
An investigation into the constitution of absolute value inequalities by Grade 12 students in a selection of Western Cape State schools as displayed in students' solutions to a baseline test problem.

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COMPULSORY DECLARATION

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ABSTRACT

This dissertation is an investigation into the constitution of absolute value inequalities in Grade 12 students' solutions to the problem $|2x - 3| < 4$ in a baseline Mathematics test conducted in seven schools populated by students from working-class families in the Western Cape of South Africa. This study is located within the general problematic of the constitution of school Mathematics in pedagogic settings and the methodology draws from a study by Davis (2013a) who examined Grade 11 students' treatment of linear inequalities. This study uses a grounded theory approach (Glaser & Strauss 1967; 1999) as well as Weber's (1964) theory of ideal type categories to organize students' solutions into ideal type categories through comparative analysis. The methodology also draws from Bernstein's (1996) notion of the pedagogic device and in particular his evaluative rule and recontextualising rule as well as methodology by Davis (2010a; 2010b; 2010c; 2011a; 2011b; 2013b) for describing students' mathematical activity in terms of operations, domains and codomains.

The production and analysis of the data is in two parts: part 1 analyses the recontextualisation of the topic of absolute value inequalities in the field of Mathematics in the relevant curriculum documents and a selection of relevant textbooks. Part 1 of the analysis informs part 2 which analysed the recontextualisation of the topic in students' solutions using ideal-type categories. From the analysis of students' solutions to the test item, three different levels of categories using three different sets of criteria emerged. The first level categorized attempted solutions in terms of how the notion of absolute value was maintained or disrupted, the second level categorized attempted solutions in terms of how the notion of order with regards to the logical connectives was maintained or disrupted and the third level categorized attempted solutions in terms of how the notion of order with regards to the order relations was maintained or disrupted.

The results of this study confirm the general findings in the literature which show that students' treatment of inequalities is heavily influenced by their experiences of solving equations- as evidenced by students who inserted an equality symbol into their solution of an absolute value inequality problem. Another finding in the literature confirmed in this study is that one of the most common errors in students' solutions to absolute value inequalities is related to their inappropriate use or non-use of logical connectives. One of the most striking findings of this study is that the majority of students immediately treated the absolute value inequality as a linear expression or as two separate linear expressions, suggesting that for most students, the notion of absolute value is absent in their conceptions of absolute value inequalities. This study also found that the majority of students' computational activity consisted of operation-like manipulations such as "switching" which reverses the spatial orientation of the inequality symbol under certain conditions, thus constituting the topic, absolute value inequalities, as a combination of basic arithmetic and "operations" on symbols.

TABLE OF CONTENTS

Declaration	1
Abstract	2
Table of contents	3
Acknowledgements	6
List of figures	7
List of tables	9
Chapter 1	10
Introduction	
1.1 The state and context of Mathematics education in South Africa	10
1.2 Identifying the problem	10
1.3 This study	11
1.4 An overview of the thesis	12
Chapter 2	14
A Review of the Literature	
2.1 Introduction	14
2.2 Teaching methods	14
2.3 Conceptions of inequalities and the equation/inequality connection	15
2.4 The procedural/conceptual distinction in the literature	17
2.5 Solving inequalities using a graphical approach	18
2.6 Absolute value inequalities	19
2.7 Conclusion	21
Chapter 3	22
A General Methodology	
3.1 Glaser and Strauss' grounded theory approach	22
3.2 Weber's ideal type categories	22
3.3 Bernstein's pedagogic device and the constitution of Mathematics	23
3.4 Operations, domains and codomains	25

Chapter 4.....26

Methodology for the Production and Analysis of Data

4.1 Research Design	26
4.1.1 Sample	26
4.1.2 Data Collection	27
4.2 Procedures for production and analysis of data	29
4.2.1 Part 1: The recontextualisation of the topic in the curriculum and textbooks	29
4.2.2 Part 2: The recontextualisation of the topic in students' solutions to a test item	30
4.3 Reliability and validity	31
4.4 Ethical considerations	32

Chapter 5.....33

Part 1: The recontextualisation of the topic in the curriculum and textbooks

5.1 Absolute value inequalities in the field of Mathematics	33
5.1.1 The definitions of absolute value in the field of Mathematics	33
5.1.2 Inequalities and order relations in the field of Mathematics	34
5.1.3 Absolute value inequalities and logical connectives in the field of Mathematics	35
5.2 The recontextualisation of absolute value inequalities in the curriculum	36
5.3 The recontextualisation of absolute value inequalities in the textbook	36
5.3.1 The definitions of absolute value in the textbook	36
5.3.2 Inequalities in the textbook	39
5.3.3 Methods for solving absolute value equations and graphs in the textbook	40
5.3.4 Methods for solving absolute value inequalities in the textbook	41
5.4 Conclusion	45

Chapter 6.....46

Part 2: The recontextualisation of the topic in students' solutions

6.1 Ground for ideal type categories	46
6.1.1 Determining the essential features of students' solutions	46
6.1.2 The notion of absolute value in absolute value inequalities	47
6.1.3 The notion of order in absolute value inequalities	48
6.1.4 Three levels of category	49
6.2 The ideal type categories	50
6.2.1 "A" categories: the notion of absolute value	50
6.2.2 "L" categories: the notion of order in terms of logical connectives	56
6.2.3 "O" categories: the notion of order in terms of order relations	61
6.2.4 Combination codes and zero categories	66
6.3 Re-grading of solutions	67
6.4 Conclusion	71

Chapter 7	72
Presentation of the Results	
7.1 The A level categories (notion of absolute value)	72
7.1.1 A level categories results	72
7.1.2 A level categories across the school	73
7.2 The L level categories (logical connectives)	74
7.2.1 L level categories results	74
7.3 The O level categories (order relations)	75
7.3.1 O level categories results	75
7.4 The combination codes	76
7.4.1 Combination codes results	76
7.5 Two-way analysis results	77
7.5.1 A and L categories	77
7.5.2 A and O categories	78
7.5.3 L and O categories	79
7.6 Results after regarding	79
7.7 Conclusion	80
Chapter 8	81
Discussion of the Results	
8.1 Common findings and discrepancies with the literature	81
8.1.1 The absence of a graphical approach in students' solutions	81
8.1.2 Reflecting on the equation/inequality connection in this study	81
8.1.3 The logical connectives problem	82
8.2 The most significant findings in this study presented through a discussion of students' computational activity	83
8.2.1 A discussion of the notion of absolute value in category A5, L4 and L5 solutions	83
8.2.2 A discussion of "switching" in category O2, O3, O4 and O5 solutions	84
8.2.3 A discussion of the notion of order in category L2, L3, L4 and L5 solutions	85
8.3 Conclusion	85
Chapter 9	86
Conclusion	
9.1 Overview of the study	86
9.2 Summary of the results	87
9.3 Limitations and potential of this study	88
Bibliography	89
Appendix A	96
Appendix B	147

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LIST OF FIGURES

Figure 5.1 The distance definition.....	37
Figure 5.2 The squaring method.....	38
Figure 5.3 An example implicitly derived from the square definition.....	38
Figure 5.4 Rules for solving inequalities in the textbook.....	39
Figure 5.5 Sketching absolute value functions.....	40
Figure 5.6 Absolute value equations.....	41
Figure 5.7 Algebraic method for solving absolute value inequalities.....	42
Figure 5.8 Graphic method for solving absolute value inequalities.....	43
Figure 5.9 Distance method for solving absolute value inequalities.....	44
Figure 5.10 A general rule using double inequalities.....	44
Figure 6.1 Three levels of category (script 23).....	49
Figure 6.2 Category A1 example (script 10).....	51
Figure 6.3 Category A1 example (script 14).....	51
Figure 6.4 Category A2 example (script 4).....	52
Figure 6.5 Category A2 example (script 25).....	53
Figure 6.6 Category A3 example (script 30).....	53
Figure 6.7 Category A3 example (script 35).....	53
Figure 6.8 Category A4 example (script 45).....	54
Figure 6.9 Category A5 example (script 39).....	55
Figure 6.10 Category A5 example (script 37).....	55
Figure 6.11 Category A5 example (script 94).....	55
Figure 6.12 Category A5 example (script 54).....	55
Figure 6.13 Category L1 example (script 35).....	57
Figure 6.14 Category L1 example (script 38).....	58

Figure 6.15 Category L2 example (script 29).....	58
Figure 6.16 Category L2 example (script 40).....	58
Figure 6.17 Category L3 example (script 5).....	59
Figure 6.18 Category L3 example (script 9).....	59
Figure 6.19 Category L4 example (script 10).....	60
Figure 6.20 Category L5 example (script 27).....	60
Figure 6.21 Category L5 example (script 25).....	60
Figure 6.22 Category O1 example (script 33).....	62
Figure 6.23 Category O2 example (script 36).....	62
Figure 6.24 Category O2 example (script 98).....	62
Figure 6.25 Category O2 example (script 10).....	63
Figure 6.26 Category O2 example (script 100).....	63
Figure 6.27 Category O3 example (script 19).....	64
Figure 6.28 Category O3 example (script 63).....	64
Figure 6.29 Category O3 example (script 61).....	64
Figure 6.30 Category O3 example (script 44).....	64
Figure 6.31 Category O4 example (script 39).....	65
Figure 6.32 Category O4 example (script 54).....	65
Figure 6.33 Category O5 example (script 51).....	65
Figure 6.34 Category O5 example (script 49).....	65
Figure 6.35 Re-grading of solutions (script 12).....	68

LIST OF TABLES

Table 6.1 A summary of the A level categories.....	56
Table 6.2 A summary of the L level categories.....	61
Table 6.3 A summary of the O level categories.....	66
Table 6.4 MSEP marking criteria.....	67
Table 6.5 Maximum possible scores for L and O categories.....	70
Table 6.6 Maximum possible scores for combinations of L and O categories.....	70
Table 7.1 Distribution of students' attempted solutions by A categories and school (Count).....	72
Table 7.2 Distribution of students' attempted solutions by A categories and school (%).....	73
Table 7.3 Distribution of students' attempted solutions by L categories and school (Count).....	74
Table 7.4 Distribution of students' attempted solutions by L categories and school (%).....	75
Table 7.5 Distribution of students' attempted solutions by O categories and school (Count).....	75
Table 7.6 Distribution of students' attempted solutions by O categories and school (Count).....	76
Table 7.7 Distribution of students' attempted solutions by A and L level categories (Count).....	77
Table 7.8 Distribution of students' attempted solutions by A and L level categories (%).....	78
Table 7.9 Distribution of students' attempted solutions by A and O level categories (Count).....	78
Table 7.10 Distribution of students' attempted solutions by A and O level categories (%).....	78
Table 7.11 Distribution of students' attempted solutions by L and O level categories (Count).....	79
Table 7.12 Distribution of students' attempted solutions by L and O level categories (%).....	79
Table 7.13 Results before and after re-grading.....	80

CHAPTER 1

Introduction

1.1 The state and context of Mathematics education in South Africa

In 2015, the World Economic Forum's Global Information Technology report ranked South Africa as last in Mathematics and Science education out of 143 countries (Dutta, Geiger & Lanvin 2015), and in 2014 South Africa was ranked as last out of 148 countries (Bilbao-Osorio, Dutta & Lanvin 2014). The 2015 report also ranked South Africa's education system as 139th out of 143 countries in 2015 and it is not the only report that points to South Africa's education system, and particularly Mathematics and Science education, as in crisis. South Africa has performed poorly in national benchmarking tests for Science and Mathematics such as the Annual National Assessment (ANA). The national average percentage mark for the Grade 9 Mathematics ANA was 13% in 2012, 14% in 2013 and 11% in 2014. International benchmarking tests, such as the Trends in International Mathematics and Science Study (TIMSS), also paint a dim picture of the state of Mathematics education in South Africa. An analysis of South Africa's results in the TIMSS over the last 20 years (Reddy, Zuze, Visser, Winnaar, Juan, Prinsloo, Arends & Rogers 2015) showed that in 2003, only 10,5% of South African learners in Grade 8 were able to demonstrate knowledge of the most basic skills in Mathematics and Science. In 2011, 24% of Grade 9 learners were able to demonstrate knowledge of the most basic skills in Mathematics and Science, a significant improvement but from a very low base. Reddy *et al.* (2015) also showed that there is a strong correlation between the Human Development Index (HDI)¹ and Mathematics achievement and that South African students perform lower than expected based on South Africa's HDI.

1.2 Identifying the problem

In the current National Curriculum Statement for Grades 10 to 12 in South Africa, the Curriculum and Assessment Policy Statement (Department of Basic Education 2011), linear inequalities are studied in Grade 10 and both linear and quadratic inequalities are studied in Grade 11 as part of the topic "Equations and Inequalities" which is a sub-topic of "Algebra". However, algebra, and specifically the topic of algebraic inequalities, is a particular problem for many South African Mathematics students. In February 2007, a project known as the Mathematics and Science Education Project (MSEP) conducted baseline Science and Mathematics tests in ten secondary schools around Cape Town. One of the problems the MSEP report revealed was students' poor performance in Algebra. In the Algebra section of the baseline tests in Grade 10 only 17.6% of the responses were correct, in Grade 11 only 9.5% of the responses were correct and in Grade 12

¹ The Human Development Index is a measure that captures both the social and economic aspects of

only 28% of the responses were correct (Mathematics and Science Education Project 2007). Using some of the data generated by MSEP, Davis (2014) found that a particularly weak area in school Mathematics is linear inequalities; only 2.5% of the hundreds of Grade 11 students who wrote the MSEP baseline Mathematics test were able to solve a basic linear inequalities problem.

We are now eight years on from the MSEP study in 2007 and the problem has clearly not been solved. The key findings of the Department of Basic Education's Diagnostic Report for the 2013 ANA for Grade 9 Mathematics stated that students had difficulty with (algebraic) equations and algebraic fractions (Department of Basic Education 2013). The Department of Basic Education's NSC 2014 Diagnostic Report (Department of Basic Education 2014) which analyzed the final Grade 12 examinations in 2014, found that one of the main problems Grade 12 students face in Mathematics is poor algebraic skills. In particular, the NSC 2014 report found that Grade 12 students are unable to solve algebraic inequalities.

Clearly, there are problems worth addressing in South Africa's Mathematics education, particularly with regard to Algebra and inequalities. This study is born out of the need to better understand these problems.

1.3 This study

This study is situated within the general problematic of the constitution of school Mathematics in pedagogic settings. Davis (2010b:100) posits that the content that is constituted as Mathematics in a school setting is not always congruent with the content institutionalized in the field of production. This study is therefore specifically concerned with describing and analyzing the content that actually emerges as Mathematics in a pedagogic context without making assumptions based on the content indicated by the topic in the field of production.

This study is a replication of a study by Davis (2013a) who examined Grade 11 students' treatment of linear inequalities through an analysis of their responses to a test item as well as the way in which linear inequalities are treated in the national curriculum and state approved Mathematics textbooks. Davis (2013a) found that the topic of linear inequalities is treated as an extension of equations and that insufficient attention is given to the notion of numerical order, which is central to solving linear inequality problems.

One of the aims of this study is to use and refine the methodology and analytical framework laid out by Davis (2013a) in order to examine the ways in which students treat *absolute value* inequalities. It is expected that the results of this study will be similar to those of Davis (2013a) but with some differences and will therefore add to the validity of Davis' findings as well as to his methodology. While Davis (2013a) examined Grade 11 students' treatment of linear inequalities, this study will examine Grade 12 students' treatment of absolute value inequalities.

At the time of the data collection for this study, the South African curriculum was in a state of transition. With the end of Apartheid in 1994, the government introduced an interim syllabus to replace the racist Apartheid curriculum that was designed to reinforce the divisions between racial groups. Essentially, the content of the interim syllabus was based on the old curriculum but it was edited so that every student in South Africa was offered the same Grade 12 examination (Engelbrecht & Harding 2008:57). In 2006 the interim syllabus was replaced with the National Curriculum Statements (NCS) (Department of Education 2003).

Up until the end of the Senior Certificate Examination (SCE) and the interim syllabus and the introduction of the National Curriculum Statements (NCS) (Department of Education 2003) into schools in 2006, absolute value inequalities were part of the Grade 11 and Grade 12 Higher Grade Mathematics curriculum. Although the topic is currently excluded from the CAPS curriculum (Department of Basic Education 2011), the value of studying Grade 12 students' treatment of this topic now is twofold; firstly topics come in and out of curriculums and so it is likely that this topic will be reincorporated at some point and secondly, since absolute value inequalities are a branch of the topic of inequalities, it is possible to re-use and adapt the methodology and analytical framework that Davis (2013a) used to study linear inequalities and confirm their validity if the findings are the same.

The focal research question of this study is: *what is constituted as the content of the topic absolute value inequalities by Grade 12 Mathematics students at seven Western Cape schools populated by students from working class families?*

In order to answer this question, the following sub questions will also be answered:

1. How is the topic of absolute value inequalities defined in the field of Mathematics?
2. How is this topic re-contextualised through the South African curriculum statements and state-approved textbooks?
3. What are the specific ways in which beginning Grade 12 students at seven Western Cape schools treat absolute value inequalities?
4. What, if any, is the relationship between the textbook and curriculum treatment of absolute value inequalities and students' treatment of the topic?
5. How does students' treatment of absolute value inequalities compare with the treatment of the topic in the field of Mathematics?

1.4 An overview of the thesis

Chapter 1 introduces the context of Mathematics education in South Africa and the problem being addressed in this study, which is a replication of a study by Davis (2013a) and which is concerned with discussing what is constituted as absolute value inequalities in Grade 12 students' solutions to a baseline test item. Chapter 2 presents a review of the relevant literature and situates this study in relation to the existing research on the teaching and learning of

inequalities in general as well as the teaching and learning of absolute value and absolute value inequalities. Chapter 3 presents the general methodology of this study which draws on the grounded theory of Glaser & Strauss (1967); Glaser (1992) and Glaser & Strauss (1999), Weber's (1964) theory of ideal types, Bernstein's (1996) theory of the pedagogic device and Davis's methodology for the description of the constitution of Mathematics (2010a; 2010b; 2011a; 2011b; 2013b).

Chapter 4 presents a framework for the production and analysis of the data, which is presented in Chapters 5 and 6 in two parts. Part one of the analysis of the data describes the treatment of the topic of absolute value inequalities in the field of Mathematics followed by the recontextualisation of the topic in a selection of relevant textbooks as well as in the relevant curriculum documents. In part two, the recontextualisation of the topic in students' solutions is analysed using ideal-type categories along with a description of the criteria used to recognize the categories. The results of the analysis in Chapter 6 are presented in Chapter 7 and discussed in Chapter 8 followed by a conclusion in Chapter 9.

CHAPTER 2

A Review of the Literature

The research focus of this study is centered on describing the constitution of absolute value inequalities as displayed in Grade 12 students' solutions to an absolute value inequality problem. In order to position this study in relation to the existing literature in the field of Mathematics education, a review of the literature on the teaching and learning of inequalities in general is presented, followed by a review of the literature that is specific to the teaching and learning of absolute value and absolute value inequalities. Although the majority of the literature on the teaching and learning of inequalities is not specific to absolute value inequalities, it is still relevant to this review because it is relevant to inequalities in general.

2.1 Introduction

Across the literature it is agreed that inequalities are one of the most useful and important topics in pure and applied Mathematics and yet one of the least understood topics in secondary school Mathematics. Despite this, Halmaghi (2011:67) notes that prior to 1999 there was very little research around inequalities in Mathematics education. In response to a research paper on inequalities by Tsamir, Almog & Tirosh (1998) presented at the 22nd Psychology of Mathematics Education (PME) conference in 1998, a call for more research on inequalities was made at the 23rd PME Conference in 1999. As a result of this call, an abundance of research was presented at the 28th PME in 2004 and this research forms a significant portion of the current literature on the teaching and learning of inequalities (Boero & Bazzini 2004; Kieran 2004; Sackur 2004; Tsamir, Tirosh & Tiano 2004). Other papers that were written as a result of the 23rd PME conference call and which form a valuable part of the literature on inequalities are the two response papers presented at the 28th PME (Tall 2004; Radford 2004) and the preliminary reports presented at the 25th PME (Boero, Bazzini & Garuti 2001; Garuti, Bazzini & Boero 2001; Bazzini & Tsamir 2001).

2.2 Teaching methods

Up until the 22nd PME in 1998, the majority of the literature around the teaching and learning of inequalities (of all types) consisted of teachers reflecting, without research support, on teaching methods for overcoming students' difficulties in solving problems involving inequalities (including absolute value inequalities) (Halmaghi 2011). Methods such as the sign chart method and number line method were proposed and developed by McLaurin (1985); Wagster (1986) and Dobbs & Peterson (1991) while Scott (1975); Sink (1979); Arcidiacono (1983); Dreyfus & Eisenberg (1985); Vandyk (1990) and Parish (1992) argued that inequalities should be taught using a graphical approach, either by teaching students to draw and then compare functions or using technology such as computers and graphical calculators. Of the methods that were specific

to absolute value problems, Horak (1994) proposed a method for solving absolute value equations using a graphical calculator while Brumfiel (1980) proposed that students should be taught to solve absolute value inequalities using five differing definitions of absolute value.

More recent literature on the teaching and learning of inequalities is focused on studying students' conceptions of linear, quadratic, rational and square root inequalities as well as their strategies for solving these types of inequalities (rather than on expounding preferred teaching methods) and common errors (e.g. Tsamir *et al.* 1998; Bazzini & Tsamir 2001; 2003; Tsamir & Bazzini 2004; Halmaghi 2011). While most of the studies are concentrated on students' conceptions and treatment of inequalities, Tsamir *et al.* (2004) looked at how teachers respond to students' errors when solving quadratic inequalities and Abramovich & Ehrlich (2007) studied how prospective teachers can be taught to use technology to overcome students' misconceptions of inequalities. This study is concerned with describing students' treatment of absolute value inequalities rather than prescribing solution or teaching methods and therefore this review is specifically focused on studies related to students' conceptions of inequalities.

2.3 Conceptions of inequalities and the equation/inequality connection

The common argument running throughout the literature is that students treat inequalities algorithmically- using a memorized set of transformations rather than an understanding of equivalence and order and that they are taught to solve inequalities using the same procedure that they learnt to solve equations, with an added rule that when they multiply or divide both sides of the expression by a negative number, they "switch the sign"(e.g. Sfard & Linchevski 1994; Tsamir *et al.* 1998; Sackur & Maurel 2000; Bazzini & Tsamir 2001; Garuti *et al.* 2001; Bazzini & Tsamir 2003; Tsamir & Bazzini 2004; Bazzini & Tsamir 2004; Halmaghi 2011; Davis & Gripper 2012; Davis 2013a). In their interviews with sixteen to seventeen year old Italian and Israeli students, Bazzini & Tsamir (2003:3) found that students solved inequalities using equations as an algorithmic "prototype" model of the form "do the same operation with the same numbers on both sides". Another study by Tsamir & Bazzini (2004), which examined students' solutions to inequalities that result in a "single-value" (i.e. $5x^4 \leq 0 \rightarrow x = 0$), found that students would not accept a single value such as $x = 0$ as a solution to an inequality and their reasoning revealed two "intuitive beliefs" about inequalities (explicitly mentioned by students in interviews). Firstly, that "inequalities result in inequalities" and secondly, that "solving inequalities and solving equations are the same process" (Tsamir & Bazzini 2004:808-809). In studies by Davis & Gripper (2012) and Davis (2013a) of the treatment of linear inequalities in the solutions of Grade 10 students in South Africa as well as in textbooks, they found that the "rules for solving inequalities" presented in the textbook were focused on the spatial orientation of the inequality symbol. For example, "Multiplying or dividing both sides of an inequality by a negative number reverses the inequality sign".

Although most of the literature agrees that the equation/inequality connection is one of the main sources of students' difficulties in solving inequalities, Kieran (2004) proposed that the connection might be used to benefit students studying inequalities. In Kieran's (2004) study of eighth Grade Japanese students who were introduced to inequalities through a contextual inequality problem, it was found that the students approached the problem first as an equation problem and then adapted the equation's solution to find the solution to the inequality. Kieran (2004:146) therefore suggests that there is a close relationship between the notion of equality and inequality and therefore "the didactical challenge is to find ways to help students beware of the traps of the equation/inequality connection in their transformational work with symbols, while they still enjoy its benefits".

According to Garuti *et al.* (2001:1), one of the reasons for students' confusion of equations and inequalities is that in many countries inequalities are primarily dealt with in an algorithmic manner (rather than graphical) and as a "subordinate topic" related to equations. For example, in Italy and Israel (Bazzini & Tsamir 2001) and similarly in France (Sackur & Maurel 2000), students tend to solve inequalities of all types algorithmically- using algebraic transformations that are suitable when dealing with equations but that ignore the special properties of inequalities. In South Africa, Davis (2013a) notes that not only are inequalities taught as a sub-topic of equations but students are explicitly taught to solve inequalities in the same way that they solve equations (using algebraic manipulations) but with certain exceptions (for example, when multiplying or dividing by a negative number, "change" the inequality sign around). This approach is mathematically problematic as it ignores the notion of equivalence implicit in equations and the notion of numerical order implicit in inequalities (Davis 2013a).

Tsamir *et al.* (1998) found that the students in their studies tended to treat inequalities as equations without any regard for differences in the meaning denoted by different symbols- some of them even changing the inequality sign to an equal sign and solving the resulting equation. They conclude that for students who confuse the notion of equation and the notion of inequality, "the given algebraic expression activates only a procedural sense, disconnected from the denotation which stands behind" (Tsamir *et al.* 1998:3). As Lim (2006:45) protests in his study of students' mental acts in solving inequalities and equations, an overemphasis on procedures and rules for manipulating symbols in algebra has been at the expense of "the semantic and structural aspects of algebra".

The idea that students are solving inequalities without an understanding of the notion of inequality is posited by a number of studies. Bazzini & Tsamir (2001) investigated students' responses to two types of tasks: standard inequality tasks, which were similar to the tasks they had encountered in class (for example, solve $(a - 5)x > 2a - 1$), and non-standard inequality tasks which required them to think in a different way about the same mathematical concept. They found that students could not extend their knowledge of inequalities to the non-standard tasks, indicating that students rely on learnt procedures to solve inequalities rather than a

mathematical understanding of the concept. They therefore argue that although students can solve inequalities algorithmically, they lack a deeper, theoretical understanding, “doing algebra is not just formal manipulation, but rather a competence, which deeply involves understanding” (Bazzini & Tsamir 2001:8). Bazzini & Tsamir (2003) relate the findings in their earlier study (Bazzini & Tsamir 2001) to Fischbein’s (1993) theory of formal, intuitive and algorithmic knowledge in mathematical thinking and they argue that when solving inequalities, students rely on intuitive knowledge, “which is accepted directly and confidently as being obvious, imparting the feeling that no justification is required” and algorithmic knowledge, which is “the ability to use theoretically justified procedures” instead of formal knowledge, which is based on propositional thinking (Bazzini & Tsamir 2003:1).

