau\\ &\geq \int_t^T\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)d\tau,\end{aligned}$$

and

$$\int_t^T\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)d\tau=(e(t),e(t))-\int_t^T\left(-\frac{de(\tau)}{d\tau},e(\tau)\right)d\tau$$

since $e(T)=0$. Therefore we have that,

$$\int_0^T\left(\frac{-de(\tau)}{d\tau},e(\tau)\right)d\tau\geq(1/2)(e(t),e(t))\geq 0. \quad (1.9.8)$$

Therefore we finally get that from (1.9.7) and (1.9.8)

$$\int_0^T[(\mathcal{L}(v_1-v_2),v_1-v_2)+\lambda([v_1-u^*]_+-v_2-u^*]_+,v_1-v_2)]d\tau\geq C\|v_1-v_2\|^2_{L^2(0,T;H^1_{0,\varpi}(\Omega))}. \quad (1.9.9)$$

Also we have that $\mathbb{A}(v,\omega;t)+\lambda([v-u^*]_+,w)$ is continuous in both v and w, hence

$$\begin{aligned}|\mathbb{A}(v,\omega;t)+\lambda([v-u^*]_+,w)| &\leq |\mathbb{A}(v,\omega;t)|+|\lambda([v-u^*]_+,w)|\\ &\leq M\|v\|\|w\|+\|\lambda([v-u^*]_+,w)\|\\ &\leq Q\|v\|\|w\|.\end{aligned}$$

Please note that the case for $k>1$ has been considered in [32], though in one dimension, the results for the three-dimensional problem remains similar without any modifications. □

Furthermore, the idea is to show that the solution u_λ of the penalty problem (1.9.1) converges to the solution u of the variational inequality problem (1.7.12). We show that this happens as $\lambda \to \infty$ with order $\mathcal{O}(\lambda^{-k/2})$ in a proper norm [32, 39]. Before that, error bounds for $[u_\lambda - u^*]_+$ is established and the results are used to bound $(u_\lambda - u)$ [32, 39].

Lemma 1.9.1. Let u_λ be the solution to (1.9.1). If $u_\lambda \in L^p(\Theta)$, then there exists a positive constant C, independent of u_λ and λ, such that

$$\|[u_\lambda - u^*]_+\|_{L^p(\Omega)} \leq C/\lambda^k, \tag{1.9.10}$$

$$\|[u_\lambda - u^*]_+\|_{L^\infty(0,T;L^2(I))} + \|[u_\lambda - u^*]_+\|_{L^2(0,T;H^1_\omega(\Omega))} \leq C/\lambda^{k/2}, \tag{1.9.11}$$

where k is the parameter and $p = 1 + 1/k$.

Proof. The proof follows from [32, 39] where the one and two factor cases were considered. Let C be a positive constant, independent of u_λ and λ. We simplify the notation by letting

$$\phi(\cdot, t) = [u_\lambda - u^*]_+ \ \in H^1_\varpi(\Omega), \quad \text{a.e. in } (0, T).$$

Let us set $v = \phi$ in (1.9.1). Then we have

$$\left(\frac{-du_\lambda}{dt}, \phi\right) + \mathbb{A}(u_\lambda, \phi; t) + \lambda(\phi^{1/k}, \phi) = (f, \phi) \quad a.e. \ in \ (0, T). \tag{1.9.12}$$

Subtracting $\left(-\frac{du^*}{dt}, \phi\right) + \mathbb{A}(u^*, \phi; t)$ from both sides of (1.9.12) yields

$$\left(-\frac{d(u_\lambda - u^*)}{dt}, \phi\right) + \mathbb{A}(u_\lambda - u^*, \phi; t) + \lambda(\phi^{1/k}, \phi) = (f, \phi) + \left(\frac{du^*}{\partial t}, \phi\right) - \mathbb{A}(u^*, \phi; t). \tag{1.9.13}$$

We use the fact that f is sufficiently smooth. Now, integrating both sides of (1.9.13) from t to T and using the Holder's inequality and the coercive property of $\mathbb{A}$, we get

$$\begin{aligned} &\frac{1}{2}(\phi(t), \phi(t)) + C\int_t^T \|\phi(\tau)\|^2_{1,\varpi} d\tau + \lambda \int_t^T \left(\phi(\tau)^{1/k}, \phi\right) d\tau \\ &\leq \int_t^T (f(\tau), \phi(\tau))d\tau + \beta\int_t^T e^{\beta t}(V_0 - V^*, \phi(\tau))d\tau - \int_t^T \mathbb{A}(u^*(\tau), \phi(\tau); \tau)d\tau \\ &\leq C\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)} d\tau\right)^{1/p} + \beta\int_t^T e^{\beta t}(V_0 - V^*, \phi(\tau))d\tau - \int_t^T \mathbb{A}(u^*(\tau), \phi(\tau); \tau)d\tau \end{aligned} \tag{1.9.14}$$

We use the fact that $\Big|V_0(s) - V^*(s)\Big|$ is uniformly bounded above by a positive constant and $\beta = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \rho_{12}\sigma_1\sigma_2 + \rho_{13}\sigma_1\sigma_3 + \rho_{23}\sigma_2\sigma_3)$ to obtain

$$\begin{aligned} &\frac{1}{2}(\phi(t), \phi(t)) + C\int_t^T \|\phi(\tau)\|^2_{1,\varpi} d\tau + \lambda \int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)} d\tau \\ \leq \ & C\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)} d\tau\right)^{1/p} - \int_t^T \mathbb{A}(u^*(\tau), \phi(\tau); \tau)d\tau. \end{aligned} \tag{1.9.15}$$

Then for the integrand

$$-\int_t^T \mathbb{A}(u^*(\tau),\phi(\tau);\tau)d\tau = \int_t^T (A\nabla u^* + \mathbf{b}u^*, \nabla\phi(\tau))\, d\tau - \int_t^T (cu^*, \phi(\tau))d\tau. \quad (1.9.16)$$

Using Green's theorem, we get

$$-\int_t^T (\mathbf{b}u^*, \nabla\phi(\tau))\, d\tau = \int_t^T \int_\Omega \nabla\cdot \mathbf{b}u^*\phi(\tau)d\Omega d\tau - \int_t^T \int_\Gamma u^*\cdot n\phi(\tau)d\Gamma d\tau, \quad (1.9.17)$$

where $\Gamma = \partial\Omega$ is the boundary of Ω. Now we let $\Omega_1 = \{0 < x < K/w_1, 0 < y < K/w_2, 0 < z < K/w_3, K - w_1x - w_2y - w_3z > 0\}$ and $\Omega_2 = \Omega\backslash\overline{\Omega}_1$. Also let Γ_0 denote the interface of Ω_1 and Ω_2. Which implies that Γ_0 has two opposite orientations: Γ_0^+ when the orientation is in the direction of $\partial\Omega_1$, and Γ_0^- when the orientation is in the direction of $\partial\Omega_2$. Now, consider the integrand $(A\nabla u^*, \nabla\phi(\tau))$ in (1.9.16). Then for $\phi \in H^1_{0,\varpi}(\Omega)$, note that $\phi = 0$ on Γ, following the lines in [39] we have

$$\begin{aligned} -(A\nabla u^*, \nabla\phi(\tau)) &= -\int_\Omega (A\nabla u^*)^T \nabla\phi d\Omega = -\int_{\Omega_1} (A\nabla u^*)^T \nabla\phi d\Omega - \int_{\Omega_2} (A\nabla u^*)^T \nabla\phi d\Omega \\ &= -\int_{\Gamma_0^+} A\nabla u^*\cdot n\phi ds + \int_{\Omega_1} \nabla\cdot(A\nabla u^*)\phi d\Omega - \int_{\Gamma_0^-} A\nabla u^*\cdot n\phi ds \\ &\quad + \int_{\Omega_2} \nabla\cdot(A\nabla u^*)\phi d\Omega \\ &= -\int_{\Gamma_0^+} (A\nabla u^*_- - A\nabla u^*_-)\cdot n\phi ds + \int_\Omega \nabla\cdot(A\nabla u^*)\phi d\Omega \end{aligned}$$

where n denotes the unit outward normal direction of the boundary segments and ∇u^*_- and ∇u^*_+ denote, respectively, the values of ∇u^* evaluated at the on the left and right sides of Γ_0^+. Since $u^* = e^{\beta t}(V_0 - V^*)$, we have that

$$\nabla u^* = e^{\beta t}(\nabla V_0 - \nabla V^*).$$

Furthermore, since $V_0 \in H^2(\Omega)$, ∇V_0 is continuous on Ω and hence,

$$\nabla u^*_- - \nabla u^*_+ = e^{\beta t}[(\nabla V_0 - \nabla V^*)_- - (\nabla V_0 - \nabla V^*)_+] = e^{\beta t}(-w_1, -w_2, -w_3)^T.$$

More so, the unit outward-normal vector to Γ_0^+ is

$$\begin{aligned} n &= \frac{\nabla(K - w_1x - w_2y - w_3z)}{\|\nabla(K - w_1x - w_2y - w_3z)\|} \\ &= \frac{(-w_1, -w_2, -w_3)^T}{(w_1^2 + w_2^2 + w_3^2)^{1/2}}. \end{aligned}$$

Then we have that

$$\begin{aligned} -(A\nabla u^*, \nabla\phi(\tau)) &= -\int_{\Gamma_0^+} e^{\beta t}\frac{(w_1, w_2, w_3)A^T(w_1, w_2, w_3)^T}{(w_1^2 + w_2^2 + w_3^2)^{1/2}}\phi ds + \int_\Omega \nabla\cdot(A\nabla u^*)\phi d\Omega \\ &\le C\int_\Omega \phi(\tau)d\Omega, \end{aligned}$$

since A is positive definite, ϕ is non-negative and $\nabla\cdot(A\nabla u^*)$ is bounded above on Ω.

Hence,

$$-\int_t^T A\nabla u^*, \nabla\phi(\tau)d\tau \leq C\int_t^T\int_\Omega \phi(\tau)d\Omega d\tau \leq C\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau\right)^{1/p}. \qquad (1.9.18)$$

Then from (1.9.17) we have,

$$-\int_t^T (\mathbf{b}u^*, \nabla\phi(\tau))\,d\tau \leq C\int_t^T\int_\Omega \phi(\tau)d\Omega d\tau \leq C\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau\right)^{1/p}, \qquad (1.9.19)$$

because $\nabla\cdot\mathbf{b}u^*$ is bounded above on Ω.

Substituting all the above inequalities in (1.9.14) and (1.9.19), we have

$$\frac{1}{2}(\phi(t),\phi(t))+\int_t^T \|\phi(\tau)\|^2_{1,\varpi}d\tau+\lambda\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau \leq C\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau\right)^{1/p}. \quad (1.9.20)$$

This implies that

$$\lambda\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau \leq C\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau\right)^{1/p}, \text{ a.e. in } (0,T).$$

It then follows that,

$$\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau\right)^{1/p} \leq C\lambda^{-1/(p-1)} = C\lambda^{-k}, \qquad (1.9.21)$$

since $p = 1 + 1/k$.

Then from (1.9.20) and (1.9.21), we have

$$\frac{1}{2}(\phi(t),\phi(t)) + \int_t^T \|\phi(\tau)\|^2_{1,\varpi}d\tau \leq C\left(\int_t^T \|\phi(\tau)\|^p_{L^p(\Omega)}d\tau\right)^{1/p} \leq \frac{C}{\lambda^k},$$

from which we get

$$(\phi(t),\phi(t))^{1/2} + \left(\int_t^T \|\phi(\tau)\|^2_{1,\varpi}d\tau\right)^{1/2} \leq \frac{C}{\lambda^{k/2}}, \quad \text{a.e. } \in (0,T). \qquad (1.9.22)$$

□

The Lemma 1.9.1 has established the bounds between $[u_\lambda - u^*]_+$. We will use this to show that the solution u_λ of the penalty problem (1.9.1) converges to the solution u of the variational inequality problem (1.7.12).

Theorem 1.10. *Assume that [32, Assumption 4.1] and the assumptions in [32, Lemma 4.1] are fulfilled. Then, there exists a constant $C > 0$, independent of u, u_λ, λ such that*

$$\|u - u_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|u - u_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \leq \frac{C}{\lambda^{k/2}}, \tag{1.10.1}$$

where k is the same parameter we are using in (1.8.2).

Proof. The proof follows from [32, 33, 39], where the one and two factor cases were considered. We particularly let

$$\phi(t) = [u_\lambda(t) - u^*(t)]_+.$$

Then we can decompose $u - u_\lambda$ as

$$\begin{aligned} u - u_\lambda &= u - u^* - (u_\lambda - u^*) \\ &= u - u^* + [u_\lambda - u^*]_- - [u_\lambda - u^*]_+ \\ &=: r_\lambda - \phi, \end{aligned}$$

where

$$[u_\lambda - u^*]_- = -\min\{u_\lambda - u^*, 0\}, \quad r_\lambda = u - u^* + [u_\lambda - u^*]_-. \tag{1.10.2}$$

It therefore follows that,

$$(\phi^\alpha, [u_\lambda - u^*]_-) = [u_\lambda - u^*]_+^\alpha [u_\lambda - u^*]_- \equiv 0, \quad \text{for } \alpha > 0. \tag{1.10.3}$$

We set $v = u - r_\lambda$ in (1.7.12) and $v = r_\lambda$ in (1.9.1), we have

$$\left(-\frac{du}{dt}, -r_\lambda\right) + \mathbb{A}(u, -r_\lambda; t) \geq (f(t), -r_\lambda), \tag{1.10.4}$$

$$\left(-\frac{du_\lambda}{dt}, r_\lambda\right) + \mathbb{A}(u_\lambda, r_\lambda; t) + \lambda(\phi^{1/k}, r_\lambda) = (f, r_\lambda) \tag{1.10.5}$$

Subtracting (1.10.4) from (1.10.5), we have

$$\left(-\frac{d(u_\lambda - u)}{dt}, r_\lambda\right) + \mathbb{A}(u_\lambda - u, r_\lambda; t) + \lambda(\phi^{1/k}, r_\lambda) \geq 0 \tag{1.10.6}$$

Using (1.10.3), we have,

$$\begin{aligned} (\phi^{1/k}, r_\lambda) &= (\phi^{1/k}, u - u^* + [u_\lambda - u^*]_-) \\ &= (\phi^{1/k}, u - u^*) + (\phi^{1/k}, [u_\lambda - u^*]_-) \\ &= (\phi^{1/k}, u - u^*) \leq 0. \end{aligned} \tag{1.10.7}$$

The last inequality is due to the fact that $\phi \geq 0$ and $u - u^* \leq 0$. That is

$$\left(-\frac{d(u_\lambda - u)}{dt}, r_\lambda\right) + \mathbb{A}(u_\lambda - u, r_\lambda; t) \leq 0,$$

since $-\lambda(\phi^{1/k}, r_\lambda) \geq 0$.

Using (1.10.3), the above inequality becomes

$$\left(-\frac{dr_\lambda}{dt}, r_\lambda\right) + \mathbb{A}(r_\lambda, r_\lambda; t) \leq \left(-\frac{d\phi}{dt}, -r_\lambda\right) + \mathbb{A}(\phi, r_\lambda; t) \tag{1.10.8}$$

Integrating both sides of (1.10.8) from $\tau = t$ to $\tau = T$, and then using $(\phi, [u_\lambda - u^*]_-) = 0$ and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} & \frac{1}{2}\Big(r_\lambda(t), r_\lambda(t)\Big) + \int_t^T \mathbb{A}(r_\lambda(\tau), r_\lambda(\tau); \tau) d\tau \\ \leq \; & \int_t^T \left(-\frac{d\phi(\tau)}{d\tau}, r_\lambda\right) d\tau + \int_t^T \mathbb{A}(\phi(\tau), r_\lambda(\tau); \tau) d\tau \\ \leq \; & (\phi(t), r_\lambda(t)) + \int_t^T \left(\phi, \frac{dr_\lambda}{d\tau}\right) d\tau + \int_t^T \mathbb{A}(\phi(\tau), r_\lambda(\tau); \tau) d\tau \\ \leq \; & \|\phi\|_{L^\infty(0,T;L^2(\Omega))} \|r_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + C\|\phi\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \|r_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \\ & + \int_t^T \left(\phi, \frac{dr_\lambda}{d\tau}\right) d\tau, \end{aligned}$$

for all $t \in (0, T)$.

Since $\phi \cdot [u_\lambda - u^*]_- = 0, \;$ for almost all $t \in (0,T)$, $u_\lambda \in L^p(\Theta)$ and $\dfrac{du}{dt} \in L^{k+1}(\Theta)$, (see [32, Assumption 4.1]) it follows that,

$$\begin{aligned} \int_t^T \left(\phi, \frac{dr_\lambda}{d\tau}\right) d\tau &= \int_t^T \left(\phi(\tau), \frac{du(\tau)}{d\tau}\right) d\tau - \beta \int_t^T e^{\beta\tau} \left(\phi(\tau), V_0 - V^*\right) d\tau \\ &\leq C\|\phi\|_{L^p(\Theta)} \left(\left\| \frac{du(\tau)}{d\tau} \right\|_{L^q(\Theta)} + \|V_0 - V^*\|_{L^q(\Theta)} \right) \leq \frac{C}{\lambda^k} \end{aligned} \tag{1.10.9}$$

where $p = 1 + 1/k$ and $q = k + 1$, so that $1/p + 1/q = 1$.

Now, using the coerciveness property of $\mathbb{A}$, and following line by line the [39], we have

$$\begin{aligned} & \Big(\|r_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|r_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\Big)^2 \\ & \leq C\left(\frac{1}{2}\|r_\lambda\|^2_{L^\infty(0,T;L^2(\Omega))}\|r_\lambda\|^2_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right) \\ & \leq C[(\|\phi\|_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}) \cdot (\|r_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|r_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}) + \lambda^{-k}] \\ & \qquad \leq C[\lambda^{-k/2}(\|r_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|r_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}) + \lambda^{-k}]. \end{aligned} \tag{1.10.10}$$

Clearly, we see that the above is of the form

$$z^2 \leq C\rho^{1/2} z + C\rho,$$

we rewrite as

$$(z-(1/2)C\rho^{1/2})^2 \leq (C+C^2/4)\rho.$$

We see this clearly implies that

$$z \leq C\rho^{1/2}$$

where $C>0$ a generic constant. Replacing y with $\|r_\lambda\|_{L^\infty(0,T;L^2(\Omega))}+\|r_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}$ and ρ with λ^{-k} respectively, we have

$$\|r_\lambda\|_{L^\infty(0,T;L^2(\Omega))}+\|r_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \leq \frac{C}{\lambda^{k/2}} \tag{1.10.11}$$

Finally, we use the triangle inequality to obtain

$$\begin{aligned}
&\|u-u_\lambda\|_{L^\infty(0,T;L^2(\Omega))}+\|u-u_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\\
&\leq \Big(\|\phi\|_{L^\infty(0,T;L^2(\Omega))}+\|\phi\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\big)+\big(\|r_\lambda\|_{L^\infty(0,T;L^2(\Omega))}+\|r_\lambda\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\Big)\\
&\leq \frac{C}{\lambda^{k/2}}.
\end{aligned}$$

□

Remark 1.10.1. For simplicity of notation, we will set $u_\lambda = u$ in the rest of the document.

1.10.2 Conclusion

In this chapter, our main interest was to give some background information on option pricing. We have considered the Black-Scholes model proposed in [9] and stated the Black-Scholes partial differential equation that is known to govern European options. The closed-form solution for the one-dimensional case was first proposed in [9]. Further in the chapter, we have stated the well known so-called linear complimentarity problem (LCP) ([32, 38]) that governs the price of American options. The LCP involves the Black-Scholes differential operator with associated constraints. We have reformulated the LCP into a more conservative variational form. It is well documented that the variational problem is equivalent to the LCP [32, 38] and it has a unique solution. We have shown the unique solvability of the three-dimensional variational problem. We have approximated the solution of the LCP by adding a power penalty term [32]. The resulting non-linear PDE is fairly easy to discretize in higher dimensions as we will see in the next section. We further referred to [32, 33] where the solution of the penalised problem was shown to converge to the solution of the original variational inequality problem.

2. Mimetic and fitted-mimetic finite difference methods for option pricing in one dimension

In our attempt to numerically solve the multidimensional option problem, for both European and American options, we present a novel spatial discretization technique which is based primarily on the so-called mimetic finite difference method (MFDM). As already mentioned, the option problem is known to have a discontinuity at $S = E$, where in general $S \in \Omega \subset \mathbb{R}^d, d \geq 1$. Also, the Black-Scholes differential operator which governs the option problem is a degenerate operator near the boundary of the domain at $S = 0$. This causes the PDE to lose important underlying properties when the stock price is close to zero. As a result, classical finite difference methods may fail to give accurate approximations near this boundary.

The fitted mimetic finite difference method is adopted here to handle the discontinuity and the degeneracy of the Black-Scholes PDE. The standard MFDM as a spatial discretization technique stems from the well-known support operator methods (SOM). The standard MFDM therefore is a design tool for discrete models that tends to preserve very important properties (i.e conservation laws, solution symmetries, asymptotic limits etc.) of the underlying differential equation. The method further adopts important properties of other advanced discretization techniques. Indeed, like the finite volume method [25, 5, 43], it also works on general grids (polygonal and polyhedral meshes) and like the finite element methods [22, 34], the standard MFDM also has a fast-growing convergence [34]. For general meshes (distorted meshes, meshes on non-convex and skewed cells, and complex domain meshes), the method remains robust while ensuring high-quality numerical results are guaranteed by accurately preserving solution features [15, 28, 29, 34]. An added advantage of the method is that there is flexibility in its construction to tackle a very challenging problem. Furthermore, we conisder the fitted scheme proposed in [43], a scheme well known to handle the degeneracy of the Black-Scholes differential operator. The combination of the two methods, the standard MFDM and the fitted scheme, result in our novel numerical technique called the fitted mimetic finite difference method (FMFDM).

In this chapter we aim to solve the one dimensional European and American option problems. Indeed our main contribution in the thesis begins with this chapter. The chapter is organised as follows, in section 2.1 we introduce the support operator method (SOM) for elliptic PDEs. We present the standard MFDM and FMFD for the Black-Scholes PDE in section 2.2, and the convection term is approximated using the first order upwind technique. We further prove the existence and uniqueness of the resulting semi-discrete solutions corresponding to our two novel schemes. The section is ended with the consistency of the fluxes for the two schemes. In section 2.9, we perform the full discretization by using the standard implicit Euler scheme for time discretization. We prove the convergence of the standard implicit time-stepping scheme. We conclude the chapter by presenting some numerical experiments to sustain our theoretical results and summarise our findings. Those experiments show that our two novel schemes outperform the standard finite difference and the fitted finite volume method [43] in terms of accuracy. The content of the chapter is published in [4].

2.1 Support Operator Method

In this section, we consider the so called Support Operator Method (SOM). The fact remains that most partial differential equations can be formulated in terms of invariant differential operators of gradient ∇, divergence $\nabla\cdot$ and curl $\nabla\times$. The SOM takes advantage of this and gives an approach for spatial discretisation by constructing discrete analogs of these differential operators [41, 45]. Generally, the continuum operators tend to satisfy certain important differential and integral identities. The SOM helps to construct the discrete operators which satisfy discrete versions of those important differential and integral identities. Generally, conservation laws, adjoint relationships and solution symmetries are some properties which we would like the discrete operators to mimic. For illustrative purposes, let us consider the following elliptic problem

$$-\nabla\cdot(K\nabla u(x)) = \mathbf{F}, \qquad x\in\Omega\subset\mathbb{R}^n,\ n\in\mathbb{N}. \tag{2.1.1}$$

where $K>0$, a material property tensor, is a bounded invertible matrix function of x and $\mathbf{F}$ is a forcing function.

Consider the operator $\mathbf{A}: H\to H$, defined by

$$\mathbf{A}u = -\nabla\cdot(K\nabla u(x)), \quad x\in\Omega \tag{2.1.2}$$

which has the properties

$$(\mathbf{A}u, v)_H = (u, \mathbf{A}v)_H, \quad (\mathbf{A}u, u)_H > 0. \tag{2.1.3}$$

Then (2.1.1) becomes

$$\mathbf{A}u = \mathbf{F}. \tag{2.1.4}$$

Now (2.1.1) can be rewritten as the following first-order system

$$\begin{cases} \nabla\cdot w = \mathbf{F} \\ w = -K\nabla u, \end{cases} \tag{2.1.5}$$

which is equivalent to

$$w - \mathbf{G}u = 0,\ \mathbf{D}w = \mathbf{F}, \tag{2.1.6}$$

where the operators $\mathbf{G}$ and $\mathbf{D}$ are defined as

$$\begin{cases} \mathbf{G}u = -K\nabla u \ \text{ on } \ \Omega \\ \mathbf{D}w = \nabla w \ \text{ on } \ \Omega \end{cases} \tag{2.1.7}$$

We note here that (2.1.6) is the new formulation of (2.1.1) with the continuous gradient ∇ and divergence $\nabla\cdot$ operators that we would like our discrete operators to mimic. Furthermore, note that from (2.1.7) we have

$$\mathbf{A} = \mathbf{D}\mathbf{G}. \tag{2.1.8}$$

Let, $H = L^2(\Omega)$ be the space of scalar functions u that are smooth on the set Ω equipped with inner product

$$(u, v)_H = \int_\Omega uv d\Omega, \quad u, v \in H, \tag{2.1.9}$$

and $\mathbf{H} = (L^2(\Omega))^n$ equipped with inner product

$$(w, z)_{\mathbf{H}} = \int_\Omega (K^{-1}w, z) d\Omega, \quad w, z \in \mathbf{H} = \left(L^2(\Omega)\right)^n;\ n \in \mathbb{N}. \tag{2.1.10}$$

The inner product (2.1.10) is weighted by the inverse of K and this formulation preserves the duality between the divergent and gradient operators [41, 45]. Since the matrix K is bounded and positive definite, it implies that its inverse, K^{-1}, is also bounded and positive definite. That is for a discontinuous K, which often is the case in mixed finite element formulations, (2.1.10) still remains an inner product. Thus, $(\cdot, \cdot)$ is the standard inner product of $\mathbb{R}^n$.

Most importantly, the following properties are fulfilled ([41, 45])

$$(\mathbf{D}w, u)_H = (w, \mathbf{G}u)_{\mathbf{H}} \tag{2.1.11}$$

$$(\mathbf{D}w, 1)_H = 0, \tag{2.1.12}$$

$$\mathbf{A} = \mathbf{D}\mathbf{G} = \mathbf{D}\mathbf{D}^* \tag{2.1.13}$$

$$\mathbf{A} = \mathbf{A}^* > 0, \tag{2.1.14}$$

where 1 is the constant function with value 1, $\mathbf{A}^*$ and $\mathbf{D}^*$ represent the adjoints of the operators $\mathbf{A}$ and $\mathbf{D}$.

The properties (2.1.11) - (2.1.13) are the important properties of the continuum operators that we want our discrete operators in the next section to mimic.

2.2 Semi-discrete problem and mimetic method for elliptic problems

Our aim is to build a mimetic finite difference method to discretize the diffusion part of our continuous problem (1.8.2) with $\lambda = 0$ (it works similarly for all λ). As we just mentioned, the corresponding discrete operators will preserve the properties (2.1.11) - (2.1.13). The domain $\Omega = [0, X_{\max}]$ is divided into $N+1$ non-overlapping intervals $\mathcal{T} = (I_i)_{0 \le i \le N}$, such that $I_i = (x_i, x_{i+1})$, $i = 0, 1, ..., N$, with $0 = x_0 < x_1 < \cdots < x_{N+1} = X_{\max}$ and set $h_i = x_{i+1} - x_i$, $h_{N+1} = 0$, $h = \max_{0 \le i \le (N+1)} h_i$. Now for $i = 0, 1, ..., N+1$, we set $l_i = x_{i+1/2} - x_{i-1/2}$ with $l = \max_{0 \le i \le (N+1)} l_i$, where the midpoints are given by $x_{i+\frac{1}{2}} = \dfrac{x_{i+1} + x_i}{2}$ for $i = 1, ..., N$. We also set $x_{-\frac{1}{2}} = x_0$ and $x_{N+\frac{3}{2}} = x_{N+1}$ [1]. Note the family $(\Omega_i)_{0 \le i \le N+1}$ is another partition of Ω with $\Omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, that we will call dual partition of $(I_i)_{0 \le i \le N}$.

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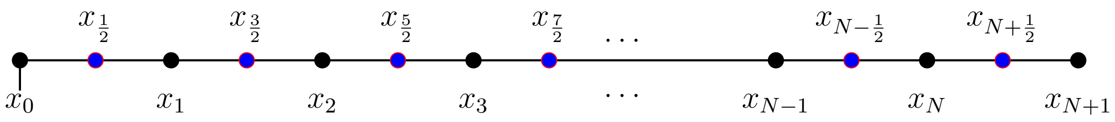
[1] Function evaluation at $x_{-\frac{1}{2}}$ or $x_{N+\frac{3}{2}}$ is understood as evaluation at $x_0 = 0$ or at $x_{N+1} = X_{\max}$



(a) Mimetic grid

Let us define $V_h := span\{\phi_i\}_{i=0}^{N+1} \cap H^1_{0,\varpi}(\Omega)$ as the set of continuous piecewise functions with respect to the partition $\{I_i\}_{i=0}^{N}$, such that $v_h = \sum_{i=0}^{N+1} v_h(x_i)\phi_{x_i}, \quad v_h \in V_h$ with $\phi_{x_i}(x_j) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker symbol. The interpolation operator $I_h : C(\overline{\Omega}) \longrightarrow V_h$ is defined by

$$I_h v(x_i) := v(x_i); \quad i = 0, 1, ..., N+1 \text{ then } I_h v = v_h = \sum_{i=0}^{N+1} v(x_i)\phi_{x_i}. \tag{2.2.1}$$

For the purpose of errors analysis we define the following appropriate norms and semi-norms on V_h as

$$\|v_h\|_{0,h} := \sqrt{(v_h, v_h)_h} = \left(\sum_{i=0}^{N+1} l_i v_h(x_i)^2\right)^{1/2}, \tag{2.2.2}$$

for the discrete $L^2(\Omega)$ norm, and

$$\|v_h\|_{1,\mathcal{T}} := \left(\sum_{i=0}^{N} \frac{x^2_{i+\frac{1}{2}}}{h_i}(v_{i+1} - v_i)^2\right)^{1/2}, \tag{2.2.3}$$

for weighted discrete H^1- semi-norm, and

$$\|v_h\|^2_{1,h,\mathcal{T}} = \|v_h\|^2_{1,\mathcal{T}} + \|v_h\|^2_{0,h}, \tag{2.2.4}$$

for the weighted discrete $H^1_{0,\varpi}(\Omega)$-norm on V_h. Indeed it is easy to show that $\|\cdot\|_{1,\mathcal{T}}$ is a semi-norm in V_h since $\dfrac{x^2_{i+\frac{1}{2}}}{h_i} > 0$. We demonstrate this below. Firstly, we need to show that for any constant α

$$\|\alpha v_h\|_{1,\mathcal{T}} = |\alpha|\|v_h\|_{1,\mathcal{T}}. \tag{2.2.5}$$

That is, from the left hand side (lhs) of (2.2.5), we have

$$\begin{aligned}
\|\alpha v_h\|_{1,\mathcal{T}} &= \left(\sum_{i=0}^{N} \frac{x^2_{i+\frac{1}{2}}}{h_i}(\alpha(v_{i+1} - v_i))^2\right)^{1/2} \\
&= \left(\sum_{i=0}^{N} \frac{x^2_{i+\frac{1}{2}}}{h_i}\alpha^2(v_{i+1} - v_i)^2\right)^{1/2} \\
&= |\alpha| \left(\frac{x^2_{i+\frac{1}{2}}}{h_i}(v_{i+1} - v_i)^2\right)^{1/2} \\
&= |\alpha|\|v_h\|_{1,\mathcal{T}}.
\end{aligned} \tag{2.2.6}$$

Also, we need to show that

$$\|u_h + v_h\|_{1,\mathcal{T}} \le \|u_h\|_{1,\mathcal{T}} + \|v_h\|_{1,\mathcal{T}}. \tag{2.2.7}$$

That is, from the lhs of (2.2.7) and using the Minkowski inequality, we have

$$\begin{aligned}
\|u_h + v_h\|_{1,\mathcal{T}} &= \left[\sum_{i=0}^{N} \frac{x^2_{i+\frac{1}{2}}}{h_i}\left((u_{i+1}+v_{i+1}) - (u_i+v_i)\right)^2\right]^{1/2} \\
&= \left[\sum_{i=0}^{N} \left(\frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}}\left((u_{i+1}-u_i) + (v_{i+1}-v_i)\right)\right)^2\right]^{1/2} \\
&= \left[\sum_{i=0}^{N} \left(\frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}}(u_{i+1}-u_i) + \frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}}(v_{i+1}-v_i)\right)^2\right]^{1/2} \\
&\le \left[\sum_{i=0}^{N} \left(\frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}}(u_{i+1}-u_i)\right)^2\right]^{1/2} + \left[\sum_{i=0}^{N} \left(\frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}}(v_{i+1}-v_i)\right)^2\right]^{1/2} \\
&\le \left[\sum_{i=0}^{N} \frac{x^2_{i+\frac{1}{2}}}{h_i}(u_{i+1}-u_i)^2\right]^{1/2} + \left[\sum_{i=0}^{N} \frac{x^2_{i+\frac{1}{2}}}{h_i}(v_{i+1}-v_i)^2\right]^{1/2} \\
&= \|u_h\|_{1,\mathcal{T}} + \|v_h\|_{1,\mathcal{T}}.
\end{aligned} \tag{2.2.8}$$

2.2.1 Discrete inner products

Here, we define the discrete analogs of the two continuous inner products (2.1.9) and (2.1.10) which effectively select a quadrature rule on each cell to approximate the integrals. Let HC and $\mathbf{HC}$ be the sets of functions of $H = L^2(\Omega)$ or $\mathbf{H} = (L^2(\Omega))^n \quad n = 1$ respectively, constants in the partition $\mathcal{T}$ ((I_i) for HC and (Ω_i) for $\mathbf{HC}$). The discrete $L^2(\Omega)$ norm defined in HC for (2.1.9) is given by

$$(U, V)_{HC} = \sum_{i=0}^{N+1} U_i V_i l_i \quad U, V \in HC. \tag{2.2.9}$$

Since the discrete information for fluxes are located at the cell centers, we adopt the midpoint rule for the inner product (2.1.10). Hence the discrete $(L^2(\Omega))^n$, $n = 1$ norm in $\mathbf{HC}$ is defined as

$$(W, Z)_{\mathbf{HC}} = \sum_{i=0}^{N+1} \frac{W_{i+1/2} Z_{i+1/2}}{K_{i+\frac{1}{2}}} h_i, \quad W, Z \in \mathbf{HC}, \tag{2.2.10}$$

where $K_{i+\frac{1}{2}}$ is defined before at the cell $i + \frac{1}{2}$. For $K_{i+\frac{1}{2}} \neq 0$, the discrete inner product (2.2.10) is well defined.

Note that in HC and $\mathbf{HC}$, we also define the following standard inner products

$$[U, V]_{HC} = \sum_{i=0}^{N+1} U_i V_i, \quad U, V \in HC, \tag{2.2.11}$$

and

$$[W, Z]_{\mathbf{HC}} = \sum_{i=0}^{N+1} W_{i+1/2} Z_{i+1/2} h_i. \tag{2.2.12}$$

It is important to note here that in HC and $\mathbf{HC}$ the two inner products are linked by

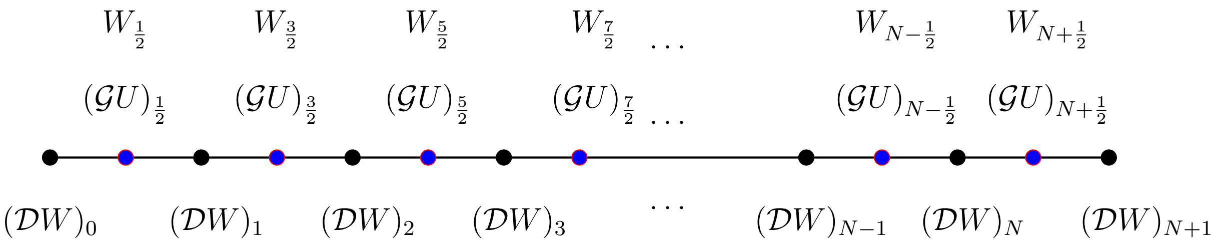
$$(U, V)_{HC} = [\mathcal{M}U, V]_{HC}, \qquad (W, Z)_{\mathbf{HC}} = [\mathcal{S}W, Z]_{\mathbf{HC}}, \tag{2.2.13}$$

where $\mathcal{M} = \mathrm{diag}(l_0, l_1, ..., l_i, ..., l_{N+1})$ and $\mathcal{S}$ is defined in (2.2.21) in the next section. Please refer to [41, 45] for more details.

2.2.2 The discrete divergence and discrete flux

In this section, we look at the discrete version of the divergence operator $\mathbf{D}$. Let us denote by $\mathcal{D}$ the discrete divergence operator. Then we define $\mathcal{D}$ as

$$(\mathcal{D}W)_i = \frac{W_{i+1/2} - W_{i-1/2}}{l_i}, \quad i = 0, 1, ..., N+1. \tag{2.2.14}$$



(b) Discrete-Mimetic finite difference grid

We can easily check, if W vanishes at the boundaries, that

$$(\mathcal{D}W, 1)_{HC} = 0, \tag{2.2.15}$$

which is the divergence property of the discrete divergence $\mathcal{D}$, mimicking the continuous divergence. Let us show this here. From the left hand side of (2.2.15), we have

$$\begin{aligned}
(\mathcal{D}W, 1)_{HC} &= \sum_{i=0}^{N+1} \left(\frac{W_{i+\frac{1}{2}} - W_{i-\frac{1}{2}}}{l_i} \right) l_i = \sum_{i=0}^{N+1} \left(W_{i+\frac{1}{2}} - W_{i-\frac{1}{2}} \right) \\
&= \sum_{i=0}^{N+1} W_{i+\frac{1}{2}} - \sum_{i=1}^{N+1} W_{i-\frac{1}{2}} - W_{-\frac{1}{2}} = \sum_{i=0}^{N} W_{i+\frac{1}{2}} + W_{N+\frac{3}{2}} - \sum_{i=1}^{N+1} W_{i-\frac{1}{2}} - W_{-\frac{1}{2}} \\
&= \sum_{i=0}^{N} W_{i+\frac{1}{2}} + W_{N+\frac{3}{2}} - \sum_{i=0}^{N} W_{i+\frac{1}{2}} - W_{-\frac{1}{2}} \\
&= W_{N+\frac{3}{2}} - W_{-\frac{1}{2}} = 0, \quad \text{by definition of the grid.}
\end{aligned}$$

For $n = 1$, $\mathbf{G} = -K\dfrac{d}{dx}$. Let us determine the discrete version of $\mathbf{G}$ denoted by $\mathcal{G}$ that mimic the continuous version properties as we have already mentioned. Indeed, $\mathcal{G}$ should fulfil the following propriety

$$(\mathcal{D}W, U)_{HC} = (W, \mathcal{G}U)_{\mathbf{HC}}. \tag{2.2.16}$$

We then expand (2.2.16) as below

$$\left(\sum_{i=0}^{N+1} U_i(\mathcal{D}W)_i\right) l_i = \sum_{i=0}^{N+1} \frac{W_{i+1/2}(\mathcal{G}U)_{i+1/2}}{K_{i+\frac{1}{2}}} h_i. \tag{2.2.17}$$

Further expanding (2.2.17) we have that

$$\sum_{i=0}^{N+1}\left(U_i - \left[\frac{h_i}{K_{i+\frac{1}{2}}}\right](\mathcal{G}U)_{i+\frac{1}{2}}\right)W_{i+\frac{1}{2}} - \sum_{i=0}^{N+1} U_i W_{i-\frac{1}{2}} = 0,$$

so

$$\sum_{i=0}^{N+1}\left(U_i - \left[\frac{h_i}{K_{i+\frac{1}{2}}}\right](\mathcal{G}U)_{i+\frac{1}{2}}\right)W_{i+\frac{1}{2}} - \sum_{i=1}^{N+1} U_i W_{i-\frac{1}{2}} - U_0 W_{-1/2} = 0. \tag{2.2.18}$$

Now, re-indexing any terms with $i - \frac{1}{2}$ to $i + \frac{1}{2}$, using the fact that $h_{N+1} = 0$, we have

$$\sum_{i=0}^{N}\left(-(U_{i+1} - U_i) - (\mathcal{G}U)_{i+\frac{1}{2}}\left[\frac{h_i}{K_{i+\frac{1}{2}}}\right]\right)W_{i+\frac{1}{2}} + U_{N+1}W_{N+\frac{3}{2}} - U_0 W_{-1/2} = 0. \tag{2.2.19}$$

Now, this is done to fully concentrate the fluxes at the $i^{th}-$node, to enhance the mimicking property at each $i^{th}-$node. We have that (2.2.19) holds for all U in HC such that $U_0 = U_{N+1} = 0$. Hence we can determine $\mathcal{G}$ by solving for $(\mathcal{G}U)_{i+1/2}$ which gives

$$(\mathcal{G}U)_{i+1/2} = -\left(\left[\frac{h_i}{K_{i+\frac{1}{2}}}\right]\right)^{-1}(U_{i+1} - U_i), \quad i = 0, ..., N. \tag{2.2.20}$$

We let $\mathcal{A}_h$ denote the discrete diffusion operator obtained by forming the composition of the discrete divergence and gradient operator $\mathcal{D}$ and $\mathcal{G}$ respectively. This by construction, $\mathcal{D} : \mathbf{HC} \to HC$ and $\mathcal{G} : HC \to \mathbf{HC}$ is given by $\mathcal{A}_h : HC \to HC$ with $\mathcal{A}_h = \mathcal{D}\mathcal{G}$. The coefficient $\mathcal{S}$ is therefore defined by

$$\mathcal{S} = \operatorname{diag}\left(\left[\frac{h_0}{K_{\frac{1}{2}}}\right], ..., \left[\frac{h_i}{K_{i+\frac{1}{2}}}\right], \left[\frac{h_N}{K_{N+\frac{1}{2}}}\right]\right) \tag{2.2.21}$$

Remark 2.2.3. As can be observed, the mimetic finite difference method depends on the discrete inner products (2.2.9) and (2.2.10). These discrete inner products are indeed the approximations of the integrals of continuous functions on the grid. Here we made use of the rectangle rule, but the trapezoidal rule or any other quadrature rule can be also used. Of course, the accuracy of the mimetic method would depend on the quadrature rule used to derive these discrete inner products.

2.2.4 Mimetic Finite Difference Scheme for Black Scholes PDE

Our goal here is to discretize the Black Scholes PDE (1.8.2) with $\lambda = 0$. The mimetic finite difference method will be used for the diffusion part while the first order upwind-finite difference scheme will be used for the convection term. Using the continuous operators $\mathbf{A}, \mathbf{D}$, and $\mathbf{G}$, we rewrite (1.8.2) with $\lambda = 0$ as

$$\begin{cases} \dfrac{-\partial u}{\partial t} + \mathbf{D}w - \dfrac{\partial}{\partial x}[bxu] + cu = f(t) \\ \mathbf{G}u := w = -K\dfrac{\partial u}{\partial x}, \\ \mathbf{A} = \mathbf{DG}, \quad K = ax^2, \end{cases} \tag{2.2.22}$$

where $a = \dfrac{1}{2}\sigma^2,\ b = r - \sigma^2,\ \ c = 2r + \beta - \sigma^2 > 0;\ \beta = \sigma^2.$

We partition $I := (0, X_{\max})$ into $N + 1$ sub-intervals as we did previously for elliptic problems with dual partition, then we have that,

$$U_i(t) \approx U(x_i, t),\ h_i = x_{i+1} - x_i,\ i = 0, 1, ..., N.$$

Set $w_{i+\frac{1}{2}} := \dfrac{w_{i+1} + w_i}{2}$, $w_{i-\frac{1}{2}} := \dfrac{w_i + w_{i-1}}{2}$ we can easily $w_{i+\frac{1}{2}} \approx w(x_{i+\frac{1}{2}}, t), \quad i = 0, 1, .., N.$ Then the discrete mimetic operators (prime and derived) are given by

$$(\mathcal{D}w)_i = \frac{w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}}}{l_i}, \quad \text{for } i = 0, ..., N + 1, \tag{2.2.23}$$

and

$$(\mathcal{G}U)_{i-1/2} = -\left(\left[\frac{h_{i-1}}{K_{i-\frac{1}{2}}}\right]\right)^{-1}(U_i - U_{i-1}),\ i = 1, ..., N + 1 \tag{2.2.24}$$

Now, we have that the discrete operator $\widehat{\mathcal{A}}_h$ is given by

$$\widehat{\mathcal{A}}_h U_h[i] = (\mathcal{D}\mathcal{G})U_h[i] = \left(\frac{-\left[\dfrac{K_{i+\frac{1}{2}}}{h_i}\right](U_{i+1} - U_i) + \left[\dfrac{K_{i-\frac{1}{2}}}{h_{i-1}}\right](U_i - U_{i-1})}{l_i}\right),\ i = 1, ..., N, \tag{2.2.25}$$

or

$$\widehat{\mathcal{A}}_h U_h[i] = \alpha_i U_{i+1} + \beta_i U_i + \gamma_i U_{i-1},\ i = 1, ..., N, \tag{2.2.26}$$

where

$$\alpha_i = \left[-\frac{K_{i+\frac{1}{2}}}{h_i l_i}\right], \quad \beta_i = \left[\frac{K_{i+\frac{1}{2}}}{h_i l_i} + \frac{K_{i-\frac{1}{2}}}{l_i h_{i-1}}\right], \quad \gamma_i = -\left[-\frac{K_{i-\frac{1}{2}}}{l_i h_{i-1}}\right]. \tag{2.2.27}$$

Applying the first order upwind finite difference for the convective term in the sense of finite volume method yields

$$-\frac{\partial}{\partial x}(bxu) \approx \frac{-x_{i+1/2}\left(b^+U_{i+1}+b^-U_i\right)+x_{i-1/2}\left(b^+U_{i-1}+b^-U_i\right)}{h_i}, \tag{2.2.28}$$

where $b^+ = \max\{b, 0\}$ and $b^- = \min\{b, 0\}$. Then

$$\widehat{\mathcal{B}}_h U_h[i] = \Gamma_i U_{i+1} + \Upsilon_i U_i + \chi_i U_{i-1}, \tag{2.2.29}$$

where

$$\Gamma_i = \frac{-b^+x_{i+1/2}}{h_i},\quad \Upsilon_i = \frac{-b^-x_{i+1/2}+b^-x_{i-1/2}}{h_i},\quad \chi_i = \frac{b^+x_{i-1/2}}{h_i};\ i = 1, ..., N \tag{2.2.30}$$

Now from (2.2.26) and (2.2.29), we have that

$$\widehat{\mathcal{C}}_h U_h[i] = \widehat{\mathcal{A}}_h U_h[i] + \widehat{\mathcal{B}}_h U_h[i] = (\alpha_i+\Gamma_i)U_{i+1} + (\beta_i+\Upsilon_i)U_i + (\gamma_i+\chi_i)U_{i-1},\ \ i = 1, ..., N. \tag{2.2.31}$$

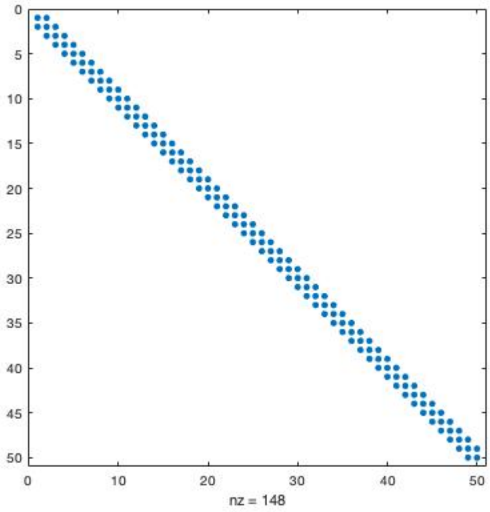


Figure 2.1: The matrix $\widehat{\mathcal{C}}_h$ structure for the mimetic finite difference method

Then by using the transformation $t = T - t$, we have

$$\begin{cases} \dfrac{dU_h}{dt} + \widehat{\mathcal{C}}_h U_h = f_h(t), \quad t \in [0, T], \\ \\ U_h(0) = U_h^* \end{cases} \tag{2.2.32}$$

2.2.5 Fitted Mimetic Finite Difference Scheme

As already mentioned, the Black-Scholes differential operator is known to be degenerate towards the boundary and hence special techniques are required to handle the degeneracy [32, 43]. In [32, 43], the authors proposed a so-called fitted scheme to tackle the degeneracy of the PDE. In this section near $x = 0$ $(i = 1)$, the sum of diffusion and convective flux is approximated using the fitted scheme and far from $x = 0$ $(i > 1)$, the diffusion flux and the convective flux will be approximated as in the previous section using respectively the standard mimetic finite difference and the upwind finite difference. This combination will yield our novel scheme called Fitted Mimetic Finite Difference Scheme.

As the case $(i > 1)$ is already covered in the previous section, we will only focus on the case $(i = 1)$. In fact we need to approximate the flux at $x_{1/2}$ with fitted finite volume method to handle the degeneracy of the Black-Scholes differential operator.

