# **Carroll Physics in General Relativity:**
# Understanding the Carroll Limit of the Singularity Theorem

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## Plagiarism Declaration

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**Abstract**

The Carroll limit of general relativity is an ultra-relativistic limit that is obtained by taking the speed of light ($c$) to zero. This is in contrast to the Galilean limit, where you can resolve it by taking the $c \to \infty$ limit. This limit was formulated in the study of the Carroll group as an ultra-relativistic limit of the Poincaré group, as such, one can exploit the various Carroll symmetries that arise. In this thesis, we study the Carroll limit of general relativity and its applications to Cosmology and the Singularity Theorem. We begin by reviewing the representation theory of the Carroll group and its applications to non-relativistic physics. We then study the Carroll scalar field and its properties in the Carroll limit, as well as how the Friedmann equations reduce in the small $c$ expansion in comparison to the evolution of the scalar field in our Standard Cosmology. We then study how fluids evolve in the Carroll regime, which gives us insight into the early universe and its dynamics. This study allows us to explore how non-relativistic matter, as well as the cosmological constant, would evolve in the Carroll limit. To understand radiation in the Carroll limit, we take a purely Classical route and study Maxwell's Theory of electromagnetism in a curved background. That then allows us to study the singularity theorem and how singularities are affected by taking this limit, which prompts us to look into a relatively recent theorem known as the BGV theorem, which looks at singularities from the perspective of geodesic incompleteness in contrast to the Hawking and Penrose ideas on the singularity theorem.

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# Contents


**1 Introduction** **5**

**2 Carroll Representation Theory** **10**

**3 The Carroll Scalar Field** **15**
3.1 Relativistic Scalar Field . . . . . 15
3.2 Inflation and Perfect Fluids . . . . . 18

**4 Friedmann Equations for the Carroll Scalar Field** **22**
4.1 The FLRW Universe . . . . . 22
4.2 Friedmann Equations through the Carroll Expansion . . . . . 28
4.3 Solving the Friedmann Equations . . . . . 31

**5 Fluids in Carroll Spacetime** **36**
5.1 Solving the Friedmann Equations in the Early Universe . . . . . 39
5.2 The Cosmological Constant . . . . . 42

**6 Carrollian Maxwell Theory** **44**
6.1 Maxwell Theory from the Action Principle . . . . . 44
6.2 Maxwell Theory in the FLRW Universe . . . . . 47

**7 Singularity Theorem** **50**
7.1 Carroll Geodesics . . . . . 51
7.2 The Singularity Theorem . . . . . 52
7.2.1 Trapped and Anti-Trapped Surfaces . . . . . 54
7.3 Borde-Guth-Vilenkin (BGV) Theorem . . . . . 56

**8 Conclusions** **58**

**A Appendices** **60**
A.1 Numerical Solutions to the Friedmann Equations . . . . . 60

# List of Figures

1.1 This figure illustrates how different kinematical groups relate to each other. The DeSitter group (D) is the invariance group of an empty expanding universe, where by taking the flat-space limit (i.e. the radius of the curvature $a(t) \to 0$) denoted $f$ gives you the Poincaré group $P$. You can resolve the Galilean group $G$ by taking the Newtonian/Galilean limit ($c \to \infty$) denoted $n$, where these three are currently the orthodox physical universes [1]. We see that we can resolve the Carroll group $C$ by taking the $c \to 0$ limit denoted $s$. The Newtonian universe with a curved space-time $N$, the Static group $S$, and the Para-Poincaré and Para-Galilei groups ($P'$ and $G'$ respectively) are explained in more detail in Bacry and Lévy-Leblond's article [2]. . . 7

4.1 (a) Evolution of the scalar field using the Carrollian expansion for the fixed Compton Wavelength showing the solution to the first order correction and (b) the Carroll limit taken on the approximate solution to the Scalar Field. . . . . . . . . . . . . . . . . . . 32

4.2 (a) Evolution of the scalar field using the Carrollian expansion for the fixed $E_0$ condition showing the numerical solution to the first order correction and (b) the Carroll limit taken on the approximate solution to the Scalar Field. . . . . . . . . . . 34

4.3 (a) Contrasting the first-order solutions for different conditions applied to the potential and (b) verification of the initial conditions applied to the numerical solutions showing a divergence in the scalar field solutions when they are not initiated correctly. 35

# Chapter 1

# Introduction

Carroll Mathematics was formulated in 1965 and 1966 by J.M. Lévy-Leblond and N.D. Sen Gupta, respectively, and independently [3][4]. Although credit for the formulation is now shared between both Lévy-Leblond and Sen Gupta, the former's paper forms the basis of what we know today as "Carrollian Physics". The Carroll group was seen more as a curious formulation, as mentioned by Lévy-Leblond [5] having studied in detail the Galilei group and its relation to the Poincaré/Lorentz groups. Seeing no direct importance to the Carroll group, he thus published the paper in a French journal far removed from all the relevant mathematics and theoretical physics of the time. However, curiosity in this formulation comes naturally when one considers what they know about the special theory of relativity and its limits. We know that given the Lorentz transformations,

$$t' = \gamma \left( t - \frac{vx}{c^2} \right) \tag{1.1}$$

$$x' = \gamma \left( x - vt \right), \tag{1.2}$$

with $\gamma = (1 - \beta^2)^{-1/2}$, $\beta = v/c$ as the boost parameter and $v$ as the relative velocity between two inertial reference frames. We can resolve the Galilean transformations that preceded them by simply requiring the velocity to be really small with respect to the speed of light $c$ (i.e., $v << 1$), with the implicit condition that the velocity be also small relative to large time-like intervals (i.e., $x << t$). A much more intuitive approach, however, would be instead to consider the limit as the speed of light $c$ goes to infinity, which satisfies both the aforementioned conditions and results in,

$$t' = t \tag{1.3}$$

$$x' = \left( x - vt \right). \tag{1.4}$$

Taking this approach of the speed of light going to infinity raises the natural question of what happens if we instead take the limit as the speed of light goes to zero. Applying this to the Lorentz transformations, we recognise issues that may arise when taking this limit in Eq. (1.1), so we employ a technique used by de Boer et al. [6] where we introduce the Carroll boost parameter $b$ as,

$$\beta \equiv cb. \tag{1.5}$$

This then allows us to reformulate the Lorentz transformations using the Carroll boost parameter, and if we then take the limit as $c \to 0$, we first get $\gamma \to 1$, and as a result, get our Carroll transformations,

$$t' = (t - bx) \tag{1.6}$$
$$x' = x. \tag{1.7}$$

We know that the Galilean limit renders time absolute while space remains relative. This is effectively the opposite of the Carroll limit, as we can see from the transformations. When boosting from one inertial reference frame to another, we find, in this instance, space to be absolute whilst time remains relative. It was from this implication that the Carroll limit could only be applied to large space-like intervals with two events possibly being totally causally disconnected that Lévy-Leblond initially concluded the Carroll limit as an unphysical limit of the existing theories at the time. In fact, the term "Carroll" that christened this theory was more so paying homage to the ideas presented by Lewis Carroll in his fictional universe of Alice in Wonderland[1]. It would be a few more decades before scientists began to see the usefulness of this limit and the subsequent theories that followed.

The Carroll limit's consequences for contemporary high-energy physics, cosmology, and quantum gravity are among the main reasons to take it into consideration. Much of our knowledge of relativistic field theories is based on the fundamental and finite speed of light assumption. But when we take the ultra-relativistic Carroll limit, we get a very different spacetime structure where space-like hypersurfaces are no longer coupled to causal dynamics, and temporal evolution is frozen. We refer to this limit as "ultra-relativistic" counter-intuitively because when the speed of light is really small, any motion will be motion at or near the speed of light. Having the Carroll limit question accepted ideas of locality and causality, it is fascinating because it offers a testing ground for open questions in which traditional ideas of spacetime have proven not entirely useful. The Carroll limit has also been used in several domains, such as holography in non-AdS spacetimes through Bondi-Metzner-Sachs (BMS) symmetries at null infinity and asymptotic symmetries governing black hole horizons [8–10]. Carrollian structures naturally emerge in the description of boundary dynamics in asymptotically flat spacetimes, as recent research has shown through explicit constructions of flat space holography where boundary correlators match Carrollian field theory predictions [11]. This link offers a solid theoretical framework for future research into the application of Carrollian physics outside of classical mechanics.

For just over three decades after the original papers on the Carroll group had been published, most citations received were mainly from papers dealing with abstract group theory, special relativity, and electromagnetism[5], mainly discussing the curious mathematical nature of the theory. In fact, Lévy-Leblond himself published a paper with H. Bacry [2] essentially fitting the Carroll group into an overall structure and classification of logically and physically possible kinematical groups as shown in Fig. 1.1. This was remarked by F. Dyson in an article chronicling a particular set of historically "missed opportunities" in physics and mathematics, based on a series of Gibbs lectures of the same title [1]. From a mathematical point of view, it seemed the classification of these kinematical groups in this manner would have led to the prediction of the expansion of the universe 20 years prior to Hubble's observations, as well as easing the discovery of general relativity by providing a postulate for spacetime curvature [1]. And ultimately, the Carroll group might have

[1] "'Well, in our country,' said Alice, still panting a little, 'you'd generally get to somewhere else if you run very fast for a long time, as we've been doing.' 'A slow sort of country!' said the Queen. 'Now, here, you see, it takes all the running you can do to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!'" - from *Through the Looking-Glass* by Lewis Carroll [7].

been formulated much earlier as well so as to mathematically complete the kinematical groups, with the timeline of the development of Carroll physics brought forth by a couple of decades.

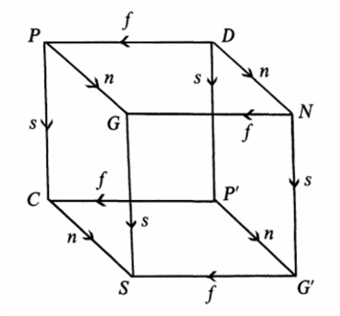



Figure 1.1: This figure illustrates how different kinematical groups relate to each other. The DeSitter group (D) is the invariance group of an empty expanding universe, where by taking the flat-space limit (i.e. the radius of the curvature $a(t) \to 0$) denoted $f$ gives you the Poincaré group $P$. You can resolve the Galilean group $G$ by taking the Newtonian/Galilean limit ($c \to \infty$) denoted $n$, where these three are currently the orthodox physical universes [1]. We see that we can resolve the Carroll group $C$ by taking the $c \to 0$ limit denoted $s$. The Newtonian universe with a curved spacetime $N$, the Static group $S$, and the Para-Poincaré and Para-Galilei groups ($P'$ and $G'$ respectively) are explained in more detail in Bacry and Lévy-Leblond's article [2].

A number of significant advancements in theoretical physics are responsible for the renewed interest in Carrollian physics. The study of non-Lorentzian geometries, especially concerning non-relativistic gravity and string theory, led to one of the first re-examinations of the Carroll group. For example, Bagchi et al. [12] drew surprising linkages between ultra-relativistic and non-relativistic physics by highlighting the appearance of Carrollian symmetries in two-dimensional conformal field theories.

Furthermore, the study of inflationary theories and cosmology has made the Carroll limit especially pertinent. According to recent research, some hypothetical scenarios can be recast in terms of Carrollian field theories. This is coming from a few examinations that Carrollian physics has the potential to develop alternative explanations of the dynamics of the early universe, namely inflation, as tackled by de Boer et al. [6]. This has strengthened Carrollian physics' place in contemporary theoretical physics by prompting studies into whether it might shed light on the nature of cosmic acceleration and dark energy.

The 2000s saw a larger increase in citations for both Lévy-Leblond and Sen Gupta's papers, with much of the works now focusing on General Relativity, Field Theory, and Gravitation, most notably in Flat holography, inflation, and as a general ultra-relativistic limit. It is with this birth of Carrollian physics that we find relevant avenues to explore. Specifically, we initially follow the arguments presented by J. de Boer et al. [6] in Chapters 3,4 where the energy-momentum tensors of a perfect fluid imply the equation of state $P + \rho = 0$ in the Carroll limit. This is relevant for inflation and dark energy, and as such, we study the scalar field and how it evolves in the Carroll regime with the background given by the Friedmann-Lamaître-Robertson-Walker (FLRW) metric in contrast to the flat Minkowski metric. Another possible use of the Carroll limit in cosmology is gravitational waves [13]. According to standard general relativity, gravitational waves travel across enormous cosmic distances at the speed of light, carrying information about their origins.

However, in the Carroll limit, where $c \to 0$, the dynamics of gravitational waves change drastically. The behaviour of primordial gravitational waves, which are remnants of early universe physics, is intriguingly called into doubt by this. A background of stochastic gravitational waves originating from quantum fluctuations in the early universe is predicted by standard inflationary models [14]. These waves leave observable traces in the polarisation of the cosmic microwave background (CMB) as they interact with the expanding spacetime. The suppression of time-like evolution may have altered the generation and propagation of these primordial gravitational waves if the early universe had a Carrollian phase.

Given the implications we concluded upon the Carroll limit on the free scalar field, we then study how other fluids evolve in the Carroll regime in Chapter 5. We study the cosmological constant in the Carroll limit, and then, using a semi-classical approach, we approach non-relativistic matter in thermal equilibrium and out of thermal equilibrium. The role of the cosmological constant, $\Lambda$, which controls the universe's accelerating expansion, is a key component of contemporary cosmology. The cosmological constant is frequently understood in the context of standard relativity as vacuum energy acting as a repulsive force on cosmic scales. However, changes to the spacetime symmetries that form the basis of general relativity offer an alternative perspective on dark energy. The equation of state in a Carrollian framework naturally approaches $P = -\rho$, simulating the impact of dark energy in conventional cosmology. This implies that cosmic acceleration could be geometrically explained in a universe with Carrollian dynamics without requiring the use of a distinct vacuum energy component. It may be suggested that dark energy is an emergent aspect of the underlying spacetime structure rather than a distinct physical entity if the universe's expansion is controlled by Carrollian symmetries on a massive scale.

In addition to providing testable predictions that differ from those of traditional $\Lambda$-Cold Dark Matter ($\Lambda$CDM) cosmology, this viewpoint could open up new possibilities for comprehending late-time cosmic acceleration. Anisotropies in the CMB or minor changes in the evolution of large-scale structure are two examples of how general relativity might be altered. Future observations, like those from the Vera C. Rubin Observatory and the Euclid mission, may shed important light on whether Carrollian processes contribute to cosmic expansion, Primordial Gravitational Waves, and Carrollian Limits. This will elucidate how we should treat the Carroll limit in the early universe, as dealing with radiation in comparison becomes a nontrivial feat. The general idea is that we want to study the singularity theorem in the Carroll limit and, because of the radiation domination in the early universe, how it evolves can provide meaningful insight into the idea.

That being said, to further understand radiation in the Carroll limit, we will approach it from a purely classical perspective, studying the covariant formulation of Maxwell's theory in this regime in Chapter 6. With the curved FLRW background, we will be able to analyse the evolution of radiation as well as gain an intuition of whether or not we can resolve a singularity when traversing backward in time. We will then study the singularity theorem, first looking at geodesic congruences in the Carroll limit leading up to the Raychaudhuri equation. Furthermore, understanding the particular conditions needed to resolve singularity and whether or not this limit violates said conditions will prove necessary beyond needing to satisfy the Raychaudhuri equation.

In addition to its use in holographic and high-energy settings, the Carroll limit provides a distinctive viewpoint on how singularities behave in general relativity. Penrose and Hawking's and other classical singularity theorems are based on certain energy conditions and geodesic-focusing arguments that are intrinsically connected to relativistic causal structures. By taking the Carrollian limit, where time evolution is degenerate, and geodesics stay frozen, one can enquire whether the conventional singularity theorems still apply or if singularities are "softened" into less severe forms of incompleteness.

Relativistic field theories, which presume a homogeneous and isotropic distribution of matter and energy, are frequently used to simulate the early universe. However, in extreme circumstances close to the Big Bang, where energy densities approach the Planck scale ($\rho \sim c^5/\hbar G^2$), and quantum gravity effects become important, the relativistic causality assumption breaks down. Proposed quantum gravity frameworks, such as loop quantum cosmology and string gas cosmology [15], suggest that spacetime itself acquires a discrete or emergent structure at these scales, invalidating the smooth manifold description of general relativity. In such regimes, the Carroll limit offers a geometric alternative: by suppressing time evolution and freezing spatial hypersurfaces, it naturally regularizes divergences associated with singularities. For instance, Donnay et al. [9] demonstrate that Carrollian symmetries govern horizon dynamics in black hole spacetimes. Whether the Carroll limit can be used as a substitute method to comprehend such horizons is an intriguing subject. One might wonder if the initial singularity could be understood as a Carrollian phase of the universe since the Carrollian limit effectively eliminates ordinary causal propagation by making time evolution degenerate.

Such a method might offer a novel viewpoint on inflationary theory's beginning circumstances and horizon issues. Large-scale correlations might arise without superluminal inflationary expansion if the universe experienced a Carrollian period in its early history, which would freeze time-like evolution. There are similarities between this concept and other theories that alter causality in the early universe, like Hořava-Lifshitz gravity and causal set theory.

# Chapter 2

# Carroll Representation Theory

The structure of symmetries in physics is one area of mathematical physics that is deeply studied and provides an elegant way to represent particular physical notions. The idea of relativity, being this "theory of covariance", is one that needs to admit a group of transformations corresponding to a particular change in reference frames. This is because physical quantities and the laws that govern them need to maintain the same form for different groups of observers. Such a group theoretical perspective for Einsteinian relativity was insisted upon and popularized by Paul Langevin as early as 1911 [5].

When dealing with Special Relativity, the Poincaré group is our group of note. This is the group that leaves the fundamental structure of Minkowski spacetime invariant, consisting of the group of spacetime translations denoted $\mathbb{R}^{1,3}$ and the group of Lorentz Transformations denoted $SO^{+}(1,3)$[1], by taking into consideration only one temporal and three spatial dimensions. The spacetime in question is endowed with the Minkowski metric,

$$ds^2 = -c^2 dt^2 + d\boldsymbol{x}^2, \tag{2.1}$$

whereby the Lorentz group elements $\Lambda$ can be represented as $4 \times 4$ matrices and are required to satisfy the the following condition [16],

$$\Lambda^T \eta \Lambda = \eta, \tag{2.2}$$

with $\eta = \text{diag}(-1,1,1,1)$ as the matrix representation of the Minkowski metric. For a general $n \times n$ real matrix, we have $n \times n$ real elements as well as $\frac{n(n+1)}{2}$ constraint equations. Subtracting the number of constraint equations from the number of real elements leaves us with the number of independent elements, which can be associated with physical symmetries. In our case, the Lorentz matrix gives us six independent elements, with the associated symmetries being three spatial rotation parameters (in one of the three planes formed by our spatial coordinates) and three boost parameters (which we can think of as hyperbolic rotations in a plane formed by the time component and one of the spatial components). With these available symmetries, as well as the Minkowski metric, then we know what the spatial rotations will look like. The matrix representation

[1] We refer to this group technically as the "restricted Lorentz group", as it is the component of the Lorentz group continuously connected to the identity and preserves the orientation of space and direction of time [16].

of the spatial rotations in the $yz$-plane, $xz$-plane, and $xy$-plane respectively are,

$$\Lambda_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}, \Lambda_{xz}(\phi) = \begin{pmatrix} \cos\phi & 0 & 0 & \sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\phi & 0 & 0 & \cos\phi \end{pmatrix}, \Lambda_{xy}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and for the boosts in our three spatial directions, we have their matrix representations as,

$$\Lambda_{xt}(\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Lambda_{yt}(\beta) = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Lambda_{zt}(\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

with $\gamma$ and the Lorentz boost parameter $\beta$ as defined in Eq. 1.1. The condition Eq. 2.2 can then be quickly verified; we call the Lorentz matrices isometries of the Minkowski metric[2].

