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Categorical semi-direct products in varieties of groups with multiple operators

by

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Dedication

Lovingly dedicated to

My wife Antonine

and

our children Joshua, Joseph, Jeremiah and Joy.

Declaration

Thesis title: Categorical semi-direct products in varieties of groups
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Contents

Abstract

The notion of a categorical semidirect product was introduced by Bourn and Janelidze as a generalization of the classical semidirect product in the category of groups. The main aim of this work is to study the general properties of semidirect products of groups with operators, describe them in various classical varieties of such algebraic structures and apply the results to homological algebra and related areas of modern algebra.

The context in which the study is done is a semiabelian category (that is, a pointed, Barr-exact and Bourn-protomodular category). The main result in the thesis is the construction of the semidirect product in a variety Ω-**RLoop** of right Ω-loops as the product of underlying sets equipped with the Ω-algebra structure. A variety of right Ω-loops is a variety that is pointed, has a binary $+$ (not necessarily associative or commutative) and a binary $-$ satisfying the identities $0+x=x$, $x+0=x$, $(x+y)-y=x$ and $(x-y)+y=x$. Thus, Ω-**RLoop** is a generalization of the variety of Ω-groups introduced by Higgins and the results obtained are valid for varieties of Ω-loops.

We also describe precrossed and crossed modules in the variety Ω-**RLoop**. The theory of crossed modules developed is independent of that developed by Janelidze for crossed modules in an arbitrary semiabelian category and gives simplified explicit formulae for crossed modules in Ω-**RLoop**. Finally, we mention that our constructions agree with the known ones in the familiar algebraic categories, specifically the categories of groups, rings and Lie algebras.

Historical Background

In this section we give some historical background information about the categorical study of the properties of groups including the theory of semidirect products.

The setting of this problem is inspired by earlier developments in category theory. In Saunders Mac Lane's paper "Duality for groups" [28] published in 1950, the problem of studying categorically the properties of groups was considered for the first time. In the paper Mac Lane mentioned that there should be a "carefully chosen" set of axioms that allows for the various group-theoretical constructions and results. Mac Lane began by studying the case of abelian groups and this led to the theory of abelian categories. Recall [14] that an abelian category is pointed, has binary products and coproducts, has kernels and cokernels and is such that every monomorphism is a kernel and every epimorphism is a cokernel. Examples of abelian categories include abelian groups and modules but not groups in general.

In [7] Bourn introduced the notion of a protomodular category, whose outstanding example was the category of groups. Semiabelian categories, introduced in [19] by Janelidze, Marki and Tholen and published in 2002, provide a framework for a categorical treatment of groups (or more generally multioperator groups in the sense of Higgins [16]). Semiabelian categories are Barr-exact [1] categories with a zero object and binary coproducts in which the Short Five Lemma holds (we recall this terminology in the next chapter). In [8], Bourn and Janelidze introduced a categorical notion of semidirect product for categories where pulling back a split epimorphism along any morphism gives a monadic functor. Their definition has proved useful in studying semidirect products in many algebraic contexts. In this thesis we will work in a semiabelian setting. Such a setting is supplied by the theory of groups with multiple operators in the sense of Higgins (see [16]).

Chapter 0

Introduction

In [8], Bourn and Janelidze developed a categorical notion of semidirect product which involves the notion of action. They also showed that this notion is a generalization of the classical semidirect product in the category of groups. Let us first recall the definition of the semidirect product of groups. Consider two (not necessarily abelian) groups (B,+) and (X,+) with B acting on X, that is, there is a map $h : B \times X \longrightarrow X$ written as $h(b, x) = bx$ satisfying the conditions

$$\begin{aligned} 0x &= x, \\ b(b'x) &= (b + b')x, \\ b(x + x') &= bx + bx'. \end{aligned} \tag{0.1}$$

The semidirect product of B and X, denoted by $B \ltimes X$ or $B \ltimes (X, h)$, is the set $B \times X$ with the operation defined by

$$(b, x) + (b', x') = (b + b', x + bx'),$$

for all $b, b' \in B$ and $x, x' \in X$.

One of the main notions central to the definition and construction of the categorical semidirect product is the theory of monads and their algebras. The relevant background on this topic can be found in chapter 5 of [26].

This thesis has the following as its main objectives:

(a) To study the categorical notions of object actions (see [5]) and semidirect products in varieties of groups with multiple operators.

(b) To describe internal categorical structures in the varieties in (a). In particular, we shall describe precrossed modules, crossed modules and star-multiplicative graphs. Actions and semidirect products of crossed modules will also be investigated.

0.1 The semiabelian context

As pointed out earlier, the context in which our results are developed is that of semiabelian categories. Semiabelian categories were introduced in [19] so as to capture among other things the homological properties of the categories of groups, rings, Lie algebras, (pre-)crossed modules and similar non-abelian structures. A category **C** is semiabelian if it is finitely complete and cocomplete, pointed, Barr-exact and Bourn-protomodular [7]. Pointed means that it has a zero object: an initial object that is also terminal. A Barr-exact [1] category is regular (finitely complete with pullback-stable regular epimorphisms) and such that every internal equivalence relation is a kernel pair. A pointed category is Bourn-protomodular when it has pullbacks along split epimorphisms and the Split Short Five Lemma holds: this means that given a diagram

$$\begin{tikzcd}
K[f] \arrow[r, "\ker f"] \arrow[d, "\kappa"] & A \arrow[r, bend left, "f"] \arrow[d, "\alpha"] & B \arrow[l, "s"] \arrow[d, "\beta"] \\
K[f'] \arrow[r, "\ker f'"] & A' \arrow[r, "f'"] & B' \arrow[l, bend left, "s'"]
\end{tikzcd} \tag{0.2}$$

such that both squares commute, $fs = 1_B$ and $f's' = 1_{B'}$, the morphisms κ and β being isomorphisms implies that α is an isomorphism.

0.2 Semi-direct product

If $\mathbf{C}$ is a semiabelian category, a point over an object B of $\mathbf{C}$ is a triple (A, α, β) where $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow A$ are morphisms in $\mathbf{C}$ with $\alpha\beta = 1_B$. The points over B form a category $Pt(B) = Pt_{\mathbf{C}}(B)$ when we define a morphism

$$f : (A, \alpha, \beta) \longrightarrow (A', \alpha', \beta')$$

to be a morphism $f : A \longrightarrow A'$ for which $\alpha' f = \alpha$ and $f\beta = \beta'$ (see [7]). Upon choosing for each point (A, α, β) a definite kernel $\kappa : \ker A \longrightarrow A'$ of α, we get a functor

$$U : Pt(B) \longrightarrow \mathbf{C} \tag{0.3}$$

sending (A, α, β) to $\ker \alpha :$ this functor has a left adjoint sending X to

$$(B + X, [1, 0] : B + X \longrightarrow B, \iota : B \longrightarrow B + X)$$

(where ι is the coproduct injection), and is monadic (see [8]). We shall write T^B or $B\flat(-)$ for the corresponding monad on $\mathbf{C}$, its value at X being the kernel $B\flat X$ of

$$[1, 0] : B + X \longrightarrow B.$$

T^B-algebras will be called B-objects. Given a T^B-algebra (X,ξ), the corresponding action $\xi : B\flat X \longrightarrow X$ of the monad $B\flat(-)$ on the object X is called a B-action on X (see [5]).
The following definition generalizes the notion of semidirect product for groups.

0.2.1 Definition. *([8]) Let B be an object of a semiabelian category* $\mathbf{C}$ *and* T^B *the corresponding monad on* $\boldsymbol{C}$ *as in Equation 0.3. A semidirect product* $B \ltimes (X,\xi)$ *of the object B and a B-algebra* (X,ξ) *is the object in* $Pt(B)$ *corresponding to* (X,ξ) *under the canonical equivalence* $Pt(B) \approx C^{T^B}$ *of* T^B*-algebras.*

0.3 The results

The main result is:

0.3.1 Theorem. *Let* $\mathbf{C}$ *be a fixed variety of universal algebras which has, among its operations, a binary +, a binary − and a nullary 0 satisfying the identities*

$$\begin{aligned} x+0&=x,\\ 0+x&=x,\\ (x-y)+y&=x,\\ (x+y)-y&=x. \end{aligned} \tag{0.4}$$

Given an object B and a T^B*-algebra* (X,ξ)*, the semidirect product* $B \ltimes (X,\xi)$ *is the set-theoretical (cartesian) product* $B\times X$ *equipped with the following* Ω*-algebra structure:*

$$\omega((b_1,x_1),...,(b_n,x_n)) = (\omega(b_1,...,b_n), \xi(\omega(x_1+b_1,...,x_n+b_n)-\omega(b_1,...,b_n))) \tag{0.5}$$

for each n-ary operation $\omega\in\Omega$ *and* $for\ all$ $b_1,...,b_n\in B,\quad x_1,...,x_n\in X.$

To prove the theorem we use the following approach: For an arbitrary split extension

$$\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, bend left, "\alpha"] & B \arrow[l, bend left, "\beta"] \end{tikzcd}$$

in $\mathbf{C}$ we exhibit a bijection between A and $B \times X$ by defining $\varphi(b,x) = \kappa(x) + \beta(b)$ and $\psi(a) = (\alpha(a), \kappa^{-1}(a - \beta\alpha(b)))$ as shown in the bottom square (ii) in the following diagram.

$$\begin{tikzcd}
B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"] \arrow[dr, phantom, "\texttt{(i)}"] & B + X \arrow[r, bend left, "{[1,0]}"] \arrow[d, "{[\beta,\kappa]}"] & B \arrow[l, "\iota_1"'] \arrow[d, equal] \\
X \arrow[r, "\kappa"] \arrow[d, equal] \arrow[dr, phantom, "\texttt{(ii)}"] & A \arrow[r, bend left, "\alpha"] \arrow[d, bend right, "\psi"'] & B \arrow[l, "\beta"'] \arrow[d, equal] \\
X \arrow[r, "<0{,}1>"'] & B \times X \arrow[u, bend right, "\varphi"'] \arrow[r, bend right, "\pi_1"'] & B \arrow[l, "<1{,}0>"]
\end{tikzcd} \tag{0.6}$$

Then we observe that the bijection φ gives the semidirect product if and only if the top square (i) in the diagram commutes. Using the commutativity of the square (i) we calculate the operations on $B \times X$.

Using Theorem 0.3.1 we give the following description of crossed modules in $\mathbf{C}$:

0.3.2 Proposition. *A crossed module in* $\mathbf{C}$ *can be equivalently defined as a quadruple* (B, X, ξ, δ) *in which* $(B, X, \xi) \in$ *Act*$(\mathbf{C})$ *and* $\delta : X \longrightarrow B$ *is a morphism such that for an* n*-ary operation* $\omega \in \Omega$

(a)

$$\omega(\delta(x_1)+b_1,...,\delta(x_n)+b_n)-\omega(b_1,...,b_n)$$
$$=\delta(\xi(\omega(x_1+b_1,...,x_n+b_n)-\omega(b_1,...,b_n))) \tag{0.7}$$

(b)

$$\xi(\omega(x_1'+(\delta(x_1)+b_1),...,x_n'+(\delta(x_n)+b_n))-\omega(\delta(x_1)+b_1,...,\delta(x_n)+b_n))+$$
$$\xi(\omega(x_1+b_1,...,x_n+b_n)-\omega(b_1,...,b_n))$$
$$=\xi(\omega((x_1'+x_1)+b_1,...,(x_n'+x_n)+b_n)-\omega(b_1,...,b_n)) \tag{0.8}$$

for all $b_1,...,b_n,b_1',...,b_n'\in B$ *and* $x_1,...,x_n,x_1',...,x_n'\in X$.

The following definitions of a precrossed module and a crossed module in an arbitrary semiabelian category were given by Janelidze in [18].

0.3.3 Definition. *An internal precrossed module in* $\mathbf{C}$ *is a 4-tuple* (B,X,ξ,δ) *with* (B,X,ξ) *in* $Act(\boldsymbol{C})$ *and* $\delta : X \longrightarrow B$ *a morphism in* $\mathbf{C}$ *such that the diagram*

$$\begin{tikzcd} B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B+X \arrow[d, "{[1,\delta]}"] \\ X \arrow[r, "\delta"] & B \end{tikzcd} \tag{0.9}$$

commutes. A morphism $(B,X,\xi,\delta)\longrightarrow(B',X',\xi',\delta')$ *is a morphism* (g,h) : $(B,X,\xi,\delta)\longrightarrow(B',X',\xi',\delta')$ *in* $Act(\boldsymbol{C})$ *such that* $g\delta=\delta'h$.

0.3.4 Definition. *An internal crossed module in* $\mathbf{C}$ *is an internal precrossed* *module* (B, X, ξ, δ) *in* $\boldsymbol{C}$ *for which the diagram*

$$\begin{tikzcd}
(B+X)\flat X \arrow[r, "{[1_B,\delta]\flat 1_X}"] \arrow[d, "{[1_{B+X},\iota_2]^\sharp}"'] & B\flat X \arrow[d, "\xi"] \\
B\flat X \arrow[r, "\xi"'] & X
\end{tikzcd} \tag{0.10}$$

commutes. Here $[1_{B+X}, \iota_2]^\sharp$ *is the unique morphism such that* $\kappa_{B,X}[1_{B+X}, \iota_2]^\sharp = [1_{B+X}, \iota_2]\kappa_{B+X,X}$.

The main observation is:

0.3.5 Remark. *In the category* $\mathbf{C}$ *of right* Ω*-loops, the description of crossed modules using Equation 0.8 is much simpler than Definition 0.3.4 and gives the usual (Peiffer) condition for crossed modules in groups much more easily.*

0.4 Structure of the thesis

The following is a brief outline of the contents of the thesis.
Chapter 1: contains the necessary categorical and universal algebra background needed for the statements and proofs of the theorems.
Chapter 2: In this chapter we construct the semidirect product in the variety $\mathbf{C}$ of right Ω-loops. We then apply the construction to describe precrossed and crossed modules and star-multiplicative graphs in $\mathbf{C}$. In the last section of the chapter we construct semidirect products of crossed modules in $\mathbf{C}$.
Chapter 3: This chapter is devoted to examples and some observations on the results obtained in the study. In particular, we show that our notions of semidirect product and crossed modules in $\mathbf{C}$ cover the classical examples of groups, rings and Lie algebras.

Chapter 1

Preliminary notions and results.

This aim of this chapter is to keep the thesis reasonably self-contained by fixing the notation and introducing the concepts which will be used later in the work.

Section 1.1 is a brief review of the theory of adjoint functors. In Section 1.2 we recall notions and theorems on aspects of monads including a statement of Beck's Theorem. The principal sources for these two sections are books by Barr and Wells [2], S. Mac Lane [26] and F. Borceux [3] to which the reader is referred for further details. Relevant notions from universal algebra are presented in section 1.3. In Section 1.4 we review the basic elements about the semidirect product, pointing out that every split epimorphism in the category of groups is a semidirect product projection. In Section 1.5 we introduce the notion of an internal category.

It is well-known that the concept of an internal category in **Groups** is equivalent to that of a cat^1-group and to that of a crossed module of groups (see, for example, J.L. Loday [24]). These equivalences are reviewed in Section 1.6 and Section 1.7. We describe the appropriate categorical context in which we will be working semiabelian categories in section 1.8. Such a setting is supplied by the varieties

Ω-groups or, more generally, varieties of Ω-loops.

1.1 Adjunctions.

See F. Borceux [3] or S. Mac Lane [26].

1.1.1 Definition. *Let $\mathbb{X}$ and $\mathbb{A}$ be categories. An adjunction from $\mathbb{X}$ to $\mathbb{A}$ is triple*

$$(F, U, \varphi) : \mathbb{X} \to \mathbb{A}$$

where F and U are functors

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "U"] & \mathbb{X} \arrow[l, bend left, "F"] \end{tikzcd}$$

while φ is a function that assigns to each pair of objects $X \in \mathbb{X}, A \in \mathbb{A}$ a bijection of sets

$$\varphi = \varphi_{X,A} : hom_{\mathbb{A}}(F(X), A) \cong hom_{\mathbb{X}}(X, U(A))$$

which is natural in X and A.

1.1.2 Remark. *Given such an adjunction we write $F \dashv U$; the functor F is said to be a left adjoint for U, while U a right adjoint for for F.*

1.1.3 Theorem. *Given functors*

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "U"] & \mathbb{X} \arrow[l, bend left, "F"] \end{tikzcd},$$

specifying an adjunction $(F, U, \varphi) : \mathbb{X} \to \mathbb{A}$ is equivalent to specifying natural transformations $\eta : 1_{\mathbb{X}} \to UF$ and $\varepsilon : FU \to 1_{\mathbb{A}}$ such that the diagrams

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FUF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd}$$

$$\begin{tikzcd} U \arrow[r, "\eta U"] \arrow[dr, equal] & UFU \arrow[d, "U\varepsilon"] \\ & U \end{tikzcd}$$

commute. These η *and* ε *are defined by* $\eta_X = \varphi_{X,F(X)}(1_{F(X)})$ *and* $\varepsilon_A = \varphi^{-1}{}_{U(A),A}(1_{U(A)})$ *for all* $X \in \mathbb{X}$ *and* $A \in \mathbb{A}$*. Conversely, given* η *and* ε*, the map* $\varphi_{X,A}$ *can be calculated by* $\varphi_{X,A}(f) = U(f)\eta_X$ *or* $\varphi^{-1}_{X,A}(u) = \varepsilon_A F(A)$*, for all* $f : F(X) \to A$ *and* $u : X \to U(A)$*.*

Hence we also denote the adjunction (F, U, φ) by $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$.

1.2 Monads and algebras.

The main references here are [26] and [2].