2.4 The procedural-conceptual distinction in the literature

Sfard & Linchevski (1994:306) found that most of the students in their study treated both equations and inequalities as “meaningless strings of symbols to which certain well-defined procedures are routinely applied”. The idea that students can solve inequalities using ““meaningless” rules and procedures (Lim 2006:45) and in “a purely algorithmic manner” (Boero & Bazzini 2004:140) rather than with a conceptual understanding of the mathematical concepts involved occurs frequently throughout the literature on the teaching and learning of inequalities, as seen in the discussion above. This distinction between a procedural understanding and a conceptual understanding of Mathematics is widely used in the field of Mathematics education and it is relevant to this study because it influences the way in which researchers think about the constitution of Mathematics in relation to students.

As Davis (2013) has previously argued, the procedural-conceptual distinction has been critiqued² by Radford (2004) who in his response to the papers on inequalities in the 2004 PME conference argues that syntax and meaning cannot be opposed- “Every sign has a meaning. Otherwise, it cannot be a sign. Conversely, every meaning is an abstract entity –“a general” (Otte 2003)- which finds instantiation in signs only.” (Radford 2004:165) From this perspective, there is no such thing as a sign or a procedure without meaning in the mind of the reader. However, this meaning is necessarily, as Strawson (1950:328) says, “not the set of things or the single thing it may correctly be used to refer to: the meaning is the set of rules, habits, conventions for its use in referring.” There is therefore always some form of meaning or concept behind every procedure and therefore the idea of “meaningless manipulations” no longer makes sense.

Instead of dismissing any activity that does not agree with the field of production as “meaningless”, in order to describe what is constituted as Mathematics it is necessary to ask *what* are the concepts represented by interactions with procedures for solving inequalities in the

² A further critique of the procedural-conceptual distinction can be found in Chitsike (2011).

classroom and are they aligned to equivalent concepts in the field of the production of Mathematics? It is therefore necessary to consider any operational activity as meaningful in the mind of the student and therefore as contributing to the constitution of Mathematics.

2.5 Solving inequalities using a graphical approach

Another common finding across the literature is that inequalities should be taught graphically. Verikios & Farmaki (2008:201) propose that purely symbolic and algebraic methods for solving inequalities should only be introduced once students' have a functional understanding (using graphs) of inequalities. Tsamir *et al.* (1998) found that although students used one of three approaches in solving inequalities: algebraic manipulations, drawing the graphs of the functions and using a number line, of the three approaches the majority of the students used the algebraic method which was the least successful method while the graphical method was the most successful. Boero *et al.* (2001) also argue for the functional approach in the teaching of inequalities. They investigated the use of "grounding metaphors" in teaching inequalities using a teaching experiment with 13-14 year olds as well as Ph.D. students in Mathematics engaged in structurally similar tasks. They found that the dominant manner of teaching treats inequalities as a "special case of equations" and avoids the concept of function and instead they propose that inequalities should be taught "as special cases of comparison of functions" (Boero *et al.* 2001:3). Garuti *et al.* (2001:2) point out that although inequalities are most often solved using algebraic procedures in school Mathematics, the functional aspect of inequalities is crucial in Mathematicians' work and they therefore also argue for a functional approach rather than an algebraic approach in teaching. Abramovich & Ehrlich (2007) studied how prospective teachers can be taught to use technology (graphing calculators) to overcome misconceptions caused by the "Einstellung effect" whereby students extrapolate their knowledge of equations to solve inequalities and found that the use of graphing technology could act as a tool to overcome the common mistreatment of inequalities as equations.

However, in response to the growing use of graphical methods and graphical calculators for solving inequalities in school, Sackur (2004:148) proposes that there is a need to study the problems that result from changing a "problem in algebra into a problem on graphs." Using Duval's (2000) theory of semiotic registers in Mathematics, Sackur (2004) posits that the same mathematical object represented in different semiotic registers is not comprised of the same meaning in both registers. Solving an inequality graphically requires shifting between four different semiotic registers (algebraic, functional, graphical bi-dimensional and graphical mono-dimensional) (Sackur 2004:149). Sackur (2004:151) therefore warns that the graphical approach can present new difficulties for students and "it should not be taken for granted that when "solving graphically" students learn the same Mathematics as when "solving algebraically"". Tall (2004:159) points out that the end goal for teaching students to solve inequalities should inform the approach. Treating inequalities graphically might help students to solve a given inequality, but if we want them to become "fluent in meaningful manipulation of symbolism" then treating

them graphically will be pointless and an algebraic approach would be far more appropriate. Tall (2004) therefore suggests that different approaches will have different positive attributes suitable for different needs.

2.6 Absolute value inequalities

Now that the literature on the teaching and learning of inequalities in general has been discussed, the literature that specifically focuses on the teaching and learning of *absolute value inequalities* (and which is therefore particularly relevant to this study) is presented. There is far less literature that is specific to absolute value inequalities than there is on other types of inequalities and while the majority of the literature in this area consists of research on students' conceptions of and approaches to solving absolute value inequalities, there are some papers that reflect solely on teaching methods. For example, Brumfiel (1980:24) suggests that students should be taught five different definitions of absolute value and that they should investigate using all five to solve a particular problem. Stupel (2013:594) agrees that students can benefit from understanding and applying different definitions but adds that certain definitions are appropriate for different types of problems and that students should investigate when a definition is most appropriate.

Studies on students' treatment of absolute value inequalities support the findings of studies on general inequalities, particularly with regard to the benefits of using a graphical approach (Scott 1975; Sink 1979; Arcidiacono 1983; Horak 1994). Sierpinska, Bobos & Pruncut (2011) report on a teaching experiment where three different approaches to absolute value inequalities ("procedural", "theoretical" and "visual") were taught to undergraduate students at a university with the aim of finding out which methods would promote "theoretical thinking" of absolute value inequalities. Sierpinska *et al.* (2011) use the term "theoretical thinking" to refer to an ability to investigate problems and think reflectively on the solutions, to think in terms of mathematical definitions and proofs and to be sensitive to mathematical notation and logic. They argue that their model of "theoretical thinking" is not based on the procedural-conceptual distinction as defined by Porter & Masingila (2000:172), who differentiate between procedural errors as errors in syntax and procedures and conceptual errors as "the selection of inappropriate procedures, misinterpretation of mathematical terms, and errors in logic". Rather, Sierpinska *et al.*'s (2011) model incorporates aspects of both procedural and conceptual thinking. They found that students who were taught using a "visual" approach, where a solution technique for solving absolute value was derived from the formal definition of absolute value and supported by a visual representation of the problem, were more likely to develop a theoretical understanding of absolute value inequalities than students who were taught using a "procedural" or "theoretical" approach, where the former consisted of the formal definition followed by a series of steps to follow and the latter consisted of the same solution technique but derived from the formal definition. Sierpinska *et al.* (2011) also found that students solving absolute value inequalities tended to make the same mistake of treating the inequalities as equations.

There are two recent studies that are particularly relevant to this review because they are specifically concerned with how students treat absolute value inequalities. The first is by Gagatsis & Panaoura (2014) who investigated students' conceptions of the notion of absolute value and their ability to apply the notion in standard problems, such as equations and inequalities involving absolute value that students might ordinarily have encountered, and non-standard problems, such as "impossible" absolute equations which have no solution. Similarly to the studies on students' treatment of inequalities in general, they found that students tended to rely on algorithmic methods³, even in situations where it was not appropriate such as when solving the obviously impossible inequality $|x - 4| < -2$. They posited that the prevalence of algorithmic methods in the teaching of absolute value has resulted in a belief that the absolute value symbol must be "taken away" in order to solve an equation or inequality.

The second study is by Almog & Ilany (2012) who examined the correct and incorrect ways that students solved absolute value inequalities, as well as the frequency and sources of their misconceptions, and found that students struggle significantly with this topic. Expecting that students would rely heavily on algebraic manipulations, they intentionally chose problems that were solvable without algebraic manipulation, with solutions that would be immediately obvious to anyone with a particular understanding of absolute value (similar to Gagatsis & Panaoura (2014)). Most of their findings confirmed the results of other studies on different types of inequalities previously discussed, such as the generalization of methods for solving equations to inequalities, difficulty with single value solutions and the belief that inequalities result in inequalities. In terms of results that are specific to absolute value inequalities, they found that students have difficulty with logical connectives and with the idea that the expression inside an absolute value could be negative or zero.

One of the main problems associated with solving absolute value inequalities, highlighted by Almog & Ilany (2012), is a misunderstanding of logical connectives. Almog & Ilany (2012:353) found that students either did not put a logical connective between the inequality expressions in their solutions or they used OR when they should have used AND, and vice versa. Other researchers, such as Tsamir *et al.* (1998); Tsamir & Almog (2001) and Kroll (1986) had similar findings.

Both Almog & Ilany (2012), Gagatsis & Panaoura (2014) examine students' responses to a variety of absolute value inequality problems and rely mainly on questionnaires and interviews with students. A more in depth analysis of the different ways in which students solve absolute value inequalities is needed in order to determine what is constituted as absolute value inequalities and therefore in this study, not only are students' solutions analyzed, but the recontextualisation of absolute value inequalities in the textbook and in the curriculum is described and discussed

³ The term "algorithmic" is used here in the sense that it is used by the authors.

alongside a discussion of the topic in the field of Mathematics. Finally, the methodology in this study is unique from other studies in that it treats all students' activity as meaningful and not only that which is congruent with the topic in the field of Mathematics.

2.7 Conclusion

Overall, as Almog & Ilany (2012) reflect, students have not effectively acquired the mathematical concepts involved in solving absolute value inequalities. One of the questions that Almog & Ilany (2012) ask at the end of their study on students' treatment of absolute value inequalities (in Israel) is, "do such results appear in other countries?". This study aims to add to the literature on students' treatment of absolute value inequalities in South Africa and in particular at schools in South Africa populated by students from working class families. In addition, the methodology for analysis used in this study differs from that used in the current literature on absolute values. The methodology used in this study will not only provide an organized discussion of the common ways in which by students approach solving absolute value inequalities but it will also contribute to a better understanding of what is constituted as absolute value inequalities in students' treatment of the topic.

Chapter 3 presents the general methodology for this study.

CHAPTER 3

A General Methodology

The aim of this chapter is to present a general methodology that will lay a foundation for the production and analysis of data, the aim of which is to examine the ways in which the topic of absolute value inequalities is treated by Grade 12 students from seven schools populated by students from working class families in the Western Cape as well as by the relevant textbooks and the curriculum. The grounded theory approach by Glaser & Strauss (1967; 1999) and Glaser (1992) will be discussed as well as Weber's (1964) theory of ideal type categories, Bernstein's (1996) theory of the pedagogic device and methods of descriptions of mathematical activity in school Mathematics from Davis (2010a; 2010b; 2011a; 2011b; 2013b).

3.1 Glaser and Strauss' grounded theory approach

In order to determine what is constituted as Mathematics, this study uses a grounded theory approach as a general methodology for the analysis of students' solutions to a test item. The notion of "grounded theory" was first introduced by Glaser & Strauss (1967) as a sociological method that is wholly concerned with the *discovery of theory* from data and while they present their approach in opposition to the overemphasis of *verification of theory* in social research, they also maintain that the verification of theory is not-unimportant and should go hand in hand with theory generation. They propose that theory should emerge from the data through the generation of categories by comparative analysis. Many "events" in the data are compared until a pattern of similar events appears and is given "a conceptual name as a category" (Glaser 1992:40). Another study in Mathematics education which uses a similar approach is by Dowling (2013). Grounded theory is appropriate in this study because the study is situated in the constitution of Mathematics, which is concerned with describing and analyzing the content that actually emerges as Mathematics in a pedagogic context without making assumptions based on the content indicated by the topic in the field of production.

3.2 Weber's ideal type categories

Glaser & Strauss (1967:22) remind us that "the general notion of comparative analysis was developed by our sociological forefathers- Weber, Durkheim, Mannheim" and in furthering the methodology developed above, Weber's (1964) theory of "ideal types" is now introduced.

Weber defines an ideal type as follows:

An ideal type is formed by the one-sided *accentuation* of one or more points of view and by the synthesis of a great many diffuse, discrete, more or less present and occasionally absent *concrete individual* phenomena, which are arranged according to those one-sidedly emphasized viewpoints

into a unified *analytical* construct... In its conceptual purity, this mental construct... cannot be found empirically anywhere in reality. (Weber 1949:90 in Ritzer 1992:222)

According to Ritzer (1992:222), “at its most basic level, an *ideal type* is a concept constructed by a social scientist... to capture the essential features of some social phenomenon”. An ideal typical category emerges through empirical observation of cases of a particular phenomenon in the real world but it is not “ideal” in the sense that it represents perfection- it is not meant to include every characteristic of a single case. Rather, it serves as an idea to capture the characteristics that are common between all the cases.

Weber’s theory of the ideal type as a conceptual tool for analyzing information is grounded in his belief that “all knowledge is acquired through a process of abstraction from reality itself” (Hekman 1983:20). It is therefore essential that the features of an (abstract) ideal type emerge from the real world rather than an abstract theory. It is equally important that the criteria which determine the selection of the “essential” characteristics of phenomena are grounded in the “specific theoretical goals” of the study. As Hekman observes, “ideal types are not the product of the whim or fancy of a social scientist, but are logically constructed concepts” (1983:32).

Davis (2013a) uses ideal typical categories to sort out and analyze Grade 11 students solutions to a problem on linear inequalities based on certain common features that emerged from the data, grounded in a set of assumptions made about the data. Davis posited that since the topic of inequalities in the field of Mathematics involves order relations over the real numbers, the successful solution of a linear inequality problem depends on the degree to which the solution explicitly or implicitly observes order relations over the real numbers (2013a:6,12). From this position, Davis generated seven ideal typical categories, selecting their “essential features” based on the degree of order relation observation. This study will construct ideal type categories to analyze students’ solutions of absolute value inequalities in a similar manner to Davis (2013a).

3.3 Bernstein’s pedagogic device and the constitution of Mathematics

The theoretical framework of this study also draws from Bernstein’s (1996) theory of the pedagogic device and in particular the “recontextualising rule” and the “evaluative rule”. Bernstein (1996:39) describes the pedagogic device as the general principles that underlie “the transformation of knowledge into pedagogic communication” through a discussion on who gets what knowledge and how this knowledge is distributed. He describes three hierarchically related rules of the device: the distributive rule, the recontextualising rule and the evaluative rule. The distributive rule regulates *who* gets what knowledge, the recontextualising rule regulates *how* and *what* knowledge is distributed and Bernstein places particular emphasis on evaluation (the evaluative rule) as being at the base of all pedagogy, saying that it is “the key to pedagogic practice” and “condenses the meaning of the whole device” (Bernstein 1996:50).

According to Bernstein (1996:33), all pedagogic discourse is constructed by a “recontextualising principle which selectively appropriates, relocates, refocuses and relates other discourses to constitute its own order”. Knowledge from the field of production is therefore recontextualised through the recontextualising rule into the pedagogic discourses in schools. Through the recontextualising rule, the definitions, axioms and propositions in the field of Mathematics serve as the basis for school Mathematics (although they may only be implicit in curricula and textbooks and in the Mathematical activity of teachers and students) (Davis 2013a). These fundamental elements of Mathematics are objective and do not change as they are appropriated from the field of Mathematics to school Mathematics and the recontextualising rule therefore provides a steady foundation for a discussion of the constitution of Mathematics. Recontextualisation from the field of Mathematics into curricula and textbooks is one level of recontextualisation. Another level, occurs at the level of pedagogic practice when teachers recontextualise curricula and textbooks. We can also consider what students produce as a recontextualisation of what is produced as Mathematics in classrooms.

Davis (2013b) draws on Bernstein’s (1996) notion of the pedagogic device, and in particular the role of the evaluative rule, to posit that in any pedagogic situation there is an idea that *ought* to be transferred and then there is the content that is *actually* realized- or what is constituted as Mathematics in that context. Davis (2013b) argues that in order to determine what is constituted as Mathematics in any pedagogic situation it is necessary to find a way of describing the content that is *actually* realized rather than what one assumes is happening or thinks *should* be happening and thereby dismissing any activity that does not accord with the field of Mathematics as meaningless. Davis (2013b:1) notes that for Bernstein, all pedagogy is evaluative and must therefore transmit certain criteria that mark out what is considered to be legitimate knowledge or activity in the pedagogic context and how such knowledge should be realized. This “evaluative criteria” is what gives the outside observer access to the content that is actually being realized, rather than the “ought” to be realized, and evaluation is therefore key in determining what knowledge is constituted as Mathematics in a pedagogic context. Evaluation in a pedagogic context can refer to teacher and student productions, teacher and student talk and the textbook. This study is specifically concerned with the productions of students, as instantiated in their solution to an absolute value inequality problem.

According to Davis (2013b:56), we can only really understand what is being transmitted to students in a pedagogic context through “observationally adequate accounts of the mathematical activity of teachers and their students, starting with more precise descriptions of their computational activity”. Davis uses the term “computational activity” to refer to the mathematical activity of teachers and students which entails the operations, domains and codomains they operate over. Further theoretical resources are needed for describing the computational activity in students productions and for this reason I now introduce Davis’s methodology (Davis 2010a; 2010b; 2011a; 2011b; 2013b)

3.4 Operations, domains and codomains

Davis (2013b:34) notes that a description of the constitution of Mathematics requires that one suspends any ideas of the content that is expected to emerge and instead pays attention to the specificities of the operational activity, particularly unexpected operational features. Since mathematical activity in a pedagogic context is necessarily concerned with operations on objects, Davis (2011a) posits that it is possible to determine what is constituted as Mathematics in pedagogic settings through observationally adequate descriptions of the operations, domains and codomains used by teachers and students in their computational activity. Describing what it is that students or teachers are *doing* will lead to a description of the objects they are operating on.

Davis (2011a:4) says that the definition of a mathematical operation is that it must be a function:

An operation, $*$, is defined in general terms as a function of the form $*: D_1 \times D_2 \times \dots \times D_k \rightarrow C$ where the sets D_j are the domains of the operation, the set C is the codomain of the operation; the fixed non-negative integer k , which indicates the number of arguments, is the arity of the operation.

According to Davis (2011a), this definition of operations as functions is so essential to Mathematics that if the operations in a mathematical activity are not functions, it is not Mathematics that accords with the field of Mathematics. Since a function generates a unique output for a given input, operations will always behave in a stable, predictable way and the objects of an operation are the elements of the domain and codomain of the function. This definition of operations as functions is essential in a discussion of the constitution of Mathematics as it makes it possible for us to recognize “operations” (referred to as *operation-like manipulations*) that do not necessarily behave as functions and are therefore external to the field of Mathematics (Davis 2010c). For example, a common operation-like manipulation used in teaching equations is “change sides, change sign”. Since numbers do not “change sides” in an equation, the domain here is the set of character strings rather than numbers and the operation-like manipulation is external to the field of Mathematics. A description of the domains and codomains, operations and operation-like manipulations in students’ computational activity gives us access to what is constituted as Mathematics.

Chapter 4 presents the methodology for the production and analysis of the data taking into account the general methodology outlined in this chapter.

CHAPTER 4

Methodology for the Production and Analysis of Data

The aim of this chapter is to present a description of the methods that led to the production and analysis of data from the test scripts of Grade 12 students from schools in the Western Cape. I will begin with a description of the sample and the data collection process followed by the procedures used for the production and analysis of data.

4.1 RESEARCH DESIGN

4.1.1 Sample

The Grade 12 learners whose solutions to a test item are analyzed in this study are from seven secondary schools in the greater Cape Town area who participated in a joint Mathematics and Science project between the University of Cape Town and the Western Cape Education Department. The project, known as the Mathematics and Science Education Project (MSEP), was conducted in ten secondary schools around Cape Town over a period of five years with one of the aims being to develop and activate intervention programmes in the schools selected for the project. At the beginning of the project, baseline tests were written by students in Grades 10, 11 and 12 at the beginning of the academic year to ascertain students' knowledge of the previous years' curricular content. Only Higher Grade Mathematics students in Grade 12 wrote the Grade 12 Mathematics baseline test. Ten secondary schools in the Western Cape took part in the project, five of which were selected by the project administrators for long-term intervention programmes whilst the other five were selected as control schools. According to the MSEP Baseline Survey Report (2007), both the project schools and the control schools have similar characteristics and all ten are Dinaledi schools, which are essentially schools selected by the South African Department of Education to have a Mathematics and Science focus as part of a long-term strategy to improve student performances in international Mathematics and Science examinations (Davis 2013a). No further details as to the sampling method are given. Since this study is interested generally in schools in the Western Cape and the data was collected at the start of the project (before any intervention programmes were implemented), for the purposes of this study the project schools and control schools will be examined together. Of the ten schools involved in the project, two of the schools did not have any Higher Grade Mathematics students in Grade 12 and the test scripts from one of the schools were lost in an office move, leaving seven schools as the sample.

The schools in this study, here forth referred to as S1, S2, S3, S4, S5, S6 and S7, are all populated by students from working class families living in townships around Cape Town, South Africa. At the time the data in this study was collected, South Africa was thirteen years into its democracy and the curriculum was still in a state of transition.

During Apartheid, separate residential areas were demarcated for the different “racial groups”⁴ under the Group Areas Act and separate schools existed in each area for that racial group, for example, “White” schools were in “White” residential areas and “African” schools were in “African” residential areas etc. (Hoadley 2005:6). Six of the seven schools in this study are situated in townships on the outskirts of Cape Town, which during the apartheid-era were demarcated as so called “African” areas, and one of them is situated in a former “White” suburb in Cape Town but is currently populated by students from working class families situated in townships formerly demarcated for “African” and “Coloured” people.

4.1.2 Data Collection

In order to examine the specific ways in which Grade 12 students treat absolute value inequalities, this study analyzes beginning Grade 12 student responses to a single test item on absolute value inequalities. Examining students’ solutions to the test item will help us to understand how students approach absolute value inequalities without teacher or peer assistance.

The topic of absolute value inequalities was included in Grade 11 as part of the Higher Grade Senior Certificate Examination (SCE) in the Interim Core Syllabus (Department of Education 1995) but in 2006 it was removed from the national curriculum entirely with the introduction of the National Curriculum Statements (Department of Education 2003). The 2007 Grade 12 cohort were therefore the last group to write the SCE and therefore the last group to study absolute value inequalities at a secondary school level. This study will examine responses from students who were in Grade 12 in 2007.

In February 2007, the Mathematics and Science Education Project (MSEP) conducted baseline Mathematics and Science tests at the ten secondary schools described above. As part of the project, students in Grades 10, 11 and 12 were tested at the beginning of the academic year on their knowledge of Mathematics at the previous Grade’s level (Grade 11s were tested on Grade 10 content and Grade 12s were tested on Grade 11 content etc.). The purpose of the baseline tests was to assess students’ knowledge of the Mathematics and Science content from the previous year in order to identify the weakest areas and focus strategies to improve these areas. For the purposes of this study, MSEP has granted permission for the access and use of the Grade 12 Mathematics baseline test scripts and this study will analyze students written responses to the test item, “solve for x if $|2x - 3| < 4$ ” which was the only absolute value inequality item in

⁴ “Racial groups” are socially and historically constructed categories that were imposed on South African citizens under Apartheid law. Although it is recognized that these categories were part of an oppressive social system, it is necessary to refer to them here as the legacy of the Apartheid system, particularly in terms of education, continues to shape schooling in South Africa. The terms “African”, “White”, “Indian” and “Coloured” are all used with capital letters.

the baseline test. This test item was chosen in order to study students' responses to an absolute value inequality.

One of the limitations of this study is that it is not possible to conduct interviews with the students around their solutions since the data was collected several years ago. This study will therefore be limited to a discussion and analysis of student's solutions as they are written in the test scripts as well as a discussion of the topic as it exists in the relevant textbooks and curriculum documents.

According to Bernstein's (1996) pedagogic device, knowledge in the field of production is recontextualised through the recontextualising rule as pedagogic discourse. A second level of recontextualisation occurs where the content in the textbooks and curricula is recontextualised in teachers' pedagogic discourse and a third level occurs where pedagogic discourse is recontextualised in students' solutions. Therefore, in order to understand what content has been recontextualised in students' solutions it is necessary to examine how the topic has been recontextualised from the field of Mathematics into pedagogic discourse. The analysis of students' solutions (for which the data collection is described above) is therefore preceded by an analysis of the recontextualisation of the topic from the field of Mathematics into textbooks and curriculum documents.

Absolute value inequalities were taught in Grade 11 as part of the Higher Grade Interim Core Syllabus and then removed from the national curriculum in 2008 and therefore this study will focus on a selection of Grade 11 textbooks used in South African schools in 2007 as well as the Interim Core Syllabus documents (Department of Education 1995) for Grade 11 Mathematics. According to Davis (2013a), at the time of the MSEP study all ten of the schools involved used the Mathematics textbook, "Classroom Mathematics" by Laridon, Brink, Fynn, Jawurek, Kitto, Myburgh, Pike, Rhodes-Houghton & van Rooyen (2000) and therefore the majority of the analysis is focused on this textbook.

4.2 PROCEDURES FOR PRODUCTION AND ANALYSIS OF DATA

The analysis section of this study consists of two parts: (1) the recontextualisation of the topic in the curriculum and textbooks (presented in Chapter 5); and (2) an analysis of students' solutions to an absolute value inequality problem (presented in Chapter 6).

4.2.1 Part 1: The recontextualisation of the topic in the curriculum and textbooks

Part 1 of the analysis is presented in Chapter 5 and presents an analysis of the recontextualisation of the topic absolute value inequalities in the relevant textbooks and curriculum documents.

Mathematics in schools is recontextualised from Mathematics in the field of production and therefore, in order to analyze the recontextualisation of the topic "absolute value inequalities" in the curriculum and textbooks, it is necessary to identify the fundamental axioms, definitions and propositions related to the topic in the field of Mathematics. For this purpose, key texts in the field of Mathematics (Beckenbach & Bellman 1961; Mac Lane 1986; Mac Lane & Birkhoff 1999; Tanton 2005; Stewart & Tall 2015) were consulted. Absolute value inequalities entail the notion of absolute value as well as the notion of inequalities. Therefore, in order to look at the formal treatment of absolute value inequalities in the field of Mathematics, it is necessary to look at the formal definition of absolute value as well as the formal treatment of inequalities in general and then to examine how they are applied in combination.

Following this, the curriculum and textbooks are examined in order to identify how the fundamental features of absolute value inequalities identified in the field of Mathematics are taken up in the curriculum and textbooks. Firstly, the definitions of absolute value provided in the textbook are examined and compared to the definitions found in the field of Mathematics. Secondly, the textbooks' treatment of inequalities in general is (briefly) discussed. Thirdly, the textbooks' methods for solving absolute value inequalities are then analysed, and particular attention is given to the definitions implicitly or explicitly referenced in the methods and the computational resources employed in the methods. This analysis provides a picture of how the topic is recontextualised at the level of school Mathematics.

Since it is not possible to know the degree to which teachers used the textbooks in their classrooms, it would not be valid to assume that students' methods are a result of the treatment of the topic in the textbooks. However, since the curriculum only lists the content to be taught and does not specify sequencing or how a particular topic should be taught and textbooks, on the other hand, are very specific in this regard, teachers often rely on textbooks to interpret the curriculum.