When bringing spacetime translations back in the mix, we get the Poincaré group denoted $ISO(1,3)$ with group elements $P = (\Lambda, a)$ that can act on points in Minkowski space as [16],

$$x \to x' = \Lambda x + a, \tag{2.3}$$

with $a$ being the shift parameter that can take any real value. This then means the Poincaré group will have 10 free parameters in 3+1 dimensions, that is, the six free parameters of the Lorentz group combined with the free parameters of the four possible spacetime translations.

Both the Lorentz and Poincaré groups are Lie groups. These are groups whereby their elements can be written in the form,

$$A = e^{ig_A v^A} \tag{2.4}$$

with $g_A$ said to be the generator of a group element A with a parameter $v^A$ [17]. Using a Taylor expansion, we can write the group element as,

$$A = \mathbb{1} + ig_A v^A + \frac{1}{2!}(ig_A v^A)^2 + \frac{1}{3!}(ig_A v^A)^3 + \cdots \tag{2.5}$$

We can use the expansion to read off the generators for each Lorentz group element, and we can also differentiate the group element with respect to the parameter to get an explicit way to compute the generators.

$$\frac{\partial A}{\partial v^A} = 0 + ig_A + \frac{1}{2!}2(ig_A v^A) + \frac{1}{3!}3(ig_A v^A)^2 + \cdots \tag{2.6}$$

$$\therefore g_A = i\frac{\partial A}{\partial v^A}\Big|_{v^A=0}. \tag{2.7}$$

The generators of the Lorentz group can then be computed directly from Eq. 2.7 and are thus given by the following matrices,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[2] This means they do not change the form of the metric, which will work to strengthen the claim that the Poincaré group is an isometry of Minkowski spacetime.

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

where the $J_i$ are the generators of the spatial rotations and the $K_i$ are the generators of the boosts. The generators of a Lie group form a special relation known as the Lie algebra of the group. An algebra $\mathfrak{g}$ with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a Lie algebra if it satisfies the following three criteria:

- Antisymmetry: $[x, y] = -[y, x]$,
- Bilinearity: $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z], \quad [z, \alpha x + \beta y] = \alpha[z, x] + \beta[z, y]$,
- The Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,

for $x, y, z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{R}$ [18]. Therefore, we can verify by matrix multiplication that the generators of the boosts $K_i$ and the generators of the rotations $J_i$ form the commutation relations,

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{2.8}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k, \tag{2.9}$$

with the mixed commutators as,

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \tag{2.10}$$

which then makes up the Lie algebra of the Lorentz group. The Poincaré group then has the same Lie algebra structure as the Lorentz group, with the spacetime translations taking the form $P_\mu = i\partial_\mu$ and $[P_\alpha]^\mu_\nu = i\delta^\mu_\nu\partial_\alpha$ when acting on the scalar field and vector field respectively [16]. We can verify for both these forms of the translation operator that they satisfy the algebra,

$$[P^\mu, P^\nu] = 0. \tag{2.11}$$

The covariant representation of Lorentz Lie Algebra can be found by introducing an antisymmetric tensor $M_{\mu\nu}$ and acting infinitesimally on a spacetime coordinate $x^\rho$. This will show that it relates to the boost and rotation generators respectively, as follows:

$$M_{0i} = M_{i0} = K_i \tag{2.12}$$
$$M_{ij} = \varepsilon_{ijk}J_k. \tag{2.13}$$

The antisymmetry ensures that we maintain the six independent elements associated with the symmetries of the Lorentz group. Therefore, the Lorentz Lie Algebra can be more compactly written as,

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(-\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}). \tag{2.14}$$

The mixed commutators with the translation operator will then be given by,

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\rho\nu}P_\mu - \eta_{\rho\mu}P_\nu). \tag{2.15}$$

This now gives us the full Lie algebra structure of the Poincaré group and the natural question that can arise from this is how the Poincaré group relates to the Galilean group as it was the predecessor. As a result, we uncover the well-studied limit of the Poincaré group where we resolve the Lie algebra of the Galilean group when taking the limit as speed of light goes to infinity ($c \to \infty$) of the Poincaré algebra. The clearest example we can give of the Galilean limit applied to the Poincaré group is in the form of the commutation relation between the boost generators and the spatial component of the translation operators,

$$\begin{aligned}[K_i, P_j] &= [M_{0i}, P_j] \\ &= i(\eta_{ji}P_0 - \eta_{j0}P_i) \\ &= i\eta_{ji}P_0 = i\eta_{ji}(i\partial_0) = -\frac{1}{c}\frac{\partial}{\partial t}\delta_{ji}. \end{aligned} \tag{2.16}$$

Where in the above, we used the form of the spacetime translations acting on a scalar field, we arrive at a similar conclusion if we act on a vector field as well. From this commutation relation, we can see that if we take the limit as $c \to \infty$, the bracket Eq. 2.16 will vanish, which matches what we know about this relation in the Galilean group. We can reverse engineer the entire Galilean algebra structure in this manner and subsequently reproduce the Galilean transformations as in Eq. 1.3.

Using this approach, we now see how one can take the Carroll limit (the limit as $c \to 0$) on the Poincaré algebra to generate the entire Carroll Lie algebra structure, as Lévy-Leblond and Sen Gupta did in their initial papers. For example, we can look at the commutation relation between the temporal component of the spacetime translations and the boost generators,

$$\begin{aligned}[K_i, P_0] &= [M_{0i}, P_0] \\ &= i(\eta_{0i}P_0 - \eta_{00}P_i) \\ &= i\eta_{00}P_i = i(-c^2)(i\partial_i) = c^2\partial_i. \end{aligned} \tag{2.17}$$

It is then clear that in the Carroll limit, the relation will vanish, and, similarly to the Galilean group, we can find the entire Carroll algebra structure in this manner. From this, we can state the full algebras of our kinematical groups, and they are then given by [3]:

| **Carroll** | **Poincaré** | **Galilée** |
|---|---|---|
| $[J_i, J_j] = i\varepsilon_{ijk}J_k$ | $[J_i, J_j] = i\varepsilon_{ijk}J_k$ | $[J_i, J_j] = i\varepsilon_{ijk}J_k$ |
| $[J_i, K_j] = i\varepsilon_{ijk}K_k$ | $[J_i, K_j] = i\varepsilon_{ijk}K_k$ | $[J_i, K_j] = i\varepsilon_{ijk}K_k$ |
| $[K_i, K_j] = 0$ | $[K_i, K_j] = -i\varepsilon_{ijk}J_k$ | $[K_i, K_j] = 0$ |
| $[J_i, P_j] = i\varepsilon_{ijk}P_k$ | $[J_i, P_j] = i\varepsilon_{ijk}P_k$ | $[J_i, P_j] = i\varepsilon_{ijk}P_k$ |
| $[K_i, P_j] = i\delta_{ij}P_0$ | $[K_i, P_j] = i\delta_{ij}P_0$ | $[K_i, P_j] = 0$ |
| $[J_i, P_0] = 0$ | $[J_i, P_0] = 0$ | $[J_i, P_0] = 0$ |
| $[K_i, P_0] = 0$ | $[K_i, P_0] = iP_i$ | $[K_i, P_0] = iP_i$ |
| $[P_i, P_j] = 0$ | $[P_i, P_j] = 0$ | $[P_i, P_j] = 0$ |
| $[P_i, P_0] = 0$ | $[P_i, P_0] = 0$ | $[P_i, P_0] = 0$ |

From this, we can compute all the transformations mentioned in the previous chapter. We can see from the various Lie algebra structures that the boost terms and the spacetime translation terms are of the most significance when considering these limits. To understand this better, we can further analyse the Lorentz transformations and their properties in the Carroll limit.

In the Carroll limit, we have the Minkowski metric reduce to $ds^2 = -c^2dt^2 + d\boldsymbol{x}^2 \to d\boldsymbol{x}^2$, from which we can tell that the light rays slow down or cease to move in space as the light cone closes, resulting in a larger space-like interval. If we consider the Lorentz transformation on the energy and momentum of a particle[6],

$$p' = \gamma\left(p - \beta\frac{E}{c}\right) \tag{2.18}$$

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta p\right), \tag{2.19}$$

taking the Carroll limit yields,

$$p' = p - bE \tag{2.20}$$

$$E' = E. \tag{2.21}$$

As we computed for the Carroll algebra, we can see that $[K_i, P_0] = [K_i, H] = 0$, i.e., the Hamiltonian will be Carroll boost invariant and as such matches with the computed transformation of the energy of a particle in the Carroll limit. And from the Carroll transformations 1.6,1.7 themselves, we can see how the velocities of these particles would transform,

$$\begin{aligned} v'^i &= \frac{dx'^i}{dt'} \\ &= \frac{dx^i}{dt - bdx^i} = \frac{dx^i}{dt}\frac{1}{1 - b\frac{dx^i}{dt}} \\ &= \frac{v^i}{1 - bv}. \end{aligned} \tag{2.22}$$

The consequence of this transformation is the existence of two types of Carroll particles: The first type is a particle of zero velocity, where if it initially has zero velocity, then any finite Carroll boost cannot give it a non-zero velocity; these are potentially massive particles. The second type of particles are the so-called "tachyonic" particles, which have an initial velocity, and as such, the Carroll boosts can give the particle any velocity, which should make them massless.

This coincides again with the conclusion of the energy of such particles. If a particle has non-zero energy ("massive"), we can perform a Carroll boost to go to a frame where the particle has no momentum ("zero velocity"). And if the particle has zero energy ("massless"), then the momentum $P_i$ is Carroll boost invariant and can obtain any velocity. This is an intriguing consequence of the Carroll group and part of the reason it was not viewed as a physical idea but rather a mathematical artefact in its early years [5]. Although our goal is not to discuss the Carroll particles as that has been done in detail by Jan de Boer et al. [6] and Kamenshchick et al. [19], we gain a better intuition from those conclusions. Because we will be studying how fluids evolve in the Carrollian regime, the idea of matter and radiation potentially freezing out in the Carroll limit gives us an idea of how to approach that particular problem in the following chapters. For now, since we wish to study cosmology in the Carroll limit, we will need a field theory perspective in this regime, and we will begin that in the following chapter.

# Chapter 3

# The Carroll Scalar Field

In our study of Carrollian cosmology, we start by exploring the evolution of a scalar field in the Carroll limit. In standard relativistic field theory, the scalar field is a fundamental object that can be used to describe a variety of physical phenomena. In modern cosmology, scalar fields are essential, especially when considering the early universe. Scalar fields play a crucial role in the standard model of cosmology by driving inflation, a phase of rapid expansion that resolves important issues like the horizon problem, flatness problem, and monopole problem. A single scalar field, $\phi$, also referred to as the inflaton evolves in a potential $V(\phi)$. In an expanding universe, the Klein-Gordon equation and Friedmann equations govern the dynamics of the inflaton field. The universe expands rapidly during inflation because the inflaton's potential energy outweighs its kinetic energy, resulting in a roughly constant energy density, mimicking the cosmological constant $\Lambda$ [20]. Conversely, the Carrollian limit alters the character of temporal evolution, which has significant ramifications for the dynamics of inflation. The Carroll scalar field has a modified behaviour where temporal evolution is effectively inhibited rather than the usual slow-roll evolution. This shift implies that inflation might manifest itself very differently in a Carrollian setting, perhaps depending more on spatial dynamics than on time-dependent development.

## 3.1 Relativistic Scalar Field

The relativistic scalar field Lagrangian density is given by [6],

$$\mathcal{L} = \frac{1}{2c^2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_i \phi)^2 - V(\phi), \tag{3.1}$$

where the general potential $V(\phi)$ represents interactions in our field theory. We wish to study the free scalar field in the Carroll limit and, more specifically, understand the evolution of the scalar field by looking at it first in the flat Minkowski background with the metric Eq. 2.1. We choose to use a different metric signature (i.e., the $(+---)$ convention) for the free scalar field for convenience, and most notably, this convention makes the mass-shell constraint take a clearer form ($p^2 = m^2c^4$) instead of a negative energy convention. When dealing with General Relativity, it then becomes convenient to use the other convention $(-+++)$, due to the more intuitive geometric interpretations with a positive spatial metric. We can extract the equations of motion from the

Lagrangian using the Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right), \tag{3.2}$$

giving rise to the usual relativistic Klein-Gordon equations,

$$\frac{1}{c^2}\ddot{\phi} - \nabla^2 \phi + V(\phi) = 0. \tag{3.3}$$

We wish to study the behaviour of the scalar field in the Carroll limit. To do so, we will take the leading and sub-leading terms of Eq. 3.1 in the limit as the speed of light goes to zero ($c \to 0$) or rather an expansion in the small $c$ regime, resulting in Carroll invariant Lagrangians, i.e. Lagrangians that are invariant under the Carroll boost transformations. For our scalar field, we take the expansion,

$$\phi = \phi_0 + c^2 \phi_1 + \mathcal{O}(c^4). \tag{3.4}$$

When we plug this expansion into the Lagrangian, it results in,

$$\mathcal{L} = c^{-2} \left[ \frac{1}{2}\dot{\phi}_0^2 + c^2 \left( \dot{\phi}_0 \dot{\phi}_1 - \frac{1}{2}\partial_i \phi_0 \partial_i \phi_0 \right) + \mathcal{O}(c^4) \right], \tag{3.5}$$

for a vanishing potential, we take $V(x) = 0$ to start in order to get an intuition of the Carroll limit for the free massless scalar field. By defining the expansion of the Lagrangian to be [6]

$$\mathcal{L} = c^{2\Delta - 2}(\mathcal{L}_0 + c^2 \mathcal{L}_1 + \mathcal{O}(c^4)), \tag{3.6}$$

for some $\Delta$, we can conclude from our computation that

$$\mathcal{L}_0 = \frac{1}{2}\dot{\phi}_0^2, \qquad \mathcal{L}_1 = \dot{\phi}_0 \dot{\phi}_1 - \frac{1}{2}\partial_i \phi_0 \partial_i \phi_0, \tag{3.7}$$

with $\Delta = 0$. We can show that these are both Carroll invariant Lagrangians by verifying that they transform into total derivatives under a Carroll boost. Recalling that a Carroll boost gives Eq. 1.6 and 1.7, a scalar field transforms as $\delta\phi = \phi(x', t') - \phi(x, t) \approx \vec{b} \cdot \vec{x}\dot{\phi}(x, t)$ thus, when we vary the zeroth order terms, $\delta\phi_0 = (\vec{b} \cdot \vec{x})\dot{\phi}_0$, this results in

$$\begin{aligned} \delta\mathcal{L}_0 &= \dot{\phi}_0 \delta\dot{\phi}_0 \\ &= \dot{\phi}_0 (\vec{b} \cdot \vec{x}) \ddot{\phi}_0 = C \dot{\phi}_0 \ddot{\phi}_0 \\ &= C \frac{\partial}{\partial t} \left( \frac{1}{2}\dot{\phi}_0^2 \right), \end{aligned} \tag{3.8}$$

which, as a total derivative, is invariant under Carroll boosts. Similarly for $\mathcal{L}_1$ we can take,

$$\begin{aligned} \delta\mathcal{L}_1 &= \delta\dot{\phi}_0 \cdot \dot{\phi}_1 + \dot{\phi}_0 \delta\dot{\phi}_1 - \vec{\partial}\phi_0 \vec{\partial}(\delta\phi_0) \\ &= (\vec{b} \cdot \vec{x})\ddot{\phi}_0 \cdot \dot{\phi}_1 + \dot{\phi}_0 \left[ (\vec{b} \cdot \vec{x})\ddot{\phi}_1 + \vec{b} \cdot \vec{\partial}\phi_0 + t\vec{b} \cdot \vec{\partial}\dot{\phi}_0 \right] - \vec{\partial}\phi_0 \left( \vec{b}\dot{\phi}_0 + (\vec{b} \cdot \vec{x})\vec{\partial}\dot{\phi}_0 \right) \\ &= C \left[ \frac{\partial}{\partial t}(\dot{\phi}_0 \dot{\phi}_1) \right], \end{aligned} \tag{3.9}$$

for some constant $C$. We can now add back interactions by utilizing a quadratic potential. With these interactions, we gain insight from the study of Carroll particles, as mentioned previously, wherein the Carroll limit, a particle with zero velocity will still have rest energy, while you can also have particles with zero energy. As such, in the Carroll limit, we can choose to keep $E_0 \equiv mc^2$ fixed or keep the Compton wavelength $\lambda^{-1} = \mu \equiv mc/\hbar$ fixed [6] to mirror those conditions. As we did with the Lagrangian, we start by substituting the expansion of the scalar field into the potential,

$$\begin{aligned}
V(\phi) &= \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2 \\
&\frac{1}{2}\left(\frac{m^2c^4}{\hbar^2}\right)c^{-2}\left(\phi_0 + c^2\phi_1\right)^2 \\
&= \frac{1}{2}\omega_0^2 c^{-2}\left(\phi_0^2 + 2c^2\phi_0\phi_1 + c^4\phi_1^2\right),
\end{aligned} \tag{3.10}$$

for $\omega_0 = E/\hbar$. We can then introduce the potential back into the expansion of the Lagrangian density, and the resultant zeroth order and first-order terms will be,

$$\mathcal{L}_0 = \frac{1}{2}\dot{\phi}_0^2 - \frac{1}{2}\omega_0^2\phi_0^2 \tag{3.11}$$

$$\mathcal{L}_1 = \dot{\phi}_0\dot{\phi}_1 - \frac{1}{2}\partial_i\phi_0\partial_i\phi_0 - \omega_0^2\phi_0\phi_1. \tag{3.12}$$

When we use the Euler Lagrange equations to compute the equations of motion associated with $\mathcal{L}_0$ and $\mathcal{L}_1$, we arrive at the following equations,

$$\ddot{\phi}_0 + \omega_0^2\phi_0 = 0 \tag{3.13}$$

$$\ddot{\phi}_1 + \omega_0^2\phi_1 = \partial^2\phi_0 \tag{3.14}$$

For these equations of motion, we find they can be represented as plane wave solutions with the zeroth order solution in the form $\phi_0 = e^{i\vec{k}\cdot\vec{x}+i\omega_0 t}$, and the first order solution given by $\phi_1 = \frac{i}{2\omega_0}\vec{k}^2 t e^{i\vec{k}\cdot\vec{x}+i\omega_0 t}$ verifying results by [6]. We can see from the solutions that $\phi_1$ has the same frequency term as $\phi_0$ but with a shifted amplitude resulting from the source term in the differential equation. We can speak more about the consequences of the first-order solutions once we consider them in a curved background in the next chapter. From the solutions, though, if we keep expanding further according to Eq. 3.4 in the $\phi_n$, we will fully reconstruct the relativistic solution of the scalar field.