1.2.1 Definition. *A* monad $T = (T, \eta, \mu)$ *in a category* X *consists of a functor* $T : \mathbb{X} \to \mathbb{X}$ *and two natural transformations* $\eta : 1_{\mathbb{X}} \to T$, $\mu : T^2 \to T$ *which make the following diagrams commute:*

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

1.2.2 Theorem. *Every adjunction* $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$ *gives rise to a monad* $(UF, \eta, U\varepsilon F)$ *in the category* $\mathbb{X}$*, and every monad arises this way.*

1.2.3 Definition. *Let* $T = (T, \eta, \mu)$ *be a monad in a category* $\mathbb{X}$. *A* T*-algebra is a pair* (X, ξ) *in which* X *is an object in* $\mathbb{X}$ *and* $\xi : T(X) \to X$ *is a morphism making the diagram*

$$\begin{tikzcd} T^2(X) \arrow[r, "\mu_X"] \arrow[d, "T(\xi)"'] & T(X) \arrow[d, "\xi"'] & X \arrow[l, "\eta_X"'] \arrow[dl, equal] \\ T(X) \arrow[r, "\xi"'] & X & \end{tikzcd}$$

commute. A morphism $h : (X, \xi) \to (X', \xi')$ *of* T*-algebras is a morphism* $h : X \to X'$ *making the diagram*

$$\begin{tikzcd} T(X) \arrow[r, "T(h)"] \arrow[d, "\xi"'] & T(X') \arrow[d, "\xi'"] \\ X \arrow[r, "h"'] & X' \end{tikzcd}$$

commute. We denote the category of T*-algebras by* $\mathbb{X}^T$*; it also is called the Eilenberg-Moore category of* T.

1.2.4 Theorem. *There is an adjunction*

$$(F^T, U^T, \eta^T, \varepsilon^T) : \mathbb{X} \to \mathbb{X}^T$$

in which the U^T *and* F^T *are given by*

$$U^T : \begin{tikzcd} (X, h) \arrow[r, mapsto] \arrow[d, "f"'] & X \arrow[d, "f"'] \\ (X', h') \arrow[r, mapsto] & X' \end{tikzcd}$$

and

$$F^T : \begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & (T(X), \mu_X) \arrow[d, "T(f)"'] \\ X' \arrow[r, mapsto] & (T(X'), \mu_{X'}) \end{tikzcd}$$

respectively, $\eta_T = \eta$, *and* $\varepsilon^T_{(X,h)} = h$ *for each algebra T-algebra* (X, h). *The monad in* $\mathbb{X}$ *determined by this adjunction is the given monad* (T, η, μ).

1.2.5 Examples. *(a) Let* G *be a group. Then for any set* X *the assignments*

$$TX = G \times X, \qquad \eta_X : X \to G \times X, \quad x \longmapsto (1, x),$$

$$\mu_X : G \times (G \times X) \longrightarrow G \times X, \quad (g_1, (g_2, x)) \longmapsto (g_1 g_2, x)$$

for $x \in X$ *and* $g_1, g_2 \in G$, *define a monad* (T, η, μ) *on the category* ***Sets***. *A T-algebra is then a set* X *together with a function* $h : G \times X \to X$ *such that writing* gx *for* $h(g, x)$, *we get the usual conditions that* $(g, x) \mapsto gx$ *defines an action of the group* G *on the set* X. *Hence in this case T-algebras are just the group actions.*

(b) Let R *be a ring with unit. Then for each abelian group* A *the assignments*

$$TA = R \bigotimes A, \qquad \eta_A : A \longrightarrow R \bigotimes A, \; a \mapsto 1 \bigotimes a,$$

$$\mu_A : R \bigotimes (R \bigotimes A) \longrightarrow R \bigotimes A, \quad r_1 \bigotimes (r_2 \bigotimes a) \longmapsto r_1 r_2 \bigotimes a$$

for $a \in A$, $r_1, r_2 \in R$ *define a monad on* ***Ab****. The category* $\boldsymbol{Ab}^T$ *of T-algebras is the category* ***R-Mod****, of left R-modules.*

1.2.6 Theorem. *Let* $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$ *be an adjunction. Then*

- $T = (UF, \eta, U\varepsilon F)$ *defines a monad on* X,
- *there is a functor* $K : \mathbb{A} \to \mathbb{X}^T$, $\quad K(A) = (U(A), U(\varepsilon_A))$ *with* $U^T K = U$ *and* $KF = F^T$ *as depicted in the commutative diagram*

$$\begin{tikzcd}
& \mathbb{A} \arrow[dl, shift left, "U"] \arrow[dr, dashed, "K"] & \\
\mathbb{X} \arrow[ur, shift left, "F"] \arrow[rr, shift right, "F^T"'] & & \mathbb{X}^T \arrow[ll, shift right, "U^T"']
\end{tikzcd}.$$

1.2.7 Definition. *Let* $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$ *and* $T = (T, \eta, \mu)$ *be the corresponding monad as in Theorem 1.2.6 . Then*

(a) the functor $K : \mathbb{A} \to \mathbb{X}^T$, $\quad K(A) = (U(A), U(\varepsilon_A))$ is called the comparison functor;
(b) the functor $U : A \to \mathbb{X}$ is said to be monadic if the functor $K : \mathbb{A} \to \mathbb{X}^T$ above is a category equivalence.

Before formulating Beck's theorem we need the following definitions (see [26])

1.2.8 Definition. *(a) A fork is a diagram of the form*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

with $hf = hg$.

(b) A parallel pair $d, c : A \to B$ is called reflexive if there is an arrow e such that $de = ce = 1_B$;

$$\begin{tikzcd} A \arrow[r, bend left=50, "d"] \arrow[r, bend right, "c"'] & B \arrow[l, "e"'] \end{tikzcd}$$

(c) A split coequalizer diagram (or split, fork or contractible coequalizer diagram [2]) is a diagram

$$\begin{tikzcd} A \arrow[r, bend left=50, "d"] \arrow[r, bend right, "c"'] & B \arrow[l, "e"'] \arrow[r, "p"] & C \arrow[l, bend left, "i"] \end{tikzcd}$$

which satisfies the conditions

$$pd = pc,\ de = 1_B,\ pi = 1_C,\ ce = ip.$$

1.2.9 Remark. *A coequalizer diagram is said to be absolute if it is preserved by any functor. Since a split fork is defined by equations involving only composites and identities, it is preserved by any functor. Hence every split fork is an absolute coequalizer diagram.*

1.2.10 Definition. *Let $U : \mathbb{A} \to \mathbb{X}$ be a functor and let $r, p : A \to B$ be a parallel pair of morphisms in $\mathbb{A}$. Then this pair is called a U-split pair if there is a split coequalizer diagram*

$$\begin{tikzcd} U(A) \arrow[r, bend left, "U(r)"] \arrow[r, bend right, "U(p)"'] & U(B) \arrow[l, "t"'] \arrow[r, "q"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

in $\mathbb{X}$.

We now state Beck's monadicity criterion (see, for example, [26],[2])

1.2.11 Theorem (Beck's Monadicity Theorem). *Let $\mathbb{A}$ and $\mathbb{X}$ be categories. A functor $U : \mathbb{A} \to \mathbb{X}$ is monadic if and only if the following conditions hold:*

(a) U has a left adjoint;

(b) U reflects isomorphisms;

(c) $\mathbb{A}$ has coequalizers of U-split pairs and U preserves them.

1.2.12 Remark. *Let $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$ and $T = (UF, \eta, U\varepsilon F)$ be as in Theorem 1.2.6. Suppose that for every T-algebra (X, ξ) the pair $(\varepsilon_{F(X)}, F(\xi))$ has a coequalizer in $\mathbb{A}$. Then the comparison functor has a left adjoint whose object part $L(X, \xi) = Q$ is defined via the coequalizer diagram*

$$\begin{tikzcd} FUF(X) \arrow[r, bend left, "\varepsilon_{F(X)}"] \arrow[r, bend right, "F(\xi)"'] & F(X) \arrow[r, "\pi_{(X,\xi)}"] & L(X, \xi) \end{tikzcd} \tag{1.1}$$

For a morphism $g : (X, \xi) \to (X', \xi')$ one then defines L(g) to be the unique *morphism making the right-hand square of the diagram*

$$\begin{tikzcd} FUF(X) \arrow[r, bend left, "\varepsilon_{F(X)}"] \arrow[r, bend right, "F(\xi)"'] \arrow[d, "FUF(g)"'] & F(X) \arrow[r, "\pi_{(X,\xi)}"] \arrow[d, "F(g)"] & L(X,\xi) \arrow[d, dotted, "L(g)"] \\ FUF(X') \arrow[r, bend left, "\varepsilon_{F(X')}"] \arrow[r, bend right, "F(\xi')"'] & F(X') \arrow[r, "\pi_{(X',\xi')}"] & L(X',\xi') \end{tikzcd}$$

commute.

1.2.13 Remark. *The pair* $(\varepsilon_{F(X)}, F(\xi))$ *involved in the coequalizer diagram* (1.1) *is a reflexive coequalizer since* $\varepsilon_{F(X)}F(\eta_X) = 1_{F(X)}$ *and* $F(\xi)F(\eta_X) = 1_{F(X)}$*; that is,* $\varepsilon_{F(X)}$ *and* $F(\xi)$ *are split epimorphisms with a common splitting* $F(\eta)$*.*

In Theorem 1.2.11 condition (c) can be replaced with the requirement of existence of reflexive coequalizers in $\mathbb{A}$ preserved by U. We have:

1.2.14 Theorem (Beck's Monadicity Theorem; reflexive form). *A functor* $U : \mathbb{A} \to \mathbb{X}$ *is monadic if*

(a) *U has a left adjoint;*

(b) *U reflects isomorphisms;*

(c) ***A*** *has coequalizers of reflexive pairs;*

(d) *U preserves coequalizers of reflexive pairs.*

1.3 Varieties of universal algebras.

In this section we recall some notions from universal algebra to be used in the sequel. For general reference the reader is referred to [12], [13] and [15].

1.3.1 Definition. *Let* Ω *be a set equipped with a map* $\ell : \Omega \longrightarrow \{0, 1, 2, ...\}$, *and let* $\Omega_0 = \ell^{-1}(0)$, $\quad \Omega_1 = \ell^{-1}(1)$, $\quad \Omega_2 = \ell^{-1}(2), ...$ *; such a pair* (Ω, ℓ) *is called a signature. An* (Ω, ℓ)*-algebra is a pair* (A, ν) *in which* A *is a set, and*

$$\nu = (\nu_\omega : A^{\ell(\omega)} \longrightarrow A)_{\omega \in \Omega}$$

is a family of operations on A.

1.3.2 Remark.

1. *We will say that* A *is the underlying set of* (A, ν), *and that* ν *is the algebra structure of* (A, ν) , *or that* ν *is an algebra structure on* A.

2. *We will use the abbreviations* $\Omega = (\Omega, \ell)$, $\quad A = (A, \nu)$ *and* $\omega(a_1, a_2, ..., a_n) = \nu_\omega(a_1, a_2, ..., a_n)$ *to simplify the notation.* (Ω, ℓ)*-algebras are also called universal algebras.*

1.3.3 Definition. *If* A *and* B *are* Ω*-algebras, a homomorphism is a mapping* $f : A \longrightarrow B$ *with*

$$f(\omega(a_1, a_2, ..., a_n)) = \omega(f(a_1), f(a_2), ..., f(a_n))$$

for every $n = 0, 1, 2, ...,$ *for every* $\omega \in \Omega_n$, *and for every* $a_1, a_2, ..., a_n$ *in* A.

The category of all Ω-algebras and their homomorphisms will be denoted by Ω-**Alg**.

1.3.4 Definition. *Given an* Ω*-algebra* A *and subset* X *in* A, *we say that* X *is a subalgebra of* A *if* X *is closed in* A *under all operations of the* Ω*-algebra structure; that is to say,*

$$a_1, a_2, ..., a_n \in X \Rightarrow \omega(a_1, ..., a_n) \in X$$

for every $n = 0, 1, 2, ...,$ *for every* $\omega \in \Omega_n$, *and for every* $a_1, ..., a_n$ *in* A. *In this case* X *itself becomes an* Ω*-algebra under the induced operations.*

1.3.5 Definition. *Let* $(A_i)_{i\in I}$ *be a family of* Ω*-algebras. The (cartesian) product* $\prod_{i\in I} A_i$ *of the family* $(A_i)_{i\in I}$ *is defined as the product of the underlying sets, with the* Ω*-algebra structure defined componentwise; that is,*

$$\omega((a_{1i})_{i\in I}, ..., (a_{ni})_{i\in I}) = (\omega(a_{1i}, ..., a_{ni}))_{i\in I}. \tag{1.2}$$

We will write

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times ... \times A_n$$

when the family is presented as a sequence $A_1, A_2, ..., A_n$. In particular, for the product $A \times B$ of two algebras A and B we have

$$\omega((a_1, b_1), ..., (a_n, b_n)) = (\omega(a_1, ..., a_n), \omega(b_1, ..., b_1)). \tag{1.3}$$

1.3.6 Definition. *A variety of universal algebras is a full subcategory of the category of all* Ω*-algebras (for fixed* Ω*) closed under products, subalgebras and quotient algebras.*

We now illustrate our definition of a variety of universal algebras by presenting a number of examples, many of which will be used later.

1.3.7 Example (Semigroups and monoids). *1. A semigroup is an algebra* $(A, \cdot)$ *with one binary operation* $\cdot$ *such that* $(a.b).c = a.(b.c)$ *for all* $a, b, c \in A$.

2. A monoid is an algebra $(M, ., 1)$ *with a binary operation* $.$ *and a nullary operation 1 satisfying the identities* $(a.b).c = a.(b.c)$ *and* $a.1 = 1.a = a$.

1.3.8 Example (Groups). *A group* G *is an* Ω*-algebra with* $\Omega = \{., ^{-1}, 1\}$ *where* $.$ *is binary, 1 is nullary and* $^{-1}$ *is unary. Corresponding to the group axioms we have the following identities:*

G1. $(a.b).c = a.(b.c)$

G2. $a.1 = 1.a = a$

G3. $a.a^{-1} = a^{-1}.a = 1$

If $a.b = b.a$ *for all* $a, b \in G$ *the group is said to be abelian.*

1.3.9 Example (Rings). *A ring* $(R, +, \cdot, -, 0)$ *is an algebra with two binary operations +,. and a nullary operation 0 such that*

(a) $(R, +, -, 0)$ *is an abelian group*

(a) $(R, .)$ *is a semigroup*

(c) $a.(b + c) = a.b + a.c$ *and* $(b + c).a = b.a + c.a$ *for all* $a, b, c \in R$.

A ring with unit is an algebra $(R, +, \cdot, -, 0, 1)$ *such that* $(R, +, ., -, 0)$ *is a ring and (R,.,1) is a monoid.*

1.3.10 Example (Quasigroups and loops). *A quasigroup is an algebra* $(Q, \cdot, \backslash, /)$ *with three binary operations which satisfy the laws:*

$$x.(x\backslash y) = y, \quad x\backslash(x.y) = y \tag{1.4}$$

$$(x/y).y = x, \quad (x.y)/y = x \tag{1.5}$$

A loop is a quasigroup $(Q, \cdot, \backslash, /)$ *which admits an identity element 1 satisfying* $1.x = x.1 = x$. *A right quasigroup is a set with multiplication and right division / satisfying the identities* (1.5). *A right loop is a right quasigroup with an identity element 1.*

1.3.11 Example (varieties of Ω-groups). *A variety of* Ω*-groups [16] is a variety that has among its operations and identities those of the variety of groups and for any n-ary operation* ω, $\omega(1, 1, ..., 1) = 1$, *where 1 denotes the unit of the group operation. The varieties of groups, abelian groups, (non-unital) rings, commutative algebras, Lie algebras, precrossed modules and crossed modules are all examples of* Ω*-groups.*

1.3.12 Example (Ω-loops). *A variety* $\mathbf{C}$ *of* Ω*-loops [16] is a variety that has among its operations the binary operations* $.,\backslash,/$ *for which the loop identities are satisfied in* $\mathbf{C}$, *and for any* n*-ary operation* $\omega \in \Omega$, *one has the identity* $\omega(1,...,1) = 1$, *where 1 denotes the unit of the loop operation. Note that the variety of* Ω*-groups is a subvariety of the variety of* Ω*-loops.*

1.3.13 Definition and Examples. *[32] A variety* $\mathbf{C}$ *of universal algebras is Maltsev if its operations allow to define a ternary operation* $p(x,y,z)$ *satisfying the identities* $p(x,x,y) = y$ *and* $p(x,y,y) = x$.

Among the examples of Maltsev varieties are groups, where a Maltsev operation is given by $p(x,y,z) = x.y^{-1}.z$, *abelian groups, rings, Lie algebras and crossed modules. More generally, every variety of* Ω*-groups is Maltsev. The variety of quasigroups is also Maltsev; a Maltsev operation is given by* $p(x,y,z) = (x/(y\backslash y)).(y\backslash z)$.

Let us recall a classical result on Maltsev varieties (see [29], [32]).

1.3.14 Theorem. *Let* $\mathbf{C}$ *be a variety. Then the following conditions are equivalent:*

1. $\mathbf{C}$ *is a Maltsev variety;*
2. *any reflexive homomorphic relation is a congruence;*
3. *for any congruences* R *and* S *on any algebra* A, $R \bigvee S = R \circ S$.

1.3.15 Definition. *[10] A finitely complete* $\mathbf{C}$ *category is is a Maltsev category if any internal reflexive relation in* $\mathbf{C}$ *is an equivalence relation.*

1.4 Semidirect products.

In this section we review the basic elements about the semidirect product of groups. We will use additive notation for all groups.

1.4.1 Definition. *Let B be a group. Then a B-group is a group X together with a homomorphism $\varphi : B \to Aut(X)$,written as $\varphi(b)(x) = bx$.*
Equivalently, a B-group can be defined as a pair (X, m) consisting of a group X and an action map $m : B \times X \to X$ written as $m(b, x) = bx$ and satisfying the axioms
$0x = x$,$b(b'x) = (b + b')x$, $b(x + x') = bx + bx'$.

1.4.2 Remark. *The equalities $b0 = 0$, $b(-x) = -bx$ for all elements $b \in B$ and $x \in X$ are easy consequences of the definition.*

1.4.3 Proposition. *Let B be a group and X be a B-group.The set $B \times X$, equipped with the addition*

$$(b, x) + (b', x') = (b + b', x + bx'),$$

is a group, called the semidirect product of B and X with action φ. It is denoted by $B \ltimes X$ or $B \ltimes (X, \varphi)$ (or $B \ltimes_\varphi X$).

Proof. Straightforward verifications (see [6]):
The addition is easily shown to be associative, $-(b, x) = (-b, -((-b)x))$ and the unit for addition is $(0, 0)$. □

We shall need the following classical result.