An analysis and description of the treatment of the topic in the relevant textbooks and curriculum documents can therefore give us a better idea of the possible treatment of the topic

in South African classrooms in general and the methods of solving absolute value inequalities in the textbook will inform the analysis of students' solutions dealt with in Part 2.

4.2.2 Part 2: The recontextualisation of the topic in students' solutions to a test item

Part 2 of the analysis is presented in Chapter 6 and presents an analysis of the recontextualisation of the topic in students' solutions to an absolute value inequality problem using the grounded theory approach (Glaser & Strauss 1967) and Weber's (1964) theory of ideal type categories described in Chapter 3.

The first step was to sort the test scripts for analysis. There were originally 133 Grade 12 Mathematics scripts collected by MSEP but unfortunately 17 of these scripts, all belonging to one of the ten original schools in the study, were lost in an office move. Of the 116 scripts that remained there were 15 in which the question on absolute value inequalities was left blank – these were removed leaving 101 student solutions for analysis across seven schools.

It is significant that ideal type categories should emerge from what can be observed empirically in the data and similarly, in grounded theory, the theory should emerge from the data through comparative analysis. Therefore the next step was to examine students' solutions for particular phenomena and common characteristics.

Following this, the next step was to determine the criteria that would govern the selection of the essential features of the ideal type categories. It is important that these criteria are grounded in the relevant theoretical goals of the study and are logically constructed concepts (Hekman 1983:32).

The main goal of this study is to describe what is constituted as absolute value inequalities by Grade 12 Mathematics students in the Western Cape, or in other words, how these students treat absolute value inequalities. According to (Davis 2010c; 2013a), the key to determining what is actually realised in a pedagogic context is evaluation, which can refer to the textbook, student and teacher talk and students and teacher productions. This study is specifically concerned with student productions and it was therefore important that the ideal type categories should essentially capture the *common methods* used by students to solve an absolute value inequality problem. Since students' solutions are a recontextualisation of the topic in the pedagogic discourse (i.e. the textbook), which is itself a recontextualisation of the topic in the field of Mathematics, the fundamental axioms, theorems and definitions related to absolute value inequalities in the field of Mathematics as well as the recontextualisation of the topic in the textbook and in the curriculum (Chapter 5) inform the criteria for the selection of the essential features of the ideal type categories.

Together with the common characteristics identified among the solutions, the criteria generated from the discussion of the notion of absolute value inequalities in the field of Mathematics as well

as the analysis of the methods in the textbook enabled the construction of the ideal-type categories.

It is important to note here that ideal type categories are not ideal in the sense that they perfectly describe every case, but they should capture the essential features that are common to all the cases. In forming ideal categories our interest is in common approaches to the solution of absolute value inequalities and this is to be distinguished from common errors(Davis 2013a).

Therefore, there may be methods in this study that are intrinsically correct but very few cases where the method is employed and the correct final solution is reached.

All of the (101) attempted solutions were then examined and allocated to an ideal-type category. The solutions were also re-graded to examine the degree to which the ideal categories capture the correctness or incorrectness of student's responses.

The use of ideal type categories is what makes it possible to analyze the types of responses that students produce and to determine whether or not there are common approaches. An analysis of the results of the categories in Chapter 7 and the discussion of the results in Chapter 8 will examine the degree to which the students' solution methods uphold or are congruent with the notion of absolute value inequalities in the field of Mathematics as well as the prevalence of the different methods across schools.

4.3 RELIABILITY AND VALIDITY

According to Maxwell (1992:283), validity pertains to the relationship between an account of a situation and some sort of objective reality outside of the account. Dowling & Brown (2012:23) describe it as the relationship between theoretical variables and empirical variables. This study is reliable because the analysis is guided by properties and definitions drawn from the field of Mathematics. Since the fundamental elements of Mathematics are objective and do not change as they are appropriated from the field of Mathematics to the school classroom, they form a steady foundation for this analysis.

Dowling & Brown (2012:23) define reliability as a measure of consistency- if another researcher carried out the same coding process would they get the same results? The reliability of this study depends on the logical construction of ideal type categories, according to the theoretical goals of the study, and the reasoning behind the selection of their essential features. These things must be made explicit in order for the ideal categories to be used as a reliable and valid tool for analyzing phenomena.

Since there are only 101 attempted solutions to the test item across seven schools, it is difficult to generalize the findings beyond the data set. However, the aim of this study is rather to extend the methodology developed by Davis (2013a) and to add to the dialogue around the constitution of

Mathematics in pedagogic settings, in particular with regards to the topic of absolute value inequalities.

4.4 ETHICAL CONSIDERATIONS

Ethical clearance for the fieldwork in the schools was given at the time of the Mathematics and Science Education Project and permission has been granted by the project for the use of the Grade 12 Mathematics baseline scripts in this study, provided that the names of schools and students are protected.

In the next chapter, I present the Part 1 of the analysis of the data.

CHAPTER 5

Part 1: The recontextualisation of the topic in the curriculum and textbooks

This chapter will present part 1 of the data analysis described in Chapter 4 beginning with a discussion of absolute value inequalities in the field of Mathematics and the recontextualisation of the topic in the South African curriculum documents and relevant textbooks.

5.1 ABSOLUTE VALUE INEQUALITIES IN THE FIELD OF MATHEMATICS

Absolute value inequalities entail the notion of absolute value as well as the notion of inequality. Therefore, in order to look at the formal treatment of absolute value inequalities in the field of Mathematics, it is necessary to look at the formal definition of absolute value as well as the formal treatment of inequalities in general and then to examine how they are applied in combination.

5.1.1 The Definitions of Absolute Value in the Field of Mathematics

Stewart & Tall (2015:33) state that the absolute value of some Real number x , notated as $|x|$, “tells us how large or small x is, ignoring whether it is negative or positive” while Tanton (2005:4) describes the absolute value of a Real number as “the positive version of that number”. These short descriptions essentially sum up the two principal ways in which the notion of absolute value can be thought of in the field of Mathematics: firstly as a non-negative number and secondly as the magnitude of a number, “how large or small”. These two meanings for absolute value are related to three formal but different definitions in the field of Mathematics.

The first meaning of absolute value, which states that the absolute value of a number can be thought of as the “positive version of that number”, can be written algebraically as follows:

$$(1) \quad \text{For } A \in \mathbb{R}, \quad |A| = \begin{cases} A & \text{if } A \geq 0 \\ -A & \text{if } A < 0 \end{cases}$$

The “algebraic definition⁵” of absolute value above is widely used and referred to in the field of Mathematics (Stewart & Tall 2015:32; Tanton 2005:4; Mac Lane 1986:99).

A second definition of absolute value is the “square definition⁶” which is commonly used by Mathematicians (Tanton 2005:4) and is similarly based on the idea that the absolute value of a number will always be positive. The square definition in (2) uses the notion that the positive square root of A^2 will be positive for any value of $A \in \mathbb{R}$:

⁵ This is my own term for the definition of absolute value in (1).

⁶ This is my own term for the definition of absolute value in (2).

$\mathbb{R}|a < b$, if $c < 0$, then $ac > bc$; and if $c > 0$ then $ac < bc$ ". Tall (2004:160) in Davis (2013a:6) reminds us that the treatment of all inequalities in formal Mathematics involves the axioms for an ordered field F , "for example, by specifying a subset P of F that has simple properties (if $a \in P$, then one and only one of these holds: $a \in P$, $-a \in P$ or $a = 0$; if $a, b \in P$ then $a + b, ab \in P$.)"

5.1.3 Absolute value inequalities and logical connectives in the field of Mathematics

It follows from the definitions of absolute value that the solution set which satisfies an absolute value inequality may be a combination of two sets. For example, the solution set to the absolute value inequality $|x| \geq a$ is the set of all real numbers x with $x \geq -a$ OR $x \leq a$. The use of the logical connective OR implies that the solution set of $|x| \geq a$ consists of a disjoint union of the set of real numbers x with $x \geq -a$ and the set of real numbers x with $x \leq a$. Since $|x| \geq 0$, there are no values of x which would satisfy the absolute value inequality $|x| \leq a$ where $a < 0$ and so the solution set of $|x| \leq a$ where $a \geq 0$ is the set of all real numbers x with $x > -a$ AND $x \leq a$. The use of the logical connective AND implies that the solution set consists of the intersection of the set of real numbers x with $x > -a$ and the set of real numbers x with $x \leq a$. In both cases the solution set is a combination of two sets and the sets are combined using one of the logical connectives, "AND" or "OR"⁸. Absolute value inequalities over the real numbers are therefore intrinsically linked to compound statements and logical connectives in propositional logic where a compound statement can be defined as a statement that consists of more than one statement combined with the use of logical connectives (Tanton 2005:514). For the purposes of this study we are dealing exclusively with the logical connectives AND and OR.

A possible source of confusion with regards to a discussion on compound inequalities might occur where a compound inequality is written in the form $a < x < b$ where $b > a$ as opposed to writing it as two statements connected with the logical connective, "AND". For example, the compound statement $x > -2$ AND $x < 2$ is equivalent in meaning to the compound statement $-2 < x < 2$ even though the statements are different at the level of expression since both represent the *intersection* of the set $x > -2$ with $x < 2$. In order to differentiate between the two expressions I use the term "compound inequality" to refer to two inequality statements which are connected with one of the logical connectives, "AND" or "OR", and I use the term "double inequality" to refer to an inequality of the form $-a < x < a$. I also use the term "single inequality" to refer to an inequality that consists of a single statement (that is not compound) for example $x > -2$.

⁸ The words "and" and "or" are used as logical connectives in Mathematics to denote "the intersection of two sets" and "the union of two sets" respectively but they take on slightly different meanings where they are used as conjunctions in English. To avoid confusion in this regard I use uppercase letters, e.g. "AND/OR", for the mathematical meaning and lower case letters, e.g. "and/or", for their usual meaning as English conjunctions.

$$(2) \quad |A| = \sqrt{A^2}$$

The third definition of absolute value comes from an understanding of absolute value as “magnitude” and says that:

(3) $|A|$ is the distance of A from the origin on the Real number line and similarly, $|a - b|$ is the distance between a and b on the Real number line (Tanton, 2005:4).

The “distance definition⁷” of absolute value in (3) is used in Mathematics especially with regard to absolute value inequalities and expressing the notion of nearness between two numbers or the approximation of a number (Mac Lane 1986:99). For example, Mac Lane (1986:99) says that absolute values:

“formulate the notion of magnitude without direction and are convenient for expressing ideas of “nearness”. Thus a is near to b when $|a - b|$ is small. Inequalities and absolute values are the customary formal tools for expressing ideas of approximation.”

The distance notion of absolute value is used in different branches of formal Mathematics. For example, in complex numbers, the absolute value of a complex number, given by $|z|$, is called the modulus of z and represents the *distance* between the complex number z and the origin (Tanton 2005:86).

Even the square definition of absolute value, $|A| = \sqrt{A^2}$, derives from the distance formula. The distance formula states that, by the theorem of Pythagoras, the distance between two points in a two-dimensional plane with co-ordinates $(x_1; y_1)$ and $(x_2; y_2)$ is given as follows:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Tanton (2005:142) reminds us that this formula can be generalized to a space with any number of dimensions. For example in a one dimensional space, which can be conceptualized as the Real number line, the distance between any two points x_1 and x_2 is $d = \sqrt{(x_1 - x_2)^2}$ which by the square definition of absolute value is $|x_1 - x_2|$.

5.1.2 Inequalities and order relations in the field of mathematics

Tanton (2005:267) defines an inequality as “a mathematical statement in which one quantity or expression is greater than or less than another” where symbols from the set $\{<, >, \leq, \geq\}$ are used and which satisfies some of the order properties of the real numbers such as the laws of trichotomy, transitivity, addition and multiplication, for example Davis (2013a:8) reminds us that from the order properties of the Real numbers we can deduce that “given non-zero $a, b, c \in$

⁷ This is my own term for the definition of absolute value in (3).

5.2 THE RECONTEXTUALISATION OF ABSOLUTE VALUE INEQUALITIES IN THE CURRICULUM

According to a draft copy of the interim core syllabus for Higher Grade Mathematics (Department of Education 1995), the topic of absolute value is introduced for the first time in Standard 9 (Grade 11). The curriculum lists the topics without specifying the sequence in which they should be taught and the first topic listed in Standard 9 Higher Grade Mathematics is “Algebra” (Department of Education 1995:12). Within algebra, the second topic listed is “absolute value” after “a brief intuitive review of the real numbers” and before “functions”. Within the topic of “absolute value” there are two sub-headings: the first is “Definition” and the second is “ $|x - a|$ (insert symbol) b ”. Since no other explanatory notes or methods are provided, we deduce that the definition of absolute value should be taught as well as absolute value equations/inequalities of the form “ $|x - a| * b$ ” where the operation $*$ is from the set $\{<, \leq, >, \geq, =\}$.

Quadratic equations and inequalities are listed together as a single topic, after functions and linear programming (Department of Education 1995:13). Linear equations and inequalities are also listed together in Standard 8 (Grade 10).

5.3 THE RECONTEXTUALISATION OF ABSOLUTE VALUE INEQUALITIES IN THE TEXTBOOK

5.3.1 The definitions of absolute value in the textbook

The textbook used by the MSEP schools at the time of the study, *Classroom Mathematics* by Laridon *et al.* (2000), presents the notion of absolute value using three different definitions, implicitly and explicitly, which are referred to in this study as “the algebraic definition”, “the distance definition” and “the square definition”⁹ respectively due to their similarity to the definitions of absolute value in the field of Mathematics. The “algebraic definition” is used explicitly to introduce the notion of absolute value as follows (Laridon *et al.* 2000:111):

- (1) The absolute value (or modulus) of any real number A is written $|A|$ and is given by:

$$|A| = \begin{cases} A & \text{if } A \geq 0 \\ -A & \text{if } A < 0 \end{cases}$$

The algebraic definition is used in many other school Mathematics textbooks, both those published before the topic was taken out of the interim syllabus for Higher Grade Mathematics (i.e. before 2006) such as Ladewig, Potgieter & Pretorius (1995:34); Shapiro, Schnell & Erasmus (1976:35); Gonin, Du Plessis, Kuyler, De Jager, Hendricks, Hawkins, Slabber & Archer (1991:63); De Jager, Fitton & Blake (1997:78) and those published after 2006 for example Clarke, Evans,

⁹ These are my own terms of description.

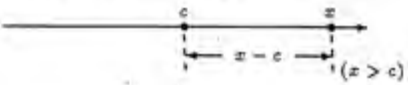
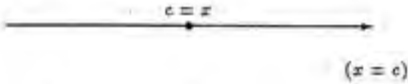
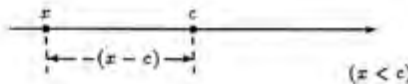
Horsley, Mparutsa & Sproule (2007:17). Even university textbooks for first year Mathematics students, such as Stewart (2006), use this definition for absolute value.

After the algebraic definition is presented, Laridon *et al.* (2000:112) link it to the notion of distance (figure 5.1). They posit that since distance is always “positive or zero”, the distance between two points x and c on the real number line can be written as $x - c$ if $x - c > 0$ and $-(x - c)$ if $x - c < 0$ and therefore from the algebraic definition, the distance between x and c can be written as $|x - c|$. Therefore, instead of defining the notion of absolute value in terms of distance, the distance definition is recontextualised by Laridon *et al.* (2000) in terms of the algebraic definition.

Distance

When we measure *distance* we require the answer to be non-negative (i.e. positive or zero) irrespective of the direction involved.

Examples

1. If we travel 50 km from Cape Town to Stellenbosch, then surely the *distance* from Stellenbosch back to Cape Town will also be 50 km (assuming we do not alter the route!).
2. Suppose c is a fixed point on the number line and x is a variable point:
 - a) If x is to the right of c , then the *distance* from c to x is $x - c$. 
 - b) If x coincides with c , then the *distance* from c to x is $x - c = c - c = 0$. 
 - c) If x is to the left of c , then the *distance* from c to x is $c - x = -(x - c)$. 

We could summarise this by:

The distance from c to x is $\begin{cases} x - c & \text{if } x - c \geq 0 \\ -(x - c) & \text{if } x - c < 0 \end{cases}$

(Convince yourself that the rule in the box says the same thing as in Example 2 a), b) and c) above) which is the same as saying:

The distance from c to x is $|x - c|$.

Figure 5.1 “The distance definition” (Laridon *et al.*, 2000:112)

Although Laridon *et al.* (2000) do not use the distance definition explicitly, it is a common definition of absolute value found in other school Mathematics textbooks (Gonin *et al.* 1991:62; Shapiro *et al.* 1976:35; Ladewig *et al.* 1995:34; De Jager *et al.* 1997:77). The university textbook, Stewart (2006), defines absolute value in terms of distance as follows:

- (2) $|A|$ is the distance from A to 0 on the real number line. Since distance is always positive or 0, $|A| \geq 0$ for any value of A .

The third (square) definition (3) of absolute value is not used by Laridon *et al.* (2000) in their introduction and explanation of the concept of absolute value but it can be found implicitly at the end of the chapter on absolute value inequalities in a short cut method and as a final note on the topic (see figures 5.2 and 5.3 below).

(3)
$$|A| = \sqrt{A^2}$$

More short cuts

You can, and should, use short cuts if you understand them. However, if you simply learn them as tricks, without understanding, you will apply them when they do not hold.

1. **Solutions by squaring**

If $C \geq 0$, then since both sides of $|A| = C$, $|A| < C$ and $|A| > C$ are non-negative, we may square both sides to get equivalent relations $A^2 = C^2$, $A^2 < C^2$ and $A^2 > C^2$.

If $C < 0$, the relations obtained on squaring are not equivalent to the originals. For example, for no A is $|A| < C$ if $C < 0$, but $A^2 < C^2$ for many values of A . (Exercise: check the other two.)

Figure 5.2 "The squaring method" (Laridon *et al.*, 2000:127)

A final note: $\sqrt{x^2} = |x|$ (Not x or $\pm x$!)
e.g. $\sqrt{3^2} = 3$ and $\sqrt{(-3)^2} = 3$

Figure 5.3 "An example implicitly derived from the square definition" (Laridon *et al.*, 2000:128)

In figure 5.2 the square definition is not explicitly used but it is implied by the method of squaring both sides of an inequality (which is referred to in this study as the "squaring method") and therefore the square definition is recontextualised as the squaring method in the textbook. At the end of the section on "more short cuts" (figure 5.2), "a final note" (figure 5.3) gives an example which is implicitly derived from the square definition. The squaring method is also used in many other school Mathematics textbooks, often as an alternative method for solving absolute value inequalities and equations rather than as a definition of absolute value (De Jager *et al.* 1997:83; Ladewig *et al.* 1995:37; Shapiro *et al.* 1976:35; Gonin *et al.* 1991:63; Clarke *et al.* 2007:17). Since the definition is implicit in the method, the method lends itself to the use of an operation-like manipulation in which squaring "removes" the absolute value sign.

5.3.2 Inequalities in the textbook

Inequalities are introduced for the first time in Grade 10 and therefore the Grade 11 textbook does not discuss the topic in great detail. However, since the notion of absolute value inequalities entails the notion of absolute value as well as the notion of inequalities, it is necessary at this point to briefly discuss the textbooks' treatment of inequalities in general.¹⁰

This discussion of the textbook's treatment of inequalities draws from a similar discussion in Davis (2013a) and Davis & Gripper (2012) of the treatment of linear inequalities in a Grade 10 textbook (Pike, Barnes, Jawurek, Kitto, Laridon, Myburgh, Rhodes-Houghton, Scheiber, Sigabi & Wilson 2011)¹¹. (Pike *et al.* 2011). Davis (2013a) and Davis & Gripper (2012) found that the textbook presented a list of "rules" for solving inequalities such as "Multiplying or dividing both sides of an inequality by a negative number reverses the inequality sign". Davis (2013a:8) reminds us that from the order properties of the Real numbers we can deduce that "given non-zero $a, b, c \in \mathbb{R} | a < b$, if $c < 0$, then $ac > bc$; and if $c > 0$ then $ac < bc$ ". The propositions from the field of Mathematics are therefore recontextualised as "rules" in the textbook that are employed as operation-like manipulations that "switch" the inequality symbol around under certain conditions.

The textbook used by the schools at the time of the study (Laridon *et al.* 2000) also presents a set of "Rules of arithmetic" for solving inequalities (see figure 5.4). Although the two rules on their own are from the order properties of the Real numbers, the second rule is restated in the third line as an operation-like manipulation which changes ("switches") the inequality sign under certain conditions. Davis & Gripper (2012).

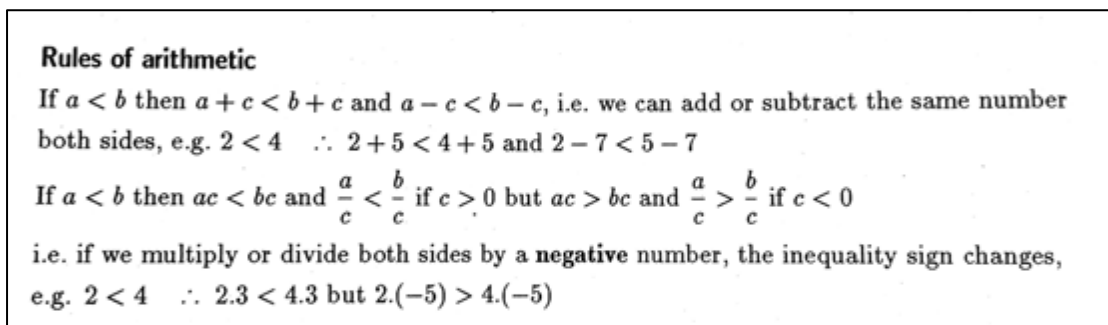


Figure 5.4 "Rules for solving inequalities in the textbook" (Laridon *et al.* 2000:44)

¹⁰ A more in depth discussion of the treatment of linear inequalities in the South African curriculum and textbooks can be found in Davis (2013a) and Davis and Gripper (2012).

¹¹ It is highly likely that this textbook was used by the students in this study in their Grade 10 year since it is from the same series and publisher and is by similar authors to the Grade 11 textbook analysed in this study (Laridon *et al.* 2000) and since, according to Davis (2013a), it was used by all the schools in the MSEP study at the time the data for this study was collected.

5.3.3 Methods for solving absolute value equations and graphs in the textbook

The topic of absolute value inequalities in the textbook (Laridon *et al.* 2000:121-129) is preceded by the topic of absolute value equations and absolute value graphs (within the same chapter) and the textbook links the previous sections to absolute value inequalities with the following introduction:

“If you have done [the absolute value equations] exercise you should find what follows easy. Here it is essential to pay extra careful attention to the conditions arising in the two usual cases.” (Laridon *et al.* 2000:121)

A solution method using “the two usual cases” was first introduced in order to sketch absolute value graphs and to solve absolute value equations and so, in order to understand where the solution methods for absolute value inequalities come from, it is essential to first examine how the textbook treats absolute values in the context of equations and graphs.

The Laridon *et al.* (2000) method for sketching graphs of absolute value functions considers two cases which are directly derived from the algebraic definition of absolute value (definition 1) (figure 5.5).

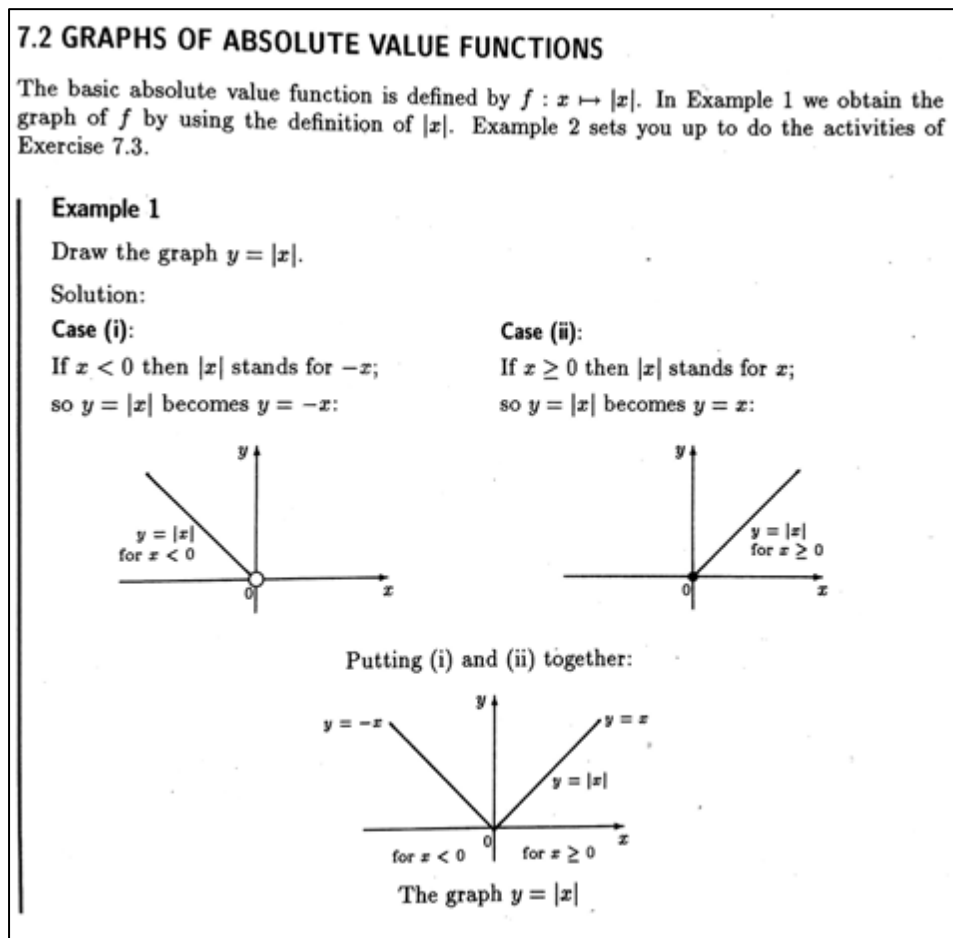


Figure 5.5 “Sketching absolute value functions” (Laridon *et al.*, 2000:113)

For example, in order to sketch the graph of $y = |A|$ the following two cases are first considered:

Case 1: If $A > 0$ then $y = A$

Case 2: If $A < 0$ then $y = -A$

The two functions $y = A$ and $y = -A$ are sketched separately and these sketches are “combined” on the same axis (figure 5.5).

The textbook treats absolute value equations in a similar manner by considering the two cases derived from the algebraic definition. The solutions to each “case” are checked against the initial “if” statement and then written as a compound equation at the end with the logical connector OR (see figure 5.6).