From our addition of the interactions, if we otherwise keep the Compton Wavelength fixed, $\mu = mc/\hbar$, we can perform similar computations, and as such, our potential can be written as,

$$\begin{aligned}
V(\phi) &= \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2 \\
&= \frac{1}{2}\mu^2\phi^2 \\
&\frac{1}{2}\mu^2\left(\phi_0^2 + 2c^2\phi_0\phi_1 + c^4\phi_1^2\right) \\
&= \frac{1}{2}\mu^2 c^{-2}\left(c^2\phi_0^2 + 2c^4\phi_0\phi_1 + c^6\phi_1^2\right),
\end{aligned} \tag{3.15}$$

this results in our Lagrangian looking like this,

$$\mathcal{L} = c^{-2}\left[\frac{1}{2}\dot{\phi_0}^{\,2} + c^2\left(\dot{\phi_0}\dot{\phi_1} - \frac{1}{2}\partial_i\phi_0\partial_i\phi_0 - \frac{1}{2}\mu^2\phi_0^2\right) + \mathcal{O}(c^4)\right]. \tag{3.16}$$

So therefore, in the expansion of $\phi$ around small $c$, the zeroth order and first order Lagrangians will be given by,

$$\mathcal{L}_0 = \frac{1}{2}\dot{\phi}_0^2 \tag{3.17}$$

$$\mathcal{L}_1 = \dot{\phi}_0\dot{\phi}_1 - \frac{1}{2}\partial_i\phi_0\partial_i\phi_0 - \frac{1}{2}\mu^2\phi_0^2. \tag{3.18}$$

We can similarly retrieve the equations of motion from the Euler-Lagrange equations, which will result in the following differential equations,

$$\ddot{\phi}_0 = 0 \tag{3.19}$$

$$\ddot{\phi}_1 = \partial^2\phi_0 - \mu^2\phi_0. \tag{3.20}$$

For the zeroth order solution, we can write it in general as $\phi_0 = f(\vec{x} + g(\vec{x})t)$. To get the behaviour of our first order solutions, we can take the zeroth order solutions to be plane spatial waves ($\phi_0 \sim e^{i\vec{k}\cdot\vec{x}} + \dots$) and setting $\ddot{\phi}_1 = 0$ we get,

$$\begin{aligned}\partial^2\phi_0 &= \mu^2\phi_0 \\ -\left|\vec{k}\right|^2 e^{i\vec{k}\cdot\vec{x}} &= \mu^2 e^{i\vec{k}\cdot\vec{x}} \\ \therefore k^2 &= -\mu^2.\end{aligned} \tag{3.21}$$

From this, we know that having $\ddot{\phi}_1$ to be non-zero will introduce back the interaction between $\phi_0$ and $\phi_1$ as described in [6] and therefore the solution for the first order scalar field will behave as $k^2 + \mu^2 > 0$. This non-trivial solution allows us to use the relativistic dispersion relation to verify that the condition in which the Compton wavelength is fixed refers to a relativistic particle with vanishing energy. On the other hand, we have shown that if we keep $E_0$ fixed, that corresponds to a relativistic particle with non-zero energy, both of these as a manifestation of the Carroll regime.

## 3.2 Inflation and Perfect Fluids

In the study of the relativistic scalar field, we want to use this opportunity to also discuss the cosmological implications of taking the Carroll limit, and that discussion starts with understanding the energy-momentum tensor. The energy-momentum tensor for a perfect fluid is given by,

$$T^{\mu\nu} = (\rho c^2 + P)u^\mu u^\nu - Pg^{\mu\nu}, \tag{3.22}$$

where $\rho$ is the energy density, $P$ is the pressure, and $u^\mu$ is the four velocity such that $u_\mu u^\mu = -c^2$. We can write the energy-momentum tensor in terms of the scalar field as [21],

$$T^{\mu\nu} = 2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} - g^{\mu\nu}\mathcal{L}. \tag{3.23}$$

The general form of the Lagrangian density for the scalar field is given by,

$$\mathcal{L} = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi), \tag{3.24}$$

where we still maintain using the Minkowski metric Eq. 2.1. The energy-momentum tensor can then be computed as,

$$\begin{aligned} T^{\mu\nu} &= 2\left[\frac{\delta}{\delta g_{\mu\nu}}\left(\frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right) - g^{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right)\right] \\ &= \left(-g^{\alpha\mu}g^{\nu\beta}\right)\left(\partial_\alpha\phi\partial_\beta\phi\right) - g^{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right) \\ &= \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right), \end{aligned} \tag{3.25}$$

(3.26)

where we varied the inverse metric using the Kronecker delta
$0 = \delta(\delta^\rho_\lambda) = \delta(g^{\rho\sigma}g_{\sigma\lambda}) = (\delta g^{\rho\sigma})g_{\sigma\lambda} + g^{\rho\beta}\delta g_{\beta\lambda} \quad \Leftrightarrow \quad \delta g^{\rho\sigma} = -g^{\rho\beta}g^{\sigma\lambda}\delta g^{\beta\lambda}$.

Looking at the homogeneous part of the scalar field, we take into account only its time derivatives in this scenario. Therefore, the time-time component of the energy-momentum tensor after contracting it with the metric ($T^0_0$) will be,

$$T^0_0 = -\frac{1}{2}\frac{1}{c^2}\dot{\phi}^2 - V(\phi). \tag{3.27}$$

And for a perfect fluid, the ($T^0_0$) component will be $-\rho$, so therefore, the energy density of the scalar field as given by the energy-momentum tensor will be

$$\rho = \frac{1}{2c^2}\dot{\phi}^2 + V(\phi). \tag{3.28}$$

Similarly, we get the pressure to be given by the ($T^i_j$) component of the energy-momentum tensor as $P\delta^i_j = T^i_j$ so we can therefore find the pressure to be,

$$P = \frac{1}{2c^2}\dot{\phi}^2 - V(\phi). \tag{3.29}$$

Using the equation of state $P = \omega\rho$, we can use the computed energy density and pressure to find the equation of state parameter $\omega$ to be,

$$\omega = \frac{\frac{1}{2c^2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2c^2}\dot{\phi}^2 + V(\phi)}. \tag{3.30}$$

If we use the conjugate momentum fields $\pi_\phi = \frac{1}{c^2}\dot{\phi}$, then we can rewrite the equation of state parameter as,

$$\omega = \frac{\frac{1}{2}c^2\pi_\phi^2 - V(\phi)}{\frac{1}{2}c^2\pi_\phi^2 + V(\phi)} = -1 + \frac{\pi_\phi^2}{V}c^2 + \mathcal{O}(c^4), \tag{3.31}$$

where we took the expansion to illustrate that when we take the Carroll limit, the equation of state parameter will be $\omega = -1$. This then implies that in the Carroll limit, the equation of state is necessarily $P = -\rho$, which we associate with dark energy in our standard cosmology. Because the kinetic term is modest in relation to the potential energy $V(\phi)$ during slow-roll inflation, the equation of state $P \approx -\rho$ is roughly constant, which promotes faster expansion.

Nevertheless, the kinetic term gets closer to zero in the Carroll limit when the scalar field's time derivatives disappear. This results in an identical equation of state, $P = -\rho$, indicating that the Carrollian scalar field functions similarly to a pure cosmological constant. The Carrollian framework naturally leads to a de Sitter-like expansion without the need for potential fine-tuning, in contrast to classical inflation, where slow-roll requirements are required to maintain acceleration.

An intriguing question is brought up by this result: might the speed limit of the universe be understood as an emergent Carrollian regime? If the universe underwent a transient Carroll phase before transitioning to a relativistic expansion, it would provide an alternative explanation for early-universe inflation without invoking a traditional inflaton potential.

Inflationary models typically assume the slow-roll approximation, which holds when the inflaton field evolves slowly compared to the Hubble expansion. This is quantified by the slow-roll parameters [22],

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \quad \eta = M_{\rm Pl}^2\frac{V''(\phi)}{V(\phi)}, \tag{3.32}$$

where $M_{\rm Pl} = (\hbar c/8\pi G)^{1/2}$ is the Planck mass, and $V'(\phi) \equiv dV/d\phi$ , $V''(\phi) \equiv d^2V/d\phi^2$. The slow-roll conditions are,

$$\epsilon \ll 1, \quad |\eta| \ll 1. \tag{3.33}$$

Under these conditions, we know the Klein-Gordon equation of motion simplifies to,

$$3H\dot{\phi} \approx -V'(\phi), \tag{3.34}$$

and the Friedmann equation reduces to,

$$H^2 \approx \frac{8\pi G}{3}V(\phi). \tag{3.35}$$

The success of inflation depends on how long the exponential expansion lasts. This is quantified by the number of e-foldings which we can compute using,

$$N = \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi, \tag{3.36}$$

where if we write the integral in the field space of the inflaton [22], we get

$$N \approx \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon}}\frac{d\phi}{M_{\rm Pl}} \approx 60. \tag{3.37}$$

Therefore, for inflation to solve the horizon and flatness problems, we require at least $N \approx 60$ e-foldings before the end of inflation.

The slow-roll paradigm makes precise predictions for the power spectrum of scalar (density) perturbations. The ability of inflationary theory to produce the primordial density fluctuations that give rise to cosmic structure is one of its main achievements. The rapid expansion causes quantum fluctuations in the inflaton field to be extended to superhorizon sizes, giving rise to these fluctuations. Today, the power spectrum of the gauge-invariant curvature perturbation $\mathcal{R}$ is most commonly used to parametrise scalar perturbations produced during inflation [20], therefore the power spectrum is given by

$$\mathcal{P}_{\mathcal{R}}(k) = 2\pi^2 A_s k^{-3} \left(\frac{k}{k_p}\right)^{n_s-1}, \tag{3.38}$$

where $A_s$ is the variance of curvature perturbations in a logarithmic wavenumber interval centred around the pivot scale $k_p$ and $n_s$ is the spectral index [20]. The spectral index is given by,

$$n_s - 1 = -4\epsilon - 2\eta. \tag{3.39}$$

These slow-roll parameters are quite crucial and convenient to understanding an inflationary model, and there are extensive studies to relate them to fundamental entities, hence studying the scalar field and the potential in much detail. Numerous observations work to place constraints on these parameters, with current data favouring small values of $\epsilon$ and $r$, in agreement with slow-roll inflation.

In the Carroll limit, where time evolution is essentially frozen ($\dot{\phi} = 0$), the slow-roll conditions could take on a different meaning than their usual form. Since $H\dot{\phi} \approx -V'(\phi)$ is dominated by the friction term, inflation could potentially persist indefinitely in the Carrollian regime. This raises fundamental questions about whether "Carrollian inflation" could provide an alternative explanation for early universe expansion. A better understanding of the Carroll limit in the early universe would be needed before any such predictions can be made.

Some observations support inflationary predictions by studying the primordial power spectrum and determining that it is almost scale-invariant. In contrast to relativistic inflation, fluctuations are not damped since the typical Hubble friction term $3H\dot{\phi}$ vanishes. Rather, it is anticipated that spatial dynamics will control quantum fluctuations, which could result in a different kind of structure development. An anisotropic power spectrum, in which fluctuations rely more heavily on spatial coordinates than on time, could be predicted by Carrollian inflation. The cosmic microwave background (CMB) may bear clearer answers, although that is not in the scope of this project. Furthermore, the Carroll limit offers an alternate mechanism for inflation since it naturally generates a de Sitter-like phase without the need for a precisely calibrated slow-roll potential because it enforces $P \approx -\rho$. To ascertain whether the Carrollian inflation model can accurately replicate the known large-scale structure of the universe, more research is necessary. To get a detailed understanding of the Friedmann equations and the subsequent effects imposed by the Carroll limit, we shift our focus to studying this regime in a curved background, in contrast to the flat Minkowski background in this chapter.

# Chapter 4

# Friedmann Equations for the Carroll Scalar Field

Having looked at the evolution of the Carroll scalar field in flat Minkowski spacetime, we now wish to explore it in the context of our standard cosmology. We assume that the entire universe is smooth on the basis that the matter and radiation distributions in our observable universe are spatially homogeneous and isotropic [23].

## 4.1 The FLRW Universe

This model of the universe is described using the FLRW metric, which is given by,

$$ds^2 = N^2c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right], \tag{4.1}$$

where $N$ is the Lapse function, a Lagrange multiplier commonly used in the Arnowitt-Deser-Misner (ADM) formalism of general relativity. The Lapse function typically captures the rate of change of proper time concerning coordinate time caused by the non-uniformity of the gravitational field, and it is usually accompanied by the shift vector $N^i$. However, in the FLRW metric, the shift vector is set to zero. This is because the FLRW metric is homogeneous and isotropic, and as such, the shift vector is not needed. When taking the Carroll limit, though, the temporal component of the metric is affected, so the Lapse function ensures our metric is not over-constrained. The time-dependent scale factor $a(t)$ describes the expansion of the universe, and $k$ encodes the spatial curvature of the universe. The curvature can be positive, negative, or zero, corresponding to a closed, open, or spatially flat universe, respectively. The metric is also written in spherical coordinates, where $d\Omega$ is the solid angle element.

Let us take $k = 0$ for now, representing a spatially flat universe, resulting in our metric being,

$$g_{\mu\nu} = \text{diag}(N^2c^2, -a^2, -a^2r^2, -a^2r^2\sin^2\theta). \tag{4.2}$$

We can write the Lagrangian of the scalar field as,

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{4.3}$$

where the general volume element $\sqrt{-g}$, is the square root of the determinant of the metric, which in our case we can compute to be $\sqrt{-g} = Nca^3r^2\sin\theta$. Given the principal nature of our spacetime being homogeneous and isotropic, the spatial derivatives in the Lagrangian will vanish, and thus the Lagrangian will be given by,

$$\mathcal{L} = r^2\sin\theta a^3\left[\frac{1}{2Nc}(\partial_t\phi)^2 - NcV(\phi)\right]. \tag{4.4}$$

We also require our action to include the Einstein-Hilbert action such that the total action will be given by the combination of the scalar field and gravitational actions,

$$\begin{aligned} S_{\text{total}} &= S_\phi + S_{\text{EH}} \\ &= \int d^4x\sqrt{-g}L_\phi + \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R. \end{aligned} \tag{4.5}$$

We can compute the Ricci scalar for FLRW spacetime with the inclusion of the Lapse function, this will be $R = \frac{6(\dot{a}^2+a\ddot{a})}{c^2N^2a^2}$, so the gravity Lagrangian will then be,

$$\begin{aligned} \mathcal{L}_{\text{grav}} &= \frac{c^4}{16\pi G}\sqrt{-g}R \\ &= \frac{3c^4}{8\pi G}r^2\sin\theta\frac{a\dot{a}^2 - a^2\ddot{a}}{Nc}r^2\sin\theta. \end{aligned} \tag{4.6}$$

Therefore, we have the total Lagrangian density as,

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_\phi + \mathcal{L}_{\text{grav}} \\ &= r^2\sin\theta\left[\frac{3c^4}{8\pi G}\left(\frac{a\dot{a}^2}{N} + \frac{a^2\ddot{a}}{N}\right) + a^3\left(\frac{\dot{\phi}^2}{2Nc} - NcV(\phi)\right)\right]. \end{aligned} \tag{4.7}$$

The second term of the scalar field Lagrangian can be integrated by parts in order to remove the extra $\ddot{a}$ term, assisted by the time dependence of the scale factor. We then get the Lagrangian to be,

$$\mathcal{L} = A\left[-\frac{3c^4}{8\pi G}\left(\frac{a\dot{a}^2}{N}\right) + a^3\left(\frac{\dot{\phi}^2}{2Nc} - NcV(\phi)\right)\right], \tag{4.8}$$

with $A = r^2\sin\theta$. We can now compute the equations of motion from this new Lagrangian using the Euler-Lagrange equation by first taking the variation with respect to $a(t)$:

$$\frac{\partial\mathcal{L}}{\partial a} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{a}}\right). \tag{4.9}$$

This results in the equation,

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = 8\pi G\left(\frac{1}{2c^4}\dot{\phi}^2 - \frac{N^2}{c^2}V(\phi)\right). \tag{4.10}$$

And if we then take the variation with respect to the Lapse function,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial N} &= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{N}}\right) \\ \Rightarrow \frac{\partial\mathcal{L}}{\partial N} &= 0, \end{aligned} \tag{4.11}$$

we get the equation,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\left(\frac{1}{2c^2}\dot{\phi}^2 + V(\phi)\right). \tag{4.12}$$

We can recognise that the two equations 4.10 and 4.12 are simply just the Friedmann equations for the scalar field. We have seen from chapter 3 how the scalar field evolves in Minkowski space, as well as the effects of the Carroll limit in that scenario; we now wish to investigate how that compares to taking the Carroll limit in FLRW spacetime and how the scalar field evolves in this scenario. We can contrast the equations we computed with the usual way to get Friedmann equations using Einstein's field equations. Einstein's field equations are given by [21],

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{4.13}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, $R$ is the Ricci scalar and $\Lambda$ is the cosmological constant. Maintaining the assumption that the matter content of the universe can be modelled as a perfect fluid, we can use the energy-momentum tensor of a perfect fluid given by Eq. 3.22. The 00 component of the Einstein equations gives:

$$R_{00} - \frac{1}{2}Rg_{00} + \Lambda g_{00} = \frac{8\pi G}{c^4}T_{00}. \tag{4.14}$$

We first compute the Ricci tensor components for the FLRW metric. The time-time component of the Ricci tensor is,

$$R_{00} = -3\frac{\ddot{a}}{a}. \tag{4.15}$$

The Ricci scalar is given by:

$$R = -\frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{4.16}$$

Substituting into the Einstein equation:

$$-3\frac{\ddot{a}}{a} + \frac{3}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)c^2 + \Lambda c^2 = \frac{8\pi G}{c^4}\rho c^2. \tag{4.17}$$

Simplifying:

$$3\frac{\dot{a}^2}{a^2} + \Lambda c^2 = \frac{8\pi G}{c^2}\rho. \tag{4.18}$$

Rearranging:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\rho + \frac{\Lambda c^2}{3}. \tag{4.19}$$

This is the **first Friedmann equation**, which governs the expansion rate of the universe. Next, we derive the acceleration equation from the spatial components of Einstein's equations. The spatial components of the Ricci tensor are:

$$R_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)g_{ij}. \tag{4.20}$$

Similarly to the first Friedmann equation, we can use the spatial components of the Einstein equation and simplify it in order to get:

$$-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \Lambda c^2 = \frac{8\pi G}{c^4} P. \tag{4.21}$$

Substituting Eq. 4.19 we then end up with,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3P\right) + \frac{\Lambda c^2}{3}. \tag{4.22}$$

This is the **second Friedmann equation**, which describes the acceleration of the universe.

The Friedmann equations govern the dynamics of the universe, linking the expansion rate $H$ to the energy content $\rho$ and pressure $P$. These equations form the foundation of modern cosmology, describing the evolution of a universe dominated by radiation, matter, or dark energy. In the Carrollian limit ($c \to 0$), modifications to these equations provide insights into alternative gravitational theories and possible deviations from standard cosmological evolution.