1.4.4 Theorem. *Every split epimorphism of groups is, up to isomorphism, a semidirect product projection.*

Proof. Let

$$\begin{tikzcd} A \arrow[r, bend left, "\alpha"] & B \arrow[l, bend left, "\beta"] \end{tikzcd}, \qquad \alpha\beta = 1_B$$

be a split epimorphism in the category of groups. We will show that $A \cong B \ltimes X$ where $X = ker(\alpha)$ and the action of B on X is given by $bx = \beta(b) + x - \beta(b)$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, bend left, "\alpha"] \arrow[dr, dashed, "\theta"'] & B \arrow[l, bend left, "\beta"] \arrow[d, bend left, "\iota_1"] \\ & & B \times X \arrow[u, bend left, "\pi_1"] \end{tikzcd}$$

where κ is the inclusion of X in A, $\pi_1(b, x) = b$ and $\iota_1(b) = (b, 0)$. Define the map

$$\theta : A \to B \times X$$

by

$$\theta(a) = (\alpha(a), a - \beta\alpha(a)).$$

We need to show that θ is an isomorphism. We see that

$$\begin{aligned} \theta(a) + \theta(a') &= (\alpha(a), a - \beta\alpha(a)) + (\alpha(a'), a' - \beta\alpha(a')) \\ &= (\alpha(a) + \alpha(a'), a - \beta\alpha(a) + \beta\alpha(a) + a' - \beta\alpha(a') - \beta\alpha(a)) \\ &= (\alpha(a) + \alpha(a'), a + a' - \beta\alpha(a') - \beta\alpha(a)) \\ &= (\alpha(a + a'), a + a' - \beta\alpha(a + a')) = \theta(a + a'). \end{aligned}$$

Next, define $\phi : B \times X \to A$ by $\phi(b, x) = x + \beta(b)$. We have to show that θ and ϕ are inverses of each other.

$$\begin{aligned} \phi(\theta(a)) &= \phi(\alpha(a), a - \beta\alpha(a)) \\ &= a - \beta\alpha(a) + \beta(\alpha(a)) \\ &= a - \beta\alpha(a) + \beta\alpha(a) = a. \end{aligned}$$

This shows that $\phi\theta = 1_A$.

$$
\begin{aligned}
\theta(\phi(b,x)) &= \theta(x+\beta(b)) \\
&= (\alpha(x+\beta(b)), x+\beta(b)-\beta\alpha(x+\beta(b))) \\
&= (\alpha(x)+\alpha\beta(b), x+\beta(b)-\beta\alpha(x)-\beta\alpha\beta(b)) \\
&= (0+b, x+\beta(b)-0-\beta(b)) = (b,x).
\end{aligned}
$$

That is, $\theta\phi = 1_{B\times X}$. This proves that θ is an isomorphism. Hence $A \cong B \ltimes X$ as required. □

1.5 Internal categories.

For the general theory of internal categories, see [3] and [26].

Note for future reference:

1.5.1 Definition. *An internal category* $C = (C_0, C_1, d, c, e, m)$ *in a category* $\mathbb{C}$ *with pullbacks is given by a diagram*

$$
\begin{tikzcd}
C_2 \arrow[r, bend left, "\pi_2"] \arrow[r, "m"] \arrow[r, bend right, "\pi_1"'] & C_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & C_0 \arrow[l, "e"']
\end{tikzcd}
$$

in which

(a) C_0 *and* C_1 *are objects in* $\mathbb{C}$ *called the object-of-objects and the object-of-morphisms respectively;*

(b) d *and* c *are morphisms from* C_1 *to* C_0 *called the domain and codomain in* C *respectively;*

(c) e *is a morphism from* C_0 *to* C_1 *called the identity in* C*;*

(d) $de = 1_{C_0} = ce$;

(e) $C_2, \quad \pi_1, \quad \pi_2$ *are given by the pullback*

$$\begin{tikzcd} C_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd}$$

;

(f) m *is a morphism from the pullback* $C_2 = C_1 \times_{C_0} C_1$ *to* C_1;

(g) $dm = d\pi_2, \quad cm = c\pi_1$;

(h) composition m *satisfies the associative and unit laws, namely* $m(1_{C_1} \times m) = m(m \times 1_{C_1}), \qquad m\langle 1_{C_1}, ed\rangle = 1_{C_1} = m\langle ec, 1_{C_1}\rangle$.

1.5.2 Definition. *An internal functor* $f : C \to B$ *between two internal categories* C, B *in* $\mathbb{C}$ *is given by two morphisms* $f_0 : C_0 \to B_0$, $f_1 : C_1 \to B_1$ *in* $\mathbb{C}$ *which make the following diagrams commute:*

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "f_1 \times f_1"] \arrow[d, "m"'] & B_1 \times_{B_0} B_1 \arrow[d, "m'"] \\ C_1 \arrow[r, "f_1"'] & B_1 \end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[r, "e"] \arrow[d, "f_0"] & C_1 \arrow[d, "f_1"] \\ B_1 \arrow[r, bend left, "d'"] \arrow[r, bend right, "c'"'] & B_0 \arrow[r, "e'"] & B_1 \end{tikzcd}$$

Composition of internal functors is defined in the obvious way. Hence one obtains $\mathbf{Cat}(\mathbb{C})$, the category of all internal categories and internal functors in $\mathbb{C}$.

1.5.3 Examples. *(a) An internal category in **Sets** is an ordinary (small) category.*

(b) An internal category in the category of groups is an ordinary category (C_0, C_1, d, c, e, m), equipped with group structures on C_0 and C_1, such that the maps d, c, e and m are group homomorphisms; the same is true for all varieties of universal algebras.

(c) An internal category in the category of topological spaces is an ordinary category (C_0, C_1, d, c, e, m) equipped with topologies on C_0 and C_1, such that the maps d, c, e and m are continuous.

1.5.4 Remark. *In [17] it is observed that for any composable pair (g,f)=*

$$x \xrightarrow{f} y \xrightarrow{g} z$$

in a Maltsev variety, we have $gf = p(g, 1_y, f)$, *where p is the Maltsev operation.*

A description of internal categories and internal groupoids in a Maltsev variety is also given in [17] and [20]. The main result is the following theorem:

1.5.5 Theorem. *Let*

$$G \;=\; \begin{tikzcd} G_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & G_0 \arrow[l, "e"'] \end{tikzcd}$$

be an internal reflexive graph in a Maltsev variety $\mathbf{C}$. *The following conditions are equivalent:*

(a) there exists an internal groupoid in $\mathbf{C}$ *where whose underlying internal reflexive graph is G;*

(b) there exists a unique internal groupoid in $\mathbf{C}$ *whose underlying internal reflexive graph is G;*

(c) there exists an internal category in $\mathbf{C}$ *whose underlying internal reflexive graph is G;*

(d) there exists a unique internal category in $\mathbf{C}$ *whose underlying internal reflexive graph is G;*

(e) for any $\omega \in \Omega_n$ *and* $(g_1, f_1), ..., (g_n, f_n) \in (G_1 \times_{G_0} G_1)$,

$$p(\omega(g_1, ..., g_n), \omega(1_{y_1}, ..., 1_{y_n}), \omega(f_1, ..., f_n)) = \omega(p(g_1, 1_{y_1}, f_1), ..., p(g_n, 1_{y_n}, f_n)), \tag{1.6}$$

where $y_i = d(g_i) = c(f_i)$ $(i = 1, ..., n)$, *and* p *is the Maltsev operation.*

1.6 Crossed modules.

Crossed modules were introduced by J.H.C Whitehead [34] in his investigation of the algebraic structure of relative homotopy groups.

1.6.1 Definition. *A crossed module* (A, B, δ) *of groups is a homomorphism* $\delta : A \to B$ *together with an action of the group* B *on* A, *written as* ba *satisfying the axioms*

(a) $\delta(a)a' = a + a' - a$

(b) $\delta(ba) = b + \delta(a) - b$

for all $a, a' \in A$ *and* $b \in B$.

1.6.2 Definition. *A morphism of crossed modules* $(f_0, f_1) : (A, B, \delta) \to (A', B', \delta')$ *is a pair of homomorphisms* $f_0 : A \to A'$ *and* $f_1 : B \to B'$,*such that*

(a) $f_1\delta = \delta' f_0$

(b) $f_0(ba) = f_1(b)f_0(a)$

for all $a \in A$ *and* $b \in B$*.*

We will denote by **XMod**, or by **XMod**(**Groups**), the category of crossed modules and morphisms between them.

1.6.3 Examples. *(a) Any normal subgroup* $N \trianglelefteq G$ *gives rise to a crossed module, namely the inclusion map* $i : N \to G$ *equipped with the conjugation action of* G *on* N*. On the other hand, given any crossed module* $(H, G, \delta), Im(\delta) = \delta(H)$ *is a normal subgroup of* G*.*

(b) $(G, Aut(G), \delta)$ *is a crossed module where* δ *assigns to each element* $g \in G$*, the inner automorphism of* G*,* $\delta(g) : x \to gxg^{-1}$ *for all* $x \in G$*, and* $Aut(G)$ *acts on* G *by* $\alpha g = \alpha(g)$ *for* $\alpha \in Aut(G)$ *and* $g \in G$*.*

(c) $(A, G, 0)$ *is a crossed module where* A *is* G*-module and the boundary operator* δ *is the zero map.*

The following relationship between the categories of crossed modules and internal categories in groups is well known (see [11], [31])

1.6.4 Proposition. *There is an equivalence of categories between* ***XMod(Groups)*** *and* ***Cat(Groups)****.*

Proof. We define a functor

$$F : \textbf{XMod(Groups)} \longrightarrow \textbf{Cat(Groups)}$$

$$\begin{tikzcd} F(X,B,f) \; = \; C_1 \times_{(c,d)} C_1 \arrow[r, "m"] & C_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & C_0 \arrow[l, "e"'] \end{tikzcd}$$

where

$$C_0 = B, \quad C_1 = B \ltimes X,$$

and $d(b,x) = b, \quad e(b) = (b,0), \quad c(b,x) = f(x) + b$. Composition of morphisms in $((B \ltimes X)_0, (B \ltimes X)_1, d, c, e, m)$ is given by $m((f(x)+b, x'), (b,x)) = (b, x'+x)$. We also define a functor

$$T : \mathbf{Cat(Groups)} \longrightarrow \mathbf{XMod(Groups)}$$

$$\begin{tikzcd} T(C_1 \times_{(c,d)} C_1 \arrow[r, "m"] & C_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & C_0) \arrow[l, "e"'] \end{tikzcd} \; = \; (ker(d), C_0, h)$$

where $h(u) = c(u)$ and C_0 acts on $ker(d)$ via conjugation $gu = e(g) + u - e(g)$. A routine calculation shows that both functorial constructions are mutually inverse: that is, $FT \cong 1_{\mathbf{Cat(Groups)}}$ and $TF \cong 1_{\mathbf{XMod(Groups)}}$. $\square$

1.7 Cat^1-groups.

Cat^1-groups (or 1-cat groups) were introduced by Loday [24] as algebraic models of homotopy 2-types, along with other analogous higher dimensional notions.

1.7.1 Definition. *(a) A cat^1-group consists of a group G together with two endomorphisms $s, t : G \to G$ such that $st = t, \quad ts = s$ and $\quad [ker(s), ker(t)] = 0$.*

(b) A morphism of cat^1-groups $(G, s, t) \to (G', s', t')$ is a group homomorphism $f : G \to G'$ such that $s'f = f's$ and $\quad t'f = f't$.

1.7.2 Remark. *Since $ker(s) = \{x - s(x) \mid x \in G\}$ and $ker(t) = \{x - t(x) \mid x \in G\}$ the requirements of Definition 1.7.1(1) are equivalent to $st(x) = t(x)$, $ts(x) = s(x)$, $x - s(x) + y - t(y) = y - t(y) + x - s(x)$ for all $x, y \in G$. Consequently, the category of cat^1-groups is a variety of universal algebras.*

We shall denote the category of cat^1-groups and their morphisms by **Cat^1-groups**

1.7.3 Proposition. *[24] There is an equivalence of categories between* ***Cat(Groups)*** *and* ***Cat^1-groups****.*

Proof. We define the functors R and Q

(a)

$$R : \mathbf{Cat(Groups)} \longrightarrow \mathbf{Cat^1\text{-}groups}$$

$$R\left(\begin{tikzcd} C_1 \times_{(c,d)} C_1 \arrow[r, "m"] & C_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & C_0 \arrow[l, "e"'] \end{tikzcd}\right) = (C_1, ed, ec)$$

(b)

$$Q : \mathbf{Cat^1\text{-}groups} \longrightarrow \mathbf{Cat(Groups)}$$

$$Q(G, s, t) = \begin{tikzcd} C_1 \times_{(c,d)} C_1 \arrow[r, "m"] & C_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & C_0 \arrow[l, "e"'] \end{tikzcd}$$

where $C_1 = G$, $C_0 = \{g \in G | s(g) = g\}$, $d(g) = s(g)$, $c(g) = t(g)$ *and e is the inclusion map.*

$\square$

1.8 Protomodular and semiabelian categories.

Our study of semidirect products will be done in the appropriate setting of semiabelian categories.

1.8.1 Definition. *[1] We say that a category* **C** *is regular if it satisfies the conditions*

1. **C** *has finite limits;*

2. **C** *has coequalizers of kernel pairs;*

3. *the pullback of a regular epimorphism along any morphism is regular epimorphism.*

1.8.2 Remark. *Condition* (3) *means that in a pullback diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

if q is a regular epimorphism then p is a regular epimorphism as well.

The main result in regular categories is the following theorem.

1.8.3 Theorem. *In a regular category, every morphism factors as a regular epimorphism followed by a monomorphism. Moreover, such a factorization is unique up to an isomorphism.*

Proof. See [1], [6]. □

Let us also recall the following definition.

1.8.4 Definition. *An equivalence relation* $< r_0, r_1 >: R \longrightarrow X \times X$ *on an object X of a category* **C** *is effective if the coequalizer q of* $< r_0, r_1 >$ *exists and* $< r_0, r_1 >$ *is the kernel pair of q.*

1.8.5 Definition. *[1] A category* **C** *is Barr-exact when:*

1. **C** *is regular*

2. *every equivalence relation in* **C** *is effective; that is, every equivalence relation in* **C** *is a kernel pair relation.*

1.8.6 Examples. *1. Any variety of universal algebras is Barr-exact. In particular, every variety of Ω-groups is Barr-exact. In a variety the regular epimorphisms are the surjective algebra homomorphisms.*

*2. The category **Set** of sets is Barr-exact.*

Recall (see, for instance [6],[7]) that when a category $\mathbf{C}$ has finite limits, any morphism $p : E \longrightarrow B$ in $\mathbf{C}$ determines a pullback functor $p^{\star} : Pt_{\mathbf{C}}(B) \longrightarrow Pt_{\mathbf{C}}(E)$.

1.8.7 Definition. *[7] A category is protomodular when*

1. $\mathbf{C}$ *admits pullbacks*

2. *for every morphism* $p : E \longrightarrow B$ *in* $\mathbf{C}$*, the pullback functor* $p^{\star} : Pt_{\mathbf{C}}(B) \longrightarrow Pt_{\mathbf{C}}(E)$ *reflects isomorphisms.*

1.8.8 Proposition. *[4, 8] Let* $\mathbf{C}$ *be a category with pullbacks and a zero object* 0*. The following statements are equivalent.*

1. $\mathbf{C}$ *is protomodular.*

2. *The split short five lemma holds in* $\mathbf{C}$*, that is; given a diagram*

$$\begin{tikzcd}
A \arrow[r, "\kappa"] \arrow[d, "f"] & B \arrow[r, bend left, "p"] \arrow[d, "g"] & C \arrow[l, "s"] \arrow[d, "h"] \\
A' \arrow[r, "\kappa'"] & B' \arrow[r, "p'"] & C' \arrow[l, bend left, "s'"]
\end{tikzcd} \qquad (1.7)$$

where all squares are commutative squares, $\kappa' f = gk$, $p'g = hp$, $gs = s'h$*, and where moreover*

$ps = 1_C, \quad p's' = 1_{C'}, \quad \kappa = ker(p), \quad \kappa' = ker(p'),$
if f *and* h *are isomorphisms then* g *is an isomorphism as well.*

1.8.9 Definition. *[19] A category* $\mathbf{C}$ *is semi-abelian when*

(a) $\mathbf{C}$ *is pointed;*

(b) $\mathbf{C}$ *is finitely complete;*

(c) $\mathbf{C}$ *is finitely cocomplete;*

(d) $\mathbf{C}$ *is Barr-exact;*

(e) $\mathbf{C}$ *is protomodular.*

The following characterization of algebraic theories with the property that the corresponding category is semi-abelian was obtained by Bourn and Janelidze:

1.8.10 Theorem. *[9] A variety of universal algebras* $\mathbf{C}$ *is semi-abelian if and only if its theory has a unique constant 0, binary terms* $t_1, t_2, ..., t_n$ *and a* $(n+1)$*-ary term t satisfying the identities* $t(t_1(x,y), ..., t_n(x,y), y) = x$ *and* $t_i(x,x) = 0$ *for each* $i = 1, ..., n$.

1.8.11 Examples.

1. *All abelian categories are semi-abelian. Recall that abelian categories are Barr-exact categories which are additive.*

2. *Every variety of* Ω*-groups is semi-abelian. The examples we are interested in in this study include: the categories of groups, rings, commutative rings, Lie algebras, precrossed modules and crossed modules.*

For more information on protomodular and semi-abelian categories the reader is referred to [4, 6, 19].