7.3 EQUATIONS INVOLVING ABSOLUTE VALUE

Example 1
Solve for x : $|x - 1| = 3$

Solution:
As always, consider two cases:

<p>(i) If $x - 1 < 0$, i.e. $x < 1$ then $x - 1$ is $-(x - 1)$ so $-(x - 1) = 3^*$ $\therefore x - 1 = -3$ $\therefore x = -2$</p>	<p>(ii) If $x - 1 \geq 0$, i.e. $x \geq 1$ then $x - 1$ is $x - 1$ so $x - 1 = 3^{**}$ $\therefore x = 4$</p>
---	---

<p>Check: The line marked * holds only when $x < 1$, so the final answer is acceptable only if it is less than 1.</p> <p>Since $-2 < 1$ the answer is acceptable. (You might also like to substitute into the given equation.)</p>	<p>Check: The line marked ** holds only if $x \geq 1$, so the final answer is acceptable only if it is greater than or equal to 1.</p> <p>Since $4 \geq 1$ the answer is acceptable. (You might also like to substitute into the given equation.)</p>
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$\therefore x = -2$ or $x = 4$

Figure 5.6 “Absolute value equations” (Laridon *et al.*, 2000:119)

5.3.4 Methods for Solving Absolute Value Inequalities in the Textbook

Laridon *et al.* (2000) demonstrates three main methods for the solution of absolute value inequalities and one other method which is demonstrated towards the end of the chapter as a “short cut” (see figure 5.2). In addition, a general rule for solving absolute value inequalities of the form $|A| \leq b$, $b > 0$ is embedded in one of the exercises. All of the textbook’s methods can be shown to be related to one of the three definitions of absolute value in the field of Mathematics.

Method 1: Algebraic Cases

The first method (figure 5.7) for solving absolute value inequalities in Laridon *et al.* (2000) uses the “two usual cases”- derived from the algebraic definition and first introduced for solving absolute value equations and graphs. I have labeled parts of the method as “steps” for ease of reference. In step 1, AND statements are expressed as “if... then” statements where the condition statement preceded by “if” and the result statement preceded by “then” are both linear inequalities and the two “if... then” statements are connected by OR¹². In step 2 the textbook solves the inequalities for each case. Since both cases present a solution to the initial inequality, the union of the two sets produces the final double inequality in step 3. It is not made explicit to the student in the textbook’s solution (figure 5.7) why a double inequality is used in the final transformation rather than an OR compound inequality and instead the operation-like manipulation “combining” is used.

The textbook uses a number line to illustrate how the compound inequalities in step 2 can be written as double inequalities but the textbook only uses the number line for method 1 in the first example and not when method 1 is used in subsequent examples so we can assume that the use of the number line at this point is to support the expressing of the two compound inequalities as double inequalities rather than as a necessary part of the method.

Example 1
Solve for x : $|x - 3| \leq 2$
Solution:
Method 1

<p>(i) If $x - 3 < 0$, i.e. $x < 3$ then $-(x - 3) \leq 2$... ① $\therefore -x + 3 \leq 2$ $\therefore -x \leq -1$ $\therefore x \geq 1$</p> <p>But ① holds only for $x < 3$ so we have $x \geq 1$ and $x < 3$:</p> <div style="text-align: center;"> <p style="text-align: center;">i.e. $1 \leq x < 3$</p> </div>	or	<p>(ii) If $x - 3 \geq 0$, i.e. $x \geq 3$ then $x - 3 \leq 2$... ② $\therefore x \leq 5$</p> <p>But ② holds only for $x \geq 3$ so we have $x \geq 3$ and $x \leq 5$:</p> <div style="text-align: center;"> <p style="text-align: center;">i.e. $3 \leq x \leq 5$</p> </div>
<p>Combining (i) and (ii) we get: $1 \leq x \leq 5$</p>		

Figure 5.7 “Algebraic method for solving absolute value inequalities” (Laridon *et al.* (2000:122))

¹² The first statement of the “if...then” statements in the textbook’s method does not imply the second statement (i.e. $x < 3$ does not imply that $-(x - 3) \leq 2$), rather both statements must be simultaneously true and should therefore be expressed as an AND statement.

Method 2: Graphical

The second method demonstrated by the textbook is graphical (figure 5.8). The problem being solved is “solve for x : $|x - 3| \leq 2$ ” and the solution method involves finding the points of intersection of the function $y = |x - 3|$ and the function $y = 2$. Both functions are sketched on the same system of axes (the method demonstrated by the textbook for sketching an absolute value function is derived from the algebraic definition- see figure 5.5) and the order relation “less than or equal to” is treated in spatial terms, i.e. “on” or “below”. The x -values for the points where the two functions intersect are found algebraically by treating the two “cases” (see figure 5.5) of the absolute value function as two straight-line functions, $y = -x + 3$ and $y = x - 3$, and finding their points of intersection with the second function, $y = 2$ algebraically. The solution is written as a compound inequality.

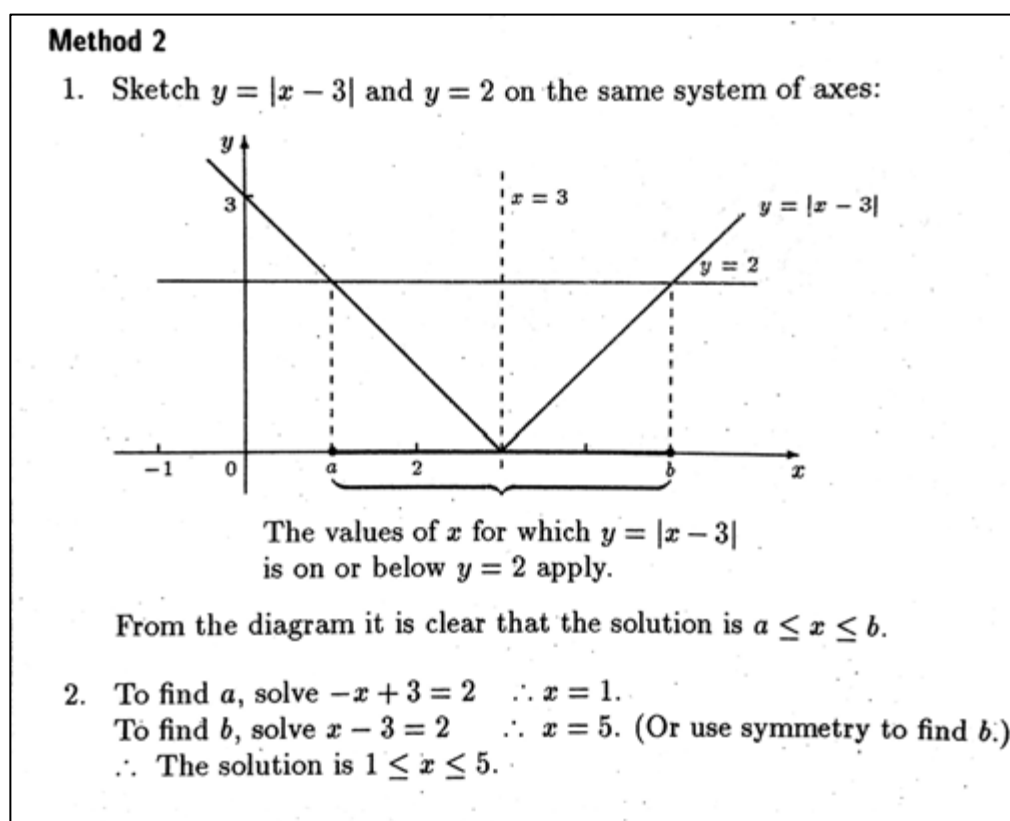


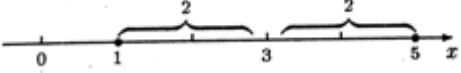
Figure 5.8 “Graphic method for solving absolute value inequalities” (Laridon *et al.* (2000:122)

Method 3: Number line and distance

In the third solution method demonstrated by the textbook (figure 5.9), the distance definition of absolute value, given by the textbook (see figure 5.1) to be “the distance from c to x is $|x - c|$ ” is restated as x is the “distance from c ” (the general terms x and c are used here but the example in figure 5.9 uses specific values). In terms of an inequality, i.e. $|x - c| \leq b$, this can be read as “ x is any distance from c that is less than or equal to b .”

Method 3

We could also use the concept of $|A|$ indicating 'distance from' to solve in a similar manner to Method 2. $|x - 3|$ means 'distance from 3', so $|x - 3| \leq 2$ means that the allowable distances of x from 3 on the number line are 2 or less than 2:



\therefore The solution is $1 \leq x \leq 5$.

Figure 5.9 "Distance method for solving absolute value inequalities" (Laridon *et al.* (2000:123)

Method 4: Squaring

The squaring method, which implicitly uses the square definition of absolute value, is introduced by the textbook at the end of the chapter as a "short cut" (see figure 5.2). The textbook gives the condition that this method can only be used for absolute value inequalities where both sides of the inequality are positive. According to the textbook, squaring both sides of an absolute value inequality such as $|x| < a$ (where $a > 0$) gives $x^2 < a^2$ which can be solved as a quadratic inequality. What is implicit in this method is the square definition of absolute value which says that $|x| = \sqrt{x^2}$. If we apply this to $|x| < a$ we can write an equivalent expression as $\sqrt{x^2} < a$ which is equivalent to $x^2 < a^2$. In the case of an absolute value inequality with a linear x term, the squaring method transforms the absolute value into a quadratic inequality.

A general rule using double inequalities

A general rule for solving certain absolute value inequalities is developed in the first exercise where students are asked to show (using any of the methods above) that an absolute value inequality of the form $|A| \leq b, b > 0$ can be written as the double inequality $-b \leq A \leq b$ (figure 5.9). Students are then asked to apply this method to similar absolute value inequalities of the form $|A| \leq b, b > 0$. In figure 5.10 (i) is the general rule and in (ii) the rule is applied by students.

2. (i) Show that if $|A| \leq b, b > 0$, then $-b \leq A \leq b$.
(ii) Use the result in (i) to solve:

a) $ x - 4 < 3$	b) $ x + 1 \leq 12$
c) $ x + 7 > 2$	d) $ x - 10 \geq 5$
e) $\frac{2}{ x - 2 } \geq 1$	

Figure 5.10 "A general rule using double inequalities" (Laridon *et al.* (2000:125)

5.4 Conclusion

The analysis in part 1 tells us that of the three definitions of absolute value in the field of Mathematics, only the algebraic definition is explicitly used in the textbook and the other two definitions are recontextualised as methods for solving absolute value inequalities. None of the methods in the textbook explicitly refer to the definitions of absolute value or to the notion of order, nor is the concept of logical connectives dealt with in relation to absolute value inequalities. Since the curriculum and the textbook are the main resources available to teachers and students, it is likely that the topic is constituted by the solution methods in the classroom rather than the definitions in the field of Mathematics. Furthermore, the order properties of the Real numbers from the field of Mathematics are recontextualised as operation-like manipulations that “switch” the inequality sign under certain conditions.

CHAPTER 6

Part 2: The recontextualisation of the topic in students' solutions

This chapter will present Part 2 of the results of the data analysis described in Chapter 4. Chapter 3 posited that it is evaluation that gives us access to what is constituted as Mathematics in a pedagogic context (Davis 2010c; 2013a) and although evaluative activity can refer to a broad array of activities, this study is particularly concerned with students' productions as instantiated in their solutions to an absolute value inequality problem. Part 2 of the data analysis is therefore concerned with an analysis of students' solutions using ideal-type categories. It was also posited in Chapter 3 that students' activity is a recontextualisation of the content relayed through pedagogic discourse. Therefore the analysis of the topic in the field of Mathematics and its recontextualisation in the curriculum and the textbook informs the ground for the construction of the ideal-type categories in this chapter and the aim of this chapter is to present a description of this ground followed by a description of the ideal type categories that emerged from students' solutions.

6.1 GROUND FOR IDEAL TYPE CATEGORIES

6.1.1 Determining the essential features of students' solutions

The purpose of creating ideal type categories for analysis is, according to Ritzer (1992:222), "to capture the essential features of some social phenomenon". This study is interested in what is constituted as absolute value inequalities by Grade 12 mathematics students. Only two out of 116 students correctly solved a relatively basic absolute value inequality problem and therefore the "phenomena" analyzed in this study are the different ways in which Grade 12 students treat absolute value inequalities.

Here are some of the main features that emerged from an analysis of the common characteristics among students' solutions:

- (1) Students solutions began with either two mathematical statements or one mathematical statement.
- (2) In the case of two statements, the logical connectors AND and OR were used in different ways or not used at all.
- (3) In a number of the solutions the inequality sign " $<$ " was inappropriately substituted with the sign " $>$ " and vice versa, and/or an equality symbol replaced the inequality symbol or

was introduced with the inequality symbol to create mathematically incoherent¹³ statements.

In Davis' (2013a) study on students' treatment of linear inequalities, Davis posited that "the successful solution of the problem "Solve for x in the inequality $2 - x > 6$ " requires that the order relations pertaining to real numbers (or any subset of the reals being used) be observed explicitly or implicitly" Davis (2013a:12). The extent to which the order relations were maintained or disrupted in each solution was one of the key features that Davis used to form his ideal type categories. From the discussion of the constitution of the topic of absolute value inequalities in the field of Mathematics (discussed in Chapter 5) it could be said that in order to correctly solve an absolute value inequality, the notion of absolute value and the notion of order must be preserved. The preservation or disruption of either of these notions can therefore be considered as an essential feature of students' solutions.

6.1.2 The notion of absolute value in absolute value inequalities

Davis(2013a:12-15) differentiated between explicit and implicit observations of the order relations in solution methods and in the same way this study will examine the explicit and implicit observation of the notion of absolute value. The notion of absolute value is bound up in its three definitions in the field of Mathematics, which have already been examined in Chapter 5. Where a student explicitly refers to one of the definitions of absolute value in their reasoning, we can say that the notion of absolute value is observed explicitly. However, unless the solution method requires explicit attention to the definition or the student provides commentary explaining their reasoning, this is not something that can necessarily be seen in students' written solutions.

Since there were no cases of students' solutions that explicitly made use of any of the definitions in their methods or who annotated their working, we are left with solution methods which either ignore the notion of absolute value or observe it implicitly. A solution method may implicitly observe a definition of absolute value, for example method 1 (algebraic cases) in the textbook (Laridon *et al.* 2000) uses two cases which are explicitly derived from the algebraic definition of absolute value (discussed in Chapter 5). Where a student produces a series of statements in which the solution set to an absolute value inequality is expressed correctly without explicitly referring to the definition of absolute value and without commentary, we can say that the notion of absolute value is implicit in their solution. This leaves us with solutions where the notion of absolute value is disrupted and we define these as solutions that disrupt one of the definitions of absolute value.

¹³ The term "mathematically incoherent" is used by Davis (2013a) to refer to expressions which do not make any sense mathematically, for example the expression $x < +2 = 5$ is mathematically incoherent.

The algebraic definition of absolute value considers two possibilities for the absolute value of a number x where $x \in \mathbb{R}$: either $x \geq 0$ or $x < 0$. The absolute value of x depends on which of these statements is true, for example if $x \geq 0$ then $|x| = x$ and if $x < 0$ then $|x| = -x$. The solution to an absolute value inequality must therefore account for both possibilities and we can say that the notion of absolute value, as it is defined algebraically, would be disrupted by a solution method which fails to consider both when $x \geq 0$ and when $x < 0$.

The distance definition of absolute value says that $|x|$ is the distance of x from 0 on the Real number line. Since distance is neither negative nor positive, a distance x from 0 could be measured from 0 to a positive number on the number line or from 0 to a negative number and have the same value in both cases. The solution to an absolute value inequality must therefore also account for both “directions” on the number line and we can therefore say that the notion of absolute value, as it is defined by the distance definition, would be disrupted by a solution method which fails to consider both the distance to a positive number and the distance to a negative number on the number line.

The square definition of absolute value says that $|x| = \sqrt{x^2}$ and we can say that the notion of absolute value, as it is defined by the square definition, is implicitly maintained by a solution method which expresses $|x|$ as $\sqrt{x^2}$ or an equivalent expression.

6.1.3 The notion of order in absolute value inequalities

For Davis (2013a:13-14) the order relations are observed explicitly by a student who uses a solution method that requires “explicit attention to the order relations, or if he/she explains a statement in terms that demonstrate such reasoning”. Implicit observation of the order relations can be attributed where a student produces a “series of true statements involving order relations” where the order relations are expressed through method appropriate notation. At a school level, method appropriate notation may be encoded through “the scriptural presence of symbols from the collection $\{<, >, \neq, \approx, \leq, \geq, \not\leq, \not\geq, =, \neq\}$, or their equivalents”(Davis 2013a:14). For Davis, order relations are disrupted through the inappropriate uses of the notation as well as the non-use of the notation. There is an infinite number of ways in which inappropriate use of the notation might occur and non-use of the notation or equivalent notation will occur in solutions that ignore the order relations (Davis 2013a:15).

This study will also consider the inappropriate or non-use of method appropriate notation as a disruption of the order relations. In the case of absolute value inequalities there is an additional manner in which order may be disrupted as a result of the fact that the solution set to an absolute value inequality can be written as a compound statement. Compound statements consist of more than one statement connected by logical connectives or equivalent notation and in school mathematics the logical relationship between two statements is expressed using one of the logical connectives from the set {AND, OR}. Therefore the inappropriate or non-use of the logical

connectives or equivalent notation to express the relationship between two sets will also indicate a disruption of the order relations.

6.1.4 Three levels of category

It is worth noting that the notion of absolute value and the notion of order as they are defined above are not necessarily dependent on each other. For example, in the students' solution for the absolute value inequality $|2x - 3| < 4$ in figure 6.1, the inequality symbols are substituted with equality symbols which disrupts the order relations. There is also no logical connective used to represent the relationship between the two statements and as a result they are treated separately throughout the solution. We can therefore say that the notion of order is also disrupted by the absence of a logical connective because the separate solutions include values that do not satisfy the initial inequality.

However, if we ignore the disruption of the order relations, we can argue that the notion of absolute value is maintained according to the algebraic definition. The original inequality is transformed into two statements and the expression $|2x - 3|$ is transformed into " $2x - 3$ " and " $-2x + 3$ " respectively. We can say that the first statement implies one of the conditions of the algebraic definition, i.e. if $2x - 3 > 0$ then $|2x - 3| = 2x - 3$ and the second statement implies the other condition, i.e. if $2x - 3 < 0$ then $|2x - 3| = -2x + 3$. Since both conditions are considered, we can say that the notion of absolute value is maintained even though the order relations are disrupted.

$2x - 3 = 4 \text{ if } x > 0$	$-2x + 3 > 4 \quad x > 0$
$2x = 4 + 3$	$-2x = 4 - 3$
$2x = 7$	$\frac{-2x}{-2} = \frac{1}{-2}$
$x = \frac{7}{2}$	$x = \frac{1}{-2}$
Figure 6.1: Three levels of category (script 23)	

We can also see from the example in figure 6.1 that the order relations can be disrupted in two separate ways. The first way is through the non-use or inappropriate use of the symbols from the set $\{<, >, \neq, \approx, \leq, \geq, \not\leq, \not\geq, =, \neq\}$, and the second way is through the non-use or the inappropriate use of the logical connectives from the set $\{\text{AND, OR}\}$. It is conceivable that in one example the order relations might be disrupted in the first way and in another example both forms of disruption of order may occur in the same solution. Davis (2013a) reminds us that "instances of uses of the notation that participate in the extension of the idea of inappropriate use can be anything at all" and we can therefore expect that there will be a number of methods by which the order relations might be disrupted either through the non-use or inappropriate use of the

symbols from the set $\{<, >, \lessgtr, \gtrless, \leq, \geq, \leqneq, \geqneq, =, \neq\}$ and/or through the non-use or the inappropriate use of the logical connectives. Add to this the notion of absolute value and the infinite number of ways in which it too might be disrupted and it is clear that if we were to try and construct one set of categories that captured every permutation we might end up with a separate category for each solution.

Therefore, in order to avoid unnecessary complexity, this analysis will examine the solutions by categorizing them at three levels, using three different sets of criteria. The first level will examine the disruption of the notion of absolute value in solution methods, the other two levels will examine the disruption of the order relations in solution methods.

The first level of category (Categories A1, A2, A3, A4 and A5) will look at each solution in terms of the notion of absolute value as it is maintained (implicitly or explicitly) or disrupted by a solution method and therefore a breakdown in the notion of absolute value in a solution method will be considered as an essential feature of these categories. At the second level of category (Categories L1, L2, L3, L4 and L5) solutions will be analyzed in terms of the notion of order as it is maintained or disrupted through the inappropriate or non-use of logical connectives and at the third level of category (Categories O1, O2, O3, O4 and O5), solutions will be analyzed in terms of the notion of order as it is maintained or disrupted through the inappropriate or non-use of symbols from the set $\{<, >, \lessgtr, \gtrless, \leq, \geq, \leqneq, \geqneq, =, \neq\}$.

6.2 THE IDEAL TYPE CATEGORIES

Where examples of students' solutions are used, these have been typed rather than inserted as pictures of the actual work in the interest of clarity. Each script was given a unique reference number and this number is indicated next to the figure number in the caption. The expression being solved in every example is $|2x - 3| < 4$ and where students rewrote the expression in their solution this is shown for the sake of accuracy of representation. Details such as lines, arrows and other working have also been shown for this reason (see figure 6.4 as an example).

6.2.1 "A" Categories: the notion of absolute value

Category A1:

Category A1 solutions can be recognized when an absolute value inequality of the general form $|x| < y$ is transformed into two linear inequality statements of the form " $x < y$ and $-x < y$ ". (This is not to be confused with the compound inequality " $x < y$ AND $-x < y$ " since the presence/absence of a logical connective between the two statements will be discussed in the second level of categories.) I refer to this solution method as an algebraic method because it is implicitly derived from the algebraic definition of absolute value which says, in general, that if $x \geq 0$ then $|x| = x$ and if $x < 0$ then $|x| = -x$. If we apply this definition to an inequality of the form $|x| < y$ it must be that $x < y$ (if $x \geq 0$) or that $-x < y$ (if $x < 0$). We can therefore say that

the notion of absolute value is maintained implicitly by this method since the value of $|x|$ is considered for both when $x \geq 0$ and when $x < 0$. Figure 6.2 and 6.3 below are both examples of category A1 solutions.

$$\begin{array}{c}
 |2x - 3| < 4 \\
 -(2x - 3) < 4 \quad 2x - 3 < 4 \\
 -2x + 3 < 4 \quad 2x < 7 \\
 -2x < 1 \quad x < \frac{7}{2} \\
 x > -\frac{1}{2} \\
 -\frac{1}{2} < x < \frac{7}{2}
 \end{array}$$

Figure 6.2: Category A1 example (script 10)

$$\begin{array}{c}
 |2x - 3| < 4 \\
 2x - 3 < 4 \quad \text{or} \quad -(2x - 3) < 4 \\
 2x < 4 + 3 \quad 2x - 3 > -4 \\
 x < \frac{7}{2} \quad 2x > -1 \\
 x < \frac{7}{2} \quad x > -\frac{1}{2}
 \end{array}$$

Figure 6.3: Category A1 example (script 14)

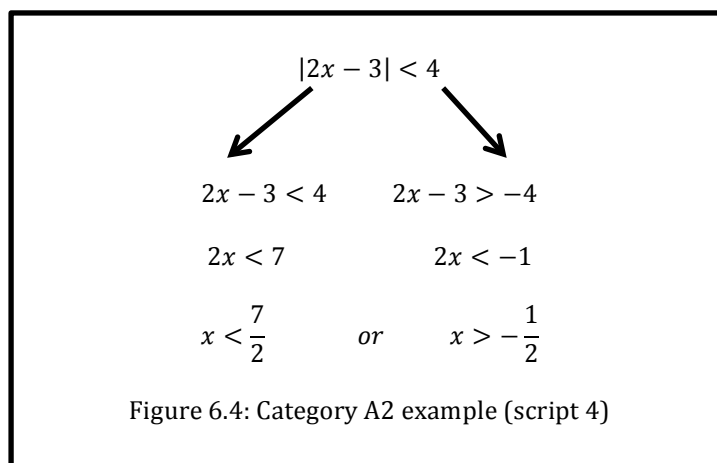
The key features of a category A1 solution are the two linear inequality statements in the initial transformation and the presence of “ $-x$ ” in the one of the initial statements. This is important to note in order to differentiate category A1 solutions from category A2 solutions.

This method is most similar to the textbook’s “method 1” discussed in Chapter 5 and it is likely that the students in category A1 were attempting to use it. However, one of the most obvious differences between the textbook’s solution method and what students actually did in category A1 solutions lies in the “if... then” statements that were absent in students’ solutions. This is discussed further in the next level of categories, which deal with the treatment of logical connectives.

Category A2:

Category A2 solutions can be recognized when an absolute value inequality of the general form $|x| < y$ is transformed into two linear inequality statements of the form “ $x < y$ ” and “ $x > -y$ ” respectively and this is also not to be confused with the compound inequality “ $x < y$ AND $x > -y$ ”. I refer to this solution method as the distance method since it is implicitly derived from the distance definition of absolute value, which says in general that $|x|$ is the distance from x to 0 on the Real number line. Since we have already determined that this distance can be measured from 0 to a positive number or from 0 to a negative number, we can apply this notion to a general inequality of the form $|x| < y$ and say that x is any value such that the distance from x to 0 is smaller than the value of y . If the distance is measured from 0 to a positive number, x must be smaller than $+y$ ($x < y$) and if the distance is measured from 0 to a negative number, x must be greater than $-y$ ($x > -y$). We can say that the notion of absolute value is maintained implicitly

by this solution method because both “directions” on the number line are considered. Figure 6.4 below is an example of a category A2 solution.



Category A1 and A2 are rather similar in that for both methods the initial transformation produces two linear inequality statements where one is of the general form $x < y$. We can therefore expect that there may be instances where a student attempts to employ one of these methods but makes an error and produces a solution that does not seem to fit either category perfectly. As a result, it is necessary to better define how a solution is recognized as category A2 rather than category A1 and vice versa.

The main difference between the appearance of an A1 solution and the appearance of an A2 solution is the presence of the statement, " $-x < y$ " for A1 solutions and " $x > -y$ " for A2 solutions. We differentiate between A1 and A2 solutions because although these statements are equivalent in meaning, the form of the statement is linked to the implicit definition of absolute value used and the operations involved in A1 and A2 solutions are different. Since this level of category is concerned with the notion of absolute value rather than the notion of order, we will not consider the choice of the order relation as a key feature and rather let the presence of " $-x$ " indicate an A1 solution, since the $-x$ is derived from the algebraic definition, and the presence of " $-y$ " indicate an A2 solution, since the $-y$ is derived from the distance definition.

There were two attempted solutions (see figure 6.5 as an example) where there was neither " $-x$ " nor " $-y$ " in the second part of the initial statement, however the order relation " $<$ " was replaced by " $>$ " in the second part of the initial statement and as this is a feature of the distance method both solutions were categorized as category A2.

