Mathematical, cognitive and didactic elements of the multiplicative conceptual field investigated within a Rasch assessment and measurement framework

by

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ABSTRACT

The transition from whole numbers to rational numbers and then to real numbers constitutes an important development in higher primary and lower secondary schooling (Usiskin, 2005). The transition takes place alongside the acquisition of a constellation of concepts, requiring multiplication, and its inverse division. The problems encountered by learners who have not made the transition manifest in inability to execute problems where fraction and ratio concepts, and proportional reasoning, are involved. These findings have emerged consistently in large-scale assessment and in classroom observations over the past 30 years (Hart, 1981; Mullis et al, 2004).

The production of meaningful outcomes requires extensive effort in ensuring that the elements deemed critical for informing validity of assessment are seriously considered. It is asserted in this thesis that five intersecting domains are required for ensuring composite validity. These domains comprise firstly a plausible and workable view of the *nature of mathematics*, and its concepts and operations, adequate for education purposes. Secondly, building on the notion that validity includes the inferences and actions to be taken from the test results, a theory of the *acquisition of mathematical concepts*, has to be considered. Thirdly, and aligned with the acquisition of mathematical concepts is an *approach to teaching*, which encompasses the fourth element, the *role of language and representation*. Finally the *type and function of assessment* has to be envisaged against the background of the knowledge domain mathematics, the acquisition of concepts, and an approach to teaching. The quality of the assessment instrument and assessment processes, embracing also the analysis and subsequent inferences and actions proposed, may be gauged from the perspective of the above five elements.

Given serious attention to validity, building upon the five components, there is the potential to locate threshold concepts that may constitute obstacles to mathematical development. The identification of such threshold concepts may inform the construction of didactical sequences that ensure the transition to more advanced mathematical concepts and processes that allow for greater generalisation (Meyer & Land, 2005).

The main research question may be expressed as;

> How may the essential elements of a framework including mathematical, cognitive, and didactic elements, and applied in the multiplicative conceptual field, address the challenges in mathematics education, and inform the curriculum and the validity of assessment processes.

This thesis aims to model the assurance of construct validity, drawing primarily on the theory of conceptual fields (Vergnaud, 1988). It is proposed in this thesis, that this theory best attends to the essential features of mathematics learning and teaching, by providing a framework that is *mathematical*, by making explicit the structural links across concepts, and by tracing the filiations and thresholds along the mathematical path from early arithmetic to advanced mathematics. From a *cognitive* perspective, the concepts-in-action and theorems-in-action provide the building blocks, which teachers may use to help learners transform current thinking into generalisable concepts and actions. The development along a mathematical path requires analysis of both problem situations, in which the desired concepts and theorems are embedded, and the observation of the current concepts-in-action and theorems-in-action with which students engage the problem situations. From a *didactic* perspective, critical elements required for the scaffolding of concepts through natural *language*, *diagrams* or *symbols* are noted, and from an *evaluative* perspective, critical elements pertaining to the assessment of mathematical concepts are offered.

The specific mathematical focus of this thesis is the multiplicative conceptual field (Vergnaud, 1983), which involves multiplication and division, fractions, ratio, rate and proportion, probability and percent, and the associated cognitive components. The multiplicative conceptual field circumscribes the additive conceptual field and is nested within the algebraic conceptual field. From a mathematical perspective, the question is asked "What features of mathematics are central to developing mathematical proficiency and therefore inform assessment and analysis?" From a psychological perspective the question is "How can the extant implicit and local ideas of learners be transformed into mathematics concepts and theorems that can be generalised to many classes of problems?" Pertaining specifically to the multiplicative conceptual field, the question is asked "What plausible hierarchies of mathematical situations, mathematical concepts and cognitive processes can facilitate the acquisition of the complex array of related constructs within the multiplicative conceptual field?"

The thesis locates itself within the South African mathematics education milieu. It uses the work of Vergnaud to argue for an extension to dominant constructivist and Bernsteinian approaches here, specifically in the multiplicative conceptual field.

The research design involved constructing an assessment instrument, including apposite items, deemed to be realisations of the multiplicative conceptual field, depicting a range of difficulty targeted at two particular sets of learners. The items were selected from the TIMSS 2003 set of released items (IEA, 2005), which constituted the problem situations. Learners from Grades 7, 8 and 9 (ages 12 to 15) from schools considered to be well functioning within the South African context, and encompassing the range of demographics, were tested. Application of the Rasch measurement model (Rasch, 1960/80) provides indicators of the successful, or otherwise, construction of a measuring instrument, thereby encouraging a reflective approach to the instrument and the construct being tested. Given a successful measure of the construct, the Rasch model configures the location of both item difficulty and learner proficiency on the same scale. This ordering of items enabled the analysis of hierarchical conceptual strands selected within the multiplicative conceptual field, with detailed attention to factors such as mathematical structure and representation that it is suggested contribute to the level of difficulty. Subsets of learners at selected focus points along the scale enabled additional insights through interview discussions into the mastery of concepts at particular levels and into the associated errors. The resulting matrix, of learner proficiency and item complexity, provides an overview of the concepts attained and those yet to be mastered, thereby yielding insights into each learner's current zone of proximal development (Vygotsky, 1962).

The need for a specific theory of mathematics education assessment is proposed as a response to the complexity inherent in both mathematics itself, and to the multiple paths to attaining proficiency. The outcome of this research study exhibits the potential of a model, drawing on the theory of conceptual fields and applying the Rasch measurement model, to provide diagnostic insights into the plausible mathematical development of individual learners along proposed mathematical pathways.

The focus here is on the multiplicative conceptual field, where the critical threshold concepts, rational number and proportional reasoning are the foundations required for further mathematics. Through attention to the composite notion of construct validity, taking seriously the five elements deemed to ensure construct validity, within a Rasch measurement framework, it is envisaged that the outcomes, in addition to being meaningful, may inform consequent action at the different levels of an education system.

The thesis concludes with reflections upon the theory of conceptual fields, and associated constructs, the domain of rational number and the construct validity of the instrument itself. The theoretical and empirical work in this thesis provides the foundation for teaching design experiments, extensions to the assessment instrument, and for replication of the analyses applied in this thesis in formative assessment settings.

Keywords: multiplicative conceptual field; Rasch measurement model; assessment; rational number; proportional reasoning; threshold concepts, mathematics education

List of Abbreviations

DBE	Department of Basic Education
DOE	Department of Education
C2005	Curriculum 2005
CAPS	Curriculum and Assessment Policy Statement
CSMS	Concepts in Secondary Mathematics and Science
GET	General Education and Training
GNI	Gross National Income
ICCAMS	Increasing Confidence and Competence in Algebraic and Multiplicative Structures
IEA	International Association for the Evaluation of Educational Achievement
NCS	National Curriculum Statement
NMAP	National Mathematics Advisory Panel
RME	Realistic Mathematics Education
STEM	Science, Technology, Engineering and Mathematics
TIMSS	Trends in International Mathematics and Science Study

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1.1 Mathematical knowledge

There can be no doubt that learning mathematics at any phase of schooling is demanding on the part of both learner and teacher, and in many instances there is neither the enduring retention of concepts taught nor the ability to apply skills in relevant contexts. Sfard notes that the "particular intricacy of mathematical thinking, the ubiquitous, sometimes insurmountable difficulty experienced by those who learn it and the resulting persistent lack of success in teaching the subject" are as puzzling as they are conspicuous (Sfard, 1991, p. 2). Sfard's observations are not new. Henri Poincaré, the French mathematician and philosopher, makes a similar observation.

> One … fact must astonish us, or rather would astonish us if we were not too much accustomed to it. How does it happen that there are people who do not understand mathematics? If the science only invokes the rules of logic, those accepted by all well-formed minds … how does it happen that there are so many people impervious to it? (Poincaré, 1952, p. 49, French original was published in 1908, cited Sfard, 1991, p. 2)

While these observations may resonate with the thinking of many in the general education community, there are many pockets of research which show promising avenues for achieving mathematical competence, notably the work of Vergnaud (1979; 1983; 1988; 1990; 1994; 1997; 1998; 2009). He proposes that establishing "the links between ordinary arithmetical situations and the relevant mathematical concepts" provides an important avenue for achieving mathematical proficiency, and therefore is "the most challenging question in mathematics" (1979, p. 263).

The starting point for this research study is the construction of the learner as a "scientist" who is engaged with his or her environment from the early years, both in and out of the classroom (Bannister & Fransella, 1986). The individual is constantly hypothesising and experimenting, accepting only what makes sense and rejecting the concepts and processes that represent "insults" to his intelligence, or that cannot be usefully accommodated into existing structures (Skemp, 1971).

One of the particular difficulties of learning mathematics is that learners encounter and have to make sense of the mathematics and the associated representations that have been condensed from centuries of experience and the thinking of geniuses. The requirement in learning mathematics is for learners to make these concepts their own. This acquisition is a highly challenging task that may mirror the historical development of concepts and requires the cyclical process of interiorisation, condensation and reification (Sfard, 1995), or some such similar process.

Pursuing the "scientist" construction of the learner, the essential requirements for learning mathematics and for the role of teachers may be hypothesised. Drawing from Piaget (1970), Vygotsky (1962), Vergnaud, (1979; 1988; 1994; 2009), Skemp (1971), Sfard (1991; 1995) and Meyer and Land (2005), some reasonable working hypotheses are proposed. It is the stated view of Skemp (1971) and Sfard (1991), among others, that the learning of mathematics is dependent on a teacher or textual resources as proxy for the teacher, and successful learning is highly dependent on the teaching method used. Skemp (1971) asserts that it is only a genius that could construct the rational number system without a carefully constructed didactic intervention.

In addition to the primacy conceded to the teacher, appropriate texts, and didactic design, the role of evaluation is regarded as important (see also Davis, 2001). Evaluation is the teacher's tool for assessing the acquisition of concepts, indicating a probable location of learners along a developmental path, and providing information about recent progress and potential challenges for learners; it also performs the critical role of assuring learners of their current progress and of signifying proximate imperatives for continued learning (Long, Wendt & Dunne, 2011).

This thesis seeks to contribute to the broad problem of the individual's developing mathematical knowledge, comprising the complex interaction of both mathematical tasks and learners' developing cognitive structures, in the restricted but important cluster of concepts, fraction, ratio, rate, proportion, probability and percent, elements of the multiplicative conceptual field. The *multiplicative conceptual field* is defined as all situations that can be analysed as simple and multiple proportion problems and for which one needs to multiply and divide. It therefore embraces rational number and proportional reasoning (Vergnaud, 1988, p. 141). The term multiplicative conceptual field is used to refer to the entire teaching and learning domain together with the situations where the associated mathematics concepts are found; rational number refers to the mathematical domain. The composite term rational number and proportional reasoning is used to refer to the complex interrelationship.

The breadth and consequence of the theory of conceptual fields provides the framework within which the multiplicative conceptual field may be understood. The role of assessment in contributing to the process is envisaged within this framework. This framework therefore provides the background against which the multiplicative conceptual field and the assessment thereof are envisaged.

1.1.1 Towards a framework

Pring (2000) asserts that "(e)ducational practices, therefore, and the pursuit of educational policies, cannot be understood except within a system of thought – the theoretical framework which makes them intelligible as practices and as policy" (p. 128). Theory, according to Pring (2000), refers to the articulation of a "framework of beliefs and understandings which is embedded in the practices which we engage in" (p. 77). In this study the term *theoretical framework* is used to denote the critical elements of mathematics education that enable understanding of curriculum, policies, and research programmes. Because the term "theory" has connotations of greater depth, for example Piagetian theory, or the theory of conceptual fields, we use the term *framework,* implying a *set of requirements* or an organisational lens with which to interrogate both the curriculum and prevalent mathematics education movements.

The function of a theory is firstly to bring into focus the explicit constructs which enable the researcher to make sense of what is implicit in the empirical landscape and then to order, categorise and analyse aspects of this landscape (Pring, 2000). The explicit system of thought, here presented as a framework, makes policies and practices intelligible. In order for the theory to be resilient, it should provide a framework which is neither too rigid that it cannot engage new insights, nor too flexible so that it becomes everything to everyone. The second function of an explicit theoretical framework is to facilitate critical scrutiny of the theory itself, the consequent claims and deductions, and the practices advocated. The pivotal role of firstly engaging and developing theory in response to crisis educational situations, and secondly to make this theory explicit, is regarded as a necessary first step in response to the challenges that confront mathematics education and that therefore provide the context for this thesis.

A theory complex and comprehensive enough to explain the process of learning and teaching mathematics requires engagement with both the epistemology of mathematics itself as the object of study, and with the questions in psychology that relate to cognitive development. However its guiding theory must ultimately be concerned with problems and questions of mathematics education. The constraints and specific challenges of mathematics education that result in both an epistemology of mathematics education and associated critical questions are the first concern of the theoretical framework (1). Secondly, this framework has to attempt at least to explain how learners acquire new mathematical concepts (2), to provide some explanation for the role of instruction in the learning process (3), to articulate the potential role of language and symbols in the acquisition of concepts (4), and finally to make explicit the purpose of assessment in the education process (5).

1.2 Theoretical framework

The nature of mathematics education, and the related field of assessment, requires engagement with both mathematical and educational theory, and with the field of measurement. Mathematics, and the cognitive structures required to master mathematics, is complex, and therefore the theory required to explain the conceptualisation process is necessarily complex (Vergnaud, 1994, p. 43). The theory of conceptual fields, it is claimed in this thesis, meets this requirement essentially by invoking a network structure of concepts, representations and situations in which concepts are found. In addition to building a comprehensive theory around the theory of conceptual fields, the view is taken in this thesis that a theory of psychosocial measurement is an important requirement for research. These two theories, independently and in relation to each other provide the framework for this thesis.

1.2.1 Theory of conceptual fields

The theory of conceptual fields has a mathematical framework, but in addition draws on a psychological perspective, notably the acquisition of concepts building on the work of Piaget, and the function of instruction building on the work of Vygotsky. In this thesis, other constructs believed to provide insights into the teaching and learning of mathematics are incorporated.

The mathematical background for this study is provided by the unfolding number systems, in particular the emergence of the rational number system from the natural number system. The mathematical concept, rational number, and proportional reasoning, the cognitive counterpart, constitute the overarching concepts. Subtopics within rational number, that is fraction measure, ratio, proportion and rate, percent and probability are the specific elements of interest[1]. Multiplication, with its inverse division, is the primary mathematical operation underpinning all these topics. These topics are deemed elements of a field[2], because they form a network of related constructs, they are singly and in combination, embedded in problem situations. Supporting this notion of a network of constructs Kieren (1976) argues that "a variety of experiences of rational numbers are necessary for synthesising the rational number concepts", and developing the associated cognitive structures. Kieren (1976) states that control of the mathematical ideas associated with rational number, and the ability to use these ideas, is a "result of a thorough understanding of the many different interpretations of rational numbers" (1976, p. 102).

The notion of threshold concepts, critical concepts providing the conceptual gateway to higher mathematics and which, by corollary, inhibit mathematical progress where learners have not gained mastery (Meyer & Land, 2005), has been applied in relation to the theory of conceptual fields. Another sphere of complementary theoretical insights is the notion of a *didactic contract*, which is attributed to Brousseau (1997) and discussed in Ben-Zvi and Sfard (2007).

One of the challenges for mathematics education is that mathematical concepts are rooted in situations and problems (Vergnaud, 1988, pp. 141-142). Consequently from a conceptual perspective, a single concept may be applied in many different problem situations, and one situation or problem may require many distinct concepts. Another reality that must be faced from a cognitive perspective is that a single concept does not develop in isolation but invariably develops in relationship with other concepts. Because the landscape of mathematical knowledge acquisition is complex, the theoretical framework from which researchers work must be complex (Vergnaud, 1994, p. 46). It is necessary therefore to study conceptual fields.

[1] The focus on multiplicative structures allowed the inclusion of items requiring pre-algebra concepts.
[2] The use of the term field here is distinct from the notions of a field in higher mathematics.

A conceptual field is conceived as a set of problem situations, the solution of which requires mastery of several concepts of different, yet related, natures (Vergnaud, 1988, p. 142). Vergnaud explicitly acknowledges the existence of an established body of mathematical knowledge, comprising concepts and theorems[3]. This knowledge, "when restricted to a particular domain or content, is described as a conceptual field in which concepts and relationships are inherent" (Zaskis & Liljedahl, 2002, p. 96). Research into a conceptual field requires the study of a "bulk" [4] of concepts which develop in relation to other concepts, "through several kinds of problems and with the help of several wordings and symbolisms" (Vergnaud, 1988, p. 142).

The particular focus of the study is on an aspect of the mathematics curriculum described by Vergnaud (1983) as the *multiplicative conceptual field* that is critical in Grades 7 to 9, the lower secondary school. While the multiplicative conceptual field is the focus, its conceptual intersections with the additive and algebraic conceptual fields necessitate engaging somewhat with all three fields, within the framework of the theory of conceptual fields. This multiplicative focus is important for equipping learners with essential mathematical skills that enable the shift from additive reasoning that takes place in the early years of schooling, to algebraic reasoning that is essential for higher mathematics (Zazkis & Liljedahl, 2002).

In accord with Piaget, Vergnaud (1998) asserts that "(y)oung children have reliable intuitive knowledge about space, quantities, order relationships, and characteristics" (p. 167). This knowledge, Vergnaud claims, is the root of mathematics as a rational enterprise – but the intuition has to be analysed in mathematical terms, as mathematical knowledge cannot be reduced to any other conceptual framework (1990).

The role of the teacher is to transform the learner's intuitive and localised conceptions that can be applied to a single problem into generalised and explicit concepts that can be applied to a class of problems. The selection of situations that are within the learners' zone of proximal development (Vygotsky, 1962) and will therefore enable the extension of their existing schemes is essential. This transformation is assisted through natural language and

[3] The connotations of theorem in French may include algorithms, results and the sequence of statements of relationship in addition to the formal sequences of statement and proof.
[4] Vergnaud uses the term "bulk" where he might have used the term "set" because set has a very specific meaning in mathematics.

the use of diagrammatic representation. Learner conceptions are assessed through problem situations, as it is only when a concept is operationalisable, that it can be regarded as a "true" concept (Vergnaud, 1979, p. 263).

The above elements have obvious implications for teacher education. The relationship between subject knowledge, and how best to teach this knowledge (pedagogical content knowledge), as proposed by Shulman (1986), is the challenge of teacher education.[5] While the general ideas presented in this study have applicability in teacher education, mathematics teacher education will not be the direct focus of this study.

The strength of the theory of conceptual fields is that it provides a coherent approach to research in mathematics at school level, in terms of:

- a plausible and workable view of the *nature of mathematics*, suitable for education purposes, and specifically a notion of *mathematical concepts*, comprising a set of situations, a set of invariants and a set of representations (Vergnaud, 1988) (1);
- a *theory of developing cognition*, that provides a plausible account of how children acquire mathematical concepts, building on the work of both Piaget and Vygotsky (2);
- an integration of epistemological elements, concept characteristics and cognitive development, into an approach to teaching (3); which together with
- a clearly articulated theory of the *role of language and representation* required to mediate mathematical concepts (4); has
- implications for *assessment*, the purpose of which is to gauge learners' development along a mathematical path in order to design further learning experiences (5).

These five elements constitute the framework that informs the thesis and are the base from which further theory is elaborated.

1.2.2 Educational measurement

As early as 1969, at the first International Congress of Mathematics Education, held in France, Begle noted the necessity of turning mathematics education into an empirical science in the hope of identifying problems and charting progress. He advocated following "a carefully correlated pattern of observation and speculation, the pattern so successfully followed in the physical and natural sciences" intertwined with theory building (p. 342, cited by Fischbein, 1990, p. 5). Begle's request that mathematics education be turned into an empirical science has been implemented. However, the paucity of valid measures for

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[5] These issues are the focus of Adler and research associates, working on the Quantum Project, University of the Witwatersrand, Johannesburg

the particular purpose for which tests are meant, is one of the problems encountered in empirical research (Mislevy, 2008). The pervasive commonness of testing may account for the fact that questions regarding the validity of the testing process are ignored by many in the education field. The routine nature of tests and exams, and the use of tests in much educational research, are met by some sectors of the mathematics education community either by outright rejection of testing (Nichols & Berliner, 2005, 2008), or a wariness that is based on intuitions and experience, for example Paul Black's *Testing: Friend or Foe* (1998). Others, notably the school improvement lobby, look to testing to provide an indicator of improved school functioning.

At the core of testing is the measurement of some clearly defined attribute. Possibly because of the familiarity of the general public with measurement, the notion of a unit of measurement for mass, length, time or velocity, is taken for granted without consideration of the fundamental theoretical requirements. Thorndike, regarded as the father of educational measurement, noted in 1904 that "(i)f one attempts to measure so simple a thing as spelling, one is hampered by the fact that there exist no units in which to measure" (1904, p. 7, cited in Wright, 1997).

The critique levelled at psychometric research, even that which exhibits the highest quality of scientific evidence, is that there is insufficient theoretical and methodological groundwork supporting the measurement objectives of research, with the consequence that the inferences and actions based on the data are not linked to theoretical rationales and therefore cannot answer the questions being asked by practitioners and education officials (Mislevy, 2008). A theoretically grounded critique of the practice of attaching numbers where there is no indication of a quantitative attribute is advanced by Michell (1999; 2008). This critique is elaborated in Chapter 5, *Assessment and measurement*.

The Rasch measurement model provides measures which may "qualify as fundamental measures", based on the concept of measurement, as defined in the classical theory of measurement used in the physical sciences (Wright & Stone, 1999, p. 5). The model functions as a tool to check the coherence of the construct or topic of interest, the assessment instrument itself, the test developer and the data that is generated. Once the researcher is satisfied that invariant measures have been obtained on a unidimensional scale, other analyses can be conducted on the measurement data. Without the existence of invariant comparisons, the inferences and actions arising from pseudo-measurement and test data may be called into question. It is for the reasons just outlined that the Rasch model for measurement[6] was deemed both suitable and necessary for this research.

1.3 Problem statement

The improvement of mathematics education in schools is at the top of the political and education agendas in most countries. The learning of mathematics has generally been regarded as important but more recently the impact of large scale comparative studies has increased the focus on mathematics and science (see Plomp & Howie, 2006). The waves of reform over the past 100 years, and more importantly the last 20 to 30 years, with the stated purpose of providing access to mathematics for a larger percentage of the population, have resulted in repeated revision of curricula in many countries.

1.3.1 South African curriculum context

In South Africa, as in other countries, notably Australia, New Zealand and Canada, an outcomes based education system was to have been the vehicle for improved performance (Department of Education (DOE), 1997). The envisaged Curriculum 2005, implemented in 1998, embodied aspects which could be described as outcomes based, in that the curriculum requirements were expressed as outcomes. However, the adoption of an outcomes based curriculum does not portend the adoption of a particular pedagogy (see Andrich, 2002). In the case of South Africa, the mathematics pedagogy that was advocated in the early 90s included elements of an approach based on socio-constructivist[7] ideas, but in addition, included elements from ethnomathematics and critical mathematics, perceived at the time to be liberating epistemologies (Laridon, Mosimege, & Mogari, 2005).

A shortcoming of this implementation, Curriculum 2005, was that while aspects of the curriculum reflected a progressive pedagogy, with more attention to the conditions of responsiveness where teachers may be alerted to differences in the ways in which learners "engage with the context and content of learning" (Meyer and Land, 2005, p. 380, see also

[6] The Rasch family of measurement models include the dichotomous model, originally devised by Georg Rasch. The partial credits Rasch model was developed by Wright and Masters (1992). In this study the partial credits Rasch model was applied using the RUMM software programme.

[7] The terms socio-constructivist, constructivism and the problem-solving approach are elaborated in Chapter 4.

Brodie & Long, 2004), there was serious misunderstanding of the didactic function of the teacher (Human, 2009a), and hence a lack of coherence within the curriculum implementation. In particular the curriculum did not make explicit the theoretical underpinnings of elements of a pedagogy based on a constructivist approach to learning and its relation to mathematics (James-Long, 1995). This deficiency resulted in teacher confusion in classrooms where there was no, or little, professional support. In general there was widespread confusion concerning the constructs *outcomes-based education* (the focus on endpoints), *socio-constructivism* (a pedagogy drawing on Piagetian and Vygotskyan insights), a *relativist epistemology* (explored by Von Glasersfeld, 1995), *critical mathematics* (Skovsmose, 1994) and *ethnomathematics* (Gerdes, 2001). The reaction to the ensuing confusion entailed a rejection of alternative approaches, and advocated a return to a traditional form of teaching, where the learner is regarded foremost as the recipient of extant knowledge, presented by the teacher or the text. Successive reforms resulted in a less radical shift in terms of the curriculum content and less profound change in terms of the pedagogy, but still maintaining elements which, in theory, purported to embrace constructivism.

The recent proposed changes to the National Curriculum Statement (NCS) in South Africa, as reported in the *Review of the National Curriculum Statement* appears to have been driven by complaints from disgruntled teachers, anxiety about the results of systemic testing, and the inevitable political prerogatives and imperatives to implement change (Dada et al., 2009, p. 5). The task group commissioned to make recommendations to the Minister of Education conducted a survey of teachers and stakeholders (Dada et al., 2009) without (in the view of this thesis) the safety net of a coherent theory which keeps in mind the nature of mathematics and the acquisition and development of mathematical proficiency. The consternation about poor results appear to have directed thought away from theoretical reflection, and away from carefully planned research concerning the central focus of education, namely from learning and teaching. The consequence, the Curriculum and Assessment Policy Statement (CAPS), attempts to address perceived flaws in the curriculum, supposedly the lack of content specificity.

Subsequent structural and technical revisions have been invoked to address the perceived problem of under-specificity in the curriculum, in particular the reintroduction of a sharp boundary between everyday knowledge and school knowledge and the central control over pacing, sequencing and coverage of the curriculum (Dada et al., 2009). The *technical-professional*[8] preoccupations with the format of the curriculum document and the pacing of specified subject topic elements by teachers in the classroom (see Dada et al., 2009, p. 36) are presumably believed adequate to address and ensure the political requirement of reporting gains in achievement, as reflected by systemic test scores. However the core principles underpinning mathematics, the nature of the objects of mathematics and the processes involved, have not yet received attention, and in addition the central question of how children acquire and develop mathematical proficiency has been ignored.

The lack of explicit theory and explicit records of the research process informing the recommendations in this review (Dada et al., 2009) inhibits constructive critique. While some of the recommendations are pragmatic and necessary, others have caused consternation in the mathematics education community, which is concerned, in the case of mathematics teaching and learning, about the lack of any theoretical rationale informed by mathematics (see submissions to the Ministerial Project Committee[9]). The empirical evidence[10] gleaned from several systemic assessments that are minimally informed by theory appears to be driving the curriculum process.

The recent proposed reforms of the South African curriculum have parallels in other countries notably Australia (see Andrich, 2009). The stated aim is to provide much more specificity to the content of the curriculum. This aim can be rephrased as replacing the "language of outcomes-based education" with more specificity of content topics, which is assumed will make it easier for teachers to translate topics into classroom practice. The hidden agenda may be that a syllabus of this specificity is easier to monitor: the mistaken view may be that quality tests will be easier to design with a clearly stipulated curriculum.

In the current political climate in South Africa, the perceived social and economic imperatives are deemed reason enough for the government to exert varied measures[11] of control and guidance on the curriculum (see Dada et al., 2009). This control is manifested

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[8] Van den Akker, 2003, distinguishes between three aspects of the curriculum, the substantive, which deals with central issues of what knowledge is valuable and should be taught, the technical professional issues which deal with the implementation of the curriculum, and the socio-political issues.

[9] The expectation is that the submissions will be made public by the Department of Basic Education.

[10] In a seminar series at the University of Cape Town (UCT) in 1995, Basil Bernstein noted that research uninformed by theory was tantamount to "gossip".

[11] While there has been consultation with some professionals, the perspective taken in this thesis is that the consultation needs to embrace a wider range of thought concerning the coherence and purpose of the curriculum as a whole.

in its intended form through curriculum documents and texts, in the implemented form by advocating teaching methods deemed to be suitable, and in the evaluative form through high stakes systemic assessment.

The waves of curriculum reform, instigated by the government, and implemented by the education departments through directives to schools and teachers, are followed by further waves of systemic testing. This scenario, far from creating a supportive and constructive environment, may evoke either the fear of failure on the part of the teachers or the desperate drive for success. These consequences may impact the teacher-learner relationship, and may inhibit the central goal of education[12] which is the development of the learner. Andrich (2009), in the *Review of the Curriculum Framework for curriculum, assessment and reporting purposes in Western Australian schools* includes comment on the *national curriculum* and *national assessment*. He reports the consternation of teachers at the constant barrage of information directed at them which, though well-intentioned, creates confusion. Andrich recommends that both the positive and the negative impact of the systemic assessment programme currently being implemented should be closely monitored.

Concurrently with the implementation of a more progressive curriculum involving socio-constructivist notions in some sectors of the South African mathematics education landscape over the past twenty years, problem solving has played a critical role in the mathematics education in many countries across the world. Learners in these countries performed relatively well in international comparative achievement studies such as Trends in International Mathematics and Science Study (TIMSS) (Human, 2009a, p. 305). The underlying question to be asked, against the background of South African confusion and the type of mathematics education promoted in high-achieving countries, is "how attainable is (the goal) to establish a real emphasis on problem solving in South African classrooms"[13] (Human, 2009a, p. 305).

[12] The view taken in this thesis is that while there are multiple functions of education, notably socialisation, qualification and individuation (see Biesta, 2009), the necessary motivation and energy are derived from attention to the individual development.

[13] "(D)ie onderliggende vraag (is) hoe haalbaar (dit is) om 'n daadwerklike klem op probleemoplossing in Suid-Afrikaanse klaskamers te vestig" (Human, 2009, p. 315).

1.3.2 Global concern over lack of progress

The need to improve mathematics teaching and learning is high on the agenda of many countries, including Australia (as intimated in the previous section), the United Kingdom, the United States and France, as highlighted in the sections which follow.

A research study in the United Kingdom, namely the “Increasing Confidence and Competence in Algebraic and Multiplicative Structures” (ICCAMS) study (Hodgen, Küchemann, Brown & Coe, 2009), aims at improving the numbers of learners taking higher level mathematics courses and entering the Science, Technology, Engineering and Mathematics (STEM) courses at tertiary level. In ICCAMS, the comparison of multiplicative and algebraic concept acquisition within a current cohort of learners in 2008 and a 1980s cohort, was conducted using original test items from the Concepts in Secondary Mathematics and Science (CSMS) Study (Hart, 1981). The results showed little improvement across the cluster of concepts tested. There was a notable improvement in decimal fractions. This change is attributed by Hodgen et al. (2009) to the United Kingdom’s conversion to the metric system.

In the United States concern over the school mathematics education resulted in the government establishing the National Mathematics Advisory Panel (NMAP) (2008)[14] to conduct a review of research, in order to inform the government of actions to be taken. Usiskin (2005), also in the United States, identifies the problem as in the lower secondary phase, Grades 6 to 10, the critical *transition years* where pivotal mathematical concepts are learned (or not learned, with consequent failure to achieve higher levels of mathematics).

From France, Duval (2006) reports that in national assessment surveys in 1993 and 1997 only “one student in three [at the beginning of secondary school] appeared to have grasped the functioning of the decimal system” and was hence able to “succeed with a set of items about the simplest operations of multiplication and division of decimals” (p. 106).

The perspective taken in this thesis is that the attempts by any government to implement quick fixes in the form of pre-packaged materials, rigorous control over the curriculum and the introduction and use of high stakes testing, may in the short term provide an

[14] A major limitation of this study is that finely crafted qualitative studies in mathematics education were not considered to be scientifically rigorous (Cobb & Jackson, 2008; Thompson, 2008), and hence completely ignored.

appearance of improvement in some respects. However, the long term development of concepts may be put in jeopardy unless attention is given to substantive mathematics content and to the question of how specific mathematical concepts are acquired.

1.3.3 Perceived factors influencing under-performance

In research from the 1980s by Hart (1981; 1984; 1989), and of current relevance, the following cluster of factors are identified: learners' lack of prerequisite knowledge, the use of incorrect methods, the clash of informal methods with formal methods, and the fact that formal methods, though learnt, had not been retained beyond three months by the majority of children (Hart, 1989). From the teaching perspective, a problem appears to be the inability of teachers to understand the mathematics other than as a procedure to be executed. The consequence is that learners exhibited distorted meanings, for example, by using phrases such as "take away the minus sign" (Hart, 1989).

Similar instruction-induced problems are noted by Davis and Johnson (2007), where, in worked examples the question "what 'step' to do next" takes precedence over reflecting on the mathematical context (p. 133). A related problem is the lack of attention to the particular mathematical object, consequently leading to such practices as operating on integers as though they are whole numbers (Davis, 2010). When adding a positive and a negative integer, one such procedure is "Find the difference, and then add the sign of the "bigger". The problem here is two-fold, the stripping of the integer of its directional character, and the incorrect use of "bigger", when applying operations on directed numbers.

Davis and Johnson (2007) hypothesise that many of the difficulties observed in mathematical classes, at least in working class schools in the Western Cape region of South Africa, are related to the teaching of mathematical procedures with no reference to the *underlying mathematical objects*, resulting in the lack of stability of "mathematical" notions that are perceived as context dependent. "(B)ecause the "mathematics" (so) constituted will be generally inconsistent and hence unstable", slow pacing results (p. 133). Given this hypothesis, that the problem of slow pacing is a result of the prevalent pedagogy, the current insistence on *curriculum coverage* and *faster pacing,* as advocated in the review of the curriculum (Dada et al., 2009, p. 20), without the necessary professional support providing insight into mathematics, may be fruitless.

According to Parker and Leinhardt (1995), an additional hazard is that teachers and learners use the different meanings embedded in each particular problem situation intuitively, but the conceptual differences are not made explicit neither between the learner's different intuitions, nor between the conceptually distinct mathematical objects. The hidden conceptual differences, for example, the varied uses of the term percent which mask the underlying referents, cause difficulties for learners. In a similar vein, Davis (2010) notes that the "lack of specific attention to the (mathematical) object upon which operations are being performed is implicated in producing low levels of competence" (p. 380).

The most pervasive challenge however, as hypothesised in this thesis and explicated in Chapter 3, may be the conceptual shift required from natural numbers to rational numbers and to real numbers (Skemp, 1971; Usiskin, 2005; Vamvakoussi & Vosniadou, 2007). Vamvakoussi and Vosniadou claim that the transition from natural to rational numbers requires a radical conceptual shift, which takes learners beyond the mistaken notion that new numbers have been added to the existing set of numbers, to the idea that rational numbers constitute a different object (p. 266). This conceptual shift directly affects the transition from additive to multiplicative structures and therefore has serious implications for teaching and the curriculum.

Kieren (1976) noted that most school curricula treat rational numbers "as objects of computation" and "the 'algebraic' aspects of the operations on rationals are lost". These algebraic aspects that become manifest when operating on rational numbers, include the "face to face" confrontation with equivalence and the facts that operations work for axiomatic reasons, and that properties such as an inverse are invoked (1976, p. 102). This observation is still of consequence in current national curricula in South Africa.

It would appear from these studies (Skemp, 1971; Hart, 1989; Parker & Leinhardt, 1995; Usiskin, 2005; Vamvakoussi & Vosniadou, 2007; Davis & Johnson, 2007; Davis, 2010) that the problems with teaching and learning mathematics are in fact endemic in many mathematics classrooms. The acceptance of this condition, even in schools catering for working class children, is incompatible with the intelligence that is exhibited by children in other equally demanding fields, but where the learners' interest is engaged. The challenge extended to teachers by Parker and Leinhardt (1995) in the domain of percent, but equally applicable to the related concepts in the multiplicative conceptual field, is to provide “the most encompassing and principled mathematical presentation” of the concepts to be learned, to present “the deepest, most flexible representations (those that call for a clear flagging of relationships)”, to give due attention to the “lengthy integration and clarification of neighbouring concepts (fractions, ratio, decimals)”, and finally, to encourage the “active involvement of students in the development of ideas” (1995, p. 472).

Evident in the challenge extended by Parker and Leinhardt, is that the active engagement of learners with the mathematical concepts to be learned, has to be taken seriously. The active cognitive engagement would answer in part the statement by Vergnaud (1979) that most difficulties confronting learners are “difficulties in the concepts involved and not in the calculations” (p. 103), requiring therefore active engagement with the concept. Another consideration for Vergnaud is that a thorough understanding of concepts takes place gradually over a long period of time and always in relation to other concepts. He proposes that the lack of a smooth transition from primary to high school might be averted by establishing the “link between ordinary mathematical situations and the relevant mathematical concepts” (Vergnaud, 1979, p. 263) earlier along the mathematical path. This proposed early intervention may avert in part the problems experienced in the Hart research noted previously. The establishment of this link is important on two counts, firstly for developing “an ability to solve ordinary arithmetical problems” and secondly for a “better understanding of some more sophisticated concepts” (Vergnaud, 1979, p. 263).

The perspective taken in this thesis is that it is the learner’s unfolding cognitive ability, in interaction with carefully designed mathematical experiences that enables him or her to assimilate and accommodate mathematical concepts. This dialectical process is fundamental to establishing a personal pathway that builds on local intuitions to formulate explicit and general theorems, and so engage more productively in his or her immediate environment. This conscious cognitive engagement underpins the process of transforming the initial intuitive and implicit knowledge used for a class of problems, into explicit and generalisable knowledge that can be applied to more than one situation (Vergnaud, 1988; 1990). The challenge of this particular mathematical phase, Grade 7 to 9, can be encapsulated as the mastery of the interrelated elements of the multiplicative conceptual field (Vergnaud, 1983; 1988).

It is within this conceptual field, that *threshold concepts*, such as rational number, that require radical conceptual shifts, are encountered, and that if successfully mastered, provide the conceptual gateway to higher levels of mathematics (Meyer & Land, 2005). According to Meyer and Land (2005), a threshold concept is defined as that which transforms the "internal view of (the particular) subject matter or part thereof" (p. 373). The notion is linked to the idea of 'troublesome'[15] knowledge, which defines "critical moments of irreversible conceptual transformation in the educational experiences of learners, and their teachers" (p. 373). These conceptual gateways may be

- transformative – "occasioning a significant shift in the perception of a subject",
- irreversible – "unlikely to be forgotten", "unlearned only through considerable effort", and
- integrative – "exposing the previously hidden interrelatedness of something" (Meyer & Land, 2005, p. 373).

The related topics ratio, proportion and percent demand extensive understanding of multiplicative structures, which then provides the essential ground work for the conceptualisation of the concept of rational number. This concept, understood to be a threshold concept, opens up the conceptual gateway for higher order topics such as algebra and functions, essential for science and technology. In addition these concepts have applications in many routine daily activities and many occupations.

1.4 Research focus

The backdrop against which this research study takes place is the broader South African education landscape. The characteristics of this landscape do in some respects mimic those of the wider international mathematics education community with forces impelling different approaches to education. The theoretical requirement is to make explicit a framework that accounts at least in part for the complex nature of mathematical learning and teaching. Because of the nature of mathematics, any attempt to contribute to the field of mathematics education must locate the mathematical focus within the larger network of mathematical concepts. The multiplicative conceptual field is located within and across the unfolding number systems. Particular note is given to threshold concepts requiring mastery

[15] The term 'troublesome knowledge' is attributed to Perkins (1999), cited in Meyer and Land (2005)

in the transition towards more complex systems. In addition to the mathematical location, the nested nature of the additive, multiplicative and algebraic conceptual fields is given attention.

Against the mathematical landscape a deeper investigation is conducted of elements of the multiplicative conceptual field, fractions, decimals, percent, ratio, proportion, rate and pre-algebra concepts, with particular attention to the lower secondary school, Grades 7 to 9, in the South African context. The location of these key concepts as elements of a set of interrelated concepts that constitute the multiplicative conceptual field, enables the deeper study and analysis of the challenges to the learner that are inherent in acquiring and mastering these concepts.

The aim of mathematics teaching, according to Vergnaud (1988), is to educate all students to a level in mathematics and science "that enables them to understand the structure of functions and the different modes of combining functions" (p. 159), these notions being essential for understanding calculus. The multiplicative conceptual field offers the "most fruitful and simplest domain to develop this understanding" (Vergnaud, 1988, p. 159). The mathematical analysis of situations, concepts and cognitive processes with a view to informing the teaching and learning of pivotal concepts is a necessary step towards this aim for mathematics education (Vergnaud, 1998).

In the South African context it is imperative that greater numbers of learners become proficient in the mathematics required for tertiary level courses, and for all learners to be proficient in mathematical applications required for 21st century living. The particular challenges that occur at Grades 7 to 9, (or perhaps Grades 6 to 10), "… constitute the most important developments in a person's mathematical (experience)" (Usiskin, 2005, p. 4). During these transition years, conceptual shifts are required "from whole number to real number", "from a number to a variable", "from informal description to formal definition of mathematical ideas" and "from a view of mathematics as a set of memorised facts to seeing mathematics as interrelated ideas accessible through a variety of means" (p. 4). The transition from whole number to real number requires mastery of the intermediate rational number system, including a conception of irrational numbers, where the mathematical ideas, or objects, become more abstract. Also at this stage the number of mathematics topics increases and there are more relationships to incorporate into emerging cognitive structures and more topics and relationships to be called upon to solve problems.

It is also during this phase in the South African curriculum that learners make the decision to continue with formal mathematics, or to continue with the parallel course mathematical literacy. Learners taking the notionally less demanding mathematical literacy will nevertheless require a good grasp of the key mathematics principles, at the centre of which lie the conceptual structures, the cognitive strategies and associated problem situations of the multiplicative conceptual field. It is in this context that this thesis seeks to make a partial contribution by focusing on the teaching and learning of mathematics in the transition years, in particular within the related concepts that constitute elements of the multiplicative conceptual field. The explications of the elements which constitute this field underpin both the informal formative assessment which is continuous in the classroom, and the more formal assessment programmes, providing systemic information.

1.4.1 Research questions[16]

The research questions, following an overarching question, fall into eight broad but related categories; first, issues impacting on the South African education landscape, the socio-political issues around the curriculum, as well as mathematics education trends; second, issues relating to the broader mathematical context; third, questions aligned with learning and teaching; fourth, questions concerning measurement; fifth, a focus on the individual elements of the network of related concepts; and the last category, but the most extensive, the empirical investigation of learner proficiency in relation to the multiplicative conceptual field, which includes both item response analysis and the analysis of interviews. The final set of questions reflects on the existing study and proposes future directions. The *overarching question* pertains to the essential elements of a framework for informing assessment of mathematics.

Question 1: How may the essential elements of a framework including mathematical, cognitive, and didactic elements address problems in the

[16] As noted by a critical reader, the questions outlined in this section are not of equal importance in the study and neither are they all fully answered. Rather the questions frame the mathematics education environment and signal the issues which are important. Where the particular section is not elaborated pointers are given to the relevant research or to future directions.

multiplicative conceptual field, and inform both curriculum development and the validity of assessment processes?

Guiding the design of this research study is the obvious fact that the learning and teaching of mathematics exists within the larger context of socio-political decisions (Van den Akker, 2003), stipulated in South Africa by the National Department of Education, concerning the curriculum and assessment. Therefore in order to address the more focused learning and teaching of particular mathematical concepts, the larger South African educational context has to be understood, or at least engaged. The overarching research question, with respect to the South African context, the socio-political, curricula and mathematics education developments, explored in Chapter 2, is:

Question 2 What socio-political, epistemological and educational perspectives influence curriculum policy, mathematics education and research, particularly in the South African educational environment?

Sub-questions follow:

2.1 What are the prevailing mathematics education epistemologies informing mathematics education in South Africa?

2.2 How does the current National Curriculum Statement (Mathematics), Grade R-9, address these proposed requirements? Where are the existing mismatches between the curriculum, and the proposed requirements for a comprehensive curriculum?

Further questions focus more specifically on prevalent educational theories, and their contributions, in terms of the proposed requirements for a workable theory.

2.3 What predominant educational theories have informed mathematics education in South Africa over the past 30 years? What does each of the theories offer in terms of the proposed framework?

After the socio-political context, the next obvious consideration is that any mathematical topic, for example the multiplicative conceptual field, is located in a mathematical context. The corresponding set of questions, explored in Chapter 3, apply to the mathematical context of this study.

Question 3 What principles and thought processes guide the unfolding of number systems and are central to developing mathematical proficiency?

3.1 What episodes from history may provide insights into the teaching and learning of elements of the multiplicative conceptual field, as demarcated in this study?

3.2 What particular threshold concepts should be anticipated in teaching and learning at this level?

3.3 What features of mathematics are fundamental to teaching and learning mathematics?

Some critical aspects of the nature of learning and teaching mathematics will be explored in Chapter 4. The primary aim of this chapter is to present a model for mathematics education that can be investigated, debated and expanded. The overarching question follows;

Question 4: What are the elements of a framework required to inform mathematics teaching and learning and which are essential to an assessment process?

The first of the intersecting aspects relates to the development of mathematics knowledge, together with its essential features, some general aspects of which are explored in Chapter 3. However aspects specific to learning and teaching are explored in Chapter 4. The question then arises as to how the mathematical concepts are acquired by learners. We look to the indicators of mathematical development in learners, comprising the cognitive domain. From an understanding of how learning develops we explore the didactic implications that support the learning process. Clearly semiotic factors, such as the role of language, representation and symbol, are critical elements impacting on the epistemological domain and are therefore given attention. And finally, since it would seem feasible that evaluation in the broadest sense of the term is central to the learning process, we consider forms of assessment. The questions follow:

4.1 What are the central factors in the development of mathematical knowledge that provide insight into individual development?

4.2 How do the first natural constructions observed in children progressively develop into more complex structures?

4.3 How can the learning process be nurtured, stimulated and accelerated by teaching? What are the essential elements of the *didactic contract*? What factors may constitute disruptions to this development?

4.4 What is the relationship of mathematical concepts to language, representation and symbol? How do language, representation and symbol interrelate to both lead and support cognitive development?

4.5 How may assessment assist teaching and learning? What unexpected effects of assessment may impact on teaching and learning?

Any attempt to test or assess individuals on any mathematical topic requires engagement with measurement principles. In the context of measurement explored in Chapter 5, the question is asked:

Question 5 How may mathematics proficiency be assessed so as to inform learning and teaching?

Following Webb (1992) and Schoenfeld (2007, p. 59) two sub-questions are explored in this chapter.

5.1 What are some of the challenges facing assessment in mathematics?

5.2 What does it mean for a student to be proficient in mathematics? How can we measure proficiency in mathematics?

The next sub-questions relate to the concept of fundamental measurement. The first of these questions explores the essential features of the Rasch measurement model. The second relates specifically to the construction and analysis of an assessment instrument. These questions embrace issues around measurement in the social sciences, the validity and reliability of results, and the inferences and actions that may be taken as a consequence of testing.

5.3 What are the essential features of fundamental measurement? How are these features satisfied in the Rasch measurement model?

5.4 How may a model be conceived that draws on both mathematics education theory and fundamental measurement?

The focus of Chapter 6 is on the multiplicative conceptual field, where each of the elements of interest in this study has some common mathematical structure, but also has unique distinct characteristics. The primary question is stated below and the subquestions which follow are explored.

Question 6: What critical elements need to be considered in the learning and teaching of elements of the multiplicative conceptual field?

6.1 What categories of problem situations provide opportunities for learners to develop critical multiplicative structures? How can we analyse the complexity of mathematical problem situations in this domain and classify them meaningfully?

6.2 What are the mathematical structures underpinning specific key concepts in the multiplicative conceptual field, namely rational number (and its subconstructs), percent and probability? How does the historical development of particular constructs make explicit the underlying concepts and the cognitive requirements?

6.3 How may rational number sense and proportional reasoning be facilitated? What particular threshold concepts require specific thought and attention on the part of teachers? What representations and symbols may assist the teaching and learning of the particular concepts?

6.4 How does the National Curriculum Statement (NCS) reflect elements of the multiplicative conceptual field? Which elements identified in this study are present and in which grades?

The questions in Chapter 6, framed within the theory of conceptual fields, provide the model against which the assessment data of the empirical phase are analysed. These data reflect the proficiency of learners on a set of items designed to assess content within the multiplicative conceptual field. The interviews with selected learners provided additional information on the acquisition of specific concepts (Chapters 7 and 8). A further set of questions relates to the empirical phase of the research study where learner performance on items selected from TIMSS 2003 (IEA, 2005) selected for pertinence within the multiplicative conceptual field are investigated from a quantitative and qualitative perspective (see Chapters 7 and 8).

Question 7: What insights can be gained from an investigation of the multiplicative conceptual field when assessed within a South African context and analysed within a Rasch measurement framework?

7.1 How are measurement principles, requirements within the Rasch measurement framework, applied in this research study?

7.2 What elements of the analytic framework are used to describe individual items and sub strands?

7.3 What procedures underpin item analysis?

7.4 How are items ranked in terms of *difficulty level*? How can these rankings be explained in terms of context, situation, mathematical structure, mode of presentation, number range and value, and response process?

7.5 How do the mathematical concepts, in the multiplicative conceptual field, correlate with the proficiency levels of learners as exhibited on the constructed instrument?

7.6 What threshold concepts can be identified within this analysis?

7.7 What adaptations, additions and changes are suggested for the instrument by the empirical data?

The term “difficulty level” has been used in this thesis implying a level to be attained. In response to a question on the use of the term level by a critical reader, the term “difficulty locale” has been identified as more appropriate to the research design. Likewise the term “proficiency locale” has been identified as a more appropriate term than the currently commonly used “proficiency level”. However, given the stage of the thesis and the unfamiliarity of the term, the terms *difficulty level* and *proficiency level* have been retained for the current document.

Question 8: What threshold concepts are acquired at the different proficiency levels exhibited by learners? Which concepts have been mastered and which are yet to be explored?

8.1 What strategies does the learner use to make sense of the problem? Does the learner attempt to understand the problem context? Or does the learner abandon natural sense making?

8.2 What strategies and procedures are used by the learner (concepts-in-action and theorems-in-action) to engage the mathematical problem situations identified within the multiplicative conceptual field? What relationships between variables are identified? Upon which implicit concepts and theorems does each procedure rely?

Finally Chapter 9 consolidates the general findings from Chapters 2 to 8, proposes theoretical insights into the multiplicative conceptual field, into the requirements and process of measurement and identifies the implications for curriculum and assessment policy, and in some measure for teaching and learning. In addition there is reflection on the methodology used for this study, and directions for further research. The questions guiding Chapter 9 follow:

Question 9 What are the implications for curriculum policy, mathematics education, assessment and future research?

9.1 What insights in the area of the multiplicative conceptual field may contribute to our knowledge and therefore can be useful for mathematics education?

9.2 What elements constitute a model for the assessment of the multiplicative conceptual field? How may this model inform assessment practice?

9.3 What elements of the research design may be improved?

9.4 What future research emanates from this study?

1.5 Research design

The design of this research study included both a theoretical component and an empirical component. Key decisions relating to the empirical dimension were guided by the notion that further development of the theory of conceptual fields requires a cycle of empirical investigation, exploration and verification, and reflection on the theory. The consequent requirement for this study was that it would be located at the confluence of *theoretical and empirical* and *quantitative and qualitative* methodologies. Empirical data obtained from a theoretically informed research design provide pointers and directions within which to further explore and to focus theoretical and qualitative explorations.

1.5.1 Literature review

The research literature informing this study covers related topics, ranging across mathematics, education and measurement theory. The selection of literature was in some cases the result of a formal search citing key concepts. In addition to targeting specific topics, the "snowball" approach to accessing relevant literature proved productive. Initial searches into the topics ratio and rational number uncovered key research by the Rational Number Project (Behr & Harel, 1990). This project referred to the *multiplicative conceptual field* (Vergnaud, 1983), the term recognised as a composite construct useful in researching, teaching and learning the collection of concepts involving multiplication and division. This discovery led to other work by Vergnaud on the theory of conceptual fields in the English language literature.

The number of journals pertaining to mathematics education is extensive, the number of doctoral and masters theses presented at conferences is also extensive. In addition there are many evaluative studies which are conducted not necessarily building on established mathematics education theory, but which have an influence on policy, for example the NMAP (2008) in the United States. It was necessary therefore to make a selection of research studies according to somewhat loosely defined features, but whose approach to the mathematical education problems have elements in common. A distinction is made in the following section between two approaches to mathematical proficiency, described here to make explicit the elements of mathematics education research which guided the selection of literature in this study.

It is generally accepted that true understanding of rational number, and hence mathematical proficiency with rational number constructs, takes time to develop (Kieren, 1976; Vergnaud, 1988, 1994; Greer, 1992, 1994; Olivier, 1992; Lamon, 2007; Vamvakoussi & Vosniadou, 2007). Engagement with the many problem situations in the domain of rational number, including ratio, rate, percent and probability problem contexts, provides the basic structures for an understanding of rational number. Another notion that no mathematics educator will dispute is that in order to be successful at algebra, the features and operations of rational number have to be understood (see the NMAP report, 2008). For some though, notably Wu (2001), the preparation for algebra appears straightforward.

> With the proper infusion of precise definitions, clear explanations and symbolic computations, the teaching of fraction can eventually hope to contribute to mathematics learning in general and the learning of algebra in particular (Wu, 2001, p. 7).

The question that has to be asked is why this particular definitional and abstract approach has not worked for the majority of children in the past. Research conducted by Hart (1981; 1984) shows that students do not necessarily learn what their teachers teach. The review of research by Parker and Leinhardt (1995) on percent, presents a picture of confusion in response to the attempt to teach and learn computations and procedures, if separated from contexts and meaning.

The work of Lamon (1994, 2007), building on the work of Vergnaud and others explores further implications through fine-grained empirical research on rational number and proportional reasoning over at least 20 years. In 2006 she completed a longitudinal teaching design experiment, with children Grades 3 to 6, building on a conceptual analysis of the components of rational number and the central multiplicative structures that she believes to be pivotal when developing higher order mathematical structures (see Lamon, 2007). The experimental groups worked through carefully designed problems posed by the teacher who encouraged the students to engage with the mathematical tasks and to communicate their thinking. The mathematical activity consisted mostly of "group problem solving, reporting and then individually writing and revising solutions for homework" (p. 653). The experimental groups were not taught rules or algorithms[17]. The

[17] The details with regard to the development of mathematical proficiency in the Lamon study are still to be explored.

control group followed a standard curriculum which involved being taught procedures and algorithms, as per the model generally used for teaching rational number concepts.

After two years of the four year study, the experimental group apparently lagged behind the control group who were already performing fraction computations. However after four years the experimental group surpassed the control group, and their "representations showed they were performing meaningful operations" (p. 660), a prerequisite for learning algebra.

The varied and often conflicting recommendations, emanating either from substantive research, whose features can be drawn from Lamon's work, or from quick-fix solutions[18], lacking critical features, led to the decision to restrict the survey of literature to those studies that have the following characteristics:

- Mathematics, together with its underlying structure, and the complexity of relationships, is seriously considered rather than treated as equivalent to any other discipline.
- Conceptual development and analysis of learner strategies is investigated in relation to the mathematical concepts.
- The researchers have engaged with, and built on mathematics concepts and previous mathematics education research.
- The findings are plausible in the light of mathematics, conceptual development, and previous research and these findings suggest the foundations for future research.

Focus areas for the literature survey

A selection of research studies, adhering to the criteria above and influencing mathematics education in South Africa, from both a psychological perspective, namely social constructivism, and from a sociological perspective, particularly the research informed by Bernstein and others, is assembled and considered in order to understand more fully the South African mathematics education landscape[19]. This analysis is reported in Chapter 2, *The South African context*.

[18] Kanjee (2007) observes that the education community in South Africa is "data rich but information poor". Research studies which provide data, without having located either the study or the findings in the mathematics literature, are considered peripheral to substantial change.

[19] Most of the authors in this first section are South African, reflecting important work being conducted in some mathematics education departments.

A second body of literature focused specifically on *mathematics,* (Adler, 1958; Kitcher, 1983; Piaget, 1952; Skemp, 1971) and an overview of domains within mathematics (Dantzig, 2007). This literature is integrated into Chapter 3, *Unfolding number systems*.

A third selection was the literature available on the theory of conceptual fields in the mathematics education literature, restricted though to that written in English (Vergnaud, 1979, 1983, 1988, 1990, 1994, 1997, 1998, 2009). The theory of conceptual fields builds on the work of both Piaget and Vygotsky. In order to fully comprehend this theory it was necessary to explore key studies reported by Piaget, in particular in relation to mathematics, and some aspects of Vygotsky's work. As the multiplicative conceptual field is located on the path toward the algebraic conceptual field, a few critical issues in the learning of algebra are also explored (Kieren, 1990; Sfard, 1991, 1995).

The idea of a mathematical scheme (see Piaget, 1970; Skemp, 1971; Vergnaud, 1988, and others) was central in this study. For the acquisition of concepts at more advanced levels, the work of Sfard (1995) was invoked. Her work also contributed to initial thoughts on instruction, in particular the notion of a didactic contract (Brousseau, 1997; Ben-Zvi & Sfard, 2007). The notion of threshold concepts (Meyer & Land, 2005) proved to complement the theoretical investigation and to provide a more nuanced description of the common occurrence of cognitive hurdles or epistemological obstacles occurring along learners' mathematical journeys. This exploration is presented in Chapter 4, *the theory of conceptual fields.*

The literature specifically focused on ratio, proportion and percent and more generally on the multiplicative conceptual field, provided insights into the development of competencies and concepts in this domain. Of particular note is the work on multiplication and division by Greer (1992) and Vergnaud (1983; 1988), fractions by Carraher (1996), rational number by Kieren (1976), percent by Parker and Leinhardt (1995), and probability by Shaughnessy (1992). Parker and Leinhardt (1995), in particular, provide an extensive overview of the topic percent, including epistemological and historical antecedents, a detailed account of its mathematical structure, attention to how children learn and to the misconceptions apparent in their incorrect handling of problem situations, and some insight into more and less successful teaching methodologies. This research also highlighted obstacles to the learning of these multiplicative concepts. Synopses of developmental levels and insights that inform this study are provided in Chapter 6, *the multiplicative conceptual field*.

Because the Rasch measurement model is intrinsic to the research design of this study, it was necessary to engage with the literature on educational measurement and the Rasch model. Important related literature on assessment, testing, and the key notion of validity was investigated (Andrich, 1988; Andrich & Marais, 2008; Black, 1998; Messick, 1989; Webb, 1992; Wright & Stone, 1979, 1999). This theme is presented in Chapter 5, *Assessment and measurement*.

Studies using a Rasch measurement model, notably Ryan and Williams (2006) on teacher development, and Misailidou and Williams (2003) on the errors in relation to proportional reasoning, provided insight into both the topic area and the application of Rasch methodology. Of particular note is the work by Van Wyke and Andrich (2006) which reports on the application of the Rasch model to systemic testing in Western Australia, and the implications for test development.

1.5.2 Empirical investigation of the multiplicative conceptual field

Given the qualitative and quantitative interrelationship required for this study, a compelling feature of a Rasch measurement approach was the complementarity in the roles of quantitative and qualitative explorations in investigating phenomena. The Rasch measurement model unites "theory and qualitative understanding, with quantification and measurement to their mutual benefit, leading to new and better understandings of many psychological [and educational] phenomena" (Styles, 1999, p. 19).

The feature of the Rasch model that is however of most significance for this study is the requirement that fundamental measurement as expected in the physical sciences should be constructed and applied in the social sciences. This requirement for measurement in the social sciences is not new. In the 1920s Thurstone (1925, cited in Andrich, 2002, also Andrich, 1989) laid down the pre-requisites for measurement in the social sciences as follows:

- Items should be located on a continuum, or scale.
- The locations of items (on a common unidimensional frame) should be invariant across different subpopulations which are being measured by the items.
- The locations of items on a continuum should satisfy the requirement of additivity.

In the 1950s Georg Rasch, a Danish mathematician, developed a statistical model that embodied these requirements (Rasch, 1960/1980). The prior empirical requirement for use of this model is that the designer of instruments has an explicit understanding of the latent trait or construct[20] of interest. Then the instrument must be designed and confirmed as exhibiting the pre-requisites.

In this study the latent construct is conceived as *proficiency in solving problems within the multiplicative conceptual field*. The objective is to explore a method to construct and refine an instrument made up of items that operationalise this trait. This practical application of the model permits the discovery and amplification of any item anomalies which are inconsistent with the general expectations of the instrument. The researcher is required to further investigate any particular item shown to be “misfitting”, and, if necessary, adapt or eliminate the item, while at the same time seeking to identify a plausible explanation of the item misfit in terms of its own characteristics. Likewise data from learners who for some or other reason do not perform as expected can be temporarily eliminated from the analysis, on the basis of explicitly recorded and reasoned arguments, for the purposes of refining the instrument and establishing probabilistic estimates of item difficulty and learner ability on the same scale.

Test instrument development

An instrument adhering to the principles of the Rasch measurement framework[21] was developed. Items, realisations of the construct of interest, were drawn from the TIMSS 2003 set of released items (IEA, 2005). The resulting instrument was used to further investigate the topics of interest in this study, and to locate learners who were on the path to acquisition of these multiplicative concepts at points on a common scale.

[20] The terms construct, variable, and trait are used to denote an underlying or latent characteristic. In this thesis the term “construct”, or “underlying construct” is used mostly. The latent trait, or variable of interest may mean the same thing.

[21] The term Rasch measurement framework is used to signal that both Rasch analysis and the principles which inform the model are considered in this study.

Participants

The participants in the study comprised 330 learners in Grades 7, 8 and 9 at two schools and indirectly involved their teachers[22]. The initial participation was restricted to one period of 40 minutes when the test was conducted. In addition selected groups of learners, in Grade 8 at the time of testing, were invited to make themselves available for interview purposes.

Interviews

The purpose of conducting interviews was to gain additional insight into learners' conceptions and cognitive processing within the multiplicative conceptual field, but also to enable the refinement of the instrument for supporting development in this conceptual field. Modifying the instrument in order to improve operationalisation of the construct is one of the outcomes inherent in the Rasch measurement framework.

From each school, students exhibiting proficiency at high, intermediate and low levels,[23] as derived from the person locations established by the Rasch analysis, were interviewed.[24] The interviews took place some months after the initial testing. The students were interviewed in groups of two to five, except for the lower group at School B, where one person arrived for the interview. A subsequent interview was set up for another two in this group.

The focused investigation of items at specified locales was intended to enable the identification of various possible errors. The interface between the theoretical component and the empirical component occurs with the analysis of items in conjunction with the analysis of student responses in the interviews.

[22] Written permission was requested from the principals, the teachers concerned and the learners' parents. In the request the right to participate and the right to withdraw at any point was emphasised.

[23] See footnote 17 for an explanation of the term "level".

[24] A list of learners invited to an interview was submitted. It was expected that learners uncomfortable with participating would not attend the interviews.

1.6 Summary: A prospective pathway

Chapter 1 provides a brief overview of the factors impacting on mathematics education within the South African educational context, both in mathematics education and educational measurement, which inform this research study. In addition a summary of the literature review foci, essential features of the research design and the research questions, are included in this study.

In accord with Dewey, education in this thesis is regarded as "the development of the distinctively human capacities of 'knowing', 'understanding', 'judging' [and] 'behaving intelligently'" (Pring, 2000, p. 12). The intricacy of mathematical thinking, and the difficulties and challenges experienced by learners and teachers are acknowledged. The development of the individual's mathematical knowledge, comprising the complex interaction of mathematical tasks and learners' developing cognitive structures, is the challenge to which this thesis seeks to contribute in the cluster of concepts fraction, ratio, rate, proportion, probability and percent, elements of the multiplicative conceptual field.

It is proposed in this thesis that the deep exploration of mathematics knowledge (Chapter 3) and attention to the learning and teaching of mathematics in general (Chapter 4), provides the necessary background for the investigation of a particular topic or set of topics, which in this study comprises the multiplicative conceptual field (Chapter 6). It is further proposed that the establishment of valid measures is necessary for informing both continuous classroom interaction and more formal testing (see Chapter 5). An analysis of all the items in terms of the factors contributing to comparative difficulty is presented (Chapter 7, Appendix B). Selected focus points along the scale allowed the selection of students at specified proficiency levels for interviews, enabling finer nuances of conceptual development to be described (Chapter 8). The problem addressed in this study, and the study itself is located within the South African policy and educational context. Attention therefore is given to curriculum policy and to what is perceived to be some influential perspectives informing mathematics education in Chapter 2.

The imperatives of mathematics education vary from individual to individual, ranging from teaching and learning to the measurement of educational objectives. This thesis attempts to cover a broad range, meeting these imperatives to varied degrees.

2 The South African context: Contrasting perspectives in mathematics education

2.1 Mathematical, curriculum and educational perspectives

The philosophical context and the mathematics education landscape including prevalent theories and methodologies that influence mathematics education in South Africa, provide the contextual background for this research study.

The debates surrounding what constitutes mathematical knowledge have preoccupied some of the greatest thinkers of our time, and it may seem presumptuous to introduce this topic. However the perspective is taken in agreement with Vergnaud (1997) that high level epistemological questions such as “the empirical and non-empirical roots of mathematical knowledge, intuition and formalism, the nature of mathematical proofs, and the relationship of mathematics to logic, or about the possibility of proving the consistency of mathematics” (p. 7) cannot be ignored because researchers and teachers generally have unarticulated views which impact on their research and teaching. These unarticulated, or inadequately articulated, views are also implicit in curriculum statements. In order to make explicit these critical notions *predominant epistemologies guiding mathematics education* are briefly described.

With the predominant epistemologies of mathematics education as background, the elements of the framework for mathematics education, proposed in Chapter 1 (Section 1.2.1), are used to analyse the *South African National Curriculum Statement* (DOE, 2002), noting the contributions that address the five components of the mathematics education framework as well as the omissions. Some historical background to the National Curriculum Statement (NCS) is provided. Current proposed revisions to the curriculum as the Curriculum and Assessment Policy Statement (CAPS) (October 2009), are noted. The further objective is to use the framework to probe research areas that have influenced mathematics education policy, and that have the potential to influence the direction of mathematics education. Contributions from psychology, in particular an approach informed by "constructivism"[25] are influential in mathematics education internationally (Ginsburg & Seo, 1999) and are perceived to have dominated the local landscape[26] (Ensor & Galant, 2005). Several theoretical and empirical papers which together provide some insight into constructivism, together with *realistic mathematics education* (RME), and a problem-solving approach, are discussed in terms of the five elements required from a mathematics education framework.

The contributions from sociology, notably the research that has been influenced by Bernstein, and some of which is currently being invoked in support of curriculum changes, are discussed in terms of their potential to contribute to a comprehensive theory of mathematics education.

2.1.1 Research questions

The overarching research question focuses on the broad trends which influence curriculum and mathematics education. Further questions focus more specifically on predominant educational perspectives, their strengths and weaknesses, and the proposed requirements for an operational framework.

Question 2 What socio-political, epistemological and educational perspectives influence curriculum policy, mathematics education and research?

2.1 What are some of the prevailing mathematics education epistemologies informing mathematics education in South Africa? (Section 2.2)

2.2 How does the current National Curriculum Statement (Mathematics), Grade R to 9, address the proposed requirements for a comprehensive framework for informing mathematics education? Where are the existing mismatches between the curriculum, and the proposed requirements for a comprehensive curriculum? (Section 2.3.1)

2.3 What predominant educational theories have informed mathematics education in South Africa over the past 30 years? What does each of the theories offer in terms of a) an explanation of mathematics and its objects, b) descriptions of how children acquire mathematical concepts, c) the function of instruction, d) the role of language and representation, and e) the purpose of assessment? (Section 2.3.2 & 2.3.3)

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[25] The term "constructivism" as used here covers variations of a reform pedagogy where attention is given to learner cognition.

[26] The statement by Ensor and Galant (2005) that a constructivist approach has dominated the South African education landscape should be qualified. In many classrooms teachers may have picked up ideas that they believe will further their children's learning, but a whole-hearted adherence and understanding of constructivism takes place only in isolated instances around the country.

2.2 Epistemologies of mathematics education

The necessity for raising epistemological issues is that implicit understandings invariably inform both the researcher's work and the teacher's approach to instruction (Ernest, 1991). Many issues, such as the relationship of mathematics to everyday experience and the importance for mathematics of other disciplines such as physics, have their roots in epistemology (Vergnaud, 1990). While epistemologies guiding mathematics education draw from either mathematics or psychology, or from both mathematics and psychology, pertinent questions that are directly related to the educational context are inevitably added. Vergnaud (1990) notes that;

> Mathematics education takes place in a certain society, a certain institution, a certain classroom, with such different aims as the education of future mathematicians and the education of rank-and-file citizens. These social constraints on mathematics education do not modify the nature of mathematics per se, but they have strong implications for the way teachers see the teaching of mathematics and mathematics itself (p. 15).

Additional complications that have a direct influence on teaching are that not only do "students' representations of mathematics differ from those of teachers", but teachers' representations vary "according to their views of mathematics, psychology and society" (Vergnaud, 1990, p. 15). It is noted by Davis (2010) that lack of understanding of fundamental mathematical ideas on the part of teachers often distorts the mathematical representations.

It can be argued that the predominant epistemologies underpinning mathematics education[27] are constructivism, formalism and intuitionism (Vergnaud, 1990).

2.2.1 Constructivism

It is generally agreed that Piaget and his collaborators provided the foundations for the central notion that "competencies and conceptions are constructed by the students" in relation to existing cognitive structures, rather than being adopted ready made into a cognitive system (Vergnaud, 1990, p. 22). The roots of constructivism are from within psychology[28]. Piaget did however collaborate with mathematicians, such as Evert

[27] A fourth epistemology namely that of information processing has been noted (Andrich, 2002; Vergnaud, 1990) but will not be discussed here.

[28] The educational ideas of Dewey, in the early twentieth century, also contributed to the problem solving approach (Hiebert et al., 1996).

Beth[29], in investigating core mathematical ideas (Inhelder, p. viii, in Piaget & Garcia, 1989). The critique of constructivism ventured by Vergnaud is that researchers do not specify clearly enough “the physical and social conditions under which knowledge construction takes place” (1990, p. 22). This critique has in part been remedied by more recent writers notably Hiebert et al. (1996), and Ginsberg and Seo (1999).

2.2.2 Complementary epistemologies

Alternative epistemologies influencing mathematics education, and complementary to constructivism, namely formalism and intuitionism are concerned primarily with mathematics and not with psychology. The formalist conception as applied in mathematics education was the preferred epistemology in the 1960s and 1970s, and was exemplified in the “modern mathematics” movement, which saw the introduction of formal set theory into the school mathematics curriculum in many countries including South Africa (Human, 2009a). Underpinning a formalist conception is the notion that all mathematics must be “expressed formally without ambiguity” (Vergnaud, 1990, p. 20). The ultimate goal of mathematics, therefore, is to “reduce mathematics to the syntactic coherence of formal and symbolic systems” (Vergnaud, 1990, p. 20). Although the formalist epistemology is less obvious in the South African environment, elements of this conception are present in “curriculum” research. The term “disciplinary knowledge” or “the knowledge structure” (Muller, 2007) presents a formalist conception of mathematics education. The distinction between the “esoteric domain” and the “public domain” made by Dowling (1998) is based on formalist epistemology. The objection to ethnomathematics by Horsthemke and Schäfer (2007) may be interpreted to assume a formalist conception of mathematics. Davis (2010), responding to the lack of mathematical rigour in classroom practice, advocates syntactic coherence at least in the presentations of teachers, and advocates observing the pre-eminence of the mathematical object in relation to the encompassing mathematics knowledge structure, thereby affirming a formalist conception.

[29] Piaget’s collaboration with Beth culminated in the book “Epistemology of Mathematics” (Beth & Piaget, 1966), written from two perspectives, one psychological and the other mathematical. The final chapter acknowledged the major achievement of the human race to produce mathematics.

In the view of Vergnaud (1990) the formalist conception, on its own, lacks a theory of both the acquisition and development of concepts, and does not account for the intuitive nature of some mathematical concepts. The fundamental understanding of a mathematical concept is limited in that the “formal and axiomatised knowledge can only be the last developed state of a student’s knowledge” (Vergnaud, 1990, p. 21). The question raised by Vergnaud and of relevance in this thesis is whether mathematics can be taught via definitions and axioms, as exemplified in an advanced course for teachers (Usiskin, Peressini, Marchisotto & Stanley, 2003), or whether it is better taught through encountering a wide range of situations, with the help of an appropriate, though less formal, explication of mathematical concepts at various phases of mathematical development.

Scheffler (1965) questions the depiction of mathematics in the learning environment as an essentially deductive process. From the perspective of a finished product, one observes deductive proof as a chain of statements that are “held together by necessity, the strongest conceptual glue: if the premises are true, the conclusion cannot fail of truth” (p. 74). However the generation of the proof is characterised by trial and error, ingenuity and creativity. Grasping the purpose for the proof might be a motivating factor in learning mathematics.

The importance of expressing mathematics clearly and unambiguously is an important goal but may be counterproductive if it comes at the expense of learners’ first hesitant attempts to develop conceptions and processes that make meaning, or at the expense of ingenuity and creativity. The need for formalist conceptions is perceived as critical when arguing the case for a correct though unfamiliar procedure within any mathematical community, at any level, including a Grade 1 classroom! The requirement for more advanced mathematics, in the secondary and tertiary sectors of the education system is a greater emphasis on a formalist conception, with the associated algorithmic structures that in particular circumstances support thinking (see Berger, 2006).

In contrast with the formalist conception, the intuitionist point of view in mathematics education as argued by Fischbein (1987, cited in Vergnaud, 1990, p. 21) is that thinking would be impossible if we could not rely on immediate and self-evident intuitions. Some defining characteristics of intuitions are that it is difficult for learners to act against intuitions; they continue for extended periods of time; they are general and global; and they are necessary for action. Some intuitions inhibit the acquisition of knowledge by exhibiting characteristics of premature closure or overconfidence. In particular, *first intuitions* are often given greater attention resulting in the enduring primacy effect, and are often difficult to replace. Another classification is that of *overgeneralised intuitions*, such as "multiplication makes bigger and division makes smaller". These characteristics of intuitions provide some explanation for the difficulty of acquiring more advanced concepts when earlier more primitive concepts appear to contradict more advanced concepts. Meyer and Land (2005) warn against introducing naïve versions of complex concepts as these versions may inhibit the engagement of a more difficult threshold concept necessary for advancing to a higher understanding. Other categories defined from the intuitionist conception (as presented by Fischbein 1987, cited in Vergnaud, 1990) are *correct mathematical intuitions* such as transitivity of equivalence relations.

The critique advanced by Vergnaud (1990) is that psychologists, and mathematics educators, should analyse the content and development of intuitions in order to render these intuitions explicit. A further requirement for "intuitionists" is to make explicit the links from intuitions to mathematical concepts and therefore to gauge these intuitions against mathematical proficiency.

2.2.3 Epistemology of emergence

The purpose of schooling in modern Western societies, that students acquire knowledge of pre-existing practices and events, followed by tests to ascertain whether the student has acquired a correct or adequate understanding, is being challenged by an emergent epistemology. The challenge to current educational practices is that "becoming educated" is not perceived as the "understanding of a finished universe, or even about *participating* in a finished and stable universe", education is rather "the result of participating in the creation of an unfinished universe" (Osberg, Biesta & Cilliers, 2008, p. 215).

The implications of an emergent epistemology for the learning and teaching of mathematics are still to be explored, but in the opinion of the writer the approach to knowing as a co-construction or a response which "adds something (which was not present anywhere before it appeared) to what came before" (p. 213), rather than the less authentic stance of "pleasing the teacher", may be a more comfortable position for the youth for whom the world is radically changing. Learning the *Theorem of Pythagoras* may be experienced as a co-construction with Pythagoras, or a response with Pythagoras to an interesting problem.

2.3 Mathematics education landscape

The theoretical framework (Chapter 1, Section 1.2.1) provided a structure for making explicit the theoretical underpinnings of mathematics education. The elements of the structure constitute a reasoned view on mathematical knowledge, the acquisition of mathematics concepts, the function of instruction, the role of language and symbols, and the purpose of assessment. These elements are used to interrogate the curriculum, a powerful predictor of what teachers will teach and what will be tested, noting both some potential contributions and the omissions.

The first research category identified within mathematics education research are studies that fall under the constructivist or socio-constructivist approach, which includes the problem-solving approach and the realistic mathematics education approach (Section 2.3.2). The second research category comprises the sociological research informed by the theory of Bernstein (1996), the major contribution of which has been providing the theoretical constructs with which to analyse the complexity of mathematics education, and a methodological approach to structure research studies (Section 2.3.3).

2.3.1 Curriculum, the vision for mathematics instruction

By reflecting the predominant voice of the mathematics education subject and pedagogic specialists, the curriculum has the potential to impact the education system. The curriculum is the structure that informs and guides teaching and learning. This belief is encapsulated in the following extract.

> Curriculum is the most fundamental structure for [the daily mathematical] experiences [of children]. It is a kind of underlying "skeleton" that gives characteristic shape and direction to mathematics instruction in educational systems around the world … [The daily

> experiences of children] do not occur randomly or by accident. They are instead deliberately based on visions of what education should be, ideas of how to create the formative experiences of education, and intended patterns of opportunities that organize the potential for those experiences. … The plan that expresses these aims and intentions, which takes them from vision to implementation, and serves as the broad course that runs through formal schooling, is curriculum (Schmidt, McKnight, Valverde, Houang and Wiley, 1996, p. 141).

In the light of this expressed function of the curriculum, the question needs to be asked whether the stated curriculum reflects a coherent theory of mathematics education able to guide the teacher in planning daily experiences for children learning mathematics.

The South African curriculum history has in part reflected international trends but has also been influenced by particular national circumstances. The radical changes in the national curriculum over the past 20 years were meant to address the challenge of improving mathematics teaching and learning, in parallel with the radical political changes that were taking place in the transition to democracy (Wendt, 2011).

Ensor and Galant (2005), from a sociological perspective, observe that the officially titled Curriculum 2005 (C2005), described as transformational outcomes-based education (OBE), proceeded to loosen two sets of boundaries within the mathematics knowledge structure at school level in the early nineties. The first set, relating to mathematical content, involved the blurring of the boundaries between school mathematics and everyday mathematics, and the blurring across subject areas (Ensor & Galant, 2005). As noted by Ensor and Galant (2005), the useful application of school-learned mathematics to everyday life at that time was the subject of many studies, three of which were particularly influential, Carraher, Carraher and Schliemann (1985) investigating the mathematics of street children, Lave (1988) investigating the mathematics of shoppers, and Walkerdine (1988) investigating the relationship of the everyday to mathematics. These studies were interpreted as showing school mathematics to be irrelevant as the mathematics required for everyday purposes was learnt in everyday contexts. A second set of blurred boundaries emerged in a learner-centred pedagogy which required the teacher to be a mediator of knowledge rather than a transmitter of knowledge. The "new" pedagogies at the time, manifesting in an engagement with the cognitive processes of learners, had been influenced by a constructivist perspective.

Laridon, Mosemege and Mogari, (2005, p. 133) present a slightly different analysis. ‘Curriculum reconstruction for a society in transition’ was the theme of the Political Dimensions of Mathematics Education Conference (PDME 2) in 1993 at which thirteen conference papers “with connections to ethnomathematics, and a similar number on curriculum” were presented (Laridon et al., 2005, p. 133). According to Laridon et al., (2005), many delegates at this conference had a subsequent direct influence in shaping a new curriculum for South Africa. The major influences for C2005 were the genetic epistemology of Piaget, radical constructivism of Von Glasersfeld, social constructivism and socio-cultural constructivism, which incorporated the ideas of Vygotsky.

In the view of Laridon et al., (2005), the approach to the development of mathematical knowledge, influenced by Piaget, Von Glasersfeld, Ernest and Vygotsky, is “located in the fallibilist[30] paradigm” (p. 136). A further development, building on the ideas of Piaget and Vygotsky, and drawing on a ‘fallibilist’ epistemology of mathematics, was an approach to teaching and learning, where the teacher was no longer the dispenser of knowledge but the mediator of an activity-based pedagogy, through which learners gained mathematical knowledge (Laridon, et al., 2005). This relativist epistemology, debated among academics, could be argued to have had little effect at classroom level. The understanding of a constructivist pedagogy, on the other hand, resulted in greater learner participation and a pedagogical approach known as “learner-centred teaching” in isolated pockets, somewhat effective in well resourced schools, or in lower socio-economic communities where there was professional support (Murray, Olivier & Human, 1993).

The implementation of C2005 was investigated by a task team consisting of both academics and teachers, who advised the national Department of Education to provide clearer curriculum guidelines for subject areas and to provide grade-by-grade outcomes rather than 3-year phase outcomes (Chisholm, et al., 2000). This change was instituted in 2001.

[30] The fallibilist view rejects the view that mathematical knowledge “consists of certain and unchallengeable truths” (Ernest, 1991, p. 7) and maintains that “mathematical truth is fallible and corrigible, and can never be regarded as beyond revision and correction” (Ernest, 1991, p. 18).

Though the purpose of a national curriculum is essentially to provide “a plan for learning”, this plan is formulated differently at the national, school, classroom and individual levels. In addition the perspectives, *socio-political*, referring to decision making by stakeholders, *technical-professional*, referring to the construction and implementation of the curriculum, and the *substantive*, focussing on what knowledge is worth teaching, each have to be considered (Van den Akker, 2003). The *substantive* aspect, the nature of the knowledge to be taught, how best learners acquire this knowledge, and how mathematics proficiency is best assessed is the focus of this thesis.

The curriculum is able to contribute to the fundamental structure of an education system from the *substantive* perspective, by making explicit the epistemological underpinnings of mathematics, providing plausible developmental paths consisting of broad topics, the listing of objectives in some form, outcome statements or content topics, and describing the key elements of the pedagogy required to support the acquisition of concepts. In addition, by providing some notion of the key concepts and their development across the grades, the curriculum from the *technical-professional* perspective is able to provide support for the process of somewhat hierarchically ordered mathematical development.

The explicitly stated *view of mathematics* in the National Curriculum Statement (NCS) (DOE, 2002, p. 4) is that mathematics is “a human activity that involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves” and that mathematics is “a product of investigation by different cultures. It is a purposeful activity in the context of social, political and economic goals and constraints”. This statement is further elaborated: “mathematical ideas and concepts build on one another to provide a coherent structure” (ibid.). These statements depict a view of mathematics drawing on both a constructivist and formalist view of mathematics education.

In the curriculum document (DOE, 2002) there are statements regarding the *outcomes* of education, and some reference to the process of *acquisition of concepts*, but no explicit articulation, for example

- “Through [observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves] … new mathematical ideas and insights are developed” (p. 4).

- Learners [are required to] "develop deep conceptual understanding" and "acquire specific knowledge and skills" (p. 5).

The description in the glossary of "outcomes based education" is that it is a "process and achievement-oriented, activity-based and learner-centred education process" (DOE, 2002, p. 102). This somewhat vague statement does not provide insight into, *how instruction supports the process of learning*. In the document there is no specific reference to the textbook. The role of the teacher is described as that of a facilitator and mediator of knowledge, in contrast to that of a transmitter of knowledge, but no explanation is presented of how the learning experience may be facilitated.

In terms of *semiotics* (language, representation and symbol), the curriculum document states explicitly that "mathematics uses its own specialized language that involves symbols and notations for describing numerical, geometric and graphical relationships" (p. 4). It also encourages learners to "participate in mathematics lessons and to express their mathematical ideas" (p. 3). The processes involved in providing meaning for mathematical language and symbols are not made explicit.

Of the five aspects regarded by this thesis as being essential to a theory of learning, the *assessment* aspect of the curriculum appears the most detailed (DOE, 2002, p. 3). The requirements for assessment are to:

- accommodate divergent contextual factors,
- provide indications of learner achievement in an effective and efficient manner,
- ensure learners integrate and apply skills,
- make judgments about (learners' own) progress, and to
- set goals for progress and further learning.

Continuous assessment is regarded as the process which "supports development of learners", "provides feedback", "allows for integration", "uses a variety of strategies" and "allows for summative assessment" (2002, p. 95). The fact is that assessment, without a coherent theory of how learners acquire knowledge and without links to knowledge development, does not necessarily support the process of teaching and learning and may inadvertently interfere with the learning process.

Core elements in the National Curriculum Statement

It is acknowledged that it is not possible for any curriculum document to cover in any detail a comprehensive theory of mathematics education and that the teachers have access to workshops, courses and other resources which add to their professional knowledge. It is noted that without adequate recourse to the deeper mathematical ideas underpinning such statements as “mathematics is a human activity”, teachers may underestimate the complexity entailed in the learning and teaching of mathematics. The teacher, without access to a plausible theory of learning and teaching, may misunderstand the careful balance between mediation and transmission.

The solution to a skeletal curriculum may be in a comprehensive textbook which provides for the teacher the detail regarding the content of the curriculum and, implicitly, some direction regarding pedagogy. In some sectors of the education community, the use of textbooks has been assumed to be counter to reform in that the restriction of mathematics to one text may not provide for the needs of all the learners or not suit the particular teacher. As there is no explicit reference to the use of a text book in the current curriculum documents, teachers may interpret this absence to mean that the textbook has no function. This interpretation has worked against learning in that the very teachers and learners for whom the textbook may have been a most valuable resource, have been deprived of the resource (Taylor, 2008). In contrast it must be noted that from a constructivist perspective it is the learner *engagement with the ideas* (in the textbook or elsewhere) that results in learning.

The engagement with concepts, the interaction with different facets of any concept, relating these facets to existing concepts and the rebuilding of the concept in a way that relates to the socially accepted understanding, is not elaborated in a theory of instruction in the NCS. There is no explicit statement regarding the importance of the process of transforming the intuitive and implicit knowledge into explicit and generalisable knowledge (Vergnaud, 1990). In sociological terms, this process would be described as the process of inducting learners into esoteric knowledge, and providing learners with the “generative principles” governing mathematics knowledge (Dowling, 1998).

National policy documents are either supported or challenged by predominant education theories. In South Africa, constructivist notions could be seen to have influenced

C2005. The current proposed revisions[31], notably the Curriculum and Assessment Policy Statement (CAPS), are strongly influenced by a sociological perspective judging by the references in the *Report of the Task Team* (Dada, et al., 2009).

2.3.2 A constructivist perspective on learning and instruction

Arguably, the most important contributions from psychology to mathematics education have come from Piaget, who together with collaborators conducted substantial theoretical and empirical work on mathematical topics (Piaget & Garcia, 1989). Piaget's view that mathematical conceptions and competences are produced through the activity of the child is widely accepted throughout the mathematics education community. It is important to note that Piaget did not directly contribute to the domain of instruction or the role of the teacher. An unclear point in the work of Piaget according to Vergnaud (1990) is the relationship between problem situations and the development of competences. The function of assessment, except in an informal evaluative sense, was not the concern of Piaget.

Vygotsky (1962) makes a complementary contribution in that he focused attention on the relationship between implicit reasoning in arithmetic and the explicit and formal reasoning in algebra, and focused attention on the role of language and representation. Another valuable contribution is that of the zone of proximal development[32], which refers to the span between the current independent problem-solving ability of the child and his or her potential development, as determined through assisted problem solving mediated by the teacher or peers.

Papers presenting the general constructivist approach[33] at the primary level are those of Hiebert, Carpenter, Fennema, Fuson, Human, Murray & Olivier (1996), who describe the central elements of a problem-solving approach, Ginsberg and Seo (1999), who

[31] At the time (July, 2010) strong control was exerted on the curriculum development process. A more collaborative approach emerged through the submission process which enabled the voice of the mathematics education community to be heard.

[32] The explanation given by Vygotsky (1962) is "(t)he discrepancy between a child's actual mental age and the level he reaches in solving problems with assistance indicates the zone of his proximal development" (p. 102). Recent definitions have moved with the times.

[33] In most cases the authors referenced are South African. Where outside authors are referenced it will be noted. Ginsberg and Seo (1999), not South African, are reviewed as their views are representative of a broad constructivist approach.

maintain a strong mathematical focus and Human (2009a) who presents an overview of the problem-solving approach over the past 30 years.

Additional contributions informative of the constructivist approach, evident in the work of South African researchers, Fraser, Murray, Hayward and Irwin (2004), who report on a teaching design experiment focused on fractions, are included. Research from the slightly different perspective of the Realistic Mathematics Education movement, is that of Van Etten and Smit (2005), whose research study focuses on algebra in the senior phase. Although, discussed under the umbrella of constructivism, realistic mathematics education may differ with regard to the implications of the constructivist epistemology for the classroom.

The essence of constructivism is the notion of an intelligent individual who actively engages with his or her world. The central assumption is that learners interpret new information in terms of prior knowledge. The existing cognitive structures develop in order to accommodate the new knowledge. The comprehension of new subject matter is "a function of knowledge construction and transformation rather than acquisition and accumulation" (Loyens and Gijbels, 2008, cited in Renkl, 2009).[34]

Of note is the work of Brodie (2007), also drawing on theories of situated cognition (Boaler, 2002, amongst others), who interrogates the notion of learner participation in secondary school classrooms. The focus of this work is on mathematical reasoning, which is cultivated in a classroom pedagogy that acknowledges the engagement and contributions of all the learners as central to learning mathematics.

In the view of this thesis, the greatest contribution of a constructivist view is in the area of learners' acquisition of mathematics concepts, one of the five elements deemed to be important.

A view of mathematics

Views on mathematics can be found across the range from radical constructivism where an objective reality is questioned (Von Glasersveld, 1995) and a fallibilist approach,

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[34] Neither Renkl (2009), nor Loyens and Gijbels (2008) are South African or commenting on the South African context, however their views are regarded as common within a constructivist grouping.

which incorporates the ethnographic idea that mathematics is differently conceived in different communities (Laridon et al., 2005), to the view that mathematics is an established body of knowledge comprising both mathematical objects and processes, exhibiting internal coherence and a measure of hierarchical order. In general, from a constructivist perspective the view is that mathematics is a hierarchical knowledge structure, a discipline to be acquired by learners. The historical development of mathematics is central to providing insights into mathematics. This approach is exemplified by Gierden (2009) who explores the origin of multiplication tables for instruction purposes.

Ginsburg and Seo (1999) observe that the quantitative features in the environment are pervasive - even Shakespeare's plays and poetry have a plethora of quantitative references. They concede that a constructivist epistemology may take the different view that man constructs these mathematical relationships and concepts, but either way learners encounter many mathematical concepts in their everyday lives.

Implicit in the *problem-solving pedagogy*, explored by Fraser et al. (2004), is an acknowledgement of the existence of the prior mathematical structure of fractions, the components of which are to be acquired by learners. The distinction is made between knowledge that can be "discovered" on one's own, logico-mathematical knowledge, and social knowledge which is transmitted by the teacher or acquired through other resources. The expressed aim in *realistic mathematics education* is to lead learners to higher levels of mathematics, explicitly acknowledging an established body of knowledge that is to be acquired through both *horizontal* and *vertical mathematising*. For both these mathematics education movements the aim is for students to acquire abstract and generalisable knowledge of mathematics.

A related view is presented by Human, Hofmeyer, Human, Makae, and Van Koersveld (2010) who, in order to upgrade their students' mathematical proficiency to the level required of engineering courses at tertiary institutions, pay attention to the mathematical modelling of real world situations, that is to make sense of the problems, and to identify the variables and the relationships between the variables.

Acquisition of mathematical knowledge

The importance of acknowledging the intelligent thought exhibited by children in the *acquisition of mathematical knowledge* is central to the problem-solving approach. Dewey's observation is that "thoughtful but ordinary methods of solving problems share fundamental features with the more refined methods of scientists, and the differences are in degree and not in kind" (Dewey, 1910, cited in Hiebert et al., 1996, p. 12).

Hiebert et al. (1996) advance the view that "children should be allowed to make the subject [mathematics] problematic" (p. 12). They emphasised the problem-solving component of mathematics, working on the assumption that this component is essential for engaging learners' inquisitiveness and therefore cognitive receptiveness to new ideas. The principle of problematising is closely associated with Dewey's notion of reflective enquiry. Two characteristics of reflective inquiry are noted; the first is that the purpose of problematizing is to understand their experiences more fully; the second defining feature is that it also involves "a willingness to endure a condition of mental unrest and disturbance" (Dewey, 1910, p. 13, cited in Hiebert et al., 1996).

For Human et al. (2010, p. 9), the acquisition of the critical concepts entailed in engineering-type courses, that of identifying variables and relationships, and understanding the property of covariance, would be more easily attained if students had engaged with variables, and with strategic thinking, in response to problem situations, much earlier in their mathematical careers. This engagement would circumvent some of the difficulties encountered in engineering-type courses.