Note that to find a new approximation at $(DW)_1$, we require the fluxes at $x_{\frac{1}{2}}$ and $x_{\frac{3}{2}}$, i.e. $(\mathcal{G}U)_{\frac{1}{2}}$ and $(\mathcal{G}U)_{\frac{3}{2}}$ respectively. Let us now integrate the Black Scholes PDE (2.2.22) across $x_{\frac{1}{2}}$ and $x_{\frac{3}{2}}$,

$$-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \frac{\partial u}{\partial t} dx - \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \frac{\partial}{\partial x}\left[ax^2\frac{\partial u}{\partial x} + bxu\right] dx + \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \left(cu - f(x,t)\right) dx = 0, \tag{2.2.33}$$

and using the midpoint rule, we have that

$$-l_1\frac{dU_1}{dt} - \left[ax^2\frac{\partial u}{\partial x} + bxu\right]_{x_{1/2}}^{x_{3/2}} + \left(cU_1 - f(x_1,t)\right) l_1 = 0, \tag{2.2.34}$$

which gives

$$-\frac{dU_1}{dt} - \frac{1}{l_1}\left[ax^2\frac{\partial u}{\partial x} + bxu\right]_{x_{1/2}}^{x_{3/2}} + \left(cU_1 - f(x_1,t)\right) = 0. \tag{2.2.35}$$

Then from (2.2.35) we have that

$$-\frac{dU_1}{dt} - \frac{1}{l_1}\left[x_{3/2}\Phi(u)|_{x_{3/2}} - x_{1/2}\Phi(u)|_{x_{1/2}}\right] + \left(cU_1 - f(x_1,t)\right) = 0. \tag{2.2.36}$$

where $\Phi(u)$ is defined by

$$\Phi(u) := ax\frac{\partial u}{\partial x} + bu. \tag{2.2.37}$$

Note that the problem is not at $x_{3/2}$ and using (2.2.24), $x_{3/2}\Phi(u)|_{x_{3/2}}$ can be approximated as

$$\begin{aligned} x_{3/2}\Phi(u)|_{x_{3/2}} &\approx (-\mathcal{G}U)_{3/2} + x_{3/2}\Big(b^+U_2 + b^+U_1\Big) \\ &= \left[\frac{K_{3/2}}{h_1} + b^+x_{3/2}\right] U_2 + \left[b^-x_{3/2} - \frac{K_{3/2}}{h_1}\right] U_1. \end{aligned}$$

Let us approximate $x_{1/2}\Phi(u)|_{x_{1/2}}$ using the fitted technique. As in [43], we consider the following two-point boundary value problem

$$\begin{aligned} (axv' + bv)' &= C_1, \quad x \in (0, x_1) && (2.2.38)\\ v(0) &= U_0, \quad v(x_1) = U_1, && (2.2.39) \end{aligned}$$

where C_1 is an unknown constant to be determined. Integrating (2.2.38) once, we have that

$$axv' + bv = C_1 x + C_2$$

Now, using the condition $v(0) = U_0$, we have that $C_2 = bU_0$ and hence

$$\Phi(v) = axv' + bv = C_1 x + bU_0. \tag{2.2.40}$$

Following [43], we have

$$(\Phi(v))|_{x_{1/2}} = (axv' + bv)_{x_{1/2}} = \frac{1}{2}[(a+b)U_1 - (a-b)U_0], \tag{2.2.41}$$

Then (2.2.40) reduces to

$$v = U_0 + (U_1 - U_0)x/x_1, \quad x \in [0, x_1]. \tag{2.2.42}$$

Remember that $-\left[\frac{\partial}{\partial x}\left(ax^2\frac{\partial u}{\partial x} + bxu\right) - cu\right] = -\mathbf{D}\Phi(u) + cu$, so the following approximation can be used where the divergence operator $\mathbf{D}$ is approximated by $\mathcal{D}$

$$\begin{aligned} &-\left[\frac{\partial}{\partial x}\left(ax^2\frac{\partial u}{\partial x} + bxu\right) - cu\right]|_{x_1} \approx -\frac{x_{3/2}(\Phi(u))|_{x_{3/2}} - x_{1/2}(\Phi(u))|_{x_{1/2}}}{l_1} + cU_1 \\ &= -\frac{\left[\frac{K_{3/2}}{h_1} + b^+ x_{3/2}\right]U_2 + \left[b^- x_{3/2} - \frac{K_{3/2}}{h_1}\right]U_1 - \frac{x_{1/2}}{2}[(a+b)U_1 - (a-b)U_0]}{l_1} + cU_1 \\ &= -\left[\frac{K_{3/2}}{h_1 l_1} + \frac{b^+ x_{3/2}}{l_1}\right]U_2 + \left[\frac{K_{3/2}}{h_1 l_1} + \frac{a x_{1/2}}{2l_1} - \frac{b^- x_{3/2}}{l_1} + \frac{b x_{1/2}}{2l_1} + c\right]U_1 - \frac{(a-b)x_{1/2}}{2l_1}U_0. \end{aligned} \tag{2.2.43}$$

Combining with the mimetic finite difference approximation at x_i, $i > 1$ yields our novel scheme called the fitted mimetic finite difference method. Let us set $U_H = (U_1, U_2, ..., U_N)$, and using the transformation $t = T - t$, we therefore need to solve, using the fitted mimetic finite difference method, the following system

$$\begin{cases} \frac{dU_H}{dt} + \mathcal{C}_H U_H = f_h(t), \quad t \in [0, T], \\ \\ U_H(0) = U_H^*(0), \end{cases} \tag{2.2.44}$$

where with (2.2.31), we have

$$\begin{cases} \mathcal{C}_H U_H[1] = -\left[\dfrac{K_{3/2}}{h_1 l_1} + \dfrac{b^+ x_{3/2}}{l_1}\right] U_2 + \left[\dfrac{K_{3/2}}{h_1 l_1} + \dfrac{a x_{1/2}}{2l_1} - \dfrac{b^- x_{3/2}}{l_1} + \dfrac{b x_{1/2}}{2l_1} + c\right] U_1, \\ \mathcal{C}^h U^h[i] = \widehat{\mathcal{C}_H} U_H[i] \quad i > 1, \end{cases} \tag{2.2.45}$$

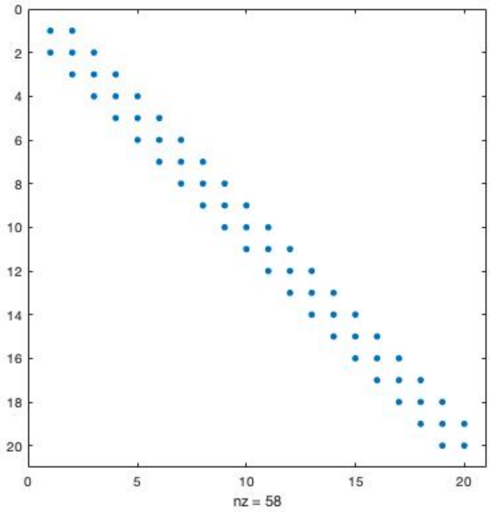


Figure 2.2: The matrix $\mathcal{C}^h$ structure for the fitted mimetic finite difference method

2.3 Discrete representation of the exact solution on mimetic grid

Consider the mimetic grids as we have defined in our previous section. The goal is to provide the discrete representation of the exact solution of (2.2.22) on $\mathcal{T}$, useful in our error analysis. Indeed remember that

$$-\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(ax^2\frac{\partial u(x,t)}{\partial x} + bxu(x,t)\right) + cu(x,t) = f(x,t). \tag{2.3.1}$$

Then in terms of the continuous mimetic operators, we have that

$$-\frac{\partial u(x,t)}{\partial t} + \mathbf{D}\left(\mathbf{G}u(x,t) - bxu(x,t)\right) + cu(x,t) = f(x,t). \tag{2.3.2}$$

Let us denote

$$\Phi(u(x,t),x,t) = -\mathbf{G}u(x,t) + bxu(x,t) = \frac{\partial}{\partial x}(ax^2 u) + bxu(x,t) \tag{2.3.3}$$

so

$$-\frac{\partial u(x,t)}{\partial t} - \mathbf{D}(\Phi(u,t)) + cu(x,t) = f(x,t). \tag{2.3.4}$$

Integrating (2.3.2) within the control volume $\Omega = (x_{i-1/2}, x_{i+1/2})$ yields

$$-\dot{u}(x_i,t)l_i - \left[\Phi(u(x_{i+\frac{1}{2}},t), x_{i+\frac{1}{2}},t) - \Phi(u(x_{i-\frac{1}{2}},t), x_{i-\frac{1}{2}},t)\right] + cl_i u(x_i,t) = f(x_i,t)l_i. \tag{2.3.5}$$

Then, multiplying (2.3.5) with an arbitrary real number v_i, and summing the results, we get

$$\begin{gathered}-\textstyle\sum_{i=1}^{N} \dot{u}(x_i,t)l_i v_i - \sum_{i=1}^{N}\left[\Phi(u(x_{i+\frac{1}{2}},t), x_{i+\frac{1}{2}},t) - \Phi(u(s_{i-\frac{1}{2}},t), x_{i-\frac{1}{2}},t)\right] v_i \\ +\textstyle\sum_{i=1}^{N} cl_i u(x_i,t)v_i = \sum_{i=1}^{N} f(x_i,t)l_i v_i.\end{gathered} \tag{2.3.6}$$

Now, we will consider, for any $v \in C(\overline{\Omega})$, a lumping operator $\mathcal{P}_h : C(\overline{\Omega}) \to L^{\infty}(\Omega)$ defined by

$$\mathcal{P}_h v|_{\Omega_i} := v(x_i), i = 1, ..., N.$$

The operator $\mathcal{P}_h$ gives the framework to consider a projection of the continuous functions $v \in C(\overline{\Omega})$ unto the mimetic grid, and v satisfies homogeneous Dirichlet boundary conditions. Furthermore we have that $\mathcal{P}_h v|_{\Omega_0} = \mathcal{P}_h v|_{\Omega_{N+1}} = 0$. Hence using the lumping operator we have

$$(-\dot{u}(t), v)_h + \mathcal{A}(u(t), v; t) = (f(t), v)_h, \quad \text{for all } v \in C(\overline{\Omega}), \text{ with } v(0) = v(X_{\max}) = 0, \tag{2.3.7}$$

where

$$\mathcal{A}(u(t), v; t) := -\sum_{i=1}^{N}\left[\Phi(u(x_{i+\frac{1}{2}},t), x_{i+\frac{1}{2}},t) - \Phi(u(x_{i-\frac{1}{2}},t), x_{i-\frac{1}{2}},t)\right]\mathcal{P}_h v(x_i) + (cu(t), v)_h, \tag{2.3.8}$$

and

$$(w, v)_h = (\mathcal{P}_h w, \mathcal{P}_h v) = \sum_{i=1}^{N} w_i v_i l_i, \quad w, v \in C(\overline{\Omega}).$$

Furthermore, in the case of the American option, (2.3.7) becomes

$$(-\dot{u}(t), v)_h + \mathcal{A}(u(t), v; t) + \Big(\Psi(u(t), t), v)_h = (f(t), v)_h, \quad \text{for all } v \in C(\overline{\Omega}), \tag{2.3.9}$$

with $v(0) = v(X_{\max}) = 0$, where $\mathcal{A}(u(t), v; t)$ is as defined in (2.3.8) and $\Psi(v,t) = \lambda\Big[v - u^*\Big]_+^{1/k}$ is the penalty term.

2.3.1 Variational formulation of Mimetic Method

In this section, we establish the key result which implies the unique solvability of the mimetic method in the discrete operators. From (2.3.5) using the mimetic approximation yields

$$\begin{cases} -\dfrac{dU_i}{dt} l_i - (\mathcal{D}\left[\Phi_h(U_h)\right])_i + cU_i(t)l_i = f_i(t)l_i, \quad i = 0, 1, ..., N+1, \\ u(s_i) \approx U_i, \end{cases} \tag{2.3.10}$$

where

$$\begin{aligned}\Phi_h(U_h,t)|_{x_{i+\frac{1}{2}}} &\approx -(\mathcal{G}U)_{i+\frac{1}{2}} + x_{i+\frac{1}{2}}\Big(b^+U_{i+1} + b^-U_i\Big)\\ &= \left[\frac{K_{i+\frac{1}{2}}}{h_i}\right](U_{i+1}-U_i) + x_{i+\frac{1}{2}}\Big(b^+U_{i+1} + b^-U_i\Big), \qquad (2.3.11)\\ \Phi_h(U_h,t)|_{x_{i-\frac{1}{2}}} &\approx -(\mathcal{G}U)_{i-\frac{1}{2}} + x_{i-\frac{1}{2}}\Big(b^+U_{i-1} + b^-U_i\Big)\\ &= \left[\frac{K_{i-\frac{1}{2}}}{h_{i-1}}\right](U_i-U_{i-1}) + x_{i-\frac{1}{2}}\Big(b^+U_{i-1} + b^-U_i\Big). \qquad (2.3.12)\end{aligned}$$

Then multiplying (2.3.10) by arbitrary numbers v_i and summing over Ω_i, we have

$$\begin{aligned}&-\sum_{i=1}^{N} l_i\frac{dU_i}{dt}v_i - \sum_{i=1}^{N}\Big[(\mathcal{G}U)_{i+\frac{1}{2}} + x_{i+\frac{1}{2}}\Big(b^+U_{i+1}+b^-U_i\Big)\Big]v_i\\ &+\sum_{i=1}^{N}\Big[(\mathcal{G}U)_{i-\frac{1}{2}} + x_{i-\frac{1}{2}}\Big(b^+U_{i-1}+b^-U_i\Big)\Big]v_i + \sum_{i=1}^{N} l_i(cU_i)v_i = \sum_{i=1}^{N} l_i f_i(t)v_i. \qquad (2.3.13)\end{aligned}$$

We write (2.3.13) in variational form as

$$-\Big(\dot{U}_h, v_h\Big)_h + \mathbf{a_h}(U_h, v_h; t) = \Big(f(t), v_h\Big)_h, \quad \text{for all } \; U_h, v_h \in V_h. \qquad (2.3.14)$$

where

$$\mathbf{a_h}(U_h(t), v_h; t) := \sum_{i=1}^{N}\Big[\Phi_h(U_h,x,t)\Big]_{x_{i+\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\mathcal{P}_h v_h(x_i) + (cu(t), v_h)_h\,, \qquad (2.3.15)$$

$$\Big[\Phi_h(U_h,x,t)\Big]_{x_{i+\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \Phi_h(U_h,t)|_{x_{i+\frac{1}{2}}} - \Phi_h(U_h,t)|_{x_{i-\frac{1}{2}}}. \qquad (2.3.16)$$

In the case of the American option, that is when $\lambda \neq 0$, we have

$$-\Big(\dot{U}_h, v_h\Big)_h + \mathbf{a_h}(U_h, v_h; t) + \Big(\Psi(U_h,t), v_h\Big)_h = \Big(f(t), v_h\Big)_h, \quad \text{for all } \; U_h, v_h \in V_h, \quad (2.3.17)$$

where $\Psi(v,t) = \lambda\Big[v - u^*\Big]_+^{1/k}$ is the penalty term.

In this sequel of this chapter, we will use the following mesh regularity.

Assumption 2.3.2. We assume that there exists a constant $c_0 > 0$ such that

$$c_0^{-1}h_{i+1} \leq h_i \leq c_0 h_{i+1}, \;\; i = 0, 1, ..., N. \qquad (2.3.18)$$

Note that condition (2.3.18) in Assumption 2.3.2 implies that

$$c_0^{-1}l_{i+1} \leq l_i \leq c_0 l_{i+1}, \;\; i = 0, 1, ..., N. \qquad (2.3.19)$$

To show that there exist a unique solution for (2.3.14), we need to show that the bilinear form $\mathbf{a_h}(\cdot,\cdot)$ is coercive and continuous. The following theorem below gives the coercivity of $\mathbf{a_h}(\cdot,\cdot)$.

Theorem 2.4. *Under the regularity of the mesh Assumption 2.3.2, there exists a constant $C > 0$ independent of h such that, for all $v_h \in V_h$, we have*

$$\mathbf{a_h}(v_h, v_h) = \mathbf{b_h^1}(v_h, v_h) + \mathbf{b_h^2}(v_h, v_h) \geq C\|v_h\|_{1,h}^2 \tag{2.4.1}$$

where

$$\mathbf{b_h^1}(v_h, v_h) = -\sum_{i=1}^{N} \left[(\mathcal{G}v)_{i+\frac{1}{2}} - (\mathcal{G}v)_{i-\frac{1}{2}}\right] v_i \tag{2.4.2}$$

$$\mathbf{b_h^2}(v_h, v_h) = -\sum_{i=1}^{N} \left[x_{i+1/2}\Big(b^+ v_{i+1} + b^- v_i\Big) - x_{i+\frac{1}{2}}\Big(b^+ v_{i-1} + b^- v_i\Big)\right] v_i + \sum_{i=1}^{N} l_i(cv_i)v_i. \tag{2.4.3}$$

Proof. Throughout the proof, C will represent a positive constant independent of h which may change from line to line. We firstly prove that

$$\mathbf{b_h^1}(v_h, v_h) \geq C\|v_h\|_{1,\mathcal{T}} \tag{2.4.4}$$

Then from (2.4.2), we have that,

$$\begin{aligned}
\mathbf{b_h^1}(v_h, v_h) &= -\sum_{i=1}^{N} \left[(\mathcal{G}v)_{i+\frac{1}{2}} - (\mathcal{G}v)_{i-\frac{1}{2}}\right] v_i \\
&= -\sum_{i=1}^{N} \left[\frac{ax^2_{i+\frac{1}{2}}}{h_i}(v_{i+1} - v_i) - \frac{ax^2_{i-\frac{1}{2}}}{h_{i-1}}(v_i - v_{i-1})\right] v_i \\
&= \sum_{i=1}^{N} \left[-\frac{ax^2_{i+\frac{1}{2}}}{h_i} v_{i+1}v_i + \left(\frac{ax^2_{i+\frac{1}{2}}}{h_i} + \frac{ax^2_{i-\frac{1}{2}}}{h_{i-1}}\right) v_i^2 - \frac{ax^2_{i-\frac{1}{2}}}{h_{i-1}} v_{i-1}v_i\right].
\end{aligned}$$

Remember that $v_0 = v_{N+1} = 0$. Now we use the following identities

$$-v_{i+1}v_i = \frac{1}{2}\left((v_{i+1} - v_i)^2 - v_{i+1}^2 - v_i^2\right) \tag{2.4.5}$$

and

$$-v_i v_{i-1} = \frac{1}{2}\left((v_i - v_{i-1})^2 - v_i^2 - v_{i-1}^2\right) \tag{2.4.6}$$

$$\begin{aligned}
\mathbf{b_h^1}(v_h, v_h) &= \sum_{i=1}^{N} \left[\frac{ax^2_{i+\frac{1}{2}}}{2h_i}\left((v_{i+1} - v_i)^2 - v_{i+1}^2 - v_i^2\right) + \left(\frac{ax^2_{i+\frac{1}{2}}}{h_i} + \frac{ax^2_{i-\frac{1}{2}}}{h_{i-1}}\right) v_i^2\right] \\
&\quad + \sum_{i=1}^{N} \left[\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}\left((v_i - v_{i-1})^2 - v_i^2 - v_{i-1}^2\right)\right] \\
&= \sum_{i=1}^{N} \left[\frac{ax^2_{i+\frac{1}{2}}}{2h_i}(v_{i+1} - v_i)^2 + \frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}(v_i - v_{i-1})^2 + \left(\frac{ax^2_{i+\frac{1}{2}}}{2h_i} + \frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}\right) v_i^2\right] \\
&\quad + \sum_{i=1}^{N} \left[-\frac{ax^2_{i+\frac{1}{2}}}{2h_i} v_{i+1}^2 - \frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}} v_{i-1}^2\right].
\end{aligned}$$

Now we consider the following expansions

$$\begin{aligned}
-\frac{a}{2}\sum_{i=1}^{N}\frac{x^2_{i-\frac{1}{2}}}{h_{i-1}}v^2_{i-1} &= \frac{a}{2}\left[-\frac{x^2_{\frac{1}{2}}}{h_0}v_0^2-\frac{x^2_{\frac{3}{2}}}{h_1}v_1^2-\frac{x^2_{\frac{5}{2}}}{h_2}v_2^2-\cdots-\frac{x^2_{N-\frac{3}{2}}}{h_{N-2}}v^2_{N-2}-\frac{x^2_{N-\frac{1}{2}}}{h_{N-1}}v^2_{N-1}\right]\\
&= \frac{a}{2}\left[-\frac{x^2_{\frac{3}{2}}}{h_1}v_1^2-\frac{x^2_{\frac{5}{2}}}{h_2}v_2^2-\cdots-\frac{x^2_{N-\frac{3}{2}}}{h_{N-2}}v^2_{N-2}-\frac{x^2_{N-\frac{1}{2}}}{h_{N-1}}v^2_{N-1}\right]\\
&= \frac{a}{2}\left[-\frac{x^2_{\frac{3}{2}}}{h_1}v_1^2-\cdots-\frac{x^2_{N-\frac{3}{2}}}{h_{N-2}}v^2_{N-2}-\frac{x^2_{N-\frac{1}{2}}}{h_{N-1}}v^2_{N-1}-\frac{x^2_{N+\frac{1}{2}}}{h_N}v^2_N\right]+\frac{ax^2_{N+\frac{1}{2}}}{2h_N}v^2_N\\
&= -\sum_{i=1}^{N}\frac{ax^2_{i+\frac{1}{2}}}{2h_i}v_i^2+\frac{ax^2_{N+\frac{1}{2}}}{2h_N}v^2_N,\quad \text{since}\, v_0=0,
\end{aligned}$$

and

$$\begin{aligned}
-\frac{a}{2}\sum_{i=1}^{N}\frac{x^2_{i+\frac{1}{2}}}{h_i}v^2_{i+1} &= \frac{a}{2}\left[-\frac{x^2_{\frac{3}{2}}}{h_1}v_2^2-\frac{x^2_{\frac{5}{2}}}{h_2}v_3^2-\cdots-\frac{x^2_{N-\frac{1}{2}}}{h_{N-1}}v^2_{N}-\frac{x^2_{N+\frac{1}{2}}}{h_N}v^2_{N+1}\right]\\
&= \frac{a}{2}\left[-\frac{x^2_{\frac{3}{2}}}{h_1}v_2^2-\frac{x^2_{\frac{5}{2}}}{h_2}v_3^2-\cdots-\frac{x^2_{N-\frac{3}{2}}}{h_{N-2}}v^2_{N-1}-\frac{x^2_{N-\frac{1}{2}}}{h_{N-1}}v^2_{N}\right]\\
&= \frac{a}{2}\left[-\frac{x^2_{\frac{1}{2}}}{h_0}v_1^2-\frac{x^2_{\frac{3}{2}}}{h_1}v_2^2-\cdots-\frac{x^2_{N-\frac{3}{2}}}{h_{N-2}}v^2_{N-1}-\frac{x^2_{N-\frac{1}{2}}}{h_{N-1}}v^2_{N}\right]+\frac{ax^2_{\frac{1}{2}}}{2h_0}v^2_1\\
&= -\sum_{i=1}^{N}\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}v_i^2+\frac{ax^2_{\frac{1}{2}}}{2h_0}v^2_1,\quad \text{since}\, v_{N+1}=0.
\end{aligned}$$

Then substituting the above expansions, we have that

$$\begin{aligned}
\mathbf{b}^{\mathbf{1}}_{\mathbf{h}}(v_h,v_h) &= \sum_{i=1}^{N}\left[\frac{ax^2_{i+\frac{1}{2}}}{2h_i}(v_{i+1}-v_i)^2+\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}(v_i-v_{i-1})^2+\left(\frac{ax^2_{i+\frac{1}{2}}}{2h_i}+\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}\right)v_i^2\right]\\
&\quad -\sum_{i=1}^{N}\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}v_i^2+\frac{ax^2_{\frac{1}{2}}}{2h_0}v_1^2-\sum_{i=1}^{N}\frac{ax^2_{i+\frac{1}{2}}}{2h_i}v_i^2+\frac{ax^2_{N+\frac{1}{2}}}{2h_N}v_N^2\\
&= \sum_{i=1}^{N}\left[\frac{ax^2_{i+\frac{1}{2}}}{2h_{i+\frac{1}{2}}}(v_{i+1}-v_i)^2+\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-\frac{1}{2}}}(v_i-v_{i-1})^2\right]+\frac{ax^2_{\frac{1}{2}}}{2h_0}v_1^2+\frac{ax^2_{N+\frac{1}{2}}}{2h_N}v_N^2\\
&\geq C\|v_h\|^2_{1,\mathcal{T}}.
\end{aligned}$$

Now we need to also prove that,

$$\mathbf{b}^{\mathbf{2}}_{\mathbf{h}}(v_h,v_h)\geq C\|v_h\|^2_{0,h}. \tag{2.4.7}$$

Now from (2.4.3), we have that,

$$
\begin{aligned}
\mathbf{b}_{\mathbf{h}}^{\mathbf{2}}(v_h, v_h) &= \sum_{i=1}^{N} \Big[-x_{i+\frac{1}{2}}\Big(b^+ v_{i+1} + b^- v_i\Big) + x_{i-\frac{1}{2}}\Big(b^+ u_{i-1} + b^- v_i\Big)\Big] v_i + c\sum_{i=1}^{N} l_i v_i^2 \\
&= \sum_{i=1}^{N} -x_{i+\frac{1}{2}}\Big(b^+ v_{i+1} + b^- v_i\Big)v_i + \sum_{i=1}^{N} x_{i-\frac{1}{2}}\Big(b^+ v_{i-1} + b^- v_i\Big)v_i + c\sum_{i=1}^{N} l_i v_i^2 \\
&= \sum_{i=1}^{N} -x_{i+\frac{1}{2}}\Big(b^+ v_{i+1} + b^- v_i\Big)v_i + \sum_{i=0}^{N-1} x_{i+\frac{1}{2}}\Big(b^+ v_i + b^- v_{i+1}\Big)v_{i+1} + c\sum_{i=1}^{N} l_i v_i^2 \\
&= x_{1/2}\Big(b^+ v_0 + b^- v_1\Big)v_1 + \sum_{i=1}^{N-1} x_{i+\frac{1}{2}}\Big[b^-\left(v_{i+1}^2 - v_i^2\right)\Big] \\
&\quad -x_{N+\frac{1}{2}}\Big(b^+ v_{N+1} + b^- v_N\Big)v_i + c\sum_{i=1}^{N} l_i v_i^2 \\
&= -b^- x_{1/2}\left(v_0^2 - v_1^2\right) - b^- \sum_{i=1}^{N-1} x_{i+\frac{1}{2}}\Big(v_i^2 - v_{i+1}^2\Big) \\
&\quad -b^- x_{N+\frac{1}{2}}\Big(v_N^2 - v_{N+1}^2\Big) + c\sum_{i=1}^{N} l_i v_i^2 \\
&= -b^- \sum_{i=0}^{N} x_{i+\frac{1}{2}} + c\sum_{i=1}^{N} l_i v_i^2 \Big(v_i^2 - v_{i+1}^2\Big).
\end{aligned}
$$

Now, remember that $c = 2r - \beta - \sigma^2 > 0, \quad \beta := \sigma^2$. Then we have that,

$$
\begin{aligned}
\mathbf{b}_{\mathbf{h}}^{\mathbf{2}}(v_h, v_h) &= -b^- \left[\sum_{i=0}^{N} \Big(x_{i-\frac{1}{2}} + l_i\Big)v_i^2 - \sum_{i=0}^{N} x_{i+\frac{1}{2}} v_{i+1}^2\right] + c\sum_{i=1}^{N} l_i v_i^2 \\
&= -b^- \left[\sum_{i=-1}^{N-1} \Big(x_{i+\frac{1}{2}} + l_{i+1}\Big)v_{i+1}^2 - \sum_{i=0}^{N} x_{i+\frac{1}{2}} v_{i+1}^2\right] + c\sum_{i=1}^{N} l_i v_i^2 \\
&= -b^- \left[x_{-\frac{1}{2}} v_0^2 + \sum_{i=-1}^{N-1} l_{i+1} v_{i+1}^2 - x_{N+\frac{1}{2}} v_{N+1}^2\right] + c\sum_{i=1}^{N} l_i v_i^2 \\
&= -b^- \sum_{i=0}^{N} l_i u_i^2 + c\sum_{i=1}^{N} l_i v_i^2 \geq C\|v_h\|_{0,h}^2. \qquad (2.4.8)
\end{aligned}
$$

That is,

$$\mathbf{b}_{\mathbf{h}}^{\mathbf{2}}(v_h, v_h) \geq C\|v_h\|_{0,h}^2. \qquad (2.4.9)$$

Then from (2.4.1), we have that

$$\mathbf{a}_{\mathbf{h}}(v_h, v_h) = \mathbf{b}_{\mathbf{h}}^{\mathbf{1}}(v_h, v_h) + \mathbf{b}_{\mathbf{h}}^{\mathbf{2}}(v_h, v_h) \geq C_1\|v_h\|_{1,\mathcal{T}}^2 + C_2\|v_h\|_{0,h}^2 = C\|v_h\|_{1,h}^2. \qquad (2.4.10)$$

$\square$

2.4.1 Coercivity of Fitted Mimetic Method

We follow similar arguments as in the previous subsection to prove the coercivity of the fitted problem. Here we have that

$$-\left(\dot{U}_H, v_h\right)_h + \mathbf{A_h}(U_H, v_h) = (f(t), v_h)_h, \text{ for all } v_h \in V_h \subset H^1_{0,w}(\Omega) \tag{2.4.11}$$

Note that the difference between the standard mimetic scheme and the fitted mimetic scheme is that the convection flux plus the diffusion flux is approximated at $x_{1/2}$ as

$$\begin{aligned} x_{1/2}\Phi(U_H)|_{x_{1/2}} &= \frac{x_{1/2}}{2}[(a+b)U_1 - (a-b)U_0] \\ &= \frac{ax_{1/2}}{2}[U_1 - U_0] + \frac{bx_{1/2}}{2}[U_1 + U_0] \\ &= -(\mathcal{G}U)_{\frac{1}{2}} + \frac{bx_{1/2}}{2}[U_1 + U_0]. \end{aligned} \tag{2.4.12}$$

So for the fitted mimetic scheme, we have

$$\mathbf{A_h}(U_H, v_h) = \mathbf{B^1_h}(U_H, v_h) + \mathbf{B^2_h}(U_H, v_h) \tag{2.4.13}$$

$$\mathbf{B^1_h}(U_H, v_h) = -\sum_{i=1}^{N}\left[(\mathcal{G}U)_{i+\frac{1}{2}} - (\mathcal{G}U)_{i-\frac{1}{2}}\right]v_i \text{ with } (\mathcal{G}U)_{\frac{1}{2}} = -\frac{ax_{1/2}}{2}[U_1 - U_0]. \tag{2.4.14}$$

$$\begin{aligned} \mathbf{B^2_h}(U_H, v_h) &= \frac{bx_{1/2}}{2}U_1v_1 - \sum_{i=1}^{N} x_{i\frac{1}{2}}\left[b^+U_{i+1} + b^-U_i\right]v_i + \sum_{i=2}^{N} x_{i-\frac{1}{2}}\left[b^+U_{i-1} + b^-U_i\right]v_i \\ &\quad + \sum_{i=1}^{N} l_i(cU_i)v_i. \end{aligned} \tag{2.4.15}$$

Theorem 2.5. *Under Assumption 4.2.2, there exists a positive constant $C > 0$, independent of h , such that for all $v_h \in V_h$, we have*

$$\mathbf{A_h}(v_h, v_h) \geq C\|v_h\|^2_{1,h}. \tag{2.5.1}$$

Proof. Again, throughout the proof, C will represent a positive constant independent of h which may change from line to line. We firstly prove that

$$\mathbf{B^1_h}(v_h, v_h) \geq C\|v_h\|_{1,\mathcal{T}}. \tag{2.5.2}$$

Then from (2.4.14)), we have that,

$$\begin{aligned} \mathbf{B^1_h}(v_h, v_h) &= -\sum_{i=1}^{N}\left[(\mathcal{G}v)_{i+\frac{1}{2}} - (\mathcal{G}v)_{i-\frac{1}{2}}\right]v_i \\ &= -\frac{ax^2_{\frac{3}{2}}}{h_1}v_2v_1 + \left[\frac{ax^2_{\frac{3}{2}}}{h_1} + \frac{x_{\frac{1}{2}}}{2}a\right]v_1^2 - \sum_{i=2}^{N}\left[(\mathcal{G}v)_{i+\frac{1}{2}} - (\mathcal{G}v)_{i-\frac{1}{2}}\right]v_i. \end{aligned}$$

Then from the second term we have that,

$$\begin{aligned}
-\sum_{i=2}^{N}\left[(\mathcal{G}v)_{i+\frac{1}{2}}-(\mathcal{G}v)_{i-\frac{1}{2}}\right]v_i &= -\sum_{i=2}^{N}\left[\frac{ax^2_{i+\frac{1}{2}}}{h_i}(v_{i+1}-v_i)-\frac{ax^2_{i-\frac{1}{2}}}{h_{i-1}}(v_i-v_{i-1})\right]v_i \\
&= \sum_{i=2}^{N}\left[-\frac{ax^2_{i+\frac{1}{2}}}{h_i}v_{i+1}v_i+\left(\frac{ax^2_{i+\frac{1}{2}}}{h_i}+\frac{ax^2_{i-\frac{1}{2}}}{h_i}\right)v_i^2-\frac{ax^2_{i-\frac{1}{2}}}{h_{i-1}}v_{i-1}v_i\right].
\end{aligned}$$

Now we use the following identities

$$-v_{i+1}v_i=\frac{1}{2}\left((v_{i+1}-v_i)^2-v_{i+1}^2-v_i^2\right), \tag{2.5.3}$$

and

$$-v_iv_{i-1}=\frac{1}{2}\left((v_i-v_{i-1})^2-v_i^2-v_{i-1}^2\right). \tag{2.5.4}$$

Let, $P=-\sum_{i=2}^{N}\left[(\mathcal{G}v)_{i+\frac{1}{2}}-(\mathcal{G}v)_{i-\frac{1}{2}}\right]v_i$, then

$$\begin{aligned}
P &= \sum_{i=2}^{N}\left[\frac{ax^2_{i+\frac{1}{2}}}{2h_i}\left((v_{i+1}-v_i)^2-v_{i+1}^2-v_i^2\right)+\left(\frac{ax^2_{i+\frac{1}{2}}}{h_i}+\frac{ax^2_{i-\frac{1}{2}}}{h_{i-1}}\right)v_i^2\right] \\
&\quad+\sum_{i=2}^{N}\left[\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}\left((v_i-v_{i-1})^2-v_i^2-v_{i-1}^2\right)\right]
\end{aligned}$$

$$\begin{aligned}
P &= \sum_{i=2}^{N}\left[\frac{ax^2_{i+\frac{1}{2}}}{2h_i}(v_{i+1}-v_i)^2+\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}(v_i-v_{i-1})^2+\left(\frac{ax^2_{i+\frac{1}{2}}}{2h_i}+\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}\right)v_i^2\right] \\
&\quad+\sum_{i=2}^{N}\left[-\frac{ax^2_{i+\frac{1}{2}}}{2h_i}v_{i+1}^2-\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}v_{i-1}^2\right].
\end{aligned}$$

Now we consider the following expansions

$$-\frac{a}{2}\sum_{i=2}^{N}\frac{x^2_{i-\frac{1}{2}}}{h_{i-\frac{1}{2}}}v_{i-1}^2 = -\sum_{i=1}^{N}\frac{ax^2_{i+\frac{1}{2}}}{2h_i}v_i^2+\frac{ax^2_{N+\frac{1}{2}}}{2h_N}v_N^2 \tag{2.5.5}$$

and

$$-\frac{a}{2}\sum_{i=2}^{N}\frac{x^2_{i+\frac{1}{2}}}{h_i}v_{i+1}^2 = -\sum_{i=2}^{N}\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}v_i^2+\frac{ax^2_{\frac{3}{2}}}{2h_{\frac{3}{2}}}v_2^2. \tag{2.5.6}$$

Substituting the (2.5.5) and (2.5.6) yields

$$\begin{aligned}
P &= \sum_{i=2}^{N}\left[\frac{ax^2_{i+\frac{1}{2}}}{2h_i}(v_{i+1}-v_i)^2+\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}(v_i-v_{i-1})^2+\left(\frac{ax^2_{i+\frac{1}{2}}}{2h_i}+\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}\right)v_i^2\right] \\
&\quad-\sum_{i=1}^{N}\frac{ax^2_{i+\frac{1}{2}}}{2h_i}v_i^2+\frac{ax^2_{N+\frac{1}{2}}}{2h_N}v_N^2-\sum_{i=2}^{N}\frac{ax^2_{i-\frac{1}{2}}}{2h_{i-1}}v_i^2+\frac{ax^2_{\frac{3}{2}}}{2h_{\frac{3}{2}}}v_2^2
\end{aligned}$$

$$
\begin{aligned}
P &= \sum_{i=2}^{N}\left[\frac{ax_{i+\frac{1}{2}}^2}{2h_i}(v_{i+1}-v_i)^2+\frac{ax_{i-\frac{1}{2}}^2}{2h_{i-1}}(v_i-v_{i-1})^2+\left(\frac{ax_{i-\frac{1}{2}}^2}{2h_{i-1}}\right)v_i^2\right] \\
&\quad -\frac{ax_{\frac{3}{2}}^2}{2h_1}v_1^2-\sum_{i=2}^{N}\frac{ax_{i+\frac{1}{2}}^2}{2h_i}v_i^2+\frac{ax_{N+\frac{1}{2}}^2}{2h_N}v_N^2+\frac{ax_{\frac{3}{2}}^2}{2h_1}v_2^2 \\
&= \sum_{i=2}^{N}\left[\frac{ax_{i+\frac{1}{2}}^2}{2h_i}(v_{i+1}-v_i)^2+\frac{ax_{i-\frac{1}{2}}^2}{2h_{i-1}}(v_i-v_{i-1})^2\right]-\frac{ax_{\frac{3}{2}}^2}{2h_1}v_1^2+\frac{ax_{N+\frac{1}{2}}^2}{2h_N}v_N^2+\frac{ax_{\frac{3}{2}}^2}{2h_1}v_2^2.
\end{aligned}
$$

Then substituting P into $\mathbf{B_h^1}$, we have that

$$
\begin{aligned}
\mathbf{B_h^1}(v_h,v_h) &= -\frac{ax_{\frac{3}{2}}^2}{h_1}v_2v_1+\left[\frac{ax_{\frac{3}{2}}^2}{h_1}+\frac{ax_{\frac{1}{2}}}{2}\right]v_1^2+\sum_{i=2}^{N}\left[\frac{ax_{i+\frac{1}{2}}^2}{2h_i}(v_{i+1}-v_i)^2+\frac{ax_{i-\frac{1}{2}}^2}{2h_{i-1}}(v_i-v_{i-1})^2\right] \\
&\quad -\frac{ax_{\frac{3}{2}}^2}{2h_1}v_1^2+\frac{ax_{N+\frac{1}{2}}^2}{2h_N}v_N^2+\frac{ax_{\frac{3}{2}}^2}{2h_1}v_2^2, \\
&= \frac{ax_{\frac{3}{2}}^2}{2h_1}\left[(v_2-v_1)^2-v_1^2-v_2^2\right]\left[\frac{ax_{\frac{3}{2}}^2}{2h_1}+\frac{ax_{\frac{1}{2}}}{2}\right]v_1^2 \\
&\quad +\sum_{i=2}^{N}\left[\frac{ax_{i+\frac{1}{2}}^2}{2h_i}(v_{i+1}-v_i)^2+\frac{ax_{i-\frac{1}{2}}^2}{2h_{i-1}}(v_i-v_{i-1})^2\right]-\frac{ax_{\frac{3}{2}}^2}{2h_1}v_1^2+\frac{ax_{N+\frac{1}{2}}^2}{2h_N}v_N^2+\frac{ax_{\frac{3}{2}}^2}{2h_1}v_2^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B_h^1}(v_h,v_h) &= \frac{ax_{\frac{3}{2}}^2}{2h_1}\left[(v_2-v_1)^2\right]+\frac{ax_{\frac{1}{2}}}{2}v_1^2 \\
&\quad +\sum_{i=2}^{N}\left[\frac{ax_{i+\frac{1}{2}}^2}{2h_i}(v_{i+1}-v_i)^2+\frac{ax_{i-\frac{1}{2}}^2}{2h_{i-1}}(v_i-v_{i-1})^2\right]+\frac{ax_{N+\frac{1}{2}}^2}{2h_N}v_N^2 \\
&= \sum_{i=1}^{N}\left[\frac{ax_{i+\frac{1}{2}}^2}{2h_i}(v_{i+1}-v_i)^2\right]+\sum_{i=2}^{N}\left[\frac{ax_{i-\frac{1}{2}}^2}{2h_{i-1}}(v_i-v_{i-1})^2\right]+\frac{ax_{\frac{1}{2}}}{2}v_1^2+\frac{ax_{N+\frac{1}{2}}^2}{2h_N}v_N^2, \\
&\geq C\|v_h\|_{1,\mathcal{T}}^2.
\end{aligned}
$$

Now from (2.4.15), we have that,

$$
\mathbf{B_h^2}(v_h,v_h) = \frac{bx_{1/2}}{2}v_1^2-\sum_{i=1}^{N}x_{i+\frac{1}{2}}\left[b^+v_{i+1}+b^-v_i\right]v_i+\sum_{i=2}^{N}x_{i-\frac{1}{2}}\left[b^+v_{i-1}+b^-v_i\right]v_i+\sum_{i=1}^{N}l_icv_i^2. \tag{2.5.7}
$$

Note that comparing with the standard mimetic scheme, we only have the term $-\frac{bx_{1/2}}{2}v_1v_1 = -\frac{bx_{1/2}}{2}(v_1)^2$ for the fitted scheme instead of $-b^-x_{1/2}v_1^2$ which corresponds to the standard mimetic. Following line by line what we did for the standard mimetic method for (2.5.7), we have

$$
\begin{aligned}
\mathbf{B}^{\mathbf{2}}_{\mathbf{h}}(v_h, v_h) &= \frac{bx_{1/2}}{2}v_1^2 - \sum_{i=1}^{N} x_{i+\frac{1}{2}}\left[b^+ v_{i+1} + b^- v_i\right] v_i + \sum_{i=2}^{N} x_{i-\frac{1}{2}}\left[b^+ v_{i-1} + b^- v_i\right] v_i + \sum_{i=1}^{N} l_i c v_i^2 \\
&= \frac{bx_{1/2}}{2}v_1^2 - \sum_{i=1}^{N} x_{i+\frac{1}{2}}\left[b^+ v_{i+1} + b^- v_i\right] v_i + \sum_{i=1}^{N-1} x_{i+\frac{1}{2}}\left[b^+ v_{i} + b^- v_{i+1}\right] v_{i+1} + \sum_{i=1}^{N} l_i c v_i^2 \\
&= \frac{bx_{1/2}}{2}v_1^2 + \sum_{i=1}^{N-1} x_{i+\frac{1}{2}} b^-\left(v_{i+1}^2 - v_i^2\right) - x_{N+\frac{1}{2}} b^- v_N^2 + \sum_{i=1}^{N} l_i c v_i^2.
\end{aligned}
\tag{2.5.8}
$$

$$
\begin{aligned}
\mathbf{B}^{\mathbf{2}}_{\mathbf{h}}(v_h, v_h) &= \left[\frac{bx_{1/2}}{2}v_1^2 + \sum_{i=1}^{N} x_{i+\frac{1}{2}} b^-\left(v_{i+1}^2 - v_i^2\right)\right] + \sum_{i=1}^{N} l_i c v_i^2 \\
&= \mathbf{B}^{\mathbf{2}}_{\mathbf{h1}}(v_h, v_h) + \mathbf{B}^{\mathbf{2}}_{\mathbf{h2}}(v_h, v_h)
\end{aligned}
\tag{2.5.9}
$$

Then for $\mathbf{B}^{\mathbf{2}}_{\mathbf{h1}}(v_h, v_h)$, we consider two cases when $b \geq 0$ and $b < 0$.

<u>1st case:</u> $b \geq 0$ if $b > 0$, then $b^- = 0$, so we have

$$
\mathbf{B}^{\mathbf{2}}_{\mathbf{h1}}(v_h, v_h) = \frac{bx_{1/2}}{2}v_1^2 > 0.
\tag{2.5.10}
$$

<u>2nd case:</u> $b < 0$, we have

$$
\begin{aligned}
\mathbf{B}^{\mathbf{2}}_{\mathbf{h1}}(v_h, v_h) &= \frac{bx_{1/2}}{2}v_1^2 + \sum_{i=1}^{N} x_{i+\frac{1}{2}} b^-\left(v_{i+1}^2 - v_i^2\right) \\
&= \frac{bx_{1/2}}{2}v_1^2 - b^-\left(\sum_{i=1}^{N}\left(x_{i-\frac{1}{2}} + l_i\right)v_i^2 - \sum_{i=1}^{N} x_{i+\frac{1}{2}} v_{i+1}^2\right) \\
&= \frac{bx_{1/2}}{2}v_1^2 - b^-\left(\sum_{i=0}^{N-1}\left(x_{i+\frac{1}{2}} + l_{i+1}\right)v_{i+1}^2 - \sum_{i=1}^{N} x_{i+\frac{1}{2}} v_{i+1}^2\right) \\
&= \frac{bx_{1/2}}{2}v_1^2 - b^-\left(\left(x_{\frac{1}{2}} + l_1\right)v_1^2 + \sum_{i=1}^{N} l_i v_i^2 - x_{N+\frac{1}{2}} v_{N+1}^2\right) \\
&= \left(\frac{bx_{1/2}}{2} - b^- x_{\frac{3}{2}}\right)v_1^2 - b^-\sum_{i=1}^{N} l_i v_i^2. \\
\mathbf{B}^{\mathbf{2}}_{\mathbf{h1}}(v_h, v_h) &\geq 0.
\end{aligned}
\tag{2.5.11}
$$

Thus for any b, we have

$$
\mathbf{B}^{\mathbf{2}}_{\mathbf{h1}}(v_h, v_h) \geq 0.
\tag{2.5.12}
$$

Then from (2.5.9) and using (2.5.12), we have

$$\mathbf{B_h^2}(v_h, v_h) \geq \sum_{i=1}^{N} l_i c v_i^2 = C\|v_h\|_{0,h}^2 \tag{2.5.13}$$

Then we have that

$$\mathbf{A_h}(v_h, v_h) \geq C\|v_h\|_{1,h}^2. \tag{2.5.14}$$

□

Remark 2.5.1. Note that using the coercivity properties in (2.4.1) and (2.5.1), with the fact that the linear mapping $v \longrightarrow (f, v)_h$ and the two bilinear forms $\mathbf{a_h}(\cdot,\cdot)$ and $\mathbf{A_h}(\cdot,\cdot)$ are continuous in V_h and $V_h \times V_h$ respectively, the existence and uniqueness of the discrete solution u_h is ensured for both the mimetic and fitted mimetic methods in (2.3.14) and (2.4.11) for European options. The proof is done exactly as for the continuous case (see [22, Theorem 1.33]).

2.5.2 Consistency of the Fluxes

In this section, we prove the consistency of the fluxes. We can now state the following important theorem

Theorem 2.6. *Let Φ be the operator defined in (2.3.10) and $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be such that $\Phi'(w, \cdot, t) \in L^2(\Omega)$ for all $t \in (0, T)$. Let Φ_h be the approximation of Φ using mimetic finite difference method (see (2.3.11)-(2.3.12)) or fitted mimetic method (see (2.3.11)-(2.3.12)) with $\Phi_h(w_h(x_{\frac{1}{2}}), x_{\frac{1}{2}}, t) = \frac{x_{\frac{1}{2}}}{2}\left[(a+b)w_1 - (a-b)w_0\right]$. Then under Assumption 2.3.2, there exists a constant $C > 0$ independent of w and h such that the following estimate holds:*

$$|\Phi_h(w_h, x_{i+\frac{1}{2}}, t) - \Phi(w, x_{i+\frac{1}{2}}, t)| \leq C\int_{x_i}^{x_{i+1}} \left[|\Phi'(w, \cdot, t)| + |w'| + |w|\right] dx, \quad i = 0, 1, ..., N, \tag{2.6.1}$$

where $w_h = I_h w$.