1.8.12 Definition. *[5] Let* $\boldsymbol{C}$ *be a pointed category with binary coproducts and kernels of split epimorphisms. The category of internal actions, denoted by Act(*$\boldsymbol{C}$*) is defined as follows: the objects are triples* (B, X, ξ)*, where* $\xi : B\flat X \longrightarrow X$ *makes the diagram*

$$\begin{tikzcd}
B\flat(B\flat X) \arrow[r, "\mu_X^B"] \arrow[d, "1\flat\xi"'] & B\flat X \arrow[d, "\xi"'] & X \arrow[l, "\eta_X^B"'] \arrow[dl, equal] \\
B\flat X \arrow[r, "\xi"'] & X &
\end{tikzcd} \tag{1.8}$$

commute. Here μ_X^B *is defined by the exactness of the rows*

$$\begin{tikzcd}
0 \arrow[r] & B\flat(B\flat X) \arrow[r, "\kappa_{B,B\flat X}"] \arrow[d, "\mu_X^B"'] & B+(B\flat X) \arrow[r, "{[1_B,0]}"] \arrow[d, "{[\iota_1,\kappa_{B,X}]}"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B\flat X \arrow[r, "\kappa_{B,X}"'] & B+X \arrow[r, "{[1_B,0]}"'] & B \arrow[r] & 0
\end{tikzcd} \tag{1.9}$$

while the object $B\flat X$ *is the kernel,* $\kappa_{B,X} : B\flat X \longrightarrow B+X$*, of the morphism* $[1_B, 0] : B+X \longrightarrow B$*. See [18] for details. A morphism* $(B, X, \xi) \longrightarrow (B', X', \xi')$ *in Act(*$\boldsymbol{C}$*) is a pair* (g, h) *where* $g : B \longrightarrow B'$ *and* $h : X \longrightarrow X'$ *are morphisms in* $\mathbf{C}$ *with* $h\xi = \xi'(g\flat h)$

The notion of a crossed module was extended to the semiabelian context by G. Janelidze.

1.8.13 Definition. *[18] An internal precrossed module in* $\mathbf{C}$ *is a* 4*-tuple* (B, X, ξ, δ) *with* (B, X, ξ) *in Act(*$\boldsymbol{C}$*) and* $\delta : X \longrightarrow B$ *a morphism in* $\mathbf{C}$ *such that the diagram*

$$\begin{tikzcd}
B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B+X \arrow[d, "{[1,\delta]}"] \\
X \arrow[r, "\delta"] & B
\end{tikzcd} \tag{1.10}$$

commutes. A morphism $(B, X, \xi, \delta) \longrightarrow (B', X', \xi', \delta')$ *is a morphism* $(g, h) : (B, X, \xi) \longrightarrow (B', X', \xi')$ *in Act(*$\boldsymbol{C}$*) such that* $g\delta = \delta' h$*.*

1.8.14 Definition. *[18] An internal crossed module in* $\mathbf{C}$ *is an internal pre-crossed module* (B, X, ξ, δ) *in* $\boldsymbol{C}$ *for which the diagram*

$$\begin{tikzcd}
(B+X)\flat X \arrow[r, "{[1_B,\delta]\flat 1_X}"] \arrow[d, "{[1_{B+X},\iota_2]^\sharp}"'] & B\flat X \arrow[d, "\xi"] \\
B\flat X \arrow[r, "\xi"'] & X
\end{tikzcd} \tag{1.11}$$

commutes. Here, $[1_{B+X}, \iota_2]^\sharp$ *is the unique morphism such that* $\kappa_{B,X}[1_{B+X}, \iota_2]^\sharp = [1_{B+X}, \iota_2]\kappa_{B+X,X}$ *; that is, the unique morphism* $[1_{B+X}, \iota_2]^\sharp$ *for which the left hand square in the diagram*

$$\begin{tikzcd}
(B+X)+X \arrow[d, "{[1_{B+X},\iota_2]}"'] & (B+X)\flat X \arrow[l, "\kappa_{B+X,X}"'] \arrow[r, "{[1_B,\delta]\flat 1_X}"] \arrow[d, "{[1_{B+X},\iota_2]^\sharp}"'] & B\flat X \arrow[d, "\xi"] \\
B+X & B\flat X \arrow[l, "\kappa_{B,X}"] \arrow[r, "\xi"'] & X
\end{tikzcd} \tag{1.12}$$

commutes.

Chapter 2

Semi-direct products in varieties of right Ω-loops.

In this chapter we construct the semidirect product in a variety of right Ω-loops, that is, a pointed variety of universal algebras with a binary $+$ and a binary $-$ satisfying the identities

$$x + 0 = x, \;\; 0 + x = x, \;\; (x - y) + y = x, \;\; (x + y) - y = x$$

$x, y \in$ Ω-**RLoop**, the variety of right Ω-loops (see [33] for further details on the theory of quasigroups and loops). Section 2.2 contains the main result of the chapter, namely, the construction of the semidirect product in the variety Ω-**RLoop**.

In Section 2.3 we describe precrossed modules and crossed modules in the variety of right Ω-loops. In Section 2.4 we describe star-multiplicative graphs, then give a necessary and sufficient condition for a star-multiplicative graph to extend to an internal category in Ω-**RLoop**.

In the last section we use Loday's description of Cat^1-algebras to define and construct semidirect products of crossed modules in $\mathbf{R}$-$\Omega loop$.

2.1 Remarks on monadicity.

We start by recalling from [6] the following definition.

2.1.1 Definition. *Let* $\mathbf{C}$ *be a category and* $B \in \mathbf{C}$ *a fixed object. The category* $Pt_{\mathbf{C}}(B)$ *of points over* B *is defined as follows:*

(a) an object is a triple (A, α, β) *consisting of an object* $A \in \mathbf{C}$ *and arrows* $\alpha : A \longrightarrow B$ *and* $\beta : B \longrightarrow A$ *with* $\alpha\beta = 1_B$ *,*

(b) a morphism $f : (A, \alpha, \beta) \longrightarrow (A', \alpha', \beta')$ *is an arrow* $f : A \longrightarrow A'$ *in* $\mathbf{C}$ *such that* $\alpha' f = \alpha$ *and* $f\beta = \beta'$*; we will then briefly say that the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & A' \arrow[d, "\alpha'"] \\ B \arrow[u, bend left, "\beta"] \arrow[r, equal] & B \arrow[u, bend right, "\beta'"'] \end{tikzcd}$$

commutes.

We will make use of the following fact (see [4], [6]).

2.1.2 Proposition. *Let* B *be a fixed group. The category* $Pt_{\textbf{\textit{Groups}}}(B)$ *is equivalent to the category* B*-**Groups** and their morphisms.*

Proof. There is a functor $F : B\text{-}\mathbf{Groups} \longrightarrow Pt_{\mathbf{Groups}}(B)$ which sends $((B, X, \varphi), \quad \varphi : B \longrightarrow Aut(X))$ to a diagram of the form

$$\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, bend left, "\alpha"] & B \arrow[l, bend left, "\beta"] \end{tikzcd}$$

where $A = (A, \alpha, \beta)$ is the classical semi-direct product of B with (X, ξ), α is the product projection, β is defined by $\beta(b) = (b, 0)$, and κ is the kernel of α. Define a functor $G : Pt_{\mathbf{Groups}}(B) \longrightarrow B$-**Groups** as follows;

$$G\left(\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, bend left, "\alpha"] & B \arrow[l, bend left, "\beta"] \end{tikzcd}\right) = (B, X, \varphi) \tag{2.1}$$

where $\varphi(b)(x)$ is the unique element $x' \in X$ such that $\kappa(x') = \beta(b) + \kappa(x) - \beta(b)$. Then $GF(B, X, \varphi) = (B, X, \psi)$ where $\psi(b)(x) = x'$ such that

$$(0, x) = (b, 0) + (0, x) + (-b, 0) = (b, bx) + (-b, 0) = (0, bx), \quad bx = \varphi(b)(x).$$

Therefore, $\psi(b)(x) = \varphi(b)(x)$ $(\ for\ all\ b \in B\ and\ x \in X)$ and $\psi = \varphi$. This shows that $GF(B, X, \varphi) = (B, X, \psi)$.

Conversely,

$$FG\left(\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, bend left, "\alpha"] & B \arrow[l, bend left, "\beta"] \end{tikzcd}\right) = \begin{tikzcd} X \arrow[r, "i"] & B \ltimes X \arrow[r, bend left, "\pi_1"] & B \arrow[l, bend left, "\iota_1"] \end{tikzcd} \tag{2.2}$$

where $i(x) = (0, x)$, $\iota_1(b) = (b, 0)$, $\pi_1(b, x) = b$ and

$$\varphi(b)(x) = x' \in X\ with \quad \kappa(x') = \beta(b) + \kappa(x) - \beta(b) \tag{2.3}$$

For

$$E = \left(\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, bend left, "\alpha"] & B \arrow[l, bend left, "\beta"] \end{tikzcd}\right),$$

we define $\eta_E : E \longrightarrow FG(E)$ as the diagram

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[r, bend left, "\alpha"] \arrow[d, "u"] & B \arrow[l, "\beta"] \arrow[d, equal] \\ X \arrow[r, "i"] & B \ltimes_{\varphi} X \arrow[r, "\pi_1"] & B \arrow[l, bend left, "\iota_1"] \end{tikzcd} \tag{2.4}$$

where φ is as in (2.3), and $u(a) = (\alpha(a), x)$, where $\kappa(x) = a - \beta\alpha(a)$.
In (2.4), we have

(a) $u\kappa(a) = (\alpha\kappa(x), x')$ where $\kappa(x') = \kappa(x) - \beta\alpha(x) = \kappa(x)$; that is, $u\kappa(x) = (0, x)$.

(b) $\pi_1 u(a) = \alpha(a)$.

(c) $u\beta(b) = (\alpha\beta(b), x)$, where $\kappa(x) = \beta(b) - \beta\alpha\beta(b) = 0$; that is, $u\beta(b) = (b, 0)$.

(d) $u(a + a') = (\alpha(a + a'), x)$, where $\kappa(x) = a + a' - \beta\alpha(a') - \beta\alpha(a)$.

On the other hand, $u(a) + u(a') = (u(a), y) + (u(a'), y')$ where $\kappa(y) = a - \beta\alpha(a)$ and $\kappa(y') = a' - \beta\alpha(a')$.
We have

$$(\alpha(a), y) + (\alpha(a'), y') = (\alpha(a) + \alpha(a'), y + \alpha(a)y')$$

and so to prove that $u(a+a') = u(a)+u(a')$, we have to prove that $x = y+\alpha(a)y'$, or, equivalently, that

$$\kappa(x) = \kappa(y) + \kappa(\alpha(a)y'). \tag{2.5}$$

We have

$$\kappa(x) = a + a' - \beta\alpha(a') - \beta\alpha(a),$$
$$\kappa(y) = a - \beta\alpha(a).$$

$$\kappa(\alpha(a)y) = \beta(a) + \kappa(y') - \beta\alpha(a) \quad (by\ (2.3)).$$

Therefore, to prove (2.5) is to prove that

$$a + a' - \beta\alpha(a') - \beta\alpha(a) = a - \beta\alpha(a) + \beta\alpha(a) + a' - \beta\alpha(a') - \beta\alpha(a)$$

which is obvious.
(a), (b), (c) and (d) imply that η_E is an isomorphism. $\square$

The following theorem will be needed in the sequel.

2.1.3 Theorem. *Let* $\mathbf{C}$ *be a pointed category with finite limits and finite coproducts. For an object* $B \in \boldsymbol{C}$*, the functor*

$$U : Pt_{\boldsymbol{C}}(B) \longrightarrow \boldsymbol{C}, \quad (A, \alpha, \beta) \longmapsto ker(\alpha)$$

has a left adjoint

$$F : \boldsymbol{C} \longrightarrow Pt_{\boldsymbol{C}}(B)$$

defined by $F(X) = (B + X, [1, 0], \iota_1)$ *where* $[1, 0]$ *is the morphism* $B + X \longrightarrow B$ *induced by the identity* $1 : B \longrightarrow B$ *and the zero map* $0 : X \longrightarrow B$*, and* ι_1 *is the coproduct injection* $\iota : B \longrightarrow B + X$*.*

Proof. For an object X in $\mathbf{C}$, we have

$$UF(X) = ker(B + X \xrightarrow{[1,0]} B) = B\flat X.$$

We define $\eta_X : X \longrightarrow B\flat X$ as the unique morphism with $\kappa_X \eta_X = \iota_2$ as in the diagram

$$\begin{tikzcd}
B\flat X \arrow[r, "\kappa_X"] \arrow[d, equal] & B + X \arrow[r, bend left, "{[1,0]}"] & B \arrow[l, "\iota_1"'] \arrow[d, equal] \\
B\flat X & X \arrow[l, dashed, "\eta_X"] \arrow[u, "\iota_2"'] \arrow[r, "0"'] & B.
\end{tikzcd}$$

In order to establish the universal property

$$\begin{tikzcd}
X \arrow[d, "\eta_X"'] \arrow[dr, "u"] & \\
UF(X) \arrow[r, "U(f)"'] & ker(\alpha)
\end{tikzcd}$$

$$F(X) \overset{f}{\dashrightarrow} (A, \alpha, \beta).$$

of η_X, we need to prove that for every morphism $u : X \longrightarrow ker(\alpha)$ (for (A, α, β) in $Pt_{\mathbf{C}}(B)$), there exists a unique morphism $f : B + X \longrightarrow A$ such that

$$\alpha f = [1, 0], \quad f\iota_1 = \beta, \quad U(f)\eta_X = u, \tag{2.6}$$

where $U(f)$ is the unique morphism $B\flat X \longrightarrow ker(\alpha)$ making the diagram

$$\begin{tikzcd} B\flat X \arrow[r, "\kappa_X"] \arrow[d, "U(f)"'] & B + X \arrow[d, "f"] \\ ker(\alpha) \arrow[r, "ker(\alpha)"'] & A \end{tikzcd} \tag{2.7}$$

commute. Since $U(f)\eta_X = u$ if and only if $(ker(\alpha))U(f)\eta_X = (ker(\alpha))u$, the conditions of f can be replaced with

$$\alpha f = [1, 0], \quad f\iota_1 = \beta, \quad f\kappa_X\eta_X = ker(\alpha)u \tag{2.8}$$

or with

$$\alpha f = [1, 0], \quad f\iota_1 = \beta, \quad f\iota_2 = (ker(\alpha))u. \tag{2.9}$$

Since

$$B \xrightarrow{\iota_1} B + X \xleftarrow{\iota_2} X \tag{2.10}$$

is a coproduct diagram, there exists a unique f satisfying the second and third equality of (2.9). Therefore, we only need to prove that the first two equalities imply the first equality of (2.9). Again, since (2.10) is a coproduct diagram, this means we have to prove the implication

$$(f\iota_1 = \beta \; and \; f\iota_2 = (ker(\alpha))u) \Longrightarrow \alpha f\iota_1 = [1, 0]\iota_1 \; and \quad \alpha f\iota_2 = [1, 0]\iota_2.$$

This is a straightforward calculation:

$$\alpha f\iota_1 = \alpha\beta = 1 = [1, 0]\iota_1 \; and \; \alpha f\iota_2 = \alpha(ker(\alpha))u = 0u = 0 = [1, 0]\iota_2.$$

□

2.1.4 Theorem. *[6], [8] Let* $\mathbf{C}$ *be a semiabelian variety. Then the kernel functor*

$$U : Pt_{\mathbf{C}}(B) \longrightarrow \mathbf{C}, \quad (A, \alpha, \beta) \longmapsto ker(\alpha)$$

is monadic, that is, there exists an equivalence of categories

$$Pt_{\boldsymbol{C}}(B) \approxeq \boldsymbol{C}^{T^B}, (A, \alpha, \beta) \longmapsto (ker(\alpha), \xi)$$

between the category of points over B *and the category of* T^B*-algebras.*

By Theorem 2.1.3, U has a left adjoint. By Beck's theorem, it suffices to show that

(1) U reflects isomorphisms;

(2) $Pt_{\mathbf{C}}(B)$ has coequalizers of reflexive pairs;

(3) U preserves coequalizers of reflexive pairs.

In order to prove the result, we need some lemmas:

2.1.5 Lemma. *Suppose the left hand part of the diagram*

$$\begin{tikzcd}
A' \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[d, bend left, "\alpha'"] & A \arrow[r, "p"] \arrow[d, bend left, "\alpha"] & C \arrow[d, bend left, "\gamma"] \\
B \arrow[u, bend left, "\beta'"] \arrow[r, equal] & B \arrow[u, bend left, "\beta"] \arrow[r, equal] & B \arrow[u, bend left, "p\beta"]
\end{tikzcd} \tag{2.11}$$

is given and that it represents two parallel morphisms in $Pt_{\boldsymbol{C}}(B)$*. Then if the diagram*

$$\begin{tikzcd}
A' \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & A \arrow[r, "p"] & C
\end{tikzcd} \tag{2.12}$$

is a coequalizer diagram, then so is Diagram 2.11 in $\boldsymbol{C}$ *for a uniquely defined* γ *in* $Pt_{\boldsymbol{C}}(B)$.

Proof. Define $\gamma : C \longrightarrow B$ as the unique morphism with $\gamma p = \alpha$ (this is possible since $\alpha f = \alpha' = \alpha g$); note that $\gamma p \beta = \alpha\beta = 1_B$. We need to prove that for every morphism $q : (A, \alpha, \beta) \longrightarrow (X, \varphi, \psi)$ in $Pt_{\mathbf{C}}(B)$ such that $qf = qg$, there exists a unique morphism $u : (C, \gamma, p\beta) \longrightarrow (X, \varphi, \psi)$ with $q = up$. Since p is the coequalizer of (f, g) in $\mathbf{C}$, we only need to prove that

$$q = up \Rightarrow (up\beta = \psi \ \wedge \ \varphi u = \gamma).$$

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[rr, bend left, dashed, "q"] \arrow[d, bend left, "\alpha"] & C \arrow[r, "u"] \arrow[d, bend left, "\gamma"] & X \arrow[d, bend left, "\varphi"] \\ B \arrow[u, bend left, "\beta"] \arrow[r, equal] & B \arrow[u, bend left, "p\beta"] \arrow[r, equal] & B \arrow[u, bend left, "\psi"] \end{tikzcd} \tag{2.13}$$

We have

$$up\beta = q\beta = \psi \quad and \quad \varphi up = \varphi q = \alpha = \gamma p.$$

Since p is an epimorphism, the last equality implies $\varphi u = \gamma$.