The role of the teacher

The distinctive characteristic of a *constructivist approach* is that engagement with mathematics starts with a problem situation rather than the discipline. From a constructivist perspective the teacher is responsible for *setting tasks* and *providing information*. Firstly the teacher draws on knowledge of the subject in order to "select tasks that encourage students to wrestle with key ideas" and secondly she draws on the knowledge of students' thinking to select tasks that are relevant and link with concepts and skills the students already possess (Hiebert et al., 1996, p. 16).

A *functional* or *structural* approach to mathematics informs a problem solving approach in different ways. A functional approach requires learners to become involved in exploring, trying out methods, reflecting on answers, communicating their findings and listening to others. The teacher's role is to "build the social and intellectual community in the classroom" by monitoring the adequacy of learner methods and searching for better ones (Hiebert et al., 1996, p. 16). The structural approach involves reflecting on mathematical experiences and "representing and organizing knowledge internally in ways that highlight relationships between pieces of information" (p. 17).

Realistic mathematics education theory uses the terms *horizontal* and *vertical mathematisation* to describe the phases of problem solving, where Hiebert et al. use a functional and structural approach. *Horizontal mathematisation* involves translating realistic problems into mathematical terms; *vertical mathematisation* involves attention to the mathematical relations "established through (finding) the solution and (the subsequent) reflection on a range of carefully selected problems" in order to develop higher order mathematical skills (Van Etten & Smit, 2005, p. 57). Van Etten and Smit (2005) note that the specific mathematics to be mastered is the first priority in the selection of potential sources for meaningful problems. At the end of the mathematising process a concept, definition or result is refined to a more formal format, with guidance from the teacher.

Ginsberg and Seo (1999) make the claim that "children's informal mathematics contains on an implicit level many of the mathematical ideas that teachers want to promote on a formal and explicit level" (p. 113). The principles of their approach emanate from two assumptions, firstly that "the environment is deeply mathematical in structure" and that "the mind comprehends the world in mathematical terms (among others)" (p. 115). The role of the educator therefore is to "uncover the mathematical ideas immanent in human thinking" and help children to "formalize these ideas (or to modify them, when there are serious misconceptions)" (p. 115). This transition is explained as follows;

> "If the child's solution implies a notion of associativity, then at some point, the child needs to appreciate in an explicit and formalized (symbolic) way the general concept of associativity, its application to various circumstances, and its place in the network of mathematical ideas" (p. 126).

Ginsberg and Seo (1999) note that an unfortunate tendency to downplay the mathematical content has resulted from the unduly singular focus on learner constructions. They aver that mathematics instruction should draw on “an integrated perspective, from both constructivist and mathematical points of view” (Ginsberg & Seo, 1999, p. 114).

The disputed view in a ‘learner-centred’ curriculum (in the South African context) that teachers do not provide necessary mathematical knowledge is qualified by Hiebert et al. (1996), who state that the teacher is “obligated to share relevant information with students as long as it does not prevent students from problematizing the subject” (p. 16). There is obviously a fine line between providing the correct level of information which encourages and facilitates progress and, by contrast, unwittingly for most teachers, effectively telling learners the mathematical equivalent to the end of the Harry Potter novel, thereby limiting their excitement and experience of achievement.

The role of the textbook is not foregrounded but serves as a resource for mathematical situations from which the teacher makes a selection. In the South African context the textbook, provided the mathematics is coherent, may also serve as a knowledge resource for the teacher; mathematics is being learned by teachers in many cases concurrently with teaching. As Human (2009a) reminds us “(t)here can be no doubt that teaching for problem-solving, or problem-centred mathematics teaching makes considerable demands on the teacher and that explicit and specific knowledge is demanded” (p. 314, own translation[35]). These extra demands require a reconceptualisation of what it means for learners to learn mathematics and what it means to teach mathematics.

Language, diagrammatic representation and symbols

Language, diagrammatic representation and symbols obviously have an important function in the acquisition of concepts. For example the following situation, which describes a language- and discourse-rich classroom, is a common experience in classrooms set up for problem-solving pedagogy.

[35] Daar kan min twyfel wees dat onderrig vir probleemoplossing en probleemgesentreerde wiskundeonderrig beduidende en spesifieke eise aan leerkragte stel, en dat bepaalde kundighede vereis word, (Human, 2009a, p. 314).

> When a group of learners repeatedly tackle problems of a similar type, and compare their individual attempts at solving the problem in groups (through discussion), they gradually progress to less time-consuming strategies that represent the same thinking (Human, 2009a, p. 310, own translation[36]).

Gierden (2009) describes how the standard learning of the multiplication tables can be transformed through identifying, describing and analysing number patterns on a multiplication board. Intrinsic to this activity is the notion that children learn through engaging problem situations, and that diagrammatic representations assist in providing the stepping stones to more abstract mathematics language and symbols.

The importance of language for expressing and clarifying mathematical ideas is the focus of classroom-based research by Brodie (2007).[37] She has taken an aspect of the new curriculum which calls on learners to "participate in mathematics lessons and to express their mathematical ideas", as central in her research. This participation in mathematics lessons is regarded by Brodie as important as it maintains the attention of learners, enables learners "to express and clarify ideas", "provides information to teachers about what learners know and don't know, and [provides information on] how learners are thinking and trying to make sense of ideas" (p. 3).

Besides Brodie's work much attention is paid to the role of language in multilingual mathematics classes (Adler, 2001; Setati, 2005; Setati, Molefe & Langa, 2008; Webb & Webb, 2008; Bohlman & Pretorious, 2009). The interesting and critical language issues constitute an important body of research but fall outside the scope of this thesis. The topic of language, representation and symbols is picked up again in Chapter 5 and will therefore not be elaborated further here, except to present a slightly different viewpoint offered by a mathematics teacher at tertiary level.

Berger (2006) contradicts a position espoused by many mathematics educators that the construction of mathematical meaning precedes the use of the symbol. Berger examines the process experienced by a student at university as that student makes meaning of a

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[36] "Wanneer 'n groep leerlinge herhaaldelik soortgelyke probleme aanpak, en hulle individuele werk vergelykend bespreek in groepe, skakel hulle geleidelik oor na minder tydrowende metodes wat dieselfde denke verteenwoordig" (Human, 2009a, p. 310).

[37] It is noted here that Brodie's work, and that of others mentioned, exhibits complexity and nuance in learning and teaching that may not be collapsed into a single category. However, for the purposes of this thesis these studies are close enough to be considered alongside studies defined as "building on constructivist ideas".

mathematical object presented in the form of a definition. Berger rejects the neo-Piagetian perspective that takes Piaget's work regarding elementary mathematics as appropriate to advanced mathematical thinking, as that notion of meaning preceding symbol extrapolated to higher mathematics, does not resonate with what she experiences in her university classroom. The central drawback of these theories according to Berger is that;

> they [the neo-Piagetian studies] are rooted in a framework in which conceptual understanding is regarded as deriving largely from interiorised actions; the crucial role of language (or signs) and the role of social regulation and the social constitution of the body of mathematical knowledge is not integrated into the theoretical framework (2006, p. 15)

A further observation is that a child does not spontaneously develop concepts independent of their meaning in the social world. The social world with already-established definitions of different words, and provided in authoritative texts, determines the way a child's concept develops. Berger makes the comparison of a child using a new word with learning in undergraduate mathematics courses. "The child uses a new word for communication purposes before the child has a full understanding of the word" (Berger, 2006, p. 16). In the same way "a student is expected to construct a concept whose use and meaning is compatible with its use in the mathematics community" (2006, p. 16).

The notion of a *pseudo concept* to some degree explains the use of "mathematical signs (in algorithms, definitions, theorems, problem solving, and so on) in effective ways that are commensurate with that of the mathematical community even though the student may not fully 'understand' the mathematical object" (Berger, 2006, p. 18). When the pseudo-concepts and the links between the pseudo-concepts become part of a "culturally-recognised and consistent system of hierarchical knowledge", they can be described as true concepts.

The function of assessment

The *role of assessment* within a constructivist paradigm would be primarily that of a continuous process of formative assessment to provide information on the learner's progress to both learner and teacher. Hiebert et al. (1996), assert that "(f)eedback on the appropriateness of the methods and solutions comes from the logic of the subject rather than from master/teacher" (p. XX). The function of assessment from a mathematical perspective would be to gauge the mathematics concepts evident in the learners' working (Ginsberg & Seo, 1999).

Schoenfeld (2007), having considered many aspects of problem solving, and taken seriously the notion of thinking mathematically, questions what it means for a student to be proficient in mathematics. The obvious answer is that students should have a sound knowledge of the subject matter. This answer is not so clear when proficiency is defined differently. The general view is that students should be able to do computations and solve the equivalent "word sums", as is common practice in the Standardised Achievement Tests-9 (SAT-9). An alternative instrument, the Balanced Assessment test was designed by Schoenfeld and colleagues to assess a broader understanding of proficiency. A comparison of proficient and non-proficient performances on the two different tests showed that doing well on the Balanced Assessment test is a good guarantee of doing well on SAT-9. The converse, however, is not true: "(O)ne third of the students declared proficient on the SAT-9 exam were declared to be not proficient on the Balanced Assessment test" (p. 63).

For proponents of a general constructivist approach, or the problem-solving approach, not only is the learner's ability to apply their knowledge to problem situations assessed, but attention is also paid to the strategies students use, to meta-cognitive processes and to the beliefs and dispositions of the students. The notion of a *productive disposition*, that is the "habitual inclination to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy" (Kilpatrick, Swafford & Findell, 2001, p. 5), has been offered. More recently the potential of particular "habits of mind" (Cuoco, Goldenberg & Mark, 1996, 2010), has been advanced by mathematics education practitioners in the learning of mathematics.

Summary

In summary, the existence of an established body of mathematical knowledge, including both concepts and processes, is evident. The focus on the detail of the constructs and relationships between constructs in mathematics, and the contexts and representations by which the novice in mathematics can be introduced to that complexity, is what is required from the teacher of mathematics at school level.

The proviso that it is necessary to problematise mathematical knowledge for the purpose of acquiring deeper mathematical knowledge is distinctive of the problem-solving approach. The notion of an intelligent child actively making sense of his or her environment whether it occurs in the public domain or in relation to abstract mathematics is perhaps the greatest contribution attributed to Piaget, Dewey and Vygotsky. The role of the teacher and of instruction is to assist learners through providing the mathematical situations and experiences which provoke their inquisitiveness and engage reflective enquiry which then results in deep knowledge. The teacher's role is crucial for this constructivist theory of mathematics education in which language and representation is the critical feature, and which promotes the development of concepts through making thinking explicit. Human states that …

> If constructivism has a message for teachers, it is that they should accept that learners construct their own knowledge, and try to influence these constructions, not by trying to close them out, but by gaining control over them. To be able to do this the teacher needs to know what learners construct and what influences these constructions. This in turn means that teachers should notice what learners think and do, and take it seriously and teach by building on learners' ideas. One thing that this means is that teachers should often move away from the apparent safety of the front space of the classroom (including the chalkboard) and move around between learners and interact with them individually and in small groups (Human, 2009b, p. 1).

Alternative tests, the *Balanced Assessment* programme, have been considered as alternatives to currently established summative assessment (Schoenfeld, 2007).

It makes sense that in the early years of learning mathematics, the building blocks on which mathematic concepts are constructed comprise the learners' everyday experience. But, as learners progress along the mathematical path, there is more and more mathematics that, together with everyday experiences, is able to provide hand holds to climb to the next level of mathematical development. A description of a constructivist approach at higher levels has different implications than in the earlier grades. At higher levels, mathematical experience needs to take into account prior knowledge in order to have an empowering effect in the solving of real world problems and in developing higher level, more abstract sets of tools and objects.

2.3.3 A sociological perspective on pedagogic practice

In this section a complementary paradigm that investigates mathematics learning and teaching from a sociological perspective, informed by Bernstein's theory of pedagogic discourse (1996), is presented. The framework constituted by the five elements, a view of knowledge, the acquisition of concepts, the purpose of instruction, language representation and symbols, and the role of assessment (see Chapter 1, Section 1.2.1), is applied to the research loosely bound in this category. The important ideas developed in this body of research are in many respects different from a constructivist perspective, nevertheless an attempt is made to group the ideas according to similar categories.

Bernstein's purpose in developing theoretical tools was to create an awareness of the unintended but avoidable reproduction of the inequalities in social class through pedagogic practices (Bernstein, 1996). How mathematical (and non-mathematical) knowledge is structured, distributed and the forms it takes, how it is acquired, transmitted, and assessed, and the impact of these factors on producing social difference were his concerns, and the concerns of researchers who have developed his work (Davis, 2001; Dowling, 1998; Ensor, 2000; Muller, 2007; Reeves & Muller, 2005; Taylor, Muller & Vinjevold, 2003). A contribution to the understanding of how language and symbols are used in the acquisition of mathematics is made by Dowling (1998).

The Study of Pedagogic Operatory Space (SPOS) group (Davis, Jaffer, Johnson and Roberts, among others) has sociological roots that can be traced to Bernstein (1996) and other eminent theorists. It is noted here that their work is driven primarily from the mathematical perspective, the desire to grant learners and students access to truly powerful mathematics (Davis & Johnson, 2007; Davis 2010; Jaffer, 2010). The conviction held is that access to the organised structure of mathematics will be a more cognitively economical route than wading through trumped up everyday explanations of mathematical processes such as “Take the *x* to the other side” (Davis, 2010).

Recently the focus of study for the SPOS group has been on the nature of mathematical objects and operations, and the use (and misuse) of mathematical objects and symbols (Davis & Johnson, 2007; Davis, 2010), the orientation towards text (Jaffer, 2010), the role of language in mathematics classrooms (Roberts, 2010), and in general attention to teachers' evaluation of learners' criteria for the reproduction of mathematics. The empirical work is located in working class schools in the Western Cape, the theoretical tools are developed by taking a mathematical lens to the pedagogical practices. Their findings and the theoretical tools that have been developed have application to mathematics education generally, and provide potential insights for formal assessment practice.

What follows is a brief somewhat cursory overview of contributions of researchers working within, or building upon a Bernsteinian framework. These studies may range from using Bernsteinian analytic tools to building theory which has applicability in mathematics education. The major contribution is noted as providing analytic concepts, with which to analyse the complexity of pedagogical situations. This overview does not claim to be comprehensive, but serves to highlight the potential for contributing towards a theoretically informed and practically executable mathematics education in South Africa. It also serves to provide the background against which the theory of conceptual fields is introduced.

Mathematics knowledge structures

In general Bernsteinian sociology has concerned itself with broad ideas about knowledge structures, rather than focus on the finer details of mathematical knowledge. Of particular note is the distinction between vertical and horizontal knowledge structures and the associated discourses. The distinction between a so-called horizontal knowledge structure (mathematics) and a vertical knowledge structure (physics) rests upon the degree of coherence within the structure. "A vertical discourse takes the form of a coherent, explicit, systematically principled structure, hierarchically organized …" whereas "a horizontal discourse consists of local, segmentally organized, context-specific and dependent strategies for maximizing encounters with persons and habitats" (Bernstein, 1996, p. 171). Keeping with Bernstein's distinction, Muller questions why, in the discipline of mathematics, there is a proliferation of new languages which become separate fields of study, whereas science progresses through the integration of different fields of study through an integrated structure (Muller, 2007, pp. 70-71). Muller's purported evidence for mathematics consisting of horizontal knowledge structures with discrete fields of knowledge is that there are different languages, manifesting perhaps in the distinction between pure mathematics and statistics.

Knowledge structures can be further classified as having weak and strong grammars. Mathematics is classified as having a strong grammar, recognisable in that mathematics texts are not understandable outside of the discipline. Some mathematics education texts however, can equally easily be read by a psychology, sociology or political science student, and therefore may be described as having a weak grammar.

Another theme that concerned Bernstein was the differences between school mathematical practices and out-of-school mathematical practices (Bernstein, 1996). The issue of relevance of the mathematical curriculum for school mathematics as opposed to mathematics as a discipline, occupied the attention of the sociologists in the late 1990s and early 2000s (Ensor & Galant, 2005; Muller & Taylor, 1995; Taylor & Vinjevold, 1999) and is of fairly recent concern (Muller, 2007). This concern that the inclusion of the everyday in the mathematics classroom is counterproductive to learning mathematics is found in several reports, notably the Review of the Task Team (Dada et al., 2009, p. 13).

Dowling (1998) investigated the phenomenon that school (academic) knowledge structures and everyday knowledge structures are different and that therefore acquisition of knowledge in different sites is differently unfolded. The significant contribution of Dowling, which has relevance to the South African mathematics education, is the analysis of school mathematics texts. In particular he observed the differences between higher level mathematics texts and lower level texts. The higher level texts, which included generalisable mathematics principles, provided students with access to the recognition and realisation rules that allow access to disciplinary knowledge. The "everyday" texts, designed for learners with supposedly less ability, exhibited a horizontal knowledge structure with topics presented in texts with little connection to the abstract ideas which, in Dowling's view, provide children with access to the regulative discourse of powerful mathematics. Of particular importance was the categorisation of mathematical statements into statements that belonged in the public domain or belonged in the esoteric domain. This distinction had application within Curriculum 2005, where the boundaries between the esoteric domain (real mathematics) and the public domain (everyday mathematics) in the view of Bernsteinian sociologists were being obscured. A further criticism by Dowling (1998) was that seeing mathematics in everyday situations, and claimed in ethnomathematics, is only possible with a mathematical 'gaze'. For middle-class children this 'gaze' is fostered by parents.

Knowledge acquisition

Another focus of Bernsteinian research is concerned with how learners gain access to knowledge structures within highly formalised systems (Dowling, 1998). How do learners gain access to the generative principles? And how do they acquire the recognition and realisation rules appropriate to the knowledge domain? Muller and Taylor (1995) draw on the work of Walkerdine (1988) to answer this question. They note progressive shifts from the everyday to formal mathematics through conceptually linked steps. O'Halloran (2007) investigates the problem that some children successfully learn the discourse of mathematics while others are not able to gain mastery of even elementary mathematics. She maintains that "cognitive ability is an acceptable guise for making invisible the political practices which enable wealth and status to be one major factor for success at school" (O'Halloran, 2007, p. 231). However, she comes to the same conclusion as that of the cognitive psychologists, that there is an "urgent need to develop theoretical and practical approaches for developing effective teaching strategies, particularly for teachers working with disadvantaged students" (O'Halloran, 2007, p. 232).

Organisation of learning experiences

Research into how learning experiences are organised is the business of the sociology of education. Bernstein introduced his analytic constructs to explain the power and control relations in pedagogical practice. *Classification* refers to power relations, and *framing* refers to relations of control. These two features characterise both the instructional dimensions (concerning knowledge distribution) and the regulative discourse (concerning rules of behaviour) (Bernstein, 1996). The relation between spaces, teacher space and learner space, can be described in terms of classification (power) in that strong classification results in clearly defined spaces, literally with the teacher at her desk and the learners at theirs; weak classification would result in unclear boundaries between spaces, more communication between the teacher and learners and even with the initiative for actions being taken by the learners. Stronger framing characterises teacher-centred pedagogy, whereas weaker framing characterises learner-centred pedagogy (Morais, 2002). The discursive rules that define teacher-learner relationships are selection, sequencing, pacing and the evaluation criteria.

With regard to mathematics, the discursive rules govern the selection of content, the sequencing of particular topics across grades, the pacing of learning experiences and the use of evaluation criteria for informing both learners and teachers of what is expected of them in regard to the particular knowledge structure. Applying these concepts to empirical research investigating the opportunity to learn (OTL) in Grade 6 mathematics classrooms, in both middle class and working class schools, Reeves and Muller (2005) note that the curriculum has some influence in directing the learners' exposure to mathematics. The curricular pacing in many of the working-class schools was particularly slow, with teachers presenting lessons that should have been taught in earlier years. This slow or delayed exposure, it is hypothesised, could be attributed to the inability of teachers to organise and pace instruction appropriately. Commenting on the ability of the curriculum to act as a guide to ensure curriculum coverage, Reeves and Muller advocate more guidance in regard to curriculum coverage and pacing.

Attention to curriculum coverage and pacing constitutes what Van den Akker (2003) identifies as the *technical-professional* aspect of curriculum. Working from the *substantive* perspective which engages knowledge at a deeper level, Davis and Johnson (2007) note that it is "lack of stability of notions constituted as mathematical notions" that produces mathematics that is context dependent and which therefore make it difficult for students to transfer knowledge to other classes of problems (p. 134). The lack of ability to generalise across contexts is attributed to "a lack of explicit attention to the object upon which operations are performed (and is therefore) implicated in producing low levels of competence" (Davis, 2010, p. 380). These difficulties result in slow pacing. The problem identified by Davis and Johnson (2007) and extrapolated in other writings (Davis, 2010), is the concern that analysis of learners' mathematical activity must be conducted from a mathematical perspective, rather than from any other disciplines. This concern is central to the work of Vergnaud (1988).

Morais, Neves and Pires (2004) report on an intervention, informed by Vygotsky's ideas about the child as an active learner within a social context, and the role of the teacher as the designer of social contexts to provide learning experiences. The intervention adheres to the Vygotskyan principle that learning precedes development, and in particular structures the teaching within the zone of proximal development, the conceptual distance from current functioning, in which learners will benefit from instruction or help from more advanced peers (Morais, Neves & Pires, 2004).

Using Bernsteinian constructs to provide the language of description required to classify the data, and statistical methods to analyse the data, Morais, Neves and Pires (2004) show how the characteristics of a particular pedagogy can be a factor in the success of children from all socio-economic levels. The most important and obvious characteristic of successful teachers is that they possess scientific knowledge. Given a knowledgeable teacher, their findings showed that the pedagogic practice which "promotes a high level of scientific development in primary school constituted a *mixed pedagogic practice* with weak boundaries between teachers' and children's spaces (weak classification between spaces) and open communication relations between teacher-child and child-child (weak framing at the level of hierarchical rules)" constitute a successful pedagogy (2004, p. 14). The following characteristics of a successful science classroom were observed:

- explicit evaluation criteria (strong framing),
- weak pacing of learning (weak framing),
- strong interdisciplinary relations (weak classification between various subject matters),
- high level of conceptual demand, and
- high level of investigative proficiency.

The findings of this study, while located in physical science, have application for mathematics teaching.

Semiotics

An example of attention to semiotics, within a Bernsteinian framework is the research of Dowling (1998). The work of Davis (2010; 2011) is also noted as it deals in depth with mathematics, the nature of its objects and operations and the relation of object, meaning and symbol.

Evaluation

The most significant contribution to assessment in education is the notion that the evaluation criteria of any subject area must be made explicit. Making the evaluation criteria explicit is necessary for gaining access to the esoteric domain (Dowling, 1998). Certainly the work of Andrich (2002) shows how the hierarchical development of a subject topic can be analysed into component parts which serve both teaching and assessment. With this analysis the evaluative criteria are closely related to the subject discipline and are made explicit, enabling both the examiner and the teacher to have insight into what is valued in the particular knowledge area. The idea that intuitive and local notions be transformed through scaffolding into explicit and general ideas is central to the work of Vergnaud (1988, 1990).

Summary

In summary, the problem as formulated by sociologists, of the Bernstein school, is how various symbolic systems shape the world and how disadvantage is perpetuated. The theory developed by Bernstein and his students makes visible the elements of a pedagogy that promotes, or alternatively restricts, access to the principles of the knowledge structure concerned.

The wide brush strokes concerning knowledge structures, notably the distinction between mathematics and science, without a thorough investigation of the particular knowledge structure (Bernstein, 1996; Muller, 2007), can be contrasted with Kitcher (1983), who makes a philosophically reasoned claim for an empiricist view of mathematics, in part based on the similarities in mathematical development and scientific development. Kitcher, also, in defence of an empiricist view[38], traces the history of calculus. This empiricist view may certainly be challenged by a view of mathematics that is somewhat removed from everyday experience, but the detailed scholarship involved provides the novice with access into the world of mathematical thought. By contrast broad brush strokes, without engagement with the discipline itself,

[38] The perspective taken in this thesis is not to make a philosophical argument in favour of particular epistemologies or views about mathematics knowledge, but instead to accept generally reasonable views that provide insight into mathematics and inform the teaching and learning of mathematics.

provide little insight into the discipline and may present a confusing picture for policy makers and planners who may not have engaged directly with mathematics themselves.

The focus on *technical-professional* aspects of the curriculum, that is the selection, sequencing and pacing of the curriculum, is necessary, though these aspects refer to surface level features of classroom practice. It appears that the sociological research purportedly based on Bernstein's work may have influenced the recently implemented Foundations for Learning Campaign (DBE, 2008) where strong framing of the curriculum, as far as what is to be taught each term and the precise sequence in which it is taught, is stipulated, with testing at the end of each term. The same implicit theory is evident in the recent recommendations to the Minister of Education (Dada et al., 2009, p. 20) and in the recommendations regarding the formatting of the proposed curriculum revision. The selection, sequencing and pacing of particular concepts may be useful but only as an adjunct to focus on the development of mathematical concepts. The decisive distinction imposed between the everyday and school mathematics is, in the view of this thesis, problematic.

The work of Morais, Neves and Pires (2004) exemplifies the use of a language of description developed from Bernsteinian theory in order to make explicit the lens through which the empirical data are being interrogated. They invoke Vygotskyan theory to design the pedagogy in their research intervention. Their findings are interesting in that they support, for the most part, the pedagogy advocated by the proponents of constructivism, as related by Human (2009b).

Davis (2010) avers that the focus on "the [mathematical] objects and operations brought into play by the regulative criteria of pedagogic discourse in mathematics classrooms enables [the researchers] to force an explicit recognition of that which remains indiscernible at the level of written school mathematics" (2010, p. 385). This statement encapsulates the critical features of the research methodology that is used by the SPOS group (Davis & Johnson, 2007). The application of theoretical tools elaborated from the work of Bernstein and others, to mathematics education, in order to investigate how mathematics is constituted in the classroom provides deeper insights than focussing only on features such as curriculum coverage.

2.4 Summary: Epistemology, curriculum and education in context

Against the description of predominant epistemologies of mathematics education (Section 2.2, **Question 2.1**), and structured according to the framework, the curriculum vision as expressed in the curriculum documents, and in the writings of predominantly South African writers, has been explored, noting the contributions to mathematics education. From this exploration, (and in answer to the **Question 2.2**), it would appear that a mix of epistemologies informs the current curriculum documents, in which references to the underlying principles guiding mathematics are at best implicit. The guidelines concerning how children learn, the role of the teacher, how language, representation and symbols interrelate in the learning of mathematics, and how we judge whether mathematical learning has taken place, are partly in place, making way for further input through pre-service and in-service training.

The predominant education influences, (described in Section 2.3, and answering **Question 2.3**), may appear in some respects to present contrasting perspectives. However, where attention is seriously given to the subject matter at hand, notably the concepts, objects and operations that constitute mathematics, there is consensus.

The background context for this thesis has been the search for a comprehensive theory of mathematics education, which provides a plausible account of acquiring new mathematical concepts, and which then relates this process to the learning and teaching of mathematics. Because both the field of mathematics and the cognitive structures required to master the field, are complex, the theory required to explain the conceptualisation process must be complex (Vergnaud, 1994, p. 43). Central to any theory informing or explaining mathematics education, is attention to the core principles and practices in interaction with the mathematical objects, concepts and theorems to be found in the knowledge domain of mathematics. The following chapter attends in more depth to the mathematical ideas underpinning this research study (Chapter 3), and which inform the theory of conceptual fields (Chapter 4).

3 Threshold concepts in the unfolding number systems

In this chapter the journey that learners take into and through unfolding number systems is explored. The threshold concepts encountered in the transitions from one number system to the next, demand radical new perspectives in relation to both the reconceptualised mathematics objects and the associated operations.

3.1 From intuitive notions into explicit knowledge

The individual's acquisition of mathematical knowledge is the central problem to which this thesis seeks to contribute in the focused context of the constellation of concepts, composed of rational number, percent and probability, and the associated construct, proportional reasoning. In essence the problem confronting mathematics education is one of conceptualisation. What is the process through which learners of mathematics develop ever-increasing proficiency and levels of abstraction?

Theoretical requirements for a framework are proposed (Chapter 1, Section 1.2.1) and used to analyse the current curriculum documents. These requirements are used to interrogate various theoretical perspectives, from a psychological perspective, a broadly defined constructivism, and from a sociological perspective, a cluster of papers building on the work of Bernstein (Chapter 2, Section 2.3.3). It is against this background that the theory of conceptual fields is presented as a plausible and comprehensive framework from which the problem of conceptualisation can be explored. While the theory embraces both a mathematical and a psychological perspective, these perspectives being inseparable in the conceptualisation process, the focus in this chapter is on core themes and concepts from a mathematical perspective.

The first critical component from the mathematics perspective is that even elementary concepts in arithmetic deal with important mathematical concepts. The problems that learners encounter in the early years are hypothesised to be conceptual in nature and not purely computational. The challenge for mathematics education in the view of Vergnaud is "(t)o establish the link between ordinary arithmetical situations and the relevant mathematics concepts" (1979, p. 263). Establishing this link performs two functions, that of developing the ability to solve arithmetical problems in the early years and that of enabling a better understanding of more advanced concepts.

The requirement to track the mathematical path leads to the second important component. Vergnaud, (1997) maintains that research into the acquisition of mathematical concepts requires both a mathematics perspective and a psychological perspective, but research from an educational perspective demands two interlinked and mathematically related imperatives. These imperatives are that mathematical concepts be "traced to students' competences and the way students progressively master situations", and that these competences be "analysed carefully with the help of well-defined mathematical concepts and theorems"[39] (Vergnaud, 1997, p. 8).

Vergnaud (1994) insists that the analysis of concepts and processes must be from a mathematical perspective as no linguistic or logical system other than mathematics can provide the "concepts sufficient to conceptualise the [mathematical] world and help us meet the situations and problems that we experience" (1994, p. 42). It is the precision of symbolic representation and well-defined concepts in mathematics that conveys both the essential aspects of the mathematical situation and the schemes used by the learner of mathematics. Natural language descriptions or levels of abstraction such as Bloom's Taxonomy (1956), and various adaptations, though essential for challenging the cognitive demand associated with educational activities, are inadequate. Subject-specific constructs are required for teaching and assessment (Andrich, 2002), and for mathematics in particular.

Thirdly, it is important to note that the conception of mathematics underpinning the theory of conceptual fields is that concepts have their roots in empirical problems which through successive abstractions have become the powerful concepts that characterise mathematics today. Vergnaud (2009) distinguishes between the *predicative form* of mathematical knowledge which is comprised of linguistic and symbolic expressions that are clearly defined and have the authority of the *collective of the creators of mathematical knowledge,*[40] and the *operational form* which is found in action on the physical and social world.

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[39] As previously, see Section 1.5, it is noted that the meaning of theorem in the French milieu is more fluid and may be understood as a sequence of coherent statements that lead to a result or inference.
[40] This phrase is attributed to Anna Sfard, 1995.

The predicative form is the established objective structure. It is this inner core of mathematical content with which the learner must engage, and then internalise, in order to gain access to the power of mathematics.

At the same time this objective structure that is depicted in most school mathematics texts, is acquired through its operational form, a process of abstracting from problems and solution-seeking in contexts that are not only strictly mathematical. Meaning is attributed in this process of abstraction through connecting the learner's immanent concepts, evolved initially in everyday experiences with mathematical constructs through manageable bites that aggregate towards concepts and theorems.

Inevitably in the learning and teaching of mathematics, issues arise that impinge on the epistemological terrain, notably on the perspectives from intuitionist and formalist conceptions of mathematics (discussed in Chapter 2, Section 2.2.2). The frustrated response from a learner "I don't understand", or "Why do we have to learn mathematics?" may have its roots in the epistemological bases of mathematics. From the teaching perspective, it is necessary to consider both the strength of intuitions in guiding or adapting learner's processing of problems (from an intuitionist view), and the critical role of logic and formal structures (from the formalist conception), for convincing a putative mathematical community (if only the class community) of the correctness of one's reasoning.

The central idea informing this chapter is that the power of mathematics lies in the process of transforming the intuitive and implicit knowledge first used in solving a class of problems into explicit and generalisable knowledge[41] that can be applied to problems of a similar mathematical structure with greater efficiency. This theme can be recognised throughout the development of mathematics.

Usiskin, et al., (2003) contrast two mutually supportive approaches that can be taken in teaching the theory of real numbers and which can be extended to most mathematical topics. One approach is to regard the real number system as complete with definitions, axioms and internal logic, and therefore engagement with mathematics becomes

[41] This notion of transforming intuitive knowledge into explicit and generalisable knowledge is attributed to both Vergnaud (1990) and to Vygotsky (1962; 1978), on whose theories Vergnaud had built.

engagement with aspects of the developed system. The second is to construct the real number system from rational numbers as was done historically. The second approach describes the links between rational and irrational numbers, decimal numbers and the geometry of the number line. The two approaches are integrated in this chapter so as to take into consideration both the "historical and conceptual evolution" of these topics and the current system of definitions, axioms and theorems (Usiskin, et al., 2003, p. 1).

3.1.1 Research questions

The purpose of this chapter is to explore in some depth the concepts that learners are required to learn along the path as they negotiate the transitions from the very early development of number sense through to natural numbers, integers and fractions, rational numbers and real numbers. This pathway is followed by many curriculum planners, textbook writers and teachers seeking a logical route towards mathematical proficiency. The purpose of retracing the path in this chapter is to remind ourselves, the mathematics education community, of some of the threshold concepts with which learners have to engage in order to develop proficiency in the multiplicative conceptual field.

In agreement with Pring (2000), the perspective taken in this thesis is that the "teaching of a particular concept in mathematics can be understood only within a broader picture of what it means to think mathematically". In addition the "significance and value [of the mathematical topic] can be understood only within the wider evaluation of the mathematics programme … the broader picture of a range of interconnected activities" (p. 27). By locating the particular focus of this research study, namely the topics fraction, ratio, probability and percent, elements of the multiplicative conceptual field, in the broader mathematical programme, the individual concepts are permitted to exhibit their greater meaning and profundity.

The following main question and sub questions guide this chapter:

Question 3. What episodes and thought processes guide the unfolding of number systems and are central to developing mathematical proficiency?

3.1 What insights from history may provide insights into the teaching and learning of elements of the multiplicative conceptual field, as demarcated in this study?

(Section 3.2)

3.2 What particular threshold concepts should be anticipated in teaching and learning at this level? (Section 3.3)

3.3 What features of mathematics are fundamental to teaching and learning mathematics? (Section 3.4)

The above three questions will be discussed under the headings "Historical and epistemological context" (Section 3.2), "Unfolding number systems" (Section 3.3) and "Factors guiding the development of mathematics" (Section 3.4).

3.2 Historical and epistemological context

Steinbring (1998), amongst others, advocates a deeper understanding of mathematics as a requirement for teaching mathematics at school level. Steinbring's model for teaching stresses the importance of understanding the changing nature of mathematical knowledge. For Steinbring,

> the teacher has to become aware of the specific epistemological status of the students' mathematical knowledge … has to be able to diagnose and analyse students' construction of mathematical knowledge and has to compare those constructions with what was intended to be learned (Steinbring, 1998, p. 234).

Steinbring describes the professional knowledge required by teachers as not so much the use of a product but a process. Thus it may be averred that a transformed understanding of the nature of mathematics as a dynamic rather than static science, may bring about a pedagogically sensitive approach to the problem of interactively constituting the meaning of mathematical objects. This view is in alignment with that of Vergnaud (1990), in which learners' immanent mathematical knowledge is transformed from an implicit and local form into progressively higher levels of abstraction. It is noted here that the concepts confronted by the mathematicians at the frontier of mathematical advances, are therefore also the thresholds for learners as they engage with these concepts,

It is the experience of many mathematics teachers that understanding some concepts proves exceptionally difficult; these concepts present a teaching challenge. It is often the case that these concepts have been pivotal in the development of mathematics historically, and consequently, the individual understanding of these problematic concepts is necessary to provide the solid foundation for proceeding to higher levels of mathematics study. The term threshold concept has been used by Meyer and Land (2005) to describe such a challenge and will be used throughout this chapter to alert the reader to specific nodes which are regarded as difficult but which, once mastered, open the door to the development of higher concepts.

3.2.1 Critical links to history

The necessity for teachers, both high school and primary school teachers, to become familiar with the epistemological underpinnings of mathematics in relation to historical incidents arises because children, even in the early grades, will express ideas that echo historical episodes.

The concept of even and odd numbers raises some questions in children's minds and has fascinating historical origins. A thought-provoking question is raised in the well-known video in which a child, Shea, insists that 6 *is* both "even" and "odd" (Ball, 1993). The associated mathematical task on even and odd numbers was designed to encourage mathematical thought in a class of Grade 3 children.

The conceptual difficulty for Shea was that the even number 6 could also be arranged into an odd grouping, for example the arrangement could be 3 groups of 2, perhaps as an array, and therefore in his mind could perhaps be classified as both odd and even. The number 6 was correctly identified by Shea as even, but his assertion that it was also odd came from well-reasoned thought processes. In the video, another child tries to convince Shea of his "error". His line of reasoning has an antecedent in history, and is not so much erroneous as at a different level of thinking.

The Pythagoreans (McLeish, 1992) made a secondary classification after the initial even-odd classification. They recognised that the even numbers like 6, 10, 12, 14, were structurally different[42] from some even numbers, for example 2, 4, 8, and 16. The Pythagoreans classified the former numbers as even-odd, because when iteratively divided by two they did not resolve to a unitary answer, for example 6 divided by two gives three and cannot be further divided by two to give an answer of one. Similarly 10 divided by two gives five and cannot be successively divided by two to give one. Four

[42] The boy Shea (Ball, 1993) notes the numbers 6, 10, and 22, for example, are different from other even numbers.

(4) and eight (8) on the other hand, when successively divided by two, give one (see also Long, 2006).

Another threshold concept is that of place value. This notion, which took centuries to develop, liberates children from their cardinal ties. Place value is thought to require specific stepping stones, and is gradually built up with successive developments from tens, then twenties, and so on (see Foundations for Learning, DBE (2008)). An example of an unexpected awareness of a number system was observed in Emily who, after her first week at school, stated resolutely "Do you know that you only have to know the numbers from 1 to 99, and then you know all the numbers that there are". This awareness that these numbers can be used in a never-ending sequence, one assumes is the result of a teacher's approach to learning and teaching numbers that extended learners' thinking beyond a hundred. The Shea and Emily of these anecdotes, serve to illustrate the varied emergence of concepts in different children as they encounter mathematical situations both in and out of school and the multiple layers of learning required for achieving mastery of mathematics even in the early years, and that these emergences may have their antecedents in history.

3.2.2 Rational and Irrational numbers

The extension of the number system from natural numbers to rational numbers provided challenges for early mathematicians, as they do for learners today. The idea that numbers that were not whole should constitute a number system was eventually accepted.

The next challenge for the collective mathematics community of earlier times and for learners today, was the concept of irrational numbers. Many mathematics teachers will vouch for the difficulty that learners have in identifying π as irrational[43], but then identifying $\frac{22}{7}$, or 3.14, the approximations of π, as rational. Many reasons for this error can be posited. One reason is misunderstanding the process of estimation, in which the "rounded" result of a decimal number is confused with the original number, rather than

[43] Both π and e are special cases of the irrationals known as transcendental numbers. These numbers are neither algebraic nor reducible to algebraic irrationals (Dantzig, 2007).

being understood as the transformation of the original number into an estimate for practical purposes. A second reason may be the difficulty of conceptualising an irrational number. This problem is most often addressed by providing a definition of a rational number as $\frac{a}{b}$, where both *a* and *b* are integers, and then asserting a rule such as: "*Just remember the definition of rational numbers. Check whether the number fits this definition. If the number does fit, it is rational: if the number does not fit the definition, it is irrational.*"

This rule appears simple until it is observed how many learners, even high achieving learners, will forget this injunction and regard $\frac{22}{7}$ as the number π, and therefore incorrectly classify $\frac{22}{7}$ as irrational. A third related possibility is that such threshold concepts are acquired through an operational-structural cycle (Sfard, 1995) that necessitates careful didactic planning.

The discovery of irrational numbers and incommensurable magnitudes, and the consequent resolution of this mathematical crisis, is described by Eves as follows:

> The discovery of the existence of irrational numbers upset (another) intuitive belief held by the early Greeks. Given any two line segments, common sense seemed to dictate that there must be some third line segment, perhaps very, very, small that can be marked off a whole number of times into each of the two given segments. Indeed, almost anyone today who has not yet learned otherwise intuitively feels the same way (Eves, 1980, p. 45).

The startling fact that numbers like $\sqrt{2}$ could not be written as a ratio of any two whole numbers and the equally startling fact that "line segments exist that have no common measure" caused great consternation in the Pythagorean community as far back as 300 BC (Eves, 1980, p. 53). The reason for the consternation was that the foundation on which the body of mathematics known at the time was built, assumed numbers to be rational. The theory of proportion, which provided the foundation for other theorems, was premised on the notion of rational numbers. The historical account of how Eudoxus resolved the crisis through formulating a definition independent of commensurability and incommensurability is a telling account of how a mathematician was able to use creativity and ingenuity to solve a problem confronting the mathematical community of a particular era (Eves, 1980, p. 54).

The resolution of the irrational number dilemma is not easily presented at school level, beyond telling learners that the *number in question* cannot be written as an integer over an integer, and therefore is *not* a rational number. Perhaps this outcome is all that can be achieved at school level, and it may not be necessary for school learners to understand the logic underpinning the development of evolving number systems. According to Eves, "the incommensurable case was relegated to an appendix, to be covered at the instructor's discretion, and sometimes it was omitted entirely, as being beyond the rigor of the course" (1980, p. 57). In some South African textbooks an attempt is made to introduce these complex ideas.[44]

The omission of this crisis of non-rationals, and its historical resolution by Eudoxus, from the high school curriculum, is in the view of this thesis an educational blunder, in that it constitutes a conceptual gap in the understanding of both rational and irrational numbers. Irrational numbers are often introduced as non-terminating, non-repeating decimals. The fact that decimal numbers were invented centuries later than the discovery of irrational numbers circumvents the very crisis, and consequent understanding of irrational numbers and incommensurability, which ought to be a precursor to the understanding of non-terminating and non-repeating decimal numbers. This approach specifically ignores the fact that incommensurable magnitudes *will result in* non-recurring decimals. The discovery of incommensurable magnitudes preceded the discovery of decimal numbers, and it may make sense to investigate the phenomenon in a similar order.

This incomplete understanding could be remedied by recourse to history, invoking geometry and visual representation (see Hockman, 2005), and by these means provide insight into the concept of an irrational number. The irrational number is perhaps a threshold concept that is very difficult to understand and demands that learners tolerate a level of dissonance while re-evaluating prior notions.

[44] The University of Chicago Schools Mathematics Programme (1999), Grade 4-6 Teachers Manual gives a detailed explanation. Some South African text books give serious attention to the topic.

The view presented here is that the concept of an irrational number is a threshold concept with which learners must engage and that presenting a naïve form[45] of irrational number may temporarily stall mathematical development in that the larger mathematical picture is ignored. It is suggested here that the recounting of an historical episode could provide some conceptual ballast to the problem. An approach which engaged learners in the problem encountered by the ancient Greeks may ensure that this threshold concept is confronted and later transformed into a true concept (that is connected to a web of interrelated concepts within the body of mathematical knowledge).

This critical mathematical node illustrates the distinction between formalists and intuitionists in a way that makes sense to the novice mathematician. The intuitionists' position, according to Dantzig (2007), is that in order for a concept to "gain admission into the realm of mathematics it is not enough for a concept to be 'well defined', it must [also] be *constructible*" (p. 236). This view provides a rationale for the difficulty of conceptualising the notion of an irrational number, that given any two line segments, there may not be a third line segment that divides evenly into each of the line segments (Eves 1980, p. 45). The formalist position of devising a new body of axioms that can account for the new "concept" compels one to think in terms of infinity, and the infinitesimal, and therefore provides some insight into the developing nature of mathematics. The mathematical system has to evolve to accommodate new findings; the concepts of infinity and the infinitesimal are objects of mathematical interest, as they are of general human interest.

3.2.3 Probability

Another critical moment pertinent to this study is the emergence of probability theory (Johnson & Mowry, 1998, p. 13). In the gambling houses in France, prior to 1654, a very popular game was played. The player bet that he could roll a die four times in a row without getting a 6. If he rolled the die four times without getting a 6, he collected his stake. If a 6 came up he lost. Of course the game was stacked in favour of the house, in other words it was more likely to get a "6" in four throws of the dice than not (since

[45] Meyer and Land (2005) aver that naïve forms of threshold concepts may in fact inhibit the acquisition of the concept.

$\left(\frac{5}{6}\right)^4 < \frac{1}{2}$). But around 1654, Antione Gombard, the Chevalier de Mere applied his mind to a more complicated problem. He had lost money by betting that he could roll one pair of sixes in twenty four rolls of a pair of dice ($\frac{1}{2} < \left(\frac{35}{36}\right)^{24}$). He challenged the French mathematician Blaise Pascal (1623 – 1662) to provide a mathematical explanation. Pascal's journey to solving this problem led to the beginnings of the formal study of probability.

This seemingly trivial challenge resulted in a new language, a new set of axioms and mathematical statements and a new set of questions. The resolution of apparent conflicts was achieved by changes in one or more components of mathematical practice (Kitcher, 1983).[46] In this case establishing a different mode of reasoning, probabilistic reasoning, which requires a prospective view on what the outcomes may obey, given an unlimited future of repeated observations, was the new salient theoretical device.

3.2.4 Insights from history

The perspective taken in this thesis is that epistemological and historical aspects of mathematics are critical for providing the teacher with a vast armoury that may be helpful in transforming learners' intuitive understandings into powerful mathematical concepts. Given this view that encountering episodes such as these historical dilemmas results in deep understanding, the question arises concerning the implications for assessment.

For the remainder of this chapter, the focus will be on the second of the approaches advocated by Usiskin et al. (2003), that is on understanding the structure of number systems, definitions, axioms and internal logic, however bringing in historical information where appropriate.

[46] Kitcher (1983) outlines *mathematical practice* (p. 163) as consisting of 5 components, a *language*, a set of accepted *statements*, a set of accepted principles for *reasonings*, a set of *questions* selected as important and a set of *meta-mathematical views* (including standards for proof and definition and claims about the scope and structure of mathematics). It is these components that may change given rigorous debate and consensus.

3.3 Unfolding number systems

The broad theme underpinning mathematics knowledge that of evolving number systems, beginning in primary school and continuing into the high school, provides the background canvas against which concepts may be learnt. Each new number system does not simply mean the inclusion of some new numbers. It requires a conceptual shift entailing new definitions, axioms and internal logic as currently agreed upon by the community of mathematicians. The explanation presented here attempts to provide an overview of mathematical knowledge from an educational perspective, against which particular problem situations can be compared and within which particular staging posts of learners' mathematical journeys can be located.

In the unfolding of "new" number systems[47], some changes take place in the transition from one system to the next in terms of the questions (that prompted the development of the new system), the language (changes in symbolic notation), the mathematical statements (definitions and axioms), and rules of operations (Kitcher, 1983).

Embedded in these changes are major transition points that may temporarily stall the learner in his or her mathematical development, but once mastered provide access to fluency with the expanded number set. The statement by Davis (2010) that the "points of failure of the [mathematical] operation with respect to classes of existing mathematical objects are often generative of new classes of objects as well as modified conceptions of extant classes of mathematical objects, even if they go by the same names" (p. 380), supports the educational consequentiality of this notion. These transition points, here identified as threshold concepts, will be noted explicitly.

3.3.1 Central notions in the unfolding of the number systems

While development is continuous from the first encounters with mathematical ideas, Usiskin (2005) avers that "the particular developments in mathematical thinking that take place in the Grades 7 to 10 phase of schooling constitute the most important developments in a person's mathematical schooling" (p. 4). Among the transitions learners are required to make at this phase, the most important conceptual shift is "from

[47] For a fuller and more extensive account see Skemp (1971), Adler (1958), Eves (1990) and others.

whole number to real number". This unfolding development of number systems from natural numbers, to integers and fractions, to rational numbers, real numbers and finally complex numbers is described by Skemp (1971) as an outstanding example of assimilation and accommodation.[48] Each new system retains elements of the previous system, but introduces new notation, new meanings to operations, and contributes additional rules.

> In each new [number] system there are sub-sets which are isomorphic[49] with earlier systems. This [isomorphism] allows us to move freely from one number system to another, and also to mix systems provided that each one is operated according to its own methods. The overall result is a conceptual system of enormous power and flexibility (Skemp, 1971, p. 226).

It is this unfolding structure of number systems, where aspects of the system, for example the fraction notation used for both fraction and ratio are deceptively alike and yet subtly different, that is the Rubicon for most learners of mathematics as they make the transition from whole number to rational numbers (see also Long, 2009). For learners of mathematics the immediate challenge is assimilation of these diverse aspects into existing schemas, and the accommodation of existing schemas to incorporate new concepts.

Historically, each conceptual evolution necessitated by the evolution of a number system constituted a difficult cognitive accommodation, and therefore significant conceptual shift in the mathematical community of its time (Skemp, 1971). It can be assumed that a similar cognitive shift has to come about in learners before deep understanding is possible. Sfard (1995) proposes that "it is because of the inherent properties of knowledge itself, because of the nature of the relationship between its different levels, that similar recurrent phenomena can be traced throughout its historical development and its individual reconstruction" (p. 15).

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[48] The concepts of *schema, assimilation* and *accommodation* will be described in detail in Chapter 4. Knowledge is *assimilated* into a *schema*. The *schema* however has to *accommodate* to the new information integrating it coherently with prior belief or information.

[49] Two number systems are isomorphic if 1) there is a mapping of one into (a subset of) the other that puts them into one-to-one correspondence, and 2) under this mapping, sums and products are preserved (Tapson, 1999). "An isomorphism means that if we do corresponding operations on corresponding elements of the two sets concerned, then the results also correspond (Skemp, 1971, p. 199)

Similar observations regarding the structure of knowledge underpin the notion of a *threshold concept* which presents particular difficulty in teaching and learning but which once acquired, results in a shift in perspective (Meyer & Land, 2005). Concepts that are thought to provide particular challenges are noted in this chapter. In particular the specific differences that distinguish similar concepts in different number systems but that are masked by the use of the same symbols necessitate analysis on the part of the researcher and the teacher, and conceptual work on the part of the student.

3.3.2 From number sense to a number system

It is commonly thought that mathematics begins with counting. The mathematical concepts required prior to the mastery of counting however, are the invariance of wholes, conservation of quantity, one-to-one correspondence, and cardinal and ordinal value (Piaget, 1952). These concepts, seemingly simple, support the understanding of higher order mathematical concepts and can be described in mathematical terms. In psychological terms these concepts may be termed pre-operational concepts, and the stage at which learners grapple with these concepts is arguably prior to understanding and performing operations.

The mastery of both *grouping into sets* and *one-to-one correspondence* (also termed the transitive property) precedes counting and both cardinal and ordinal value. Cardinal value precedes addition and subtraction. The perspective advanced in this thesis, following Vergnaud, is that relating the concepts underpinning these early developments to advanced mathematical concepts provides insight into how higher-order concepts might be acquired.

Conservation of quantity

Piaget (1952)[50] asserts that "conservation of something is postulated as a necessary condition for any mathematical understanding" (p. 3). This statement is elaborated in the following quote;

[50] Piaget's aim in The Child's Conception of Number (1952) is to investigate how the sensory motor schemata of assimilating intelligence are organised in operational systems on the plane of thought. While the focus is not on how the child learns mathematical concepts, Piaget necessarily unfolds important mathematical ideas.

> In a word whether it be a matter of continuous or discontinuous qualities, of quantitative relations perceived in the sensible universe, or of sets of numbers conceived by thought, whether it be a matter of the child's earliest contact with number or of the most refined axiomatizations of any intuitive system, in each and every case the conservation of something is postulated as a necessary condition for any mathematical understanding (p. 3).

The concept of invariance, observed in the conservation of quantity, includes both a qualitative dimension and a quantitative dimension. In order to make a qualitative judgement about two quantities, two relations are possible, "symmetrical relations expressing resemblances" and "asymmetrical relations expressing differences" (Piaget, 1952, p. 10). Because comparative judgement about qualities, such as 'more than', or 'less than' necessarily include asymmetrical relations of difference, qualitative judgement becomes the germ of quantitative relations.

At the first stage of conservation, the child's judgement takes the form of a *gross unidimensional quantity*; the child considers only one attribute, for example height. At this stage the judgement is made on the grounds of general perception, the child notes that the water is higher in one of two different glasses. An intermediary second step is the *separate comparison over each of two dimensions*, both height and width, termed by Piaget, the "logical multiplication of relationships". The *attainment of invariance*, the third stage, is exhibited when the child is able to consider both height and width, taking into account the differences in both, and making some comparative quantitative judgement. In Piaget's terms, invariance requires the "co-ordination of the relations and the mathematical composition of parts and proportions" (Piaget, 1952, p. 18).

Piaget uses the term 'intensive quantity' to describe a magnitude that is not susceptible to addition, with the implication that it does not comprise parts (1952, p. 244). For the transition to occur from intensive to extensive quantity, that is conceptualisation of a magnitude susceptible to addition, an intervention is required. This intervention is suggested by Piaget to be the notion of the 'unit' that leads to extensive quantification in the form of arithmetical partition or proportions. Piaget (1952) explains further:

> (P)roportion comes into being through the combination of equality and asymmetrical relationship. There is arithmetical partition [and hence proportion] as soon as elements of a whole can be equated with one another and yet remain distinct (p. 23).

The conservation of *discontinuous quantities*, such as beads, has parallels with continuous quantities. At the first stage even though the beads are discrete quantities the child perceives their quantity to expand and contract with the size of the container. This perception does not change with the learner placing the beads one by one into one container, even when the researcher matches this sequence by placing one by one into another container at the same time as the learner. In the intermediate stage, the child tends to think that the discrete quantities have been preserved if the containers are the same size and shape, or if the quantities are built up one by one. However the belief that quantity has been conserved when the shape of the container is changed does not yet hold.

The move to the third stage begins with the co-ordination of relations, “the height is more but the width is less” (Piaget, 1952, p. 35). When the child is able to “co-ordinate height and cross-section in a ‘multiplication of relations’ that lead to intensive quantification”, then he is capable of “equating the differences and therefore achieving extensive quantification” (p. 35). It is in this third stage that one-to-one correspondence implies lasting equivalence. The child is able to hold the idea of one-to-one correspondence even if the objects are in differently shaped containers.

Intuitively, the judgement is made that the increased height and decreased width, because they are perceived as made up of parts, is equivalent.

It is at this stage that conservation of quantity is attained; learners are able to compare quantitatively and are able to hold at least two ideas in their thought, that of height *and* width. The earliest conceptions of proportional reasoning, linked to the property of invariance, are evident in Piaget’s work with young children, and constitute a critical threshold concept.

One-to-one correspondence

The development of *one-to-one correspondence*, that is assigning to every object of one collection a unique object of the other collection, is achieved after the property of invariance. The first step towards one-to-one correspondence is the comparison of sets, guided by perception. This strategy becomes inadequate when the sets change in density, length or width. Essentially an explicit ordering of each set of discrete objects, and some form of parallel arrangement, is necessary for a one-to-one correspondence.

The second stage is characterised by intuitive qualitative correspondence, guided by some level of reasoning, but freed from linear orderings. It is only at the third stage that the qualitative correspondence is transformed to a numerical character and therefore results in one-to-one correspondence.

When qualitative correspondence is transformed to a numerical character, one-to-one correspondence becomes a "transitive property … which enables the judgement as to the inclusion in, or exclusion from a class of matching sets" (Skemp, 1971, p. 148). This concept is acquired through engaging with practical counting activities. Again, even in an activity one might regard as simple, the precursors to higher mathematical thinking are taking place.

Cardinal and ordinal numbers, and counting

The cardinal value rests on the principle of correspondence. The cardinal value of one set can be matched against the cardinal value of another set. Without counting we are able to judge whether the cardinal values of two sets are equal. For example, each person at the table has his or her own plate; the cardinal value of plates matches the cardinal value of people. We judge there to be the same number of people as plates. The next step requires the creation of models for each collection (Dantzig, 2007). The model for two-ness could be the "wings of a bird". In fact the model for seven-ness in isiZulu, *isikhombisa*, is derived from the word *ukukhomba* which means *to point*. The right hand pointing finger is the seventh finger when counting starts with the left hand. There is a parallel in many other South African indigenous languages, where the number name for seven is related to the expression *to point*. The number word which is the model for the particular cardinal set, may once have been tied to the concrete model as in the isiZulu seven, but the numbers now are distinct from concrete referents[51] (see Zaslavsky, 1973).

An important transition for the early learner from the first steps in counting (one-to-one correspondence) is the understanding that the last count of discrete elements of a set (for example 1, 2, 3, *4*) is an ordinal number, but it also represents the size or cardinal value

[51] The phenomenon of a separate concrete referent is evident in that native isiZulu speakers are not initially aware of the association between number seven and the pointing finger. A similar observation may be made for native English speaker who do not recognise that the word twenty is derived from two tens.

of the collection (the count for the entire set). The question "To which family of sets does this set belong?" yields the answer, "The family of sets with a cardinal number 4" (Adler, 1958).

The particular complexity of our number system is that cardinal value and ordinal number use the same number names and the same number symbols. In essence the cardinal value relies on the principle of correspondence: the ordinal concept on the principle of succession. To find the cardinal value of a set, we find a model collection with which we can match the set. The essence of the ordinal concept is based on the assumption that "we can pass from any number to its successor" (Dantzig, 2007, p. 9).

The precondition for counting is that the models aligned to the sets of objects are arranged in "an ordered sequence which progresses in the sense of growing magnitude, the natural sequence: one, two, three …" (Dantzig, 2007, p. 8). As Dantzig explains:

> Once this system is created, *counting a collection* means assigning to every member a term in the natural sequence in *ordered succession* until the collection is exhausted. The term of the natural sequence assigned to the last member of the collection is called the *ordinal number* of the collection. … The *ordinal* system acquires existence when the first few number words have been committed to memory in their *ordered succession*, and a phonetic scheme has been devised to pass from any larger number to its *successor* (p. 8, emphasis in the original).

The notion of ordinal and cardinal number constitute an example of system efficiency where more advanced mathematicians will understand the nuance of meaning but novices may not. Young children grasp the differences by adapting schemes to accommodate various experiences. The case of a young child of three or four chanting the counting numbers, but then not being able to answer the question, "How many are there?" is indicative of the complexity, and the dual nature, of number. The elementary mathematics that underpin the concepts, conservation of quantity, one-to-one matching, counting, and ordinal and cardinal number, are the foundations for advanced understanding of higher order mathematics concepts.

3.3.3 Natural number systems

From the empirical roots in agriculture and commerce, counting has been transformed into an ordered system – the natural number system. Cardinal numbers are "properties of actual sets"; natural numbers are "abstract symbols whose entire meaning lies in the formal rules by which we manipulate them" (Adler, 1958, p. 35). There are two binary operations defined on the natural number system, addition and multiplication. The properties of these operations are embodied in the addition and multiplication tables. The cardinal numbers are however isomorphic with the natural numbers; they behave and can be used in the same way.

Adler (1958) presents a system of axioms for the natural number system (not including 0). A set of elements is called a natural number system if it has the following characteristics:

- It contains an element called 1.
- For every number in the natural number system, there is another member (and only one) called its successor.
- Two distinct members do not have the same successor.
- There is no member of the system that has 1 as its successor.
- If a set of elements belonging to the system contains 1, and, for each member that the set contains, the set also contains the corresponding successor, then this set contains the whole system (Adler, 1958, p. 33).

A number system is any collection of objects on which two binary operations called addition and multiplication are defined, such that addition is commutative and associative (i and ii), multiplication is commutative and associative (iii and iv) and multiplication is distributive with respect to addition (v). The five laws follow;

i. $a + b = b + a$

ii. $(a + b) + c = a + (b + c)$

iii. $a \times b = b \times a$

iv. $(a \times b) \times c = a \times (b \times c)$

v. $(b + c) \times a = b \times a + c \times a$

The numbers learnt in the first few years of schooling are restricted to the set of numbers 1; 2; 3 … . Adding or multiplying any two elements of the natural number system gives another natural number, i.e. addition and multiplication are closed with respect to the natural number system. The limitation of the natural number system in its inability to accommodate subtraction led to the construction of the integers. Division is defined in terms of multiplication and, as with subtraction, is a limited operation in the natural number system. The outcome of $8 \div 5$ is not a whole number. There is no natural number which when multiplied by 5 gives a product of 8. The need therefore arises for an extended system in which the resulting number from $8 \div 5$ is an element of the system. This problem is accommodated by the rational number system.

The knowledge that one is working within a system remains somewhat hidden in the early years as only one system is known. The introduction of another system, for example the Mayan system, with base 20, may provoke a cognitive dissonance that may result in the features of the decimal system becoming less opaque. The concept of number being an element of a number system constitutes a threshold understanding.

A further "troublesome" number is zero. The complexity and therefore efficiency of the *symbol zero* in the natural number system, is that it represents both the count of an empty set and a place holder in representation of number. While to most people, the concept of zero now presents no problem, some insight into the historical discovery of zero may throw light on some of the difficulties highlighted by children's errors with this abstract concept.

An explanation of the discovery of zero presented by Dantzig (2007) arises from the common use of the abacus, or counting board, for calculating. The number 1023, for example, could be represented on the abacus by having one bead in the thousands column, an empty space in the hundreds column, two beads in the tens column and three beads in the units column. A permanent record could not be made without a symbol for that empty column. According to Dantzig (2007) "(t)he concrete mind of the ancient Greeks could not conceive the void as a number, [and neither could it] endow the void with a number" (p. 31).

The ancient Hindu people could also not provide a symbol for nothing. They did however provide a term which meant empty or blank. The word *sunya*, meaning empty or blank, later extended its meaning to have the connotation of void, or zero, and became the turning point in mathematical development "without which the progress of modern science, industry or commerce would be inconceivable" (Dantzig, 2007, p. 35).

The symbol for zero has many functions: it is a place holder, the starting place for a natural number, the reference point which distinguishes negative from positive numbers on a number line, and a symbol of the count of an empty set. Zero is also the unique number which when added to an arbitrary number yields as the sum the same arbitrary number. It is therefore both the zero element and the additive identity. But when an arbitrary number is multiplied by zero, the resulting product is zero (Dantzig, 2007).

The number one also has "troublesome" aspects and can therefore be described as a threshold concept, where a true understanding constitutes something of a conceptual shift. According to McLeish (1992), Aristotle would not concede the status of number to *one* as it did not constitute a heap. A complicating factor was that one is *the unit* used to count. The idea that one could be both a number and the unit was inconceivable. *One* is now accepted as the *unity element* for multiplication: Multiplying any number by one leaves the number unchanged making "1" the multiplicative identity.

3.3.4 Integers

In order to accommodate subtraction both a zero and negative numbers are required. A negative number is defined in terms of addition, and one number is the negative of another number if their sum is zero.

The isomorphic properties of positive integers and the natural numbers enable a mathematician to work with both systems (Skemp, 1971). The problem arises when the move to working in the different systems is not made explicit, for example the notation indicating negative numbers appears to be the same as the subtraction operation sign. Another difficulty arises with the multiplication concept, in that multiplying two negatives yields a positive answer. This change of sign is counter intuitive until learners understand that they are working with a different system which has a different set of rules based on an axiomatic system.

The point made by Skemp (1971) is that

> … because the integers behave so similarly to the natural numbers … we can mix the two systems rather freely. But 'mix' can mean either 'intermingle' or 'confuse'; and when as beginners we mix natural numbers with integers, it is more often with the latter meaning [confuse] (p. 207).

Davis (2010), amongst others, has observed that "tricks" are used to circumvent engaging with integers in high school mathematics classes. The isomorphism of operations between integers and natural numbers means that it is possible to get correct answers without engaging with the concept of an integer, but that approach means evading an important threshold concept, that of integers, with the concomitant evasion of the insights necessary for more advanced work.

3.3.5 Rational number system

In both the natural number system and integers, division is a limited operation. The rational number system allows for division by almost any number. Note that we may not divide by zero. Simply put, in addressing $\frac{a}{0}$ there is no number that when multiplied by 0 will yield *a*. In addressin $\frac{0}{0}$, since any number multiplied by zero will yield zero, there is no inherent uniqueness of meaning. So for different reasons both $\frac{0}{0}$, and $\frac{a}{0}$ are undefined within the context of the rational number system.

The standard definition of a rational number is that it can be written as the quotient of two integers. The consequence of this definition means that numbers written as quotients $\frac{8}{5}$ or $\frac{3}{2}$ are rational numbers, but it also follows that any integer, *k*, is a rational number as it can be written as $\frac{k}{1}$. This subset within the rational number system is isomorphic to the system of integers. Usiskin et al. (2003) make the point that "the definition of a 'rational number' does not require that a rational number *must* be written as a quotient of integers, but only that it can be written that way" (p. 20). Rational numbers are "not determined by how they look, but by how they *can* look". Because of this isomorphic property the rational number, for example, $\frac{a}{1}$ (where *a* is an integer), may be thought of as the same as an integer. A rational number is, however, differently defined as the characteristic property of any proportion, and is represented by any of its ratios. A rational number may be the division of a numerator defined in terms of multiplication. In the rational number system $\frac{a}{b}$ always has a meaning if $b \neq 0$ and if $b \times x = a$, then *x* always has a solution.

The operations of multiplication and division required for dealing with rational numbers are more complex. Multiplication must be more than repeated addition and division must be more than repeated subtraction. The representation of these operations requires different models, for example area models, rate factor models and size change models (Usiskin, 2005).

The challenge to learners is to understand that the rational numbers are dense on the number line, that is between any two rational numbers a third rational number can always be found. The number line model depicts this property of density of rational numbers. By contrast there are points on the number line that cannot be represented by a rational number.

The rational number system is inadequate as we have seen for the solution of $\sqrt{2}$, $\sqrt{3}$ and in fact the root of any prime. The transcendental numbers, π, and *e*, special cases of irrational numbers can not be accommodated by the rational number system. These *irrational numbers*, though located by a point on the number line, cannot be represented as a ratio of integers. The concept of an irrational number is regarded, in the view of this thesis, as a threshold concept which requires cognitive accommodation and the accompanying ability to tolerate dissonance prior to grasping its significance.

3.3.6 Real number system

It can be shown that the real number system, which includes both rational and irrational numbers, complies with all the requirements of a number system (see Skemp, 1971, pp. 216-223). Addition of real numbers (including irrational numbers) is both commutative and associative, multiplication of real numbers is both commutative and associative, and multiplication is distributive with respect to addition. The same concerns that arose in the transition from natural numbers to rational numbers arise in the transition from rational numbers to real numbers. The *reals* and the *rationals* are in many respects isomorphic and the same operations apply, thereby making for both a powerful and an economic system of numbers.

3.3.7 Complex number system

The simplest equation with no solution within the real number system is $x^2 + 1 = 0$. The initial response to the solution $x = \sqrt{-1}$ was resisted as the square of any real number is always positive. Bombelli in 1572 found it was possible to use complex numbers in the solution of an equation and proceeded to establish rules for these entities. It was much later that the usefulness of these numbers was established (Dantzig, 2007, p. 191).

The development of this new number, the complex number, followed an analogous cycle to previous "strange objects" such as a fraction, a negative number and an irrational number. At first a new number type is treated as a process but with restricted use in computation. In the next phase the new number type is tolerated and used for particular manipulations, and finally the new found number is treated as a fully-fledged mathematical object (Sfard, 1991, p. 14).

The purpose of exploring further developments in mathematics, such as the real and complex numbers, has the objective of locating the focus of this research study in a larger context, and of anticipating future thresholds. In these cycles of extending constructs, development and systematisation occur amongst both the *collective of the creators of mathematical knowledge* and in individual advancement through thresholds of understanding. The individual's acquisition of mathematical concepts will be taken further in Chapter 5.

3.4 Factors guiding developing number systems

It could be speculated that mathematical modelling[52] is the original mathematical experience. The earliest mathematicians, in attempting to solve a practical problem, may have started thinking about the elements of that problem and the relationships between different parts of the problem. If this thought had then been separated from the problem and been written down using some set of symbols, then such a distillation would represent a simplified form of mathematical modelling. A next step in the

[52] Mathematical modelling may be defined as making sense of real life situations, identifying the variables and relationships, expressing the variables and relationships mathematically, solving the mathematical problem and verifying the answer.

mathematical process would be to apply the particular mathematical model to a different problem and verify its solution. Modelling and application constitute important sets of mathematical processes required in the modern world.

Progress in mathematics requires that through trial and error and inductive processes, the rough concepts formed in relation to particular problems, are then transformed into abstract mathematical concepts and theorems. Deductive reasoning provides the conceptual means of convincing oneself and one's community of the correctness of one's thinking. This process reaches back as far as Aristotle, and earlier.

3.4.1 Mathematical reasoning

Most of the progress discussed in this section on *unfolding number systems* has been historically achieved through logical reasoning. A question central to mathematics education arises as to whether the reasoning required in mathematics is different from the reasoning required in everyday experience. The logical steps passed down through time and captured by Aristotle are that one defines a premise without ambiguity, then through a step by step logical process incorporates further information or assumptions and arrives at a conclusion. The mathematical equivalent of this form of reasoning, deductive reasoning, begins with definitions, postulates and axioms and then through applying rules of logic, new statements are derived and their validity or correctness demonstrated. Euclidean geometry is an example of a deductive science (see Eves, 1990, for a historical perspective).

Scientific investigation, on the other hand, proceeds through inductive reasoning, which advances from the particular to the general. While this process is not considered adequate for mathematics, it is the starting point for many explorations which then lead to the further need for deductive reasoning and proof. In mathematical problem solving it is only in the final product where the axiomatic organisation of the solution is achieved and presented. In Bombelli's case, an idea emerged in his mathematical work and he ventured a hypothesis that complex numbers could be used to find a solution to a particular problem. His experimenting provided him with further understanding and he eventually arrived at a solution to his problem. His solution then had to be communicated to the community of mathematicians of the time, in a language and with a set of reasonings, with which he was able to argue for the construction of the number $\sqrt{-1}$, and therefore for a new expanded set of numbers.

In general the stages of problem solving begin with comprehending the problem, making a plan, executing the plan and then verifying the solution against the original problem (see Polya, 1945). In Vergnaud’s terms the learner engages schemes that have previously served him or her in similar situations. After finding some probable route to a solution, a systematic method may be explored, which, if successful, eventually becomes an algorithm for the particular problem. This process requires personal involvement, and an intuitive insight into the respective concepts and procedures and generally a great deal of hard work. The activity requires both formal definitions and theorems, and algorithmic and intuitive processes.

3.4.2 Algebra

Dantzig (2007), drawing an analogy between arithmetic and algebra, notes that just as the lack of a symbol for nought or zero restricted arithmetic, the lack of a general notation restricted algebra to a haphazard collection of rules that could be applied to specific equations.

The introduction of letters liberated algebra in three significant ways; firstly the symbol was liberated from the concrete object; secondly the symbolic expression allowed operations; and thirdly the symbol played a role in the generalisation of the number concept. When working with the natural number system, there are restrictions for what is possible (Dantzig, 2007, p. 90). Of the equations below[53] where x is a natural number, A, B, and C are admissible in the natural number system:

A) $x+2=8$ B) $3x=6$ C) $x^2=16$

D, E, and F are not.

D) $x+6=3$ E) $2x=7$ F) $x^2=8$

In order to accommodate D, the system of integers is required. To solve E, one requires the introduction of fractions. Together D and E can be accommodated in the rational

[53] These equations are adapted from Dantzig, (2007, p. 91).

number system. The solution to F requires a further extension of the rational number system to accommodate irrational numbers. When only literal equations are considered as in G, H, and I, symbolic representations of solutions are possible.

G) $\begin{array}{l} x+b=a \\ x=a-b \end{array}$ H) $\begin{array}{l} bx=a \\ x=a/b \end{array}$ I) $\begin{array}{l} x^n=a \\ x=\sqrt[n]{a} \end{array}$

The specific role of symbols "exemplifies the power of modelling: to extract functional relationships as a model of the situation, to operate within the model without paying attention to the external meaning of such operations, and to interpret afterwards the results of such operations" (Vergnaud, 1990, p. 20).

Further important transitions noted by Usiskin as essential within the transition years, besides the transition from natural numbers to real numbers, are the transitions "from a number to a variable", "from inductive arguments to deductive ones", "from informal description to formal definition of mathematical ideas" and "from a view of mathematics as a set of memorised facts to seeing mathematics as interrelated ideas accessible through a variety of means" (2005, p. 4). These transitions are all realised in the successful mastery of algebra.

3.5 Summary: Central factors in mathematical development

In this chapter the focus has been primarily on mathematics as seen from the perspective of an established body of knowledge. From this exploration (and in answer to **Question 3.1**), it is suggested that the epistemological and historical aspects of mathematics, may provide teachers with insight into meta-aspects of mathematics that are helpful in guiding learning and understanding the difficulties learners may encounter. According to Sfard (1991) the cycle the individual follows is broadly the process of historical development in the transition from operational understanding to structural understanding. Some of the difficulties are inherent in the symbolic system, and specifically the fact that the same symbols are used for different concepts. The teacher needs to be alerted to these threshold nodes within mathematical development.

The focus of this study is centrally on the mathematical topics fractions, ratio, rate, proportion and percent, encompassed within the multiplicative conceptual field. These topics may be perceived as framed within the number systems that have unfolded historically, precipitated by the need for more comprehensive systems. The expanding number systems characterise distinct developments in mathematics (in answer to **Question 3.2**), that include threshold concepts, notably the conservation of quantity, one-to-one correspondence, the transition from number sense to a number system, and the transition from natural numbers to rational number, which, once attained, provide the conceptual gateway to more complex concepts. It is this structure with identified nodes, against which problem situations can be compared and within which particular markers of learners' mathematical journeys can be located.

The transition from natural numbers to rational numbers is not made explicit in the curriculum, where it is simply assumed that higher level concepts are introduced to solve more difficult problems that cannot be answered within the earlier frameworks. This lack of explication hides the emerging necessity for a new set of numbers, comprising a new number system. The distinct developments of mathematical concepts and processes, objects and operations, that constitute difficult nodes, it is suggested here (in answer to **Question 3.3**), could underpin the structure of a national curriculum, and be made explicit in supporting documents for teachers.

Another development not made explicit in curriculum documents is the transition from working with numbers to reasoning with abstract symbols, which constitute the foundational ideas for algebra. It would seem that this transition has to be carefully designed in both the curriculum and in learning sequences. This area is outside the scope of this thesis, though noted as a critical research area.

Mathematical development requires the learning and teaching of concepts with the particular attention to the core ideas of mathematics but also to the level of development of the children, and the process of acquiring concepts. A framework that is at core mathematical informs the *theory of conceptual fields*. Other domains, critical to learning and teaching mathematics, are also considered in Chapter 4.

4 Theory of conceptual fields: Essential domains informing teaching and learning

4.1 Embracing the complexity in learning mathematics

A journey through history alerts one to the power of mathematical ideas. The mathematical giants of former times struggled with problems, drew on mathematics concepts and reasoning skills, and came up with the ideas which we value today. These ideas, without the vantage point of history, may to the school children appear uninteresting or to teachers simply material that has to be learned (by rote if necessary). Doll (2005), draws on the ideas of Whitehead (1929, 1967), one of the 20th century's great mathematicians, who warns against the uncritical adoption of ideas, against "'inert ideas' … that are merely received into the mind without being utilized, or tested, or thrown into fresh combinations" (Doll, 2005, p. 36). He expands on the first of Whitehead's three phases of an education cycle, that of playing with ideas (romance), of exactness (precision), and of abstracting general principles (generalisation). He explains that the reason for "throwing ideas (and facts) into fresh combinations" is to help students understand the relationships, interconnections, and patterns, and so make the knowledge their own[54] (Doll, 2005, p. 27). Without this first phase, the next two, precision and generalisation, are almost impossible.

While the focus of this study is the multiplicative conceptual field, the larger context of mathematics education frames this study. It is with the broader context in mind that one of the central themes guiding this thesis has been the search for a theoretical framework that structures the different components impacting on the mathematics education of students in a coherent manner.

Because both the field of mathematics and the cognitive engagement required to master the field are complex, the theory required to explain the learning of mathematics must be complex (Vergnaud, 1994, p. 43). A theory of mathematics education, therefore, in

[54] The challenge to provide children with ideas that are liberating and inspiring, and the injunction for children to take these ideas and throw them into fresh combinations, points to that function of schooling termed *individuation*, which is essentially about a "competence manifesting in autonomous thinking and acting" (Biesta, 2009, p. 8).

addition to acknowledging the richness of history, requires a complexity that has to embrace …

> … in one theoretical glance, the whole development of situations progressively mastered, [the development] of the concepts and theorems required to operate efficiently in those situations, [the development] of the words and symbols that can effectively represent these concepts and operations to students and then relate [these developments] specifically to the cognitive level of the child (Vergnaud, 1994, p. 44).

The theory that is being proposed is not entirely new to the South African mathematics education landscape. Elements will be recognised in the predominant theories found in the current mathematics education discourse (see Chapter 2, Sections 2.3.2 and 2.3.3). There are, however, some aspects of the theory of conceptual fields, which in the view of this thesis will add to the existing landscape.[55]

An essential notion, informing this theory, is that mathematical proficiency is developed through *encountering situations or problems* that are carefully designed for the purpose of learning. These constructed situations serve two purposes, the first is to illustrate a concept by providing a context, at the cognitive level of the child, and the second is to expand the existing conceptual structures (schemes) of the child through extending the complexity of the mathematical situation beyond the child's current level of mastery.

The relationship of a mathematical problem situation to a mathematical concept is complex: "a concept's meaning does not come from one situation only but from a variety of situations, and reciprocally, a situation cannot be analysed with one concept alone, but rather with several concepts, forming systems" (Vergnaud, 2009, p. 86). The requirement therefore is to research a conceptual field, "defined as a set of situations, the mastering of which requires mastery of several mathematical concepts of different natures" (Vergnaud, 1988, p. 141).

According to Vergnaud (1997), the epistemological and conceptual underpinnings of a class of problems which make up the multiplicative conceptual field and the cognitive processes required to solve these problems, have to be "analysed carefully with the help of well-defined mathematical concepts and theorems" (p. 8). The mathematics even

[55] The theory is presented not as a constraint on thought concerning mathematics education but as a tool to understand the complexity and as a text to be critiqued in order that elements that are weak be improved, adapted or discarded, and elements that are strong elaborated.

from early grades can be described using "explicit concepts and theorems"[56] (Vergnaud, 1997, p. 8). The cognitive processes and *yet-to-be* concepts of the learner have to be described, according to Vergnaud, as "theorems-in-action", that is implicit understandings of mathematics, and as "concepts-in-action", that is the categories inherent in a problem situation that are identified by the learner.

The importance of describing both the concepts and the cognitive processes in precise mathematical language is that a mathematical path may be explicated and embody both increasing complexity of tasks, and cognitive progression. Such a path is required by teachers and educational planners to monitor student progress and to provide students with mathematically rich and appropriate situations, in which they are able to extend their existing schemes within their current zones of proximal development (Vergnaud, 1998, p. 180).

Vergnaud (1997) insists that mathematics is a knowledge domain rather than a language (p. 7). Language is, however, critical to mathematics, in that words and descriptions that represent concepts and operations at particular cognitive levels are necessary for the students' journey to holistic understanding and, in the case of the teacher, for scaffolding concepts and developing ways of reasoning.

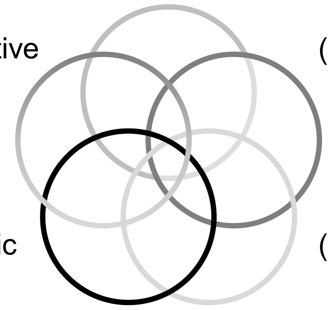
4.1.1 Components of the theory

The domains[57] are initially separately described and explored in the order shown in Figure 4.1, namely the *conceptual domain* (I), the *cognitive domain* (II), the *didactic domain* (III), the *semiotic domain* (IV) and the *evaluative domain* (V). A useful metaphor may be that of a searchlight, focused in turn on the different elements, though inevitably highlighting aspects of the intersections with other domains.

[56] This form of description may seem an unnecessarily difficult task but it will be demonstrated in the data analysis section of this thesis, that mathematics has the power to order and therefore to simplify problems, and to admit formal representation.

[57] The original idea of this articulation came from Radford, in Fauvel & van Maanen, 2000, p. 144.

Figure 4.1: Articulation of domains impacting on teaching and learning



The substance of the domains has been drawn primarily from Vergnaud, but elaborated by drawing on other theorists. These domains are sketched as follows.

I. The *conceptual domain*[58] refers to mathematical concepts and situations. This domain is understood to be embedded in the larger *historical and epistemological domain* that in some sense defines what constitutes knowing mathematics.

II. The *cognitive domain* refers to the acquisition of mathematics concepts through perception, intuition and reasoning. The concept promoted by Dewey, Piaget and Vygotsky, of an intelligent child, engaging current schemes in response to particular situations, is central to the cognitive domain.

III. The *didactic domain* encompasses didactic design and instruction, the purpose of which is to transform intuitive, implicit and localised knowledge of the novice learner into formal, explicit and generalisable knowledge.

IV. The *semiotic domain* refers to the creation of meaning through the reference context, natural language and representation. Of particular importance is the collective function of natural language, diagrammatic representation and symbolic representation in articulation with the didactic, the conceptual and the cognitive domains.

[58] The historical and epistemological elements were the concern of Chapter 3. Some elements, having direct relevance for the cognitive and didactic domains, are repeated in order to highlight connections between the conceptual and both the cognitive and didactic domains.

V. The *evaluative domain* refers to the role of evaluation in teaching and learning. At one level this role comprises formative evaluation, but at another level it refers to the formal scientific investigation of the construct of interest and the measurement of *proficiency levels* in learners and *difficulty levels* of problem situations. This domain draws on the conceptual, the cognitive, and the semiotic domains, and has application in the didactic domain.

4.1.2 Research questions

The first priority is to present a plausible explanation of how the canon of mathematical knowledge came into existence, together with its current essential features, located in the *conceptual domain*. In that light, the question arises as to how mathematical concepts are acquired by learners, a question located in the *cognitive domain*, where we look to the first indicators of mathematical processes in learners. From our understanding of how learning develops we explore the *didactic implications* that support the process. Clearly *semiotic factors*, such as the role of language, representation and symbol, are critical elements drawing on the conceptual domain, informing learning and impacting on teaching. And finally, since it would seem feasible that *evaluation* in the broadest sense of the term is central to the learning process, we consider forms of assessment that impact on both classroom and systemic practices. The primary question and the related sub questions follow:

Question 4 What are the elements of a framework required to inform mathematical teaching and learning and which are essential to an assessment process?

4.1 What are the central factors in the development of mathematical knowledge that provide insight into individual development? (Section 4.2)

4.2 How do the first natural constructions observed in children progressively develop into more complex structures? (Section 4.3)

4.3 How can the learning process be nurtured, stimulated and accelerated through teaching? What are the essential elements of the *didactic contract*? What factors may constitute disruptions to learning? (Section 4. 4)

4.4 What is the relationship of mathematical concepts to language, representation and symbol? How do language, representation and symbol interrelate to both lead and support cognitive development? (Section 4.5)

4.5 How may assessment assist teaching and learning? What unexpected effects may impact on teaching and learning? (Section 4.6)

4.2 Conceptual domain

Underpinning the theory of conceptual fields is the view that mathematical concepts have "their first roots in the action on, and in the representation of, the physical and social world,[59] even though there may be a great distance today between that pragmatic and empirical source, and the sophisticated concepts of contemporary mathematics" (Vergnaud, 1998, p. 167). The power of mathematics however, lies in the process of transforming the intuitive and implicit knowledge first used in solving a class of problems, into explicit and generalisable knowledge that can be applied to more than one situation (Vergnaud, 1990). Evident in this statement is that both the strength of intuitions in guiding or adapting students' processing of problems, and the critical role of logic and formal structures are essential.

Vergnaud (1997) highlights the paradox that arises from the fact that mathematicians strive to be

> … precise, complete and parsimonious when they write definitions, whereas psychologists try to understand how concepts are progressively shaped, by different kinds of situations and competences and by different kinds of linguistic representations and symbols (p. 5).

The resolution of this contrast in perspectives is one of the challenges addressed by the theory of conceptual fields. From the mathematics perspective there is a distinction between the *predicative form* of knowledge, the formal product determined by the historical and contemporary collective of mathematical knowledge creators, and the *operational form*, the creative process, using skills and processes drawing from predicative knowledge, but not confined by this form.

In this section we address the following question.

Question 4.1 What are the central factors in the development of mathematical knowledge that provide insight into individual development?

In order for individuals to engage with this body of mathematics knowledge, at least two factors need to be considered. Firstly, the nature of mathematical knowledge and

[59] The historical development of mathematical concepts is important, especially with regard to the problem situations in which they arose.

the features of mathematical concepts must be explored.[60] Secondly the body of knowledge has to be ordered in a partially hierarchical form. In this study the first of these factors are drawn from Vergnaud's definition of a mathematical concept as a "triple of sets". For the second of these factors the notion of nested conceptual fields is used.

4.2.1 Mathematical concept as a "triple of sets"

Vergnaud (1997) asserts that a mathematical definition, though expressed succinctly in the predicative form, is acquired within "a system of concepts and true propositions without which this derivation [of a definition] would be impossible" (p. 5). He maintains that concept development (and concept analysis) requires a "triple of sets", namely *situations*, *invariants* and *representations*, each of which require distinctive analysis, in order to characterise particular concepts. The requirement for the acquisition of concepts [in the conceptual field theory] is a set or "bulk" of *situations* which provide the context for the learner to become familiar with the concept, but which also require particular mathematical structures to solve the inherent problem.

Invariants are the underlying mathematical and cognitive components. From the mathematical perspective, invariants include a set of concepts and theorems that make it possible to report and analyse the "bulk" of mathematical concepts requiring multiplicative structures. From the psychological perspective, invariants are mathematical categories, objects, properties, relationships and schemes, relatively stable but transformable notions that learners use when engaging with a problem. In the cognitive domain these notions find form in concepts-in-action and theorems-in-action.

Symbolic representation refers to the part played by language, tables, diagrams, and the language of algebraic formulae, in the recognition and selection of relevant mathematical objects, of their properties and relationships, and in the progressive elaboration of differentiated explicit concepts. Various representations illustrate different ways of making explicit the same hidden mathematical structure at different

[60] The nature of mathematics has challenged the great thinkers; an attempt to formulate a working description which is a requirement for teaching and learning is presented here.

levels of abstraction. These perspectives are dealt with in Section 5.5, devoted to language and representation, the semiotic domain.

The three categories, situations, invariants, and representations, together characterise a mathematical concept. These same categories are necessary to understand how students master more and more complex situations, more and more profoundly and reliably (Vergnaud, 1994, p. 57).

4.2.2 Conceptual fields

A conceptual field is a construct that is broad enough to accommodate the complexity of the related concepts and processes required to solve a “bulk” of problem situations, but should also function as a manageable research domain. The multiplicative conceptual field, the primary concern in this thesis, embraces the additive conceptual field and is embedded within the algebraic conceptual field. Attention is given to the intersecting fields.

The *additive conceptual field* comprises the contexts and situations for which additive structures (addition and subtraction) are required. These structures include both counts and measures, situations of comparison, with the related concepts of order and equality, and situations of combination, including concatenation, addition and subsequent comparison.

Vergnaud (1997) following Piaget, notes that there is distinct cognitive development that takes place from counting procedures to additive operations. These operations start with the concept of cardinal numbers which are then combined or separated. With appropriate scaffolding the combination of two cardinal numbers, say from two apples combined with four apples, makes six apples, can be abstracted to the addition operation that asserts 2 + 4 maps onto 6. The placing of these mappings onto a number line or an addition table transforms the concrete cardinals to the abstract system of natural numbers. Learning number bonds, as represented in an addition table, is in essence a process of abstracting from cardinal numbers, and in so doing, constructing the natural number system (see Chapter 3, Section 3.3.2, also Adler, 1958; Dantzig, 2007).

The *multiplicative conceptual field* is conceptualised as “all situations that can be analysed as simple and multiple proportion problems and for which one usually needs to multiply or divide. Several kinds of concepts are tied to those situations in addition to the thinking required to master them" (Vergnaud, 1988, p. 141). Concepts include *inter alia*, multiplication and division, fraction, ratio, rate, rational number, linear functions, vector spaces and dimensional analysis. As with the additive conceptual field, the development of proficiency in the multiplicative conceptual field begins in the early grades and continues through high school and further.

The *algebraic conceptual field* includes variables, formulae, equations and functions. The introduction of algebra, as we have shown with the evolving number systems, modifies the cognitive status of arithmetic (Vergnaud, 1997, see also Chapter 3, Section 3.4.2). The difference between arithmetic and algebra in problem solving is that "algebra uses a formal detour, where arithmetic uses a sequence of intuitive choices" (p. 26).

The power of symbolism to liberate mathematics from the restrictions placed on it by number has profound implications for mathematics knowledge. For the algebraic novice the polyvalence of signs presents great difficulty, for example a subtraction sign has at least four meanings.[61] However, the power of algebra is in the fact that it provides the tools for solving problems that are otherwise unsolvable or only solvable with difficulty (Vergnaud 1997, p. 26).

Here the notion of a threshold concept, identified in the critical transitions along the path of mathematical development, is used. A threshold concept is described by Meyer and Land (2005) as a 'conceptual gateway' or 'portal' that once acquired leads to "previously inaccessible … way(s) of thinking about something" (p. 373). We suggest that core elements of the multiplicative conceptual field, may be considered threshold concepts that serve as portals to the algebraic conceptual field.

4.2.3 Some factors in development of mathematics knowledge

Some key factors in the development of mathematical knowledge that provide insight into individual development, (in answer to **Question 4.1**) are that a mathematical concept is not necessarily easy to define or to acquire, but that development of a concept

[61] "Take away", "find the difference", "how many more", "how many less" constitute some uses of the sign. The subtraction sign also indicates a value less than zero.

requires investigating a triplet of sets, as discussed in Section 4.2.1. This conceptual complexity, together with the fact that a problem situation invariably involves more than one concept, and that a concept can be applied to many different contexts, makes it necessary to delineate a conceptual field for both teaching and learning, and research. As has been noted in Section 4.2.2, elements of the multiplicative conceptual field, that serve as stepping stones to the algebraic conceptual field, may be considered “threshold concepts”.

4.3 Cognitive domain

Cognitive processes identified in the learning of mathematics cannot be separated from the conceptual aspects; they are at least intersecting arenas, but may also be pictured as two sides of the same coin, or as a co-construction in an emergent understanding of mathematics. We look to the first indicators of mathematical processes in learners’ development and seek to explain how progressive learning results in high level cognitive structures. The question of interest is:

Question 4.2 How do the first natural constructions observed in children progressively develop into more complex structures?

In order to answer this question, we draw firstly on Piagetian theory for a model of cognitive development, and its accelerated form *learning*. For insights into the development of higher order mathematics concepts, the work of Sfard (1991, 1995), from both a mathematical and a cognitive perspective, and drawing on historical and epistemological features of mathematics, is invoked. The work of Sfard is extended, drawing on the work of Meyer and Land (2005), in particular the notion of threshold concepts. These potentially powerful ideas are used to elaborate aspects of the theory of conceptual fields, notably a model of the conceptualisation process, when engaging with mathematical situations (Vergnaud, 1998, p. 181), and presented in this section.

4.3.1 The subject and the external world

Central to Piaget’s theory is that “in order to know objects, the subject must act upon them, and therefore transform them; he must displace, connect, combine, take apart, and reassemble them” (Piaget, 1970, p. 704). Cognitive or epistemological relations are established through a set of structures that are “progressively constructed by continuous interaction between the subject and the external world" (Piaget, 1970, p. 703). For Piaget, a scheme[62] is a starting point for the child, a "universal that is efficient for a whole range of situations that can generate different sequences of action, information gathering or control, depending on the specific characteristics of each particular situation" (Vergnaud, 1998, p. 172). A scheme includes goals and expectations, rules to generate actions, operational invariants and possibilities for inference. The notion of scheme is useful for hypothesising a mechanism for engaging with problem situations.

Development and learning

Piaget distinguishes between development and learning; learning being the aspect of cognitive development that is facilitated and accelerated by experience (1970). He avers that the "fundamental psychogenetic connections" that are generated in the process of development cannot be reduced to empirical associations (1970, p. 706). The process to be understood is the creative process by which the learner manages to construct and invent, and not how learners repeat or copy.

Piaget lists the classical factors of development as comprising *maturation*, *experience of the physical environment*, *action of the social environment*, and *self-regulation*. When these factors are facilitated or accelerated towards a specific goal, the process of learning is engaged.