Proof. For the proof, we consider both Mimetic and Fitted-Mimetic cases. For the Mimetic case, the proof presented is for all $i = 0, 1, ..., N$. In what follows, we consider the case 1 (mimetic). Remember from (2.3.10), that for $i = 0, 1, ...N$, we have

$$-\frac{dU_i}{dt} l_i - \left[\Phi(U)|_{x_{i+1/2}} - \Phi(U)|_{x_{i-1/2}}\right] + (cU_i - f_i(t))\, l_i = 0, \tag{2.6.2}$$

where

$$\Phi(w, x, t) := ax^2\frac{\partial w}{\partial x} + bxw \tag{2.6.3}$$

and

$$\Phi_h(w_h, x_{i+\frac{1}{2}}, t) := a x^2_{i+\frac{1}{2}} \frac{w_{i+1} - w_i}{h_i} + b x_{i+\frac{1}{2}} w_{i+1}. \tag{2.6.4}$$

Then from (2.6.3) and (2.6.4), we have

$$\begin{aligned}
\Phi_h(w_h, x_{i+\frac{1}{2}}, t) - \Phi(w, x_{i+\frac{1}{2}}, t) &= a x^2_{i+\frac{1}{2}} \frac{w_{i+1} - w_i}{h_i} + x_{i+\frac{1}{2}} \left(b^+ w_{i+1} + b^- w_i\right) \\
&\quad - a x^2_{i+\frac{1}{2}} w'(x_{i+\frac{1}{2}}) - b x_{i+\frac{1}{2}} w(x_{i+\frac{1}{2}}) \\
&= a x^2_{i+\frac{1}{2}} \left[\frac{w_{i+1} - w_i}{h_i} - w'(x_{i+\frac{1}{2}})\right] \\
&\quad + x_{i+\frac{1}{2}} \left[\left(b^+ w_{i+1} + b^- w_i\right) - w(x_{i+\frac{1}{2}})\right] \\
&= : A_1 + A_2.
\end{aligned}$$

Indeed using the Sobolev embedding theorem, as we are in dimension 1, $H^2(\Omega) \hookrightarrow C^1(\Omega)$, using the Taylor expansion with integral remainder, we have

$$w(z) = w(x_{i+\frac{1}{2}}) + (z - x_{i+\frac{1}{2}}) w'(x_{i+\frac{1}{2}}) + \int_{x_{i+\frac{1}{2}}}^{z} (z - x) w''(x) dx. \tag{2.6.5}$$

Then when we set $z = x_{i+1}$, we have that

$$w(x_{i+1}) = w(x_{i+\frac{1}{2}}) + (x_{i+1} - x_{i+\frac{1}{2}}) w'(x_{i+\frac{1}{2}}) + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - x) w''(x) dx, \tag{2.6.6}$$

and also set $z = x_i$, we have that

$$w(x_i) = w(x_{i+\frac{1}{2}}) + (x_i - x_{i+\frac{1}{2}}) w'(x_{i+\frac{1}{2}}) - \int_{x_{i+\frac{1}{2}}}^{x_i} (x_i - x) w''(x) dx. \tag{2.6.7}$$

Then we have that

$$\begin{aligned}
\frac{w(x_{i+1}) - w(x_i)}{h_i} &= \frac{1}{h_i} \left[(x_{i+1} - x_i) w'(x_{i+\frac{1}{2}}) + \int_{x_i}^{x_{i+\frac{1}{2}}} (x_i - x) w''(x) dx\right] \\
&\quad + \frac{1}{h_i} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - x) w''(x) dx
\end{aligned}$$

$$\frac{w(x_{i+1}) - w(x_i)}{h_i} - w'(x_{i+\frac{1}{2}}) = \frac{1}{h_i} \left[\int_{x_i}^{x_{i+\frac{1}{2}}} (x_i - x) w''(x) dx + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - x) w''(x) dx\right]$$

We use the following inequalities

$$\left|\int_{x_i}^{x_{i+\frac{1}{2}}} (x_i - x) w''(x) dx\right| \leq \frac{h_i}{2} \int_{x_i}^{x_{i+1}} |w''(x)| dx \tag{2.6.8}$$

$$\left|\int_{x_{i+\frac{1}{2}}}^{x_i+1}(x_{i+1}-x)w''(x)dx\right| \leq \frac{h_i}{2}\int_{x_i}^{x_{i+1}}|w''(x)|dx. \tag{2.6.9}$$

Then we have that,

$$\begin{aligned}\left|\frac{w(x_{i+1})-w(x_i)}{h_i}-w'(x_{i+\frac{1}{2}})\right| &\leq \frac{1}{h_i}\left[\frac{h_i}{2}\int_{x_i}^{x_{i+1}}|w''(x)|dx+\frac{h_i}{2}\int_{x_i}^{x_{i+1}}|w''(x)|dx\right]\\ &\leq \frac{1}{2}\int_{x_i}^{x_{i+1}}|w''(x)|dx.\end{aligned}$$

Hence we have that

$$ax^2_{i+\frac{1}{2}}\left|\frac{w(x_{i+1})-w(x_i)}{2h_i}-w'(x_{i+\frac{1}{2}})\right| \leq \int_{x_i}^{x_{i+1}}\frac{ax^2_{i+\frac{1}{2}}}{2}|w''(x)|dx. \tag{2.6.10}$$

Now we consider the following from Assumption 1, for $x_i \leq x \leq x_{i+1}$, we have

$$\begin{aligned}\left(\frac{x_{i+\frac{1}{2}}}{x}\right)^2 \leq \left(\frac{x_{i+\frac{1}{2}}}{x_i}\right)^2 = \left(\frac{x_{i+1}+x_i}{2x_i}\right)^2 = \left(1+\frac{h_i}{2x_i}\right)^2 = \left(1+\frac{h_i}{2(x_{i-1}+h_{i-1})}\right)^2\\ \leq \left(1+\frac{h_i}{2h_{i-1}}\right)^2 \leq \left(1+\frac{c}{2}\right)^2.\end{aligned} \tag{2.6.11}$$

Then from (2.6.11) we have that

$$\begin{aligned}ax^2_{i+\frac{1}{2}}\left|\frac{w(x_{i+1})-w(x_i)}{h_i}-w'(x_{i+\frac{1}{2}})\right| &\leq \int_{x_i}^{x_{i+1}}\frac{ax^2_{i+\frac{1}{2}}}{2x^2}|x^2w''(x)|dx\\ &\leq a\left(1+\frac{c}{2}\right)^2\int_{x_i}^{x_{i+1}}|s^2w''(x)|dx.\end{aligned}$$

Now from (2.6.3), we have that

$$ax^2w'' = \Phi'(w,x,t)-(2a+b)xw'-bw. \tag{2.6.12}$$

Then we have that

$$\begin{aligned}ax^2_{i+\frac{1}{2}}\left|\frac{w(x_{i+1})-w(x_i)}{h_i}-w'\left(x_{i+\frac{1}{2}}\right)\right| &\leq \left(1+\frac{c}{2}\right)^2\int_{x_i}^{x_{i+1}}|\Phi'(w,x,t)-(2a+b)xw'-bw|dx\\ &\leq \left(1+\frac{c}{2}\right)^2\int_{x_i}^{x_{i+1}}\left[|\Phi'(w,x,t)|+|(2a+b)||w'|+|bw|\right]ds\\ &= \left(1+\frac{c}{2}\right)^2\left(\int_{x_i}^{x_{i+1}}\left[|\Phi'(w,x,t)|+|b||w|\right]dx+\int_{x_i}^{s_{i+1}}|(2a+b)||w'|ds\right)\end{aligned}$$

$$|A_1| \leq C\left(\int_{x_i}^{x_{i+1}}\left[|\Phi'(w,x,t)|+|w'|+|w|\right]dx\right).$$

Now to estimate $A_2 = x_{i+\frac{1}{2}}\left(b^+ w_{i+1} + b^- w_i - bw\left(x_{i+\frac{1}{2}}\right)\right)$. Remember that

$$b^+ = \max\{b,0\} = \begin{cases} b, & \text{if } b>0, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad b^- = \min\{b,0\} = \begin{cases} 0, & \text{if } b>0, \\ b, & \text{otherwise.} \end{cases} \tag{2.6.13}$$

For case one, $b > 0 \Rightarrow \; b^+ = b$ and $b^- = 0$. We apply the formula

$$w_{i+1} - w\left(x_{i+\frac{1}{2}}\right) = \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} w' dx, \tag{2.6.14}$$

and then we have

$$bx_{i+\frac{1}{2}}\left|w_{i+1} - w\left(x_{i+\frac{1}{2}}\right)\right| \leq b\left|x_{i+\frac{1}{2}}\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} w' dx\right| \leq C\int_{x_i}^{x_{i+1}} |w'|dx. \tag{2.6.15}$$

Also, for $b < 0 \Rightarrow \; b^+ = 0$ and $b^- = b$. Then we have

$$bx_{i+\frac{1}{2}}\left|w_{i} - w\left(x_{i+\frac{1}{2}}\right)\right| \leq b\left|x_{i+\frac{1}{2}}\int_{x_{i+\frac{1}{2}}}^{x_{i}} w' dx\right| \leq C\int_{x_i}^{x_{i+1}} |w'|dx. \tag{2.6.16}$$

That is

$$|A_2| \leq C\int_{x_i}^{x_{i+1}} |w'|dx. \tag{2.6.17}$$

Therefore we have that

$$\begin{aligned} |\Phi\left(w_h, x_{i+\frac{1}{2}}, t\right) - \Phi\left(w, x_{i+\frac{1}{2}}, t\right)| = |A_1 + A_2| &\leq |A_1| + |A_2| \\ &\leq C\left(\int_{x_i}^{x_{i+1}} \left[|\Phi'(w,x,t)| + |w'| + |w|\right] dx\right). \end{aligned}$$

Therefore,

$$|\Phi\left(w_h, x_{i+\frac{1}{2}}, t\right) - \Phi\left(w, x_{i+\frac{1}{2}}, t\right)| \leq C\left(\int_{x_i}^{x_{i+1}} \left[|\Phi'(w,x,t)| + |w'| + |w|\right] dx\right). \tag{2.6.18}$$

Now for the fitted mimetic case, it is sufficient to consider the flux approximation at $i = 0$, hence we have that

$$\begin{aligned} &\left|\Phi_h\left(w_h\left(x_{\frac{1}{2}}\right), x_{\frac{1}{2}}, t\right) - \Phi\left(w\left(x_{\frac{1}{2}}\right), x_{\frac{1}{2}}, t\right)\right| \\ &= \left|\frac{x_{\frac{1}{2}}}{2}\left[(a+b)w_1 - (a-b)w_0\right] - ax_{\frac{1}{2}}^2 w'\left(x_{\frac{1}{2}}\right) - bx_{\frac{1}{2}} w\left(x_{\frac{1}{2}}\right)\right| \\ &= \left|\frac{x_{\frac{1}{2}}}{2}\left[(a+b)\left(\frac{w_1 - w_0}{h_0}h_0 + w_0\right) - (a-b)w_0\right] - ax_{\frac{1}{2}}^2 w'\left(x_{\frac{1}{2}}\right) - bx_{\frac{1}{2}} w\left(x_{\frac{1}{2}}\right)\right| \\ &= \left|x_{\frac{1}{2}}^2 (a+b)\left(\frac{w_1 - w_0}{h_0}\right) + bx_{\frac{1}{2}} w_0 - ax_{\frac{1}{2}}^2 w'\left(x_{\frac{1}{2}}\right) - bx_{\frac{1}{2}} w\left(x_{\frac{1}{2}}\right)\right|, \text{ since } h_0 = 2x_{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \left| a x_{\frac{1}{2}}^2 \left(\frac{w_1 - w_0}{h_0} - w'\left(x_{\frac{1}{2}}\right) \right) + b x_{\frac{1}{2}} \left(w_0 - w\left(x_{\frac{1}{2}}\right) \right) + b x_{\frac{1}{2}}^2 \left(\frac{w_1 - w_0}{h_0} \right) \right| \\
&\leq \left| a x_{\frac{1}{2}}^2 \left(\frac{w_1 - w_0}{h_0} - w'\left(x_{\frac{1}{2}}\right) \right) \right| + \left| b x_{\frac{1}{2}} \left(w_0 - w\left(x_{\frac{1}{2}}\right) \right) \right| + \left| b x_{\frac{1}{2}}^2 \left(\frac{w_1 - w_0}{h_0} \right) \right| \\
&\qquad =: D_{10} + D_{11} + D_{12}.
\end{aligned}
$$

Now, we estimate D_{10} by using a similar argument as with the general case, that is

$$
\left(\frac{w_1 - w_0}{h_0} - w'\left(x_{\frac{1}{2}}\right) \right) = \frac{1}{h_0} \int_{x_0}^{x_{\frac{1}{2}}} w''(x)(x_0 - x)dx + \frac{1}{h_0} \int_{x_{\frac{1}{2}}}^{x_1} w''(x)(x_1 - x)dx. \tag{2.6.19}
$$

Hence,

$$
D_{10} = a x_{\frac{1}{2}}^2 \left| \frac{w_1 - w_0}{h_0} - w'\left(x_{\frac{1}{2}}\right) \right| \leq \frac{a x_{\frac{1}{2}}^2}{2} \int_{x_0}^{x_1} |w''| dx \leq C \int_{x_0}^{x_1} \left[|\Phi'(w)| + |w'| + |w| \right] dx.
$$

Now we consider the estimate of D_{11}

$$
D_{11} = b x_{\frac{1}{2}} \left| w_0 - w\left(x_{\frac{1}{2}}\right) \right| \leq b x_{\frac{1}{2}} \int_{x_0}^{x_1} |w'| dx \leq C \int_{x_0}^{x_1} |w'| dx.
$$

Also, the estimate of D_{12}

$$
D_{12} = b x_{\frac{1}{2}}^2 \left| \frac{w_1 - w_0}{h_0} \right| \leq b x_{\frac{1}{2}}^2 \int_{x_0}^{x_1} |w'| dx \leq C \int_{x_0}^{x_1} |w'| dx.
$$

Therefore,

$$
\left| \Phi_h \left(w_h\left(x_{\frac{1}{2}}\right), x_{\frac{1}{2}}, t \right) - \Phi\left(w\left(x_{\frac{1}{2}}\right), x_{\frac{1}{2}}, t \right) \right| \leq C \int_{x_0}^{x_1} \left[|\Phi'(w)| + |w'| + |w| \right] dx. \tag{2.6.20}
$$

□

2.7 Full Discrete Problem

We subdivide the time interval $[0, T]$ in M subdivisions. We consider here the implicit scheme for time integration although the general $\theta-$Euler method can be used. Note that when $\lambda \neq 0$, (2.3.14) can be written as

$$
-\left(\dot{U}_h, v_h \right)_h + \mathbf{a_h}(U_h, v_h; t) + \left(\Psi(U_h, t), v_h \right)_h = \left(f(t), v_h \right)_h, \quad \text{for all} \;\; U_h, v_h \in V_h. \tag{2.7.1}
$$

where

$$
\mathbf{a_h}(U_h(t), v_h; t) := \sum_{i=1}^{N} \left[\Phi_h(U_h, x, t) \right]_{x_{i+\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{P}_h v_h(x_i) + (cu(t), v_h)_h \,, \tag{2.7.2}
$$

$$
\left[\Phi_h(U_h, x, t) \right]_{x_{i+\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \Phi_h(U_h, t)|_{x_{i+\frac{1}{2}}} - \Phi_h(U_h, t)|_{x_{i-\frac{1}{2}}}, \tag{2.7.3}
$$

and $\Psi(U_h,t) := \lambda\Big[U_h - U^*\Big]_+^{1/k}$ is the penalty term in the case of the American option and $\Psi(U_h,t) \equiv 0$ in the case of the European option.

The full discretization of (2.7.1), after using the transformation $t = T - t$, with the implicit Euler method can be formulated as : Find a sequence $U_h^1,\dots,U_h^M \in V_h$ such that for $m \in \{0,\dots,M-1\}$

$$\Bigg\{ \left(\frac{U_h^{m+1}-U_h^m}{\Delta t}, v_h\right)_h + \mathbf{a_h}\Big(U_h^{m+1}, v_h; t_{m+1}\Big) + \Big(\Psi(U_h^{m+1},t_{m+1}), v_h\Big)_h = \Big(f^{m+1}, v_h\Big)_h \tag{2.7.4}$$

with U_h^0 given and $\mathbf{a_h}$ is the bilinear form for the mimetic method given in (2.7.1). Similarly, when the fitted mimetic difference method is applied for the the spatial discretization, the full discretization after using the transformation $t = T - t$, is formulated as follows: Find a sequence $U_H^1,\dots,U_H^M \in V_h$ such that for $m \in \{0,\dots,M-1\}$

$$\Bigg\{ \left(\frac{U_H^{m+1}-U_H^m}{\Delta t}, v_h\right)_h + \mathbf{A_h}\Big(U_H^{m+1}, v_h; t_{m+1}\Big) + \Big(\Psi(U_H^{m+1},t_{m+1}), v_h\Big)_h = \Big(f^{m+1}, v_h\Big)_h \tag{2.7.5}$$

with U_H^0 given and $\mathbf{A_h}$ is the bilinear form of the fitted mimetic method given in (2.4.11).

We now present the main convergence result for the method proposed in this thesis.

Theorem 2.8. *Let ζ_h^m be the numerical solution of the fully discretized scheme using the mimetic method (2.7.4) ($\zeta_h^m = U_h^m$ for the mimetic method) or the fitted mimetic method (2.7.5) ($\zeta_h^m = U_H^m$ for fitted mimetic method). Let u be the unique solution of (2.3.6). If $u \in H^1\Big(0,T;H^1(\Omega)\Big) \cap H^2\Big(0,T;L^2(\Omega)\Big)$ and $\Phi(u,\cdot,\cdot) \in C(0,T,H^1(\Omega))$, then there exists a positive constant C, independent of h, Δt, M, and N such that*

$$\big|\big|u(t_m) - \zeta_h^m\big|\big|_{0,h} \le C(h+\Delta t). \tag{2.8.1}$$

Remark 2.8.1. Note that for European options, the existence and uniqueness of ζ_h^m in (2.7.4) and (2.7.5) is ensured using the well known Lax-Milgram Theorem [22], since for every fixed time t_{m+1} the bilinear forms $(\cdot,\cdot)_h + \Delta t\mathbf{a_h}(\cdot,\cdot)$ and $(\cdot,\cdot)_h + \Delta t\mathbf{A_h}(\cdot,\cdot)$ are V_h-coercive.

The proof of Theorem 2.8 need the following lemma, similar to [30, Lemma 2].

Lemma 2.8.2. There exist two constants C_3 and C_4 independent of h such that the coefficient $\tau_{i+\frac{1}{2}} = \dfrac{x_{i+1/2}^2}{h_i}$ used in (2.2.3) for the discrete $H^1_{0,w}(\Omega)-$norm and its inverse are bounded as follows:

$$\tau_{i+\frac{1}{2}} \le C_3, \qquad \frac{1}{\tau_{i+\frac{1}{2}}} \le C_4 h_i \qquad\qquad i = 0,\dots,N. \tag{2.8.2}$$

Proof. Indeed, remember that $\Omega = [0, X_{\max}]$, then we have

$$\tau_{i+\frac{1}{2}} = \frac{x_{i+1/2}^2}{h_i} = \frac{x_{i+1/2}^2}{x_{i+1}-x_i} = \frac{\frac{x_{i+1/2}^2}{x_{i+1}}}{1-\frac{x_i}{x_{i+1}}} \le \frac{X_{\max}}{1-\frac{x_i}{x_{i+1}}}. \tag{2.8.3}$$

By setting $\nu = \frac{x_i}{x_{i+1}}$ $i = 1, ..., N$, we have $0 < \nu < 1$, and therefore using the Taylor expansion we have

$$\frac{1}{1-\nu} = 1 + \nu + \mathcal{O}(\nu^2). \tag{2.8.4}$$

Then, there exists $C > 0$ such that

$$\frac{1}{1-\nu} < 2 + C, \tag{2.8.5}$$

and finally

$$\tau_{i+\frac{1}{2}} \leq C_3, \ \text{ with } \ C_3 < X_{\max}(2 + C). \tag{2.8.6}$$

Let us prove the second estimation. Indeed we have

$$\frac{1}{\tau_{i+\frac{1}{2}}} = \frac{h_i}{x^2_{i+1/2}} \tag{2.8.7}$$

It will be sufficient to prove that, there exists a constant $M > 0$ such that

$$\frac{1}{x_{i+1/2}} \leq M, \tag{2.8.8}$$

to conclude the existence of $C_4 > 0$ such that

$$\frac{1}{\tau_{i+\frac{1}{2}}} \leq C_4 h. \tag{2.8.9}$$

Indeed we have

$$\frac{1}{x_{i+\frac{1}{2}}} = \frac{1}{X_{\max} \cdot \dfrac{x_{1+\frac{1}{2}}}{X_{\max}}} = \frac{1}{X_{\max}} \times \frac{1}{Z_1}, \text{ with } Z_i = \frac{x_{i+\frac{1}{2}}}{X_{\max}}. \tag{2.8.10}$$

We can rewrite as

$$\frac{1}{1-\xi} = \frac{1}{1-(1-Z_i)} = \frac{1}{Z_1}, \text{ where } \xi = 1 - Z_1. \tag{2.8.11}$$

As $0 < \xi < 1$, Taylor expansion allows to have

$$\frac{1}{1-\xi} = 1 + \xi + \mathcal{O}(\xi^2). \tag{2.8.12}$$

So, there exist a constant $M_1 > 0$ such that

$$\frac{1}{1-\xi} < 2 + M_1. \tag{2.8.13}$$

and therefore

$$\frac{1}{x_{i+1/2}} \leq M, \quad \text{with } M = \frac{2 + M_1}{X_{\max}}, \tag{2.8.14}$$

and the proof of the lemma is completed. □

Now we provide the proof of Theorem 2.8.

Proof. The proof we provide here follow in the same lines as [5, Theorem 7]. Indeed, we give a summary of the key steps here. Furthermore, it is important to mention that the standard mimetic and the fitted mimetic proofs remain the same. So we will only consider the fitted mimetic case where $\zeta_h^m = U_H^m$. Indeed it is easy to see that

$$\|u(t_m) - U_H^m\|_{0,wh} \quad \leq \quad \|u(t_m) - I_h u(t_m)\|_{0,h} + \|I_h u(t_m) - U_H^m\|_{0,h} \tag{2.8.15}$$

where $C > 0$ is a constant independent of both h and Δt, and I_h is the interpolation operator defined in (2.2.1).

The following results establishes the bound of $\|u(t_m) - I_h u(t_m)\|_{0,h}$. If $u(t) \in H^2(\Omega)$, then there exists a positive constant $C > 0$ independent of u, t and h such that the following estimate holds (please see [25, Theorem 3.25] for more details)

$$\|u(t_m) - I_h u(t))\|_{0,w} \leq Ch^2 |u(t)|_2, \tag{2.8.16}$$

where $|\cdot|_2$ denotes the semi-norm of $H^2(\Omega)$. Furthermore, if $u(t) \in C([0,T], H^2(\Omega))$, then there exists $C = C(u,T)$ such that

$$\|u(t_m) - I_h u(t_m))\|_{0,h} \leq Ch, \qquad \forall t \in [0,T]. \tag{2.8.17}$$

What remains therefore is to estimate $\mathbf{Q}^m := I_h u(t_m) - U_H^m$, in the discrete L^2-norm. In what follows, we have done the transformation $t = T - t$. Let

$$\mathbb{T} = \left(\frac{\mathbf{Q}^{m+1} - \mathbf{Q}^m}{\Delta t_m}, v_h \right)_h + \mathbf{A_h}(\mathbf{Q}^{m+1}, v_h; t_{m+1}), \tag{2.8.18}$$

be an expression, with $\mathbf{A_h}$ the bilinear form associated to the fitted mimetic method.

Then expanding (2.8.18) yields

$$\begin{aligned} \mathbb{T} \;&=\; \left(\frac{I_h u(\tau_{m+1}) - I_h u(t_m)}{\Delta t_m}, v_h \right)_h + \mathbf{A_h}(I_h u(\tau_{m+1}), v_h; t_{m+1}) - \left(\frac{U_H^{m+1} - U_H^m}{\Delta t_m}, v_h \right)_h \\ &\quad -\mathbf{A_h}(U_H^{m+1}, v_h; t_{m+1}) \end{aligned} \tag{2.8.19}$$

Now, using (2.3.9) and (2.3.17), and by simple algebraic manipulations, we have

$$\begin{aligned} \mathbb{T} \;&=\; -(\dot{u}(t_{m+1}), \mathcal{P}_h v_h) + (\dot{u}(t_{m+1}), \mathcal{P}_h v_h) + \left(\frac{I_h u(t_{m+1}) - I_h u(t_m)}{\Delta t_m}, v_h \right)_h \\ &\quad +\mathcal{A}(u(t_{m+1}), v_h; t_{m+1}) - \mathcal{A}(u(t_{m+1}), v_h; t_{m+1}) + \mathbf{A_h}(I_h u(t_{m+1}), v_h; t_{m+1}) \\ &\quad -\mathbf{A_h}(U_H^{m+1}, v_h; t_{m+1}) - \left(\frac{U_H^{m+1} - U_H^m}{\Delta t_m}, v_h \right)_h + \Big(\Psi(u(t_{m+1}), t_{m+1}), \mathcal{P}_h v_h \Big) \\ &\quad -\Big(\Psi(u(t_{m+1}), t_{m+1}), \mathcal{P}_h v_h \Big) + \Big(\Psi(U_H^{m+1}, t_{m+1}), v_h \Big)_h - \Big(\Psi(U_H^{m+1}, t_{m+1}), v_h \Big)_h \end{aligned}$$

$$\begin{aligned}
&= \left[\left(\frac{I_hu(t_{m+1}) - I_hu(t_m)}{\Delta t_m}, v_h\right)_h + (\dot{u}(t_{m+1}), \mathcal{P}_h v_h)\right] \\
&\quad + \Big[\mathbf{A_h}(I_hu(t_{m+1}), v_h; t_{m+1}) - \mathcal{A}(u(t_{m+1}), v_h; t_{m+1})\Big] \\
&\quad + \Big[\Big(\Psi(U_H^{m+1}, t_{m+1}), v_h\Big)_h - \Big(\Psi(u(t_{m+1}), t_{m+1}), \mathcal{P}_h v_h\Big)\Big] + \Big[\theta(f^{m+1}, \mathcal{P}_h v_h) - (\theta f^{m+1}, v_h)_h\Big] \\
&= Y_1^m + Y_2^m + Y_3^m + Y_4^m. \qquad (2.8.20)
\end{aligned}$$

The estimation of Y_1^m is done exactly as in [5, (54)] and hence we have:

$$Y_1^m = (w^m, \mathcal{P}_h v_h), \qquad (2.8.21)$$

with

$$w^m := \frac{1}{\Delta t}\left(\mathcal{P}_h u(t_{m+1}) - \mathcal{P}_h u(t_m)\right) + \dot{u}(t_{m+1}). \qquad (2.8.22)$$

Then

$$|Y_1^m| \leq \|w^m\|_0 \|v_h\|_{0,h}. \qquad (2.8.23)$$

Hence,

$$\|w\|_0 \leq \frac{1}{\Delta t}\int_{t_m}^{t_{m+1}} \|(\mathcal{P}_h - I)\dot{u}(s)\|_0 ds + \int_{t_m}^{t_{m+1}} \|\ddot{u}(s)\|_0 ds. \qquad (2.8.24)$$

Let us estimate $|Y_2^m|$: which is given by

$$Y_2^m = \mathbf{A_h}(u(t_{m+1}), v_h; t_{m+1}) - \mathcal{A}(u(t_{m+1}), v_h; t_{m+1}). \qquad (2.8.25)$$

This leads us to consider statements of the form

$$\delta_2(w, v_h; s) := \mathbf{A_h}(w, v_h; s) - \mathcal{A}(w, v_h; s).$$

Then we have by definitions (2.3.8) and (2.3.15) that

$$\begin{aligned}
\delta_2(w, v_h; s) &= -\sum_{i=1}^{N} \left[\Phi_h(I_hw, x, s) - \Phi(w, x, s)\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{P}_h v_h(x_i) + \left[(cw, v_h)_h - (cw, \mathcal{P}_h v_h)\right] \\
&= \sum_{i=0}^{N} \left[\Phi_h(I_hw, x, s) - \Phi(w, x, s)\right](v_{i+1} - v_i) + \left[c((\mathcal{P}_h - I)w, \mathcal{P}_h v_h)\right],
\end{aligned}$$

then

$$\begin{aligned}
|\delta_2(w, v_h; s)| &= \left|\sum_{i=0}^{N} \left(\Phi_h(I_hw, x, s) - \Phi(w, x, s)\right)\Big|_{x_{i+\frac{1}{2}}} (v_i - v_{i+1})\right| + |c((\mathcal{P}_h - I)w, \mathcal{P}_h v_h)| \\
&:= \delta_{21} + \delta_{22}.
\end{aligned}$$

Now let us estimate δ_{21};

$$\begin{aligned}\delta_{21} &= \left|\sum_{i=0}^{N} \left(\Phi_h(I_h w, x, s) - \Phi(w, x, s)\right)\Big|_{x_{i+\frac{1}{2}}} (v_i - v_{i+1})\right| \\ &\leq \left|\left(\Phi_h(I_h w, x_{\frac{1}{2}}, s) - \Phi(w, x_{\frac{1}{2}}, s)\right)(v_1 - v_0)\right| \\ &\quad + \left|\sum_{i=1}^{N} \left(\Phi_h(I_h w, x_{i+\frac{1}{2}}, s) - \Phi(w, x_{i+\frac{1}{2}}, s)\right)(v_{i+1} - v_i)\right|. \qquad (2.8.26)\end{aligned}$$

Before we consider the terms above, let us see the following expansion of $|v_1|$

$$|v_1| = \left(\frac{\sqrt{h_0}}{h_0}\right)\sqrt{h_0}|v_1| \qquad (2.8.27)$$

Then using (2.8.14), for $i = 0$ in Lemma 2.8.2, we have

$$|v_1| = \left(\frac{\sqrt{h_0}}{h_0}\right)\sqrt{h_0}|v_1| \leq M\sqrt{h_0}\left(\sqrt{h_0}|v_1|\right). \qquad (2.8.28)$$

Now from the first term (2.8.26), and using the Cauchy Schwartz inequality and (2.8.28), we have

$$\begin{aligned}\left|\left(\Phi_h(I_h w, x_{\frac{1}{2}}, s) - \Phi(w, x_{\frac{1}{2}}, s)\right)(v_1 - v_0)\right| &\leq \left|\Phi_h(I_h w, x_{\frac{1}{2}}, s) - \Phi(w, x_{\frac{1}{2}}, s)\right| |v_1| \\ &\leq C_1 \left\{\int_{x_0}^{x_1} 1^2 dx\right\}^{1/2} \left\{\int_{x_0}^{x_1} \left[|\Phi'| + |w'| + |w|\right]^2 dx\right\}^{1/2} |v_1|, \\ &\leq C_1\sqrt{h_0} \left\{\int_{x_0}^{x_1} \left[|\Phi'| + |w'| + |w|\right]^2 dx\right\}^{1/2} \left(\sqrt{h_0}\left(h_0 v_1^2\right)^{1/2}\right),\end{aligned}$$

where C_1 is an arbitrary positive constant independent of h. Then

$$\left|\left(\Phi_h(I_h w, x_{\frac{1}{2}}, s) - \Phi(w, x_{\frac{1}{2}}, s)\right)(v_1 - v_0)\right| \leq C_1 h_0 \left\{\int_{x_0}^{x_1} \left[|\Phi'| + |w'| + |w|\right]^{1/2} dx\right\} (h_0 v_1^2)^{1/2}. \qquad (2.8.29)$$

Now before we consider the second term, let us expand on the term $(\sum_{i=1}^{N} |v_{i+1} - v_i|)$, by using simple algebraic manipulations

$$\sum_{i=1}^{N} |v_{i+1} - v_i| = \sum_{i=1}^{N} \frac{\sqrt{h_i}}{x_{i+\frac{1}{2}}} \frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}} |v_{i+1} - v_i|. \qquad (2.8.30)$$

Then from Lemma 2.8.2 we have

$$\sum_{i=1}^{N}|v_{i+1}-v_i| = \sum_{i=1}^{N}\frac{\sqrt{h_i}}{x_{i+\frac{1}{2}}}\,\frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}}|v_{i+1}-v_i| \le M\sqrt{h}\sum_{i=1}^{N}\left(\frac{x_{i+\frac{1}{2}}}{\sqrt{h_i}}|v_{i+1}-v_i|\right). \tag{2.8.31}$$

Then from the second term of (2.8.26), using (2.8.31), and for some $C_2 > 0$ independent of h, we have

$$\begin{aligned}
&\left|\sum_{i=1}^{N}\left(\Phi_h(I_hw, x_{i+\frac{1}{2}}, s) - \Phi(w, x_{i+\frac{1}{2}}, s)\right)(v_{i+1}-v_i)\right| \\
&\qquad \le \left|\sum_{i=1}^{N}\left(\Phi_h(I_hw, x_{i+\frac{1}{2}}, s) - \Phi(w, x_{i+\frac{1}{2}}, s)\right)\right|\left|\sum_{i=1}^{N}(v_{i+1}-v_i)\right| \\
&\le C_2\sqrt{h}\left(\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left[|\Phi'|+|w'|+|w|\right]dx\right)\left(\sum_{i=1}^{N}\frac{x^2_{i+\frac{1}{2}}}{h_i}|v_{i+1}-v_i|\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\left|\sum_{i=1}^{N}\left(\Phi_h(I_hw, x_{i+\frac{1}{2}}, s) - \Phi(w, x_{i+\frac{1}{2}}, s)\right)(v_{i+1}-v_i)\right| \\
&\qquad \le C_2\sqrt{h}\left(\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left[|\Phi'|+|w'|+|w|\right]dx\right)\|v_h\|_{1,\mathcal{T}} \\
&\le C_2\sqrt{h}\left\{\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}1^2dx\right\}^{1/2}\left\{\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left[|\Phi'|+|w'|+|w|\right]^2dx\right\}^{1/2}\|v_h\|_{1,\mathcal{T}} \\
&\quad \le C_2h\left\{\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left[|\Phi'|+|w'|+|w|\right]^2dx\right\}^{1/2}\|v_h\|_{1,\mathcal{T}}.
\end{aligned}$$

Then we have that,

$$\begin{aligned}
&\left|\sum_{i=1}^{N}\left(\Phi_h(I_hw, x_{i+\frac{1}{2}}, s) - \Phi(w, x_{i+\frac{1}{2}}, s)\right)(v_{i+1}-v_i)\right| \\
&\le C_2h\left\{\int_{x_1}^{x_{N+1}}\left[|\Phi'|+|w'|+|w|\right]^2dx\right\}^{1/2}\|v_h\|_{1,\mathcal{T}}
\end{aligned} \tag{2.8.32}$$

Hence from (2.8.29), (2.8.32), using the fact that $h_0 \le cl_1$ (from Assumption (2.3.2)), noting the fact $C_2 > 0$ may change from line to line, that we have that

$$\delta_{21} \leq C_1 h_0 \left\{ \int_{x_0}^{x_1} \left[|\Phi'| + |w'| + |w| \right]^{1/2} dx \right\} (h_0 v_1^2)^{1/2}$$
$$+C_2 h \left\{ \int_{x_1}^{x_{N+1}} \left[|\Phi'| + |w'| + |w| \right]^2 dx \right\}^{1/2} \|v_h\|_{1,\mathcal{T}}$$
$$\leq C_2 h \left[\left\{ \int_{x_0}^{x_{N+1}} \left[|\Phi'| + |w'| + |w| \right]^2 dx \right\}^{1/2} \right] \left\{ l_1 v_1^2 + \|v_h\|_{1,\mathcal{T}}^2 \right\}^{1/2}$$
$$\leq C_2 h \left[\left\{ \int_{x_0}^{x_{N+1}} \left[|\Phi'| + |w'| + |w| \right]^2 dx \right\}^{1/2} \right] \|v_h\|_{1,h}$$
$$\leq C_2 h \left[|\Phi'|_1 + \|w\|_1 \right] \|v_h\|_{1,h}.$$

It must be noted that, $\|\cdot\|_1$ and $|\cdot|_1$, are the $H^1(\Omega)$ norm and semi norm, respectively. Also, δ_{22} is exactly as estimated in [5]. That is, for $C_3 > 0$, we have

$$\begin{aligned} \delta_{22} &:= |c((\mathcal{P}_h - I)w, \mathcal{P}_h v_h)| && (2.8.33) \\ &\leq C_3 h \|w\|_1 \|v_h\|_{0,h}\,. && (2.8.34) \end{aligned}$$

Then combining δ_{21} and δ_{22}, we have that

$$|\delta_{21}| \leq Ch \left[\|\Phi\|_1 + \|w\|_1 \right] \|v_h\|_{1,h},$$

where $C > 0$ and independent of h.

Hence,

$$\begin{aligned} |Y_2^m| &\leq |\delta_{21}(u(t_{m+1}), v_h; t_{m+1})| \\ &\leq Ch \left[\|\Phi(u(t_{m+1}); \cdot; t_{m+1})\|_1 \right] \|v_h\|_{1,h} \\ &\quad + C_h \left[\|u(t_{m+1})\|_1 \right] \|v_h\|_{1,h} \\ &\leq Ch \|v_h\|_{1,h}. \end{aligned}$$

Let us estimate Y_3^m. Adding and subtracting the term $\Big(\Psi(I_h u(t_{m+1}, t_{m+1}), v_h\Big)_h$, we have

$$\begin{aligned} Y_3^m &= \Big(\Psi(U_H^{m+1}, t_{m+1}), v_h\Big)_h - \Big(\Psi(u(t_{m+1}, t_{m+1}), \mathcal{P}_h v_h\Big) \\ &= Y_{31}^m(v_h) + Y_{32}^m, && (2.8.35) \end{aligned}$$

with

$$\begin{aligned} Y_{31}^m &:= \Big(\Psi(U_H^{m+1}, t_{m+1}), v_h\Big)_h - \Big(\Psi(I_h u(t_{m+1}, t_{m+1}), v_h\Big)_h, \\ Y_{32}^m &:= \Big(\Psi(I_h u(t_{m+1}, t_{m+1}), v_h\Big)_h - \Big(\Psi(u(t_{m+1}, t_{m+1}), \mathcal{P}_h v_h\Big). && (2.8.36) \end{aligned}$$

We will estimate the term Y_{31}^m later on in the proof, let us consider the estimate of the term Y_{32}^m. Furthermore in this proof, we assume the linear penalty function $\Psi(v,\cdot) = \lambda\big[v - u^*\big]_+$. Now, since $\mathcal{P}_h I_h = \mathcal{P}_h$ on $C(\overline{\Omega})$, we have

$$\begin{aligned} Y_{32}^m &= \Big(\Psi(I_h u(t_{m+1}, t_{m+1}), v_h\Big)_h - \Big(\Psi(u(t_{m+1}, t_{m+1}), \mathcal{P}_h v_h\Big) \\ &= \Big((\mathcal{P}_h - I)\Psi(u(t_{m+1}, t_{m+1}), \mathcal{P}_h v_h\Big). \end{aligned} \tag{2.8.37}$$

Then it follows that

$$|Y_{32}^m| \leq Z_1 \|v_h\|_{0,h}, \tag{2.8.38}$$

where

$$Z_1 = \|(\mathcal{P}_h - I)\Psi(u(t_{m+1}, t_{m+1})\|_0. \tag{2.8.39}$$

Now, the function $\xi \mapsto \big[\xi\big]_+$ is Lipschitz-continuous with a Lipschitz-constant equal to one and everywhere differentiable except at the origin. Therefore we have $\dfrac{\partial}{\partial x}\Psi(w,\cdot) \in H^1(\Omega)$ whenever $(w - u^*) \in H^1(\Omega)$, and $|\Psi(w,\cdot)|_1 \leq \lambda |w - u^*|_1$. It then follows that

$$Z_1 \leq \lambda h |u(t_{m+1}) - u^*(t_{m+1})|_1. \tag{2.8.40}$$

Therefore, substituting the Z_1 into (2.8.38), we have that there exists a positive constant C, independent of h and m, such that

$$|Y_{32}^m| \leq C\lambda h \|v_h\|_{0,h}. \tag{2.8.41}$$

Now, the estimate of Y_4^m also follows from [5] and hence we have that

$$\begin{aligned} |Y_4^m| &\leq \left(\left\|(I - \mathcal{P}_h)\left(f^{m+1} - f^m\right)\right\|_0 + Ch|f(t_m)|_1\right) \|v_h\|_{0,h} \\ &\leq C_4 h \|v_h\|_{0,h} \end{aligned}$$

Then combining the estimates of $|Y_i^m|,\ i = 1, 2, 3, 4$, we have that

$$\left(\frac{\mathbf{Q}^{m+1} - \mathbf{Q}^{m+1}}{\Delta t_m}, v_h\right)_h + \mathbf{A_h}\left(\mathbf{Q}^{m+1}, v_h; t_{m+1}\right) \leq Y_{31}^m + D^m(\Delta t_m, h)\|v_h\|_{1,h} \tag{2.8.42}$$

where

$D^m(\Delta t_m, h) := D_h^m(\Delta t_m, h) + Ch$

Let us consider, a test function $v_h := \mathbf{Q}^{m+1}$, then from LHS of (2.8.42) we have that

$$\left(\frac{\mathbf{Q}^{m+1}-\mathbf{Q}^m}{\Delta t_m}, v_h\right)_h + \mathbf{A_h}\left(v_h, v_h; t_{m+1}\right) \geq \frac{1}{2\Delta t_m}\left[\|\mathbf{Q}^{m+1}\|^2_{0,h} - \|\mathbf{Q}^m\|^2_{0,h}\right] + C\|\mathbf{Q}^{m+1}\|^2_{1,h}.$$

Then we have from (2.8.42) that

$$\frac{1}{2\Delta t_m}\left[\|\mathbf{Q}^{m+1}\|^2_{0,h} - \|\mathbf{Q}^m\|^2_{0,h}\right] + C\|\mathbf{Q}^{m+1}\|^2_{1,h} \;\leq\; Y^m_{31}(v_h) + D^m(\Delta t_m, h)\|\mathbf{Q}^{m+1}\|_{1,h}. \quad (2.8.43)$$

Let us consider the function Ψ in more detail as we estimate the term $Y^m_{31}(v_h)$. For $(x,t) \in \Omega \times (0,T)$, and $v, w \in C(\overline{\Omega})$, we can write

$$\begin{aligned} v - w &= v - u^* - (w - u^*) \\ &= [v-u^*]_+ - [w-u^*]_+ - \big([v-u^*]_- - [w-u^*]_-\big), \end{aligned} \quad (2.8.44)$$

where $[\xi]_- = \max\{0, -\xi\}$. Then following exactly the lines in [5], we have

$$\begin{aligned} &\Big(\Psi(v,\cdot) - \Psi(w,\cdot)\Big)(v-w) = \lambda\Big([v-u^*]_+ - [w-u^*]_+\Big)^2 \\ &\quad -\Big([v-u^*]_+ - [w-u^*]_+\Big)\Big([v-u^*]_- - [w-u^*]_-\Big) \\ &= \lambda\Big([v-u^*]_+ - [w-u^*]_+\Big)^2 - \underbrace{[v-u^*]_+[v-u^*]_-}_{=0} + \underbrace{[v-u^*]_+[w-u^*]_-}_{\geq 0} \\ &\quad + \underbrace{[w-u^*]_+[v-u^*]_-}_{\geq 0} - \underbrace{[w-u^*]_+[w-u^*]_-}_{=0} \\ &\geq \lambda\Big([v-u^*]_+ - [w-u^*]_+\Big)^2. \end{aligned} \quad (2.8.45)$$

Then it implies that

$$\begin{aligned} Y^m_{31}(v_h) &= \big(\Psi(U_H^{m+1}, t_{m+1}) - \Psi(I_h u(t_{m+1}), t_{m+1}), v_h\big) \\ &\leq -\lambda\left\|\big[U_H^{m+1} - u^*(t_{m+1})\big]_+ - \big[I_h u(t_{m+1}) - u^*(t_{m+1})\big]_+\right\|^2_{0,h}. \end{aligned} \quad (2.8.46)$$

Then following exactly the lines of [5], we have

$$\begin{aligned} &\|\mathbf{Q}^M\|^2_{0,h} + 2\lambda\sum_{m=0}^{M-1}(\Delta t_m)\left\|\big[U_H^{m+1} - u^*(t_{m+1})\big]_+ - \big[I_h u(t_{m+1}) - u^*(t_{m+1})\big]_+\right\|^2_{0,h} \\ &\qquad \leq \|\mathbf{Q}^0\|^2_{0,h} + \frac{1}{2C}\sum_{m=0}^{M-1}(\Delta t_m)[D^m(\Delta t_m, h)]^2. \end{aligned} \quad (2.8.47)$$

We then consider the estimate of the term

$$[D^m(\Delta t_m,h)]^2 \quad \leq \quad 2\left\{[D_h^m(\Delta t_m,h)]^2+C^2h^2\right\},$$

with

$$\sum_{m=0}^{M-1}(\Delta t_m)[D_h^m(\Delta t_m,h)]^2 \leq 2\left[h^2\|\dot{u}(s)\|^2_{L^2(0,T;H^1(\Omega))}+(\Delta t)^2\|\ddot{u}(s)\|^2_{L^2(0,T;L^2(\Omega))}\right]. \tag{2.8.48}$$

Then from (2.8.47)-(2.8.48), we have

$$\begin{aligned}
&\|\mathbf{Q}^M\|^2_{0,h}+2\lambda\sum_{m=0}^{M-1}(\Delta t_m)\left\|\left[U_H^{m+1}-u^*(t_{m+1})\right]_+-\left[I_hu(t_{m+1})-u^*(t_{m+1})\right]_+\right\|^2_{0,h}\\
&\quad\leq \|\mathbf{Q}^0\|^2_{0,h}+\left\{2\left[h^2\|\dot{u}(s)\|^2_{L^2(0,T;H^1(\Omega))}+(\Delta t)^2\|\ddot{u}(s)\|^2_{L^2(0,T;L^2(\Omega))}\right]+C^2h^2\right\}\\
&\quad\quad\leq \|\mathbf{Q}^0\|^2_{0,h}+C\left(h^2+(\Delta t)^2\right).
\end{aligned}$$

Finally, we consider $U_H^0=I_hu(t_0)$, we have $\|\mathbf{Q}^0\|^2_{0,h}$, which yields

$$\|I_hu(t_m)-U_H^m\|_{0,h}\leq C\left(h+(\Delta t)\right). \tag{2.8.49}$$

Therefore from (2.8.17) and (2.8.49), we have

$$\|u(t_m)-U_H^m\|_{0,h}\leq C\left(h+(\Delta t)\right). \tag{2.8.50}$$

□

2.9 Numerical Tests

2.9.1 Test 1

We run all numerical simulations on a 8 GB 1600 MHz DDR3, MacBook Pro. We simulate our results using Matlab R2017b-(student use). For the simulations, we consider the European put option parameters, $E=100$, $r=0.1$, $\sigma=0.3$, $T=1$, $S=3K$. We choose the space to be $(0,1000)$ and time intervals to be $(0,1)$. They are subdivided into $N=10000$, $M=1000$ subintervals respectively. The Figures below illustrate our results for the various spatial techniques considered coupled with the implicit

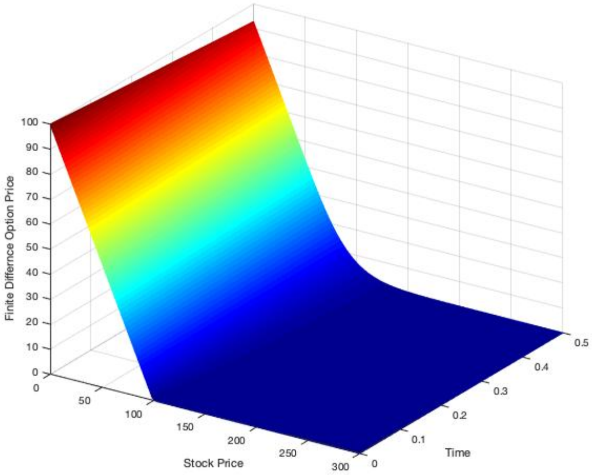


Figure 2.3: The numerical solution for Finite difference with implicit method and associated parameters, $E = 100$, $r = 0.1$, $\sigma = 0.3$, $T = 0.5$, $S = 3E$.

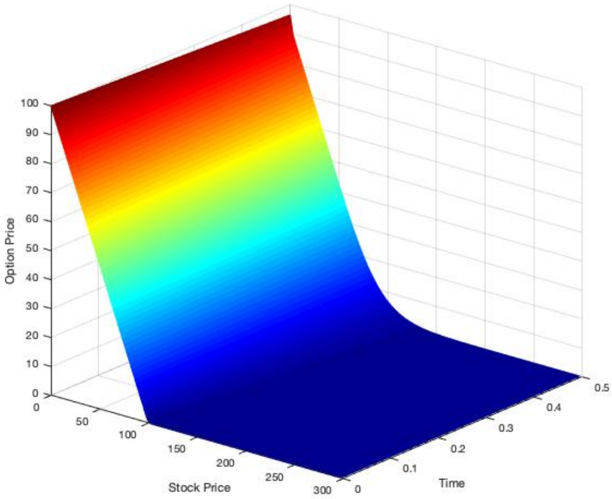


Figure 2.4: The numerical solution for Fitted finite volume with implicit method and associated parameters, $E = 100$, $r = 0.1$, $\sigma = 0.3$, $T = 0.5$, $S = 3E$.

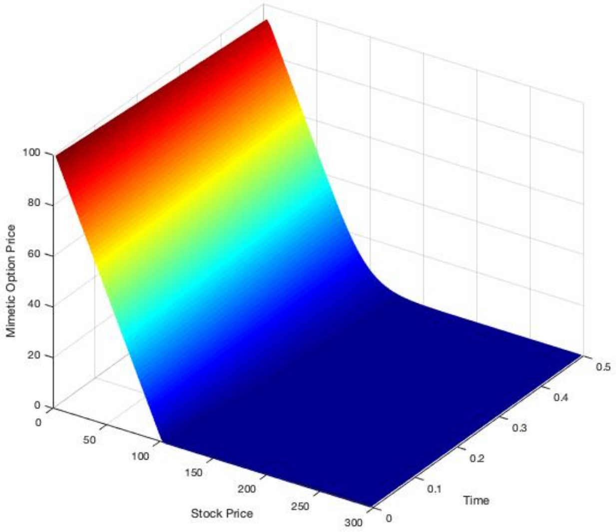


Figure 2.5: The numerical solution for Mimetic method with implicit method and associated parameters, $E = 100$, $r = 0.1$, $\sigma = 0.3$, $T = 0.5$, $S = 3E$.

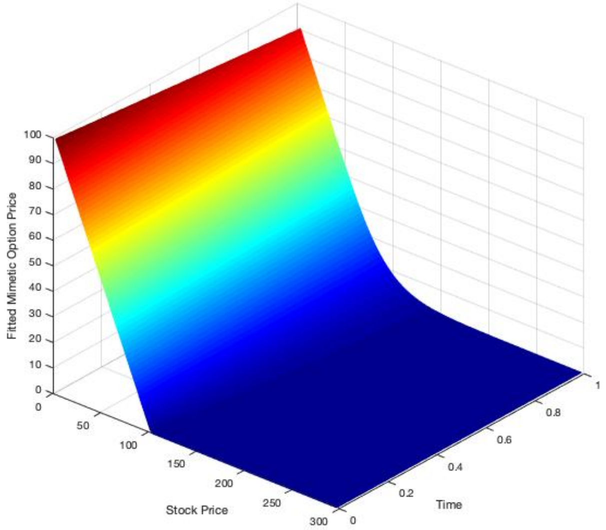


Figure 2.6: The numerical solution for Fitted mimetic method with implicit method and associated parameters, $E = 100$, $r = 0.1$, $\sigma = 0.3$, $T = 0.5$, $S = 3E$.