□

2.1.6 Lemma. *Suppose*

$$\begin{tikzcd} A \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[d, bend left, "\alpha"] & A' \arrow[l, "e"'] \arrow[r, "p"] \arrow[d, bend left, "\alpha'"] & A'' \arrow[d, bend left, "\alpha''"] \\ B \arrow[u, bend left, "\beta"] \arrow[r, equal] & B \arrow[u, bend left, "\beta'"] \arrow[r, equal] & B \arrow[u, bend left, "\beta''"] \end{tikzcd} \tag{2.14}$$

is a reflexive coequalizer diagram in $Pt_{\mathbf{C}}(B)$, *where* $\mathbf{C}$ *is a semiabelian variety. Let* $E = \{(d(a), c(a)) \mid a \in A\}$ *and* $\pi_i : E \longrightarrow A'$ $(i = 1, 2)$ *be the projections.*

Then E is an equivalence relation on A' and

$$\begin{tikzcd} E \arrow[r, bend left, "\pi_1"] \arrow[r, bend right, "\pi_2"'] & A' \arrow[r, "p"] & A'' \end{tikzcd} \tag{2.15}$$

is a coequalizer diagram.

Proof. Since $\mathbf{C}$ is Maltsev, E is an equivalence relation. We make diagram (2.15) a reflexive coequalizer diagram in $Pt_{\mathbf{C}}(B)$ as follows:

$$\begin{tikzcd} E \arrow[r, bend left, "\pi_1"] \arrow[r, bend right, "\pi_2"'] \arrow[d, bend left, "\alpha'\pi_1"] & A' \arrow[r, "p"] \arrow[d, bend left, "\alpha'"] & A'' \arrow[d, bend left, "\alpha''"] \\ B \arrow[u, bend left, "\overline{\beta}"] \arrow[r, equal] & B \arrow[u, bend left, "\beta'"] \arrow[r, equal] & B \arrow[u, bend left, "\beta''"] \end{tikzcd} \tag{2.16}$$

where $\overline{\beta}(b) = (d\beta(b), c\beta(b))$ and $\alpha'\pi_1 = \alpha'\pi_2$ since, for $(d(a), c(a)) \in E$, we have $\alpha'\pi_1(d(a), c(a)) = \alpha' d(a) = \alpha(a) = \alpha' c(a) = \alpha\pi_2(d(a), c(a))$. Then diagram (2.16) is exact in $Pt_{\mathbf{C}}(B)$; that is, p is the coequalizer of (π_1, π_2) and (π_1, π_2) is the kernel pair of p. $\square$

2.1.7 Lemma. *$U = ker : Pt_{\boldsymbol{C}}(B) \longrightarrow \boldsymbol{C}$ preserves regular epimorphisms.*

Proof.

$$\begin{tikzcd} A' \arrow[rr, "p"] \arrow[dr, "\alpha'"] & & A'' \arrow[dl, bend left, "\alpha''"] \\ & B \arrow[ul, bend left, "\beta'"] \arrow[ur, "\beta''"'] & \end{tikzcd}$$

is a regular epimorphism in $Pt_{\mathbf{C}}(B)$ if and only if p is surjective. We need to show that this makes the induced map $ker(\alpha') \longrightarrow ker(\alpha'')$ surjective too. Indeed, for $a'' \in ker(\alpha'')$, we have $p(a') = a''$ for some $a' \in A'$ since p is surjective.
$\alpha'(a') = \alpha'' p(a') = \alpha''(a'') = 0 \implies a \in ker(\alpha')$ shows that $a' \in ker(\alpha')$. $\square$

2.1.8 Lemma. *$U = ker : Pt_{\boldsymbol{C}}(B) \longrightarrow \boldsymbol{C}$ preserves coequalizers of reflexive pairs.*

Proof. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "\alpha"] & E \arrow[rr, bend left, "\pi_1"] \arrow[rr, bend right, "\pi_2"'] \arrow[d, bend left, "\alpha'\pi_1"] & & A' \arrow[r, "p"] \arrow[d, bend left, "\alpha'"] & A\text{''} \arrow[d, bend left, "\alpha''"] \\
B \arrow[u, bend left, "\beta"] \arrow[r, equal] & B \arrow[u, bend left, "\overline{\beta}"] \arrow[rr, equal] & & B \arrow[u, bend left, "\beta'"] \arrow[r, equal] & B \arrow[u, bend left, "\beta''"]
\end{tikzcd} \tag{2.17}$$

in $Pt_{\mathbf{C}}(B)$, where q is defined by $q(a) = (d(a), c(a))$ and its image

$$\begin{tikzcd}
ker(\alpha) \arrow[r, "q^*"] & ker(\alpha'\pi_1) \arrow[r, bend left, "{\pi_1}^*"] \arrow[r, bend right, "{\pi_2}^*"'] & ker(\alpha') \arrow[r, "p^*"] & ker(\alpha'')
\end{tikzcd} \tag{2.18}$$

under the functor $U = ker : Pt_{\mathbf{C}}(B) \longrightarrow \mathbf{C}$. We observe that;

(a) p^* and q^* are surjective by Lemma 2.1.7

(b) $({\pi_1}^*, {\pi_2}^*)$ is the kernel pair of p^* (since $ker : Pt_{\mathbf{C}}(B) \longrightarrow \mathbf{C}$ preserves limits and (π_1, π_2) is the kernel pair of p by Lemma 2.1.6.)

(c) since $({\pi_1}^*, {\pi_2}^*)$ is the kernel pair of p^* and p^* is a regular epimorphism,

$$\begin{tikzcd}
ker(\alpha'\pi_1) \arrow[r, bend left, "{\pi_1}^*"] \arrow[r, bend right, "{\pi_2}^*"'] & ker(\alpha') \arrow[r, "p^*"] & ker(\alpha'')
\end{tikzcd} \tag{2.19}$$

is a coequalizer diagram.

(d) since the diagram (2.19) is a coequalizer diagram, and q^* is surjective,

$$\begin{tikzcd} ker(\alpha) \arrow[r, bend left, "d^*"] \arrow[r, bend right, "c^*"'] & ker(\alpha') \arrow[r, "p^*"] & ker(\alpha'') \end{tikzcd} \tag{2.20}$$

(where $d^* = {\pi_1}^* q^*$ and $c^* = {\pi_2}^* q^*$) is also a coequalizer diagram.

□

To complete the proof of Theorem 2.1.4, we need to show that U reflects isomorphisms. Suppose $f : (A, \alpha, \beta) \longrightarrow (A', \alpha', \beta')$ is a morphism in $Pt_{\mathbf{C}}(B)$. Consider the diagram

$$\begin{tikzcd} U(A,\alpha,\beta) \arrow[r, "\kappa_A"] \arrow[d, "U(f)"] & A \arrow[r, bend left, "\alpha"] \arrow[d, "f"] & B \arrow[l, "\beta"] \arrow[d, equal] \\ U(A',\alpha',\beta') \arrow[r, "\kappa_{A'}"] & A' \arrow[r, "\alpha'"] & B' \arrow[l, bend left, "\beta'"] \end{tikzcd} \tag{2.21}$$

If $U(f)$ is an isomorphism then by the split short five lemma, f is an isomorphism as well. Therefore, U reflects isomorphisms.
Thus U is monadic.

The notion of a categorical semi-direct product was introduced by Bourn and Janelidze.

2.1.9 Definition. *[8] Let* $\mathbf{C}$ *be a semi-abelian category.*

(a) A B*-algebra is an algebra for the monad* T^B *corresponding to the monadic functor* $U : Pt_{\mathbf{C}}(B) \longrightarrow \mathbf{C}$.

(b) The semidirect product $B \ltimes (X, \xi)$ of a B-algebra (X, ξ) and the object $B \in \boldsymbol{C}$ is the (domain part of the) pointed object

$$\begin{tikzcd} A \arrow[r, bend left, "\alpha"] & B \arrow[l, "\beta"] \end{tikzcd}, \quad \alpha\beta = 1_B$$

corresponding to (X, ξ) under category equivalence $Pt_{\boldsymbol{C}}(B) \simeq \boldsymbol{C}^{T^B}$.

2.2 The semi-direct product in a variety of right Ω-loops.

Our next objective is to construct the semi-direct product in a variety of right Ω-loops. The main result is the following:

2.2.1 Theorem. *Let* **C** *be a fixed pointed variety of universal algebras which has among its operations a binary $+$, a binary $-$ and a nullary 0 satisfying the identities*

$$x + 0 = x, \tag{2.22}$$

$$0 + x = x, \tag{2.23}$$

$$(x - y) + y = x, \tag{2.24}$$

$$(x + y) - y = x. \tag{2.25}$$

Given an object B and a T^B-algebra (X, ξ), the semidirect product $B \ltimes (X, \xi)$ is the set-theoretical (cartesian) product $B \times X$ equipped with the following Ω-algebra structure:

$$\omega((b_1, x_1), ..., (b_n, x_n)) = (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) \tag{2.26}$$

for each n-ary operation $\omega \in \Omega$ and for all $b_1, ..., b_n \in B$, $x_1, ..., x_n \in X$.

Proof. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[d, shift right, "\psi"'] \arrow[r, bend left, "\alpha"] & B \arrow[l, "\beta"] \arrow[d, equal] \\
X \arrow[r, "<0{,}1>"'] & B \times X \arrow[u, shift right, "\varphi"'] \arrow[r, bend right, "\pi_1"'] & B \arrow[l, "<1{,}0>"']
\end{tikzcd} \tag{2.27}$$

where $\alpha\beta = 1_B$ and $\kappa = ker(\alpha)$. Let the maps φ and ψ be defined as follows;
$\varphi : B \times X \longrightarrow A, \quad (b, x) \longmapsto \kappa(x) + \beta(b).$
$\psi : A \longrightarrow B \times X, \quad a \longmapsto (\alpha(a), \kappa^{-1}(a - \beta\alpha(a)))$. We then have

$$\psi\varphi(b, x) = \psi(\kappa(x) + \beta(b)) = (b, \kappa^{-1}((\kappa(x) + \beta(b)) - \beta(b)) = (b, x),$$
$$\varphi\psi(a) = \varphi(\alpha(a), \kappa^{-1}(a - \beta\alpha(a))) = (a - \beta\alpha(a)) + \beta\alpha(a) = a.$$

Therefore $\psi\varphi = 1_{B\times X}$ and $\varphi\psi = 1_A$. This shows that A is in bijection with $B \times X$.
We note that every element $B + X$ can be presented as a $2n$-ary term

$$t(\iota_1(b_1), ..., \iota_1(b_n), \iota_2(x_1), ..., \iota_2(x_n)) = t(b_1, ..., b_n, x_1, ..., x_n) \quad (b_1, ..., b_n \in B,\ x_1, ..., x_n \in X)$$

where ι_1 and ι_2 are the coproduct injections $B \longrightarrow B + X$ and $X \longrightarrow B + X$ respectively. According to this notation

$$T^B(X) = \{t(b_1, ..., b_n, x_1, ..., x_n)|\ t(b_1, ..., b_n, 0, ..., 0) = 0\}, \quad b_1, ..., b_n \in B,\ x_1, ..., x_n \in X.$$

A straightforward calculation shows that given $(A, \alpha, \beta) \in Pt_{\mathbf{C}}(B)$, the commutative diagram

$$\begin{tikzcd}
B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"] & B + X \arrow[d, "{[\beta,\kappa]}"] \arrow[r, bend left, "{[1,0]}"] & B \arrow[l, "\iota_1"] \arrow[d, equal] \\
X \arrow[r, "\kappa"] & A \arrow[r, "\alpha"] & B \arrow[l, bend left, "\beta"]
\end{tikzcd} \tag{2.28}$$

defines the corresponding T^B-algebra (X, ξ).
If $\omega \in \Omega$ is an n-ary operation, then

$$\omega((b_1, x_1), ...(b_n, x_n)) = \psi(\omega(\varphi(b_1, x_1), ..., \varphi(b_n, x_n))) \quad (\textit{using } \psi\varphi = 1_{B\times X})$$
$$= \psi(\omega(\kappa(x_1)+\beta(b_1), ..., \kappa(x_n)+\beta(b_n))) = (\alpha(\omega(\kappa(x_1)+\beta(b_1), ..., \kappa(x_n)+\beta(b_n))),$$
$$\kappa^{-1}[\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n)) - \beta\alpha\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n))])$$
$$= (\omega(b_1, ..., b_n), \kappa^{-1}[\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n)) - \omega(\beta(b_1), ..., \beta(b_n))])$$
$$= (\omega(b_1, ..., b_n), \kappa^{-1}[\beta, \kappa](\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))$$
$$= (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) \quad (2.29)$$

by the commutativity of the left-hand square in (2.28) $\square$

2.2.2 Remark. *The identities* (2.22), (2.23), (2.24) *and* (2.25) *are actually not only sufficient but also necessary for our definition of* φ *and* ψ *to determine bijections inverse to each other.*

Indeed: since the diagram

$$\begin{tikzcd}
X \arrow[r, "<0,1>"] \arrow[d, equal] & B \times X \arrow[d, "\varphi"] \arrow[r, bend left, "\pi_1"] & B \arrow[l, "<1,0>"] \arrow[d, equal] \\
X \arrow[r, "\kappa"] & A \arrow[u, bend left, "\psi"] \arrow[r, "\alpha"] & B \arrow[l, bend left, "\beta"]
\end{tikzcd} \quad (2.30)$$

(*where* $\varphi(b, x) = \kappa(x) + \beta(b)$ and $\psi(a) = (\alpha(a), \kappa^{-1}(a - \beta\alpha(a)))$, commutes, we have

$$\varphi < 0, 1 >= \kappa \Leftrightarrow \varphi(0, x) = \kappa(x) \Leftrightarrow \kappa(x) + 0 = \kappa(x).$$

From the diagram

$$\begin{tikzcd}
X \arrow[r, "1_X"] & X \arrow[r, "0"] & 0 \arrow[l, bend right, "0"']
\end{tikzcd} \quad (2.31)$$

in $\mathbf{C}$ we conclude that $\quad \forall_{C\in\mathbf{C}}\,\forall_{x\in C}\ x+0=x.$

Next, we also have $\alpha\varphi=\pi_1 \Leftrightarrow \alpha(\kappa(x)+\beta(b))=b \Leftrightarrow 0+b=b \Rightarrow \quad \forall_{C\in\mathbf{C}}\,\forall_{x\in C}\ 0+x=x.$

$\varphi\psi(a)=a \Leftrightarrow \varphi(\alpha(a),\kappa^{-1}(a-\beta\alpha(a)))=a \Leftrightarrow ((a-\beta\alpha(a))+\beta\alpha(a)=a$. Now consider the diagram

$$\begin{tikzcd} B \arrow[r, "<0{,}1>"] & B\times B \arrow[r, "\pi_1"] & B \arrow[l, bend right, "<1{,}1>"'] \end{tikzcd}. \tag{2.32}$$

For $a=(b_1,b_2)$, we have

$$\begin{aligned}(a-\beta\alpha(a))+\beta\alpha(a))=a &\Leftrightarrow ((b_1,b_2)-\beta\alpha(b_1,b_2))+\beta\alpha(b_1,b_2)=(b_1,b_2)\\ &\Leftrightarrow ((b_1,b_2)-(b_1,b_1))+(b_1,b_1)=(b_1,b_2)\\ &\Leftrightarrow (0,b_2-b_1)+(b_1,b_1)=(b_1,b_2)\\ &\Leftrightarrow (0+b_1,(b_2-b_1)+b_1)=(b_1,b_2)\\ &\Leftrightarrow 0+b_1=b_1,\quad (b_2-b_1)+b_1=b_2\\ &\Rightarrow \forall_{C\in\mathbf{C}}\,\forall_{x,y\in C}\,(x-y)+y=x.\end{aligned}$$

and this proves the identity (2.24).

$$\begin{aligned}\psi\varphi=1_{B\times X} &\Leftrightarrow (0+b,\kappa^{-1}((\kappa(x)+\beta(b))-\beta(0+b)))=(b,x)\\ &\Leftrightarrow (b,\kappa^{-1}((\kappa(x)+\beta(b))-\beta(b)))=(b,x)\\ &\Leftrightarrow (\kappa(x)+\beta(b))-\beta(b)=\kappa(x)\end{aligned}$$

Using the same commutative diagram (2.32), we get

$$
\begin{aligned}
(\kappa(x)+\beta(y))-\beta(y)=\kappa(x)& \\
&\Leftrightarrow ((0,x)+(y,y))-(y,y)=(0,x)\\
&\Leftrightarrow ((0+y)-y \quad (x+y)-y)=(0,x)\\
&\Rightarrow \forall_{C\in \mathbf{C}} \quad \forall_{x,y\in C} \quad (x+y)-y=x
\end{aligned}
$$

This shows that the identity (2.25) holds.

2.2.3 Observation. *For any variety of universal algebras satisfying the conditions of the above Theorem 2.2.1, we have*

$$
\begin{aligned}
(0,x)+(b,0) &= (0+b,\xi((x+0)+(0+b)-(0+b)))\\
&= (b,\xi((x+b)-b)) = (b,\xi(x)) = (b,x).
\end{aligned} \tag{2.33}
$$

A straightforward calculation shows that for the case of groups (with $\Omega = +,$) Equation (2.26) yields the following usual formula for semi-direct products:

$$
(b_1,x_1)+(b_2,x_2)=(b_1+b_2,x_1+\xi(b_1+x_2-b_1)) \tag{2.34}
$$

for all $b_1,b_2\in B$ and $x_1,x_2\in X$.

In the next three sections we apply the construction of the semi-direct product in this section to study internal categorical structures in varieties of right Ω-loops.

2.3 Crossed modules in a variety of right Ω-loops.

In Section 1.6 of Chapter 1, we reviewed the theory of crossed modules in the category of groups. We now formulate the notion of a crossed module in $\mathbf{C}$, where $\mathbf{C}$ is the variety of right Ω-loops.

2.3.1 Proposition. *A precrossed module in* $\mathbf{C}$ *can equivalently be defined as a quadruple* (B, X, ξ, δ) *in which* $(B, X, \xi) \in Act(\boldsymbol{C})$ *and* $\delta : X \longrightarrow B$ *is a morphism such that for an* n*-ary operation* $\omega \in \Omega$,

$$\begin{aligned}\omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n) - \omega(b_1, ..., b_n)& \\ = \delta(\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))& \qquad (2.35)\end{aligned}$$

for all $x_1, x_2, ..., x_n \in X$, $b_1, b_2, ..., b_n \in B$.