Maturation refers primarily to biological and neurological features that are outside the scope of this thesis. *Experience of the external physical environment* has the components *exercise*, *physical experience* and *logico-mathematical experience*. For analytic purposes these components, though intersecting in the learning of mathematics, are discussed separately;

- Exercise is defined as an action that can be consolidated by active repetition. To the extent that exercise involves only assimilation into existing structures, the resultant learning can be regarded as the consolidation of existing cognitive structures. To the extent that the exercise requires accommodation of existing structures, the exercise may result in new learning about the external environment.

[62] The concept of scheme was introduced by Kant (1724 – 1804), and used by several psychologists in the early 20th Century, but it was Piaget who was the first to provide "concrete and convincing examples of its significance with his descriptions of early development" (Vergnaud, 2009, p. 84).

- Physical experience involves extracting information from objects themselves through a process of empirical abstraction.
- Logico-mathematical experience plays an essential role at all levels of cognitive development, as it is here that information is derived from the subject's own actions, and therefore implies "auto-regulation" of these actions (Piaget, 1970, p. 725). Logico-mathematical experience also plays an important role where a situation demands more than logical deduction and computation, and it plays a crucial part where the learner is required to develop new deductive tools.

The third component in development is the *action of the social environment*. Piaget makes two claims in respect of the social environment. Firstly, he claims that stages of development are "accelerated or retarded in their average chronological ages according to the child's cultural and educational environment" and secondly, the influence of particular learning experiences structured by the educational and social environment will only have some effect if the learner is capable of assimilating them, through already acquired and adequate structures. In fact "what is taught … is effectively assimilated only when it gives rise to active reconstruction or even reinvention by the child" (1970, p. 721).

The first three factors, *maturation*, *experience of the physical environment* and *the impact of social and educational factors*, cannot, on their own, account for "how the fundamental structures of intelligence (can) appear and evolve with all those [structures] that later derive from them" (Piaget. 1970, p. 724). Piaget offers *equilibration* as the self-regulating process that brings into action a "set of active reactions to external disturbances" (p. 725).

The learning process, perturbation and equilibration

The function of *perturbation* arising from awareness of a current limited understanding when confronted by a new problem in context, and *equilibration*, a revised understanding after resolution of the problem, are deemed to be essential in the learning process (Craig, 2007). When encountering a new concept or procedure, the learner attempts to assimilate the new concept into his existing cognitive structures. This encounter triggers an activity, which if it leads to an expected result, causes the person not to differentiate the particular concept (or procedure) from those concepts (or procedures) previously encountered. This situation at best may result in the consolidation of previous processes in response to such activities (Craig, 2007).

If at some point the individual realises that the concept or procedure encountered is not something previously encountered and therefore he or she initially has no mechanism from which to draw appropriate action, this void results in disequilibrium or perturbation (explained in Craig, 2007). The subject then attempts to assimilate the new item. The assimilation process may result in a false assimilation if the individual has not identified the item as different (Piaget, 1970). For example a learner may apply an existing concept of addition of whole numbers to the addition of numerators and denominators of fractions, or apply an additive understanding rather than a multiplicative understanding to a percentage ratio.

The learner may recognise, or be confronted with the idea that the new item requires a different set of responses, applicable to the new concept fraction. This realisation sets in motion the accommodation of existing structures, the potential learning of mathematics.

The process of assimilation of new concepts into existing structures, and accommodation of existing structures to incorporate new initially dissonant concepts, is central to Piaget’s theory. These two functions exist to a varying extent in every activity. When assimilation outweighs accommodation, there is the danger that only the characteristics of the object consistent with the subject’s immediate needs are taken into account. When the balance tips in favour of accommodation, the concern is the stability of the existing cognitive structures. *Equilibration* is the term applied to the resolution of assimilation and accommodation processes.

4.3.2 Operational-structural relations

Sfard (1991) draws on an historical analysis of concept development to account for the process through which learners of mathematics develop ever-increasing proficiency and levels of abstraction. Drawing on insights into the nature of mathematical knowledge, distilled from the historical development of concepts, Sfard formulates the critical question, “How does mathematical abstraction differ from other kinds of abstraction in its nature, in the way it develops, (and) in its functions and applications?” (p. 2).

Sfard maintains that an essential component of mathematical ability is “being capable of somehow ‘seeing’ … invisible (mathematical) objects”. She hypothesises that it is the absence of this capacity which is the reason that “mathematics appears practically impermeable" to so many individuals who might otherwise exhibit "well-formed" thinking ability (p. 3). Taking the argument further she posits "a deep ontological gap" between the two types of mathematical conceptions, an *operational* and a *structural* conception (p. 4). An operational conception is comprised of processes, algorithms and actions that are "dynamic, sequential and detailed". A structural conception[63] is characterised by the ability to manipulate the concept as a "static structure" that is both instantaneous and integrative (p. 4). In the process of forming concepts, the operational conception necessarily precedes the structural conception as the formation of a "structural conception is a lengthy, often painfully difficult process" (p. 16). It is this structural conception that culminates in precise and complete definitions.

Following an historical account of the development of the number system, Sfard (1991), like Dantzig (2007), notes that the "development of the notion of number was a cyclic process, in which approximately the same sequence of events could be observed time and again", whenever a new type of number was in the process of being conceptualised (Sfard, 1991, p. 13). In each of these cycles, three phases could be identified: a pre-conceptual stage, an operational stage and a structural phase. For the present purposes we observe that there is a phase of becoming familiar with a concept, a stage where the focus is on the processes involved, and finally a stage when the "processes performed on already accepted abstract objects have been converted into compact wholes" (p. 14).

This mathematical process is reflected in the cognitive domain by the processes *interiorisation*, *condensation* and *reification*[64]. In the first stage, interiorisation, the learner "gets acquainted with the processes which will give rise to a new concept", the learner then becomes fluent in executing these processes, and finally the process becomes interiorised – the new concept can get compared and analysed without the process being performed (Sfard, 1991, p. 18).

The second phase, condensation, "is a period of "squeezing" lengthy sequences of operations into more manageable units" (Sfard, 1991, p. 19). Condensation enables

[63] This operational-structural notion is also supported by Douady (1997) who uses the terms personalised and institutionalised knowledge to refer to and distinguish between knowledge in use, the tool concept, and knowledge that is easily retrievable, the object concept.

[64] Whitehead's three stages of romance, precision and generalisation can be compared with Sfard's conception (see Section 4.1).

easier manipulations of processes - processes may be combined, compared and generalised. This phase continues as long as “the new entity remains tightly connected to a process”. It is only when the notion is conceived “as a fully-fledged object” that it can be said that reification, the third phase, has been achieved. Sfard notes that this reification is an ontological shift; the object is now seen in a new light. While interiorisation and condensation are gradual quantitative changes, reification requires a qualitative shift, in that “the new entity becomes detached from the process that produced it” and now derives its meaning from becoming a “member of a certain category” (Sfard, 1991, p. 19).

True to the cyclic character of mathematical development, the point at which reification of the concept is attained is the same point where an interiorisation of higher level concepts occurs, whose beginnings are now operations performed on the new object.

4.3.3 Threshold concepts

The cycle proposed by Sfard, may in some respects correspond to the way Meyer and Land (2005) envisage engagement with *threshold concepts*. A threshold concept, for example an irrational number, requires initially tolerating a sense of confusion in preparation for entering into a new conceptual space.[65] This engagement with, and mastery of threshold concepts, has at least three features, which may be

- transformative – “occasioning a significant shift in the perception of a subject”,
- irreversible – “unlikely to be forgotten”, and “unlearned only through considerable effort” and
- integrative – “exposing the previously hidden interrelatedness of something” (Meyer & Land, 2005, p. 373)

4.3.4 From schemes and situations to generalisable concepts

A core notion underlying the theory of conceptual fields, is that “(s)tudents’ conceptions and competencies develop over long periods of time, through experience with a large number of situations, both in and out of school” (Vergnaud, 1988, p. 141). Concepts are formed by children and adults by dealing with different classes of situations.

[65] This new conceptual space may be liked to Sfard’s notion of reification.

The concepts scheme, concept-in-action and theorem-in-action, form part of the category invariants. Vergnaud uses the term, scheme, to refer to a relationship that is not changed by a designated mathematical operation or different context. The mathematical term invariant may form part of the notion scheme, though in this Vergnaud sense, the term invariant refers to the continuous and evolving features, a concept-in-action, or a theorem-in-action, that comprise cognition.

Schemes, concepts-in-action and theorems-in-action

Vergnaud, building on Piaget's notion of scheme as a starting point for the child, describes a scheme as a "universal" that may be applied to a range of situations (Vergnaud, 1998). The "concept of scheme is essential to any theory of cognition because it articulates into [an entity] both its behavioural and representational features: rules of action and operational invariants" (Vergnaud, 1997, p. 27).

Vergnaud defines an algorithm as a special type of scheme.[66]

> An algorithm is an effective rule or an effective set of rules to solve a [particular] class of problems. This set of rules makes it possible to find a solution to any problem of the class in a finite number of steps, if such a solution exists, or to show that there is no solution (Vergnaud, 1998, p. 171).

In the theory of conceptual fields, algorithms[67] may be some of the formal mathematical outcomes of what began as a scheme.

Concepts-in-action and theorems-in-action are important components of schemes (Vergnaud, 1998). Concepts-in-action involve the identification of different categories that enable the subject to cut the real world into distinct elements and aspects, and to pick up the most adequate selection of information according to the situation and scheme involved. A concept-in-action is the personalised equivalent of a mathematical concept, and is required to identify or select information, to characterise representation and to inform action. Concepts-in-action in this research study comprise counts,

[66] An algorithm is a scheme, but not all schemes are algorithms (Vergnaud, 1998).

[67] The use of algorithms does not imply understanding. The efficient and creative use of algorithms, as opposed to a routine and unthinking use of algorithms, is discussed in Hockman (2005).

durations, distances, areas, volumes, consumption, proportions, scalar ratios and functions.[68]

From a mathematical perspective, a theorem is a proposition held to be true on the basis of logical reasoning beginning with some axioms whose validity is simply assumed. From a cognitive perspective a theorem-in-action is the set of "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (Vergnaud, 1988, p. 144). Theorems-in-action are the first intuitive base invoked when students are confronted with a problem. In this way theorems-in-action exhibit the link between the learners' thought processes and mathematical knowledge.

Identifying theorems-in-action provides a way to make a better diagnosis of what students know and do not know, in order to make suitable situations available which will then "compel" them to consolidate their knowledge. The critical feature of theorems-in-action is that they provide a way to analyse students' intuitive strategies and help students transform intuitive knowledge into explicit knowledge. They therefore constitute a tool for the long term development of students in a particular field providing that they can be discerned sufficiently by the teacher to become a focus for setting new problems.

From knowledge-in-action to mathematical object

Encountering a mathematical problem situation triggers the invocation of an existing scheme for the learner. This dialectic relationship between situations and schemes stimulates, propels and consolidates learning. The context of the problem situation can assist in the learning process in two ways; firstly, a familiar context triggers an existing scheme that may or may not be appropriate, and secondly, a familiar context, together with, for example, an extended number range, or change in number system such as introducing decimals, may provide the stimulus for the development of a more complex inclusive scheme. Through engaging with "contrasting situations" the features of a concept or process may be identified and clarified (Vergnaud, 2009, p. 86).

[68] These concepts are discussed in the thesis, though not all elements are included in the test items.

When confronted with a new situation, perhaps a new number range, different types of number, or more complex mathematical structure, the learners "use previous knowledge shaped by experience with simpler situations to adapt to a new situation" (Vergnaud, 1988, p. 141). In the first engagement with a situation, the knowledge is implicit and used in action. The learner may choose adequate operations but may not be able to express adequate reasoning for the choice. This initial process, identified by Vergnaud, is indicative of alpha behaviour, described by Piaget (see Piaget, 1985, cited in Craig, 2007), in which the learner remains with the strategies he knows; for example, using additive constructs when the student has not yet accommodated multiplicative constructs. Students may enter a situation in which they have no scheme available. In such a circumstance students will "call on schemes in the neighbourhood" (Vergnaud, 1998, p. 173), to try to decompose and recombine them, in order to form new schemes with or without the help of teachers or other students. This process constitutes beta behaviour.

In order for this initial knowledge to develop from the "intuitive and implicit" stage, the knowledge has to be made explicit and expressed in symbolic form (natural language, schemas, diagrams and formal mathematical notation and sentences) (Vergnaud,1994). When a class of problems is solved, that is, the learner has developed an efficient scheme to deal with nearly all problems of the class, then cognitive development occurs: the learner has advanced to what Piaget terms gamma behaviour (Piaget, 1985). The original problematic character of this class of problems falls away, the originally novel items are no longer disturbing; the problematic nature is replaced by mastery of the class of problems (Vergnaud, 1994). The newly acquired scheme enables the student to tackle new situations and objects, and to understand new properties and relationships (Vergnaud, 1994, p. 42). This gamma behaviour involves revised understandings of former concepts, properties and relationships, and is characterised by rich and extendable mental constructs (Craig, 2007).

4.3.5 An integration of key ideas

To conclude this section, theorising how the first natural constructions observed in children progressively develop into more complex structures (answering **Question 4.2**) it appears that a stage of perturbation, dissonance or unrest is inevitable, which then requires focused attention to the problem at hand. The resolution of perturbation, in Piaget's language, requires the processes of assimilation, accommodation and the resolution in equilibration. When assimilation outweighs accommodation, the situation may present as learners applying, for example, additive reasoning to a problem that requires a new multiplicative structure. When accommodation is weighted too heavily, without the conservative stabilising influence of assimilation, the learner may become confused and anxious.

Vergnaud calls on the notion of scheme (from a cognitive perspective) which when engaged with particular (mathematical) situations, results in the learner calling on concepts-in-action and theorems-in-action. The identification of these cognitive and potentially true mathematical processes is what in the view of Vergnaud is needed to transform intuitive understandings to generalisable and explicit understandings.

Vergnaud (1998) claims that "(n)either Piaget nor Vygotsky realised how much cognitive development depends on situations and on the specific conceptualisations that are required to deal with them" (p. 181). This limitation is the main reason why Vergnaud "on the basis of both Piaget and Vygotsky's legacies" developed the theory of conceptual fields (p. 181). This theory, according to Vergnaud, and in the view of this thesis, provides "a more fruitful approach to cognitive development" because it provides a framework that refers "to the contents of knowledge themselves" (Vergnaud, 1994, p. 41) and therefore attempts to provide more "concrete guidelines for teaching" (Vergnaud, 1998, p. 181).

Sfard (1991 notes that learning mathematics generally requires an operational phase prior to a structural conception. True to the cyclic character of mathematical development, the point at which reification of the concept is attained is the same point where an interiorisation of higher level concepts must later occur, but whose beginnings are operations performed on the new object.

4.4 Didactic domain

From the base of our understanding of how learning develops we explore some *didactic implications* that may offer insight into the teaching and learning of mathematics. We focus on the question below.

Question 4.3. How can the learning process be nurtured, stimulated and accelerated? What are the essential elements of the 'didactic contract'? What elements constitute disruptions to this development?

4.4.1 Nurturing the learning process

While acknowledging that "no one, in place of the student, can grasp the meaning of a problem (and eventually its solution), make sense of a mathematical sentence or develop a new mathematical scheme to be part of the student's repertoire", the role of the teacher is essential (Vergnaud, 1994, p. 44). In addition to having advanced mathematical knowledge, it is necessary for the teacher to make the links between "the knowledge that underlies ordinary competences and the knowledge that is involved in science" (Vergnaud, 1997, p. 6).

The teaching cycle described by Vergnaud (1988) begins with the provision of fruitful and staged situations, the elements of which provide opportunities for engagement by learners. Teachers observe the *schemes* learners use, comprising *concepts-in-action* and *theorems-in-action,* in simple situations. The theorems-in-action used in attempts to solve the problem, are the first intuitive formulations that teachers can use to extend and formalise students' concepts.

By describing and analysing cognitive processes, expressed in *theorems-in-action,* the teacher is able, as a starting point, to take account of the intuitive knowledge, describe this intuitive knowledge and analyse the relationship between the intuitive theorem-in-action and the explicit mathematical theorem. This analysis allows the teacher or researcher to "diagnose what students know and do not know so that teachers may offer students situations that will enable them to consolidate their knowledge, increase it, recognise its limits, and eventually overtake it" (Vergnaud, 1988, p. 149). By objectifying the theorems-in-action, teachers can help students to extend the use of these relationships to more complex situations (Vergnaud, 1988, p. 148).

The important theoretical insight raised by Vergnaud (1998) is that the link between knowledge-in-action and knowledge-in-text can only be established through operational invariants: schemes, concepts-in-action and theorems-in-action. “When operational invariants are expressed and involved in systems of concepts and symbols, their cognitive status changes …”. By making the “relevant properties of the mathematical objects and operations involved in action” explicit, it becomes possible to “analyse their connections, and to demonstrate that a certain set of results, for a certain class of situations is effective” (p. 176). The operational invariants initially experienced as schemes, through being expressed and articulated by the student, and with explicit links to objects and operations, may become algorithms.

Situations (involving mathematical concepts), verbal and symbolic mediations, and scaffoldings of all kinds can be used by teachers to help children learn the objective set of mathematical content. The appropriation process, by which students make social knowledge their own personal knowledge, is enabled with the help of teachers, parents and peers, in fact by focused teaching in the zone of proximal development (Vygotsky, 1962).

4.4.2 The didactic contract

In order for meta-learning to happen, there has to be a *learning-teaching agreement*, a didactic contract, that includes at least three aspects, the *leading discourse* (that which the teacher knows and the learner does not), agreement on the *respective roles (*it must be clear who has the knowledge to be acquired) and the *nature of the expected change* (the change is in the direction from learner discourse to teacher discourse) (Ben-Zvi & Sfard, 2007, pp. 127-128). Acknowledgement of the leading discourse results in the establishment of routines. While an agreed set of rules may be negotiated, and conventions loosened, the focus must nevertheless remain on the learner acquiring the new discourse in order to honour the contract. Agreement on roles requires that the teacher feels responsible for bringing about change in the learner’s discourse, while the learner accepts the leadership of the teacher. The “necessary course of discursive change” inevitably follows the path of “thoughtful imitation” in enacting a rule, in an attempt to understand the “inner logic” of the new discourse (Ben-Zvi & Sfard, 2007, p. 128).

Part of the function of the teacher may be to design or orchestrate a set of classroom sequences, which are linked with the specific purpose of initially establishing meaning and secondly establishing powerful knowledge. The term *didactic engineering* refers to the design of a set of learning sequences that is both a *product*, resulting from an in-depth exploration of the concept, and a *process*, resulting from the adaptation of the original product in “the dynamic conditions of the classroom” (Douady, 1997, p. 373).

Disruptions to a learning cycle may occur on two major fronts, firstly mathematical experiences that are provided for the learner might not include the appropriate mathematical concepts to be learned, and secondly the learners for some reason might not engage with the task at hand. The suitability of the situations may be too far from the learner’s current level of proficiency, either much too difficult or much too easy. The ideal difficulty level required lies within the zone of proximal development that is at the level where the learner can benefit from instruction.

4.4.3 The teacher’s role

In summary, for the teaching process to be effective[69], for learning to be nurtured, stimulated and accelerated, (in answer to **Question 4.3**), the process requires a deep knowledge of the subject, covering both “the historical and conceptual evolution” of school mathematics and the many connections between the mathematical topics (Usiskin et al., 2003, p. 3). In addition a “bulk” of situations is required, in order to give meaning to the concepts to be learnt (Vergnaud, 1994). The focus is, however, on making the link from the learner’s current conceptions to explicit mathematical concepts.

The didactic contract is constituted as an implicit understanding of the teaching-learning relationship required for the teaching of mathematics by both teachers and learners. This contract being accepted and honoured is a necessary adjunct to the teacher having requisite subject knowledge, as without engagement by the learner, the teacher’s knowledge and the careful didactic engineering that goes into designing teaching sequences may be ineffective.

[69] The ideas presented here are by no means comprehensive. The principles and practices noted in Chapter 2, Sections 2.3.2 and 2.3.3 contribute to the didactic domain.

A disruption to productive learning may be, firstly, insufficient mathematical knowledge on the part of the teacher, which results in the teacher not able to identify and plan mathematical situations appropriate to the mathematical task and the learner's current level of proficiency, but also that the teacher does not recognise the yet-to-be concepts immanent in the learner schemes. The second disruption to learning may be the lack of engagement by learners and teachers, in other words both parties reneging on the implicit didactic contract.

4.5 Semiotic domain

The role of language, representation and symbol in both developing and communicating thought has concerned philosophers for centuries. Osberg, et al. (2008) challenge the predominant representation epistemology underlying education, which they suggest is the notion that what is learnt in school is a mirror of what is "out there", and that learners have to replicate what is being presented to them. The alternative view which has the potential to make education and therefore learning a more dynamic and interactive process is that what we have are models of how we think the world might work. These models are, however, only our collective best shot; our knowing of reality is constantly developing. They therefore advocate an emergent epistemology where reality, and our understanding of it, is being co-created. Both the historical development of mathematics and the accounts of learning presented here support the notion of learning as a process of finding more complex and creative ways of interacting with reality, in this case interacting with what might be termed a mathematical universe.

The question of concern for this thesis is how the acquisition and development of mathematical concepts and processes may be supported through language, diagrammatic and symbolic representation.

The question of concern in this section is the following.

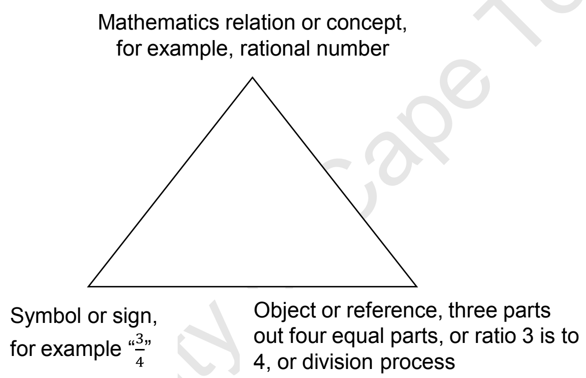
Question 4.4 What is the relationship of mathematics concepts to language, representation and symbol? How do language, representation and symbol interrelate to both lead and support cognitive development?

4.5.1 The status of knowledge

In order to have the status of knowledge, the implicit notions of the learner, the operational invariants, comprising concepts-in-action and theorems-in-action, have to be "transformed into words and symbols", and "transformed into real scientific concepts and theorems" that can be used and be presented in some form and communicated to others (Vergnaud, 1998, p. 175). Vergnaud proposes that schemes are engaged by problem situations, and through expressing concepts-in-action and theorems-in-action, and through representations, real scientific concepts may be developed.

The "epistemological triangle" (see Figure 4.2) depicts the relationship of the triad, mathematical *concept*, *referent or object* and *sign or symbol* (Steinbring, 1998, 2006).

Figure 4.2: Epistemological triangle



adapted from Steinbring, 1998, p. 174

The most important aspect of this representation triangle is that learning happens between the vertices, along the sides of the triangles, rather than at the vertices. It is in the relationship between symbol or sign, referent and mathematical concept that learning and then further abstraction of concepts develops[70]. From referent to mathematical concept, the mediating mechanism is the scheme, made up of concepts- and theorems-in-action. From mathematical concept to sign or symbol, some mediating representation may connect the two vertices.

[70] This conception of learning brings into focus the role that carefully planned contexts can play in both the engagement of learners' existing schema and in transforming intuitive and informal knowledge into explicit and generalisable knowledge.

Vergnaud advocates the use of diagrams in teaching topics such as simple, and multiple, proportion. Various representations, natural language, tables, diagrams, and the language of algebraic formula illustrate different ways of making explicit the same hidden mathematical ideas, thereby providing an external representation. Diagrams exhibiting the mathematical structure of problems can be invoked to mediate from natural language to algebraic symbolism at different levels of abstraction (Vergnaud, 1979).

The strategic use of diagrams enables the learner to extract the essential elements of the problem, provided that there is a match between the thinking of the learner and the use of symbolic representation. A symbolic representation may be currently too abstract for a particular learner and therefore not immediately assist that learner in solving a problem. For successful learning, the use of more differentiated systems as intermediary steps towards more abstract systems is required.

4.5.2 Developmental stages towards greater abstraction

Steinbring (2006) identifies developmental stages of relationship between the elements of the triad; mathematical concept, representation and symbol (see Figure 4.2). The initial stage is described as *empirical*, where there is a direct reference from symbol to concept and reference. This stage, while a necessary starting point, is not conducive to mathematical thinking as this one-to-one identification restricts the concept's abstract power (Steinbring, 2006, p. 137). This problem may be observed when a first understanding of percent as a part-whole fraction is presented and becomes fixed as the concept of percent, thereby making the ratio conception difficult to accommodate.

The second stage that of a *structural* relationship, is more conducive to moving between representations. This stage may be mediated through relational diagrams, such as multiplication charts (see Gierden, 2009), or decimal charts (Steinbring, 2006), or diagrams showing referent relationships in percent problems (Parker & Leinhardt, 1995). The structural relationship is more conducive to moving between representations. A third phase occurs when conceptual relations are generated and organised into a *system of theoretical relationships*.

Semiotic representations of various forms are an essential requirement for "the development of mathematical thought" (Duval, 2006, p. 106). It is this "changing [of] the representation register" that becomes the threshold of mathematical understanding for learners "at each stage of the curriculum" (Duval, 2006, p. 128).

4.5.3 Language, an elaborated social system

Piaget and Inhelder (1969) compare cognitive processes that develop through experience, with the learning of language which "has already been elaborated socially and contains a notation for an entire system of cognitive instruments (relationships, classifications, etc.) for use in the service of thought" (Piaget & Inhelder, 1969; 2000, p. 87). With the development of mathematical concepts the learner finds him or herself confronted with an existing knowledge structure that has an elaborate system of concepts, procedures and representations that has to be acquired.

In similar vein, both Vygotsky (1962) and Radford (1998) claim "communication and interaction" as the two most important dimensions of knowledge formation, in contrast to "individual-centred cognitive accounts". Radford elaborates on the role of activity in the socio-cultural approach, which is "underpinned by the idea that the very essence of the individual resides in her social nature". The individual does not think "only in unity and contact with nature or in immediate contact with it, but through the arsenal of conceptions that her culture makes available to her" (Radford, 1998, p. 280).

4.5.4 Mathematical concepts, language and transformation of identity

Threshold concepts may be acquired by different means: natural language, classroom discourse, or from disciplinary texts that involve symbolic language (Meyer and Land, 2005). But whatever the means of generation, the "shift in perspective" is almost invariably accompanied by an elaboration of the discourse encountered, the result of which is that "new thinking is brought into being, expressed, reflected upon and communicated" (p. 374)[71]. This process inevitably results in a shift in the learner's identity which arises as a result of the interaction of new perceptions and the development of the associated language (p. 374).

[71] This elaboration and reordering of ideas, aligns with the notion expressed by Whitehead (1927) that ideas be thrown into fresh combinations (see Doll, 2005).

4.5.5 Summary: Language precision and mathematics

It is claimed in this study (in part answer to **Question 4.4**) that the integrative nature and power of mathematics manifests its precision and descriptive efficiency in both its formal logical structure and in symbol, which in turn generate further transformation and extension. This logical structure of mathematics and its elements are only intelligible within a domain that regards language as the foundation of fluent organisation and communication. The use of diagrams, tables and symbols may support the dialectic relationship through making explicit the mathematical ideas. It is however, this precision of language that allows enriched verbal and social efficacy, but also provides the gateway into the precision of thought which constitutes mathematical reasoning.

This description is valid not only in the abstract, but applies to the development of mathematical proficiency. Language is therefore essential to mathematical development. The transformation from operational invariants, concepts-in-action and theorems-in-action through explanatory situations explicating mathematical relationships, diagrams and language are the means towards developing scientific concepts and theorems concepts, relationships and structures as abstract essences of logical processes.

4.6 Evaluative domain

In light of the view of mathematics presented, insights into the learning process and some thoughts about the essential function of teaching, how should we understand assessment and testing?

In the predominant view of education, assessment and testing may serve a formative, a certification, or an accountability purpose. In this section we focus on the formative purpose where gauging the mathematical development of learners is the focus.

Question 4.5 How may assessment assist teaching and learning? What unexpected effects of assessment may impact on teaching and learning?

At the classroom level evaluation is the teacher’s tool for assessing the acquisition of concepts, indicating a probable location of learners along a developmental path, and providing information about recent progress and potential challenges. Vergnaud (2009) insists that it is important to analyse “the continuities and discontinuities of development in mathematics" in order to envisage and construct the experiences that are likely to propel students further into engaging with the complexity of the multiplicative conceptual field (Vergnaud, 2009, p. 94) or alternatively inhibiting progress. Assessment is also necessary to "understand the filiations and jumps in student knowledge" (2009, p. 84). It is therefore necessary to analyse and classify mathematical situations and cognitive procedures, while at the same time being conscious of the mathematical path taken by the student, and the future learning that is to take place.

Prior to any assessment programme it is necessary to have a model that provides "guidance about the ways in which a pupil might progress in learning, linked to a clear conception of the curriculum" (Black, 1998, p. 26). The establishment of a model in the case of this study is informed by the analysis of elements within the multiplicative conceptual field (see Chapter 6).

Vergnaud insists that researchers who investigate mathematical competencies "cannot be satisfied with the view that mathematical words and sentences, as they appear in textbooks or in the teachers' comments and explanations, would be a sufficient criterion to evaluate students' competencies". Their competency is to be measured by "their activity in novel situations, where they have to adapt their cognitive resource and face a problem never met before" (Vergnaud, 2009, p. 88).

The learners' knowledge may be implicit in the sense that the learner can use this knowledge in action by choosing adequate operations, but cannot necessarily express or communicate their reasoning or it may be explicit in that learners can express ideas in symbolic form, (natural language, schemes and diagrams, formal sentences etc.).

It is pertinent for this study that the theory of conceptual fields provides the tools that enable both the items and the learner responses to be analysed from a mathematical point of view. The difficulty of an item depends on at least three aspects of the problem it is intended to exhibit: the situation or context, the invariants (concepts and theorems) and the representation (natural language, diagrams and symbols). For the analysis of learner responses the following assumption is invoked: learners approach a problem situation with a scheme; the scheme, made up of essential components, concepts-in-actions and theorems-in-action can be analysed mathematically; and finally the learner may use different forms of representation, natural language, a diagram, an idea in the learners' mind or an algebraic formulation. These learner conceptions that may be gauged in terms of development along a mathematical path.

These operational invariants, concepts-in-action and theorems-in-action, provide applicable tools for the analysis of learners' reasoning about problems. Vergnaud (1990) claims that theorems-in-action constitute the best tool for describing the long-term development of students' competences in a given conceptual field. They are the best tool for tracing filiations and jumps in the understanding of a conceptual field, and therefore for analysing the relationship between implicit intuitive knowledge and the explicit mathematical theorems and symbolisms yet to be acquired. These concepts enable the researcher/teacher to characterise cognitive differences between two different competences: either between two contrasting situations or between two ways of dealing with the same situation.

4.6.1 Assessment for learning

Assessment practices may assist the learning process if, firstly, the learner responses are able to be interpreted within a mathematical model against which development is explained in some measure. The challenge presented by Vergnaud is to interpret learner responses through a mathematical lens. Assessment not attending to these notions may, at most, provide information to teachers, but at worst disrupt a teaching programme.

4.7 Summary: Consequences for educational research and measurement

The purpose of this chapter was to explore the intersecting domains that impact on teaching and learning (in part answer to the general **Question 4**). The objective was two-fold, firstly to understand in greater depth the different domains impacting on teaching and learning mathematics, and secondly, to make explicit the notions underpinning an approach to assessment, which it is hoped will support the individual's acquisition of mathematical knowledge in the multiplicative conceptual field, in particular the focused context of the related concepts within the multiplicative conceptual field.

The requirement for mathematics education researchers to establish "the link between ordinary arithmetical situations and the relevant mathematics concepts" (Vergnaud, 1979, p. 263) mirrors the project of Piaget and his collaborators in investigating the mathematical development in young children from a mathematical perspective, but also is intricately related to developing cognition, and therefore also from a psychological perspective.

The *theory of conceptual fields* accommodates mathematical complexity through collecting mathematics concepts and related problem situations into aggregated contexts for directed experience and discovery. By identifying the situations through which learners engage in order to develop their schemes, the inherent difficulties of the mathematics may be accessed, explored and mastered. In order to understand the complex relationships between mathematical concepts that are built on earlier concepts and that have structural links to several other concepts, it is necessary to conduct research within conceptual fields.

However, for evidence of acquisition we need to take account of the words of Vergnaud (2009):

> Researchers who address the development of mathematical competencies cannot be satisfied with the view that mathematical words and sentences, as they appear in the textbooks or in teachers' comments and explanations, would be a sufficient criterion to evaluate students' competencies. The test of their activity in situations is essential, particularly in novel situations, when they have to adapt their cognitive resources and face a problem never met before (p. 88).

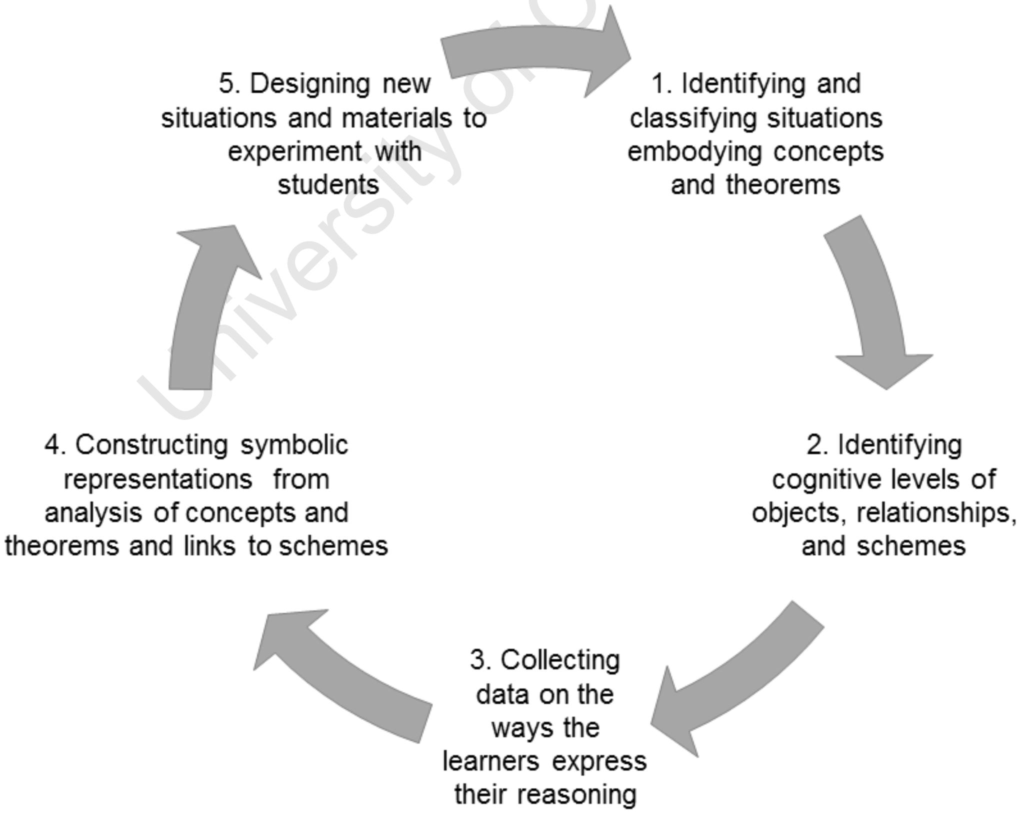
From the perspective of threshold concepts, Meyer and Land (2005) propose that "methods of observation and enquiry" that allow the exploration of "variations in students' experiences of threshold concepts" be devised, in order to achieve greater insight into "the acquisition of threshold concepts", from the perspective of teaching and learning (p. 385).

The requirement from a research perspective, according to Vergnaud (1979), is to analyse mathematical problem situations in terms of the mathematical concepts and theorems required to solve the problems. From the cognitive development perspective, we may require analysis of the developmental complexity of the learner's responses, in terms of mathematics concepts-in-action and theorems-in-action (p. 264). The purpose of this analysis is to inform and structure didactic situations at the appropriate level for the learner, so that they might develop mathematical proficiency in their current proximal zone of mathematical development.

Vergnaud (1988, p. 149) proposes a canonical approach to research in mathematics education (see Figure 4.3), which involves;

1) Identifying and classifying situations that are likely to elicit the concepts and theorems in the topic of interest, and which are at the cognitive level of the learner defining the conceptual domain.
2) Identifying levels of objects, relationships, and schemes (concepts-in-action and theorems-in-action) currently employed by learners in engaging with the problem situations, spanning the cognitive domain.
3) Collecting data on the ways the learners articulate their reasoning, identifying the links between the conceptual and cognitive domains as expressed by learners in the didactic domain.
4) Constructing symbolic representations by observing and analysing the use of concepts and theorems (from the mathematical concept perspective) and schemes that learners use (from the psychological/cognitive perspective).
5) Designing new situations and materials to experiment with students to inform the didactic perspective.

Figure 4.3: The research cycle



Source: Vergnaud, 1988, p. 149

The task of devising methods for designing items, constructing and administering assessment instruments, and analysing the data through methods appropriate for mathematics education, is the focus of Chapter 5.

5 Assessment and measurement: A discussion of core requirements

5.1 From mathematics to measurement

Underpinning assessment in any context, the classroom or large-scale programmes, is a model of mathematical development, and a model of the associated cognitive development, though these models may be implicit rather than explicit. The assessment process may provide essential information for both the student and the teacher towards the goal of developing increasingly complex conceptual structures. However, the quality of any assessment activity depends on the extent of the theoretical work underpinning the model of education to be assessed. “Assessment activity” also refers also to assessment-in-process as described by Steinbring (see Section 3.2, p. 69), though the focus here is primarily on written assessment.

This research study focuses on a phase of mathematical development and a content domain, the multiplicative conceptual field, which in its complexity and depth is understood to be critical for learner development. The study embraces two fields of study, mathematics education and measurement theory. *Theoretical and qualitative* investigation and the application of *quantitative and measurement* techniques, interactively support understanding of this mathematical field, and provide outcomes building on both quantitative and qualitative input.

Consideration of the intrinsically developmental nature of mathematics, in particular the unfolding character of number systems, provides the background to research in this subdomain of mathematics. The idea that the central features of any knowledge domain should inform assessment is critical to the concept of measurement as interpreted within the Rasch measurement framework (Rasch, 1960/80).

5.1.1 Research questions

In order to provide a context for the empirical phase of this research study the challenges facing assessment in mathematics are highlighted (Section 5.2). Against this background, the central elements of a theory of mathematics assessment, a perspective proposed by Webb (1992), are discussed. The use of various assessment models, together with their assumptions, advantages and limitations are considered (Section 5.3).

The view held in this thesis is that central to the design, construction and administration of any test, is the notion of *fundamental measurement*, which when applied to psychosocial constructs, should adhere in important ways to the measurement of constructs as applied in physics, for example the requirement for invariant measures. The Rasch model is a mathematical model which is premised on the requirement for invariant measures within a specified frame of reference (Section 5.4). The potential coherence of a specific mathematics assessment model, drawing insights from the theory of conceptual fields and from the Rasch measurement framework is explored (Section 5.5).

The primary question on the assessment of mathematics proficiency is followed by four subquestions.

Question 5 How may mathematics proficiency be assessed so as to inform learning and teaching?

5.1 What are some of the challenges facing assessment in mathematics? (Section 5.2)

5.2 What does it mean for a student to be *proficient in mathematics*?[72] How can we *measure proficiency* in mathematics? (Section 5.3)

5.3 What are the essential features of *fundamental measurement*? How are these features satisfied in the Rasch measurement model? (Section 5.4)

5.4 How may a model for assessment be designed that draws on both mathematics education theory and on fundamental measurement? (Section 5.5)

5.2 Assessment and evaluation

The purposes of assessment can be assigned to the following categories;

- *Feedback for teaching and learning*. The term formative assessment is generally used in this context (Black, 1998; Black & Wiliam, 1998).
- *A form of communication*. In the sense that assessment items and test forms provide teachers and students with information about the level of knowledge expected, one purpose may be that of communication (Webb, 1992, p. 663).

[72] This question was proposed by Webb (1992), Kilpatrick, Swafford and Findell (2001) and Schoenfeld (2007).

- *Learner performance*. Learner performance is assessed in order to measure mastery of a topic and to judge competence.
- *Educational programme evaluation*. The purpose of the assessment task may be to gauge the effectiveness of a programme and the relative effectiveness of teaching. This purpose serves an accountability function and is therefore the rationale underpinning school improvement research.
- *System evaluation*. Systemic assessment is the term generally used for providing information on the system as a whole.[73]

Assessment is considered by many as central to learning, especially in the practice of *formative assessment* (Black & Wiliam, 1998), or in the more finely crafted *regulation of learning* (Perrenoud, 1998). Following Bernstein, Davis (2001) uses the term evaluation to refer to "teacher-student interactions as well as questions, problems, tests [and] examinations"; evaluation in this sense being the key to all pedagogic practice (p. 2). In this thesis the term *mathematical assessment* refers to "the comprehensive accounting of an individual's or a group's functioning within mathematics or in an application of mathematics" (Webb, 1992). This definition specific to mathematics is built on the general definition provided by Wood (1987), where

> assessment is regarded as providing a comprehensive account of an individual's functioning in the widest sense – following on a variety of evidence, qualitative as well as quantitative, and therefore going beyond the testing of cognitive levels by pencil-and-paper techniques which, for many people [apparently constitute] measurement (Wood, 1987, p. 2, cited in Webb, 1992, p. 663).

The term assessment will be used in this chapter to refer to the many practices in which learning is monitored.

In the context of the Rasch measurement framework, the term *measurement* has a specific meaning, aligned to what may be termed fundamental measurement in the physical sciences: that is, a construct is defined, for which there is a "natural unit" of measurement, specific to a particular frame of reference (Humphry, 2005).

The term *test* refers to a quantitative assessment tool that follows a systematic procedure for observing and describing performance behaviour with the aid of fixed categories and numerical variables (Webb, 1992, citing Cronbach, 1984). Following Webb (1992), the term *evaluation* in this chapter refers to "the systematic collection of evidence to help

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[73] While the stated objective may be *the system as a whole*, the purpose is often reduced to a judgement on the teachers alone.

make decisions" regarding students' learning, materials development and a programme, but also has a connotation of value, worth or utility.

5.2.1 Large-scale assessment and learning

Large-scale assessment is premised upon acquiring information across large groups of students relatively efficiently, for the purposes of "providing accountability information about schools and districts" (Taylor, 1994). Further aims are noted as "identifying needs", "allocating resources", and "providing instructional diagnosis" (Nasser Abu-Alhija, 2007). According to Taylor (1994), the selection of a model of assessment is critical to the achievement of these purposes. She identifies a psychometric model, which she describes as a "measurement"[74] model, based on the assumption of an underlying trait, and which is norm-referenced (p. 236), and a standards[75] model, which is criterion-referenced and which attempts to broaden the type of assessment to include performance assessment (p. 242).

The differences between assessment for accountability purposes and for informing teaching and learning, in terms of the objective and mode of testing, may in some senses mean that they are diametrically opposed (Webb, 1992; Gipps, 1994; Black, 2003, French, 2003; Nichols & Berliner, 2005, 2008). While attempts have been made to broaden the scope of international large-scale assessments for multiple purposes, the main purpose is fundamentally different from that of assessment for teaching and learning. Assessment on a large scale must inevitably reduce the knowledge domain to those aspects of the domain which are amenable to administration on a large scale and which are relatively easily scored and analysed.

In addition the test has to be targeted at a general hypothetical group taking into account the commonalities across countries, rather than the specifics of individual, school programme or country that may have a distinctive educational focus.

Lamon (2007) alludes to the limitations of standardised tests.

[74] In this research study the term "measurement" has a specific meaning taken from the classical theory of measurement.

[75] The term "standards" is sometimes conflated with the curriculum contents. In other senses it means benchmarks of achievement.

> We habitually look to standardised examinations to measure success in spite of the fact that they reflect a different set of values, include more procedural than conceptual content, are often developed for different purposes, and are problematic in that the structure and arrangement of content they assess is inconsistent with the nature and growth of knowledge in the multiplicative conceptual field (p. 646).

Systemic assessment has a role in appraising the needs of education systems. For large-scale international assessment, the purpose is to provide access to large data sets, allow achievement and contextual comparisons, and enable subsequent secondary analyses. In addition to signalling system-wide needs, the tests serve a communication function as an adjunct to school assessment, in that they may at least present examples of professionally constructed tests and test items (Andrich, 2009, p. 29). In the South African context, systemic tests provide information to education departments about schools that are functioning poorly, and where there are remarkable deficits, for example recently introduced curriculum topics, for purposes of planning interventions.

TIMSS 2003 and South African participation

South Africa participated in the Trends in International Mathematics and Science Study (TIMSS),[76] a study with approximately 40 participating countries, in 1995, 1999 and 2003. While in 1995 South Africa was the only African country taking part, other African countries have subsequently participated (see Appendix A). A comparison with the five other African countries, four of which have Gross National Incomes (GNI) lower than South Africa, but which are ranked higher in terms of achievement, highlights the educational challenges faced by the South African education system. It must be noted that for some of the countries listed, schooling is not compulsory for all children, while the South African system aims at 100% access in the General Education and Training (GET) band up to Grade 9.

Participation in studies such as the TIMSS fulfils two important aims, namely the broad based accountability of education systems, and the formative function of alerting teachers to a general standard agreed on by the representatives of the participating countries. But unless the results gained from the testing process are interpreted and

[76] In 1995, the Third International Mathematics and Science Study (TIMSS) was conducted. In 1999, the title was changed to TIMSS-Repeat. From 2003, the title has been Trends in International Mathematics and Science Study (TIMSS).

transformed into strategies for implementation, much of the value is lost (Venter, Long & Dunne, in process). The question also arises as to how the results are interpreted, and what effects the particular interpretation may have on determining curriculum policy. In the South African context, the results of international assessments are attributed in the National Department of Basic Education Review, to the negative impact of the “new curriculum” (Dada et al., 2009, p. 5) and perceived to have been partly responsible for “distrust in the [current] education system” (p. 7).

In order to understand in more depth the poor results of the South African cohorts over the three TIMSS cycles, a secondary analysis was conducted with a focus specifically on items which tested multiplicative structures. The items, selected from the TIMSS set of released items, were located across the content domains *Number, Algebra, Geometry, Measurement* and *Data*, the major subdomains in the TIMSS Frameworks. The results were sourced from the TIMSS data banks (see Appendix A). While most items came from the *Number* domain, and the subdomain of *ratio, proportion and percent*, other items were selected from other subdomains. The performance of the South African samples in two recent cycles, TIMSS 1999 and TIMSS 2003, on items designed to test the content domain of *ratio, proportion and percent* and aligned cognitive processes, confirmed that South African students in general found these items difficult (Mullis, et al., 2004; Long, 2006).

Additional contextual information reported by the National Country Representative (see Mullis et al., 2004) indicated that South African teachers found this subdomain difficult to teach and therefore only taught this topic to the more able students (see Appendix A). In both cycles, TIMSS 1999 and TIMSS 2003, the national reports revealed that the low performance in mathematics in general could be attributed to paucity of properly functioning schools and associated poor teaching (Howie, 2001; Reddy, 2006). In a separate but related study, Howie (2002) investigated the link between language proficiency and mathematics achievement. The correlation, correcting for socio-economic differences, was shown to be high. Preliminary selected item analyses indicated that the exceptionally low performance in the content domain *ratio, proportion and percent*, might also be attributed to the complexity of this conceptual field (see Long, 2006).

Participation in the TIMSS studies, by South Africa[77] in particular, has been the subject of critique by mathematics education researchers on the grounds that other “political necessities and needs” should take priority, and that the premise of such tests, that there is a direct link between the teaching of mathematics and the performance of students, is allegedly unwarranted (Keitel, 2005, p. 340). A valid critique may well be that the sophisticated nature of the TIMSS research requires levels of interpretation in order to be helpfully interpreted by policy makers and education officials.[78]

The design of large-scale international studies indicates a comparative objective. The achievement comparisons may be used across countries to inform the education systems of weaknesses. There are, however, only tenuous links from the large-scale testing to strategies for classroom implementations and teaching. This fact has to be considered from both points of view; the education department looking for evidence to support particular reform programmes, and researchers investigating teaching and learning.

5.3 A theory of mathematics assessment

The term paradigm shift is used by Gipps (1994) to describe the shift from psychometric testing, with roots in psychological and intelligence testing, to educational assessment, where the focus is on developing cognitive skills. Webb (1992) proposes a further shift towards a specific theory of mathematics assessment. This domain specificity in regard to assessment, he avers, is needed to respond to the complexity inherent in mathematics itself and to identify the composite concepts and theorems required for attaining proficiency. Webb argues that “the nature of mathematics itself and the pedagogical approaches to teaching mathematics warrant specific assessment techniques in the area of mathematics” (Webb, 1992, p. 662). He explains further:

> The calculus, algebra, and number system are axiomatic systems that require a knowledge and deep understanding of axioms, operations and theorems. The power of mathematics accrues when it is used to abstract a situation and then mathematical manipulations are used to gain further knowledge of the phenomenon (p. 662).

[77] South Africa took part in TIMSS 1995, 1999, 2003. Participation in TIMSS 2007 was halted. The reason given was that the reform programme needed time to consolidate (Blaine, 2007). Participation in 2011 is in process.

[78] In the South African context there were requests made for the results to be communicated using less complex statistical terms (Reddy, 2006).

A theory of mathematics assessment, it is proposed, could draw together the information currently known about mathematics and its relation to teaching and learning, and therefore enable researchers and those involved with assessment to make reasoned decisions concerning assessment for a specified purpose. The specification of particular content knowledge in the formulation of tests is subject to the particular view of what is important in the teaching and learning of mathematics. This specification has implications for the type of responses expected in an assessment programme. Together, mathematical content, item design, and response type impact on reliability and validity. A further consideration is the inferences and actions to be taken in response to the results (Messick, 1989), in particular how the results are disseminated.

Drawing from Webb (1992), some of the implicit assumptions that underpin current assessment models are highlighted, and their potential contributions and limitations in relation to mathematics education discussed.

5.3.1 Conceptions of mathematics

Different perspectives have been presented on the nature of mathematics in this thesis, which may impact at a fundamental level. However three conceptions may be perceived to impact differently on assessment. The first is that mathematics consists of a collection of topics that can be taught and learnt independently. For example, the topic of place value may be thought of as independent of number sense. The second is the functional or operational aspect, where the expectation is that students solve problems in context. This understanding is exemplified in Vergnaud's statement that assessment should be through solving problems in context, as a concept is not a true concept unless it is operationalisable (Vergnaud, 1979, p. 263). The third is that of a "structured body of knowledge [comprised] of interdependent elements". Therefore, "for students to know mathematics", they need to know "the elements [of mathematics,] concepts, skills, properties and principles" and the interrelationship among these elements (Webb, 1992, p. 665). This third conception aligns with the conceptual fields approach.

A composite approach to mathematical proficiency, advocated by Kilpatrick et al. (2001), includes five interconnected strands, which together may describe proficiency, and each interactively promotes the development of the other strands. These strands include *conceptual knowledge*, *procedural fluency*, *strategic competence*, *adaptive* *reasoning* and *a productive disposition* (p. 144). Conceptual knowledge refers to "an integrated and functional grasp of mathematical ideas" (p. 118); procedural knowledge refers to knowing when and how to "perform procedures flexibly, accurately and efficiently" (p. 121); strategic competence embodies a modelling function, which refers to "the ability to formulate mathematical problems, represent them, and solve them" (p. 124); and adaptive reasoning refers "to the capacity to think logically about the relationships" (p. 129) between variables. The fifth strand, a productive disposition, refers to "the tendency to see sense in mathematics" (p. 131).

The emphasis on any of these aspects, singly or in combination, determines an approach to testing. For a detailed examination of different approaches, see Webb, (1992, pp. 665-667). Here, selected approaches will be grouped for the purpose of providing insight into TIMSS and the formulation of the test instrument design, administration and analysis in this research.

5.3.2 Critical elements for an assessment programme

The critical elements for the formulation of any assessment programme, class tests, systemic testing, large-scale studies, or even examinations for qualification purposes are: *content selection*, *response format*, *situational and administrative factors*, the *type of analysis* and the *interpretation and meaning of results*.

Content specification and response format

Some categories of this critical dimension drawn from Webb (1992) are identified.

The *topic approach* is exemplified in the TIMSS conceptual frameworks, where there are five major topics, Number, Algebra, Measurement, Geometry and Data, subtopics (see Table 5.1), and further descriptions for each subtopic (see Mullis et al., 2003). Another approach is to stipulate *cognitive* domains. The cognitive domains in TIMSS are identified as *knowing facts, procedures and concepts*, *applying knowledge and conceptual understanding*, and *reasoning.* The combined use of topics and cognitive domains results in a *content-by-behaviour* test specification matrix.

Table 5.1: TIMSS Frameworks

Number	Algebra	Measurement	Geometry	Data
whole numbers	patterns	attributes and units	lines and angles	data collection and organisation
fractions and decimals	algebraic expressions	tools, techniques and formulas	two- and three-dimensional	data representation
integers	equations and formulas		congruence and similarity	data interpretation
ratio, proportion and percent	relationships		locations and spatial	uncertainty and probability
			symmetry and transformations	

Source: Mullis et al., 2003, pp. 11-22

The underlying assumption of the *topics approach* and the *content-by-behaviour* matrix is firstly, that mathematics knowledge can be separated into distinct topics, and secondly, that behaviour, for example, *knowing*, *applying* and *reasoning*, are distinct and can be attributed *a priori* to a test item without regard for a student's cognitive level, or a particular approach to solving problems.

Distinct from a topic and behaviour approach is the *process approach*, in which problem-solving approaches and higher-order thinking skills are identified. In some cases paper-and-pencil tests may be used, but generally interviews and observations are conducted in order to identify actions and processes, and to make thought processes explicit.

In a *conceptual approach*, a conceptual field is generated by specific criteria, for example the multiplicative conceptual field is identified by common multiplicative structures. The key elements of the field include problems situations, operations of thought and symbolic representations, but in addition this approach considers the interrelationships "between problems and situations and [the] student's thinking in addressing them" (Webb, 1992, p. 667). The assumption underlying conceptual fields is that "a small number of symbols and symbolic statements can be used to represent a vast array of different problem situations" (Romberg, 1987, cited in Webb, 1992, p. 667). The strength of this approach is that it can be used to "map what a student knows within a knowledge domain and to track the maturation of concepts within that domain" (Webb, 1992, p. 667). In this approach extensive work is required to specify the elements from both a mathematical and a cognitive perspective, for example by specifying the increasing complexity of multiplicative structures (see Greer, 1992, Vergnaud, 1983).

Schoenfeld (2007) uses the notion of proficiency articulated by Kilpatrick et al. (2001), to formulate a *Balanced Assessment* test[79] based on the National Council for Teachers of Mathematics (NCTM) standards.

With the advent of item response theory *statistical methods* are increasingly being used to guide the selection of items or tasks. Typically, a larger selection of items is trialled, analysed, and then a subset selected for their observed efficacy in distinguishing levels of student ability, and conformity with particular criteria. An assumption underlying the use of statistical methods for this purpose is that the mathematical proficiency of interest can be located along a linear continuum. A continuum implies measurement of some sort, in that along the continuum, relationships, including "more than" and "less than", are always and consistently implied.

In some test designs where the need is to cover a large extent of the curriculum, but not simultaneously overburden any particular student, a *matrix design* is used. This design involves the use of several booklets, each of which covers only a subset of the complete set of available items, or question components, but which have concurrently some overlapping content structures. This design is used in TIMSS, where eight booklets are distributed across the test population. A simpler matrix design with core common items, and extra items distributed across all the booklets is used in this study.

The *response format* varies across each of the content specification types. Inevitably large scale and systemic studies tend to use mostly a *multiple-choice* format. For process testing approaches, the response format cannot be restricted to short answers but

[79] This test is not known to the author at the time of writing, but this test and others formulated on the same framework will surely inform future testing programmes.

rather must include extended processes, which then require observation in terms of the theoretically defined phases required of problem solving (see for example, Polya, 1945).

Situation and administration

In addition to the specification of content, the assessment *situation* and mode of *administration* have to be considered. Webb (1992) elaborates on each of these criteria (see pp. 668-672). For the purpose of this thesis the following is noted:

> If knowledge and learning are situated in specific contexts, the situations used to assess them must have authenticity relative to the practice about which interpretations are being made (Webb, 1992, p. 668).

The administration of the test, whether it is conducted by the teacher or the unknown fieldworker, may have an effect on student performance, especially on very young children, for whom the test format and testing situation may be unfamiliar.

Analysis of results

The results of most school assessment programmes are presented as a quantified aggregate for each learner. In class tests for example, one may attain 67% for mathematics. A qualitative dimension may be added in that a teacher may provide specific feedback concerning the learner's progress.[80] For systemic assessment, results may be aggregated per province, per district, per school, per class, or at the individual level. For the purposes of this chapter, it is noted that the reliability of reporting at individual level in the case of large-scale testing is questionable, as errors may arise from a number of sources and causes, one of which is the variability inherent in any student from day to day. There may also be errors in the marking, scoring, data capture and so on. On the other hand, aggregation at a district level may not be useful for informing educational strategies at school level.

Webb (1992) notes that assessment of mathematics programmes may require specialised procedures for assessing their validity.

> (T)he traditional summary statistics of reliability, item-total biserial correlation coefficients, and mean proportion correct can be misleading. The psychometric perspective

[80] For an interesting study on the effects of different types of feedback, see Butler (1988).

> of measuring knowledge does not always coincide with assessment derived from a cognitive or information-processing approach (p. 668).

Interpretation and inference

Cobb and Jackson (2008), in a critique of large-scale experimental studies, note that the knowledge claims made refer in general to "an abstract, collective, individual or statistical aggregate that is constructed by combining the measures of psychological attributes of the participating students (for example, measures of mathematics achievement). This statistically constructed individual *does not correspond* to any particular student" (p. 574, italics added). The aggregated data provide little information about the progression for any particular learner along a path to greater proficiency.

In agreement with Cobb and Jackson (2008), Lamon (2007) and Webb (1992), it is noted that the type of assessment in general to which we turn for evidence of mathematics proficiency is not necessarily aligned with the teaching and learning of mathematics that is regarded as necessary for mathematical development. The links from the tests to classroom practice need to be rendered explicit.

A warning against the impact of testing, in particular premature testing, may be observed by the example of a study reported by Lamon (2007). The object of her study was a teaching intervention based on the development of cognitive skills within the multiplicative conceptual field. When tested at the end of the second year (Grade 4) the experimental group was lagging behind a control group, but by the end of the fourth year (Grade 6) significant achievements were noted in the experimental group which outweighed the control groups. Without conviction on the part of the teachers and outside professional support, the teachers may after two years have lost courage to continue with the reform programme.

5.3.3 Core notions for assessment

The challenges facing assessment in many countries are great (in part answer to **Question 5.1**). The overarching challenge concerns the construct validity of the entire testing process. Most pertinent is the question of what constitutes mathematical proficiency. Whether the large scale and systemic assessment programmes can inform developmental progress is a question raised by Lamon (2007) and cited in Section 5.2.1.

The purpose of outlining a *theory of mathematics assessment* (in part answer to **Question 5.2**) is to make explicit the issues involved in the specific assessment of mathematics, taking into account the nature of mathematics, the underlying reasons for selecting a particular set of item content, the type of items and response format, the test situation and administration, and the type of analysis and inferences that can be drawn from the whole process.

We note three points arising from this section. Firstly, the approach to content selection in this study is a *conceptual fields* approach. The challenge in this approach is to develop a model or models against which the results of any individual student may be mapped. Secondly, the formulation of an assessment programme needs to consider all the aspects impacting on the assessment of proficiency, and thirdly that the inferences made are aligned with the type of testing.

5.4 Measurement and the Rasch model

Against the background of general assessment, aspects of the Rasch measurement model are presented. The view that measurement[81], when applied to the social sciences, should reflect the fundamental principles of the *classical theory of measurement* as applied in the physical sciences is central to Rasch measurement. The ideas foundational to classical measurement therefore provide the context and determine the features of the model which make it appropriate for application in this study. The claim that the application of the Rasch model, in conjunction with the theory of conceptual fields, may contribute to a theory of assessment in mathematics education, is explored.

The particular strength of the Rasch model, based on measurement principles, is that it provides "individual-centred statistical techniques" in which "each individual is characterised separately" (Rasch, 1960/1980, p. xx). In this section, the features of the probabilistic model most appropriate for transforming scholastic data into measures, and other selected aspects of the Rasch model, are discussed.

[81] The term measurement is used widely with respect to assessment and research. Advocates of a classical theory of measurement aver that clearly defining the construct of interest and devising a measurement instrument (composed of items or questions) that is then both theoretically and empirically scrutinised for construct validity, and for adherence to measurement requirements, is not only good science, but also in the interests of social justice (Andrich & Marais, 2008; Bond & Fox, 2007; Wright, 1997).

5.4.1 Measurement

Measurement in the physical sciences has a long history dating back at least as far as the cubit measure, mentioned in the Noah's Ark story. Some early developments in measurement can also be traced to commerce and politics as far back as 723 AD when taxes were introduced in the Middle East, and close to 800 years ago, when the Magna Carta stipulating standard measures for length, area, volume and weight was proclaimed in England (Wright, 1997). The modern measure for weight or mass is of interest in that to establish the mass of an object, it is necessary to consider gravity. The Newtonian definition force = mass × acceleration ($F = m \times a$), can be reconstituted as

mass = force/acceleration or acceleration = force/mass.

In order to understand acceleration, we have a two-way frame of reference, with complementary concepts defined in terms of each other, that is, *constitutive definitions* for force and mass (Andrich, 1988, p. 18). Measures for mass and other physical qualities are generally accepted without reference to the extensive theoretical work involved in their construction. It is perhaps because of this long history and the smooth functioning system of physical measures in the modern world, that the principles underlying measures and measurement are not often rendered explicit, except to the fortunate student of physics. The family of Rasch models, applied to the psychosocial sciences, seeks to adhere to the principles of classical measurement.

Variable construction

In any assessment situation in the psychosocial sciences, including education, where students are ranked or a judgement is made; implicit measurement of a construct is effected, *even when* the approach may be qualitative in nature. As with physical measures of mass, where the complementary elements force and acceleration are involved, a measure of attainment in a test involves the complementary parameters of person ability[82] and test difficulty which together govern the resultant measure.

[82] The term ability is used to denote attainment or proficiency.

In order to be scientific about the measure, rather than randomly assigning numbers to individuals[83], theoretical work into the nature of the construct is required. In order to operationalise magnitudes of a particular construct, a process termed *variable construction* is required (Andrich, 1988, p. 9; Wright & Stone, 1999, p. 37). This process converts knowledge of the construct, for example mathematical proficiency, into items which operationalise the construct and from which we may obtain quantitative measures.

The process of variable construction requires understanding the property, or characteristic of interest, in order to map the variable or construct onto a line. When the idea of measurement is applied to psychosocial constructs, for example in "measuring" mathematical achievement, it is necessary to "force the qualitative variations into a scholastic linear scale of some kind" (Thurstone, 1928, pp. 534-535). Where there is a continuum of some sort, that is where the comparative notions of "more than" and "less than" apply, the assumption is that of an underlying unidimensional construct.

Andrich and Marais (2008) remind us that it is a *quality of the object* that is being measured and not the object itself. We measure *the length of the table,* rather than the table, as *the table* has other qualities, for example, width, or mass, or density, some of which may be the focus of other means of quantification. When measuring scholastic achievement we are measuring, for example, the manifestations of the trait *mathematical proficiency*, and not the property[84] itself (Andrich, 1988, p. 14). We may of course unwittingly be measuring reading ability in addition to mathematical proficiency: this complication may, if undue reading difficulty affects outcomes, constitute noise or *construct-irrelevant variance* (Messick, 1989, p. 7),.

Unidimensional construct

Measurement in the physical sciences requires a unidimensional construct, a single property or characteristic, common in varying degrees to all the entities in the

[83] Wright (1997) reminds us that assigning numbers to phenomena is not sufficient for measurement.

[84] Where these properties reside, and whether they exist is not of concern to Andrich (1988); the concern is rather that the identification of properties are useful and serve some educational purpose.

measurement instrument. In physics for example we cannot measure both length and weight concurrently towards one summative end result. The basic assumption underlying measurement in the psychosocial sciences, and made explicit in the Rasch model, is that a relatively stable latent trait, a characteristic of an individual not directly observable, underlies test results and can be measured.

By determining the probability of success and of failure for each item and person combination on an instrument with items that are believable realisations of the construct, measures of the latent trait may be estimated. While in the physical sciences, measurement requires a theoretically defined unidimensional construct, the requirement in the social sciences is more complex.

This notion of unidimensionality has been challenged by Gipps (1994, p. 6) in that the underlying attribute in educational contexts is invariably more complex. Andrich (2006) acknowledges that the restriction of a scale to one-dimension when the domain being measured is actually multidimensional may in some senses be counterproductive. A theoretical construct, though unidimensional in a sense, may also vary in breadth. For example the construct may be defined as *proficiency with multiplication tables* (narrow construct) or it may be defined as *proficiency in the multiplicative conceptual field* (a broader construct).

The balance between establishing a unidimensional construct with greater reliability and establishing a broader construct with greater construct validity is the trade-off in educational measurement (see Andrich, 2006). Andrich refers to Cronbach (1951), who considers a similar issue “as a trade-off between *fidelity,* the precision of a single variable, and *bandwidth*, the incorporation of variables other than the main variable in the measurement” (p. 4, italics in original). Andrich elaborates further, arguing that dimensionality is “conceptually and empirically a *relative*, and not an *absolut*e concept” (p. 38, italics added). The question arising in any measurement situation is “How unidimensional is the scale?” In essence the dimensionality is inherent in the specified frame of reference, rather than in the requirements of data analysis (Andrich, 2006).

It is in order to address this tension between unidimensionality of scale, and breadth of construct, that Andrich (2006) draws an analogy with the geographical notion of fractals. The “two key concepts of fractal geometry are those of *roughness* and *self-*

similarity, [which] may be useful in summarizing this multidimensionality in social measurement, both numerically and conceptually". The key notion is that both roughness and self-similarity may happen "at different units of scale" (p. 5).

In the case of mathematics, we may envisage an encompassing mathematical ability as a unidimensional construct. But at finer gradations of scale, for example the *multiplicative conceptual field*, the conceptual requirements of *rational number sense* and *proportional reasoning* may constitute a different scale, though aligned with the first. Both items of varying conceptual difficulty, and students of varying proficiency exhibited in, for example, *rational number sense* and *proportional reasoning*, may be located on the scale. At a finer level still, a multiplicative scale focusing on simplest understandings, linked to place value, may constitute a focus with a finer natural unit defined according to conceptual and cognitive levels, but targeting the lower levels of the mathematical proficiency scale.

The point is that the *roughness* constituted by the broad construct of mathematical ability, may at finer levels of scale be constituted by *self-similarity* (Andrich, 2006). The mathematics observed in a focused instance, a cameo of mathematical activity, may reflect elements of more extended or more advanced mathematical thinking. The question is really whether a student solving a proportional problem at the beginning of her mathematical path, is exhibiting elements of the trait that is required to solve the proportionality problems at higher levels. Vergnaud (1988) suggests this continuity, and therefore insists that the mathematical path be analysed and described.[85]

From a measurement perspective, Michell (2008) challenges researchers to investigate the quantitative nature of the variables under consideration, in order to justify the claim that the concepts can be measured!

Frame of reference and observation framework

The situation in which psychosocial measurement takes place is necessarily contrived. The controlled situation which directs observations may be termed the *observation*

[85] The implications for measurement are that a set of linked instruments of varying difficulty range be constructed along the continuum.

framework. This framework may take the form of a test, a questionnaire or a classroom observation schedule. The observation framework occurs within a *frame of reference*. The frame of reference is "specified to circumscribe the application and interpretation of the construct" (Andrich, 1988, p. 10).

In this research study the multiplicative conceptual field constitutes the frame of reference. The observation framework consists of the test instruments (a well-defined response context), which includes particular items (a class of items) to be administered to the Grades 7, 8 and 9 students (a class of persons). The follow-up interviews constitute a second, though related, observation framework. As with physical measurement, the frame of reference "must be rendered as explicit as possible to ensure that intended observations, and not others, are made, to ensure consistency of observations and to permit replications of the procedure" (Andrich 1988, p. 10).

The cyclical process of investigating the multiplicative conceptual field, constructing the test instrument, administering the test, analysing the data and then refining the instrument constitutes the process of variable construction, and is the essential and iterative precursor to successful measurement.

5.4.2 Mathematical models

In order to obtain measures from observations in controlled situations it is necessary to transform the observations by invoking a model of some kind. A *deterministic model* implies the exact prediction of an outcome, for example, in Newtonian physics the model,

$$F = ma$$

is robust enough to account for measures of force, mass and acceleration, and in this case measurement error is minimal (Andrich, 1988, p. 11). In cases where the model does not function as expected, for example, as was found in specific instances by Einstein, more theoretical work is conducted and a new theory emerges, the theory of relativity, and a more refined model is developed. A paradigm shift may take place in response to anomalies found in an existing model that challenge current assumptions.

By contrast a probabilistic model is applied when it is expected: firstly, that a model will not account for all the expected outcomes of a particular situation, secondly, that "differences in outcome in ideal replications" cannot be ignored, and thirdly, that the same outcome will not necessarily result from the same known circumstances (Andrich, 1988, p. 11). In the case of scholastic achievement it is generally accepted that a student may perform well on one day and not the next, or that she will solve a problem at a specified level of difficulty within her range on one day, but may on the same day or other days, make a mistake on another problem of equal difficulty. It is therefore in the interests of fairness and honesty to apply a probabilistic model in order to account for patterns within multiple indeterminate situations.

In probabilistic models, the probability of each outcome is calculated as a proportion of the total count of all possible observations of replications of an identical procedure. The probability, for example denoted by symbol π with $0 < \pi < 1$, may also be specified in terms of odds θ, where $\theta = \frac{\pi}{1-\pi}$, and $\pi = \frac{\theta}{\theta+1}$, or $\pi = \frac{\theta}{1+\theta}$. "The estimate of $\pi(\theta)$ as a ratio of appropriate frequencies of outcomes depends on the outcomes being *replications governed by the same value* π. If they were not governed by the same value", then we could not establish the probability (Andrich, 1988, p. 12).

Fundamental measurement

Georg Rasch's mathematical discovery that links the testing of psychosocial constructs to fundamental measurement is the distinguishing feature of the Rasch models. The import is that "with the careful applications of the Rasch models, and by invoking the knowledge available for constructing sound tests, and questionnaires, it is possible to *attempt* to construct measurements of a fundamental kind in standard test and questionnaire exercises" (Andrich, 1988, p. 16, emphasis in original).

Measurements are essentially ratios, and hence they are real numbers or scalars, obtained from the multiplicative ratio comparison of one or more magnitudes, or extents, against a single but essentially arbitrary unit. For any magnitude, or extent, to be uniquely measured we need to have explicit knowledge of the unit against which it is to be compared. Thus a measurement of 17.3 cm implies a ratio of $\frac{17.3}{1}$ between the extent in question and the chosen unit, 1cm. The unit, implying an extent, or magnitude, for example a metre, has been previously uniquely characterised so as to admit multiplicative comparisons with other extents of length, for example, the kilometre and millimetre. Multiplicative or ratio comparison of magnitudes or extents, have a property of ratio invariance that permits the use of a single unique real number for any paired comparison of extents regardless of the units.

In effect the equation

$$\frac{17.3\,cm}{1.0\,cm} = \frac{173\,mm}{10\,mm} = \frac{17.3}{1} = 17.3$$

Equation 1

exhibits the arbitrariness of the unit and the uniqueness of the ratio.

An additional requirement of measurement is the property of *additivity*, defined as the consistency of differences between equally spaced items. For example, if the difference between the location of Item A and location of Item B is x, and the difference between location Item B and location of Item C is y (in the same direction), then the difference from location Item A and location Item C, should be equal to $x + y$. On the grounds of the ratio and additivity properties, we accept the one-to-one correspondence between any measures, for example length measures, metres, or mass measures, kilograms, and the real number system. This requirement is taken as given in physical measures.

An additional requirement, the criterion of *invariance*,[86] stipulates the necessity for “the instrument to operate the same way (invariantly) across groups”, that is the item locations should be invariant across groups within the test’s range of application (Andrich & Marais, 2008, p. 1:6). Andrich and Marais note that invariance is important not only for “science and generality, but where humans are concerned, for social justice and accurate diagnoses” (p. 1:7).

In the next section we provide background information about the Rasch measurement model, describe some central statistical and mathematical ideas and highlight the features of the model which have a bearing on this research study.

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86 The criterion of invariance refers to the notion that measurement of an object is affected by the characteristic being measured and not by any other features of the object. Thus in educational and psychological measurement we require that the particular construct should not be dependent on the person who designs the test, or the students who take the test (Rasch, 1960/80, p. xv).

5.4.3 The development of the Rasch model

The development of the first probabilistic models by Georg Rasch, a Danish mathematician, arose in response to a practical problem facing the Danish government regarding the retrospective evaluation of reading programmes for school underachievers (Rasch, 1960/1980). The question posed to Rasch was whether the adults, who had been provided with supplementary instruction as children, had benefited from this programme.

Because the children of interest had taken tests of differing difficulty, and not all children had taken the same tests, it was necessary to find some valid means of making comparisons. The immediate problem requiring statistical analysis was to place the tests of varying difficulty, taking the form of *texts*, and the children of varying ability, on a common valid scale (Rasch, 1960/1980). In order to solve this problem Rasch instituted a special design of chained or linked sets of tests (See Table 5.2) to collect new data.

> The design held that the texts students read should not be so difficult that they become frustrated and distracted, nor so easy that they were not challenged. The number of errors should be in the region of 5% to 10%. This [criterion] meant that the same [common] text could not be given to all students (Andrich, 2005, p. 4)

Five test forms, previously administered over an extended period to the adults of interest when they were children, were re-administered to different children at this later time in Grades 2 to 7, specifically four children from each of the selected grades in 24 schools. Each test was administered to at least two grades. From Grades 3 to 7, each grade (other than Grade 2) was given at least two tests.

Table 5.2: Plan for linking test forms on the same scale

	Grade					
Test	**2**	**3**	**4**	**5**	**6**	**7**
ORF	X	X				
ORU		X		X		
ORS			X	X	X	
OR5			X	X	X	X
OR6					X	X

Source: Rasch, 1960/80, p. 5

Rasch approached the problem by taking the person-test parameter, the average number of errors made on a particular text by a particular person, and decomposing this parameter according to the following reasoning. Since persons of greater ability make fewer errors than persons of lesser ability on any relevant test, and since persons of the same ability will make a greater number of errors on more difficult texts, Rasch chose to decompose the average number of errors into two explanatory components, a numerical *reading ability* measure for the person, and a numerical *text difficulty* measure for the text (Rasch, 1960/1980).

This pair of components permitted that the average number of errors for a specified person-test combination could be hypothesised to be the ratio of text difficulty, D_t, to the level of person ability, B_v, where $B_v > 0$. Equation 2 stipulates this relationship.

$$Ave\ [x_{vt}] = \frac{D_t}{B_v}$$

Equation 2

where the expression signifies the average number of errors that a person labelled v of ability B_v, would be expected to make on a test, labelled t, but with a difficulty of D_t. This structure implicitly assumes that no other factors impose any effects on that average, and this assumption is then open to retrospective testing and confirmation.

Independence within parameter sets

Another key requirement of the process of measuring in a specified frame of reference is the functional *independence of parameters*. This requirement is explained as follows:

> The comparison between two stimuli [items] should be independent of which particular individuals were instrumental for the comparison; and it [this particular comparison] should also be independent of which other stimuli [items] within the considered class were or might also have been compared.
>
> Symmetrically, a comparison between two individuals should be independent of which particular stimuli [items] within the class considered were instrumental for the comparison; and it should be independent of which other individuals were also compared on the same or other occasion (Rasch, 1961, p. 332, cited in Humphry, 2005, p. 14).

Intuitively, in a social science testing situation, one would expect that the difficulty value of an item should be intrinsic to the item and not dependent on the particular persons writing the test, providing that the test was properly targeted to the group.

Similarly, a person's ability on the underlying construct should be intrinsic to the person and not be dependent on or influenced by the choice of specific test to be written.

From the expression, Equation 2, relating the average number of errors for a person-test combination, encompassing person ability and test difficulty, it is possible to take another step which is to draw a comparison between two texts, 1 and 2, of difficulty measure D_1 and D_2, written by the same person, *v*, of ability measure B_v (see Andrich & Marais, 2008). The ratio comparison of the average number of errors on Text 1, *Ave* $[x_{v1}]$, and Text 2, *Ave* $[x_{v2}]$, when simplified exhibits the comparison of the two difficulty measures, with no reference to the person measure.

$$\frac{Ave\ [x_{v1}]}{Ave\ [x_{v2}]} = \frac{\frac{D_1}{B_v}}{\frac{D_2}{B_v}} = \frac{D_1}{B_v} \cdot \frac{B_v}{D_2} = \frac{D_1}{D_2}$$

Equation 3

This consequence, Equation 3, is important for the criterion required of measurement, the *independence of parameter sets*. This functional independence, exhibited in Equation 3, essentially means that the instrument used to measure is not itself affected by the objects being measured. In the Danish study the relative difficulty of the texts is independent of the particular student against whose average performances they are compared. The ratio of the average numbers of errors for the two texts is the same for every person regardless of the person's ability.

Likewise by comparing two persons on a text of the same difficulty, a similar result emerges. The simplified equation shows the ratio comparison of the average number of errors to be independent of the text measure. The relative number of errors of the two persons is independent of the difficulty of any text on which they are compared.

$$\frac{Ave\ [x_{1t}]}{Ave\ [x_{2t}]} = \frac{\frac{D_t}{B_1}}{\frac{D_t}{B_2}} = \frac{D_t}{B_1} \cdot \frac{B_2}{D_t} = \frac{B_2}{B_1}$$

Equation 4

This notion of parameter independence, that the two variables *person ability*, *B*, and *text difficulty*, *D*, are theoretically separate, makes sense intuitively. However the breakthrough by Rasch was to make this property explicit in a mathematical formula and to make connections to the then current statistical models. We note that these ideas also address a *test t*, which may generally comprise many words or items. Moreover, these independences carry over into tests comprised of individual items, which may even in their simplest form be dichotomous. As expressed by Andrich and Marais (2008):

> This equation and estimation can be generalised to comparisons of all items and so the relative difficulty of the tests can be obtained independently of the students (providing the persons are well-targeted to give information) (p. 1:4).

One consequence of the functional independence of parameters is that the inclusion or exclusion of any person and the inclusion or exclusion of any item never has any effect on the nature of the pervasive comparisons on the remaining persons or items respectively (Wright & Stone, 1999). Matching the ability level of students to difficulty level of items means that the information on relative difficulty level can be more accurately estimated, through judging empirically on student performance.

The mathematical model underpinning the Rasch model

The central proposition for the Rasch model[87] is that the response of a student to a dichotomous item, that is an item for which there are only two responses, one correct and the other incorrect, is a function of both the item difficulty and the person ability. The probability of a person achieving success on a particular item is entirely determined by the difference between the difficulty of the item and the student's ability. The principle is:

> (A) person having a greater ability than another person should have the greater probability of solving any item of the type in question, and similarly, one item being more difficult than another means that for any person the probability of solving the second item is the greater one (Rasch, 1960/80, p. 117).

The probability is high that a poorly performing person would answer only the easiest of questions correctly. The probability is also high that the strongly performing persons will be able to answer all the easy questions and in addition some of the more difficult questions correctly. Equivalently, the probability is high that a person of strong

[87] The discussion here will concern the dichotomous model. Extensions of the model have been derived from this model for partial credit scoring by Masters (1982), and rating scales by Andrich (1978).

performance makes errors mainly on the most difficult items. The total score obtained by a person on a test is therefore a function of person ability and item difficulty.

The starting point of the model is that the latent trait, or theoretical construct, may be conceived as a single dimension, or continuum. Along this dimension both items, in terms of their difficulty, and persons, in terms of their ability, may be located. Following the convention (used in Rasch, 1960/1980 and Andrich & Marais, 2006), item difficulty is denoted by D and person ability is denoted by B when referring to multiplicative parameters. The probability of attaining a correct answer to an item is governed by the difference between ability level of the person and difficulty level of the item.

Following Andrich (1988), but adapting the terminology to the present study, the process is explained as follows:

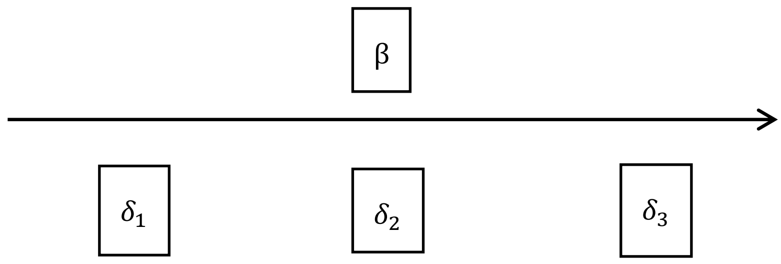
> Let the [proficiency or ability] of the person *v* be characterised by the variable B*v*, and the [difficulty level of the item i] be characterised by the variable D_i. The symbols B and D, respectively, represent the property of [proficiency or ability] pertaining to the person, and [difficulty value of the item] in general terms.
>
> If we introduce the dichotomous discrete variable *X*, which takes on the value $x = 0$, [for an incorrect response] and value $x = 1$, [for a correct response], then the observed response is governed (not determined) by B_v and D_i.
>
> We write that the value *x* will be observed with a particular probability which is a function of only B_v and D_i in the following way:
>
> $\Pr\{X = x\} = \varphi\ (B_v, D_i)$, where φ (phi) is some function and B_v and D_i are the parameters in the model. In order to estimate their values from observed data, it is necessary to specify the function φ (pp. 15-16).

From latent trait to measures

In the explanation that follows below (adapted from Andrich & Marais, 2006, pp. 60-63), symbols δ (delta) and β (beta) denote the log transformed ability and difficulty measures, $\beta = \ln B$, $\delta = \ln D$.

A diagrammatic representation of the latent trait, with the location of person ability at β, and items at δ_1, δ_2 and δ_3, respectively of low difficulty, moderate difficulty and high difficulty relative to the person is presented in Figure 5.1. Given a person of ability at β as measured on the latent trait, success on item δ_1 has a high probability (greater than $\frac{1}{2}$), as the ability is located above the difficulty of the item. Success on item δ_3 is less likely (less than $\frac{1}{2}$) as the difficulty of the item is located above the ability level of the person. Success on item δ_2 is expected to be 50%.

Figure 5.1: Person ability and item difficulty on a continuum



Adapted from Andrich & Marais, 2006

When a person v answers an item i which is scored dichotomously as right/wrong, the person's score on the item will be x_{vi}, in terms of the person's ability β_v and the item's difficulty δ_i. Because the outcome of an item of scholastic achievement cannot be predicted precisely, it makes sense to state this relationship of ability and difficulty as a probability of a correct response. It follows that if a person's ability is above the item's difficulty we would expect the probability of the person being correct to be greater than 0.50. Presented mathematically;

If $(\beta_v - \delta_i) > 0$ then $\mathrm{P}\{x_{vi} = 1\} > 0.5$

If a person's ability is below the item's difficulty we would expect the probability of a correct response to be less than 0.50. Presented mathematically;

If $(\beta_v - \delta_i) < 0$ then $\mathrm{P}\{x_{vi} = 1\} < 0.5$

In the case where the person's ability and the item's difficulty are at the same point on the scale we would expect that the probability of a correct response is 0.50. The equality relationship can be likened to the equal probability of heads or tails with the toss of a fair coin. Presented mathematically;

If $(\beta_v - \delta_i) = 0$ then $P\{x_{vi} = 1\} = 0.5$

These model assumptions (Andrich & Marais, 2006, p. 63) relate the probability of a correct response to the difference between the person's ability and the item's difficulty.

The probability of a correct response ranges from 0 to 1, and can be expressed

$$0 \leq P\{x_{vi} = 1\} \leq 1$$

Relation 5

The difference between the ability and the difficulty can range between $-\infty$ and $+\infty$, and can be expressed as

$$-\infty < (\beta_v - \delta_i) < +\infty$$

Relation 6

The transformation of the difference between ability and difficulty to an exponent of the base *e* is possible, because $y = e^x$ is a strictly increasing function.

$$e^{-\infty} < e^{(\beta_v - \delta_i)} < e^{+\infty}$$

Relation 7

This inequality relationship results in the expression

$$0 < e^{(\beta_v - \delta_i)} < +\infty$$

Relation 8

"With a further adjustment we can obtain the expression that has the limits of zero and one, and therefore could perhaps be the formula for the probability of a correct response" for a dichotomous item (Andrich & Marais, 2006, p. 63).

$$0 < \left\{\frac{e^{(\beta_v - \delta_i)}}{1 + e^{(\beta_v - \delta_i)}}\right\} < 1$$

Relation 9

For person *v* and item *i*, the probability of a correct response is therefore governed by

$$P\{x_{vi} = 1 | \beta_v, \delta_i\} = \frac{e^{(\beta_v - \delta_i)}}{1 + e^{(\beta_v - \delta_i)}}$$

Equation 10

We read the left hand side of the equation as "the probability of person *v* being correct on item *i* given the person's ability, β_v, and the item's difficulty, δ_i" (Andrich & Marais, 2006, p. 63).

The complementary probability of the same person being incorrect on any particular item of equal difficulty would be:

$$P\{x_{vi} = 0|\beta_v, \delta_i\} = 1 - P\{x_{vi} = 1|\beta_v, \delta_i\}$$
$$= 1 - \{\frac{e^{(\beta_v - \delta_i)}}{1 + e^{(\beta_v - \delta_i)}}\}$$
$$= \frac{1}{1 + e^{(\beta_v - \delta_i)}}$$

Equation 11

We express both Equation 10 and Equation 11 simultaneously as

$$P\{X = x_{vi}|\beta_v, \delta_i\} = \frac{e^{x_{vi}(\beta_v - \delta_i)}}{1 + e^{(\beta_v - \delta_i)}}$$

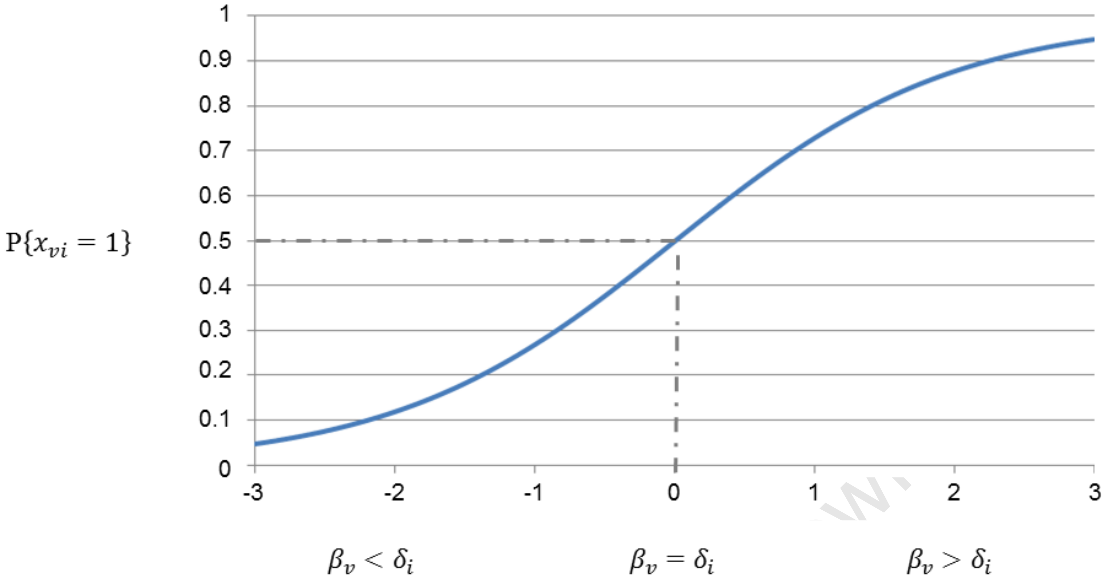
Equation 12

When x_{vi} = 1, the response is correct and the formula converts to Equation 10. When x_{vi} = 0, the formula converts to Equation 11.

The associated graph (Figure 5.2) depicts the (conditional) probability of a success on the vertical axis against the difference (β_v - δ_i) on the horizontal axis.

The item characteristic curve represents the probability that a person *v*, with given ability β_v, responds correctly to an item *i* with difficulty level δ_i. If ability level β_v is greater than the difficulty level δ_i, then the probability of the person answering the item correctly will be greater than 0.5, and if a person's ability is less than the difficulty level, then the probability of answering the item correctly is less than 0.5. For ability level β_v much greater than the difficulty level δ_i, the probability will tend to 1. For ability level β_v much less than the difficulty level δ_i, the probability of a correct response will tend to zero.

Figure 5.2: Probability of success given (β_v - δ_i) for persons of varying ability, β_v, on item with difficulty, δ_i



Estimating person ability and item difficulty

The Simple Logistic Model for dichotomous responses follows directly from the requirement for invariant comparisons (see Andrich, 1988, p. 44). The process of parameter estimation involving estimating item difficulty, termed *item calibration*, and estimating person abilities, termed *person measurement*, which together locates students and items on the same continuum, is explained in various sources, namely Wright and Stone, (1979), Andrich (1988), Andrich and Marais (2006) and Bond and Fox (2007). Andrich (1988, p. 44), lists three methods (conditional probability, joint estimation and pair-wise comparison), which each have strengths and limitations, and within a margin achieve a similar result. The process of item calibration involves empirically estimating the properties of items from the responses of a sample of persons (for details see Andrich, 1988, pp. 46-61). Likewise the process of person measurement is achieved through comparing the responses to items.

The presupposition before applying the Rasch model is that each item, and the test as a whole, has a contextual validity that satisfies the critique of the experts or professionals in the discipline concerned. Moreover the test must be suitably targeted for the range of abilities of any intended set(s) of learners. Given that set of conditions, and under the further requirements of the local independence of k items, and the independence of N

persons engaging with the test, as stipulated by Rasch, we obtain from collecting all the terms of Equation 12 into one product, the following:

$$\prod_{v=1}^{N}\prod_{i=1}^{k} P\{x_{vi} = X|\beta_v, \delta_i\} = \prod_{v=1}^{N}\prod_{i=1}^{k}\frac{e^{x_{vi}\,(\beta_v-\delta_i)}}{1+e^{(\beta_v-\delta_i)}} = \frac{e^{\sum x_{v\cdot}\,\beta_v}\; e^{-\sum x_{\cdot i}\,\delta_i}}{\prod\prod(1+e^{(\beta_v-\delta_i)})}$$

Equation 13

The import of Equation 13 is that the person v and item i locations will be determined by the corresponding total score $x_{v\cdot}$ on all items attained by the person v, $0 \le x_v \le k$, and by the sum $x_{\cdot i}$, $0 \le x_{\cdot i} \le N$, of all N person scores on the specific item i.

The manner in which those locations are derived for the measurement scale is complicated and not discussed here. This mathematical machinery of the Rasch model extends from dichotomous items to polytomous items (with suitable grading memoranda), and hence to composite scores accumulated from several subsets. Rasch methods involve a series of post hoc tests on the validity of the model assumptions, and adaptive strategies to ensure the necessary modifications for compliance with Rasch model assumptions.

Thus although Rasch originally worked on the reading ability and text difficulty issue using error counts, his later work gives rise to methods which handle ability and difficulty by counting scores based on ability totals.

5.4.4 Validity

Because of the requirement for unidimensionality of a measurement variable, and the requirement for establishing a line of enquiry that constitutes a continuum of hierarchical development, Rasch analyses incorporate tests of fit which assess "the degree to which persons and items fit our idea of an ideal unidimensional line" (Bond & Fox, 2007, p. 34).

Where an item does not function as expected, for example, if students with high scores on the test as a whole, tend to answer incorrectly on a specified item, this item may be highlighted. Essentially the test (of fit) highlights items where the observed scores deviate markedly from that pattern which is required by the model. A qualitative investigation of items by researchers and subject experts is expected to check whether the items are eliciting the expected construct, thereby supporting construct validity. The quantitative support for the qualitative investigation, which arises from fit statistics provided by the Rasch model, helps one determine whether, by the current version of the instrument, the researcher has completed a task of sufficient quality to claim that "item estimations may be held as meaningful quantitative summaries of the observations" (Bond & Fox, 2007, p. 35).