To confirm the theoretical results obtained in (2.8.49), we present the following graphs. Note here that we run the time graph with the following parameters: $E = 1$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, $S = 3E$, $N = 1000$ and $M = 4000$. The space graph with the following parameters: $E = 1$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, $S = 5E$, $N = 10000$ and $M = 1000$.

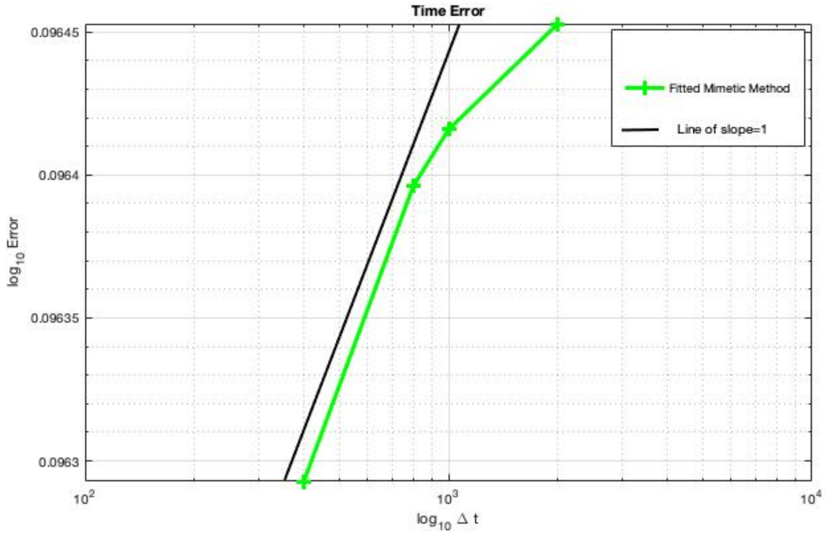


Figure 2.7: Time error vs the time step. The graph shows that the practical error of convergence is 1, which is in agreement with our theoretical result

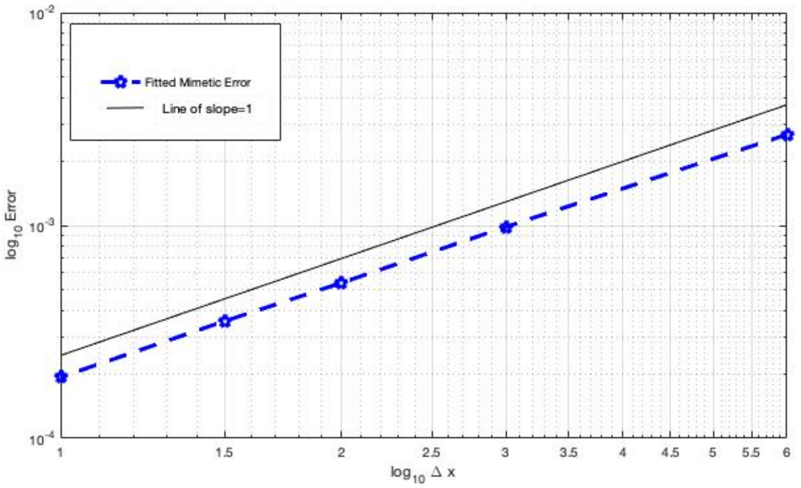


Figure 2.8: Fitted mimetic space error vs the space step. The graph shows that the error of convergence is 1, which is in agreement with our theoretical result.

2.9.2 Test 2

The table below shows the L^2 relative error for the mimetic finite difference method compared with the Fitted finite volume method [43] for the European put option. Let us recall from (1.5.3) the analytical solution for the European call option as given in [16]. Here we provide the analytical solution for the put option problem

$$P(s,t) = Ee^{-r(T-t)}N(d_2) - sN(d_1) \tag{2.9.1}$$

where

$$N(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e\,\frac{y^2}{2}\,dy$$

$$d_1 = \frac{\ln(s/E) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{2.9.2}$$

$$d_2 = \frac{\ln(s/E) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{2.9.3}$$

Let U represent all numerical solutions, $U^{analytic}$ represent the analytical solution and $mes(\Omega_i)$ is the measure of the interval Ω_i. Then we define the L^2-norm adopted for the relative error computation by

$$Error = \frac{\sqrt{\sum_{i=1}^{N} h_i(U_i - U_i^{analytic})^2}}{\sqrt{\sum_{i=1}^{N} h_i(U_i^{analytic})^2}}. \tag{2.9.4}$$

Therefore the relative error for the European put option is given by Table 2.1 below:

N	Finite Difference	Fitted Finite Volume	Mimetic FDM	Fitted Mimetic FDM
100	0.0130	0.0077	0.0063	9.8522e-04
200	0.0093	0.0068	0.0032	3.5554e-04
500	0.0059	0.0052	0.0013	9.1060e-05
1000	0.0042	0.0042	6.3691e-04	3.2327e-05
2000	0.0030	0.0029	3.1876e-04	1.1453e-05
3000	0.0024	0.0024	2.1257e-04	6.2355e-06
5000	0.0019	0.0019	1.2758e-04	2.9016e-06
10000	0.0013	0.0013	6.3800e-05	1.0270e-06

[1] $T = 1, r = 0.1, E = 100, M = 1000, \sigma = 0.3, X_{\max} = 3E.$

Table 2.1: This table showing L^2- relative error for the various spatial discretization methods

From the Table 2.1 the mimetic methods outperforms the standard finite difference and the fitted finite volume methods. In particular the fitted mimetic finite difference scheme presented more accurate results than the standard mimetic scheme. This shows the importance of the fitted scheme to handling the degeneracy of the Black-Scholes differential operator.

Furthermore, the CPU timings (in seconds) for all the methods are given in the Table 2.2 below:

N	Finite Difference CPU	Fitted FV CPU	Mimetic FDM CPU	Fitted MFDM CPU
100	0.039	0.036	0.037	0.032
200	0.049	0.046	0.049	0.043
500	0.102	0.064	0.102	0.061
1000	0.163	0.096	0.161	0.103
2000	0.312	0.257	0.310	0.185
3000	0.531	0.371	0.527	0.407
5000	1.078	1.064	1.072	0.875
10000	3.996	3.019	3.264	2.791

[1] $T = 1, r = 0.1, E = 100, M = 1000, \sigma = 0.3, X_{\max} = 3E.$

Table 2.2: This table shows the CPU time for the various spatial discretization methods

The CPU time of all the methods are very close as can be observed in Table 2.2. This is not surprising because the during the construction of the methods, we observed that the matrix representations have the same structure.

2.9.3 Conclusion

In this chapter, we have considered the so-called mimetic finite difference method applied to the degenerate Black-Scholes equation which governs option pricing. We have presented the fitted technique to handle the degeneracy near the boundary at zero. The unique solvability of the resulting semi-discrete problem was considered as the coercivity of the bilinear form in appropriate norms was presented. We have proved the consistency of the fluxes. We have furthermore established an error bound of order $\mathcal{O}(h + \Delta t)$ for the full discrete solution. We have presented numerical results to confirm the theoretical results.

From our numerical results, the novel fitted mimetic finite difference method was more accurate than the standard mimetic, standard finite difference and the fitted finite volume methods. We consider the application of the standard mimetic and fitted mimetic methods as a contribution in this chapter as neither has been previously applied to problems of this nature in finance (to the best of our knowledge). In the next chapter, we consider the two dimensional option pricing problem.

3. Mimetic and fitted-mimetic finite difference methods for pricing a two dimensional penalised American option problem

This chapter further extends our contribution in this thesis to the two-dimensional option problem. In chapter 2, we showed that the mimetic methods outperformed the fitted finite volume method and the standard finite difference method in one spatial dimension. Here, we consider the two dimensional put option problem. We construct the two-dimensional mimetic grid to follow the finite volume grid. We recall here that in chapter 2, the construction of MFDM followed from the method's design principles studied by earlier authors [15, 28, 29, 34, 41, 45] and the appropriate references therein. From structuring the underlying PDEs into conservative form (i.e. with invariant operators such as grads, divs and curls) to constructing discrete analogs of continuous operators and inner products as a result of the support operator methods. Indeed, the tools associated to the mimetic method are then discrete spaces equipped with discrete inner products, derived discrete operators built from duality relationships involving discrete operators, and finally the duality principle that connects important properties for the resulting discrete model [34, 41, 45]. In [40], the structural design framework for developing mimetic techniques are well summarized.

This chapter is organised as follows, in section 3.1 we consider an extension of the support operator method (SOM) for the two-dimensional case. We further present MFDM and FMFD in section 3.1. We also adopt the upwind scheme to discretize the convection term. Then in section 3.1, we consider a full discretization of the scheme with the standard implicit method. We perform numerical experiments in 3.1 to support the theoretical results.

3.1 Semi-discrete problem and mimetic method for elliptic problems

As seen in chapter 2, where we considered a general one-dimensional elliptic problem given by (2.1.1), in two dimensions therefore, we define an operator $\mathbf{A} : H \to H$ such that

$$\mathbf{A}u = -\nabla \cdot (\mathbf{K}\nabla u(x, y)) = f, \quad (x, y) \in \Omega \subset \mathbb{R}^2 \tag{3.1.1}$$

with boundary condition

$$u(x, y) = 0, \quad (x, y) \in \partial\Omega,$$

and properties

$$(\mathbf{A}u, v)_H = (u, \mathbf{A}v)_H, \quad (\mathbf{A}u, u)_H > 0, \quad \mathbf{F} = f. \tag{3.1.2}$$

Clearly (3.1.1) becomes

$$\mathbf{A}u = \mathbf{F}. \tag{3.1.3}$$

which can be rewritten as the following first-order system

$$\begin{cases} \nabla \cdot \mathbf{w} = \mathbf{F} \\ \mathbf{w} = -\mathbf{K}\nabla u. \end{cases} \tag{3.1.4}$$

Note that all the properties defined in section 2.1 of chapter 2 are also satisfied here similarly. Furthermore, note that the diffusion matrix tensor $\mathbf{K}$ is diagonal. Therefore, we intend to mimic the properties (2.1.11) - (2.1.13) of the continuum problem in the discrete case.

The domain $\Omega = [0, X] \times [0, Y]$ is divided into $(N_x+1) \times (N_y+1)$ non-overlapping intervals $\mathcal{T} = (I_i \times I_j)_{0 \le i \le Nx+1, 0 \le i \le Ny+1}$, such that $I_i = (x_i, x_{i+1}),\ i = 0, 1, ..., N_x$, with $0 = x_0 < x_1 < \cdots < x_{N_x+1} = X$ and $I_j = (y_j, y_{j+1}),\ j = 0, 1, ..., N_y$, with $0 = y_0 < y_1 < \cdots < y_{N_y+1} = Y$. We set $h_{x_i} = x_{i+1} - x_i$, with $h_x = \max_{0 \le i \le (N_x+1)} h_{x_i}$ and $h_{y_j} = y_{j+1} - y_j$, with $h_y = \max_{0 \le j \le (N_y+1)} h_{y_j}$.

Now, we define the following mid-points $x_{i-\frac{1}{2}} = \dfrac{x_i + x_{i-1}}{2}$, $x_{i+\frac{1}{2}} = \dfrac{x_i + x_{i+1}}{2}$ for $i = 1, ..., N_x$, $y_{j-\frac{1}{2}} = \dfrac{y_j + y_{j-1}}{2}$ and $y_{j+\frac{1}{2}} = \dfrac{y_j + y_{j+1}}{2}$ for $j = 1, ..., N_y$. We also set $x_{-\frac{1}{2}} = x_0$, $x_{N_x+\frac{3}{2}} = x_{N_x+1}$, $y_{-\frac{1}{2}} = y_0$ and $y_{N_y+\frac{3}{2}} = y_{N_y+1}$. Now for $i = 0, 1, ..., N_x + 1$, we set $l_{x_i} = x_{i+1/2} - x_{i-1/2}$, $l_x = \max_{0 \le i \le (N_x+1)} l_{x_i}$. We also set $l_{y_j} = y_{j+1/2} - y_{j-1/2}$, $l_y = \max_{0 \le j \le (N_y+1)} l_{y_j}$ for $j = 0, 1, ..., N_y + 1$. Furthermore, we set $x_{-\frac{1}{2}} = x_0$ and $x_{N_x+\frac{3}{2}} = x_{N_x+1}$ [1]. Similarly, $y_{-\frac{1}{2}} = y_0$ and $y_{N_y+\frac{3}{2}} = y_{N_y+1}$ [2]. Note the family $(\Omega_i \times \Omega_j)_{0 \le i \le Nx+1, 0 \le i \le Ny+1}$ is another partition of Ω with $\Omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $\Omega_j = (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$. We will call this the dual partition of the family $(I_i \times I_j)_{0 \le i \le Nx+1, 0 \le i \le Ny+1}$.

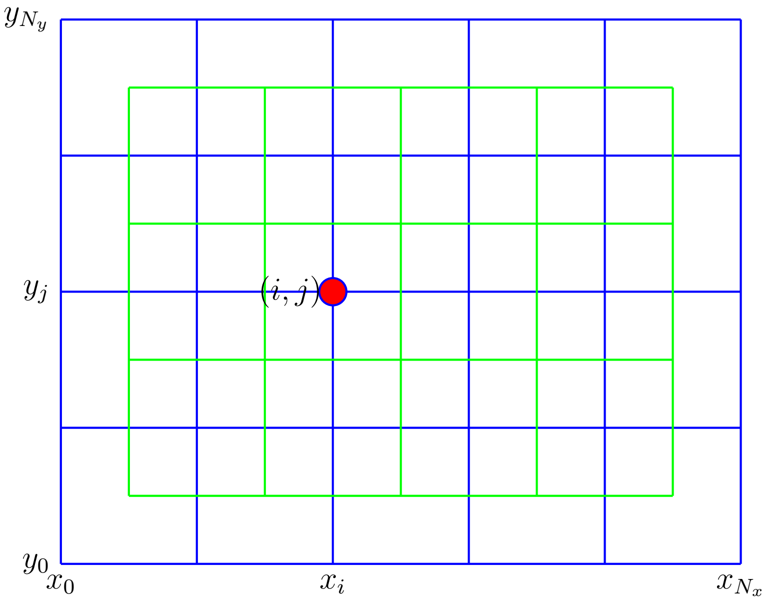


Figure 3.1: Two dimensional Mimetic finite difference grid

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[1] Function evaluation at $x_{-\frac{1}{2}}$ or $x_{N_x+\frac{3}{2}}$ is understood as evaluation at $x_0 = 0$ or at $x_{N_x+1} = X$

[2] Function evaluation at $y_{-\frac{1}{2}}$ or $y_{N_y+\frac{3}{2}}$ is understood as evaluation at $y_0 = 0$ or at $y_{N_y+1} = Y$

We define $V_h := span\left\{\phi_{x_i,y_j}\right\}_{0\leq i\leq Nx+1, 0\leq i\leq Ny+1} \cap H^1_{0,w}(\Omega)$ as the set of continuous piecewise with respect to the partition $(I_i \times I_j)_{0\leq i\leq Nx+1, 0\leq i\leq Ny+1}$ such that $v_h = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} v_h(x_i,y_j)\phi_{x_i,y_j}$. The interpolation operator I_h is defined by $I_h : C(\overline{\Omega}) \longrightarrow V_h$

$$\begin{aligned} I_h v_h(x_i,y_j) &:= v(x_i,y_j),\ i=0,1,...,N_x+1,\ j=0,1,...,N_y+1,\ \text{then} \\ I_h v = v_h &= \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} v_h(x_i,y_j)\phi_{x_i,y_j}. \end{aligned}$$

3.1.1 Discrete Inner products

As in chapter 2, we define discrete analogs of the two continuous inner products (2.1.9) and (2.1.10) by selecting a quadrature rule on each cell to approximate the integrals. Let HC and $\mathbf{HC}$ denote the discrete spaces of discrete scalar and vector functions for $H = L^2(\Omega)$ and $\mathbf{H} = (L^2(\Omega))^n, n = 2$ respectively, in the partition $\mathcal{T}$. Then the discrete L^2 norm defined in HC is given by

$$(U,V)_{HC} = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} l_{x_i} l_{y_j} U_{i,j} V_{i,j}. \tag{3.1.5}$$

Again, we choose the midpoint rule for the inner product (2.1.10) since the discrete information for fluxes are located at the cell centers. Let the Cartesian components of the tensor $\mathbf{K}$ be given by K_{xx} and K_{yy}. Then, $(\mathbf{K}^{-1})_{xx}$, and $(\mathbf{K}^{-1})_{yy}$ are the associated Cartesian components of the tensor $\mathbf{K}^{-1}$, the inverse of $\mathbf{K}$. For any vectors, $\mathbf{W}, \mathbf{Z}$, we represent their components as $\mathbf{W} = (W_x, W_y)$ and $\mathbf{Z} = (Z_x, Z_y)$.

$$\left(\mathbf{K}^{-1}\mathbf{W}, \mathbf{Z}\right) = (\mathbf{K}^{-1})_{xx} W_x Z_x + (\mathbf{K}^{-1})_{yy} W_y Z_y. \tag{3.1.6}$$

For Simplicity, we choose the following,

$$W_{x_{i+\frac{1}{2},j}} = W_{i+\frac{1}{2},j},\ W_{y_{i,j+\frac{1}{2}}} = W_{i,j+\frac{1}{2}},\ Z_{x_{i+\frac{1}{2},j}} = Z_{i+\frac{1}{2},j},\ Z_{y_{i,j+\frac{1}{2}}} = Z_{i,j+\frac{1}{2}}.$$

Hence we have that, the discrete $(L^2(\Omega))^n, n = 2$ norm defined in $\mathbf{HC}$ is given by

$$(\mathbf{W},\mathbf{Z})_{\mathbf{HC}} = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} h_{x_i} h_{y_j} \left[(\mathbf{K}_{xx}^{-1})_{i+\frac{1}{2},j} W_{i+\frac{1}{2},j} Z_{i+\frac{1}{2},j} + (\mathbf{K}_{yy}^{-1})_{i,j+\frac{1}{2}} W_{i,j+\frac{1}{2}} Z_{i,j+\frac{1}{2}}\right] \tag{3.1.7}$$

3.1.2 The Discrete Divergence and Discrete flux

In this section, we present the discrete version of the divergence operator $\mathbf{D}$. The resulting discrete divergence is denoted by $\mathcal{D}$ is given by

$$(\mathcal{D}W)_{i,j} = \left(\frac{W_{i+\frac{1}{2},j} - W_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{W_{i,j+\frac{1}{2}} - W_{i,j-\frac{1}{2}}}{l_{y_j}} \right) \tag{3.1.8}$$

$i = 0, 1, ..., N_x + 1$ and $j = 0, 1, ..., N_y + 1$.

As we mentioned earlier in chapter 2, for W vanishing at the boundary of the domain, we have that

$$(\mathcal{D}W, 1)_{HC} = 0, \tag{3.1.9}$$

which is the divergence property of the discrete divergence $\mathcal{D}$ which mimics the continuous divergence.

It is important to note here that for $n = 2$, $\mathbf{G} = -\mathbf{K}\nabla$, and we choose the flux tensor $\mathbf{K} = \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix}$, a 2×2 diagonal matrix tensor.[3]

In this case therefore, $\mathbf{K}^{-1} = \begin{pmatrix} \dfrac{1}{k_{11}} & 0 \\ 0 & \dfrac{1}{k_{22}} \end{pmatrix}$.

Now, we determine the discrete version of $\mathbf{G}$ denoted by $\mathcal{G}$ that mimics the continuous version properties we have already mentioned. Indeed, $\mathcal{G}$ must satisfy the following property [41, 45],

$$(\mathcal{D}W, U)_{HC} = (W, \mathcal{G}U)_{\mathbf{HC}}. \tag{3.1.10}$$

We then expand (3.1.10) as below,

$$\begin{aligned} \sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y+1} U_{i,j} (\mathcal{D}W)_{i,j} l_{x_i} l_{y_j} = & \sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y+1} \left(K^{-1}_{xx_{i+\frac{1}{2},j}} W_{i,j+\frac{1}{2}} W_{i+\frac{1}{2},j} (\mathcal{G}U)_{i+\frac{1}{2},j} \right) \\ & + \sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y+1} \left(K^{-1}_{yy_{i,j+\frac{1}{2}}} W_{i,j+\frac{1}{2}} (\mathcal{G}U)_{i,j+\frac{1}{2}} \right) h_{x_i} h_{y_j}. \end{aligned} \tag{3.1.11}$$

Which then leads to

$$\begin{aligned} & \sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y+1} U_{i,j} \left\{ \left(\frac{W_{i+\frac{1}{2},j} - W_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{W_{i,j+\frac{1}{2}} - W_{i,j-\frac{1}{2}}}{l_{y_j}} \right) \right\} l_{x_i} l_{y_j} \\ = & \sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y+1} \left(K^{-1}_{xx_{i+\frac{1}{2},j}} W_{i+\frac{1}{2},j} (\mathcal{G}U)_{i+\frac{1}{2},j} + K^{-1}_{yy_{i,j+\frac{1}{2}}} W_{i,j+\frac{1}{2}} (\mathcal{G}U)_{i,j+\frac{1}{2}} \right) h_{x_i} h_{y_j}. \end{aligned} \tag{3.1.12}$$

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[3] In the next chapter, we will consider the case where the flux tensor $\mathbf{K}$ is a full matrix tensor.

Then grouping the like terms of (3.1.12), we have,

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1}\left[l_{y_j}U_{i,j} - K^{-1}_{xx_{i+\frac{1}{2},j}} h_{x_i}h_{y_j}(\mathcal{G}U)_{i+\frac{1}{2},j}\right]W_{i+\frac{1}{2},j} - \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} l_{y_j}U_{i,j}W_{i-\frac{1}{2},j} +$$
$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1}\left[l_{x_i}U_{i,j} - K^{-1}_{yy_{i,j+\frac{1}{2}}} h_{x_i}h_{y_j}(\mathcal{G}U)_{i,j+\frac{1}{2}}\right]W_{i,j+\frac{1}{2}} - \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} l_{x_i}U_{i,j}W_{i,j-\frac{1}{2}} = 0. \quad (3.1.13)$$

Then we can rewrite the terms of (3.1.13) as follows

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1}\left[l_{y_j}U_{i,j} - K^{-1}_{xx_{i+\frac{1}{2},j}} h_{x_i}h_{y_j}(\mathcal{G}U)_{i+\frac{1}{2},j}\right]W_{i+\frac{1}{2},j} - \sum_{i=1}^{N_x+1}\sum_{j=0}^{N_y+1} l_{y_j}U_{i,j}W_{i-\frac{1}{2},j}$$
$$- \sum_{j=0}^{N_y+1} l_{y_j}U_{0,j}W_{-\frac{1}{2},j} + \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1}\left[l_{x_i}U_{i,j} - K^{-1}_{yy_{i,j+\frac{1}{2}}} h_{x_i}h_{y_j}(\mathcal{G}U)_{i,j+\frac{1}{2}}\right]W_{i,j+\frac{1}{2}}$$
$$- \sum_{i=0}^{N_x+1}\sum_{j=1}^{N_y+1} l_{x_i}U_{i,j}W_{i,j-\frac{1}{2}} - \sum_{i=0}^{N_x+1} l_{x_i}U_{i,0}W_{i,-\frac{1}{2}} = 0. \quad (3.1.14)$$

Now, re-indexing any terms with $i-\frac{1}{2}$ to $i+\frac{1}{2}$, and $j-\frac{1}{2}$ to $j+\frac{1}{2}$, and making using of the fact that $h_{N_x}=0$ and $h_{N_y}=0$, then (3.1.14) becomes

$$\sum_{i=0}^{N_x}\sum_{j=0}^{N_y+1}\left[-l_{y_j}(U_{i+1,j}-U_{i,j}) - K^{-1}_{xx_{i+\frac{1}{2},j}} h_{x_i}h_{y_j}(\mathcal{G}U)_{i+\frac{1}{2},j}\right]W_{i+\frac{1}{2},j} - \sum_{j=0}^{N_y+1} l_{y_j}U_{0,j}W_{-\frac{1}{2},j}$$
$$+ \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y}\left[-l_{x_i}(U_{i,j+1}-U_{i,j}) - K^{-1}_{yy_{i,j+\frac{1}{2}}} h_{x_i}h_{y_j}(\mathcal{G}U)_{i,j+\frac{1}{2}}\right]W_{i,j+\frac{1}{2}} - \sum_{i=0}^{N_x+1} l_{x_i}U_{i,0}W_{i,-\frac{1}{2}}$$
$$+ \sum_{j=0}^{N_y+1} l_{y_j}U_{N_x+1,j}W_{N_x+\frac{3}{2},j} + \sum_{i=0}^{N_x+1} l_{x_i}U_{i,N_y+1}W_{i,N_y+\frac{3}{2}} = 0. \quad (3.1.15)$$

Note that, the technique is done to fully concentrate the fluxes at the $(i,j)^{th}-$node to enhance the mimicking property at that node. We further note here that, (3.1.15) holds for all U in HC such that $U_{0,j}=U_{N_x+1,j}=0$ and $U_{i,0}=U_{i,N_y+1}=0$.

Hence solving for $(\mathcal{G}U)_{i+1/2,j}$ gives

$$(\mathcal{G}U)_{i+\frac{1}{2},j} = -\left(\left[\frac{l_{y_j}k_{11_{i+\frac{1}{2}}}}{h_{x_i}h_{y_j}}\right]\right)(U_{i+1,j}-U_{i,j}) \quad (3.1.16)$$

for $i=0,...,N_x$ and $j=0,...,N_y+1$,
and solving for $(\mathcal{G}U)_{i,j+1/2}$,

$$(\mathcal{G}U)_{i,j+\frac{1}{2}} = -\left(\left[\frac{l_{x_i}k_{22_{j+\frac{1}{2}}}}{h_{y_j}h_{x_i}}\right]\right)(U_{i,j+1} - U_{i,j}) \tag{3.1.17}$$

for $i = 1, ..., N_x$ and $j = 1, ..., N_y + 1$.

Let $\mathcal{A}_h$ denote the discrete diffusion operator obtained by forming the composition of the discrete divergence and gradient operator $\mathcal{D}\,\text{and}\,\mathcal{G}$ respectively. Indeed by construction, $\mathcal{D} : \mathbf{HC} \to HC$ and $\mathcal{G} : HC \to \mathbf{HC}$ is given by $\mathcal{A}_h : HC \to HC$. Furthermore, let $z_{i,j} = (i-1) \times N_x + j$ then we have the vector U_h is given by

$$U_h = (U_{11}, U_{12}, ..., U_{1,N_y}, U_{21}, ..., U_{2,N_y}, ..., U_{N_x,1}, ..., U_{N_x,N_y})^T. \tag{3.1.18}$$

Then $\mathcal{A}_h$ becomes

$$\begin{aligned} \mathcal{A}_h U_h[z_{i,j}] &= (\mathcal{D}\mathcal{G})U_h[z_{i,j}] \\ &= \frac{(\mathcal{G}U)_{i+\frac{1}{2},j} - (\mathcal{G}U)_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{(\mathcal{G}U)_{i,j+\frac{1}{2}} - (\mathcal{G}U)_{i,j-\frac{1}{2}}}{l_{y_j}} \end{aligned} \tag{3.1.19}$$

for $i = 1, 2, ..., N_x$ and $j = 1, 2, ..., N_y$, where A_h is a $(N_x)^2 \times (N_y)^2$ tridiagonal block matrix and U_h is a vector.

3.1.3 Mimetic Finite difference Scheme for penalised option

In this section, our aim is to discretise the penalised put option problem (1.8.2). It is important to note that, the only difference between the European and American options is the penalty term $\lambda[u^* - u]_+^{1/k}$. Hence the discrete scheme obtained also holds for the European option, i.e when the penalty parameter $\lambda = 0$. The mimetic finite difference method will be used for the diagonalised diffusion term (see $\mathbf{K}_1$ below), while using central difference for the mixed diffusion term (see $\mathbf{K}_2$ below), and the first order upwind finite difference scheme for the advection terms. Let $u_\lambda = u$ for simplicity, then we have that

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \mathbf{D}w - \nabla \cdot [\mathbf{K}_2 \nabla u + \mathbf{b}u] + cu + \lambda[u^* - u]_+^{1/k} = f(x, y, t) \\ \mathbf{G}u := \mathbf{w} = -\mathbf{K}_1 \nabla u \\ \mathbf{A} = \mathbf{DG}, \end{cases} \tag{3.1.20}$$

where

$$\mathbf{K}_1 = \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sigma_1^2 x^2 & 0 \\ 0 & \frac{1}{2}\sigma_2^2 y^2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 0 & k_{12} \\ k_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\rho\sigma_1\sigma_2 xy \\ \frac{1}{2}\rho_{12}\sigma_1\sigma_2 xy & 0 \end{pmatrix},$$

$$\mathbf{b} \;=\; \begin{pmatrix} b_1 x \\ \\ b_2 y \end{pmatrix} = \begin{pmatrix} (r-\sigma_1^2-\frac{1}{2}\rho\sigma_1\sigma_2)x \\ \\ (ry-\sigma_2^2-\frac{1}{2}\rho\sigma_1\sigma_2)y \end{pmatrix},\; c = 3r-(\sigma_1^2+\sigma_2^2+\rho\sigma_1\sigma_2),\; \text{and } \beta := \sigma^2.$$

Remark 3.1.4. Unlike in chapter 2, we will assume here that $r \geq \sigma_1^2$ and $r \geq \sigma_2^2$. This is a practical assumption based on the work in [35].

We then partition $I_i := (0,X)$ into N_x+1 and $I_j := (0,Y)$ into N_y+1 subintervals respectively, with dual partitions as we did for the elliptic problems (see illustration in section 2.1). Then we have that,

$U_{i,j} \approx U(x_i,y_j,t),\; l_{x_i} = x_{i+1/2}-x_{i-1/2},\; l_{y_j} = y_{j+1/2}-y_{j-1/2}, h_{x_i} = x_{i+1}-x_i,\; h_{y_j} = y_{j+1}-y_j,$

for $i = 0,1,...,N_x$ and $j = 0,1,...,N_y$. Let us set $w_{i+\frac{1}{2},j} := \dfrac{w_{i+1,j}-w_{i,j}}{2}$, $w_{i-\frac{1}{2},j} := \dfrac{w_{i,j}-w_{i-1,j}}{2}$, $w_{i,j+\frac{1}{2}} := \dfrac{w_{i,j+1}-w_{i,j}}{2}$, $w_{i,j-\frac{1}{2}} := \dfrac{w_{i,j}-w_{i,j-1}}{2}$. We can easily see that $w_{i,j} \approx w(x_i,y_j,t)$, for $i = 0,1,...,N_x$ and $j = 0,1,...,N_y$. That is the discrete mimetic operators (prime and derived) are given by

$$(\mathcal{D}w)_{i,j} = \frac{w_{i+\frac{1}{2},j}-w_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{w_{i,j+\frac{1}{2}}-w_{i,j-\frac{1}{2}}}{l_{y_j}},\;\; i = 0,...N_x+1,\;\; j = 0,...,N_y+1, \tag{3.1.21}$$

and

$$(\mathcal{G}U)_{i+\frac{1}{2},j} = -\left(\left[\frac{l_{y_j}k11_{i+\frac{1}{2}}}{h_{x_i}h_{y_j}}\right]\right)(U_{i+1,j}-U_{i,j})\;\; i = 0,...N_x,\;\; j = 0,...,N_y+1, \tag{3.1.22}$$

$$(\mathcal{G}U)_{i,j+\frac{1}{2}} = -\left(\left[\frac{l_{x_i}k22_{j+\frac{1}{2}}}{h_{x_i}h_{y_j}}\right]\right)(U_{i,j+1}-U_{i,j})\;\; i = 0,...N_x,\;\; j = 0,...,N_y+1. \tag{3.1.23}$$

Then the discrete operator $\widehat{\mathcal{A}}_h$ is given by

$$\begin{aligned}\widehat{\mathcal{A}}_h U_h[z_{i,j}] = (\mathcal{D}\mathcal{G})U_h[z_{i,j}] &= \frac{(\mathcal{G}U)_{i+\frac{1}{2},j}-(\mathcal{G}U)_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{(\mathcal{G}U)_{i,j+\frac{1}{2}}-(\mathcal{G}U)_{i,j-\frac{1}{2}}}{l_{y_j}} \\ &= \frac{-\left(\dfrac{l_{y_j}k11_{i+\frac{1}{2}}}{h_{y_j}}\right)\dfrac{U_{i+1,j}-U_{i,j}}{h_{x_i}} + \left(\dfrac{l_{y_j}k11_{i-\frac{1}{2}}}{h_{y_j}}\right)\dfrac{U_{i,j}-U_{i-1,j}}{h_{x_{i-1}}}}{l_{x_i}} \\ &+ \frac{-\left(\dfrac{l_{x_i}k22_{j+\frac{1}{2}}}{h_{x_i}}\right)\dfrac{U_{i,j+1}-U_{i,j}}{h_{y_j}} + \left(\dfrac{l_{x_i}k22_{j-\frac{1}{2}}}{h_{x_i}}\right)\dfrac{U_{i,j}-U_{i,j-1}}{h_{y_{j-1}}}}{l_{y_j}}\end{aligned} \tag{3.1.24}$$

for $i = 1, ..., N_x$, $j = 1, ..., N_y$, or

$$\widehat{\mathcal{A}}_h U_h[z_{i,j}] = \alpha_{i,j} U_{i+1,j} + \beta_{i,j} U_{i,j+1} + \gamma_{i,j} U_{i,j} + \Gamma_{i,j} U_{i-1,j} + \delta_{i,j} U_{i,j-1} \tag{3.1.25}$$

where

$$\alpha_{i,j} = \left[\frac{-l_{y_j} k_{11_{i+\frac{1}{2},j}}}{h_{y_j} h_{x_i} l_{x_i}}\right], \quad \beta_{i,j} = \left[\frac{-l_{x_i} k_{22_{i,j+\frac{1}{2}}}}{h_{y_j} l_{y_j} h_{x_i}}\right], \quad \Gamma_{i,j} = \left[\frac{-l_{y_j} k_{11_{i-\frac{1}{2},j}}}{l_{x_i} h_{x_{i-1}} h_{y_j}}\right]$$

$$\gamma_{i,j} = \left[\frac{l_{y_j} k_{11_{i+\frac{1}{2}}}}{h_{x_i} h_{y_j} l_{x_i}} + \frac{l_{y_j} k_{11_{i-\frac{1}{2}}}}{l_{x_i} h_{x_{i-1}} h_{y_j}} + \frac{l_{x_i} k_{22_{j+\frac{1}{2}}}}{h_{y_i} l_{y_j} h_{x_i}} + \frac{l_{x_i} k_{22_{j-\frac{1}{2}}}}{l_{y_i} h_{y_{i-1}} h_{x_i}}\right], \quad \delta_{i,j} = \left[\frac{-l_{x_i} k_{22_{j-\frac{1}{2}}}}{h_{y_j} h_{y_{j-1}} h_{x_i}}\right]. \tag{3.1.26}$$

Also from (3.1.20) we have that

$$-\nabla \cdot (\mathbf{K}_2 \nabla u + \mathbf{b} u) = -\left[\nabla_x (k_{12} \nabla_y u) + \nabla_y (k_{21} \nabla_x u) + \nabla_x (b_1 x u) + \nabla_y (b_2 y u)\right]$$

$$= -\left[(k_{12} + k_{21}) \nabla_{xy} u + (\frac{1}{2}\rho\sigma_1\sigma_2 x + b_1 x) \nabla_x u + (\frac{1}{2}\rho\sigma_1\sigma_2 y + b_2 y) \nabla_y u + (b_1 + b_2) u\right], \tag{3.1.27}$$

where $\nabla_x = \dfrac{\partial}{\partial x}$, $\nabla_y = \dfrac{\partial}{\partial y}$ and $\nabla_{xy} = \dfrac{\partial^2}{\partial x \partial y}$.

Applying the central difference to the mixed diffusion term [13], and the first order upwind finite difference method advection term [42] of (3.1.27), we have that

$$(k_{12} + k_{21}) \nabla_{xy} u \approx (k_{12} + k_{21})_{i,j} \left[\frac{U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1}}{4 h_{x_i} h_{y_j}}\right], \tag{3.1.28}$$

$$\left(\frac{1}{2}\rho\sigma_1\sigma_2 + b_1\right) x \nabla_x u \approx (r - \sigma_1^2) \left[\frac{x_{i+1/2} U_{i+1,j} - x_{i-1/2} U_{i,j}}{h_{x_i}}\right], \tag{3.1.29}$$

and

$$\left(\frac{1}{2}\rho\sigma_1\sigma_2 + b_2\right) y \nabla_y u \approx (r - \sigma_2^2) \left[\frac{y_{j+1/2} U_{i,j+1} - y_{j-1/2} U_{i,j}}{h_{y_j}}\right]. \tag{3.1.30}$$

We assume in this work that $r \geq \sigma_1^2, \sigma_2^2$, and therefore

$$-\nabla \cdot (K_2 \nabla u + \mathbf{b} u) + \mathbf{c} u \approx -(k_{12} + k_{21})_{i,j} \left[\frac{U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1}}{4 h_{x_i} h_{y_j}}\right]$$

$$- (r - \sigma_1^2) \left[\frac{x_{i+1/2} U_{i+1,j} - x_{i-1/2} U_{i,j}}{h_{x_i}}\right] - (r - \sigma_2^2) \left[\frac{y_{j+1/2} U_{i,j+1} - y_{j-1/2} U_{i,j}}{h_{y_j}}\right] + r U_{i,j}. \tag{3.1.31}$$

Then we have that,

$$\begin{aligned}\widehat{\mathcal{B}}_h U_h[z_{i,j}] &= -\frac{(k_{12}+k_{21})_{i,j}}{4h_{x_i}h_{y_j}}U_{i+1,j+1} + \frac{(k_{12}+k_{21})_{i,j}}{4h_x h_{y_j}}U_{i-1,j+1} + \frac{(k_{12}+k_{21})_{i,j}}{4h_{x_i}h_{y_j}}U_{i+1,j-1} \\ &-\frac{(k_{12}+k_{21})_{i,j}}{4h_{x_i}h_{y_j}}U_{i-1,j-1} + \left[\left(\frac{(r-\sigma_1^2)\,x_{i-1/2}}{h_{x_i}}\right) + \left(\frac{(r-\sigma_2^2)\,y_{j-1/2}}{h_{y_j}}\right) + r\right]U_{i,j} \\ &+ \left(\frac{(r-\sigma_1^2)\,x_{i+1/2}}{h_{x_i}}\right)U_{i+1,j} + \left(\frac{(r-\sigma_2^2)\,y_{j+1/2}}{h_{y_j}}\right)U_{i,j+1}\end{aligned} \tag{3.1.32}$$

or

$$\begin{aligned}\widehat{\mathcal{B}}_h U_h[z_{i,j}] = \Pi_{i,j}U_{i+1,j+1} + \Lambda_{i,j}U_{i-1,j+1} + \Upsilon_{i,j}U_{i+1,j-1} + \eta_{i,j}U_{i-1,j-1} + \chi_{i,j}U_{i,j} \\ +\zeta_{i,j}U_{i+1,j} + \varepsilon_{i,j}U_{i,j+1}\end{aligned} \tag{3.1.33}$$

where

$$\begin{aligned}\Pi_{i,j} = -\frac{(k_{12}+k_{21})_{i,j}}{4h_{x_i}h_{y_j}},\ \Lambda_{i,j} = \frac{(k_{12}+k_{21})_{i,j}}{4h_{x_i}h_{y_j}},\ \Upsilon_{i,j} = \frac{(k_{12}+k_{21})_{i,j}}{4h_{x_i}h_{y_j}},\ \eta_{i,j} = -\frac{(k_{12}+k_{21})_{i,j}}{4h_{x_i}h_{y_j}}, \\ \chi_{i,j} = -\left[\frac{1}{h_{x_i}}\left((r-\sigma_1^2)\,x_i\right) + \frac{1}{h_{y_j}}\left((r-\sigma_2^2)\,y_j\right) - r\right],\ \zeta_{i,j} = \frac{1}{h_{x_i}}\left((r-\sigma_1^2)\,x_i\right), \\ \varepsilon_{i,j} = \frac{1}{h_{y_j}}\left((r-\sigma_2^2)\,y_j\right)\end{aligned} \tag{3.1.34}$$

Now from (3.1.25) and (3.1.33), we have that

$$\begin{aligned}\widehat{\mathcal{C}}_h U_h[z_{i,j}] = \widehat{\mathcal{A}}_h U_h[z_{i,j}] + \widehat{\mathcal{B}}_h U_h[z_{i,j}] = \Pi_{i,j}U_{i+1,j+1} + \Lambda_{i,j}U_{i-1,j+1} + \Upsilon_{i,j}U_{i+1,j-1} + \delta_{i,j}U_{i,j-1} \\ +\eta_{i,j}U_{i-1,j-1} + (\gamma_{i,j}+\chi_{i,j})U_{i,j} + (\alpha_{i,j}+\zeta_{i,j})U_{i+1,j} + (\beta_{i,j}+\varepsilon_{i,j})U_{i,j+1} + \Gamma_{i,j}U_{i-1,j}\end{aligned} \tag{3.1.35}$$

for all $i = 1, 2, ..., N_x$ and $j = 1, 2, ..., N_y$.

Now, using the transformation $t = T - t$ we have

$$\begin{cases} \dfrac{dU_h}{dt} + \widehat{\mathcal{C}}_h U_h + \lambda\left[U_h^* - U_h\right]_+^{1/k} = f_h(t), \qquad \forall t \in [0, T], \\ \\ U_h(0) = U_h^* \end{cases} \tag{3.1.36}$$

3.1.5 Fitted mimetic finite difference scheme

To handle the degeneracy of the Black-Scholes differential operator near the boundary $x, y = 0$, we resort to the special fitted technique proposed in [43]. In this section near $x, y = 0$, the sum of the diffusion and advection flux is approximated using the fitted finite volume scheme, see

[12] . Far from $x, y = 0$ $(i > 1, j > 1)$ however, the diffusion flux and advection flux will be approximated as in section 3.1 using respectively, the standard mimetic finite difference and the upwind finite difference. This combination will yield our novel scheme called the Fitted mimetic finite difference scheme.

As the case $(i > 1, \text{ and} j > 1)$ is already covered in the previous section 3.1, we will only focus on the cases $(i = 1, \text{ and} j = 1)$, $(i = 1, \text{ and} j > 1)$ and $(i > 1, \text{ and} j = 1)$, as those are the degenerate regions.

Case I $(i, j = 1)$

We need to approximate the flux at $x_{1/2}$ and $y_{1/2}$ with the fitted finite volume method to handle the degeneracy of the Black-Scholes differential operator. Indeed, to find a new approximation at $(\mathcal{D}W)_{1,1}$, we require the fluxes at $(x_{\frac{1}{2}}, y_1), (x_{\frac{3}{2}}, y_1)$ and $(x_1, y_{\frac{1}{2}}), (x_1, y_{\frac{1}{2}})$, i.e. $(\mathcal{G}U)_{\frac{1}{2},1}$, $(\mathcal{G}U)_{\frac{3}{2},1}$, $(\mathcal{G}U)_{1,\frac{1}{2}}$ and $(\mathcal{G}U)_{1,\frac{3}{2}}$ respectively. Now let $\mathcal{R}_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$.

Now integrating (3.1.20) across $\mathcal{R}_{1,1} = [x_{\frac{1}{2}}, x_{\frac{3}{2}}] \times [y_{\frac{1}{2}}, y_{\frac{3}{2}}]$ we have

$$\begin{aligned}
-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} \frac{\partial u}{\partial t} dxdy - \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_x \cdot (k_{11} \nabla_x u + k_{12} \nabla_y u + b_1 x u)\, dxdy \\
- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_y \cdot (k_{21} \nabla_x u + k_{22} \nabla_y u + b_2 y u)\, dxdy \\
+ \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\Big[cu + \lambda [u^* - u]_+^{1/k} - f(t) \Big]\, dxdy = 0. \qquad (3.1.37)
\end{aligned}$$

and using the midpoint rule, to approximate the first and last terms of (3.1.37), we obtain

$$\begin{aligned}
-R_{1,1} \frac{dU_{1,1}}{dt} - \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_x \cdot (k_{11} \nabla_x u + k_{12} \nabla_y u + b_1 x u)\, dxdy \\
- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} \int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_y \cdot (k_{21} \nabla_x u + k_{22} \nabla_y u + b_2 y u)\, dxdy \\
&+ R_{1,1} [c U_{1,1} + \lambda [U^*_{1,1} - U_{1,1}]_+^{1/k} - f_{1,1}(t)] = 0, \qquad (3.1.38)
\end{aligned}$$

where $R_{1,1} = l_{x_1} l_{y_1}$ is the area of a control volume around the point (x_1, y_1), with $U_{x_1,y_1} = U_{1,1}$, $U^*_{x_1,y_1} = U^*_{1,1}$ and $f_{x_1,y_1} = f_{1,1}$. Let us define the following

$$\Phi(u) := \frac{1}{2} \sigma_1^2 x \nabla_x u + (r - \sigma_1^2 - \rho \sigma_1 \sigma_2) u = \mathrm{a}_1 x \nabla_x u + b_1 u \qquad (3.1.39)$$

and

$$\Psi(u) := \frac{1}{2}\sigma_2^2 y\nabla_y u + (r - \sigma_2^2 - \rho\sigma_1\sigma_2)u = \mathsf{a}_2 y\nabla_y u + b_2 u, \tag{3.1.40}$$

where $\mathsf{a}_1 = \frac{1}{2}\sigma_1^2$, $\mathsf{a}_2 = \frac{1}{2}\sigma_2^2$, and b_1, b_2 are as already defined. Then the second term of (3.1.38) can be approximated by

$$\begin{aligned} -\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} \nabla_x\cdot(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u)\,dxdy &= -\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} \nabla_x\cdot(x\Phi(u) + k_{12}\nabla_y u)\,dxdy \\ &\approx l_{y_1}\left[x\Phi(u) + k_{12}\nabla_y u\right]\Big|_{(x_{\frac{1}{2}},y_1)}^{(x_{\frac{3}{2}},y_1)}, \end{aligned} \tag{3.1.41}$$

and the third term of (3.1.38) can be approximated by

$$\begin{aligned} -\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} \nabla_y\cdot(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u)\,dxdy &= -\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} \nabla_y\cdot(y\Psi(u) + k_{21}\nabla_x u)\,dxdy \\ &\approx l_{x_1}\left[y\Psi(u) + k_{21}\nabla_x u\right]\Big|_{(x_1,y_{\frac{1}{2}})}^{(x_1,y_{\frac{3}{2}})}. \end{aligned} \tag{3.1.42}$$

Recall that

$$[x\Phi(u)]_{(x_{\frac{1}{2}},y_1)}^{(x_{\frac{3}{2}},y_1)} = x_{3/2}\Phi_{\frac{3}{2},1}(u) - x_{1/2}\Phi_{\frac{1}{2},1}(u). \tag{3.1.43}$$

Note here that the problem is not at $(x_{3/2}, y_1)$ and hence using (3.1.16), $x_{3/2}\Phi(u)|_{x_{3/2},y_1}$ can be approximated as

$$\begin{aligned} x_{3/2}\Phi(u)|_{x_{3/2},y_1} &\approx (-\mathcal{G}U)_{3/2,1} + b_1 x_{3/2}U_{1,1} \\ &= \left[\frac{l_{y_1}k_{11_{\frac{3}{2}}}}{h_{x_1}h_{y_1}}\right]U_{2,1} + \left[b_1 x_{3/2} - \frac{l_{y_1}k_{11_{\frac{3}{2}}}}{h_{x_1}h_{y_1}}\right]U_{1,1}. \end{aligned}$$

Let us now consider $x_{1/2}\Phi(u)|_{x_{1/2},y_1}$ using the fitted technique [12, 43]. We consider the following two-point boundary value problem:

$$(\mathsf{a}_1 x\nabla_x v + b_1 v)' = C_1, \quad x \in (0, x_1) \tag{3.1.44}$$

$$v(0, y_1) = U_{0,1}, \quad v(x_1, y_1) = U_{1,1}, \tag{3.1.45}$$

where C_1 is an unknown constant to be determined. Integrating (3.1.44) once, we have that

$$\mathsf{a}_1 x\nabla_x v + b_1 v = C_1 x + C_2$$

Now, using the condition $v(0, y_1) = U_{0,1}$, we have that $C_2 = b_1 U_{0,1}$ and hence

$$\Phi_0(u) := \mathsf{a}_1 x\nabla_x v + b_1 v = C_1 x + b_1 U_{0,1}. \tag{3.1.46}$$

Then, solving (3.1.46) analytically yields

$$v(x,y_1)=\begin{cases} U_{0,1}+\dfrac{C_1x}{a_1+b_1}+C_3x^{b_1/a_1}, & b_1/a_1\neq -1,\\[2ex] U_{0,1}+\dfrac{C_1x}{a_1}x\ln x+C_3x, & b_1/a_1=-1,\end{cases} \tag{3.1.47}$$

where C_3 is a constant which depends on t and $x \in (0, x_1)$. To determine the constants C_1 and C_3, let us consider the case when $b_1/a_1 \neq -1$. Now, when $b_1/a_1 \geq 0$, using $v(0,y_1) = U_{0,1}$ implies that $C_3 = 0$. Otherwise if $b_1/a_1 < 0$, C_3 is arbitrary, and so we choose $C_3 = 0$. Also, using $v(x_1,y_1) = U_{1,1}$, we obtain $C_1 = \dfrac{1}{x_1}(a_1+b_1)(U_{1,1}-U_{0,1})$.