Exploring the physical implications of the Friedmann equations, we start with the first one,

$$H^2 = \frac{8\pi G}{3c^2}\rho + \frac{\Lambda c^2}{3}, \tag{4.23}$$

which dictates the expansion rate of the universe. The term $H^2$ represents how fast the universe is expanding at a given time. And the physical interpretations for each component of this equation are given as follows [22]:

- $\rho$ is the energy density of the universe, which can include contributions from radiation, matter, and dark energy.
- We initially took $k = 0$ where the $k$-term would represent the effect of spatial curvature. For our consideration, we deal with a spatially flat universe however, it can also be considered to be open ($k < 0$) or closed ($k > 0$).
- $\Lambda c^2/3$ is the contribution from the cosmological constant, which is associated with vacuum energy and drives accelerated expansion.

This shows that the expansion rate is directly tied to the energy content. During different epochs, different components dominate,

- Radiation-dominated era: $\rho \propto a^{-4}$, meaning that in the early universe, radiation was the primary driver of expansion.
- Matter-dominated era: $\rho \propto a^{-3}$, indicating that galaxies and dark matter governed cosmic dynamics at intermediate times.
- Dark energy-dominated era: If $\Lambda$ is non-zero, it eventually dominates, leading to accelerated expansion.

The second Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho c^2 + 3P) + \frac{\Lambda c^2}{3}, \tag{4.24}$$

determines whether the universe is accelerating or decelerating. The key term here is $(\rho c^2 + 3P)$, which governs the effect of different energy components:

- If $P = 0$ (dust-like matter), the equation reduces to a decelerating scenario.
- If $P = \rho/3$ (radiation), the universe also decelerates.
- If $P = -\rho$, which results in accelerated expansion.

This last case corresponds to dark energy or a de Sitter universe, where $P = -\rho$ produces exponential expansion.

A de Sitter universe is a solution to the Friedmann equations in which a positive cosmological constant dominates [20]. Setting $\rho = \frac{\Lambda}{8\pi G}$ and $P = -\rho$, the second Friedmann equation simplifies to:

$$\frac{\ddot{a}}{a} = \frac{2\Lambda}{3}. \tag{4.25}$$

This implies that the universe undergoes accelerated expansion, with the scale factor evolving as

$$a(t) = a_0 e^{Ht}, \quad H = \sqrt{\frac{\Lambda}{3}}. \tag{4.26}$$

This exponential expansion occurs in two key cosmological contexts,

- Early Universe Inflation: Inflationary models assume an early de Sitter-like phase to resolve the horizon and flatness problems.
- Late-Time Acceleration: Observations suggest that the present-day universe is undergoing accelerated expansion due to dark energy, which behaves like a cosmological constant.

In a Carrollian framework, where time evolution is altered ($c \to 0$), the traditional interpretation of inflation and dark energy comes under scrutiny, as depicted in the previous chapter. The Friedmann equations provide a mathematical framework for understanding cosmic expansion. The first equation governs how the expansion rate evolves, while the second determines acceleration. In the presence of a cosmological constant, these equations predict de Sitter expansion, which is crucial in both inflationary models and late-time acceleration. The Carrollian modifications to these equations present an alternative perspective on the role of time in cosmic evolution, warranting further exploration.

Going back to the Friedmann equations in the standard relativistic setting, we can say that in the Carrollian limit, it is reasonable to treat derivatives with respect to time as becoming suppressed, leading to a fundamental restructuring of these equations. If the Hubble parameter is no longer dynamically evolving in the same way, then the rate of expansion may either freeze, remain arbitrary, or be dictated purely by boundary conditions. Several consequences arise from this,

- The usual matter-dominated and radiation-dominated epochs may not evolve in the standard way since their energy densities rely on the scale factor evolving in time.
- Structure formation mechanisms could be significantly altered, as density perturbations require a temporally evolving background to grow via gravitational collapse.
- The absence of conventional time evolution raises questions about how cosmic acceleration (e.g., de Sitter expansion) would manifest in a Carrollian setting.

Having seen the effects of exponential expansion, a natural question is whether a de Sitter-like phase still emerges. One possibility is that cosmic expansion is encoded in spatial rather than temporal variations. That is, instead of evolution in time, the expansion may be governed by a gradient of expansion across space, leading to an anisotropic version of de Sitter expansion.

Alternatively, if time derivatives completely vanish, then the universe may appear "static" in the Carrollian framework, leading to an interpretation where the universe remains in an eternally fixed state unless external perturbations induce transitions between different solutions. This suggests that Carrollian cosmology could provide an alternative explanation for the near-de Sitter nature of the current universe, where the observed accelerated expansion may be interpreted as an emergent feature rather than the result of vacuum energy.

Several theoretical challenges arise when applying the Carroll limit to cosmology:

- **How do density perturbations evolve?** In standard cosmology, the growth of large-scale structures relies on the time evolution of density fluctuations. If time derivatives are suppressed, alternative mechanisms must be proposed to explain the cosmic structure.
- **Can inflation occur in a Carrollian setting?** Inflationary models rely on a scalar field rolling down a potential, which inherently depends on time evolution. If the inflaton remains "frozen" due to the Carroll limit, then inflationary dynamics need to be reinterpreted.
- **Is there an observational distinction?** If Carrollian modifications to the Friedmann equations produce effects that differ from standard cosmology, there may be observational consequences, such as distinct imprints in the cosmic microwave background (CMB) or altered galaxy clustering statistics.

These considerations suggest that while the Carrollian limit offers a novel way to rethink cosmic evolution, further work is needed to reconcile it with observational constraints. Our goal lies more with tackling the scalar field dynamics in the FLRW background in the hopes of addressing the second point. To gain some intuition on the full implications of the Carroll limit, we must analyse the modified Friedmann equations. If time derivatives vanish or are highly suppressed, the standard form of the equations must be rewritten. Taking the limit $c \to 0$, the standard Friedmann equations become,

$$\frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{4.27}$$

$$0 = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}. \tag{4.28}$$

The first equation suggests that the curvature term must balance the energy density and cosmological constant, leading to constraints on allowed spatial geometries. The second equation implies that the pressure term is directly constrained by $\Lambda$, which could indicate that only particular types of fluids (e.g., vacuum energy with $P = -\rho$) are permissible in this limit. An alternative approach is to consider small deviations from the strict Carroll limit by introducing a short time evolution term. One possibility is to write,

$$H^2 + \epsilon\dot{H} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{4.29}$$

where $\epsilon$ is a small parameter that quantifies deviations from the strict Carrollian case. This allows for a gradual restoration of time evolution effects, potentially reconciling Carrollian cosmology with observational constraints.

The Carrollian limit forces a fundamental reconsideration of cosmic evolution by suppressing time evolution in the Friedmann equations. While this challenges conventional cosmological models, it opens new possibilities for interpreting cosmic acceleration and structure formation. Further studies, particularly numerical simulations of the modified Friedmann equations, are essential for assessing the viability of Carrollian cosmology. To consolidate our understanding of this limit, we take the approach of the Carrollian expansion as showcased in Chapter 3 to see if this matches the intuition we have laid out.

## 4.2 Friedmann Equations through the Carroll Expansion

As we have seen from our study of the scalar field, we can use its expansion $\phi \approx \phi_0 + c^2\phi_1$ around the small $c$ limit, and so when we substitute this into our scalar field Lagrangian in the FLRW background 4.8, whilst also keeping $E_0$ fixed as our condition for the potential, the expanded Lagrangian looks like,

$$\mathcal{L} = A\left[-\frac{3c^4}{8\pi G}\left(\frac{a\dot{a}^2}{N}\right) + a^3\left(\frac{1}{2Nc}(\dot{\phi}_0^2 + 2c^2\dot{\phi}_0\dot{\phi}_1 + \mathcal{O}(c^4)) - \frac{1}{2}N\omega_0^2c^{-1}(\phi_0^2 + 2c^2\phi_0\phi_1 + \mathcal{O}(c^4)\right)\right]. \tag{4.30}$$

If we then partition this in factors of $c$ and using the expansion of the Lagrangian,

$$\mathcal{L} = c^{2\Delta-1}(\mathcal{L}_0 + c^2\mathcal{L}_1 + \mathcal{O}(c^4)), \tag{4.31}$$

for $\Delta = 0$, we can get the separate equations for $\mathcal{L}_0$ and $\mathcal{L}_1$ as,

$$\mathcal{L}_0 = A\left(\frac{6}{N}a\dot{a}^2 + \frac{6}{N}a^2\ddot{a} + \frac{1}{2N}a^3\dot{\phi}_0^2 - \frac{N}{2}\omega_0^2\phi_0^2\right) \tag{4.32}$$

$$\mathcal{L}_1 = A\left(\frac{1}{N}a^3\dot{\phi}_0\dot{\phi}_1 - N\omega_0^2\phi_0\phi_1\right). \tag{4.33}$$

We then employ the Euler-Lagrange equation once more to get the dynamics of our zeroth and first-order Lagrangians. Starting with the zeroth order Lagrangian, we vary with respect to the zeroth order scalar field and its derivative and end up with

$$\ddot{\phi}_0 = -3\frac{\dot{a}}{a}\dot{\phi}_0 - \frac{N^2\omega_0^2}{a^3}\phi_0. \tag{4.34}$$

And if we also vary the first order Lagrangian with respect to the first order scalar field, we get,

$$\ddot{\phi}_1 = -3\frac{\dot{a}}{a}\dot{\phi}_1 - \frac{N^2\omega_0^2}{a^3}\phi_1. \tag{4.35}$$

If we vary the Lagrangians with respect to $a$ and $\dot{a}$, we do not get the Friedmann equations we have come to expect, which tells us that we are missing a crucial part of this computation, that is, we need to expand the scale factor as well because of its time dependence. So let us rather substitute the expansion of the scale factor as well as the expansion of the scalar field straight into the Friedmann equations we have already computed Eq. 4.10 and 4.12. The expansion of the scale factor is given by,

$$a(t) = a_0 + c^2a_1 + \mathcal{O}(c^4), \tag{4.36}$$

so, therefore, its time derivatives will be $\dot{a} \approx \dot{a}_0 + c^2\dot{a}_1$ and $\ddot{a} \approx \ddot{a}_0 + c^2\ddot{a}_1$, and hence in order to get the ratios that we see in the Friedmann equations (or FLRW Cosmology in general) we can compute the ratios of the expansion resulting in,

$$\frac{\dot{a}^2}{a^2} \approx \frac{\dot{a}_0^2}{a_0^2} + 2c^2\frac{\dot{a}_0^2(a_0\dot{a}_1 - \dot{a}_0 a_1)}{a_0^3} \tag{4.37}$$

$$\frac{\ddot{a}^2}{a} \approx \frac{\ddot{a}_0^2}{a_0} + c^2\frac{a_0\ddot{a}_1 - \ddot{a}_0 a_1}{a_0^2}. \tag{4.38}$$

We can now fully expand our scalar field Friedmann Equations, starting with Eq. 4.10. We know that the derivative of the scalar field expanded will be $\dot{\phi}^2 = (c^2(\dot{\phi}_0 + c^2\dot{\phi}_1 + \dots))^2$, and if we choose to keep $E_0$ fixed and as such substitute the expansion of the scalar field into the potential getting it to be,

$$V(\phi) = \frac{1}{2}\omega_0^2 c^2(\phi_0^2 + 2c^2\phi_0\phi_1 + \mathcal{O}(c^4)). \tag{4.39}$$

The zeroth and first-order equations resulting from the expansion of the Friedmann equation are therefore given by,

$$2\frac{\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2} = 4\pi G(\dot{\phi}_0^2 - N^2\omega_0^2\phi_0^2) \tag{4.40}$$

$$\frac{\ddot{a}_1}{a_0} + \frac{\dot{a}_0\dot{a}_1 - \ddot{a}_0 a_1}{a_0^2} - \frac{\dot{a}_0^2 a_1}{a_0^3} = 4\pi G(\dot{\phi}_0\dot{\phi}_1 - N^2\omega_0^2\phi_0\phi_1). \tag{4.41}$$

Similarly, for the Friedmann equation 4.12, we utilize the same procedure with the same condition for the potential, so therefore, after the full expansion of the equation, the zeroth and first-order equations are,

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{4\pi G}{3}(\dot{\phi}_0^2 - \omega_0^2\phi_0^2) \tag{4.42}$$

$$\frac{\dot{a}_0\dot{a}_1}{a_0^2} - \frac{\dot{a}_0^2 a_1}{a_0^3} = \frac{4\pi G}{3}(\dot{\phi}_0\dot{\phi}_1 - \omega_0^2\phi_0\phi_1). \tag{4.43}$$

If we instead choose to now keep the Compton wavelength fixed instead of $E_0$, then we most certainly get a different potential, and its expansion will be,

$$V(\phi) = \frac{1}{2}\mu^2 c^4(\phi_0^2 + 2c^2\phi_0\phi_1 + \mathcal{O}(c^4)). \tag{4.44}$$

So substituting the new potential into the expansion of the first Friedmann equation 4.10, we get the zeroth and first-order equations to be,

$$2\frac{\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2} = 4\pi G\dot{\phi}_0^2 \tag{4.45}$$

$$\frac{\ddot{a}_1}{a_0} + \frac{\dot{a}_0\dot{a}_1 - \ddot{a}_0 a_1}{a_0^2} - \frac{\dot{a}_0^2 a_1}{a_0^3} = 2\pi G(2\dot{\phi}_0\dot{\phi}_1 - N^2\mu^2\phi_0^2). \tag{4.46}$$

Then similarly for the second Friedmann equation 4.12, we get,

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{4\pi G}{3}\dot{\phi}_0^2 \tag{4.47}$$

$$\frac{\dot{a}_0\dot{a}_1}{a_0^2} - \frac{\dot{a}_0^2 a_1}{a_0^3} = \frac{2\pi G}{3}(2\dot{\phi}_0\dot{\phi}_1 - \mu^2\phi_0^2). \tag{4.48}$$

Now, before we can get any useful results from the new Friedmann equations we just derived for the zeroth and first-order terms of the scalar field and the scale factor, we recognize that there are too many unknowns in our equations, and both the scalar field and scale factor are heavily coupled. So, to make our lives easier, we can go back to our scalar field equations of motion and expand those to first order, and then from there, we will have enough equations to play with to solve for the zeroth and first-order terms.

From our Lagrangian 4.8, we can get the equations of motion first before expanding our scalar field and scale factor. Using the Euler-Lagrange equation, we get

$$-Nca^3\frac{\partial V}{\partial \phi} = \frac{1}{Nc}(3a^2\dot{a}\dot{\phi} + a^3\ddot{\phi}). \tag{4.49}$$

We already obtained the expansions for $\phi$ as well as $a(t)$, so to get the expansion of the derivative of potential, we take,

$$V(\phi) = \frac{1}{2}\left(\frac{m^2c^2}{\hbar^2}\phi^2\right) \tag{4.50}$$

$$\frac{\partial V}{\partial \phi} = \frac{m^2c^2}{\hbar^2}\phi \tag{4.51}$$

$$= \omega_0^2(\phi_0 + c^2\phi_1 + \mathcal{O}(c^4)). \tag{4.52}$$

Now that we have all our expansions, we can expand our equations of motion fully, and we get that in the zeroth order, we have

$$\ddot{\phi}_0 + 3\frac{\dot{a}_0}{a_0}\dot{\phi}_0 + N^2\omega_0^2\phi_0 = 0 \tag{4.53}$$

And for the first order, we have,

$$\ddot{\phi}_1 + 3H_0\dot{\phi}_1 + 3\left(H_1 - 2H_0\frac{a_1}{a_0}\right)\dot{\phi}_0 + N^2\omega_0^2\phi_1 = 0. \tag{4.54}$$

If we instead change the potential and choose to keep the Compton Wavelength fixed, our potential will look like

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 \tag{4.55}$$

$$\frac{\partial V}{\partial \phi} = \mu^2\phi \tag{4.56}$$

$$= \mu^2(\phi_0 + c^2\phi_1 + \mathcal{O}(c^4)), \tag{4.57}$$

so, going back to our equations of motion and expanding once more, we get the zeroth order equation of motion to be,

$$\ddot{\phi}_0 + 3\frac{\dot{a}_0}{a_0}\dot{\phi}_0 = 0, \tag{4.58}$$

and at first order we have,

$$\ddot{\phi}_1 + 3H_0\dot{\phi}_1 + 3H_1\dot{\phi}_0 + N^2\mu^2\phi_0 = 0. \tag{4.59}$$

## 4.3 Solving the Friedmann Equations

Now that we have these equations, we can solve both the zeroth order and first-order Friedmann equations for the scalar field. For simplification, if we take a vanishing potential (i.e., $\omega_0 \to 0$), utilizing equations 4.42 and 4.53, we get a simple differential equation for the zeroth order scalar field,

$$\ddot{\phi}_0 + 3H_0\dot{\phi}_0 = 0, \tag{4.60}$$

with the zeroth order Hubble parameter given by $H_0 = \frac{\dot{a}_0}{a_0}$. Given our vanishing potential, we get the Hubble parameter to be simply $H_0^2 = (4\pi G/3)\dot{\phi}_0^2$, simplifying our differential equation further to being,

$$\ddot{\phi}_0 + \sqrt{12\pi G}\dot{\phi}_0^2 = 0, \tag{4.61}$$

where integrating twice, we get the solutions for the zeroth order scalar field to be given by,

$$\phi_0(t) = \frac{1}{\sqrt{12\pi G}}\ln(\sqrt{12\pi G}t + A) + B, \tag{4.62}$$

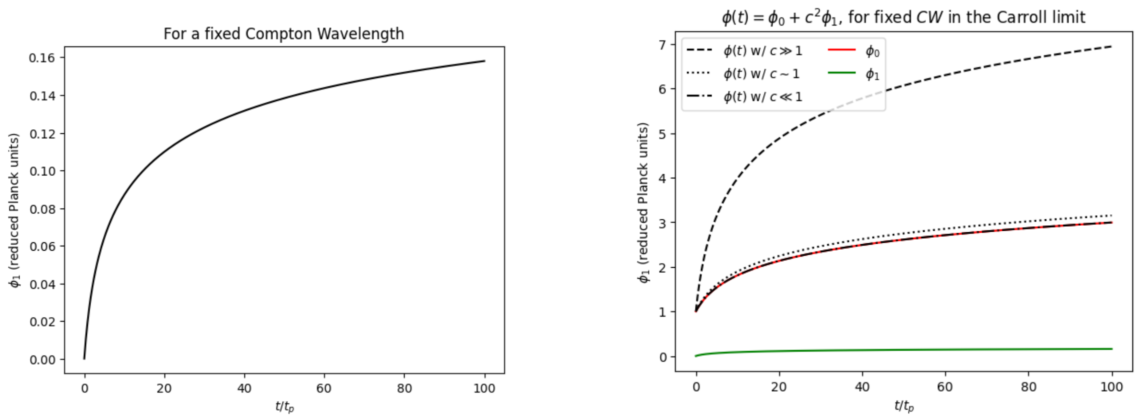
with A and B as mere integration constants. This logarithmic solution very much resembles the complete solution of the scalar field as shown in [6], which is to be expected from the leading order (LO) term. When keeping the Compton Wavelength fixed, with a vanishing potential, our solution for the leading order term of the scalar field is not altered. Given this scalar field solution, we can verify how the zeroth order scale factor behaves, where we can compute it using its derivative $\dot{\phi}_0 = 1/(\sqrt{12\pi G}t + A)$, we get

$$\begin{aligned} \frac{\dot{a}_0}{a_0} &= \sqrt{\frac{4\pi G}{3}} \cdot \frac{1}{\sqrt{12\pi G}t + A} \\ \therefore a_0(t) &= C\,(t + A)^{\frac{1}{3}}. \end{aligned} \tag{4.63}$$

From these solutions, we are now able to tackle the next-to-leading (NLO) order solution for the scalar field and the scale factor. First, considering the situation of the fixed Compton wavelength, the equations 4.48 and 4.59, we can use them to get the following nonlinear differential equation,

$$\ddot{\phi}_1 + 3H_0\dot{\phi}_1 + \frac{4\pi G}{H_0}\dot{\phi}_0\dot{\phi}_1^2 = 0. \tag{4.64}$$

This is simply a separable differential equation in $\dot{\phi}_1$, so this would essentially require us to integrate twice. Using the built-in Python function `quad` in the `scipy.integrate` package; we can numerically integrate the equation whereby we get the solution for the first order scalar field to be given by Figure 4.1a. This exhibits the similar logarithmic trend found in the zeroth order solution of the scalar field. When we take the LO and NLO solutions to approximate the full solution of the relativistic scalar field $\phi = \phi_0 + c^2\phi_1$, we can vary the value of $c$ to illustrate taking the Carroll limit, and as expected, by Figure 4.1b we can see that in the Carroll limit, we approach the zeroth order solution.