Proof. A precrossed module in $\mathbf{C}$ corresponds to a reflexive graph

$$\begin{tikzcd} B \ltimes X \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\gamma"'] & & B \arrow[ll, "\beta"'] \end{tikzcd} \qquad (2.36)$$

with $\alpha\beta = 1_B = \gamma\beta$, $\quad \alpha(b, x) = b$, $\quad \beta(b) = (b, 0)$.
Since γ is a homomorphism and $(b, x) = (0, x) + (b, 0)$, we have
$\gamma(b, x) = \gamma(0, x) + \gamma(b, 0) = \gamma(0, x) + b$. This shows that γ is completely determined by $\gamma(0, x)$, $for\ all\ x \in X$. We introduce the morphism $\delta : X \longrightarrow B$ defined by $\delta(x) = \gamma(0, x)$ and then $\gamma(b, x) = \delta(x) + b$ for each $b \in B$ and each $x \in X$.
For an n-ary operation $\omega \in \Omega$,

$$\gamma(\omega((b_1, x_1), ..., (b_n, x_n))) = \omega(\gamma(b_1, x_1), ..., \gamma(b_n, x_n)) = \omega(\delta(x_1)+b_1, ..., \delta(x_n)+b_n) \qquad (2.37)$$

$$\begin{aligned}\gamma(\omega(b_1, ..., b_n), \xi(\omega(x_1+b_1, ..., x_n+b_n)-\omega(b_1, ...b_n))) = \omega(\gamma(b_1, x_1), ..., \gamma(b_n, x_n))& \\ = \delta(\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) + \omega(b_1, ..., b_n)& \qquad (2.38)\end{aligned}$$

for $x_1, ..., x_n \in X$ and $b_1, ..., b_n \in B$. Equating equation (2.37) and equation (2.38), we get equation (2.35). $\square$

The next proposition gives the Peiffer condition in the definition of a crossed module.

2.3.2 Proposition. *A precrossed module* (B, X, ξ, δ) *can equivalently be defined as a crossed module if and only if*

$$\begin{aligned}\xi(\omega(x'_1 + (\delta(x_1) + b_1), ..., x'_n + (\delta(x_n) + b_n)) - \omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n))+ \\ \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)) \\ = \xi(\omega((x'_1 + x_1) + b_1, ..., (x'_n + x_n) + b_n) - \omega(b_1, ..., b_n))\end{aligned} \tag{2.39}$$

for all $b_1, ..., b_n \in B, \quad b'_1, ..., b'_n \in B, \quad x_1, ..., x_n \in X, \quad x'_1, ...x'_n \in X.$

Proof. Consider the diagram

$$(B \ltimes X) \times_B (B \ltimes X) \xrightarrow{m} (B \ltimes X)$$

where m denotes composition of morphisms and

$$(B \ltimes X) \times_B (B \ltimes X) = \{((b', x'), (b, x)) \mid b' = \delta(x) + b\},$$

Let

$$b \xrightarrow{(b,x)} b' = \delta(x) + b \xrightarrow{(b',x')} \delta(x') + b'$$

be a composable pair of morphisms. Since $\mathbf{C}$ is Maltsev, composition is given by

$$\begin{aligned} m((b', x'), (b, x)) = p((b', x'), 1_B, (b, x)) = p((b', x'), (b', 0), (b, x)) \\ = (p(b', b', b), p(x', 0, x)) = ((b' - b') + b, x' + x) = (b, x' + x) \quad (\textit{by Remark } 1.5.4).\end{aligned}$$

For an n-ary operation ω, we then have

$$\begin{aligned} m(\omega((b'_1, x'_1), ..., (b'_n, x'_n)), \omega((b_1, x_1), ..., (b_n, x_n))) \\ = m((\omega(b'_1, ..., b'_n), \xi(\omega(x'_1 + b'_1, ..., x'_n + b'_n) - \omega(b'_1, ..., b'_n))), \\ (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ...x_n + b_n) - \omega(b_1, ..., b_n)))) \\ = (\omega(b_1, ..., b_n), \xi(\omega(x'_1 + b'_1, ..., x'_n + b'_n) - \omega(b'_1, ..., b'_n))+ \\ \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) \ (\textit{using Equation} \quad 0.5)\end{aligned} \tag{2.40}$$

and

$$
\begin{aligned}
&\omega(m((b_1',x_1')(b_1,x_1)),...,m((b_n',x_n')(b_n,x_n)))\\
&\qquad\qquad = \omega((b_1',x_1')(b_1,x_1),...,(b_n',x_n')(b_n,x_n))\\
&\qquad = \omega((b_1,x_1'+x_1),...,(b_n,x_n'+x_n)) = (\omega(b_1,...,b_n),\\
&\xi(\omega((x_1'+x_1)+b_1,...,(x_n'+x_1)+b_n)-\omega(b_1,...,b_n))) \quad (\textit{by composition in } \mathbf{C})
\end{aligned}
\tag{2.41}
$$

for $b_1,...,b_n \in B, \quad b_1',...,b_n' \in B, \quad x_1,...,x_n \in X, \quad x_1',...,x_n' \in X$ with $b_i' = \delta(x_i)+b_i, \quad i=1,...,n.$ Since m is a homomorphism, we obtain

$$
\begin{aligned}
&m(\omega((b_1',x_1'),...,(b_n',x_n')),\omega((b_1,x_1),...,(b_n,x_n)))\\
&\qquad = \omega(m((b_1',x_1')(b_1,x_1)),...,m((b_n',x_n')(b_n,x_n)))\\
&\qquad\quad = \omega((b_1',x_1')(b_1,x_1),...,(b_n',x_n')(b_n,x_n)).
\end{aligned}
\tag{2.42}
$$

From equation (2.40) and equation (2.41), we get

$$
\begin{aligned}
&\xi(\omega(x_1'+b_1',...,x_n'+b_n')-\omega(b_1',...,b_n'))+\\
&\qquad\qquad \xi(\omega(x_1+b_1,...,x_n+b_n)-\omega(b_1,...,b_n))\\
&\quad = \xi(\omega((x_1'+x_1)+b_1,...,(x_n'+x_n)+b_n)-\omega(b_1,...,b_n))
\end{aligned}
\tag{2.43}
$$

which is equivalent to 2.39. □

The following example shows that when $\mathbf{C} = \mathbf{Grp}$, the crossed modules are exactly the crossed modules of Whitehead [34].

2.3.3 Example of Groups. *For $\omega = +$, we have*

$$\begin{aligned}
&\omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n) - \omega(b_1, ..., b_n) \\
&= \delta(\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) \quad (\textit{precrossed module condition}) \\
&\Rightarrow \delta(x_1) + b_1 + \delta(x_2) + b_2 = \delta(\xi(x_1 + b_1 + b_2 - b_2 - b_1)) + b_1 + b_2 \\
&\Leftrightarrow \delta(x_1) + b_1 + \delta(x_2) = \delta(\xi(b_1 + x_2 - b_1)) + b_1 \\
&\Leftrightarrow b_1 + \delta(x_2) = \delta(b_1 x_2) + b_1 \\
&\Leftrightarrow \delta(b_1 x_2) = b_1 + \delta(x_2) - b_1
\end{aligned} \tag{2.44}$$

To check the Peiffer identity, we take $\omega = +$. Then

$$\begin{aligned}
&\xi(\omega(x'_1 + b'_1, ..., x'_n + b'_n) - \omega(b'_1, ..., b'_n)) + \\
&\quad \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)) \quad (\textit{Peiffer condition}) \\
&= \xi(\omega((x'_1 + x_1) + b_1, ..., (x'_n + x_1) + b_n) - \omega(b_1, ..., b_n)) \\
&\Rightarrow \xi(x'_1 + b'_1 + x'_2 + b'_2 - b'_2 - b'_1) + \xi(x_1 + b_1 + x_2 + b_2 - b_2 - b_1) \\
&= \xi(x'_1 + x_1 + b_1 + x'_2 + x_2 + b_2 - b_2 - b_1) \\
&\Leftrightarrow \xi(x'_1 + b'_1 + x'_2 - b'_1) + \xi(x_1 + b_1 + x_2 - b_1) = \\
&\quad \xi(x'_1 + x_1 + b_1 + x'_2 + x_2 - b_1) \\
&\Leftrightarrow x'_1 + b'_1 x'_2 + x_1 + b_1 x_2 = x'_1 + x_1 + b_1(x'_1 + x_2) \\
&\Leftrightarrow b'_1 x'_2 + x_1 = x_1 + b_1 x'_2 \\
&\Leftrightarrow (\delta(x_1) + b_1) x'_2 = x_1 + b_1 x'_2 - x_1 \quad (\textit{using} \quad b'_1 = \delta(x_1) + b_1) \\
&\Leftrightarrow \delta(x_1)(b_1 x'_2) = x_1 + b_1 x'_2 - x_1 \\
&\Leftrightarrow \delta(x_1)(x') = x_1 + x' - x_1 \quad (\textit{when} \quad b_1 x'_2 = x').
\end{aligned} \tag{2.45}$$

As an immediate consequence of Proposition 2.3.1 and Proposition 2.3.2, we observe that

2.3.4 Corollary. *If $\boldsymbol{C}$=Ω-RLoop, the equivalence $XMod(\boldsymbol{C}) \approx Cat(\boldsymbol{C})$ holds.*

2.4 Star-multiplication in the variety of right Ω-loops.

In this section a description of star-multiplicative graphs in the category Ω-**Rloop** is given.

2.4.1 Definition. *[18] A multiplicative graph is a reflexive graph*

$$\begin{tikzcd} A \arrow[r, bend left, "\alpha"] \arrow[r, "\gamma"'] & B \arrow[l, bend left, "\beta"] \end{tikzcd} \tag{2.46}$$

together with a binary multiplication, that is, a morphism $A \times_B A \xrightarrow{m} B$ *such that the following diagram commutes;*

$$\begin{tikzcd} A \arrow[r, "\langle 1, \beta\alpha \rangle"] \arrow[dr, equal] & A \times_B A \arrow[d, "m"] & A \arrow[l, "\langle \beta\gamma, 1 \rangle"'] \arrow[dl, equal] \\ & A & \end{tikzcd} \tag{2.47}$$

2.4.2 Definition. *[18] Let* $S = (A, B, \alpha, \beta, \gamma)$ *be an internal reflexive graph in a pointed category* $\mathbf{C}$ *with finite limits, and* $\kappa : X \longrightarrow A$ *a (fixed) kernel of* α*. The graph* S *is said to be star-multiplicative if there exists a unique morphism* $s : A \times_B X \longrightarrow X$ *making the diagram*

$$\begin{tikzcd} X \arrow[r, "\langle \kappa, 0 \rangle"] \arrow[dr, equal] & A \times_B X \arrow[d, dashed, "s"] & X \arrow[l, "\langle \beta\gamma\kappa, 1 \rangle"'] \arrow[dl, equal] \\ & X & \end{tikzcd} \tag{2.48}$$

commute; here $A \times_B X$ *is defined as the pullback*

$$\begin{tikzcd} A \times_B X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "\gamma\kappa"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd} \tag{2.49}$$

We now describe star-multiplicative graphs in Ω-**RLoop**.

2.4.3 Theorem. *An internal crossed module* (B, X, ξ, δ) *in* Ω-***RLoop*** *corresponds to a star-multiplicative graph under the equivalence*

$$PXMod(\Omega\text{-}\boldsymbol{RLoop}) \approx RG(\Omega\text{-}\boldsymbol{RLoop}) \tag{2.50}$$

if and only if

$$\begin{aligned} &\xi(\omega(\delta(x'_1) + x_1, ..., \delta(x'_n) + x_n) - \omega(x_1, ..., x_n)) \\ &\qquad = \omega(x'_1 + x_1, ..., x'_n + x_n) - \omega(x_1, ..., x_n). \end{aligned} \tag{2.51}$$

for all $x_1, ..., x_n \in X,\ x'_1, ..., x'_n \in X$.

Proof. Consider a diagram in Ω-**RLoop** of the form

$$\begin{tikzcd} X \arrow[r, "<\kappa{,}0>"] & A \times_B X \arrow[r, bend left, "\pi_2"] & X \arrow[l, bend left, "<\beta\gamma\kappa{,}0>"] \end{tikzcd}. \tag{2.52}$$

If $\bar{\xi} : X\flat X \longrightarrow X$ is the X-action on X corresponding to this split epimorphism, then $\bar{\xi}$ making the diagram

$$\begin{tikzcd} X\flat X \arrow[r, "\kappa_{X,X}"] \arrow[d, "\bar{\xi}"'] & X + X \arrow[d, "{[<\beta\gamma\kappa,1>,<\kappa,0>]}"] \\ X \arrow[r, "<\kappa{,}0>"'] & A \times_B X \end{tikzcd} \tag{2.53}$$

commute; that is to say that

$$< \kappa, 0 > \bar{\xi} = [< \beta\gamma\kappa, 1 >, < \kappa, 0 >]\kappa_{X,X} \tag{2.54}$$

This is equivalent to the requirement that the diagram

$$\begin{tikzcd} X\flat X \arrow[r, "\kappa_{X,X}"] \arrow[d, "\bar{\xi}"'] & X+X \arrow[d, "{[<\beta\gamma\kappa,\kappa>]}"] \\ X \arrow[r, "\kappa"'] & A \end{tikzcd} \tag{2.55}$$

commutes.

By chasing this diagram, we get

$$\begin{aligned}
&\kappa\bar{\xi} = [\beta\gamma\kappa, \kappa]\kappa_{X,X} \\
&\Longleftrightarrow \kappa\bar{\xi}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\
&= [\beta\gamma\kappa, \kappa]\kappa_{X,X}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\
&\Longleftrightarrow \bar{\xi}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\
&= [\beta\gamma\kappa, \kappa](\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\
&\Longleftrightarrow \bar{\xi}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\
&= \omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)
\end{aligned} \tag{2.56}$$

An internal crossed module (B, X, ξ, δ) will then correspond to a star-multiplicative graph under the equivalence

$$PXMod(\Omega\text{-}\mathbf{RLoop}) \approx RG(\Omega\text{-}\mathbf{RLoop}) \tag{2.57}$$

if and only if the diagram

$$\begin{tikzcd} X\flat X \arrow[r, "\bar{\xi}"] \arrow[d, "\delta\flat 1"'] & X \arrow[d, equal] \\ B\flat X \arrow[r, "\xi"'] & X \end{tikzcd} \tag{2.58}$$

commutes. By the commutativity of this diagram, we have

$$\begin{aligned}
&\xi(\delta\flat 1)(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\
&\qquad = \xi(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\
&\qquad \Longleftrightarrow \xi(\omega(\delta(x_1') + x_1, ..., \delta(x_n') + x_n) - \omega(x_1, ..., x_n)) \\
&\qquad = \omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)
\end{aligned} \tag{2.59}$$

□

2.4.4 Corollary. *In the category* ***Grp*** *of groups every star-multiplicative graph extends to an internal category structure.*

Proof. In equation 2.4.3 let ω be the binary $+$. Then

$$\begin{aligned}
&\xi((\delta(x_1') + x_1) + (\delta(x_2') + x_2) - (x_1 + x_2)) \\
&\qquad = (x_1' + x_1) + (x_2' + x_2) - (x_1 + x_2) \\
&\qquad \Longleftrightarrow \xi(\delta(x_1') + x_1 + \delta(x_2') - x_1) = x_1' + x_1 + x_2' - x_1
\end{aligned} \tag{2.60}$$

Since this equation holds for all $x_1, x_1', x_2, x_2' \in X$, we can substitute $x_1' = 0$. We then have

$$\xi(x_1 + \delta(x_2') - x_1) = x_1 + x_2' - x_1; \tag{2.61}$$

that is, $\delta(x_1).x_2' = x_1 + x_2' - x_1$, (the Peiffer condition for crossed modules in groups).
This confirms (see [18], Remark 4.7) that every star multiplicative graph in the category of groups is an internal category. □

2.5 Actions and semi-direct products of crossed modules.

In this section we will define the notion of an action of one crossed module on another and use it to construct semidirect products of crossed modules in the variety of right Ω-loops.

2.5.1 Theorem. *Let* $\mathbf{C}$ *be an abstract category with equalizers and* $\mathbf{C}^M$ *be the category of* M*-actions in* $\mathbf{C}$*, where M is the monoid* $M = \{s, t : st = t,\, ts = s\}$*. Then*

$$\boldsymbol{RGraphs}(\mathbf{C}) \approx \mathbf{C}^M, \tag{2.62}$$

where $\boldsymbol{RGraphs}(\mathbf{C})$ *denotes the category of reflexive graphs in* $\mathbf{C}$.

Proof. Let us define functors

$$U : \mathbf{RGraphs}(\mathbf{C}) \longrightarrow \mathbf{C}^M$$
$$F : \mathbf{C}^M \longrightarrow \mathbf{RGraphs}(\mathbf{C})$$

where U sends a reflexive graph

$$\begin{tikzcd} G_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & G_0 \arrow[l, "e"'] \end{tikzcd} \tag{2.63}$$

to (G_1, ed, ec) (where ed and ec are the monoid operations) and F sends an M-object (C, s, t) to the diagram

$$\begin{tikzcd} C \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & Eq(s, 1_C) \arrow[l, "k"'] \end{tikzcd}$$

where $Eq(s, 1_C)$ denotes the equalizer of s and 1_C and d,c are induced by s,t respectively as depicted in the commutative diagram

$$\begin{tikzcd}
C \arrow[d, dashed, "d"'] \arrow[dr, "s"] & & \\
Eq(s, 1_C) \arrow[r, "k"] & C \arrow[r, bend left, "1_C"] \arrow[r, bend right, "s"'] & C \\
C \arrow[u, dashed, "c"] \arrow[ur, "t"'] & &
\end{tikzcd} . \tag{2.64}$$

Therefore, the equations kd=s and kc=t define s and t.