Furthermore, when $b_1/a_1 = -1$, from (3.1.47), we have that $v(0,y_1) = U_{0,1}$ is satisfied for any C_1 and C_3. That is, solutions with such C_1 and C_3 are not unique. We choose $C_1 = 0$, and $v(x_1,y_1) = U_{1,1}$, which then gives $C_3 = (U_{1,1} - U_{0,1})/x_1$.

Therefore, (3.1.46) becomes

$$(\Phi(v))|_{x_{1/2},y_1}=(\mathsf{a}_1x\nabla_x v+b_1v)_{x_{1/2},1}=\frac{1}{2}[(\mathsf{a}_1+b_1)U_{1,1}-(\mathsf{a}_1-b_1)U_{0,1}], \tag{3.1.48}$$

for both $b/a \neq -1$ and $b/a = -1$.

Then (3.1.46) reduces to

$$v=(U_{1,1}-U_{0,1})x/x_1,\quad x\in[0,x_1]. \tag{3.1.49}$$

Then from (3.1.43) and (3.1.48), we have that

$$[x\Phi(u)]^{(x_{\frac{3}{2}},y_1)}_{(x_{\frac{1}{2}},y_1)}\approx\left[\frac{l_{y_1}k_{11_{\frac{3}{2}}}}{h_{x_1}h_{y_1}}\right]U_{2,1}+\left[b_1x_{3/2}-\frac{l_{y_1}k_{11_{\frac{3}{2}}}}{h_{x_1}h_{y_1}}\right]U_{1,1}-\frac{x_{1/2}}{2}[(\mathsf{a}_1+b_1)U_{1,1}-(\mathsf{a}_1-b_1)U_{0,1}]. \tag{3.1.50}$$

We follow a similar argument as before and establish that (3.1.42), can be approximated as

$$[y\Psi(u)]^{(x_1,y_{\frac{3}{2}})}_{(x_1,y_{\frac{1}{2}})}\approx\left[\frac{l_{x_1}k_{22_1}}{h_{y_1}h_{x_1}}\right]U_{1,2}+\left[b_2y_{3/2}-\frac{l_{x_1}k_{22_1}}{h_{y_1}h_{x_1}}\right]U_{1,1}-\frac{y_{1/2}}{2}[(\mathsf{a}_2+b_2)U_{1,1}-(\mathsf{a}_2-b_2)U_{1,0}]. \tag{3.1.51}$$

Remember that from (3.1.38), we have

$$\begin{aligned}&-\nabla_x\cdot(\Phi(u)+k_{12}\nabla_y u)-\nabla_y\cdot(\Psi(u)+k_{21}\nabla_x u)+cu\\[2ex] &=-\mathbf{D}\left[(\Phi(u)+k_{12}\nabla_y u)+(\Psi(u)+k_{21}\nabla_x u)\right]+cu.\end{aligned} \tag{3.1.52}$$

Then considering by definition that $\mathbf{D}$ is approximated by $\mathcal{D}$, we obtain

$$
\begin{aligned}
&[-\nabla_x\cdot(\Phi(u)+k_{12}\nabla_y u)-\nabla_y\cdot(\Psi(u)+k_{21}\nabla_x u)+cu]\,|x_1,y_1 \approx\\
&\frac{-\left(x_{3/2}\Phi_{\frac{3}{2},1}(u)-\frac{x_{\frac{1}{2}}}{2}[(\mathsf{a}_1+b_1)U_{1,1}-(\mathsf{a}_1-b_1)U_{0,1}]\right)}{l_{x_1}}+cU_{1,1}\\
&-k_{12_{1,1}}\frac{[U_{2,2}-U_{2,0}-U_{0,2}+U_{0,0}]}{2h_{x_1}h_{y_1}}-\frac{\left(y_{3/2}\Psi_{1,\frac{3}{2}}(u)-\frac{y_{\frac{1}{2}}}{2}[(\mathsf{a}_2+b_2)U_{1,1}-(\mathsf{a}_2-b_2)U_{1,0}]\right)}{l_{y_1}},\\
&=\frac{-\left(\left[\frac{l_{y_1}k_{11_{\frac{3}{2}}}}{h_{y_1}}\right]\frac{U_{2,1}-U_{1,1}}{h_{x_1}}-\frac{x_{\frac{1}{2}}}{2}[(\mathsf{a}_1+b_1)U_{1,1}-(\mathsf{a}_1-b_1)U_{0,1}]\right)}{l_{x_1}}-\left[\frac{k_{12_{1,1}}}{2h_{x_1}h_{y_1}}\right][U_{2,2}-U_{2,0}]\\
&-\left[\frac{k_{12_{1,1}}}{2h_{x_1}h_{y_1}}\right][U_{0,0}-U_{0,2}]-\frac{\left(\frac{l_{x_1}k_{22_{\frac{3}{2}}}}{h_{x_1}}\frac{U_{1,2}-U_{1,1}}{h_{y_1}}-\frac{y_{\frac{1}{2}}}{2}[(\mathsf{a}_2+b_2)U_{1,1}-(\mathsf{a}_2-b_2)U_{1,0}]\right)}{l_{y_1}}+cU_{1,1},
\end{aligned}
\tag{3.1.53}
$$

where $U_{1,0}, U_{2,0}, U_{0,2}, U_{0,1}$ and $U_{0,0}$ are solutions obtained from the 1D problem.

Now remember that $z_{i,j}=(i-1)*N_x+j$ for $U_H=(U_{1,1},U_{1,2},...,U_{1,N_y},...,U_{N_x,1},U_{N_x,2},...,U_{N_x,N_y})$,

Then have that

$$
\begin{aligned}
C_HU_H[z_{1,1}]=&-\left[\frac{k_{12_{1,1}}}{2h_{x_1}h_{y_1}}\right]U_{2,2}-\left[\frac{l_{y_1}k_{11_{\frac{3}{2}}}}{h_{x_1}l_{x_1}h_{y_1}}\right]U_{2,1}-\left[\frac{l_{x_1}k_{22_{\frac{3}{2}}}}{l_{y_1}h_{x_1}h_{y_1}}\right]U_{1,2}\\
&+\left[\frac{l_{y_1}k_{11_{\frac{3}{2}}}}{h_{x_1}l_{x_1}h_{y_1}}+\frac{l_{x_1}k_{22_{\frac{3}{2}}}}{l_{y_1}h_{x_1}h_{y_1}}+\frac{x_{\frac{1}{2}}}{2l_{x_1}}(\mathsf{a}_1+b_1)+\frac{y_{\frac{1}{2}}}{2l_{y_1}}(\mathsf{a}_2+b_2)+c\right]U_{1,1}.
\end{aligned}
\tag{3.1.54}
$$

Case II $(i=1, j>1)$

Again, we approximate the flux along $x_{1/2}$ and $y_j, j>1$ with the fitted finite volume method to handle the degeneracy. By so doing, we find a new approximation $(DW)_{1,j}$, which will require the fluxes at $(x_{\frac{1}{2}},y_j),(x_{\frac{3}{2}},y_j)$ and $(x_1,y_{j+\frac{1}{2}}),(x_1,y_{j-\frac{1}{2}})$, i.e. $(\mathcal{G}U)_{\frac{1}{2},i}$, $(\mathcal{G}U)_{\frac{3}{2},i}$, $(\mathcal{G}U)_{1,j+\frac{1}{2}}$ and $(\mathcal{G}U)_{1,j-\frac{1}{2}}$ respectively.

Again, integrating (3.1.20) across $\mathcal{R}_{1,j}=[x_{\frac{1}{2}},x_{\frac{3}{2}}]\times[y_{j-\frac{1}{2}},y_{j+\frac{1}{2}}]$, and following a similar argument as is case 1, we have

$$\begin{aligned}-R_{1,j}\frac{dU_{1,j}}{dt} &- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \nabla_x\cdot(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u)\,dxdy\\ &- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \nabla_y\cdot(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u)\,dxdy\\ &+R_{1,j}[cU_{1,j} + \lambda[U^*_{1,j} - U_{1,j}]_+^{1/k} - f_{1,j}(t)] = 0,\end{aligned} \tag{3.1.55}$$

where $R_{1,j} = l_{x_1}l_{y_j}$ is the area of a control volume around the point (x_1, y_j), $j > 1$. We approximate the second and third terms of (3.1.55) by

$$-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \nabla_x\cdot(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u)\,dxdy \approx l_{y_1}\,[x\Phi(u) + k_{12}\nabla_y u]|_{(x_{\frac{1}{2}},y_j)}^{(x_{\frac{3}{2}},y_j)}, \tag{3.1.56}$$

and

$$-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} \nabla_y\cdot(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u)\,dxdy \approx l_{x_1}\,[y\Psi(u) + k_{21}\nabla_x u]|_{(x_1,y_{j-\frac{1}{2}})}^{(x_1,y_{j+\frac{1}{2}})}. \tag{3.1.57}$$

Note that our focus here will be to apply the fitted scheme to (3.1.56). The standard mimetic, central difference and the first order upwind will be used to approximate the terms of (3.1.57).

Again, recall that

$$[x\Phi(u)]_{(x_{\frac{1}{2}},y_j)}^{(x_{\frac{3}{2}},y_j)} = x_{3/2}\Phi_{\frac{3}{2},j}(u) - x_{1/2}\Phi_{\frac{1}{2},j}(u). \tag{3.1.58}$$

and hence using (3.1.16), $x_{3/2}\Phi(u)|_{x_{3/2},y_j}$ can be approximated as

$$\begin{aligned}x_{3/2}\Phi(u)|_{x_{3/2},y_1} &\approx (-\mathcal{G}U)_{3/2,j} + b_1 x_{3/2}U_{1,j}\\ &= \left[\frac{l_{y_j}k_{11_{\frac{3}{2}}}}{h_{y_j}h_{x_1}}\right]U_{2,j} + \left[b_1 x_{3/2} - \frac{l_{y_j}k_{11_{\frac{3}{2}}}}{h_{y_j}h_{x_1}}\right]U_{1,j}.\end{aligned}$$

To find the approximation for $x_{1/2}\Phi(u)|_{x_{1/2},y_j}$ using the fitted technique, we consider again the following two-point boundary value problem:

$$(\mathsf{a}_1 x\nabla_x v + b_1 v)' = C_1, \quad (x,y) \in (0,x_1)\times(0,Y) \tag{3.1.59}$$

$$v(0,y_j) = U_{0,j}, \quad v(x_1,y_j) = U_{1,j}, \tag{3.1.60}$$

where C_1 is an unknown constant to be determined.

Now, integrating (3.1.59) once, we have that

$$\mathsf{a}_1 x\nabla_x v + b_1 v = C_1 x + C_2$$

Now, using the condition $v(0,y_j) = U_{0,j}$, we have that $C_2 = b_1 U_{0,j}$ and hence

$$\Phi_0(u) := \mathsf{a}_1 x \nabla_x v + b_1 v = C_1 x + b_1 U_{0,j}. \tag{3.1.61}$$

Then (3.1.61) is solved analytically to yield

$$v(x,y_j) = \begin{cases} U_{0,j} + \dfrac{C_1 x}{a_1 + b_1} + C_3 x^{b_1/a_1}, & b_1/a_1 \neq -1, \\ \\ U_{0,j} + \dfrac{C_1 x}{a_1} x \ln x + C_3 x, & b_1/a_1 = -1, \end{cases} \tag{3.1.62}$$

where C_3 is a constant which depends on t and $x \in (0, x_1)$.

We determine the constants C_1 and C_3 by firstly considering the case when $b_1/a_1 \neq -1$. When $b_1/a_1 \geq 0$, using $v(0,y_j) = U_{0,j}$ implies that $C_3 = 0$. When $b_1/a_1 < 0$, C_3 is arbitrary. We then choose $C_3 = 0$. Now using $v(x_1,y_j) = U_{1,j}$ yields $C_1 = \dfrac{1}{x_1}(a_1 + b_1)(U_{1,j} - U_{0,j})$.

Now, when $b_1/a_1 = -1$, from (3.1.62), we have that $v(0,y_j) = U_{0,j}$ is satisfied for any C_1 and C_3. That is, solutions with such C_1 and C_3 are not unique. We choose $C_1 = 0$, and $v(x_1,y_j) = U_{1,j}$, which then gives $C_3 = (U_{1,j} - U_{0,j})/x_1$.

Therefore, (3.1.61) becomes

$$(\Phi(v))|_{x_{1/2},y_j} = (\mathsf{a} x \nabla_x v + \mathsf{b} v)_{x_{1/2},j} = \frac{1}{2}[(\mathsf{a}_1 + b_1)U_{1,j} - (\mathsf{a}_1 - b_1)U_{0,j}], \tag{3.1.63}$$

for both $b_1/a_1 \neq -1$ and $b_1/a_1 = -1$.

Again following that the divergent operator $\mathbf{D}$ is approximated by $\mathcal{D}$, we can approximate the flux in (3.1.55)

$$\mathbf{D}\left[\Phi(u) + \Psi(u)\right]|_{x_1,y_j} \approx -\frac{1}{l_{x_1}}\left(x_{3/2}\Phi(u)|_{\frac{3}{2},j} - \frac{x_{\frac{1}{2}}}{2}[(\mathsf{a}_1 + b_1)U_{1,j} - (\mathsf{a}_1 - b_1)U_{0,j}]\right)$$

$$-\frac{1}{l_{y_j}}\left((\mathcal{G}U)_{1,j+\frac{1}{2}} - (\mathcal{G}U)_{1,j-\frac{1}{2}} + b_2 y_{j+\frac{1}{2}} U_{i,j+1} - b_2 y_{j-\frac{1}{2}} U_{i,j}\right)$$

and

$$-\nabla \cdot \left[K_2 \nabla u\right]|_{x_1,y_j} \approx -\left[k_{21_{1,j}} \frac{U_{2,j+1} - U_{0,j+1} + U_{2,j-1} + U_{0,j-1}}{2h_{x_1} h_{y_j}}\right] \tag{3.1.64}$$

$$
\begin{aligned}
&\left[\mathbf{D}\left[\Phi(u)+\Psi(u)\right]|_{x_1,y_j}\right] - \nabla\cdot\left[K_2\nabla u\right]\Big|_{x_1,y_j} \approx \\
&-\frac{\left(\left[\dfrac{l_{x_1}k_{22_{j+\frac{1}{2}}}}{h_{x_1}}\right]\dfrac{U_{1,j+1}-U_{1,j}}{h_{y_j}} - \left[\dfrac{l_{x_1}k_{22_{j-\frac{1}{2}}}}{h_{x_1}}\right]\dfrac{U_{1,j}-U_{1,j-1}}{h_{y_{j-1}}}\right)}{l_{y_j}} + cU_{1,j} \qquad (3.1.65)\\
&-\frac{\left(b_2 y_{j+\frac{1}{2}}U_{i,j+1} - b_2 y_{j-\frac{1}{2}}U_{i,j}\right)}{l_{y_j}} - \left[k_{21_{1,j}}\frac{U_{2,j+1}-U_{0,j+1}+U_{2,j-1}+U_{0,j-1}}{2h_{x_1}h_{y_j}}\right].
\end{aligned}
$$

Then we have

$$
\begin{aligned}
C_H U_H[z_{1,j}] = &-\left[\frac{k_{12_{1,j}}}{2h_{x_1}h_{y_j}}\right]U_{2,j+1} - \left[\frac{l_{y_j}k_{11_{\frac{3}{2}}}}{h_{y_j}h_{x_1}l_{x_1}}\right]U_{2,j} - \left[\frac{k_{12_{1,j}}}{2h_{x_1}h_{y_j}}\right]U_{2,j-1}\\
&-\left[\frac{l_{x_1}k_{22_{j+\frac{1}{2}}}}{h_{x_1}h_{y_j}l_{y_j}} + \frac{b_2 y_{j+\frac{1}{2}}}{l_{y_j}}\right]U_{1,j+1} + \left[\frac{l_{x_1}k_{22_{j-\frac{1}{2}}}}{l_{y_j}h_{y_{j-1}}h_{x_1}}\right]U_{1,j-1} \qquad (3.1.66)\\
&+\left[\frac{l_{y_j}k_{11_{\frac{3}{2}}}}{h_{y_j}h_{x_1}l_{x_1}} + \frac{l_{x_1}k_{22_{j+\frac{1}{2}}}}{h_{x_1}h_{y_j}l_{y_j}} + \frac{l_{x_1}k_{22_{j-\frac{1}{2}}}}{h_{x_1}h_{y_{j-1}}l_{y_j}} + \frac{x_{\frac{1}{2}}}{2l_{x_1}}(\mathsf{a}_1+b_1) + \frac{b_2 y_{j-\frac{1}{2}}}{l_{y_j}} + c\right]U_{1,j}.
\end{aligned}
$$

Case III $(i > 1, j = 1)$

This case follows similarly to the previous cases I and II. The only difference is we swap the variables and repeat the process exactly as we did for case II.

Hence we have that

$$
\begin{aligned}
C_H U_H[z_{i,1}] = &-\left[\frac{k_{12_{i,1}}}{2h_{x_i}h_{y_1}}\right]U_{i+1,2} - \left[\frac{l_{x_i}k_{22_{\frac{3}{2}}}}{h_{y_1}h_{x_i}l_{y_1}}\right]U_{i,2} - \left[\frac{k_{12_{i,1}}}{2h_{x_i}h_{y_1}}\right]U_{i-1,2}\\
&-\left[\frac{l_{y_1}k_{11_{i+\frac{1}{2}}}}{h_{y_1}h_{x_i}l_{x_i}} + \frac{b_1 x_{i+\frac{1}{2}}}{l_{x_i}}\right]U_{i+1,1} + \left[\frac{l_{y_1}k_{11_{i-\frac{1}{2}}}}{l_{x_i}h_{x_{i-1}}h_{y_1}}\right]U_{i-1,1} \qquad (3.1.67)\\
&+\left[\frac{l_{x_i}k_{22_{\frac{3}{2}}}}{h_{x_i}h_{y_1}l_{y_1}} + \frac{l_{y_1}k_{11_{i+\frac{1}{2}}}}{h_{y_1}h_{x_i}l_{x_i}} + \frac{l_{y_1}k_{11_{i-\frac{1}{2}}}}{h_{y_1}h_{x_{i-1}}l_{x_i}} + \frac{y_{\frac{1}{2}}}{2l_{y_1}}(\mathsf{a}_2+b_2) + \frac{b_1 x_{i-\frac{1}{2}}}{l_{x_i}} + c\right]U_{i,1}.
\end{aligned}
$$

Now combining the mimetic approximation of (3.1.36) at (x_i, y_j), $i > 1, j > 1$, with (3.1.54), (3.1.66) and (3.1.67), yields our novel scheme called the fitted mimetic finite difference method.

Therefore for $U_H = (U_{1,1}, U_{1,2}, ..., U_{1,N_y}, ..., U_{N_x,1}, U_{N_x,2}, ..., U_{N_x,N_y})$, and considering the transformation $t = T - t$, we need to solve in the case of the fitted mimetic method the following system,

$$\begin{cases} \dfrac{dU_H}{dt} + \mathcal{C}_H U_H + F(U_H, t) = 0, \quad t \in [0, T], \\ \\ U_H(0) = U_H^* \\ \\ F(U_H, t) = \lambda \left[U_H^* - U_H\right]_+^{1/k} - f_H(t). \end{cases} \tag{3.1.68}$$

where

$$\begin{cases} C_H U_H[z_{1,1}], \ \text{ is as given in } (3.1.54) \\ C_H U_H[z_{1,j}], \ j > 1, \ \text{ is as given in } (3.1.66) \\ C_H U_H[z_{i,1}], \ i > 1, \ \text{ is as given in } (3.1.67) \\ C_H U_H[z_{i,j}] = \widehat{\mathcal{C}}_h U_h[z_{i,j}], \ i > 1, j > 1 \end{cases} \tag{3.1.69}$$

with $z_{i,j} = (i-1) \times N_x + j$.

3.1.6 Time discretization using standard implicit schemes

We subdivide the time interval $[0, T]$ in M subdivisions. That is, $0 = t_0 < t_1 < ... < t_M = T$, such that $\Delta t = t_{m+1} - t_m$, for $m = \{0, 1, ..., M\}$. We adopt the stable time discretization method mostly used, that is the following Euler$-\theta-$methods for (3.1.36) and (3.1.68), representing the semi-discrete solutions for the standard mimetic finite difference method and the fitted mimetic finite difference method, respectively. Then using the transformation $t = T - t$ we have

$$\begin{cases} \dfrac{U_h^{m+1} - U_h^m}{\Delta t} = \theta G(U_h^{m+1}, t_{m+1}) + (1-\theta) G(U_h^{m+1}, t_{m+1}) \\ \\ U_h(0) = U_h^*, \qquad\qquad 0 < \theta \leq 1, \end{cases} \tag{3.1.70}$$

with $G(U_h(t), t) = -\left(\widehat{\mathcal{C}}_h U_h(t) + F(U_h(t), t)\right)$, and

$$\begin{cases} \dfrac{U_H^{m+1} - U_H^m}{\Delta t} = \theta G(U_H^{m+1}, t_{m+1}) + (1-\theta) G(U_H^m, t_m) \\ \\ U_H(0) = U_H^*, \qquad\qquad 0 < \theta \leq 1, \end{cases} \tag{3.1.71}$$

with $G(U_H(t), t) = -\left(\widehat{\mathcal{C}}_H U_H(t) + F(U_H(t), t)\right)$. The scheme (3.1.70) is order 2 in time when $\theta = 1/2$ and order 1 if $\theta \neq 1/2$.

3.2 Numerical Tests

In this section, we conduct some numerical experiments to find the solution of the option. We will consider two tests. Firstly, we will consider the solution for the European put option problem and the second test will consider the solution for the American put option.

3.2.1 Test 1: Error for European options

We consider the case when the penalty parameter $\lambda = 0$ in (1.8.1). Indeed, this case corresponds to the solution of the European put option. There exits a closed form solution to the two-dimensional Black-Scholes PDE when the coefficents are constant, and this is given in [16] as below

$$\begin{aligned} P(S_1, S_2, E, T) &= Ee^{-rT}\Big(1 - M(-y_1 + \sigma_1\sqrt{T}, -y_2 + \sigma_2\sqrt{T}; \rho)\Big) \\ &\quad -s_1 e^{-rT} M(y_1, d; \rho_1) - s_2 e^{-rT} M(y_2, -d; \rho_2), \end{aligned} \tag{3.2.1}$$

where

$$\begin{aligned} d &= \frac{\ln(s_1/s_2) + \left(b_1 - b_2 + \dfrac{\sigma_1^2}{2}\right)T}{\sigma\sqrt{T}}, \\ y_1 &= \frac{\ln(s_1/E) + \left(b_1 + \dfrac{\sigma_1^2}{2}\right)T}{\sigma\sqrt{T}}, \quad y_2 = \frac{\ln(s_2/E) + \left(b_2 + \dfrac{\sigma_2^2}{2}\right)T}{\sigma\sqrt{T}}, \\ \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - \rho\sigma_1\sigma_2}, \quad \rho_1 = \frac{\sigma_1 - \rho\sigma_2}{\sigma}, \quad \rho_2 = \frac{\sigma_2 - \rho\sigma_2}{\sigma}, \end{aligned}$$

and

$$M(a, b; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a}\int_{-\infty}^{a} \exp\left(\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dxdy. \tag{3.2.2}$$

Let U_h be the numerical solution and U_h^{analytic}. Indeed, to compute the relative error, we use the L^2-norm given by

$$\text{Error} = \frac{\sqrt{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \text{meas}(\mathcal{R}_{i,j})(U_{i,j} - U_{i,j}^{\text{analytic}})^2}}{\sqrt{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \text{meas}(\mathcal{R}_{i,j})(U_{i,j}^{\text{analytic}})^2}}, \tag{3.2.3}$$

where $\text{meas}(\mathcal{R}_{i,j})$ is the measure of the cell $\mathcal{R}_{i,j}$. Then we present the figures for the mimetic solution and the analytical solution below

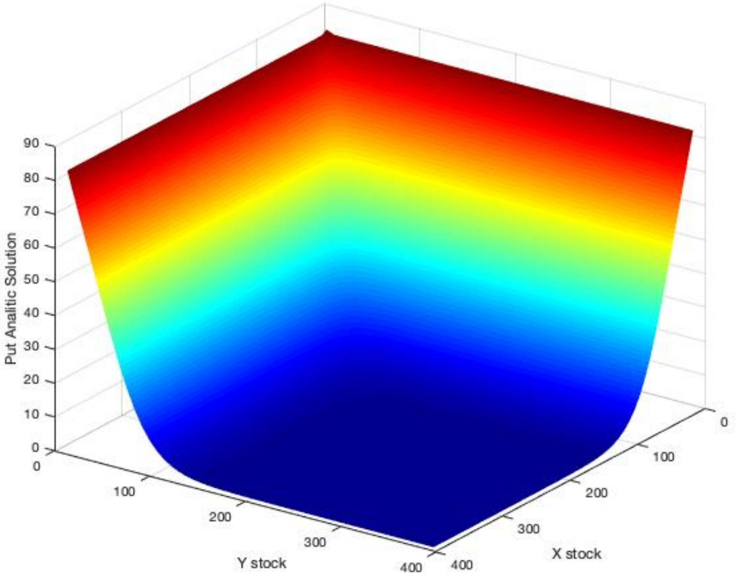

Figure 3.2: The Analytical Solution for the European option.

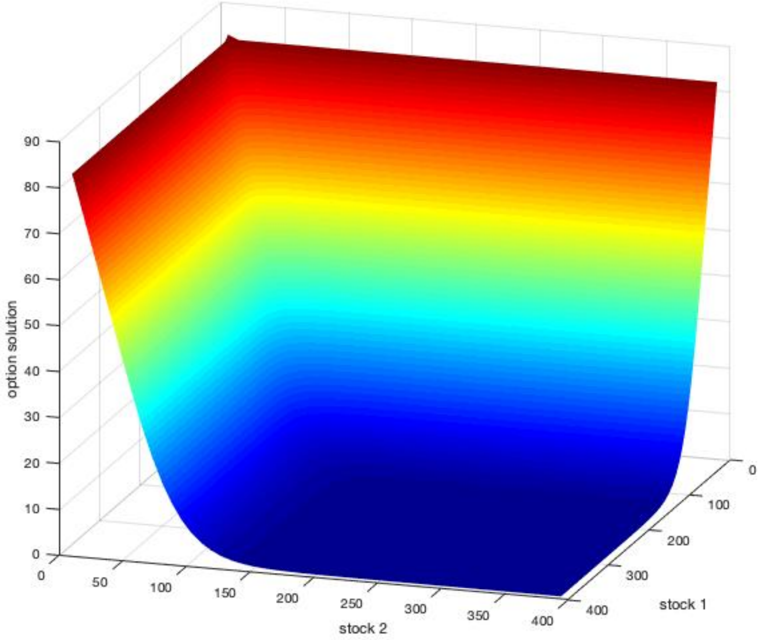

Figure 3.3: Mimetic Solution for the European option with parameters: $T = 1,\ r = 0.1,\ E = 1,\ X_{\max} = Y_{\max} = 4E,\ \sigma_1 = \sigma_2 = 0.2, \rho = 0.4, \alpha_1 = \alpha_2 = 0.5$

N	Fitted Finite Volume Error	Mimetic FDM Error	Fitted Mimetic FDM Error
10×10	0.1095	0.0061	0.0056
15×15	0.0733	0.0059	0.0055
30×30	0.0369	0.0057	0.0053
50×50	0.0222	0.0057	0.0053
75×75	0.0148	0.0056	0.0052
100×100	0.0111	0.0056	0.0052

[1] $T = 1,\ r = 0.1,\ E = 1\ X_{\max} = Y_{\max} = 4E,\ \sigma_1 = \sigma_2 = 0.2, \rho = 0.4, \alpha_1 = \alpha_2 = 0.5,\ \ \Delta t = T/100.$

Table 3.1: This table shows the two-dimensional L^2-relative error for the various spatial discretization methods for the European option.

From Table 3.1, we can observe the accuracy of the mimetic methods compared to the finite volume method. The table further shows the importance of the fitted scheme as the fitted mimetic scheme outperforms the standard mimetic scheme.

Now, the CPU timings (in seconds) for all three methods are given in the following table below:

N	Fitted Finite Volume CPU time (sec)	Mimetic FDM CPU time (sec)	Fitted Mimetic FDM CPU time (sec)
10×10	0.751	0.744	0.326
15×15	1.314	1.126	0.583
30×30	3.211	2.699	1.951
50×50	6.944	6.733	5.492
75×75	15.211	14.944	12.749
100×100	30.013	27.110	26.271

Table 3.2: This table shows the two-dimensional CPU time (sec) for the various spatial discretization methods with the following parameters: $T = 1,\ r = 0.1,\ E = 1,\ X_{\max} = Y_{\max} = 4E,\ \sigma_1 = \sigma_2 = 0.2, \rho = 0.4, \alpha_1 = \alpha_2 = 0.5,\ \text{and}\ \Delta t = T/100.$

As mentioned before, the CPU time of all the methods are very close as can be observed in Table 3.2. This is not surprising because the during the construction of the methods, we observed that the matrix representations have the same structure.

3.2.2 Test 2: Error for American put options

The Figures 3.4, 3.5 and 3.6 are the American put option solutions for the various methods with the following parameters: $NX = NY = 50,\ E = 100,\ r = 0.07,\ \sigma_1 = 0.3,\ \sigma_2 = 0.3\ T = 1,$ $X_{\max} = Y_{\max} = 4E,\ \Delta t = T/100$, and penalty parameters $\lambda = 100,\ k = 0.5$.

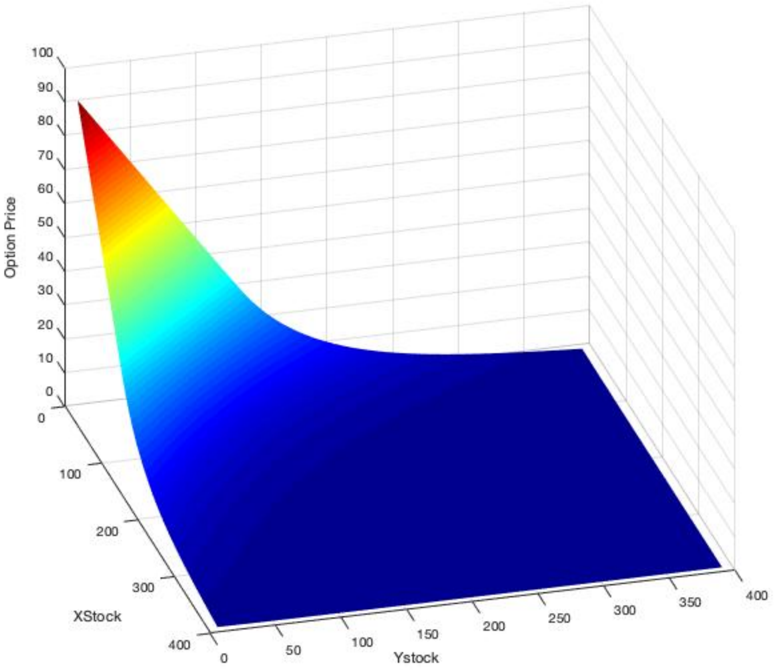

Figure 3.4: Fitted Finite Volume Method Solution for American put option

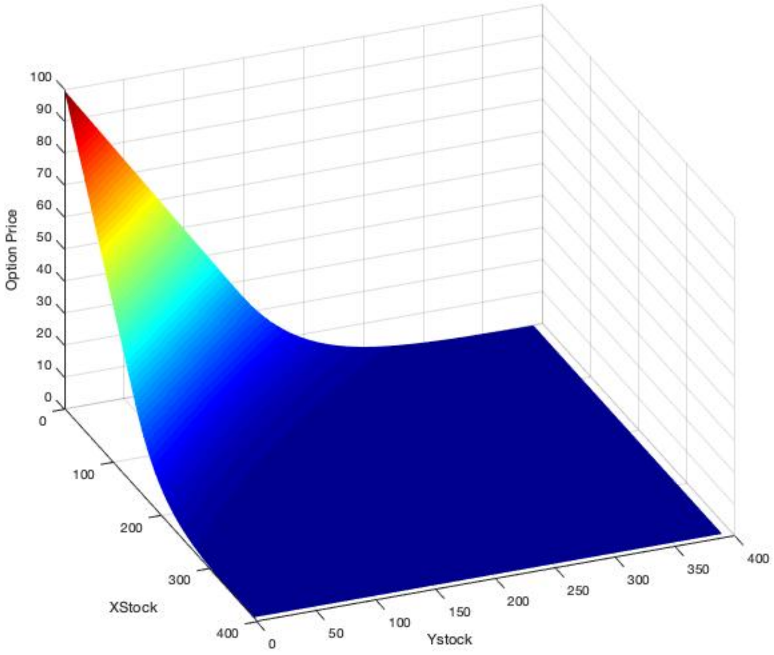

Figure 3.5: Mimetic Method Solution for American put option

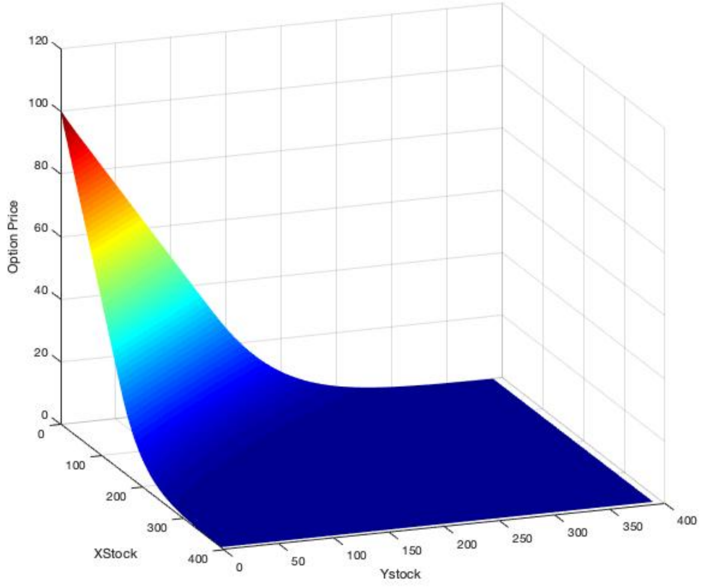

Figure 3.6: Fitted Mimetic Method Solution for American put option

Since American options in general have no analytical solution even when the Black-Scholes operator has constant parameters, we consider using a reference solution to compute the relative error. In our case, we choose the fitted mimetic scheme (see Figure 3.6 above) as the reference solution. The reason for this choice is the accuracy of the fitted scheme as seen in the previous test of European options and even for the one dimensional case in chapter 2. The fitted mimetic scheme presented more accurate results in comparison to the standard mimetic and the fitted finite volume methods. The fitted mimetic method therefore acts as an "exact solution", and the other methods are numerical solutions. The relative error is computed as shown in the table below:

N	Fitted Finite Volume Error	Mimetic FDM Error
10×10	0.1428	0.0241
20×20	0.1058	0.0052
30×30	0.0711	0.0033
50×50	0.0698	0.0022
75×75	0.0592	0.0018
85×85	0.0600	0.0017
100×100	0.0570	0.0016

[1] $T = 1,\ r = 0.1,\ E = 1\ X_{\max} = Y_{\max} = 4E,\ \sigma_1 = \sigma_2 = 0.2,\ \rho = 0.4,\ \alpha_1 = \alpha_2 = 0.5,\ tol = 10e-7, \epsilon = 10e-4,\ \Delta t = T/100$, and penalty parameters: $\lambda = 100,\ k = 2$.

Table 3.3: This table shows the two-dimensional L^2- relative error for the various spatial discretization methods for the American put option.

We also provide CPU timings (in seconds) for the methods given in Table 3.3 above

N	Fitted Finite Volume CPU time (sec)	Mimetic FDM CPU time (sec)
10×10	0.369	0.352
20×20	0.714	0.688
30×30	1.826	1.721
50×50	11.935	11.529
75×75	15.881	14.672
85×85	19.769	18.948
100×100	28.951	26.869

[1] $T = 1,\ r = 0.1,\ E = 1\ X_{\max} = Y_{\max} = 4E,\ \sigma_1 = \sigma_2 = 0.2,\ \rho = 0.4,\ \alpha_1 = \alpha_2 = 0.5,\ tol = 10e-7, \epsilon = 10e-4,\ \Delta t = T/100$, and penalty parameters: $\lambda = 100,\ k = 2$.

Table 3.4: This table shows the two-dimensional CPU time (in seconds) for two spatial discretization methods for the American put option.

It can be observed from Table 3.3 that the mimetic methods are more accurate when compared to the fitted finite volume method for pricing the American option problem. Furthermore, we observe in Table 3.4that the CPU time of the two methods are very close. As remarked earlier, this is not surprising because the during the construction of the methods, we observed that the matrix representations have the same structure.

Note that we used the Newton method with tolerance $tol = 10e - 7$, to solve the non-linear full discrete solutions in (3.1.70) and (3.1.71), with initial guess U_h^m. Remember that $[U^{*m} - U_h^m]_+^{1/k} = \max\left\{[U_h^{*m} - U_h^m]^{1/k}, 0\right\}$, that is for $\epsilon > 0$, we have

$$[U_h^{*m} - U_h^m]_+^{1/k} = \begin{cases} [U_h^{*m} - U_h^m]^{1/k}, & \text{if } U_h^{*m} - U_h^m \geq \epsilon \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.4}$$

3.2.3 Conclusion

In this chapter, we have considered the fitted mimetic finite difference method applied to the two dimensional degenerate Black-Scholes differential operator governing option pricing. We have presented the support operator method (mainly here as an extension of the one-dimensional case in chapter 2) which underlies the construction of the standard mimetic finite difference method. We have further applied the mimetic method to discretize the diagonal diffusion term of the option PDE in space. Indeed, to handle the degeneracy near the boundary at zero of the Black-Scholes differential operator, we have proposed the fitted scheme. The novel combined scheme, fitted mimetic finite difference method out-performed the standard mimetic and fitted finite volume methods when numerical experiments were conducted for European options. Furthermore, we have adopted the fitted mimetic method as a reference solution to compute the relative error for American put options. The choice was based on the accuracy of the fitted mimetic scheme for European options. We have observed that the proposed mimetic schemes (standard mimetic finite difference method and fitted mimetic finite difference method) were more accurate when compared to the fitted finite volume method proposed by [43]. As we had mentioned earlier, using the standard mimetic finite difference and the novel fitted mimetic finite difference methods for the option problem are the contributions we made in this chapter.

4. Mimetic and fitted mimetic finite difference methods applied to the full diffusion term of a two dimensional penalised option problem

In the previous chapters, we proposed accurate spatial discretization schemes (standard mimetic finite difference and novel fitted mimetic finite difference methods) to solve the degenerate option problem. In chapter 2, we developed the one-dimensional cases of the standard mimetic and novel fitted mimetic finite difference methods. Further in chapter 3, we extended our results to the two-dimensional option problem. Indeed, during the construction of the support operator method, we considered a diagonal matrix tensor of the flux term. This is generally considered in the mimetic applications in the literature [28, 29, 34, 40, 41, 45]. We showed in chapter 3 that the resulting standard mimetic finite difference and fitted mimetic schemes were more accurate when compared to the existing fitted finite volume method [12, 43]. In chapter 4, we will consider the full diffusion matrix tensor (that is a 2×2 matrix) in the design of the mimetic finite difference method. To fully mimic the important properties associated with the underlying flux term, we consider the full matrix coefficient [34, 45]. This distinguishes chapter 4 from chapter 3. Therefore proposing the standard mimetic finite difference and fitted mimetic finite difference methods for the full diffusion matrix tensor remain our contribution in this chapter.

We organise chapter 4 as follows, in section 4.1 we recall the formulation of the elliptic problem in chapter 3. We further proceed in section 4.1 to construct both the MFDM and FMFD and show the existence and uniqueness of all the resulting semi-discrete solutions. We furthermore present the flux consistency of these semi-discrete solutions. Then in section 4.2, we consider a full discretization of the scheme with the standard implicit stepping schemes. We perform some numerical experiments to show the accuracy of the proposed schemes.

4.1 Semi-discrete problem and mimetic method for elliptic problems

The construction of the mimetic finite difference method here remains the difference between chapter 3 and chapter 4. Our aim still remains to discretize the diffusion part of the continuous problem (3.1.1) with the corresponding discrete operators that mimic the properties (2.1.11) - (2.1.13) as reference from chapter 2.

Now, as in chapter 3, we define an operator $\mathbf{A} : H \to H$ such that

$$\mathbf{A}u = -\nabla \cdot (\mathbf{K}\nabla u(x, y)) = f, \quad (x, y) \in \Omega \subset \mathbb{R}^2 \tag{4.1.1}$$

with boundary condition

$$u(x,y) = 0, \;\; (x,y) \in \partial\Omega,$$

and properties

$$(\mathbf{A}u, v)_H = (u, \mathbf{A}v)_H, \quad (\mathbf{A}u, u)_H > 0, \;\; \mathbf{F} = f. \tag{4.1.2}$$

Clearly (4.1.1) becomes

$$\mathbf{A}u = \mathbf{F}. \tag{4.1.3}$$

which can be rewritten as the following first-order system

$$\begin{cases} \nabla \cdot \mathbf{w} = \mathbf{F} \\ \mathbf{w} = -\mathbf{K}\nabla u. \end{cases} \tag{4.1.4}$$

Note here that $\mathbf{K}$ here is a full (2×2) diffusion matrix and its components are given by K_{xx}, K_{xy}, K_{yx} and K_{yy}. The domain specifications are as given in section 3.1. However on the back of the analysis we wish to do here, we define $V_h := span\left\{\phi_{x_i,y_j}\right\}_{0\leq i\leq Nx+1, 0\leq i\leq Ny+1} \cap H^1_{0,\varpi}(\Omega)$ as the set of continuous piecewise with respect to the partition $(I_i \times I_j)_{0\leq i\leq Nx+1, 0\leq i\leq Ny+1}$ such that $v_h = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} v_h(x_i,y_j)\phi_{x_i,y_j}$. The interpolation operator I_h is defined by $I_h : C(\overline{\Omega}) \longrightarrow V_h$

$$\begin{aligned} I_h v_h(x_i,y_j) \;\; &:= \;\; v(x_i,y_j), \; i = 0,1,...,N_x+1, \; j = 0,1,...,N_y+1, \; \text{then} \\ I_h v = v_h \;\; &= \;\; \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} v_h(x_i,y_j)\phi_{x_i,y_j}. \end{aligned}$$

Furthermore, we define the following appropriate norms and semi-norms on V_h as

$$\|v_h\|_{0,h} := \sqrt{(v_h, v_h)} = \left(\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i} l_{y_j} v_{i,j}^2\right)^{1/2}, \tag{4.1.5}$$

for the discrete $L^2(\Omega)$ norm, and

$$\|v_h\|_{1,h_x} := \left(\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{y_j} \frac{\sigma_1^2 x_{i+\frac{1}{2}}^2}{2h_{x_i}}(v_{i+1,j} - v_{i,j})^2\right)^{1/2}, \;\; i = 0,1,...,N_x, \; j = 0,1,...,N_y, \tag{4.1.6}$$

$$\|v_h\|_{1,h_y}^2 := \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i} \frac{\sigma_2^2 y_{j+\frac{1}{2}}^2}{2h_{y_j}}(v_{i,j+1} - v_{i,j})^2, \;\; i = 0,1,...,N_x, \; j = 0,1,...,N_y, \tag{4.1.7}$$

for the weighted discrete H^1-semi-norm, and

$$\|v_h\|_{1,h}^2 = \|v_h\|_{1,h_x}^2 + \|v_h\|_{1,h_y}^2 + \|v_h\|_{0,h}^2, \tag{4.1.8}$$

for the weighted discrete $H^1_{0,\varpi}(\Omega)-$norm on V_h. Indeed it is easy to show that $\|\|_{1,h_x}$ and $\|\|_{1,h_y}$ are semi-norms in V_h since $l_{y_j}\dfrac{\sigma_1^2 x^2_{i+\frac{1}{2}}}{2h_{x_i}} > 0$ and $l_{x_i}\dfrac{\sigma_2^2 y^2_{j+\frac{1}{2}}}{2h_{y_j}} > 0$, respectively.

Discrete Inner products

Indeed, to define discrete analogs of the two continuous inner products (2.1.9) and (2.1.10) in two dimensions, we select the appropriate quadrature rule to approximate the integrals. Let HC and $\mathbf{HC}$ denote the discrete spaces of discrete scalar and vector functions for $H = L^2(\Omega)$ and $\mathbf{H} = (L^2(\Omega))^n$, $n = 2$ respectively, in the partition $\mathcal{T}$. Then the discrete L^2-norm defined in HC is given by

$$(U, V)_{HC} = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} l_{x_i} l_{y_j} U_{i,j} V_{i,j}. \tag{4.1.9}$$

Recall that the Cartesian components of the tensor $\mathbf{K}$ is given by K_{xx}, $K_{xy} = K_{yx}$, K_{yy}. Then, $(\mathbf{K}^{-1})_{xx}$, $(\mathbf{K}^{-1})_{xy} = (\mathbf{K}^{-1})_{yx}$, $(\mathbf{K}^{-1})_{yy}$ are the associated Cartesian components of the tensor $\mathbf{K}^{-1}$, the inverse of $\mathbf{K}$. Again, for any vectors, $\mathbf{W}, \mathbf{Z}$, we represent their components as $\mathbf{W} = (W_x, W_y)$ and $\mathbf{Z} = (Z_x, Z_y)$. Hence we have that,

$$\left(\mathsf{K}^{-1}\mathsf{W}, \mathsf{Z}\right) = (\mathbf{K}^{-1})_{xx} W_x Z_x + (\mathbf{K}^{-1})_{xy}(W_x Z_y + W_y Z_x) + (\mathbf{K}^{-1})_{yy} W_y Z_y. \tag{4.1.10}$$

Then using the midpoint rule, we define the discrete vector inner product of (2.1.10) as

$$\begin{aligned}(\mathbf{W}, \mathbf{Z})_{\mathbf{HC}} = {} & \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} h_{x_i} h_{y_j} \left[(\mathbf{K}^{-1}_{xx})_{i+\frac{1}{2},j} Z_{x_{i+\frac{1}{2},j}} W_{x_{i+\frac{1}{2},j}} + (\mathbf{K}^{-1}_{xy})_{i,j+\frac{1}{2}} Z_{y_{i,j+\frac{1}{2}}} W_{x_{i+\frac{1}{2},j}}\right] \\ & + \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} h_{x_i} h_{y_j} \left[(\mathbf{K}^{-1}_{xy})_{i+\frac{1}{2},j} Z_{x_{i+\frac{1}{2},j}} W_{y_{i,j+\frac{1}{2}}} + (\mathbf{K}^{-1}_{yy})_{i,j+\frac{1}{2}} Z_{y_{i,j+\frac{1}{2}}} W_{y_{i,j+\frac{1}{2}}}\right]\end{aligned} \tag{4.1.11}$$

4.1.1 The Discrete Divergence and Discrete Flux

The discrete divergence of $\mathbf{D}$ denoted by $\mathcal{D}$ is given by

$$(\mathcal{D}W)_{i,j} = (\mathcal{D}_1 W)_{i,j} + (\mathcal{D}_2 W)_{i,j} = \frac{W_{i+\frac{1}{2},j} - W_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{W_{i,j+\frac{1}{2}} - W_{i,j-\frac{1}{2}}}{l_{y_j}} \tag{4.1.12}$$

$i = 0, 1, ..., N_x + 1$ and $j = 0, 1, ..., N_y + 1$, and as we mentioned earlier in chapter 3, for W vanishing at the boundary of the domain,

$$(\mathcal{D}W, 1)_{HC} = 0. \tag{4.1.13}$$

Indeed, what follows distinguishes this chapter from chapter 3. For $n = 2$, $\mathbf{G} = -\mathbf{K}\nabla$, where $\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$.