(a) (b)

Figure 4.1: (a) Evolution of the scalar field using the Carrollian expansion for the fixed Compton Wavelength showing the solution to the first order correction and (b) the Carroll limit taken on the approximate solution to the Scalar Field.

The solutions presented in Figure 4.1 illustrate the evolution of the scalar field under the Carrollian limit, contrasting with the standard relativistic evolution in conventional cosmology. To understand the significance of this behaviour, it is useful to compare it with the evolution of a standard inflaton field in a cosmological setting.

In standard slow-roll inflation, the inflaton field $\phi$ follows the equation of motion [22],

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{4.65}$$

where the Hubble friction term $3H\dot{\phi}$ plays a crucial role in suppressing oscillations and allowing the field to slowly roll down its potential. In contrast, Figure 4.1a demonstrates that in the Carrollian regime, where time evolution is significantly altered, the dominant term governing evolution is different. The logarithmic trend observed in the zeroth-order solution indicates that the field undergoes a form of constrained evolution where we could have spatial gradients dictate behaviour. Moreover, in Figure 4.1b, the limiting behaviour effectively freezes the field's evolution. From the standard scenarios where the field eventually settles into the minimum of its potential, further studies could be performed to test Carrollian effects. For our case, the logarithmic solution persists indefinitely, suggesting that such a Carrollian scalar field could have underlying effects on the inflationary mechanism.

Taking a similar approach for the case whereby we keep $E_0$ fixed, we can now try to solve for the NLO term. Given our solution for $a_0$, we find that the zeroth order Hubble parameter will then be given by,

$$H_0(t) = \frac{\sqrt{4\pi G}}{A + 6\sqrt{\pi G}t}. \tag{4.66}$$

It is clear that the first-order Hubble parameter is non-trivial, so if we look at the direct expansion of $H(t) \approx H_0 + c^2 H_1$ using the expansion of the scale factor, we get

$$\begin{aligned} H = \frac{\dot{a}}{a} &= \frac{\dot{a}_0 + c^2\dot{a}_1}{a_0 + c^2 a_1} \\ &= \frac{\dot{a}_0 + c^2\dot{a}_1}{a_0 + c^2 a_1} \cdot \frac{a_0 - c^2 a_1}{a_0 - c^2 a_1} \\ &= \frac{\dot{a}_0 a_0 + c^2(a_0\dot{a}_1 - \dot{a}_0 a_1)}{a_0^2}, \end{aligned} \tag{4.67}$$

where we neglected all terms of order $c^4$ in the computation. Knowing what the zeroth order Hubble parameter looks like, as it is of order $c^0$, we can deduce that the first order Hubble parameter (at order $c^2$) will then be given by,

$$H_1(t) = \frac{a_0\dot{a}_1 - \dot{a}_0 a_1}{a_0^2}. \tag{4.68}$$

From Eq. 4.43, we can try to simplify it now with a reasonable way to deal with the scale factor using the expansion of the Hubble parameter,

$$\begin{aligned} \frac{\dot{a}_0}{a_0}\left(\frac{\dot{a}_1}{a_0} - \frac{\dot{a}_0 a_1}{a_0^2}\right) &= \frac{4\pi G}{3}\phi_0\dot{\phi}_1 \\ \Rightarrow H_0 H_1 &= \frac{4\pi G}{3}\phi_0\dot{\phi}_1 \\ H_1 &= \frac{4\pi G}{3H_0}\phi_0\dot{\phi}_1, \end{aligned} \tag{4.69}$$

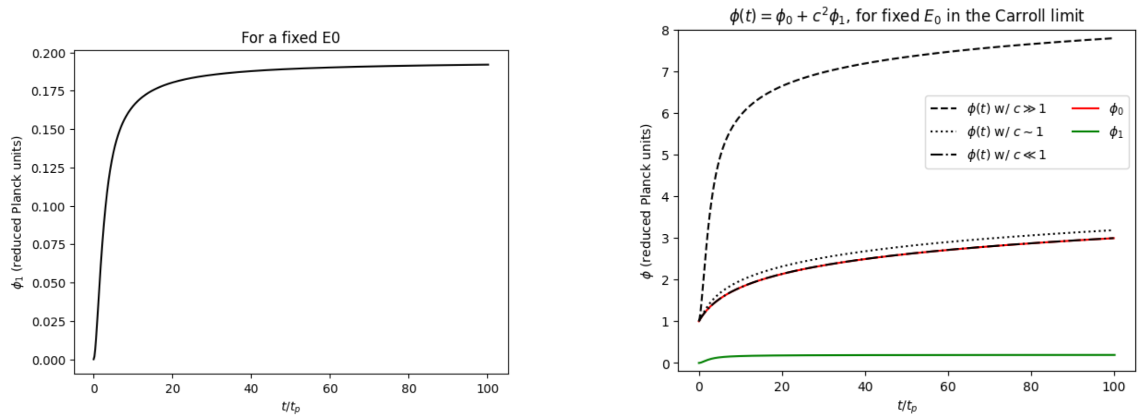
and plugging that into Eq. 4.54, we get the differential equation,

$$\ddot{\phi}_1 + 3H_0\dot{\phi}_1 + 3\left(\frac{4\pi G}{3H_0}\phi_0\dot{\phi}_1 - 2H_0\frac{a_1}{a_0}\right)\dot{\phi}_0 = 0. \tag{4.70}$$

The equation above is not easy to handle, so in an effort to carve out what the solution can look like, we can make some assumptions to simplify it. Looking at 4.66, we understand that if we take the early time limit, $H_0^-1$ will vanish, and we find the scale factor to be identical to the zeroth order scale factor. This early time limit is not completely random, as we understand from our study of the free scalar field in Minkowski spacetime that in the Carroll limit, we get dark energy solutions attracting inflationary solutions. Looking ahead as well to our upcoming study of the singularity theorem, this further justifies us taking the early time limit. We also opt for using reduced Planck units, where we take $8\pi G = t_p = 1$, and as such, the differential equation will be given by,

$$\ddot{\phi}_1 + 3H_0\dot{\phi}_1 + 3\left(\frac{t_p^2}{6H_0}\phi_0\dot{\phi}_1 - 2H_0\frac{a_1}{a_0}\right)\dot{\phi}_0 = 0. \tag{4.71}$$

We can use the late time limit to gauge that our approximations to the solution are reasonable, and in this limit, the first order Hubble parameter will vanish as $H_1 = \frac{a_0\dot{a}_1 - \dot{a}_0 a_1}{a_0^2} = 0$. We also get




(a) (b)

Figure 4.2: (a) Evolution of the scalar field using the Carrollian expansion for the fixed $E_0$ condition showing the numerical solution to the first order correction and (b) the Carroll limit taken on the approximate solution to the Scalar Field.

the zeroth order Hubble parameter $H_0$ to vanish therefore, for the first order scalar field, we can simplify the differential equation to be,

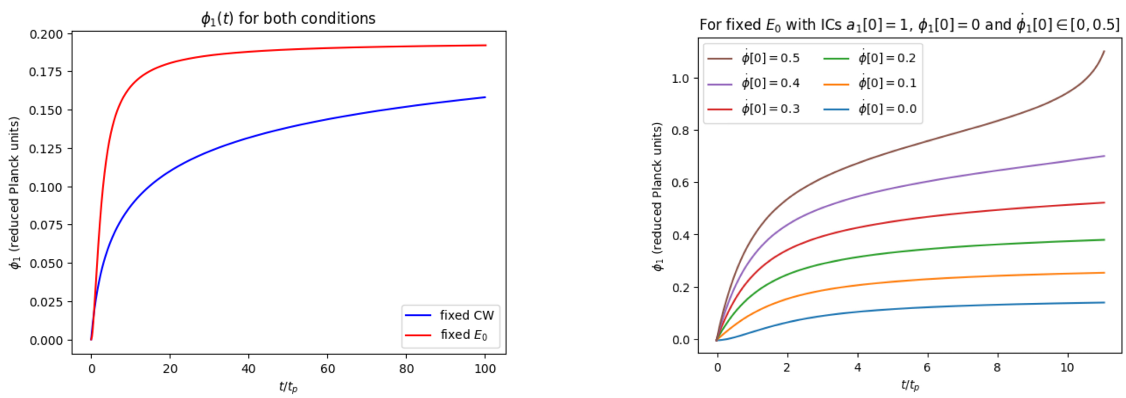
$$\ddot{\phi}_1 + \frac{1}{H_0}\phi_0\dot{\phi}_1 = 0, \tag{4.72}$$

which is a simpler equation to solve by a few substitutions, resulting in the solution for the first order scalar field at late times to be, $\phi_1(t) = \int \exp[\int dt\phi_0/H_0]$. These early and late time limits assist us in getting a sense of the boundary conditions we can impose when trying to get the full solutions for the first-order scalar field. Using a 4th order Runge-Kutta technique to numerically solve for the first order scalar field with the condition of a fixed $E_0$, we get Figure 4.2a, and when using it to get the complete scalar field solution, we get Figure 4.2b. In standard cosmology, the first-order corrections in a perturbative expansion of the inflaton field typically introduce corrections that lead to reheating mechanisms or modifications to the power spectrum of fluctuations. Here, the first-order solution reveals a crucial modification to the Hubble parameter,

$$H_1 = \frac{4\pi G}{3H_0}\phi_0\dot{\phi}_1. \tag{4.73}$$

This correction term is indicative of how perturbations in the Carrollian framework are influenced by the altered causal structure. Unlike conventional inflation, where small perturbations can grow and form the seeds of large-scale structures, Carrollian perturbations evolve under a fundamentally different constraint. This suggests that in a Carrollian universe, density fluctuations might behave differently than expected from standard perturbation theory, whose consequences could be intriguing topics for future studies.

The simplifications made to the equations for a fixed $E_0$ did not affect the overall trend of the solutions, and we can put the solutions side by side with that of a fixed Compton Wavelength shown in Figure 4.3a to illustrate this. The choice of initial conditions also affects how the solutions behave, and to verify that choice of initial conditions, we can show a range of solutions resulting from a different choice of initial conditions, specifically the choice of $\dot{\phi}_1(0)$. By Figure 4.3b, we can see that we get diverging solutions when we further increase the starting point of the $\dot{\phi}_1$. In our scalar field model, this acts as a friction/damping term, and we can assume that in the Carroll limit (i.e., as we move backward in time), the damping term should go to zero as well, which was the intuition gained from our choice of initial conditions.




(a) (b)

Figure 4.3: (a) Contrasting the first-order solutions for different conditions applied to the potential and (b) verification of the initial conditions applied to the numerical solutions showing a divergence in the scalar field solutions when they are not initiated correctly.

When combined, these plots show an evolution of the scalar field that is different from that of conventional inflationary cosmology. Specifically, the Carrollian framework's lack of dynamical temporal development suggests that a purely spatial paradigm would control the structures that form in such a universe. This stands in sharp contrast to the typical inflationary concept, in which the cosmic web and galaxies are seeded by quantum fluctuations generated by the field evolution. Whether Carrollian disruptions can be reconciled with the known structure of the universe is one important question that emerges from this approach.

Furthermore, numerical solutions indicate that the Carrollian limit does not result in an immediate singularity but rather in an indefinitely frozen state. This is similar to certain cyclic or emergent cosmological models in which the universe does not originate from a singularity but rather moves through many phases. Investigating whether this scenario can be expanded to incorporate transitions between distinct Carrollian phases may provide a new window into understanding the early universe. The numerical solutions present convincing evidence for revisiting the function of scalar fields in cosmology, giving us an additional mechanism to further probe this period of cosmology. While traditional inflation uses slow-roll circumstances to create an essentially de Sitter phase, Carrollian evolution approaches these criteria differently, leading to a qualitatively different mechanism for cosmic expansion. Future research could investigate if perturbations in this model provide observable signals that distinguish it from conventional inflation, as well as the consequences for late-time cosmology and dark energy models.

# Chapter 5

# Fluids in Carroll Spacetime

Having looked at the solution for the Friedmann equations for the free scalar field in the Carroll limit, we wish to do the same for the other fluids. The goal is to understand how matter, radiation, and the cosmological constant all evolve in Carroll spacetime in the FLRW background, and we will do so by studying the energy densities of these fluids and solving the resulting Friedmann equations. This is to serve our later study on the singularity theorem, so in that effort, we wish to understand these factors in the context of the early universe.

The early universe had an extreme environment characterised by high temperatures, dense matter, and rapid expansion. Understanding its thermodynamics necessitates the application of statistical mechanics and relativistic fluid dynamics. This section delves deeply into equilibrium thermodynamics in the early universe, laying the groundwork for further work on the effects caused by Carrollian physics. The thermodynamics of the early universe is intricately linked to statistical mechanics, which governs the distribution, interactions, and evolution of cosmic elements. The equilibrium paradigm given here serves as a foundation for investigating how Carrollian dynamics affect these concepts.

For a system in thermal equilibrium, the probability distribution of particles over energy states is given by the Boltzmann factor [24],

$$f(E) = e^{-E/k_B T}, \tag{5.1}$$

where $E$ is the energy of the state, $k_B$ is the Boltzmann constant, and $T$ is the temperature. For a gas of relativistic or non-relativistic particles, the number density $n$ is obtained from the partition function,

$$n = g_* \int \frac{d^3p}{(2\pi)^3} f(E), \tag{5.2}$$

where $g_*$ is the degeneracy factor, and $p$ is the momentum. Because of the high energy conditions of the early universe, particles were in thermal equilibrium due to fast interactions, resulting in Maxwell-Boltzmann, Bose-Einstein, or Fermi-Dirac distributions, depending on their nature.

As briefly discussed in previous chapters, the equation of state relates the pressure $P$ to the energy density $\rho$. In cosmology, the two most significant equations of state are [22],

- **Radiation-dominated universe:** $P = \frac{1}{3}\rho$ (valid for relativistic particles).
- **Matter-dominated universe:** $P \approx 0$ (valid for non-relativistic particles).

These equations arise naturally from the thermodynamic relations for ideal gases and are crucial in determining the evolution of the universe. A key property of the universe's expansion is entropy conservation. The entropy density is given by [25],

$$s = \frac{\rho + P}{T}. \tag{5.3}$$

It can be found that for a radiation-dominated universe,

$$s \propto gT^3, \tag{5.4}$$

where $g$ is the effective number of relativistic degrees of freedom. Since entropy is conserved in an adiabatic expansion,

$$gT^3a^3 = \text{constant}, \tag{5.5}$$

this relation governs the cooling of the universe over a period of time. As the universe expands and cools, reaction rates decrease below the Hubble expansion rate $H$, resulting in decoupling [26]. The decoupling condition is provided by,

$$\Gamma \sim H, \tag{5.6}$$

where $\Gamma$ is the interaction rate. Two key decoupling events[26]:

- **Neutrino Decoupling:** Neutrinos decouple from the cosmic plasma around $T \approx 1$ MeV, leading to a relic neutrino background.
- **Photon Decoupling:** Occurring at $T \approx 3000$ K, allowing photons to free-stream and form the cosmic microwave background (CMB).

The photon distribution in thermal equilibrium follows Planck's law,

$$u(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}, \tag{5.7}$$

where $\nu$ is the frequency of radiation. The integral of this distribution over all frequencies gives the total energy density of radiation,

$$u(T) = \frac{aT^4}{c}, \tag{5.8}$$

where $a$ is the radiation constant, given by,

$$a = \frac{8\pi^5 k_B^4}{15h^3c^3}. \tag{5.9}$$

This relationship, known as the Stefan-Boltzmann law, plays a crucial role in cosmological thermodynamics [25]. The peak of this spectrum follows Wien's law:

$$\lambda_{\text{peak}}T = \text{constant}. \tag{5.10}$$

This explains the cooling of the cosmic microwave background over time as $T \propto 1/a$. This law implies that as the universe cools, the peak wavelength of the cosmic microwave background (CMB) radiation shifts toward longer wavelengths, moving from high-energy photons in the early universe to microwaves today.

As the universe expands, the scale factor $a(t)$ grows, affecting the temperature and energy distribution of radiation. The redshift $z$ is defined as,

$$1 + z = \frac{a_0}{a}, \tag{5.11}$$

where $a_0$ is the present-day scale factor. Since the wavelength of radiation is stretched by cosmic expansion,

$$\lambda \propto a(t). \tag{5.12}$$

This directly affects the temperature evolution of blackbody radiation. Since energy density scales as $T^4$,

$$T \propto \frac{1}{a}. \tag{5.13}$$

This relation explains how the CMB temperature evolved from approximately 3000 K at the time of photon decoupling to its current value of $T_0 \approx 2.725$ K [25]. The CMB provides the most compelling observational evidence of blackbody evolution in an expanding universe. COBE, WMAP, and Planck satellite measurements show that the CMB has a nearly perfect blackbody spectrum with variations of less than one part in $10^5$. This extraordinary agreement with theoretical predictions lends evidence to the validity of traditional cosmological models [27].

Furthermore, precise temperature measurements in the CMB have provided vital insights into the density fluctuations of the early universe, constraining dark matter, dark energy, and cosmic inflation. Wien's law and blackbody radiation offers essential tools for comprehending the universe's thermal history. One important prediction that has been verified by measurements is that the expansion of spacetime causes the peak wavelength of the CMB to redshift while maintaining the blackbody spectrum. Deep insights into the mechanics of the early universe and the essential characteristics of cosmic evolution are still being gained from the study of these events. The number density of particles with a chemical potential $\mu$ follows:

$$n = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{(E-\mu)/k_B T} \pm 1}. \tag{5.14}$$

The baryon asymmetry, crucial for the observed matter-antimatter imbalance, is quantified by the baryon-to-photon ratio [28],

$$\eta_B = \frac{n_B - n_{\bar{B}}}{n_\gamma} \approx 6 \times 10^{-10}. \tag{5.15}$$

This small asymmetry, likely generated through baryogenesis mechanisms, is essential for the formation of galaxies and cosmic structures.

The early universe saw a variety of phase transitions and interactions, which formed the matter composition we see today. A critical component of this evolution is the presence of a chemical potential, which determines particle abundance and interactions. Looking at the role of the chemical potential in equilibrium thermodynamics, we want to see how it relates to the baryon asymmetry of the universe (BAU).