The distinctive feature of the Rasch model, namely that the data are required to fit the model, and that an absence of fit requires the attention of the researcher, has profound implications for research. As explained by Andrich;

> In the Rasch approach the data are required to subscribe to aspects of validity usually required by scientists (Duncan, 1984, p. 398), and in addition, to conform to the chosen model. That [validation] task needs to be carried out by researchers who understand the substance of the variables (Andrich, 1989, p. 16).

While the statistical analyses may indicate misfitting items, decisions regarding the elimination or adaptation of items are the domain of the researcher who understands the subject matter[88]. Andrich (1988) notes that

> items should not simply be discarded – they should be studied and hypotheses generated to explain why they do not accord with the model, as do other items. Discarding items simply on statistical criteria creates risks of capitalising on sampling errors of various kinds and thereby reduces the chances of a general application of the test (pp. 62 - 63).

The empirical information that is gained from administering test items "should be incorporated into the construction and modifications of other items" and contribute to further understanding of the variable or particular construct of interest (Andrich, 1988, p. 63).

5.4.5 Reliability

In addition to item fit indices, indicating whether or not a specific item conforms with measurement expectations, the Rasch model provides two indices, an *item reliability index* and a *person reliability index*, which enable one to determine whether there are

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88 The degree of fit of items in the test constructed for the purposes of this investigation will be discussed in relation to this study in Chapter 7, Section 7.2.5.3.

enough items spread sufficiently uniformly along the continuum and also whether there is enough spread of ability among persons, for the proposed measurement instrument to be adequately precise in the particular context to which it is applied.

The *person reliability index* is a measure of the extent to which, given the same students on a parallel set of items, measuring the same construct on the parallel test, these students would exhibit the same ordering of proficiency. Person reliability requires a large enough spread of person proficiency measures across the sample so that a hierarchy of construct development can be demonstrated, that is there is an appropriate range of item difficulties present to permit a robust cohesion of distinctions between the array of person proficiencies (Bond & Fox, 2007).

The *item reliability index* is a measure of the extent that the items would retain the same ordering of item difficulties, given a different but equivalent sample of students. High item reliability indicates that we have a developed line of inquiry in which there is an appropriate range of item difficulty (Bond & Fox, 2007).

5.4.6 Core ideas underpinning the Rasch model

The essential features of fundamental measurement are that of the unidimensionality, the invariance of measures, and additivity. If a theoretical investigation of a construct has resulted in a theoretical line of inquiry against which learners of greater proficiency and lesser proficiency can be ordered, and items of greater difficulty and lesser difficulty may be ordered, then the requirement for fundamental measurement may have been met.

The Rasch models "provide a theoretical base for the invariant comparison (of measures) within a given frame of reference" (Humphry, 2005, p. 3). In essence the construct being measured is defined, a probabilistic process is invoked to define natural units of measurement that are independent of both the construct and the persons being measured, and both student ability and item difficulty are located on a common continuum. In order to estimate locations on a common continuum that fit the criteria of invariance for a particular frame of reference and the observed patterns in the data, we apply Rasch analysis (Rasch, 1960/1980; Humphry & Andrich, 2008).

The particular strength of the class of models developed by Rasch, and subsequently by associates, enables “individual-centred statistical techniques” where each individual is characterised separately and where the “comparisons of individuals become (statistically) independent of which particular instrument” is used, and the comparisons of items are (statistically) independent of which particular individuals responded to the items (Rasch, 1960/1980, p. xx). This innovation in comparative measurement, extending a now familiar notion in the physical sciences to an analogue in educational testing, provides an answer to the severe criticism of applications of statistics in psychological research. The criticism arose because conventional statistics of the time could only provide distributional information on groups of individuals,[89] and hence no methods for reliable comparison of measurements of individuals.

The theoretical requirement from a research and measurement perspective for this study is to analyse mathematical situations in terms of the concepts and theorems required to solve problems, and from the cognitive development perspective to conduct an analysis of the developmental complexity of students’ responses (Vergnaud, 1979, p. 264). The conceptual fields approach, with the core focus on the multiplicative conceptual field, is from one perspective complex, in that it embraces situations (mathematical contexts), in addition to concepts and theorems, cognitive processes, and representational forms, and from another perspective clear-cut, since the multiplicative structures that underpin the many contexts, concepts and procedures can be described mathematically according to the variables, and the relationships between variables, present in the conceptual field.

Qualitative research into the separate components of the multiplicative conceptual field reveals the mathematical complexity of some constructs, for example the concept rate, and proportional reasoning. A review of the multiplicative conceptual field (in Chapter 6) explores concepts and interrelationships within this field. However, empirical verification of the construct, defined clearly and realised in a measurement instrument, may challenge current thinking about the construct, and enable further development of the field.

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[89] The details of this story may be read in the foreword by Benjamin Wright in Rasch (1960/1980).

5.5 Validity of assessment practices

In order to ensure validity in the assessment of mathematical proficiency, in particular to ensure that the inferences and actions from the outcomes may inform teaching and learning, a number of factors must be considered.

The first step in ensuring validity is to start with a clearly defined attribute (Wright & Stone, 1979). In this study this attribute is defined as proficiency within the multiplicative conceptual field. The next step in theory formation is to formulate how this attribute relates to other attributes. In this study the theoretical work involved establishing the links between the elements within the multiplicative conceptual field. Thus in order to achieve test validity, a great deal of attention is required in the test construction stage of the process. This emphasis is somewhat counter to current procedures where a great deal of attention is given to the post hoc analysis of the test data and the pursuit of correlations. Humphry (2011) echoes this concern, maintaining that theoretical work is required to connect the measurement of mental attributes and phenomena to substantive theories (p. 2). The direction of test construction is therefore from theory to the measurement instrument.

The process for the construction of a test with substantial validity requires that theoretical work in the field concerned is conducted in order to formulate the defining features of the construct. Secondly, the ideal circumstance would require a model which outlines the interrelationships between component parts of the theory. The next requirement, the third stage, is an analysis or prediction of response behaviour where the research is cognisant of the attribute variations that may produce corresponding variations in the measurement outcomes. And finally, the fourth stage requires the linking of the knowledge of variations of cognitive process responses to item responses.

The requirement for a theory of assessment tailored to mathematics (Webb, 1992) may be met in some respects by the Rasch measurement model and its requirement for clarity on the construct to be measured. Certainly the requirement that the construct of interest be understood provides support for the detailed investigation of specific mathematical topics.

The Rasch model provides a mechanism to obtain empirical verification that a coherent theoretical line of enquiry has been pursued and developed, one that demands rigour in definition of the construct, in the selection of items and in the construction of the test instrument. The outcomes of such a process are that the theoretical construct as initially defined is checked for unidimensionality[90] in its empirical manifestation through the test. Items which do not fit as expected are checked for anomalies. An item is problematic when students of high ability as measured by the entire test tend to answer incorrectly while students of low ability answer correctly. The expected functioning of students is also investigated: the profile of any anomalous students, who function in unexpected ways are examined,[91] with the purpose of gaining greater insight into the construct and learning of the construct, in the case of educational research.

The property of the Rasch model that locates both item difficulty and student ability as measurements on the same linear continuum enables the (post hoc) targeting of items at any specified difficulty level, in order to investigate the underlying concepts and theorems embedded in the item. It also enables the selecting of students at specified locations on the common scale, for further targeted research and enquiry.

The application of the Rasch model to the study of multiplicative conceptual fields is the focus of Chapter 7. Analysis of the multiplicative conceptual field follows in Chapter 6.

[90] The breadth of construct and the effect on the reliability and validity of measurements is discussed at length in Andrich (2006).

[91] Anomalous performances may of course be a function of instruction, or lack of competent instruction. These outcomes may also arise from traumatic situations affecting a student, or from students copying some or all test answers.

6 The multiplicative conceptual field

6.1 Mathematical structure and developmental consequences

In providing a framework for multiplicative structures, Greer (1992) draws on two themes: firstly on the range of situations that are modelled by *multiplication and division*, and secondly on the "long course of development of multiplicative concepts", in particular "the radical conceptual restructuring" that is required to extend the early notions of these concepts beyond the integer domain (Greer, 1992, p. 288).

These two themes resonate with the view of the learning of mathematics as the transforming of implicit and local intuitions that arise in response to one class of problems into explicit and generalised concepts that can be applied to many different problems (Vergnaud, 1990). The modelling cycle, identifying variables and relationships in problems across the range of situations that exist in the social and natural environment, and describing them mathematically, is critical to mathematical development in both the individual and the knowledge domain. The intrinsically developmental nature of mathematics has the consequence that concepts which arise in a limited domain such as natural numbers, are extended as a response both to encountering more complex problems, and to the application of formal analysis.

The purpose of this chapter is to render explicit the complexity of the interrelated elements within the multiplicative conceptual field, both from the perspective of engagement with problem situations, and from the perspective of a conceptual analysis of this mathematical domain. The operations, multiplication and division, are the base elements of this conceptual field. Fraction, ratio, rate, percent and probability are subtopics that have common mathematical structures which locate them in the domain of rational number, and hence the multiplicative conceptual field, but the subtopics may also be perceived as distinct in that these concepts have arisen in response to social and economic situations; their development has ensued through formal mathematical analysis.

The elements within the multiplicative conceptual field of concern in this study are, firstly, multiplication and division, the core concepts underpinning the constructs in this field. Inevitably, division of numbers for which the divisor is not a factor leads to the concept of a fraction. This move places multiplicative structures firmly in the rational number system, although as Skemp (1971) points out, the natural number system and the rational number system are in some respects isomorphic. This isomorphism makes some concepts transferable from one system to another but others demand conceptual restructuring on the part of the learner (Greer, 1992), which takes into account the "new" mathematical object, a rational number.

A concern in this thesis and echoed by Lamon (2007) is that "several mathematical constructs [in the rational number domain] are ill-defined, and their relationships to one another have become obfuscated as terminology has been used indiscriminately" (p. 633). The terms rational number and fraction are often used interchangeably as though synonymous (Usiskin, 2005; Lamon, 2007), and compounding the complexity, the terms fraction and ratio intersect in use. The definitions of ratios and rates have also been blurred in several ways. In addition the term proportional reasoning is sometimes inappropriately applied, where there is obviously no reasoning and no proportion involved but only the routine execution of a cross-multiplication rule (Lamon, 2007; Lesh, Post & Behr, 1988). Hence clarification of terms is necessary both for research and to support teachers in their understanding and teaching of this domain.

When discussing the entire teaching and learning domain together with the situations that make the associated mathematics accessible, the term *multiplicative conceptual field* is used; when dealing specifically with the mathematical domain the term *rational number* will be used. The composite term *rational number and proportional reasoning* will be used in some cases to signal instances where the complex interrelationship of these two mathematical ideas is apposite.

Drawing on research, elements in the domain of rational number are described by clarifying the mathematical definitions and terms of subconstructs, notably, *multiplication and division* (Section 6.2), and the composite subconstructs within *rational number*, including *proportional reasoning* (Section 6.3). Also in this section we clarify some conceptual distinctions such as *rational number and fraction*, *fraction and ratio*, and *ratio and rate*. From this base the specialised areas of *percent* (Section 6.4) and *probability* (Section 6.5) are discussed. We note that underpinning these terms and concepts are *multiplicative structures* which link these topics horizontally, but also link these concepts longitudinally to more advanced mathematical concepts, namely function and calculus. The discussion in the case of each subtopic focusses on definitions, on the interrelations between subtopics, on the problem situations in which these concepts are encountered, and on the core mathematical structure. In some cases historical insights may be invoked to illustrate a concept. In addition the early development of the concept may be discussed to provide insight into the educational context.

Algebra is arguably one of the greatest achievements in mathematics and the high point of mathematics education. The ultimate goal of school mathematics teaching, in the view of this thesis, is the attainment of proficiency with the tools of algebra for the greater proportion of South Africa's students. The position taken is that developing algebraic thinking is not separate from developing rational number sense or proportional reasoning, but builds on the foundation of the multiplicative conceptual field. A discussion of algebra, however, is beyond the scope of this thesis. In Section 6.6, the topic of proficiency in the multiplicative conceptual field, the foundation for algebra, is discussed.

Against this background, there is reference to the explicit statements in the National Curriculum Statement (NCS) concerning rational number as a topic in itself, and also reference to elements in the curriculum, notably multiplication and division, fractions, ratio, rate, percent, proportional reasoning and probability, concepts whose core is multiplicative structures. In this overview of the curriculum we note the critical elements to learning mathematics present in the curriculum, and suggest areas for which more elaboration is required (Section 6.7).

In summary, the implications for curriculum development, didactic decisions, ,assessment and research are discussed in Section 6.8.

6.1.1 Research questions

The research questions framing this chapter are in a sense similar to the questions framing each of the earlier chapters. Here however, the questions focus specifically on the demarcated domain of multiplicative conceptual fields.

Question 6 What critical factors need to be considered in the learning and teaching of elements within the multiplicative conceptual field?

6.1 What categories of problem situations provide opportunities for learners to develop critical multiplicative structures? How can we analyse the complexity of mathematical problem situations in the multiplicative domain and classify them hierarchically, but meaningfully? (Section 6.2)

6.2 1. What are the mathematical structures underpinning specific key concepts in the multiplicative conceptual field, namely rational number (and subconstructs) (Section 6.3), percent (Section 6.4) and probability (Section 6.5)?

2. What specific difficulties are encountered in this cluster of concepts? (Sections 6.3, 6.4, 6.5)

3. How may teaching of these subtopics be facilitated? (Sections 6.3, 6.4, 6.5)

4. What strategies may support teaching and learning? (Sections 6.3, 6.4, 6.5)

5. How does the historical development of particular constructs provide insight into the underlying concepts and cognitive development? (Sections 6.3, 6.4, 6.5)

6.3 How may rational number sense and proportional reasoning be facilitated? What particular threshold concepts require specific thought and attention on the part of teachers? What representations and symbols may assist the teaching and learning of the particular concepts? (Section 6.6)

6.4 How does the National Curriculum Statement reflect elements of the multiplicative conceptual field? Which elements identified in this study are present, and in which grades? (Section 6.7)

6.2 Multiplication and division

From a mathematical point of view, multiplication (and division) of positive integers and rational numbers may be considered relatively simple. However research reveals the psychological complexity that exists behind the mathematical simplicity (Greer, 1992, p. 276). The complexity can be explained from multiple perspectives: firstly multiplication and division are two-dimensional constructs where addition and subtraction easily admit only one dimension; secondly, the constructs are models for a wide variety of situations; and thirdly, the transition from multiplication and division of whole numbers to rational numbers requires cognitive restructuring (Greer, 1992; Vamvakoussi & Vosniadou, 2007).

An early conception of multiplication, described by Euclid (365–300 BC) as the operation of a "repetition number" on a cardinal number, is essentially the notion of repeated addition (Fowler, 1999, p. 14-16). This early conceptualisation of a mathematical concept is based on concrete activity where "the relationship between the aspects of reality being modelled and the mathematical objects used to model them is relatively direct" (Greer 1994, p. 68). This direct attachment of concept to concrete referent inevitably restricts more abstract conceptions. Historically the multiplication concept was extended from natural numbers to integers, rational numbers, real numbers and further, through both encountering practical problems and through formal theoretical analysis.

6.2.1 Problem situations

Children's earliest encounter with multiplication in the school curriculum is the situation in which there are a number of equal groups, with the same cardinal number in each group. For example,

Five children each have three cakes. How many cakes altogether?

In this problem, the number of children is *the multiplier* that operates on the number of cakes, *the multiplicand* is the cake, the implicit unit of interest. This situation can, depending on the position of the unknown, be categorised as a multiplication problem or as one of two division problems. The three possibilities are *multiplication, partitive division*, and *quotitive division* (see Table 6.1). The three variations differ in complexity.

Table 6.1: Multiplication and division

Class of problem	Multiplication problem	Partitive division (divide by multiplier)	Quotitive division (divide by multiplicand)
Equal groups	5 children each have 3 cakes. How many cakes altogether?	15 cakes are shared equally between 5 children. How many cakes does each child get?	If you have 15 cakes, to how many children can you give 3 cakes each?

At least two of the variations described in Table 6.1 can be applied to the eight additional situations,[92] expressed mathematically by $a \times b = c$ (see Table 6.2).

Table 6.2: Types of situation modelled by $a \times b = c$.

Situations modelled by multiplication and division	
Type	**Problem situation**
Equal groups (or repeated addition)	How many wheels are there on four bicycles? Or, more explicitly stated: 4 bicycles each have 2 wheels, how many wheels altogether?
Grouping	How many groups of 5 beads can be obtained from 15 beads?
Sharing	There are 15 apples to share between 3 children. How many apples does each child get?
Proportional sharing	Zanele collects 1 bottle for every 3 bottles Mishack collects. If Zanele collects 9, then how many bottles does Mishack collect?[1]
Multiplicative comparison	John picks 3 times as many apples as Mary. If Mary picks 4 apples, how many apples does John pick?
Rate	Allie drinks 3 cups of milk every day, how many cups will she drink in a week?
Rectangular area	Robert plants 3 rows of cabbages, with 5 cabbages in each row. How many cabbages does he have altogether?
Cartesian product	Anna has a blue, a red and a green shirt. She has a black and a white skirt. How many different combinations can she wear?

[1] See Item 1 in the test constructed for this study.

Additional complexity to the situations described in Table 6.2 may be introduced by changing the problem to a division problem. For example, in the rectangular area problem situation, the variation, *Robert plants 15 cabbages in three rows. How many cabbages are in each row?* changes the problem from multiplication to division. The first experiences of proportional reasoning, namely proportional sharing, are found in these situations, and extended through encountering varied situations.

6.2.2 Extension to rational numbers

With more complex problems, and inevitably with division where the divisor is not a factor of the dividend, the requirement is for parts of wholes, namely fractions. A further need for a number system that goes beyond the natural numbers arises in the context of measurement, where the numbers do not represent counts but measures, and

[92] See Greer (1992); Murray et al., (1993); Teachers' Guide for the Development of Learning Programmes, (DOE, 2003, p. 66).

where decimal notation is commonly used. This inevitable extension to rational numbers has a formal component in terms of an evolving number system and a modelling component in that a wider range of applications are possible.

The situations modelled by multiplication and division, and identified by Greer (1992), (*equal groups, equal measures, rate, measure conversion, multiplicative comparison, part-whole, multiplicative change, Cartesian product, rectangular area* and *product of measures),* are extended in the rational number domain (see Table 6.3) (Greer, 1992, pp. 277-280). The complexity of the problems is again determined by whether the problem situation is expressed as a multiplication, or in some cases as either of two division problems, *partitive* or *quotitive division*. An additional complexity is that the associated representations for the problem types may differ (see Greer, 1992, pp. 281-282).

The first seven categories, in Table 6.3, consist of "simple direct proportion between two measure spaces,[93] M_1 and M_2" (Vergnaud, 1983, p. 129) and are clustered by Vergnaud (1983, 1988) in the category *isomorphism of measures*.

Table 6.3: Situations modelled by multiplication and division

Multiplicative situations

Integers		**Rational numbers**		**Vergnaud (1988)**
Equal groups and proportional sharing	1 2	Proportional sharing Equal measures	1 2	Isomorphism of measures
Rate	3	Rate	3	
		Measure conversion	4	
Multiplicative comparison	5	Multiplicative comparison	5	
		Part-whole	6	
		Multiplicative change	7	
Rectangular array and rectangular area		Rectangular area	9	Product of measures
Cartesian product	8	Product of measures	10	
			11	Multiple proportion

adapted from Greer, 1992, p. 280

[93] The term measure space is taken from the work of Vergnaud and used to identify two sets of variables that are related both horizontally and vertically.

The other two distinct subtypes within multiplicative structures are defined as *product of measures*, consisting of the “Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3” (1983, p. 134), and *multiple proportion*, which has characteristics similar to the *product of measures*, in that “a measure-space M_3 is proportional to two independent measure-spaces M_1 and M_2” (Vergnaud 1983, p. 138). These multiplicative strategy subtypes provide common structures which serve the purpose of linking arithmetical concepts to more complex mathematical concepts.

6.2.3 Multiplicative structures

Each of the three subtypes within multiplicative structures, *isomorphism of measures*, *product of measures* and *multiple proportion*, has subclasses of theorems or theorems-in-action which are explained below. The purpose of explaining these subclasses is twofold, firstly the theoretical analysis leads to an understanding of how multiplicative structures are connected to functional relationships, and secondly this analysis is used in both the item analyses and the interviews conducted with selected learners about corresponding test problems.

Isomorphism of measures

The class *isomorphism of measures* is defined by the relationship of four quantities, one of which is missing. The two examples in Figure 6.1 exhibit the same multiplicative structure, though the mathematical context of the second example is more complex, for example per hour has to be interpreted as “in one hour” (see Vergnaud, 1988).

Figure 6.1: Representation of “isomorphism of measures”

Example 1 Anna buys 4 cars. Each car costs R5. How much will she pay?		**Example 2** Jim cycles 10.5 km per hour. How far will he travel in 3 hours?	
M_1 (cars)	**M_2 (cost)**	**M_1 (km)**	**M_2 (hours)**
1	5	10.5	1
4	x	x	3

Source: Vergnaud, 1988

Vergnaud (1983) and associate researchers, identify three subclasses, *multiplication, division (two types)* and the *rule of three* within the *isomorphism of measures* subtype.

The basic structure can be depicted as the direct proportion between two measure spaces. The term measure space has been used by others, notably Lamon (2007). Shield and Dole (2008) use the same representation but refer to measure fields or variables. The same representation is used by Olivier (1992). Other representations equally serve the purpose of making explicit the underlying mathematical structure, see Olivier (1992) and Shield and Dole (2008), amongst others. The measure spaces represent the variety of objects or magnitudes that may be compared.

In the categories that follow (see Figure 6.2), the same pattern is found, although the position of the unknown may change or be differently expressed.

Figure 6.2: Structure of "isomorphism of measures"

General case		Multiplication		Multiplication	
M_1	M_2	M_1	M_2	M_1	M_2
a	c	1	b	1	a
b	d	c	bc	b	x

The concept of direct proportion *within measure spaces*, that is *a is to c* and *b is to d* depicts the *general case* for multiplication and division problems. The direct proportion may also be *between measure spaces*, that is *a is to b* and *c is to d*. The unknown may be situated in either of the measure-spaces, and in any of the four positions.

The concept of direct proportion precedes the representation $\frac{a}{c} = \frac{b}{d}$, which precedes the notion that $\frac{a}{c} \times cd = \frac{b}{d} \times cd$, and therefore may result in the cross product $ad = bc$, and variations such as $a = \frac{bc}{d}$, or $d = \frac{bc}{a}$. The cross product solution is termed the *rule of three*, a mathematical procedure taught to merchants to calculate proportions in earlier times, and still taught today (Parker & Leinhardt, 1995). While this concept is efficient, it is open to erroneous procedures if not built on solid conceptual foundations.

The subclass multiplication includes two theorems, the *binary law of composition*, and the *unary operation* (see Figure 6.3). The *binary law of composition* involves identifying and extracting two variables that exist in a multiplicative relationship from the problem, $c \times b = x$, for example, *Malibongwe buys four rolls of fencing (c) at a cost of R300 (b) per roll*. The multiplicative structure can be expressed with M_1 being fencing, and M_2, the cost. The *unary operation* differs from the binary law of composition in that the *within measure-space* relationship or the *between measure-space* relationship is observed.

1: *If one roll costs R300 (b), then 1 × 4 (c) will cost R300 × 4.*

In this case × 4 (in the general case **×** ***c***) is the *scalar operator* and the operation is *within measure spaces*.

2: *One roll at 1 × R300 (a), therefore 4 rolls at 4 × R300.*

In this case × R300 (or in the general case × ***b***) is the *function operator* and the operation is *between measure spaces*.

Figure 6.3: Binary law of composition and unary operation

Rule of Three (general case)		**Binary law of composition**		**Unary operation**	
M_1	M_2	M_1	M_2	M_1	M_2
a	b	1	b		
c	d	c	x		

$$\begin{tikzcd} 1 \arrow[r, "\times b"] \arrow[d, bend right, "\times c"'] & b \arrow[d, bend left, "\times c"] \\ c \arrow[r, "\times b"'] & x \end{tikzcd}$$

The division subclass includes both *partitive division*, which comprises the notion of sharing, and *quotitive division* which comprises the notion of grouping. In partitive division the problem is to find the unit value. For example, *If four rolls of wire cost R1 200, what is the cost of 1 roll?* This problem is solved by applying the scalar operator /*c* to magnitude *d*.

Figure 6.4: Partitive division and quotitive division

Partitive division	**Quotitive division**
M_1 M_2	M_1 M_2

$$\begin{tikzcd} 1 \arrow[d, bend right, "/c"'] & x \arrow[d, bend left, "/c"] \\ c & d \end{tikzcd}
\qquad
\begin{tikzcd} 1 & b \arrow[l, "/b"'] \\ x & d \arrow[l, "/b"] \end{tikzcd}$$

In quotitive division the problem is to find x knowing d, $f(x)$, and b, $f(1)$. *Malibongwe has R1 200 (d) and each roll costs R300 (b). How many rolls can he buy?* This question is solved by inverting the direct function operator b and applying it to d. Note that the position of the unknown x is what distinguishes the two types of division problems.

The solution to *isomorphism of measure* problems may be obtained through determining the scalar operator or the function operator, in which case the operation is multiplication, or it may be to determine the inverse of the scalar operator or the function operator, in which case the operation is division.

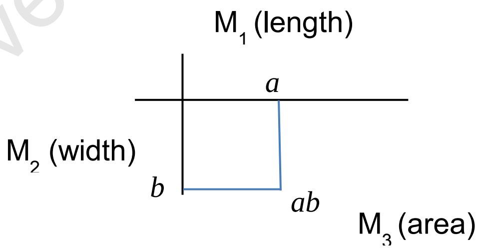
Product of measures

The *product of measures* comprises the Cartesian composition of two measure spaces, including the related division type question, as well as the double isomorphism subclass. The distinctive feature of the product of measures is the following;

1 unit (of length) × 1 unit (of length) = 1 unit (of area).

Example 1. *A vegetable garden has length 5 metres and width 4 metres. The area is 20 square metres*. This relationship can be expressed in the following diagram, where the length *a* units is multiplied by width, *b units*, and results in *area block* comprising *ab square units.*

Figure 6.5: Composition of measures, multiplication (*ab* is unknown)

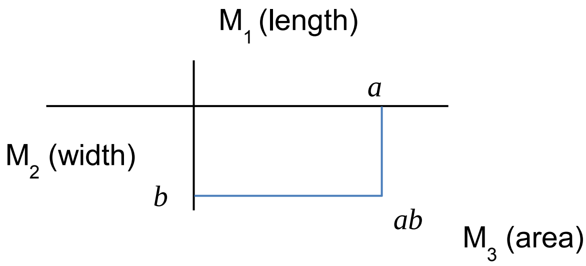


In this *composition of measures* problem, length is proportional to area when width is held constant and width is proportional to area when length is held constant. The width and the length are inversely proportional to each other when area is held constant.

Area (m^2) /width (m) = length (m)

The division variation of product of measures arises when one is given the product measure and one elementary measure, and the task is to find the value of the other elementary measure. For example: *The area of the vegetable garden is 20 square metres* *and the length is 5 metres. Calculate the width.* The dimension of the quantity to be found (the width) is the quotient of the dimensions of the product (the area) by the dimension of the other elementary measure (the length).

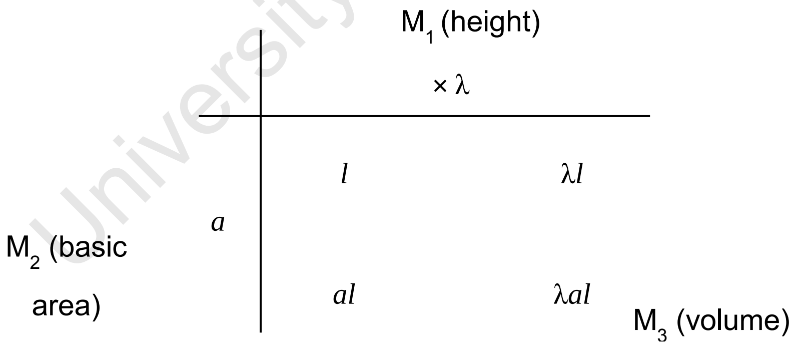
Figure 6.6: Composition of measures, division (either *a* or *b* is unknown)



Double isomorphism

The subclass *double isomorphism* (Vergnaud, 1983, p. 137) has the essential elements of the *product of measure* schema, but is overlaid by the proportional relationship created by multiplying one of the elementary measures. For example: *The base area of a container is 6 square metres, the height is 8 metres. If the height is multiplied by λ, the volume will proportionally be multiplied by λ.*

Figure 6.7: Double isomorphism



Multiple proportion

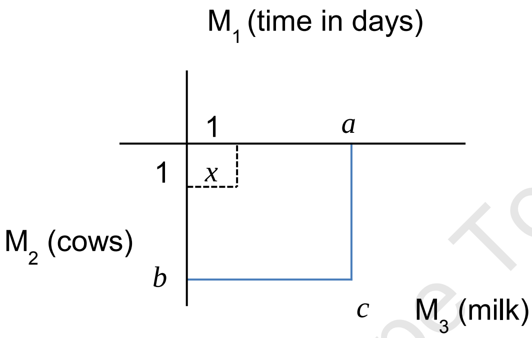
Multiple proportion arises when a variable, M_1, is proportional to two independent variables, M_2 and M_3. This relationship may require only multiplication, or it may require one or other division subclass. For example, *Six families each need 16 metres of fencing at cost R8 per metre.* The multiplicative relationship results in $a \times b \times c = x$.

Multiple proportion; first type division

A variation of the division relationship can be expressed as *partitive division*, where the task is to find the unit value. For example: *Find the average milk production per cow per day when 12 cows (b) produce 600 litres (c) over 5 days (a).* The division relationship results in $a \times b \times x = c$.

In Figure 6.8, *x is equal to the amount of milk produced by one cow in one day.*

Figure 6.8: Multiple proportion, finding the unit value

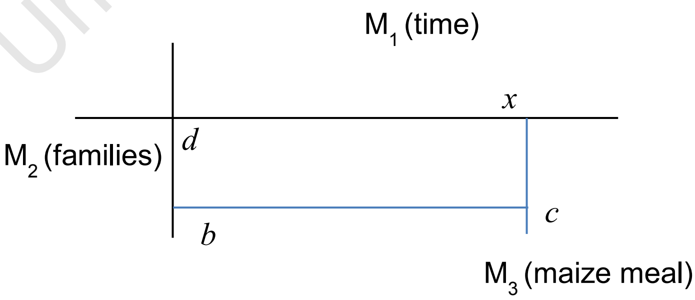


Multiple proportion; second type division

Second type division requires finding *x* given *f (x,a) = b and f (1,1)*. For example;

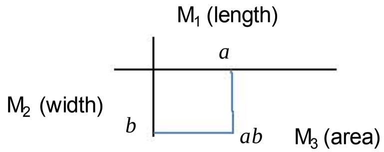
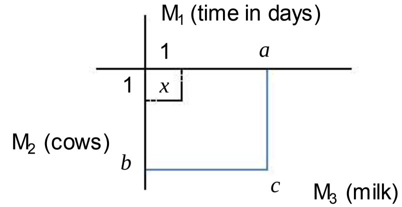
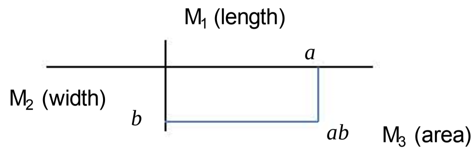
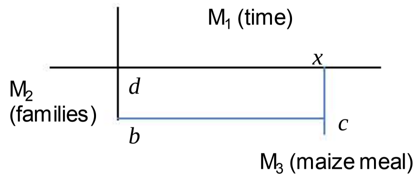
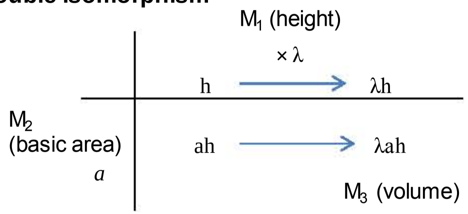
A cooperative has 550 kg of maize meal (c). A family consumes 2.5 kg per week (d). There are 22 families (b). How long will the meal last? This problem results in $b \times x \times d = c$.

Figure 6.9: Multiple proportion, second type division



A summary of the multiplicative subtypes identified in problem situations with related diagrams (see Vergnaud, 1983) is presented in Table 6.4. The diagrams may appear too abstract to be immediately comprehended. The salient point, however, is that both teachers and learners, by whatever means available, are required to identify the variables and relationships, within these problem types.

Table 6.4: Multiplicative structure subtypes

Isomorphism of measures (2 measure-spaces)	Product of measures (two measure-spaces (M_1 and M_2) in to a third (M_3))	Multiple proportion (M_1 proportional to two independent spaces, M_2 and M_3)
Multiplication: *Binary law of composition* $b \times c = x$ 4 rolls fencing at R300 each $\begin{array}{cc} \mathbf{M_1} & \mathbf{M_2} \\ \text{(fencing)} & \text{(price)} \\ 1 & b \\ c & x \end{array}$ *Unary operation* × c - scalar operator × d - function operator $\begin{array}{ccc} \mathbf{M_1} & & \mathbf{M_2} \\ 1 & \xrightarrow{\times b} & b \\ \times c \downarrow & & \downarrow \times c \\ c & \xrightarrow{\times b} & x \end{array}$	**Cartesian composition:** (1 unit of length) × (1 unit of length) = 1 unit of area  Length is proportional to area when width is held constant. Width is proportional to area when length is held constant. Width is inversely proportional to length when area is held constant.	**Multiplication:** $a \times b \times c = x$ Example: 6 families each need 10 metres of fencing at cost R8 per metre. **Division**: *Partive division* Find a unit value. Example: Average milk production when 12 cows (b) produce 600 litres (c) over 5 days (a). 
Division: *Partive division* Find the unit value Solved by applying scalar operator $/c$ to magnitude d $\begin{array}{cc} \mathbf{M_1} & \mathbf{M_2} \\ 1 & x \\ /c \downarrow & \downarrow /c \\ c & d \end{array}$ *Quotitive division* Find x knowing f(x), f(1) Solved by inverting the direct function operator and applying it to b $\begin{array}{ccc} \mathbf{M_1} & & \mathbf{M_2} \\ 1 & \xleftarrow{/b} & b \\ x & \xleftarrow{/b} & d \end{array}$	**Division:** Given the product measure and one elementary measure, find the value of the other  The dimension of the quantity to be found is the quotient of the dimensions of the product by the dimension of the other elementary measure. Area (m^2) /width (m) = length (m)	*Quotitive division* Finding x given f(x,a) = b and f(1,1) Example: A cooperative has 550 kg of maize meal (c). A family consumes 2.5 kg per week (d). There are 22 families (b). How long will the meal last? 
Rule of three: General case Multiplication and division problems are simple cases of the rule of three problem. $\begin{array}{cc} \mathbf{M_1} & \mathbf{M_2} \\ a & b \\ c & x \end{array}$	**Double isomorphism** 	

Diagrams such as these, or perhaps initially simpler diagrams may assist this process. The importance of describing problem situations in terms of measure-spaces allows the functional nature of the situations to be identified. The increase in difficulty of problem situations is achieved through introducing magnitudes of different types, through increasing the number range or introducing fractions, or decimal fractions.

6.2.4 Building the base for rational number

It has been noted in Section 6.2 (in part answer to **Question 6.1**) that there are at least ten situations which can be modelled using multiplication and division, by $a \times b = c$, but in addition there are variations on the problem situations that may be encountered.

The complexity of problem situations (in answer to **Question 6.2**) may be classified in terms of number domain, for example natural numbers or rational numbers, but in addition the underlying mathematical structure. The transition from working with natural numbers to working with rational numbers requires some conceptual restructuring.

6.3 Rational number[94]

The rational number is a logico-mathematical construct, which has a proportional nature, represented by the ratio of two integers, for which there are equivalent representations for the same number. It is a number, though it may represent a magnitude.

The main problem in mastering the multiplicative conceptual field, according to Vergnaud (1983), is that "rational numbers are numbers and that entities involved in multiplicative structures are not pure numbers but measures and relationships". And while rational numbers are well-defined as "an equivalence class of ordered pairs of whole numbers [integers]" (p. 160), the terms fraction and ratio are not clearly defined but applied loosely depending on the context, and refer sometimes to a concept and other times to the particular notation.

[94] See also Chapter 3, Section 3.3.5

Through a logical analysis of rational number, Kieren (1976) identified subconstructs of rational number, the most frequent references are *part-whole, measure, quotient, operator* and *ratio.* According to Freudenthal (1983) the subconstructs of rational number are the learning sites in which children gain insight into the essential nature of rational number.

6.3.1 Rational number sub constructs

Full descriptions of the subconstructs are provided by Kieren (1976)[95] and Sowder, Philippe, Armstrong and Schappelle (1998, pp. 8-11). Brief statements follow.

- Part-whole refers to the partitioning of a continuous quantity or a set of discrete objects into equal-sized parts. These parts can be compared, added or subtracted.
- Measure refers to the number assigned to some measurable quantity, specifying size in terms of units.
- A rational number $\frac{a}{b}$ may represent the quotient *a/b*; that is *a* and *b* are integers satisfying the equation $a = bx$, where x is the scalar operator.
- A rational number acting as an operator maps a set or region multiplicatively into another set or region (Kieren, 1980, cited in Sowder et al., 1998, p. 11).
- A ratio interpretation is invoked when two quantities are compared multiplicatively: a ratio can be expressed as an ordered pair (the ratio of boys to girls is 3:2, or $\frac{3}{2}$), or as a relation between a quantity and a unit (2 loaves of bread cost R10, bread is R5 per loaf). The second case, where the units of the comparison are different, is termed by some a rate.

Historically and in common usage a fraction means “a small part” or “a little piece”. The earliest conceptual hurdle to understanding fractions is the ability to divide or partition objects and quantities into equal sized parts. In this sense fractions are used for “a fractional part of a whole” (Vergnaud, 1983, p. 160), and satisfy the description of the *part-whole* construct (Kieren, 1976). Fractional numbers (of the part-whole type) have two parts; the denominator indicates the number of parts into which the whole has been divided; the numerator, the extent of the collection of parts of particular interest.

[95] Additional interpretations given by Kieren (1976) are equivalence classes of fractions, and decimal fractions which form a natural extension of whole numbers via the Hindu-Arabic numeration system.

An important conceptual threshold is that the fraction indicates the relationship between the collection of parts and the whole, and not simply the size or magnitude of the fragment. Fraction parts apply equivalently to wholes constituted by discrete and continuous quantities (see also Figure 6.12).

The second of Kieren's sub constructs, *fraction as measure***,** describes a magnitude that "cannot be expressed by a whole number of units" (Vergnaud, 1983, p. 160). This sub construct links directly to decimal fractions and to systems such as the metric system of measures.

To comprehend the third fraction subconstruct, *quotient*, requires linking the fraction to divided quantities. The quotient *a/b* represents the relationship between the total amount (*a*) and for example, the number of people sharing the amount (*b*), or the total amount and the share per person (*b*). The conceptual development required to make meaning is supported by linking quotients to divided quantities, through encountering problem situations in which this quotient relationship is embedded.

A *fractional operator* is the fourth sub construct identified by Kieren (1976). The subconstructs, fraction measure and fraction operator are distinct in a sense, though by eliminating magnitudes and dimensions may be regarded as the same rational number. A fractional operator such as $\frac{4}{5}$ acting on a magnitude 10, may be perceived as follows: 10 is divided by 5 into groups of 2. Two (2) is then multiplied by 4, to obtain 8. In similar mode, $\frac{4}{5}$ acting on 1 metre results in $\frac{4}{5}$ metre. The fractional operator $\frac{5}{2}$ on R160 is a fractional measure of the cost for 5 people that results from applying the fractional operator $\frac{5}{2}$ to the cost for two people (R160). The aforementioned subconstruct *operator* is conceptually more difficult than the fraction part-whole concept.

The *ratio* meaning, the fifth subconstruct, of a fraction is observed in an "ordered pair of symbols $\frac{p}{q}$" and "a relationship linking two magnitudes of the same kind" (Vergnaud, 1983, p. 160). While for these two situations ratio and fraction are both used, the term ratio (or rate) is mostly used for linking magnitudes of different kinds.

As noted by Vergnaud (1983), Usiskin (2005) and Lamon (2007), the terminology used in the rational number domain may be confusing. We therefore devote some attention to some confusing notions; namely the term fraction, the terms rational number and fraction, often used interchangeably; the terms fraction and ratio, often overlapping; and the distinction and commonality of the terms ratio and rate.

Lamon (2007) presents some multiple meanings that can be attributed to the *fraction symbol*, denoting the rational number, for example $\frac{3}{4}$ (see Table 6.5).

Table 6.5: Meanings attributed to the fraction symbol

Interpretations of $\frac{3}{4}$	Meaning	Classroom activities
Part-whole comparison "3 parts out of 4 equal parts"	$\frac{3}{4}$ means three parts out of four equal parts of the unit, with equivalent fractions found by thinking of the parts in terms of larger or smaller chunks $\frac{3}{4} = \frac{12}{16} = \frac{1.5}{2}$ (whole pies) (quarter pies) (pairs of pies)	Producing equivalent fractions and comparing fractions
Measure "3 ($\frac{1}{4}$-units)" – a unit count	$\frac{3}{4}$ means a distance of 3 ($\frac{1}{4}$-units) from 0 on the number line or 3 ($\frac{1}{4}$-units) of a given area	Reading meters and gauges
Operator "$\frac{3}{4}$ of something"	$\frac{3}{4}$ is a rule that tells us how to operate on a unit. Multiply by 3 and divide by 4, or divide by 4 and multiply the result by 3.	Area models for multiplication and division.
Quotient "3 divided by 4"	$\frac{3}{4}$ is the amount each person receives when 4 people share a 3-unit aggregate	Partitioning
Ratio "3 exemplars of A compared with 4 exemplars of A"	3:4 is a relationship in which 3 A's are in a multiplicative comparison to 4 A's rather than an additive sense.	Appropriate situations
Frequency rate "3 exemplars of A per 4 exemplars of B"	Ratio gets modified into rate and then the fraction has the complete units/ref.unit as additional information	Appropriate situations
Reduced fraction	The fraction $\frac{3}{4}$ is reduced from $\frac{6}{8}, \frac{9}{12}, \dots$	Simplifying fractions
Latent proportion	Moving to $\frac{6}{8}, \frac{9}{12}, \dots$	
Probability	An expected long-term frequency	

extended from Lamon, 2007, p. 654

These multiple meanings require investigation on the part of the novice mathematician. The first five meanings correspond to the rational number subconstructs identified by Kieren (1976). Others have been generated in this study.

Rational number and fraction

In many situations the term rational number and fraction are used interchangeably. In order to embrace the meaning of the term fraction as a particular notation *and* as a rational number construct, Lamon (2007, p. 635) offers the following definitions.

- Fractions are bipartite symbols, a specific form for writing numbers using two integers: the symbol, $\frac{a}{b}$, refers to a vertical notational system, a symbol involving two integers written with a horizontal bar between them.
- Fractions are rational numbers, of the form $\frac{a}{b}$; a and b may be positive or negative integers, with the restriction that $b \neq 0$.

Because fractions are introduced to children long before integers, a and b of the term $\frac{a}{b}$, are initially restricted to whole numbers, therefore *fractions* and the notion of *part-whole* have become synonymous for children and for some teachers. This understanding of fraction is a limiting factor and a construct that in later years requires conceptual restructuring (Lamon, 2007, p. 635). She therefore prefers to classify the term "fraction" as synonymous with the rational number subconstruct *measure,* but retains the use of the word "fraction" for the type of notation where two symbols are separated by a bar. Sowder et al. (1998) support the notion of fraction as measure, as it "provides a foundation for reconfiguration of a unit", for example reconfiguring three $\frac{1}{4}$ *units,* as one $\frac{3}{4}$ *unit* (Kieren, 1995, cited in Sowder et al., 1998).

The conceptual restructuring of fractions that follows the unfolding number systems is explained in Table 6.6. For current purposes the values of a/b are restricted to positive numbers. The danger of identifying fractions as only part whole and restricted to natural numbers is that this naïve definition stalls the development of more advanced understanding.

Table 6.6: Evolving concept of fraction

Fragments of 1 $\frac{a}{b}$, where $a < b$	Natural numbers Part-whole understanding
Greater than 1 (includes ratio concept) $0 < \frac{a}{b}$, where a and b are unrestricted rationals	Rational numbers, where a and b are both positive or both negative
Greater than 1 (includes irrationals) $0 < \frac{a}{b}$, where a and b are unrestricted reals	Real numbers, where a may be irrational for example, π.

Listed among the many different meanings attributed to the symbol ¾ (see Table 6.5) are the ratio and rate meanings. The similarities and differences between ratio and rate are discussed below.

Rates and ratios

A common distinction between ratios and rates that pervades thinking is that a ratio is a multiplicative comparison of same type quantities, whereas a rate is a multiplicative comparison of distinct type quantities. The Greeks restricted their use of the term ratio to magnitudes of the same type. In Euclid, Book V, Definition 3, (cited in Fowler, 1979, p. 812) it is stated;

> A ratio (logos) is a sort of relation in respect of size between two magnitudes of the same kind.

An alternative terminology is introduced in Euclid's Definition 6.

> Let [four] magnitudes which have the same ratio be called proportional (analogon)

Freudenthal (1983) distinguishes between "an internal or within ratio if its constituent magnitudes shared the same measure space" and "an external or between ratio [that is] composed of magnitudes from different measure spaces". This notion is linked to Vergnaud's *isomorphism of measures*, where a proportion problem can be conceived as a *within measure space* problem or a *between measure space* problem. For example the problem in Figure 6.10 can be solved using between measure or within measure space relationships.

Figure 6.10: Ratio and functional relationships in proportional reasoning problems

Jim cycles 10 km in 2 hours. How far will he travel in 3 hours?

M_1 (kilometres)	M_2 (hours)
10	2
x	3

A *within measure* calculation would be "2 (hours) is to 3 (hours), therefore 10 (kilometres) is to *x* kilometres". The constant coefficient would be a scalar number, 3/2 or 1.5. A *between measure* space calculation would be "2 (hours) is to 10 (kilometres), therefore 3 (hours) is to *x* (kilometres)", the operator from M_1 to M_2 being ÷ 5, or × 1/5. This relationship may be described as a functional relationship, $f(t) = kt$, where $f(t)$ is the distance corresponding to the time t and k is the *constant of proportionality.*

Since $f(2) = 10 = 5 \times 2$, the constant k satisfies $k = 5$

So that $f(3) = 5 \times 3 = 15$

In order to establish the constant of proportionality one has to find the unit rate, in this case 5 kilometres per unit hour.

A *within measure* space calculation involves finding a scalar operator. In order to find the scalar operator, the multiplicative relationship between 2 and 3 has to be found. Since 3 is one and a half times 2, or $\frac{3}{2} \times 2$, the scalar operator, the constant k, is $\frac{3}{2}$. Thus $10 \times \frac{3}{2} = 15$.

This problem can therefore be approached either as a *between measure* problem, using equal rates, or as a *within measure* problem, using equal ratios.

Following Vergnaud's line of thinking allows that the concept of ratio or rate is not so much in the problem as in the means of solving the proportionality problem. Thompson (1994), similarly suggests that the difference between ratio and rate is in the mental construction rather than the problem situation. He therefore defines a ratio as the "result of comparing two quantities multiplicatively" (p. 190). A ratio becomes a rate "when one reconceives the situation as being that the ratio applies generally outside the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate (i.e. reflected it to a mental operation)" (p. 192). For Thompson, a rate is therefore "a reflectively abstracted constant ratio" (p. 192). Thompson locates ratio and rate strictly in the approaches taken to a proportionality problem meaning that the distinction is not so much in the problem as in the approach to the problem.

Schwartz (1988) distinguishes two types of mathematical quantities, those of an extensive quantity, such as counts, measures and values, and those of an intensive quantity where the resulting quantity comprises a relation between two quantities. Intensive quantities can be of two types, internal ratios which relate to quantities of the same type (people per people, or litre per litre), and external ratios which relate quantities of different types (kilometres per hour, or litres per hour). Internal ratios are scalars and have no units, (the units cancel out in a dimensional analysis); external ratios maintain their units. This terminology corresponds to the *within measures* (internal ratios) and *between measures* (external ratios).

In proportion and comparative problems, there are two categories of ratio problems, the first invokes one quantity as a part of the other quantity (inclusive case) and the second case occurs when there is "no obvious inclusion relationship (exclusive case)" (Vergnaud, 1983 p. 163). Important differences between inclusive and exclusive fractions are given in Table 6.7.

Table 6.7: Inclusive and exclusive fractions and ratios

Inclusive fractions or ratios	**Exclusive fractions or ratios**
Fractions or ratios are smaller than one This statement is true, except when the set is compared to itself (when it is one whole). For example, 3/25 of the people at the play were children.	Fractions or ratios greater than, less than or equal to 1 *The shop increased its prices by 20%. The price is now 120/100 of the original price.*
Fractions are not reducible to whole numbers, except in when the combined parts are equal to one.	Exclusive fractions can be either reducible to a whole number, as in the case of 6/2 or non-reducible, as in the case of 22/7.
Inclusive fractions have no inverse for the mathematical initiate because it is somewhat meaningless to consider the whole as a fraction of its parts.	There is a "natural" inverse, *The highway takes 4/5 of the time travelled on the country road. Inversely, the country road takes 5/4 of the time traveling on the highway.*
Inclusive fractions may be made meaningful either by sharing operations or more generally by subset-set proportions.	Exclusive fractions involve comparisons, such as set-set comparisons.

Inclusive examples (taken from test in this study) are as follows; *Four-fifths of the teacher's books are novels* (Item 16). *Three-fifths (or 60%) of the class are girls* (Item 26). An example of an exclusive ratio relationship may be *The distance covered in 25 minutes is 5/4 of the distance covered in 20 minutes* (Item 7). *The price R800 increased by 20%* (Item 8).

A problem identified by Vergnaud (1983, p. 164) is that the inclusive model is usually the first model presented to learners. The comparisons and ratios between any two quantities of the same kind are a more powerful model than inclusive fractions, providing a more general foundation for scalar operators and ratios, and therefore should be presented concurrently in the early years of learning mathematics.

6.3.2 Operations on fractions

Fractions use the same notation as natural numbers but the concept of a fraction (notably that it embodies a relationship between the numerator and the denominator), and the procedures involved with the notation, are different.

Fraction addition means combining quantities, though it is fraction parts being added. Similarly fraction parts can be subtracted. The multiplication and division of fractions in general refers to the scalar operator subconstruct. Multiplying fractions and dividing fractions appear to be the same operation as multiplying whole numbers and in fact use the same symbol, but the procedures in some conceptions are different.

Multiplicative structures applied to fraction measures

The analysis applied to multiplicative structures, also applies to fractions and ratio. The fraction notion, sharing a whole into parts involves "direct proportion between the shares and the magnitude to be shared (isomorphism of measures)" (Vergnaud, 1983, p. 161). As stated previously, the magnitude concerned can be either discrete or continuous. In the case of a discrete magnitude the elements can be counted, but in the case of a continuous magnitude the measure, for example area or mass, may not be known, and therefore must be expressed as a fractional quantity. The relationship between fractions and measure spaces is shown in Figure 6.11.

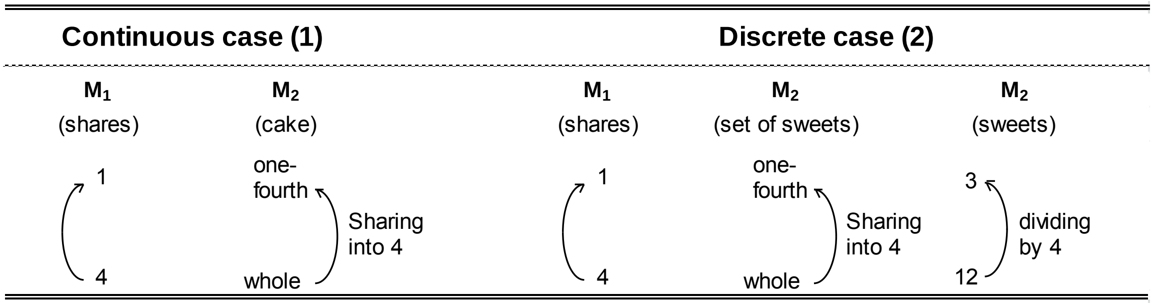
Example 1 (continuous case): *A cake is shared among four children. How much does each one get?*

Example 2 (discrete case): *Twelve sweets are shared between four children. How many sweets does each child get?*

Partitive division can be recognised in the discrete case of fractions, but as Vergnaud (1983, p. 161) notes, this structure is not easily recognised in the case of continuous magnitudes. For example, *How many 1/5s are there in six wholes*?

A scalar operator links two quantities of the same kind, and being the quotient of two quantities of the same dimension and expressed in the same unit, it has no dimension and no unit, as in 1 (share) to 4 (shares). This comparison is multiplicative rather than additive and is the shift that students must make. A function operator links two quantities of different kinds, for example ¼ cake per 1 person.

Figure 6.11: Measure spaces applied to fractions



6.3.3 Synthesis of rational number

The meaning of concepts and theorems come from working with magnitudes and quantities. Vergnaud (1983) considers three elements necessary for the synthesis of rational number, the conceptualisation of *fractional addition and subtraction*, of the interconnected nature of *fractional multiplication and division*, and of the *infinite character of each number class*. The "meaning of addition and subtraction [of rational numbers] comes from [working with] fractional quantities"; the meaning of the interconnected nature of multiplication and division (concatenation) comes from [working with] "fractional scalar operators and ratios"; and the conceptualisation of the "infinite character of each rational-number class comes from fractional function operators and ratios" (Vergnaud, 1983, p. 165). In summary;

> (t)he synthesis of all three aspects [of rational number, addition, multiplication and the equivalence relation] can occur only if measures, scalar operators and function operators

> lose their dimensional aspects and [lose] the distinction between element and relationship, and if the concept of rational numbers as pure numbers is built up (p. 165).

Some similarities may be observed between Vergnaud's analysis and that of Kieren (1976). The insight provided here is making clear the differences between quantities and operators, and the distinction between scalar and functional operators. In essence the analyses, from both Vergnaud and Kieren, describe the complexity involved in synthesising the predicative form of rational number.

Having made these distinctions we may conclude the following:

- All rational numbers may be written in fraction form, using the conventions of fraction notation.
- Not all numbers written in fraction form are necessarily rational numbers, for example $\frac{\pi}{2}$.
- Rational numbers may be written in fraction form but they may also be written in other forms, namely a decimal form or a percent form, adhering to the conventions of the notational system.

We may now contrast a standard definition for rational numbers.

- Terminating decimals are rational numbers.
- Non-terminating, repeating decimals are rational.
- Non-terminating and non-repeating decimals are not rational.

The definition represents the distillation of the operational experiences into the predicative form, but on its own does not elicit the myriad forms and meanings of the concept rational number. A grounded understanding of the notion of rational (and irrational) numbers is necessary to make sense of this definition.

Rational number sense, regarded as an essential benchmark along the mathematical path towards algebra and calculus, can be described as having an intuitive feel for the relative sizes of rational numbers, as well as having the ability to estimate, and to think qualitatively and multiplicatively. The added ability of moving "flexibly between interpretations and representations", to make decisions, to reflect on the sense making of one's decisions and therefore to make reasonable judgements, are the qualities required for solving proportion problems (Lamon, 2007, p. 636).

6.3.4 Proportional reasoning

The topic of proportional reasoning is of importance in this study because it is the development of true or complete proportional reasoning that is required to master more demanding mathematics and is essential for modelling complex constructs. The concept of proportional reasoning is used in many senses and coupled indiscriminately with ratio, with rational number, or assumed to be the cognitive equivalent of the proportion concept (Lamon, 2007).

According to Lesh et al. (1988), the "evolution of proportional reasoning is characterised by a gradual increase in local competence" (p. 103). Proportionality is initially mastered in small and restricted classes of problem settings; competence is then gradually extended to larger classes of problems. Piaget has been challenged for his stage theory, in particular for his assertions that proportional reasoning is an advanced stage of reasoning only attainable in adolescent years. Carraher (1996), for example, has noted that proportional reasoning when linked to magnitudes is evident in young children.

The important stages of developing proportional reasoning, have been described by Piaget, Karplus, Hart and others (cited in Lesh et al. (1988, pp. 104-105). These stages are described in Table 6.8.

Table 6.8: Stages of proportional reasoning

Stage 1	Children in this first stage do not understand the problem and therefore ignore part of the data.
Stage 2	Students notice the relationships but are able to relate them only in a qualitative manner. For example, that 3>2 is associated with x>10.
Stage 3	Early attempts at quantifying relationships, involve constant additive differences rather than multiplicative relationships. For example, 3 = 2 + 1, so that they infer x = 10 + 1.
Stage 4	The earliest use of multiplicative reasoning is based on a sort of "pattern recognition and replication" strategy, which some have called a "build-up" strategy (Hart, 1984).
Stage 5	Piaget's "logical proportions" indicate a level of thought in which a specific multiplicative relationship is noticed between two terms; this relationship is then applied to the other two terms.

According to Piaget et al. (cited in Lesh et al., 1988) the first stage is pre-proportional because "children have the intuition that the differences change with the size of the numbers and that the change may be multiplicative in nature, but they do not necessarily realize that they need to consider the constantly increasing differences between the related terms of each rate pair, that is, of each ratio" (Lesh et al., 1988, p. 105). According to Piaget, pre-proportionality is brought about by a coordinating function, whereas proportionality is based on reversible operations.

The development of adolescents' proportional reasoning can therefore be summarised as transforming from a "global compensatory strategy (often additive in nature)" initially, to a multiplicative strategy, but without generalisation to all categories, and eventually to the "final formulation of a law of proportions" (Piaget, 1968, cited in Lesh et al., 1988, p. 105).

Lamon (2007) concurs that proportional reasoning is a long term developmental process. She proposes that proportional reasoning is "a consequence of understanding rational number". Traditionally, the term proportional reasoning has been tied to two types of problem, "comparison" and "missing value" which have collectively been located under the umbrella term "ratio and proportion". This restriction to two problem types is perceived as a limited conceptualisation by both Lesh et al. (1988) and Lamon (2007).

Olivier (1992), (drawing on the work of both Tourniaire, 1986, and Van den Brink & Streefland, 1979), advises that children should be introduced to situations requiring proportional thinking early, initially by handling situations intuitively and informally. Progressive development of these skills, focusing on the reasoning processes, should precede the formal introduction of notation and algorithms.

From the ratio relationship, $\frac{a}{b} = \frac{c}{d}$, for example, *a kilometres is to b kilometres*, and *c hours to d hours*, will covary; the ratio *a is to b*, and *c* is to *d*, will remain invariant. In a rate relationship *a kilometres to c hours* and *b kilometres to d hours* will covary, the rate *a/c* and *b/d* will remain invariant. In Figure 6.12, *10 kilometres to 15 kilometres and 2 hours to 3 hours* will covary, the ratio 10/15 (2/3) remains invariant. Similarly, *10 kilometres to 2 hours*, and *15 kilometres to 3 hours* will covary, the rate 5 remains constant.

Figure 6.12: Scalar and functional relationships in proportional problems

Jim cycles 10 km in 2 hours. He travels 15 kilometres in 3 hours.

Measure space 1	Measure space 2
Kilometres	**Hour**
10	2
15	3

The following explanation is provided by Lamon (2007).

> (P)roportional reasoning means supplying reasons in support of claims made about the structural relationships among four quantities, (say *a*, *b*, *c*, *d*), in a context simultaneously involving covariance of quantities and invariance of ratios or products: this [reasoning] would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities (p. 638).

The cognitive requirement which follows is to recognise "the constant ratio between elements of the same measure space", for example kilometres with kilometres, and the "functional relationship between measure spaces", for example kilometres per hour, as shown in Figure 6.12.[96]

The construct proportionality is according to Lamon (2007) far more demanding than rational number sense or proportional reasoning. The ability to use proportionality to model the complex relationships of real world situations "develops only as one studies higher mathematics and science" (p. 640). The larger concept of proportionality is developed through "interaction with mathematical and scientific systems that involve the invariance of a ratio or a product" (p. 640).

Proportionality is by definition an equality of ratios, referring to the underlying structure, in which a special invariant (constant) relationship exists between two covarying quantities (quantities that are inherently linked in a context and are changing together). The mathematical model is a linear function of the form $y = kx$, where k is a constant of proportionality, y is a constant multiple of x. Proportionality and rational number sense are interrelated (Lamon, 2007, p. 638).

[96] Vergnaud notes that diagrams depicting the relationship of elements in a measure space are efficient in that they "forget" non-essential features of the situation and concentrate upon the relevant elements and relationships (1988, p. 148).

The particular challenge to be addressed from an educational perspective is that any "constant of proportionality" is context specific and changes representation in the many situations in which it appears too implicitly to be readily identified, but rather as "a structural element" that has to be discovered (Lamon, 2007, p. 638). Among its many guises the following, drawing from Lamon (2007, pp. 638-639) are listed;

- In symbols, it is a constant, for example in $y = kx$
- In a linear graph, it is the slope, $y = mx$
- In a multiplicative matrix table, in a particular row or column, it is the difference between an entry and the one before it for example,

2	4	6	8
3	6	9	12
4	8	12	16

- In rate situations, it is a constant rate.
- In ratio situations it is hidden and implicit in an equality for example, in the equality 2:5 and 4:10, the hidden constant, or scalar operator is 2.5.
- In reading maps, it is a scale factor.
- In determining sales tax, it is a percentage.
- In rolling dice, it is a theoretical probability.

Though the situations and contexts are numerous, the similar underlying mathematical structure may be recognised by the proficient student, while the less proficient student may be equally confused by each problem. The necessity is for "conceptual coordination of all the contributing domains", namely the rational number subconstructs as contexts for discovery and application (Lamon, 1994, p. 90). Following Vergnaud, the obvious recommendation is to introduce, early in the mathematics education of children, a variety of situations in their simpler formulations, which both illuminate a construct, for example rate, and then progressively extend understanding to a more abstract conception. This transformation to a more abstract conception may be assisted by attention to the common underlying mathematical structures.

Lamon (1994) describes this domain, the understanding of rational number concepts and the development of proportional reasoning as "a critical juncture at which many types of mathematical knowledge are called into play and a point beyond which a student's understanding in the mathematical sciences will be greatly hampered" without proper mastery (p. 90). This notion resonates with Lesh et al. (1988), who propose that proportional reasoning is the “capstone of children’s elementary school arithmetic” and the “cornerstone of all that is to follow” (p. 94).

6.3.5 Functional relationship and link to calculus

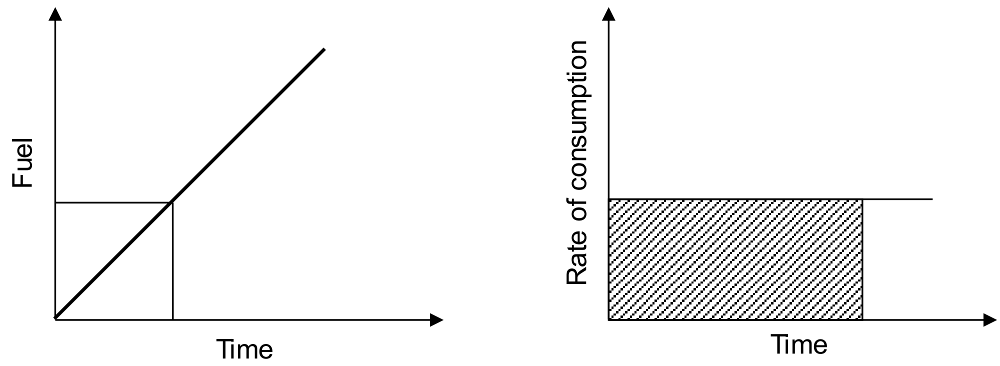
Carraher (1996) advocates placing fractions, ratios and proportions in an algebraic context and establishing clear links from these concepts to functions even in early years. He notes that even when working with a physical model of fractions, such as half an apple, the fractional numeral, 1/2 , is actually shorthand for the complete functional description which is A = 1/2 B, or $y = 1/2\ x$. The parameter numeral, 1/2, is a constant of proportionality.

Vergnaud (1988) points out that the underlying mathematical structure for many classes of multiplication problems is a functional relationship. Schwartz (1988, cited in Greer, 1992, p. 284) alerts us to the fact that “the graphical representations of the relationships between the three quantities of interest are in fact representations of the essential ideas of the differential and integral calculus respectively”, namely slopes and areas under curves (see Figure 6.13).

Figure 6.13: Links between multiplication, division and calculus

A machine uses 2.4 litres of gasoline for every 30 hours of operation. How many litres will the machine use in 100 hours?

Amount (in litres) = 2.4/30 × time (in hours) f(t) = kt f (30)= 2.4/30×100



Source: Schwartz, 1988, cited in Greer, 1992

Olivier (1992) notes that “(t)he concept of ratio only becomes useful when it is used to describe the functional relationship between the elements of two sets (variables)” (1992, p. 305). Defining ratio therefore as “as a property belonging to two sets … a special kind of relationship between two sets, namely a relationship that is described by formulae of the type $y = kx$" highlights the functional relationship. The term proportional reasoning then "refers to reasoning in a system of two variables between which there exists a linear functional relationship of the type $y = kx$" (Olivier, 1992, p. 305).

6.3.6 Considering salient features

As with the topic "multiplication and division" there are many situations to which rational number subconstructs apply. These situations provide the learning sites in which children gain insight into rational number. The challenge of engaging with the many different subconstructs and different representations, fractions, ratios, decimals etc., may be counterbalanced by attention to the underlying mathematical structure, and therefore grouping together classes of problems (in part answer to **Question 6.2.1**).

Kieren (1976) warns teachers and researchers that failing to take into account the "salient features of each interpretation [of rational number]" they may have "encountered difficulties that could logically have been anticipated" (p. 103). The specific difficulties encountered may be summarised as the challenge of recognising the different interpretations, the variables involved and the relationships between variables (**Question 6.2.2**).

The solution proposed for learning and teaching by Vergnaud (**Question 6.2.3**) is that, from the learner's perspective, operational invariants are expressed and explored in many different situations. From the teacher's perspective the concepts and theorems underlying learners' schema are rendered explicit. Kieren (1976) advocates investigating similarities and differences of different rational number interpretations, and thereby having more chance of generalisation.

Kieren (1976), like Vergnaud (1983), notes that the different interpretations, or subconstructs are interdependent, and "with appropriately defined operations and relations" are isomorphic. The diagrammatic representation in the form of measure spaces, or other representations may assist in transforming intuitive and local understandings, built on working with measures and quantities and various subconstructs, into the mathematical concept, rational number (**Question 6.2.4**).

6.4 Percent

Parker and Leinhardt (1995), in a review of research from the previous 60 years, conclude that students have displayed "inadequate performance and in some cases utter confusion" when confronted with percent problem tasks (Parker & Leinhardt, 1995, p. 422). Numerous research studies have investigated the problem and proposed instructional methodologies that have had little impact. Parker and Leinhardt therefore propose that the problem with percent arises partly from the complexity of the concept itself. It includes all the complexity of rational number, namely that there is a formulation pertaining to percent for each of the rational number subconstructs, part-whole, ratio, operator, measure and quotient (Behr, Harel, Post & Lesh, 1992), which intersect and have "multiple interpretations depending on the context" (Parker & Leinhardt, 1995, p. 435).

The roots of the concept lie in both formal mathematics that can be traced to the Greeks, in that percent is a proportion, and in the market place, where the establishment of abacus schools can be traced to Italy in the 13th century (Parker & Leinhardt, 1995). However it was in the 16th and 17th centuries at the height of the Renaissance, that mathematics, in particular measurement, evolved to solve problems that arose in relation to trade. Meticulous calculations with weights and measures were required in exchanges, either in the interest of fairness or of favouring one party! (Radford, 2003, p. 28).

The burgeoning of trade required mathematical services in the form of abacus teachers. Abacus schools were set up to train teachers. Texts by the abacus masters included collections of problems, and methods of dealing with each particular problem. However, it appears that in the 16th Century trade and industry, mathematics or at least arithmetic was essentially a useful tool required to solve its quantitative problems. Some of the abacus masters became noted mathematicians, and some order and structure was sought for the many problems. But even in the 19th century and early 20th century arithmetic texts within the United States, the usefulness of percent, rather that the underlying mathematical structure, was foregrounded. There were as many as 20 separate percent-related topics each with its own explanation of rules and solution method featured in these texts (Steiner, 1946, cited in Parker & Leinhardt, 1995). The aspect of mathematics that responds to the needs of the marketplace was clearly being exemplified. This theme of functionality rather than meaning continues in the classroom today.

Progress in the formal analysis was nevertheless being made: the set of topics that arose directly from applications eventually became subsumed under the general mathematical construct percent.

6.4.1 Mathematical Structure

The percent construct has core similarities with rational number, and as such embodies the rational number subconstructs identified by Kieren (1976), such as part-whole, fraction measure, quotient, ratio and operator. In addition the percent construct has the definitive features of a specialised language.

A common practice is to define percent as a quantity out of 100. In the view of Parker and Leinhardt (1995) this oversimplified definitional approach is problematic in that it confines students to a part-whole conception of percent. Van Engen (1960) distinguished between a fraction (a measure of a fragment that can be added to other fragments), and a rate pair. Using this distinction, he placed percent firmly in the category of a rate pair. As Nelson (1969) pointed out, however, a percent has both a *fraction measure* meaning, as represented by a *subset–set comparison* model (for example, 25% of a group of people), and a ratio meaning as represented by a *set-disjoint set comparison* model (for example, a specified quantity is 25% of another quantity). The *subset-set comparison* aligns with the term inclusive fraction, or ratio, while the *set-set comparison* aligns with the exclusive fraction or ratio (Vergnaud, 1983, p. 163).

The two types of mathematical quantities, distinguished by Schwartz (1988), those of an extensive quantity, such as counts, measures and values, and those of an intensive quantity where the resulting quantity comprises a relation between two quantities, also apply to percent type problems. Intensive quantities can be of two types, internal ratios which relate to quantities of the same type (people per people), and are therefore scalar quantities, and external ratios which relate quantities of different types (for example kilometres per hour) and therefore maintain their composite units, relate to different conceptions of percent. This terminology corresponds to the *within measures* (internal ratios) and *between measures* (external ratios) (see Section 6.3.1).

Usiskin and Bell (1983, cited in Parker & Leinhardt, 1995, p. 437) distinguish six categories to which they assign number status: *counts, measures, ratio comparisons, constants, locations* and *codes*; percent was given the status of a *ratio comparison* number. A *comparative index* was the term given by Behr, Harel, Post and Silver (1983). Having a common base of 100 means percent is a standardised expression of a ratio that can be ordered linearly, and so easily compared. This attribute of ordering allows comparisons, but can also be confusing when the referent for the percent is not made explicit. The necessity in every comparison is to know the reference counts or quantities, for example 70% of learners may have passed matric in 2010 as opposed to 60% in 2009, but the question is *70% of what count*.

The commonplace strategy of converting percent to fraction form or to decimal form means effectively that the "hypothetical unit of reference" changes from 100 in the percent system to 1 in the decimal system, and if converted to a "fraction" measure, to any number of reference units (on account of a rational number having equivalent forms) (Risacher, 1992, cited in Parker & Leinhardt, 1995, p. 436).

Percent notation is in some senses isomorphic with numbers: percentages are extensive quantities that can be added if they are different portions of the *same whole*. But if the underlying referent quantities are different a conversion has to take place back to the referent quantities before the two percents can be additively combined in a meaningful way. Such additions require the referents to be expressed in common units in order to admit meaningful addition, before converting back to a percent.

Quantities in fraction notation are not easily ordered on account of the infinite number of denominators. The decimal system, in contrast to fractions, has a natural ordering of extensive quantities based on place value. The power of the percent concept, as with decimals, is in its "ability to provide a similar natural ordering of some intensive quantities, in particular those that are created by forming internal ratios" (Parker & Leinhardt, 1995, p. 438). The percent concept and notation enables us to;

- Locate on a scale from 0 to 100 the size of a part as it relates to its whole.

- Locate on an unbounded scale, a depiction of a multiplicative relationship between two referent quantities of the same type and unit, and,
- Compare the magnitudes of several such comparative ratios quickly based on the natural ordering of the decimal numeration system.

The nine comparative mathematical contexts are presented in Table 6.9.

The first categorisation that percent has both a “fraction” part-whole meaning and a ratio meaning, has been discussed. The fraction type problem is essentially finding a part of the whole, or the whole if given a part. The ratio type problems have been classified by Parker and Leinhardt (1995), into 8 distinct classes of problems, building on the work of the Rational Number Project (Behr, et al., 1983). The major structural contexts are that of *change* within a referent group and the *comparison* between two referent groups. The minor contexts[97] show that the solving of percent type problems require careful attention to the context, to the reference quantities and to the percent language.

The underlying mathematical structure of percent situations, like that of rational number, can be represented as the direct comparison of measure spaces, as shown in Table 6.9 (last two columns).

The variation in the use of percent, for example percent has common uses as a statistic to report the relationship between known segments of data, or as a functional operator used as a standardised expression for income and sales tax, makes it essential to analyse the concise language, and to identify the underlying structure.

[97] For diagrammatic representations of these types, see Parker and Leinhardt (1995, p. 441).

Table 6.9: Comparative contexts

Type	Major context	Minor context	Examples	Mathematical structure	
				M_1 (%)	M_2 (R)
Fraction			Find 25% of R60.	100 25	60 x
Ratio	**Change within a referent group**	Absolute change as % within 100%	**Decrease to %** (A shirt cost R15. It now costs R10. What is the % decrease?)	100 x	15 5
			Increase to % (A shirt cost R10. It now costs R15. What is the % increase?)	100 x	10 5
		Relative change from 100% to new %	**Decrease by %** (A shirt cost R20. It was marked 50% off. What is the new price?	100 50	20 x
			Increase by % (A shirt cost R20. This year it increased by 25%. What is the new price?)	100 125	20 x
	Compare two referent groups	Compare Q1 to Q2. The sizes of two distinct sets, Q1 (girls) and Q2 (boys) are considered	**Smaller is % of** (32 girls and 8 boys. What is the percent relationship between the counts of girls and the counts of boys? The number of boys is 25% of the number of girls).	100 x	32 8
			Greater is % of (32 girls and 8 boys. The number of girls is 400% of the number of boys.	100 x	8 32
		Differ by % The difference in size of the two distinct sets or objects is compared to one of the sets.	**Smaller by %** (24 girls, 8 boys. By what percentage less does the full count of boys differ from the full count of girls?)	100 x	24 16
			Greater by % (24 girls, 8 boys. By what percentagemore does the full count of girls differ from the full count of boys?)	100 x	8 16

adapted, Parker & Leinhardt, 1995, p. 439

6.4.2 The language of percent

The topic percent has a language in which the underlying referents are usually implicit, and the language is not always semantically appropriate, for example the common expression "more than", as in "Dan earns 10% *more than* Jack", means in essence "proportionally *more than*". Those who are

> fluent in the language of percent understand that the underlying referents exist even if they are only implied, and are able to identify these referents and establish the proportional relationships between them (Parker & Leinhardt, 1995, p 445).

The great advantage of conciseness in percent language lies in the compression of statements which for novices may have been clearer in fraction form, for example, "one and a half times the size of" may be simpler that "150% of" (the whole), or "50% more than" (the whole). The danger is that the subtle differences in language and the compressed nature are not easy to interpret, especially for the child who may still be struggling with proportion concepts.

Parker and Leinhardt (1995) identify three language related areas which they understand to be problematic in the learning and teaching of percent. The first is the conciseness of the language. While this brevity is the greatest advantage of percent language, there is often ambiguity about the underlying referents. The second problem is in the use of the word "of", as in natural language the preposition "of" implies "a part of". The use therefore of this preposition becomes problematic for percents greater than a 100, in that this fractional understanding may cause interference with an application of percent in a ratio context. The third problem is that additive language is used where a multiplicative relationship is implied. The three critical considerations when interpreting percent language (Kinney, 1958, cited in Parker & Leinhardt, 1995, pp. 447-448) are the following;

- A number is both an additive component of another larger number and a fractional proportion of another number, for example, 10 is both an additive component of 30, (10 + 20 = 30), and a multiplicative component, 10 is a $\frac{1}{3}$ of 30.
- A number is both an additive holder of other smaller numbers and a scalar multiple of other numbers, for example 15 is both the additive holder of 10 + 5, and a scalar multiple of 10; $\frac{3}{2} \times 10$ or $1\frac{1}{2} \times 10$, or 1.5×10, are all equal to 15.

- A number differs from another number both additively (15 is 5 more than 10, and 10 is 5 less than 15) and proportionally (15 is $\frac{1}{2}$ [of 10] more than 10 and 10 is a $\frac{1}{3}$ [of 15] less than 15).

For each of the above multiplicative relationships percent has a specialised language (see Table 6.10)

Parker and Leinhardt (1995) highlight the power and efficiency of the percent language, but they also explain the difficulties inherent in the topic. The words *base*, *percent* (often called the rate), and *percentage* are used to define the elements of a percent equation. They use the term *percent* to refer to the concept; the *percent* to the rate, as in 14% VAT; the *base* as the reference quantity; the word *percentage* to the target quantity, as in the resulting VAT quantity as determined by the rate 14%. They note that the two words, *percent* and *percentage* are often interchanged in common usage.

Table 6.10: From common language to the language of percent

	Fraction measure or ratio	Percent
Fractional portions	10 is a $\frac{1}{4}$ of 40	10 is 25% of 40
Scalar multiples	$2\frac{1}{2} \times 10$ is equal to 25 or $\frac{2}{5}$ of 25 is equal to 10	250% of 10 is 25 or 40% of 25 is 10
Relative difference	15 is 5, or $\frac{1}{2}$ [of 10], more than 10 or 10 is 5, or a $\frac{1}{3}$ [of 15] less than 15	15 is 50% more than 10 or 10 is $33\frac{1}{3}$% less than 15

adapted, Parker & Leinhardt, 1995

6.4.3 Tasks and problems

Parker and Leinhardt (1995) note that percent tasks are an important part of the curriculum for the higher primary[98] school, and have importance in applications in the high school. The typical percent tasks found in textbooks and tests can be classified as *conversions, exercises, shading tasks*, or *problems*. These common tasks, a description

[98] The "Intermediate Phase" is used in South African schools instead of Higher Primary.

and an example, and the underlying rational number subconstruct are presented in Table 6.11.

Conversions involve changing between the three notational systems, namely fractions, decimals and percent. This task generally relates to the part-whole understanding of percent. Even when the percent is greater than 100, the decimal fraction greater than 1, and the fraction is mixed, this conversion can still be a whole-part conception, depending on the context and the language used. An *exercise* requires finding one of three possible unknowns in a percent equation. A *shading task* requires recognising a proportion, then either shading a proportion of a continuous quantity or shading some proportion of a set of discrete objects. A *problem task* requires a student to extract relevant information from a problem situation, provide a mathematical model, then solve the problem by applying the model to the situation in which percent is embedded.

Parker and Leinhardt (1995, p. 428) found the following errors that students make with percent type tasks:

- Drop the percent sign. Students regard the % sign as a label such as is found in a measure unit, for example a centimetre that they assume they can drop and pick up again later.
- Abandon natural sense making. Students' answers point to the fact that they have no clear understanding of the nature of numbers or the legitimacy of operations.
- Strong part-whole notion of percent causes serious misconceptions. These misconceptions arise from the notion that "percents greater than 100 [are] counterintuitive, since a part cannot exceed the whole" (Parker & Leinhardt, 1995, p. 428).

Table 6.11: Common percent type tasks (adapted, Parker & Leinhardt, 1995)

Type	Description, and example	Sub construct
Conversions	Convert between fraction, decimal and percent notation, for example, Write 20% as a fraction.	Part-whole
Exercise (Case 1)	Finding the percentage, for example, "15% of 120 = ___".	Operator sub-construct
Exercise (Case 2)	Finding the percent, for example, "___% of 120 = 18".	Ratio sub-construct
Exercise (Case 3)	Finding the base, for example, "15% of ___ = 18".	Ratio sub-construct
Shading	A continuous quantity, for example, Shade 50% of the shape. Discrete objects, for example, Shade 25% of the (set of) beads.	Part-whole
Problems	Percent embedded in an application context, extract relevant information, mathematize and then solve.	Modelling, multiple constructs

6.4.4 A concise language with important consequences

The underlying mathematical structure for percent problems may be likened to the general problems encountered with rational number subconstructs (**Question 6.2.1**), however the language of percent is concise and therefore not easily understood.

The topic of percent is usually coupled with fractions and decimals. The teaching of percent is in general conducted through the teacher providing an instructional method. The testing of percent is invariably checking to see whether particular methods have been remembered. The errors arise from problems of interpretation of the context, misunderstanding of the percent language and lack of underlying proportional reasoning ability (**Question 6.2.2**).

Percent therefore has all the complexity of the rational number subtopics of fraction, ratio and rate, but then also carries its own complexity. Perhaps it is because of this complexity that teachers resort to teaching routines. In addition to the scaffolding that routines may provide, attention to the underlying structure and the language of percent is critical if errors are to be avoided (**Question 6.2.3**).

As with rational number subconstructs, attention to the mathematical objects, in particular the distinctions between a quantity, or a measure, and a scalar operator (ratio) or function operator (rate) subconstruct may assist learners and teacher to investigate common structures and differences which may lead to generalisations (**Question 6.2.4**).

6.5 Probability

In this section some aspects of the topic probability within the mathematics education context are discussed. In addition, we explore aspects of probability and statistics that contribute to the general complexity of teaching and learning and therefore have implications for the learning of concepts, instructional strategies and modes of assessment. Attention is also given to historical factors which have impacted on the development of probability theory.

Probability is defined in its classical form as the proportion of all replications of an unchanging but repeatable process that will yield a particular outcome or a particular set of outcomes within the set of all possible outcomes. For example, a roulette wheel with 38 numbers, 00, 0, 1, 2, … 36, if appropriately balanced, is expected to yield the number 19 on 1/38 of all spins. It is also expected to yield an odd number (that is any of the 18 possible odd numbers, assuming 00 and 0 are neither even or odd) on 18/38 of all spins. This proportion is then applied to any (and every) additional spin of the wheel.

In comparison with geometry and algebra, *probability and statistics* is a relatively new branch of mathematics, evolving from counting and gambling problems in the mid-17th century, and from the necessity to use empirical data for inferences that report uncertainty in some explicit way. Shaughnessy (1992) notes that the discipline statistics is required for every field of enquiry, for both an informed citizenry and responsible consumers (p. 466), and should therefore be the focus of both research and teaching and learning in the school curriculum. A further requirement of teaching in the interests of social responsibility may be to provide students with situations that exhibit how misconceptions of probability can lead to erroneous decision making, such as in the work of the legal and medical professions[99] (p. 482). In contrast, learners can be led to an appreciation of the role of assumptions in inferences from observed retrospective proportions to unobserved prospective proportions, and the importance of having comparable references in those proportions. Appreciation of the contrast may alert students to common misconceptions that can impact on their lives.

6.5.1 Mathematical structure

The topic of probability, involving the frequencies of specific outcomes within replications of an experiment, builds on an understanding of ratio and proportional reasoning. The percent language and notation may be used. In data analysis *fraction measures*, *ratios*, and *proportions* may have either *retrospective* or *prospective* orientations. The contrast between looking back in time on data fragments and summarising a finite set of observed and recorded data by counts, averages and ratios (retrospective) versus looking ahead in time and envisaging what pattern the prospective totality of an unlimited future of repeated observations might bring, needs to be rendered explicit, if learners are to master probability notions (T. Dunne, Personal communication, 3rd January, 2010).

The inclusion of probability items in this thesis was based on the grounds that the calculations required multiplicative structures, in particular the concept of ratio. While the number of items was relatively small (just three items, roughly 8% of the instrument), the topic nevertheless warrants consideration as part of the constellation of constructs constituting the multiplicative conceptual field.

6.5.2 Historical factors

Some philosophical and historical factors impact on both the research and the teaching and learning of probability. Shaughnessy (1992) describes an epistemological schism between two traditions. The *rationalist tradition*, instituted by Plato and followed by Descartes, regards the attainment of knowledge as the result of pure reason, so that absolute truths are discovered by the mind. The *empiricist tradition* is represented by Aristotle, who was concerned with the concrete and the particular, and continued by

[99] Steven J. Gould (2002) "The median isn't the message" provides an interesting account of misconceptions.

Locke, who fundamentally denied the notion of absolute truth. In the empiricist view, knowledge is the result of sense perception, and it is from sense perception that theory is built inductively. This distinction may relate to the predicative and operational forms of knowledge noted by Vergnaud (1990).

A further complication impacting on the teaching of probability is the dual meaning that is attached to the word probability. One meaning is to indicate “a degree of belief” and the other is to calculate “stable [long-term] frequencies for random events” (Hacking, 1975, cited in Shaughnessy, 1992, p. 468). The objective (repeated but random outcomes) and subjective (degree of belief) nuances contribute differently to teaching and learning. The “seeds of statistical decision theory, based on stable frequencies of random events” (Shaughnessy, 1992, p. 469), must take seriously the existing schemas students use in order to lead students to mature statistical decision theory (albeit in simple contexts within school education).

6.5.3 The acquisition of probabilistic concepts

Piaget and Inhelder (1951, 1975) identify three stages in the development of probability concepts. In the first stage children under the age of seven are unable “to distinguish between necessary and possible events”, and would “predict the most frequently observed event, with total disregard for the population proportion” (cited in Shaughnessy, 1992, p. 479). In these scenarios the link between ratio and probability is not understood. In the second stage[100] (up to fourteen) “the child recognises the distinction between necessary and possible events, but has no systemic approach for generating a list of possibilities” (p. 479). In the third stage (over the age of 14) the student “begins to develop facility with combinatorial analyses, and understands probability as the limit of relative frequency” (p. 479). This development shows the concept of ratio to be critical to the understanding of probability.

The large scale study in England investigating students’ understanding of probability (Green 1988, cited in Shaughnessy, 1992) supports Piagetian stages, though according to Shaughnessy (1992), it is not clear how much exposure the students had to instruction

[100] The age of the learners in this study range from 12 to 15 years. This locates them in the second stage of development. This point will be pursued in Chapter 7.

prior to the study. Piaget, it must be noted, clearly stated that the stages are modified in relation to social and educational factors and experiential procedures (Piaget & Inhelder, 1969).

A study by Fischbein and Gazit (1984) investigates the role of instruction on shaping intuitions. This experimental study compared an experimental class which received instruction on probability theory, with the control group that received no instruction. The experimental group was given a questionnaire (QA) post instruction on the formal learning to which they had been exposed. Both the experimental and the control group were given a questionnaire (QB) post instruction for which no formal study was required but which was designed to test intuitions. The study yielded some interesting findings and further hypotheses.

The finding that resonated with other research into rational number and proportional reasoning and the related concept of percent, is that a major source of errors is the use of additive procedures where multiplicative procedures are appropriate. In addition it was found that on the two proportional reasoning questions in QB, the experimental group performed worse that the control group. The experimental group also performed poorly on the proportional reasoning question in QA, which Fischbein and Gazit note is in line with the findings of similar studies.

Fischbein and Gazit (1984) venture the following hypothesis,

> Probabilistic thinking and proportional reasoning are based on two distinct mental schemata. A progress obtained, as an effect of instruction, in one direction does not imply an improvement on the other. Certainly, probability computations may require ratio comparisons and calculations but probability as a specific mental attitude, does not, necessarily, imply a formal understanding of proportion concepts (Fischbein & Gazit, 1984, p. 24).

Some research has shown that young children can identify correct probability estimates, while their proportional reasoning skills are undeveloped (Goldberg, 1966; Yost, Siegal & Andrew, 1962, cited in Fischbein & Gazit, 1984). The hypothesis is that there is a common core of skills, notably ratio skills, but there are intuitions and technical procedures that make those skills distinct. “By emphasising (via systematic instruction) specific probability viewpoints and procedures one may disturb the subject’s proportional reasoning, [that may] still [be] fragile in many adolescents” (Fischbein & Gazit, 1984, p. 23).

6.5.4 A distinctive reasoning

A major source of errors noted in the literature is the use of additive procedures where multiplicative procedures are appropriate. The common features underpinning both rational subconstructs and probability may provide some insight for learning and teaching, however the distinctive reasoning that requires prospective thinking is essential. This topic is beyond the scope of this research study.

6.6 Proficiency in the multiplicative conceptual field

In answer to Question 6, the following observations are offered:

The range of situations modelled by multiplication and division may in one sense appear overwhelming and present a confusing picture. On the other hand the power of mathematics is demonstrated in that these many situations are modelled by one operation, multiplication (and its inverse division).

The transition from working with natural numbers to working with rational numbers requires some conceptual restructuring. For this transition to occur with minimum disruption it is necessary for the differences encountered in the rational number system to be made explicit. Two properties of fractions, or fractional numbers, that are different from natural numbers, are equivalence and order. In both these cases cognitive shifts from natural numbers are required. The property of equivalence denotes that there are an infinite number of fractions equivalent to any fraction, and the property of order denotes that between any two distinct fractions there are an infinite number of fractions (Smith, 2002). An understanding of equivalence and order is essential for a conceptual understanding of numerical conversions.

A problem identified by Vergnaud (1983, p. 164) is that the model first presented to learners is the inclusive fraction model. The comparisons and ratios between any two quantities of the same kind, the exclusive model is the a more powerful model than the inclusive model, providing a more general foundation for scalar operators and ratios, and therefore should be presented concurrently in the early years of learning mathematics.

From a mathematical point of view we have extremely efficient notation. The fraction symbol $\frac{1}{4}$ applies to many different situations. However in order to attach a rich meaning to the symbol, a great deal of experience with situations and contexts which provide meaning for the symbol is necessary. The ultimate aim of these two very important mathematical features, the efficient notation and the multiple situations is that learners understand the concept of invariance of equivalences, sustained by covariances of elements within ratios.

From a cognitive perspective, *rational number sense* develops through engagement with the sub-topics, fraction, ratio and rate, percent and probability. *Proportional reasoning* is, according to Lamon (2007), acquired as a consequence of developing rational number sense. Lamon (2007) presents the following reasoning: “proportions arise in the study of rational numbers as a natural expression of their equivalence” and therefore “as one develops rational number sense through various experiences with many personalities of rational number, one learns to reason proportionally” (p. 640). The ultimate aim of developing proportional reasoning is to understand the concept of invariance (as when a mathematical structure remains unchanged) and covariance (as when two quantities are linked to each other in a way such that when either one changes, the other also changes).

In the view of this thesis, delineating the multiplicative conceptual field, with the elaborated definition of a mathematical concept as embodying situations, operational invariants and representations, in addition to the complex interrelationships linking elements such as fractions, ratio, rate, proportional reasoning, percent and probability provides a conceptual space for learning and teaching. This strategy provides the necessary tools for analysis of this multiplicative area of mathematics. The function of teaching in this theory of conceptual fields is the transforming of local intuitions concerning these multiple situations into generalisable procedures that can be applied to many situations.

The two contrasting approaches to teaching this area of mathematics are through teaching procedures and hoping that the procedures can then be applied to problems. The preferred approach is to develop rational number sense through encountering many situations, that can be compared and contrasted, and through which learners apply their knowledge in action. Through expressing their thoughts and reasoning, learners become aware of common structures. From the teachers' perspective, the need is firstly to deal with the concepts intuitively and informally, and then through making explicit the implicit concepts and theorems used by learners, to assist learners to develop more generalisable procedures.

6.7 Curriculum development

The curriculum, according to Schmidt et al. (1996), is fundamental in structuring educational experiences for children. The curriculum, they aver, does not happen by accident but is "deliberately based on visions of what education should be" (p. 141). The purpose for reviewing the curriculum in the light of significant information from the mathematics education research on the multiplicative conceptual field is to investigate through a new lens both the strengths and shortcomings of the existing curriculum.