$2x - 3 < 4$	$2x - 3 > 4$
$2x < 7$	$2x > 7$
$x < \frac{7}{2}$	$x > \frac{7}{2}$

Figure 6.5: Category A2 example (script 25)

Category A3:

Solutions in category A3 can be recognized when an absolute value inequality is expressed as a double inequality in the initial transformation. A (correct) category A3 solution of the general form $|x| < y$ will produce a double inequality statement of the form $-y < x < y$. The solution method is implicitly derived from the distance definition of absolute value in a similar manner to the distance method described for Category A2 solutions with the primary difference being that instead of the compound inequality $x < y$ AND $x > -y$, the equivalent double inequality $-y < x < y$ is used in order to describe a distance from x to 0 which is smaller than $+y$ and greater than $-y$. We can say that the notion of absolute value is maintained implicitly in Category A3 solutions in the same way that it is maintained by Category A2 solutions that use the distance method. Although the statement $-y < x < y$ is equivalent to the statement $x < y$ AND $x > -y$, we differentiate between Category A3 solutions and Category A2 solutions because the operations involved in each method are different.

Figure 6.6 and 6.7 below are examples of Category A3 solutions where we can say that the transformation of the absolute value inequality as a double inequality suggests that the students have attempted to use the solution method described for Category A3 solutions. Although the solution in figure 6.6 disrupts the notion of absolute value as it is defined by the distance definition (the negative parameter (-4) is replaced with the positive parameter $(+4)$), if the (intrinsically correct) method were to be applied correctly then the notion of absolute value would be preserved.

$4 < 2x - 3 < 4$ $\frac{4 + 3}{2} < x < \frac{4 + 3}{2}$ $\frac{7}{2} < x < \frac{7}{2}$ $3\frac{1}{2} < x < 3\frac{1}{2}$
--

Figure 6.6: Category A3 example (script 30)

$-4 < 2x - 3 < 4$ $-1 < 2x < 7$ $-\frac{1}{2} < x < \frac{7}{2}$ $\therefore x > -\frac{1}{2} \text{ and } x < \frac{7}{2}$

Figure 6.7: Category A3 example (script 35)

Category A4:

Category A4 solutions can be recognized when the initial transformation employed by the student on an absolute value inequality of the general form $|x| < y$ produces a quadratic inequality of any of the following different forms: " $x^2 < y^2$ " or " $x^2 < y$ " or " $x < y^2$ " since squaring is used in all three forms. Only the first of these inequalities ($x^2 < y^2$) is able to produce the full solution set for the initial absolute value inequality but I include the other forms because although incorrect, they are examples of an attempt to apply an intrinsically correct solution method.

The method for category A4 solutions is implicitly derived from the square definition of absolute value which states that $|x| = \sqrt{x^2}$ and therefore an absolute value inequality of the general form $|x| < y$ can be rewritten as $\sqrt{x^2} < y$. The notion of absolute value, as it is defined by the square definition, is implicit in category A4 solutions because none of the students explicitly equated $|2x - 3|$ with $\sqrt{(2x - 3)^2}$, rather they went straight from $|2x - 3| < 4$ to $(2x - 3)^2 < 4^2$ in their solutions. So the solutions in this category are using a "squaring method" that stands in place of the square definition of absolute value. Figure 6.8 below is a typical example of how this method appears in student solutions.

$$\begin{aligned}(2x - 3)^2 &< 4^2 \\ 4x^2 - 12x + 9 &< 16 \\ 4x^2 - 12x - 7 &< 0 \\ (2x + 1)(2x - 7) &< 0 \\ 2x > -1 \quad 2x &< 7 \\ x > -\frac{1}{2} \quad x &< \frac{7}{2}\end{aligned}$$

Figure 6.8: Category A4 example (script 45)

There were two solutions in category A4 (scripts 39 and 37) where the right hand side of the absolute value inequality $|2x - 3| < 4$ is squared and the left hand side is treated as a non-absolute value expression, so that it reads $2x - 3 = 16$ (figure 6.9) or $2x - 3 < 16$ (figure 6.10). In figure 6.9 it can be seen that the student was most likely attempting to use a version of the squaring method. However, in squaring the left hand side, the absolute value symbol is removed and so the student is using an operation-like manipulation that "removes" the absolute value symbol, producing a linear expression.

$$(|2x - 3|)^2 = (4)^2$$

$$2x - 3 = 16$$

$$2x = 19$$

$$x = \frac{19}{2}$$

Figure 6.9: Category A4 example (script 39)

$$2x - 3 < 16$$

$$2x < 19$$

$$x < \frac{19}{2}$$

Figure 6.10: Category A4 example (script 37)

Category A5:

Category A5 solutions can be recognized where an absolute value expression is treated as a linear expression, therefore disrupting the notion of absolute value. The initial transformation of an absolute value inequality of the general form $|x| < y$ may produce a linear inequality of the form $x < y$ (figure 6.11) or a linear equation of the form $x = y$ (figure 6.12), but since both expressions are linear and the notion of absolute value is ignored, both are treated as Category A5.

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$x < \frac{7}{2}$$

Figure 6.11: Category A5 example (script 94)

$$2x - 3 = 4$$

$$2x - 3 - 4$$

$$2x = 3 + 4$$

$$\frac{2x}{2} = \frac{7}{2}$$

Figure 6.12: Category A5 example (script 54)

Table 6.1 below summarises the A level categories. A short description for each category is given which will be used to refer to the categories in subsequent chapters. In every A category the notion of absolute value is either maintained or disrupted and where it is maintained the definition of absolute value is either implicit or explicit and this is also summarized for each category in table 6.1.

Table 6.1: A summary of the A level categories

Category	Description	Notion of absolute value maintained/ disrupted	Definition of absolute value implicit/explicit/absent
A1	Algebraic Method	Maintained	Implicit
A2	Distance Method	Maintained	Implicit
A3	Double Inequality Method	Maintained	Implicit
A4	Squaring Method	Maintained	Implicit
A5	Linear Expression	Disrupted	Absent

6.2.2 “L” Categories: the notion of order in terms of logical connectives

It was noted in the description of category A1 solutions that although they are similar to the textbook’s method 1 there were no cases where students successfully included the “if...then” statements in their solutions.

It is therefore relevant to note here, before a discussion of the L categories which relate to logical connectives, how the “if...then” statements are related to the use of logical connectives when solving an absolute value inequality.

Method 1 in the textbook (derived from the algebraic definition) initially transforms the absolute value inequality into an OR compound inequality with two “if...then” statements. For example, using the textbook’s method 1, the first transformation of the absolute value inequality $|2x - 3| < 4$ would be as follows:

$$\text{If } 2x - 3 > 0 \text{ then } 2x - 3 < 4 \qquad \text{OR} \qquad \text{if } 2x - 3 \leq 0 \text{ then } -(2x - 3) < 4$$

The use of OR to indicate a union in the statement above is mathematically correct since both statements cannot be true at the same time. However, without the conditional (if) statements in the first transformation, the use of OR is no longer appropriate. For example, the notion of order would be disrupted by the statement $2x - 3 < 0$ OR $-(2x - 3) < 0$ in a solution to the inequality $|2x - 3| < 4$ because the solution set contains values which do not satisfy the absolute value inequality.

Therefore, for the notion of order to be maintained in solutions that fall into category A1, the logical connective AND¹⁴ must be used to connect the statement $2x - 3 < 4$ and the statement $-(2x - 3) < 0$ since both statements must be true at the same time (intersection) in order to satisfy the inequality $|2x - 3| < 4$.

Category L1:

Category L1 solutions can be recognized when the initial absolute value inequality ($|2x - 3| < 4$) is treated appropriately as either a double inequality or as an equivalent compound inequality (with the logical connective AND) in every transformation.

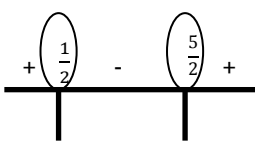
In the last transformation of figure 6.13 below it is clear that the substitution of the double inequality with a compound inequality with the logical connective AND (or vice versa) does not disrupt the order relations because these are equivalent expressions and therefore the solution set described by both expressions is identical.

$$\begin{aligned} -4 < |2x - 3| < 4 \\ -1 < 2x < 7 \\ -\frac{1}{2} < x < \frac{7}{2} \\ \therefore x > -\frac{1}{2} \text{ and } x < \frac{7}{2} \end{aligned}$$

Figure 6.13: Category L1 example (script 35)

Solutions which use the “squaring method” (category A4) produce a quadratic inequality when correctly applied for which the solution set is also expressed as a compound statement connected by AND or a double inequality when solved correctly. Therefore we can also expect to recognize category L1 solutions in which the initial absolute value inequality is transformed into a quadratic inequality and then into a compound or equivalent double inequality. Figure 6.14 below is an example of a solution that uses the “squaring method” and falls into category L1 (the inequality is solved on the left hand side). Although the notion of order is disrupted in the final line of the solution, it is disrupted by the inappropriate use of the order relations rather than the inappropriate use of a logical connective (a disruption of this manner is captured in O-level categories).

¹⁴ This only applies if the if/then condition statement is not used.

$$\begin{aligned}
 & (2x - 3)^2 < 4 \\
 & 4x^2 - 12x + 9 < 4 \\
 & 4x^2 - 12x + 9 - 4 < 0 \\
 & 4x^2 - 12x + 5 < 0
 \end{aligned}$$


$$\begin{aligned}
 & (2x - 1)(2x - 5) \\
 & 2x - 1 = 0 \\
 & 2x = 1 \\
 & x = \frac{1}{2} \quad x = \frac{5}{2}
 \end{aligned}$$

$$\frac{1}{2} > x > \frac{5}{2}$$

Figure 6.14: Category L1 example (script 38)

The notion of order is not disrupted by the inappropriate or non-use of the logical connectives or equivalent notation throughout the solution in category L1.

Category L2:

Category L2 solutions can be recognized when the initial absolute value inequality is treated as a quadratic, double or compound inequality using AND throughout but order is disrupted by the insertion of the logical connective OR in the final transformation. The notion of order is disrupted by the use of OR instead of AND or an equivalent double inequality because OR denotes the union of two sets rather than their intersection (denoted by AND and double inequalities) and therefore values are included in the solution set which do not satisfy the initial inequality $|2x - 3| < 4$. Figures 6.15 and 6.16 below are typical examples of category L2 solutions.

$$\begin{aligned}
 & -4 + 3 < 2x < 4 + 3 \\
 & -1 < 2x < 7 \\
 & 2x < 7 \text{ or } 2x > -1
 \end{aligned}$$

Figure 6.15: Category L2 example (script 29)

$$\begin{aligned}
 & (2x - 3)(2x - 3) < 4 \\
 & 2x^2 - 6x + 5 < 0 \\
 & (2x - 5)(x - 1) < 0 \\
 & x < 5 \text{ or } x < 1
 \end{aligned}$$

Figure 6.16: Category L2 example (script 40)

Category L4:

Category L4 solutions can be recognized when the initial absolute value inequality is transformed into two separate inequalities which, in the final transformation, are transformed into a double inequality or a compound inequality with the logical connective AND or OR. Category L4 solutions can produce a full solution set where the expression produced in the final transformation is a double inequality or a compound inequality with the logical connective AND. However the absence of a logical connective in the preceding statements disrupts the order relations. Where the final transformation is a compound inequality with the logical connective OR the order relations are disrupted further as values are included in the solution set which do not satisfy the initial inequality. Figure 6.19 below is a typical example of a category L4 solution.

$$\begin{array}{ll} -(2x - 3) < 4 & 2x - 3 < 4 \\ -2x + 3 < 4 & 2x < 7 \\ -2x < 1 & x < \frac{7}{2} \\ x > \frac{1}{-2} & \\ & -\frac{1}{2} < x > \frac{7}{2} \end{array}$$

Figure 6.19: Category L4 example (script 10)

Category L5:

Category L5 solutions can be recognized when the initial absolute value inequality is treated as two separate, single inequalities and produces two separate solution sets. The absence of a logical connective disrupts the order relations. Figures 6.20 and 6.21 below are typical examples of category L5 solutions. In Figure 6.20 the initial use of OR is not taken into account as the expression is thereafter treated as two separate inequalities, suggesting that the absolute value inequality is thought of as two separate linear inequalities by the student.

$$\begin{array}{l} |2x - 3| < 4 \\ 2x - 3 < 4 \text{ or } 2x - 3 < -4 \\ \frac{2x}{2} < \frac{7}{2} \quad \frac{2x}{2} < \frac{-1}{2} \\ x < \frac{7}{2} \quad x < -\frac{1}{2} \\ \longrightarrow \quad \longrightarrow \end{array}$$

Figure 6.20: Category L5 example (script 27)

$$\begin{array}{ll} 2x - 3 < 4 & 2x - 3 > 4 \\ 2x < 7 & 2x > 7 \\ x < \frac{7}{2} & x > \frac{7}{2} \end{array}$$

Figure 6.21: Category L5 example (script 25)

Table 6.2 below summarises the L level categories. A short description for each category is given, which will be used to refer to the categories in subsequent chapters, and a summary of where the order relations are maintained or disrupted is included.

Table 6.2: A summary of the L level categories.

Category	Description	Order relations disrupted/maintained
L1	AND or double inequality present throughout.	Order relations maintained throughout.
L2	AND or double inequality throughout until final transformation when OR is used.	Order relations disrupted.
L3	OR present throughout without if-then statements.	Order relations disrupted.
L4	Two linear expressions treated separately throughout until the final transformation.	Order relations disrupted.
L5	Two linear expressions treated separately throughout.	Order relations disrupted.

6.2.3 "O" Categories: the notion of order in terms of order relations

Solutions are categorized at the O level according to the degree to which they either maintain or disrupt the notion or order through the appropriate, inappropriate or non-use of the symbols from the set $\{<, >, \leq, \geq, \neq, =, \approx\}$ or their equivalents.

Category O1:

Category O1 solutions can be recognized when the inequality symbol is present in every transformation employed by the student. The notion of order is explicitly or implicitly maintained throughout the solution through the method appropriate use of the inequality symbol. In order to establish whether the notion of order is explicitly used by the student, we would need to interview the student. Figure 6.22 below is a typical example of a category O1 solution.

$$-4 < 2x - 3 < 4$$

$$-1 < 2x < 7$$

$$\frac{-1}{2} < x < \frac{7}{2}$$

Figure 6.22: Category O1 example (script 33)

Category O2:

Category O2 solutions can be recognized when the inequality symbols are present in every transformation and the order relations are maintained up until the point where either an expression of the form “ $a > b$ ” is illegitimately substituted with “ $a < b$ ” or the other way around (I call this operation-like manipulation “switching” as it appears as though the inequality symbol is reversed). Switching could also occur in a double or compound inequality where one or both of the inequality symbols are illegitimately reversed. Order is disrupted in this solution method because the solution set is replaced by a set that does not satisfy the initial inequality.

I also include in this category solutions where “ $-a > b$ ” is substituted with “ $a > -b$ ” (or vice versa) thereby disrupting the notion of order. For example, where a student writes $-2x < 1 \rightarrow \frac{-2x}{-2} < \frac{1}{-2} \rightarrow x > -\frac{1}{2}$, order is disrupted in the middle step since $\frac{-2x}{-2} > 0$ and $-\frac{1}{2} < 0$ and therefore $\frac{-2x}{-2} > \frac{1}{-2}$.

Figure 6.23 below is a typical example of a category O2 solution where in the fourth transformation the symbols $>$ have been replaced with the symbols $<$, producing a double inequality which describes the intersection of the set $x < -\frac{1}{2}$ with the set $x > \frac{7}{2}$ which is an empty set. In figure 6.24 the second transformation replaces the inequality symbol $<$ with \geq , which is an example of “switching”. The use of the symbol \geq also changes the order relation from a total order to a partial order and therefore disrupts the order relation.

$$\begin{aligned} -4 < 2x - 3 < 4 \\ +3 - 4 < 2x < 4 + 3 \\ -1 < 2x < 7 \\ \frac{-1}{2} > x > \frac{7}{2} \\ -\frac{1}{2} > x > 3\frac{1}{2} \end{aligned}$$

Figure 6.23: Category O2 example (script 36)

$$\begin{aligned} 2x - 3 < 4 \\ 2x \geq 4 + 3 \\ \frac{2x}{2} \geq \frac{7}{2} \\ x \geq 3,5 \end{aligned}$$

Figure 6.24: Category O2 example (script 98)

The operation-like manipulation, “switching”, is probably a misapplication of the common rule taught in schools for solving inequalities that says, “when multiplying or dividing by a negative number the inequality signs must be turned around” or “change sides, change sign” which is discussed in the literature (Chapter 2) and in the textbook’s treatment of inequalities (Chapter 5). The students in figures 6.23 and 6.24 are likely using a variation of this rule whereby “switching” occurs because the negative “number”¹⁵ (−3) is transposed across the inequality sign (changes sides).

There are two solutions that have been categorized as category O2 but are treated as exceptions for different reasons. In figure 6.24 the notion of order is maintained in terms of the treatment of the inequality symbols until the last transformation where the separate linear inequalities are combined and the inequality symbols are “switched”. One explanation for why a student might make this mistake is that they have an idea of what the solution to this type of inequality should look like (ie. a double or compound inequality) and that x must be “between” two values. The student’s activity is therefore regulated by an iconic idea of what the solution should look like rather than the order relations.

The second exception is shown in figure 6.26. It is categorized as category O2 because it is similar to figure 6.25 in that the linear inequality is transformed into a compound inequality through an operation-like manipulation of “taking the 3 over and changing the sign” and it is likely that the student is also regulated by an iconic idea of what the solution should look like. However, there is no switching of the inequality sign in this example.

$$\begin{array}{ll}
 -(2x - 3) < 4 & 2x - 3 < 4 \\
 -2x + 3 < 4 & 2x < 7 \\
 -2x < 1 & x < \frac{7}{2} \\
 x > \frac{1}{-2} & \\
 -\frac{1}{2} < x > \frac{7}{2} &
 \end{array}$$

Figure 6.25: Category O2 example (script 10)

$$\begin{array}{l}
 |2x - 3| < 4 \\
 2x < 4 + 3 \\
 -3 < 2x < 4 \\
 -5 < x < 2
 \end{array}$$

Figure 6.26: Category O2 example (script 100)

¹⁵ I use the word number in inverted commas because numbers do not “change sides” and the students are therefore operating on the character string rather than numbers.

Category O3:

Category O3 solutions can be recognized when the inequality symbol is maintained in every transformation except for the final transformation where it is replaced with an equality symbol. A single value, which does not satisfy the initial inequality as it is not an element of the solution set, replaces the solution set and as a result the order relations are disrupted. Figures 6.27 and 6.28 below are typical examples of category O3 solutions.

$$2x - 3 < 4 \text{ or } 2x - 3 > 4$$

$$2x < 4 + 3$$

$$2x = \frac{7}{2}$$

Figure 6.27: Category O3 example (script 19)

$$2x > 4 + 3$$

$$2x > 4 + 3$$

$$\frac{2x}{2} > \frac{7}{2}$$

$$x = 3,5$$

Figure 6.28: Category O3 example (script 63)

Figure 6.29 is categorized as O3 because an inequality symbol is maintained in every transformation except for the final transformation where it is replaced with an equality symbol, however it also includes switching of inequality signs in the second transformation, hence it is considered an exception in category O3 as it could also be categorised as O2. Figure 6.30 is also an exception because instead of equality symbols replacing the inequality symbols, a partial order replaces the total order and order is disrupted as a result.

$$|2x - 3| < 4$$

$$2x > 4 + 3$$

$$2x > 7$$

$$x > \frac{7}{2}$$

$$x = 3,5$$

Figure 6.29: Category O3 example (script 61)

$$(2x - 3)^2 < (4)^2$$

$$4x^2 - 12x + 9 < 16$$

$$4x^2 - 12x - 7 < 0$$

$$(2x + 3)(2x - 2)$$

$$x \leq \frac{-3}{2} \text{ or } x \geq 1$$

Figure 6.30: Category O3 example (script 44)

Category 04:

Category 04 solutions can be recognized when the inequality sign is immediately replaced with an equality sign and the subsequent transformations are performed on an equation rather than an inequality. Here, the operation-like manipulation “swopping¹⁶” which takes an inequality sign as its input and produces an equal sign as its output is employed in Category 04 solutions. The order relations are disrupted from the beginning although the final solution might be identical to that of Category 03 solutions. Figures 6.31 and 6.32 below are typical examples of category 04 solutions.

$$(|2x - 3|)^2 = (4)^2$$

$$2x - 3 = 16$$

$$2x = 19$$

$$x = \frac{19}{2}$$

Figure 6.31: Category 04 example (script 39)

$$2x - 3 = 4$$

$$2x - 3 - 4$$

$$2x = 3 + 4$$

$$\frac{2x}{2} = \frac{7}{2}$$

Figure 6.32: Category 04 example (script 54)

Category 05:

Category 05 solutions can be recognized when the inequality sign is maintained throughout the transformations, but an equality sign is also introduced in such a way that the expression becomes mathematically nonsensical and the order relations break down as a result. Figures 6.33 and 6.44 below are typical examples of category 05 solutions.

$$2x - 3 < 4$$

$$2x - 3 - 4 > 0$$

$$2x - 7 > 0$$

$$\frac{2}{2}x = \frac{7}{2} > 0$$

$$x \leq \frac{7}{2}$$

Figure 6.33: Category 05 example (script 51)

$$2x - 3 < 4$$

$$2x = 3 < 4$$

$$2x = < 7$$

$$\frac{2x}{7} = < \frac{7}{7}$$

$$\frac{2}{7}x = 1$$

Figure 6.34: Category 05 example (script 49)

¹⁶ My own term.

Table 6.3 below summarises the O level categories. A short description for each category is given, which will be used to refer to the categories in subsequent chapters, and a summary of where the order relations are maintained or disrupted is included.

Table 6.3: A summary of the O level categories.

Category	Description	Order relations maintained/disrupted
01	Inequality symbol present in every transformation.	Order relations maintained throughout.
02	Inequality symbols present in every transformation until they are switched.	Order relations disrupted.
03	Equality sign replaces inequality sign in the final transformation.	Order relations disrupted.
04	Equality sign replaces inequality sign throughout.	Order relations disrupted.
05	Equality sign introduced- nonsensical expression.	Order relations disrupted.

6.2.4 Combination codes and zero categories:

Once every script was categorized at an A level, an L level and an O level, it was given a “combination code” in the form [A category][L category][O category]. For example, a solution that fell into category A1, L3 and O2 would be given a combination code of 132.

Solutions that fall into category A5 cannot fall into any L categories because the L level looks at the way double or compound inequalities are treated and in category A5 solutions there is only a single inequality. A5 solutions are therefore given a code of L0 for the L level so that these solutions can be included in an examination of the combinations. For example a category A5 solution that also falls into category O1 would be given the combination code 501. This allows us to look at every solution as a whole but with an idea of what is happening at each level. Solutions in categories A1, A2, A3 and A4 fall into one of the five L categories and one of the five O categories ($4 \times 5 \times 5 = 100$ possible combination codes) and solutions in category A5 must fall

into category L0 and one of the five O categories ($1 \times 1 \times 5 = 5$ possible combination codes). There are therefore 105 possible combination codes.

6.3 RE-GRADING OF SOLUTIONS

The re-grading of student solutions in this study is designed to be similar to that of the original test designers as well as the approach used by Davis (2013a). The original marking criteria used by the test designers is shown in table 6.4 below. In the original criteria, a score of 4 is given for a correct solution and a score of 3 is given for a careless error whereas scores of 2 and 1 are given for conceptual errors. A score of 0 is given where there is no evidence of understanding and X is given where the question is not attempted. Since Davis (2013a) also uses MSEP data, he too designs the re-grading of the solutions in his study to be similar to that of the original test designers. Davis (2013a:18) retains the maximum score of 4 for a correct solution but awards the minimum score of 0 to any solution that uses a method that implies a failure to maintain the notion of order, “irrespective of the correctness of the arithmetic computations employed in the solution”. Where a solution uses a method that, in principle, does not disrupt the order relations, Davis (2013a) deducts one mark for every computational error.

Table 6.4: MSEP marking criteria (Mathematics and Science Education Project 2007:8)

CODE	DESCRIPTIONS
4	The question was correctly answered
3	Answer had a careless error (including errors in language and/or presentation where applicable)
2	Answer had one conceptual error
1	Answer had two or more conceptual errors
0	No evidence of understanding resulting in incorrect answer
X	The question was not attempted

I found the original marking criteria (table 6.4) to be problematic since there are no clear definitions of or guidelines for identifying careless versus conceptual errors. For example, there were only two solutions that were given a score of 3 by the original researchers, one of which is shown in figure 6.35.

26. Solve for x if $|2x-3| < 4$.

$+ \quad 2x - 3 < 4$	$-2x + 3 < 4$
$2x < 7$	$-2x < 1$
$x > \frac{7}{2}$	$x > -\frac{1}{2}$

4
3

Figure 6.35: Re-grading of solution (script 12)

The original researcher gave the solution in figure 6.35 a score of 4 (indicated on the left hand side) which was then changed by a moderator to a score of 3 (indicated on the right hand side). The score of 3 indicates that the moderator found there was one “careless” error (but no conceptual errors) and the moderator circled the last transformation on the left hand side, where the student illegitimately replaced the inequality symbol “<” with the inequality symbol “>”, indicating that this is where the careless error occurs.

If we interpret a conceptual error to be one where the student is regulated by a concept or idea that is not congruent with the topic in the field of Mathematics and a careless error to be one where the student is regulated by the correct concepts but makes a mistake in a particular instance, then labeling this error as careless rather than conceptual assumes that the observer knows exactly which concepts are regulating the students activity. However, since no interviews with students were conducted in the original study, there is no way of knowing which concepts are regulating the students’ activity purely from the written work. For example, the student might be regulated by a completely different rule which says, “when dividing both sides of the inequality, the inequality sign is turned around” which would be a conceptual error. Since it is not possible in this study to interview students about their thinking in their solutions, it is impossible to differentiate between a careless error and a conceptual error and we therefore must consider all errors as conceptual.

Davis (2013a) takes a slightly different approach by doing away with the conceptual/careless distinction and instead awards a score of 0 to any solution that employs a method that disrupts the order relations and for solutions where the order relation is maintained, one mark is deducted for every “computational error”. Since there is no further elaboration for what is meant by a “computational error” we assume it to be any error that does not disrupt order relations.

Davis (2013a) also awards a score of 1 to solutions that observe the order relation up to a point, for example a student who writes $x = \frac{1}{2}$ as the final statement following $2x - 1 < 0 \rightarrow 2x < 1 \rightarrow x < \frac{1}{2}$ will get a score of 1 since the order relations are maintained implicitly up until the last statement.

This study takes a similar approach to Davis (2013a) but since it is particularly concerned with students' treatment of absolute value inequalities and therefore students' treatment of the notions of absolute value and order, this study is particularly concerned with errors that disrupt either of these notions. In the re-grading of students' solutions in this study a maximum score of 4 is awarded for correct answers and for a solution to be correct it must fall into either category A1, A2, A3 or A4 as well as category L1 and O1 since the notion of absolute value is maintained as well as the notion of order by this combination. A score of 0 is automatically awarded to any solution that disrupts either the notion of absolute value throughout (any solution in category A5) or the notion of order throughout (any solution in categories O4, O5, L3, L4 or L5). For example, a solution which is in category A1, O1 and L4 is awarded a score of 0 since category L4 indicates that the notion of order is disrupted throughout the solution. Where a solution maintains the order relations up to a point (any solution in categories L2 or O2 or O3) a maximum score of 2 can be awarded. However, a solution that falls into either L2 *and* O2 or L2 *and* O3 will be awarded a score of 0 since the order relations are disrupted in two separate ways.