In statistical mechanics, the chemical potential $\mu$ appears in the distribution functions for particles. For a system in thermal equilibrium, the number density $n$ of particles follows [24],

$$n = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{(E-\mu)/k_B T} \pm 1}, \tag{5.16}$$

where we use the plus sign for fermions (satisfying Fermi-Dirac statistics) and the minus sign for bosons (satisfying Bose-Einstein statistics). For non-relativistic particles, where $E \approx mc^2 + p^2/2m$, the number density simplifies to:

$$n \approx g \left(\frac{mT}{2\pi}\right)^{3/2} e^{(\mu - m)/k_B T}. \tag{5.17}$$

This relation is crucial in determining how particle abundances evolve in the early universe.

As the universe expands, particle interactions determine whether a species remains in thermal equilibrium. If reaction rates are much greater than the Hubble expansion rate $H$, equilibrium is maintained. Otherwise, particles "freeze out," leading to the relic abundances observed today. As the universe expands, particle interactions determine whether a species maintains thermal equilibrium. If reaction rates are significantly higher than the Hubble expansion rate $H$, equilibrium is preserved. Otherwise, particles "freeze out", resulting in the current relic abundances. For a system in equilibrium, the chemical potentials of interacting particles must satisfy the following,

$$\sum_i \mu_i = 0, \tag{5.18}$$

where the sum is taken over all reactants and products in a given process.

The baryon asymmetry Eq. 5.15 implies that, for every billion photons in the cosmic microwave background, there exists a small excess of baryons. This imbalance is essential for the existence of matter-dominated structures in the universe, including galaxies, stars, and planets. The generation of baryon asymmetry requires three fundamental conditions, known as the Sakharov conditions [29],

1. **Baryon number violation**: Processes must exist that change the total baryon number $B$.
2. **C and CP violation**: Charge conjugation (C) and charge-parity (CP) symmetry must be violated to differentiate between matter and antimatter.
3. **Departure from thermal equilibrium**: Equilibrium processes obey detailed balance, preventing asymmetry generation. Thus, an out-of-equilibrium phase is required.

Grand unified theories (GUTs), leptogenesis, and electroweak baryogenesis are among the theoretical possibilities that satisfy these requirements. According to the Standard Model, electroweak symmetry breaks during a phase transition at high temperatures. If there is enough CP violation during this transition, interactions between the Higgs field and sphaleron processes can produce a baryon imbalance [30].

Observations of the CMB and BBN place strict constraints on baryon asymmetry. Measurements of deuterium and helium abundances confirm the small excess of baryons over antibaryons. Chemical potential plays a crucial role in determining particle abundances in the early universe, influencing processes such as thermal freeze-out and baryogenesis. The observed baryon asymmetry requires beyond-Standard-Model physics. Understanding these mechanisms not only sheds light on cosmic evolution but also provides connections to ongoing particle physics experiments [31].

## 5.1 Solving the Friedmann Equations in the Early Universe

Now that we have a solid foundation on the important concepts that come into play in the early universe, we can move to our attempt to solve the Friedmann equations when taking into account the relevant fluids. The Friedmann equations, not including the cosmological constant and taking a spatially flat universe ($k = 0$), are given by,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho \tag{5.19}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho c^2 + 3P). \tag{5.20}$$

The energy density of weakly-interacting gas particles, with factors of $c$ restored, is given by [23],

$$\rho = \frac{g}{2\pi^2}\int_{mc^2}^{\infty}\frac{1}{(\hbar c)^3}\frac{\left(E^2 - m^2c^4\right)^{1/2}}{\exp[(E-\mu)/T] \pm 1}E^2 dE. \tag{5.21}$$

with $g$ as the internal degrees of freedom for a species of mass $m$ and chemical potential $\mu$ at a temperature $T$. To study radiation in this scenario, we want to integrate over the wavelength $\lambda$ instead of the Energy $E$ because, in the Carroll limit ($c \to 0$), we lose the time component of the metric, so it is not entirely clear what time translations mean in this context as they pertain to the Energy. So using $E = \hbar c/\lambda$, we have $dE = -(\hbar c/\lambda)d\lambda$ therefore,

$$\rho = \frac{g}{2\pi^2}\int_0^{\frac{\hbar c}{mc^2}}\frac{\left(\left(\frac{\hbar c}{\lambda}\right)^2 - m^2c^4\right)^{1/2}}{\exp\left[\left(\frac{\hbar c}{\lambda} - \mu\right)/T\right] \pm 1}\frac{1}{\lambda^4}d\lambda. \tag{5.22}$$

Since in the relativistic limit, we have $m \to 0$, we can simplify the integral, and when we consider the Bose-Einstein species, we get,

$$\text{(BOSE)} \;\to \rho = -\frac{g}{2\pi^2}\frac{T}{(\hbar c)^3}\int_{-\frac{\mu}{T}}^{\infty}\left(T^3x^3 + 3T^2\mu x^2 + 3T\mu^2 x + \mu^3\right)/\left(e^x - 1\right)dx \tag{5.23}$$

$$= -\frac{g}{2\pi^2}\frac{T}{\hbar c^3}\left[T^3\frac{\pi^4}{15} + 6T^2\mu\zeta(3) + 3T\mu^2\frac{\pi^2}{6} + \dots\right], \tag{5.24}$$

whereby the solution diverges. However, for the Fermi species, we get,

$$\text{(FERMI)} \;\to \rho = -\frac{g}{2\pi^2}\frac{T}{(\hbar c)^3}\int_{-\frac{\mu}{T}}^{\infty}\left(T^3x^3 + 3T^2\mu x^2 + 3T\mu^2 x + \mu^3\right)/\left(e^x + 1\right)dx \tag{5.25}$$

$$= -\frac{g}{2\pi^2}\frac{T}{\hbar c^3}\left[T^3\frac{7\pi^4}{120} + \frac{9}{2}T^2\mu\zeta(3) + 3T_\mu^2\frac{\pi^2}{12} + \mu^3\ln(2)\right]. \tag{5.26}$$

Attempting to find converging solutions from these equations proved quite difficult (i.e. the $c \to 0$ limit leads to a divergence). This could be illustrating the fact that the relativistic Bose-Fermi distribution in thermal equilibrium is not well defined, so to further understand radiation in the Carroll limit, we chose a purely classical approach. It is, therefore, in the following chapter that we will tackle classical Maxwell Theory, which will provide us with more reasonable results on how radiation needs to behave in the Carroll regime.

In dealing with non-relativistic matter in thermal equilibrium, however, we find the energy density to be [23],

$$\rho = mc^2 g \left(\frac{mc^2 T}{2\pi}\right)^{3/2} \exp\left[-\left(mc^2-\mu\right)/T\right] \tag{5.27}$$

$$\text{and } P = gT \left(\frac{mc^2 T}{2\pi}\right)^{3/2} \exp\left[-\left(mc^2-\mu\right)/T\right]. \tag{5.28}$$

When we now look at the Friedmann equations we have,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3\sqrt{8}} \left(\frac{m^5 g}{\pi}\right)^{1/2} c^3 T^{3/2} e^{-\left(mc^2-\mu\right)/T}. \tag{5.29}$$

Using Wien's law for blackbody distribution, we get that the temperature is inversely proportional to the scale factor, i.e., $T/T_0 = a_0/a \Rightarrow T = T_0/a$ for $a_0 = 1$. We can now use this fact to find the behaviour of the scale factor, which we can get by utilizing equations 5.19 and 5.20, and solving the following,

$$\dot{a}^2 = \frac{8}{3\sqrt{8}} \left(\frac{m^5 g T_0^3}{\pi}\right)^{1/2} c^3 a^{1/2} e^{-a\left(mc^2-\mu\right)/T_0} \tag{5.30}$$

$$\ddot{a} = -\frac{4\pi G}{3c^2} \left(mc^2 a + 3T_0\right) g \left(\frac{mc^2 T_0}{2\pi}\right)^{3/2} \frac{1}{\sqrt{a}} e^{-a\left(mc^2-\mu\right)/T_0} \tag{5.31}$$

We can recognize from the equations themselves in both cases that Carroll limit causes the scale factor to freeze out as $a(t) \propto$ constant.

Understanding that in the early history of the universe, there were notable events that departed from thermal equilibrium, it would be helpful to look into the evolution in that scenario as well. When out of thermal equilibrium, the energy density for non-relativistic matter whose out of thermal equilibrium is given by [23],

$$\rho_{\text{out}} \sim n_0 mc^2 \left(\frac{a_0}{a}\right)^3, \tag{5.32}$$

where $n_0$ is the relic density at freeze out and $m$ is the rest mass energy. The Friedmann equations for the matter out of equilibrium will then become,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi a}{3c^2} n_0 mc^2 \left(\frac{a_0}{a}\right)^3, \tag{5.33}$$

from which the solution gives us the behaviour of the scale factor out of equilibrium,

$$a(t) = \left(6\pi G n_0 m\right)^{1/3} t^{2/3}. \tag{5.34}$$

It is imperative that we are careful about the initial conditions when dealing with the scale factor out of equilibrium, so to handle the energy density for the matter out of equilibrium, we take $a_0$ to be the late time boundary condition for the energy density in thermal equilibrium. So for non-relativistic dust $\psi$,

$$\rho_\psi(a) = \rho_\psi\left(a_0\right) \left(\frac{a_0}{a}\right)^3 \exp(-t/\tau), \tag{5.35}$$

with $\tau$ as the mean lifetime of $\psi$ where in thermal equilibrium we had 5.27, so we can set $\rho = \rho_\psi(a(t_f))$ and that allows us to get $\rho_\psi(a_0) \sim c^5$. Therefore, the energy density we get is then,

$$\rho_\psi(a) = c^5 \left(\frac{a_0}{a}\right)^3 \exp(-t/\tau). \tag{5.36}$$

If we now incorporate the energy density for the matter out of equilibrium in the Friedmann equations 5.19, we can solve the equation for the scale factor getting,

$$a(t)_{\text{out}} \sim ct^{2/3}, \tag{5.37}$$

whereby if we take the Carroll limit, we find the scale factor vanishes, so for matter out of equilibrium in this regime, it seems not to contribute to the expansion of the universe. Taking this into account, radiation and the cosmological constant can potentially be the only contributors to the expansion of the universe in the Carroll regime. We look into the cosmological constant next.

## 5.2 The Cosmological Constant

The cosmological constant was introduced by Einstein in 1917 to allow for a static universe and was later recognized as the driver of the universe's accelerated expansion. Historically, it is typically included in the Einstein Hilbert action or subsequently in the Einstein Field Equations, where we had the cosmological constant $\Lambda$, which is a constant energy density of a vacuum that fills space homogeneously and isotropically, leading to a de Sitter-like expansion when dominant. This section explores how the cosmological constant behaves in Carrollian spacetimes and examines its implications for early universe cosmology and the nature of vacuum energy. The energy density and pressure for the cosmological constant are commonly written as,

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \tag{5.38}$$

$$P_\Lambda = -\rho_\Lambda. \tag{5.39}$$

When we only have the vacuum energy present, we can use the Friedmann Equations to solve for the scale factor. The Friedmann equations for the cosmological constant are then,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho_\Lambda \tag{5.40}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho_\Lambda c^2 + 3P_\Lambda). \tag{5.41}$$

The solution to the Friedmann equations for the cosmological constant is then,

$$a(t) = a_0 e^{H_0 t}, \tag{5.42}$$

where $H_0 = \sqrt{\Lambda/3}$ is the Hubble constant. The scale factor grows exponentially with time in the presence of a cosmological constant. In the Carroll limit, however, we find that the scale factor freezes out as $a(t) \propto$ constant with the Hubble parameter vanishing. The role of $\Lambda$ is quite curious in this scenario as it could act as a modifier of spatial curvature rather than a driver of acceleration.

We could also have a possibility that the observed cosmic acceleration is an emergent phenomenon resulting from non-local effects rather than direct time evolution. $\Lambda$ could be interpreted as a purely geometric quantity affecting spatial correlations rather than dynamic evolution. This Carrollian interpretation of $\Lambda$ does present a theoretical departure from standard cosmology. However, it also offers potential avenues for exploring the mysteries of the cosmological constant and the role it plays in early universe cosmology. We know the cosmological constant plays a fundamental role in standard cosmology as the driver of late-time acceleration. However, in the Carroll limit, its role could be reconsidered. Instead of governing the rate of expansion, $\Lambda$ may influence spatial curvature in a fundamentally different way. Further exploration of this concept could yield new insights into the nature of dark energy and the fundamental geometry of spacetime. The freezing out of the scale factor when taking the Carroll limit in the outline scenarios in this chapter leaves us to consider how radiation could be the potential sole contributor to the expansion of the universe. Therefore, we need to look into the energy density and pressure of radiation in the context of the classical Maxwell theory in the Carroll regime, which we tackle in the following chapter.

# Chapter 6

# Carrollian Maxwell Theory

Before proceeding to the Carrollian limit, it is necessary to lay a more solid basis for Maxwell's equations in their standard relativistic version. This section delves into the brief derivation of Maxwell's equations from an action principle and the role of gauge invariance in classical field theory. These conversations deepen our grasp of how the electromagnetic field acts in a Lorentz-invariant framework, making the transition to the Carroll limit clearer.

## 6.1 Maxwell Theory from the Action Principle

The principle of least action allows us to formulate the classical field equations of electromagnetism. The four-potential equation $A_\mu = (\phi, \mathbf{A})$ describes the electromagnetic field, with $\phi$ representing the scalar potential and $\mathbf{A}$ representing the vector potential. The field strength tensor is defined as follows [32],

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.1}$$

The action for the electromagnetic field in a vacuum is given by,

$$S = \int d^4x \left( -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right). \tag{6.2}$$

To derive Maxwell's equations, we perform a variation of the action with respect to the potential $A_\mu$,

$$\delta S = \int d^4x \left( -\frac{1}{4\mu_0} \delta(F^{\mu\nu} F_{\mu\nu}) \right). \tag{6.3}$$

Using the variation of the field strength tensor,

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu, \tag{6.4}$$

the variation of the action becomes,

$$\delta S = \int d^4x \left( -\frac{1}{2\mu_0} F^{\mu\nu} \partial_\mu \delta A_\nu \right). \tag{6.5}$$

Integrating by parts and assuming that variations vanish at the boundary, we obtain

$$\delta S = \int d^4x \left(\partial_\mu F^{\mu\nu} \delta A_\nu\right). \tag{6.6}$$

For the action to be stationary under arbitrary variations $\delta A_\nu$, we must have,

$$\partial_\mu F^{\mu\nu} = 0. \tag{6.7}$$

These equations describe the dynamics of the electromagnetic field in a source-free environment. The inhomogeneous Maxwell equations, which include charge and current densities, arise when coupling the field to matter via the four-current $J^\nu$,

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{6.8}$$

This formulation demonstrates how Maxwell's equations naturally arise from a variational principle, emphasizing their close relationship to fundamental symmetries.

We also know that electromagnetism exhibits gauge symmetry, meaning that the physical content of the field is invariant under transformations of the form,

$$A_\mu \to A_\mu + \partial_\mu \Lambda. \tag{6.9}$$

This gauge freedom allows for the imposition of specific conditions, such as the Lorenz gauge,

$$\partial^\mu A_\mu = 0, \tag{6.10}$$

which simplifies the equations governing the field. Gauge invariance is also fundamental in the formulation of quantum electrodynamics (QED), where it underpins charge conservation via Noether's theorem. We will not be tackling the Carroll limit of QED in this project. However, that is an interesting avenue that can be explored further.

From the action, we can extrapolate what the Lagrangian density will look like, so it will be given by,

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta}, \tag{6.11}$$

where $F^{\alpha\beta}$ is the Faraday tensor, which we can formulate first in the flat Minkowski background. Given the Minkowski metric 2.1, with the 4-velocity $u^\mu$ such that $u_\mu u^\mu = -c^2$, we can define the electric and magnetic fields using the Faraday tensor [33] as,

$$E_\mu = u^\nu F_{\mu\nu} \qquad B_\mu = \frac{1}{2} \eta_{\mu\nu\rho\sigma} u^\sigma F^{\nu\rho}, \tag{6.12}$$

with $\eta_{\mu\nu\rho\sigma} = -\frac{\sqrt{-g}}{c} \epsilon_{\mu\nu\rho\sigma}$ as the generalized Levi-Civita symbol, for $\epsilon_{0123} = 1$ and $\sqrt{-g} = c$. Using this definition, we can construct the flat Faraday tensor where, using the definition of the electric field, we find $F_{0i} = -E_i$ and $F^{0i} = \frac{1}{c^2} E^i$. From the definition of the magnetic field we find that $F^{ij} = -\epsilon^{ijk} B_k$ and $F_{ij} = -\epsilon_{ijk} B^k$. Therefore, when we write the Faraday tensor in terms of the electric and magnetic fields, we get the following matrix representations,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E^1/c^2 & -E^2/c^2 & -E^3/c^2 \\ E^1/c^2 & 0 & -B^3 & B^2 \\ E^2/c^2 & B^3 & 0 & -B^1 \\ E^3/c^2 & -B^2 & B^1 & 0 \end{pmatrix} \qquad F_{\alpha\beta} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \tag{6.13}$$

We can get the equations of motion with $\nabla_\mu F^{\mu\nu} = 0$ and use the Bianchi identity $\nabla_{[\sigma} F_{\mu\nu]}$ to verify that the Faraday tensor we computed gives valid Maxwell's equations which we get to be,

$$\partial_i F^{i0} = 0 \qquad \frac{1}{c^2}\frac{\partial}{\partial t}E^j = \epsilon^{jik}\partial_i B_k \tag{6.14}$$

$$\partial_i F^{jk} = 0 \qquad \frac{\partial}{\partial t}B^l = \epsilon^{ijl}\partial_i E_j. \tag{6.15}$$

With that, we can already verify that we can get the Lagrangian in the same form as shown in Jan de Boer [6] in the following manner,

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4\mu_0}(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}) \\ &= \frac{1}{\mu_0}\left(\frac{1}{2c^2}(E_i)^2 - \frac{1}{4}(F_{ij})^2\right).\end{aligned}$$

In order to generally compute the energy-momentum tensor, we may vary the Lagrangian with respect to the metric [34],

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} + 2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}, \tag{6.16}$$

so we have,

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} &= -\frac{1}{4\mu_0}\frac{\delta}{\delta g_{\mu\nu}}(g_{\sigma\rho}g_{\alpha\beta}F^{\rho\alpha}F^{\sigma\beta}) \\ &= -\frac{1}{4\mu_0}\left[\frac{1}{2}g_{\alpha\beta}(F^{\nu\alpha}F^{\mu\beta} + F^{\mu\alpha}F^{\nu\beta}) + \frac{1}{2}g_{\sigma\rho}(F^{\rho\mu}F^{\sigma\nu} + F^{\rho\nu}F^{\sigma\mu})\right],\end{aligned}$$

and plugging the variation back into the energy-momentum tensor gives us the general expression,

$$T^{\mu\nu} = -\frac{1}{4\mu_0}g_{\alpha\beta}\left[g^{\mu\nu}g_{\sigma\rho}F^{\rho\alpha}F^{\sigma\beta} + (-)2(F^{\mu\alpha}F^{\nu\beta} + F^{\nu\alpha}F^{\mu\beta})\right]. \tag{6.17}$$

Applying the Minkowski metric to Eq. 6.17 and computing the energy-momentum components, we find them to be,

$$\begin{aligned}T^{00} &= \frac{1}{\mu_0 c^2}\left(-\frac{1}{2}(E_i)^2 - \frac{1}{2}(B_i)^2\right) \\ T^{0i} &= \frac{1}{\mu_0 c}(\vec{E}\times\vec{B})_i \\ T^{ij} &= \frac{1}{\mu_0}\left[\frac{1}{2}\eta^{ij}(E_i)^2 - \frac{1}{2}\eta^{ij}(B_i)^2 + E^iE^j + B^iB^j\right].\end{aligned}$$

These energy-momentum components can be run through a Carroll expansion technique of the same nature as what we performed for the scalar field. The consequences of that are the separation of the magnetic and electric sectors with these sectors at different orders of $c$ in the expansion. A more detailed discussion on the magnetic and electric sectors is given by de Boer et al. [6], and what we wish to take from this is the technique to apply it to the curved FLRW background.