The National Curriculum Statement (DOE, 2002) [101], like curriculum documents around the world, is divided into strands[102] for the sake of convenience. The strands are *Numbers, operations and relationships* (1), *Patterns, functions and algebra* (2), *Space and shape* (3), *Measurement* (4) and *Data handling* (5). Some topics can be located easily into precisely one of the strands. The multiplicative conceptual field however, straddles all five strands, as multiplicative structures are core to *number*, provide the foundation for *function and algebra,* are critical in *measurement,* have links to *space and shape* (geometry) and are related to the concepts *probability and data handling.*

In the first strand, *Numbers, operations and relationships,* attention is given to the operations multiplication and division. It is also in this strand that the topic of rational numbers and the subconstructs, fraction, ratio, rate and percentage are covered. We observed in the previous sections, that the route that leads to algebra, and the algebraic conceptual field, begins with the multiplicative conceptual field. A natural outcome of

[101] The document used here is the rearrangement of the NCS by Scheiber (2010). This document is written in an easily accessible format.
[102] These separate domains have been called "learning outcomes" (see DOE, 2002), presumably for the reason that one wants to see these elements of mathematics as products of education. "Content area" is currently preferred by the Department of Basic Education is. The term "strand" is used here.

the rational number sub-constructs is also the concept of function which is located in the *Patterns, function and algebra* strand. The *Theorem of Pythagoras* includes the concept of angles in the triangle, the ratio of sides of the triangle and the concept of square area units, and is found in the *Space and shape* strand. Similar triangles, for which the same ratio of lengths for pairs of sides is the defining property, is also found in this strand. A rate is a relationship of measures, (for example, distance and time give us kilometres per hour), and is located in the *Measurement* strand. And then finally we have the construct probability, which sometimes seems to be in a category of its own. However probability certainly builds on the foundation of ratio and proportional reasoning, and together with relative frequencies of data is located in *Data handling*.

In *Numbers, operations and relations*, the introduction and progression of the rational number subconstructs is in line with most curricula. The introduction of multiplication and division proceeds from working in a low number range to a higher number range, and the transition is made from natural numbers to rational numbers. The topics of ratio and rate are introduced in Grade 4, however here there is no obvious progression across grades. For each of the Grades 4, 5, and 6, the statement is essentially the same except that the contextual examples are different (National Curriculum Statement, DOE, 2002).

In the succeeding Grades 7 and 8, the statement "Solves problems that involve ratio and rate" is the sole mention of ratio. In Grade 9 the statement is extended "Solves problems that involve ratio, rate and proportion (direct and indirect)" (see Table 6.13).

Table 6.12: National Curriculum Statement (2002): Excerpt, Intermediate Phase

Grade 4	**Grade 5**	**Grade 6**
Solves problems that involve:	Solves problems that involve:	Solves problems that involve:
• Comparing two or more quantities of the same kind (ratio)	• Comparing two or more quantities of the same kind (ratio)	• Comparing two or more quantities of the same kind (ratio)
• Comparing two or more quantities of a different kind (rate, eg. kg/Rand)	• Comparing two or more quantities of a different kind (rate, eg. learners/teacher)	• Comparing two or more quantities of a different kind (rate, eg. wages/day)

Table 6.13: National Curriculum Statement (2002): Excerpt, Senior Phase

Grade 7	Grade 8
Solves *problems* involving time including: relating time, distance and speed (7.4.1, emphasis added)	Solves *more complex problems* involving time including: relating time, distance and speed (8.4.1, emphasis added)

In the Foundation Phase, Grades R (Reception Year) to Grade 3, there are no references to proportion, however one might see potential for the development of proportional concepts in "Describes observed patterns" (see DOE, 2002) and in "Estimates, measures, compares and orders 3-D objects, namely mass, length and capacity" (DOE, 2002).

The initial introduction to percent in Grade 6 is "Recognises and uses equivalent forms of fractions, decimals and percent" (see DOE, 2002). The next percent statement is "Finding the percentage of whole numbers". These introductory topics provide teachers with only two percent problem situations, namely part-whole and as scalar operator. The focus in Grade 7 is on "Recognising and using equivalent forms of the rational numbers including common fractions, decimals and percents" (DOE, 2002) which may be interpreted as conversions between fraction notation, decimal notation and percent notation. Limiting the number value to common fractions, $\frac{a}{b}$, with $a < b$, however, may support the part-whole concept of percent that according to Parker and Leinhardt (1995) is a limiting factor to later learning of percent as an unbounded ratio. The next statement in Grade 7 is "Finding percentages" (DOE, 2002). In Grade 8 the statement is the same as Grade 7 (DOE, 2002). This use of the word *percentage* may mean the target quantity as in "What is 50% of 20?" or it may mean the ratio, "What percentage is 30 marks out of 90 marks?"[103]

The *Patterns, function and algebra* strand focuses on:

- Describing patterns and relationships through the use of symbolic expressions, graphs and tables
- Identifying and analysing regularities and change in patterns and relationships that enable learners to make predictions and solve problems

[103] We may accept that in a curriculum document, there is not the space to clarify terms. Clarification is the function of a supporting document or a textbook.

This focus lends itself to multiplicative thinking and proportional reasoning, although there is no specific mention of any of the subconstructs of interest. The topic of proportional reasoning, accepted generally as being the capstone of the primary school and the cornerstone of high school mathematics (Lesh et al., 1988), is not mentioned. This topic should ideally undergird this strand, but is conspicuously absent.

The *Space and shape* strand for Grade 8 includes similarity (see DOE, 2002), and in Grade 9 includes the enlargement and reduction of geometric figures (see DOE, 2002). In the *Measurement* strand, in Grade 8, there is reference to π and to its historical development. This entry is perhaps the cue for introducing the idea of evolving number systems that have responded to society's need, and to give attention to the crisis that was precipitated by the discovery that not every magnitude could be represented by a rational number. Also, beginning in Grade 8, the Theorem of Pythagoras is introduced as an aid to solving problems.

The section on probability, in the *Data Handling* strand, beginning in Grade 4 with the language of probability, and culminating in relative frequency experiments and theoretical probability in Grade 9, follows a logical progression, which takes into account the development of intuition.

The National Curriculum Statement is, in the view of this thesis, an adequate supporting document. There are particular topics that could be rendered more explicit, but that is probably the role of the textbook. The first problem, however, is that current teacher knowledge underlying these complex topics may be inadequate. This situation might be expected given the complexity of the topic and the confusion around some of the subtopics. The second problem is the absence of specificity of how the learning of these topics may be guided. There is clear evidence that teaching through definitions, and through giving rules and demonstrating algorithms, is not enough to develop proportional reasoning or to develop rational number sense.

Evidence of the fact that little attention is given to topics involving *ratio, proportion and percent* in the classroom is provided in the TIMSS 2003 Report (see Mullis, et al., 2004), in which the National Research Coordinator (NRC) for South Africa reports that teachers had not taught the topics clustered into the subdomain *ratio, proportion and percent*, or had only taught these topics to more able learners (see Appendix A). Given, firstly, the lack of direct curriculum support and the complexity of the topic this outcome is expected. This situation is not unique to South Africa; Lamon (2007) notes that, in general, American teachers are "not prepared to teach content other than part-whole fractions" (p. 632).

6.8 Summary: Curriculum development, didactic implications, assessment and research

Given the interrelated nature of elements of the multiplicative conceptual field (summarised in Section 6.6), the challenge for researchers and the designers of test instruments is to structure assessment frameworks and test instruments that recognise this network, as noted in Chapter 4, Section 4.7. But prior to any assessment programme, attention has to be given to the curriculum. Kieren (1976)[104] suggests that both short-term and long-term objectives be considered when developing the curriculum. He also proposes that networks of sequences of instructional activities be designed, piloted, and verified, in anticipation of classroom use.

The investigation of the cluster of concepts, the rational number subconstructs, percent and probability, which have multiplicative structures as their core, has provided some interesting insights that may be considered in teaching and learning. These insights include the following;

- Rational number sense and proportional reasoning develops over time with appropriate experiences.
- The research reminds us that the development of higher order constructs, such as functions and algebra, have their roots in early arithmetic experiences, highlighting the importance of meaning making in the very early years, notably the Foundation Phase.
- The confusion of terms and the ambiguity of concepts may contribute to the poor teaching and performance in the rational number domain (Parker & Leinhardt, 1995). The relations between some of the constructs, for example, rational number, fraction measure, and fraction notation, have been clarified by both Usiskin (2005) and Lamon (2007). These classifications may avoid confusion in the classroom.
- Lamon (2007) has proposed that the mastery of the rational number subconstructs and the development of proportional reasoning occur interactively. There is

[104] Kieren (1976) suggests a curriculum process with associated cognitive structures, and offers instructional sequences.

agreement generally that proportional reasoning is a threshold concept (Lesh et al., 1988; Lamon, 2007).

- Because, as Vergnaud (1988) states, a problem situation is very rarely made up of one concept and a single concept requires multiple situations and contexts to be fully described, it is necessary to research conceptual fields. The specific focus on identified concepts such as ratio, in addition to the interrelationships between concepts is necessary (see Sowder, et al., 1998, for this approach in teacher education).
- The notion of measure spaces with specified units, made explicit in diagrammatic representations, render explicit the many relationships involved in rational number comparisons.

The above findings have implications for teaching and learning the related concepts within the multiplicative conceptual field. The literature that has been surveyed in this chapter and the analysis of the constructs certainly provides some clear indications of what is required for the successful teaching and learning of mathematics, in particular the critical threshold domain of rational number and proportional reasoning.

The problem situations in this research study are restricted to the category labelled isomorphism of measures. The reason for extending the discussion to situations of greater complexity is based on the principle that it is necessary to keep in perspective the future development of the construct, in the interest of locating current concepts along a developmental continuum.

Vergnaud proposes an approach to research in mathematics education generally, and which applies specifically to the multiplicative conceptual field (1988, p. 149). The first task is to identify and classify situations on the topic of interest at the cognitive stage of the learner. The selection of items from TIMSS 2003, with slight adaptation, constructing and administering assessment instruments, and analysing the data is explained in Chapter 7.

7 Exploration of data within the Rasch measurement framework

7.1 Understanding complexity through application of the Rasch model

The theoretical groundwork presented in earlier chapters and the empirical research informed by the Rasch model, are used interactively to provide an analysis of the particular mathematical constructs, embedded within the multiplicative conceptual field, and depicted as items in the instrument, namely, fraction measure (1), ratio, proportion and rate (2), percent (3), probability (4) and pre-algebra (5) constructs. In order to direct attention to selected arrays of concepts, within the multiplicative conceptual field, clusters of items grouped within a specified mathematical construct are discussed separately. These separations are for purposes of analysis and explanation only, as there is a conceptual blurring of the boundaries of categories within the problem situations, precisely as is to be expected within a bulk of problem situations comprising a conceptual field.

The aim of this chapter is to describe the data analysis process, in particular to describe the application of assessment, and measurement principles (Section 7.2), to highlight aspects of the analytic framework, based on the analysis of the *multiplicative conceptual field* (Section 7.3) and present an approach to data analysis, that is assisted by the application of Rasch measurement (Section 7.4). Summary descriptions and tables of the strands will be presented in the body of the thesis. The detail of each item analysis, and strand by level summaries, are provided in Appendix B.

In the summary analysis the strands are integrated and discussed according to several difficulty locations[105]. Level descriptors[106] drawing from the five strands, and relating to the indicated proficiency levels, are presented (see Section 7.10).

[105] The levels were empirically defined after administering the test items to the cohort of learners in this study.
[106] Level descriptors may be more aptly described as descriptions of difficulty locales. However, the term "level descriptors is the place holder term that will be elaborated in the subsequent sections.

7.1.1 Research questions

The primary question and subquestions guiding this chapter, are presented below.

Question 7 What insights can be gained from an investigation of the multiplicative conceptual field when assessed within a South African context and analysed through the lens of the Rasch measurement framework?

7.1 How are measurement principles, requirements within the Rasch measurement framework, applied in this research study? (Section 7.2)

7.2 What elements of an analytic framework are used to describe individual items and substrands? (Section 7.3)

7.3 What procedures underpin the item analysis? (Section 7.4)

7.3 How are items ranked in terms of difficulty level? How can these rankings be explained in terms of context, situation, mathematical structure, mode of presentation, number range and value, and response process? (Sections 7.5 – 7.9)

7.4 How do the mathematical concepts, within the multiplicative conceptual field, correlate with the proficiency levels of learners as exhibited on this test? (Section 7.10)

7.5 What threshold concepts can be identified within this analysis? (Section 7.10)

7.6 What adaptations, additions and changes are suggested for the instrument by reflection on the empirical data? (Section 7.10)

7.2 Methodology for the empirical investigation

From a methodological perspective the question underpinning the empirical phase of this research study is how to investigate scientifically the concepts and skills required for proficiency in the multiplicative conceptual field, in a selection of South African students, for the purpose of investigating development of the concepts inherent in this field.

7.2.1 Test development within a Rasch measurement framework

The first necessity in establishing a measure of proficiency is, according to Wright and Stone (1979), to define the construct of interest, and within this clearly defined domain, to establish an “explicit understanding of the line of inquiry” (Bond & Fox, 2007). In the case of this study, the requirement for establishing a line of inquiry is met by the theoretical investigation of the multiplicative conceptual field. (See also Chapter 6, Section 6.5)

A second requirement is to develop an instrument which should provide reliable information about the domain to be tested. This requirement necessitates developing an instrument[107] that accurately targets the population to be tested in terms of difficulty level of items. The construction and administration of items which are "believable realizations" of the particular mathematical domain and which elicit signs of the proficiency variable in the person, determine the efficacy of the test (Wright & Stone, 1979, p. 3). In this study the items selected from the TIMSS 2003 pool of released items (IEA, 2005) were judged to include the subconstructs of rational number, and invoke proportional reasoning in the solution.

The third and fourth requirements are to demonstrate that the items when taken by suitable persons are consistent with parameter independence expectations, and that the patterns of student responses are consistent with parameter independence expectations (Wright & Stone, 1979). It is at this point that the Rasch model provides the possibility of empirical verification. Finding unexpected responses in either items or students requires an investigation of the *items* and also an investigation of the *student* profile on the test as a whole. The procedures and actions taken to deal with misfitting items are explained in Section 7.2.5.

The theoretical position proposed by Vergnaud (1988), that both the mathematical concepts inherent in mathematical situations and the associated cognitive structures that students engage in response to problems, need to be analysed using mathematical terminology, resonates with a defining feature of the Rasch Measurement Model (Rasch, 1960/1980), that transforms both item difficulty and student proficiency to locations on the same scale. The concepts in the mathematical situation align with the item difficulty locations and the cognitive processes align with the location of students, along the common scale of developing proficiency.

7.2.2 Participants

The requirement from an empirical perspective was an analysis of student competence and the identification of consequential errors which would provide an essential insight into the broader understanding of teaching and learning within this conceptual field. In order to assess the performance apart from the adverse school factors in many South African schools, two

[107] The development of an instrument implies the development of a measuring device that ideally makes the resulting outcome meaningful.

well-functioning schools, though serving different socio-economic groups, were selected as sources for students.

In addition to testing Grade 8s, the grades either side, Grades 7 and 9, were included in order to track or at least signal the developing mathematical competences across this three year phase. All the students in two schools, at Grades 7, 8 and 9, in 16 classes, were tested (see Table 7.1). The selected group comprised 330 students; 140 attended School A (only girls) and 190 attended School B (boys and girls in equal proportions). This selection procedure is classifiable as *purposive sampling,* where the researcher has "handpicked the cases to be included in the sample on the basis of (the researcher's) judgement", for their suitability for the purposes of the particular research study (Cohen, Manion & Morrison, 2000, p. 103).

Table 7.1: Number of students by schools and grades

	Grade 7	Grade 8	Grade 9
School A	22 (1 class)	51 (2 classes)	67 (3 classes)
School B	33 (2 classes)	84 (4 classes)	73 (4 classes)
Total	55	135	140

7.2.3 Test formulation

The decision to develop a test instrument was informed by the efficiency of obtaining access to a large, reasonably representative body of data; the testing phase being the staging post for further investigation. The administration of a test instrument could establish, partly, the empirical verification of the theoretical development of the particular constellation of concepts *fraction, ratio, rate and proportion, probability* and *percent*, elements of the multiplicative conceptual field. It was envisaged that the analysis of responses to the items would provide insight into the differential development of the concepts within students and enable the identification of associated difficulties. Furthermore analysis of individual student responses to items of increasing difficulty would provide insight into the acquisition of concepts of increasing complexity.

Content specification and Instrument construction

Subtopics within the TIMSS Mathematical Frameworks (Mullis et al., 2003) were identified as closely matched to the multiplicative conceptual field. Items considered appropriate for testing in this conceptual field were selected from the released items from content domains and the subdomains listed in Table 7.2.

Table 7.2: Selection from TIMSS content domains and subdomains

Number	Algebra	Geometry	Measurement	Data
ratio, proportion and percent	patterns, equations and formulas	congruence and similarity	tools, techniques and formulas	data interpretation
fractions and decimals				uncertainty and probability

Source: Mullis et al., 2003

Selections were made that covered a range of content and cognitive skills (see Table 7.3). The initial selection of items was drawn from the content domain, *Number*, and the subdomain *ratio, proportion and percent*, and *fractions and decimals*. In addition other items judged to require multiplicative concepts and skills were selected from other content domains; *Algebra*, and the subdomains, *equations and formulas* and *patterns*; *Geometry*, and subdomain, *congruence and similarity*; *Measurement*, and subdomain *tools, techniques and formulas*; and *Data*, and subdomains *data interpretation*, and *uncertainty and probability*. The items were distributed roughly evenly across the cognitive domains identified by TIMSS 2003.

The criteria for the selection of items were that:

- the set of items would cover a range of content within the multiplicative conceptual field and should cover a range of cognitive skills,
- there would be a close match between the National Curriculum Statement (NCS) and the content of the test items,
- both multiple-choice and constructed-response formats should be used, and that
- the items would exhibit a range of difficulty.

A combination of multiple-choice format and constructed-response format was used, with roughly two thirds, 25 of 36, (69%) of the items presented in multiple-choice format and about one third, 11 of 36, (31%) of the items presented in constructed-response format. Six of the 11 constructed-response items required two responses, making a total of 42 possible points[108] for the planned test items.

Table 7.3: Items from TIMSS framework: content and cognitive domain

[108] One of the items could not be used due to an oversight in proofreading of the final instrument. This error left 35 items with a maximum total score of 41 points.

		Cognitive domain			Total
Content domain	**Subcategory**	**Knowing facts, procedures, concepts**	**Solving routine problems**	**Reasoning**	
Number	Fractions and decimals	3mc, 1cr*	3mc	1mc	8
	Ratio, proportion and percent	3mc	3mc	1cr	7
Algebra	Patterns		1cr	1mc, 3cr	5
	Equations and formula	2mc			2
Measurement	Tools, techniques, formula		1cr	1cr	2
Geometry	Congruence	1mc			1
Data	Data Interpretation		2cr	4mc,1cr	7
	Uncertainty and probability		1mc	2mc	3
Total		10	11	14	35

* cr - constructed response, mc - multiple choice

In order to judge the difficulty levels, and to ensure that the test was appropriately targeted for the population to be tested, the percentage correct on each of the items was checked for the South African sample tested in TIMSS 2003, and where items were common, from TIMSS 1999. Because the South African population performed poorly on the selected items, there was a concern that the test instrument would be too difficult. To cater for this possible outcome, six items from TIMSS 2003 Grade 4 bank of items (IEA, 2005) were included. The addition of easier items did in fact generate balance in terms of difficulty level required for this research study population. The fact that the previous percentage correct statistics were accessible meant that the demand for correct targeting could be met reasonably well.

The language and names in the TIMSS 2003 items were, in some cases, adapted to make them applicable to the South African context. The main changes were name changes and some terminology that may have been unfamiliar.

7.2.4 Test situation, administration and scoring

A major goal in the test design was to maximise content coverage of the mathematical topic in order to ensure that each of the students responded to sufficient items to provide a reliable measure. A further consideration was that each student should not be overburdened by too many items and that the test should be completed within 40 minutes, which was the duration best suited to a school period. Note that the purpose of the test was not to gauge the ability of individual students but rather to investigate the multiplicative conceptual field and ascertain general response patterns through the investigation of responses, and errors, in this field.

The test instrument was structured as four parallel forms (1, 2, 3 and 4). The matrix design included twelve common items in all four booklets to provide points of comparability, and the 24 additional items were distributed across the four booklets. The test therefore comprised four booklets, each containing 12 core items and 6 extra items. The 24 non-core items were checked for difficulty level against the percent correct in the TIMSS 2003 data for the South African cohort and balanced according to difficulty level across the tests. This procedure was adopted to ensure that all four versions of the test had approximately the same degree of difficulty, and similar ranges of item difficulties.[109] While the difficulty level was distributed throughout the booklet, some easier items were situated at the beginning of the test so as to encourage students. The balance of items was slightly different owing to some items admitting a maximal score of one point, and some others, two points.

Another feature of the test design was that the items were differently ordered across the four test booklets. The purpose of the different orderings was to obtain a fair coverage of every item rather than having a common set at the end of a particular test booklet, where there was greater chance of fatigue and non-response. A further advantage of having the items differently arranged and including different items in each booklet, was that there was less danger of "copying".[110] The tests were administered to the 16 classes during a mathematics class period of about 40 minutes, in some cases by the researcher, and in other cases by the class teacher. The time was judged to be adequate to complete the 18-item test. The booklets were evenly distributed across the classes, with roughly equal numbers of students answering each of the four booklets in each class.

The multiple-choice items were scored according to an answer sheet, developed on the basis of the TIMSS 2003 Scoring Guides. The constructed-response items were marked by a subject expert who made judgements based on the scoring guides for each item.

[109] The differences were accounted for later using statistical methods.

[110] In the Rasch measurement framework "copying" and guessing are perceived to be possible problems of the test construction. These behaviours may arise when a test is pitched at too high a level for the learner.

7.2.5 Data Analysis

In addition to qualitative investigation of item responses and the overall working of the test, Rasch analysis was applied for the reasons outlined previously (see Section 7.2.1). One purpose was to investigate to what extent the instrument could be judged to exhibit construct validity, and whether the test responses per item and per student would conform to the expectations of a valid and coherent instrument. A related purpose was to estimate current proficiency levels.[111]

7.2.5.1 Rasch analysis

The analysis was conducted using RUMM 2020 software for Rasch analysis (Andrich, Sheridan & Luo, 2005),[112] The Partial Credit model (Masters, 1982), one of the family of Rasch measurement models, is able to accommodate both dichotomous data (multiple-choice, with one correct answer) and polytomous data (where an item is scored 0, 1, 2), and also to accommodate missing data.

The RUMM programme enabled further analysis of the items by partitioning the group of students attempting an item into a chosen number of class intervals of student proficiency. The number of class intervals can be set as high as 10. In this study five class intervals were judged to be most appropriate for the analysis of roughly 330 student performances, with roughly 66 students at the five focus points for core items. Where the non-core items were administered to a smaller group of students, roughly 83 students, only four class intervals were used, with roughly 21 at each of four focus points. The output of the distractor selections made within multiple-choice items was obtained in both tabular and graphic form.

When applying any of the Rasch models for measurement, the first task is to check that the data fit the model. If the data do not fit the model, adjustments have to be made to the instrument or to the data. With the initial analysis run, one item was problematic, and was removed for the purpose of analysis. A subsequent run without the item resulted in a good fit. A further adjustment was made to each of three items initially found to be problematic. These adjustments and other special procedures are described in Section 7.2.5.2 and 7.2.5.3.

[111] As explained in Section 1.4.1 Footnote 17 the term level has been used throughout this study to demarcate different groupings of items and learner proficiencies. The terms “difficulty locale” and “proficiency locale” have been identified as more suitable descriptions of the groupings of the items and the learners. For the current document the terms “difficulty level” and “proficiency level” have been retained.

[112] A prior analysis was conducted using Conquest (Wu, Adams & Wilson, 1998).

7.2.5.2 Special procedures

For the analysis, the multiple-choice items and the constructed-response items were treated differently. In the multiple-choice items missing data was scored as a blank space. For the constructed-response items the following logic was applied. *If the item was omitted by the student it was given a zero. An item was considered "not reached" if the two items preceding the item were not answered and none of the following items had been completed.* In this case the item data is taken to be missing. The reason for treating the two item types (multiple choice and constructed-response) differently is that in the South African TIMSS 2003 data and in this study, responses to constructed-response items were few.[113] presumably because they have high difficulty levels.

7.2.5.3 Misfitting items

The Rasch analysis has statistical procedures for detecting items which do not function as expected. **Item 19** (see Figure 7.1), a multiple-choice item scored dichotomously, was misfitting according to more than one of the fit statistics. Further investigation of this item was required as it presented an anomaly within the instrument. The item[114] was, however, included for purposes of qualitative analysis as it was deemed to be an important item for the overall design of the test instrument.[115]

Figure 7.1: Probability item (Item 19)

In a school there were 1 200 learners (boys and girls). A sample of 100 learners was selected at random, and 45 boys were found in the sample. Which of these (answers below) is most likely to be the number of boys in the school?

A. 450	B. 500	C. 540	D. 600

From a mathematics education point of view, it may be that the associated sub-topic probability was new in the curriculum at the time of the study and perhaps not thoroughly taught so that a great deal of guessing occurred along the whole spectrum of student abilities. An additional insight is provided by Fischbein and Gazit (1984), who hypothesised that while

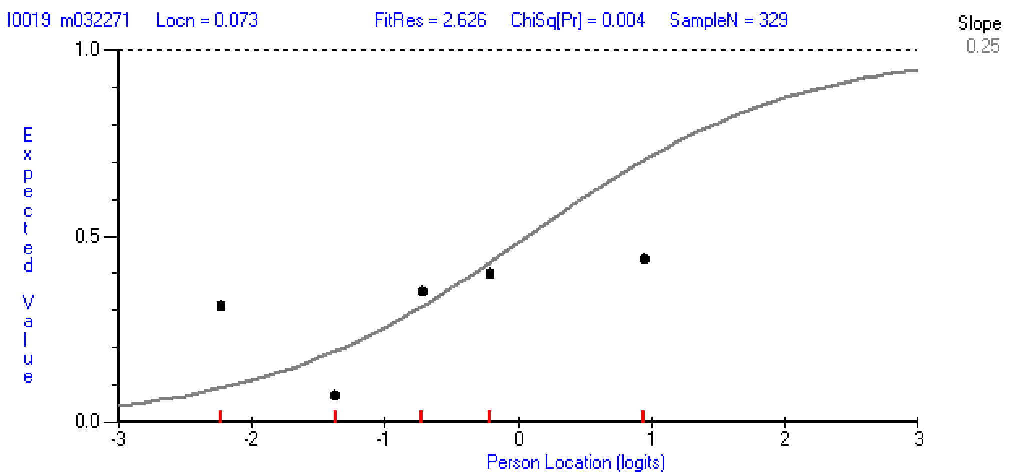
[113] This differential treatment may have impacted slightly on quality of the data, and would be reconsidered in subsequent studies.
[114] This item would most certainly be adapted for inclusion in subsequent test formulations.
[115] The item was therefore removed for the purposes of item calibration and person location specification, but reinstated for the conceptual analysis.

proportional reasoning and probabilistic reasoning have common core concepts such as ratio, the two types of reasoning differ in important respects.

The item characteristic curve for **Item 19** (Figure 7.2) provides a graphic display of the functioning of the item. In this graphical display the location of the item is fixed. The location of 0.073 (see top left corner, Locn = 0.073), indicates that the item is slightly more difficult than the item mean. The person location is shown on the horizontal axis and the probability of a successful response[116] on the vertical axis. The curve indicates the theoretical probability of a response to this item. The expectation is that students of less ability (to the left on the horizontal axis) should perform less well than students of higher ability (located to the right on the horizontal axis). The black dots represent the means of the person locations of the five class intervals of roughly equal frequencies, obtained from the partitioning of those learners who responded to the item.

The fit residual is the "standardised difference between the *observed number* of persons in the group who have the number correct and the *expected number* according to the model" (Andrich & Marais, 2006). From the graphical display it can be observed that the means of the lowest group and the highest group do not differ greatly. This outcome is unexpected according to the model. The high fit residual at 2.626 (see FitRes = 2.626, top and slightly left of centre) indicates misfitting of the item.

Figure 7.2: Item 19 Characteristic curve



The Chi Square probability is "calculated from the discrepancies between the observed means in the class intervals and the expected values according to the model" (Andrich & Marais,

[116] For dichotomous responses, scored 0 or 1, the probability of a success is equivalent to the expected value of the score.

2008). If the probability is less than 0.05, the implication is that the discrepancy between the observed mean and the expected value is larger than might be reasonable if chance alone were the explanation. The Chi Square probability of 0.004 indicates that the item underfits the model expectations. A deeper study of each such item helps to ascertain the reasons for the misfit. The item has a very low discrimination index (0.15), reflecting only a mild increase in expected value against person location. Those students with high proficiency as measured by this test have an almost equal probability of having the item correct as students of low proficiency. This outcome is consistent with a guessing strategy dominating responses.

A second item, **Item 35**, a constructed-response item, scored polytomously with 0, 1, or 2, was shown to be anomalous (see Figure 7.3). The responses to the item did not provide the ordinal relationship with ability we would expect from a polytomous item. An investigation of **Item 35** gives an indication of why the item might not function as expected. An additional complexity of the item is that the scoring memo indicated 2 marks *correct with working*, 1 mark for giving only *the correct answer*, and also 1 mark for an answer giving *working with correct reasoning but with an error in the calculation.*

Figure 7.3: Item 35 Description

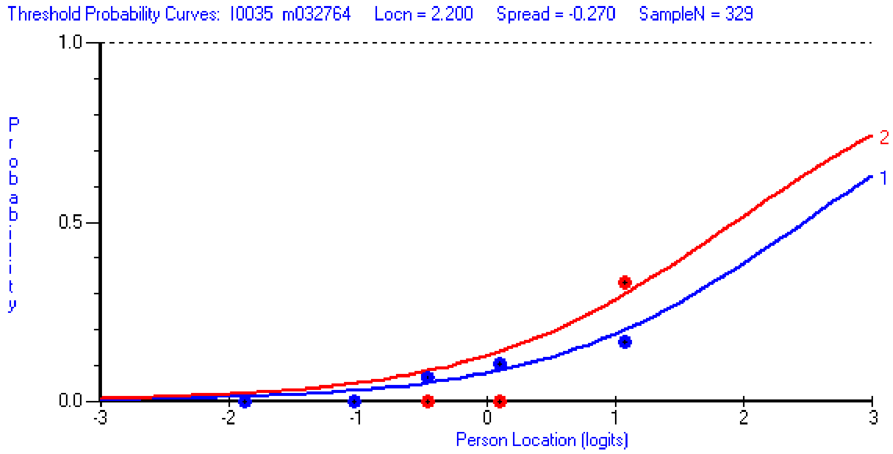
Dalene signed up for Plan B, and the cost for one month's service was 75 zeds. How many minutes did she talk that month? *Show your work.*

Plan	Monthly Fee	Rate per minute		Free minutes per month
		Day (8 am – 6 pm)	Night (6 pm – 8 am)	
Plan A	20 zeds	3 zeds	1 zed	180
Plan B	15 zeds	2 zeds	2 zeds	120

A rationale for this marking scheme in the TIMSS 2003 setting may be that proficient students would be able to perform this calculation using an accepted procedure. Less proficient students when confronted with a difficult problem may solve the problem intuitively and give the correct answer, but not feel confident enough to write out their working. This explanation resonates with lower South African achievement on TIMSS 2003.

As can be seen in the plot below (see Figure 7.4), **Item 35** was located on the horizontal scale at a point estimate 2.2, where according to the model those students with an ability estimate of 2.2 have a 50% chance of getting this item correct. This item is therefore considered difficult.

Figure 7.4: Item 35 Threshold Probability Curves for scores 1 and 2



The evidence from the empirical data is that the middle category, and hence the scoring of 1 point, occurred less frequently with this cohort of students, and we would normally expect more students in the lower region below 2.2 to attain a partially correct answer than a correct answer. If we compare the location of students for whom the probability of attaining 2 points is 0.5 with location of learners for whom the probability of attaining 1 point is 0.5, we find the latter at a higher location. Clearly this situation is problematic. The expectation is that one point should be easier to attain than two points.

For any further testing this item would need some adaptation, for example the specification of the two intended steps could be clarified. This pattern for the item may however be particular to this cohort, as the item may have functioned better for the wider range of ability and the greater number of learners in TIMSS 2003.

Figure 7.5 shows that for students located from -3 to about 2.2, there is a greater likelihood of attaining zero, the blue line. From about 2.2 there is a greater likelihood of the learner attaining 2 points. The likelihood of the student attaining one point should be easier than attaining 2 points. The separate scoring of 1 and 2 in this item does not function as expected. Because of this unexpected functioning of this item, it was decided in a later analysis to collapse the two scoring categories. The resulting item category curve, after rescoring and collapsing the categories, is a better fit to the model. Compare Figure 7.4 and Figure 7.6. The Item Category Curve exhibits a better fit after rescoring.

Figure 7.5: Item 35 Category probability curves

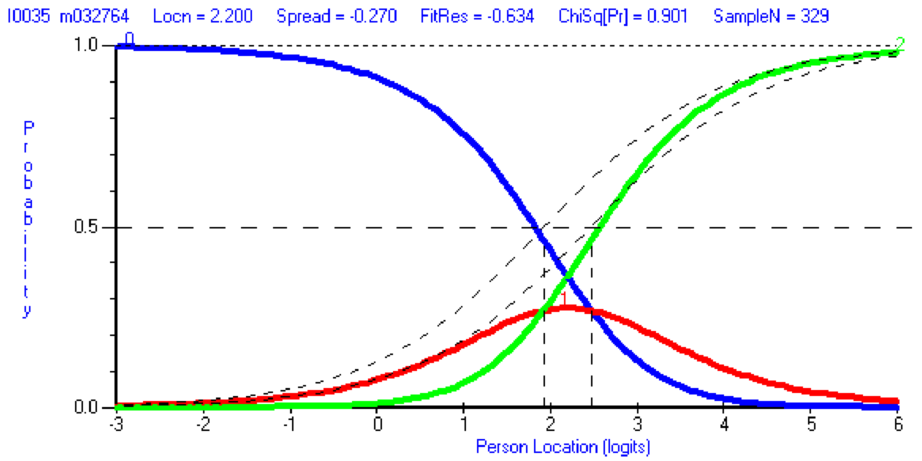
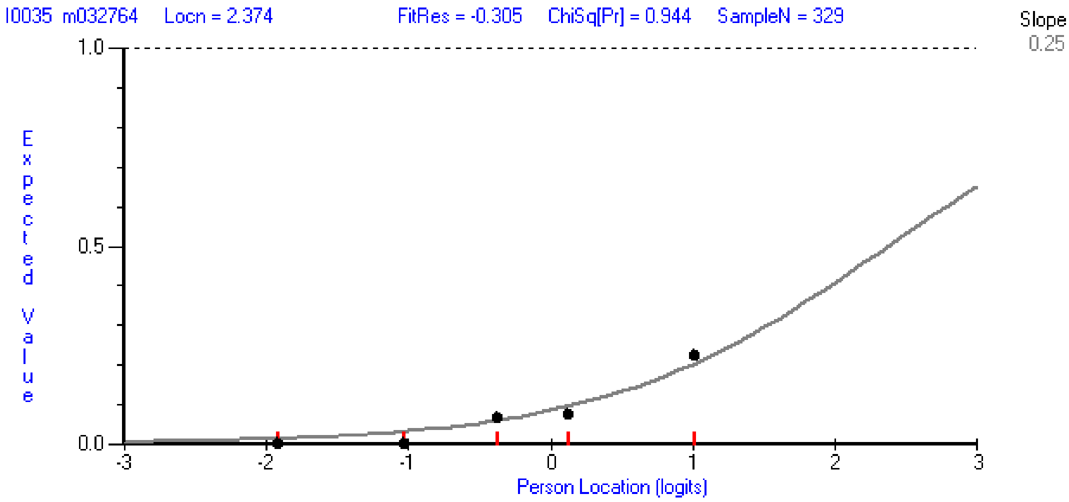


Figure 7.6: Item 35 Category curve after rescoring



Likewise a similar adjustment was instituted for Item 29, (see Figure 7.7 and Figure 7.8).

Figure 7.7: Item 29 Category probability curves

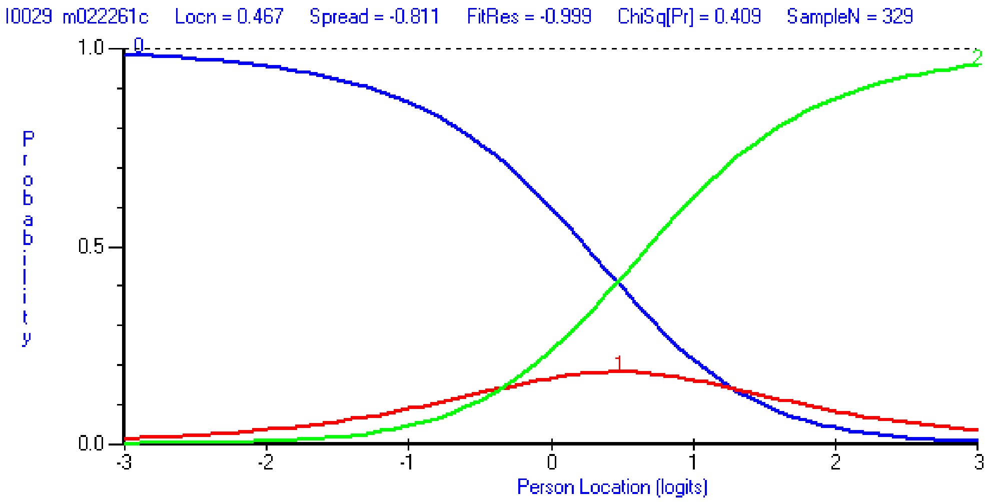
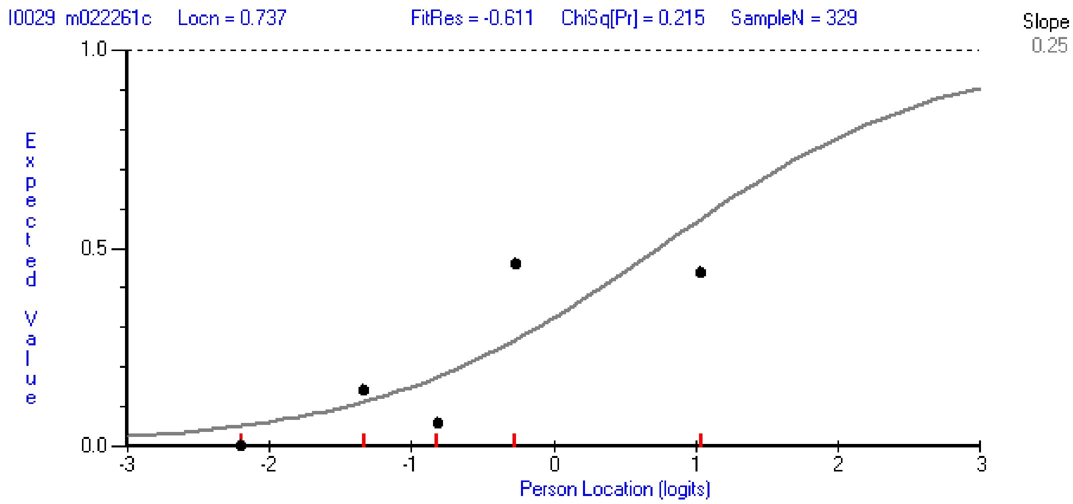


Figure 7.8: Item 29 Post-hoc adjustment



A similar adjustment was applied to **Item 29** (see Figure 7.7 and 7.8) and **Item 33**. The post-hoc adjustment in the scoring of Items 29, 33 and 35 resulted in a better fit. Note that the rescoring of the three items occurred after the item analyses had been completed. With the adaptation of these three items, the comparative location of items changed as follows:

Item 29 0.467 changed to 0.737 (difficulty increased slightly)

Item 33 -0.296 changed to -0.833 (difficulty decreased slightly)

Item 35 2.200 changed to 2.374 (difficulty increased slightly)

The locations of the other 32 items changed marginally after these three item adjustments by less than 0.02. Because the change was marginal and because the graphs and tables had already been generated the statistics calculated in the first analysis for the 32 items have been retained.

7.2.5.4 Test statistics

The Rasch analysis was run on all 35 items (RUN 1). One item had been rejected prior to this run where a typing error had slipped through in one of the four test booklets. The 12 common items were potentially answered by 330 learners, with the remaining 24 distributed across four groups with roughly 83 learners in each group. It was envisaged that each learner would answer 18 questions. However, one group would now only be scored on 17 items due to removal of one item. Some items were scored 0, 1 and 2, which meant slight variations. After 99 iterations, only 3 parameters converged in the original data set. Item 19 (discussed above) was eliminated for the second run.

In the initial run of 35 accepted items, the single student who achieved a perfect score was eliminated because according to the model, neither a zero nor a perfect score provides any information on the relative difficulty of the items. Thus, if a student had not achieved success on any of the items, that student would have been eliminated. In fact no students were eliminated on the grounds of attaining a zero. The same logic applied to items: An item which every student answered correctly or every student answered incorrectly would not give information on the relative abilities of the students tested. However, none of the items needed to be eliminated for the purposes of item calibration.

The analysis was run a second time (RUN 2) without **Item 19**. The locations arising from this run were used for generating most of the graphs and tables. **Item 19** was placed back on the Item-Person map at a location estimated from RUN 1.

After 118 iterations all 40 parameters converged. The test of fit used 5 class intervals. The model sets the item mean at 0. The person mean, determined in relation to the item mean, was –0.550, the item standard deviation was 2.081 and the standard deviation for persons, 1.253. The person separation index was 0.745 and the Chi Square probability 0.04722. This statistic indicated a fair degree of fit of the data to the model.

For a reason still to be explored, the items that were scored 0, 1 and 2, did not function as expected. There were disordered thresholds in three cases, meaning that obtaining a 2 was more likely than obtaining a 1.[117] These problematic items **Items 29, 33,** and **35** were rescored, having collapsed the marking to either right or wrong, to 0 or 1. A third analysis (RUN 3 RESCORE A), with this adjustment, resulted in the locations of most of the items varying but by less than 0.02 logits. The adjustment made with rescoring **Items 29, 33,** and **35**, led to slight changes, with the item standard deviation (SD) being 1.863. The person mean (-0.366) shifted closer to the item mean, indicating a better targeting and with the standard deviation now at 1.215. The person separation index, slightly lowered to 0.727, and the Chi Square probability lowered to 0.033092. Because the analysis had been done prior to RUN 3, the earlier graphs and therefore the locations for all items, except these three items, were retained.

Another anomaly was observed in **Items 27, 28,** and **29.** These three questions related to the same stem, with **Item 27** the easiest, and scored 0, 1 and 2, **Item 28** somewhat more difficult,

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[117] For detailed discussion on disordered thresholds, possible reasons and remedial action see Van Wyke and Andrich (2006).

though allocated only one mark, and **Item 29** the most difficult with scores 0, 1 and 2. **Item 27** was initially ranked at a higher difficulty than **Item 28**. With a second rescore (RUN 3 RESCORE B), **Items 27** and **29** were adapted. This rescoring brought **Item 27** in line with what was expected and found its rightful place at a lower difficulty location.

The above discussion and results provide an example of investigating the instrument itself for unsatisfactory functioning and, if necessary, making post-hoc adjustments. For any subsequent testing these items will be checked and perhaps adapted. The point is that the items are not sacrosanct. They are a means for measuring a construct and if they do not function adequately for that purpose, they have to be adapted for future testing, or in some instances dropped.

Table 7.4: Statistical comparisons depicting adjustments

Statistic	**RUN 2**	**RUN 3** RESCORE A	**RUN 3** RESCORE B
Item mean	0	0	0
Item standard deviation	2.081	2.091	1.863
Person mean	-0.550	-0.536	-0.366
Person standard deviation	1.253	1.276	1.215
Person separation index	0.745	0.743	0.727
Chi Square probability	0.04722	0.033092	0.062485

7.2.5.5 Person – item locations

As can be seen in Figure 7.8 (also, Figure 7.9, and Figure 7.10),[118] the spread of both items and persons is wide and the associated regions overlap substantially. The instrument would become more sensitive to the development of proficiency in this mathematical field for the context of learners of this study if further items were developed around the -3 logit location and between location 1 and 2 logits.

[118] Note the adjustments to items change the picture slightly, but notably at the extremes, for example the learner located at the extreme above the 7 logit location with readjustment attained the total score and was therefore removed from this analysis.

Figure 7.9: Person-Item location distribution (RUN 2)

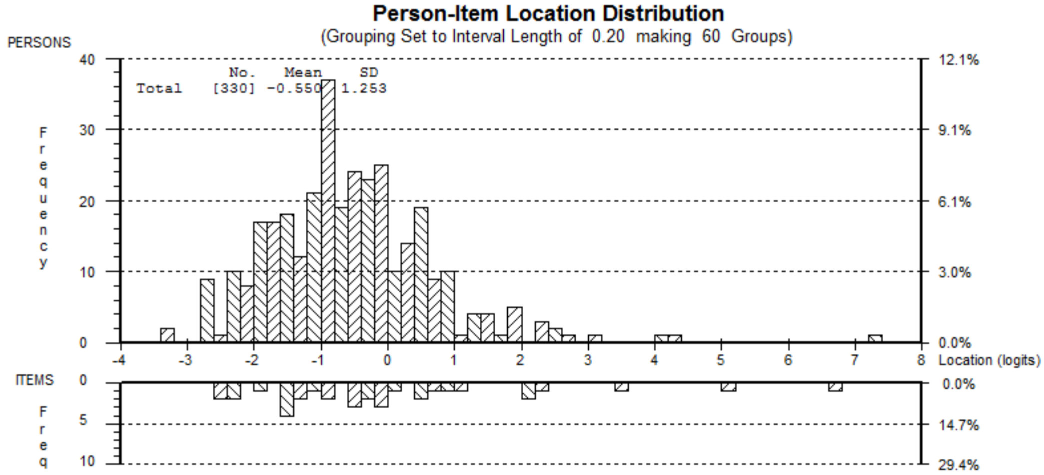


Figure 7.10: Person-Item location distribution (RUN 3, RESCORE A)

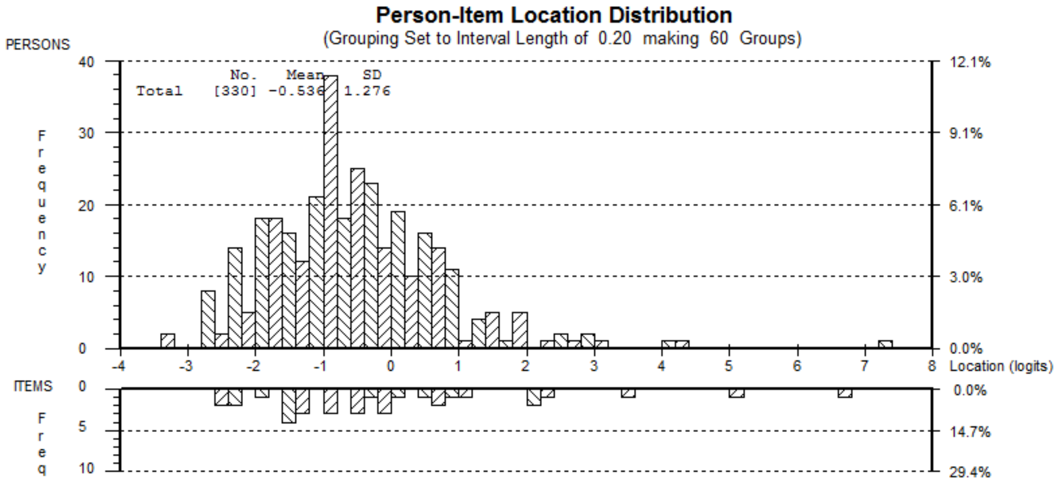
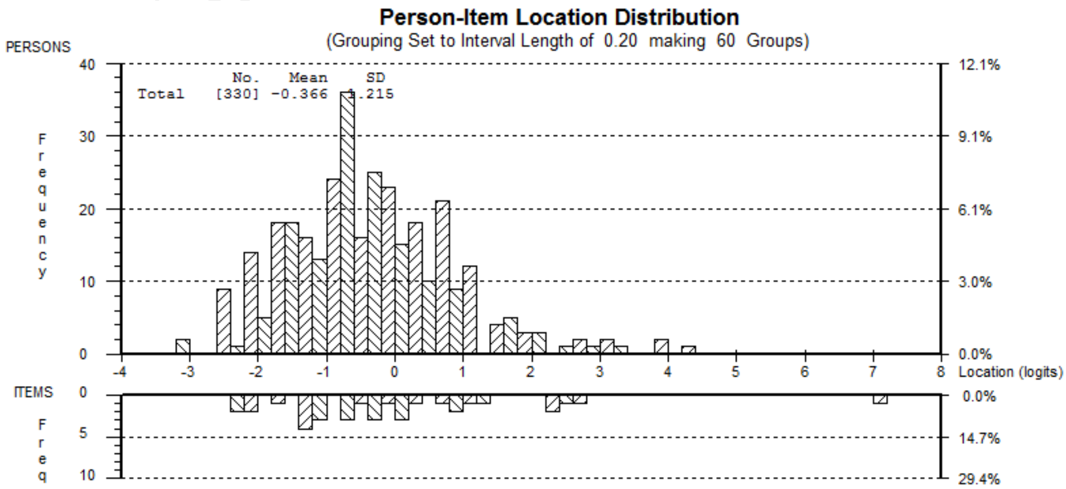


Figure 7.11: Person-Item location distribution (RUN 3, RESCORE B)



The functioning of the test instrument in this research study for the purposes defined was fairly good. The statistical indices indicate that the constructed test satisfies the unidimensionality requirement in the sense that dimensionality may be complex rather than singular (see Chapter 5). For specific treatment of unidimensionality, see Andrich (2006). The difficulty level of the items was spread across the range of student proficiency thereby providing an adequate initial measure of reliability or precision.

From the person-item distribution, students at particular focus points indicating graded levels of proficiency were selected for interviews. The items to be used in the interviews were selected for their pertinence in terms of the multiplicative conceptual field, in particular the concept ratio and the cognitive construct proportional reasoning.

7.3 Analytic framework for item analysis

The items selected for the test instrument were chosen on the basis of including multiplicative structures. A first level analytic category was premised on items falling predominantly into one or other subtopic, namely *fraction measure (1), ratio, rate and proportion (2), percent (3), probability (4)* and then items which required *pre-algebraic reasoning (5)*. Given the nature of problem situations in mathematics it is inevitable that one or more subtopic, operation or symbolic notation is found in each problem situation (item). Where non-uniqueness applied, most items have been categorised into only the associated major subtopic, but occasionally an item applies to each of two subtopics (see Table 7.5).

Vergnaud (1988), drawing from empirical work in the multiplicative conceptual field, provided analytic categories that determine the complexity of items (Vergnaud, 1988, p. 148). These categories have been adapted for use in this study as follows:

- The context in which the problem is presented
- The type of situation modelled by multiplication and division, which intersects with the rational number subconstructs
- The mathematical structure (from a mathematical perspective, concepts and theorems)
- The mode of representation of the problem, that is, natural language, natural numbers, fraction notation, decimal or percent notation, diagrammatic form, or symbolic notation.
- The number range and value in the problem situation, that is, the range of natural numbers; 0 to 30; 31 to 100; 101 to 1000; large numbers though in multiples of 100 or 1000; fraction values, 1/2s and 1/4s; 1/3s; decimal numbers or percent language.
- Cognitive processes (from a psychological perspective), include concepts-in-action (categories to organise relevant information) and theorems-in-action (propositions from which inferences are made). These critical constructs will be discussed in Chapter 8. In the

current chapter the category, *response processes*, is used to classify the type of activity that may have been applied to solve the problem.

7.3.1 Contextual factors

Problem situations, or test items in the case of this study, may involve an everyday context which enables the learner to identify the elements of the situation, or may invoke a mathematical context, in which the elements are mathematical objects and operations. At a second level the particular context may be familiar, for example an athletics track, or it may be unfamiliar.

7.3.2 Type of situation

The classification of *situations* modelled by multiplication and division are described in Chapter 6 (see Section 6.2.1, Table 6.3). The situations listed are *proportional shares (1), equal measures (2), rate (3), measure conversion (4), multiplicative comparison (5), part-whole (6), multiplicative change (7), Cartesian product (8), rectangular area (9),* and *product of measures (10).* As explained previously, Vergnaud (1988) collapses these measures into *isomorphism of measures* (1 to 7), *product of measures* (8 to 10) and *multiple proportion*.

Against this framework of situations requiring multiplicative structures, Parker and Leinhardt (1995) distinguish nine *comparative problem classes* for percent that overlap somewhat with the classes of multiplicative structures listed in Table 6.2. Drawing from the comparative problem contexts for percent (see Table 6.6 in Chapter 6, Section 6.2.3.1), three major categories that encompass percent problems are identified, namely *fraction measure*, which we may define as either equal measure, or proportional part, *multiplicative comparison*, and *multiplicative change*. Four categories are identified for each of *multiplicative comparison* and *multiplicative change* type situations.

The plethora of meanings attributed to the fraction symbol may be identified in different types of problem situations. (For explanations see Chapter 6, Table 6.4, Section 6.2.2). These meanings intersect with and extend the ratio subconstructs identified by Kieren (1976) and have been extended in this study.

The type of situation modelled by multiplication and division, and the particular meaning attributed to the fraction symbol is closely linked to the underlying mathematical structure.

7.3.3 Mathematical structure

Vergnaud (1988) insists on identifying the underlying mathematical structure as this identification may lead to possibilities for generalisation. Many of the problem types exhibiting a multiplicative structure can be depicted as comparisons between and within measure spaces. Measure spaces are discussed in detail in Chapter 6, Section 6.2.1.3. Where possible the links will be made from a current problem type to the more abstract mathematical structure.

As discussed in Section 7.3.2, Vergnaud (1983) identifies three distinct subtypes within multiplicative structures, *isomorphism of measures*, that consist of "simple direct proportion between two measure spaces, M_1 and M_2" (p. 129), *product of measures*, that consist of the "Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3" (p. 134), and *multiple proportion*, which has characteristics similar to the *product of measures*, in that "a measure-space M_3 is proportional to two independent measure-spaces M_1 and M_2" (p. 138). The items discussed in this chapter are restricted to *isomorphism of measures*, the direct proportion of two measure spaces, where the subclass of problems includes *multiplication* and *division* (*partitive* and *quotitive*) and the *rule of three* (see Figure 6.2).

Fractions, ratio and rational numbers may also be described as the direct proportion between two measure spaces. Sharing a whole into parts involves "direct proportion between the shares, and the magnitude to be shared" (Vergnaud, 1983, p. 161). In the case of a discrete magnitude the elements can be counted, but in the case of a continuous magnitude the measure, the area for example, is not known, and therefore must be expressed as a fractional quantity (see Figure 6.10).

A particular problem may also be classified in terms of rational number subconstructs that present within that problem, part-whole, measure, ratio, operator and quotient (Kieren, 1976, see also Sowder et al., 1998). The operator construct has two forms in the Vergnaud (1983) analysis. A *scalar operator* links two quantities of the same kind and being the quotient of two quantities of the same dimension, expressed in the same unit, it has no dimension and no unit. A *function operator* (as defined here) links items of different dimensions, and therefore has a composite unit. This composition is multiplicative rather than additive. The learner therefore has to make the shift from an additive contrast to a multiplicative contrast.

In proportion and comparative problems, there are two categories of ratio problems, the *inclusive case* for example, *Three-fifths (or 60%) of the class are girls* (Item 26), and the *exclusive case***,** for example, *The price R800 is increased by 20%* (Item 8). (Vergnaud, 1983, p. 163).

As noted in Chapter 6, percent problems include all the complexity of rational number, but in addition have a concise language that in some cases obscures the underlying relationships. As was shown in Chapter 6 (Table 6.6), percent problems may also be represented by the direct comparison of measure spaces, that is characterising *percent* as one *measure space* and *the quantity* or magnitude to which the percent is being applied, as the *second measure space*. In addition to the analysis that presents the problem as the direct comparison between two measure spaces, we classify the percent problem contexts as a *fraction part, change relationship* or as a *comparison relationship* (see Chapter 6, Table 6.6). The characteristics of multiplicative structures are identified in problem situations and learner responses. Describing problem situations in terms of measure-spaces allows the identification of the variables, and the relationship between variables in problem situations.

7.3.4 Mode of representation

The representational form, *natural language*, *pictures*, *diagrams*, *tables*, *graphical forms* and the use of abstract *symbols* are all factors contributing to the complexity of the problem. As to be expected this category overlaps with the category *situation type* as well as mathematical structure, for example the fraction symbol may refer to a situation type, part-whole.

7.3.5 Number range and value

Both the number range and the number value contribute to the difficulty level of a problem. Numbers below 100, for example, may be easier for learners to work with than numbers in the thousands. In addition there may be more occasions for errors in the calculations. However operations with unit thousands, for example "20 thousand divide by 5" may be no more difficult than "20 divide by 5", provided there is an elementary understanding of place value. Division item difficulty is affected by the number values of the dividend and divisor, for example whether or not there is a common factor.

Natural numbers at a particular level of mathematical development, may present less difficulty than fractions or decimal numbers. Again the difficulty of division with decimal numbers is determined by the presence or absence of a common factor.

7.3.6 Response processes and procedures

Vergnaud (1979) states that while the study of task difficulty is the first priority, the specification of the complexity of procedures and behaviours, is also necessary. There are many ways to solve a problem situation or item, some of which are more mathematically sophisticated than others.

> For the same task, the same problem or the same situation, pupils will offer a variety of procedures. There is not only one way of getting the right answer, there is not only one wrong answer and there is not only one way of obtaining the same wrong answer. … Among the successful (responses) there are differences which show different degrees of generality (Vergnaud, 1979, p. 266).

The procedures exhibited by learners, either right or wrong, cannot be regarded as equivalent from a cognitive point of view. Some analysis in terms of procedures and processes is undertaken in the analysis of distractors, albeit inferred procedures gleaned from both the literature and experience.

The research conducted by Vergnaud and associate researchers (1983, pp. 140-145) elicited a variety of procedures (pp. 145-149). The variety of procedures and consequent analysis are discussed in detail in Chapter 8, as these categories are critical for the analysis of the learner responses in the interviews. The *correct procedures* are classified into five subcategories, *scalar, scalar decomposition*, *function, unit value*, *rule of three*, (pp. 145-146). The incorrect strategies were also classified into groups (pp. 147-149), which in many cases are based on some elements of the corresponding correct strategy (p. 147). Here we note the importance of measure spaces for making explicit the underlying mathematical structure and in some cases venture an inference about the strategy that may be used.

The particular errors found by Parker and Leinhardt (1995), are “drop the percent sign”, abandoning “natural sense making” and exhibiting misconceptions relating to a part-whole rather than a ratio conception (p. 428). These categories are drawn on in the item analyses.

The cognitive domains defined by Mullis et al. (2004) as *knowing facts, procedures and concepts* (1), *solving routine problems* (2) and *reasoning* (3) are used in TIMSS 2003. This categorisation, which has its antecedents in Bloom’s taxonomy, is found in some respects difficult to apply in the categorisation of mathematics problems. The mistakenly perceived hierarchical arrangement, in the sense that *reasoning* is regarded as demanding greater cognitive engagement than *knowing facts,* is countered in TIMSS documentation where it is stated that the cognitive domain does not correlate directly with the difficulty level (Mullis et al., 2003, p. 25). The perspective taken in this thesis is that the difficulty level of items is interactively determined by other factors such as those already listed, namely the mathematical structure, the context, mode of presentation, number value and number range. The category title, *response processes and procedures*, is reserved for the type of activity, mathematical in form, that may be applied to solve the problem.

7.4 Item analysis

Informing the item analysis is the Person-Item map,[119] or Wright map, after Benjamin Wright (see Figure 7.12) where the items are arranged in levels of difficulty. In principle there may be many relevant and useful ways of partitioning the continuum of values along which the item difficulties are represented. For the purposes of analysis in this study, bands of one logit width have been demarcated. The items are grouped according to difficulty bands one to seven, according to logit band location. Items located below -2 logits are classified as **Level 1** items for the purposes of this research study. **Items 1, 2, 13** and **17** are located at this level. Items from -2 and up to, but not including -1, are classified as **Level 2** items. **Items 4, 6, 11, 12, 20, 22, 23** and **24** are clustered at this level. The pattern continues up the Person Item map, to Level 7.

The items, arranged in hierarchical difficulty order in the Person-Item Map, are shown in Table 7.5, where the empirical levels are derived from the locations on the Person-Item map. Here the items are shown in five substrands, categorised for analysis purposes. Note that **Items 4, 11,** and **24** are listed in two strands where this duplication makes sense from a conceptual perspective, but only discussed under the heading of major pertinence. **Item 14** is also listed in two strands but discussed in both from slightly different perspectives.

119 The construction of the map is partially explained in Chapter 5; more detail can be found in the Rasch literature (e.g. Wright & Stone, 1979, Van Wyke & Andrich, 2006). The person-item map was used in Wright and Stone (1979), under the heading “map of the variable” (pp. 119-120). The term “Wright map” was used by Wilson (2005, pp. 90-110).

Person-Item Map[120]

```
PERSONS [locations=estimated proficiency]      ITEMS [locations=estimated difficulty]
                                      o |
7.0                                     |
                                        | 34
                                        |
6.0                                     |
                                        |
                                        |
                                        |
5.0                                     | 26
                                        |
                                        |
                                        |
                                      o |
4.0                                   o |
                                        |
                                        |
                                        | 27
                                        |
3.0                                   o |
                                        |
                                      o |
                                      o |
                                     oo | 35
2.0                                     | 30 31
                                    ooo |
                                      o |
                                     oo |
                                     oo |
1.0                                   o | 28
                                  ooooo | 7
                                  ooooo | 32
                             oooooooooo | 10 29
                                ooooooo |
0.0                               ooooo | 18 19
                          ooooooooooooo | 9 8 15
                           oooooooooooo | 33 16
                           oooooooooooo | 5 25 14
                             oooooooooo |
-1.0                ooooooooooooooooooo | 3 21
                            ooooooooooo | 23
                                 oooooo | 4 24
                              ooooooooo | 20 6 12 11
                              ooooooooo |
-2.0                          ooooooooo | 22
                                   oooo |
                                  ooooo | 2 17
                                      o | 13 1
                                  ooooo |
-3.0                                    |
                                      o |
                                        |
```

Items of greater difficulty level and learners of greater proficiency are located at the top of the map. The symbol “o” represents two learners.

Two learners are located at proficiency level above 7.0 logits. **Item 34** was estimated as the most difficult, with **Item 26**, and **Item 27**, the next most difficult respectively.

The item mean is set at zero, with progressively more difficult items above zero and progressively easier items below zero.

The average learner location, calibrated in relation to the items is slightly less than the item mean in this study.

Items of lowest difficulty level and learners of lesser proficiency are located at the bottom of the map.

The symbol “o” represents two learners. Two learners are located at proficiency level below -3.0 logits. **Items 1, 2, 13** and **17** were estimated as the least difficult.

Figure 7.12: Person-Item map (Wright map)

[120] This person-item map depicts the output from Run 2. The location of Item 19 was estimated from Run 1.

Table 7.5: Items presented in substrands of the multiplicative conceptual field.

Empirical levels	1 (-∞ , -2)	2 [-2, -1)	3 [-1, 0)	4 [0, 1)	5 [1, 2)	6 [2, 3)	7 [3,∞)	Total (repeat items)
Fraction	**13**; **17**	**22**; 4; **24**; 11	**14**; **16**	**18**; **32**				7 (+ 3)[1]
Ratio, rate, proportion	**1**; **2**	**20**	**3; 5**; **9**; **15**; <u>**33**</u>	**10**		**30**; **31**; <u>**35**</u>	<u>**34**</u>	12
Percent		**4**	**8**	**7**			**26**	4
Probability		**11**	**14**	**-19**				3
Algebra		**6**; **12**; **23**; 24	**21**; **25**	<u>**29**</u>	<u>**28**</u>		<u>**27**</u>	9 (+ 1)
Total	4	8 (+ 3)	10 (+ 1)	6	1	3	3	35

Bold text: discussed in "row" section
<u>Underlined text</u>: items clustered for discussion purposes
[1] The number in brackets indicates counts of items assigned to two categories. For example, Items 4 and 11 are listed in Fractions but discussed in the substrands, Percent and Probability, respectively. Item 14 is discussed in both Fraction and Probability.

7.4.1 Item by strand analysis

For each strand, a summary of items is presented in table form indicating the *problem context*, *type of situation*, the *mathematical structure*, *notation and representation*, *number range and value*, and inferred *response processes and procedures* (see Summary of Fraction Items, Table 7.8). In the interest of conserving space, the following labels are used, **Context, Situation, Mathematical structure, Notation, Range** and **Response.**

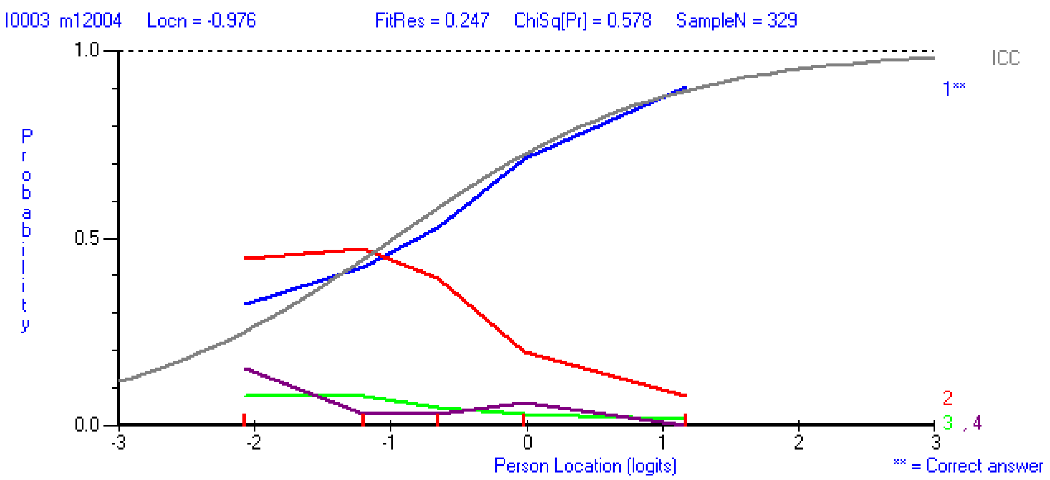
7.4.2 Item analysis

In addition to a mathematical analysis of each of the items, additional information is provided by distractor analyses. The RUMM software (Andrich, Sheridan and Luo, 2005) used to perform the analysis provides a graph showing the frequency of the choice of response options for learners located at different proficiency levels. An example of Item 3 in reduced format (see Figure 7.13) and the associated multiple choice distractor graph (Figure 7.14) is presented for purposes of explication. For the purposes of analysis the group of learners is divided into quintile groups with approximately 65 learners in each quintile group for most of the items or quartile groups for some items, especially when fewer learners encountered the item. Some 12 core items were administered to 330 learners; the other 24 items were distributed across 4 groups (see matrix design, Chapter 4, Section 4.5.3).

Figure 7.13: Item 3 Description

3. Thabang can run 4 times around the track in the same time that Tshepo can run 3 laps. When Thabang has run 12 laps, how many laps has Tshepo run?
A. 9 B. 11 C. 13 D. 16

Figure 7.14: Item 3 Multiple choice distractor plots by quintile group



In the graph depicting the distractor analysis for **Item 3**, the overall score locations are exhibited on the horizontal axis (see Figure 7.14). The vertical axis shows the probability of attaining a correct response. The original quintile groups for each item reflect and categorise the results of learners on the test as a whole. The mean overall score of learners in each ordinal group is marked on the horizontal axis by a vertical red tick.

The **grey line** indicates the model expectation for this problem. The **blue line** represents **Option A**. In the case of this problem the blue line indicates the learners who answered correctly, that is learners who chose **Option A**. Note that the highest quintile, whose mean is located above one logit (short vertical red tick furthest to the right), the estimated probability of obtaining a correct answer is 90%. For the second highest quintile located just below zero, the probability of a correct response falls to about 70%. For the third highest quintile, located just below -0.5 logits, the probability of a correct response is just over 50%. For the next two quintiles, in descending order, the probability of a correct response is just over 40%, then just over 30%. A similar analysis may be conducted by following the **red line**, that is those learners who selected the incorrect **Option B** for Item 3, the green line for those learners who selected incorrect **Option C**, and the purple line for those learners who selected incorrect **Option D**. The information in Figure 7.14 is presented in tabular form below (see Table 7.6).

Table 7.6: Item 3 Inferred procedures for multiple choice responses

Option	Total	Line	Inferred procedure	Mean locations of quintile groups				
				-2.078	**-1.194**	**-0.657**	**-0.023**	**1.170**
A. 9	58%	blue	Correct ratio reasoning 4:3::12:**9**	32%	42%	53%	71%	90%
B. 11	32%	red	Additive reasoning, 3 is one less than 4, 11 is one less than 12. 4:3::12:**11**	45%	47%	39%	20%	8%
C. 13	5%	green	Interchanging the ratio pairs, using additive reasoning 4:3::**13**:12	8%	8%	5%	3%	2%
D. 16	5%	purple	Interchanging the ratio pairs, multiplicative reasoning 3:4::12:**16**	15%	3%	3%	6%	0%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

The **choice of distractor** is given in the first column, the **total percentage** of learners selecting each option in the second column, the **colour code** of the **line graph** in the third column and the **inferred mathematical procedure** in the fourth column. The **mean locations of the quintile groups** are provided in the last five, or in some cases four, columns. These locations correspond to the vertical red ticks on the horizontal axis. Thereafter, the percentages (rounded) of the learners, within each quintile group and selecting each of the distractors, is presented in columns directly below the location.

The logic for the order of the distractors is as follows. The correct distractor is positioned in the top row. Thereafter the order is determined by the most common distractor. In some cases the distractor selected by the highest quartile or quintile group, is selected for ease of comparison. The logic for this arrangement is that the distractor that ensnares learners in the highest group could potentially be the distractor most close to the correct answer.

This disaggregated information provides the researcher, and teacher, with more detailed information. The single statistic, that 58% of the class selected the correct response, is replaced with more useful information about different subgroups, for example correct item performance ranges from 90% (the highest quartile), to 32% (the lowest quartile). The additional information about the options selected provides insight into what kinds or conceptual errors may have been made. For example, learners in the highest quartiles choosing B may have been careless and selected the first option, or there may be an incomplete conception of multiplicative thinking.

For four of the items, two from each of the *Ratio, proportion and rate*, and the *Percent* strand interviews were conducted to provide more detailed information. The analysis presented here for 35 items, remains at the level of a conceptual analysis of items and the analyses provided by the Rasch model, however some inferences about response processes are included.

Guidelines for reading the analysis

In summary, for each substrand the following analyses are conducted and presented.

1. *International comparison***.** A *comparison of the item percent correct* with the TIMSS 2003 International, the TIMSS 2003 South Africa, and this research study cohort.
2. *Hierarchical development and analysis of substrand***.** For each substrand as a whole, we present
   - A person-item map indicating the calibrated difficulty levels of items, and estimates of learner location.
   - A summary analysis of items for each substrand in terms of analytic categories.
3. *Item analysis and proficiency level per subgroups per substrand.*[121] (See Appendix D). For each item, or group of items, the following analysis is presented,
   - A description and comparison in terms of the analytic categories of most interest,
   - A description of the underlying mathematical structure,
   - A distractor analysis noting the inferred rationale for both the correct and incorrect selections or answers, in conjunction with an analysis of quartile or quintile groups,
   - Identification of critical findings and possible threshold constructs which distinguish the group who answer correctly and by inference understand the concepts involved, from the learners for whom knowledge of the concepts has not yet been demonstrated.
4. A *summary of the themes* observed across the strand.
5. A final *summary analysis of level descriptors* and associated errors by quartile groups.

[121] The Item analysis and proficiency level subgroups per substrand are located in Appendix C2. The *strand by level* analysis for each substrand in this Appendix will provide the supporting data for the statements in the summary findings in discussion of each substrand.

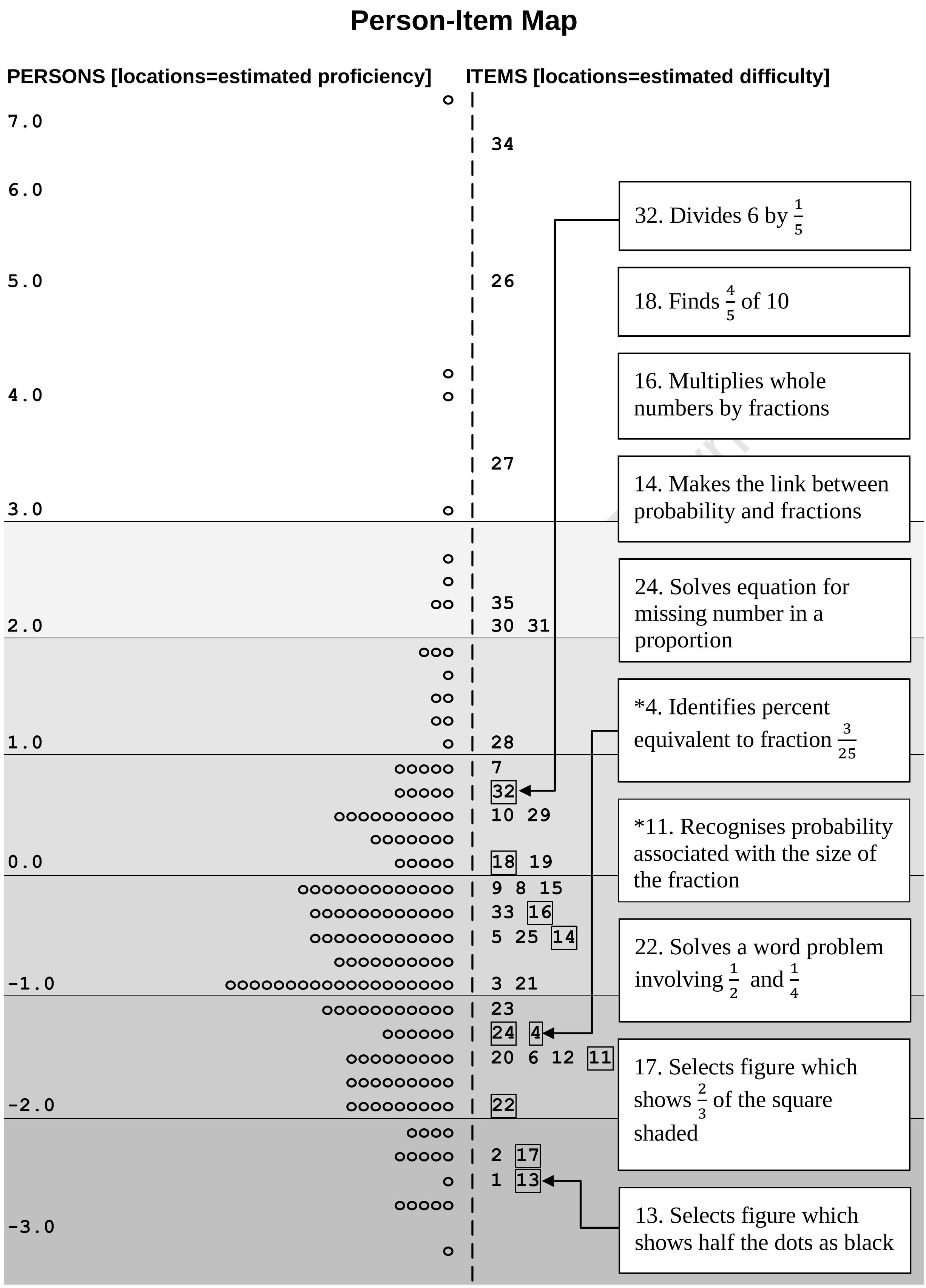
7.5 Fraction item analysis

In the test instrument, ten items were identified for the fraction substrand. A comparison of these items in terms of aggregated percentage correct across TIMSS 2003 International, the TIMSS 2003 South Africa, and the group tested in this study, is presented in Table 7.7. Three items, Items 13, 17 and 22, sourced from the TIMSS Grade 4 item bank, do not have South African data. The Grades 7, 8 and 9 learners in this study, on average, perform at an equivalent level to the TIMSS international percentage for the easier items, but for the more difficult items these learners exhibit less proficiency than the international cohort. All items are located on the *Person-Item Map* (see Figure 7.14), where both item difficulty and learner ability are located on the same scale. *A Summary Analysis of Fraction Items* is provided in Table 7.8.

Table 7.7: TIMSS 2003, TIMSS SA 2003[122] and Research Group comparison

No	Itemcode in TIMSS	Item description	Percent correct			
			TIMSS 2003 int.	TIMSS 2003 SA	Study group %	n
13	M011001	Recognises one-half of a set of objects given sets of objects with a fraction of the set shaded			85	(83)
17	M012044	Recognises a familiar fraction represented by a figure with shaded parts			79	(84)
22	M012119	Solves a word problem involving $\frac{1}{2}$ and $\frac{1}{4}$			77	(81)
11	M022252	Given the set of possible outcomes expressed as fractions of all outcomes, recognizes that probability is associated with size of fraction	60	34	61	(82)
4	M032570	Identifies a percent equivalent to a given fraction with a denominator, a factor of 100	55	33	62	(330)
24	M012040	Solves equation for missing number in a proportion	68	26	68	(83)
14	M022146	When given the possible number of outcomes and the probability of successful outcomes, solves for the number of successful outcomes	50	26	47	(83)
16	M022004	Solves a multi-step problem involving multiplication of whole numbers by fractions	48	27	43	(83)
18	M012001	Finds $\frac{4}{5}$ of a region divided into 10 equal parts	49	16	33	(84)
32	M022156	Solves a one-step word problem involving division of a whole number by a unit fraction	38	7	21	(84)

[122] Items 13, 17 and 22 were selected from the TIMSS 2003 Grade 4 items. There are therefore no South African data.



o = **2 Persons;** * = Items are discussed in other strands as well.

Figure 7.15: Person-Item location highlighting “fraction” items

The analysis of items across the levels is summarised under the headings *critical findings*, the *discussion of existing theory*, and *assessment instrument recommendations*. For detailed item analysis, distractor analyses, and strand by level summaries see Appendix B.

7.5.1 Critical findings: Fraction items at Levels 1, 2, 3 and 4

The empirical hierarchy, resulting from administering the items and conducting a Rasch analysis, indicates items using natural language and diagrams to be easiest (**Items 13** and **17**). More difficult are simple additions and subtractions with 1/2s and 1/4s (**Item 22**), and then conversions to equivalent fractions (**Items 24, 14** and **16**). Multiplying fractions and dividing fractions were found to be most difficult (**Item 18** and **32**).

Items at **Level 1** (Grade 4 items) were successfully answered by about 80% of learners (see Appendix B, Fraction item analysis. Proficiency at this level indicates mastery of fraction concepts, understood as parts of wholes, where the whole may be a continuous quantity or a set of discrete objects. At **Level 2**, proficiency with addition of simple fractions and equivalence concepts is indicated; these items were successfully answered by just over 60% of learners. At **Level 3**, just over 40% of learners showed ability to multiply fractions, and at **Level 4** roughly 30% of learners were found to have correctly answered a problem requiring division by fractions. In **Items 22, 18** and **16**, the errors suggest that a large proportion of learners only consider the numerator in fraction problems, evidence of little understanding of the concept of a rational number, in that the ratio of numerator to denominator is not considered (see Appendix B, for detailed analysis by level).

7.5.1.1 Confirmation and elaboration of theory

Greer (1992) describes the transition from multiplication of natural numbers to multiplication of rational numbers as a critical transition. It is noted in this research study that the use of fraction notation makes an item more difficult than an item in which natural language is used for example **Items 13, 1, 2,** and **17** (locations -2.584 to -1.877). The five items higher on the difficulty scale, **Items 22, 20, 6, 12,** and **11** (locations -1.557 to -1.402) used fraction notation and according to the empirical results were found to be more difficult than the natural language of the first four items (Levels 1 and 2, see Figure 7.14).

The everyday contexts, where the use of *half* is written as natural language was easier than items where fraction notation was used. In terms of problem type, recognition of fractions was easier than adding, which was easier than comparing fractions. Equivalence relations and multiplicative (and division) relations were most difficult for this group of learners. For this cohort, the multiplication of rational numbers is a concept not fully acquired by the majority of learners.

7.5.1.2 Assessment instrument recommendations

While this test covered a number of ‘fraction’ concepts there were omissions, notably the ordering of fractions, and the addition and subtraction of more complex fractions. Moreover, decimal fractions were not specifically selected and therefore were only found in one item. The inclusion of decimal fractions would certainly have extended the difficulty level of the ‘fraction’ questions. Of the items in this strand it is only **Items 16** and **18** (levels 3 and 4) and **Item 32** that provided a challenge for learners in the highest quartile. In this test there were no fraction items that extended learners beyond what was defined as **Level 4**, although fraction notation and concepts are embedded in the other strands.

Table 7.8: Fraction items (Levels 1, 2, 3 and 4)

	Description	Context	Type of situation	Mathematical structure		Notation	Number range	Response process
Level 1 [-∞, -2]	13. Selects the figure in which half the dots are black?	Diagram (dice-like)	Part-whole comparison of discrete quantities	$\frac{3}{6}=\frac{1}{2}$	M_1 M_2 1 3 2 6	Diagram, natural language	Halves, ≤ 10	Recognises 3 parts of six as a half
	17. Selects figure which shows $\frac{2}{3}$ of the square shaded?	Shaded shapes	Part-whole comparison of continuous quantities	2 out of 3 equal parts equal to $\frac{2}{3}$	M_1 M_2 2 2 3 3	Diagram, natural language	$\frac{2}{3}$	Recognises 2 shaded parts from 3 in a geometric shape
Level 2 [-2, -1]	22. Solves a problem, adding and subtracting $\frac{1}{2}$ and $\frac{1}{4}$	Every day, sharing cake	Part-whole, inclusive fraction Addition of fractions	$\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$		Natural language, fraction notation	$\frac{1}{2},\frac{1}{4}$	Adds fraction measures
	24. Solves equation for missing number in a proportion	Symbolic	Multiplicative comparison Quotitive division	$\frac{12}{n}=\frac{36}{21}$, $\frac{a}{c}=\frac{ax}{cx}$	M_1 M_2 12 36 x 21	Fraction notation, equation form	≤ 63 (in answer)	Identifies multiplicative operator in equivalence relation
Level 3 [-1, 0]	14. 30 students. 1 in 5 probability that student less than 13 years	Everyday context, probability	Multiplicative change Ratio comparison Partitive division	$\mathrm{p}=\frac{1}{5}$ Count = 30 $\frac{1}{5}=\frac{x}{30}=6$	M_1 M_2 1 x 5 30	Natural language, fraction notation	≤ 30	Recognises equivalence relation, applies multiplicative operator
	16. Multiplying whole numbers by fractions	Parts of whole quantities	Multiplicative change Partitive division	$(\frac{4}{5}$ of $45)$ - $(\frac{2}{3}$ of $45)$	M_1 M_2 4 x 5 45	Diagram, natural language	$\frac{2}{3},\frac{4}{5}$ ≤ 45	Applies operator subconstruct
Level 4 [0,1]	18. Finds $\frac{4}{5}$ of 10	Finding a part of a whole	Multiplicative change Partitive division	$\frac{3}{10}+\frac{x}{y}=\frac{4}{5}$ $\frac{4}{5}\times\frac{2}{2}$ $\frac{3}{10}+\frac{x}{y}=\frac{8}{10}$	M_1 M_2 4 x 5 10	Diagram, natural language	≤ 10 $\frac{4}{5}$	Applies fraction equivalence, identifies multiplicative operator
	32. Division of 6 by $\frac{1}{5}$	Measuring context	Multiplicative change Quotitive division	6 kg / $\frac{1}{5}$	M_1 M_2 1/5 1 x 6	Fraction notation, units	≤ 30 $\frac{1}{5}$	Recognises part-whole relationship, multiplies by the reciprocal

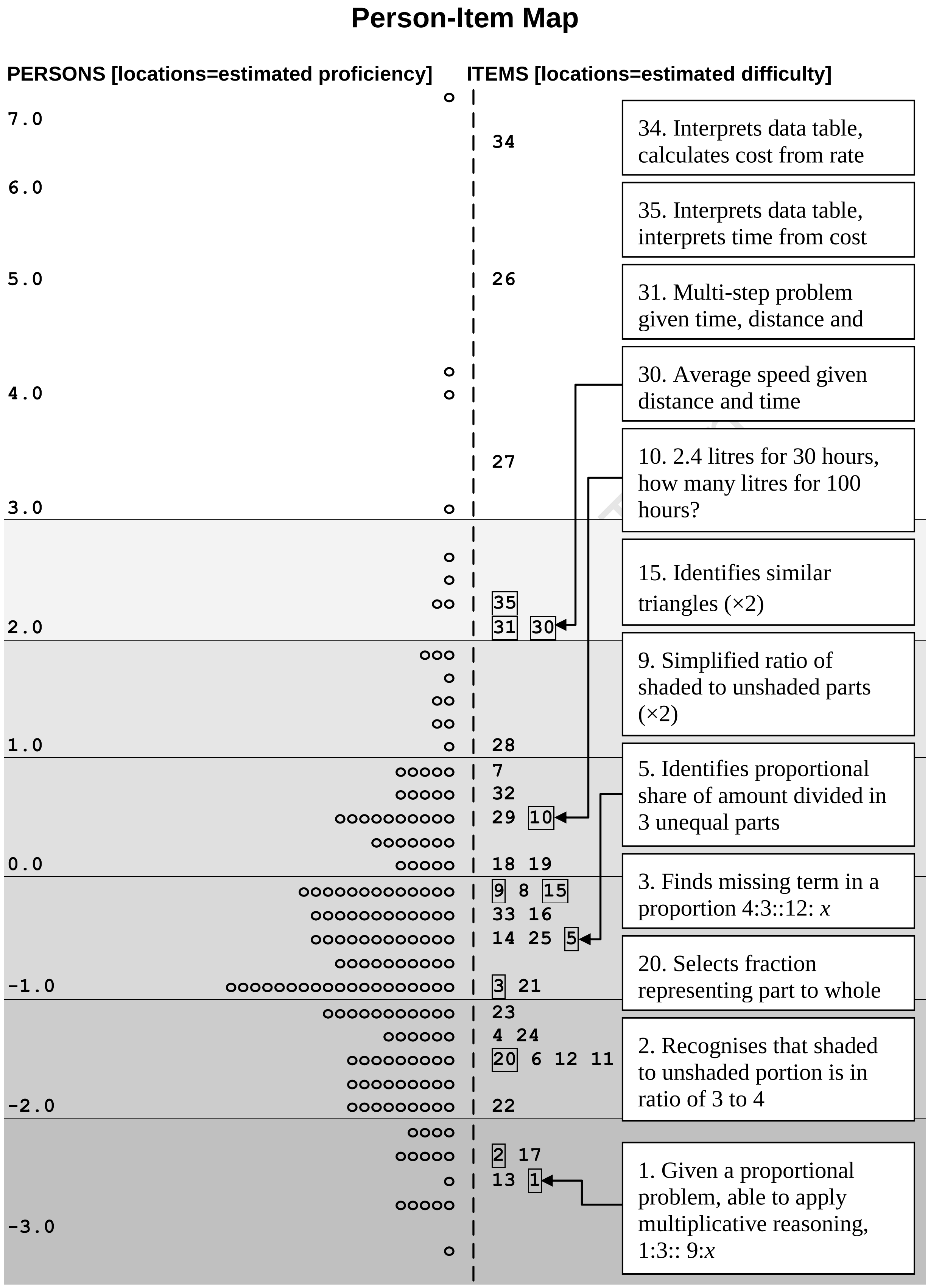
7.6 Ratio, proportion and rate item analysis

In the test instrument, 12 items were identified to include ratio, proportion and rate concepts. The comparisons with the TIMSS results are provided in Table 7.9. For the easiest three items shown in the comparison, **Items, 20, 3,** and **5**, the percentage correct for the study group is slightly higher than both the international percentage correct and the South African percentage correct. For the next two items, **Item 9** and **15**, the percentage correct in the study group drops but is still within 5% of the international percentage correct. For **Items 10, 30,** and **31** the international percentage correct is higher. **Items 35** and **34** were found to be very difficult for all three groups. The items are listed on the *Person-Item map* (see Figure 7.15) and in the *Summary Analysis of Ratio Items* (see Tables 7.10 and 7.11).

Table 7.9: TIMSS 2003[123], TIMSS SA 2003 and Research Group comparison

No	Itemcode in TIMSS	Item description	Percent correct			
			TIMSS 2003 int.	TIMSS 2003 SA	Study group %	n
1	M031108	Solves a word problem involving simple proportional reasoning			79	(330)
2	M011016	Recognises a figure that illustrates a simple ratio			69	(82)
20	M012041	Selects a fraction representing the comparison of part to whole, given each two parts in a word problem setting	52	21	67	(84)
3	M012004	Solves a word problem by finding the missing term in a proportion	48	47	57	(330)
5	M032727	Identifies proportional share of an amount divided into three unequal parts	45	32	49	(330)
9	M032447	Determines the simplified ratio of shaded to unshaded parts of a shape	42	16	39	(330)
15	M032261	Identifies a triangle similar to a specific triangle given the lengths of all sides	42	22	37	(83)
10	M032533	Solves a word problem with decimals involving a proportion	48	17	28	(330)
30	M032649a	Solves a word problem to find average speed	36	7	13	(83)
31	M032649b	Solves a multistep problem involving time distance and average speed	18	2	11	(83)
35	M032764	Interprets the data from a table to make calculations to solve a problem	6	0	6	(81)
34	M032763	Interprets the data from a table to make calculations to solve a problem	0	0	1	(81)

[123] Items 1 and 2 were selected from the TIMSS 2003 Grade 4 items. There is therefore no South African data as South Africa did not participate at this level.



o = 2 Persons

Figure 7.16: Person-Item distribution with “ratio, proportion and rate” items

7.6.1 Critical findings: Ratio, rate and proportion items at Levels 1 to 7

The multiple-choice ratio items were located from logits -3 to 1. The constructed-response items were located above 2 logits. The constructed-response format contributed to the difficulty level for the cohort of learners involved in this study, or conversely the multiple-choice format contributed to the lesser difficulty of items. Learners' responses to items at **Level 1** (Grade 4 items) indicated that learners in the highest quartile group had an understanding of simple ratio, but the lowest quartile had little idea of the multiplicative relationship required of ratio problems (see Appendix B, Ratio item analyses) . At **Level 2**, learners in the top three quartiles exhibited some idea of part-whole relationships. At **Level 3**, the ability to reason proportionally and project the ratio from one set of quantities onto another set was managed by the highest quartile groups, but the responses of learners at the lower quartiles suggest that learners flounder when more than one step is required in a problem. The introduction of decimal fractions, and division, where there was no common factor characterised the **Level 4** item, were generally found to be difficult. At **Levels 5, 6** and **7**, the items included concepts of unit rate, ratio, and the proportionality constant, and again were only managed by a few learners.

7.6.1.1 Confirmation and elaboration of theory

Additive reasoning is the inferred response attributed to the selection of false options in the **Items 3** and **9** (see Appendix B, Table 23, 27). The learner responses to **Item 10** (Level 4) indicate that 65% of the highest group is able to calculate rates, but on the whole about 30% of the cohort is able to calculate correctly. The items involving reasoning proportionally in relation to time, distance and speed, and items requiring multiple steps posed problems for this cohort. A clear structure in which to align the problem components and hence enable learners to locate the variables and relationships has been presented by Olivier (1992), and Dole (2008), amongst others. Vergnaud (1983) advocates diagrams which hold the essential information and discard the inessential information.

7.6.1.2 Assessment instrument recommendations

While this test includes a number of ratio concepts, proportional reasoning and rate concepts, there are omissions. For diagnostic purposes items of difficulty level between **Level 4** and the constructed-response items at **Level 6** and **7**, need to be designed, with specific attention to theorised levels of proportional reasoning from the mathematics education literature.