To establish an equivalence of categories, we must show that

$$FU \simeq 1_{\mathbf{RGraphs(C)}} , \tag{2.65}$$

$$UF \simeq 1_{\mathbf{C}^M} . \tag{2.66}$$

For the reflexive graph (2.63), we have

$$FU\left(\begin{tikzcd} G_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & G_0 \arrow[l, "e"'] \end{tikzcd} \right) = F(G_1, ed, ec) = \begin{tikzcd} G_1 \arrow[r, bend left, "\bar{d}"] \arrow[r, bend right, "\bar{c}"'] & Eq(ed, 1_{G_1}) \arrow[l, "k"'] \end{tikzcd} . \tag{2.67}$$

Now consider the diagram

$$\begin{tikzcd}
G_1 \arrow[r, bend left, "\bar{d}"] \arrow[r, bend right, "\bar{c}"'] \arrow[d, equal] & Eq(s, 1_{G_1}) \arrow[l, "k"'] \arrow[r, "k"] & G_1 \arrow[d, equal] \arrow[dr, "1_{G_1}"] & \\
G_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & G_0 \arrow[l, "e"'] \arrow[r, "e"] \arrow[u, dashed, "\varphi"'] & G_1 \arrow[r, "ed"'] & G_1 \quad .
\end{tikzcd} \tag{2.68}$$

Since k is an equalizer it is a monomorphism and hence

$$\begin{aligned} k\bar{d} &= ed = k\varphi d \\ \implies \bar{d} &= \varphi d, \end{aligned} \tag{2.69}$$

$$\begin{aligned} k\bar{c} &= ec = k\varphi c \\ \implies \bar{c} &= \varphi c. \end{aligned} \tag{2.70}$$

We must show that φ is an isomorphism. To do this we must show that

$$\begin{tikzcd}
G_0 \arrow[r, "e"] & G_1 \arrow[r, bend left, "1_{G_1}"] \arrow[r, bend right, "ed"'] & G_1
\end{tikzcd} \tag{2.71}$$

is an equalizer diagram.

We will use the fact that for the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & Z \\
X' \arrow[u, "\varphi"] \arrow[ur, "f\varphi"'] & &
\end{tikzcd} \tag{2.72}$$

with $f = Eq(g,h)$, the equalizer of g and h,

$$\varphi \quad is \quad an \quad isomorphism \iff f\varphi = Eq(g,h) \tag{2.73}$$

From the diagram

$$\begin{tikzcd}
X \arrow[d, dashed, "g"'] \arrow[dr, "f"] & & \\
G_0 \arrow[r, "e"] & G_1 \arrow[r, bend left, "1_{G_1}"] \arrow[r, bend right, "ed"'] & G_1
\end{tikzcd} , \tag{2.74}$$

given the morphism f such that f=edf, we take g=df as the unique morphism for which the equalizer condition is satisfied. Then eg=edf=f.

Conversely, if eg=f, then $deg = df \iff g = df$. This shows that the diagram (2.71) is an equalizer diagram. We have now established the isomorphism in Equation (2.65).

Using the definition of U and F and Diagram (2.64), we get

$$UF(C,s,t) = U\left(\begin{tikzcd}
C \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & G_0 \arrow[l, "k"']
\end{tikzcd}\right)$$
$$= (C, kd, kc).$$

Since s=kd and t=kc, we have the isomorphism $UF \simeq 1_{\mathbf{C}^M}$. This completes the proof of the category equivalence. $\square$

By close analogy with Loday's original definition [24] of Cat^1-groups we give the following definition of a Cat^1-loop.

2.5.2 Definition. *A Cat^1-loop (L, s, t) is a right Ω-loop L with 2 endomorphisms $s, t : L \longrightarrow L$, such that*

(L1) $ts = s,\ st = t,$

(L2) $[\ker s, \ker t] = 0.$

2.5.3 Remark. *A morphism of Cat^1-loops $(L, s, t) \longrightarrow (L', s', t')$ is a homomorphism of right Ω-loops $f : L \longrightarrow L'$ such that $s'f = fs$ and $t'f = ft$*

The resultant category of Cat^1-loops will be denoted by Cat^1-**RLoop**.

When $\mathbf{C}$ is a variety of right Ω-loops,

$$Cat(\mathbf{C}) \approx Cat^1(\mathbf{C}) \tag{2.75}$$

is a subvariety of $\mathbf{C}^M$, where $M = \{1, s, t\}$ is the monoid in which st=t and ts=s. Similarly, we get $Cat^2(\mathbf{C}) \subset (\mathbf{C}^M)^M$, $Cat^3(\mathbf{C}) \subset ((\mathbf{C}^M)^M)^M$ and this process can be extended to $Cat^n(\mathbf{C})$. by induction.

Therefore, Cat^n-loops can be obtained. Consequently, the theory of semidirect products developed in Section 2.2 applies to semidirect products of crossed modules.

2.5.4 Definition and Proposition. *Let $X = (X_0, X_1, d_X, c_X, e_X, m_X)$ and $B = (B_0, B_1, d_B, c_B, e_B, m_B)$ be internal categories in the category of internal categories in Ω-**RLoop**. A semidirect crossed module with $B_0 \ltimes \ker(d_B)$ acting on $X_0 \ltimes \ker(d_X)$ is the quadruple*

$$((B_0 \ltimes \ker(d_B), s_B, t_B), (X_0 \ltimes \ker(d_X), s_X, t_X), \hat{\xi}, \hat{\delta})$$

where

(a) $B_0 \ltimes \ker(d_B)$ *is the semidirect with* B_0*-action* $\xi_B : X_0 \flat \ker(d_B) \longrightarrow \ker(d_B)$,

(b) $X_0 \ltimes \ker(d_X)$ *is the semidirect product with* X_0*-action* $\xi_X : X_0 \flat \ker(d_X) \longrightarrow \ker(d_X)$,

(c) $\delta : (X_0 \ltimes \ker(d_X), s_X, t_X) \longrightarrow (B_0 \ltimes \ker(d_B), s_B, t_B)$ *is a morphism of cat*^{1}*-loops,*

(d) $\xi : (B_0 \ltimes \ker(d_B)(\flat(X_0 \ltimes \ker(d_X)) \longrightarrow (X_0 \ltimes \ker(d_X))$ *is the* $(B_0 \ltimes \ker(d_B))$*-action.*

Proof. There is an isomorphism

$$X \cong (X_0, X_0 \ltimes \ker(d_X), \bar{d}_X, \bar{c}_X, \bar{e}_X, \bar{m}_X)$$

$$\begin{tikzcd} = (X_0 \ltimes M) \times_{X_0} (X_0 \ltimes M) \arrow[r, "\bar{m}_X"] & (X_0 \ltimes M) \arrow[r, bend left, "\bar{d}_X"] \arrow[r, bend right, "\bar{c}_X"'] & X_0 \arrow[l, "\bar{e}_X"'] \end{tikzcd}$$

where $M = \ker(d_X), \bar{d}_X(x,m) = x, \bar{c}_X(x,m) = \delta_X(m) + x, \delta_X = \bar{d}_X|_M, \bar{e}_X(x) = (x,0), \bar{m}_X((\bar{\delta}_X(m) + x, m'), (x,m)) = (x, m' + m)$. Moreover, $(X_0, M, \xi_X, \delta_X)$ is a crossed module with X_0-action $\xi_X : X_0 \flat M \longrightarrow M$. $X_0 \ltimes M$ denotes the semidirect product with underlying set $X_0 \times M$ and Ω-algebra structure

$$\begin{aligned} &\omega((x_1, m_1), ..., (x_n, m_n)) \\ &\qquad = (\omega(x_1, ..., x_n), \xi_X(\omega(m_1 + x_1, ..., m_n + x_n) - \omega(x_1, ..., x_n))) \end{aligned} \tag{2.76}$$

for all $(x_1, m_1), ..., (x_n, m_n) \in M$ and for each n-ary operation $\omega \in \Omega$.
Similarly,

$$B \cong (B_0, B_0 \ltimes \ker(d_B), \bar{d}_B, \bar{c}_B, \bar{e}_B, \bar{m}_B)$$

$$\begin{tikzcd} = (B_0 \ltimes P) \times_{B_0} (B_0 \ltimes P) \arrow[r, "\bar{m}_B"] & (B_0 \ltimes P) \arrow[r, bend left, "\bar{d}_B"] \arrow[r, bend right, "\bar{c}_B"'] & B_0 \arrow[l, "\bar{e}_B"'] \end{tikzcd}$$

where $P = \ker(d_B), \bar{d}_B(b,p) = b, \bar{c}_B(b,p) = \delta_B(p) + b, \delta_B = \bar{d}_B|_P, \bar{m}_B((\bar{\delta}_B(p) + b, p'), (b,p)) = (b, p' + p)$. As a crossed module $(B_0, P, \xi_B, \delta_B)$ has B_0-action

$\xi_B : B_0 \flat P \longrightarrow P$. $B_0 \ltimes P$ denotes the semidirect product with underlying set $B_0 \times P$ and Ω-algebra structure

$$\begin{aligned}\omega((b_1, p_1), ..., (b_n, p_n))&\\ &= (\omega(b_1, ..., b_n), \xi_B(\omega(p_1 + b_1, ..., p_n + b_n) - \omega(b_1, ..., b_n)))\end{aligned} \tag{2.77}$$

for all $(b_1, p_1), ..., (b_n, p_n) \in B \times P$ and for each n-ary operation $\omega \in \Omega$.
Recall that, via the equivalence

$$Cat^1\textbf{-RLoop} \approx XMod(\Omega\textbf{-RLoop}),$$

a crossed module (B, X, ξ, δ) corresponds to the Cat^1-loop $(B \ltimes X, s, t)$, where $s(b, x) = (b, 0), t(b, x) = (\delta(x) + b, 0)$ for all $(b, x) \in B \ltimes X$. The cat^1-loop associated with the crossed module $(X_0, M, \xi_X, \delta_X)$ is $(X_0 \ltimes M, s_X, t_X)$, where $s_X(x, m) = (x, 0)$, $t_X(x, m) = (\delta_X(m)+x, 0)$. The crossed module $(B_0, P, \xi_B, \delta_B)$ has associated cat^1-loop $(B_0 \ltimes P, s_B, t_B)$, where $s_B(b, p) = (b, 0)$ and $s_B(b, p) = (\delta_B(p) + b, 0)$. Now consider the $(B_0 \ltimes P, s_B, t_B)$-action of cat^1-loops $\hat{\xi} : (B_0 \ltimes P, s_B, t_B) \flat (X_0 \ltimes M, s_X, t_X) \longrightarrow (X_0 \ltimes M, s_X, t_X)$. Using this action we can form the crossed module

$$\Big((B_0 \ltimes P, s_B, t_B), (X_0 \ltimes M, s_X, t_X), \hat{\xi}, \hat{\delta}\Big)$$

where

$$\hat{\delta} : (X_0 \ltimes M, s_X, t_X) \longrightarrow (B_0 \ltimes P, s_B, t_B)$$

is a morphism of cat^1-loops. By 2.35 and 2.39 the following identities hold; $\hat{\delta} s_X = s_B \hat{\delta}$, $\quad \hat{\delta} t_X = t_B \hat{\delta}$ and for each n-ary operation $\omega \in \Omega$,

(a)

$$\begin{aligned}\omega\left(\hat{\delta}(x_1, p_1) + (b_1, m_1), ..., \hat{\delta}(x_n, p_n) + (b_n, m_n)\right)&\\ = \omega\left((b_1, m_1), ..., (b_n, m_n)\right)&\\ = \hat{\delta}\left(\hat{\xi}(\omega((x_1, p_1) + (b_1, m_1), ..., (x_n, p_n) + (b_n, m_n)))\right)&\end{aligned} \tag{2.78}$$

for all

$$(x_1,p_1),...,(x_n,p_n) \in (X_0 \times P),$$
$$(b_1,m_1),...,(b_n,m_n) \in B_0 \times M. \tag{2.79}$$

(b)

$$\begin{aligned}
&\hat{\xi}(\omega((x_1',p_1') + (\hat{\delta}(x_1,p_1) + (b_1,m_1)),...,(x_n',p_n') + (\hat{\delta}(x_n,p_n) + (b_n,m_n))) \\
&\qquad - \omega(\hat{\delta}(x_1,p_1) + (b_1,m_1),...,\hat{\delta}(x_n,p_n) + (b_n,m_n)) \\
&+ \hat{\xi}(\omega((x_1,p_1) + (b_1,m_1),...,(x_n,p_n) + (b_n,m_n)) - \omega((b_1,p_1),...,(b_n,m_n)))) \\
&= \hat{\xi}(\omega(((x_1',p_1') + (x_1,p_1)) + (b_1,m_1),...,((x_n',p_n') + (x_n,p_n)) + (b_n,m_n)) \\
&\qquad - \omega((b_1,m_1),...,(b_n,m_n)))
\end{aligned} \tag{2.80}$$

for all

$$(x_1,p_1),...,(x_n,p_n) \in X_0 \times P,$$
$$(x_1',p_1'),...,(x_n',p_n') \in X_0 \times P,$$
$$(b_1,m_1),...,(b_n,m_n) \in B_0 \times M. \tag{2.81}$$

$\square$

2.5.5 Proposition. *The semidirect product* $(B_0 \ltimes P)\ltimes(X_0 \ltimes M)$ *has underlying set* $(B_0 \times P) \times (X_0 \times M)$ *with* Ω*-algebra structure*

$$\begin{aligned}
&\omega(((b_1,p_1),(x_1,m_1)),...,((b_n,p_n),(x_n,m_n))) \\
&\quad = (\omega((b_1,p_1),...,(b_n,p_n)), \xi(\omega((x_1,m_1) + (b_1,p_1),...(x_n,m_n) + (b_n,p_n)) \\
&\qquad - \omega((b_1,p_1),...,(b_n,p_n))))
\end{aligned} \tag{2.82}$$

for all $(b_1,p_1),...,(b_n,p_n) \in B_0 \times P$ *and* $(x_1,m_1),...(x_n,m_n) \in X_0 \times M.$

Proof. This is a straightforward application of Theorem 2.2.1. $\square$

2.5.6 Example (Groups). *Let* $\omega = +$. *Then the formula* (2.82) *for the semidirect product of crossed modules of groups becomes*

$$\begin{aligned}
&((b_1, p_1), (x_1, m_1)) + ((b_2, p_2), (x_2, m_2)) \\
&= ((b_1, p_1) + (b_2, p_2), \xi(((x_1, m_1) + (b_1, p_1)) + ((x_2, m_2) + (b_2, p_2)) - ((b_1, p_1) + (b_2, p_2)))) \\
&= ((b_1 + p_2, p_1 + \xi_B(b_1 + p_2 - b_1)), (x_1, m_1) + \xi((b_1, p_1) + (x_2, m_2) - (b_1, p_1)))
\end{aligned} \tag{2.83}$$

where $((b_1, p_1), (x_1, m_1)), ((b_2, p_2), (x_2, m_2)) \in (B_0 \ltimes P) \ltimes (X_0 \ltimes M)$ *and* $\xi_B : B_0 \flat P \longrightarrow P$ *and* $\xi : (B_0 \ltimes P) \ltimes (X_0 \ltimes M) \longrightarrow (X_0 \ltimes M)$ *are the actions.*

Chapter 3

General remarks and examples

In this chapter we shall apply the theory developed in Chapter 2 to give examples and make some general remarks.

We show that our constructions agree with the known ones in the familiar algebraic categories, specifically the categories of groups, rings and Lie algebras.

3.1 Remarks

Let us begin by highlighting the following observations from the previous chapter.

(a) In [18] Janelidze extends the notion of a crossed module to an arbitrary semiabelian category, which describes internal categories and is defined via condition 1.11. Using condition 1.11 and the description of the monad $B\flat(-)$ in **Grps**, one can show that condition 1.11 gives the usual condition (see [18], Example 3.10).

(b) Semidirect products of Ω-loops have the same underlying sets as products.

This helps to describe the theory of crossed modules independently of condition 1.11; they are defined via 2.39 instead.

(c) Condition 2.39 is much simpler than 1.11 and gives the usual condition for groups much more easily.

(d) Condition 2.39 is intermediate between 1.11 and the usual condition for groups. However, there is no visible way of deducing 2.39 from 1.11; it was possible because we knew a good description of the monad $B\flat(-)$ in **Grps**.

Thus, we arrive at independent theory of crossed modules in the variety of right Ω-loops.

3.2 Categories of groups with operations and categories of interest.

Before we give the examples, let us recall the notion of a category of groups with operations (See [30] and [31]). Let $\mathbf{C}$ be a category of groups with a set of operations Ω and a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group identities and the following conditions hold. If Ω_i is the set of i-ary operations in Ω_1 then:

1. $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

2. the group operations $0,\ -, +$ are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $* \in \Omega_2$, then Ω_2' contains $*^\circ$ defined by $x *^\circ y = y * x$. Assume further that $\Omega_0 = \{0\}$;

3. for each $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;

4. for each $\omega \in \Omega'_1$ and $* \in \Omega'_2$, $\mathbb{E}$ includes the identities $\omega(x+y) = \omega(x) + \omega(y)$ and $\omega(x) * y = \omega(x * y)$.

In the context of the variety of right Ω-loops, we shall focus on the case when all additional operations satisfy at least one of the following conditions:

(a) they are unary and loop homomorphism;

(b) they are binary and distributive under the loop structure.

Such categories are called in [30] *categories of interest*.

Categories of interest are semiabelian and so include many of the familiar algebraic categories. For example, groups, rings, associative algebras, modules over a ring and Lie algebras can be interpreted as categories of interest. In the cases of the categories of groups and modules we take $\Omega'_2 = \emptyset$. For associative algebras with multiplication represented by $*$ take $\Omega'_2 = \{*, *^\circ\}$. In the case of Lie algebras we take $\Omega'_2 = \{[\,], [\,]^\circ\}$ where $[a,b]^\circ = [b,a] = -[a,b]$.

Given a split extension

$$\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, "\alpha"] & B \arrow[l, bend right, "\beta"'] \end{tikzcd}, \qquad \alpha\beta = 1_B\ ,$$

in the category of interest $\mathbf{C}$ we can define a semidirect product $B \ltimes X$ using the results of Chapter 2. Using Theorem 2.2.1 and the binary $+$, we have

$$(b_1, x_1) + (b_2, x_2) = (b_1 + b_2, x_1 + \xi(b_1 + x_2 - b_1)) \tag{3.1}$$

for all $b_1, b_2 \in B$ and $x_1, x_2 \in X$.