## 6.2 Maxwell Theory in the FLRW Universe

We now pivot to putting Maxwell's Theory in the FLRW background; we do this so we can exploit the measure of the scale factor provided in this cosmology, especially in our study of radiation. The FLRW metric in conformal time is given by

$$g_{\mu\nu} = a^2(\eta)\text{diag}(-c^2, 1, 1, 1). \tag{6.18}$$

We take this conformal route because, in an expanding spacetime, light propagation is best studied in conformal time [22]. We follow a similar procedure as in flat spacetime to get the curved Faraday tensor using the definitions eq. 6.12. For the 4-velocity $u^\mu$ with the same conditions, we choose $u^0 = 1/a$, and we can verify the validity of this 4-velocity by checking if it satisfies the geodesic equation,

$$\begin{aligned}\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\alpha}u^\nu u^\alpha &= \frac{1}{a}\frac{du^\mu}{d\eta} + \Gamma^\mu_{\nu\alpha}u^\nu u^\alpha \\ &= \frac{1}{a}\frac{d}{d\eta}(\frac{1}{a}) + \frac{a'}{a}(\frac{1}{a})^2 \\ &= 0.\end{aligned}$$

So we have our 4-velocity as $u^\mu = (1/a, \vec{0})$ and $u_\mu = (-c^2 a, \vec{0})$. So for the Faraday tensor in terms of the electric and magnetic fields, we have $F^{0i} = \frac{E^i}{c^2 a}$, $F_0 i = -aE_i$ and, $F^{ij} = -\frac{1}{a^3}\epsilon^{ijk}B_k$, $F_{ij} = -a\epsilon_{ijk}B^k$, resulting in the matrix representation,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1/c^2a & -E^2/c^2a & -E^3/c^2a \\ E^1/c^2a & 0 & B^3 & -B^2 \\ E^2/c^2a & -B^3/a^3 & 0 & B^1/a^3 \\ E^3/c^2a & B^2/a^3 & -B^1/a^3 & 0 \end{pmatrix}, \quad F_{\mu\nu} = \begin{pmatrix} 0 & aE^1 & aE^2 & aE^3 \\ -aE^1 & 0 & aB^3 & -aB^2 \\ -aE^2 & -aB^3 & 0 & aB^1 \\ -aE^3 & aB^2 & -aB^1 & 0 \end{pmatrix}. \tag{6.19}$$

We can now use this tensor to get Maxwell's equations in curved spacetime, with the nonzero Christoffel symbols given by

$$\Gamma^0_{00} = \frac{a'}{a}, \quad \Gamma^0_{ij} = \frac{1}{c^2}\frac{a'}{a}\delta_{ij}, \quad \Gamma^i_{0j} = \frac{a'}{a}\delta^i_j$$

. Using the equation of motion and the Bianchi identity as we did in the flat case, we can find Maxwell's equations in curved spacetime with the result being,

$$\partial_i F^{i0} = 0 \qquad -\frac{a}{c^2}\partial_0(aE^i) = \epsilon^{ijk}\partial_j B_k \tag{6.20}$$

$$\partial_i F^{jk} = 0 \qquad \partial_0(aB^i) = \epsilon^{ijk}\partial_j(aE_k). \tag{6.21}$$

We can use these equations to now better understand the curved electric and magnetic sectors; we do this by taking the curls of Faraday's Law and Ampere-Maxwell Law in the equations in the following manner,

$$\begin{aligned}\epsilon_{lmi}\partial^m(aB^i)' &= \epsilon^{lmi}\partial_m\epsilon^{ijk}\partial_j(aE_k) \\ (a\epsilon_{lmi}\partial^m g_{ij}B^i)' &= \epsilon_{lmi}\epsilon^{ijk}\partial^m\partial_j(aE_k) \\ ((aE^j)')' &= c^2\partial^2(aE_l)\end{aligned}$$

and for the magnetic field,

$$\epsilon_{lmi}\partial^m(\epsilon^{ijk}\partial_j B_k) = -\epsilon_{lmi}\partial^m \frac{a}{c^2}(aE^i)'$$
$$a\left(\frac{(aB^j)'}{a^2}\right)' = c^2\partial^2(B_l).$$

The result is a wave equation for both the electric and magnetic fields, and to work better with the equations, we can scale the fields with the scale factor. For the electric field, $E_l = \frac{1}{a}E_l$, therefore that wave equation becomes $(E^l)'' = c^2\partial^2 E_l$, similarly for the magnetic field we get $a\left(\frac{1}{a^2}B_l'\right)' = c^2\partial^2(B_l)$.

We can use the following ansatz for our wave equations, $E = E_0 e^{i(k\cdot z-\omega t)}$ and $B = B_0 e^{i(k\cdot z-\omega t)}$, for which if we apply these to the two Gauss' Laws, we get vanishing amplitudes in the direction of the wave propagation $(E_0)_z = (B_0)_z = 0$. From Faraday's Law, after scaling our fields with the scale factor, we get $\partial_0(B) = \nabla\times E$, where if we insert our ansatz, we can get a relation between the amplitudes of the fields which will be given by $E_0 \sim (\omega/k)B_0$ where $\omega/k = c$.

Having verified that the electric and magnetic fields remain in phase and mutually perpendicular even in the curved FLRW background, keeping the same relation, we can discuss how the energy and momentum behave as a result of these fields. By the energy-momentum equation for a perfect fluid Eq. 3.22, we can relate the pressure and energy density to the components of the energy-momentum tensor we compute. From eq. 6.17, we can compute the energy-momentum components in the FLRW background in cosmic time to be,

$$T^{00} = -\frac{1}{\mu_0}\frac{1}{c^2}a^2(t)\left[-\frac{1}{2}(E^i)^2 - \frac{1}{2}a^2(t)(B^i)^2\right] \tag{6.22}$$

$$T^{0i} = \frac{a^2}{\mu_0 c}(\vec{E}\times\vec{B})^i \tag{6.23}$$

$$T^{ij} = -\frac{1}{\mu_0}\frac{1}{a^2}\left[\delta^{ij}(E^i)^2 - a^2E^iE^j - a^2(\delta^{ij}(B^i)^2 + a^2B^iB^j)\right], \tag{6.24}$$

and these can be recast into conformal time using the relation,

$$d\eta = \frac{dt}{a(t)}. \tag{6.25}$$

From these, we can see that when we take the limit as $a(t)\to 1$, we return to the energy-momentum tensor components computed in the flat space. We can now compute the energy density and pressure from the energy-momentum tensor, with the energy density being given by $\rho = a^2c^2T^{00}$ and the pressure being given by $p = g_{ij}T^{ij}$.

To round up our argument on understanding the evolution of radiation in the FLRW background, we can use the energy density and pressure to compute the equation of state parameter $\omega = p/\rho$. We can then use the Friedmann equations to understand the evolution of the scale factor and the energy density of the radiation. We can compute the Friedmann equations from Einstein's Field Equations as shown in Chapter 3. We can get the Friedmann equations to be,

$$3\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{c^4}T_{00}$$
$$-\frac{2}{c^2}\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = -\frac{8\pi G}{c^2}T_{ij}.$$

So now, actually taking our energy-momentum components to be those computed in the electromagnetism context, we can trace the consequences of the Carroll limit. We first see that the Carroll limit forces a divergence in the evolution of the electric and magnetic fields, and in the equation of state $P = -\rho$ is only satisfied when the energy density and pressure either both vanish or both approach $\pm\infty$, the former giving a better physical interpretation for our electric and magnetic sectors. The scale factor limit assists us here because the electric and magnetic fields have different dependencies. Therefore, the more we go backward in time, the faster the magnetic field will die off, leaving the electric field frozen in the Carroll limit, acting as a cosmological constant in the early universe.

# Chapter 7

# Singularity Theorem

We now wish to study Singularity theorems and how they may be affected by the Carroll limit. Singularity theorems are crucial to our knowledge of spacetime structure in general relativity. They define the conditions under which singularities (regions of infinite curvature or geodesic incompleteness) are unavoidable in solutions to Einstein's field equation. The most well-known of these are the Penrose-Hawking singularity theorems, which show that, under certain physically acceptable assumptions, singularities must form during gravitational collapse and cosmic evolution [35]. Prior to the formulation of these theorems, Raychaudhuri, Komar, and Geroch established the framework for a rigorous mathematical analysis of singularities. In this section, we present a thorough theoretical background on singularity theorems, including their reasons, fundamental mathematical structures, and the conditions necessary for their validity. Then, we will go through the Carroll limit of geodesic congruences, as that will play a big role in our study of the Raychaudhuri equations, a fundamental part of the singularity theorem.

Einstein's general theory of relativity, developed in 1915, transformed our understanding of gravity by defining it as the curvature of spacetime caused by energy and momentum. Early solutions to Einstein's field equations, such as the Schwarszchild solution, demonstrated the presence of singularities near $r = 0$ when curvature invariants diverge [36]. However, these answers were initially interpreted as mathematical artefacts rather than physical predictions. The contemporary study of singularities began with the work of Oppenheimer and Snyder (1939) [37], who demonstrated that the gravitational collapse of a dust cloud causes the development of a black hole. The question then arose: are singularities generic properties of general relativity, or are they simply the result of very symmetric solutions? This question remained unanswered until the development of rigorous singularity theorems.

In the 1950s and 1960s, groundbreaking work by Raychaudhuri, Komar, and Geroch developed a more broad framework for investigating singularities. Raychaudhuri's equation was a helpful tool for analysing geodesic congruences and their focussing qualities, revealing the conditions under which singularities must emerge. This prepared the path for the seminal work of Roger Penrose (1965) [38] and Stephen Hawking (1970) [39], who developed singularity theorems that apply to both black hole generation and cosmic evolution.

## 7.1 Carroll Geodesics

There are a few preliminary ideas we need to comb through before exploring singularity theorems in depth. First, we want to study the geodesics for a massive particle in the FLRW background. From the given FLRW metric in Eq. 4.1, we can use the geodesics equation given by [34]

$$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{7.1}$$

for the proper time $\tau$ where $ds^2 = c^2 d\tau^2$. With the non-zero Christoffel symbols already computed, we can proceed in analysing the equation. In our geodesic equation, if we choose our free variable to be $\lambda = 1$ and compute the geodesic, we get

$$\frac{d^2r}{d\tau^2} = -2\left(\frac{\dot{a}}{a}\right)\frac{dr}{d\tau}\frac{dt}{d\tau} \tag{7.2}$$

which when we write $dt/d\tau$ in terms of $dr/d\tau$, we end up with,

$$\frac{d^2r}{d\tau^2} = -\frac{2}{c^2}\dot{a}^2\left(\frac{dr}{d\tau}\right)^3, \tag{7.3}$$

from which we can conclude that as you take the Carroll limit ($c \to 0$), then the derivative of r with respect to the conformal time also vanishes, hence, in the Carroll limit, particles are fixed to their comoving world lines.

For null geodesics, we want to solve for the null vector field $k^\alpha$ such that $k^\alpha\nabla_\alpha k^\beta = 0$ and $k^\alpha k_\alpha = 0$. If we have some arbitrary four-vector $k^\alpha = (k^0, k^1, 0, 0)$, when taking $0 = k^\alpha g_{\alpha\beta}k^\beta = c^2(k^0)^2 - a^2(k^1)^2$, we end up with,

$$k^0 = \pm\frac{a(t)}{c}k^1. \tag{7.4}$$

Now for the affine condition $k^\alpha\nabla_\alpha k^\beta = 0$, when we start with $\beta = 0$ we get,

$$\begin{aligned} 0 &= k^\alpha\nabla_\alpha k^0 \\ &= k^0\nabla_0 k^0 + k^1\nabla_1 k^0 \\ &= k^0\partial_0 k^0 + \frac{a\dot{a}}{c^2}(k^1)^2, \end{aligned}$$

substituting what we got previously and solving for $k^1$, we get the differential equation,

$$\frac{\partial}{\partial t}k^1 = -2Hk^1, \tag{7.5}$$

which when we solve, we get $k^1 = e^{-2Ht} = \frac{1}{a^2}$, and therefore means $k^0 = \frac{1}{ac}$, and thus we get our null vector field to be,

$$k^\alpha = \left(\frac{1}{ac}, \frac{1}{a^2}, 0, 0\right). \tag{7.6}$$

We reach the same vector if we choose $\beta = 1$ as well, so in order to verify that this is indeed the most general we can get our null vector field to be, we introduce an arbitrary function of time,

$$k^\alpha = f(t)\left(\frac{1}{ac}, \frac{1}{a^2}, 0, 0\right). \tag{7.7}$$

Now, if we try and solve for $f(t)$ using the affine condition, we get that $f(t)$ is merely a constant and has no time dependence. That allows us to be able to scale the null vector field with a factor of c, so we instead get,

$$k_1^\alpha = \left(\frac{1}{ac}, \frac{1}{a^2}, 0, 0\right) \tag{7.8}$$

$$k_2^\alpha = \left(\frac{1}{a}, \frac{c}{a^2}, 0, 0\right). \tag{7.9}$$

Now that we have our null vector fields, we can consider the expansion of $a(t)$ given by $a(t) = a_0 + c^2 a_1 + \mathcal{O}(c^4)$. So when we expand each component, we get,

$$\frac{1}{ca} \approx \frac{1}{ca_0} - c\frac{a_1}{a_0^2} \tag{7.10}$$

$$\frac{1}{a^2} \approx \frac{1}{a_0^2} - 2c^2\frac{a_1}{a_0^3} \tag{7.11}$$

So up to order $c^2$, we have our first null vector as

$$k_1^\alpha = \left(\frac{1}{ca_0} - c\frac{a_1}{a_0^2}, \frac{1}{a_0^2} - 2c^2\frac{a_1}{a_0^3}, 0, 0\right), \tag{7.12}$$

and we can verify that using the null condition with $k_1^\alpha k_{1\alpha} = 0$. Similarly, for $k_2^\alpha$. The conclusion we can draw here is that in the Carroll limit, not all geodesics will be frozen. Especially considering how we can scale the geodesics using a factor of $c$, it is intriguing to consider how this affects the singularity theorem.

## 7.2 The Singularity Theorem

We are interested in an initial singularity, as such,h we first look at the Penrose-Hawking theorem, where an initial singularity can be defined as the existence of a past-directed null geodesic which is inextensible beyond a finite range of its affine parameter [40]. The theorem thus states that there exists an initial singularity in a connected spacetime if the following conditions are satisfied:

1. There exists a noncompact Cauchy surface,
2. there exists an anti-trapped surface,
3. and the null energy condition $T_{\alpha\beta}l^\alpha l^\beta \geq 0$ is satisfied for a null vector $l^\alpha$.

The first condition we assume to hold for the purposes of this project. We can discuss the second condition in more detail.

We, therefore, need to speak more about the third condition (the energy condition) before we can speak more about how the Carroll limit affects the theorem. The Raychaudhuri equation usually illustrates the form in which the energy condition will take.

The Raychaudhuri equation for null congruences is given by [34],

$$\frac{d}{d\tau}\theta_l = -R_{\alpha\beta}l^\alpha l^\beta - \frac{1}{2}\theta_l^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \frac{1}{2}\rho, \tag{7.13}$$

where $\theta_l$ is the expansion scalar, describing the rate of change of volume of a geodesic bundle, $\sigma_{ab}$ is the shear tensor, representing shape distortions without volume change, $\omega_{ab}$ is the rotation tensor, representing rotational effects, $\rho$ is the energy density and $l^\alpha$ is the tangent vector to the null geodesic. To obtain this equation, we start with the geodesic deviation equation,

$$\frac{D^2\xi^\mu}{d\lambda^2} = -R^\mu{}_{\nu\rho\sigma}u^\nu\xi^\rho u^\sigma, \tag{7.14}$$

where $\xi^\mu$ represents the deviation vector between nearby geodesics. The expansion scalar is defined as,

$$\theta = \nabla_\mu u^\mu. \tag{7.15}$$

The full Raychaudhuri equation can be obtained by differentiating along the geodesic flow and decomposing $\nabla_\nu u_\mu$ into the trace, symmetric, traceless, and antisymmetric portions. For spacetimes that meet the null energy condition (NEC), $R_{\mu\nu}k^\mu k^\nu \geq 0$, the focussing term in the Raychaudhuri equation ensures that initially converging geodesics attain a singularity in finite affine parameter.

The Raychaudhuri equation for timelike congruences is given by [34],

$$\frac{d}{d\tau}\theta_v = -R_{\alpha\beta}v^\alpha v^\beta - \frac{1}{3}\theta_v^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \frac{1}{2}\rho - \frac{1}{2}p, \tag{7.16}$$

and the Einstein Field Equation is also given by,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{7.17}$$

.

We can, therefore, use the Einstein Field Equation to find the Ricci tensor and the Ricci scalar and substitute them in the Raychaudhuri equation to find the evolution of the expansion scalar as,

$$\frac{d}{d\tau}\theta_l = -\left(\frac{8\pi G}{c^4}T_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}R\right)l^\alpha l^\beta - \frac{1}{2}\theta_l^2 \tag{7.18}$$

$$= -\frac{8\pi G}{c^4}T_{\alpha\beta}l^\alpha l^\beta - \frac{1}{2}\theta_l^2. \tag{7.19}$$

Singularity theorems rely on certain energy conditions that restrict the behaviour of matter in spacetime. The most commonly used conditions are [21],

- Null Energy Condition (NEC): $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any null vector $k^\mu$.
- Weak Energy Condition (WEC): $T_{\mu\nu}u^\mu u^\nu \geq 0$ for any timelike vector $u^\mu$.
- Strong Energy Condition (SEC): $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)u^\mu u^\nu \geq 0$.
- Dominant Energy Condition (DEC): $T_{\mu\nu}u^\mu$ is a non-spacelike vector.

These energy conditions ensure that geodesic congruences behave in a manner consistent with the presence of attractive gravity, a crucial requirement for proving singularity theorems.

Here the NEC implies that $\frac{d}{dt}\theta_l \leq 0$. This implies that the expansion scalar of a null geodesic congruence is non-increasing.