Table 7.10: Ratio, rate and proportion items (Levels 1, 2, and 3)

	Description	Context	Type of situation	Mathematical structure		Notation	Number range	Response process
Level [-∞ , -2)	1. Given a proportional problem, apply multiplicative reasoning	Collecting bottles	Proportional sharing	1 : 3 :: 9 : ?	M_1 M_2 1 9 3 x	Natural language, whole numbers	Less than 30	Identify multiplicative relationship
	2. Recognizes the ratio of shaded to unshaded portions	Shaded blocks	Multiplicative comparison	3 : 4 :: 6 : 8	M_1 M_2 3 b 4 d	Natural language, diagram	Less than 30	Identifies multiplicative operator
Level [-2, -1)	20. Selects fraction representing part to whole ratio	Birthday	Part-whole comparison Inclusive ratio	16 of total (16 + 14) $\frac{16}{16+14} + \frac{16}{30}$	M_1 (time) M_2 (children) first half 16 total 30	Fraction notation	≤ 30	Selects part-whole relationship
Level 3 [-1, 0)	3. Finds missing term in proportion 4:3::12:x	Athletic field	Multiplicative comparisons	4 : 3 :: 12 : x 4 : 3 :: 4(×3) : 3(×3) $a : b = ax : bx$	M_1 M_2 4 12 3 x	Natural language, whole numbers	≤ 16	Identifies multiplicative operator
	5. 45 000 zeds shared in unequal proportions	Family sharing of money	Proportional sharing	2 : 3 : 4 $\frac{4}{9}$ of 45 000	M_1 M_2 4 x 9 45	Natural language	≤ 45 000 (1 000)	Identifies proportional relationship
	9. Simplified ratio of shaded to unshaded parts	Geometric figure	Part-part ratio Multiplicative comparison	10 : 6 :: 5 : 3 $a : b :: \frac{a}{b} : \frac{b}{x}$	M_1 M_2 10 b 6 d	Pictorial	≤ 16	Recognises equivalence relation
	15. Identifies similar triangles	Similar triangles	Multiplicative comparison	8 : 10 : 12 8a : 10a : 12a	8 x 10 y 12 z	Pictorial	≤ 24	Recognises proportional relation

Table 7.11: Ratio, rate and proportion items (Levels 4, 5 and 7)

	Description	Context	Type of situation	Mathematical structure		Notation	Number range	Response process
Level 4 [0, 1)	10. 2.4 litres to 30 hours, how many for 100 hours?	Capacity-time	Proportion Multiplicative comparison Isomorphism of measures (general case)	$f(x) = \frac{2.4}{30}$	M_1 (capacity) / M_2 (hours) 2.4 / 30 x / 100	Decimal notation	≤ 100	Identifies proportional relationship and multiplicative operator
Level 5 [1, 2]	30. Average speed given distance and time.	Car rally, Distance, time, speed	Rate calculation Isomorphism of measures (general case)	$\frac{160\text{ km}}{2.5\text{h}}$ = 64km/h $\frac{d}{t} = s$	M_1 (hours) / M_2 (km) 2.5 / 160 1 / x	Decimals	≤ 160	Identifies proportional relationship and calculates unit rate
	31. Multi-step problem given time, distance and speed.	Car rally, distance, time and speed	Rate calculation Isomorphism of measures (general case)	(160 – 40) kilometres (2.5 – 1) hours $\frac{120}{1.5}$ = 80 km/h	M_1 (km) / M_2 (hours) 120 / 1.5 x / 1	Decimals	≤ 160	Identifies proportional relationship and multiplicative operator
	35. Interprets data table, interprets time from cost.	Telephone costs	Interpretation of data	$\frac{75\text{z} - 15\text{z}}{2\text{z}}$ + 120z		Data table	≤ 180	Interprets table
Level 7 [3, ∞	34. Interprets data, calculates cost from rate	Telephone costs	Interpretation of data	Plan A 300m (used) - 180m (free) = 120m (to pay) 120m × 1z (night rate) = 120z 120z + 20z = 140z Plan B (similar to Plan A)		Natural language Reading a table	≤ 180	Interprets table

7.7 Percent item analysis

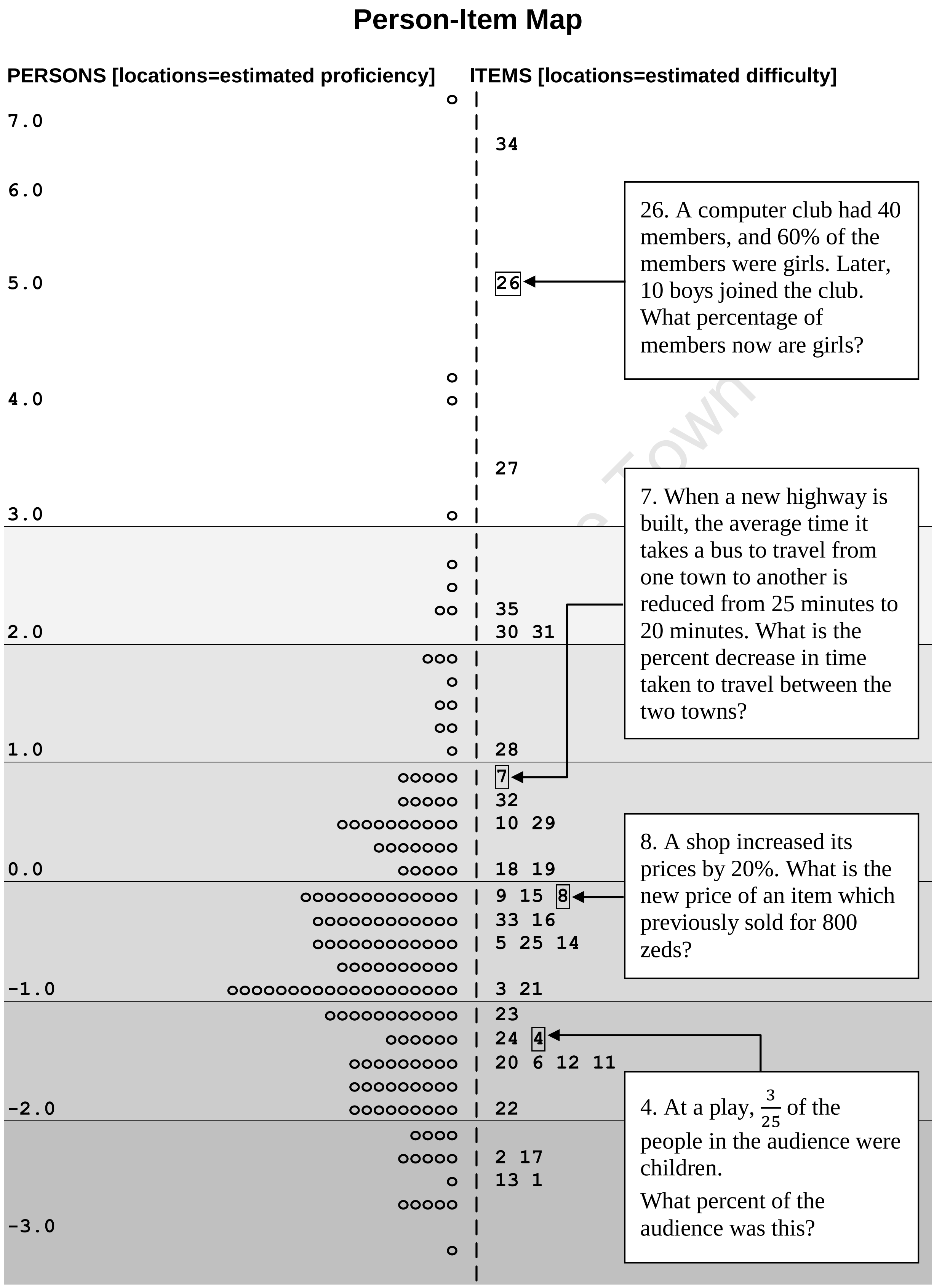
Four items on the concept of percent, including fraction, ratio and rate, located at graded levels of difficulty are discussed. Of the four items, it is only **Item 4** where there is comparable performance between the study group and the TIMSS 2003 International percentage correct; the performance on the remaining three items is considerably lower than TIMSS 2003 (see Table 7.12). The higher proficiency on the items of lower difficulty, and lower proficiency on the items of greater difficulty observed within the percent items, constitutes a similar pattern to both the *fraction* and the *ratio, rate and proportion* strands.

Three of the items (**Items 4, 8, and 7**) were in multiple-choice format for which 4 or 5 response options were provided. **Item 26** was presented in a constructed-response format. These items are shown in Figure 7.17, and Table 7.13.

Table 7.12: TIMSS 2003, TIMSS South Africa and Research Group comparison

No	Itemcode in TIMSS	Item description	Percent correct			
			TIMSS 2003 int.	TIMSS 2003 SA	Study group %	n
4	M032570	Identifies a percent equivalent to a given fraction with a denominator that is a factor of 100	55	33	63	(330)
8	M032228	Calculates the new price of an item given the percent increase in price	50	19	38	(330)
7	M022139	Finds the percent change given the original and the new quantities	32	22	23	(330)
26	M032233	Solves a multi-step non-routine problem involving percent	12	2	8	(330)

The four items represent progressively increasing levels of difficulty and by implication, complexity. As with the previous two strands, the defined levels and the items located in these levels are shown in Figure 7.15. Table 7.13 provides a mathematical analysis taking into account the *context*, the *type of situation*, the *notation and representation*, the *number range*, the *mathematical structure* and the *response process*. The reasoning behind the responses to two of the 'percent' items, **Item 8** and **Item 26**, was explored in subsequent interviews (see Chapter 8).



o = 2 Person

Figure 7.17: Person-Item location distribution for ‘percent’ items

7.7.1 Critical findings: Percent items at Levels 2, 3, 4, and 7

From the four items that included the percent concept and percent notation, there is an indication of hierarchical difficulty levels. A fraction understanding of percent is the least difficult. Greater difficulty is found with problems that require a ratio change as in percent increase and percent decrease, or that require the calculation of a ratio comparison. Problems where the referent changes, and the requirement is to switch between percent ratio and the referent quantities, provide the greater challenge (see Appendix B, for a detailed analysis).

Confirmation and elaboration of theory

The hierarchical order in terms of difficulty affirms the theoretical insights of Parker and Leinhardt (1995), in particular the observation that focusing on percent as equivalent to a fraction, and fraction and percent as easily exchangeable, may inhibit more complex ratio understandings of percent. It may very well be the case that the teaching of fraction, decimal and percent focuses on simple conversions. It may also be the case that procedures are taught for particular categories of problem. However, when confronted with a complex problem with multiple procedures which require systematic analysis, the procedures alone are not adequate.

Assessment instrument recommendations

In terms of the assessment instrument, this cluster of items at varied difficulty levels functions fairly well; the distances between levels provide information that may inform teaching. The inclusion of multiple-choice items, of greater complexity than **Item 7** (level 4) and yet not expecting the level of problem solving required from **Item 26** (level 7), may provide useful information.

Additional items at all levels requiring a constructed response would add to the information gained from this type of analysis. The inclusion of partial-credit scoring would also add to the nuanced understanding of the percent concept.

Table 7.13: Percent items (Levels 2, 3, 4 and 7)

	Description	Context	Type of situation	Mathematical structure		Notation	Number range	Response process
Level 2 [-2, -1)	4. At a play, $\frac{3}{25}$ of the people were children. What percent was this?	People at a play	Fraction meaning, conversion from fraction to percent notation	$\frac{3}{25}=\frac{3x}{25x}=\frac{}{100}$ $\frac{a}{b}=\frac{ax}{bx}$	**M_1** (fraction) **M_2** (percent) 3 x 25 100	Fraction notation	≤ 100	Recognise equivalence relation, determine multiplicative constant
Level 3 [-1, 0)	8. A shop increased its price by 20%. What is the new price of an item previously sold for 800 zeds?	Financial context	Change, relative change from 100% to new percentage	Add 100% (of 800) + 20% (of 800) = 120% of 800	**M_1** (amount) **M_2** (percent) 800 100 x 20 y 120	Percent notation	≤ 1 000	Calculate relative change
Level 4 [0, 1)	7. On the new highway, the average time it takes a bus to travel from one town to another is reduced from 25 minutes to 20 minutes. What is the percent decrease?	Measurement (time)	Comparison, the difference in magnitude as a proportion of one of the sets	Find the difference 25 – 20 = 5 Decide on referent $\frac{5}{25}$ convert to percent = 20 %	**M_1** (fraction) **M_2** (percent) 5 x 25 100	Percent notation	≤ 100	Compare two quantities. Find the difference, recognize direction of change (decrease), identify referent
Level 7 [3, ∞)	26. A computer club had 40 members, and 60% of the members were girls. Later, 10 boys joined the club. What percentage of members now are girls?	Computer club	Percent of a quantity, referent quantity changes, comparison of Q1 and Q2	60% of 40 = 24, 40 + 10 = 50 $\frac{24}{50}=\frac{24\times 2}{50\times 2}$ = 48%	**M_1** (percent) **M_2** (people) 60 x 100 40 x 24 100 50	Percent notation	≤ 100	Recognising the two-step process, identify referent in each case and relationship between the parts

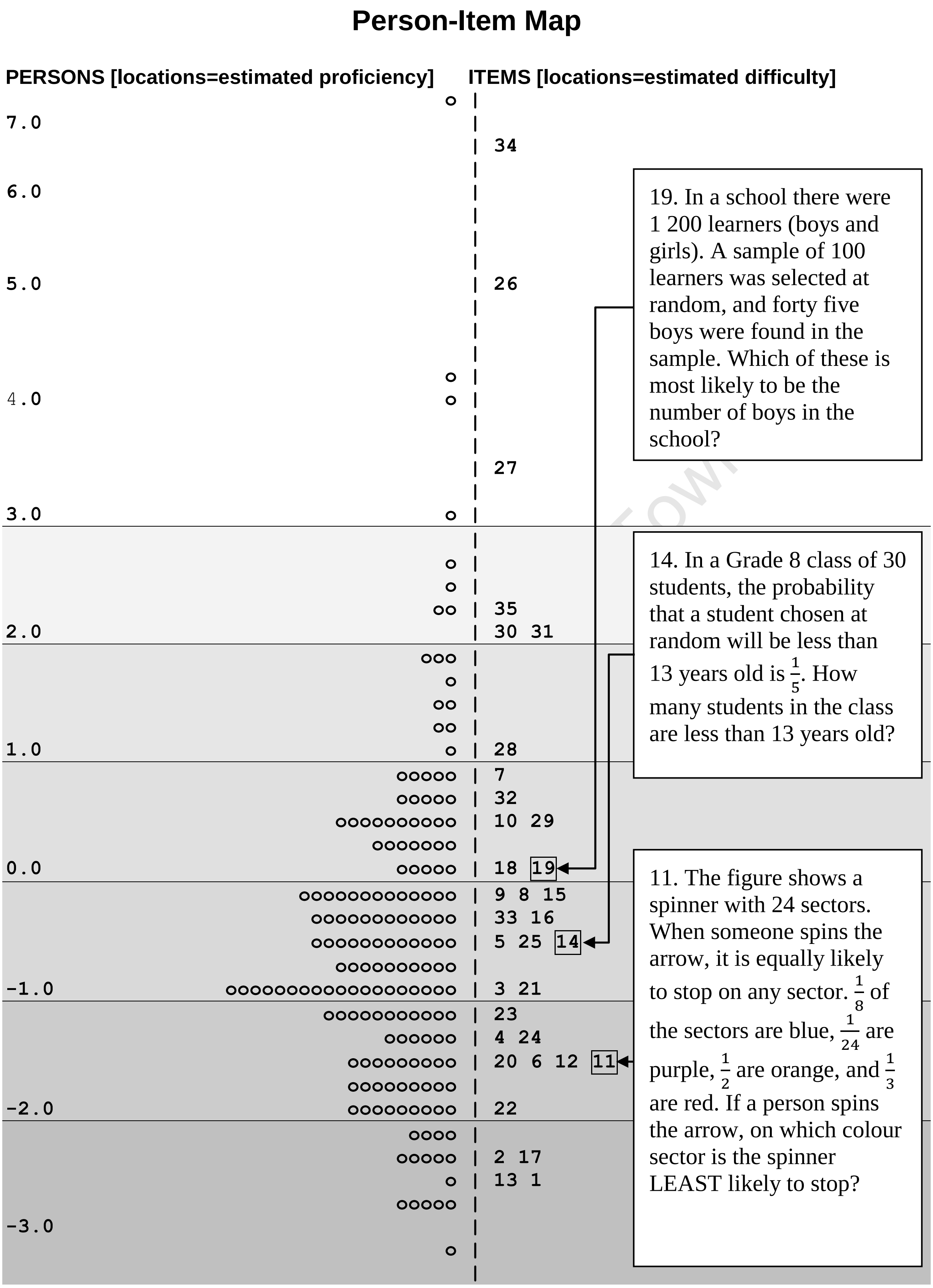
7.8 Probability item analysis

In this section three items **Items 11, 14** and **19**, that use the concept of probability, (in addition to fraction, ratio and rate concepts), located at distinct graded levels of difficulty, are investigated. Both **Items 11** and **14** were included in the Fraction substrand, and **Item 14** was discussed as a **Level 3** Fraction item. **Item 19** was highlighted as problematic on a number of misfit indicators, and was therefore removed for the item calibration process. Because of its intrinsic interest and because there were only three items invoking the probability concept, this item was reintroduced at an estimated difficulty level.

For both **Items 11** and **14**, the study group performed at a level comparable with the TIMSS international cohort (Table 7.14). As with the other substrands, the performance on more complex items drops below the international percent correct. The three items are presented at their locations on the Person-Item map (see Figure 7.18), at **Levels 2, 3** and **4**, and presented in a *Summary analysis of probability items* (see Table 7.15).

Table 7.14: TIMSS 2003, TIMSS SA 2003 and Research Group comparison

No	Itemcode in TIMSS	Item description	Percent correct			
			TIMSS 2003 int.	TIMSS 2003 SA	Study group %	n
11	M022252	Given the set of possible outcomes expressed as fractions of all outcomes, recognizes that probability is associated with size of fraction	60	34	61	(82)
14	M022146	When given the possible number of outcomes and the probability of successful outcomes, solves for the number of successful outcomes	50	26	45	(83)
19	M032271	Uses the size of a group with a given characteristic in a sample to estimate the size of a group with that characteristic in a population	47	23	31	(84)



o = 2 Persons

Figure 7.18: Person-Item distribution highlighting ‘probability’ items

7.8.1 Critical findings: Probability items at Levels 2, 3 and 4

The inclusion of this small cluster of probability items provides some information that may be further investigated. The unexpected pattern of response for **Item 19** signals attention to the item construction, the item functioning within the test and for the learner cohort. A qualitative investigation of the item does not uncover any problems, which suggests that the unexpected functioning of the item may be due to the learner cohort and educational context. A tentative hypothesis may be that the topic probability was at the time relatively new in the curriculum and had not been given much attention.

Confirmation and elaboration of theory

The difficulty levels of the items that emerged in the empirical testing of this group of learners support the theory in the following way. Firstly, the problem description for **Item 11**, the easiest of the three items, has a diagram which provides clues to the meaning of the language, and the number range is low. The terminology is familiar, for example the term "least likely" is an everyday expression. The mathematical requirement is to compare the fraction parts.

Item 14, by contrast, requires understanding fraction notation, application of the operator rational number subconstruct, and attention to the probabilistic reasoning embedded in the question. The number range is relatively low, and the fact that 30 is a multiple of 5, assists with the division calculation. The requirement for **Item 19** is to recognise the ratio relationship 45 to 100. The fraction operator, $\frac{45}{100}$, is then to applied to 1 200. The larger numbers may have compounded the difficulty of the item. The statistical terms "sample", and "at random", may not be familiar to the learners.

Assessment instrument recommendations

For a future test instrument these three items, or similar item constructions would be retained. The possibility arises that the investigation of probability, while sharing the core concepts of ratio and fraction notation, requires specific focus on difficulties encountered in the topic probability.

Table 7.15: Probability items (Levels 2, 3 and 4)

	Description	Context	Type of situation	Mathematical structure		Notation	Number range	Response process
Level 2 [-2, -1)	11. Given the set of possible outcomes expressed as fractions of all outcomes, recognizes that probability is associated with size of fraction	Spinner	Fraction meaning associated with the ratio of expected outcomes to total sample space	Each $\frac{1}{24}$ is equally likely therefore the greater fraction is more likely. $\frac{1}{2} > \frac{1}{3} > \frac{1}{8} > \frac{1}{24}$		Fraction Diagram	≤ 1	Recognises that the fraction area is related to the probability
Level 3 [-1, 0)	14. In a word problem when given the possible number of outcomes and the probability of successful outcomes, solves for the number of successful outcomes	Classroom situation	Linking probability written in fraction form with predicted frequency in the population	p = $\frac{1}{5}$ Count = 30 $\frac{1}{5}$ = 30 -> x = 6	**M_1** **M_2** 1 x 5 30	Natural language, fraction notation, counts	≤ 1	Identifies fraction-probability link, applies the multiplicative operator
Level 4 [0, 1)	19. Uses the size of a group with a given characteristic in a sample to estimate the size of a group with that characteristic in a population	Probability and sampling	From theoretical probability, determine the predicted outcome in the population	45:100 :: x:1 200 45: 100 90: 200 450: 1 000	**M_1** **M_2** 45 x 100 1 200	Natural language, whole numbers	≤ 1 200	Recognises proportional relationship

7.9 Pre-Algebra item analysis

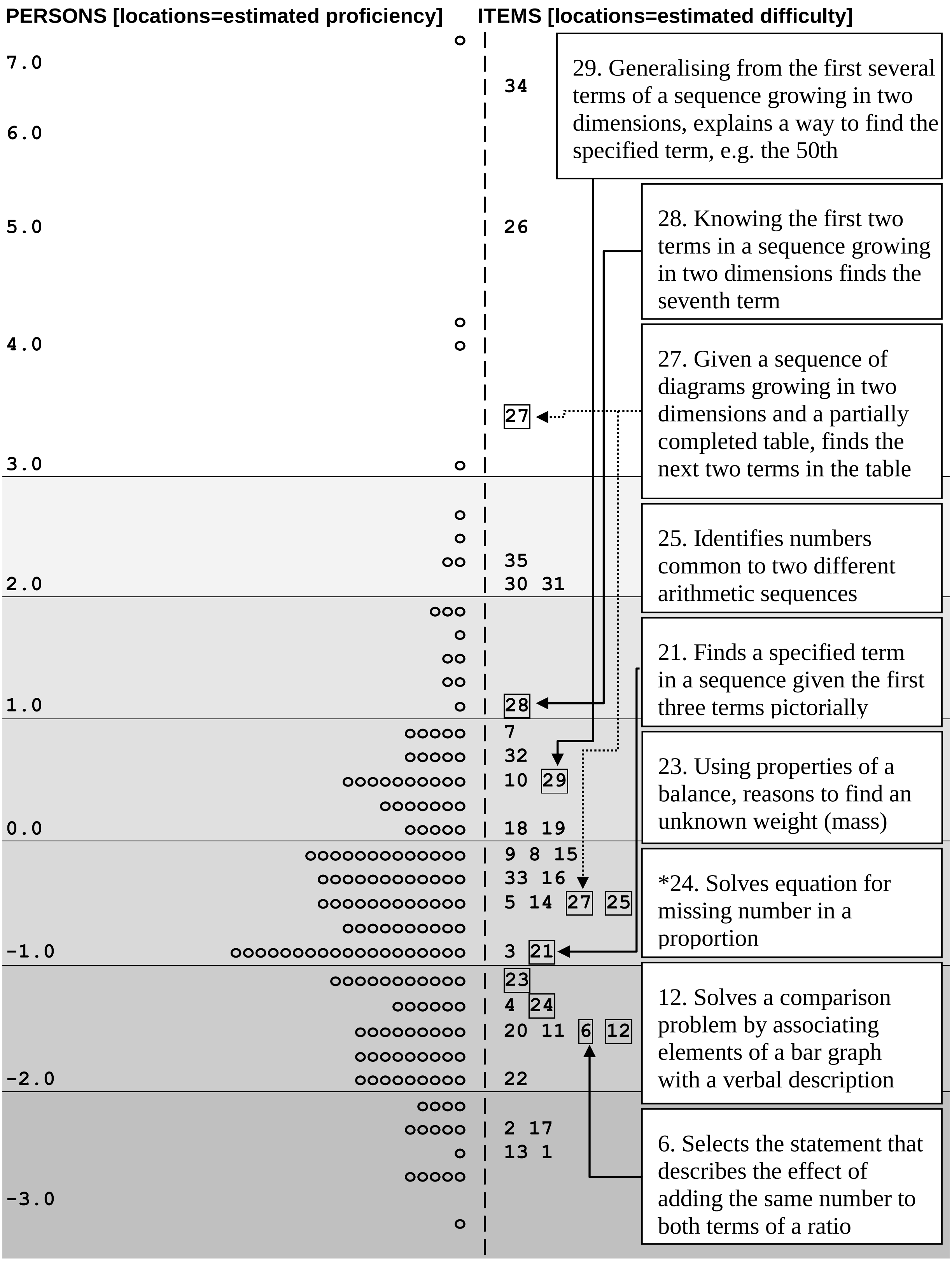
For the first six items (**Items 6, 12, 21, 23, 24** and **25**), the study group percentage correct are comparable with the international percentage correct, and higher than the South African population (see Table 7.16). For **Items 27, 28** and **29**, the results present something of an anomaly, which will be discussed in relation to the specific items. **Item 24** was discussed as part of the set of fraction items; information is included in the summary tables (Table 7.17).

Table 7.16: TIMSS 2003, TIMSS SA 2003 and Research Group comparison

No	Itemcode in TIMSS	Item description	Percent correct			
			TIMSS 2003 int.	TIMSS 2003 SA	Study group %	n
6	M022191	Selects the statement that describes the effect of adding the same to both terms of a ratio	59	23	66	(330)
12	M022189	Solves a comparison problem by associating elements of a bar graph with a verbal description	68	28	59	(82)
24	M012040	Solves equation for missing number in a proportion	64	26	68	(83)
23	M012002	Using the properties of a balance, reasons to find an unknown weight (mass)	64	28	67	(81)
21	M012017	Finds a specified term in a sequence given the first three terms pictorially	49	24	51	(84)
25	M022008	Identifies numbers common to two different arithmetic sequences	31	6	47	(330)
27*	M022261a	Given a sequence of diagrams growing in two dimensions and a partially completed table, find the next two terms in the table	37	6	40	(81)
28*	M022261b	Knowing the first two terms in a sequence growing in two dimensions finds the seventh term	18	7	0	(82)
29*	M022261c	Generalising from the first several terms of a sequence growing in two dimensions, explains a way to find the specified term, e.g. the 50th	14	1	11	(81)

* Items 27, 28 and 29 were in constructed response format and relate to the same geometric construction. Items 27 and 29 were polytomous, scoring 0, 1, or 2. The results indicated an anomaly with either 0 or 2 more likely than 1. This unexpected result led to a decision to collapse the categories. The final results presented here are the results of the second analysis with collapsed categories

Person-Item Map



o = 2 Persons; * Item 24 is also discussed in the fraction section

Figure 7.19: Person-Item distribution highlighting 'pre-algebra' items

7.9.1 Critical findings: Pre-Algebra items at Levels 2, 3, 4 and 5

In general, the group in this research study exhibited proficiency with pre-algebra items which required reasoning about comparative relationships presented in natural language. The preliminary finding (see Appendix B, Table 54) that the highest quartile maintains a high percentage at **Level 3**, but the lowest quartile shows little proficiency at **Level 3** raises questions about the uneven progression of learner groups.

7.9.1.1 Confirmation of theory

Mathematics education literature points to the difficulties learners have with algebra.[124] The difficult transition referred to by Usiskin (2005), "from reasoning with objects to reasoning with variables" (p. 4), is suggested in this study and requires further investigation. The importance of proficiency with the array of elements in the multiplicative conceptual field, incorporating the rational numbers and its various subconstructs, and proportional reasoning, are considered essential for the transition from reasoning with numbers to reasoning with variables.

The transition from natural language to fluency with mathematical symbols requires careful didactic structuring, taking into consideration that in any class there will be uneven progression. The analysis suggests that the differences in proficiency among grade cohorts increase progressively with items of greater difficulty. A danger may be that in the interests of classroom management, teachers may avoid introducing problems of greater difficulty resulting in the mathematical potential of these learners not being extended.

7.9.1.2 Assessment instrument recommendations

The importance of algebra demands an assessment instrument carefully designed against what the theory considers hierarchical development of algebraic concepts. The items, or similar items, that have been used in this instrument may form link items to different sets of instruments.

[124] The focus of this study was not on algebra. Because of the importance of algebra in the high school curriculum and the fact that the algebraic conceptual field rests on both the multiplicative conceptual field and the additive conceptual field, some attention has been given to this topic in this study.

Table 7.17: Pre-algebra items, Level 2, 3 and above

	Description	Context	Type of situation	Mathematical structure		Notation	Number range	Response process
Level 2 [-2, -1)	6. Selects the statement that describes the effect of adding the same number to both terms of a ratio	People at a meeting	Equivalence and direction change, Comparative relationships	$\frac{2}{3}+\frac{1}{3}=1$ $\frac{2}{3}(+10) > \frac{1}{3}(+10)$	$a+b=c$ $if\ a>b$ then $a+z>b+z$	Natural language	10 $\frac{1}{3}, \frac{2}{3}$	Reasoning about the direction of change
	12. Solves a comparison problem by associating elements of a bar graph with a verbal description	Bar graph Everyday objects pens, etc	Comparative relationships	Reading a bar graph and connects information with comparative language		Graphical	≤ 140	Connecting statements of comparison with the graphical representation
	24. Solves equation for missing number in a proportion	Equivalent fractions	Multiplicative comparison	$\frac{12}{n}=\frac{36}{21}$	$\frac{a}{b}=\frac{ax}{bx}$	Symbolic	≤ 63	Identifying multiplicative operator
	23. Using properties of a balance, reasons to find an unknown weight (mass)	Scale balance	Equivalent relationship	$1\text{ kg}+\frac{1}{2}\text{brk}=1\text{ brk}$ $\frac{1}{2}\text{brk}+\frac{1}{2}\text{brk}=1\text{ brk}$ $\therefore\ \frac{1}{2}\text{brk}=1\text{ kg}$ $\therefore\ 1\text{ brk}=2\text{ kg}$	If $a+b=c$ and $d+b=c$ then $a=d$	Pictorial presentation, Decimal notation	$\frac{1}{2}, 3$	Trial and error or reasoning with unknowns
Level 3 [-1, 0)	21. Finds a specified term in a sequence given the first three terms pictorially	Pictorial diagram with matches	Recognises a pattern, calculates the additive difference or the multiplicative	**1 2 3 4 …** **6 9 12** 3 + (1 × 3) = 6 3 + (2 × 3) = 9 **3 + (*n* × 3) = t**		Diagram, natural numbers	≤ 42	Counting, identifying the pattern, extending the pattern
	25. Identifies numbers common to two different arithmetic sequences	Numbers	Recognises the pattern in two sequences, with common elements	**7 11 15 19 23 …** (increase by 4) **1 10 19 28 37 …** (increase by 9) 19 + (4 × 9)		Language, natural numbers	≤ 55	Reasoning
Level 3, 4, 5	*27. Given a sequence of diagrams growing in two dimensions and a partially completed table, find the next two terms in the table	Geometric figure	Recognises a pattern, calculates the additive difference or the multiplicative	**1 2 3 4** **2 8 18 32** 1 2(1 × 1) 2 2(2 × 2) ***n* 2(*n* × 4)**		Diagram, natural numbers	≤ 55	Counting, identifying the pattern, extending the pattern
	*28. Knowing the first five terms in a sequence growing in one dimension finds the seventh term	Geometric figure	Recognises a pattern, calculates the additive difference or the multiplicative	**1 2 3 4 … 7** **2 8 18 32** 1 2(1 × 1) 2 2(2 × 2) ***n* 2(*n* × 2)** 7 2(7 × 7)		Diagram, natural numbers	≤ 98	Counting, identifying the pattern, extending the pattern
	*29. Generalising from the first several terms of a sequence growing in two dimensions, explains a way to find the specified term, e.g. the 50th	Geometric pattern	Recognises a pattern, calculates the additive difference or the multiplicative	**1 2 3 4 … 7 …** **2 8 18 32** 1 2(1 × 1) 2 2(2 × 2) ***n* 2(*n* × *n*)** 50 2(50 x 50) = 5 000		Diagram, natural numbers	≤ 5 000	Counting, identifying the pattern, extending the pattern, Generalizing the pattern

* varying information under different analyses

7.10 Summary descriptions at Levels 1 to 7

An analysis of each item by quartile groups provided an average percentage correct for each quartile group (see Table 7.18, from Item analyses, Appendix B).

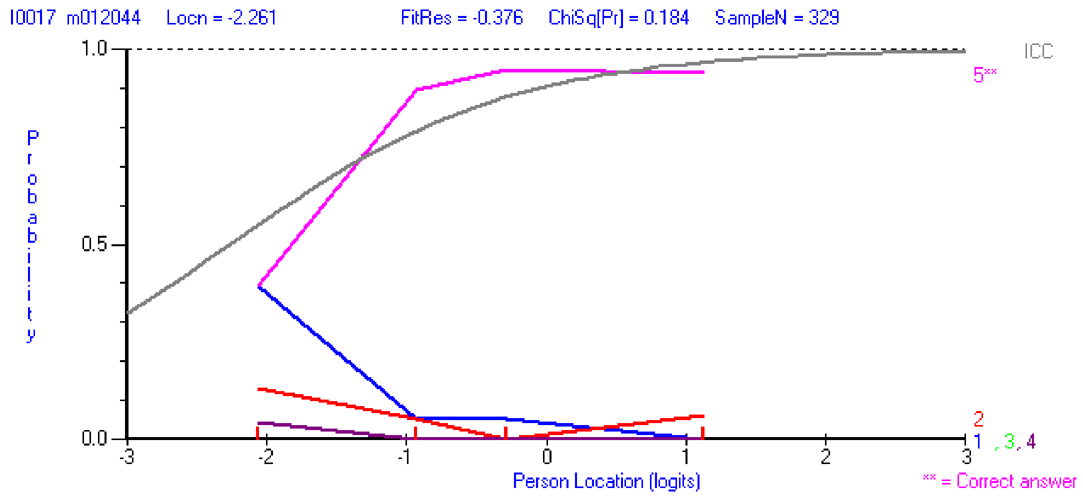
Table 7.18: Item 17. Inferred procedures for multiple-choice responses

				Mean locations of quartile groups			
Option	**Total**	**Line**	**Inferred mathematical procedure**	**-2.063**	**-0.925**	**-0.288**	**1.124**
E	79%	pink	2 out of 3 equal parts shaded	39%	89%	95%	94%
A	12%	blue	2 shaded, 3 unshaded of 5 equal parts	39%	5%	5%	0%
B	6%	red	2 of 3 unequal parts shaded	13%	5%	0%	6%
C	1%	green	2 of 3 unequal parts shaded	4%	0%	0%	0%
D	1%	purple	2 of 3 unequal parts shaded	4%	0%	0%	0%
Total[1]				**100%**	**100%**	**100%**	**100%**

[1] The percentage of learners selecting a distractor is rounded to a whole number. In some cases this rounding means the total will appear less than or greater than 100%.

This analysis, from RUMM 2020 (Andrich, Sheridan & Luo, 2005), is also presented graphically. The colour codes in Table 7.18 refer to line graphs in Figure 7.18.

Figure 7.20: Item 17 multiple-choice response by quartile group



The percent correct for all items at a particular level were collated (see Table 7.19, Item 17 is shaded). A weighted mean of percentage correct of items at each level provided a rough estimate for each quartile group, assigned to, Low quartile (Low Q), Middle-low (M Low Q), Middle-high (M High Q) and High (High Q) quartile groups, at each level. For example at Level 1, the High Quartile group are estimated to have a 94% mastery of the concepts located at Level 1, while the Low Quartile group exhibit a 53% mastery at this level.

Table 7.19: Proficiency levels (%) by substrand and quartile means: Level 1

	Item	n	Level	Substrand	Low Q	M Low Q	M High Q	High Q
Level 1	13	83	1	Fraction	68	89	89	95
	17	**84**	**1**	**Fraction**	**39**	**89**	**95**	**94**
	1	330	1	Ratio	58	84	93	98
	2	83	1	Ratio	45	90	94	90
	Mean (weighted)				**53**	**88**	**93**	**94**

From an analysis of the item, and an inference about the possible reasons for the selection of an incorrect distractor, a summary of each strand at each level was generated. From this summary, plausible descriptors of a composite of concepts at the defined levels are generated, for which learners in the different quartile groups, exhibit proficiency.

The items empirically located in the **Level 1** band include the fraction concepts of part-whole measure of both discrete and continuous quantities, and the concept of proportion as the relationship between two measure spaces, with a multiplicative operator determining the relationship within or between the measure spaces. Learners in the highest performing quartile (94%), the second highest quartile (93%), and third highest quartile average (88%) respectively, exhibit proficiency at this level, in contrast to the low performing group at just over 50% exhibit proficiency (see Table 7.20 for this and other levels).

Descriptions of the concepts inferred to have been understood at different levels by this cohort, together with the errors encountered are described below. A summary table is also presented (see Table 7.21 for summaries of the levels).

The errors inferred from the choice of incorrect options by the complements of those exhibiting proficiency, that is less than 10% for the top two groups and less than 50% of the low performing group, are the following:

- Inability to recognise the part-whole relationship within a set of discrete objects;
- Confusion between fraction measure and ratio meaning of fraction notation; and
- The application of an additive relationship rather than a multiplicative relationship.

Table 7.20: Summary: Proficiency levels by substrand and quartile means[125]

	Item	n	Level	Substrand	Low Q	M Low Q	M High Q	High Q
Level 1	13	83	1	Fraction	68	89	89	95
	17	**84**	**1**	**Fraction**	**39**	**89**	**95**	**94**
	1	330	1	Ratio	58	84	93	98
	2	83	1	Ratio	45	90	94	90
	Mean (weighted)				**53**	**88**	**93**	**94**
Level 2	22	81	2	Fraction	52	86	88	92
	24	83	2	Fraction	35	63	90	100
	20	84	2	Ratio	28	70	81	86
	4	330	2	Percentage	28	51	84	96
	11	82	2	Probability	40	75	58	84
	6	330	2	Pre-Algebra	43	61	74	90
	12	82	2	Pre- Algebra	30	50	84	89
	23	81	2	Pre- Algebra	48	67	78	85
	Mean (weighted)				**37**	**61**	**79**	**91**
Level 3	14	83	3	Fraction	11	11	70	91
	16	83	3	Fraction	22	38	55	64
	3	330	3	Ratio	31	48	65	88
	5	330	3	Ratio	24	35	55	85
	9	330	3	Ratio	9	29	55	73
	15	83	3	Ratio	19	27	53	71
	8	330	3	Percentage	13	26	45	72
	21	84	3	Pre- Algebra	0	56	58	85
	25	330	3	Pre- Algebra	16	46	52	73
	Mean (weighted)				**16**	**35**	**58**	**78**
Level 4	18	84	4	Fraction	17	13	33	71
	32	84	6	Fraction	11	17	14	43
	10	330	4	Ratio	8	19	28	65
	7	330	4	Percentage	9	16	19	42
	19	84	4	Probability	25	29	32	43
	27	81	4	Pre- Algebra	5	25	47	95
	33	81	4	Pre- Algebra	38	60	56	70
	Mean (weighted)				**14**	**22**	**28**	**58**
Level 5, 6 and 7	28	81	5	Pre- Algebra	0	5	16	58
	29	81	5	Pre- Algebra	0	9	35	40
	30	83	6	Ratio	0	0	5	41
	31	83	6	Ratio	0	5	5	27
	35	81	6	Ratio	0	4	5	22
	26	330	6	Percent	0	1	5	25
	34	81	7	Ratio	0	0	0	6
	Mean (weighted)				**0**	**3**	**9**	**29**

[125] In the analysis both quintile and quartile groups were used for distractor analyses. For aggregate purposes it proved easier to recalculate all the items using quartile groups.

Table 7.21: Association of proficiency and errors within quartile groups

	MCF concepts	Errors	LQ (-1.9)	MLQ (-1.1)	MHQ (0)	HQ (1.1)
Level 1[-3,2)	• Part-whole of discrete and continuous quantities • Both fraction and ratio meaning of fraction notation • Multiplicative relationship between sets of ratios	• Confusion between fraction measure and ratio meaning and with fraction notation	53%	88%	93%	94%
Level 2 [-2-1)	• Fraction equivalence • Part-part and part-whole ratios • Percent concept and notation • Connect probability with fraction measure • Covariant relationships • Comparative relationships	• Natural number confusion • Just add the percent sign % • Language difficulty • Ignoring part of the problem	37%	61%	79%	91%
Level 3 [-1,0)	• Rational number, operator subconstruct • Identification of ratio • Multiplicative comparison (operator construct) • Percentage increase (operator subconstruct) • Applying ratio operator construct to find the sample • Multiplicative comparison	• Confusion with operator construct • Confusion of additive and multiplicative relationship • Ignoring part of the problem	16%	35%	57%	78%
Level 4 [0, 1)	• Fraction measures, addition (subtraction), multiplication (division) • Ratio and rate concepts • Multiplication (division) of decimals • Probability and statistics concepts, "sample", "random"	• Consider only the numerator • Additive reasoning • Lack of fluency with multiplication and division • Confusion with terminology	14%	22%	28%	58%
Level 5 to 7 (> 1)	• Rate and ratio • Percent, identifying referents • Covariant relationships • Reasoning with unknowns • 2 step problems	• Confusion with percent language and referents	0%	3%	9%	29%

At **Level 2** the concepts required to answer the items correctly are fraction equivalence, part-whole and part-part ratios, comparative and covariant relationships, the percent concept and notation, and the representation of a probability in fraction notation. An estimated 90% of the highest quartile group, almost 80% of the second highest quartile group, just over 60% of the third quartile group and under 40% of the lowest quartile group indicate proficiency with these concepts (see Table 7.20 and Table 7.21). The corresponding errors inferred from the options selected are;

- Confusion of procedures for adding natural numbers with procedures for adding rational numbers, for example simply adding the numerators without considering the ratio of numerator to denominator.
- Not considering percent as a ratio between two quantities, and therefore simply appending the percent sign to an existing fraction.
- Difficulty understanding comparative language.

At **Level 3**, the concepts required are an understanding of rational number, in particular the operator subconstruct. The identification of the multiplicative operator in a proportional relationship, in percent increase and in the calculation of a probability is required. Almost 80% of learners in the highest performing group exhibit proficiency in these concepts, 55% of the second highest quartile, 36% of the next quartile group and around 20% of the lowest quartile group. The associated errors are;

- Applying an additive difference instead of a multiplicative ratio; and the
- Inability to apply the operator subconstruct.

The operations and procedures, for addition (and subtraction), and multiplication (and division) of rational numbers, both in fraction and in decimal form, are the requirements to solve items at **Level 4**. Ratio and rate concepts as applied in the proportional relationship between two sets of measure spaces are key concepts in the solution of problems at this level. In addition to the critical concepts of ratio and proportional reasoning, the knowledge of the statistical terms “sample” and “random” are necessary conditions for answering one of the items at this level. About 60% of the highest quartile exhibit proficiency at this level, 30% of the second highest quartile, just over 20% of the third highest quartile and just over 10% of the lowest performing group.

The associated conceptual errors are the following;

- Lack of fluency with multiplication and division;
- Additive rather than multiplicative reasoning; and
- Considering only the numerator when dealing with fractions.

At **Level 5, 6** and **7**, the ability to identify the covariant relationship between two sets of measure spaces and to identify the referents in the different types of percent problems, is a requirement. In addition the ability to work with problems that require multiple steps and the ability to generalise is required. In the region of 30% of the highest performing group provide evidence of proficiency with these concepts, about 10% of the second highest quartile, 5% of the third highest quartile, and close to zero of learners in the lowest quartile group (see Table 7.20 and Table 7.21). The low performance on these items suggests that the conceptual difficulties at this level are the lack of the building blocks composed of the concepts listed in the early levels.

This output is somewhat to be expected, from the Rasch model output, as the mean locations of the highest quartile group on the test as a whole is at just over 1.000 logits which is within what we have defined as **Level 5**. On the **Level 5** items this group averages around 50% (see **Item 28** and **Item 29**). For the second quartile group, whose mean is located at around -0.500 logits, which is defined in this study as **Level 3**, the performance on the **Level 3** items is 55%. The third highest quartile mean is located just below -1.000 logits that is **Level 2**, where learners show a 61% proficiency level. At the lowest quartile, whose mean is located at -1.852, that is close to **Level 1**, just over 50% are proficient at this level. The critical point though is that concepts are aligned with learners' proficiency.

7.10.1 Critical points and Threshold concepts

There are threshold concepts on the mathematical development path to proficiency in the multiplicative conceptual field that provide the conceptual gateway to higher levels of mathematics. The distinct transition from additive reasoning to multiplicative reasoning has been noted. In addition, the transition required from working with natural numbers to working with rational numbers indicates that the rational number concept can be identified as a threshold concept. Dividing by a fraction and finding a larger number is counter-intuitive for learners who have only engaged procedurally and whose understanding of number is still that of natural numbers. Most of the learners in this study cohort do not understand the procedure for dividing by a fraction, and it can be ventured that division, whether by natural numbers or fractions may be a threshold concept yet to be mastered by the majority of the cohort.

In addition one would expect that problem situations, represented here by items, should be able to be solved through problem-solving strategies not directly related to algorithms taught in the classroom (see Schoenfeld, 1985). The quantities are small enough in this case to resort to drawing or acting out. This is not the case.

7.10.2 Instrument development recommendations

The current instrument provides a fair amount of information in this area of mathematics for this cohort of learners. Omissions have been noted, namely that of fraction items of greater complexity and decimal fractions. An advantage of using TIMSS items is that these items had already been through a review process. A direct comparison of the percent correct for the TIMMS 2003 items and the Rasch difficulty locations has not been made, though it is noted here that there were no radical differences. Items that were easier for the international cohort were in general easier for the study group.Improvements of the instrument for this particular purpose of providing insight into learning in a particular field could be made by attending to the existing theoretical work and constructing items that are aligned with empirically defined levels. The process of constructing items requires cycles of piloting and review, but the importance of having items that reflect the construct of interest warrants the extra attention to the instrument.

7.10.3 Reflections on the analytic framework

The analytic framework for item analysis in this study draws on the work of Vergnaud (1983, 1988, 1994). An alternate framework, discursive in nature, is proposed by Brodie and Berger (2010) for the identification of errors in learner responses to multiple-choice items. The error categories align somewhat with the strategies identified by Hart (1981), namely *building-up strategies,* but in this case the strategies are specifically focused on multiple-choice items and the associated distractors. These strategies include *errors of routine, errors of signifiers, errors of visual mediators* and *lack of factual knowledge.* Errors of routine are identified as a *halting signal*, or a *keyword trigger*. In both these error types it is hypothesised that there is something in the distractors that distracts the learners from successfully carrying out the calculation. Errors associated with a signifier may occur as a result of a *previous reference* or to *familiarity.* Another category of error resonating with the errors in this study is *lack of factual knowledge*. Brodie and Berger (2010) hypothesise that classification of errors into these types may assist teachers in correcting errors. Following Vergnaud (1988; 1994) it is suggested that alerting teachers to the mathematical structure by describing response processes in terms of concepts-in-action and theorems-in-action is more likely to assist in the transformation of a learner's intuitive localised schema to generalisable concepts and theorems.

7.10.4 Reflections and further insights

The significance of the conclusions drawn from the process outlined in this chapter are dependent on the quality of the assessment process, from defining the construct, to the selection or creation of items, to the administration of the test, and finally the analysis of the data. In retrospect, the selection of items may have been more focused on particular strands with built-in threshold concepts. Another possibility may be to focus more closely on the construct proportional reasoning, regarded by many as the "capstone of primary school" and the "cornerstone of all that is to come" (Lesh, 1988; Lamon, 2007). However, the use of items, developed in the TIMSS 2003, has proved useful for an initial investigation. The purpose for which they were initially designed has been extended.

In keeping with the notion that learners will not perform optimally when subjected to long and inappropriately targeted tests, we propose that short appropriately targeted tests would serve the interests of children, the teachers and the whole education system best. We therefore propose that additional items are generated against the knowledge currently available in the mathematics education literature.

8 Identifying threshold concepts in reasoning behind item responses

8.1 Tracking learner competences

The particular challenge encountered by the field of mathematics education is to apprehend how learners develop competences in mathematics and progressively master concepts of increasing complexity.

According to Piaget,

> (t)he essential problem is how to characterise the important stages in the evolution of a concept or a structure or even the general perspective concerning a particular discipline, irrespective of accelerations or regressions, the impact of precursors, or "epistemological gaps" (Piaget & Garcia, 1989, p. 7).

Vergnaud extends the problem from characterising the evolution of a problem, to tracking learner competences in relation to mathematical concepts. He claims that,

> a sound psychological and educational approach to the nature of mathematical concepts requires on the one hand that these concepts be traced to students' competences and in the way students progressively master mathematical situations, and on the other hand that these competences be analysed carefully with the help of well-defined mathematical concepts and theorems (Vergnaud, 1997, p. 8).

Previous chapters have focused on "a general perspective" concerning mathematics, in particular how this general perspective can be identified in the progressive development of number systems, and how the progressive development from mathematical objects that are tied to concrete reality, to mathematical structures that are generalised into systems, which "leave behind" the concrete connections, take place. The progressive development of number systems and the formalisation of sets of numbers and associated properties is "tightly associated with the development of algebra" (Vergnaud, 1997, p. 25).

In this study the research problem is to investigate mathematical concepts and the associated cognitive conceptions in problem situations located along the path to proficiency in the multiplicative conceptual field, both from a theoretical perspective and by taking a cross-section of learners' performance on test items.

In Chapter 6, the task of describing a network of multiplicative concepts that are linked horizontally, and which develop hierarchically towards concepts of greater abstraction, was initiated. The focus of Chapter 7 was to identify in the problem situations, the embedded mathematical concepts and procedures. Test items were discussed in terms of difficulty level, along particular mathematical content strands. The associated performance of learners was linked to the item characteristics. The task in Chapter 8 is to analyse the particular competences or lack of competences in terms of mathematics concepts and theorems and to describe competence levels. The location of learners on the person-item map generated by the Rasch model serves as an estimation of levels, determined by performance on the test items.

In this aspect of study, the research problem is to investigate the development of mathematical concepts and the associated cognitive conceptions in problem situations located along the path to proficiency in the multiplicative conceptual field. The purpose of conducting learner interviews was to provide additional insight into the performance of the items and distractors[126] in the assessment instrument. A second purpose was to describe in general the competences of learners at particular location points, ascertain the concepts requiring attention, and in addition identify the zone in which optimum learning and teaching may subsequently take place.

Primarily, the Rasch model, and its application has enabled a more focused analysis. The location of items from greater difficulty to less difficulty, and the location of learners ranked in terms of those exhibiting greater proficiency to those exhibiting lesser proficiency, invoke the same scale, as depicted in Figure 8.1, and therefore facilitate focused research.

A secondary function of the application of the model is to refine the construct of interest by reflecting on the analysis, and thereby improve the instrument which attempts the measurement of this particular construct of interest.

8.1.1 Research questions

The related research questions addressed in this phase of the research are directly modelled on the questions posed by Vergnaud (1997, p. 9). The central questions are;

[126] Critique of the items and current distractors are important for future instruments informed by this study.

Question 8 What threshold concepts are acquired at the different proficiency levels exhibited by learners? Which concepts have been mastered and which are yet to be explored? How do we understand the reasoning behind the responses?

8.1 What strategies does the learner use to make sense of the problem? Does the learner attempt to understand the problem context? Or does the learner abandon natural sense making?

8.2 What strategies and procedures are used by the learner (concepts-in-action and theorems-in-action) to engage the mathematical problem situations identified within the multiplicative conceptual field? What relationships between variables are identified? Upon which implicit concepts and theorems does each procedure rely?

8.2 Research method

From the 35 test items, four items were selected from the TIMSS substrand *ratio, proportion and percent*, to provide insight into learner's reasoning and problem-solving strategies through focused discussion. The three items of lower difficulty are located at approximately -0.5 **(Item 5)**, zero **(Item 8)** and +0.5 **(Item 10),** that is clustered around the arbitrarily set item mean of zero, and slightly higher than the person mean of -0.550. The fourth item, **Item 26** (specific location 5.085) was found empirically to be much more difficult.[127] The selection of these four items was made on the grounds that the content and context of the items related to typical ratio- and proportional-reasoning type problems. The choice to avoid very easy items was deliberate as these items were not expected to elicit interesting discussion. The four items have been extensively discussed in Appendix B. Learners, Grade 8 at the time of testing, were selected from each of the two schools, at three locations, at the highest location on the scale, at around the median location, and at the lowest location. Table 8.1 exhibits the lists[128] provided to the schools of learners requested to take part in the interviews; the learners marked in italics took part in the interviews.

[127] Item 26 was a constructed-response item, requiring learners to record their calculations. The difficulty level may be higher due to learner reluctance to engage with open response items.
[128] The names have been changed to protect anonymity of participants.

Table 8.1: Composition of interview groups at Schools A and B

	Group A1		Group A2		Group A3	
School A	**learner**	**proficiency estimate[3]**	**learner**	**proficiency estimate**	**learner**	**proficiency estimate**
	Adele	4.13	*Carola*	0.18	Nandi	-0.68
	Kelly	1.86	*Shiluba*	0.26	*Zanele*	-0.82
	Jane	1.86	Denise	0.20	*Cheryl*	-0.93
	Carla	1.38	*Linda*	0.12	Lebo	-0.93
	Angela	1.41	*Kate*	0.12	Michelle	-1.23
	Group B1		**Group B2**		**Group B3**	
School B	**learner**	**proficiency estimate**	**learner**	**proficiency estimate**	**learner**	**proficiency estimate**
	Prinella	1.86	*Phaphama*	-0.52	*Mishack*	-2.19
	Anna	2.37	*Maria*	-0.20	*Mahesh*	-2.60
	Sipho	0.48	*Mpho*	-0.52	*Amukelani*	-2.72
	Thembani	0.60				
	Lerato	0.61				

[1] learners marked in '*italics*' took part in the interviews

[2] all names have been changed to ensure anonymity

In the case of School A (all girls), five learners were selected at each of the levels. It was expected that at least three of the five learners would participate. In the case of School B (both boys and girls) the selection was changed slightly in order to select at least two boys from the higher location band, Thembani (0.601) and Sipho (0.477).

The interviews took place in May 2007, approximately seven months after the initial testing in October 2006. The learners at **School A** were interviewed in groups during school. Three group interviews were conducted; five learners at the high proficiency level, four at the intermediate proficiency level[129], and two at the low proficiency level responded and were interviewed. This arrangement afforded a potential 44 written calculations and person-item interviews. The actual count was 44 written calculations (some were written during the discussion phase of the interview) and 32 recorded interviews on the items.

At **School B**, the interviews took place after school. It proved difficult to meet at the pre-arranged times. The first interview at School B was with Prinella, from the highest performing

[129] The recordings of interviews for the middle group on the last three items have been misplaced. The written responses alone however do provide some evidence as to their thinking.

group.[130] Later interviews took place with the remainder of the highest performing group (3 learners), a middle group (3 learners) and two separate interviews for the lower group. In the case of Amukelani, in the lower group, the one-on-one interview was reasonably successful. The other two in the lower group, Mahesh and Mishack, could be persuaded to attempt the items, but when the discussion started they were reluctant to participate and had to “get the taxi”. The potential item-interviews comprised therefore 40 written responses and 40 item-person interviews. The actual numbers completed were 37 written responses (some only a few numerals or marks on the paper) and 28 item-person interviews.

Learners across a breadth of performance within each school provided a range of proficiency levels. The location of learners on the learner-item map (see Figure 8.1) is marked by initials, for example Adele is located at 4.1 logits, within **Level 7** in this study, whereas Amukelani is located at -2.7 logits, within **Level 1**.

Interview process

In the first phase of the interview the learners were requested to again answer the four items from the original test, but this time to individually write down their calculations. Once they had completed this task, they were asked to explain how they had reasoned about the problem. During the first phase learners worked individually; during the second phase each learner explained to the interviewer (and to the rest of the group) how they had reasoned about the problem.

As to be expected, learners at the high proficiency levels had clear and more concise explanations. The four selected items were also relatively easy for these high proficiency learners. For the learners at the low end of the scale even the easiest of the four interview items was a challenge, as expected. In cases where the learner’s reasoning was unclear, explanations were requested. Where the learner had not understood the problem or used an incorrect procedure, the interviewer probed for information regarding the misunderstanding. In these cases the interviewer provided scaffolding in order to gauge the extent of the missing or incomplete

[130] Due to a technical problem there is no record of the interview.

concepts deemed necessary to solve the problem and the extent of the work still required for the learner to master the particular concepts.

Learner responses to items at graded difficulty level

In **School A**, the learners in the top group were mostly located at **Level 5**, with one learner located at **Level 7**; learners in the middle group were located at **Level 4**; and learners in the lower group at **Level 3**. In **School B**, the learners in the top group were mostly located at **Level 4**, with one learner located at **Level 6**; learners in the middle group at **Level 3**; and learners in the lower group at **Level 1.** Selected learner interviews therefore included learners at **Level 1** [-3, -2), **Level 3** [-1, 0), **Level 4** [0, 1), **Level 5** [1, 2), **Level 6** [2, 3) and **Level 7** (> 3). Note that the proficiency levels are aligned with the Person-Item map.

Application of the Rasch model locates items in order along a continuum. The learners are likewise located along the same continuum. The selection of particular items for the interviews sought to ensure the inclusion of ratio and proportional reasoning concepts. It is to be expected that learners located above the interview problem item location have a greater probability of solving the problems, while learners located below the level of the particular problem item will have greater difficulty. Learners located at the same level as the item have a 50% probability of success with the item.

The selection on the part of the researcher of three items clustered around the item mean was expected to have the following implications. Learners at the top end of the scale would find these items relatively easy. It is only **Item 26** located at the top end of the scale where the top learners might experience difficulty. On the other hand, for learners at the lower end of the scale, it was expected that all three items would present as relatively difficult. It was therefore from the middle groups where most information was obtained. Because there were no items at the lower end of the continuum, an attempt was made during the interviews to investigate the obstacles to solving the problems.

Table 8.2: Matrix exhibiting item difficulty level by proficiency level

	Low proficiency		Middle proficiency		High proficiency		
proficiency level	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Item (location)	[-3, - 2),	[-2, -1),	[-1, 0)	[0, 1)	[1, 2)	[2, 3)	(> 3)
Level 1 No item							
Level 2 No item							
Level 3 Item 5 (-0.585) Item 8 (-0.060)	Mishack, Mahesh, Amukelani		Cheryl, Zanele, Phaphama, Maria, Mpho	Carola, Shiluba, Linda, Kate, Thembani, Sipho	Kelly, Angela, Jane, Carla, Prinella	Anna	Adele
Level 4 Item 10 (0.438)	Mishack, Mahesh, Amukelani		Cheryl, Zanele, Phaphama, Maria, Mpho	Carola, Shiluba, Linda, Kate, Thembani, Sipho	Kelly, Angela, Jane, Carla, Prinella	Anna	Adele
Level 5 No item							
Level 6 No item							
Level 7 Item 26 (5.038)			Cheryl, Zanele, Phaphama, Maria, Mpho	Carola, Shiluba, Linda, Kate, Thembani, Sipho	Kelly, Angela, Jane, Carla, Prinella	Anna	Adele

For analysis purposes, the **high proficiency** group included learners located at **Level 5, 6** and **7**. This group is discussed in two sections, firstly **Levels 7 and 6**, (Adele, **Level 7** from **School A** and Anna, **Level 6** from **School B**), and then secondly, **Level 5,** (four learners from **School A**, and one learner from **School B**).

The **middle-high proficiency** group included learners at **Level 4**, Groups B1 and A2, and learners at **Level 3**, Groups B2 and A3, and the **low proficiency** group consisted of **Level 1**, Group B3. Table 8.2 shows the item difficulty levels on the vertical axis and the learner proficiency levels on the horizontal axis. The diagonally shaded blocks show the coincidence of learner level and item level. Note there are no learners at **Level 2**.

The interviews for both high proficiency learners and low proficiency learners were conducted on the same items. This exercise resulted in a wide range of responses. It is therefore expected that for learners at the higher end of the continuum more sophisticated and generalisable mathematical structures will be evident, but at the lower end of the continuum, the focus may be on more basic skills. Table 8.3 provides an overview of the graded analysis applied in this study.

Table 8.3: Focus area for each level group

Proficiency levels		Addition and subtraction	Multiplication and division	Fractions, decimals, operations	Percent, concept, operations	Identify variables, relationships	Fluency with algorithms
High	(L5, L6, L7)				✓	✓	✓
Middle high	(L4)			✓	✓	✓	✓
Middle low	(L3)		✓	✓	✓	✓	
Low	(L1)	✓	✓	✓	✓		

8.3 Framework for interview analyses

For reasons outlined previously the theoretical analyses of multiplicative structures, presented in Chapter 6, are based on the work of Vergnaud and associates (1983). Vergnaud justifies the analytic framework, by asserting that the mathematical potential of the theorems-in-action may be recognised by both teacher and researcher.

Three distinct subtypes are identified within multiplicative structures, *isomorphism of measures*, consisting of "simple direct proportion between two measure spaces, M_1 and M_2" (Vergnaud,1983, p. 129), *product of measures*, consisting of the "Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3" (p. 134), and *multiple proportion*, which has characteristics similar to the *product of measures*, in that "a measure-space M_3 is proportional to two independent measure-spaces M_1 and M_2" (p. 138). Each of these three subtypes has subclasses of problems which are explained in Chapter 6, Section 6.2.3. The items discussed in this chapter are restricted to *isomorphism of measures*, the direct proportion of two measure spaces, where the subclass of problems includes *multiplication* and *division* (*partitive* and *quotitive structures*) and the *rule of three* (Table 8.4).

The analysis applied to multiplicative structures, also applies to fractions, both discrete and continuous, ratio, both intensive and extensive, and percent type problems. Sharing a whole into parts involves "direct proportion between the shares and the magnitude to be shared (isomorphism of measures)" (Vergnaud, 1983, p. 161). See Chapter 6, Section, 6.3.2.1, Figure 6.11).

As noted in Chapter 6, percent problems include all the complexity of rational number, but in addition have a concise language that in some cases obscures the underlying relationships. Using Table 6.6, we infer that percent problems may also be represented by the direct comparison of measure spaces. The percent aspect is presented as one *measure space* and *the quantity* or magnitude to which the percent is being applied, as the *second measure space*. In addition to characterising percent problems as the direct comparison between two measure spaces, we classify the percent problem contexts as a *fraction part*, *change within one referent* relationship or as a *comparison between two referents* relationship (see Chapter 6, Table 6.6). The characteristics of multiplicative structures are identified in problem situations and learner responses. Describing problem situations in terms of measure-spaces allows the relationship of problem elements to be identified.

Table 8.4: Isomorphism of measures (2 measure-spaces)

Multiplication:	**M_1**	**M_2**	**Division:**	**M_1**	**M_2**
Binary law of composition $b \times c = x$ 4 rolls fencing at R300 each Solved by identifying b and c in a multiplicative relationship	(fencing) 1 c	(price) b x	*Partitive division* Find the unit value Solved by applying scalar operator $/c$ to magnitude d	$\begin{array}{c} 1 \\ \downarrow /c \\ c \end{array}$	$\begin{array}{c} b \\ \downarrow /c \\ d \end{array}$
Unary operation - scalar operator - function operator Solved by calculating scalar or function operator	**M_1**	**M_2**	*Quotitive division* Find x knowing $f(x)$, $f(1)$ Solved by inverting the direct function operator and applying it to b	**M_1**	**M_2**
	$\begin{array}{ccc} 1 & \xrightarrow{\times b} & b \\ \times c \downarrow & & \downarrow \times c \\ c & \xrightarrow{\times b} & x \end{array}$			$\begin{array}{ccc} 1 & \xleftarrow{/b} & b \\ & & \\ x & \xleftarrow{/b} & d \end{array}$	
Rule of three: General case Multiplication and division problems are simple cases of the rule of three problem, where one of a, b, c and d may be the unknown.				**M_1** a c	**M_2** b d

Some experiments designed to explore and verify the theoretical analysis developed by Vergnaud and associate researchers, were conducted (see Vergnaud, 1983, pp. 140-145). Of interest is the variety of procedures observed (pp. 145-149). For this study, we present the description of Items 5, 8 and 10 (see Table 8.5), in terms of measure spaces (Figure 8.1). These measure spaces then serve as a framework for describing the procedures (see Table 8.6). The letters *a*, and *c* refer to measure-space M_1, *b* and *d* refer to the measure-space M_2.

Table 8.5: Items in the interviews

Item 5 (simplified)	• Three brothers share 45 000 zeds in proportion to the number of children each one has. Dan has 4, Sam has 3 and Thabo had 2 children. How much money does Dan receive?
Item 8	• A shop increased its prices by 20%. What is the new price of an item which previously sold for 800 zeds?
Item 10 (simplified)	• A machine uses 2.4 litres of gasoline for every 30 hours of operation. How many litres will the machine use in 100 hours?

Figure 8.1: Items represented as measure spaces

General case		Item 5		Item 8		Item 10	
M_1	M_2	M_1 (children)	M_2 (money)	M_1 (percent)	M_2 (money)	M_1 (time)	M_2 (capacity)
a	b	9	45 000	100	800	30	2.4
c	d	4	x	120	x	100	x

The *correct procedures* in the study undertaken by Vergnaud and associates were classified into five subcategories, *scalar*, *scalar decomposition*, *function*, *unit value*, and *rule of three*, (Vergnaud, 1983, pp. 145-146). The incorrect strategies were also classified into groups (pp. 147-149), which in many cases were based on some elements of the corresponding correct strategy (p. 147).

Table 8.6*: Isomorphism of measures subcategories*

Type	Explanation	Example
Scalar:	The student calculates ($c/a = \lambda$) or ($a/c = \lambda$). This calculation may be done by division or through the missing factor procedure either explicitly or mentally. The student then calculates $x = \lambda \times b$ or $x = b \times \lambda$, or b/λ.	**Item 5**: Angela attempts a scalar procedure by finding 4/9 (c/a) of 45 000 zeds (b).
Scalar decomposition	The student decomposes the magnitude c as a linear combination of other different magnitudes.	**Item 10**: Kelly calculates (2.4 × 3) + (2.4 / 3) for the amount of fuel required by decomposing 100 hours (c) into (30 × 3 + 30/3).
Function	The student calculates $\frac{b}{a} = \lambda$ or $\frac{a}{b} = \lambda$, again through division, the missing factor procedure, and either explicitly or mentally.	**Item 10**: Adele divides 2.4 (b) by 3 (30) (a) in order to get function operator 0.8 for 10 hours.
Unit value	The student calculates $\frac{b}{a} = \lambda$, but then explains that λ, or $\frac{b}{a}$, is the unit value.	**Item 10**: Anna divides 2 400 millilitres (b) (having converted from 2.4 litres) by 30 hours (a) and gets a unit rate of 1 hour = 80 millilitres.
Rule of three	The student calculates $\frac{b \times c}{a}$ or $\frac{c \times b}{a}$.	**Item 8**: Kelly multiplies 800 (zeds) (b) × 20 (%) (c) and then divides by 100 (a).

Source: Vergnaud, 1983, pp. 145-146

The *incorrect procedures* found in Vergnaud's experiments "are based on some aspects of the indicated real situations" (Vergnaud, 1983, p. 147). These procedures are identified as *erroneous scalar, incorrect scalar decomposition, erroneous function, erroneous scalar and function, inverse, erroneous product, erroneous quotient* and *other* (see Table 8.7).

Parker and Leinhardt (1995, p. 428) found the errors that students make with percent type tasks to be simply *dropping the percent sign*, *abandoning natural sense making* and to having a strong *part-whole notion of percent,* when a ratio understanding is required.

Table 8.7: Erroneous subcategories

Type	Explanation	Example
Erroneous scalar	The student uses a scalar ratio or difference (c/a) or (a/c) or a-c and gives this part result as an answer, or multiplies by b, or adds it to b, or divides b by it, or subtracts it from b.	**Item 5**: Angela attempts a scalar procedure by finding 4/9 (c/a) of 45 000 zeds (b) but then divides instead of multiplying.
Incorrect scalar decomposition	The student decomposes the magnitude c as a linear combination of other different magnitudes. This procedure was found to be common in the current research study and will be explained in more detail with specific reference to learners.	**Item 10**: Linda multiplies 2.4 litres (b) by 3 (in an attempt to relate the 30 hours to 100 hours), to get 7.2, but then adds 10 hours.
Erroneous function	The student uses a function ratio or difference, either (b/a), or b-a, or (a/b), or a-b, and gives this ratio as an answer or applies this to c.	**Item 5**: Phaphama finds (b/a) and then instead of multiplying by c, divides by c.
Erroneous scalar and function	The student makes a calculation, $b \times c$, forgetting division by a.	
Inverse	The student uses the inverse ratio a/c instead of c/a, or c/a instead of a/c, or b/a instead of a/b.	**Item 10**: 30/100 of 2.4.
Erroneous product	The student multiplies c and a or b and a.	**Item 10**: Mahesh multiplies 100 by 30.
Erroneous quotient	The student divides c by b, or b by c which has no meaning.	**Item 5**: Amukelani incorrectly says "I divided 45 (45 000) into the number of children 9". Phaphama divides 5 000 zeds by 4, instead of multiplying.

Source: Vergnaud, 1983, p. 147

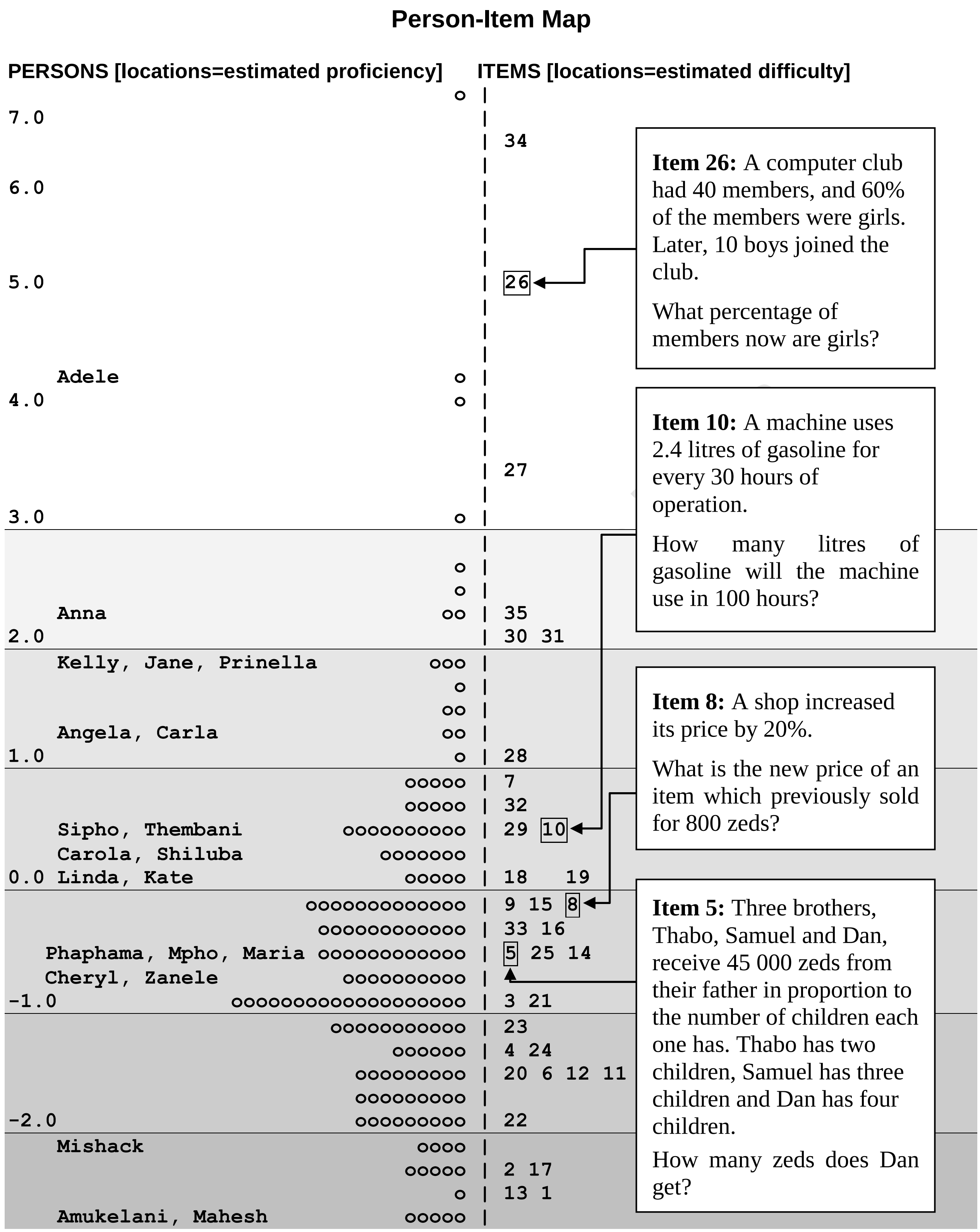


Figure 8.2: Item interviewee scale indicating item difficulty and learner ability

8.4 High proficiency learners

For learners at **Levels 5, 6** and **7**, we can assume fluency with addition and subtraction, multiplication and division, and assume that the transition has been made from natural numbers to rational numbers, and therefore fluency with fractions and decimals has been achieved. The particular focus is the identification of variables and relationships within problem situations, the application of algorithms, and percent-related concepts (see Table 8.8).

Table 8.8: Focus areas for high proficiency level

Proficiency levels		Addition and subtraction	Multiplication and division	Fractions, decimals, operations	Percent, concept, operations	Identify variables, relationships	Fluency with algorithms
High	(L5, L6, L7)				✓	✓	✓

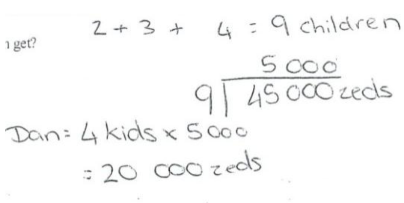
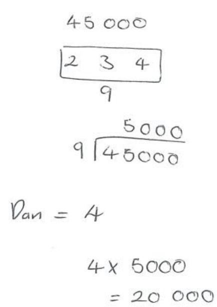
8.4.1 Levels 6 and 7: Adele (School A), Anna (School B)

According to the Rasch model, Adele's location on the scale at location 4.13 means that the probability of attaining a correct response to items at a lower level location should be fairly high. Similarly, Anna's level estimated above two logits, location 2.37, means that she should find most items relatively easy. Both Adele and Anna's written responses and verbal explanations for **Items 5** are recorded in Table 8.9.

Both Adele and Anna confidently engaged with the problems in an attempt to understand the mathematical relationships. Adele's strategy for **Item 5** can be described as finding the function operator. She has divided 45 000 by 9 to find the unit value, 1 child corresponds to 5 000 zeds. She then multiplies by 4.

Anna begins her explanation of **Item 5** with a reference to the topic domain of the item problem, "I first thought about ratio and proportion". She applies a similar procedure to Adele, which could be described as first applying a unit value approach, then a function approach.

Table 8.9: Item 5 response and analysis: Levels 6 and 7

<table>
<tr><th>Learner response</th><th>Natural language</th><th colspan="2">Mathematical structure</th></tr>
<tr><td>Adele:
</td><td>"I divided by nine and multiplied by four … "</td><td>M_1
children
9
/ 9
1
× 4
4</td><td>M_2
money
45 000
/ 9
5 000
× 4
20 000</td></tr>
<tr><td>Anna:
</td><td>"I first thought about ratio and proportion, and then I thought about adding up all the children … so I could work out how much one child would get … and then I worked it out to be five thousand. And then they asked how many zeds would Dan get … so I multiplied this by 4, and I got 20 000 … "</td><td>M_1
children
9
/ 9
1
× 4
4</td><td>M_2
money
45 000
/ 9
5 000
× 4
20 000</td></tr>
</table>

The strategy used by **Adele** for **Item 8** can be described as the *rule of three* strategy, where she multiplies $b \times c$ and divides by a.

The interviewer further investigated Adele's understanding of percent (see below).

Interviewer:	*So what is the hundred percent?*
Adele:	Because you're increasing it … .The amount is already a hundred over a hundred and then to increase it you add another 20 onto that to get the percentage.
Interviewer:	*So a hundred is the full price and the hundred and 20 is the 20% added?*

Her explanation shows that she understands that 100/100 represents the whole, and that the increase requires that the percentage be added so that the ratio is 120:100.

Anna's strategy for **Item 8** is subtly different from Adele's approach (see Table 8.10). Her written calculation mirrors the *rule of three* strategy; her verbal explanation is aligned more with a *function* strategy. As in **Item 5,** Anna begins her explanation for **Item 8**, with "I thought of interest and financial maths".

Table 8.10: Item 8 response and analysis: Levels 6 and 7

Learner response	Natural language	Mathematical structure	
Adele:		Percent	Amount
100% + 20% = 120% $\frac{120}{100}\% \times \frac{800}{1}$ = 960 zeds	"I said 120 over a hundred. I first added to get the percent. I added 20 onto the full price, the hundred, and then I timesed it by 80."	100 + 20 100 120	 800 960
		$\frac{c}{a} \times \frac{b}{1}$	
Anna:		Percent	Amount
$\frac{800}{1} \times \frac{120}{100}$ = 160 800 + 160 = 960zeds	"I thought of interest and financial maths. So I thought how much 20% of 800 is … and that is 160, and I added 160 to the 800 and got R960."	100 20 120	800 160 960
		$\frac{b}{1} \times \frac{c}{a}$	

The analysis in Table 8.11 shows both Adele and Anna's calculation and reasoning for **Item 10**. Note that both Adele and Anna use the equals sign inappropriately. Adele used two conversions for **Item 10**, firstly divide 30 by 3 to get 10, and then multiply by 10 to get a hundred. The strategy Adele uses may be described as *scalar decomposition*.

Anna's solution to **Item 10** is also correct. She uses a combination of approaches, though her first intuition is the *unit value* strategy, dividing both sides by 30. She also applies a *scalar decomposition* strategy to the problem. The unit value strategy is however the more efficient strategy and will accommodate greater number ranges, and numbers for which there is no common denominator. When challenged however, she explores another option that may be described as a *function* strategy.

Anna:	I could have multiplied it (the 800 millilitres) by ten hours … equals 800 millilitres to get 100 hours (after I found that 10 hours was equal to 800 millilitres). And that would have been 100 hours is equal to … 8 000.

Or she could have multiplied by 100 after she had calculated that 1 hour "is equal to" 80 millilitres.

Table 8.11: Item 10, response and analysis: Levels 6 and 7

Learner response	Natural language	Mathematical structure: Capacity	Mathematical structure: Duration
Adele:		**Capacity**	**Duration**
10 hours = 2,4ℓ ÷ 3 = 0,8ℓ 100 hours = 0,8ℓ × 10 = 8ℓ $3\overline{)2,4}$ = 0,8	"I said 2,4 litres, divide by 3 to get 10 hours. Then I multiplied 0,8 by 10 to get 100 hours."	2,4 $\xrightarrow{/\,3}$ 0.8 $\xrightarrow{\times\,10}$ 8.0	30 $\xrightarrow{/\,3}$ 10 $\xrightarrow{\times\,10}$ 100
Anna:		**Capacity**	**Duration**
(A) 7.2 (B) 8.0 (C) 8.4 (D) 9.6 2,4l = 2400ml (30hrs) 30hrs = 2400ml 2400 + 2400 + 2400 = 7200 $30\overline{)2400}$ = 80 1hr = 80ml 10hrs = 800ml 90hrs = 7200ml 7200 + 800 = 8000ml 100hrs = 8000ml	"2,4 litres is 2 400 millilitres which is 30 hours.	2.4 2 400	30
	I divided 2 400 millilitres by 30 hours to get 80, which means 1 hour is equal to 80 millilitres, which means 10 hours is equal to 800 millilitres,	2 400 $\xrightarrow{/\,3}$ 80 $\xrightarrow{\times\,10}$ <u>800</u>	30 $\xrightarrow{/\,3}$ 1 $\xrightarrow{\times\,10}$ <u>10</u>
	And then for 90 hours I added them (2 400 + 2 400 + 2 400).	2 400 2 400 <u>2 400</u>	
	I got 7 200.	7 200	90
	Then I added my 800 to get my 8 000, which is 8 litres."	<u>800</u> 8 000 8 litres	<u>10</u> 100

Item 26, the item ranked most difficult of the interview items, and second most difficult of all the items, provided the greatest challenge amongst items in the interview. In this problem the first step requires that the learner understands "40 children" to be "the whole", and thereafter to calculate the number of boys and girls. This step requires the *operator* subconstruct, 60/100 is applied to 40 children. The "whole" then changes to 50 children and a ratio of 24 to 50 has to be converted to percent.

Both **Adele and Anna** approach this problem with relative ease (see Table 8.12). Adele slips momentarily and writes 24/100, but corrects herself and writes 24/50. Adele uses a standard *rule of three* procedure to find the number of children, $\frac{a \times d}{c}$, and then works comfortably with the changing referents. Anna also uses the *rule of three* strategy to calculate the initial number of girls. She also used this method to calculate the final percentage. The fact that the underlying referent changes and that there is a change in percent type problem does not present any difficulties to these learners.

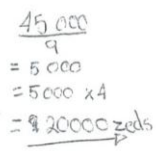
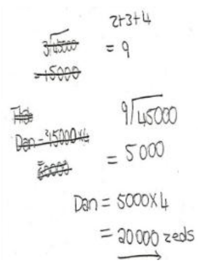
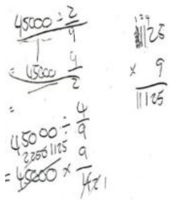
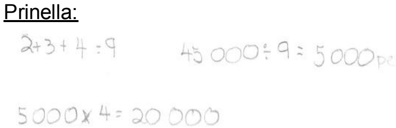
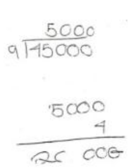
Table 8.12: Item 26, response and analysis: Levels 6 and 7

Learner response	Natural language	Mathematical structure: Percent	Mathematical structure: Amount
Adele:		Percent	Amount
$\frac{40}{100} \times \frac{40}{1} = 16$ boys $\frac{60}{100} \times \frac{40}{1} = 24$ girls; Answer: 48%	"40% boys at the beginning so that was 16 boys.	40 *(a)*	16 *(x)*
16 boys + 10 = 26 boys + 24 girls = 50 children		100 *(c)*	40 *(d)*
$\frac{24}{50} \times \frac{100}{1} = 48\%$ are girls	And then I added 10 that make 26.		16 + 10
			26 ⁄ 50 (boys)
	24 were girls that made 50 children 24 girls in 50 children made 48%."	48%	24 ⁄ 50 (girls)
Anna:		Percent	Amount
$\frac{40}{1} \times \frac{60}{100} = 24$ girls	"60% of the forty members gave me	60 *(a)*	24 *(x)*
∴ 16 boys		100 *(c)*	40 *(d)*
16 + 10 = 26 boys ∴ 50 members altogether	24 girls and 16 boys		16 ⁄ 40 (boys)
$\frac{24}{50} \times \frac{100}{1} = 48\%$	I added ten boys … and that gave me 26 boys out of the 50 members altogether … the 24 girls over the 50 … 48%"		+ 10
			26 ⁄ 50 (boys)
		100 *(a)*	50 *(b)*
		48 *(c)*	24 *(d)* (girls)

8.4.2 Level 5: Kelly, Jane, Angela, Carla (School A), Prinella (School B)

The written work and excerpts of the interviews with four of the five learners from the highest performing group at School A, and Prinella, from School B, are presented in Table 8.13.

Table 8.13: Item 5 response and analysis: Level 5 Groups A1 and B1

Learner response	Natural language	Mathematical structure
Kelly: 	"I first divided by nine then timesed by 4."	$\begin{array}{c\|cc\|c} & 9 & 45\,000 & \\ /\,9 & 1 & 5\,000 & /\,9 \\ \times 4 & 4 & 20\,000 & \times 4 \end{array}$
Carla: 	"You divide 40 000 by nine. And then you get 5 000… [Dan] has four children so four times 5 000"	$\begin{array}{c\|cc\|c} & 9 & 45\,000 & \\ /\,9 & 1 & 5\,000 & /\,9 \\ \times 4 & 4 & 20\,000 & \times 4 \end{array}$
Angela: 	"I didn't finish. I just guessed 15 000"	$\begin{array}{c\|c} 45\,000 & \\ 1\,125 & /\,4/9 \end{array}$
Prinella: 	[no recording]	$\begin{array}{c\|cc\|c} & 9 & 45\,000 & \\ /\,9 & 1 & 5\,000 & /\,9 \\ \times 4 & 4 & 20\,000 & \times 4 \end{array}$
Jane: 	[nothing verbalised]	$\begin{array}{c\|cc\|c} & 9 & 45\,000 & \\ /\,9 & 1 & 5\,000 & /\,9 \\ \times 4 & 4 & 20\,000 & \times 4 \end{array}$

Kelly produces the correct answer for **Item 5**, (Table 8.13), though she makes the mistake of using the equals sign as a separator between the mathematical statements rather than a symbol which denotes equality between the respective lines (Kieren, 1990). **Carla**, while initially misinterpreting the problem and dividing 45 000 by 3, corrects herself. Both **Jane's** and **Prinella's** written responses are also correct. The strategy these four learners use may be described as *scalar decomposition*, first dividing by 9, and then multiplying by 4. This approach may also be described as finding the *unit value*, although not explicitly. The rational number subtype is the operator subconstruct. These learners exhibit proficiency with multiplication and division of whole numbers.

Angela presented something of an anomaly[131] as she seemed not to exhibit the proficiency expected at her estimated level. In order to explore further the difficulty Angela experienced, the interviewer asked her to elaborate.

Interviewer:	*Ok, so why didn't you finish the sum? What did you find difficult about it?*
Angela:	I just forgot how to work out the ratios because we learnt how to do them. It was a long time ago. We knew about ratios, but I forgot.
Interviewer:	*It was a while ago?*
Angela:	It was last year this time.
Interviewer:	*So what do you understand about the word proportion?*
Angela:	It is the amount each person gets. It is like a fraction …

In fact Angela had attempted the scalar operator procedure by calculating 2/9 of 45 000 (Thabo's share) and then 4/9 of 45 000 (Dan's share). But she was confused about multiplication and division of fractions.

In Angela's thinking "sharing in proportion" was interpreted to be division by a fraction instead of multiplying by a fraction. It does appear that she remembers that "dividing by a fraction involves inverting and multiplying". However this procedure, while remembered, does not assist her with solving the problem.

[131] The researcher investigated Angela's script and its scoring to investigate the anomaly that Angela, according to the model, should perform better on this item. No inconsistencies were found.

In **Item 8**, similar processes are followed by all the learners (see Table 8.14). They use versions of the *rule of three*, that is $\frac{b \times c}{a}$. The response may be procedural and the underlying relationship implicit, though this inference is not clear. In **Kelly's** case she gets the correct answer for Item 8, but the interviewer is not entirely sure of her reasoning. She also repeats the error of using the equals sign as a separator rather than a symbol for equality. The interviewer follows up with a query.

Interviewer:	*... You've multiplied 800 × 20 and then you divide it by hundred ... OK, I understand what you did. Why did you multiply by 20? Is this a method that you have learned?*
Kelly:	That's how you get the percentage if you multiply it by that (pointing to 20) you get the percentage then you divide by a hundred.
Interviewer:	*So you are finding 20 parts out of a hundred.*
Kelly:	(laughs) Am I?

Angela's initial response to **Item 8** is that it is more difficult than Item 5. However, when questioned about percent, she manages to recall a procedure and calculate correctly.

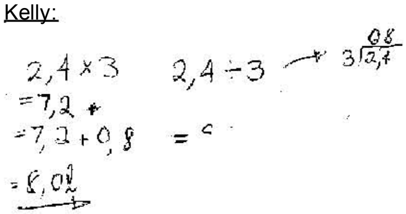
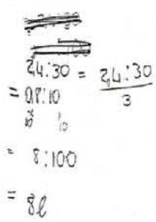
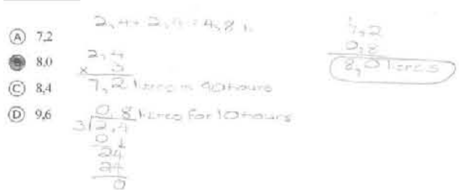
Interviewer:	*What do you understand by percent? Let me start with you Angela.*
Angela:	It is how much out of the hundred ... The amount of something from a hundred …
Interviewer	*Okay, and then when there is an increase of 20%?*
Angela	You work out 20% of the given amount ... the increase … and then to add to the amount you had before.

Angela has the procedure stored in her memory but was unable to make the links between the concepts in the problem and the procedure she had learnt.

Table 8.14: Item 8 response and analysis: Level 5, Groups A1 and B1

Learner response	Natural language	Mathematical structure
Kelly: $(800 \times 20) \div 100$ $= 16\,000 \div 100$ $= 160$ $= 160 + 800$ $= 960$ zeds	"I worked out percentage first and then I divided by hundred, and then I plussed eight hundreds to that percentage"	$800 \times 20 = 1\,600$ $1\,600 / 100 = 160$ $800 + 160$ $= 900$ $(b \times c)/a$
Carla: ~~$\frac{120}{100} \times \frac{800}{1}$~~ $\frac{120}{100} \times \frac{800}{1}$ $= 960$ zeds	[No discussion. From the written working we see that Chanelle applied the more sophisticated procedure of multiplying by 120%]	**Percent** / **Amount** $100 + 20$ / $800\ (b)$ $100\ (a)$ / $160\ (d)$ / $800 + 160$ $120\ (c)$ / 960 $\frac{c}{a} \times \frac{b}{1}$
Angela: $800 \times \frac{20}{100}$ $= \frac{8}{1} \times \frac{20}{1}$ $= 160$ $= 800 + 160 = 960$ zeds	[Initially confused about the problem, then she recalls the procedure] "You work out 20% of the given amount … the increase … and then to add to the amount you had before"	**Percent** / **Amount** $100\ (a)$ / $800\ (b)$ $20\ (c)$ / $160\ (d)$ $100 + 20$ / $800 + 160$ 120 / 960 $b = (\frac{c}{a} \times \frac{b}{1})$
Prinella: $\frac{800}{1} \times \frac{20}{100} = 160$ zeds $800 + 160 = 960$	[no recording]	**Percent** / **Amount** $100\ (a)$ / $800\ (b)$ $20\ (c)$ / $160\ (d)$ 120 / 960 $\frac{b}{1} \times \frac{c}{a}$
Jane: $\frac{100+20}{100} \times \frac{800}{1}$ 120 $\frac{800}{1} \times \frac{100+20}{100}$ 120×8 960	"I am sure mine is the same just written differently"	**Percent** / **Amount** $100 + 20$ / $800\ (b)$ $100\ (a)$ / $960\ (d)$ $120\ (c)$ $\frac{b}{1} \times \frac{c}{a}$

Table 8.15: Item 10, response and analysis: Level 5, Groups A1 and B1

Learner response	Natural language	Mathematical structure: Percent	Amount
Kelly: 	"I said 2,4 times 3 because that is what would be 90 hours and I've got 7,2. Then I said 2,4 divide by 3 (to give me ten hours) which was 0,8 so then it was 7,2 plus 0,8 equals 8,0 l"	$2.4 \xrightarrow{\times 3} 7.2$ $2.4 \xrightarrow{/\,3} 0.8$ 7.2 + 0.8 = 8.0*l*	$30 \xrightarrow{\times 3} 90$ $30 \xrightarrow{/\,3} 10$ 90 + 10 = 100h
Carla: 	[Carla did not offer an explanation here, but from the working, we see that she approached the problem from the perspective of ratio]	$2.4 \xrightarrow{/\,3} 0.8 \longrightarrow 8.0$ 8.0*l*	$30 \xrightarrow{/\,3} 10 \xrightarrow{\times 10} 100$
Angela: 	"I found the percentage, don't ask me why."	2.4 (a) x (c) $\frac{a}{1} \times \frac{b}{d}$	30 (b) 100 (d)
Prinella: 	[no recording]	$2.4 \xrightarrow{\times 3} 7.2$ $2.4 \xrightarrow{/\,3} 0.8$ 7.2 + 0.8 = 8.0*l*	$30 \xrightarrow{\times 3} 90$ $30 \xrightarrow{/\,3} 10$ 90 + 10 = 100h
Jane: 	"I moved the comma, and made it 24 and 300 and I divided by three to get 100 and then divided 24 by 3 to get eight."	$24 \xrightarrow{/\,3} 8$ 8.0*l*	$300 \xrightarrow{/\,3} 100$

The group of learners located at this level of proficiency exhibit fluency with multiplication and division, and with the decimal fractions necessary in **Item 10** (see Table 8.15). **Kelly** and **Prinella** both applied scalar decomposition strategy to **Item 10**. This strategy is also described by Vergnaud (1997) as the isomorphic property of the linear function, written mathematically as;

$f(30) = 2.4 \qquad f(100) = f(3 \times 30 + 1/3 \times 30)$

$\qquad\qquad\qquad\qquad = 3\,f(30) + 1/3 \times f(30)$

where f = the function value that assigns the corresponding capacity to the given duration.