Then it follows that the inverse of $\mathbf{K}$ is given by

$$K^{-1} = \frac{1}{k_{11}k_{22} - k_{12}k_{21}} \begin{bmatrix} k_{22} & -k_{12} \\ \\ -k_{21} & k_{11} \end{bmatrix} = \begin{bmatrix} k_1^{-1} & k_3^{-1} \\ \\ k_4^{-1} & k_2^{-1} \end{bmatrix}. \tag{4.1.14}$$

Now, we determine the discrete version of $\mathbf{G}$ denoted by $\mathcal{G}$ that mimics the continuous version properties we have already mentioned. In fact, $\mathcal{G}$ satisfies the following property

$$(\mathcal{D}W, U)_{HC} = (W, \mathcal{G}U)_{\mathbf{HC}}. \tag{4.1.15}$$

Expanding (4.1.15) we have that,

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} U_{i,j}(\mathcal{D}W)_{i,j} l_{x_i} l_{y_j} = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} \left(k^{-1}_{1_{i+\frac{1}{2},j}} (\mathcal{G}_1 U)_{i+\frac{1}{2},j} W_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}} (\mathcal{G}_2 U)_{i,j+\frac{1}{2}} W_{i+\frac{1}{2},j} \right) h_{x_i} h_{y_j}$$
$$+ \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} \left(k^{-1}_{4_{i,j+\frac{1}{2}}} (\mathcal{G}_1 U)_{i+\frac{1}{2},j} W_{i,j+\frac{1}{2}} + k^{-1}_{2_{i,j+\frac{1}{2}}} (\mathcal{G}_2 U)_{i,j+\frac{1}{2}} W_{i,j+\frac{1}{2}} \right) h_{x_i} h_{y_j}, \tag{4.1.16}$$

Then from (4.1.16) we have,

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} U_{i,j} \left[(\mathcal{D}_1 W)_{i,j} + (\mathcal{D}_2 W)_{i,j} \right] l_{x_i} l_{y_j}$$
$$= \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} \left(k^{-1}_{1_{i+\frac{1}{2},j}} (\mathcal{G}_1 U)_{i+\frac{1}{2},j} W_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}} (\mathcal{G}_2 U)_{i,j+\frac{1}{2}} W_{i+\frac{1}{2},j} \right) h_{x_i} h_{y_j}$$
$$+ \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} \left(k^{-1}_{4_{i,j+\frac{1}{2}}} (\mathcal{G}_1 U)_{i+\frac{1}{2},j} W_{i,j+\frac{1}{2}} + k^{-1}_{2_{i,j+\frac{1}{2}}} (\mathcal{G}_2 U)_{i,j+\frac{1}{2}} W_{i,j+\frac{1}{2}} \right) h_{x_i} h_{y_j}. \tag{4.1.17}$$

We address the first term of (4.1.17), that is

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} U_{i,j} \left[(\mathcal{D}_1 W)_{i,j} \right] l_{x_i} l_{y_j} = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} h_{x_i} h_{y_j} \left(k^{-1}_{1_{i+\frac{1}{2},j}} (\mathcal{G}_1 U)_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}} (\mathcal{G}_2 U)_{i,j+\frac{1}{2}} \right) W_{i+\frac{1}{2},j},$$

which becomes

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} U_{i,j} \left(\frac{W_{i+\frac{1}{2},j} - W_{i-\frac{1}{2},j}}{l_{x_i}} \right) l_{x_i} l_{y_j}$$
$$= \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} h_{x_i} h_{y_j} \left(k^{-1}_{1_{i+\frac{1}{2},j}} (\mathcal{G}_1 U)_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}} (\mathcal{G}_2 U)_{i,j+\frac{1}{2}} \right) W_{i+\frac{1}{2},j}. \tag{4.1.18}$$

Then grouping the terms of (4.1.18) yields

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1}\left[l_{y_j}U_{i,j} - h_{x_i}h_{y_j}\left(k^{-1}_{1_{i+\frac{1}{2},j}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)\right]W_{i+\frac{1}{2},j} - \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} l_{y_j}U_{i,j}W_{i-\frac{1}{2},j} = 0 \tag{4.1.19}$$

so that

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1}\left[l_{y_j}U_{i,j} - h_{x_i}h_{y_j}\left(k^{-1}_{1_{i+\frac{1}{2},j}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)\right]W_{i+\frac{1}{2},j} - \sum_{i=1}^{N_x+1}\sum_{j=0}^{N_y+1} l_{y_j}U_{i,j}W_{i-\frac{1}{2},j} - \sum_{j=0}^{N_y+1} l_{y_j}U_{0,j}W_{-\frac{1}{2},j} = 0. \tag{4.1.20}$$

Now re-indexing any terms in (4.1.20) with $i-\frac{1}{2}$ to $i+\frac{1}{2}$, and making use of the fact that $h_{N_x+1}=0$, we have

$$\sum_{i=0}^{N_x}\sum_{j=0}^{N_y+1}\left[-l_{y_j}(U_{i+1,j}-U_{i,j}) - h_{x_i}h_{y_j}\left(k^{-1}_{1_{i+\frac{1}{2},j}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)\right]W_{i+\frac{1}{2},j} - \sum_{j=0}^{N_y+1} l_{y_j}U_{0,j}W_{-\frac{1}{2},j} + \sum_{j=0}^{N_y+1} l_{y_j}U_{N_x+1,j}W_{N_x+\frac{3}{2},j} = 0. \tag{4.1.21}$$

This is done to fully concentrate the fluxes at the $i^{\text{th}}-$node, to enhance the mimicking property at each $i^{\text{th}}-$node. Furthermore, we note that (4.1.21) holds for all U in HC such that $U_{0,j} = U_{N_x+1,j} = 0$. Then (4.1.21) becomes

$$\left[-l_{y_j}(U_{i+1,j}-U_{i,j}) - h_{x_i}h_{y_j}\left(k^{-1}_{1_{i+\frac{1}{2},j}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)\right]W_{i+\frac{1}{2},j} = 0. \tag{4.1.22}$$

Furthermore from (4.1.15) we have that,

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} U_{i,j}\left[(\mathcal{D}_2W)_{i,j}\right]l_{x_i}l_{y_j} = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} h_{x_i}h_{y_j}\left(k^{-1}_{4_{i,j+\frac{1}{2}}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{2_{i,j+\frac{1}{2}}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)W_{i,j+\frac{1}{2}},$$

which leads to

$$\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} U_{i,j}\left(\frac{W_{i,j+\frac{1}{2}} - W_{i,j-\frac{1}{2}}}{l_{y_j}}\right)l_{x_i}l_{y_j} = \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} h_{x_i}h_{y_j}\left(k^{-1}_{4_{i,j+\frac{1}{2}}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{2_{i,j+\frac{1}{2}}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)W_{i,j+\frac{1}{2}}. \tag{4.1.23}$$

Now grouping the like terms from (4.1.23) yields,

$$\begin{aligned}\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1}\left[l_{x_i}U_{i,j} - h_{x_i}h_{y_j}\left(k^{-1}_{4_{i,j+\frac{1}{2}}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{2_{i,j+\frac{1}{2}}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)\right]W_{i,j+\frac{1}{2}}&\\ - \sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y+1} l_{x_i}U_{i,j}W_{i,j-\frac{1}{2}} = 0,&\end{aligned} \tag{4.1.24}$$

Then re-indexing any terms in (4.1.24) with $j-\frac{1}{2}$ to $j+\frac{1}{2}$, and using the fact that $h_{y_{N_{y+1}}} = 0$, we have

$$\begin{aligned}\sum_{i=0}^{N_x+1}\sum_{j=0}^{N_y}\left[-l_{x_i}(U_{i,j+1}-U_{i,j}) - h_{x_i}h_{y_j}\left(k^{-1}_{4_{i,j+\frac{1}{2}}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{2_{i,j+\frac{1}{2}}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)\right]W_{i,j+\frac{1}{2}}&\\ - \sum_{i=0}^{N_x+1} l_{x_i}U_{i,0}W_{i,-\frac{1}{2}} + \sum_{i=0}^{N_x+1} l_{x_i}U_{i,N_y+1}W_{i,N_y+\frac{3}{2}} = 0.&\end{aligned} \tag{4.1.25}$$

Note here that (4.1.25) holds for all U in HC such that $U_{i,0} = U_{i,N_{y+1}} = 0$. Then we have that (4.1.25) becomes

$$\left[-l_{x_i}(U_{i,j+1}-U_{i,j}) - h_{x_i}h_{y_j}\left(k^{-1}_{4_{i,j+\frac{1}{2}}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{2_{i,j+\frac{1}{2}}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right)\right]W_{i,j+\frac{1}{2}} = 0. \tag{4.1.26}$$

Then from (4.1.22) and (4.1.26), we have the following equations

$$-l_{x_i}(U_{i,j+1}-U_{i,j}) - h_{x_i}h_{y_j}\left(k^{-1}_{4_{i,j+\frac{1}{2}}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{2_{i,j+\frac{1}{2}}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right) = 0, \tag{4.1.27}$$

$$-l_{y_j}(U_{i+1,j}-U_{i,j}) - h_{x_i}h_{y_j}\left(k^{-1}_{1_{i+\frac{1}{2},j}}(\mathcal{G}_1U)_{i+\frac{1}{2},j} + k^{-1}_{3_{i+\frac{1}{2},j}}(\mathcal{G}_2U)_{i,j+\frac{1}{2}}\right) = 0. \tag{4.1.28}$$

Then solving for $(\mathcal{G}_1U)_{i+\frac{1}{2},j}$ and $(\mathcal{G}_2U)_{i,j+\frac{1}{2}}$ from (4.1.27) and (4.1.28), we obtain

$$\begin{aligned}(\mathcal{G}_1U)_{i+\frac{1}{2},j} = -\frac{l_{y_j}k_{1_{i+\frac{1}{2},j}}}{h_{x_i}h_{y_j}}\left(1 + \frac{k^{-1}_{3_{i+\frac{1}{2},j}}k_{1_{i+\frac{1}{2},j}}}{(k_{4_{i,j+\frac{1}{2}}}k^{-1}_{2_{i,j+\frac{1}{2}}} - k^{-1}_{3_{i+\frac{1}{2},j}}k_{1_{i+\frac{1}{2},j}})}\right)(U_{i+1,j}-U_{i,j})&\\ + \left(\frac{l_{x_i}k_{1_{i+\frac{1}{2},j}}k^{-1}_{3_{i+\frac{1}{2},j}}k_{4_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}(k_{4_{i,j+\frac{1}{2}}}k^{-1}_{2_{i,j+\frac{1}{2}}} - k^{-1}_{3_{i+\frac{1}{2},j}}k_{1_{i+\frac{1}{2},j}})}\right)(U_{i,j+1}-U_{i,j}),&\end{aligned} \tag{4.1.29}$$

and

$$\begin{aligned}(\mathcal{G}_2U)_{i.j+\frac{1}{2}} = &\left(\frac{l_{y_j}k_{1_{i+\frac{1}{2},j}}}{h_{x_i}h_{y_j}(k_{4_{i,j+\frac{1}{2}}}k^{-1}_{2_{i,j+\frac{1}{2}}} - k^{-1}_{3_{i+\frac{1}{2},j}}k_{1_{i+\frac{1}{2},j}})}\right)(U_{i+1,j} - U_{i,j}) \\ &- \left(\frac{l_{x_i}k_{4_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}(k_{4_{i,j+\frac{1}{2}}}k^{-1}_{2_{i,j+\frac{1}{2}}} - k^{-1}_{3_{i+\frac{1}{2},j}}k_{1_{i+\frac{1}{2},j}})}\right)(U_{i,j+1} - U_{i,j}).\end{aligned} \tag{4.1.30}$$

We let $\widehat{\mathcal{A}}_h$ denote the discrete diffusion operator obtained by forming the composition of the discrete divergence and gradient operator $\mathcal{D}\,\text{and}\,\mathcal{G}$ respectively. This by construction, $\mathcal{D} : \mathbf{HC} \to HC$ and $\mathcal{G} : HC \to \mathbf{HC}$ is given by $\widehat{\mathcal{A}}_h : HC \to HC$, with $\mathcal{A}_h = \mathcal{D}\mathcal{G}$.

Remark 4.1.2. As can be observed, the mimetic finite difference method depends on the discrete inner products in (4.1.9) and (4.1.11). Indeed, these discrete inner products are the approximations of the integrals of continuous functions on the grids. We made use of the rectangle rule here, but the trapezoidal rule or any other method could be used. Of course, the accuracy of the mimetic method depends on these discrete inner products.

4.1.3 Mimetic finite difference for penalised Black-Scholes PDE

The goal in this section is to discretise the two dimensional penalised Black Scholes PDE (4.1.31). The mimetic finite difference method will be used for the diffusion term, while the first order upwind-finite difference scheme for the convection terms. Then we have that

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \mathbf{D}w - \nabla\cdot[\mathbf{b}u] + cu + \lambda[u^* - u]_+^{1/k} = f(x,y,t) \\ \qquad\qquad \mathbf{G}u := \mathbf{w} = -\mathbf{K}\nabla u \\ \qquad\qquad \mathbf{A} = \mathbf{DG}, \end{cases} \tag{4.1.31}$$

Let us partition $I_i := (0, X)$ into $N_x + 1$ and $I_j := (0, Y)$ into $N_y + 1$ subintervals respectively, with dual partitions as we did for the elliptic problems. Then we have that,

$U_{i,j} \approx U(x_i, y_j, t),\ l_{x_i} = x_{i+1/2} - x_{i-1/2},\ l_{y_j} = y_{j+1/2} - y_{j-1/2},\ h_{x_i} = x_{i+1} - x_{i-1},\ h_{y_j} = y_{j+1} - y_j,$

for $i = 0, 1, ..., N_x$ and $j = 0, 1, ..., N_y$.

Let us set $w_{i+\frac{1}{2},j} := \dfrac{w_{i+1,j} - w_{i,j}}{2},\ w_{i-\frac{1}{2},j} := \dfrac{w_{i,j} - w_{i-1,j}}{2},\ w_{i,j+\frac{1}{2}} := \dfrac{w_{i,j+1} - w_{i,j}}{2},\ w_{i,j-\frac{1}{2}} := \dfrac{w_{i,j} - w_{i,j-1}}{2}$. We can easily see that $w_{i,j} \approx w(x_i, y_j, t)$, for $i = 0, 1, ..., N_x$ and $j = 0, 1, ..., N_y$.

That is the discrete mimetic operators (prime and derived) are given by

$$(\mathcal{D}w)_{i,j} = \frac{w_{i+\frac{1}{2},j} - w_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{w_{i,j+\frac{1}{2}} - w_{i,j-\frac{1}{2}}}{l_{y_j}},\ i = 0, ...N_x + 1,\ j = 0, ..., N_y + 1. \tag{4.1.32}$$

Now from (4.1.14), by substituting the values of K^{-1}, we have,

$$K^{-1} = \frac{1}{k_{11}k_{22} - k_{12}k_{21}} \begin{bmatrix} k_{22} & -k_{12} \\ -k_{21} & k_{11} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{\sigma_1^2 x^2 (1-\rho^2)} & \dfrac{2\rho}{\sigma_1\sigma_2 xy(\rho^2-1)} \\ \dfrac{2\rho}{\sigma_1\sigma_2 xy(\rho^2-1)} & \dfrac{2}{\sigma_2^2 y^2 (1-\rho^2)} \end{bmatrix}. \tag{4.1.33}$$

Then using the explicit formulas of (4.1.33), (4.1.29) and (4.1.30) can be rewritten as

$$(\mathcal{G}_1 U)_{i+\frac{1}{2},j} = -\frac{l_{y_j}\sigma_1^2 x^2_{i+\frac{1}{2}}}{2h_{x_i}h_{y_j}}(U_{i+1,j} - U_{i,j}) - \left(\frac{l_{x_i}\rho\sigma_1\sigma_2 x_{i+\frac{1}{2}} y_j}{2h_{x_i}h_{y_j}}\right)(U_{i,j+1} - U_{i,j}), \tag{4.1.34}$$

and

$$(\mathcal{G}_2 U)_{i,j+\frac{1}{2}} = -\frac{l_{x_i}\sigma_2^2 y^2_{j+\frac{1}{2}}}{2h_{x_i}h_{y_j}}(U_{i,j+1} - U_{i,j}) - \left(\frac{l_{y_j}\rho\sigma_1\sigma_2 x_i y_{j+\frac{1}{2}}}{2h_{x_i}h_{y_j}}\right)(U_{i+1,j} - U_{i,j}). \tag{4.1.35}$$

Now using the following ordering for the grid, ie. $z_{i,j} = (i-1)\times N_x + j, \text{ for } i = 1, .., N_x, \text{ and } j = 1, ..., N_y$ we have that

$$U_h = (U_{1,1}, U_{1,2}, ..., U_{1,N_y}, ..., U_{N_x,1}, U_{N_x,2}, ..., U_{N_x,N_y})^T. \tag{4.1.36}$$

Remember that $k_{11} = \dfrac{\sigma_1^2 x^2}{2}, k_{12} = k_{21} = \dfrac{\rho\sigma_1\sigma_2 xy}{2}$, and $k_{22} = \dfrac{\sigma_2^2 y^2}{2}$, then

$$(\mathcal{G}_1 U)_{i+\frac{1}{2},j} = -\frac{l_{y_j} k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}h_{y_j}}(U_{i+1,j} - U_{i,j}) - \left(\frac{l_{x_i} k_{12_{i+\frac{1}{2},j}}}{h_{x_i}h_{y_j}}\right)(U_{i,j+1} - U_{i,j}), \tag{4.1.37}$$

and

$$(\mathcal{G}_2 U)_{i,j+\frac{1}{2}} = -\frac{l_{x_i} k_{22_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}}(U_{i,j+1} - U_{i,j}) - \left(\frac{l_{y_j} k_{12_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}}\right)(U_{i+1,j} - U_{i,j}). \tag{4.1.38}$$

Then $\widehat{\mathcal{A}}_h$ then becomes

$$\widehat{\mathcal{A}}_h U_h[z_{i,j}] = (\mathcal{D}\mathcal{G})U_h[z_{i,j}] = \frac{(\mathcal{G}_1 U)_{i+\frac{1}{2},j} - (\mathcal{G}_1 U)_{i-\frac{1}{2},j}}{l_{x_i}} + \frac{(\mathcal{G}_2 U)_{i,j+\frac{1}{2}} - (\mathcal{G}_2 U)_{i,j-\frac{1}{2}}}{l_{y_j}} \tag{4.1.39}$$

$$\widehat{\mathcal{A}}_h U_h[z_{i,j}] = \frac{-\left[\dfrac{l_{y_j} k11_{i+\frac{1}{2},j}}{h_{y_j}}\right]\dfrac{U_{i+1,j}-U_{i,j}}{h_{x_i}} + \left[\dfrac{l_{y_j} k11_{i-\frac{1}{2},j}}{h_{y_j}}\right]\dfrac{U_{i,j}-U_{i-1,j}}{h_{x_{i-1}}}}{l_{x_i}}$$
$$+\frac{-\left[\dfrac{l_{x_i} k12_{i+\frac{1}{2},j}}{h_{x_i}}\right]\dfrac{U_{i,j+1}-U_{i,j}}{h_{y_j}} + \left[\dfrac{l_{x_{i-1}} k12_{i-\frac{1}{2},j}}{h_{x_{i-1}}}\right]\dfrac{U_{i-1,j+1}-U_{i-1,j}}{h_{y_{j-1}}}}{l_{x_i}}$$
$$+\frac{-\left[\dfrac{l_{y_j} k12_{i,j+\frac{1}{2}}}{h_{y_j}}\right]\dfrac{U_{i+1,j}-U_{i,j}}{h_{x_i}} + \left[\dfrac{l_{y_{j-1}} k12_{i,j-\frac{1}{2}}}{h_{y_{j-1}}}\right]\dfrac{U_{i+1,j-1}-U_{i,j-1}}{h_{x_{i-1}}}}{l_{y_j}}$$
$$+\frac{-\left[\dfrac{l_{x_i} k22_{i,j+\frac{1}{2}}}{h_{x_i}}\right]\dfrac{U_{i,j+1}-U_{i,j}}{h_{y_j}} + \left[\dfrac{l_{x_i} k22_{i,j-\frac{1}{2}}}{h_{x_i}}\right]\dfrac{U_{i,j}-U_{i,j-1}}{h_{y_{j-1}}}}{l_{y_j}}$$

or

$$\widehat{\mathcal{A}}_h U_h[z_{i,j}] = \alpha_{i,j}U_{i+1,j} + \beta_{i,j}U_{i,j+1} + \gamma_{i,j}U_{i,j} + \Gamma_{i,j}U_{i-1,j} + \varepsilon_{i,j}U_{i,j-1} + \eta_{i,j}U_{i-1,j+1} + \vartheta_{i,j}U_{i+1,j-1}, \tag{4.1.40}$$

where

$$\alpha_{i,j} = \left[\frac{-l_{y_j} k11_{i+\frac{1}{2},j}}{h_{x_i}h_{y_j}l_{x_i}} - \frac{k12_{i,j+\frac{1}{2}}}{h_{x_i}h_{y_j}}\right], \quad \beta_{i,j} = \left[\frac{-l_{x_i} k22_{i,j+\frac{1}{2}}}{h_{y_j}h_{x_i}l_{x_i}} - \frac{k12_{i+\frac{1}{2},j}}{h_{x_i}h_{y_j}}\right], \quad \eta_{i,j} = \left[-\frac{l_{y_{j-1}} k12_{i,j-\frac{1}{2}}}{l_{x_i}h_{x_i}h_{y_{j-1}}}\right]$$

$$\gamma_{i,j} = \left[\frac{l_{y_j} k11_{i+\frac{1}{2},j}}{h_{x_i}h_{y_j}l_{x_i}} + \frac{l_{y_j} k11_{i-\frac{1}{2},j}}{h_{x_{i-1}}h_{y_j}l_{x_i}} + \frac{l_{x_i} k22_{i,j+\frac{1}{2}}}{h_{y_j}h_{x_i}l_{y_j}} + \frac{l_{x_i} k22_{i,j-\frac{1}{2}}}{h_{y_{j-1}}h_{x_i}l_{y_j}} + \frac{k12_{i+\frac{1}{2},j}}{h_{x_i}h_{y_j}} + \frac{k12_{i,j+\frac{1}{2}}}{h_{x_i}h_{y_j}}\right]$$

$$\Gamma_{i,j} = \left[\frac{-l_{y_j} k11_{i-\frac{1}{2},j}}{h_{y_j}h_{x_{i-1}}l_{x_i}} - \frac{l_{x_{i-1}} k12_{i-\frac{1}{2},j}}{h_{x_{i-1}}h_{y_j}l_{x_i}}\right], \quad \varepsilon_{i,j} = \left[\frac{-l_{x_i} k22_{i,j-\frac{1}{2}}}{h_{y_{j-1}}h_{y_j}l_{y_j}} - \frac{l_{y_{j-1}} k12_{i,j-\frac{1}{2}}}{h_{x_i}h_{y_{j-1}}l_{y_j}}\right],$$
$$\vartheta_{i,j} = \left[-\frac{l_{x_{i-1}} k12_{i-\frac{1}{2},j}}{h_{x_{i-1}}h_{y_j}l_{y_j}}\right]. \tag{4.1.41}$$

Now applying the first order upwind finite difference to the convection terms of (4.1.31) in the sense of the finite volume method yields

$$\begin{aligned} -\nabla\cdot(\mathbf{b}u)+cu &= -b_1x\nabla_x u - b_2y\nabla_y u + ru \\ &\approx -b_1\left[\frac{x_{i+\frac{1}{2}}U_{i+1,j}-x_{i-\frac{1}{2}}U_{i,j}}{h_{x_i}}\right] - b_2\left[\frac{y_{j+\frac{1}{2}}U_{i,j+1}-y_{j-\frac{1}{2}}U_{i,j}}{h_{y_j}}\right] + rU_{i,j} \end{aligned}$$

where $\nabla_x = \dfrac{\partial}{\partial x}$, $\nabla_y = \dfrac{\partial}{\partial y}$.

Then

$$\widehat{\mathcal{B}}_h U_h[z_{i,j}] = \upsilon_{i,j}U_{i+1,j} + \chi_{i,j}U_{i,j+1} + \epsilon_{i,j}U_{i,j} \tag{4.1.42}$$

where

$$\upsilon_{i,j} = \frac{-b_1x_{i+1/2}}{h_{x_i}},\ \chi_{i,j} = \frac{-b_2y_{i+1/2}}{h_{y_i}},\ \epsilon_{i,j} = \frac{b_1x_{i-1/2}}{h_{x_i}} + \frac{b_2y_{i-1/2}}{h_{y_i}} + r. \tag{4.1.43}$$

Now from (4.1.40) and (4.1.42), we have that

$$\begin{aligned} \widehat{\mathcal{C}}_hU_h[z_{i,j}] = \widehat{\mathcal{A}}_hU_h[z_{i,j}] + \widehat{\mathcal{B}}_hU_h[z_{i,j}] = (\alpha_{i,j}+\upsilon_{i,j})U_{i+1,j} + (\beta_{i,j}+\chi_{i,j})U_{i,j+1} \\ +(\gamma_{i,j}+\epsilon_{i,j})U_{i,j} + \Gamma_{i,j}U_{i-1,j} + \varepsilon_{i,j}U_{i,j-1} + \eta_{i,j}U_{i-1,j+1} + \vartheta_{i,j}U_{i+1,j-1} \end{aligned} \tag{4.1.44}$$

for all $i = 1, 2, ..., N_x$ and $j = 1, 2, ..., N_y$.

Then using the transformation $t = T - t$, we have

$$\begin{cases} \dfrac{dU_h}{dt} + \widehat{\mathcal{C}}_hU_h + \lambda[U_h^* - U_h]_+^{1/k} = f_h(t), \quad \forall t\in[0,T], \\ U_h(0) = U_h^* \end{cases} \tag{4.1.45}$$

4.1.4 Fitted mimetic finite difference scheme

In this section near $x, y = 0$, the sum of the diffusion and convection flux is approximated using the fitted finite volume scheme. Far from $x, y = 0$ $(i > 0, j > 1)$ however, the diffusion flux and convection flux will be approximated as in the previous section using respectively, the standard mimetic finite difference and the upwind finite difference. The fitted scheme here follows similarly to what we presented in chapter 3. We give a summary of the results here.

Case I $(i, j = 1)$

Remember that $\mathcal{R}_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$. Now integrating (4.1.31) across $\mathcal{R}_{1,1} = [x_{\frac{1}{2}}, x_{\frac{3}{2}}] \times [y_{\frac{1}{2}}, y_{\frac{3}{2}}]$ we have

$$
\begin{aligned}
-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}}\frac{\partial u}{\partial t}dxdy - \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_x\cdot\left(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u\right)dxdy \\
-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_y\cdot\left(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u\right)dxdy \\
+\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\Big[cu + \lambda[u^* - u]_+^{1/k} - f(t)\Big]dxdy = 0. \qquad (4.1.46)
\end{aligned}
$$

and using the midpoint rule, to approximate the first and last terms of (4.1.46), we obtain

$$
\begin{aligned}
-R_{1,1}\frac{dU_{1,1}}{dt} - \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_x\cdot\left(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u\right)dxdy \\
-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}} &\nabla_y\cdot\left(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u\right)dxdy \\
+R_{1,1}&\Big[cU_{1,1} + \lambda\left[U_{1,1}^* - U_{1,1}\right]_+^{1/k} - f_{1,1}(t)\Big] = 0, \qquad (4.1.47)
\end{aligned}
$$

where $R_{1,1} = l_{x_1}l_{y_1}$ is the area of a control volume around the point (x_1, y_1), with $U_{x_1,y_1} = U_{1,1}$, $U_{x_1,y_1}^* = U_{1,1}^*$ and $f_{x_1,y_1} = f_{1,1}$. Let us define the following

$$
\Phi(u) := \frac{1}{2}\sigma_1^2 x\nabla_x u + (r - \sigma_1^2 - \rho\sigma_1\sigma_2)u = \mathsf{a}_1 x\nabla_x u + b_1 u \qquad (4.1.48)
$$

and

$$
\Psi(u) := \frac{1}{2}\sigma_2^2 y\nabla_y u + (r - \sigma_2^2 - \rho\sigma_1\sigma_2)u = \mathsf{a}_2 y\nabla_y u + b_2 u, \qquad (4.1.49)
$$

where $\mathsf{a}_1 = \frac{1}{2}\sigma_1^2$, $\mathsf{a}_2 = \frac{1}{2}\sigma_2^2$, and b_1, b_2 are as already defined. Then the second term of (4.1.47) can be approximated by

$$
\begin{aligned}
-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}}\nabla_x\cdot\left(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u\right)dxdy &= -\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}}\nabla_x\cdot\left(x\Phi(u) + k_{12}\nabla_y u\right)dxdy \\
&\approx l_{y_1}\left[x\Phi(u) + k_{12}\nabla_y u\right]\Big|_{(x_{\frac{1}{2}},y_1)}^{(x_{\frac{3}{2}},y_1)}, \qquad (4.1.50)
\end{aligned}
$$

and the third term of (4.1.47) can be approximated by

$$
\begin{aligned}
-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}}\nabla_y\cdot\left(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u\right)dxdy &= -\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}}\nabla_y\cdot\left(y\Psi(u) + k_{21}\nabla_x u\right)dxdy \\
&\approx l_{x_1}\left[y\Psi(u) + k_{21}\nabla_x u\right]\Big|_{(x_1,y_{\frac{1}{2}})}^{(x_1,y_{\frac{3}{2}})}. \qquad (4.1.51)
\end{aligned}
$$

Recall that

$$[x\Phi(u)]_{(x_{\frac{1}{2}},y_1)}^{(x_{\frac{3}{2}},y_1)} = x_{3/2}\Phi_{\frac{3}{2},1}(u) - x_{1/2}\Phi_{\frac{1}{2},1}(u). \tag{4.1.52}$$

For convenience in the formulation, we introduce from (4.1.34) and (4.1.35), the following

$$(\mathcal{G}_1U)_{i+\frac{1}{2},j} = (\mathcal{G}_{11}U)_{i+\frac{1}{2},j} + (\mathcal{G}_{12}U)_{i+\frac{1}{2},j}, \tag{4.1.53}$$

where

$$\begin{aligned}(\mathcal{G}_{11}U)_{i+\frac{1}{2},j} &= -\frac{l_{y_j}k_{11_{i+\frac{1}{2},j}}}{h_{x_i}h_{y_j}}(U_{i+1,j}-U_{i,j}),\\ (\mathcal{G}_{12}U)_{i+\frac{1}{2},j} &= -\left(\frac{l_{x_i}k_{12_{i+\frac{1}{2},j}}}{h_{x_i}h_{y_j}}\right)(U_{i,j+1}-U_{i,j}),\end{aligned} \tag{4.1.54}$$

and

$$(\mathcal{G}_2U)_{i,j+\frac{1}{2}} = (\mathcal{G}_{21}U)_{i,j+\frac{1}{2}} + (\mathcal{G}_{22}U)_{i,j+\frac{1}{2}}, \tag{4.1.55}$$

where

$$\begin{aligned}(\mathcal{G}_{21}U)_{i,j+\frac{1}{2}} &= -\left(\frac{l_{y_j}k_{12_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}}\right)(U_{i+1,j}-U_{i,j}),\\ (\mathcal{G}_{22}U)_{i,j+\frac{1}{2}} &= -\frac{l_{x_i}k_{22_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}}(U_{i,j+1}-U_{i,j}).\end{aligned} \tag{4.1.56}$$

Now,

$$[x\Phi(u)]_{(x_{\frac{1}{2}},y_1)}^{(x_{\frac{3}{2}},y_1)} = x_{3/2}\Phi_{\frac{3}{2},1}(u) - x_{1/2}\Phi_{\frac{1}{2},1}(u), \tag{4.1.57}$$

and $x_{3/2}\Phi(u)|_{x_{3/2},y_1}$ can be approximated as

$$\begin{aligned}x_{3/2}\Phi(u)|_{x_{3/2},y_1} &\approx (-\mathcal{G}_{11}U)_{3/2,1} + b_1x_{3/2}U_{1,1}\\ &= \left[\frac{l_{y_1}k_{11_{\frac{3}{2},1}}}{h_{x_1}h_{y_1}}\right]U_{2,1} + \left[b_1x_{3/2} - \frac{l_{y_1}k_{11_{\frac{3}{2},1}}}{h_{x_1}h_{y_1}}\right]U_{1,1}.\end{aligned}$$

To approximate $x_{1/2}\Phi(u)|_{x_{1/2},y_1}$ using the fitted technique, we follow line by line in chapter 3 and have

$$(\Phi(v))|_{x_{1/2},y_1} = (\mathsf{a}_1x\nabla_xv + b_1v)_{x_{1/2},1} = \frac{1}{2}[(\mathsf{a}_1+b_1)U_{1,1} - (\mathsf{a}_1-b_1)U_{0,1}]. \tag{4.1.58}$$

Then from (4.1.57) and (4.1.58), we have that

$$[x\Phi(u)]_{(x_{\frac{1}{2}},y_1)}^{(x_{\frac{3}{2}},y_1)} \approx \left[\frac{l_{y_1}k_{11_{\frac{3}{2},1}}}{h_{x_1}h_{y_1}}\right]U_{2,1} + \left[b_1x_{3/2} - \frac{l_{y_1}k_{11_{\frac{3}{2},1}}}{h_{x_1}h_{y_1}}\right]U_{1,1} - \frac{x_{1/2}}{2}[(\mathsf{a}_1+b_1)U_{1,1} - (\mathsf{a}_1-b_1)U_{0,1}]. \tag{4.1.59}$$

A similar argument as before establishes that the third expression of (4.1.47), can be approximated as

$$[y\Psi(u)]_{(x_1,y_{\frac{1}{2}})}^{(x_1,y_{\frac{3}{2}})} \approx \left[\frac{l_{x_1}k22_{1,\frac{3}{2}}}{h_{y_1}h_{x_1}}\right]U_{1,2}+\left[b_2y_{3/2}-\frac{l_{x_1}k22_{1,\frac{3}{2}}}{h_{y_1}h_{x_1}}\right]U_{1,1}-\frac{y_{1/2}}{2}[(\mathsf{a}_2+b_2)U_{1,1}-(\mathsf{a}_2-b_2)U_{1,0}]. \tag{4.1.60}$$

Remember that from (4.1.47), we have

$$\begin{gathered}-\nabla_x\cdot(\Phi(u)+k_{12}\nabla_y u)-\nabla_y\cdot(\Psi(u)+k_{21}\nabla_x u)+cu= \\ -\mathbf{D}\left[(\Phi(u)u+k_{12}\nabla_y u)+(\Psi(u)+k_{21}\nabla_x u)\right]+cu.\end{gathered} \tag{4.1.61}$$

Then considering by definition that $\mathbf{D}$ is approximated by $\mathcal{D}$, we obtain

$$\begin{gathered}[-\nabla_x\cdot(\Phi(u)+k_{12}\nabla_y u)-\nabla_y\cdot(\Psi(u)+k_{21}\nabla_x u)+cu]\,|x_1,y_1\approx \\ \frac{-\left(x_{3/2}\Phi_{\frac{3}{2},1}(u)-\frac{x_{\frac{1}{2}}}{2}[(\mathsf{a}_1+b_1)U_{1,1}-(\mathsf{a}_1-b_1)U_{0,1}]\right)}{l_{x_1}}+\frac{(\mathcal{G}_{12}U)_{\frac{3}{2},1}-(\mathcal{G}_{12}U)_{\frac{1}{2},1}}{l_{x_1}} \\ -\frac{\left(y_{3/2}\Psi_{1,\frac{3}{2}}(u)-\frac{y_{\frac{1}{2}}}{2}[(\mathsf{a}_2+b_2)U_{1,1}-(\mathsf{a}_2-b_2)U_{1,0}]\right)}{l_{y_1}}+\frac{(\mathcal{G}_{21}U)_{1,\frac{3}{2}}-(\mathcal{G}_{21}U)_{1,\frac{1}{2}}}{l_{y_1}}+cU_{1,1}.\end{gathered} \tag{4.1.62}$$

Then from (4.1.62), and using $z_{i,j}=(i-1)\times N_x+j$,

for $U_H=(U_{1,1},U_{1,2},...,U_{1,N_y},...,U_{N_x,1},U_{N_x,2},...,U_{N_x,N_y})$, we have that

$$\begin{gathered}C_HU_H[z_{1,1}]=-\left[\frac{l_{y_1}k11_{\frac{3}{2},1}}{h_{x_1}h_{y_1}l_{x_1}}-\frac{k12_{1,\frac{3}{2}}}{h_{x_1}h_{y_1}}\right]U_{2,1}-\left[\frac{l_{x_1}k22_{1,\frac{3}{2}}}{h_{y_1}h_{x_1}l_{y_1}}-\frac{k12_{\frac{3}{2},1}}{h_{x_1}h_{y_1}}\right]U_{1,2} \\ +\left[\frac{l_{y_1}k11_{\frac{3}{2},1}}{h_{x_1}h_{y_1}l_{x_1}}+\frac{l_{x_1}k22_{1,\frac{3}{2}}}{h_{y_1}h_{x_1}l_{y_1}}+\frac{x_{\frac{1}{2}}}{2l_{x_1}}(\mathsf{a}_1+b_1)+\frac{y_{\frac{1}{2}}}{2l_{y_1}}(\mathsf{a}_2+b_2)+\frac{k12_{1,\frac{1}{2}}}{h_{x_1}h_{y_1}}+\frac{k12_{\frac{1}{2},1}}{h_{x_1}h_{y_1}}+c\right]U_{1,1}.\end{gathered} \tag{4.1.63}$$

Case II $(i=1,j>1)$

This case also follows from the work presented in chapter 3. Integrating (4.1.31) across $\mathcal{R}_{1,j}=[x_{\frac{1}{2}},x_{\frac{3}{2}}]\times[y_{j-\frac{1}{2}},y_{j+\frac{1}{2}}]$ we have

$$\begin{aligned} -\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\frac{\partial u}{\partial t}dxdy &- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\nabla_x\cdot\left(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u\right)dxdy \\ &- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\nabla_y\cdot\left(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u\right)dxdy \\ &+ \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\Big[cu + \lambda[u^* - u]_+^{1/k} - f(t)\Big]dxdy = 0. \end{aligned} \tag{4.1.64}$$

Using the midpoint rule to approximate first and final terms, we obtain

$$\begin{aligned} -R_{1,j}\frac{dU_{1,j}}{dt} &- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\nabla_x\cdot\left(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u\right)dxdy \\ &- \int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\nabla_y\cdot\left(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u\right)dxdy \\ &+ R_{1,j}[cU_{1,j} + \lambda[U^*_{1,j} - U_{1,j}]_+^{1/k} - f_{1,j}(t)] = 0, \end{aligned} \tag{4.1.65}$$

where $R_{1,j} = l_{x_1}l_{y_j}$ is the area of a control volume around the point $(x_1, y_j),\ j > 1$.

We approximate the second and third terms of (4.1.65) by

$$-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\nabla_x\cdot\left(k_{11}\nabla_x u + k_{12}\nabla_y u + b_1 x u\right)dxdy \approx l_{y_1}\Big[x\Phi\left(u\right) + k_{12}\nabla_y u\Big]\Big|_{(x_{\frac{1}{2}},y_j)}^{(x_{\frac{3}{2}},y_j)}, \tag{4.1.66}$$

and

$$-\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\int_{y_{\frac{1}{2}}}^{y_{\frac{3}{2}}}\nabla_y\cdot\left(k_{21}\nabla_x u + k_{22}\nabla_y u + b_2 y u\right)dxdy \approx l_{x_1}\Big[y\Psi(u) + k_{21}\nabla_x u\Big]\Big|_{(x_1,y_{j-\frac{1}{2}})}^{(x_1,y_{j+\frac{1}{2}})}. \tag{4.1.67}$$

Note that our focus here will be to apply the fitted scheme to (4.1.66). The standard mimetic and the first order upwind will be used to approximate the terms of (4.1.67).

Now from (4.1.66), we have

$$\Big[x\Phi(u)\Big]_{(x_{\frac{1}{2}},y_j)}^{(x_{\frac{3}{2}},y_j)} = x_{3/2}\Phi_{\frac{3}{2},j}(u) - x_{1/2}\Phi_{\frac{1}{2},j}(u). \tag{4.1.68}$$

Then $x_{3/2}\Phi(u)|_{x_{3/2},y_j}$ can be approximated as

$$\begin{aligned} x_{3/2}\Phi(u)|_{x_{3/2},y_1} &\approx (-\mathcal{G}_{11}U)_{3/2,j} + b_1 x_{3/2}U_{1,j} \\ &= \left[\frac{l_{y_j}k_{11_{\frac{3}{2},j}}}{h_{y_j}h_{x_1}}\right]U_{2,j} + \left[b_1 x_{3/2} - \frac{l_{y_j}k_{11_{\frac{3}{2},j}}}{h_{y_j}h_{x_1}}\right]U_{1,j}. \end{aligned}$$

Then $x_{1/2}\Phi(u)|_{x_{1/2},y_j}$ is approximated using the fitted technique to give

$$(\Phi(v))|_{x_{1/2},y_j} = (\mathsf{a}_1 x\nabla_x v + b_1 v)_{x_{1/2},j} = \frac{1}{2}[(\mathsf{a}_1 + b_1)U_{1,j} - (\mathsf{a}_1 - b_1)U_{0,j}]. \tag{4.1.69}$$

Then following that the divergent operator $\mathbf{D}$ is approximated by $\mathcal{D}$, we can approximate the flux in (4.1.65)

$$-\left[\mathbf{D}\left[\Phi(u) + k_{12}\nabla_y u\right] - \mathbf{D}\left[\Psi(u) + k_{21}\nabla_x u\right] + cu\right]\Big|_{x_1,y_j} \approx$$

$$-\frac{\left(\left[\frac{l_{x_1}k22_{1,j+\frac{1}{2}}}{h_{x_1}}\right]\frac{U_{1,j+1}-U_{1,j}}{h_{y_j}} - \left[\frac{l_{x_1}k22_{1,j-\frac{1}{2}}}{h_{x_1}}\right]\frac{U_{1,j}-U_{1,j-1}}{h_{y_{j-1}}}\right)}{l_{y_j}} + cU_{1,j} - \frac{1}{l_{y_j}}\left(b_2 y_{j+\frac{1}{2}}U_{1,j+1}\right)$$
$$+\frac{1}{l_{y_j}}\left(b_2 y_{j-\frac{1}{2}}U_{1,j}\right) - \left(\left[\frac{k12_{1,j+\frac{1}{2}}}{h_{y_j}}\right]\left[\frac{U_{2,j}-U_{1,j}}{h_{x_1}}\right] - \left[\frac{l_{y_{j-1}}k12_{1,j-\frac{1}{2}}}{h_{y_{j-1}}l_{y_j}}\right]\left[\frac{U_{2,j-1}-U_{1,j-1}}{h_{x_1}}\right]\right). \tag{4.1.70}$$

That is from (4.1.70), and for $z_{i,j} = (i-1)\times N_x + j$, we have

$$C_H U_H[z_{1,j}] = \left[-\frac{l_{y_j}k11_{\frac{3}{2},j}}{h_{y_j}h_{x_1}l_{x_1}} - \frac{k12_{1,j+\frac{1}{2}}}{h_{x_1}h_{y_j}}\right]U_{2,j} - \left[\frac{l_{y_{j-1}}k12_{1,j-\frac{1}{2}}}{h_{x_1}h_{y_{j-1}}l_{y_j}}\right]U_{2,j-1} + \left[\frac{k12_{\frac{3}{2},j}}{h_{x_1}h_{y_j}}\right]U_{1,j}$$
$$+\left[-\frac{l_{x_1}k22_{1,j+\frac{1}{2}}}{h_{y_j}h_{x_1}l_{y_j}} - \frac{k12_{1,j+\frac{1}{2}}}{h_{x_1}h_{y_j}} - \frac{b_2 y_{j+\frac{1}{2}}}{2l_{y_j}}\right]U_{1,j+1} + \left[\frac{l_{x_1}k22_{1,j-\frac{1}{2}}}{h_{x_1}l_{y_j}h_{y_{j-1}}} - \frac{l_{y_{j-1}}k12_{\frac{1}{2},j}}{h_{x_1}h_{y_{j-1}}l_{y_1}}\right]U_{1,j-1}$$
$$+\left[\frac{l_{y_j}k11_{\frac{3}{2},j}}{h_{x_1}l_{x_1}h_{y_j}} + \frac{l_{x_1}k22_{1,j+\frac{1}{2}}}{h_{y_j}h_{x_1}l_{y_j}} + \frac{l_{x_1}k22_{1,j-\frac{1}{2}}}{l_{y_j}h_{y_{j-1}}h_{x_1}} + \frac{x_{\frac{1}{2}}}{2l_{x_1}}(\mathsf{a}_1 + b_1) + \frac{b_2 y_{j-\frac{1}{2}}}{l_{y_j}} + \frac{l_{y_{j-1}}k12_{1,j-\frac{1}{2}}}{h_{x_1}h_{y_{j-1}}l_{y_j}} + c\right]U_{1,j}. \tag{4.1.71}$$

Case III $(i > 1, j = 1)$

Indeed, this case follows the previous cases above. Therefore we have that

$$C_H U_H[z_{i,1}] = \left[-\frac{l_{x_i}k22_{i,\frac{3}{2}}}{h_{y_1}h_{x_i}l_{y_1}} - \frac{k12_{i+\frac{1}{2},j}}{h_{y_1}h_{x_1}}\right]U_{i,2} - \left[\frac{l_{x_{i-1}}k12_{i-\frac{1}{2},j}}{h_{y_1}h_{x_{i-1}}l_{x_i}}\right]U_{i-1,2} + \left[\frac{k12_{i,\frac{3}{2}}}{h_{y_1}h_{x_i}}\right]U_{i,1}$$
$$+\left[-\frac{l_{y_1}k11_{i+\frac{1}{2},j}}{h_{x_i}h_{y_1}l_{x_i}} - \frac{k12_{i+\frac{1}{2},j}}{h_{y_1}h_{x_i}} - \frac{b_1 x_{i+\frac{1}{2}}}{2l_{x_i}}\right]U_{i+1,1} + \left[\frac{l_{y_1}k11_{i-\frac{1}{2},1}}{h_{y_1}l_{x_i}h_{x_{i-1}}} - \frac{l_{x_{i-1}}k12_{i,\frac{1}{2}}}{h_{y_1}h_{x_{i-1}}l_{x_i}}\right]U_{i-1,j}+$$
$$\left[\frac{l_{x_i}k22_{i,\frac{3}{2}}}{h_{y_1}l_{y_1}h_{x_i}} + \frac{l_{y_1}k11_{i+\frac{1}{2},1}}{h_{x_i}h_{y_1}l_{x_i}} + \frac{l_{y_1}k11_{i-\frac{1}{2},j}}{l_{x_i}h_{x_{i-1}}h_{y_1}} + \frac{y_{\frac{1}{2}}}{2l_{y_1}}(\mathsf{a}_2 + b_2) + \frac{b_1 x_{i-\frac{1}{2}}}{l_{x_i}} + \frac{l_{x_{i-1}}k12_{i-\frac{1}{2},1}}{h_{y_1}h_{x_{i-1}}l_{x_i}} + c\right]U_{i,1}. \tag{4.1.72}$$

Now combining the mimetic approximation of (4.1.45) at $(x_i, y_j), i > 1, j > 1$, with (4.1.63), (4.1.71) and (4.1.72), yields our novel scheme called the fitted mimetic finite difference method. Therefore for $U_H = (U_{1,1}, U_{1,2}, ..., U_{1,N_y}, ..., U_{N_x,1}, U_{N_x,2}, ..., U_{N_x,N_y})$, and considering the transformation $t = T - t$, we need to solve in the case of the fitted mimetic method the following system

$$\begin{cases} \dfrac{dU_H}{dt} + \mathcal{C}_H U_H + F(U_H, t) = 0, \quad t \in [0, T], \\ \\ U_H(0) = U_H^* \\ \\ F(U_H, t) = \lambda\, [U_H^* - U_H]_+^{1/k} - f_H(t). \end{cases} \tag{4.1.73}$$

where with (4.1.45), and for $z_{i,j} = (i-1) \times N_x + j$, we have

$$\begin{cases} C_H U_H[z_{1,1}], \quad \text{is as given in } (4.1.63) \\ C_H U_H[z_{1,j}], j > 1, \ \text{is as given in } (4.1.71) \\ C_H U_H[z_{i,1}], \ i > 1 \text{ is as given in } (4.1.72) \\ C_H U_H[z_{i,j}] = \widehat{\mathcal{C}}_h U_h[z_{i,j}], \ i > 1, j > 1 \end{cases} \tag{4.1.74}$$

4.2 Variational form of mimetic method and existence of discrete solution

The goal in this section is to establish the unique solvability of both standard mimetic method and the fitted mimetic method in the discrete operators.