The non-zero components of Christoffel symbols for the FLRW metric are given by,

$$\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{c^2}, && \Gamma^0_{22} = \frac{a\dot{a}}{c^2}r^2, && \Gamma^0_{33} = \frac{a\dot{a}}{c^2}r^2\sin^2\theta, \\
&\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j, && \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j, && \Gamma^1_{22} = -r, \\
&\Gamma^1_{33} = -r\sin^2\theta, && \Gamma^2_{33} = -\sin\theta\cos\theta, && \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \\
&\Gamma^2_{12} = \Gamma^3_{13} = \Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r}.
\end{aligned}$$

Using the null geodesic given by $l^\alpha = (\frac{1}{a}, \frac{c}{a^2}, 0, 0)$, we get the expansion to be,

$$\begin{aligned}
\theta &= \frac{d}{cdt}\left(\frac{1}{a}\right) + \frac{\dot{a}}{a}\left(\frac{3}{a}\right) + \frac{1}{r}\left(\frac{2c}{a^2}\right) \\
&= -\frac{1}{c}\left(\frac{1}{a^2}\right) + 3\frac{\dot{a}}{a^2} + \frac{2c}{a^2 r} \\
&= \frac{1}{a}\left(3H - \frac{1}{ac} + \frac{2c}{ar}\right)
\end{aligned}$$

If the expansion were to vanish, then we would have $3H = \frac{1}{ac} - \frac{2c}{ar}$, which implies that $H = \frac{1}{3a}\left(\frac{1}{c} - \frac{2c}{r}\right)$, this implies that the Hubble parameter is non-negative. If we rewrite the expression as,

$$r = \frac{2c^2}{1 - 3Hac}, \tag{7.20}$$

then we can see that the radius will vanish in the Carroll limit. The consequence this has on the theorem is that the Carroll limit does not impact the energy condition but rather the anti-trapped surface, of which the Carroll limit does not entirely remove the singularity but more so softens it to a problem of geodesic incompleteness. In essence, we find that in the Carroll limit, the null rays become even more geodesically incomplete. Thus, going back to the second condition of the singularity theorem, let's discuss that in detail first before moving on to the discussion on geodesic incompleteness.

### 7.2.1 Trapped and Anti-Trapped Surfaces

Trapped surfaces play an important part in the development of singularity theorems and black hole theory. Trapped surfaces, first described by Roger Penrose in his 1965 singularity theorem, are a geometric condition that indicates the onset of gravitational collapse, which leads to a singularity. We say a compact 2-d surface is anti-trapped if the expansion of outgoing null geodesics is negative, that is, both the inward and outward-going sets of light rays emanating normally from the surface tend to diverge at every point on the surface. We say the surface is trapped if the expansion of outgoing null geodesics is positive, that is, the null geodesics orthogonal to the 2-surface converge. All iterations of the Singularity theorem require the existence of a trapped or an anti-trapped surface in some variation, most times coupled with an energy condition. For most major singularity theorems dealing with closed trapped surfaces, the past is finite because of the generating geodesics that converge, implying the existence of focusing points [41].

A trapped surface is a closed, spacelike two-surface $S$ in spacetime such that the expansion scalar $\theta$ of outgoing or ingoing null geodesics is negative,

$$\theta_+ < 0 \quad \text{and/or} \quad \theta_- < 0. \tag{7.21}$$

This condition means that even outgoing light rays are being focused inward due to extreme gravitational curvature, indicating strong gravitational collapse.

The expansion scalar $\theta$ for a congruence of null geodesics with tangent vector $k^\mu$ is given by [34],

$$\theta = \nabla_\mu k^\mu. \tag{7.22}$$

In the presence of an event horizon, the outermost trapped surface marks the boundary beyond which light cannot escape, defining the black hole's apparent horizon.

To understand how trapped surfaces arise, we analyse the evolution of null congruences using the Raychaudhuri equation:

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu. \tag{7.23}$$

If the null energy condition (NEC) holds, meaning that $R_{\mu\nu}k^\mu k^\nu \geq 0$, then the presence of a trapped surface leads to a focusing of geodesics, implying the formation of a singularity in finite affine parameter.

In gravitational collapse scenarios, trapped surfaces form as a direct consequence of increasing curvature. Consider the collapse of a spherically symmetric mass shell :

- Initially, outgoing null rays expand outward ($\theta_+ > 0$).
- As the shell collapses, curvature increases, and the expansion rate of outgoing rays slows down.
- When the mass reaches a critical density, outgoing rays begin to converge ($\theta_+ < 0$), indicating the formation of a trapped surface.

This process marks the point of no return, leading inevitably to the formation of an event horizon and singularity within the framework of classical general relativity.

In cosmological models, trapped surfaces can arise in strongly curved regions, particularly in the early universe or in scenarios involving big crunch singularities. Notably:

- In inflationary cosmology, certain inhomogeneities can lead to localized trapped surfaces.
- In the presence of a cosmological constant, horizon formation, and expansion can create apparent horizons that behave similarly to trapped surfaces.

Understanding these surfaces in a cosmological setting provides insight into the nature of singularities beyond black holes, with implications the early universe physics.

Trapped surfaces serve as a crucial mathematical tool in singularity theorems, ensuring that once a certain threshold of gravitational collapse is crossed, a singularity must inevitably form. Their presence in both black hole physics and cosmology provides deep insights into the nature of gravitational collapse, event horizons, and the possible limits of classical general relativity. Given what we see from the study of the singularity theorem, we can ask if the Carroll limit reduces the singularity to a coordinate singularity rather than a physical one, we see this directly when analysing the Raychaudhuri equation. This idea is tackled in more detail in the paper by Borde, Guth, and Vilenkin [42], which we briefly discuss in the next section.

## 7.3 Borde-Guth-Vilenkin (BGV) Theorem

A spacetime $(M, g_{\mu\nu})$ is considered geodesically complete if all geodesics may be extended infinitely. If any geodesic ends at a finite affine parameter, the spacetime is geodesically incomplete, which is a prerequisite for the presence of a singularity. Physically, this means that there are observers or particles whose paths cease after a finite amount of proper time (timelike geodesics) or affine parameter (null geodesics), suggesting a breakdown in the smooth structure of space. The BGV theorem discusses the past completeness of inflationary spacetimes. It makes a compelling case that, under common assumptions, inflationary models cannot be past-eternal, necessitating the existence of an initial singularity or a physics beyond classical general relativity. This theorem is especially important when considering the Carroll limit ($c \to 0$), where time evolution no longer plays a standard role in dynamical equations. This section delves further into the BGV theorem, its derivation, and the ramifications of singularity theorems.

According to the BGV theorem, any universe experiencing everlasting expansion has an average Hubble parameter satisfying [43],

$$H_{\text{ave}} > 0 \tag{7.24}$$

must be geodesically incomplete in relation to the past. This indicates that past-directed timelike or null geodesics cannot be extended endlessly, necessitating the existence of a past barrier or original singularity. The theorem is extensively applicable to inflationary models, especially those attempting to generate a past-eternal inflationary phase. Since most inflationary models postulate an eternally expanding universe, the BGV theorem provides a fundamental constraint, requiring either an initial singularity or a revision to classical physics in the distant past.

Given the FLRW metric, we can find the relation between the scale factor and the wavelength of a wavevector travelling along a null geodesic. The null geodesic is given by $ds^2 = 0$, which implies that $c^2dt^2 = a^2(t)dr^2$. The frequency of the wave as measured by a comoving observer is related to the time component of the wavevector as,

$$\omega = -k_\mu u^\mu, \tag{7.25}$$

where $u^\mu = (1, 0, 0, 0)$ is the four-velocity of the comoving observer in FLRW spacetime. The wavelength of the wave stretches proportionally to the scale factor $\lambda_{\text{obs}} \propto a(t_{\text{obs}})$ similarly for the emitted wavelength $\lambda_{\text{em}}$ at time of emission $t_{\text{em}}$. Since the frequency is inversely proportional to the wavelength, it follows that the frequency will be inversely proportional to the scale factor, capturing the redshift experienced by waves propagating through an expanding universe. The wave vector $k^\mu = dx^\mu/d\lambda$ satisfies $k^\mu k_m u = 0$ for null geodesics, and from Section 7.1 we found $k^\mu = (\frac{1}{a}, \frac{c}{a^2}, 0, 0)$ to be one such wavevector so we can therefore conclude that,

$$d\lambda \propto \frac{cdt}{\omega} = a(t)cdt. \tag{7.26}$$

Following [42], we can multiply both sides of the above equation with the Hubble parameter $H(\lambda)$

and then integrate over the interval $[t_i, t_f]$ to get,

$$\begin{aligned}\int_{\lambda_i}^{\lambda_f} H(\lambda)d\lambda = \int_{t_i}^{t_f} H(\lambda)a(t)cdt \leq c\\ H(\lambda_f - \lambda_i) = c\int da,\\ H = \frac{c}{\lambda_f - \lambda_i}\int da,\end{aligned}$$

therefore,by the inequality above, we can define the average Hubble parameter as,

$$H_{\text{ave}} = \frac{c(a(t_f) - a(t_i))}{\lambda_f - \lambda_i} \leq \frac{c}{\lambda_f - \lambda_i}, \tag{7.27}$$

where in the Carroll limit, we have null rays that are necessarily past incomplete because moving backward in time, the null geodesics will have a finite affine length with $H_{\text{ave}} > 0$. The Carroll limit does not allow us to traverse past the boundary of an inflation spacetime, but simply echoes the arguments put forward by [42] and [44].

The BGV theorem is closely connected to the Hawking-Penrose singularity theorems; however, unlike them, it is not dependent on energy constraints such as the null energy condition (NEC). Instead, it is entirely dependent on the kinematic features of geodesics and expansion rates. This makes it particularly relevant in inflationary models, which frequently assume unusual energy conditions. The theorem shows that inflation does not eliminate the need for an initial singularity. Instead, it implies that inflation should be integrated into a wider framework.

# Chapter 8

# Conclusions

This thesis has explored various fundamental aspects of general relativity, cosmology, and field theory, emphasizing their interconnections through the Carroll limit. The Carrollian regime provides a rich theoretical testing ground for ultra-relativistic ideas within physics, and we have shown how its effects influence how one can view well-known ideas from a different perspective. The Carroll limit was initially developed as a mathematical structure of curiosity, and over a couple of decades, it developed into an idea that could be applied to physical ideas. Given the great importance of the Poincaré group in General Relativity and Physics in general, it is a logical conclusion that a direct descendant of the group could also hold great significance in the field.

We studied the relativistic scalar field, and its formalism plays a crucial role in describing the early universe, particularly in inflationary models. We explored the dynamics of scalar fields using the Klein-Gordon equation and analysed the conditions for slow-roll inflation. In this study, we explored the Carroll expansion to see if we could get interesting results from the NLO solutions. We verified in our study of the scalar field the equation of state-imposed by taking the Carroll limit to be $P = -\rho$, which is the equation of state we usually get for dark energy in standard cosmology. This prompted our direction into further studying the Carroll limit in the early universe, and as such, we decided to continue the study of the scalar field in curved spacetime using the FLRW background.

The solutions found for the scalar field in the FLRW background indicated perturbations around the conventional inflationary models that would require further study in order to fully understand. This did indicate, however, that the early universe could show different dynamics to standard model predictions, so a further study of the evolution of fluids in Carroll spacetime was warranted. We, therefore, went the semi-classical route to study fluids in the early universe. Non-relativistic matter and the cosmological constant were found in frozen states when solving the Friedmann equations and then imposing the Carroll limit, alluding to the fact that radiation could potentially be the sole contributor to the expansion of the universe at those scales. Relativistic matter & radiation in thermal equilibrium was rather inconclusive in this context, prompting a shift in direction, so going the pure classical route, we studied Maxwell's theory, and by writing out electromagnetism in its covariant from, we got a better idea of how the energy density and pressure behaved in the form of the energy-momentum tensor. Taking the Carroll limit in the context of Maxwell's theory, we found a separation in how the electric and magnetic fields would be treated; the magnetic field vanishes faster, and the electric field acts as some sort of cosmological constant.

Having all this context around the early universe, we then proceeded with the study of the singularity theorem. The Penrose-Hawking singularity theorems established rigorous criteria for singularities to develop in spacetimes that satisfy the energy conditions of general relativity. We offered thorough derivations of the Raychaudhuri equation and its application to geodesic focussing. We studied the Carroll limit of geodesics in the FLRW background, and that provided better insight into how to approach the singularity theorem in its entirety. The ramifications of trapped surfaces in gravitational collapse were investigated, demonstrating how they point to the inevitability of singularity creation in both black holes and cosmic situations. We found that the Carroll limit does not entirely remove an initial singularity but instead softens the singularity problem to a problem of geodesic incompleteness, which we address by studying the BGV theorem.

The BGV theorem gave a kinematic justification for why inflationary spacetimes with a positive average expansion rate are past incomplete. This finding strengthens the case that inflation cannot be past-eternal and requires an initial singularity or a change in physics at extremely early eras. Our examination of this theorem highlighted its distinctiveness from energy-condition-based singularity theorems, confirming its broad applicability to inflationary models.

Much of this study of the Carroll limit of General Relativity was detailed however, it left much on the table to be explored. Further investigations into the inflationary paradigm governed by this regime could prove an interesting task. The cosmological constant still remains a big mystery in cosmology, and Carrollian physics could be a tool used to further investigate its properties. There is much to still explore with the Carroll limit of Electrodynamics, and what interesting ideas could come from this separation of the magnetic and electric sectors. Furthermore, the Carroll limit can be used to dig deeper into the singularity theorem from the perspective of geodesic incompleteness, as we found the Carroll regime having a bigger effect on the null rays.

# Appendix A

# Appendices

## A.1 Numerical Solutions to the Friedmann Equations

Showcased below are the techniques used to numerically solve the differential equations for the first-order scalar field in Chapter 4. The packages used are the usual scientific Python packages, namely `numpy` and `scipy`. The plots can therefore be generated using the `matplotlib` package.

In solving the coupled system of differential equations for the first order-scalar field with the fixed $E_0$ condition, the fourth order Runge Kutta method was implemented. This method effectively uses a weighted average of four slope estimates to achieve higher accuracy in the numerical solution compared to simpler methods like Euler's method or the second-order Runge-Kutta variants. The global truncation error of the RK4 method scales as $\mathcal{O}(h^4)$, where $h$ is the step size. For our purposes, an increase in the step size keeps the results identical, with the confidence interval computable using the stability function $R(z) = 1 + z + z^2/2 + z^3/6 + z^4/24$, where $z = h \cdot \lambda$ and $\lambda$ representing the eigenvalue of the system [45]. We find $R < 1$, which is the necessary bound we need for stable solutions. A change in the initial conditions also tests the stability of our solutions, and that is displayed clearly in Fig. 4.3b.

```
    # The Zeroth Order solutions are given by

def phi0_func(t):
    return (1/np.sqrt(3))*np.log(np.sqrt(3)*t + np.exp(np.sqrt(3)))
def a0(t):
    return (np.exp(np.sqrt(3)) + 3*t)**(1/3)
```

---

```
    # For the  fixed Compton Wavelength

def integrand(t):
    numerator = 1
    denominator = (np.exp(np.sqrt(3)) + 3 * t) * (3 + np.sqrt(3) *
        np.log(np.exp(np.sqrt(3)) + np.sqrt(3) * t))
    return numerator / denominator

def calculate_phi(t):
    result, _ = quad(integrand, 0, t)
    return result

t_values = np.linspace(0, 100, 500)

phi_values = [calculate_phi(t) for t in t_values]
```

---

```
    # For the fixed E_0 condition

a=0.0
b=100.0
N=500
dphi_IC = np.arange(0.0, 1.1, .1)
alp = []
# Initial Conditions a1(0) = 1, phi1(0) = 0, phi1_d(0) = [0, 1]
for i in range(len(dphi_IC)):
    alp.append([1.0, 0.0, dphi_IC[i]])

c = 0.1
m=3

h=(b-a)/N



def f(t,u1,u2,u3):
    a0 = (np.exp(np.sqrt(3)) + 3*t)**(1/3)
    H0 = 1/(np.exp(np.sqrt(3)) + 3*t)
    phi0 = (1/np.sqrt(3))*np.log(np.sqrt(3)*t + np.exp(np.sqrt(3)))
```

```
    d_phi0 = 1/(np.exp(np.sqrt(3)) + np.sqrt(3)*t)
    return [(u1/(3*H0))*phi0*u3, u3, -3*H0*u3 - 3*((1/(3*H0))*phi0*u3 -
        2*H0*(u1/a0))*d_phi0]

def early(t,u1,u2,u3):
    H0 = 1/(np.exp(np.sqrt(3)) + 3*t)
    return [u1*H0, u3, (6/(np.exp(np.sqrt(3))+3*t))*(1/(np.exp(np.sqrt(3)) +
        np.sqrt(3)*t))-(3/(np.exp(np.sqrt(3))+3*t))*u3]

def late(t,u1,u2,u3):
    H0 = 1/(np.exp(np.sqrt(3)) + 3*t)
    phi0 = (1/np.sqrt(3))*np.log(np.sqrt(3)*t + np.exp(np.sqrt(3)))
    return [(u1/3*H0)*phi0*u3, u3, -(1/H0)*phi0*u3]

def RKsys_3(func):
    t=a
    w=[[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],
        [0,0,0],[0,0,0],[0,0,0]]
    K1=[[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],
        [0,0,0],[0,0,0],[0,0,0]]
    K2=[[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],
        [0,0,0],[0,0,0]]
    K3=[[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],
        [0,0,0],[0,0,0]]
    K4=[[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],[0,0,0],
        [0,0,0],[0,0,0]]

    for k in range(len(alp)):
        for j in range(m):
            w[k][j] = alp[k][j]

    t_vals = [[a],[a],[a],[a],[a],[a],[a],[a],[a],[a],[a]]
    w1_vals = [[w[0][0]],[w[1][0]],[w[2][0]],[w[3][0]],[w[4][0]],[w[5][0]],
        [w[6][0]],[w[7][0]],[w[8][0]],[w[9][0]],[w[10][0]]]
    w2_vals = [[w[0][1]],[w[1][1]],[w[2][1]],[w[3][1]],[w[4][1]],[w[5][1]],
        [w[6][1]],[w[7][1]], [w[8][1]],[w[9][1]],[w[10][1]]]
    w3_vals = [[w[0][2]],[w[1][2]],[w[2][2]],[w[3][2]],[w[4][2]],[w[5][2]],
        [w[6][2]],[w[7][2]], [w[8][2]],[w[9][2]],[w[10][2]]]

    for k in range(len(alp)):
        for i in range(N+1):
        # print(w)
            for j in range(m):
                K1[k][j] = h*func(t, w[k][0], w[k][1], w[k][2])[j]
            for j in range(m):
                K2[k][j] = h*func(t+h/2, w[k][0]+K1[k][0]/2,
```

```
                w[k][1]+K1[k][1]/2, w[k][2]+K1[k][2]/2)[j]
        for j in range(m):
            K3[k][j] = h*func(t+h/2, w[k][0]+K2[k][0]/2,
                w[k][1]+K2[k][1]/2, w[k][2]+K2[k][2]/2)[j]
        for j in range(m):
            K4[k][j] = h*func(t+h, w[k][0]+K3[k][0], w[k][1]+K3[k][1],
                w[k][2] + K3[k][2])[j]
        for j in range(m):
            w[k][j] = w[k][j]+(K1[k][j]+2*K2[k][j]+2*K3[k][j]+K4[k][j])/6

        t=a+(i+1)*h

        t_vals[k].append(t)
        w1_vals[k].append(w[k][0])
        w2_vals[k].append(w[k][1])
        w3_vals[k].append(w[k][2])
return [t_vals, w1_vals, w2_vals, w3_vals]
```

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