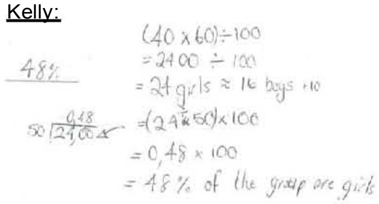
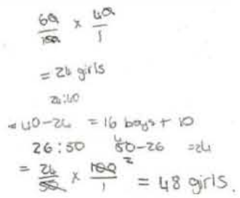
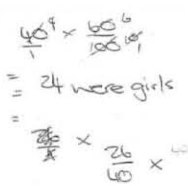
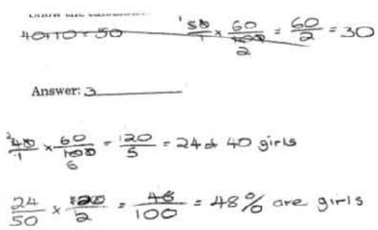
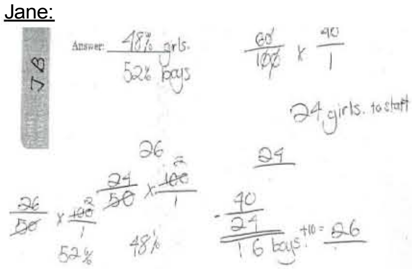
Both **Carla** and **Jane** approached this problem by applying their understanding of ratio, that is by applying a scalar operator to both measure spaces. Carla first divides both 2.4 litres and 30 hours by 3 to get 0.8 and 10, and then multiplies both by 10, to get 8 litres to 100 hours, Jane first converts both 2.4 litres and 30 hours into 24 (litres) and 300 (hours). She then applies a function operator /3 to convert 300 hours to 100 hours and 24 litres to 8 litres. This approach is effective. She is calculating *within* the measure space.

Item 26 requires a two-step calculation, first “finding the percentage of” an amount, then converting a ratio into a percent.

Carla and **Jane** use two *rule of three* algorithms to solve the problem in **Item 26** (see Table 8.16). Carla again supports her reasoning through invoking the ratio relationship. **Kelly** uses a similar strategy though sets out the operation somewhat differently (40 × 60) / 100. She makes the same logical error as she did for Items 5 and 8, using the equals sign as a separator rather than a symbol for equality.

Angela manages the first part of the two-part calculation, but then “got stuck”. However when she started to express her reasoning she was able to mentally calculate the equivalence, “24 girls in 50 made 48%”. Likewise, **Prinella** calculates the first part of the problem by applying the *rule of three* algorithm, and for the second part of the problem uses the concept of equivalent fractions.

Table 8.16: Item 26, response and analysis: Level 5, Groups A1 and B1

Learner response	Natural language	Mathematical structure: Percent	Mathematical structure: Amount
Kelly: 	[Kelly again uses the procedure 1. $(b \times c) \div a$. For the second step she changes the strategy to $\frac{a}{c} \times 100$ 2. $\frac{a}{d} \times \frac{c}{1}$]	100 (a) 60 (c) ? (a) 100 (c)	40 (b) 24 (d) 24 (b) 50 (d)
Carla: 	[Carla again uses a ratio approach to the problem. This time in addition to using fraction calculation procedures. 1. $\frac{c}{a} \times \frac{b}{1}$ 2. $\frac{b}{d} \times \frac{c}{a}$]	100 (a) 60 (c) ? (a) 100 (c)	40 (b) 24 (d) 24 (b) 50 (d)
Angela: 	"40 over 1 times by 60 over 100 and then I got stuck. I didn't know how to do the boys 1. $\frac{c}{a} \times \frac{b}{1}$ 40% boys at the beginning so that was 16 boys. I added 10 that make 26. 24 girls in 50 children made 48%."	100 (a) 60 (c) ? (a) 100 (c)	40 (b) 24 (d) 24 (b) 50 (d)
Prinella: 	[Prinella begins with the *rule of three algorithm* $\frac{b}{1} \times \frac{c}{a}$ But then uses a different strategy $\frac{24}{50} \times \frac{2}{2} = \frac{48}{100}$]	100 (a) 60 (c) ? (a) 100 (c)	40 (b) 24 (d) 24 (b) 50 (d)
Jane: 	[Applies the same strategy as Carla]	100 (a) 60 (c) ? (a) 100 (c)	40 (b) 24 (d) 24 (b) 50 (d)

8.4.3 Proficiency exhibited at Levels 5, 6 and 7

It is acknowledged that the estimates at the extremes of the scale are less stable on account of fewer learners and fewer items with which to make comparative estimations. Nevertheless, we will explore similarities and differences in the problem-solving approaches of the two groups of learners.

The group, located at **Level 5**, has a degree of proficiency over the problems selected for the interview and probably over most of the items in the test**.** According to the model, the six items, **Items 26, 27, 31, 32, 34**, and **35,** are those for which they have a less than 50% probability of attaining the correct answer.

All four items selected for the interview, were easily solved by both Adele and Anna at **Levels 6** and **7**, and by four of the five learners at **Level 5**. An intuitive understanding of the relationship between the elements of the problem ensured that the procedures they used made some sense. The errors in the procedures may present difficulties elsewhere, and should therefore be noted. In some sense, Angela, though not attaining correct answers, showed some insight into the concepts.

This group of learners, besides Angela, have a positive engagement with the problems. Angela, it seems, is not confident in her approach to solving these problems and it appears that she is trying to recall taught strategies rather than engage with the problem at hand. When she does engage with the problem, she is relatively quick at solving the problem. In general the group exhibits fluency with multiplication and division and with problems where decimal and percent concepts and notation are required.

Items 5 and **10** evoke a scalar operator approach from this group. In particular the related strategy of scalar decomposition appears to be favoured. Jane and Carla call on a ratio concept in solving the problem. The percent problems, **Item 8** and **Item 26**, are solved using the *rule of three* algorithm, which most of the group execute satisfactorily. As a whole there was evidence of fluency with *multiplication and division*, and *proportional sharing*. For Angela there appeared to be some confusion with calculating proportional shares. Her explanation was that she had forgotten “ratio and proportion” as it had been taught the previous year.

Conversions between equivalent forms of rational number representations and measures, and operations with *decimal fractions* were managed by this group. They managed three types of *percent* problems: change involving percent increase or a percent decrease, the fraction type treating percent involving a proportion of the whole and the conversion of a ratio of two magnitudes into a percent.

The mathematical skills of *identifying variables and relationships*, of using ratio understanding to find proportions, identifying *scalar and function operators*, and *finding the unit rate*, appeared to be well mastered by this group as a whole, with most learners showing a preference for one or other particular approach. Patterns of schema were observed.

Fluency with algorithms was observed, in addition to learners applying algorithms in conceptually clumsy ways, for example Kelly, and using inappropriate algorithms, for example Angela, where they were of no use at all. For the most part correct mathematical notation was applied in this group.

Kelly's particular version of the *rule of three* algorithm does not make immediate sense, but through this method she attains the correct answer. Kelly also makes the mistake of using the equals sign as a separator of mathematical statements rather than an equality symbol (Kieren, 1990) in three of four items about which she was interviewed. This misuse of the equal sign does not affect Kelly in this problem situation. It is part of a theorem-in-action, though not correct. The risk is that this error will become an obstacle for problems of greater complexity. Kelly's schema, translated into an algorithm of sorts, apparently works for her in all the situations she has confronted thus far. In order for her to develop a more sophisticated schema, that is generalisable to other problems, she has to be confronted with situations where this schema proves inadequate, and where it is necessary for her to work with "symbolic representation to make the right interpretation of the relationships and the right interpretation of data" (Vergnaud, 1979, p. 269).

The rule of three, though a well-known algorithm, according to Vergnaud, is rarely used among the students (Vergnaud, 1983) as it does not make sense to multiply 800 zeds by 20(%), for example. This procedure requires fairly sophisticated algebra to be properly understood.

While the group showed proficiency on the items tested, the progressive development of this level of learner is not to be neglected. For example, more advanced problem skills must be demanded through presenting these learners with items which include magnitudes and quantities that do not have common factors. The type of problem presented in **Item 10**, could be adapted to extend learner's problem-solving skills.

The problem encountered in **Item 10** can be interpreted through measure spaces, either a calculation *within measure spaces* or *between measure spaces*, a ratio problem or a rate problem. Within measure spaces the multiplicative operator is 10/3, the constant ratio within a covariant relationship (see Figure 8.3).

An advance on the method used by Adele would have been to use the constant coefficient property, which is the most efficient for many of the problems of this nature found in the multiplicative conceptual field. This structure may be termed a *between measure spaces* problem, where the multiplicative operator is (0.08 or 2.4/30). This interpretation makes use of the concept rate. In fact, Anna used a variation of this approach (see Figure 8.4).

Figure 8.3: Scalar operator approach to solving Item 10

$$\begin{tikzcd} 30 \arrow[d, bend right, "\times 10/3"'] & 2.4 \arrow[d, bend left, "\times 10/3"] \\ 100 & 8 \end{tikzcd}$$

Figure 8.4: Function approach to solving Item 10

Duration		Capacity
30	$\times k$	2.4
100	$\times k$	8

Where $k = \frac{f(30)}{30}$

We can interpret the responses by Adele and Anna as concepts-in-action and theorems-in-action, which if given appropriate tuition will develop into concepts and theorems (Vergnaud, 1988). The brief interaction with the interviewer indicates that the transition to this more efficient procedure is within the zone of their proximal development.

8.5 Middle-high proficiency

We assume that learners in the two groups, at the middle-high proficiency level, one group from each of the schools, have the basic concepts of addition and subtraction, and multiplication and division. We therefore focus on the transition to rational number concepts, including percent, and the ability to identify variables and relationships, and the fluency with algorithms which arise from correctly identifying variables and relationships (see Table 8.17).

Table 8.17: Focus areas for middle-high proficiency level

Proficiency levels	Addition and subtraction	Multiplication and division	Fractions, decimals, operations	Percent, concept, operations	Identify variables, relationships	Fluency with algorithms
Middle high (L4)			✓	✓	✓	✓

8.5.1 Level 4, Thembani and Sipho (School B)

This discussion focuses on **Thembani** and **Sipho**, two of the five learners requested for the interviews from School B. Anna, the analysis of whose work is presented in Section 8.2.1, was interviewed with this group. The written calculations and excerpts of the interviews for **Item 5** are provided in Table 8.18. **Thembani** correctly divides 45 000 zeds by 9 children. His written recording is somewhat cryptic, leaving off the thousands. His strategies may enable him to answer multiple-choice response type questions, however, the written recording of his thoughts is haphazard. **Sipho** presents something of an anomaly (discussed later).

Thembani is able to convert from percentages to fractions in **Item 8**. He has converted the percentage 20% to the fraction 1/5 with which it seems he is better able to work. **Sipho'**s response to Item 8 is puzzling. It appears that he has a strategy of working back from the multiple-choice answer that he intuitively thinks is correct.

Table 8.18: Item 5, response and analysis: Level 4, Group B1

Learner response	Natural language	Mathematical structure
Thembani: 9⟌45 5 4×5=20 3×5=15 2×5=10	"First of all I added all the children of Samuel, Thabo and Dan, which added up to 9. So what I said was 45 000 divided by nine equal 5. So what I said was the children I timesed by five, how many children each one had I timesed it by five. So what I did was ask, how many did Dan have. I said 4 times 5 which equals 20."	**Percent**: $9 \xrightarrow{/\,9} 1 \xrightarrow{\times 4} 4$ **Amount**: $45 \xrightarrow{/\,9} 5 \xrightarrow{\times 4} 20$
Sipho: [No attempt made]	"I didn't answer it."	

Table 8.19: Item 8, response and analysis, Level 4, Group B1

Learner response	Natural language	Mathematical structure
Thembani: 800 ÷ ~~20~~ 5 = 160 800 − 160 = 640	"I made a mistake. Instead of adding I subtracted. … maybe its because of doing the sum. I just forgot that I had to add."	$100 \xrightarrow{/\,5} 20$ $800 \xrightarrow{/\,5} 160$
Sipho:	"I needed to get the percentage so I divided 900 twice which gave me 50%, then I divided 500 twice which gave me 25%. This gave me 222. From 222 … I subtracted this from 900."	[Halving procedures though inaccurate, rather estimations]

In **Item 10**, Thembani has identified the scalar operator × 3.3, which he applies to 2.4 litres. He is able to multiply using decimal notation and understands where to insert the decimal comma. Had Thembani been working with a calculator he may have arrived at a more accurate answer. As it stands, explicit engagement with Thembani's current understanding, in order to encourage formal setting out that clarifies the relationships between the variable in the problem would be helpful to him.

Table 8.20: Item 10 response and analysis, Level 4, Group B1

Learner response	Natural language	Mathematical structure
Thembani: 2,4 × 3,3 = 24 × 33 72 720 7,92	"If like 2,4 litres of gasoline will be used for … 30 hours of operation uses 2,4 litres. So what I said since they asked for how much they used in 100 hours, I said 2,4 time 3, and what I ended to was 7,2. Since I was looking for what number could I round off, the nearest number I could find was 8,0."	**Capacity** **Duration** 100 ∕ 30 = 3.3 $2.4 \xrightarrow{\times 3.3} 7.92$ $30 \xrightarrow{\times 3} 1\ 000$ Therefore, rounding off, answer is 8.0
Sipho: [no reponse]	"I think I got it wrong."	

Thembani's response to **Item 26** was not initially comprehensible to the interviewer (see Table 8.21). Further probing was initiated.

Interviewer:	*Why did you times by 1,5?*
Thembani:	To get 100. Forty (40) by fifteen (1,5)? (speaking to himself) How can I do this? … Forty by 1,5... I am a little bit confused, Miss. But I ended up with a percentage of 60….

After listening to the explanation by Anna, Thembani attains more clarity.

Table 8.21: Item 26 response and analysis: Level 4, Group B1

Learner response	Natural language	Mathematical structure
Thembani: ~~40~~ 1 ~~learner = 1.5%~~ ~~40%~~ 56% 16+10=26B 60% - 4% = 56%	"First I mentioned that there were 50 boys and 50 girls, so there would be 20-20. So I said how would the girls end up being 60%. So I take the forty members, right. I timessed it by 1,5"	1 learner = 1.5%

Thembani:	That's what I did, (learning from Anna). So that's what I got 24. Then what I did 16 plus ten boys.

Interviewer:	*So did you get the 24?*
Thembani:	I got 16 for the boys and 24 for the girls. My original answer was correct.
Interviewer:	*16 is correct. So where did you get the 24? You got the 24 girls and the 16 boys. So then you got the 26 boys. That is perfectly right. Then how did you get from the 26 boys. … You are on the right track. So what went wrong here?*
	…
Thembani:	I must admit difficulty with this. I was trying to find a … I said 40 times 1.5 …
Interviewer:	*How did it end up to be 100%? … 40% times 1,5 … You mean 2,5*
Thembani	No
Interviewer:	*You added 1,5 instead of multiplying by 1,5?*

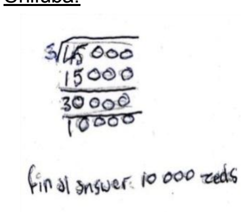
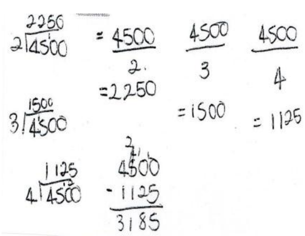
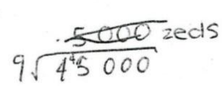
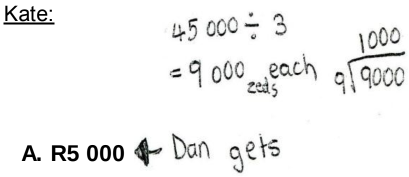
There was no resolution to this problem. The interviewer was not able to follow the logic of Thembani's reasoning though it is suspected that he had a schema which had worked for him in the past. He was now executing this schema without immediate success, and unable to communicate to the interviewer. In retrospect it seems that he was reasoning with the concept $40 \times 1.5 = 60$.

On reflection the interview with this group was unsatisfactory for two reasons, firstly there was a large divergence between Anna, who had answered these problems correctly, and the other two learners, Thembani who had maverick, self-taught procedures, and Sipho, who it seemed used estimation and intuition to achieve the correct answers. Secondly, Sipho was reluctant to take part in the interview. A review of his script written seven months earlier shows the correct choice of answer in Item 5. In the test as a whole he obtained 12 correct answers out of 17. In fact he gets 2 marks for one of the difficult items, where he has to explain the generalisation of a pattern, something not many learners answered correctly. However, three of his incorrect answers are the next three interview items! A check of the Rasch analysis shows a fit residual of 0.032, a good fit; the Guttman pattern was acceptable, that is the expected pattern of correct for easier items, leading to incorrect for more difficult items. It could be that he objected to staying after school. Nevertheless, what emerges is fairly informative even if unexpected.

8.5.2 Level 4: Shiluba, Carola, Linda and Kate (School A)

As with the previous group, we focus on the transition to rational number, the ability to identify variables and relationships, and fluency with executing algorithms (see Table 8.14). The responses were written as shown in Table 8.21.

Table 8.22: Item 5, response and analysis: Level 4, Group A2

Learner response	Natural language	Mathematical structure
Shiluba:  	"I didn't really completely understand but I got a part answer. My logic behind it … I divided 3 into the R45 000 … And this gave me R15 000. … I think I then timesed by two over one which gave me 30, which I then divided by three which then gave me R10 000."	**Amount** / **People** 45 000 / 3 (sons) 15 000 / 1 Divides 45 000 (a) by 3 (b) but then some confusion.
Carola:  	"I divided all of them" *By what?* "4 500 divide by 2 …"	**Amount** / **People** 4 500 / 3 (children) 2 250 / 1 (child) 4 500 / 3 1 500 / 1 4 500 / 4 1 125 / 1
Linda:  	"I realise I did it wrong … really bad … I worked out all of them together. All the children. And I divided them by the zeds. Then I got 5 000. Then I think you would divide the 5 000 by 4."	2 + 3 + 4 = 9 **People** / **Amount** 9 (children) / 45 000 1 / 5 000
Kate:  	"I said 45 000 divide by 3 and I got 9 000. And then I don't know what I did and I got 5 000."	**Amount** / **People** 45 000 / 3 (sons) 9 000 / 1 [incorrect division]

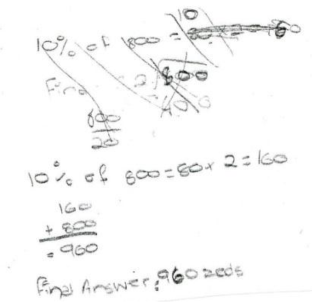
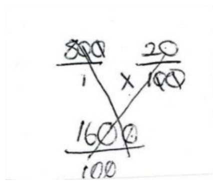
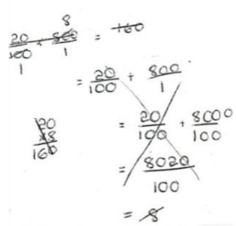
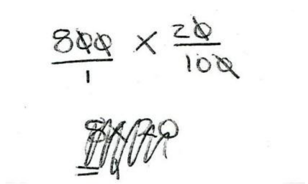
The discussion on **Item 5** begins with **Shiluba**, attempting to provide a logical explanation for her attempt at solving the problem. It appears that Shiluba unthinkingly plucks numbers from the text. **Carola**, applying a somewhat systematic approach takes the 45 000 and divides 2, then 3, and then 4, the number of children in each of the families. She presents some vague notion of a procedure for calculating proportional shares. She realises as she speaks that what she is doing does not make sense, “I don’t know why I did that”. **Linda**, though correctly adding up the number of children, dividing by 9, and obtaining 5 000, is not entirely sure that her process is correct. She says, “I divided them (the children) by the zeds”, when she means “I divided the money by the children”. She concludes that she may have to divide the 5 000 by 4. There is some evidence that her understanding of multiplication and division requires some work.

Kate first divides 3 into 45 000 (as shown in Table 8.22) and obtains the answer 9 000. She then divides 9 000 by 9, and obtains an answer of 1 000. However, she is aware that 1000 is incorrect. She may have intuitively chosen 5 000 from the list of distractors, indicating that she may have gained more understanding of the problem. The interviewer, investigating the learners’ reasoning, puts a question to the group. The interview took just under 6 minutes.

Interviewer:	*OK, so has anybody got a picture of what is happening here?*
Shiluba responds:	We were supposed to add up the children and divide it into 45 000. And then we would get 5 000.
Interviewer	*And what does that 5 000 mean. 5 000 for … Dan?*
Shiluba:	No, 5 000 for each of the children
Interviewer	*For each of the children …* (Murmurs of realisation from the group)
Shiluba	You get 15 000.
Group:	(The group corrects her). You get 20 000.
Interviewer:	*OK. Did you get that?* (Confirmation and chuckling from the group). *So where was the problem? Was it in the reading?*
Shiluba:	In the “How many zeds does Dan get?” Because I didn’t really, … I kind of shut off …
	Which is that part?
	The middle part? Ja, “The money is shared between the brothers in proportion to the number of children each one has”. ... So I disregarded the children.

The common theme across these learners is lack of confidence in their own reasoning ability. The hypothesis is that they rely on pre-set algorithms which, when it is not immediately evident how to apply any such algorithm to the problem, obstruct their reasoning. For **Carola, Linda** and **Kate**, some recall of a learnt algorithm is evident in the **Item 8** discussion (see Table 8.23), however these learners are unable to apply this algorithm fluently when confronted with a problem situation. Of the three who apply the *rule of three*, only Linda is partially successful. The other two have problems with the “cancelling” process. **Shiluba** is successful with a *scalar decomposition* strategy which may make more sense to her. This could also (less accurately) be seen as *finding the unit value*, if we understand 10% of 800 to be the unit in question.

Table 8.23: Item 8, response and analysis: Level 4, Group A2

Learner response	Comment	Mathematical structure	
		Percent	**Amount**
Shiluba: 	[After initial confusion (see scan) Shiluba realises she can find 10% of 800 quite easily. She then finds 2 × 80 to get 160, which she then ads to 800. This could be regarded as a less sophisticated process but at least she is sure of what she is doing.]	100 10 20 120%	800 80 160 800 + 160
		Percent	**Amount**
Carola: 	[Carola remembers the algorithm, $\frac{800}{1} \times \frac{20}{100}$ = 160 but then is confused about the “cancelling process”.]	100 (a) 60 (c) $\frac{b}{1} \times \frac{c}{a}$	800 (b) x
		Percent	**Amount**
Linda: 	[Linda starts with the correct algorithm, $\frac{800}{1} \times \frac{20}{100}$ = 160 but then crosses this out and tries adding instead. $\frac{20}{100} + \frac{800}{1}$ = (confusion with addition of fractions)]	100 (a) 20 (c) $\frac{c}{a} \times \frac{b}{1}$	800 (b) x
		Percent	**Amount**
Kate: 	[Kate starts with the algorithm, but is confused with “cancelling”. She crosses out 3 zeroes in the numerator but only one zero in the denominator. Still she gets 8 × 20 = 1 … (but unsure how to proceed)]	100 (a) 60 (c) $\frac{b}{1} \times \frac{c}{a}$	800 (b) x

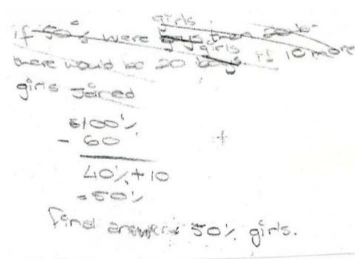
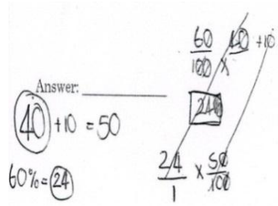
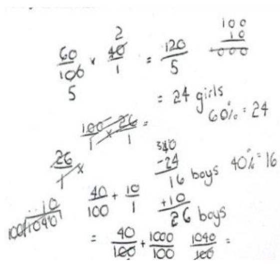
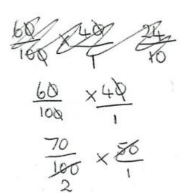
For **Item 10**, **Kate** uses an additive strategy, **Carola** uses doubling which is essentially an additive strategy (see Table 8.24). **Linda** and **Shiluba** both use a multiplicative strategy, but are floored when a number does not divide into the dividend a whole number of times.

Table 8.24: Item 10, response and analysis: Level 4, Group A2

Learner response	Comment	Mathematical structure	
Shiluba:		**Capacity**	**Duration**
30 × 3 90 hours 2,4 | 30 2,4 × 3 7,2 +8,0	[Shiluba uses a multiplication strategy. She multiplies 2, 4 by 3 and 30 by 3, and arrives at 7.2 litres and 90 hours. She suspects that the additional amount of gasoline, equivalent to the ten hours is 8, but she does not really understand the decimal notation.]	$2.4 \xrightarrow{\times 3} 7.2$ 8.0 ?	$30 \xrightarrow{\times 3} 90$ 100
Carola:		**Capacity**	**Duration**
2,4 = 30 4,8 = 60 7,6 = 90 8,0 = 100	[She uses an additive strategy to get to 90. But she makes a mistake … 4.8 + 2.4 = 7.6 (instead of 7.2). As this was a multiple choice question she may have estimated the answer to be between 7.6 and 8.4.]	2.4 4.8 7.6 8.0	30 60 90 100
Linda:		**Capacity**	**Duration**
2,4 x 3 7,2 + 10 hours = 8,0	[Linda multiplies 2.4 by 3 to get 7,2. She writes 7,2 (litres) + 10 (hours). It is not clear whether she came to the conclusion: 0.8 litres is required for 10 hours.]	$2.4 \xrightarrow{\times 3} 7.2$ + 10 hours 8.0	
Kate:		**Capacity**	**Duration**
30 + 30 30 90 2,4 x 3 7,2 ℓ in 90 hours +10 8,2	[Kate combines an additive strategy for the hours: 30 + 30 + 30 = 90 and a multiplicative strategy for the gasoline; 2.4 × 3 = 7.2. She writes 7.2 in 90 hours, but is then at a loss and she adds 7.2 litres + 10 hours. And gets 8.2 "somethings".]	2.4 × 3 7.2 + 10 hours 8.2	30 30 30 90

Three of the four in this group successfully answered the first part of **Item 26** by using the rule of three to "find the percentage of an amount" (see Table 8.25). Their approaches prove inadequate when they have to convert a ratio to a percentage, indicating that they do not understand the relationships inherent in the problem. Shiluba confuses percents and children, and erroneously adds 40% and 10 children.

Table 8.25: Item 26, response and analysis: Level 4, Group A2

Learner response	Comment	Mathematical structure
Shiluba: 	[Shiluba applied additive strategies which were not helpful.]	
Carola: 	[Carola was able to solve the first part of the problem, 60% of 40, that is 24 girls. The reference to the whole then changed. She understood that 10 boys were added to the whole, 40, to make 50.]	**Percent** / **Amount** 100 (*a*) 40 (*b*) 60 (*c*) 24 (*d*)
Linda: 	[Linda was able to find the part amount when given a percentage and a whole.]	**Percent** / **Amount** 100 (*a*) 40 (*b*) 60 (*c*) 24 (*d*)
Kate: 	[Kate managed the first part of the problem, though had difficulty with cancelling.]	**Percent** / **Amount** 100 (*a*) 40 (*b*) 60 (*c*) ? (*d*)

8.5.3 Proficiency exhibited at Level 4

The learners from **School A** engage readily with the problems, however there are conceptual gaps which prevent the learners from successfully solving the problems. It is clear that the group from School A have been taught algorithms. However, because they do not think about the relationships and variables in the problem situations (except perhaps Linda) the algorithms prove more a stumbling block than a help.

Thembani from **School B** is certainly enthusiastic when approaching problems. In contrast to the girls at School A, Thembani engages with the ratio relationships, but does not have the scaffolding of formal mathematical structures to assist the organisation of his ideas, and therefore is not able to draw on the fluency with algorithms, formal mathematical notation and methods of setting out problems. As is evident from his location on the person-item map, the multiple-choice response format may work in his favour, but his current written skills do not provide a sound foundation for further learning. He does, however, exhibit a level of skill with multiplication and division, and working with decimal and percent notation, and he is able somewhat to reason proportionally.

Fluency with multiplication and division, proportional sharing is evident in some cases. However in the case of Shiluba and Kate, there may be reason to believe that they have let go of their natural sense making when approaching problems. Carola applies an incorrect algorithm to calculate proportional shares. There is evidence that some of the learners in this group apply additive reasoning where multiplicative reasoning is required to solve the problem. They do apply the multiplication operation, however there is not much evidence of the division operation, except for **Item 5**, where the learners are dividing whole numbers by 2, 3 and 4. Lack of fluency with multiplication and division, observed in the work of Carola and Kate restricts their ability to reason with ratio and rates.

Calculation errors are evident. Carola adds incorrectly in **Item 10**. Errors are made in the “cancelling” process. Shiluba has problems recognising relationships between variables, however given her quick learning in response to discussion of **Item 5**, it would seem that this outcome may be due to lack of attention to the particular problems.

All three types of percent problems present stumbling blocks for the learners in this group. Shiluba adds percents and quantities, instead of finding the percent increase; Thembani finds the percent decrease. He also confuses the relationships in the percent problem. For those learners who have been taught algorithms to assist them in calculating percent, the variables and the relationships in the problem situations are not always clear. The mistakes made in cancelling can be attributed to lack of understanding of multiplication and division.

A next step would involve offering these learners problems with similar mathematical structure, though keeping in mind that it is the generalisation of the problem type that is the eventual goal. The concepts-in-action and theorems-in-action, in many cases incorrect, do provide the preliminary stepping stones for solving the problem. This group show som measure of insight into their own reasoning.

8.6 Middle-low proficiency

For the two groups located below the item mean, we look for evidence of proficiency with the basic concepts of multiplication and division, and addition and subtraction (see Table 8.26). The transition from working with natural numbers to working with rational numbers, including fractions, decimals and percent, and identifying variables and relationships in problem situations is given attention at this level. Fluency with algorithms is dependent on the identification of variables and relationships.

Table 8.26: Focus areas for middle-low proficiency group

Proficiency levels	Addition and subtraction	Multiplication and division	Fractions, decimals, operations	Percent, concept, operations	Identify variables, relationships	Fluency with algorithms
Middle low (L3)		✓	✓	✓	✓	

8.6.1 Level 3, Phaphama, Maria, Mpho (School B)

The group, located at the median of School B, at location points below the item mean of zero and slightly below the person mean, presented many conceptual errors. In Maria's case she was able to solve **Item 5**, and **Phaphama** engaged enthusiastically (see Table 8.27). **Mpho** did not take part in the second phase of the interview. The evidence used for **Mpho** is her written response to the items.

Table 8.27: Item 5, response and analysis: Level 3, Group B2

Learner response	Natural language	Mathematical structure Children	 Amount
Maria: 45 000 Thabo (2) 5000×2 10 000 Samuel (3) 5000×3 15000 Dan (4) 5000 +5000 +5000 +5000 20000 45 000 ÷ 9 = 5000	"I divided the 45 000 by nine and got five thousand. Then I multiplied the 5 000 by 4. Each child was going to get 5 000, so I added … so I multiplied 5 000 by 4, that is Dan's children … Then I am getting 20 000 for Dan …"	9 1 4	45 000 5 000 5 000 5 000 5 000 <u>5 000</u> <u>20 000</u>
Phaphama: 1250 4)5000 −4 10 −8 20 −20 15000	"At first I thought about dividing by three, for Thabo, Samuel and Dan, and then they would give to the children. But then that is not what the question asks … which is where I got stuck … If I said like 5 000… 9 times 5 is forty five … but what would the others get … Thabo and Dan."	 4 [*why?*] 1	45 000 = 9 × 5 000 5 000 1 250 [Thereafter, conclusion]
Mpho (own written words): *If dan get 20 000 because he has 4 children, Sameul will get 15000 and Thabo will get 10 000*	[Assumed to have calculated in her head]	9 1 4	45 000 5 000 20 000

The strategy used by **Maria** to solve **Item 5** gets her to the correct answer. Her working shows that she uses multiplication to calculate 5 000 by 2 and 3, but reverts to addition when multiplying 5 000 by 4 (5 000 + 5 000 + 5 000 + 5 000). **Mpho** it seems has worked out everything in her head. **Phaphama** initially uses some inverted strategy where he looks for the factors which will give him 45 000, and settles on 5 000 times 9. Once he had found 5 000 he proceeded to divide by 4, rather than multiply by 4.

The interviewer posed questions in order to understand **Phaphama's** reasoning. While expressing his thinking he gains some insight into the concept.

Interviewer:	*What was the five thousand? Why did you divide by nine?*
Phaphama:	Because Dan has 4 children and Samuel …. 15 … 30 … 10 … I don't know.
Interviewer	*It sounds to me you are on the right track. 45 thousand rand. How many children are there?*
Phaphama:	(Adds the children) … Nine children. Yes that is where I got the nine from.
	And in order to find out what each one gets, what can you do?
Phaphama:	Ummm . 5 000 each.
	Now let's look again at the question.
Phaphama:	5 000 divide by 4 … It is 45 000 divide by 9. … So now it is divide by 4.
	Dan has four children … so. 5 000 for each child so …
Phaphama:	Divide by 4
	I think you are on the right track so how much will that be?
Phaphama:	(Thinking) One thousand two hundred and fifty (1 250).

Phaphama wrongly divides by 4 instead of multiplying by 4. The interviewer acknowledges his partially correct theorem-in-action. While listening to Maria's explanation Phaphama compares his process with hers and notes his error.

Phaphama:	I divided by 4.
Interviewer:	*Yes, divided instead of multiplying.*

In **Item 8** (see Table 8.28), **Maria** erroneously assumes that 100 is 10% of 800 and that 20% is therefore 200 zeds. "*I thought 10% of 800 is 100 zeds. And then I added to the hundred 200 zeds.*

Table 8.28: Item 8, response and analysis: Level 3, Group B2

Maria[1]	**Phaphama**	**Mpho**
800 × 20% = 100 10% of 800 = 100 zeds 10th of 800 = 80 20% of 800 = 160 ∴ 800 + 160 = 960	800z × 20% 1000	*1000 zeds* *Because each 10% equals 100*

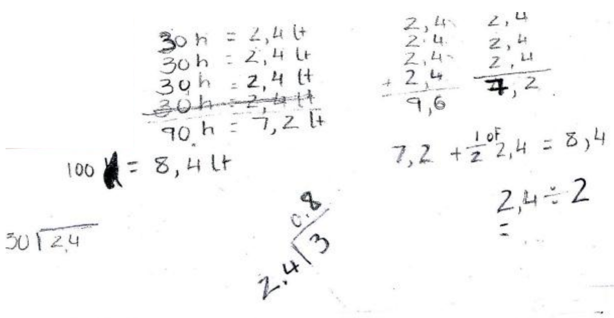
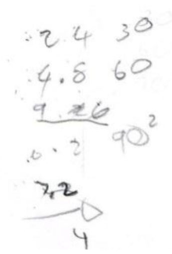
[1] Note that the second part of Maria's written record was written during the discussion stage of the interview. Similarly, responses were written by other learners from this point on.

The interviewer explores further.

Interviewer:	*Say that again. You found 10% of 800?*
Maria:	Yes
Interviewer	*And you got 100?*
Maria:	I did this in my head. I am not sure it is the right one.
	What does 10% mean? If you tell me you got 10% for a test, what does this mean?
Maria:	It means I got a tenth of the marks.
	And a tenth of that (pointing to 800).
Maria:	One hundred (100)
	What is one tenth?
Maria:	(Thinking a long time). I think it is 80.
	It is 80. Then if 10% is 80, what is 20%?
Maria	One hundred and sixty (160)
	Now would you change your answer?
Maria	It is C (referring to one of the multiple-choice distractors).
	How do you know it is C?
Maria	Because 800 and 160 is 960.
	So *you had the right idea but you made the miscalculation here (pointing to the 10% of 800 equal to 100).*

The "miscalculation" by Maria is not so much a miscalculation as a misconception. It seems that both Mpho and Phaphama have the same misconception that 800 is 80% and that a further 20% would make 1000. An impromptu lesson was given by the interviewer making use of the unit method of finding 1%, which is noted in Parker & Leinhardt (1995). After finding 1%, the interviewer asks "How do we get 20%?" Phaphama's initial response is "Add 19". When challenged he is able to recall that he can multiply 1% by 20 to get 20%. It is interesting to note that Maria uses this technique confidently in a later problem.

Table 8.29: Item 10, response and analysis: Level 3, Group B2

Maria	Phaphama	Mpho
		8,4 Because the litres in 100 hours is 8,4

As with the other groups, the difficulty experienced with **Item 10** (see Table 8.29), is that 30 is not a factor of 100. The learners have no problem calculating the amount of fuel for 90 hours. The problem arises when trying to calculate the amount of fuel for 10 hours, given the amount for 30 hours. Phaphama explains. Maria also contributes to the discussion.

Phaphama:	10 hours is 2,4 litres … 60 hours would be 4,8 …. And 90 got up to 6,2 …
Interviewer:	*How did you get 6,2?*
	I did a miscalculation somewhere here.
	Now, what was it?
	4 plus 4 is 8, and 8 plus 4 is 12.
	You forgot about carrying the 1. That would have given you 7,2. That is for ninety. You said 30 …60 …90. They are asking you for 100. So how would you get to 100?
	7,2 for 90 … (thinking) I think I would end up with 8,4 (one of the distractors).
Maria:	I also did what he did. I add the 30's and the 2,4's (in parallel). At 90 I got 7,2. And I added to the 7,2, half of 2,4.
Interviewer	*Why did you add half of 2,4?*
	Because I … wanted to get to a 100 because I only had ninety here.
	Why did you add half? Let's try and think about that. Why did you find half of 2,4?
	Silence for a while.
	If 30 hours gives you 2,4 … then for ten hours … do you see that?
	She has divided 2,4 by 2. Is she right?
	(Silence for a while.)
Phaphama:	If I divide by 2 then I get 15 (hours).
	So what can you do instead? Divide the thirty by … ?
Maria:	By 3.

	Good, and then divide this (the gasoline) by three. And what would your answer be if you divide 2,4 by 3?
Phaphama:	(divides 2,4 into 3).
	No the other way around … divide 3 into 2,4.
	(Phaphama counts 3 … 6 … 9 …)
	You are counting this up …3 ….6 …9….12.
	(Phaphama counts) … 12 … 15 … 18 … 21 … 24 … that will be 8.
	Is it 8 or point 8?
Phaphama:	It will be comma eight
	I was thinking about this. 3 can't go into 2 … .

This transcript shows difficulties encountered with decimal multiplication and division.

In a conversation with **Phaphama**, prior to engaging with **Item 26** (see Table 8.30), it emerges that he does not like percentages. He claims to like algebraic expressions, and geometry, and likes letters better than numbers. **Maria**, having now learnt the technique of finding a unit value applies this technique to the next problem.

Table 8.30: Item 26, response and analysis: Level 3, Group B2

Maria	Phaphama	Mpho
60% of 40 = ~~25~~ 40 < boys = 40% +10 = 50% Girls = 60% = 50% 48% 50% 40 members 60% 1% 4 4×6 60% 24 girls 16 + 10 = 26	24 girls 16 boys + 10	100% Because each 10 percent = 10 girls

The interviewer probes further.

Interviewer:	*OK, let's look at the last (item).*
Maria:	1% of 800 is 8, so 1% of 40 is …
	I am thinking about what you said previously, and I just said 1% of 40 is 4. Now for me to get the 60 I think I need to times it by …..6 … the answer is 24. Right? 24 of them are girls. The rest are boys.
	Good. 24 are girls, the rest are boys. So you have got the first part right. 24 are girls and 16 are boys. Now what happens?
Maria:	Now I give the ten to the boys. Then I will have 26 boys and 24 girls is …

Maria is on the right track. Phaphama tries to follow her reasoning, "How did she get to 24 … girls?" Maria is incorrect in her assertion that 1% of 40 is 4, however her common sense and estimation techniques ensure that she gets the correct answer. The interviewer presents an impromptu lesson, where both Maria and Phaphama show reasoning ability when presented with diagrams and some standard methods.

8.6.2 Level 3: Cheryl and Zanele (School A)

Cheryl and **Zanele** both attended School A. The approximate -0.90 logit location of Cheryl and Zanele, indicates that they have a slightly less than 50% probability of achieving a correct response for **Item 5**, located at the -0.50 logit level. Zanele doesn't understand the problem and therefore does not see that the first step is to *Add the number of children*, the second step is to *Divide the 45 000 by the number of children*, third step, *Multiply the amount of money per child by the number of children possessed by Dan.* The discussion follows (see also Table 8.31).

Zanele:	I got this wrong because I said you divide this by 4, because he had four children. And then it is wrong
Interviewer:	*… and now how do you think you should do it.*
	Divide by three … (does calculation]
	This is how much Samuel will get from his dad.
	Why do you say that?
	Because he has three children, and if you divide 45 000 by three you get 15 000
	You're on the way … And if Samuel is going to get 15 thousand how much is Dan going to get?
	I don't know how to work this out, because if I divide by 4 … he gets 11 thousand … . 11 thousand 3 hundred and something … .

In fact the interviewer misunderstands Zanele's attempt to make sense of the problem as focusing on a third of the children, rather than a third of the brothers. The fact that dividing by Samuel's 3 children happens to get her the correct answer is something of a red herring. Cheryl, located at the same level as Zanele, also has trouble understanding the problem. She makes the reasonable, though in general faulty, assumption that the answer has to be a round number.

Table 8.31: Item 5, response and analysis: Level 3, Group A3

Learner response	Natural language	Mathematical structure
Zanele: 45 ÷ 3 15 = 15 000	"I said you divide this by 4, because he had four children. And then it is wrong Divide by three … (does calculation] … divide by 4 … he gets 11 thousand … 11 thousand 3 hundred and something"	First attempt Divide 45 000 zeds by 4 Second attempt Divide 45 000 zeds by 3
Cheryl: 4)45 000 = 11 250 Dan gets: R 11 250 x 3)45 000 = 15 000 9)45 000 = 5 000	"What I did is I tried to divide the number into all of these … I thought to myself … um … if I divide this (pointing to 9) then each child will get 5 000. I thought it was a perfect amount that would go into that [45 000] Thabo will get 10 000, Samuel will get 15 000 and Dan has four children, he will get 20 000, and this makes 45000"	

The interviewer investigates further.

Interviewer:	*You did it more intuitively … you didn't really know how you got the 5000.*
Cheryl:	No, I didn't.
	Add the two, the three and the four.
	Seven …
	No.
	Oh, that's nine.
	And if you divided the 9 into the 45.
Zanele:	Ohhh (realisation)
Cheryl:	I know what I am doing but I can't say it.

Working with the notion that the power of mathematics lies in the process of transforming the intuitive and implicit knowledge first used in the solving of problems or a class of problems, into explicit and generalisable knowledge that can be applied to more than one situation, we see that Cheryl would need to make her intuitive and sometimes unreliable concept-in-action explicit. The extent to which her schema, made up of concepts-in-action and theorems-in-action, is generalisable, depends on whether the schema can be applied to other situations, and whether the concepts-in-action and theorems-in-action align with generally accepted mathematical concepts and theorems.

Cheryl's response to **Item 8** is correct (see Table 8.32). It appears from her explanation that at this point in her development the procedural route is somewhat separated from conceptual understanding. Unlike Adele in the high proficiency group, who understands that an increase requires multiplying by 120/100, Cheryl uses a less sophisticated variation of the *rule of three* that works.

Table 8.32: Item 8, response and analysis: Level 3, Group A3

Learner response	**Natural language**	**Mathematical structure**	
Zanele:		**Percent**	**Amount**
$\frac{20}{1\cancel{00}} \times \frac{8\cancel{00}}{1} = 160$	"What I did here was put 20 over a 100 and then I times by 800 over 1.	100 (*a*)	800 (*b*)
	And then I crossed out the noughts then I got 160.	20 (*c*)	160 (*d*)
$800 + 160 = 960$	And then what he previously sold for was 800, and so I added the 160."	120	960
Cheryl:		**Percent**	**Amount**
$\frac{\cancel{800}^{8}}{1} \times \frac{20}{\cancel{100}}$; $20 \times 8 = 160$	"I put 20 over 100 and then multiplied by 800 over 1 and then I got 160 …	100 (*a*)	800 (*b*)
$= 160$; $+800$; 960 Zeds…	and then I got stuck."	20 (*c*)	160 (*d*)

Zanele uses the same procedure as Cheryl initially but then "gets stuck". However, after Cheryl's explanation Zanele continues her calculation and succeeds in obtaining the correct answer. Her written answer in Table 8.32 took place after the discussion.

Neither of the two learners, Zanele and Cheryl, gave a response to **Item 10**. In order to ascertain just where the inability to engage with this problem lay, the interviewer probed and provided some scaffolding (see Table 8.33, and Table 8.34).

Table 8.33: Item 10, response and analysis: Level 3, Group A3

Learner response	Natural language	Mathematical structure	
		Duration	**Capacity**
<u>Zanele:</u> [no attempt made]	"I know that for 30 hours it will be 2,4 litres, then for 100 hours … . I will know the answer, but I won't know exactly how to work it out." *Which is the answer?* "I think it is 7,2 but I don't know why"	30 100	2.4 ?
<u>Cheryl:</u> [no attempt made]	"I was thinking it was 8,4. I was thinking if I could halve this, half of 30 and then try and figure out what would go into a 100. You are on the right track. " *Do you think you could work this out for 10 hours?*	$8.4 \xrightarrow{/\,2} ?$ $2.4 \xrightarrow{/\,3} 0.8 \xrightarrow{\times 10} 8.0$	$30 \xrightarrow{/\,2} 100\ ?$ $30 \xrightarrow{/\,3} 10 \xrightarrow{\times 10} 100$

Cheryl was then asked to explain to Zanele, how she now understood the problem.

Table 8.34: Item 10, response and analysis: Level 3, Group A3

Learner	Natural language	Mathematical structure	
		Duration	**Capacity**
<u>Cheryl explaining to Zanele:</u>	<u>Cheryl:</u> "You divide 30 by 3 to get 10, then you must also divide 2,4 by 3, then you get 0,8." <u>Zanele:</u> "Ah … and then you multiply by 10 … Ah, I get it."	$30 \xrightarrow{/\,3} 10 \xrightarrow{\times 10} 100$	$2.4 \xrightarrow{/\,3} 0.8 \xrightarrow{\times 10} 8.0$

Both Zanele and Cheryl attempted **Item 26**, though lack of understanding of the problem situation and the changing referent provided obstacles to solving the problem (see Table 8.35). Essentially, it seems that the concept of percent at the time of the interview was restricted to a part-whole understanding, that is "Find *x*% of A."

Table 8.35: Item 26 response and analysis: Level 3, Group A3

Cheryl	Zanele
$\frac{6\cancel{0}}{10\cancel{0}} \times \frac{4\cancel{0}}{1} = \frac{24}{\cancel{10}}$ $40-24 = \cancel{30}\ 16$ $\frac{16}{40}$ $\frac{10}{4\cancel{0}} \times \frac{10\cancel{0}}{1} = \frac{100}{4}$	$\frac{4\cancel{0}}{1} \times \frac{60}{10\cancel{0}}$ $= \frac{24\cancel{0}}{1\cancel{0}} = 24$ girls. $\frac{24}{6\cancel{0}} \times \frac{10\cancel{0}}{1}$ $= \frac{\cancel{24}0}{\cancel{6}}$ $= 40\%$

Cheryl and Zanele manage to get the first part of the problem to find the percentage, with the help of the algorithm, correct. The second part of the problem where a ratio comparison must be converted to a percentage confuses both learners.

8.6.3 Proficiency exhibited at Level 3

One of the obstacles to solving problems for this group is the lack of fluency with multiplication and division. Achieving fluency does not imply computational exercises but rather the encountering of situations which can be modelled with multiplication and division (see Chapter 5). The concept of proportional sharing does not appear to be familiar to this group of learners, who confuse multiplication and division in calculating proportional shares. This flaw was evident in the working of Phaphama, and Cheryl.

There were three types of percent problems in the interview situation. The first type to be encountered in Item 8 is that of percent increase. This type has been described as the change type. It involves a ratio understanding rather than a part whole understanding of percent. Maria, Phaphama and Mpho assumed that 800 zeds would be 80% and therefore an increase of 20% would make the total 1 000 zeds. The increase in price is in the ratio of 120 to 100. For three of the learners in this group the percent (a ratio) was confused with an amount. The second percent type was that of "finding the percentage of an amount". Learners were required to "find 60% of 40". This type was generally understood. The next type, that of converting a ratio to a percent presented difficulties for this group. One of the difficulties was with changing the referent. In Item 26 the "whole" starts with 40 children, but midway through the problem 10 boys are added to the group. The "whole" is now 50 children.

A critical part of the problem-solving process is identifying variables and relationships. With correct identification of variables and relationships, the standard algorithms function very efficiently. With incorrect identification of relationships the algorithms do not work at all. Instead of identifying critical features of the problem, Thembani selected an answer from the multiple-choice options and worked backwards (see Item 5).

8.7 Low proficiency

It is expected for this low proficiency group that while they may have elements of more advanced concepts such as knowledge of the percent concept or be able to identify variables and relationships in a problem, there will be gaps in the more basic skills.

Table 8.36: Focus area for low proficiency group

Proficiency levels		Addition and subtraction	Multiplication and division	Fractions, decimals, operations	Percent, concept, operations	Identify variables, relationships	Fluency with algorithms
Low	(L1)	✓	✓	✓	✓		

8.7.1 Level 1: Mishack, Amukelani and Mahesh (School B)

The responses were written as shown in Table 8.38. Their part explanations are provided.

Table 8.37: Item 5, response and analysis: Level 1, Group B3

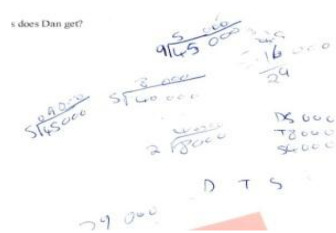
Mishack	Mahesh
45 000 ÷ 5	

Table 8.38: Item 5, response and analysis: Level 1, Group B3

Learner response	Natural language	Mathematical structure
Amukelani: [Marks three of the distractors] 5 000 15 000 20 000	"The number children altogether is 9, so I divided this 45 into the number of children … the number of children is nine … the amount of money 45, so 45 divided by nine … the answer is five."	$9 \xrightarrow{/9} 1$ $45\,000 \xrightarrow{/9} 5\,000$

For **Item 8** Amukelani has written 20% and the decimal conversion 0,20, next to the question. Mahesh knows that he has to either multiply 800 by 20 or divide 800 by 20. He gets 160. He multiplies 8 by 160 and then tries dividing 160 by 8. Mishack has some notion of the algorithm, but is also uncertain how to proceed.

Table 8.39: Item 8, response and analysis, Level 1, Group B3

Amukelani	Mishack	Mahesh
Converts 20% to 0.20	$\frac{1}{100}$ $\frac{1}{100}$ $\frac{800}{20}$	60 $\sqrt[20]{800}$ 800×20 $8 \times 160 =$ $\sqrt[8]{160}$

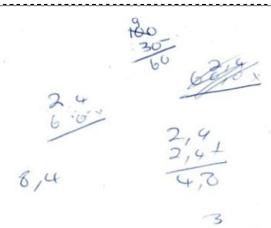
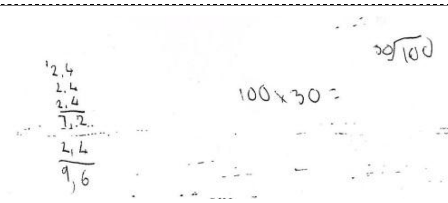
When initiating the discussion Amukelani, eagerly ventures an explanation. The interviewer thinks he should write something down before the discussion.

Amukelani:	Its going to be …
Interviewer:	*Now first try it and then we'll discuss it.*
	Increase the price by 20% meaning, Miss, that the price would be a 1 000
	Can you explain to me how you did that?
	They increased the price by 20% and the original price was R800 so an increase of 20% ….
	What is 20% of R800?
	It is R200. Oh … of R800. … It is R200
	Is there another way? Do you know how to write 20% as a fraction?
	Miss, I can write it in decimal form.

	OK, write it in decimal form.
	Miss, it is still going to give you the same thing.
	Show me
	It's going to be 0,20
	So you are going to find 0,20 of R800?
	(Long silence)

The answer of 1 000 for this problem was interesting. Like Phaphama, Amukelani makes the mistake of equating 800 with 80%. Amukelani has some facility with decimals. This answer revealed to the interviewer that Amukelani has gaps in early mathematical knowledge. He has some conception of decimals, though the concept of percent is a part-whole conception rather than ratio conception.

Table 8.40: Item 10 response and analysis: Level 1, Group B3

Amukelani	Mishack	Mahesh
[no attempt initially]		

Item 10 also presents some difficulty though **Amukelani** is very willing to engage with the problem. Lack of facility with multiplication and division hampers his progress. Mahesh employs a repeated addition strategy, considers both multiplying 100 by 30 and dividing 100 by 30, but makes no real progress. Mishack likewise makes some attempt guided by the multiple-choice answers.

Amukelani:	The machine uses 2,4 litres of gasoline for every 30 hours … of operation.
Interviewer:	*How many litres for 100 hours?*
	30 times 2 will be 4,8. 4,8 times 2 will be (works on paper). So far I am at 90.
	You are on the right track. You have got 90.
	I am at ninety because here we have 30.
	So how are you going to find the other ten?
	It will be 120.

8.7.2 Proficiency exhibited at Level 1

The learners located at this level of the scale are hampered in their problem-solving ability by a number of factors. The most important factor is that of not approaching the problem with a view to understanding how the objects and elements relate to each other. This process of understanding may be assisted with diagrams such as measure-spaces, or simply by drawing a picture. The next factor is fluency with multiplication and division, concepts which are not purely computational in nature but require conceptual engagement. Proportional sharing is another concept not properly understood by learners in this group.

Two important transitions are relevant for this group; the first is the transition from additive to multiplicative structures and the associated reasoning. Of course this transition has to take into account that additive structures are nested within multiplicative structures and therefore fluency with additive structures is a prerequisite for fluency with multiplicative structures. The second is the transition from natural to rational numbers.

8.8 Overview of four proficiency levels

The primary set of questions framing this chapter is “What threshold concepts are acquired at the different proficiency levels exhibited by learners? Which concepts have been mastered and which are yet to be explored? How do we understand the reasoning behind the responses?” (**Question 8**). It is acknowledged that these questions are broad and not easily answered. However, within the framework of the theory of conceptual fields part answers have been ventured. The overview of proficiency levels draws from the summary of each proficiency group (see Table 8.41).

It is clear from the descriptions that there is a wide variation from the high proficiency group to the low proficiency group. The associated errors noted in Table 8.41 are aligned with the concepts still to be attained. This summary provides an overview of learners across proficiency levels who took part in the interviews and may be generalised to other learners.

Table 8.41: Concepts and skills (and errors) at proficiency levels

High proficiency level	**Fluency with multiplication and division, proportional sharing** (confusing multiplication and division in calculation proportional shares) **Conversions, operations with decimal fractions** **Conversions of measurement units** **Percent (change type), percent increase** **Percent (fraction type), finding percentage of an amount restricted to part- whole** **Converting ratio relationship to a percentage** **Fluency with algorithms** (applying algorithms in conceptually clumsy ways, using inappropriate algorithms) **Correct mathematical notation** (misuse of equals sign as a separator) **Using ratio understanding to find proportions, finding unit rate** **Identifying variables and relationships** **Identifying scalar and function operators**
Middle high proficiency	**Fluency with multiplication and division, proportional sharing** (abandoning sense making, applying incorrect algorithm to calculate proportional shares) **Conversions from fraction to percent** **Multiplying decimal fractions** **Percent** (mixing objects, adding percents and quantities) **Finding percent increase** (finding percent decrease) **Reasoning with ratio and rates** (lack of fluency with multiplication and division) **Percent, finding percent – fraction type** (confuses relationships, incorrect use of algorithms, Mistakes with cancelling when applying algorithms) **Identifying variables and relationships**
Middle low proficiency	**Fluency with multiplication and division, proportional sharing** (confusing multiplication and division in calculating proportional shares, incorrect conception of proportional sharing) **Fluency of operations with decimal fractions** **Percent (change type), percent increase; Percent (fraction type), finding percentage of an amount** (confusing percent with amount, rather than ratio) **Converting ratio relationship to a percentage** (difficulty with changing referent) **Fluency with algorithms** (rote use of algorithm) **Identifying variables and relationships** (incorrect identification of relationships, working back from the answer)
Low proficiency	**Fluency with multiplication and division, proportional sharing** (misreading of the problem) **Identifying variables and relationships** (incorrect identification of relationships; working backwards from the answer), **Conversions from percent to decimals** **Addition and subtraction with decimals**

Description of concepts and skills in bold, (errors enclosed in brackets)

The composite approach to mathematical proficiency, including conceptual knowledge, procedural fluency, strategic competence, adaptive reasoning and a productive disposition (Kilpatrick et al., 2001, p. 144) has relevance in the interactions mentioned in this chapter (and partly provides an answer to **Question 8.1**). It is proposed in this thesis that conceptual knowledge and procedural fluency are interactively linked; procedural fluency cannot be attained without conceptual knowledge; conceptual knowledge may be supported by procedural scaffolding (see also, Hiebert & Lefevre, 1986; Long, 2005). In some cases the learners' lack of formal structured mathematical procedures were perceived as a disadvantage. In other cases the hurried application of an algorithm prior to understanding the problem proved a hindrance to solving the problem.

Also interwoven with conceptual knowledge and procedural fluency, is "the capacity to think logically about the relationships" between variables *(*p. 129). Supporting this process is the use of drawings, diagrams and tables, also advocated by Vergnaud (1994), which assist in identifying the important variables and relationships. It was noted in this research that the use of diagrams was helpful in learner interviews. The notion of a productive disposition, "the tendency to see sense in mathematics" (p. 131), is noted here as an important aspect of proficiency, and it is hypothesised that at the appropriate level, this tendency to see sense is evident in all learners. There was evidence for this in the learners who took part in this study. Believing that one can make sense of mathematics is the necessary first step to understanding the problem, which in the phases of solving a problem (see Polya (1945), is often bypassed.

8.9 Theoretical insights from the theory of conceptual fields

Vergnaud (1998) proposes that the link between knowledge-in-action and knowledge-in-text can only be established through operational invariants, concepts-in-action and theorems-in-action. "When operational invariants are expressed and involved in systems of concepts and symbols, their cognitive status changes …". By making the "relevant properties of the mathematical objects and operations involved in action" explicit, it becomes possible to "analyse their connections, and to demonstrate that a certain set of results, for a certain class of situations, is effective" (p. 176). Table 8.42 presents a summary of the implicit concepts and theorems underpinning the procedures identified by Vergnaud (1983) and applied here.

Table 8.42: Analysis by Vergnaud strategies applied in problem situations

	Scalar (erroneous scalar)	Scalar decomposition	Function (erroneous function)	Unit value	Rule of three (errors)	Other
Adele	I05				I08; I10; I26	
Anna	I05			I10	I08; I10; I26	
Kelly		I05; I10			I08; I26	
Jane	I10 ratio	I05			I08; I26	
Prinella	I26	I05; I10			I08; I26	
Angela	(I05)				(I08); (I10); (I26	
Carla	I10	I05			I08; I26	
Thembani	I08; I10	I05				(I26)
Sipho						
Shiluba		I10			(I08)	(I05); (I26)
Carola		I10				(I05); (I26)
Linda		(I10)	(I05 inc)		(I08 inc); (I26)	
Kate		(I10 inc)			(I08 inc); (I26)	(I05)
Maria		I05; (I10)			(I26 inc)	(I08)
Phaphama		I10				(I05); (I08); (I26)
Mpho						
Zanele			(I10 inc)		I08; (I26 inc)	(I05)
Cheryl			(I10 inc)		I08; (I26 inc)	(I05)
Mishack		I10				
Mahesh					I08	(I05 inc)
Amukelani			(I05 inc)			

erroneous procedures in brackets; inc. - incomplete
shaded rows indicate School B

The operational invariants initially experienced as schemes, through being articulated by the student, and with explicit links being made to objects and operations may become algorithms, or lead to the creative use of algorithms. In answer to Question 8.2, it is noted that the strategies and procedures used by the learner (concepts-in-action and theorems-in-action) to engage the mathematical problem situations identified within the multiplicative conceptual field have been noted through both the item analysis and the interviews.

8.10 Reflections for the instrument and interview process

In retrospect, easier items should have been included in the interview phase of the study. An item at Level 1 would have provided information on low proficiency learners' reasoning about problems at the level they currently find manageable, and would have provided more detailed information on the skills they possess and how these skills are engaged in problem solving. For the lower group items targeting basic operations are required.

For learners at the high end of proficiency only Items 26 and 34 in the test as a whole could have provided this group of learners with a challenge. A test instrument with this range of difficulty level does not provide the teacher of this high proficiency group with much information about the next steps in their development. For purposes of providing insight into their development and in order to explore the current limits of these learners, items of greater complexity, in the region of their proficiency levels and higher, demanding engagement with mathematical concepts and number values and ranges, and working with abstract procedures, are required.

In general the level of the instrument and the range of items is appropriate for the two middle proficiency groups. A refinement of items to include more decimal notation, a greater variety of percent problems and extending the multiplicative structure problems to *product of measure* problems and *multiple proportion* problems, may extend the scope of the instrument.

The initial purpose of conducting interviews was to corroborate or refute evidence gained from the testing process. However, the richness of the engagement in particular the engagement with the middle groups for whom the test items were appropriately targeted generated rich information that was regarded as important for the larger view. The decision to interview in groups was premised on the notion that the engagement of learners with their peers would elicit more interesting information through engaging in dialogues. An ideal model would be to have sets of interview items specifically targeted at the proficiency locale of the groups. The role played by the interviewer inevitably influences the results. The complex debates surrounding the conducting of clinical interviews are noted but are beyond the scope of this thesis.

9 Addressing complexity: Implications for curriculum, teaching and assessment

9.1 Answering Poincaré

The question to consider is whether we, the mathematics education research community, have found answers to Poincaré's perplexing question about the general lack of success in the learning and teaching of mathematics. In this thesis the question is applied to the topic area of rational number and proportional reasoning, regarded by the mathematics education community as a difficult but critical area, and an important foundation for functions, algebra and higher mathematics. As has been shown elsewhere difficulties with this topic area have been reported from the late 19th century to today (Chapter 6, see Kieren, 1976; Usiskin, 2007), and both tentative and bold solutions have been advocated, with varied levels of success.

In this thesis a framework, with intersecting domains, (labelled conceptual, cognitive, didactic, semiotic and evaluative) that impact on the acquisition of mathematics, has been described and explained (Chapter 1, Section 1.2.1). The mathematics education landscape, aspects of the curriculum and an overview of some research studies that can be located in the specified domains have been briefly described (Chapter 2). Some schools and select communities have benefited from orientations and philosophies which have provided teachers with mathematical and pedagogical insights (see Chapter 2, Section 2.3.2 and Section 2.3.3) but in general very little substantial change has been observed on a large scale in South Africa.

The research potential of testing learners on items which reflect in some respects the current curricular topics at particular grades with a view to informing mathematics interventions, located in the evaluative domain, is promoted by most countries. Such testing is seen as a means of improving mathematics teaching and learning (See Chapter 5, Section 5.2). This approach has not proved efficacious in South Africa. The reports coming from these testing programmes currently to the best of the author's knowledge, provide very little information other than aggregated data for the teacher and the learner. As reported in the *Review of the Task Team* (Dada et al., 2009), both systemic and international test results have caused South African teachers to lose confidence in the education system and in themselves. The route from the analysis of test results to providing teachers with specific information on particular topic areas, for individuals or clusters of learners, is currently absent in the South African systemic assessment cycles, although various research groups have conducted analyses and provided information that informs the education community. A slight variation on the theme of systemic testing has been the Annual National Assessments (ANAs) in that the tests were marked by the teachers of the learners writing the tests. Other interventions where teachers have been involved in test item analyses have been noted (see Brodie & Berger, 2010).

The curriculum changes, in part instigated by disappointing results on systemic and large scale assessment, that have taken place since 1994 (see Chapter 2, Section 2.3.1), have contributed to the confusion in the mathematics classroom. Each new wave has promised change in performance but in effect has exacerbated rather than addressed the core problem which is attention to the acquisition of mathematics.

This thesis draws on two major theoretical contributions, the theory of conceptual fields, from a mathematics education perspective (described throughout the thesis but mainly in Chapter 4), and the Rasch measurement model from a measurement and statistics perspective (described in Chapter 5 and Chapter 7). Key insights from the two theoretical perspectives independently and in conjunction, as exemplified in this research, are presented in this chapter (Section 2 and 3). Thereafter we propose insights gained from the study that may inform curriculum and assessment, and therefore have implications for teaching (Section 4). The limitations of the current research study are acknowledged (Section 5) and further research avenues are presented (Section 6). The conclusion reflects on the central concern of the thesis (7).

9.2 Insights from the theory of conceptual fields

The argument made in this thesis is that a major problem is insufficient rigorous attention to the specificities of the area of mathematics delineated by the multiplicative conceptual field, including rational number and proportional reasoning in the curriculum (see Chapter 6, Section 6.7). This conclusion is inferred from disappointing results in international tests (see Chapter 5, Section 5.2), and other reports of a general nature. Mathematics as has been described in Chapter 3 (and Chapter 5, Section 5.3.1) is an abstract self-contained series of concepts, axioms, theorems and tools, developed by communities of mathematicians and often spearheaded by individuals. The elements exist in a formal objective structure, but their foundations are acquired through engaging with real-life situations and problems (Chapter 4, Section 4.2). The importance of clustering mathematics concepts and strategies into conceptual fields, as has been described earlier in the thesis, rests on the fact that concepts are not learnt in isolation but in relation to other concepts and they may be learnt at varying speeds. The long term development of a field must continually keep in mind the further development of concepts first encountered in earlier school years.

The case has been made by Vergnaud that no psychological or socio-cognitive language can effectively address the complexity of acquiring mathematical knowledge other than mathematics itself (Chapter 4, Section 4.1). While other perspectives may contribute to the understanding of didactic and pedagogic challenges in some other disciplines, in mathematics they are necessarily insufficient. It follows that this attention to the mathematics itself applies in the selection of problem situations and in the engagement with learners' current schemes as they engage with problem situations. Any analysis of problem situations or learners' current schemes is, according to Vergnaud, to be described using mathematics concepts and theorems, and informed by a mathematical framework.

The pertinence of Vergnaud's claim, and therefore of the implications suggested, is that attention be paid by the teaching and research communities to the creation and assembling of problem situations that embody mathematical concepts and theorems to be learnt at every level of the school curriculum and for every set of mathematical themes.[132] The requirement thereafter is for teachers to select from the collective repertoire of problem situations appropriate elements that target their own learners' zone of proximal development while holding in mind the estimated current ability. These selected problem situations, appropriate for the learner or groups of learners, may then provide the impetus for the learner to engage with mathematical ideas by drawing on existing subjective schemes where available, and by demanding adaptations where existing schemes are unavailable (see Chapter 4, Section 4.3).

[132] We may note that already some such collections of problem situations exist and can be obtained from the Malati Project (Olivier, 2000), (University of Stellenbosch). Collections of similar materials are also to be found in some sets of textbooks.

Didactic support from the teacher, who ideally has a repertoire of language, diagrams and symbols to support the emerging mathematical concepts and theorems, provides the scaffolding to support the learner in transforming local and implicit insights from solving particular problems into general and explicit mathematical concepts (see Chapter 4, Sections 4.4 and 4.5). However, it is argued here that even where the teacher is perceived to have insufficient mathematical knowledge, the necessary foundations may be co-developed in the process of selecting and investigating problem situations within a supportive research or teaching community, as exemplified in the study by Brodie and Berger (2010). The rationale for learning through engagement with problems was given in Chapter 2 (Section 2.3.2). Details of the theory of conceptual fields and its pertinence in the domains regarded as essential for mathematics education are described in Chapter 4.

Of particular note is Kelly's concept of the learner (and the teacher) as a scientist who is constantly making comparisons and juxtaposing similarities and differences (see Bannister & Fransella, 1986). These inherent features of the child discussed in Chapter 4, Section 4.3, are supported in learning advanced concepts through representations (see Samson & Schäfer, 2011, amongst others).

9.3 Insights from the perspective of assessment and measurement

From a perspective developed within a Rasch measurement framework, some major difficulties of systemic assessment may be attributed to lack of apposite targeting. Critique of mathematics testing has been that there is insufficient attention to the particular construct allegedly being measured (discussed in Chapter 5). In the case of mathematics testing we may question whether adequate attention has been given to the construct mathematical proficiency as described by Kilpatrick et al. (2001) and others. The arguments for a specific theory of mathematics assessment have been made in Chapter 5. In addition the argument has been made that serious attention be given to the construct of interest prior to, and during, the construction of the instrument and that validation of the instrument in terms of unidimensionality, invariance and additivity is built into the model.

Various statistics in the Rasch measurement framework alert the researcher to anomalies that have to be addressed. However given that the instrument may be confined in its content and targeting, and item and person anomalies resolved, the invariance properties and the requisite forms of independence assure that one has an objective measurement instrument. The radical difference between the Rasch model and other statistical models is that the instrument data are required to fit the measurement model, rather than a plausible model found to fit the data as an aggregate. This measurement requirement, as has been reported in Chapter 5, is not only in the interests of good science, but is also in the interests of social justice.

9.3.1 Rasch analysis and the theory of conceptual fields

The major contribution of this thesis is an exploration of how the general insights of Vergnaud can be vindicated by Rasch methodology. This methodology provides a means to represent in a common dimension both the learners' current abilities and the mathematical challenges (as represented by items) the learners are currently required to address at their particular level of schooling. The location of both item difficulty (representing concepts and theorems) and learner proficiency (representing concepts-in-action and theorems-in-action) on a common scale, will inform an explicit matching of learners to particular problem situations (Chapter 5, Section 5.4; Chapter 7, Section 7.4).

9.3.2 Person-item map

Both the individual located at a proficiency level, and the group located across a range of levels, can be depicted on a Person-Item map (Wright map) (see Chapter 7, Figure 7.12). The strength of this depiction is that the output from the Rasch analysis is able simultaneously to describe an individual and a group level of proficiency. Similarly, individual items may be described at various levels of difficulty (difficulty locales). Items of similar mathematical structure can be generalised from any existing item location. As can be seen in the substrand analyses (Chapter 7, Section 7.4; also Appendix B), clusters of items at similar levels can be analysed in terms of concepts that are appropriate for the specified level of a particular learner, and clusters of items that are within the zone of proximal development for any particular group of similar learners, may be identified. This ability to provide evidence at both the individual level (proficiency locales) and at a general level is a defining strength of the Rasch model, which makes it particularly important for education. The objective evidence provided by the measurement model, and the qualitative insight and expertise informed by knowledge and insights of the committed teacher, function in conjunction.

In discussing the outputs from the Rasch analysis, the terms “difficulty level” and “proficiency level” have been used. For future work in this area, the terms “difficulty locale” and “proficiency locale” will be used. These terms are congruent with both the theory of conceptual fields, in which a network of concepts which are learned progressively and in relation to each other, but also congruent with Rasch measurement theory where the locations of learners are probabilistic rather than deterministic.

The vindication of the Rasch model is limited in the case of this study to the multiplicative conceptual field and specifically to the domains of rational number and proportional reasoning at Grades 7 to 9, though the efficacy of the model for other topic areas and other models is claimed. The pertinence of this model for education has been demonstrated by others, notably Van Wyke and Andrich (2006), Lamprianou and Williams (2002) and Misailidou and Williams (2003), and is applied in systemic assessment programmes in Western Australia (see Andrich, 2009; also Van Wyke & Andrich, 2006).

9.3.3 Cognitive and pedagogical insights

From a validated Rasch analysis, groups of learners at similar levels of performance ability may be allocated problems at appropriate levels of difficulty. In this study, learners were selected from the instrument output at focused levels for further investigation of the mathematical topic and its associated concepts. It has been demonstrated (see Chapter 8, sections 8.5, and Sections 8.6) that learners can be productively engaged at their particular level of proficiency, that is where they have a notional 50% probability of attaining a correct answer. These interactions proved productive in terms of confronting learners with appropriate problem situations, and therefore with mathematical concepts which they had a good chance of mastering. A tactic for obviating the difficulties associated with pitching educational experiences either too low for a high proficiency group (see Chapter 8, Section 8.4) or too high for a low proficiency group has been demonstrated in this study (see Chapter 8, Section 8.7). With the application of the Rasch methodology and by meeting the requirements of this paradigm, ineffective or inefficient mismatch may be avoided.

Educational theorists are in agreement that educational experience should be appropriately targeted in the zone where optimal learning may take place. The appropriate targeting of items to proficiency level resonates with this and other learning principles. In addition, it is at the correct level that the learner is able to engage schemes which have the potential to function more or less efficiently, but then may also be adapted to accommodate a similar problem, or one of greater complexity, but within a proximate margin. It has been the experience of this study that groups of similar proficiency levels worked well together and made progress (see Chapter 8). When there were learners of different levels, some gains were to be made in that learners could model their strategies on a more successful strategy used by another learner, or the teacher, but only if given adequate time to think through the problem (see Chapter 8).

The interaction between groups of learners or between a practitioner and a group of learners in response to a mathematical activity, when adequately planned at an appropriate level, has the potential to accelerate learning, but at the same time and more importantly, to encourage a productive disposition towards mathematics problems. The initial appropriate targeting engages learners at a point of understanding which may then be consolidated or provide a platform for further learning.

9.4 Implications for curriculum, teaching and research

Although this study did not include cycles of further testing and interviewing, the argument is made that having had some successful interventions based upon a first test instrument, other instruments of the same kind may allow the teacher to iterate the process. Subscales may be developed and modified by exchanging or replacing items at graded levels of the scale, for specific use in the case of low proficiency or high proficiency learners. The possibility also arises of taking a selection of current items in a specific focus area, for example percent, and then guided by the literature include more items to the instrument at difficulty levels along the construct. The process of drawing up such an instrument, requires from the developer or the development team reflection on the topic, which ideally requires a logical analysis of the different subconstructs, the associated errors and threshold concepts (as was discussed in Chapter 6, Section 6,4).

The potential of instruments such as the one developed for this study may be taken up by the mathematics education community, who with individual and combined expertise at different phases of the school programme and in different subtopics of mathematics, may develop a battery of such items and instruments for use in schools.

The benefits of a collective approach are not only for the learners but also for all teachers who engage with the methodology, precisely because it is constructed upon nuances of mathematics in problem situations, and addresses the manner in which both teachers and learners of mathematics perceive those problem situations. This claim is informed by the problem-solving approach as described in Chapter 2 (Section 2.3.2). The methodology described here in both general cycles of test development and analysis (as is exemplified in this thesis) and the classroom cycles of individual interview scenarios (as shown in Chapter 8), requires and brings about mathematical explicitness. In addressing real world phenomena with appropriate scaffolding of material we may assist learners to transform local and implicit concepts-in-action into explicit and generalisable concepts that can be applied to many problems with a similar mathematical structure.

The very nature of mathematics with its objective, logical and somewhat hierarchical structure, arguably makes it a subject more suited than other disciplines to the construction of instruments that have Rasch properties.[133] However, having provided an instrument with a measure of precision, the potential for educational insights and elaboration are many. For example, a creative and ordered approach to designing lesson sequences informed partially by the empirical evidence provided by the Rasch methodology, may be inferred from the close examination of the person–item map.

[133] The surprise expressed by Callingham and Bond (2007) that not many researchers in mathematics education engage with statistical models, reportedly because statistical approaches do not provide the necessary qualitative nuances at the level of the individual learner, may be addressed with the Rasch model in that the sensitivity to qualitative nuances is retained.

We argue that this approach can encourage learners and teachers, in that the Rasch model makes explicit the level at which the learner may exhibit competence, and therefore the subsequent teaching experiences by being appropriately focussed, may be productive and engage further development, rather than demoralising (see Chapter 8, Section 8.7).

The claim is also made here that competence at all levels in the mathematics education community, may be enabled or improved with an approach that involves teachers and researchers in co-constructing items and reflecting on developmental paths. The added advantage of this approach is that rather than the testing process being regarded as a black box, the explicit nature of the results will enable the teacher to make a difference at the very place where mathematics education happens, that is in the classroom. A Rasch approach will enable the teacher to be more able to scaffold support with which to engage the threshold concepts that have been noted in the mathematics education literature, and that are currently being researched in sites around the country.

9.4.1 Levels of development

As has been shown in Chapter 7, analysis of individual items, together with distractor analyses, enables empirically derived summary level descriptors that may inform the teaching of this particular mathematical area. For example, Table 9.1 (Table 7.21 repeated) provides an overview of the mathematical proficiency of this particular topic area in the current study across two schools, three grades and 16 classes. This information may inform the teacher of these learners who may firstly reflect on the big picture, and secondly on the individual results, and may therefore have additional evidence for planning lessons on this topic area and perhaps even use such a table as a resource for planning an assessment programme.

Table 9.1: Association of proficiency and errors within quartile groups

	MCF concepts	Errors	LQ (-1.9)	MLQ (-1.1)	MHQ (0)	HQ (1.1)
Level 1[-3,2)	• Part-whole of discrete and continuous quantities • Both fraction and ratio meaning of fraction notation • Multiplicative relationship between sets of ratios	• Confusion between fraction measure and ratio meaning and with fraction notation	53%	88%	93%	94%
Level 2 [-2-1)	• Fraction equivalence • Part-part and part-whole ratios • Percent concept and notation • Connect probability with fraction measure • Covariant relationships • Comparative relationships	• Natural number confusion • Just add the percent sign % • Language difficulty • Ignoring part of the problem	37%	61%	79%	91%
Level 3 [-1,0)	• Rational number, operator subconstruct • Identification of ratio • Multiplicative comparison (operator construct) • Percentage increase (operator subconstruct) • Applying ratio operator construct to find the sample • Multiplicative comparison	• Confusion with operator construct • Confusion of additive and multiplicative relationship • Ignoring part of the problem	16%	35%	57%	78%
Level 4 [0, 1)	• Fraction measures, addition (subtraction), multiplication (division) • Ratio and rate concepts • Multiplication (division) of decimals • Probability and statistics concepts, “sample”, “random”	• Consider only the numerator • Additive reasoning • Lack of fluency with multiplication and division • Confusion with terminology	14%	22%	28%	58%
Level 5 to 7 (> 1)	• Rate and ratio • Percent, identifying referents • Covariant relationships • Reasoning with unknowns • 2 step problems	• Confusion with percent language and referents	0%	3%	9%	29%

For example a teacher may note that 53% of the lowest quartile exhibits a level of proficiency at Level 1, which means those learners have a basic understanding of fractions and ratio. For teaching a topic such as percent, the teacher may want to ensure that the low proficiency group are alerted to some problems such as the confusion between a fraction and a ratio interpretation of percent. On the other hand, the high proficiency learners may need to be extended to problems involving multiple proportion, or be given problems for which the use of algebra is a necessity, or at least will be more efficient than the learner's existing schemes. Information of this kind may also inform departments of education or district structures in a more nuanced way than simply reporting aggregates.

Details of the mathematics concepts identified at each proficiency level, and the associated errors as empirically determined in this study are reported in Chapter 7 (Section 7.10, see also Table 7.21) and Chapter 8 (Section 8.8, and Tables 8.40, 8.41, and 8.42). Table 9.1(originally Table 7.1) and Table 9.2 summarise the information from Chapters 8 and 9.

9.4.2 Identifying threshold concepts

The reasons for conducting interviews have been explained in Chapter 8. The interview process highlighted some issues, firstly, that if information is to be gained from testing then the instruments must be targeted at the appropriate level. Secondly, in alignment with theories of learning proposed in this study, the process highlighted that even learners at the lowest proficiency level approach problems by applying existing schemes and adapting these schemes to make sense of the problem, sometimes applying higher order concepts, though incorrectly (for example Thembani, (Chapter 8, Section 8.5.1) and even Amukelani, (Chapter 8, Section 8.7.1)).

From knowledge of test construction and the Rasch measurement literature, it is expected that the maximal information to be gained from testing arises when the test is targeted at the current levels of performance. It was clear in this research that the selection of interview items, three of which were clustered around the middle of the scale, at the same level as the middle proficiency level learners, resulted in valuable information at this level. The interviews with learners on items that were at approximately the same scale location as the learner provided greater information about the learners' current levels of proficiency, and about the range of their proximal development (see Table 9.2, Table 8.41 repeated).

Table 9.2: Concepts and skills (and errors) by quartile group

Levels	High (L 5,6,7)	Middle high (L4)	Middle low (L3)	Low (L1, 2)
Add and subtract	**Fluency with addition and subtraction**	**Fluency with addition and subtraction**	(Mistakes, use of addition where multiplication more efficient)	(Use of addition where multiplication more efficient)
Multiply and divide	**Fluency with multiplication and division, proportional sharing** (Confusing multiplication and division in proportional shares)	**Some fluency with multiplication and division, sharing** (Lack of fluency with multiplication and division; Abandon sense making Applying incorrect algorithm)	**Some fluency with multiplication and division, sharing** (Confusing multiplication and division; Incorrect conception of proportional sharing)	**Some fluency with multiplication and division** (Misreading of the problem)
Fractions, decimals, operation	**Conversions, operations with decimal fractions Conversions of measurement units**	**Conversions from fraction to percent Multiplying decimal fractions**	**Fluency of operations with decimal fractions**	**Conversions from percent to decimals**
Percent, concept, operations	**Percent change - percent increase Percent - fraction type Converting ratio relationship to a percentage**	**Finding percent increase** (finding percent decrease) **Percent, fraction type** (Mixing objects, adding percents and quantities)	(Confusing percent with amount, rather than ratio. Difficulty changing referent when converting ratio to percent)	
Using ratio and rate concepts	**Using ratio understanding to find proportions, finding unit rate**	**Reasoning with ratio and rates** (Lack of fluency, multiplication and division)		
Identify variables and relationships	**Identifying variables and relationships Identifying scalar and function operators**	**Identifying variables and relationships** (Confuses relationships incorrect use of algorithms)	(Incorrect identification of relationships, working back from the answer)	(Incorrect identification of relationships; working backwards from the answer)
Algorithms, symbolic notation	**Fluency with algorithms** (Applying algorithms in conceptually clumsy ways, using inappropriate algorithms Misuse of equals sign)	(Incorrect use of algorithms, Mistakes with cancelling, when applying algorithms)	(Rote use of algorithm, sometimes correct)	(Rote use of algorithms)

Description of concepts and skills in bold, (errors enclosed in brackets)

However information at the high proficiency level was somewhat sparse. For the learners located at the low proficiency level the items were experienced as difficult. Table 9.2 presents a summary of the skills evident and the associated errors (in brackets), as observed in the interviews of the learners on four items. Where the difference was too great, the item was either too easy or too difficult for the learner and less information could be gleaned for insight into interventions.

Even given these flaws in the research design, much information has been acquired. With the Rasch methodology, and this research design, the analysis by quartile group and clusters of concepts provides an overview of current proficiency in this conceptual field (see Table 9.2). For example, learners at the high level of proficiency show competence at all preceding levels, though there are elements at the highest level such as fluency with algorithms, where there is work to be done.

Obviously the quality of the data depends on the prior construction of the instrument. An improvement to this instrument is the addition of items at the top end of the scale that will ensure that attention is given to the further learning of this cohort, in particular the expectation that these learners will be encouraged to use algebraic methods.

Table 9.1 and Table 9.2 provide a concept matrix, with learner proficiency levels along one axis and concepts hierarchically ordered as levels of complexity that emerged in the empirical phase of this research. For the Grade 7, 8 or 9 mathematics teacher the complexity of this map is what she confronts in every lesson. The ordering of the concepts and skills is somewhat predictable in that the work of mathematics education giants such as Kieren (1976) have conducted logical analyses of these and other topics. However, what is presented here applies to this particular group of students and may therefore provide a more nuanced landscape from which to design lesson sequences, implement in the minute by minute activity in the classroom, and against which to construct tests that may provide optimal information for individual at all levels of the proficiency ladder.

9.5 Reflections and limitations

9.5.1 Instrument development recommendations

The information to be gleaned from the item analysis can only provide the information that is explored in the design of each individual test item and included in the test instrument as a whole. It is therefore critical that the items, and that the test as a whole, adhere to the notion of a theoretically informed construct. From a theoretical perspective, the items in the test for this study should exhibit multiplicative structures. The subconstructs included were fraction, ratio, rate and proportion, percent, probability and algebra. Each of these constructs is important and could therefore be constructed as an alternative scale.

In terms of the current assessment instrument, this cluster of items at varied difficulty levels functions fairly well in that the analysis of items describes learner functioning at various levels. The inclusion of multiple-choice items, of greater complexity than Item 7 (level 4) and yet not at the level required from Item 26 (level 7) may provide useful information (see Person-Item maps, Section 7.4). Additional items requiring a constructed response at all levels would add to the information gained from analysis. The inclusion of partial response scoring might also add to the nuanced understanding of the percent concept, a ubiquitous and critical multiplicative concept.

In retrospect, it is evident that the items may have been better tailored to focus on only one narrower construct, for example simple and multiple proportion, and include the category product of measures and multiple proportion. While this instrument includes a number of ratio, rate, and proportional reasoning concepts, there are as to be expected in a test of this length, some serious omissions. For diagnostic purposes items of difficulty level between Level 4 and the constructed-response items at Level 6 and 7, need to be designed and incorporated. The topic of fractions involved only items of low difficulty level. There were some important omissions, notably the ordering of fractions, and the addition and subtraction of more complex fractions. Moreover, decimal fractions were not purposefully selected and therefore in consequence appeared only in one item. The inclusion of decimal fractions might have extended the difficulty level of the 'fraction' questions.

The subtopic percent on its own could give rise to at least ten items of graded difficulty, taking into account the different types of percent items identified by Parker and Leinhardt (1995). The items requiring an understanding of probability were not core to this instrument. However, these items could certainly provide the basis for a scale with probability type items. For a future test instrument, these three probability items, or similar constructions could be retained, but augmented by additional items that further explore the topic. The possibility arises that the investigation of probability, while sharing the core concepts of ratio and fraction notation, may require specific attention to probabilistic reasoning, which differs in some important respects from proportional reasoning.

We have noted that the topic of algebra is central to mathematical proficiency. The items in this test that were labelled pre-algebra items may provide the starting point for interviews with learners, who while initially solving the problems by means of counting, may be encouraged to explore the variables and relationships and therefore begin to develop algebraic reasoning. The importance of algebra demands an assessment instrument carefully designed against features and characteristics which in the theory are considered hierarchical developmental levels of attainment of algebraic proficiency. The items, or similar items, that have been used in this instrument may form link items to different instruments. Note that dedicated work on algebra is considered critically important, however this topic was beyond the scope of this thesis.

While the ultimate aim of this thesis is to contribute to teaching and learning, the thesis has concluded at a strategic point, after which a teaching design experiment is envisaged. The preparatory work conducted in this thesis is regarded as necessary for such an enterprise. A broad survey of the field has been partially achieved, though it is acknowledged there are many aspects, for example multiple proportion, for which more work is needed. It is also acknowledged that focused work, both theoretical and empirical is required to convert the findings of this research into a sequence of classroom lessons. In order to accomplish this step further work is required in each of the content strands.

9.5.2 Limitations of the study

Teacher education has not been directly addressed in this study, nevertheless the claim is made that the ideas that are relevant to teachers in the classroom are also relevant to teacher education, the impact of which, though one step removed, is certainly felt in the classroom.

Within the scope of this research it is not possible to do justice to the substantial mathematics education research that is being done at the 23 institutions around the country,[134] however this study may contribute in some way to integrating a focus for new directions.

Attention to the detail of mathematics concepts has in part been achieved in this study. However, further engagement with the concepts covered in this study, is required. Here it is acknowledged that insights from history may provide additional depth, and indeed may inform the acquisition of concepts (see Radford, 2000, Fauvel & Van Maanen, 2000).

The acknowledgement that in the learning of mathematics one is certain to encounter threshold concepts, which are initially obdurate but which once engaged and mastered open up new horizons, may enlighten teachers to the enormity of the task required, but also offer consolation when learners do not grasp a concept immediately.

The literature sourced from the theory of conceptual fields in this study included only that which is available in English. The writings of Vergnaud are extensive, for instance his work is published in French and Portuguese. Many aspects of the theory explored and presented in this study require greater depth, in particular the area of semiotics. This further work may build on Vergnaud's work but here the work of Radford (1998, 2003, 2004) and Steinbring (1998, 2006) is noted. In South Africa the work of Samson and Schäfer (2011) builds on the work of Radford and others.

Further implications are that for research and systemic purposes, clusters of tests at the appropriately targeted levels of proficiency, should be administered to learners. This implication

[134] A survey of cross country research has been conducted by Schäfer (in process).

raises practical issues of how learners are categorised prior to testing. A possible solution would be for teachers or researchers to administer a pre-test of say 10 graded items, from very easy to very difficult. The results of this pre-test would determine the level of test to be administered. The results from the subsequent appropriately targeted test would provide detailed information to an education department and to the teacher.

Computer adaptive testing is based on the same principle. The learner is given an item of previously determined median difficulty. If the learner answers correctly an item of greater difficulty is offered. Similarly, if the item was answered incorrectly an item of lesser difficulty is offered.

The approach of this thesis has not yet obtained a suitable model against which to gauge mathematical development and to track some suitable path for learning and teaching. Diagrammatic models have been attempted by Skemp (1971), though inevitably these models are selective. The bedevilling issue is the adequate representation of complex linkages between constructs rather than the constructs themselves in a list or a flat diagram.

The quality of the test output is determined by the theoretical work conducted prior to the construction of items and test design. The items used were considered good enough in quality, but they were constructed as items in a test designed for large scale comparison rather than individual cognitive development. For further research the items would most certainly be refined.

This testing was only conducted in two schools, chosen because they functioned relatively well. The implications for extending this type of testing across a greater number of schools is certainly that items need to be constructed to target learners exhibiting considerably less proficiency, in order to provide useful information about the South African education environment.

In this test the total number of items in use was 36, with each learner attempting a maximum of 18 items. The analysis has generated rich information at each level (see Table 9.1, and Table 9.2) that could with the appropriate didactical insight be converted into a plausible series of lessons, taking into account the different levels of proficiency within each class group. The implication of this finding is that systemic tests do not have to be excessively long to explore information about the system.

For any future testing, interview items would be staggered in difficulty level in order to accommodate learners located at different levels of proficiency. The targeting of appropriate difficulty level for interviews would certainly have elicited more information for learners at the higher end of the scale and for those at the lower end.

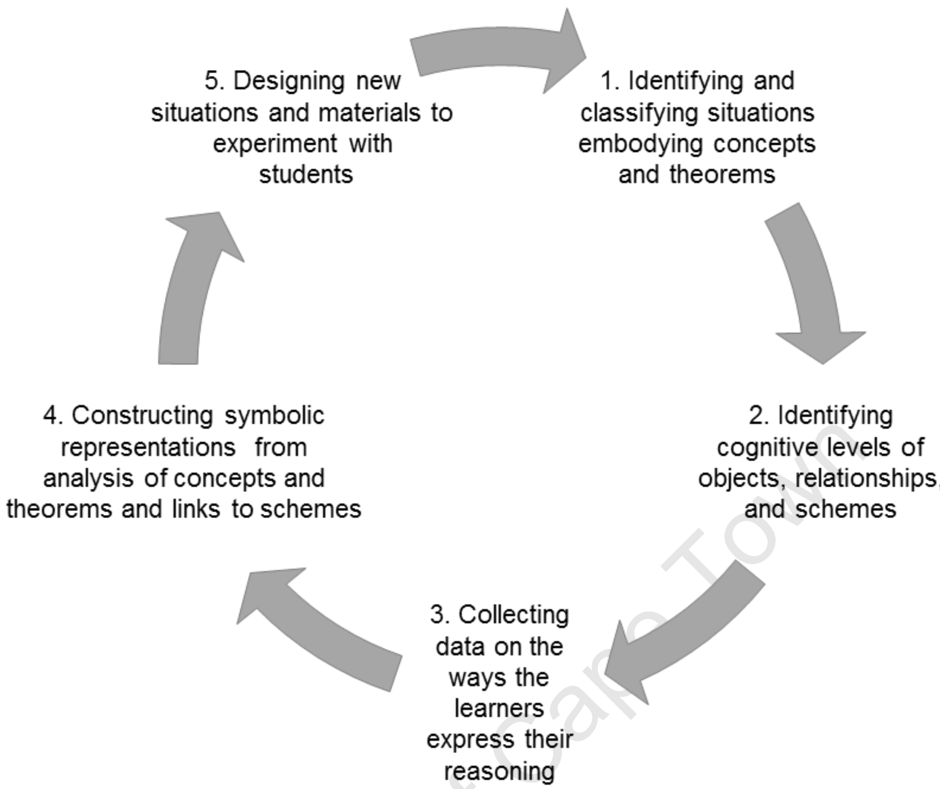
9.6 Future Research

Progress in the field of mathematics education is possible “if it proceeds at a very general level and at the same time in very concrete examples” (Steiner, 1984, p. 28, cited in Bartolini Bussi, 2007, p.63). While theoretical solutions may be proposed it is in the concrete engagement at the level of the learners guided by the teacher, that progress is made. Vergnaud (1988, p. 149) proposes a canonical approach to research in mathematics education which may inform teaching at the level of the classroom and teacher education in the sites responsible for the education of teachers.