For the binary operation $*$, we get

$$
\begin{aligned}
&(b_1, x_1) * (b_2, x_2) = (b_1 * b_2, \xi((x_1 + b_1) * (x_2 + b_2)) - b_1 * b_2) \\
&\iff (b_1, x_1) * (b_2, x_2) = (b_1 * b_2, \xi(x_1 * x_2 + b_1 * x_2 + x_1 b_2 + b_1 * b_2 - b_1 * b_2)) \\
&\iff (b_1, x_1) * (b_2, x_2) = (b_1 * b_2, \xi(x_1 * x_2 + b_1 * x_2 + x_1 * b_2)) \\
&\iff (b_1, x_1) * (b_2, x_2) = (b_1 * b_2, \xi(x_1 * x_2) + \xi(b_1 * x_2) + \xi(x_1 * b_2)) \\
&\iff (b_1, x_1) * (b_2, x_2) = (b_1 * b_2, x_1 * x_2 + b_1 * x_2 + x_1 * b_2) \qquad (3.2)
\end{aligned}
$$

where $\xi(x_1 * x_2) = x_1 * x_2$, $\xi(b_1 * x_2) = b_1 * x_2$ and $\xi(x_1 * b_2) = x_1 * b_2$.

Using these results we see that, in any category of interest, given an action $\xi : B\flat X \longrightarrow X$ of B on X, the semidirect product $B \ltimes X$ is the universal algebra whose underlying set is $B \times (X, \xi)$ and the operations are defined by

$$(b_1, x_1) + (b_2, x_2) = (b_1 + b_2, x_1 + b_1.x_2) \qquad (3.3)$$

$$(b_1, x_1) * (b_2, x_2) = (b_1 * b_2, x_1 * x_2 + b_1 * x_2 + x_1 * b_2) \qquad (3.4)$$

This same result was obtained in [30].

3.3 Groups, rings and Lie algebras

Let **C** be as in Theorem 2.2.1. Since the formula 2.26 determines the semi-direct products, and therefore all objects in $Pt_{\mathbf{C}}(B)$ (for all B in **C**), and since the category $Pt_{\mathbf{C}}(B)$ is equivalent to the category $B\flat(-)$-algebras, any $B\flat(-)$-algebra structure $\xi : B\flat X \longrightarrow X$ is determined by the values of

$$\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)) \text{ (in the notation above)}. \qquad (3.5)$$

This means that $B\flat(-)$-algebras form a variety whose basic operations can be described as the basic operations of **C** and additional operations $\omega_{b_1,...,b_n}$, where ω is an n-ary basic operation of **C** and $b_1, ..., b_n \in B$; given a $B\flat(-)$-algebra structure $\xi : B\flat X \longrightarrow X$, the operation $\omega_{b_1,...,b_n} : X^n \longrightarrow X$ is of course defined by

$$\omega_{x_1,...,x_n} = \xi(\omega(x_1 + b_1, ..., x_n + b_n) - (b_1, ..., b_n)). \tag{3.6}$$

Moreover, the identities that are axioms for **C** determine, via 2.26, the identities that the operations $\omega_{b_1,...,b_n}$ must satisfy, and it is usually a routine calculation to find those. In particular, as also confirmed by the results of [30], for groups, rings, and Lie algebras we obtain the following results respectively:

Groups

For groups the operation ω becomes a binary + (addition). Denote this binary addition by A. Then

$$\omega_{b_1,b_2} = (A_{b_1,b_2})_{b_1,b_2} \tag{3.7}$$

where $b_1, b_2 \in B$ and

$$A_{b_1,b_2}(x_1, x_2) = \xi((x_1 + b_1 + (x_2 + b_2) - (b_1 + b_2)) \tag{3.8}$$

$$= \xi(x_1 + b_1 + x_2 + b_2 - b_2 - b_1) \quad (\textit{by associativity of} +) \tag{3.9}$$

$$= \xi(x_1 + b_1 + x_2 - b_1) \tag{3.10}$$

$$= \xi(x_1) + \xi(b_1 + x_2 - b_1) \tag{3.11}$$

$$= x_1 + \xi(b_1 + x_2 - b_1) \tag{3.12}$$

If $\xi(b + x - b) = bx$, then we have $A_{b_1,b_2}(x_1, x_2) = x_1 + b_1x_2$. When A is commutative, we get

$$A_{b_1,b_2}(x_1, x_2) = A_{b_2,b_1}(x_2, x_1) \tag{3.13}$$

$$x_1 + b_1x_2 = x_2 + b_2x_1 \tag{3.14}$$

and this equalities hold for all $x_1, x_2 \in X$ and for all $b_1, b_2 \in B$.
Putting $x_2 = 0$, we get $x_1 = b_2x_1$ for all $x_1, x_2 \in X$, $b_1, b_2 \in B$. This shows that if A is commutative the B-action on X is trivial.

Rings

When the equivalence of Proposition 2.1.2 is extended to the category of rings, we have

3.3.1 Theorem. *The category Pt(**Rng**) of points over B in the category of rings is equivalent to the category Act(**Rng**) of B-actions in the category of rings; that is to say,*

$$Pt(\boldsymbol{Rng}) \approx Act(\boldsymbol{Rng}). \tag{3.15}$$

Under this equivalence, an object (A, α, β) corresponds to the quadruple

$$(B, X, \varphi : B \times X \longrightarrow X, \psi : X \times B \longrightarrow X)$$

with $X = \ker(\alpha), \varphi(b, x) = \beta(b)x$ and $\psi(x, b) = x\beta(b)$. The quadruple (B, φ, ψ) corresponds to the split epimorphism

$$\begin{tikzcd} B \ltimes (X, \varphi, \psi) \arrow[r, bend left, "\pi_1"] & B \arrow[l, bend left, "\iota_1"] \end{tikzcd}$$

where $B \ltimes (X, \varphi, \psi) = B \times X$ as additive groups and

$$(b, x)(b', x') = (bb', xx' + \varphi(b, x') + \psi(x, b')) \tag{3.16}$$

When $\omega = .$ (binary multiplication denoted by m) Equation 2.26 becomes

$$\omega_{b_1,b_2} = (m_{b_1,b_2})_{b_1,b_2} \tag{3.17}$$

Using this multiplication in place of binary group addition in Equation 3.7 yields

$$m_{b_1,b_2}(x_1, x_2) = \xi((x_1 + b_1)(x_2 + b_2) - b_1 b_2) \tag{3.18}$$

$$= \xi(x_1 x_2 + b_1 x_2 + x_1 b_2) \tag{3.19}$$

$$= \xi(x_1 x_2) + \xi(b_1 x_2) + \xi(x_1 b_2) \tag{3.20}$$

$$= (x_1 x_2) + \xi(b_1 x_2) + \xi(x_1 b_2) \tag{3.21}$$

Defining ξ as $\xi(b_1.x_2) = b_1.x_2$ and $\xi(x_1.b_2) = x_1.b_2$, we see that

$$m_{b_1,b_2}(x_1, x_2) = x_1.x_2 + b_1.x_2 + x_1.b_2 \tag{3.22}$$

The construction of the categorical semidirect product given in Theorem 2.2.1 gives

$$\omega((b_1, x_1), ..., (b_n, x_n)) = (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))$$

$$\Longleftrightarrow (b_1, x_1).(b_2, x_2) = (b_1.b_2, \xi((x_1 + b_1).(x_2 + b_2) - b_1.b_2)) \tag{3.23}$$

$$\Longleftrightarrow (b_1, x_1).(b_2, x_2) = (b_1.b_2, \xi(x_1.x_2 + b_1.x_2 + x_1.b_2)) \tag{3.24}$$

$$\Longleftrightarrow (b_1, x_1).(b_2, x_2) = (b_1.b_2, x_1.x_2 + \xi(b_1.x_2) + \xi(x_1.b_2)) \tag{3.25}$$

Equation 3.25 is the same as the formula for the classical semidirect product with the multiplication operation defined by Equation 3.16.

3.3.2 Definitions. *[23] Let B be a ring. A precrossed B-ring is a triple (B, R, δ) where R is a two-sided B-module and $\delta : R \longrightarrow B$ is a B-module morphism. (B, R, δ) is a crossed B-ring if, moreover, $\delta(r)r' = rr' = r\delta(r')$ for all $r, r' \in R$. A morphism of (pre)-crossed B-rings $f : (B, R, \delta') \longrightarrow (B, R', \delta')$ is a B-module morphism $f : R \longrightarrow R'$ such that $\delta' f = \delta$.*

By the formula for precrossed modules of right Ω-loops, we get

$$\omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n) - \omega(b_1, ..., b_n) = \delta(\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))$$

$$\Longleftrightarrow (\delta(x_1) + b_1).(\delta(x_2) + b_2) - b_1 b_2 = \delta(\xi((x_1 + b_1).(x_2 + b_2))) \tag{3.26}$$

$$\begin{aligned}\Longleftrightarrow \delta(x_1).\delta(x_2) + \delta(x_1).b_2 + b_1.\delta(x_2) &= \delta(\xi(x_1.x_2 + x_1.b_2 + b_1.x_2)) \\ &= \delta(x_1.x_2) + \delta(\xi(x_1.b_2)) + \delta(\xi(b_1.x_2))\end{aligned} \tag{3.27}$$

The Equation 3.27 is equivalent to

$$\delta(x_1).b_2 + b_1.\delta(x_2) = \delta(\xi(x_1.b_2)) + \delta(\xi(b_1.x_2)) \tag{3.28}$$

and since the equality holds for all $x_1, x_2 \in X,\ b_1, b_2 \in B,$ we deduce that

$$\delta(\xi(x_1.b_2)) = \delta(x_1).b_2, \tag{3.29}$$

$$\delta(\xi(b_1.x_2)) = b_1.\delta(x_2). \tag{3.30}$$

These two conditions define a precrossed module in the category of rings.

The next task is to check the condition for crossed modules (that is, the Peiffer condition) using Equation 2.39 of the previous chapter.

$$\begin{aligned}\xi(\omega(x'_1 + (\delta(x_1) + b_1), ..., x'_n + (\delta(x_n) + b_n)) - \omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n))+ \\ \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)) \\ = \xi(\omega((x'_1 + x_1) + b_1, ..., (x'_n + x_n) + b_n) - \omega(b_1, ..., b_n))\end{aligned} \tag{3.31}$$

$$\iff \xi((x'_1 + b'_1).(x'_2 + b'_2) - b'_1.b'_2) + \xi((x_1 + b_1).(x_2 + b_2) - b_1.b_2) \tag{3.32}$$

$$= \xi(((x'_1 + x_1) + b_1).((x'_2 + x_2) + b_2) - b_1.b_2) \tag{3.33}$$

$$\iff \xi(x'_1.x'_2 + x'_1.b'_2 + b'_1.x'_2) + \xi(x_1.x_2 + x_1.b_2 + b_1.x_2) \tag{3.34}$$

$$= \xi((x'_1 + x_1).(x'_2 + x_2) + (x'_1 + x_1).b_2 + b_1.(x'_2 + x_2)) \tag{3.35}$$

$$\iff \xi\,(x'_1.x'_2 + x'_1.(\delta(x_2) + b_2) + (\delta(x_1) + b_1).x'_2) + \xi(x_1.x_2 + x_1.b_2 + b_1.x_2) \tag{3.36}$$

$$= \xi\,((x'_1 + x_1).(x'_2 + x_2) + (x'_1 + x_1).b_2 + b_1.(x'_2 + x_2)) \tag{3.37}$$

Since Equation 3.37 is true for all $x'_1, x'_2, x_1, x_2 \in X$ and $b_1, b_2 \in B$, we can put $b_1 = b_2 = 0$. This yields

$$\xi(x'_1.x'_2 + x'_1.\delta(x_2) + \delta(x_1).x'_2) + \xi(x_1.x_2) = \xi((x'_1.x'_2 + x'_1.x_2 + x_1.x'_2 + x_1.x_2) \tag{3.38}$$

$$\Longleftrightarrow \xi(x'_1.\delta(x_2)) + \xi(\delta(x_1).x'_2) = x'_1.x_2 + x_1.x'_2. \tag{3.39}$$

Consequently, we have the Peiffer condition (again using the fact that the equality of Equation 3.39 holds for all $x_1, x_2, x'_1, x'_2 \in X$)

$$\delta(x)x' = xx' = x\delta(x') \tag{3.40}$$

for crossed modules.

Lie algebras

This example is included to demonstrate that the theory developed in the previous chapter is also valid for non-associative algebras.

3.3.3 Definition. *[25] Let R be a fixed ring (with or without unit). A Lie algebra over R is an R-module L equipped with an R-linear map $[-,-] : L \times L \longrightarrow L$ called the Lie bracket which satisfies the following two conditions:*

(a) antisymmetry: $[x, x] = 0$ *for all* $x \in L$.

(b) Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, *for all* $x, y, z \in L$.

Note that (a) implies: $[x, y] = -[y, x]$, *for all* $x, y \in L$.

Let B and X be Lie algebras. By an action of B on X we mean an R-linear map $B \times X \longrightarrow X$ satisfying

(a) $[b, b'].x = b(b'.x) - b'(b.x)$,

(b) $b.[x, x'] = [b.x, x'] + [x, b.x']$,

for all $x, x' \in X$, $b, b' \in B$. Using ω as binary Lie multiplication [ ] (denoted by m^L) in Equation 3.17, we get

$$\omega_{b_1,b_2} = m^L_{b_1,b_2}(x_1, x_2) \tag{3.41}$$

$$\begin{aligned}
\omega_{b_1,b_2}(x_1, x_2) &= \xi([x_1 + b_1, x_2 + b_2] - [b_1, b_2]) && (3.42)\\
&= \xi([x_1, x_2] + [b_1, x_2] + [x_1, b_2] + [b_1, b_2] - [b_1, b_2]) && (3.43)\\
&= \xi([x_1, x_2] + [b_1, x_2] + [x_1, b_2]) && (3.44)\\
&= \xi([x_1, x_2] + [b_1, x_2] - [b_2, x_1]) \qquad (using\ [x_1, b_2] = -[b_2, x_1]) && (3.45)\\
&= [x_1, x_2] + \xi([b_1, x_2]) - \xi([b_2, x_1])) && (3.46)
\end{aligned}$$

The semidirect product is constructed as in the case of rings using Theorem 2.2.1;

$$\begin{aligned}
\omega((b_1, x_1), ..., (b_n, x_n)) &= (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))\\
\Longleftrightarrow [(b_1, x_1), (b_2, x_2)] &= ([b_1, b_2], \xi([x_1 + b_1, x_2 + b_2] - [b_1, b_2]))\\
&= ([b_1, b_2], \xi([x_1, x_2] + [b_1, x_2] + [x_1, b_2] + [b_1, b_2] - [b_1, b_2]))\\
&= ([b_1, b_2], [x_1, x_2] + [b_1, x_2] - [b_2, x_1])
\end{aligned}$$

This confirms that $B \ltimes X$ is the R-module $B \times X$ equipped with the bracket $[(b_1, x_1), (b_2, x_2)] = ([b_1, b_2], [x_1, x_2] + [b_1, x_2] - [b_2, x_1])$.

Recall from [22] the notion of a crossed module of Lie algebras. A crossed module of Lie algebras is a Lie algebra homomorphism $\delta : X \longrightarrow B$ together with an action of B on X such that,

(a) $\delta(b.x) = [b, \delta(x)]$,

(b) $\delta(x).x' = [x, x']$,

for all $x, x' \in X, \ b \in B$.

When $\omega = [,]$, we have

$$\begin{aligned}
&[\delta(x_1) + b_1, \delta(x_2 + b_2)] - [b_1, b_2] = \delta(\xi([x_1 + b_1, x_2 + b_2] - [b_1, b_2])) \\
&\iff [\delta(x_1), \delta(x_2)] + [\delta(x_1), b_2] + [b_1, \delta(x_2)] = \delta(\xi([x_1, x_2] + [x_1, b_2] + [b_1, x_2])) \\
&\iff [\delta(x_1), \delta(x_2)] + [\delta(x_1), b_2] + [b_1, \delta(x_2)] = \delta(\xi([x_1, x_2])) + \delta(\xi([x_1, b_2])) + \delta(\xi([b_1, x_2]))
\end{aligned} \tag{3.47}$$

from which we deduce the identities

$$\begin{aligned}
\delta(x_1.x_2) &= [\delta(x_1), \delta(x_2)], \\
\delta(x_1.b_2) &= [\delta(x_1), b_2], \\
\delta(b_1.x_2) &= [b_1, \delta(x_2)].
\end{aligned} \tag{3.48}$$

The equation $\delta(b_1.x_2) = [b_1, \delta(x_2)]$ gives condition for a precrossed module in Lie algebras.

From Equation 2.39 for crossed modules in Ω-**RLoop**, we get

$$\begin{aligned}
&\xi([x_1' + (\delta(x_1) + b_1), x_2' + (\delta(x_2) + b_2)] - [\delta(x_1) + b_1, \delta(x_2) + b_2]) \\
&+\xi([x_1 + b_1, x_2 + b_2] - [b_1, b_2]) = \xi([(x_1' + x_1) + b_1, (x_2' + x_2) + b_2] - [b_1, b_2])
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
&\iff \xi([x_1', x_2'] + [x_1', \delta(x_2) + b_2] + [\delta(x_1) + b_1, x_2']) + \xi([x_1, x_2] + [x_1, b_2] + [b_1, x_2]) \\
&= \xi([x_1' + x_1, x_2' + x_2] + [x_1' + x_1, b_1] + [b_1, x_2' + x_2])
\end{aligned} \tag{3.50}$$

Since Equation 3.50 holds for all $x_1, x_1', x_2, x_2' \in X$ and for all $b_1, b_2 \in B$, we can put $b_1 = b_2 = 0$ and $x_1' = x_2 = 0$. We then get

$$\xi([0, x_2] + [0, 0] + [\delta(x_1), x_2']) + \xi([x_1, 0]) = \xi([x_1, x_2']) \tag{3.51}$$

$$\iff \xi([\delta(x_1), x_2']) = \xi([x_1, x_2']) \tag{3.52}$$

Defining ξ as $\xi([\delta(x_1), x_2']) = \delta(x_1).x_2',$ we obtain the Peiffer identity $\delta(x_1).x_2' = [\delta(x_1), x_2'].$

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