4.2.1 Coercivity of mimetic finite difference method

Let us establish the unique solvability of the mimetic finite difference method in the discrete operators. Here we will consider the case when $\lambda = 0$. It is important to note here that the coercivity follows from the one dimensional case as presented in Chapter 2. We extend the proof here to the two-dimensional problem. From (4.1.31) using the mimetic approximation yields

$$\begin{cases} -\dfrac{dU_{i,j}}{dt} + (\mathcal{D}\,[\Phi_h U_h + \Upsilon_h U_h])_{i,j} + (\mathcal{D}\,[\Psi_h U_h + \Lambda_h U_h])_{i,j} + cU_{i,j} = f_{i,j}(t) \\ u(x_{i,y_j}) \approx U_{i,j}, \end{cases} \tag{4.2.1}$$

where

$$\Phi_h(U_h, t)|_{x_{i+\frac{1}{2}}, y_j} \approx -(\mathcal{G}_{11}U)_{i+\frac{1}{2},j} + b_1 x_{i+\frac{1}{2}} U_{i+1,j} = \left[\frac{l_{y_j} k_{11_{i+\frac{1}{2},j}}}{h_{x_i} h_{y_j}}\right](U_{i+1,j} - U_{i,j}) + b_1 x_{i+\frac{1}{2}} U_{i+1,j}, \tag{4.2.2}$$

$$\Phi_h(U_h,t)|_{x_{i-\frac{1}{2}},y_j} \approx (\mathcal{G}_{11}U)_{i-\frac{1}{2},j}+b_1x_{i-\frac{1}{2}}U_{i,j} = \left[\frac{l_{y_j}k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}h_{y_j}}\right](U_{i,j}-U_{i-1,j})+b_1x_{i-\frac{1}{2}}U_{i,j}, \quad (4.2.3)$$

$$\begin{aligned}\Psi_h(U_h,t)|_{x_i,y_{j+\frac{1}{2}}} &\approx -(\mathcal{G}_{22}U)_{i,j+\frac{1}{2}}+b_2y_{j+\frac{1}{2}}U_{i,j+1} = \left[\frac{l_{x_i}k_{22_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}}\right](U_{i,j+1}-U_{i,j})+b_2y_{j+\frac{1}{2}}U_{i,j+1},\\ \Psi_h(U_h,t)|_{x_i,y_{j-\frac{1}{2}}} &\approx -(\mathcal{G}_{22}U)_{i,j-\frac{1}{2}}+b_2y_{j-\frac{1}{2}}U_{i,j} = \left[\frac{l_{x_i}k_{22_{i,j-\frac{1}{2}}}}{h_{x_i}h_{y_{j-1}}}\right](U_{i,j}-U_{i,j-1})+b_2y_{j-\frac{1}{2}}U_{i,j},\end{aligned} \quad (4.2.4)$$

$$\begin{aligned}\Upsilon_h(U_h,t)|_{x_i,y_{j+\frac{1}{2}}} &\approx -(\mathcal{G}_{21}U)_{i,j+\frac{1}{2}} = \left[\frac{l_{y_j}k_{12_{i,j+\frac{1}{2}}}}{h_{x_i}h_{y_j}}\right](U_{i+1,j}-U_{i,j}),\\ \Upsilon_h(U_h,t)|_{x_i,y_{j-\frac{1}{2}}} &\approx -(\mathcal{G}_{21}U)_{i,j-\frac{1}{2}} = \left[\frac{l_{y_j}k_{12_{i,j-\frac{1}{2}}}}{h_{x_{i-1}}h_{y_j}}\right](U_{i,j}-U_{i-1,j}),\end{aligned} \quad (4.2.5)$$

and

$$\begin{aligned}\Lambda_h(U_h,t)|_{x_{i+\frac{1}{2}},y_j} &\approx -(\mathcal{G}_{12}U)_{i+\frac{1}{2},j} = \left[\frac{l_{x_i}k_{12_{i+\frac{1}{2},j}}}{h_{x_i}h_{y_i}}\right](U_{i,j+1}-U_{i,j}),\\ \Lambda_h(U_h,t)|_{x_{i-\frac{1}{2}},y_j} &\approx -(\mathcal{G}_{12}U)_{i-\frac{1}{2},j} = \left[\frac{l_{x_i}k_{12_{i-\frac{1}{2},j}}}{h_{x_i}h_{y_{j-1}}}\right](U_{i,j}-U_{i,j-1}).\end{aligned} \quad (4.2.6)$$

Then multiplying (4.2.1) by arbitrary numbers $v_{i,j}$, and summing over $\Omega_{i,j}$, we have

$$\begin{aligned}&\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i}l_{y_j}\frac{dU_{i,j}}{dt}v_{i,j} - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{y_j}\left[(\mathcal{G}_{11}U)_{i+\frac{1}{2},j}-(\mathcal{G}_{11}U)_{i-\frac{1}{2},j}+b_1x_{i+\frac{1}{2}}U_{i+1,j}-b_1x_{i-\frac{1}{2}}U_{i,j}\right]v_{i,j}\\ &-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i}\left[(\mathcal{G}_{22}U)_{i,j+\frac{1}{2}}-(\mathcal{G}_{22}U)_{i,j-\frac{1}{2}}+b_2y_{j+\frac{1}{2}}U_{i,j+1}-b_2y_{j-\frac{1}{2}}U_{i,j}\right]v_{i,j}\\ &-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{y_j}\left[(\mathcal{G}_{12}U)_{i+\frac{1}{2},j}-(\mathcal{G}_{12}U)_{i-\frac{1}{2},j}\right]v_{i,j} - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i}\left[(\mathcal{G}_{21}U)_{i,j+\frac{1}{2}}-(\mathcal{G}_{21}U)_{i,j-\frac{1}{2}}\right]v_{i,j}\\ &\qquad+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} cl_{x_i}l_{y_j}U_{i,j}v_{i,j} = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i}l_{y_j}f_{i,j}(t)v_{i,j}\end{aligned}$$

which leads to

$$-\left(\frac{dU_h}{dt},v_h\right)_h + \mathbf{a_h}(U_h,v_h;t) = (f_h(t),v_h)_h, \forall U_h,v_h \in V_h. \quad (4.2.7)$$

where

$$\left(\frac{dU_h}{dt},v_h\right)_h = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i}l_{y_j}\frac{dU_{i,j}}{dt}v_{i,j}, \quad (4.2.8)$$

$$(f_h(t), v_h)_h = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i} l_{y_j} f_{i,j} v_{i,j}, \tag{4.2.9}$$

and

$$\begin{aligned}
\mathbf{a_h}(v_h, v_h; t) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{y_j}\left[(\mathcal{G}_{11}v)\right]_{x_{i-\frac{1}{2}},y_j}^{x_{i+\frac{1}{2}},y_j} v_{i,j} - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i}\left[(\mathcal{G}_{22}v)\right]_{x_i,y_{j-\frac{1}{2}}}^{x_i,y_{j+\frac{1}{2}}} v_{i,j} \\
&\quad -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\left[(\mathcal{G}_{21}v)\right]_{x_{i-\frac{1}{2}},y_j}^{x_{i+\frac{1}{2}},y_j} l_{y_j} + \left[(\mathcal{G}_{12}v)\right]_{x_i,y_{j-\frac{1}{2}}}^{x_i,y_{j+\frac{1}{2}}} l_{x_i}\right) v_{i,j} + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} c\, l_{x_i} l_{y_j} v_{i,j}^2 \\
&\quad -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left[b_1 l_{y_j}\left(x_{i+\frac{1}{2}} v_{i+1,j} - x_{i-\frac{1}{2}} v_{i,j}\right) + b_2 l_{x_i}\left(y_{j+\frac{1}{2}} v_{i,j+1} - y_{j-\frac{1}{2}} v_{i,j}\right)\right] v_{i,j} \\
&= B_1(v_h, v_h) + B_2(v_h, v_h) + B_3(v_h, v_h) + B_4(v_h, v_h).
\end{aligned}$$

In what is to follow, we will use the following mesh regularity.

Assumption 4.2.2. For constants $c_1, c_2, c_3, c_4 > 0$, we have that the following regularity conditions hold

$$\begin{aligned}
c_1^{-1} h_{x_{i+1}} \leq h_{x_i} \leq c_1 h_{x_{i+1}}, \quad i = 0, 1, ..., N_x, \\
c_2^{-1} h_{y_{j+1}} \leq h_{y_j} \leq c_2 h_{y_{j+1}}, \quad j = 0, 1, ..., N_y
\end{aligned} \tag{4.2.10}$$

The same hold for the second partition, such that

$$\begin{aligned}
c_3^{-1} l_{x_{i+1}} \leq l_{x_i} \leq c_3 l_{x_{i+1}}, \quad i = 0, 1, ..., N_x, \\
c_4^{-1} l_{y_{j+1}} \leq l_{y_j} \leq c_4 l_{y_{j+1}}, \quad j = 0, 1, ..., N_y.
\end{aligned} \tag{4.2.11}$$

Remark 4.2.3. Furthermore,as we have mentioned before, we will assume without loss of generality that $b_1 = r - \sigma_1^2 \geq 0$ and $b_2 = r - \sigma_2^2 \geq 0$. Please see chapter 2 and [30] where the general case was considered in one dimension.

Theorem 4.3. *Under the regularity of the mesh Assumption 4.2.2, for sufficiently small h_{x_i}, h_{y_j}, there exists a constant $C > 0$, which is independent of h_{x_i}, h_{y_j} such that, for all $v_h \in V_h$, we have*

$$\mathbf{a_h}(v_h, v_h; t) = B_1(v_h, v_h) + B_2(v_h, v_h) + B_3(v_h, v_h) + B_4(v_h, v_h) \geq C\|v_h\|_{1,h}^2. \tag{4.3.1}$$

The proof here is an similar to the work in chapter 2, and we are extending to the two dimensional problem.

Proof. Firstly, we begin by proving that

$$B_1(v_h, v_h) \geq C\|v_h\|^2_{1,h_x}. \tag{4.3.2}$$

Now we have that,

$$\begin{aligned} B_1(v_h,v_h) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}[(\mathcal{G}_{11}v)_{i+\frac{1}{2},j} - (\mathcal{G}_{11}v)_{i-\frac{1}{2},j}]l_{y_j}v_{i,j} \\ &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}}(v_{i+1,j}-v_{i,j}) - \frac{k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})\right]v_{i,j} \\ &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{h_{y_j}}\left[-\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}}v_{i+1,j}v_{i,j} + \left(\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}} + \frac{k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}}\right)v^2_{i,j} - \frac{k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}}v_{i-1,j}v_{i,j}\right]. \end{aligned} \tag{4.3.3}$$

Now we use the following expansions

$$-v_{i+1,j}v_{i,j} = \frac{1}{2}\left((v_{i+1,j}-v_{i,j})^2 - v^2_{i+1,j} - v^2_{i,j}\right), \tag{4.3.4}$$

and

$$-v_{i,j}v_{i-1,j} = \frac{1}{2}\left((v_{i,j}-v_{i-1,j})^2 - v^2_{i,j} - v^2_{i-1,j}\right). \tag{4.3.5}$$

Using (4.3.4) and (4.3.5), (4.3.3) becomes

$$\begin{aligned} B_1(v_h,v_h) = &\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}\left((v_{i+1,j}-v_{i,j})^2 - v^2_{i+1,j} - v^2_{i,j}\right) + \left(\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}} + \frac{k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}}\right)v^2_{i,j}\right] \\ &+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{h_{y_j}}\left[\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-\frac{1}{2}}}}\left((v_{i,j}-v_{i-1,j})^2 - v^2_{i,j} - v^2_{i-1,j}\right)\right], \end{aligned} \tag{4.3.6}$$

so

$$\begin{aligned} B_1(v_h,v_h) = &\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}(v_{i+1,j}-v_{i,j})^2 + \frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})^2\right] \\ &+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{h_{y_j}}\left[\left(\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}} + \frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}\right)v^2_{i,j} - \frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}v^2_{i+1,j} - \frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}v^2_{i-1,j}\right]. \end{aligned} \tag{4.3.7}$$

Now making use of the fact that $v_{0,j} = v_{N_x+1,j} = 0$, we have the following expressions,

$$\begin{aligned} -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{h_{y_j}}\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}v^2_{i-1,j} &= -\sum_{j=1}^{N_y}\frac{l^2_{y_j}}{2h_{y_j}}\left[\frac{k_{11_{\frac{1}{2},j}}}{h_{x_0}}v^2_{0,j} + \frac{k_{11_{\frac{3}{2},j}}}{h_{x_1}}v^2_{1,j} + \ldots + \frac{k_{11_{N_x-\frac{1}{2},j}}}{h_{N_x-1}}v^2_{N_x-1,j}\right] \\ &= -\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l^2_{y_j}}{2h_{y_j}}\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}}v^2_{i,j} + \sum_{j=1}^{N_y}\frac{l^2_{y_j}}{2h_{y_j}}\frac{k_{11_{N_x+\frac{1}{2},j}}}{h_{N_x}}v^2_{N_x,j}, \end{aligned}$$

and

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{2h_{y_j}}\frac{k11_{i+\frac{1}{2},j}}{h_{x_i}}v_{i+1,j}^2=-\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{2h_{y_j}}\left[\frac{k11_{\frac{3}{2},j}}{h_{x_1}}v_{2,j}^2+\frac{k11_{\frac{5}{2},j}}{h_{x_2}}v_{3,j}^2+\ldots+\frac{k11_{N_x+\frac{1}{2},j}}{h_{N_x}}v_{N_x+1,j}^2\right]$$

$$=-\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{y_j}^2}{2h_{y_j}}\frac{k11_{i-\frac{1}{2},j}}{h_{x_{i-1}}}v_{i,j}^2+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{2h_{y_j}}\frac{k11_{\frac{1}{2},j}}{h_0}v_{1,j}^2.$$

Then substituting the above expansions into (4.3.7), we get

$$\begin{aligned}
B_1(v_h,v_h)&=\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k11_{i+\frac{1}{2},j}}{2h_{x_i}}(v_{i+1,j}-v_{i,j})^2+\frac{k11_{i-\frac{1}{2},j}}{2h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})^2\right]\\
&+\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{y_j}^2}{h_{y_j}}\left[\left(\frac{k11_{i+\frac{1}{2},j}}{2h_{x_i}}+\frac{k11_{i-\frac{1}{2},j}}{2h_{x_{i-1}}}\right)v_{i,j}^2-\frac{k11_{i+\frac{1}{2},j}}{2h_{x_i}}v_{i+1,j}^2-\frac{k11_{i-\frac{1}{2},j}}{2h_{x_{i-1}}}v_{i-1,j}^2\right]\\
&=\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k11_{i+\frac{1}{2},j}}{2h_{x_i}}(v_{i+1,j}-v_{i,j})^2+\frac{k11_{i-\frac{1}{2},j}}{2h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})^2\right]\\
&+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{2h_{y_j}}\frac{k11_{N_x+\frac{1}{2},j}}{h_{N_x}}v_{N_x,j}^2+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{2h_{y_j}}\frac{k11_{\frac{1}{2},j}}{h_0}v_{1,j}^2\\
&\geq C\|v_h\|_{1,h_x}^2. \qquad (4.3.8)
\end{aligned}$$

That is

$$B_1(v_h,v_h)\geq C\|v_h\|_{1,h_x}^2. \qquad (4.3.9)$$

Now we consider the case for $B_2(v_h,v_h)$, that is

$$\begin{aligned}
B_2(v_h,v_h)&=-\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}[(\mathcal{G}_{22}v)_{i,j+\frac{1}{2}}-(\mathcal{G}_{22}v)_{i,j-\frac{1}{2}}]l_{x_i}v_{i,j}\\
&=-\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k22_{i,j+\frac{1}{2}}}{h_{y_j}}(v_{i,j+1}-v_{i,j})-\frac{k22_{i,j-\frac{1}{2}}}{h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})\right]v_{i,j}\\
&=\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\left[-\frac{k22_{i,j+\frac{1}{2}}}{h_{y_j}}v_{i,j+1}v_{i,j}+\left(\frac{k22_{i,j+\frac{1}{2}}}{h_{y_j}}+\frac{k22_{i,j-\frac{1}{2}}}{h_{y_{j-1}}}\right)v_{i,j}^2-\frac{k22_{i,j-\frac{1}{2}}}{h_{y_{j-1}}}v_{i,j-1}v_{i,j}\right].
\end{aligned}$$

Now we use the following expansions

$$-v_{i,j+1}v_{i,j}=\frac{1}{2}\left((v_{i,j+1}-v_{i,j})^2-v_{i,j+1}^2-v_{i,j}^2\right), \qquad (4.3.10)$$

and

$$-v_{i,j}v_{i,j-1} = \frac{1}{2}\left((v_{i,j}-v_{i,j-1})^2 - v_{i,j}^2 - v_{i,j-1}^2\right). \tag{4.3.11}$$

That is

$$\begin{aligned}
B_2(v_h, v_h) &= \sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k_{22_{i,j+\frac{1}{2}}}}{2h_{y_j}}\left((v_{i,j+1}-v_{i,j})^2 - v_{i,j+1}^2 - v_{i,j}^2\right)\right] \\
&+\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k_{22_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}\left((v_{i,j}-v_{i,j-1})^2 - v_{i,j}^2 - v_{i,j-1}^2\right) + \left(\frac{k_{22_{i,j+\frac{1}{2}}}}{h_{y_j}} + \frac{k_{22_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}\right)v_{i,j}^2\right] \\
&= \sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k_{22_{i,j+\frac{1}{2}}}}{2h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \frac{k_{22_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})^2\right] \\
&+\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\left[\left(\frac{k_{22_{i,j+\frac{1}{2}}}}{2h_{y_j}} + \frac{k_{2_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}\right)v_{i,j}^2 - \frac{k_{22_{i,j+\frac{1}{2}}}}{2h_{y_j}}v_{i,j+1}^2 - \frac{k_{2_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}v_{i,j-1}^2\right].
\end{aligned} \tag{4.3.12}$$

Now we consider the following expansions

$$-\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{2_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}v_{i,j-1}^2 = \sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{2_{i,j+\frac{1}{2}}}}{h_{y_j}}v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{2_{i,N_y+\frac{1}{2}}}}{h_{N_y}}v_{i,N_y}^2 \tag{4.3.13}$$

and

$$-\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{22_{i,j+\frac{1}{2}}}}{h_{y_j}}v_{i,j+1}^2 = -\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{22_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{22_{i,\frac{1}{2}}}}{h_{y_0}}v_{i,1}^2. \tag{4.3.14}$$

Then using (4.3.13) and (4.3.14), $B_2(v_h, v_h)$ we have that

$$\begin{aligned}
B_2(v_h, v_h) &= \sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\left[\frac{k_{22_{i,j+\frac{1}{2}}}}{h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \frac{k_{22_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})^2\right] \\
&+\sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\left[\left(\frac{k_{22_{i,j+\frac{1}{2}}}}{h_{y_j}} + \frac{k_{22_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}\right)v_{i,j}^2 - \frac{k_{22_{i,j+\frac{1}{2}}}}{h_{y_j}}v_{i,j+1}^2 - \frac{k_{22_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}v_{i,j-1}^2\right]
\end{aligned} \tag{4.3.15}$$

$$\begin{aligned}
B_2(v_h, v_h) &= \sum_{j=1}^{N_y}\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\left[\frac{k_{22_{i,j+\frac{1}{2}}}}{h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \frac{k_{22_{i,j-\frac{1}{2}}}}{h_{y_j}}(v_{i,j}-v_{i,j-1})^2\right] \\
&\qquad+\sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{22_{i,\frac{1}{2}}}}{h_{y_0}}v_{i,1}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{2h_{x_i}}\frac{k_{22_{i,N_y+\frac{1}{2}}}}{h_{N_y}}v_{i,N_y}^2 \\
&\qquad\geq C\|v_h\|_{1,h_y}^2,
\end{aligned} \tag{4.3.16}$$

that is

$$B_2(v_h, v_h) \geq C\|v_h\|^2_{1,h_y}. \tag{4.3.17}$$

Furthermore, we consider the term $B_3(v_h, v_h)$

$$\begin{aligned} B_3(v_h, v_h) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(l_{y_j}\,[\mathcal{G}_{21}v]_{x_{i-\frac{1}{2}},y_j}^{x_{i+\frac{1}{2}},y_j} + l_{x_i}\,[\mathcal{G}_{12}v]_{x_i,y_{j-\frac{1}{2}}}^{x_i,y_{j+\frac{1}{2}}}\right) v_{i,j} \\ &= B_{31}(v_h, v_h) + B_{32}(v_h, v_h). \end{aligned} \tag{4.3.18}$$

We then consider each term of (4.3.18). For the first term, $B_{31}(v_h, v_h)$, we have

$$\begin{aligned} B_{31}(v_h, v_h) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}[\mathcal{G}_{21}v]_{x_{i-\frac{1}{2}},y_j}^{x_{i+\frac{1}{2}},y_j}\, l_{y_j} v_{i,j} \\ &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[\left(\frac{k_{12_{i,j+\frac{1}{2}}}}{h_{y_j}}(v_{i,j+1}-v_{i,j})\right) - \left(\frac{k_{12_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})\right)\right] v_{i,j} \\ &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[-\frac{k_{12_{i,j+\frac{1}{2}}}}{h_{y_j}}v_{i,j+1}v_{i,j} + \left(\frac{k_{12_{i,j+\frac{1}{2}}}}{h_{y_j}} + \frac{k_{12_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}\right)v_{i,j}^2 - \frac{k_{12_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}v_{i,j}v_{i,j-1}\right]. \end{aligned}$$

Then following from previous expansions, we have that

$$\begin{aligned} B_{31}(v_h, v_h) &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[\frac{k_{12_{i,j+\frac{1}{2}}}}{2h_{y_j}}\left[(v_{i,j+1}-v_{i,j})^2 - v_{i,j+1}^2 - v_{i,j}^2\right]\right] \\ &+ \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[\frac{k_{12_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}\left[(v_{i,j}-v_{i,j-1})^2 - v_{i,j-1}^2 - v_{i,j}^2\right] + \left(\frac{k_{12_{i,j+\frac{1}{2}}}}{h_{y_j}} + \frac{k_{12_{i,j-\frac{1}{2}}}}{h_{y_{j-1}}}\right)v_{i,j}^2\right]. \end{aligned}$$

That is we have

$$\begin{aligned} B_{31}(v_h, v_h) &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[\frac{k_{12_{i,j+\frac{1}{2}}}}{2h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \frac{k_{12_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}(v_{i,j-1}-v_{i,j})^2\right] \\ &+ \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[\left(\frac{k_{12_{i,j+\frac{1}{2}}}}{h_{y_j}} + \frac{k_{12_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}\right)v_{i,j}^2 - \frac{k_{12_{i,j+\frac{1}{2}}}}{2h_{y_j}}v_{i,j+1}^2 - \frac{k_{12_{i,j+\frac{1}{2}}}}{2h_{y_{j-1}}}v_{i,j-1}^2\right]. \end{aligned}$$

Now, we consider the following expansions

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\frac{k_{12_{i,j+\frac{1}{2}}}}{2h_{y_j}}v_{i,j+1}^2 = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_{j-1}}}{h_{x_i}}\frac{k_{12_{i,j-\frac{1}{2}}}}{2h_{y_{j-1}}}v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}l_{y_0}}{h_{x_i}}\frac{k_{12_{i,\frac{1}{2}}}}{2h_{y_0}}v_{i,1}^2, \tag{4.3.19}$$

and

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\frac{k12_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}v_{i,j-1}^2=-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_{j+1}}}{h_{x_i}}\frac{k12_{i,j+\frac{1}{2}}}{2h_{y_j}}v_{i,j}^2+\sum_{i=1}^{N_x}\frac{l_{x_i}}{h_{x_i}}\frac{k12_{i,N_y+\frac{1}{2}}l_{y_{N_y+1}}}{2h_{y_{N_y}}}v_{i,N_y}^2. \qquad (4.3.20)$$

Then we have that

$$\begin{aligned}B_{31}(v_h,v_h)&=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[\frac{k12_{i,j+\frac{1}{2}}}{2h_{y_j}}\left(v_{i,j+1}-v_{i,j}\right)^2+\frac{k12_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}\left(v_{i,j-1}-v_{i,j}\right)^2\right]\\&+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{x_i}}\left[\left(\frac{k12_{i,j+\frac{1}{2}}}{2h_{y_j}}+\frac{k12_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}\right)v_{i,j}^2\right]-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_{j-1}}}{h_{x_i}}\frac{k12_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}v_{i,j}^2\\&+\sum_{i=1}^{N_x}\frac{l_{x_i}l_{y_0}}{h_{x_i}}\frac{k12_{i,\frac{1}{2}}}{2h_{y_0}}v_{i,1}^2-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_{j+1}}}{h_{x_i}}\frac{k12_{i,j+\frac{1}{2}}}{2h_{y_j}}v_{i,j}^2+\sum_{i=1}^{N_x}\frac{l_{x_i}}{h_{x_i}}\frac{k12_{i,N_y+\frac{1}{2}}l_{y_{N_y+1}}}{2h_{y_{N_y}}}v_{i,N_y}^2.\end{aligned}$$

Then using the Assumption 4.2.2 on the regular grid, we have

$$\begin{aligned}B_{31}(v_h,v_h)&\geq\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}}{h_{x_i}}\left[\frac{l_{y_j}k12_{i,j+\frac{1}{2}}}{2h_{y_j}}\left(v_{i,j+1}-v_{i,j}\right)^2+\frac{l_{y_j}k12_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}\left(v_{i,j-1}-v_{i,j}\right)^2\right]\\&+\sum_{i=1}^{N_x}\frac{l_{x_i}l_{y_0}}{h_{x_i}}\frac{k12_{i,\frac{1}{2}}}{2h_{y_0}}v_{i,1}^2+\sum_{i=1}^{N_x}\frac{l_{x_i}k12_{i,N_y+\frac{1}{2}}l_{y_{N_y+1}}}{2h_{x_i}h_{y_{N_y}}}v_{i,N_y}^2\geq 0. \qquad (4.3.21)\end{aligned}$$

Similarly for $B_{32}(v_h,v_h)$ we have

$$\begin{aligned}B_{32}(v_h,v_h)&=-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\left[\mathcal{G}_{12}v\right]_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}l_{x_i}\right)v_{i,j}\\&=-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{y_j}}\left[\left(\frac{k12_{i+\frac{1}{2},j}}{h_{x_i}}(v_{i+1,j}-v_{i,j})\right)-\left(\frac{k12_{i-\frac{1}{2}}}{h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})\right)\right]v_{i,j}\\&=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{y_j}}\left[-\frac{k12_{i+\frac{1}{2},j}}{h_{x_i}}v_{i+1,j}v_{i,j}+\left(\frac{k12_{i+\frac{1}{2},j}}{h_{x_i}}+\frac{k12_{i-\frac{1}{2},j}}{h_{x_{i-1}}}\right)v_{i,j}^2-\frac{k12_{i-\frac{1}{2},j}}{h_{x_{i-1}}}v_{i,j}v_{i-1,j}\right].\end{aligned}$$

Indeed, following from previous expansions we have

$$\begin{aligned}B_{32}(v_h,v_h)&=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{y_j}}\left[\frac{k12_{i+\frac{1}{2},j}}{2h_{x_i}}\left[(v_{i+1,j}-v_{i,j})^2-v_{i+1,j}^2-v_{i,j}^2\right]\right]\\&+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}l_{y_j}}{h_{y_j}}\left[\left(\frac{k12_{i+\frac{1}{2},j}}{h_{x_i}}+\frac{k12_{i-\frac{1}{2},j}}{h_{x_{i-1}}}\right)v_{i,j}^2+\frac{k12_{i-\frac{1}{2},j}}{h_{x_{i-1}}}\left[(v_{i,j}-v_{i-1,j})^2-v_{i,j}^2-v_{i-1,j}^2\right]\right],\end{aligned}$$

so that

$$B_{32}(v_h, v_h) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{l_{x_i} l_{y_j}}{h_{y_j}} \left[\frac{k_{12_{i+\frac{1}{2},j}}}{2h_{x_i}} \left(v_{i+1,j} - v_{i,j}\right)^2 + \frac{k_{12_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}} \left(v_{i-1,j} - v_{i,j}\right)^2 \right]$$
$$+ \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{l_{x_i} l_{y_j}}{h_{y_j}} \left[\left(\frac{k_{12_{i+\frac{1}{2},j}}}{2h_{x_i}} + \frac{k_{3_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}} \right) v_{i,j}^2 - \frac{k_{12_{i+\frac{1}{2},j}}}{2h_{x_i}} v_{i+1,j}^2 - \frac{k_{12_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}} v_{i-1,j}^2 \right].$$

Now, we consider the following expansions

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{l_{x_i} l_{y_j}}{h_{y_j}} \frac{k_{12_{i+\frac{1}{2},j}}}{2h_{x_i}} v_{i+1,j}^2 = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{l_{x_{i-1}} l_{y_j}}{h_{y_j}} \frac{k_{12_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}} v_{i,j}^2 + \sum_{j=1}^{N_y} \frac{l_{x_0} l_{y_j}}{h_{y_j}} \frac{k_{12_{\frac{1}{2},j}}}{2h_{x_0}} v_{1,j}^2, \quad (4.3.22)$$

and

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{l_{x_i} l_{y_j}}{h_{y_j}} \frac{k_{12_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}} v_{i-1,j}^2 = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{l_{x_{i+1}} l_{y_j}}{h_{y_j}} \frac{k_{12_{i+\frac{1}{2},j}}}{2h_{x_i}} v_{i,j}^2 + \sum_{j=1}^{N_y} \frac{l_{y_j}}{h_{y_j}} \frac{k_{12_{N_x+\frac{1}{2},j}} l_{x_{N_x+1}}}{2h_{x_{N_x}}} v_{N_x,j}^2. \quad (4.3.23)$$

Then using Assumption 4.2.2 we have

$$\begin{aligned} B_{32}(v_h, v_h) \;\; &\geq \;\; \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \frac{l_{x_i} l_{y_j}}{h_{y_j}} \left[\frac{k_{12_{i+\frac{1}{2},j}}}{2h_{x_i}} \left(v_{i+1,j} - v_{i,j}\right)^2 + \frac{k_{12_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}} \left(v_{i-1,j} - v_{i,j}\right)^2 \right] \\ &\quad + \sum_{j=1}^{N_y} \frac{l_{y_j}}{h_{y_j}} \frac{k_{12_{N_x+\frac{1}{2},j}} l_{x_{N_x+1}}}{2h_{x_{N_x}}} v_{N_x,j}^2 + \sum_{j=1}^{N_y} \frac{l_{x_0} l_{y_j}}{h_{y_j}} \frac{k_{12_{\frac{1}{2},j}}}{2h_{x_0}} v_{1,j}^2 \geq 0 \end{aligned}$$

Therefore the term $B_3(v_h, v_h)$ becomes

$$B_3(v_h, v_h) = B_{31}(v_h, v_h) + B_{32}(v_h, v_h) \geq 0. \quad (4.3.24)$$

Now we consider the term $B_4(v_h, v_h)$,

$$\begin{aligned} B_4(v_h, v_h) \;\; &= \;\; -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \left[b_1 l_{y_j} \left(x_{i+\frac{1}{2}} v_{i+1,j} - x_{i-\frac{1}{2}} v_{i,j} \right) + b_2 l_{x_i} \left(y_{j+\frac{1}{2}} v_{i,j+1} - y_{j-\frac{1}{2}} v_{i,j} \right) \right] v_{i,j} \\ &\quad + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} c v_{i,j}^2 l_{x_i} l_{y_j} \\ &= \;\; -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j} \left(x_{i+\frac{1}{2}} v_{i+1,j} - x_{i-\frac{1}{2}} v_{i,j} \right) v_{i,j} \\ &\quad - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_2 l_{x_i} \left(y_{j+\frac{1}{2}} v_{i,j+1} - y_{j-\frac{1}{2}} v_{i,j} \right) v_{i,j} + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} c v_{i,j}^2 l_{x_i} l_{y_j} \\ &= \;\; B_{41}(v_h, v_h) + B_{42}(v_h, v_h) + B_{43}(v_h, v_h). \end{aligned}$$

Then for $B_{41}(v_h, v_h)$, we have that

$$\begin{aligned}
B_{41}(v_h, v_h) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j}\left(x_{i+\frac{1}{2}}v_{i+1,j} - x_{i-\frac{1}{2}}v_{i,j}\right)v_{i,j}\\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 x_{i+\frac{1}{2}}l_{y_j}}{2}\left[(v_{i+1,j}-v_{i,j})^2 - v_{i+1,j}^2 - v_{i,j}^2\right] + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 x_{i-\frac{1}{2}} l_{y_j} v_{i,j}^2\\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j}\left[\frac{x_{i+\frac{1}{2}}}{2}(v_{i+1,j}-v_{i,j})^2 + x_{i-\frac{1}{2}}v_{i,j}^2\right] - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 x_{i+\frac{1}{2}}l_{y_j}}{2}\left[v_{i+1,j}^2 + v_{i,j}^2\right].
\end{aligned}$$

Now let us consider the following expansion,

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 x_{i+\frac{1}{2}}l_{y_j}}{2}v_{i+1,j}^2 \;=\; -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 x_{i-\frac{1}{2}}l_{y_j}}{2}v_{i,j}^2 + \sum_{j=1}^{N_y}\frac{b_1 x_{\frac{1}{2}}l_{y_j}}{2}v_{1,j}^2. \tag{4.3.25}$$

Then substituting the above expansions into $B_{41}(v_h, v_h)$, we have,

$$\begin{aligned}
B_{41}(v_h, v_h) &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j}\left[\frac{x_{i+\frac{1}{2}}}{2}(v_{i+1,j}-v_{i,j})^2 + x_{i-\frac{1}{2}}v_{i,j}^2\right] - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 x_{i+\frac{1}{2}}l_{y_j}}{2}\left[v_{i+1,j}^2 + v_{i,j}^2\right]\\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j}\left[\frac{x_{i+\frac{1}{2}}}{2}(v_{i+1,j}-v_{i,j})^2\right] + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 l_{y_j}}{2}\left[x_{i-\frac{1}{2}} - x_{i+\frac{1}{2}}\right]v_{i,j}^2 + \sum_{j=1}^{N_y}\frac{b_1 x_{\frac{1}{2}}l_{y_j}}{2}v_{1,j}^2\\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j}\left[\frac{x_{i+\frac{1}{2}}}{2}(v_{i+1,j}-v_{i,j})^2\right] - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 l_{y_j}l_{x_i}}{2}v_{i,j}^2 + \sum_{j=1}^{N_y}\frac{b_1 x_{\frac{1}{2}}l_{y_j}}{2}v_{1,j}^2.
\end{aligned}$$

A similar argument exists for $B_{42}(v_h, v_h)$.

$$\begin{aligned}
B_{42}(v_h, v_h) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_2 l_{x_i}\left(y_{j+\frac{1}{2}}v_{i,j+1} - y_{j-\frac{1}{2}}v_{i,j}\right)v_{i,j}\\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 y_{j+\frac{1}{2}}l_{x_i}}{2}\left[(v_{i,j+1}-v_{i,j})^2 - v_{i,j+1}^2 - v_{i,j}^2\right] + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_2 y_{j-\frac{1}{2}} l_{x_i} v_{i,j}^2\\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_2 l_{x_i}\left[\frac{y_{j+\frac{1}{2}}}{2}(v_{i,j+1}-v_{i,j})^2 - y_{j-\frac{1}{2}}v_{i,j}^2\right] - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 y_{j+\frac{1}{2}}l_{x_i}}{2}\left[v_{i,j+1}^2 + v_{i,j}^2\right]
\end{aligned}$$

Now let us consider the following expansion,

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 y_{j+\frac{1}{2}}l_{x_i}}{2}v_{i,j+1}^2 \;=\; -\sum_{i=1}^{N_x}\sum_{i=1}^{N_x}\frac{b_2 y_{j-\frac{1}{2}}l_{x_i}}{2}v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{b_2 y_{\frac{1}{2}}l_{x_i}}{2}v_{i,1}^2. \tag{4.3.26}$$

Then substituting the above expansion into $B_{42}(v_h, v_h)$, we have that,

$$
\begin{aligned}
B_{42}(v_h, v_h) &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_2 l_{x_i}\left[\frac{y_{j+\frac{1}{2}}}{2}(v_{i,j+1}-v_{i,j})^2 - y_{j-\frac{1}{2}}v_{i,j}^2\right] - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 y_{j+\frac{1}{2}} l_{x_i}}{2}\left[v_{i,j+1}^2+v_{i,j}^2\right] \\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 l_{x_i}}{2}\left[y_{j+\frac{1}{2}}(v_{i,j+1}-v_{i,j})^2\right] + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 l_{x_i}}{2}\left[y_{j-\frac{1}{2}}-y_{j+\frac{1}{2}}\right]v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{b_2 y_{\frac{1}{2}} l_{x_i}}{2}v_{i,1}^2 \\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 l_{x_i}}{2}\left[y_{j+\frac{1}{2}}(v_{i,j+1}-v_{i,j})^2\right] - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 l_{x_i} l_{y_j}}{2}v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{b_2 y_{\frac{1}{2}} l_{x_i}}{2}v_{i,1}^2.
\end{aligned}
$$

Finally, for $B_{43}(v_h, v_h)$ we have that

$$
B_{43}(v_h, v_h) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} c l_{x_i} l_{y_j} v_{i,j}^2 = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i} l_{y_j}\left(3r - (\sigma_1^2+\sigma_2^2+\rho\sigma_1\sigma_2)\right)v_{i,j}^2.
$$

Now remember that for $r-\sigma_1^2$, $r-\sigma_2^2 \geq 0$ (see [30] for general case) and $\rho \in [-1, 1]$, we have that

$$
\begin{aligned}
B_4(v_h, v_h) &= B_{41}(v_h, v_h) + B_{42}(v_h, v_h) + B_{43}(v_h, v_h) \\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j}\left[\frac{x_{i+\frac{1}{2}}}{2}(v_{i+1,j}-v_{i,j})^2\right] - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{(b_1+b_2) l_{y_j} l_{x_i}}{2}v_{i,j}^2 + \sum_{j=1}^{N_y}\frac{b_1 x_{\frac{1}{2}} l_{y_j}}{2}v_{1,j}^2 \\
&\quad + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 l_{x_i}}{2}\left[y_{j+\frac{1}{2}}(v_{i,j+1}-v_{i,j})^2\right] + \sum_{i=1}^{N_x}\frac{b_2 y_{\frac{1}{2}} l_{x_i}}{2}v_{i,1}^2 + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i} l_{y_j} c v_{i,j}^2.
\end{aligned}
$$

$$
\begin{aligned}
B_4(v_h, v_h) &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 l_{y_j}\left[\frac{x_{i+\frac{1}{2}}}{2}(v_{i+1,j}-v_{i,j})^2\right] + \sum_{j=1}^{N_y}\frac{b_1 x_{\frac{1}{2}} l_{y_j}}{2}v_{1,j}^2 + \sum_{i=1}^{N_x}\frac{b_2 y_{\frac{1}{2}} l_{x_i}}{2}v_{i,1}^2 \\
&\quad + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 l_{x_i}}{2}\left[y_{j+\frac{1}{2}}(v_{i,j+1}-v_{i,j})^2\right] + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i} l_{y_j}\left(\frac{4r - (\sigma_1^2+\sigma_2^2+\rho\sigma_1\sigma_2)}{2}\right)v_{i,j}^2.
\end{aligned}
$$

$$
B_4(v_h, v_h) \geq C\|v_h\|_{0,h}. \tag{4.3.27}
$$

Therefore from (4.3.9), (4.3.24), (4.3.27), we have that

$$
\mathbf{a_h}(v_h, v_h) \geq C\|v_h\|_{1,h_x} + C\|v_h\|_{1,h_y} + C\|v_h\|_{0,h}. \tag{4.3.28}
$$

□

4.3.1 Coercivity of fitted-mimetic finite difference

We follow a similar argument as in the previous subsection to prove the coercivity of the fitted scheme. Now we have that,

$$-(\dot{U}_H, v_h)_h + \mathbf{A_h}(U_H, v_h) = (f_h, v_h)_h, \forall U_H, v_h \in V_h \subset H^1_{0,w}(\Omega). \tag{4.3.29}$$

It is important to note here that the difference between the standard mimetic scheme and the fitted mimetic scheme is that, the convection flux at $(x_{1/2}, y_j)$ and $(x_i, y_{1/2})$ as

$$\begin{aligned} x_{1/2}\Phi(U_h)|_{x_{1/2},y_j} &= \frac{x_{1/2}}{2}\left[(\mathsf{a}_1 + b_1)U_{1,j} - (\mathsf{a}_1 - b_1)U_{0,j}\right] \\ &= \frac{\mathsf{a}_1 x_{1/2}}{2}[U_{1,j} - U_{0,j}] + \frac{b_1 x_{1/2}}{2}[U_{1,j} + U_{0,j}] \\ &= -(\mathcal{G}_{11}v)_{\frac{1}{2},j} + \frac{b_1 x_{1/2}}{2}[U_{1,j} + U_{0,j}], \end{aligned} \tag{4.3.30}$$

and

$$\begin{aligned} y_{1/2}\Psi(U_h)|_{x_i,y_{1/2}} &= \frac{y_{1/2}}{2}\left[(\mathsf{a}_2 + b_2)U_{i,1} - (\mathsf{a}_2 - b_2)U_{i,0}\right] \\ &= \frac{\mathsf{a}_2 y_{1/2}}{2}[U_{i,1} - U_{i,0}] + \frac{b_2 y_{1/2}}{2}[U_{i,1} + U_{i,0}] \\ &= -(\mathcal{G}_{22}v)_{\frac{1}{2},j} + \frac{b_2 x_{1/2}}{2}[U_{i,1} + U_{i,0}]. \end{aligned} \tag{4.3.31}$$

Then for the fitted mimetic scheme, we have

$$\mathbf{A_h}(U_H, v_h; t) = \mathbf{B^1_h}(U_H, v_h) + \mathbf{B^2_h}(U_H, v_h) + \mathbf{B^3_h}(U_H, v_h) + \mathbf{B^4_h}(U_H, v_h)$$

$$\mathbf{B^1_h}(U_H, v_h) = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{y_j}\left[(\mathcal{G}_{11}U)\right]_{x_{i-\frac{1}{2}},y_j}^{x_{i+\frac{1}{2}},y_j} v_{i,j}, \text{ with } (\mathcal{G}_{11}U)_{1/2,j} = \frac{\mathsf{a}_1 x_{1/2}}{2}[U_{1,j} - U_{0,j}], \tag{4.3.32}$$

$$\mathbf{B^2_h}(U_H, v_h) = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} l_{x_i}\left[(\mathcal{G}_{22}U)\right]_{x_i,y_{j-\frac{1}{2}}}^{x_i,y_{j+\frac{1}{2}}} v_{i,j}, \text{ with } (\mathcal{G}_{22}U)_{i,1/2} = \frac{\mathsf{a}_2 y_{1/2}}{2}[U_{i,1} - U_{i,0}], \tag{4.3.33}$$

$$\mathbf{B^3_h}(U_H, v_h) = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\left[(\mathcal{G}_{21}U)\right]_{x_{i-\frac{1}{2}},y_j}^{x_{i+\frac{1}{2}},y_j} l_{y_j} + \left[(\mathcal{G}_{12}U)\right]_{x_i,y_{j-\frac{1}{2}}}^{x_i,y_{j+\frac{1}{2}}} l_{x_i}\right) v_{i,j}, \tag{4.3.34}$$

and

$$\begin{aligned} \mathbf{B^4_h}(U_H, v_h) &= -\sum_{j=1}^{N_y} l_{y_j}(b_1 x_{3/2}U_{2,j} - \frac{b_1}{2}U_{1,j})v_{1,j} - \sum_{i=2}^{N_x}\sum_{j=1}^{N_y} l_{y_j} b_1\left(x_{i+\frac{1}{2}}U_{i+1,j} - x_{i-\frac{1}{2}}U_{i,j}\right)v_{i,j} \\ &\quad -\sum_{i=1}^{N_x} l_{x_i}(b_2 y_{3/2}U_{i,2} - \frac{b_2}{2}U_{i,1})v_{i,1} - \sum_{i=1}^{N_x}\sum_{j=2}^{N_y} l_{x_i} b_2\left(y_{j+\frac{1}{2}}U_{i,j+1} - y_{j-\frac{1}{2}}U_{i,j}\right)v_{i,j} \\ &\quad +\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(cU_{i,j})l_{x_i}l_{y_j}v_{i,j}. \end{aligned} \tag{4.3.35}$$

Theorem 4.4. *Under Assumption 4.2.2, there exists a positive constant $C > 0$, which is independent of h_x, h_y, l_x, l_y, such that for all $v_h \in V_h$, we have*

$$\mathbf{A_h}(v_h, v_h; t) = \mathbf{B_h^1}(v_h, v_h) + \mathbf{B_h^2}(v_h, v_h) + \mathbf{B_h^3}(v_h, v_h) + \mathbf{B_h^4}(v_h, v_h) \geq C\|v_h\|_{1,h}^2 \tag{4.4.1}$$

Proof. Now let us consider the term, $\mathbf{B_h^1}(v_h, v_h)$ and show that

$$\mathbf{B_h^1}(v_h, v_h) \geq \|v_h\|_{1,h_x}. \tag{4.4.2}$$

Then we have that,

$$\begin{aligned}
\mathbf{B_h^1}(v_h, v_h) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left[(\mathcal{G}_{11}v)_{i+\frac{1}{2},j} - (\mathcal{G}_{11}v)_{i-\frac{1}{2},j}\right] l_{y_j} v_{i,j} \\
&= -\sum_{j=1}^{N_y} l_{y_j}\left[(\mathcal{G}_{11}v)_{\frac{3}{2},j} - (\mathcal{G}_{11}v_{\frac{1}{2},j})\right] v_{1,j} - \sum_{i=2}^{N_x}\sum_{j=1}^{N_y} l_{y_j}\left[(\mathcal{G}_{11}v)_{i+\frac{1}{2},j} - (\mathcal{G}_{11}v)_{i-\frac{1}{2},j}\right] v_{i,j} \\
&= -\sum_{j=1}^{N_y}\left[\frac{l_{y_j}^2 k_{1_{\frac{3}{2},j}}}{h_{y_j}h_{x_1}}(v_{2,j} - v_{1,j}) - \frac{l_{y_j}x_{\frac{1}{2}}}{2}[\mathsf{a}_1 v_{1,j} - \mathsf{a}_1 v_{0,j}]\right] v_{1,j} \\
&\quad -\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[(\mathcal{G}_{11}v)_{i+\frac{1}{2},j} - (\mathcal{G}_{11}v)_{i-\frac{1}{2},j}\right] v_{i,j} \\
&= -\sum_{j=1}^{N_y}\left[\frac{l_{y_j}^2 k_{1_{\frac{3}{2},j}}}{h_{y_j}h_{x_1}}(v_{2,j} - v_{1,j})v_{1,j} - \frac{l_{y_j}x_{\frac{1}{2}}}{2}\mathsf{a}_1 v_{1,j}^2\right] + \mathbf{P_h}. \qquad (4.4.3)
\end{aligned}$$

Then we have that,

$$\mathbf{P_h} = -\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}}(v_{i+1,j} - v_{i,j}) - \frac{k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}}(v_{i,j} - v_{i-1,j})\right] v_{i,j}$$

$$= \sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[-\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}}v_{i+1,j}v_{i,j} + \left(\frac{k_{11_{i+\frac{1}{2},j}}}{h_{x_i}} + \frac{k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}}\right)v_{i,j}^2 - \frac{k_{11_{i-\frac{1}{2},j}}}{h_{x_{i-1}}}v_{i,j}v_{i-1,j}\right]$$

$$= \sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j,j}}}{2h_{x_i}}(v_{i+1,j} - v_{i,j})^2 + \frac{k_{11_{i-\frac{1}{2}}}}{2h_{x_{i-1}}}(v_{i,j} - v_{i-1,j})^2\right]$$

$$+\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left(\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}} + \frac{k_{1_{i-\frac{1}{2}}}}{2h_{x_{i-1}}}\right)v_{i,j}^2 - \sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}v_{i+1,j}^2 + \frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}v_{i-1,j}^2\right].$$

Now we consider the following expansions

$$-\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}v_{i+1,j}^2 = -\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}v_{i,j}^2 + \sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{\frac{3}{2},j}}}{2h_{x_1}}v_{2,j}^2 \tag{4.4.4}$$

and

$$-\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}v_{i-1,j}^2=-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}v_{i,j}^2+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{N_x+\frac{1}{2},j}}}{2h_{x_{N_x}}}v_{N_x,j}^2. \tag{4.4.5}$$

Then using (4.4.4) and (4.4.5), $\mathbf{P_h}$ becomes,

$$\begin{aligned}\mathbf{P_h} &= \sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}(v_{i+1,j}-v_{i,j})^2+\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})^2\right]\\ &\quad+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{N_x+\frac{1}{2},j}}}{2h_{x_{N_x}}}v_{N_x,j}^2+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{\frac{3}{2},j}}}{2h_{x_1}}v_{2,j}^2.\end{aligned}$$

Now, we have that $\mathbf{B_h^1}$ becomes

$$\begin{aligned}\mathbf{B_h^1}(v_h,v_h) &= -\sum_{j=1}^{N_y}\left[\frac{l_{y_j}^2k_{1_{\frac{3}{2},j}}}{h_{y_j}h_{x_1}}(v_{2,j}-v_{1,j})v_{1,j}-\frac{l_{y_j}x_{\frac{1}{2}}}{2}\mathsf{a}_1v_{1,j}^2\right]+\mathbf{P_h}\\ =\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}&\left[\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}(v_{i+1,j}-v_{i,j})^2+\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})^2\right]+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{\frac{3}{2},j}}}{2h_{x_1}}v_{2,j}^2\\ -\sum_{j=1}^{N_y}&\left[\frac{l_{y_j}^2k_{1_{\frac{3}{2},j}}}{h_{y_j}h_{x_1}}(v_{2,j}-v_{1,j})v_{1,j}-\frac{l_{y_j}x_{\frac{1}{2}}}{2}\mathsf{a}_1v_{1,j}^2\right]+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{N_x+\frac{1}{2},j}}}{2h_{x_{N_x}}}v_{N_x,j}^2,\end{aligned}$$

so

$$\begin{aligned}\mathbf{B_h^1}(v_h,v_h) &= \sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}(v_{i+1,j}-v_{i,j})^2+\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})^2\right]\\ &+\sum_{j=1}^{N_y}\left[\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{\frac{3}{2},j}}}{2h_{x_1}}\left((v_{2,j}-v_{1,j})^2-v_{1,j}^2-v_{2,j}^2\right)+\left(\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{\frac{3}{2},j}}}{2h_{x_1}}+\frac{l_{y_j}x_{\frac{1}{2}}}{2}\mathsf{a}_1\right)v_{1,j}^2\right]\\ &\quad+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{N_x+\frac{1}{2},j}}}{2h_{x_{N_x}}}v_{N_x,j}^2+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{\frac{3}{2},j}}}{2h_{x_1}}v_{2,j}^2\\ &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{i+\frac{1}{2},j}}}{2h_{x_i}}(v_{i+1,j}-v_{i,j})^2+\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\left[\frac{k_{11_{i-\frac{1}{2},j}}}{2h_{x_{i-1}}}(v_{i,j}-v_{i-1,j})^2\right]\\ &\quad+\sum_{j=1}^{N_y}\frac{l_{y_j}^2}{h_{y_j}}\frac{k_{11_{N_x+\frac{1}{2},j}}}{2h_{x_{N_x}}}v_{N_x,j}^2+\sum_{j=1}^{N_y}\frac{l_{y_j}\mathsf{a}_1x_{\frac{1}{2}}^2}{2}v_{1,j}^2\\ &\geq C\|v_h\|_{1,h_x}.\end{aligned}$$

Also, for the term $\mathbf{B_h^2}$, we have

$$\begin{aligned}
\mathbf{B_h^2}(v_h, v_h) &= -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left[(\mathcal{G}_{22}v)_{i,j+\frac{1}{2}} - (\mathcal{G}_{22}v)_{i,j-\frac{1}{2},j}\right] l_{x_i} v_{i,j} \\
&= -\sum_{i=1}^{N_x}\left[\frac{l_{x_i}^2 k22_{i,\frac{3}{2}}}{h_{x_i}h_{y_1}}(v_{i,2}-v_{i,1}) - \frac{l_{x_i}y_{\frac{1}{2}}}{2}[\mathsf{a}_2 v_{i,1} - \mathsf{a}_2 v_{i,0}]\right] v_{i,1} \\
&\quad -\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\left[(\mathcal{G}_{22}v)_{i,j+\frac{1}{2}} - (\mathcal{G}_{1}v)_{i,j-\frac{1}{2}}\right] v_{i,j} \\
&= -\sum_{i=1}^{N_x}\left[\frac{l_{x_i}^2 k22_{i,\frac{3}{2}}}{h_{x_i}h_{y_1}}(v_{i,2}-v_{i,1}) - \frac{l_{x_i}y_{\frac{1}{2}}}{2}\mathsf{a}_2 v_{i,1}^2\right] + \mathbf{P_1}. \qquad (4.4.6)
\end{aligned}$$