For any solution, 1 mark is subtracted for every computational error but not beyond a score of 0 and incomplete solutions are automatically given a score of 0.

The maximum possible scores for the O categories and the L categories are shown in table 6.5 as well as the maximum possible scores for combinations of L and O categories in table 6.6 assuming that a solution falls into either category A1, A2, A3 or A4 since any solution in category A5 is automatically awarded a score of 0 (as the notion of absolute value is disrupted).

Table 6.5: Maximum possible scores for L and O categories

	POSSIBLE SCORES					
		0	1	2	3	4
CATEGORY L	1	✓	✓	✓	✓	✓
	2	✓	✓	✓	✗	✗
	3	✓	✗	✗	✗	✗
	4	✓	✗	✗	✗	✗
	5	✓	✗	✗	✗	✗
CATEGORY O	1	✓	✓	✓	✓	✓
	2	✓	✓	✓	✗	✗
	3	✓	✓	✓	✗	✗
	4	✓	✗	✗	✗	✗
	5	✓	✗	✗	✗	✗

Table 6.6 shows the maximum possible scores for combinations of L and O categories. The lower score between the L category's maximum score and O category's maximum score for a particular solution gives the maximum combination score. For example, if a solution falls into category L2 and O4, the maximum score is 0 because the maximum score for a solution in category O4 is 0.

Table 6.6: Maximum possible scores for combinations of L and O categories

		Category O				
		1	2	3	4	5
Category L	1	4	2	2	0	0
	2	2	0	0	0	0
	3	0	0	0	0	0
	4	0	0	0	0	0
	5	0	0	0	0	0

6.4 CONCLUSION

Every solution was categorized at the A level, the L level and the O level respectively, was re-graded and allocated a combination code. Chapter 7 presents the results of this analysis.

CHAPTER 7

Presentation of the Results

This chapter will present the results of the analysis presented in Chapter 6. The results for each level of category will be presented separately. The results of the combination code analysis are also presented as well as a two-way analysis of the A and L level categories, the A and O level categories and the L level and O level categories.

7.1 THE A LEVEL CATEGORIES (Notion of absolute value)

The A level categories (notion of absolute value) differ from the L (logical connectives) and O level (order relations) categories in that categories A1 (algebraic method), A2 (distance method), A3 (double inequality method) and A4 (squaring method) are all solution methods that are capable of producing a correct solution if used correctly whereas only category L1 at the L level and O1 at the O level are capable of producing a correct solution. Every attempted solution must fall into an A level category, if there was any transformation of the initial inequality written down by a student it can be categorized at the A level and the A level categories therefore capture every students' initial approach to an absolute value inequality.

7.1.1 A level categories results

Tables 7.1 and 7.2 summarise the distribution of students' attempted solutions by A categories and school. The majority of the attempted solutions treated the absolute value inequality as a linear inequality (category A5: 52,5%), which is the only solution method at the A level of analysis that is not capable of producing a correct solution. Category A1 (algebraic method) had the second highest number of attempted solutions with 19,8% of the total, although it was still significantly less than the highest category, A5. Category A2 (distance method) had the least number of attempted solutions (7,9%) but not significantly less than categories A3 (double inequality) (8,9%) and A4 (squaring) (10,9%).

Table 7.1: Distribution of students' attempted solutions by A categories and school (Count).

School	n	A1	A2	A3	A4	A5
1	19	0	2	1	1	15
2	20	7	2	0	2	9
3	17	1	0	0	3	13
4	7	0	4	0	2	1
5	22	1	0	8	0	13
6	10	7	0	0	1	2
7	6	4	0	0	2	0
Total	101	20	8	9	11	53

Table 7.2 Distribution of students' attempted solutions by A categories and school (%).

School	n	A1	A2	A3	A4	A5
1	19	0%	10,5%	5,3%	5,3%	78,9%
2	20	35%	10%	0%	10%	45%
3	17	5,9%	0%	0%	17,6%	76,5%
4	7	0%	57,1%	0%	28,6%	14,3%
5	22	4,5%	0%	36,4%	0%	59,1%
6	10	70%	0%	0%	10%	20%
7	6	66,7%	0%	0%	33,3%	0%
Total	101	19,8%	7,9%	8,9%	10,9%	52,5%

7.1.2 A level categories across the schools

Categories A1, A2, A3 and A4 are all solution methods that are used by the textbooks and therefore the distribution of students' solutions across these categories may indicate which methods are preferred and encouraged in the classroom.

Only two out of the seven schools in this study had any solutions in category A3 (double inequality method) and eight out of the nine solutions in category A3 were from school 5, suggesting that the double inequality method was emphasized in the classroom at school 5 but is not widely used across the schools. Of the five student solutions that were given a score of greater than 0 in the re-grading, four were in category A3 and both of the two solutions that were given a score of 4 in the re-grading fell into category A3. Of the five student solutions that were given a score of greater than 0 in the re-grading, four were in category A3 and the fifth was in category A4 (squaring method) which suggests that students who transformed the absolute value inequality into a double inequality were the most likely to produce the correct solution. However, we have no way of knowing, without interviewing students, whether those producing the correct solutions are doing so explicitly using the notions of absolute value and order since it is entirely possible to generate a correct solution expressively (what is written down as the solution) without these notions being present.

Six out of the seven schools had at least one solution in Category A4 (the square method), which suggests that although it was one of the least popular solution methods, it was probably taught at most of the schools in this study.

The majority (52,5%) ignored the notion of absolute value completely and treated the absolute value inequality as a linear expression (category A5). This was not something that was found in the literature (see Chapter 2).

7.2 THE L LEVEL CATEGORIES (Logical connectives)

Since 52,5% of students' solutions fell into category A5 (linear expression) and therefore fell into category L0, it follows that only 47,5% of the total attempted solutions are available for analysis at the L level.

7.2.1 L level categories results

At the L level solutions were categorized according to their treatment of the logical connectives. Tables 7.3 and 7.4 summarise the distribution of students' attempted solution methods by L categories and school. The majority of the attempted solutions that could be categorized at the L level treated the linear expressions separately-category L5 (35,4% which is 16,8% of the total attempted solutions). Category L1 (AND or double inequality throughout) had the second highest number of attempted solutions (33,3% which is 15,8% of the total attempted solutions). Order is disrupted in solutions that fall into categories L2, L3, L4 and L5 and therefore order was disrupted at the L level through inappropriate use of logical connectives in 66,7% of students attempted solutions (31,7% of the total attempted solutions).

Categories L4 and L5 are similar in that in both categories the absolute value inequality is treated as two separate linear inequalities. Of the attempted solutions at the L level, 43,8% (20,8% of the total attempted solutions) fall into either category L4 or L5 which is over and above the solutions that treated the absolute value inequality as a linear expression (category A5). If we combine the number of solutions in category A5 (53) with the number of solutions in categories L4 (4) and L5 (17), we can see that out of a total of 101 solutions, 73,3% of students treated the absolute value inequality as a linear expression(s).

Table 7.3: Distribution of students' attempted solutions by L categories and school (Count).

School	n	L1	L2	L3	L4	L5
1	4	2	0	1	0	1
2	11	2	0	2	1	6
3	4	1	2	0	0	1
4	6	1	0	0	1	4
5	9	7	1	0	0	1
6	8	1	0	5	0	2
7	6	2	0	0	2	2
Total	48	16	3	8	4	17

Table 7.4: Distribution of students' attempted solutions by L categories and school (%).

School	n	L1	L2	L3	L4	L5
1	4	50%	0%	25%	0%	25%
2	11	18,2%	0%	18,2%	9,1%	54,5%
3	4	25%	50%	0%	0%	25%
4	6	16,7%	0%	0%	16,7%	66,7%
5	9	77,8%	11,1%	0%	0%	11,1%
6	8	12,5%	0%	62,5%	0%	25%
7	6	33,3%	0%	0%	33,3%	33,3%
Total	48	33,3%	6,3%	16,7%	8,3%	35,4%

7.3 THE O LEVEL CATEGORIES (Order relations)

Every attempted solution can be categorized at the O level, since solutions are examined in terms of their treatment of the order relation symbols from the set $\{<, <=, >, >=, \leq, \neq, \geq, \neq, =, \neq\}$ and at least one of these symbols occurred in every attempted solution.

7.3.1 O level categories results

Category O1 (inequality symbol maintained in every transformation) had the highest number of attempted solutions (47,5%) at the O level of analysis. However, since category O1 is the only category at the O level of analysis in which the order relations are not disrupted and 52,5% of the attempted solutions fell into categories O2 to O5, order is disrupted by the majority of the attempted solutions in this study. The second highest category is O2 (inequality symbols present in every transformation until they are switched) with 28,7% of the attempted solutions. Category O4 (equality sign replaces inequality sign throughout) is the third highest with 12,9% and categories O3 (equality sign replaces inequality sign in the final transformation) and O5 (equality sign introduced-nonsensical expression) have 6,9% and 4% of the attempted solutions respectively.

Table 7.5: Distribution of students' attempted solutions by O categories and school

School	n	O1	O2	O3	O4	O5
1	19	8	3	0	6	2
2	20	7	6	0	5	2
3	17	9	6	2	0	0
4	7	5	0	1	1	0
5	22	15	3	3	1	0
6	10	3	6	1	0	0
7	6	1	5	0	0	0
Total	101	48	29	7	13	4

Table 7.6: Distribution of students' attempted solutions by O categories and school (Count).

School	n	O1	O2	O3	O4	O5
1	19	42,1%	15,8%	0%	31,6%	10,5%
2	20	35%	30%	0%	25%	10%
3	17	52,9%	35,3%	11,8%	0%	0%
4	7	71,4%	0%	14,3%	14,3%	0%
5	22	68,2%	13,6%	13,6%	4,5%	0%
6	10	30%	60%	10%	0%	0%
7	6	16,7%	83,3%	0%	0%	0%
Total	101	47,5%	28,7%	6,9%	12,9%	4%

7.4 THE COMBINATION CODES

Since every attempted solution in this study must fall into an A category and an O category, and every attempted solution, except category A5 solutions, must fall into an L category: there are 105 possible combination codes for this data. For the category A5 (linear inequality) solutions that do not fall into an L category, the category L0 is assigned in order to include these solutions when examining the combination codes.

7.4.1 Combination codes results

Out of 105 possible combination codes only 26 actually occurred (see Appendix B). Of these, the most common combination code was 501 (Linear expression method, no L category, order relations maintained throughout), which accounted for 23,8% of the attempted solutions. The second most common combination code was 502 (Linear expression method, no L category, order relations disrupted by switching), which accounted for 12,9% of the attempted solutions. 30,7% of the attempted solutions fell into one of categories 502, 503, 504 or 505 which means that the majority of students who treated the absolute value inequality as a linear expression (category A5) also disrupted the notion of order (categories O2 to O5).

Since solutions that fall into one or more of categories A5, O2, O3, O4, O5, L2, L3, L4 and L5 are not capable of producing a correct solution, only solutions with the combination codes 111, 211, 311 and 411 are capable of producing a correct solution. There were no solutions with combination codes 111 (algebraic method, AND compound inequality throughout and inequality symbols maintained throughout) or 211 (distance method, AND compound inequality throughout and inequality symbols maintained throughout) but five with the combination code 411 (squaring method, AND compound inequality or double inequality throughout and inequality symbols maintained throughout) and seven with the combination code 311 (double inequality method, AND compound inequality or double inequality throughout and inequality symbols

maintained throughout). None of the five 411 solutions were complete and therefore were given a score of 0 in the re-grading. Of the seven 311 solutions, only four were given a score greater than 0 in the re-grading and two produced a correct solution (score of 4) (these were the only two correct solutions in this study).

7.5 TWO-WAY ANALYSIS RESULTS

This study has looked at how the solutions were categorized across all three levels but it is also useful to examine how they are categorized across two levels at a time as this will enable us to see, for example, how many students who used the algebraic method used the logical connective OR throughout. This two-way analysis looks at a combination of the A and L level categories, the A and O level categories and the L and O level categories separately.

7.5.1 A and L categories

Tables 7.7 and 7.8 present the distribution of students' attempted solutions by A and L level categories. None of the students who used the algebraic or distance method (category A1 and A2) were able to maintain the notion of order in terms of the logical connectives (category L1). 71,4% of the students in categories A1 and A2 treated the linear inequalities separately throughout (category L5) or up until the last transformation (category L4). The rest used OR throughout without the necessary if-then statements (category L3) but more of the students who used the algebraic method (category A1) used OR throughout without the necessary if-then statements (category L3) than those who used the distance method (category A2). Eight out of the nine students who used the double inequality method (category A3) treated the initial inequality as a double inequality or an AND compound inequality throughout and only one student disrupted order by substituting OR at the end. Of the eleven students who used the squaring method (category A4) most (eight students) maintained the notion of order (category L1), two substituted OR at the end (category L2) and one treated the two linear inequalities separately (category L5).

Table 7.7: Distribution of students' attempted solutions by A and L level categories (Count).

A-L	A1	A2	A3	A4	A5	Total
L0	0	0	0	0	53	53
L1	0	0	8	8	0	16
L2	0	0	1	2	0	3
L3	7	1	0	0	0	8
L4	3	1	0	0	0	4
L5	10	6	0	1	0	17
Total	20	8	9	11	53	101

Table 7.8: Distribution of students' attempted solutions by A and L level categories (%).

A-L	A1	A2	A3	A4	A5	Total
L0	0%	0%	0%	0%	52,5%	52,5%
L1	0%	0%	7,9%	7,9%	0%	15,8%
L2	0%	0%	1%	2%	0%	3%
L3	6,9%	1%	0%	0%	0%	7,9%
L4	3%	1%	0%	0%	0%	4%
L5	9,9%	5,9%	0%	1%	0%	16,8%
Total	19,8%	7,9%	8,9%	10,9%	52,5%	100%

7.5.2 A and O categories

Tables 7.9 and 7.10 present the distribution of students' attempted solutions by A and O level categories. Of the students in this study who treated the absolute value inequality as a single linear expression (category A5), 56,6% also disrupted order in terms of the order relations (categories O2 to O5) and only 52,1% of the students who chose a solution method that is capable of producing the correct solution (categories A1 to A4) did not disrupt order in terms of the order relations (category O1).

Table 7.9: Distribution of students' attempted solutions by A and O level categories (Count).

A-O	A1	A2	A3	A4	A5	Total
O1	6	5	8	6	23	48
O2	11	1	1	3	13	29
O3	1	1	0	1	4	7
O4	2	1	0	1	9	13
O5	0	0	0	0	4	4
Total	20	8	9	11	53	101

Table 7.10: Distribution of students' attempted solutions by A and O level categories (%).

A-O	A1	A2	A3	A4	A5	Total
O1	5,9%	5%	7,9%	5,9%	22,8%	47,5%
O2	10,9%	1%	1%	3%	12,9%	28,7%
O3	1%	1%	0%	1%	4%	6,9%
O4	2%	1%	0%	1%	8,9%	12,9%
O5	0%	0%	0%	0%	4%	4%
Total	19,8%	7,9%	8,9%	10,9%	52,5%	100%

7.5.3 L and O categories

Tables 7.11 and 7.12 present the distribution of students' attempted solutions by L and O level categories. The intersection of category L0 with the O level categories is not included in these tables because it will be the same as the intersection of category A5 with the O level categories as every solution in category L0 is also in category A5. Most of the solutions that maintained the notion of order at the L level (category L1) also maintained it at the O level (category O1). However, of all the solutions that maintained the notion of order at the O level (category O1) most of them disrupted the notion of order at the L level (category L2 to L5).

Table 7.11: Distribution of students' attempted solutions by L and O level categories (Count).

L-O	O1	O2	O3	O4	O5	Total
L1	12	3	0	1	0	16
L2	1	1	1	0	0	3
L3	2	6	0	0	0	8
L4	1	2	0	1	0	4
L5	9	4	2	2	0	17
Total	25	16	3	4	0	48

Table 7.12: Distribution of students' attempted solutions by L and O level categories (%).

L-O	O1	O2	O3	O4	O5	Total
L1	25%	6,3%	0%	2,1%	0%	33,3%
L2	2,1%	2,1%	2,1%	0%	0%	6,3%
L3	4,2%	12,5%	0%	0%	0%	16,7%
L4	2,1%	4,2%	0%	2,1%	0%	8,3%
L5	18,8%	8,3%	4,2%	4,2%	0%	35,4%
Total	52,1%	33,3%	6,3%	8,3%	0%	100%

7.6 RESULTS AFTER RE-GRADING

Most of the solutions were re-graded as 0 since they disrupted either the notion of absolute value or the notion of order. There were six solutions that were given a score of greater than 0 in the re-grading. Two solutions were given a score of 4 in both the original grading and the re-grading, one solution was given a score of 4 in the original grading and 3 in the re-grading, one solution maintained a score of 2 in both the original grading and the re-grading and two other solutions were re-graded with a score of 1 but were given a score of 4 and 2 respectively in the original grading. Table 7.13 shows how many students received a score of 0, 1, 2, 3 and 4 respectively in the original MSEP grading and in the re-grading.

Table 7.13: Results before and after re-grading

Score	No. of solutions before re-grading	No. of solutions after re-grading
0	25	95
1	18	2
2	39	1
3	2	1
4	17	2

7.7 CONCLUSION

The results of the analysis show that there is an absence of the notions of absolute value and order in the majority of students' solutions, which suggests that students' computational activity is regulated by notions outside of the field of Mathematics. The results are discussed further in Chapter 8.

CHAPTER 8

Discussion of the results

The aim of this discussion is to compare the results of this study to the general findings in the literature and to discuss any common findings or discrepancies. The most significant findings of the analysis will also be discussed in terms of the methodology presented in Davis (2010a; 2010b; 2010c; 2011a; 2011b); Davis & Gripper (2012); Davis (2013a; 2013b).

8.1 COMMON FINDINGS AND DISCREPANCIES WITH THE LITERATURE

8.1.1 The absence of a graphical approach in students' solutions

The majority of the literature on the teaching and learning of inequalities of all types and of absolute value inequalities in particular agrees that the graphical approach is by far the most effective and efficient way for students to understand inequalities and to approach solving them. However, since there was not a single instance of a student in this study who attempted to use a graphical approach to solve the given absolute value inequality, it is impossible for this study to confirm or refute this general finding in the literature.

Although the textbook (Laridon *et al.* 2000) presents a graphical approach to solving absolute value inequalities (method 2 discussed in Chapter 5) the lack of evidence of a graphical approach in students' solutions strongly suggests an absence at the level of pedagogy. This conjecture cannot be tested however as this study does not have access to what was taught in the classroom.

8.1.2 Reflecting on the equation/inequality connection in this study

Another general finding in the research literature discussed in Chapter 2 is that students' treatment of inequalities is heavily influenced by their experiences with equations. Garuti *et al.* (2001:1) reflect that a possible reason for students' confusion of equations and inequalities is that in most countries inequalities are treated as a sub-topic of equations. In the analysis (Chapter 5) of the textbook used by all the schools in the study at the time the data was collected (Laridon *et al.* 2000) as well as a selection of other relevant textbooks, this study found that equations and inequalities are treated together (often in the same chapter) without exception-which supports Garuti *et al.*'s (2001) hypothesis. Another common finding across the literature (see Chapter 2) is that students are specifically taught to solve inequalities using the same procedures they used to solve equations. This finding is confirmed in the analysis of students' methods for solving absolute value inequalities in this study (see Chapter 6) and the results of this analysis (see Chapter 7) which show that the majority of students treat absolute value inequalities as equations, which suggests that students' treatment of inequalities is heavily influenced by their experiences of solving equations. For example, an equality sign was illegitimately introduced in 23,8% of the attempted solutions (categories O3, O4 and O5), which

agrees with the general findings of the literature (see Chapter 2) that students treat inequalities as equations and in particular this confirms the findings in Davis (2013a) who found that an equality sign was illegitimately introduced in more than 50% of the attempted solutions to a linear inequality problem.

The literature also found that students are taught to solve inequalities using a rule that says, “when multiplying or dividing by a negative number, change the inequality sign around”. A similar method for solving inequalities to the one described above was found in the textbook (see Chapter 5) and in the textbook for Grade 10 students analyzed by Davis & Gripper (2012) who examined the treatment of linear inequalities in the textbook, students’ solutions and classroom teaching. Although it is beyond the scope of this study to make claims about pedagogy, there is strong evidence of an operation-like manipulation referred to as “switching” in students’ solutions where the inequality symbol appeared to be “reversed” (28,7% of the attempted solutions fell into category O2). “Switching” occurred most often when a negative number was entailed in an operation, which suggests that the students in this study were taught to reverse the spatial orientation of the inequality sign when multiplying or dividing by a negative number and this confirms the finding in the literature. Therefore, although it is not possible to generalize the results of this study to other classrooms, the findings of this study in conjunction with the analysis of pedagogy in Davis & Gripper (2012) strongly suggests that the notions of absolute value, logical connectives and order relations are not explicitly dealt with in classrooms.

8.1.3 The logical connectives problem

Another finding in the literature, which is confirmed by the results in this study, is that one of the main sources of errors in students’ treatment of absolute value inequalities is related to the inappropriate use/absence of logical connectives - only 13,9% of the total attempted solutions in this study maintained the notion of order throughout through the appropriate use of logical connectives (category L1).

Out of 101 attempted solutions, only 46 (45,5%) selected a method that had the potential to produce the correct compound or double inequality (categories A1, A2, A3 and A4) (the rest treated the absolute value inequality as a linear expression-category A5) and it is significant to this study that of these 46 attempted solutions, 23,9% used OR without the necessary if/then statements when they should have used AND (categories L2 and L3) and 45,7% transformed the absolute value inequality into two separate inequalities (categories L4 and L5). Furthermore, logical connectives are not explicitly dealt with in the textbook and the results of this study suggest that they are not explicitly taught in the classroom at all- this is therefore an area that requires further investigation of classroom teaching as well as studies that focus on students’ understanding of logical connectives. Although absolute values are currently excluded from the South African curriculum, logical connectives are entailed in the solution of quadratic equations and inequalities and this is therefore an issue that needs to be explored further.

This study also found that although the double inequality method (category A3) is the solution method that is used least across the schools (only 8,9% of the students attempted this approach), it was the most likely to produce the correct solution. A possible reason why students might have been more successful using the double inequality method (category A3) than any other method in this study is that they were less likely to make a logical connectives error- the two-way analysis results (see Chapter 7) showed that only one student who used the double inequality method (category A3) made a logical connectives error (category L2)). This is possibly because the notion of intersection (AND) is embedded in the double inequality and therefore students do not need to attend to the logical connectives¹⁷. The double inequality method therefore potentially enables students to produce the correct solution at the level of expression, but without the notions of absolute value and order relations necessarily being present.

8.2 THE MOST SIGNIFICANT FINDINGS IN THIS STUDY PRESENTED THROUGH A DISCUSSION OF STUDENTS' COMPUTATIONAL ACTIVITY

In Chapter 3 the methodology by Davis (2010a; 2011a; 2013b) for describing operational or computational activity in a school setting in terms of domains, codomains, operations and operation-like manipulations was presented. The most significant findings of the analysis (see Chapter 7) are now summarized with respect to the computational activity of students in order to discuss what is constituted as absolute value inequalities by Grade 12 students at a selection of Western Cape schools.

8.2.1 A discussion of the notion of absolute value in category A5, L4 and L5 solutions

Perhaps the most important finding in this study, which is not echoed in the literature, is that irrespective of the method taught to students in the classroom, the majority (73,3%) ignored the notion of absolute value completely and treated the absolute value inequality as either a linear expression (category A5) or two separate linear expressions (categories L4 and L5) (see Table 7.8). Furthermore, without interviews with students, there is no way of knowing whether the 26,7% of students who selected a method which implicitly maintains the notion of absolute value actually used the notion of absolute value.

In the attempted solutions that are categorized as category A5 the absolute value inequality $|2x - 3| < 4$ is transformed into one of the two linear expressions: either $2x - 3 < 4$ or $2x - 3 = 4$. Since, in the domain of the real numbers (\mathbb{R}), there is no mathematical operation such that the expression $|2x - 3| < 4$ produces either the expression $2x - 3 < 4$ or $2x - 3 = 4$

¹⁷ However, the double inequality method is only applicable when solving absolute value inequalities of the form $|A| \leq B, B \geq 0$.

for every $x \in \mathbb{R}$ (according to any of the definitions of absolute value) the transformation of an absolute value expression into a linear expression requires an existential shift from the domain of the real numbers to the domain of characters or symbols, which shall be referred to as \mathbb{X} . The operation-like manipulation (T) involved in category A5 solutions is therefore such that the object $|x| \in \mathbb{X}$ is taken as an input and the object $x \in \mathbb{X}$ is generated as an output: $T: |x| \rightarrow x$ where $|x|, x \in \mathbb{X}$. The operation-like manipulation, T , can also be described as an operation which “removes” the absolute value symbol, and since T operates over symbols rather than numbers it is external to the field of Mathematics. Since there is no evidence of the notion of absolute value regulating the activity of these students, it is highly likely that for these students the topic “absolute values” consists of basic operations together with operation-like manipulations on symbols.

8.2.2 A discussion of “swopping” and “switching” in category O2, O3, O4 and O5 solutions

Two other operation like-manipulations, “swopping” (S_1) and “switching” (S_2), are involved in attempted solutions in categories O2, O3, O4 and O5.

For attempted solutions that either immediately transformed the absolute value inequality into a linear equation (category O4) or first transformed it into a linear inequality and then into a linear equation in subsequent transformations (category O3), the operation-like manipulation S_1 (“swopping”) occurs as follows: an element from the set $I = \{<, \leq, \geq, >\}$ is taken as an input and an element of the set $E = \{=\}$ is generated as an output such that, $S_1: I \rightarrow E$.

Similarly, for attempted solutions that inappropriately “switched” the inequality sign when a “number” was transposed across the inequality sign (category O2), the operation-like manipulation S_2 can be described as follows: an element from the set $I = \{<, \leq, \geq, >\}$ is taken as an input and a different element from the set $I = \{<, \leq, \geq, >\}$ is generated as an output such that $S_2: I \rightarrow I$.

Since the symbols from the set $\{<, \leq, >, \geq, =\}$ are used interchangeably in the solutions in categories O2, O3, O4 and O5, it is likely that, for these students, the different symbols do not signify different order relations but are rather a set of interchangeable dividers between two expressions. The operation-like manipulations S_1 and S_2 operate over symbols rather than numbers and are therefore external to the field of Mathematics.

8.2.3 A discussion of the notion of order in category L2, L3, L4 and L5 solutions

The failure of the majority of the students in this study to use a logical connective to represent the logical relationship between the two linear inequalities in their solution (categories L4 and L5) suggests that for these students, “solving an absolute value inequality” is constituted as separating it into two linear inequalities and then simplifying those inequalities separately. Since there is no evidence in these solutions that the students recognize any logical relationship between the two inequalities we can say that the notion of order with regard to the logical connectives is absent for most students.