Then we have that,

$$\begin{aligned}
\mathbf{P_1} &= -\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k22_{i,j+\frac{1}{2}}}{h_{y_j}}(v_{i,j+1}-v_{i,j}) - \frac{k22_{i,j-\frac{1}{2}}}{h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})\right] v_{i,j} \\
&= \sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \frac{k22_{i-\frac{1}{2},j}}{2h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})^2\right] \\
&\quad + \sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\left[\left(\frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}} + \frac{k22_{i-\frac{1}{2},j}}{2h_{y_{j-1}}}\right) v_{i,j}^2 - \frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}} v_{i,j+1}^2 - \frac{k22_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}} v_{i,j-1}^2\right].
\end{aligned}$$

Now we consider the following expansions

$$-\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}} v_{i,j+1}^2 = -\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}} v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,\frac{3}{2}}}{2h_{y_1}} v_{i,2}^2, \qquad (4.4.7)$$

and

$$-\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}} v_{i,j-1}^2 = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}} v_{i,j}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,N_y+\frac{1}{2}}}{2h_{y_{N_y}}} v_{i,N_y}^2. \qquad (4.4.8)$$

Hence using (4.4.7) and (4.4.8), we have that $\mathbf{P_1}$ becomes,

$$\begin{aligned}
\mathbf{P_1} &= \sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \frac{k22_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})^2\right] \\
&\quad + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,\frac{3}{2}}}{2h_{y_{N_y}}} v_{i,N_y}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,N_y+\frac{1}{2}}}{2h_{y_1}} v_{i,2}^2.
\end{aligned}$$

Then we have that, $\mathbf{B}_{\mathbf{h}}^{\mathbf{2}}$ becomes

$$
\begin{aligned}
&\mathbf{B}_{\mathbf{h}}^{\mathbf{2}}(v_h, v_h) = -\sum_{i=1}^{N_x}\left[\frac{l_{x_i}^2 k22_{i,\frac{3}{2}}}{h_{x_i}h_{y_1}}(v_{i,2}-v_{i,1}) - \frac{l_{x_i}y_{\frac{1}{2}}}{2}\mathsf{a}_2 v_{i,1}^2\right] + \mathbf{P_1} \\
&= \sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \frac{k22_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})^2\right] + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,N_y+\frac{1}{2}}}{2h_{y_1}}v_{i,2}^2 \\
&\qquad -\sum_{i=1}^{N_x}\left[\frac{l_{x_i}^2 k22_{i,\frac{3}{2}}}{h_{x_i}h_{y_1}}(v_{i,2}-v_{i,1}) - \frac{l_{x_i}y_{\frac{1}{2}}}{2}\mathsf{a}_2 v_{i,1}^2\right] + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,\frac{3}{2}}}{2h_{y_{N_y}}}v_{i,N_y}^2 \\
&= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,j+\frac{1}{2}}}{2h_{y_j}}(v_{i,j+1}-v_{i,j})^2 + \sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{l_{x_i}^2}{h_{x_i}}\left[\frac{k22_{i,j-\frac{1}{2}}}{2h_{y_{j-1}}}(v_{i,j}-v_{i,j-1})^2\right] \\
&\qquad + \sum_{i=1}^{N_x}\frac{l_{x_i}^2}{h_{x_i}}\frac{k22_{i,\frac{3}{2}}}{2h_{y_{N_y}}}v_{i,N_y}^2 + \sum_{i=1}^{N_x}\frac{l_{x_i}y_{\frac{1}{2}}}{2}\mathsf{a}_2 v_{i,1}^2.
\end{aligned}
$$

We then have that

$$
\mathbf{B}_{\mathbf{h}}^{\mathbf{2}}(v_h, v_h) \geq C\|v_h\|_{1,h_y}. \tag{4.4.9}
$$

We remark here that the term $\mathbf{B}_3(v_h, v_h)$ has already been shown in the previous section and is given by

$$
\mathbf{B}_{\mathbf{h}}^{\mathbf{3}}(v_h, v_h) = \mathbf{B}_{31}(v_h, v_h) + \mathbf{B}_{32}(v_h, v_h) \geq 0. \tag{4.4.10}
$$

Considering the term $\mathbf{B}_{\mathbf{h}}^{\mathbf{4}}(v_h, v_h)$, we have

$$
\begin{aligned}
\mathbf{B}_{\mathbf{h}}^{\mathbf{4}}(v_h, v_h) &= -\sum_{j=1}^{N_y} l_{y_j}(b_1 x_{3/2}v_{2,j} - \frac{b_1}{2}v_{1,j})v_{1,j} - \sum_{i=2}^{N_x}\sum_{j=1}^{N_y} l_{y_j}b_1\left(x_{i+\frac{1}{2}}v_{i+1,j} - x_{i-\frac{1}{2}}v_{i,j}\right)v_{i,j} \\
&\quad -\sum_{i=1}^{N_x} l_{x_i}(b_2 y_{3/2}v_{i,2} - \frac{b_2}{2}v_{i,1})v_{i,1} - \sum_{i=1}^{N_x}\sum_{j=2}^{N_y} l_{x_i}b_2\left(y_{j+\frac{1}{2}}v_{i,j+1} - y_{j-\frac{1}{2}}v_{i,j}\right)v_{i,j} \\
&\quad +\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(cv_{i,j}^2)l_{x_i}l_{y_j} \\
&= \mathbf{B}_{\mathbf{h}}^{\mathbf{41}}(v_h, v_h) + \mathbf{B}_{\mathbf{h}}^{\mathbf{42}}(v_h, v_h) + \mathbf{B}_{\mathbf{h}}^{\mathbf{43}}(v_h, v_h).
\end{aligned} \tag{4.4.11}
$$

For $\mathbf{B}_{\mathbf{h}}^{\mathbf{41}}(v_h, v_h)$, we have that

$$
\begin{aligned}
\mathbf{B}_{\mathbf{h}}^{\mathbf{41}}(v_h, v_h) &= -\sum_{j=1}^{N_y} l_{y_j}(b_1 x_{3/2}v_{2,j} - \frac{b_1}{2}v_{1,j})v_{1,j} - \sum_{i=2}^{N_x}\sum_{j=1}^{N_y} l_{y_j}b_1\left(x_{i+\frac{1}{2}}v_{i+1,j} - x_{i-\frac{1}{2}}v_{i,j}\right)v_{i,j} \\
&= \sum_{j=1}^{N_y}\frac{b_1 x_{\frac{1}{2}}l_{y_j}}{2}v_{1,j}^2 - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} b_1 x_{i+\frac{1}{2}}l_{y_j}v_{i+1,j}v_{i,j} + \sum_{i=2}^{N_x}\sum_{j=1}^{N_y} b_1 x_{i-\frac{1}{2}}l_{y_j}v_{i,j}^2,
\end{aligned}
$$

so

$$\begin{aligned}\mathbf{B_h^{41}}(v_h,v_h) &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1x_{i+\frac{1}{2}}l_{y_j}}{2}\left[(v_{i+1,j}-v_{i,j})^2-v_{i+1,j}^2-v_{i,j}^2\right] \\ &\quad+\sum_{j=1}^{N_y}\frac{b_1x_{\frac{1}{2}}l_{y_j}}{2}v_{1,j}^2+\sum_{i=2}^{N_x}\sum_{j=1}^{N_y}b_1x_{i-\frac{1}{2}}l_{y_j}v_{i,j}^2. \end{aligned} \tag{4.4.12}$$

Then from the following expansion,

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1x_{i+\frac{1}{2}}l_{y_j}}{2}v_{i+1,j}^2 = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1x_{i-\frac{1}{2}}l_{y_j}}{2}v_{i,j}^2+\sum_{j=1}^{N_y}\frac{b_1x_{\frac{1}{2}}l_{y_j}}{2}v_{1,j}^2, \tag{4.4.13}$$

$\mathbf{B_h^{41}}(v_h,v_h)$ becomes

$$\mathbf{B_h^{41}}(v_h,v_h) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1l_{y_j}}{2}x_{i+\frac{1}{2}}(v_{i+1,j}-v_{i,j})^2+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1l_{y_j}}{2}\left(x_{i-\frac{1}{2}}-x_{i+\frac{1}{2}}\right)v_{i,j}^2. \tag{4.4.14}$$

A similar argument exists for $\mathbf{B_h^{42}}(v_h,v_h)$, we have that

$$\begin{aligned}\mathbf{B_h^{42}}(v_h,v_h) &= -\sum_{i=1}^{N_x}l_{x_i}(b_2y_{3/2}v_{i,2}-\frac{b_2}{2}v_{i,1})v_{i,1}-\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}l_{x_i}b_2\left(y_{j+\frac{1}{2}}v_{i,j+1}-y_{j-\frac{1}{2}}v_{i,j}\right)v_{i,j} \\ &= \frac{b_2y_{\frac{1}{2}}h_{x_i}}{2}v_{i,1}^2-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}b_2y_{j+\frac{1}{2}}h_{x_i}v_{i,j+1}v_{i,j}+\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}b_2y_{j-\frac{1}{2}}h_{x_i}v_{i,j-1}v_{i,j} \\ &= \sum_{i=1}^{N_x}\sum_{j=2}^{N_y}\frac{b_2y_{j+\frac{1}{2}}h_{x_i}}{2}\left[(v_{i,j+1}-v_{i,j})^2-v_{i,j+1}^2-v_{i,j}^2\right]+\sum_{i=1}^{N_x}\frac{b_2y_{\frac{1}{2}}h_{x_i}}{2}v_{i,1}^2 \\ &\quad+\sum_{i=1}^{N_x}\sum_{j=2}^{N_y}b_2y_{j-\frac{1}{2}}h_{x_i}v_{i,j}^2. \end{aligned}$$

Then we consider the following expansion,

$$-\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2y_{j+\frac{1}{2}}h_{x_i}}{2}v_{i,j+1}^2 = -\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2y_{j-\frac{1}{2}}h_{x_i}}{2}v_{i,j}^2+\sum_{i=1}^{N_x}\frac{b_2y_{\frac{1}{2}}h_{x_i}}{2}v_{i,1}^2. \tag{4.4.15}$$

Substituting (4.4.15) into $B_{42}(v_h,v_h)$ yields,

$$\mathbf{B_h^{42}}(v_h,v_h) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2h_{x_i}}{2}y_{j+\frac{1}{2}}(v_{i,j+1}-v_{i,j})^2+\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2(y_{j-\frac{1}{2}}-y_{j+\frac{1}{2}})h_{x_i}}{2}v_{i,j}^2. \tag{4.4.16}$$

Finally, using (4.4.14), (4.4.16) and $\mathbf{B_h^{43}}(v_h, v_h)$, we have that $\mathbf{B_h^4}(v_h, v_h)$ becomes,

$$\mathbf{B_h^4}(v_h, v_h) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\frac{b_1 h_{y_j}}{2} x_{i+\frac{1}{2}}\left(v_{i+1,j} - v_{i,j}\right)^2 + \frac{b_2 h_{x_i}}{2} y_{j+\frac{1}{2}}\left(v_{i,j+1} - v_{i,j}\right)^2\right)$$
$$- \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_1 h_{x_i} h_{y_j}}{2} v_{i,j}^2 - \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{b_2 h_{y_j} h_{x_i}}{2} v_{i,j}^2 + \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(3r - (\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2)) h_{x_i} h_{y_j} v_{i,j}^2$$
$$= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\frac{b_1 h_{y_j}}{2} x_{i+\frac{1}{2}}\left(v_{i+1,j} - v_{i,j}\right)^2 + \frac{b_2 h_{x_i}}{2} y_{j+\frac{1}{2}}\left(v_{i,j+1} - v_{i,j}\right)^2\right)$$
$$+ \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\frac{4r - (\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2)}{2} h_{x_i} h_{y_j} v_{i,j}^2\right).$$

That is

$$\mathbf{B_h^4}(v_h, v_h) \geq C\|v_h\|_{0,h}. \tag{4.4.17}$$

Therefore we have that

$$\mathbf{A_h}(v_h, v_h) \geq C\|v_h\|_{1,h_x} + C\|v_h\|_{1,h_y} + C\|v_h\|_{0,h} \tag{4.4.18}$$

□

Remark 4.4.1. As remarked earlier in chapter 2, to be precise remark 2.5.1, same is true in this case. Using the coercivity properties in (4.3.28) and (4.4.18), with the fact that the linear mapping $v \to (f, v)_h$ and the two bilinear forms $\mathbf{a_h}(\cdot,\cdot)$ and $\mathbf{A_h}(\cdot,\cdot)$ are continuous in V_h and $V_h \times V_h$, the existence and uniqueness of the discrete solution U_h is ensured for both the mimetic and fitted mimetic methods in (4.3.28) and (4.4.18). The proof is done exactly as for the continuous case (see Theorem 1.7 in chapter 1) which satisfies the conditions of Theorem 1.4.

4.4.2 Consistency of the fluxes

As we did in theorem 2.6 in chapter 2, we prove the consistency of the fluxes in this section. Indeed the following theorem is an extension of theorem 2.6 in two dimensions.

Theorem 4.5. *Let Φ and Ψ be the operators defined in (4.2.1) and $w \in H^1_{0,w}(\Omega)$. Assume that $w(\cdot, y) \in H^2((0, X)$ for all $y \in (0, Y))$, and $w(x, \cdot) \in H^2((0, Y)$ for all $x \in (0, X))$, and $\Phi_x(w, \cdot, \cdot, t), \Psi_y(w, \cdot, \cdot, t) \in (L^2(\Omega))^2$ for all $t \in (0, T)$. Let Φ_h and Ψ_h be the approximations of Φ and Ψ respectively, using the mimetic finite difference method (see (4.2.2), (4.2.4) respectively) or fitted mimetic method (see (4.2.2), (4.2.4) respectively) with $\Phi_h(w(x_{\frac{1}{2}}, y_j), x_{\frac{1}{2}}, y_j, t) = \frac{x_{\frac{1}{2}}}{2}\left[(\mathsf{a}_1 + b_1)w_{1,j} - (\mathsf{a}_1 - b_1)w_{0,j}\right]$ and $\Psi_h(w(x_i, y_{\frac{1}{2}}), x_i, y_{\frac{1}{2}}, t) = \frac{y_{\frac{1}{2}}}{2}\left[(\mathsf{a}_2 + b_2)w_{i,1} - (\mathsf{a}_2 - b_2)w_{i,0}\right]$, respectively. Then under Assumption 4.2.2, there exists constants $C_1, C_2 > 0$ independent of w, h_x, l_x, l_y and h_y such that the following estimates hold:*

$$|\Phi_h(w_h, x_{i+\frac{1}{2}}, y, t) - \Phi(w, x_{i+\frac{1}{2}}, y, t)| \leq C_1 \int_{x_i}^{x_{i+1}} \left[|\Phi_x(w, \cdot, y, t)| + |w_x(\cdot, y)| + |w(\cdot, y)|\right] dx, \tag{4.5.1}$$

for $i = 0, 1, ..., N_x$, *and*

$$|\Psi_h(w_h, x, y_{j+\frac{1}{2}}, t) - \Psi(w, x, y_{j+\frac{1}{2}}, t)| \leq C_2 \int_{y_j}^{y_{j+1}} \left[|\Psi_y(w, x, \cdot, t)| + |w_y(x, \cdot, t)| + |w(x, \cdot, t)|\right] dy, \tag{4.5.2}$$

for $j = 0, 1, ..., N_y$.

Proof. The proof considers both Mimetic and Fitted-Mimetic cases. For the Mimetic case, the proof presented is for all $i = 0, 1, ..., N$. The difference with the Fitted case however is the consideration of the approximation of the flux term.

Case I (Mimetic Method)

Now we have that,

$$\Phi(w, x, y, t) := \mathsf{a}_1 x^2 \frac{\partial w}{\partial x} + b_1 x w, \tag{4.5.3}$$

and

$$\Phi_h(w_h, x_{i+\frac{1}{2}}, y_j, t) := \mathsf{a}_1 \frac{x^2_{i+\frac{1}{2}} l_{y_j}}{h_{y_j}} \frac{w_{i+1,j} - w_{i,j}}{h_{x_i}} + b_1 x_{i+\frac{1}{2}} w_{i+1,j}. \tag{4.5.4}$$

Also,

$$\Psi(w, x, y, t) := \mathsf{a}_2 y^2 \frac{\partial w}{\partial y} + b_2 y w \tag{4.5.5}$$

and

$$\Psi_h(w_h, x_i, y_{j+\frac{1}{2}}, t) := \mathsf{a}_2 \frac{y^2_{i+\frac{1}{2}} l_{x_i}}{h_{x_i}} \frac{w_{i,j+1} - w_{i,j}}{h_{y_j}} + b_2 y_{j+\frac{1}{2}} w_{i,j+1}. \tag{4.5.6}$$

Then from (4.5.5) and (4.5.6), we have that

$$\begin{aligned}
\Phi_h(w_h, x_{i+\frac{1}{2}}, y_j, t) - \Phi(w, x_{i+\frac{1}{2}}, y_j, t) &= \mathsf{a}_1 \frac{x^2_{i+\frac{1}{2}} l_{y_j}}{h_{y_j}} \frac{w_{i+1,j} - w_{i,j}}{h_{x_i}} + b_1 x_{i+\frac{1}{2}} w_{i+1,j} \\
&\quad - \mathsf{a}_1 x^2_{i+\frac{1}{2}} w_x(x_{i+\frac{1}{2}}, y_j) - b_1 x_{i+\frac{1}{2}} w(x_{i+\frac{1}{2}}, y_j) \\
&= \mathsf{a}_1 x^2_{i+\frac{1}{2}} \left[\frac{l_{y_j}}{h_{y_j}} \frac{w_{i+1,j} - w_{i,j}}{h_{x_i}} - w_x(x_{i+\frac{1}{2}}, y_j)\right] + b_1 x_{i+\frac{1}{2}} \left[w_{i+1,j} - w(x_{i+\frac{1}{2}}, y_j)\right] \\
&=: A_1 + A_2.
\end{aligned} \tag{4.5.7}$$

As in chapter 2, using the Sobolev embedding theorem, for $w(\cdot, y) \in H^2(0, X), \forall y \in (0, Y)$, we have $H^2(0, X) \hookrightarrow C^1(0, X)$, for all $y \in (0, Y)$, and so $w_x(\cdot, y)$ exists.

Now we adopt the following Taylor expansion with integral remainder to estimate the first term,

$$w(z, y) = w(x_{i+\frac{1}{2}}, y) + (z - x_{i+\frac{1}{2}}) w_x(x_{i+\frac{1}{2}}, y) + \int_{x_{i+\frac{1}{2}}}^{z} (z - x) w_{xx}(x, y) dx. \tag{4.5.8}$$

Then when we set $z = x_{i+1}$, we have that

$$w(x_{i+1}, y) = w(x_{i+\frac{1}{2}}, y) + (x_{i+1} - x_{i+\frac{1}{2}}) w_x(x_{i+\frac{1}{2}}, y) + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - x) w_{xx}(x, y) ds, \tag{4.5.9}$$

and also set $z = x_i$, we have that

$$w(x_i) = w(x_{i+\frac{1}{2}}, y) + (x_i - x_{i+\frac{1}{2}}) w_x(x_{i+\frac{1}{2}}, y) - \int_{x_{i+\frac{1}{2}}}^{x_i} (x_i - x) w_{xx}(x, y) ds. \tag{4.5.10}$$

Then we have that

$$\begin{aligned} \frac{w(x_{i+1}, y_j) - w(x_i, y_j)}{h_{x_i}} &= w_x(x_{i+\frac{1}{2}}, y_j) + \frac{1}{h_{x_i}} \left[\int_{x_i}^{x_{i+\frac{1}{2}}} (x_i - x) w_{xx}(x, y) dx \right] \\ &+ \frac{1}{h_{x_i}} \left[\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - x) w_{xx}(x, y) dx \right], \end{aligned}$$

which leads to

$$\begin{aligned} \left[\frac{l_{y_j}}{h_{y_j}} \frac{w(x_{i+1}, y_j) - w(x_i, y_j)}{h_{x_i}} - w_x(x_{i+\frac{1}{2}}, y_j) \right] &= \left(\frac{l_{y_j}}{h_{y_j}} - 1 \right) w_x(x_{i+\frac{1}{2}}, y_j) \\ + \frac{l_{y_j}}{h_{y_j} h_{x_i}} \left[\int_{x_i}^{x_{i+\frac{1}{2}}} (x_i - x) w_{xx}(x, y) dx \right. &+ \left. \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - x) w_{xx}(x, y) dx \right], \end{aligned}$$

so that using lemma 4.2.2, we have that

$$\begin{aligned} \left[\frac{l_{y_j}}{h_{y_j}} \frac{w(x_{i+1}, y_j) - w(x_i, y_j)}{h_{x_i}} - w_x(x_{i+\frac{1}{2}}, y_j) \right] &\leq C w_x(x_{i+\frac{1}{2}}, y_j) \\ + \frac{C}{h_{x_i}} \left[\int_{x_i}^{x_{i+\frac{1}{2}}} (x_i - x) w_{xx}(x, y) dx \right. &+ \left. \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} (x_{i+1} - x) w_{xx}(x, y) dx \right]. \end{aligned} \tag{4.5.11}$$

We use the following inequalities

$$\left| \int_{x_i}^{x_{i+\frac{1}{2}}} (x_i - x) w_{xx}(x, y) dx \right| \leq \frac{h_{x_i}}{2} \int_{x_i}^{x_{i+1}} |w_{xx}(x, y)| ds \tag{4.5.12}$$

$$\left| \int_{x_{i+\frac{1}{2}}}^{x_i+1} (x_{i+1} - x) w_{xx}(x, y) ds \right| \leq \frac{h_{x_i}}{2} \int_{x_i}^{x_{i+1}} |w_{xx}(x, y)| ds \tag{4.5.13}$$

Then from (4.5.11), for $C_1 > 0$, we have

$$\left| \frac{l_{y_j}}{h_{y_j}} \frac{w(x_{i+1}, y_j) - w(x_i, y_j)}{h_{x_i}} - w_x(x_{i+\frac{1}{2}}, y_j) \right| \leq \frac{C_1}{2} \int_{x_i}^{x_{i+1}} |w_{xx}(x, y)| dx. \tag{4.5.14}$$

Hence we have that

$$\mathsf{a}_1 x^2_{i+\frac{1}{2}} \left| \frac{w(x_{i+1}, y_j) - w(x_i, y_j)}{h_{x_i}} - w_x(x_{i+\frac{1}{2}}, y_x) \right| \quad \leq \quad \int_{x_i}^{x_{i+1}} \frac{\mathsf{a}_1 C_1 x^2_{i+\frac{1}{2}}}{2} |w_{xx}(x, y)| dx. \tag{4.5.15}$$

Now we consider the following from Assumption 4.2.2, we have

$$\begin{aligned} \left(\frac{x_{i+\frac{1}{2}}}{x}\right)^2 &\leq \left(\frac{x_{i+1} + x_i}{2x_i}\right)^2 = \left(1 + \frac{h_{x_i}}{2x_i}\right)^2 = \left(1 + \frac{h_{x_i}}{2(x_{i-1} + h_{x_{i-1}})}\right)^2 \\ &\leq \left(1 + \frac{h_{x_i}}{2h_{x_{i-1}}}\right)^2 \leq \left(1 + \frac{h_{x_i}}{2h_{x_{i+1}}}\right)^2 \leq \left(1 + \frac{C_2}{2}\right)^2, \end{aligned} \tag{4.5.16}$$

then from (4.5.15) we have

$$\begin{aligned} \mathsf{a}_1 x_{i+\frac{1}{2}} \left| \frac{w(x_{i+1}, y_j) - w(x_i, y_j)}{h_{x_i}} \right| \quad &\leq \quad \int_{x_i}^{x_{i+1}} \frac{\mathsf{a}_1 C_1 x_{i+\frac{1}{2}}}{2x^2} |x^2 w_{xx}(x, y)| dx \\ &\leq \quad \frac{\mathsf{a}_1 C_1}{2} \left(1 + \frac{C_2}{2}\right)^2 \int_{x_i}^{x_{i+1}} |x^2 w_{xx}(x, y)| ds. \end{aligned}$$

Now from (2.4.7), we have

$$\mathsf{a}_1 x^2 w_{xx} \quad = \quad \Phi_x(w, x, y, t) - (2\mathsf{a}_1 + b_1) x w_x - bw, \tag{4.5.17}$$

then we have that, for $C > 0$,

$$\begin{aligned} &\mathsf{a}_1 x_{i+\frac{1}{2}} \left| \frac{w(x_{i+1}, y_j) - w(x_i, y_j)}{h_{x_i}} \right| \leq C \int_{x_i}^{x_{i+1}} |\Phi_x(w, x, y, t) - (2\mathsf{a}_1 + b_1) x w_x - b_1 w| dx \\ &\quad \leq C \int_{x_i}^{x_{i+1}} \left[|\Phi_x(w, x, y, t)| + |(2\mathsf{a}_1 + b_1)||w_x| + |b_1 w| \right] dx \\ &\quad = C \left(\int_{x_i}^{x_{i+1}} \left[|\Phi_x(w, x, y, t)| + |b_1||w| \right] dx + \int_{x_i}^{x_{i+1}} |(2\mathsf{a}_1 + b_1)||w_x| ds \right) \\ &|A_1| \leq C \left(\int_{x_i}^{x_{i+1}} \left[|\Phi_x(w, x, y, t)| + |w_x| + |w| \right] dx \right). \end{aligned}$$

Note that C is a changing positive constant.

Again,for $w(\cdot, y) \in H^2(0, X), \forall y \in (0, Y)$, we have $H^2(0, X) \hookrightarrow C^1(0, X)$, for all $y \in (0, Y)$, and so $w_x(\cdot, y)$ exists and hence we estimate A_2 by using the first order Taylor expansion

$$w_{i+1,j} - w(x_{i+\frac{1}{2}}, y_j) = \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} w_x(x, y) ds \tag{4.5.18}$$

and then we have

$$\begin{aligned} bs_{i+\frac{1}{2}} \left| w_{i+1,j} - w(x_{i+\frac{1}{2}}, y_j) \right| &\leq b \left| x_{i+\frac{1}{2}} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} w_x ds \right| \\ |A_2| &\leq C \int_{x_i}^{x_{i+1}} |w_x(x, y)| dx. \end{aligned}$$

Therefore we have that

$$\begin{aligned} |\Phi(w_h, x_{i+\frac{1}{2}}, y_j, t) - \Phi(w, x_{i+\frac{1}{2}}, y_j, t)| &= |A_1 + A_2| \leq |A_1| + |A_2| \\ &\leq C \left(\int_{s_i}^{s_{i+1}} \left[|\Phi_x(w, x, y, t)| + |w_x| + |w| \right] dx \right) \end{aligned}$$

Case II (Fitted mimetic Method)

Indeed for the case II, it is sufficient to consider the flux approximation at $i = 0$, hence we have that

$$\begin{aligned} &\left| \Phi_h(w_h(x_{\frac{1}{2}}), x_{\frac{1}{2}}, y_j, t) - \Phi(w(x_{\frac{1}{2}}, y_j), x_{\frac{1}{2}}, y_j, t) \right| \\ &= \left| \frac{x_{\frac{1}{2}}}{2} \left[(\mathsf{a}_1 + b_1) w_{1,j} - (\mathsf{a}_1 - b_1) w_{0,j} \right] - \mathsf{a}_1 x_{\frac{1}{2}}^2 w_x(x_{\frac{1}{2}}, y_j) - b_1 x_{\frac{1}{2}} w(x_{\frac{1}{2}}, y_j) \right| \\ &= \left| \frac{x_{\frac{1}{2}}}{2} \left[(\mathsf{a}_1 + b_1) \left(\frac{w_{1,j} - w_{0,j}}{h_{x_0}} h_{x_0} - w_{0,j} \right) - (\mathsf{a}_1 - b_1) w_{0,j} \right] - \mathsf{a}_1 x_{\frac{1}{2}}^2 w_x(x_{\frac{1}{2}}, y_j) - b_1 x_{\frac{1}{2}} w(x_{\frac{1}{2}}, y_j) \right| \\ &= \left| x_{\frac{1}{2}}^2 (\mathsf{a}_1 + b_1) \left(\frac{w_{1,j} - w_{0,j}}{h_0} \right) + b_1 x_{\frac{1}{2}} w_{0,j} - \mathsf{a}_1 x_{\frac{1}{2}}^2 w_x(x_{\frac{1}{2}}, y_j) - b_1 x_{\frac{1}{2}} w(x_{\frac{1}{2}}, y_j) \right|, \text{ since } h_{x_0} = 2x_{\frac{1}{2}} \\ &= \left| \mathsf{a}_1 x_{\frac{1}{2}}^2 \left(\frac{w_{1,j} - w_{0,j}}{h_{x_0}} - w_x(x_{\frac{1}{2}}) \right) + bs_{\frac{1}{2}} \left(w_{0,j} - w(x_{\frac{1}{2}}, y_j) \right) + b_1 x_{\frac{1}{2}}^2 \left(\frac{w_{1,j} - w_{0,j}}{h_{x_0}} \right) \right| \\ &\leq \left| \mathsf{a}_1 x_{\frac{1}{2}}^2 \left(\frac{w_{1,j} - w_{0,j}}{h_{x_0}} - w_x(x_{\frac{1}{2}}, y_j) \right) \right| + \left| b_1 x_{\frac{1}{2}} \left(w_{0,j} - w(x_{\frac{1}{2}}, y_j) \right) \right| + \left| b_1 x_{\frac{1}{2}}^2 \left(\frac{w_{1,j} - w_{0,j}}{h_{x_0}} \right) \right| \\ &=: D_{10} + D_{11} + D_{12} \end{aligned}$$

To estimate D_{10}, D_{11} and D_{12}, remember that for $w(\cdot,y) \in H^2(0,X), \forall y \in (0,Y)$, $H^2(0,X) \hookrightarrow C^1(0,X)$, for all $y \in (0,Y)$, and $w_x(\cdot,y)$ exists.

Then we estimate D_{10} by using a similar argument as with the general case, to obtain

$$\left(\frac{w_{1,j}-w_{0,j}}{h_{x_0}} - w_x(x_{\frac{1}{2}},y_j)\right) = \frac{1}{h_{x_0}}\int_{x_0}^{x_{\frac{1}{2}}} w_{xx}(x_0-x)dx + \frac{1}{h_{x_0}}\int_{x_{\frac{1}{2}}}^{x_1} w_{xx}(x_1-x)dx. \quad (4.5.19)$$

Hence,

$$\begin{aligned} D_{10} = \mathsf{a}_1 x_{\frac{1}{2}}^2 \left|\frac{w_{1,j}-w_{0,j}}{h_{x_0}} - w_x(x_{\frac{1}{2}},y_j)\right| &\le \frac{\mathsf{a}_1 x_{\frac{1}{2}}^2}{2}\int_{x_0}^{x_1} |w_{xx}|dx \\ &\le C\int_{x_0}^{x_1} \left[|\Phi_x(w)| + |w_x| + |w|\right]dx \end{aligned}$$

Now we consider the estimate of D_{11}

$$D_{11} = b_1 x_{\frac{1}{2}} \left|w_{0,j} - w(x_{\frac{1}{2}},y_j)\right| \le b_1 x_{\frac{1}{2}} \int_{x_0}^{x_1} |w_x|dx \le C\int_{x_0}^{x_1} |w_x|dx. \quad (4.5.20)$$

Also, the estimate of D_{12}

$$D_{12} = b_1 x_{\frac{1}{2}}^2 \left|\frac{w_{1,j}-w_{0,j}}{h_{x_0}}\right| \le \frac{b_1 x_{\frac{1}{2}}}{2}\int_{x_0}^{x_1} |w_x|dx \le C\int_{x_0}^{x_1} |w_x|dx. \quad (4.5.21)$$

Therefore,

$$\left|\Phi_h(w_h(x_{\frac{1}{2}},y_j), x_{\frac{1}{2}}, y_j, t) - \Phi(w(x_{\frac{1}{2}},y_j), x_{\frac{1}{2}}, y_j, t)\right| \le C\int_{x_0}^{x_1} \left[|\Phi_x(w)| + |w_x| + |w|\right]dx. \quad (4.5.22)$$

Now similarly from (4.5.5)) and (4.5.6), we have that

$$\begin{aligned} &\Psi_h(w_h, x_i, y_{j+\frac{1}{2}}, t) - \Phi(w, x_i, y_{j+\frac{1}{2}}, t) = a_{22} y_{j+\frac{1}{2}}^2 \frac{w_{i,j+1}-w_{i,j}}{h_{y_j}} + b_2 y_{j+\frac{1}{2}} w_{i,j+1} \\ &-a_{22} y_{j+\frac{1}{2}}^2 w_y(x_i, y_{j+\frac{1}{2}}) - b_2 y_{j+\frac{1}{2}} w(x_i, y_{j+\frac{1}{2}}) \\ &= a_{22} y_{j+\frac{1}{2}}^2 \left[\frac{w_{i,j+1}-w_{i,j}}{h_{y_j}} - w_y(x_i, y_{j+\frac{1}{2}})\right] + b_2 y_{j+\frac{1}{2}} \left[w_{i,j+1} - w(x_i, y_{j+\frac{1}{2}})\right] \\ &=: A_3 + A_4. \end{aligned}$$

This case follows exactly as the case before. As in the previous case, indeed for $w(x,\cdot) \in H^2(0,Y), \forall x \in (0,X)$, we have $H^2(0,Y) \hookrightarrow C^1(0,Y)$, for all $x \in (0,X)$, and so $w_y(x,\cdot)$ exists. Then following line by line the previous case, we have

$$|A_3| \leq C \left(\int_{y_j}^{y_{j+1}} \left[|\Psi_y(w,x,y,t)| + |w_y| + |w| \right] dy \right). \tag{4.5.23}$$

and

$$|A_4| \;\; \leq \;\; C \int_{y_j}^{y_{j+1}} |w_y| dy. \tag{4.5.24}$$

Therefore we have that

$$\begin{aligned} |\Psi(w_h, x_i, y_{j+\frac{1}{2}}, t) - \Psi(w, x_i, y_{j+\frac{1}{2}}, t)| \;\; &= \;\; |A_3 + A_4| \\ &\leq \;\; C \left(\int_{y_j}^{y_{j+1}} \left[|\Psi_y(w,x,y,t)| + |w_y| + |w| \right] dy \right). \end{aligned}$$

Again, for the fitted mimetic case, it is sufficient to consider the flux approximation at $j = 0$. This case follows from the fitted case above. Therefore we have that

$$\left| \Phi_h(w_h(x_i, y_{\frac{1}{2}}), x_i, y_{\frac{1}{2}}, t) - \Phi(w(x_i, y_{\frac{1}{2}}), x_i, y_{\frac{1}{2}}, t) \right| \leq C \int_{y_0}^{y_1} \left[|\Phi_y(w)| + |w_y| + |w| \right] dy \tag{4.5.25}$$

□

4.6 Numerical experiment

We discretize the ODE systems (4.1.45) and (4.1.73) in time. We let $t_m (m = 0, 1, 2, ..., M_T)$ be a set of partition points in $[0, T]$ satisfying $0 = t_0 < t_1 < ... < t_{M_T} = T$. Then taking into consideration the transformation $t = T - t$, we have

$$\begin{cases} \dfrac{U_h^{m+1} - U_h^m}{\Delta t} + \theta G(U_h^{m+1}, t_{m+1}) + (1-\theta) G(U_h^{m+1}, t_{m+1}) = 0 \\ \qquad\qquad U_h(0) = U_h^*, \qquad\qquad 0 < \theta \leq 1, \end{cases} \tag{4.6.1}$$

with $G(U_h(t), t) = \left(\widehat{\mathcal{C}}_h U_h(t) + F(U_h(t), t) \right)$, and

$$\begin{cases} \dfrac{U_H^{m+1} - U_H^m}{\Delta t} + \theta G(U_H^{m+1}, t_{m+1}) + (1-\theta) G(U_H^m, t_m) = 0 \\ \qquad\qquad U_H(0) = U_H^*, \qquad\qquad 0 < \theta \leq 1, \end{cases} \tag{4.6.2}$$

with $G(U_H(t), t) = \left(\widehat{\mathcal{C}}_H U_H(t) + F(U_H(t), t) \right)$. The scheme is order 2 in time when $\theta = 1/2$ and order 1 if $\theta \neq 1/2$.

4.6.1 Test 1: Error for European options

We are considering the case when the penalty parameter $\lambda = 0$ in (4.6.1), which indeed corresponds to the European option. There exits an analytical solution to the two-dimensional Black-Scholes PDE when the coefficents are constant, and this is given in [16] (see chapter 3).

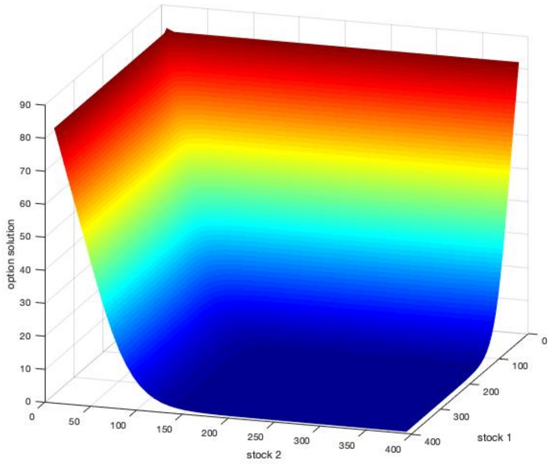

Figure 4.1: The Mimetic Solution for the European option

$N_x \times N_y$	Fitted Finite Volume Error	Mimetic Error (Diagonal Diffusion term)	Mimetic Error (Full Diffusion term)	Fitted Mimetic Error (Diagonal Diffusion term)	Fitted Mimetic Error (Full Diffusion term)
10×10	0.1095	0.0061	0.0056	0.0056	0.0052
15×15	0.0733	0.0059	0.0055	0.0055	0.0051
30×30	0.0369	0.0057	0.0053	0.0053	0.0050
50×50	0.0222	0.0057	0.0053	0.0053	0.0050
75×75	0.0148	0.0056	0.0052	0.0052	0.0049
100×100	0.0111	0.0056	0.0052	0.0052	0.0049

Table 4.1: Table showing the L^2-error of the full matrix coefficient two-dimensional European option problem with the following parameters: $E = 100$, $r = 0.07$, $\rho = 0.4$, $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $T = 1$, $X_{\max} = Y_{\max} = 4E$ and $\Delta t = T/100$.

From Table 4.2 we can observe the accuracy of the mimetic methods compared to the finite volume method. The table further shows the importance of the fitted scheme as the fitted mimetic method outperforms the standard mimetic method.

The table below shows the CPU time (in seconds) of each method presented in two dimensions.

$N_x \times N_y$	Fitted Finite Volume CPU time (sec)	Mimetic (Diagonal Diffusion term) CPU time (sec)	Mimetic (Full Diffusion term) CPU time (sec)	Fitted Mimetic (Diagonal Diffusion term) CPU time (sec)	Fitted Mimetic (Full Diffusion term) CPU time (sec)
10×10	0.751	0.744	0.562	0.326	0.318
15×15	1.314	1.126	0.770	0.583	0.559
30×30	3.211	2.699	1.887	1.951	1.822
50×50	6.944	6.733	4.168	5.483	3.971
75×75	15.211	14.944	9.188	12.749	8.794
100×100	30.013	27.110	16.704	26.271	15.014

Table 4.2: Table showing the CPU time (sec) of the full matrix coefficient two-dimensional European option problem with the following parameters: $E = 100$, $r = 0.07$, $\rho = 0.4$, $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $T = 1$, $X_{\max} = Y_{\max} = 4E$ and $\Delta t = T/100$.

In Table 4.2, we observe that the fitted mimetic with full diffusion tensor is more efficient when compared to the fitted mimetic method (diagonal diffusion tensor) and the fitted finite volume method.

4.6.2 Test 2: Error for American options

Here we perform some numerical experiments for (4.6.2) and (4.6.1).

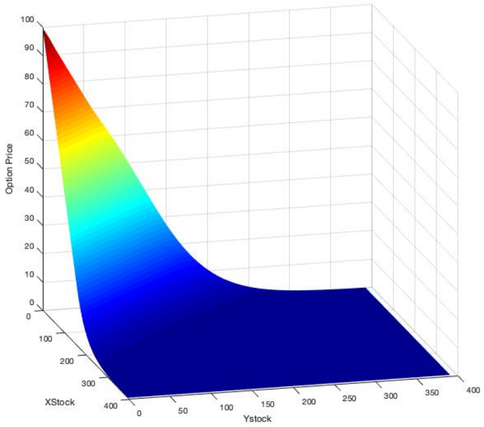

Figure 4.2: Fitted Mimetic Finite Difference method with the following parameters: $E = 100$, $r = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $T = 1$, $X_{\max} = Y_{\max} = 4E$, $\Delta t = 1/100$ and penalty parameters: $\lambda = 100$, $k = 0.5$.

As we did in chapter 3, since American options generally have no analytical solutions even for constant parameters of the Black-Scholes operator, we consider using a reference solution to compute the error. The argument remains the same as in test 2 chapter 3. Note in our comparison in test 1 above that, we considered the family of mimetic methods developed in both chapter 3 and chapter 4. We therefore choose the fitted mimetic method with full diffusion tensor matrix (developed in this chapter) as the reference solution. The reason for this choice remains as in chapter 3, since the fitted mimetic method with full diffusion tensor matrix presented more accurate results in test 1 above. The table below shows the L^2- relative error:

$N_x \times N_y$	Fitted Finite Volume Error	Mimetic Error (Diagonal Diffusion term)	Mimetic Error (Full Diffusion term)	Fitted Mimetic Error (Diagonal Diffusion term)
15×15	0.1416	0.0039	0.0036	0.0018
20×20	0.1440	0.0026	0.0025	0.0015
30×30	0.0882	0.0011	0.0010	0.0011
50×50	0.0876	7.863e-04	7.6980e-04	9.7070e-04

[1] $T = 1,\ r = 0.05,\ E = 1,\ X_{\max} = Y_{\max} = 4E,\ \sigma_1 = \sigma_2 = 0.2,\ \rho = 0.4,\ \alpha_1 = \alpha_2 = 0.5,\ tol = 10e-7,\ \epsilon = 10e-4,\ \Delta t = T/100$, and penalty parameters: $\lambda = 100, k = 2$.

Table 4.3: This table shows the two-dimensional L^2-relative error for the various spatial discretization methods for the American option

Furthermore, we present the CPU timings (in seconds) for all the methods in Table 4.3 below

$N_x \times N_y$	Fitted Finite Volume CPU time (sec)	Mimetic (Diagonal Diffusion term) CPU time (sec)	Mimetic (Full Diffusion term) CPU time (sec)	Fitted Mimetic (Diagonal Diffusion term) CPU time (sec)
15×15	0.525	0.498	0.473	0.479
20×20	0.697	0.643	0.611	0.628
30×30	1.792	1.672	1.638	1.651
50×50	11.854	10.332	8.846	9.841

[1] $T = 1,\ r = 0.05,\ E = 1,\ X_{\max} = Y_{\max} = 4E,\ \sigma_1 = \sigma_2 = 0.2, \rho = 0.4, \alpha_1 = \alpha_2 = 0.5,\ tol = 10e-7,\ \epsilon = 10e-4,\ \Delta t = T/100$, and penalty parameters: $\lambda = 100, k = 2$.

Table 4.4: This table shows the CPU time (in seconds) for the various spatial discretization methods for the American option

As in the previous test, from Table 4.3 we can observe the accuracy of the family of mimetic schemes compared to the fitted finite volume method in [43] for the American option. Furthermore, in Table 4.4, we observe that the mimetic with full diffusion tensor is more efficient when compared to the fitted mimetic (diagonal diffusion tensor), standard mimetic (diagonal diffusion tensor) and the fitted finite volume methods.

Again, we used the Newton method with tolerance $tol = 10e-7$, to solve the non-linear full discrete solutions in (4.6.1) and (4.6.2), with initial guesses U_h^m and U_H^m, respectively. Furthermore, recall that $[U^{*m} - U_h^m]_+^{1/k} = \max\left\{[U_h^{*m} - U_h^m]^{1/k}, 0\right\}$, which as we did in chapter 3, for $\epsilon > 0$, we have

$$[U_h^{*m} - U_h^m]_+^{1/k} = \begin{cases} [U_h^{*m} - U_h^m]^{1/k}, & \text{if } \ U_h^{*m} - U_h^m \geq \epsilon \\ 0, & \text{otherwise.} \end{cases} \tag{4.6.3}$$

4.6.3 Conclusion

In this chapter, we have presented both the standard mimetic and fitted mimetic finite difference methods coupled with the standard implicit time stepping scheme to fully discretize the two-dimensional penalised option problem. The differentiating feature in this chapter is the fact that we considered the full matrix tensor for the flux term when designing the mimetic scheme. In chapter 3 however, we have considered a diagonal form of this matrix tensor. We have proved the unique solvability of all semi-discrete solutions resulting from the spatial discretization. Further in the chapter, we have provided some consistency results for the flux terms of the semi-discrete solutions. To show the accuracy of the proposed methods, we provided some numerical experiments. In conclusion, we have observed that the novel fitted mimetic scheme with full matrix coefficient was more efficient when compared to the standard (full matrix tensor) mimetic method, the fitted (diagonal tensor) mimetic and the standard (diagonal tensor) mimetic methods. Nonetheless, the family of mimetic methods proposed in this thesis altogether were more accurate when compared to the fitted finite volume method proposed in [43].

5. Conclusions and Future Work

This chapter is a conclusion of the work in this thesis. We have developed accurate spatial discretization schemes for pricing vanilla options in one and two dimensions. The methods we have developed are primarily because the American option problem, in general, does not have an analytical solution. Furthermore, the methods were developed to handle the degeneracy of the Black-Scholes differential operator. The standard mimetic and fitted mimetic finite difference methods we have proposed here have proven to be very accurate spatial discretization schemes to solve the European and American option problems in one and two dimensions.

In the first chapter, we have introduced results in the literature around the formulation of the option problem. In particular, we have considered the so called linear complementarity problem (LCP) that fully encapsulates the early exercise constraint and the financial reasoning behind American option pricing. Furthermore in the chapter, the power penalty method was resorted to approximate the resulting variational formulation of the complementarity problem. The chapter further considered the convergence of the penalty method. It is important to note here that, when the penalty parameter $\lambda = 0$, the resulting PDE is associated with the European option problem. We have used this fact throughout the thesis.

Having laid all the necessary foundations for the work, we have presented our main contribution to this thesis in chapter 2. We considered a spatial discretization technique that mimics important fundamental properties of the continuous problem in the discrete case by constructing discrete analogs of these properties. The standard mimetic finite difference method has proven to be a very special spatial discretization technique in the literature. We have presented the standard mimetic method combined with the fitted scheme proposed in the literature to handle the discontinuity and degeneracy of the Black-Scholes differential operator. The resulting is a novel spatial discretization technique to solve the option problem in one dimension. We have remarked that, to the best of our knowledge, the mimetic scheme has not previously been used to solve this problem. In this chapter, the unique solvability and consistency results for all the resulting semi-discrete solutions have been presented. Furthermore, we have resorted to the standard θ time-stepping scheme to obtain fully discrete solutions. We have presented a rigorous convergence proof of the fully discrete solution (fitted mimetic method) coupled with the standard implicit time-stepping scheme. The expected first-order convergence in time for the method has been obtained in this chapter. To back our theoretical results, we have performed numerical experiments. We have observed from the results that the novel fitted mimetic finite difference method is more accurate than the standard mimetic, standard finite difference, and the fitted finite volume methods already proposed in the literature.

In chapter 3, we have extended our results from the one-dimensional case to two dimensions. The design of the standard mimetic finite difference method was similar to the one-dimensional case. Again, we have coupled the resulting fitted mimetic finite difference method with the standard implicit time stepping scheme. The approach resulted in accurate numerical methods for solving the option problem. Using numerical experiments we have been able to observe how accurate the methods are. We have remarked that we considered a diagonal matrix tensor for the flux term.

The diagonal flux term is considered in the application of mimetic methods in the literature.

In chapter 4, we have considered the two-dimensional problem with a specific emphasis on the coefficient matrix tensor of the flux term. Unlike in chapter 3 where we had considered a diagonal coefficient matrix tensor, we have considered the full matrix tensor in this chapter. The construction of the mimetic scheme is therefore affected. We used the resulting fitted mimetic method to discretize the two-dimensional option problem. Furthermore, in this chapter, we have presented the unique solvability and consistency results just as we had done in chapter 2. We then adopted the standard implicit scheme to obtain the fully discrete solutions to all the semi-discrete solutions presented earlier in the chapter. Using numerical experiments, we have shown that the fitted mimetic method (full matrix tensor) is more efficient when compared to the fitted mimetic (diagonal matrix tensor) and the fitted finite volume methods.

Therefore as a conclusion to this work, we have been able to develop accurate spatial discretization techniques coupled with a standard implicit time stepping scheme to solve the one and two-dimensional option problems (American and European options).

A potential future work resulting from this thesis would be to extend the fitted mimetic finite difference technique developed here to the three-dimensional option problem. The construction of the three-dimensional mimetic scheme remains comparable in large extent to the two dimensional scheme [27, 34]. It would be interesting to apply the novel fitted mimetic formulation to the three-dimensional option problem. Furthermore, an even more interesting idea would be to present rigorous convergence proofs for all the schemes given.

The work of *Lie et al* [26] is also of high interest to us. Trying to incorporate the schemes developed here into an open source software could help to migrate our formulation onto highly unstructured grids and other complex domains fairly easier. It is an interest we look to pursuing sooner rather than later.

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