Similarly, in the attempted solutions which replace AND with OR (category L2) or that use OR (without the if-then) conditions throughout the solution (category L3) the logical connectives are used interchangeably and inappropriately and there is no evidence that the students recognize the different relationships that the logical connectives AND and OR signify. The computational activity is therefore not regulated by the notion of an intersection or a union between two sets but rather by an iconic idea of what the solution to an absolute value inequality should look like.

8.3 Conclusion

Overall, the findings in this study agreed with the findings in the literature. In particular this study found that equations and inequalities are treated together in the textbook and in the curriculum and that the majority of students tend to confuse equations and inequalities in their solutions and to make errors with regard to their use (or non-use) of logical connectives. This study found that the majority of students’ computational activity consists of operation-like manipulations, which are external to the field of Mathematics, such as “switching” and “swopping”, and that the majority of the students in this study treated the absolute value inequality problem as a linear expression(s). This points to an absence of the notion of absolute value and order in students’ conceptions of absolute value inequalities, particularly with regard to the notion of logical connectives.

In conclusion, Chapter 9 presents a brief overview of this study as well as a summary of the results and a discussion of the potential and limitations of this research.

CHAPTER 9

Conclusion

9.1 Overview of the study

This study set out to investigate the constitution of absolute value inequalities as displayed in 101 Western Cape Grade 12 students' solutions to an absolute value inequality problem in a baseline Mathematics test. Mathematics in schools is recontextualised from Mathematics in the field of production and therefore, in order to analyze the recontextualisation of the topic "absolute value inequalities" in the curriculum and textbooks, it was necessary to identify the fundamental axioms, definitions and propositions related to the topic in the field of Mathematics. A discussion of the topic as it exists in the field of Mathematics was used to identify the recontextualisation of the topic in the curriculum and the textbook. This recontextualisation was used to identify the ideal type categories, which reflect the recontextualisation of the topic by students.

The methodology used in this study was based on a methodology originally developed by Davis (2013a) who examined the computational activity of students as displayed in their written solutions to a linear inequality problem using Weber's (1964) theory of ideal type categories and Davis' (2013b; 2010a; 2010b; 2010c; 2011a; 2011b) methodology for describing the computational activity of students. In this study, Davis' (2013a) methodology was extended to the more complex topic of absolute value inequalities. It was posited that the successful solution of an absolute value inequality problem requires that the notion of absolute value, as it is defined in the field of Mathematics, is maintained throughout the solution as well as the notion of order as it is represented by the appropriate use of the logical connectives from the set {AND; OR} and by the appropriate use of the symbols from the set $\{<, >, \neq, \leq, \geq, \nless, \ngtr, =, \neq\}$.

Since it would have been impractical to construct an ideal type category for every combination of the permutations above, three different "lenses" of analysis were used and every solution attempt was analyzed firstly in terms of the maintenance or disruption of the notion of absolute value in the solution method (A level categories), secondly in terms of the appropriate, inappropriate or non-use of the logical connectives or equivalent in the solution method (L level categories) and thirdly in terms of the appropriate, inappropriate or non-use of the order relations (O level categories) in the solution methods. Five categories for each level emerged and every solution attempt was categorized at the A level, the L level and the O level respectively. The solution attempts were also re-graded in order to get a sense of how students performed across the categories.

9.2 Summary of the results

The results of the analysis of the recontextualisation of the topic in the curriculum and textbook in Chapter 5 along with the results of the analysis of the recontextualisation of the topic in students' solutions in Chapter 6 and Chapter 7 confirmed the general findings in the literature (discussed in Chapter 2). It also found the following:

- (1) The notion of order was disrupted and the notion of absolute value was absent in the majority of students' solutions.
- (2) The variety of solution methods that emerged suggests that students across the different schools were exposed to different solution methods. However, this study found that irrespective of the method taught in the classroom, the majority of students immediately treated the absolute value inequality as a linear inequality or equation, thereby disregarding the notion of absolute value completely in their computational activity. For those students that did transform the absolute value inequality into two expressions, no logical connective was used to indicate the relationship between them and the absolute value inequality was treated as two separate linear inequalities.
- (2) The majority of students were unable to use logical connectives appropriately and the notion of logical connectives is not explicitly dealt with in the textbook. Since logical connectives are entailed in the solution of quadratic inequalities and equations, this is an issue that needs to be explored further.
- (3) The formal Mathematical definitions and propositions related to absolute value inequalities in the field of Mathematics are recontextualised in the textbook as rules such as "when multiplying or dividing by a negative, change the sign around" and solution methods which are implicitly derived from the definitions but which do not make these definitions explicit to the reader.
- (4) The majority of students' computational activity entails operation-like manipulations which do not operate over the domain of real numbers and instead use symbols as the domain, thus situating their computational activity external to the field of Mathematics. The topic, absolute values, was therefore constituted as basic operations together with "operations" on symbols

Overall, this study painted a dim picture of the state of Mathematics education in South Africa, in particular with regard to the teaching and learning of absolute value inequalities.

9.3 Limitations and potential of this study

One of the limitations of this study is that it was not possible to conduct interviews with the students whose solutions were analyzed. Student interviews would have given this study greater insight into how students think about the notion of absolute value, logical connectives and order relations and what concepts (mathematical or other) are regulating their activity.

Another limitation is that due to the size and nature of this research project, it was not possible to look at the treatment of inequalities by teachers and students in the classroom through lesson observation or to interview students. An analysis of a sample of lessons in combination with students' solutions and interviews would have given this study a more detailed picture of how absolute value inequalities are treated and therefore what is constituted as Mathematics in a pedagogic situation (Davis & Gripper (2012) and Davis (2013a) present one such analysis but with regard to linear inequalities).

This study would also have benefitted from a larger sample of data. As part of a larger study, MSEP originally collected data from ten schools but due to the relatively small number of students who were doing higher grade Mathematics in Grade 12 at the time and due to the loss of some scripts in an office move, only 116 scripts from seven schools were available. In 15 of these scripts the particular problem that this study was interested in was not attempted, which left only 101 scripts for analysis.

The main potential of this study lies in the methodology, which has been extended and developed from a methodology for a simple linear inequality to a methodology that can be used to analyze students' solutions to more complex topics and which has the potential to be extended and developed even further.

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Script 3

Script Number	14iii
Observation Number	3
School Code	2
Gender	Male
Combination Code	132
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$\begin{array}{l}
 2x-3 > 4 \quad \text{or} \quad -2x+3 < 4 \\
 2x > 7 \quad \text{or} \quad -2x < 1 \\
 x > \frac{7}{2} \quad \text{or} \quad x < -\frac{1}{2} \\
 x > \frac{7}{2} \quad \text{or} \quad x < -\frac{1}{2}
 \end{array}$$

2

Script 4

Script Number	2iii
Observation Number	4
School Code	4
Gender	Female
Combination Code	241
Original Score	4
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$\begin{array}{l}
 2x-3 < 4 \quad \text{or} \quad 2x-3 > -4 \\
 2x < 7 \quad \text{or} \quad 2x > -1 \\
 x < \frac{7}{2} \quad \text{or} \quad x > -\frac{1}{2}
 \end{array}$$

4

Script 5

Script Number	11iii
Observation Number	5
School Code	6
Gender	Male
Combination Code	132
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$\begin{array}{l}
 2x - 3 < 4 \quad \text{or} \quad -(2x - 3) < 4 \\
 \underline{2x < 7} \quad \text{or} \quad \underline{-2x + 3 < 4} \\
 \underline{x < \frac{7}{2}} \quad \text{or} \quad \underline{-2x < 1} \\
 \underline{x < \frac{7}{2}} \quad \text{or} \quad \underline{x > -\frac{1}{2}} \\
 \underline{x > \frac{1}{2}}
 \end{array}$$

Script 6

Script Number	5vi
Observation Number	6
School Code	6
Gender	Male
Combination Code	132
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$\begin{array}{l}
 |2x - 3| - 4 < 0 \\
 \underline{2x - 3 < 4 \quad \text{or} \quad -(2x - 3) < 4} \\
 \underline{2x < 4 + 3 \quad \text{or} \quad -2x < 4 - 3} \\
 \underline{x < \frac{7}{2} \quad \text{or} \quad -2x < 1} \\
 \underline{x < \frac{7}{2} \quad \text{or} \quad x > -\frac{1}{2}} \\
 \underline{x > \frac{1}{2}}
 \end{array}$$

Script 7

Script Number	24i
Observation Number	7
School Code	2
Gender	Female
Combination Code	151
Original Score	4
New Score	0

26. Solve for x if $|2x-3| < 4$.

$x = \frac{1}{3}$
 $x = \frac{5}{3}$

$$\begin{array}{l} 2x-3 < 4 \qquad -2x+3 < 4 \\ \hline 2x < 4+3 \qquad -2x < 4-3 \\ 2x < 7 \qquad -2x < 1 \\ \hline x < \frac{7}{2} \qquad x > -\frac{1}{2} \\ \hline \end{array}$$

Combine

$-\frac{1}{2} < x < \frac{7}{2}$

Script 8

Script Number	16iii
Observation Number	8
School Code	7
Gender	Female
Combination Code	142
Original Score	3
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$\begin{array}{l} 2x-3 < 4 \quad \text{and} \quad -(2x-3) < 4 \\ \hline 2x < 7 \qquad -2x+3 < 4 \\ \hline x < \frac{7}{2} \qquad -2x < 1 \\ \hline \qquad \qquad \qquad x > -\frac{1}{2} \\ \hline \frac{7}{2} < x < -\frac{1}{2} \\ \hline \end{array}$$

Script 9

Script Number	19iv
Observation Number	9
School Code	2
Gender	Male
Combination Code	131
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

4

$$2x - 3 < 4 \quad \text{or} \quad -(2x - 3) < 4$$

$$2x < 7 \quad \text{or} \quad -2x + 3 < 4$$

$$x < \frac{7}{2} \quad \text{or} \quad -2x < 1$$

$$\therefore x > -\frac{1}{2}$$

Script 10

Script Number	3iii
Observation Number	10
School Code	7
Gender	Female
Combination Code	142
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

4

$$-(2x - 3) < 4 \quad \quad \quad 2x - 3 < 4$$

$$-2x + 3 < 4 \quad \quad \quad 2x < 7$$

$$-2x < 1 \quad \quad \quad x < \frac{7}{2}$$

$$x > -\frac{1}{2}$$

$$-\frac{1}{2} < x < \frac{7}{2}$$

Script 11

Script Number	18ii
Observation Number	11
School Code	2
Gender	Male
Combination Code	251
Original Score	4
New Score	0

26. Solve for x if $|2x-3| < 4$.

$-4 < 2x-3 < 4$
 $-1 < 2x < 7$
 $-\frac{1}{2} < x < \frac{7}{2}$

4 $2x-3 < 4$ or $-2x-3 < -4$
 $2x < 7$ $-2x < -1$
 $x < \frac{7}{2}$ $x > \frac{1}{2}$

Script 12

Script Number	3i
Observation Number	12
School Code	2
Gender	Male
Combination Code	152
Original Score	3
New Score	0

26. Solve for x if $|2x-3| < 4$.

~~4~~ $2x-3 < 4$ $-2x+3 < 4$
 $2x < 7$ $-2x < 1$
 $x > \frac{7}{2}$ $x > -\frac{1}{2}$

3

Script 13

Script Number	21iii
Observation Number	13
School Code	3
Gender	Female
Combination Code	152
Original Score	4
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$\begin{array}{l|l} |2x-3| > 0 & \text{or } |2x-3| < 0 \\ \hline (2x-3) < 4 & -(2x-3) < 4 \\ \hline 2x < 4+3 & -2x+3 < 4 \\ \hline 2x < 7 & -2x < 1 \\ \hline \frac{2x}{2} < \frac{7}{2} & \frac{-2x}{-2} < \frac{1}{-2} \\ \hline x < 3\frac{1}{2} & x > -\frac{1}{2} \end{array}$$

4

Script 14

Script Number	20ii
Observation Number	14
School Code	5
Gender	Male
Combination Code	151
Original Score	4
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$\begin{array}{l|l} 2x-3 < 4 & \text{or } -(2x-3) < 4 \\ \hline 2x < 4+3 & 2x-3 > 4 \\ \hline x < \frac{7}{2} & 2x > -1 \\ \hline & x > -\frac{1}{2} \end{array}$$

4

Script 15

Script Number	6iii
Observation Number	15
School Code	7
Gender	Male
Combination Code	151
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

4

$$\begin{array}{l} 2x - 3 < 4 \quad \text{or} \quad -(2x - 3) < 4 \\ \hline 2x < 4 + 3 \quad \quad \quad -2x + 3 < 4 \\ 2x < 7 \quad \quad \quad \quad \quad -2x < 4 - 3 \\ \hline x < \frac{7}{2} \quad \quad \quad \quad \quad -2x < 1 \\ \hline \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x > -\frac{1}{2} \end{array}$$

Script 16

Script Number	2i
Observation Number	16
School Code	6
Gender	Male
Combination Code	151
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$$\begin{array}{l} 2x < 3 + 4 \quad \text{or} \quad \cancel{-(2x - 3)} < 4 \\ \hline 2x < 7 \quad \quad \quad -2x + 3 < 4 \\ \hline x < \frac{7}{2} \quad \quad \quad -2x < 4 - 3 \\ \hline \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -2x < 1 \\ \hline \quad x > -\frac{1}{2} \end{array}$$

Script 17

Script Number	14ii
Observation Number	17
School Code	7
Gender	Male
Combination Code	152
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$2x-3 < 4 \quad \text{or} \quad -(2x-3) < 4$$

$$2x < 7 \quad \quad \quad -2x+3 < 4$$

$$x < \frac{7}{2} \quad \quad \quad -2x < 1$$

$$\quad \quad \quad \quad \quad \quad \quad x < -\frac{1}{2}$$

Script 18

Script Number	27i
Observation Number	18
School Code	2
Gender	Female
Combination Code	152
Original Score	1
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$2x-3 < 4 \quad \text{if} \quad 2x-3 < 0 \quad \text{or} \quad -2x-3 < 4 \quad \text{if} \quad -2x-3 > 0$$

$$2x-3 < 4 \quad \quad \quad -2x-3 < 4$$

$$2x > 4+3 \quad \quad \quad -2x > 4+3$$

$$2x > 7 \quad \quad \quad -2x > 7$$

$$x > \frac{7}{2} \quad \quad \quad x > -\frac{7}{2}$$

Script 19

Script Number	8iii
Observation Number	19
School Code	4
Gender	Male
Combination Code	253
Original Score	0
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$2x-3 < 4 \quad \text{or} \quad 2x-3 > 4$$

$$2x < 4 + 3$$

$$x = \frac{7}{2}$$

Script 20

Script Number	4iii
Observation Number	20
School Code	6
Gender	Male
Combination Code	153
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

Handwritten work:

$$(2x-3)^2 < 16$$

$$2x^2 - 12x + 9 < 16$$

$$2x^2 - 12x - 7 < 0$$

$$2x^2 - 12x - 7 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 4(2)(-7)}}{2(2)}$$

$$x = \frac{12 \pm \sqrt{144 + 56}}{4}$$

$$x = \frac{12 \pm \sqrt{200}}{4}$$

$$x = \frac{12 \pm 10\sqrt{2}}{4}$$

$$x = \frac{3 \pm 2.5\sqrt{2}}{1}$$

$$x = 3 \pm 2.5\sqrt{2}$$

Script 21

Script Number	28i
Observation Number	21
School Code	2
Gender	Female
Combination Code	144
Original Score	0
New Score	0

26. Solve for x if $|2x-3| < 4$.

○ $2x-3 < 4$ if $2x+3 < 4$

$2(4)-3 = x$ $2(4)+3 = x$

$5 = x$ or $x = -5$

Script 22

Script Number	21iv
Observation Number	22
School Code	2
Gender	Female
Combination Code	254
Original Score	1
New Score	0

26. Solve for x if $|2x-3| < 4$.

$2x-3 < 4$ if $2x-3 = 0$ or $2x-3 < -4$

$2x = 4+3$ $2x = -4+3$

$\frac{2x}{2} = \frac{7}{2}$ $\frac{2x}{2} = \frac{-1}{2}$

$x = \frac{7}{2}$ $x = \frac{-1}{2}$

Script 23

Script Number	20iii
Observation Number	23
School Code	2
Gender	Male
Combination Code	154
Original Score	1
New Score	0

26. Solve for x if $|2x-3| < 4$.

$2x-3=4$ if $x > 0$ $-2x+3=4$ $x < 0$
 $2x=4+3$ $-2x=4-3$
 $2x=7$ $-2x=1$
 $x=\frac{7}{2}$ $x=-\frac{1}{2}$

Script 24

Script Number	9iii
Observation Number	24
School Code	6
Gender	Female
Combination Code	132
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

$2x-3 < 4$ or $-(2x-3) < 4$
 $2x-3-4 < 0$ or $-2x+3-4 < 0$
 $2x-7 < 0$ or $-2x-1 < 0$
 $\frac{2x}{2} < \frac{7}{2}$ or $\frac{-2x}{-2} < \frac{-1}{-2}$
 $x < 3\frac{1}{2}$ or $x < -\frac{1}{2}$

Script 25

Script Number	3iv
Observation Number	25
School Code	4
Gender	Male
Combination Code	251
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$2x - 3 < 4$	$2x - 3 > -4$
$2x < 7$	$2x > -1$
$x < \frac{7}{2}$	$x > -\frac{1}{2}$

Script 26

Script Number	5iii
Observation Number	26
School Code	4
Gender	Male
Combination Code	251
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

4

$2x - 3 < 4$	$2x - 3 > -4$
$2x < 7$	$2x > -1$
$x < \frac{7}{2}$	$x > -\frac{1}{2}$

Script 27

Script Number	19ii
Observation Number	27
School Code	1
Gender	Male
Combination Code	251
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

4

$$\begin{array}{l} |2x - 3| < 4 \\ \hline 2x - 3 < 4 \quad \text{or} \quad 2x - 3 < -4 \\ \hline \frac{2x}{2} < \frac{7}{2} \quad \quad \quad \frac{2x}{2} < \frac{-1}{2} \\ \hline x < \frac{7}{2} \quad \quad \quad x < -\frac{1}{2} \end{array}$$

Script 28

Script Number	22i
Observation Number	28
School Code	1
Gender	Male
Combination Code	232
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

4

$$\begin{array}{l} 2x - 3 < 4 \quad \text{OR} \quad 2x - 3 < -4 \\ \hline 2x < 4 + 3 \quad \text{OR} \quad 2x < -4 + 3 \\ \hline \frac{2x}{2} < \frac{7}{2} \quad \text{OR} \quad \frac{2x}{2} < \frac{-1}{2} \\ \hline x < \frac{7}{2} \quad \quad \quad x > -\frac{1}{2} \end{array}$$

Script 29

Script Number	8iv
Observation Number	29
School Code	5
Gender	Male
Combination Code	321
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$-4 < 2x - 3 < 4$$

$$-1 < 2x < 7$$

$$2x < 7 \text{ or } 2x > -1$$

Script 30

Script Number	1iii
Observation Number	30
School Code	5
Gender	Female
Combination Code	311
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$4 < 2x - 3 < 4$$

$$\frac{4+3}{2} < x < \frac{4+3}{2}$$

$$\frac{7}{2} < x < \frac{7}{2}$$

$$3\frac{1}{2} < x < 3\frac{1}{2}$$

Script 31

Script Number	27ii
Observation Number	31
School Code	5
Gender	Male
Combination Code	311
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$$4 < 2x - 3 < 4$$

$$4 + 3 < 2x < 4 + 3$$

$$7 < 2x < 7$$

$$\frac{7}{2} < x < \frac{7}{2}$$

Script 32

Script Number	31i
Observation Number	32
School Code	5
Gender	Male
Combination Code	311
Original Score	2
New Score	2

26. Solve for x if $|2x - 3| < 4$.

2

$$-4 < 2x - 3 < 4$$

$$-7 < 2x < 1$$

$$-3\frac{1}{2} < x < \frac{1}{2}$$

2

Script 33

Script Number	21ii
Observation Number	33
School Code	5
Gender	Male
Combination Code	311
Original Score	4
New Score	4

26. Solve for x if $|2x - 3| < 4$.

4

$$\begin{aligned} -4 < 2x - 3 < 4 \\ -1 < 2x < 7 \\ -\frac{1}{2} < x < \frac{7}{2} \end{aligned}$$

Script 34

Script Number	2v
Observation Number	34
School Code	5
Gender	Female
Combination Code	311
Original Score	4
New Score	4

26. Solve for x if $|2x - 3| < 4$.

4

$$\begin{aligned} -4 < 2x - 3 < 4 \\ -4 + 3 < 2x < 4 + 3 \\ -1 < 2x < 7 \\ -\frac{1}{2} < x < \frac{7}{2} \end{aligned}$$

Script 35

Script Number	18iii
Observation Number	35
School Code	1
Gender	Male
Combination Code	311
Original Score	4
New Score	3

26. Solve for x if $|2x-3| < 4$.

$$-4 < \cancel{2x} - 3 < 4$$

$$-1 < 2x < 7$$

$$-\frac{1}{2} < x < \frac{7}{2}$$

$$\therefore x > -\frac{1}{2} \text{ and } x < \frac{7}{2}$$

Script 36

Script Number	14i
Observation Number	36
School Code	5
Gender	Female
Combination Code	312
Original Score	4
New Score	1

$$-4 < 2x - 3 < 4$$

$$+3 - 4 < 2x < 4 + 3$$

$$-1 < 2x < 7$$

$$-\frac{1}{2} > x > \frac{7}{2}$$

$$-\frac{1}{2} > x > 3\frac{1}{2}$$

Script 37

Script Number	1ii
Observation Number	37
School Code	4
Gender	Male
Combination Code	411
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 16$$

$$2x < 19$$

$$x < \frac{19}{2}$$

Script 38

Script Number	33i
Observation Number	38
School Code	7
Gender	Male
Combination Code	412
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$(2x - 3)^2 < 4$$

$$4x^2 - 12x + 9 < 4$$

$$4x^2 - 12x + 5 < 0$$

$$4x^2 - 12x + 5 < 0$$

$\frac{12 \pm \sqrt{144 - 80}}{8} = \frac{12 \pm \sqrt{64}}{8}$

$\frac{12 \pm 8}{8}$

$\frac{20}{8} = \frac{5}{2}$

$\frac{4}{8} = \frac{1}{2}$

$\frac{1}{2} < x < \frac{5}{2}$

Script 39

Script Number	15ii
Observation Number	39
School Code	2
Gender	Female
Combination Code	414
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$(2x - 3) = 4$
 $2x - 3 = 4$
 $2x = 7$
 $x = \frac{7}{2}$

Script 40

Script Number	28iii
Observation Number	40
School Code	3
Gender	Male
Combination Code	422
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$(2x - 3)(2x - 3) < 4$
 $2x^2 - 6x + 3 < 0$
 $(2x - 5)(x - 1) < 0$
 $x < 1$ or $x < 1$

Script 41

Script Number	13iii
Observation Number	41
School Code	3
Gender	Female
Combination Code	411
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$4x^2 - 12x + 9 < 4$$

$$4x^2 - 12x + 13 < 0$$

$$(2x - 3)(2x - 5) < 0$$

Script 42

Script Number	9v
Observation Number	42
School Code	1
Gender	Male
Combination Code	411
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$|2x - 3| < 4$$

$$4x^2 - 12x + 9 < 4$$

$$4x^2 - 12x + 9 - 4 < 0$$

$$4x^2 - 12x + 5 < 0$$

Script 43

Script Number	11iv
Observation Number	43
School Code	2
Gender	Male
Combination Code	411
Original Score	0
New Score	0

26. Solve for x if $(2x-3)^2 < 4$.

0 $4x^2 - 12x + 9 < 4$

Script 44

Script Number	5iv
Observation Number	44
School Code	3
Gender	Male
Combination Code	423
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

$(2x-3)^2 < (4)^2$ 2

$4x^2 - 12x + 9 < 16$

$4x^2 - 12x - 7 < 0$

$(2x+3)(2x-7) < 0$

$x < -\frac{3}{2}$ OR $x > \frac{7}{2}$

Script 45

Script Number	9iv
Observation Number	45
School Code	4
Gender	Male
Combination Code	451
Original Score	4
New Score	0

26. Solve for x if $|2x - 3| < 4$.

4

$$(2x - 3)^2 < 4^2$$

$$4x^2 - 12x + 9 < 16$$

$$4x^2 - 12x - 7 < 0$$

$$(2x + 1)(2x - 7) < 0$$

$$2x > -1 \quad 2x < 7$$

$$x > -\frac{1}{2} \quad x < \frac{7}{2}$$

Script 46

Script Number	10iii
Observation Number	46
School Code	6
Gender	Male
Combination Code	411
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

0

$$(|2x - 3|)^2 < (4)^2$$

$$4x^2 - 6x + 9 < 16$$

$$4x^2 - 6x - 7 < 0$$

0

Script 47

Script Number	17ii
Observation Number	47
School Code	7
Gender	Male
Combination Code	412
Original Score	2
New Score	1

26. Solve for x if $|2x-3| < 4$.

2

⊗

$$\begin{aligned} & (2x-3)^2 < (4)^2 \\ & 4x^2 - 12x + 9 < 16 \\ & 4x^2 - 12x - 7 < 0 \\ & (2x-7)(2x+1) < 0 \\ & 2x < 7 \quad \text{AND} \quad 2x < -1 \\ & x < \frac{7}{2} \quad \quad \quad x < -\frac{1}{2} \end{aligned}$$

Script 48

Script Number	17iv
Observation Number	48
School Code	1
Gender	Female
Combination Code	505
Original Score	0
New Score	0

26. Solve for x if $|2x-3| < 4$.

0

$$\begin{aligned} & 2x-3 < 4 \\ & 2x < 7 \\ & 2x < 7 \\ & x < \frac{7}{2} < 4 \end{aligned}$$

Script 49

Script Number	12ii
Observation Number	49
School Code	1
Gender	Female
Combination Code	505
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$2x - 3 < 4$
 $2x - 3 < 4$
 $2x < 7$
 $\frac{2x}{2} < \frac{7}{2}$
 $x < \frac{7}{2}$

Script 50

Script Number	6i
Observation Number	50
School Code	2
Gender	Male
Combination Code	505
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$2x - 3 < 4$
 $2x - 3 < 4$
 $2x =$

Script 51

Script Number	13ii
Observation Number	51
School Code	2
Gender	Male
Combination Code	505
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$2x - 3 < 4$
 $2x - 3 - 4 > 0$
 $2x - 7 > 0$
 $2x = 7 > 0$
 $x = \frac{7}{2}$

4
2

Script 52

Script Number	2ii
Observation Number	52
School Code	1
Gender	Female
Combination Code	504
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$\frac{2x}{2} = \frac{3}{2}$
 $x = \frac{3}{2}$

2

Script 53

Script Number	5ii
Observation Number	53
School Code	1
Gender	Female
Combination Code	504
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$2x-3 < 4$$

$$-2x+3 = 4$$

$$-2x+3-4 = 0$$

$$-2x+1 \quad -2x-1 = 0$$

$$-2x = 1$$

$$-x-x = 1x$$

$$x = -1$$

Script 54

Script Number	11v
Observation Number	54
School Code	1
Gender	Female
Combination Code	504
Original Score	1
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$2x-3 = 4$$

$$2x+3-4$$

$$2x = 3+4$$

$$2x = 7$$

$$\frac{2}{2} \quad \frac{7}{2}$$

$$x = 7/2$$

Script 55

Script Number	21i
Observation Number	55
School Code	1
Gender	Female
Combination Code	504
Original Score	0
New Score	0

26. Solve for x if $|2x-3| < 4$. $|2x-3| = -2x+3 < 4$

$-2x - 3 = -4$

$-2x + 1 = 0$

$2x - 1 = 0$

$x = \frac{1}{2}$

$\frac{1}{2} \rightarrow$

Script 56

Script Number	5i
Observation Number	56
School Code	2
Gender	Female
Combination Code	504
Original Score	1
New Score	0

26. Solve for x if $|2x-3| < 4$.

$2x - 3 < 4$

$2x - 3 - 4 = 0$

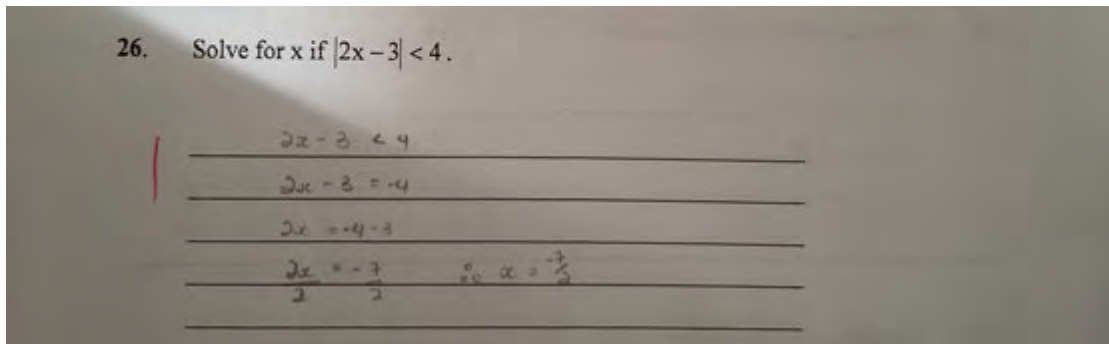
$-2x = 3 - 4$

$2x = -1$

$x = -\frac{1}{2}$

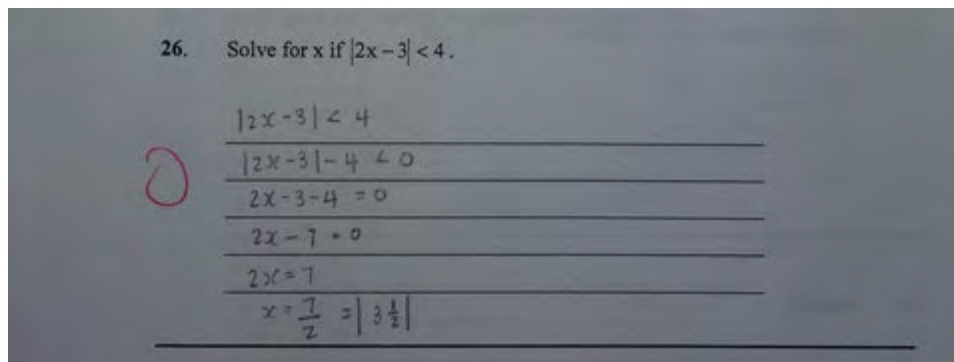
Script 57

Script Number	7ii
Observation Number	57
School Code	1
Gender	Female
Combination Code	504
Original Score	1
New Score	0



Script 58

Script Number	6vi
Observation Number	58
School Code	5
Gender	Female
Combination Code	504
Original Score	0
New Score	0



Script 59

Script Number	4iv
Observation Number	59
School Code	4
Gender	Male
Combination Code	504
Original Score	0
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$|2x-3| < 4$$

$$|2x-3| = -4$$

$$2x-3 = 4$$

$$2x = 7$$

$$x = \frac{7}{2} \text{ or } 3.5$$

Script 60

Script Number	3ii
Observation Number	60
School Code	1
Gender	Male
Combination Code	504
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

$$2x-3 = 4$$

$$2x = 4+3$$

$$2x = 7$$

$$\frac{2x}{2} = \frac{7}{2}$$

$$\therefore x = \frac{7}{2} \text{ or } 3.5$$

Script 61

Script Number	30ii
Observation Number	61
School Code	5
Gender	Female
Combination Code	503
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

~~SA~~
 $|2x - 3| < 4$
 $2x > 4 + 3$
 $2x > 7$
 $x > \frac{7}{2}$
 $x = 3\frac{1}{2} +$

Script 62

Script Number	24ii
Observation Number	62
School Code	5
Gender	Female
Combination Code	503
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

~~SA~~
 $2x - 3 < 4$
 $2x < 4 + 3$
 $2x < 7$
 $\div 2 \quad x < \frac{7}{2}$
 $x = 3\frac{1}{2}$

Script 63

Script Number	11ii
Observation Number	63
School Code	3
Gender	Male
Combination Code	503
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x < 4 + 3$$

$$2x < 4 + 3$$

$$2x < 7$$

$$x < \frac{7}{2}$$

$$x = 3, 5$$

2

Script 64

Script Number	28ii
Observation Number	64
School Code	5
Gender	Male
Combination Code	503
Original Score	2
New Score	0

*26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < -4$$

$$2x < -1$$

$$x < -\frac{1}{2}$$

2

Script 65

Script Number	15i
Observation Number	65
School Code	5
Gender	Male
Combination Code	502
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

1

$$\begin{aligned} 2x - 3 &< 4 \\ 2x &> 4 - 3 \\ 2x &> 1 \\ x &> \frac{1}{2} \end{aligned}$$

Script 66

Script Number	14iv
Observation Number	66
School Code	1
Gender	Male
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$$\begin{aligned} 2x &< 4 + 3 \\ 2x &< 7 \\ x &< \frac{7}{2} \\ \therefore x &< \frac{7}{2} \end{aligned}$$

Script 67

Script Number	13i
Observation Number	67
School Code	5
Gender	Female
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

2

$$\begin{aligned} 2x-3 &< 4 \\ 2x &< 7 \\ x &< \frac{7}{2} \end{aligned}$$

Script 68

Script Number	12iii
Observation Number	68
School Code	5
Gender	Female
Combination Code	502
Original Score	1
New Score	0

26. Solve for x if $|2x-3| < 4$.

1

$$\begin{aligned} |2x-3| &< 4 \\ 2x &< 4+3 \\ 2x &< 7 \\ x &< \frac{7-2}{2} \\ x &< \frac{5}{2} \end{aligned}$$

Script 69

Script Number	11i
Observation Number	69
School Code	5
Gender	Female
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$2x < 7$$

$$x < 7/2$$

Script 70

Script Number	10ii
Observation Number	70
School Code	3
Gender	Female
Combination Code	501
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$|2x - 3| < 4$$

$$|2x| < 4 + 3$$

$$2x < 7$$

$$x < 7/2$$

Script 71

Script Number	9ii
Observation Number	71
School Code	5
Gender	Female
Combination Code	501
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

1

$$\begin{aligned} |2x - 3| - 4 < 0 \\ |2x - 7| < 0 \\ |2x| < 7 \\ x < 7/2 \end{aligned}$$

Script 72

Script Number	9i
Observation Number	72
School Code	2
Gender	Male
Combination Code	501
Original Score	2
New Score	0

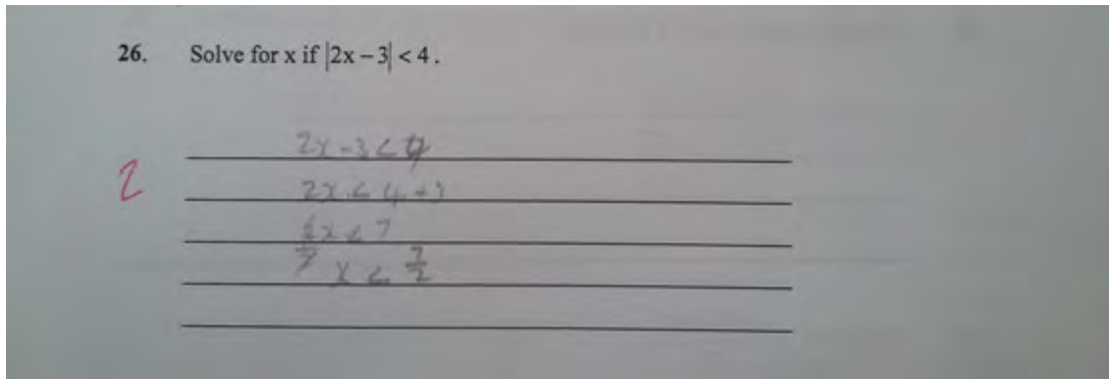
26. Solve for x if $|2x - 3| < 4$.

2

$$\begin{aligned} 2x - 3 < 4 \\ 2x < 7 \\ x < 7/2 \end{aligned}$$

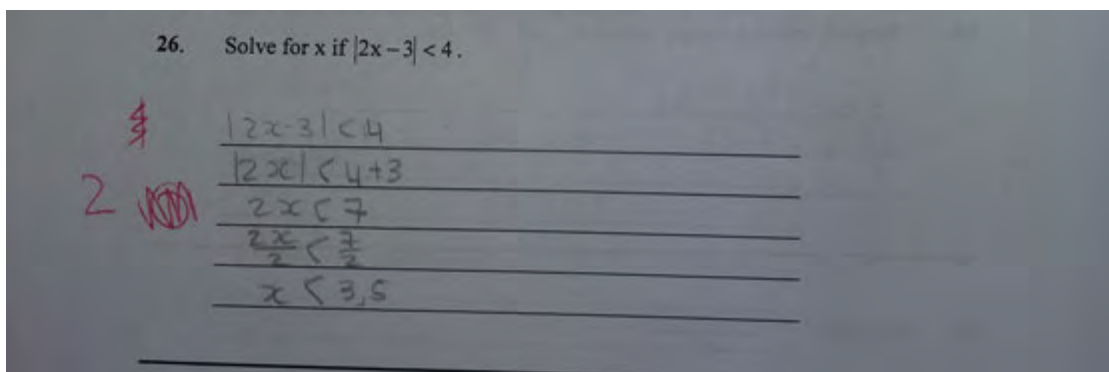
Script 73

Script Number	7i
Observation Number	73
School Code	2
Gender	Female
Combination Code	501
Original Score	2
New Score	0



Script 74

Script Number	3vi
Observation Number	74
School Code	5
Gender	Female
Combination Code	501
Original Score	2
New Score	0



Script 75

Script Number	2iv
Observation Number	75
School Code	3
Gender	Male
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

Handwritten work on lined paper:

$$2x - 3 < 4 + 3$$

$$2x < 7$$

$$x < \frac{7}{2}$$

A red '2' is written on the left margin, and a green '2' is written on the right margin.

Script 76

Script Number	1iv
Observation Number	76
School Code	3
Gender	Male
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

Handwritten work on lined paper:

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$\frac{2x}{2} < \frac{7}{2}$$

$$x < \frac{7}{2} = 3\frac{1}{2}$$

A red '2' is written on the left margin, and a green '2' is written on the right margin.

Script 77

Script Number	1i
Observation Number	77
School Code	1
Gender	Male
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$\begin{array}{l} |2x - 3| < 4 \\ 2x - 3 < 4 \\ 2x < 7 \\ x < \frac{7}{2} \end{array}$$

2

Script 78

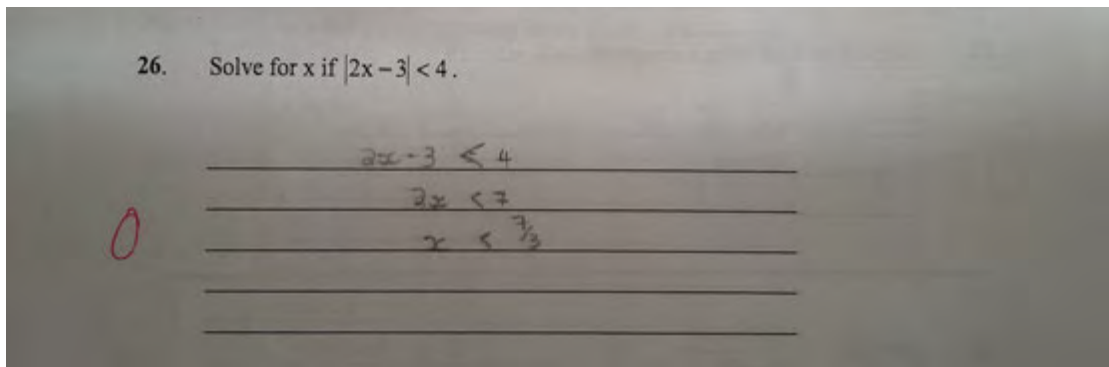
Script Number	15iii
Observation Number	78
School Code	1
Gender	Male
Combination Code	502
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$\begin{array}{l} 2x - 3 < 4 \\ 2x - 3 + 4 < 0 \\ 2x - 7 < 0 \\ 2x < 7 \\ x < \frac{7}{2} \end{array}$$

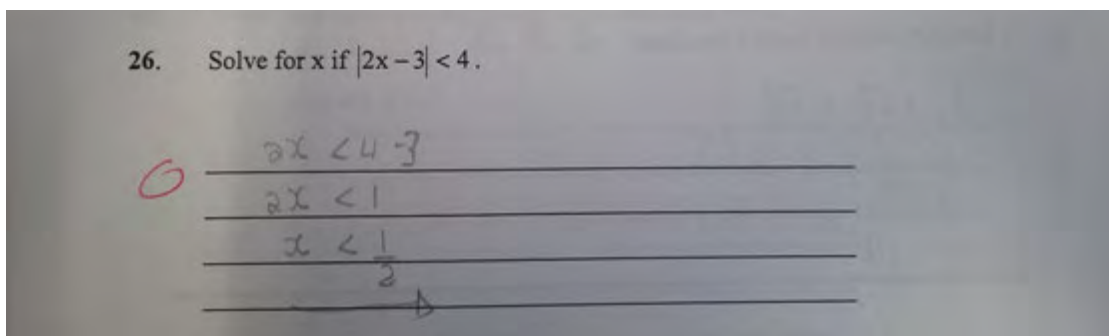
Script 79

Script Number	16ii
Observation Number	79
School Code	3
Gender	Female
Combination Code	501
Original Score	0
New Score	0



Script 80

Script Number	16iv
Observation Number	80
School Code	1
Gender	Female
Combination Code	501
Original Score	0
New Score	0



Script 81

Script Number	17i
Observation Number	81
School Code	5
Gender	Female
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$$\begin{aligned} |2x - 3| < 4 \\ (2x - 3) < 4 \\ 2x - 3 < 4 \\ 2x < 4 + 3 \\ 2x < 7 \\ x < \frac{7}{2} \end{aligned}$$

Script 82

Script Number	19iii
Observation Number	82
School Code	3
Gender	Female
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$$\begin{aligned} |2x - 3| < 4 \\ 2x < 4 + 3 \\ 2x < 7 \\ 2 \quad 2 \\ x < \frac{7}{2} \end{aligned}$$

Script 83

Script Number	20i
Observation Number	83
School Code	1
Gender	Female
Combination Code	501
Original Score	0
New Score	0

26. Solve for x if $|2x-3| < 4$.

6

$$2x-3 < 4$$

$$2x < 4+3$$

$$\frac{2x}{2} < \frac{7}{2}$$

$$x < \frac{7}{2}$$

Script 84

Script Number	22iii
Observation Number	84
School Code	3
Gender	Male
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x-3| < 4$.

2

$$|2x-3| < 4$$

$$2x < 7$$

$$x < \frac{7}{2}$$

Script 85

Script Number	22ii
Observation Number	85
School Code	5
Gender	Female
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$$2x < 4 + 3$$

$$< 7$$

$$\therefore x < \frac{7}{2}$$

Script 86

Script Number	23i
Observation Number	86
School Code	2
Gender	Male
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

2

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$\frac{2x < 7}{2}$$

$$2 < \frac{7}{2} \text{ or } 3.5$$

Script 87

Script Number	23ii
Observation Number	87
School Code	5
Gender	Male
Combination Code	501
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$\frac{2x}{2} < \frac{4+3}{2}$$

$$x < 3\frac{1}{2}$$

Z

Script 88

Script Number	25iii
Observation Number	88
School Code	3
Gender	Male
Combination Code	502
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$|2x - 3| < 4$$

$$2x - 3 - 4 > 0$$

$$2x - 7 > 0$$

$$2x < 7$$

$$x > \frac{7}{2}$$

B
D

Script 89

Script Number	4ii
Observation Number	89
School Code	1
Gender	Male
Combination Code	502
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$2x < 7$$

$$x > \frac{1}{2}$$

Script 90

Script Number	27iii
Observation Number	90
School Code	3
Gender	Male
Combination Code	501
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$\frac{2x}{2} < \frac{7}{2}$$

$$x < \frac{7}{2}$$

Script 91

Script Number	23iii
Observation Number	91
School Code	3
Gender	Male
Combination Code	502
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 4$$

$$2x < 4 + 3$$

$$x < 3\frac{1}{2}$$

2

Script 92

Script Number	6v
Observation Number	92
School Code	3
Gender	Female
Combination Code	502
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 4$$

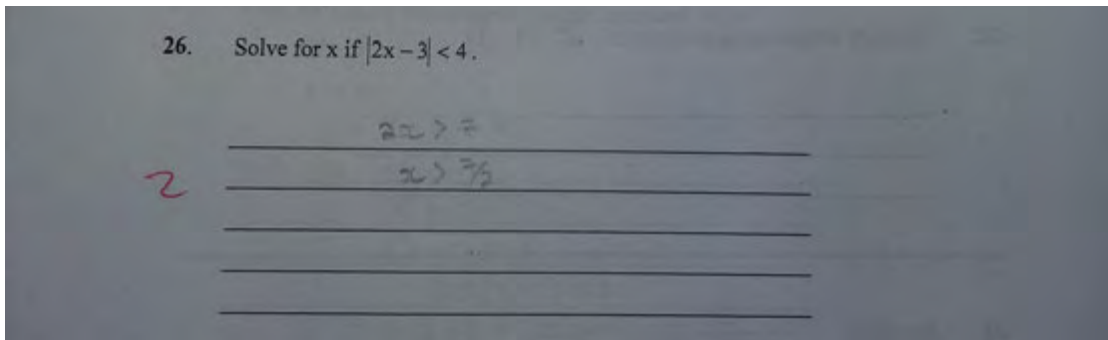
$$2x > -1$$

$$\therefore x > -\frac{1}{2}$$

0

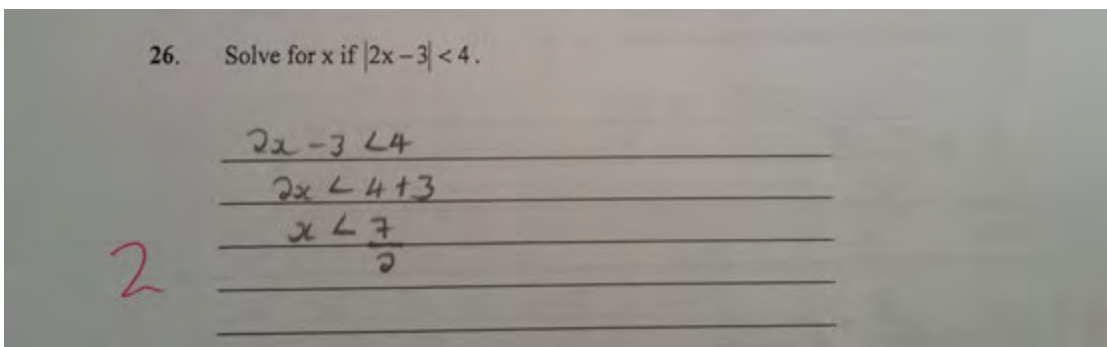
Script 93

Script Number	3v
Observation Number	93
School Code	6
Gender	Female
Combination Code	502
Original Score	2
New Score	0



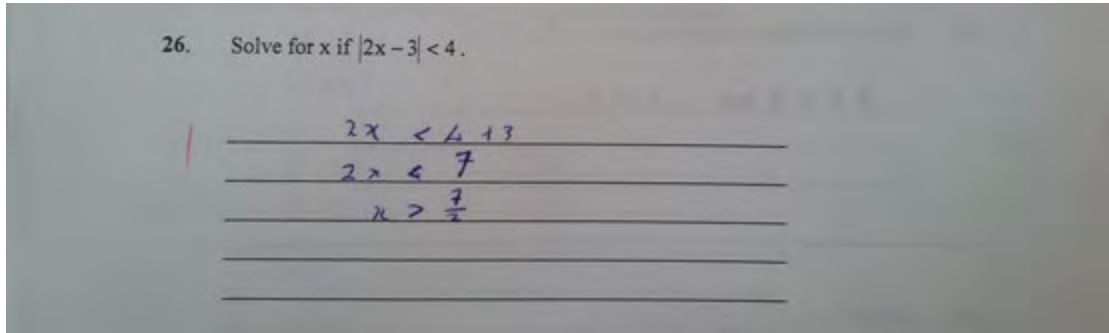
Script 94

Script Number	26ii
Observation Number	94
School Code	3
Gender	Female
Combination Code	501
Original Score	2
New Score	0



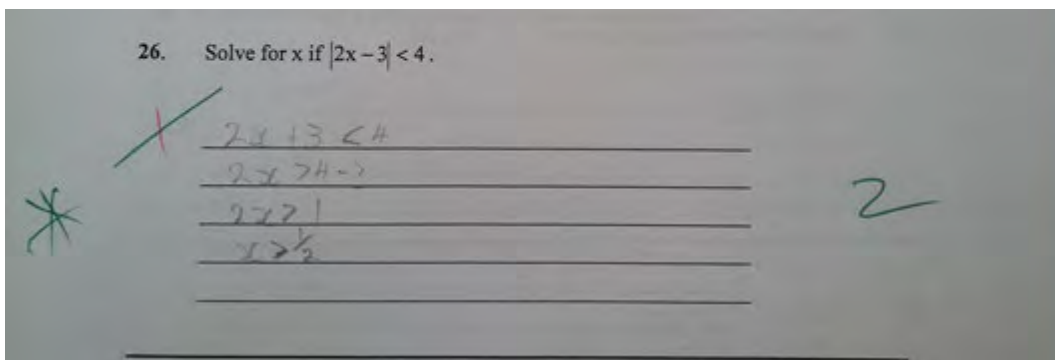
Script 95

Script Number	30i
Observation Number	95
School Code	2
Gender	Male
Combination Code	502
Original Score	1
New Score	0



Script 96

Script Number	4i
Observation Number	96
School Code	2
Gender	Female
Combination Code	502
Original Score	2
New Score	0



Script 97

Script Number	8i
Observation Number	97
School Code	6
Gender	Female
Combination Code	502
Original Score	2
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x + 3 < 4$$

$$2x < 4 - 3$$

$$2x < 1$$

$$x < \frac{1}{2}$$

Script 98

Script Number	20iv
Observation Number	98
School Code	3
Gender	Male
Combination Code	502
Original Score	1
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x - 3 < 4$$

$$2x \geq 4 + 3$$

$$\frac{2x}{2} \geq \frac{7}{2}$$

$$x \geq 3,5$$

Script 99

Script Number	13iv
Observation Number	99
School Code	1
Gender	Female
Combination Code	501
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$2x < 4$$

$$x < 2$$

Script 100

Script Number	26i
Observation Number	100
School Code	5
Gender	Male
Combination Code	311
Original Score	0
New Score	0

26. Solve for x if $|2x - 3| < 4$.

$$-|2x - 3| < 4$$

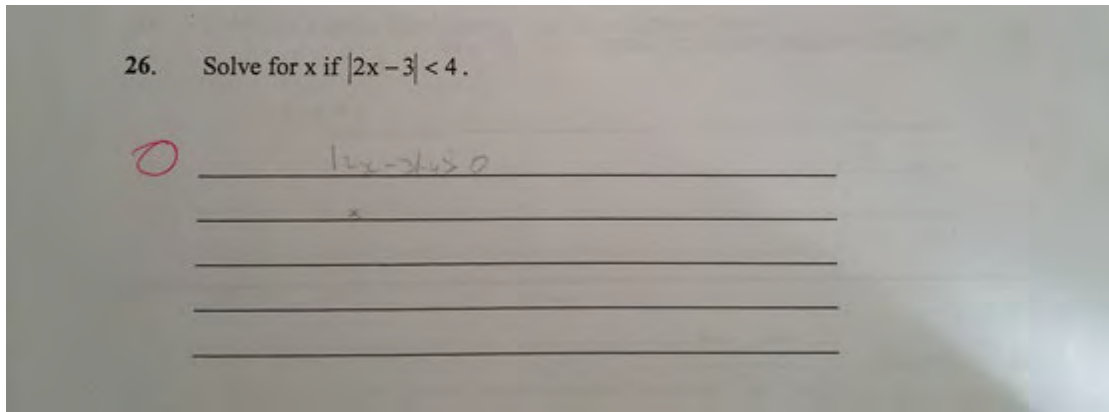
$$- 2x < 4 + 3$$

$$- 3 < 2x < 4$$

$$- 5 < 3 < 2$$

Script 101

Script Number	23iv
Observation Number	101
School Code	2
Gender	Male
Combination Code	502
Original Score	0
New Score	0



APPENDIX B

The 26 realised combination codes (out of 105 possible) as they occurred across the schools.

School	n	131	132	142	144	151	152	153
1	19	0	0	0	0	0	0	0
2	20	1	1	0	1	1	2	0
3	17	0	0	0	0	0	1	0
4	7	0	0	0	0	0	0	0
5	22	0	0	0	0	1	0	0
6	10	1	4	0	0	1	0	1
7	6	0	0	2	0	1	1	0
Total	101	2	5	2	1	4	4	1

154	232	241	251	253	254	311	312	321	411	412	414
0	1	0	1	0	0	1	0	0	1	0	0
1	0	0	1	0	1	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	0	0
0	0	1	2	1	0	0	0	0	1	0	0
0	0	0	0	0	0	6	1	1	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	2	0
1	1	1	4	1	1	7	1	1	5	2	1

422	423	451	501	502	503	504	505
0	0	0	5	2	0	6	2
0	0	0	3	3	0	1	2
1	1	0	8	4	1	0	0
0	0	1	0	0	0	1	0
0	0	0	7	2	3	1	0
0	0	0	0	2	0	0	0
0	0	0	0	0	0	0	0
1	1	1	23	13	4	9	4