A canonical approach (see Figure 9.1) involves;

- Identifying and classifying situations in the defined domain that involve the concepts and theorems in the particular topic of interest, but which are at the cognitive level of the learner.
- In the cognitive domain, identifying levels of objects, relationships, and schemes (concepts-in-action and theorems-in-action) currently employed by learners in engaging with the problem situations.
- Collecting data on the ways the learners articulate their reasoning and identifying the links between the conceptual domain as defined by mathematics in the predicative form, and the cognitive domains as expressed by learners.
- Constructing symbolic representations by observing and analysing the use of concepts and theorems (from the mathematical concept perspective) and schemes that learners use (from the psychological/cognitive perspective), to assist students in structuring their engagement processes.
- Designing new situations and materials to conduct teaching design experiments.

Figure 9.1: Research cycle



Source: Vergnaud, 1988, p. 149

The research process advanced by Vergnaud may be further enhanced through the design, validation and use of instruments informed by a Rasch measurement framework, as has been demonstrated in this study.

The current research study has not engaged at all with the didactic domain, except through references to theory, and has not stepped inside the classroom. In the preliminary stages of the current study, videos were taken of the teachers' series of lessons on "ratio and proportion". These videos were not used for a number of reasons, the most pertinent of which was the uncomfortable position of being a spectator in another teacher's classroom. A more fruitful arrangement for future studies may be the production and co-production of a series of lessons, based on a shared understanding of the field, drawing on the findings of this research, as well on other research, and a shared exploration and discovery of the elements that constitute effective mathematics teaching.

The designing and execution of teaching design experiments such as those reported in the research of Lamon (2007) and others, are envisaged. Research studies with similar research designs (Bartolini Bussi & Bazzini, 2003, p. 213) have reported that teachers

> proved to be essential not only in the careful management of classroom activity but also in the elaboration of analytic tools and of the theoretical framework in didactics of mathematics. Last but not least, while taking part in the design of experiments, the teachers were put in the condition of deepening some issues concerning the theoretical dimension of mathematics and its relationship with experiential reality. In other words, the theoretical dimension of mathematics became part of the intellectual life of teachers (Bartolini Bussi & Bazzini, 2003, p. 213).

The design of a sequence of linked teaching events provides the framework to transform the intuitive and localised schemes into powerful mathematics that can be generalised across many situations.

9.7 Conclusion

The main research question informing this thesis may be expressed as

> How may the essential elements of a framework including mathematical, cognitive and didactic elements, and applied in the multiplicative conceptual field within a Rasch Measurement Framework, address problems in mathematics education, and inform the development of the curriculum and the validity of assessment processes?

In this study the question as stated above has partially been answered across the eight chapters. In essence the insights of the theory of conceptual fields, as outlined by Vergnaud, have been combined with the theoretical soundness and empirical strength of the Rasch measurement model, first recognised and constructed by Georg Rasch, but subsequently developed by communities of researchers in the measurement field, to present a model of some complexity but with obvious advantages for the classroom, the school and education departments.

A key feature of this model is *construct validity* in its broadest conception (following Messick, 1989; Wright & Stone, 1979, 1999; Andrich & Marais, 2008), for both educational and assessment purposes, which in an elaborated form is supported by the theory of conceptual fields. However, it is only when the Rasch conditions are met that we are assured of the validity of adding scores from items within and across persons. This fact makes it imperative to use Rasch approaches when we seek evidence for educational phenomena that have the robustness of measurement rather than well-intentioned but arbitrary allocation of scores. Thus if assessment is to serve rigorously comparative and influential purposes, and assumedly influence and improve mathematics education, we may aver that a Rasch approach is optimal. The proviso remains, however, that it is the theoretical work underpinning any assessment instrument that allows its efficacy.

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AppendicesA to C

Mathematical, cognitive and didactic elements of the multiplicative conceptual field investigated within a Rasch assessment and measurement framework

by

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Appendix A: Chapter 5, Assessment and measurement

A1: Gross National Income per Capita and Achievement

	GNI per Capita (US$)	Average Scale Score 1995	1999	2003	Int. Ave 2003	Difference*
Chile	4250		392	387	467	-80
Slovak Republic	3970	534	534	508	467	41
Lithuania	3670	472	482	502	467	35
Malaysia	3540		519	508	467	41
Latvia	3480	488	505	508	467	41
Botswana	3010			366	467	-101
South Africa	2500	278	275	264	467	-203
Russian Federation	2130	524	526	508	467	41
Tunisia	1990		448	410	467	-57
Romania	1870	474	472	475	467	8
Bulgaria	1770	527	511	476	467	9
Jordan	1760		428	424	467	-43
Iran	1720	418	422	411	467	-56
Macedonia	1710		447	435	467	-32
Egypt	1470			406	467	-61
Morocco	1170		337	387	467	-80
Philippines	1030		345	378	467	-89
Indonesia	710		403	411	467	-56
Moldova	460		469	460	467	-7
Ghana	270			276	467	-191

*Difference between 2003 Average .Scale Score and the international average

A2: The implemented curriculum, reported inTIMSS 2003 Report

Number	Algebra	Measurement	Geometry	Data
■Whole numbers including place value, factorisation and the four operations	■ Numeric, algebraic and geometric patterns or sequences	■ Standard units for measures of length, area, volume, perimeter, circumference, time speed, density, angle, mass/weight	■ Angles-acute, right, straight, obtuse, reflex, complementary and supplementary	● Organising a set of data by one or more set of characteristics using a tally chart, table or graph
■ Computations, estimations or approximations involving whole numbers	■ Sums, products and powers of expressions containing variables	○Relationships among units of conversions within systems of units and for rates	■ Relationships for angles at a point, angles on a line, vertically opposite angles, angles associated with a transversal cutting parallel lines and perpendicularity	○ Sources of error in collecting and organising data
■ Common fractions	● Simple linear equations, equalities and simultaneous equations	●Use standard tools to measure length, time, speed, angle and temperature	○ Properties of angle bisectors and perpendicular bisectors of lines	○ Data collection methods
■ Decimal fractions	○ Equivalent representation of functions as ordered pairs, tables, graphs, words or equations	● Estimations of length, circumference, area, volume, weight, time, angle, and a speed in problem situations	■ Properties of geometric shapes, triangles and quadrilaterals	● Drawing and interpreting graphs, tables, pictographs, bar graphs, pie charts and line graphs
■ Representing decimals and fractions	● Proportional, linear and non-linear relationships	○ Computations with measurements in problem situations	● Properties of other polygons	○ Characteristics of data sets including mean, median, range and shape of distribution
■ Computations with fractions	○ Attributes of a graph	● Measurement formulas for perimeter of a rectangle, circumference of a circle, areas of plane figures, surface area and volume of rectangular solids and rates	● Construct or draw triangles and rectangles ofgiven dimensions	○ Interpreting data sets
■ Computations with decimals		○ Measures of irregular or compound rates	● Pythagorean theorem to find the length of a side	○ Evaluating interpretations of data with respect to completeness of interpretation
■ Integers including words, numbers, or models		○ Precision of measurements	○ Congruent figures and their corresponding measures ○Similar triangles and recall their properties	○ Simple probability including data from experiments to estimate probabilities for favourable outcomes
□ Ratios	The curriculum topics reported taught to all or almost all the students are marked as follows ■. The topics reported only taught to the more able students are marked ●. The topics not in the intended curriculum are marked as ○.		○ Cartesian plane-ordered pairs, equations, intercepts, intersections and gradient	
■ Conversions of percents to fractions and decimals			○ Relationships between two-dimensional and three-dimensional shapes	
○ Line and rotational symmetry for two-dimensional shapes○Translation, reflection, rotation and enlargement				

A3: Released items, rational number and proportional reasoning

Question code	New No.	Strand	Content description	Cognitive domain	Item description	Grade 8 scores (Int. mean)
M012001	18	Number	Fractions and decimals	Using concepts	Finds 4/5 of a region divided into 12 equal parts	Higher Int. Benchmark
M012004	3	Number	Ratio, proportion and percent	Solving routine problems	Solves a word problem by finding the missing term in a proportion	Higher Int. Benchmark
M012040	24	Algebra	Equations and formulas	Knowing facts and procedures	Solves equation for missing number in a proportion	26% (1,5) Int. av. 65% Interm. Int. Benchmark
M032570	4	Number	Ratio, proportion and percent	Using concepts	Identifies a percent equivalent to a given fraction with a denominator that is a factor of 100	Interm. Int. Benchmark
M012017	21	Algebra	Patterns	Reasoning	Finds a specified term in a sequence given the first three terms pictorially	Higher Int. Benchmark
M022189	12	Data	Data interpretation	Reasoning	Solves a comparison problem by associating elements of a bar graph with a verbal description	Interm. Int. Benchmark
M022191	6	Number	Fractions and decimals	Reasoning	Selects the statement that describes the effect of adding the same to both terms of a ratio	Higher Int. Benchmark
M022139	7	Number	Ratio, proportion and percent	Knowing facts and procedures	Finds the percent change given the original and the new quantities	Adv. Int. Benchmark
M022146	14	Data	Uncertainty and probability	Solving routine problems	In a word problem when given the possible number of outcomes and the probability of successful outcomes, solves for the number of successful outcomes	Higher Int. Benchmark
M022156	32	Number	Fractions and decimals	Knowing facts and procedures	Solves a one-step word problem involving division of a whole number by a unit fraction	Higher Int. Benchmark 7% (1.3) Int. av. 38%
M022004	16	Number	Fractions and decimals	Solving routine problems	Solves a multi-step problem involving multiplication of whole numbers by fractions	Higher Int. Benchmark
M022008	25	Algebra	Patterns	Reasoning	Identifies numbers common to two different arithmetic sequences	Adv. Int. Benchmark
M022252	11	Data	Uncertainty and probability	Reasoning	Given the set of possible outcomes expressed as fractions of all outcomes, recognises that probability is associated with size of fraction	Higher Int. Benchmark
M032228	8	Number	Ratio, proportion and percent	Solving routine problems	Calculates the new price of an item given the percent increase in price	Higher Int. Benchmark
M032533	10	Number	Ratio, proportion and percent	Solving routine problems	Solves a word problem with decimals involving a proportion	Higher Int. Benchmark
M032261	15	Geometry	Congruence and similarity	Using concepts	Identifies a triangle similar to a specific triangle given the lengths of all sides	Higher Int. Benchmark
M032271	19	Data	Uncertainty and probability	Reasoning	Uses the size of a group with a given characteristic in a sample to estimate the size of a group with that characteristic in a population	Higher Int. Benchmark
M032762	33	Data	Data interpretation	Reasoning	Interprets data from a table, draws and justifies conclusions	6% (1,2) Int. av. 21%

Question code	**New No.**	**Strand**	**Content description**	**Cognitive domain**	**Item description**	**Grade 8 scores (Int. mean)**
						Adv. Int. Benchmark
M032763	34	Data	Data interpretation	Solving routine problems	Interprets the data from a table to make calculations to solve a problem	Above Adv. Int. Benchmark
M032764	35	Data	Data interpretation	Solving routine problems	Interprets the data from a table to make calculations to solve a problem	Above Adv. Int. Benchmark
M032727	5	Number	Ratio, proportion and percent	Solving routine problems	Identifies proportional share of an amount divided into three unequal parts	Higher Int. Benchmark
M032233	26	Number	Ratio, proportion and percent	Reasoning	Solves a multi-step non-routine problem involving percents	Adv. Int. Benchmark
M032447	9	Number	Ratio, proportion and percent	Knowing facts and procedures	Determines the simplified ratio of shaded to unshaded parts of a shape	Higher Int. Benchmark
M032649a	30	Measure ment	Tools techniques and formula	Routine procedures	Solves a word problem to find average speed.	High International Benchmark
M032649b	31	Measure ment	Tools techniques and formula	Reasoning	Solves a multi-step word problem involving time distance and average speed.	High International Benchmark

Appendix B: Chapter 7, Item analyses

Analysis of Sub-strands, Fraction, Ratio, rate and proportion, Percent, Probability and Pre-Algebra by item and proficiency level exhibited.

B1: Fraction item analysis (See Section 7.5, Thesis document)

Level 1 fraction item analysis: Description and comparison

Items 13 and 17 are the easiest of four items[1]. Of the group, 83% of learners (n=83) correctly answered Item 13, and 73% (n=84) correctly answered Item 17. These two items require simple recognition of fraction parts. Both problems are presented in diagram form.

In **Item 13 (location -2.584)** (see **Fraction Item 13,** see Table 1), the learner is asked to identify which diagram depicting discrete objects represents the fraction *a half*, written in natural language. The mathematical structure can be described as a fraction of **discrete quantities.**

For **Item 17 (location -2.261)** (see **Fraction Item 17**, see Table 3) the learner is required to recognise which figure has $\frac{2}{3}$ shaded. Item 17 is also presented in diagram form, though with a **continuous whole quantity**, where the learner is required to identify that object in which a $\frac{2}{3}$ fraction is shaded, a fraction more complex than the fraction $\frac{1}{2}$. While the recognition of the continuous fraction $\frac{2}{3}$ is a more advanced cognitive skill, it is still at the level of recognising concepts, and does not require a mathematical procedure.

Distractor analyses by quartile groups

For **Item 13** the analysis of learner performance divided into 4 quartile groups (see **Fraction Item 13,** see Table 2) shows that 95% of learners in the highest quartile group selected the correct **Option B**, with 89% of the second and third quartile selecting the correct option, and 68% of learners in the lowest quartile choosing the correct Option B. These learners are assumed to have understood that 3 out of 6 dots is equivalent to *a half*.

[1] Items 13, 1, 2, and 17 (locations -2.584 to -2.261) were selected from TIMSS 2003, Grade 4.

For learners in the highest quartile, the probability of choosing **Option D**, is 5% (see purple line), choosing **Option C** is 0% (green line), and choosing **Option A** is 0%. For learners at the low end of the scale, at location – 1.852, the probability of choosing the correct **Option B** is 68% (see red line). The probability of choosing the **Option C** (3) or **Option A** (1) is 16% in each case.

Again for Item 17, four 4 quartile groups were constructed. Of the highest two quartiles 94% and 95% chose the correct **Option E**, depicting *3 equal parts, two of which are shaded*. The remaining highest quartile learners interchanged the shaded and unshaded parts of the diagram, selecting **Option B**, while the remaining second highest quartile learners confused a part-whole and a ratio understanding, **Option A**. Only 39% of the lowest quartile selected the correct option. A further 39% of the lowest quartile, and 5% of the second lowest quartile selected **Option A,** confusing a ratio and a part-whole understanding of fractions. The common error was confusing a ratio meaning of the symbol $\frac{2}{3}$ where the ratio of the dark sections to the light sections is 2:3, or $\frac{2}{3}$, with the part-whole fraction meaning.

Critical features for fractions, level 1

The finding that about 20%, of the Grades 7 to 9 learners, have not mastered the concepts required to answer Item 13 and 17 correctly is of concern. The confusion between a fraction and a ratio understanding of the quotient $\frac{2}{3}$ is worth noting. Proficiency at Level 1 indicates mastery of fraction concepts, understood as parts of wholes, where the whole may be a continuous quantity or a set of discrete objects. Recognition of fraction parts and associating the *relationship between parts* with a symbol may be described as a threshold concept that is difficult to learn and requires careful didactic planning.

Fraction Item 13 (location -2.584)(level 1)

Table 1: Item 13description

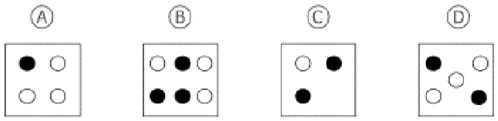
Fraction item	**Mathematical Structure**		**Response process**
13. In which figure are half the dots black? 	M_1 (fraction) 1 2	M_2 (dots) *b* *d*	Recognises 3 parts of 6 as a half, or as ratio 1 to 2.

Figure 1: Item 13 : Multiple-choice response by quartile group

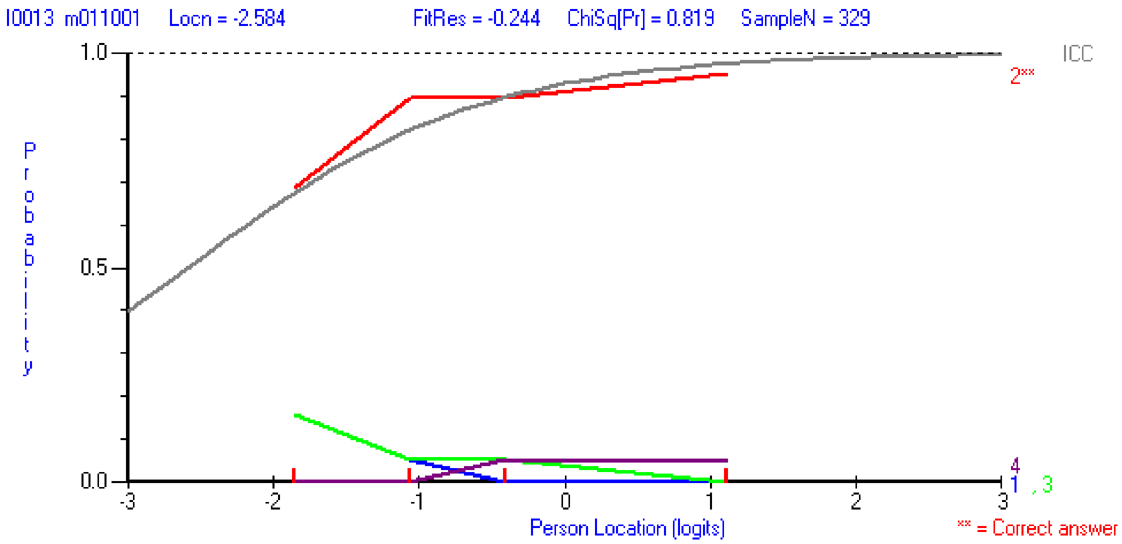


Table 2: Item 13 inferred procedures for multiple-choice responses

Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quartile groups**			
				-1.852	**-1.060**	**-0.402**	**1.117**
B	85 %	red	3 out of 6 dots equal to half	68%	89%	89%	95%
C	6.5%	green	2 shaded, one unshaded	16%	5%	5%	0%
A	5%	blue	1 shaded, 3 unshaded	16%	5%	0%	0%
D	2.5%	purple	2 out of 5 shaded	0%	0%	5%	5%
				100%	**100%**	**100%**	**100%**

Fraction Item 17 (location -2.261) (level 1)

Table 3: Item 17description

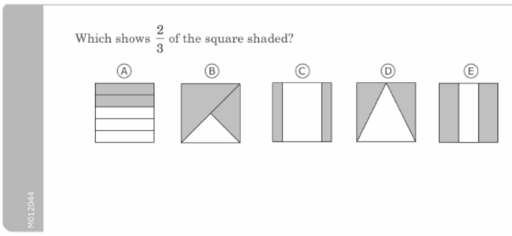
Fraction item	Mathematical structure		Response process
17. Which [figure] shows $\frac{2}{3}$ of the square shaded? 	**M_1** (fraction) 2 3	**M_2** (parts) 2 (shaded) 3 (total)	Recognises shape depicting 2 shaded parts out of 3 parts altogether.

Figure 2: Item 17: Multiple-choice response by quartile group

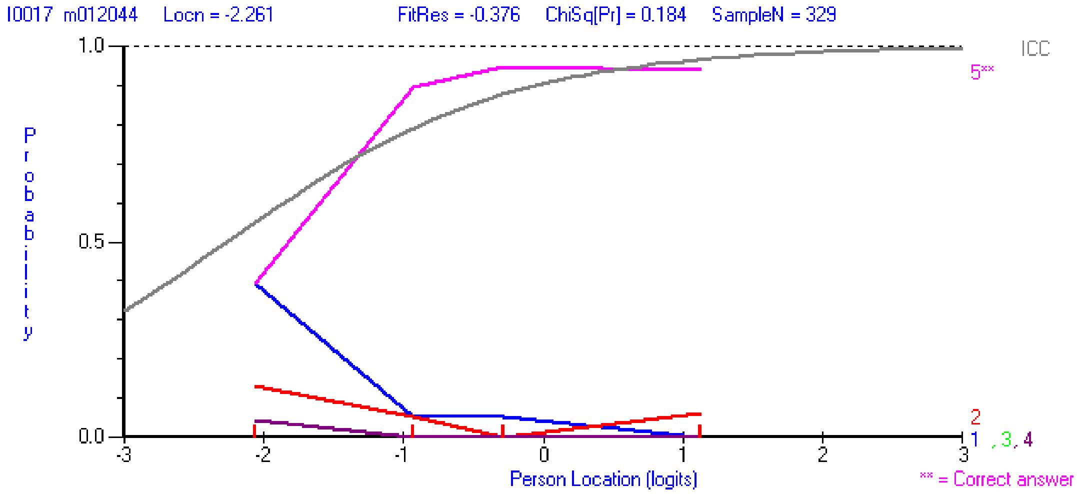


Table 4: Item 17 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-2.063	**-0.925**	**-0.288**	**1.124**
E	79%	pink	2 out of 3 equal parts shaded	39%	89%	95%	94%
A	12%	blue	2 shaded, 3 unshaded of 5 equal parts	39%	5%	5%	0%
B	6%	red	2 of 3 unequal parts shaded	13%	5%	0%	6%
C	1%	green	2 of 3 unequal parts shaded	4%	0%	0%	0%
D	1%	purple	2 of 3 unequal parts shaded	4%	0%	0%	0%
Total				**100%**	**100%**	**100%**	**100%**

Level 2 fraction item analysis: Description and comparison

Item 22 (-1.877) entails adding and subtracting $\frac{1}{2}$and $\frac{1}{4}$and depicts an everyday context, in which natural language and fraction notation are used (see Table 5). For **Item 22**, 77% answered correctly. Of particular note is that of the 23% of learners at Grades 7, 8 and 9, that selected an incorrect option, 15% of these selected **Option A** ($\frac{3}{4}$), where it is presumed learners have just added the numerators.

Item 24 (-1.358) presents a mathematical context, in which learners are required to find *n* in the equation, $\frac{12}{n} = \frac{36}{21}$ (see Table 7). Of the cohort, 68% answered Item 24 correctly, selecting **Option B**. A further 18% selected **Option A**. The inferred reason may be the simple recognition that 12 × **3** = 36, while not considering the proportional relationship 12 and *n*, and 36 and 21.

Distractor analyses by quintile groups

For **Item 22**, 91% of the highest quintile learners selected the correct answer **Option D** (*none*), and thereafter, 92% of the second highest quartile, 88% of the middle quintile, and 69% and 47% of the next lowest quintiles, respectively, selected the correct answer (see Table 6). Notable in this analysis is that 47% of the lowest quintile selected **Option A**, where it is hypothesised that the learners may have only taken the numerators into account and ignored the denominators. Of the highest quintile, 100% obtained the correct answer for **Item 24**, with 90% of the second highest quintile and 70% of the middle quintile also answering correctly (see Table 8)..This correct answer decreases to 40% with the second lowest quintile, and 33% of the lowest quintile.

Critical features for fractions, level 2

In essence **Item 22** requires recognition of the equivalence relation between fraction parts and wholes, and applying the operation of addition for fractions. Similarly **Item 24** requires recognition of the equivalence relation, but demands multiplicative structures, an application of multiplication and its inverse division. It is noticeable when examining the two distractor plots (Figures x and x) that the highest two quintiles have mastery over the concepts as indicated by their responses to these items. However the lowest two quintiles exhibit sharply reduced understanding.

The problem for **Item 24** could be interpreted as an equivalent fraction problem, where the equivalent fraction rule is used to find *n*. The TIMSS study classified this problem as the cognitive domain, *knowing facts, procedures and concepts*, which in essence assign the response to one of remembering or recognising procedures. This problem could also be interpreted as a proportional reasoning problem, 12: *n*:: 36: 21. It could also be interpreted as an equation, where the requirement is to solve for a particular unknown. It is hypothesised that the choice of Option A (3) indicates a partially correct approach. The learner recognises the number with which to multiply 12 to get 36. He or she establishes this multiplicative comparison to be 3, but then does not perceive the direction of change. This response is part way to a procedural skill but the concept of equivalence, required to support the procedural steps, is missing.

The threshold concepts which require attention include the understanding of fraction equivalence, both in terms of part-whole understanding and in terms of multiplicative relations, multiplication and division. The procedures, either cross multiplication or the rule for generating equivalent fractions may either support the understanding of proportional relationships, or be supported by an understanding of proportional relationships.

Fraction Item 22 (location -1.877) (level 2)

Table 5: Item 22description

Fraction item	Mathematical structure	Response process
22. Janis, Anna and their mother were eating cake. Janis ate $\frac{1}{2}$ of the cake. Anna ate $\frac{1}{4}$ of the cake. Their mother ate $\frac{1}{4}$ of the cake. How much was left? A. $\frac{3}{4}$ B. $\frac{1}{2}$ C. $\frac{1}{4}$ D. none	$\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$ or $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{4}=0$	Equivalence relation between fraction parts and wholes. Additive structure

Figure 3: Item 22: Multiple-choice response by quintile group

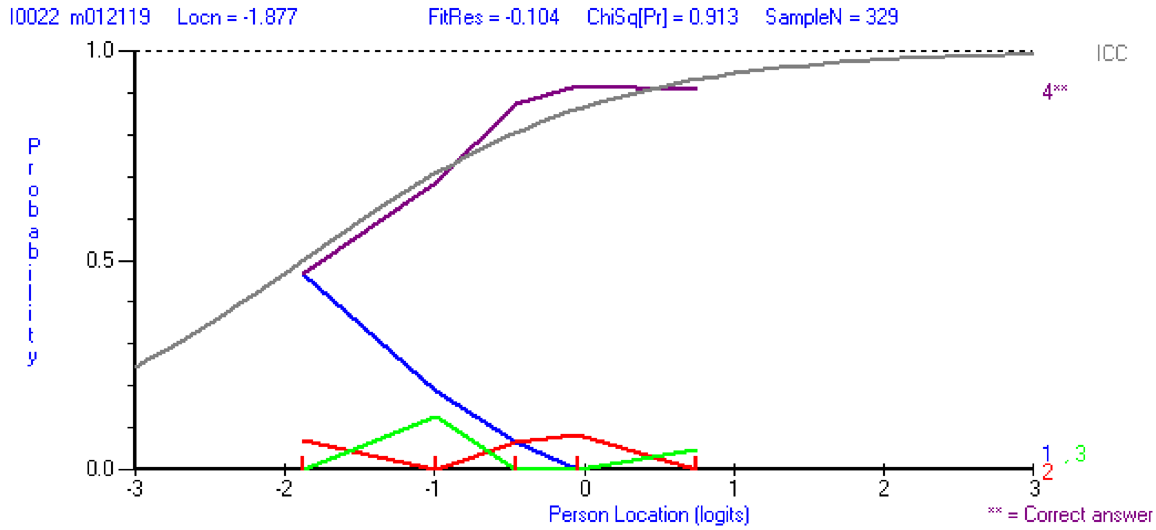


Table 6: Item 22 inferred procedures for multiple-choiceresponses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quintile groups				
				-1.880	**-0.992**	**-0.461**	**-0.044**	**0.746**
D. none	77%	purple	$\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$	47%	69%	88%	92%	91%
A. $\frac{3}{4}$	15%	blue	1+1+1 =3	47%	19%	6%	0%	5%
B. $\frac{1}{2}$	4%	red	Janis is overlooked	7%	0%	6%	8%	0%
C. $\frac{1}{4}$	4%	green	Misreading	0%	13%	0%	0%	5%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Fraction Item 24: (location -1.358) (level 2)

Table 7: Item 24description

Fraction item	Mathematical structure			Response process
24. If $\frac{12}{n}=\frac{36}{21}$, then n equals A. 3 B. 7 C. 36 D. 63	$\frac{12}{n}=\frac{36}{21}$ $\frac{a}{b}=\frac{ax}{bx}$	M_1 12 n	M_2 36 21	Recognises equivalence relation, and calculates accordingly

Figure 4: Item 24: Multiple-choice response by quintile group

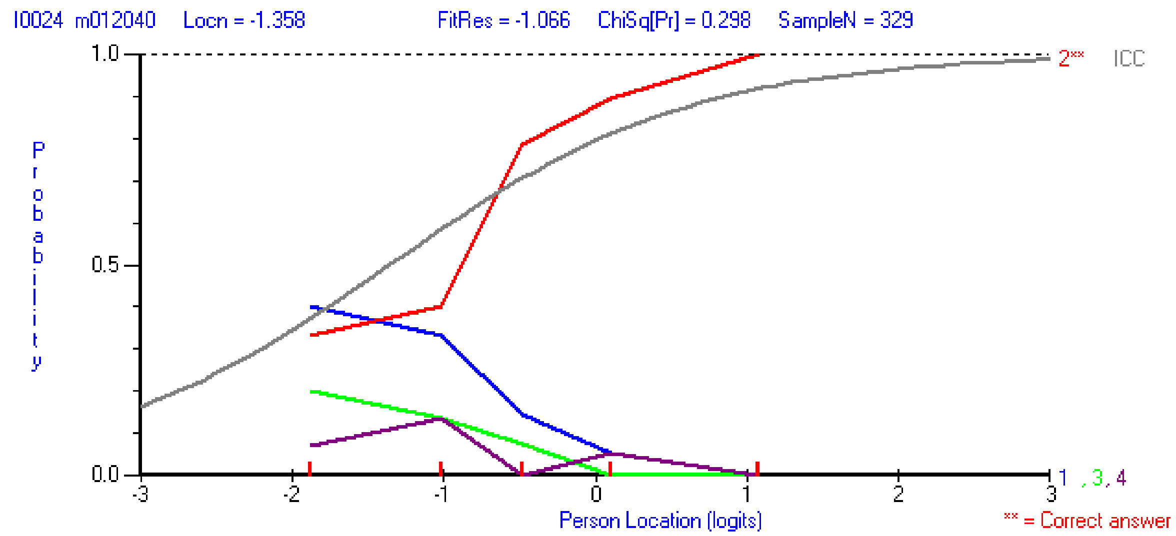


Table 8: Item 24 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quintile groups				
				-1.880	**-1.012**	**-0.478**	**0.105**	**1.081**
B. 7	68%	red	12: 7 :: 36: 21	33%	40%	79%	89%	100%
A. 3	18%	blue	12 x 3 = 36	40%	33%	14%	5%	0%
C. 36	8%	green	Symmetry fallacy	20%	13%	7%	0%	0%
D. 63	5%	purple	Multiply 21 by 3, reverse 36 to 63	7%	13%	0%	5%	0%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Level 3 and Level 4 fraction item analysis: Description and Comparison

Item 14 (location -0.416)(see Table 9)and **Item 16 (location -0.280)**(see Table 11)at **Level 3** both require multiplication by a fraction, i.e. the rational number sub-construct defined as the operator sub-construct. This multiplicative operation is understood to be more difficult than recognising fractions, adding fractions, or converting equivalent fractions. **Item 14** at first appears much easier than **Item 16** which is a multi-step problem involving multiplication of whole numbers by fractions. For both items, 45% of the study group attained the correct answer. We note that the common aggregated result of 45% hides some interesting differences in the learner responses to these items.

Both **Item 18 (location 0.072)**(see Table 13) and **Item 32 (location 0.658, 0.660 on second analysis)**(see Table 16) at **Level 4** were located above the item difficulty level set at zero. While in the same defined interval from zero to 1 on the scale, **Item 32** is located at over 0.5 logits more difficult (see Figure 7.8). For **Item 18** the task is to convert the fraction $\frac{4}{5}$to an equivalent fraction with a different denominator. The task required in **Item 32** was to find out how many$\frac{1}{5}$ kilogram portions are contained in 6 kilograms. Some children may have understood this problem to require division. Of the group as a whole, 34% and 21% answered correctly for **Item 18** and **Item 32**, respectively, located at Level 4.

Distractor analyses by quartile/quintile group

We observe in the distractor analysis that for **Item 14**, 91% of the highest quartile answered correctly, **Option E**, and 70% of the second highest quartile, however for the two lower quartiles, only 11% in both cases answered correctly (see Table 10). For Item 14, the selection of incorrect **Option C** and **Option D** are equally distributed across the lower two quartiles, 32% and 37%, the lowest quartile, and 33% and 39%, the second lowest quartile, and we infer most likely the result of a fairly intelligent guess. The selection of **Option A** and **Option B** are chosen only by learners in the lowest quartiles, 11% of each group.

This pattern of stark contrast between the highest and the lowest quartiles differs somewhat for **Item 16**(see Table 12), where the difference in the percentage correct for the highest quartiles (64% and 55%), and the lowest two quartiles (38% and 22%) is less marked. Some 64% of the highest quartile selected the correct **Option C (6)** while 18% of this group chose the incorrect **Option A (2)**, indicating lack of understanding of the fraction concept and the associated procedures required to solve the problem, requiring applying the operator sub-construct. This option was also selected by 56% and 50% of each of the lower two quartiles.

At **Level 4**, 71% of the highest quartile answered **Item 18** correctly (see Table 14), selecting **Option A (5),** 33% of the second highest quartile, and respectively, 13% and 17% of the lower quartiles. Just over 30% of the whole group selected the **Option E (1)**, with a substantial percentage across the 4 quartiles, 10% (highest quartile), 33%, 48% and 28% (lowest quartile). A reasonable hypothesis may be that the learners have only taken the numerator into account and have ignored the denominator, regarding the fraction as a natural number. **Item 32** was a constructed response item. The statistics show that 21% (n=83) responded correctly. Of the 83 learners to whom this item was administered 65 learners did not attempt an answer. The breakdown into quintiles of learners who responded to the item (see Figure 7.42), shows that just over 50% of the highest quintile group answered the item correctly. For the lower four quintile groups, the percentage correct answers hover around the 15% mark.

Critical features for fractions, Level 3 and Level 4

Both **Item 14** and **Item 16**, at **Level 3**, were categorised by the TIMSS study as *solving routine problems.* For the majority of learners in the study, these problems appeared more than a routine procedure. Item 16 required at least two steps, of the subtraction of fractions. It may be that learners consistently use calculators to solve fraction problems. As calculators were not allowed in this test administration, learners might have been at a loss as to how to proceed. The notion that even routine procedures are based on sound conceptual knowledge is affirmed in this thesis.

For **Item 18**, at **Level 4**, facility with moving between different rational number representations is required, for example, the terms $\frac{4}{5}$ and $\frac{8}{10}$ are equivalent representations for the diagram.There are at least two hypothesised approaches to solving **Item 32**, also at **Level 4**. The first is algorithmic: *To divide by a fraction (or a natural numbers one must multiply by the inverse*. 6 x 5 = 30. The second approach is to reflect on everyday experience. *How many fifths in one kilogram? Multiply by 6 to find how many fifths in 6 kilograms.*

Fraction Item 14: location -0.416 (level 3)

Table 9: Item 14 description

Fraction items	**Mathematical structure**			**Response process**
14. In a Grade 8 class of 30 students, the probability that a student chosen at random will be less than 13 years old is $\frac{1}{5}$. How many students in the class are less than 13 years old? A. Two B. Three C. Four D. Five E. Six	$P = \frac{1}{5}$ Count $= 30$ $\frac{1}{5} = \frac{x}{30}$	M_1 1 5	M_2 x 30		Applying operator sub-construct to find a proportion

Figure 5: Item 14: Multiple-choice response by quartile group

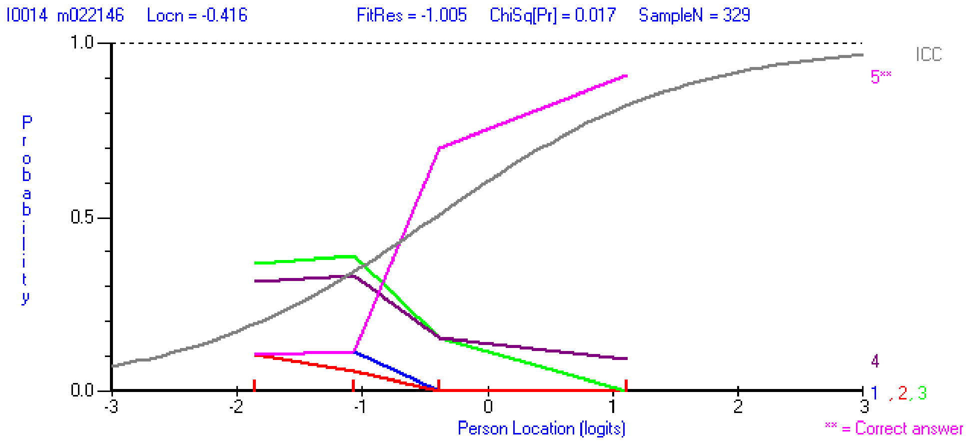


Table 10: Item 14 inferred procedures for multiple-choice responses

Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quartile groups**			
				-1.852	**-1.062**	**-0.386**	**1.117**
E. 6	45%	pink	$\frac{1}{5}$ of 30 or 1:5:: 6 :30	11%	11%	70%	91%
D. 5	22%	purple	Random inverse of 5	32%	33%	15%	9%
C. 4	23%	green	Confusion of 20 and 30	37%	39%	15%	0%
A. 2	6%	blue	Select first distractor	11%	11%	0%	0%
B. 3	4%	red	Guess	11%	6%	0%	0%
Total				**100%**	**100%**	**100%**	**100%**

Fraction Item 16: location - 0.280 (level 3)

Table 11: Item 16 description

Fraction items	Mathematical structure	Response process
16. A teacher and a doctor each have 45 books. If $\frac{4}{5}$ of the teacher's books are novels and $\frac{2}{3}$ of the doctor's books are novels, how many more novels does the teacher have than the doctor? A. 2 B. 3 C. 6 D. 30 E. 36	$(\frac{4}{5}$ of $45)$ $-(\frac{2}{3}$ of $45)$ Or $(\frac{4}{5}-\frac{2}{3})$ of 45 M_1 M_2 4 x 5 45	Operator sub-construct, adding and multiplying fractions

Figure 6: Item 16: Multiple-choice response by quartile group

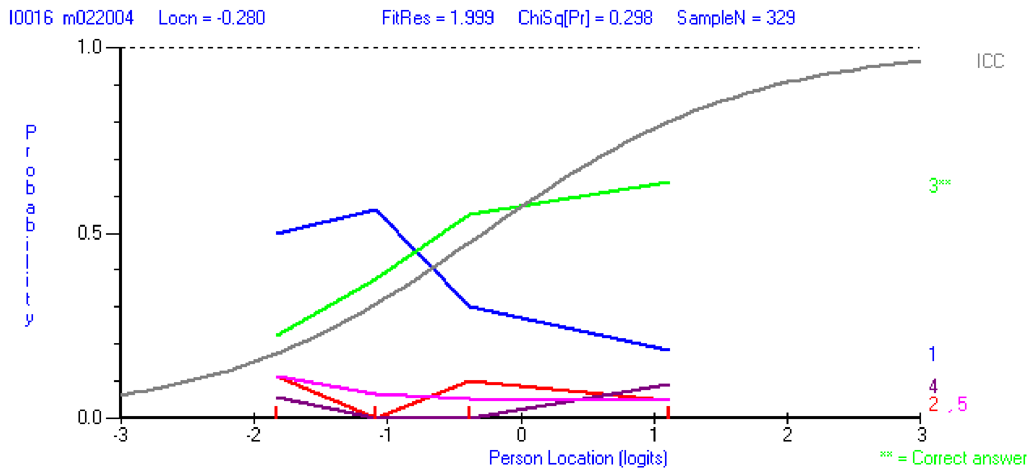


Table 12: Item 16 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-1.836	**-1.083**	**-0.386**	**1.117**
C. 6	45%	green	$(\frac{4}{5}-\frac{2}{3})$ of 45	22%	38%	55%	64%
A. 2	39%	blue	4 (fifths) – 2 (thirds) = 2	50%	56%	30%	18%
E. 36	7%	pink	$\frac{4}{5}$ of 45part answer	11%	6%	5%	5%
B. 3	9%	red	$45 \div 5 \div 3$	11%	0%	10%	5%
D.30	4%	purple	$\frac{2}{3}$ of 45part answer	6%	0%	0%	9%
Total				**100%**	**100%**	**100%**	**100%**

Fraction Item 18: location 0.072 (level 4)

Table 13: Item 18description

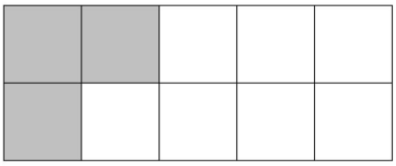
Fraction item	**Mathematical Structure**		**Response process**
18. In the figure how many MORE small squares need to be shaded so that $\frac{4}{5}$ of the small squares are shaded?  A. 5 B. 4 C. 3 D. 2 E 1	$\frac{3}{10}+-=\frac{4}{5}$ $\frac{4}{5}\times\frac{2}{2}$ $\frac{3}{10}+-=\frac{8}{10}$	M_1 M_2 4 x 5 10	Applies fraction equivalence, adds fractions.

Figure 7: Item 18: Multiple-choice response by quartile group

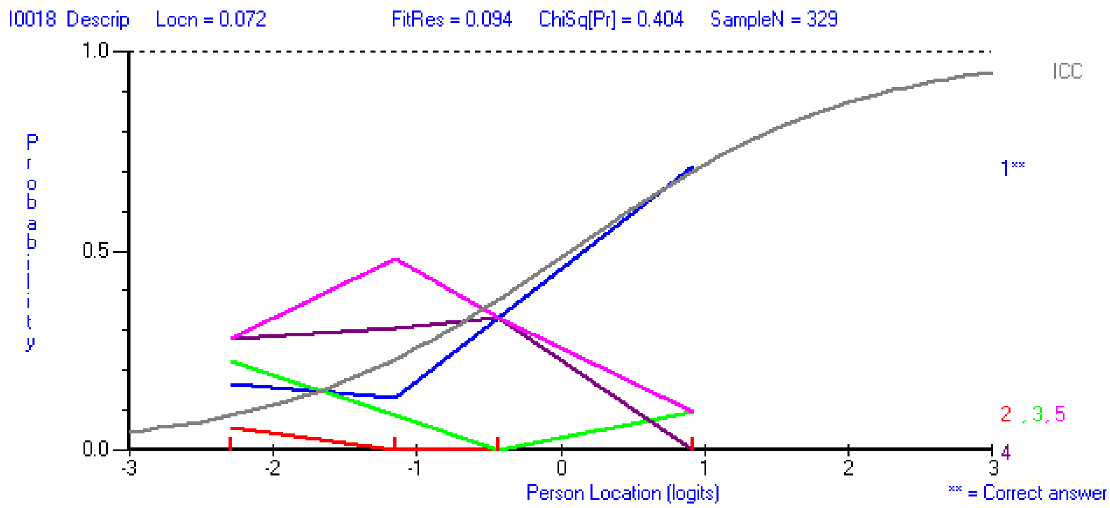


Table 14: Item 18 inferred procedures for multiple-choice responses

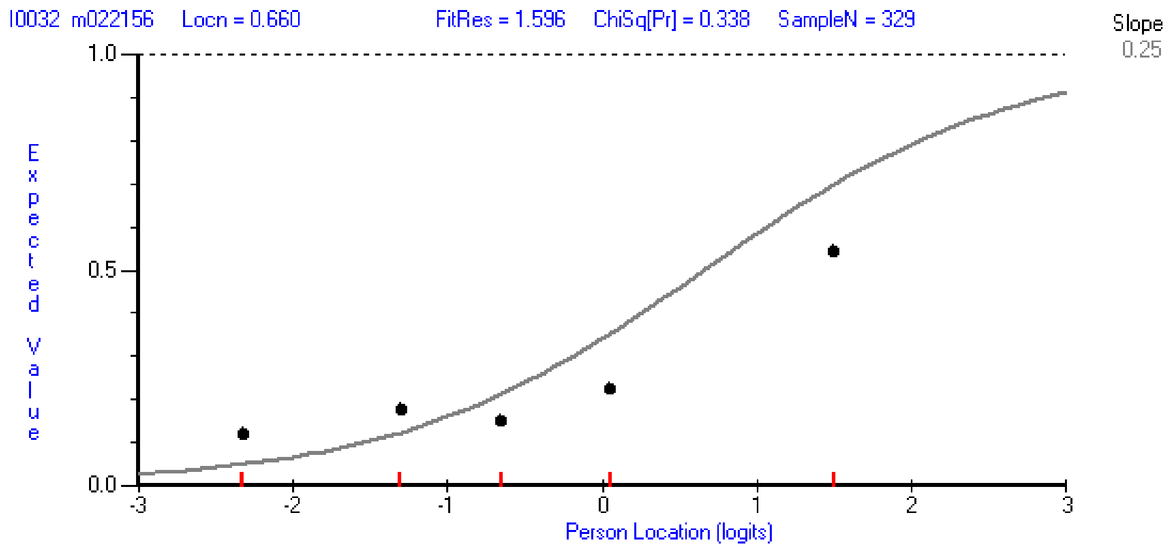
Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quartile groups**			
				-2.289	**-1.155**	**-0.428**	**0.918**
A. 5	34%	blue	$\frac{4}{5}$ of 10 = 8 3 + ? = 8	17%	13%	33%	71%
E. 1	31%	pink	3 + 1 = 4 (only numerators)	28%	48%	33%	10%
D. 2	23%	purple	3 + 2 = 5	28%	30%	33%	0%
C. 3	10%	green	?	22%	9%	0%	10%
B. 4	4%	red	?	6%	0%	0%	10%
Total				**100%**	**100%**	**100%**	**100%**

Fraction Item 32: location 0.660 (level 4)

Table 15: Item 32 description

Fraction item	Mathematical Structure	Response process
32. A scoop holds $\frac{1}{5}$ kg of flour. How many scoops of flour are required to fill a bag with 6 kg of flour? Answer ____________________	$6\text{ kg} \div \frac{1}{5}$ M_1 M_2 1/5 x 1 6	Multiplying by the reciprocal

Figure 8: Item 32: Item characteristic curve



B2: Ratio, proportion and rate item analysis (see Section 7.6, Thesis document)

Level 1 and Level 2 item analysis: Description and comparison

The two **Level 1** items selected from TIMSS Grade 4 were answered correctly by most learners, with 83% correctly answering **Item 1 (location – 2.477)**(see Table 16) and 76% overall correctly answering **Item 2 (location -2.294)**(see Table 18).**Item 1** was framed in natural language, while **Item 2** was represented pictorially. Both items required multiplicative reasoning with numbers within a small number range. **Item 20 (difficulty location -1.557)**(see Table 20), at **Level 2** and phrased in an everyday context, demands an understanding of part-whole ratio and fraction notation, where the number range is small (≤ 30).

Distractor analysis by quintile groups

Item 1 required reasoning proportionally, *one is to three, and therefore nine is to 27*. Overall the percentage answering correctly **Option D** (27) was 82% (see Table 17). For the highest class interval, located at 1.110 logits on the scale, 98% selected the correct answer. In the middle class interval, where the mean of the ability score was located at -0.670, 88% of learners selected the correct answer. For the lowest group of learners (located at -2.040), only 55% answered correctly.

Incorrect **Option B** (12) and **Option A** (3) were chosen by 22% and 23% respectively of the learners in the lowest class interval. A hypothesised reason for learners choosing **Option B** was that they increased 9 by 3, instead of multiplying 9 by 3. A hypothesised reason for choosing **Option A** was a misreading of the problem; they may have simply quoted “3” from the text. The subtlety of the relational language was not understood.

Item 2 (see Table 19) required learners to recognise squares in which the relationship of shaded to unshaded was in the proportion 3 to 4. Learners were required to recognise that the ratio 3 to 4 could be represented as 6 out of 8 squares. Overall 79% of learners in the group selected the correct **Option C** (6 out of 8), with 16% selecting **Option B** where the relationship of unshaded to shaded is 3 to 4.

Some 93% of learners in the highest quintile selected the correct answer **Option A**, with the remaining 7% of the highest quintile choosing **Option B**. For learners in the next three quintile groups the results were similar, however for the lowest quintile only 38% exhibited mastery of this relatively low level ratio concept. This outcome indicates that for the learners in the Grades 7 to 9 of this study, located in the lowest quintile, roughly 20% have an inadequate understanding of ratio and proportional reasoning.

For **Item 20**(see Table 21), 69% of the total group of learners selected the correct **Option D** ($\frac{16}{30}$). For learners in the highest quintile, 91% selected the correct answer. The other 9% of this highest quintile group selected the part-part ratio, **Option C** ($\frac{16}{14}$), instead of the part-whole option. Of the lowest quintile, 24% selected the correct answer, which was the part-whole ratio, while 47% selected the part-part ratio and 24% selected the part-part ratio, which was in the incorrect order.

Critical features for ratio, rate and proportion, level 1and 2

The two Level 1 items discussed above require the understanding of the covariant relationship inherent in a ratio. In **Item 1**, the ratio *1 to 3*, is applied within *3 is to the unknown quantity*. The hidden operator must be understood as *multiply by 3*. The relationship is multiplicative rather than additive. This threshold understanding appears to have been attained by 83% of the cohort. In **Item 2** the ratio 3 out of 4 is matched to 6 out of 8 blocks. The operation *multiply by 2* is hidden in this problem.

The concept of part-part and part-whole ratio in **Item 20**, and the distinction between these concepts, is still to be attained by 30% of this cohort. Learners are required to firstly establish the total number of children in the class and then convert the ratio relationship, of birthdays in the first half of the year to the total, into fraction notation. The increase in difficulty from Items 1 and 2, and Item 20 may be accounted for by the two steps involved in solving Item 20.

Ratio, rate and proportion Item 1: location -2.447 (level 1)

Table 16: Item 1 description

Ratio item	**Mathematical structure**			**Response process**
For every cool drink bottle Zanele collected, Modiegi collected 3. Zanele collected a total of 9 bottles, how many did Modiegi collect? A. 3 B. 12 C. 13 D. 27	1: 3:: 9: ?	**M_1** (Zanele) 1 9	**M_2** (Modiegi) 3 x	Recognise proportional covarying relationship

Figure 9: Item 1: Multiple-choice distractor plots by quintile group

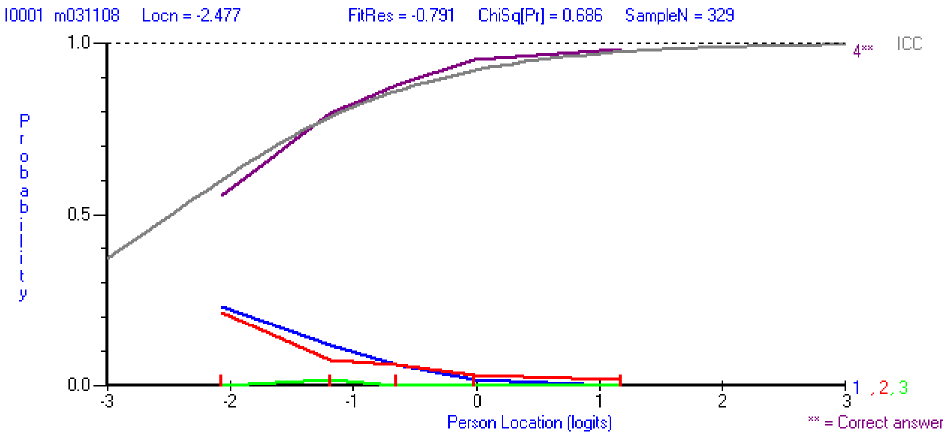


Table 17: Item 1 inferred procedures formultiple-choice responses

Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quintile groups**				
				-2.040	**-1.191**	**-0.670**	**-0.036**	**1.110**
D. 27	83%	purple	1:3:: 9:27	55%	78%	88%	96%	98%
B. 12	8%	red	1 is to 3; 9 is to 12 (add 3)	22%	8%	6%	3%	2%
A. 3	8%	blue	Misreading: Modiegi collected 3	23%	12%	6%	1%	0%
C. 13	0%	green	Add all 3 numbers	0%	2%	0%	0%	0%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

.

Ratio, rate and proportion Item 2: location -2.294 (level 1)

Table 18: Item 2 description

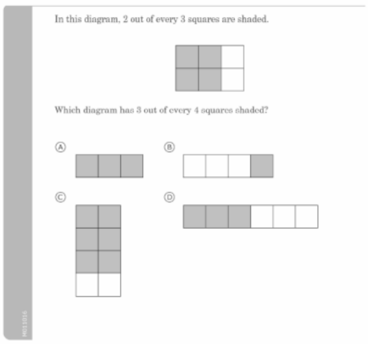
Ratio item	**Mathematical structure**		**Response process**
Which diagram has 3 out of every 4 blocks shaded? 	3 out of 4 same as 6 out of 8	**M_1** **M_2** 3 *b* 4 *d*	Recognise proportional covarying relationship

Figure 10: Item 2: Multiple-choice response by quintile group

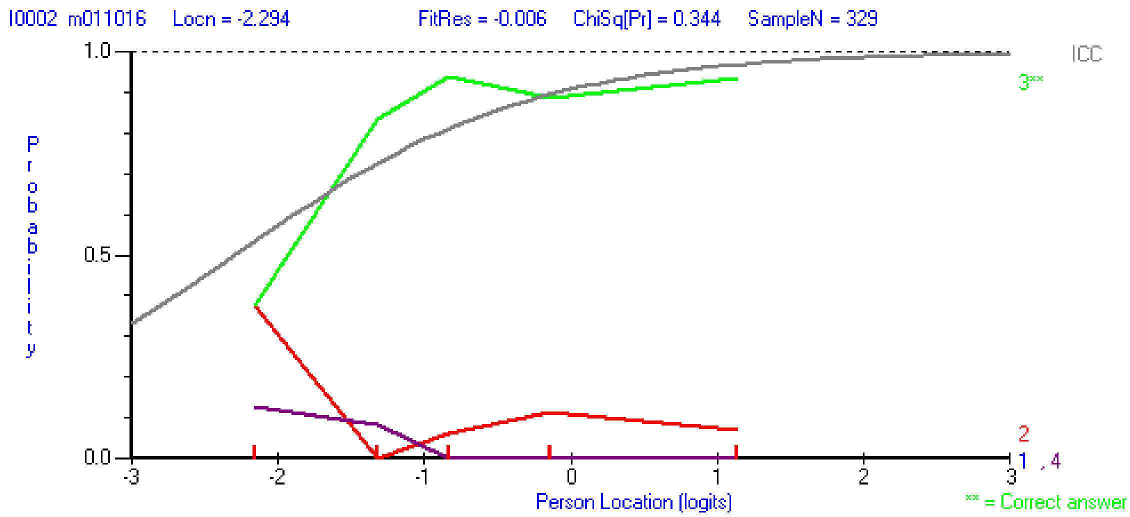


Table 19: Item 2 inferred procedures for multiple-choice responses

Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quintile groups**				
				-2.162	**-1.319**	**-0.828**	**-0.134**	**1.137**
C. 6 of 8	79%	green	Ratio 3:4 :: 6:8	38%	83%	94%	89%	93%
B. 3 of 4	16%	red	3 unshaded out of 4	38%	0%	6%	11%	7%
A. 3 blocks	4%	blue	Ratio relation ignored	13%	8%	0%	0%	0%
D. 3 of 6	4%	purple	Consider only numerator	13%	8%	0%	0%	0%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Ratio, rate and proportion Item 20: location -1.557 (level 2)

Table 20: Item 20 description

Ratio item	**Mathematical structure**		**Response process**
In a group of children, 16 have birthdays during the first half of the year, and 14 have birthdays during the second half of the year. What fraction of the group has birthdays in the first half of the year? A. $\frac{14}{30}$ B. $\frac{14}{16}$ C. $\frac{16}{14}$ D. $\frac{16}{30}$ E. $\frac{30}{16}$	16 of total (16 + 14) $\frac{16}{16+14} = \frac{16}{30}$	**M_1** (time) / **M_2**(children) 1st half / 16 total / 30	Additive relation, then part whole fraction

Figure 11: Item 20: Multiple-choice responses by quintile group

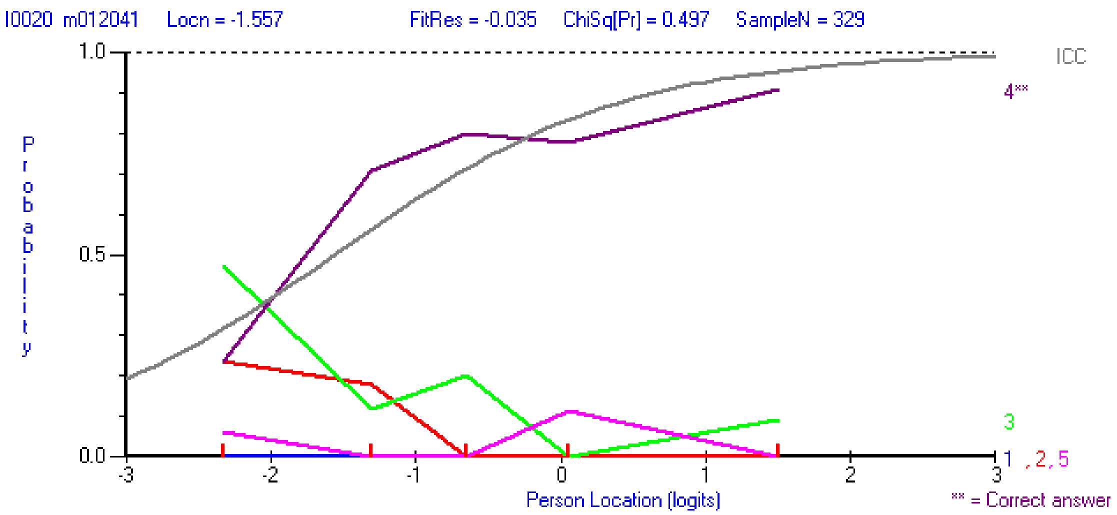


Table 21: Item 20 inferred procedures for multiple-choice responses

Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quintile groups**				
				-2.331	**-1.305**	**-0.655**	**0.050**	**1.501**
D. $\frac{16}{30}$	69%	purple	Ratio: first half to whole	24%	71%	80%	78%	91%
C. $\frac{16}{14}$	17%	green	Ratio: first to second half	47%	12%	20%	0%	9%
A. $\frac{14}{30}$	2%	blue	Ratio: second half to whole	0%	0%	0%	11%	0%
E. $\frac{30}{16}$	4%	pink	Ratio of whole to first half	6%	0%	0%	11%	0%
B. $\frac{14}{16}$	8%	red	Ratio of second to first half	24%	18%	0%	0%	0%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Level 3 and Level 4 ratio item analysis: Description and comparison

Level 3 comprised four items, **Item 3 (location -0.976)**(see Table 22),**Item 5 (-0.585)**(see Table 24),**Item 9 (-0.147)**(see Table 26) and **Item 15 (-0.045)**(Table 28).These items were answered correctly by 58% (Item 3), 50% (Item 5) 42% (Item 9) and 43% (Item 15) of the cohort. All four Level 3 items required proportional reasoning; learners are required to project the ratio between two or three quantities, onto another set.

The first two items were set in everyday contexts. **Item 3** was presented in natural language and required multiplicative reasoning. The number range is small, that is $\leq$ 16. The problem in this item was to find the missing part in the proportion statement *4 is to 3* is the same as *12 is to the unknown*. In **Item 5** the learner has to divide an amount in proportion to three unequal amounts. The problem is set in a family context involving the sharing of money. The number range is in thousands, but as these numbers are whole thousands, they are presumed to function as smaller numbers. Though the dividend is 45 000, given that the divisor is a factor of 45, in this case 9, the calculation is relatively simple. At a less sophisticated level this item becomes a two-step problem first finding the money allocated to one child and then finding the corresponding amount for the parent Dan. The number range for **Items 9** and **Item 15** is less than 30. The proportional problems are set in a geometric rather than everyday context. **Item 9** requires understanding a proportional relationship between the shaded and unshaded small triangles within the larger triangle, and **Item 15** the proportional relationship between the sides of triangles.

Item 10 (location 0.438)(Table 30), at **Level 4** is a proportion problem with a given rate of 2.4 litres per 30 hours. The task is to calculate the litres per 100 hours. This problem could be solved either using elements within measure spaces, *30 is to 100: 2.4 is to x*, or elements between measure spaces, *30 is to 2.4: 100 is to x*. Working with decimal numbers, a higher number range, and a divisor that is not a factor, makes this problem more difficult. Some 30% answered the problem correctly.

Distractor analysis by quartile or quintile group

For **Item 3**, 90% of the highest quintile group answered correctly, **Option A** (9), 71% of the second highest group, progressively decreasing to 32% of the lowest group (see Table 23). The incorrect solution, **Option B** (11) that *4 is to 3, is the same as 12 is to 11*, was selected by 32% of learners. This incorrect answer *12 is to 11* may have been arrived at by reasoning additively. This answer was chosen by 8% of the highest quintile but by 45% of the lowest quintile.

Finer analysis of **Item 5** shows that 85% of the highest quintile selected the correct distractor, **Option D** (20 000) which required calculating 4/9 of 45 000, progressively decreasing with each lower quintile group, but with only 14% of the lowest quintile answering the item correctly (see Table 25). The **Option C** (15 000)), that is incorrectly dividing the money between 3 sons, instead of 9 grandchildren, was selected by 8% of learners in the highest quintile, and by 31% of the lowest quintile. The selection of this option is hypothesised as inaccurate reading of the problem. The selection of **Option A** (5000) is understood to be partially correct, as dividing by 9 is a correct first step at a particular level of functioning.

For **Item 9**(see Table 27), 73% of the highest quartile selected the correct **Option A** (5:3). This percentage of correct answers drops sharply to only 9% in the lowest quartile. For a reason not altogether clear, 13% of the highest quartile and 51% of the lowest quartile selected the Option B (8:5). For this group 10 shaded blocks to 6 unshaded blocks simplifies to 8:5. If this response is not simply a guess, it may reflect a strategy other than the counting of the two sets of triangles.

Item 15(see Table 29)also required the conversion of a ratio. In this item learners were required to recognise that the ratio of three sides of a triangle, when multiplied by two, produces a similar triangle. The concept of a similar triangle is not in the curriculum at these grades. However it seems for the highest quartile, an intuitive understanding was evident to 71% of that group. For close on 81% of the lowest quartile, a colloquial meaning of similar, in that two of the three sides were the same, may influenced the selection.

For **Item 10**(see Table 30), at **Level 4**, 65% of the highest quartile group of learners answered correctly **Option B** (8.0). At the lower quartiles, the percentage answering correctly decreases to 28%, 19%, and 8%. At the second highest quartile there is some evidence pointing to guessing, as the choice of each of the 4 distractors hovers around 25%, although this inference cannot be verified. At the lowest quartile, the most often selected response is **Option C** (8.4), where the inferred procedure could be *multiply 30 by 3 to get ninety, and then add some more, possibly a half.* Or the procedure may be *add 30, three times, and then add some more*.

Critical features for rate, rate and proportion, Level 3 and Level 4

The requirement for solving the four **Level 3** items is to identify the existing ratio, and then through a multiplicative comparison to apply the same ratio to the new set of numbers. The requirement was either to generate the new set of numbers or to recognise the multiplicative operator. At **Level 4**, the same concepts are required, but an additional key concept is that of working with decimal fractions.

Ratio, rate and proportion Item 3: location -0.976 (level 3)

Table 22: Item 3 description

Ratio items	Mathematical structure			Response process
Thabang can run 4 times around the track in the same time that Tshepo can run 3 laps. When Thabang has run 12 laps, how many laps has Tshepo run? A. 9 B. 11 C. 13 D. 16	4:3 :: 12 : x 4:3 :: 4(×3): 3(×3) a:b = ax: bx	M_1 4 3	M_2 12 d	Identifies proportional relationship and multiplicative operator

Figure 12: Item 3: Multiple-choice responses by quintile group

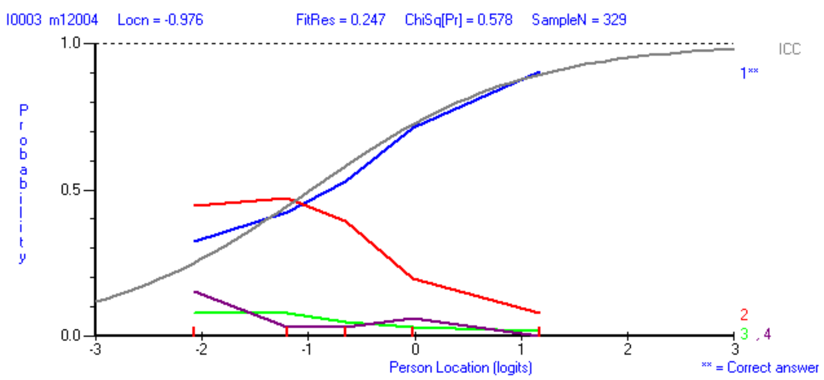


Table 23: Item 3 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quintile groups				
				-2.078	**-1.194**	**-0.657**	**-0.023**	**1.170**
A. 9	58%	blue	4:3::12:9	32%	42%	53%	71%	90%
B. 11	32%	red	4:3::12:**11** Additive	45%	47%	39%	20%	8%
C. 13	5%	green	4:3::**13**:12 Interchange additive	8%	8%	5%	3%	2%
D. 16	5%	purple	3:4::12:**16** Interchanged	15%	3%	3%	6%	0%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Ratio, rate and proportion Item 5: location -0.585 (level 3)

Table 24: Item 5 description[2]

Ratio items	Mathematical structure		Response process
Three brothers, Thabo, Samuel and Dan, receive 45 000 zeds from their father in proportion to the number of children each one has. Thabo has two children, Samuel has three children and Dan has four children. How many zeds does Dan get? A. 5000 B. 10 000 C. 15 000 D. 20 000	2 : 3 : 4 $\frac{4}{9}$ of 45000	M_1 M_2 9 45000 1 5000 4 x	Identifies proportional relationship and multiplicative operator

Figure 13: Item 5: Multiple-choice responses by quintile group

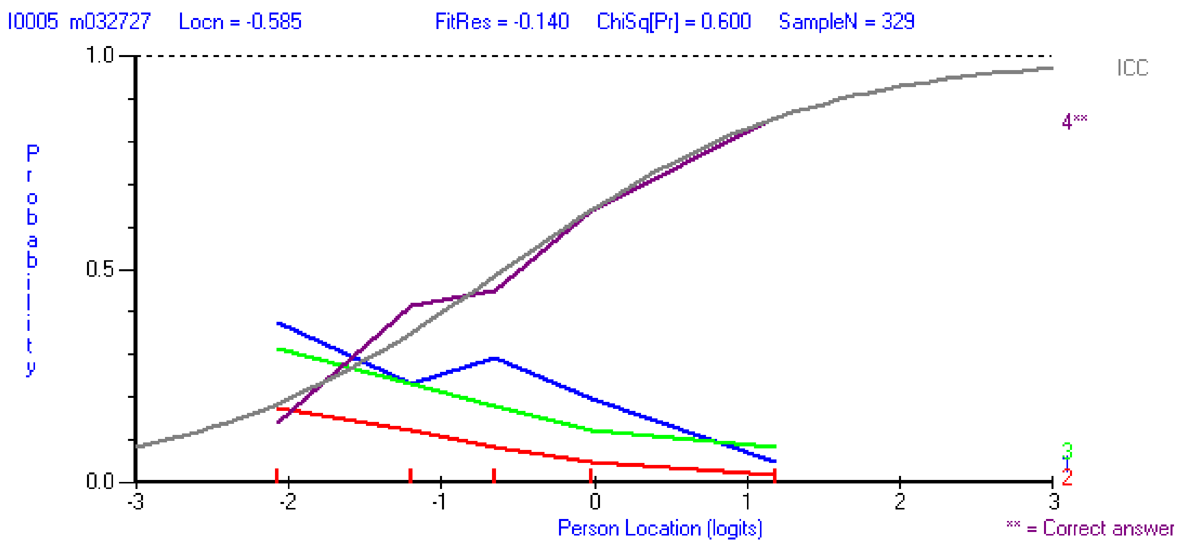


Table 25: Item 5 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quintile groups				
				-2.079	**-1.193**	**-0.656**	**-0.023**	**1.181**
D. 20 000	50%	purple	Divide by 9, multiply by 4	14%	42%	45%	64%	85%
C. 15 000	19%	green	Divide by 3 sons	31%	23%	18%	12%	8%
A. 5 000	23%	blue	Divide by 9, omits 2nd step	38%	23%	29%	20%	5%
B. 10 000	8%	red	Misinterpretation	17%	12%	8%	5%	2%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

[2] Note that interviews with learners at different abilities were conducted on this item.

Ratio, rate and proportion Item 9: location -0.147 (level 3)

Table 26: Item 9 description

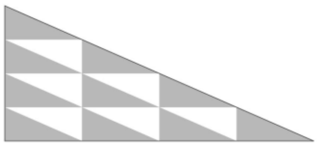
Ratio items	Mathematical structure		Response process
9. In the figure below, each of the smaller triangles has the same area. What is the ratio of the shaded area to the unshaded area?  A. 5:3 B. 8:5 C. 5:8 D. 3:5	10:6::5:3 $a{:}b{::}\frac{a}{x}:\frac{b}{x}$	**M_1** **M_2** 10 b 6 d	Recognises equivalent relation

Figure 14: Item 9: Multiple-choice responses by quartile group

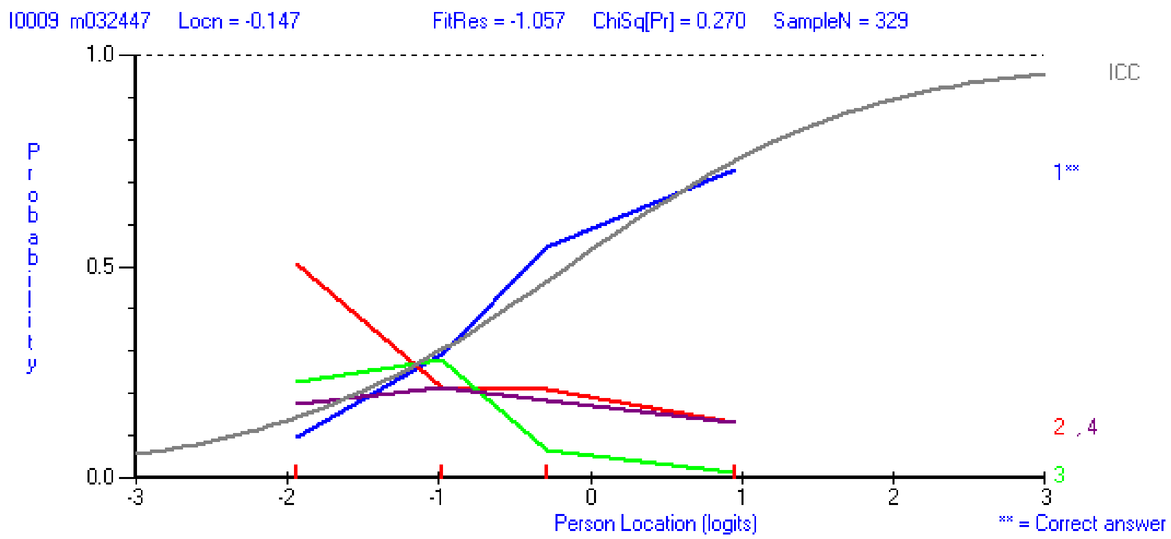


Table 27: Item 9 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematics procedure	Mean locations of quartile groups			
				-1.945	**-0.985**	**-0.284**	**0.961**
A. 5:3	42%	blue	Simplify 10:6	9%	29%	55%	73%
B. 8:5	27%	red	Subtract from 10 and 6	51%	22%	21%	13%
D. 3:5	18%	purple	Inverted ratio	17%	22%	18%	13%
C. 5:8	15%	green	Subtract from 10 and 6, invert	23%	28%	6%	1%
Total				**100%**	**100%**	**100%**	**100%**

Ratio, rate and proportion Item 15: location -0.045 (level 3)

Table 28: Item 15 description

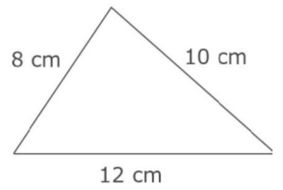
Ratio items	Mathematical structure		Response process
 Which of the following triangles* is similar to the triangle shown above? A. 10: 12: 15 B. 10: 12: 14 C. 6: 8: 12 D. 16: 20: 24 *triangles were drawn	8: 10: 12 $8a : 10a : 12a$	**M_1** **M_2** 8 x 10 y 12 z	Recognises proportional relationship, matches equivalent ratios

Figure 15: Item 15: Multiple-choice responses by quartile group

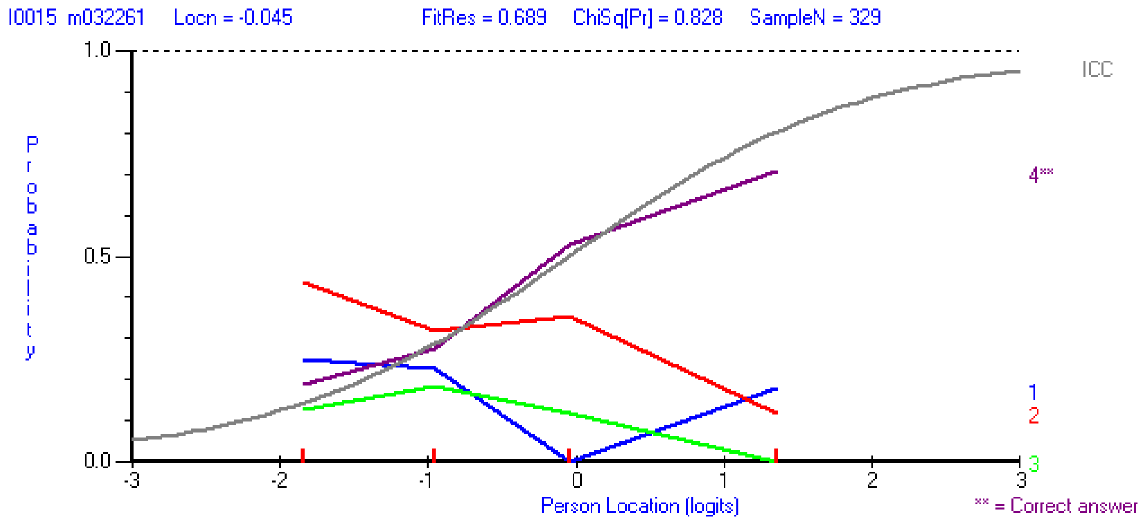


Table 29: Item 15 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-1.838	-0.955	-0.043	1.353
D. 16:20:24	43%	purple	Identify similar triangle	19%	27%	53%	71%
A.10:12:15	16%	blue	Literal meaning of similar	25%	23%	0%	18%
B. 10:12:14	31%	red	Literal meaning of similar	44%	32%	35%	12%
C. 6:8:12	11%	green	Literal meaning of similar	13%	18%	12%	0%
Total				**100%**	**100%**	**100%**	**100%**

Ratio, rate and proportion Item 10: location 0.438 (level 4)

Table 30: Item 10 description

Ratio item	Mathematical structure			Response process
A machine uses 2.4 litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use in 100 hours? A. 7.2 B. 8.0 C. 8.4 D. 9.6	$f(x) = \dfrac{2.4 \times 100}{30}$	M_1 2,4 x	M_2 30 100	Identification of proportional relationship, identifies multiplicative operator

Figure 16: Item 10: Multiple-choice responses by quartile group

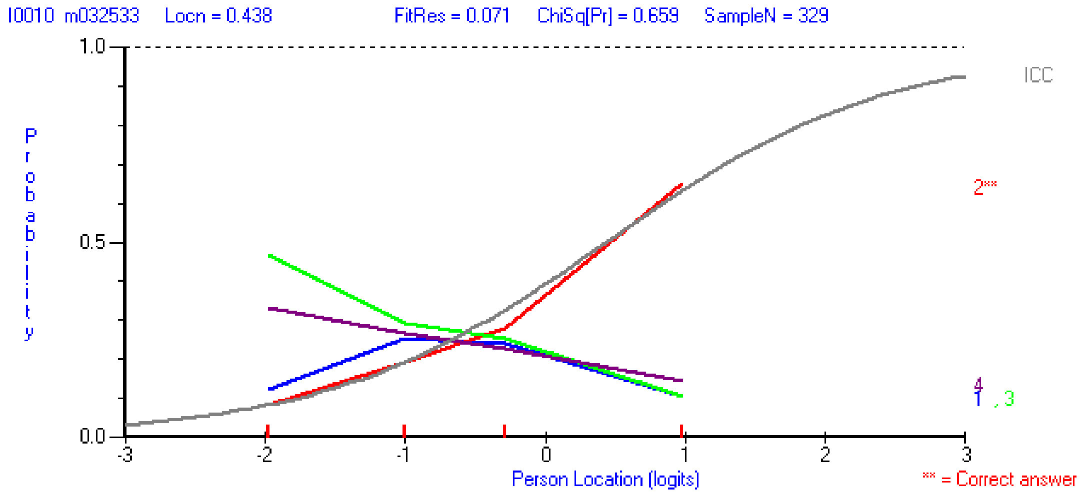


Table 31: Item 10 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-1.975	-1.006	-0.248	0.975
B. 8.0	30%	red	Divide by 30, multiply by 100 or multiply by 10/3.	8%	19%	28%	65%
D. 9.6	24%	purple	Multiply by 4	33%	27%	23%	14%
C. 8.4	28%	green	Multiply by 3 and then add half, or add 30 + 30 +30, and then some.	47%	29%	25%	10%
A. 7.2	18%	blue	Multiply by 3, or add 30, three times	12%	25%	24%	10%
Total				**100%**	**100%**	**100%**	**100%**

Level 6 and 7 ratio item analysis: Description and comparison

The four items discussed at this level were administered to about 83 learners. The hypotheses that are made are therefore tentative. However, the TIMSS 2003 International percentage means provide an interesting comparison. The International cohort results for Item 30 (36%), the remaining three, Item 31 (18%), Item 35 (6%) and Item 35 (0%), indicated that these items were found to be difficult across education systems. Of the study group 13% achieved a correct answer for Item 30; 11% selected a correct answer for Item 31; 6% for Item 35 and 1% for Item 34 (see Table 7.24, Section 1, Ratio and Rate). It appears that the concepts required for answering these items correctly are universally difficult.

Items 30 (location 2.081)and **Item 31 (location 2.147)**(see Table 32), were attached to the same problem context (see the Item description, Table 7.24). Both items had an open-response format. They involved an understanding of multiplicative reasoning, an understanding of rate, and at least two steps were needed to achieve the required answer. **Item 33 (location - 0.650)**, **Item 34 (location 6.727)** and **Item 35 (location 2.374)**(see Table 33) were also based on the same context. These items required a constructed response. The first step was to interpret the table. Following a correct reading, two calculations were involved for each of two plans. These results had to be compared.

Analysis by quintiles

An investigation of the **Item 30** item characteristic curve with quintile groups (see Figure 17) indicates that for the highest quintile, in this case the top 16 learners, only 50%, were able to answer correctly. For **Item 31** this figure drops to about 5 learners, for **Item 35**, about 4 learners and for **Item 34**, perhaps 1.

Critical features for ratio, rate and proportion, levels 6 and 7

Distance, time and speed problems are generally understood to be difficult, as are the calculation of rates. It is clear that this type of situation requires carefully constructed didactic interventions.

Ratio, rate and proportion, Item 30 (2.082), Item 31 (2.147) (level 6)

Table 32: Item 30 and Item 31 description

Ratio, rate and proportion item description			
In a car rally two checkpoints are 160 km apart. Drivers must travel from one checkpoint to the other in exactly 2.5 hours to earn maximum points.	**Mathematical structure**		**Response process**
30. A. What must the average speed be to travel the 160 km in this time?	$\frac{160\text{km}}{2.5\text{h}} = 64\text{km/h}$ $\frac{d}{t} = s$	**M_1** **M_2** 3 b 4 d	Proportional reasoning 2-step
31. B. A driver took 1 hour to travel through a 40km hilly section at the beginning of the course. What must the average speed, in kilometres per hour, be for the remaining 120km, if the total time between checkpoints is 2.5 hours?	(160 – 40) kilometres (2.5 – 1) hours $\frac{120}{1.5} = 80$ km/h	**M_1** **M_2** 3 b 4 d	Proportional reasoning 2-step

Figure 17: Item 30: Item characteristic curve, with quintile markers

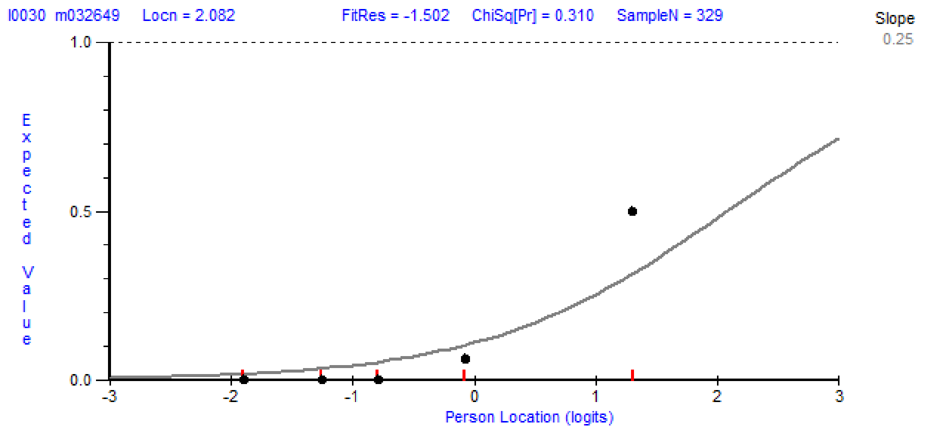
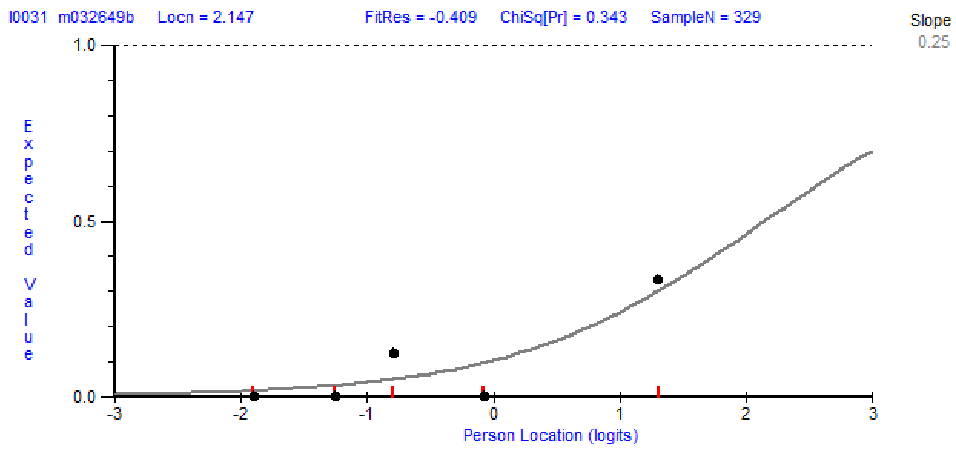


Figure 18: Item 31: Item characteristic curve, with quintile markers



Ratio, rate and proportion: Items 33 (-0.833), 35 (2.374), 34 (6.765) (levels 6, 7)

Table 33: Item 33, Item 35 and Item 34 descriptions

Ratio, rate and proportion item description

Betty, Frank and Darlene have just moved to Zedland. They need a new phone service. They received the following information from the telephone company.

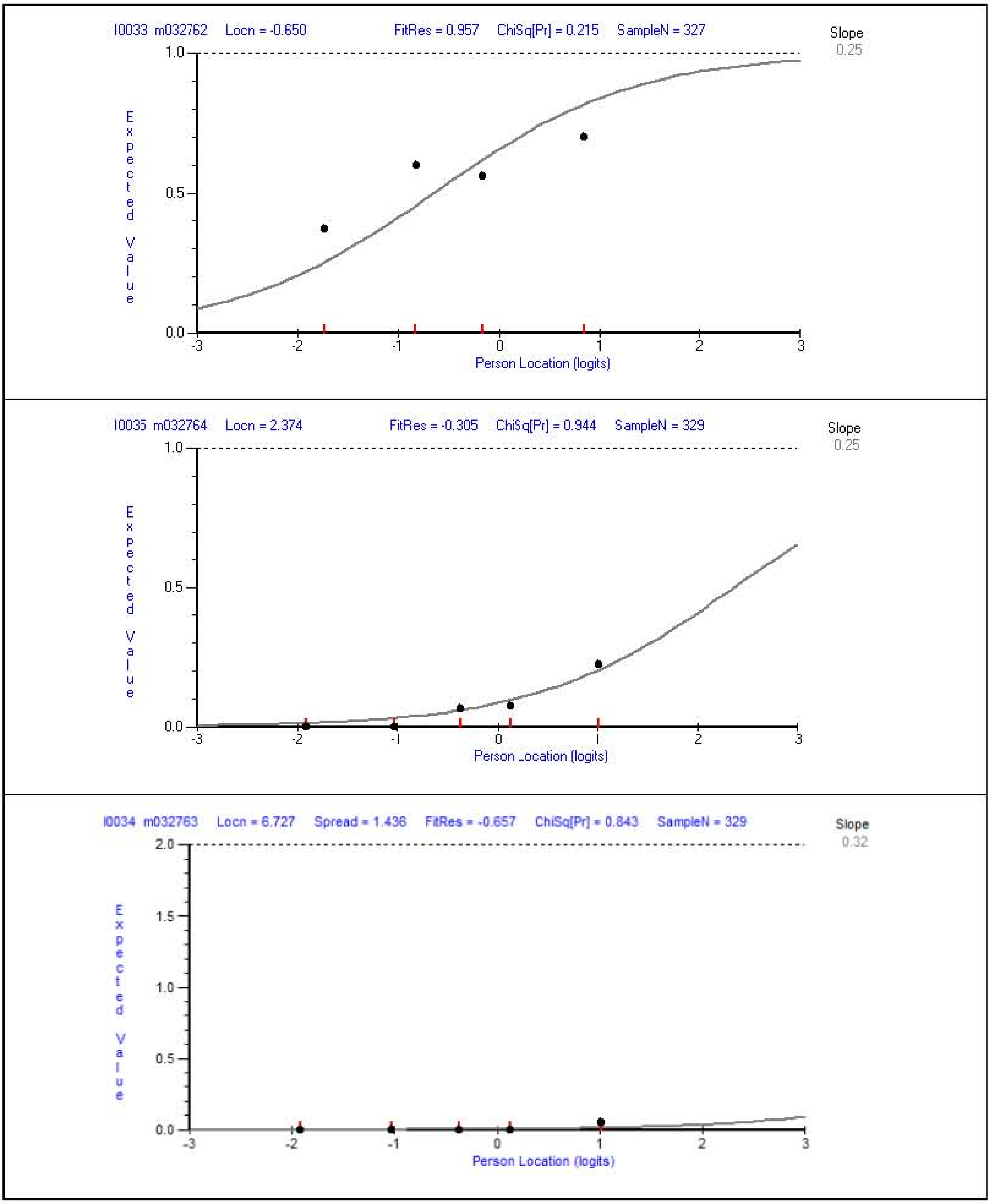
They must each pay a set fee each month and there are different rates for each minute that they talk. These rates depend on the time of day or night they use the phone and which payment plan they choose. Both plans include time for which the phone calls are free. Details of the two plans are shown below.

Plan	Monthly Fee	Rate per minute		Free minutes per month
		Day (8 am – 6 pm)	Night (6 pm – 8 am)	
Plan A	20 zeds	3 zeds	1 zed	180
Plan B	15 zeds	2 zeds	2 zeds	120

Ratio item	Mathematical structure	Response process
33. Betty talks for less than 2 hours per month. Which plan would be less expensive for her? Less expensive plan … Explain your answer in terms of both the monthly fee and free minutes.		Recognise that Plan B, the cheaper plan, has two hours-worth of free minutes
35.Dalene signed up for Plan B, and the cost for one month's service was 75 zeds. How many minutes did she talk that month? Show your work.	$\frac{75-15}{2} + 120$	Proportional reasoning 2-step
34. Frank talks for 5 hours per month at the night rate. What would each plan cost him per month? Show your work.	5 hours = 300 minutes	Proportional reasoning 2-step

Plan A	Plan B
300m (used) - 180m (free) = 120m (to pay) 120m x 1 zed (night rate) =120z 120z + 20z = 140z	300m (used) - 120m (free) = 180m (to pay) 180m x 2 zeds (night rate) = 360z 360z + 15z = 375z

Figure 19: Item characteristic curve: Item 33, Item 35 and 34



B3: Percent item analysis (see Section 7.7, thesis document)

Level 2 and 3 percent item analysis: Description and comparison

Item 4 (location -1.359), a relatively easy Level 2 item, was set in an everyday context that is familiar to most learners, namely people attending a play (see Table 34). The requirement is to convert a proportion presented in fraction notation into percent notation. A part-whole meaning of fraction, and of percent, underpinned the mathematical structure. This item was found empirically to be of greater difficulty than the previous nine items, using natural language, (Items 1, 2, 13, 17, and 22), and using fraction notation (Items 6, 11, 12 and 20).

The context for **Item 8 (location -0.060)**(see Table 36), a level 3 item, is a financial situation, which learners may encounter in shopping. A ratio understanding of percent, is required, and an understanding of percent notation. The "800 zeds" is recognised to be the whole, to which 20% of that whole is added. Interestingly, the TIMSS cognitive category for this item is *solving routine procedures*. The TIMSS classification rests on the assumption that learners have been taught and learnt the "percentage increase" procedure.

Distractor analysis by quartiles/quintiles

Observed in the **Item 4**(see Table 35) distractor analysis (see Figure 30 and Table 49) is that of the top three quintiles, 97%, 81% and 71% answered correctly. It is inferred that those learners understand the notion of the fraction $\frac{3}{25}$ being equivalent to $\frac{12}{100}$, and therefore equivalent to12%. There is a sharp decline in understanding at the lowest two quintiles. The option selected by the lowest two quintiles is inferred as *appending the percent sign*. Over 50% of the lowest two quintiles selected the responses, **Option B** (3%) and **Option C** (0.3%), where the percent sign was appended. **Option D** (0.12%) could be interpreted as partially correct in that learners have observed the ratio relationship between the numerator and the denominator, but then confused may have been confused with decimal notation.

For **Item 8**(see Table 17), some 72% of learners in the highest performing quintile, with a mean location of 1.23 logits, selected the correct answer **Option C** (960 zeds) (see Figure 31 and Table 51). Some 62% of the second highest performing quintile selected this correct option, but thereafter the percentage for the lower three quintiles decreases dramatically. For learners in the lowest quintile, at – 2.09, the selection of the incorrect **Option D** (1000z) is 57%, for the second lowest quintile, 45%, and the third lowest quintile, 40%. Of the remaining two distracters, **Option B** elicited the most responses. This preponderance may have been due to the fact that learners estimated an answer greater than 800. The choice of **Option A** could have been a result of calculating the correct percentage but then subtracting rather than adding, which in fact may indicate an understanding of percent change, but in the wrong direction.

Critical features for percent level 2 and 3

The first encountered concept of percent in many curricula is that of a part-whole fraction. The noteworthy difference from 63% correct, **Item 4** requiring a part-whole understanding, to 30% correct, **Item 8** requiring a ratio understanding, is indicative of the increase in complexity from a fraction understanding to a ratio understanding. Even for the highest performing quintile the percentage correct drops from 97% (Item 4) to 72% (Item 8). The selection of **Option D** (1000) for **Item 8** may have been due to a part-whole understanding of percentage, possibly influenced by the presence of 1000 in the distractor options. The learners at the lower end of the scale, it is hypothesised have retained the first encountered concept of percentage, that is where fractions, decimals and percents are used interchangeably.

Parker and Leinhardt (1995) remind us that percent is a complex construct. This complexity is exacerbated in teaching when the varied meanings of percent are not made explicit.

Percent Item 4: location -1.359 (level 2)

Table 34: Item 4 description

Percent item	Mathematical Structure			Response process
At a play $\frac{3}{25}$ of the audience were children. What percent of the audience was this? A. 12% B. 3% C. 0.3% D. 0.12%	$\frac{3}{25} = \frac{3x}{25x} = \frac{}{100}$ $\frac{a}{b} = \frac{ax}{bx}$	M_1 3 25	M_2 x 100	Recognise equivalence relation, determine multiplicative constant

Figure 20: Item 4: Multiple-choice response by quintile group

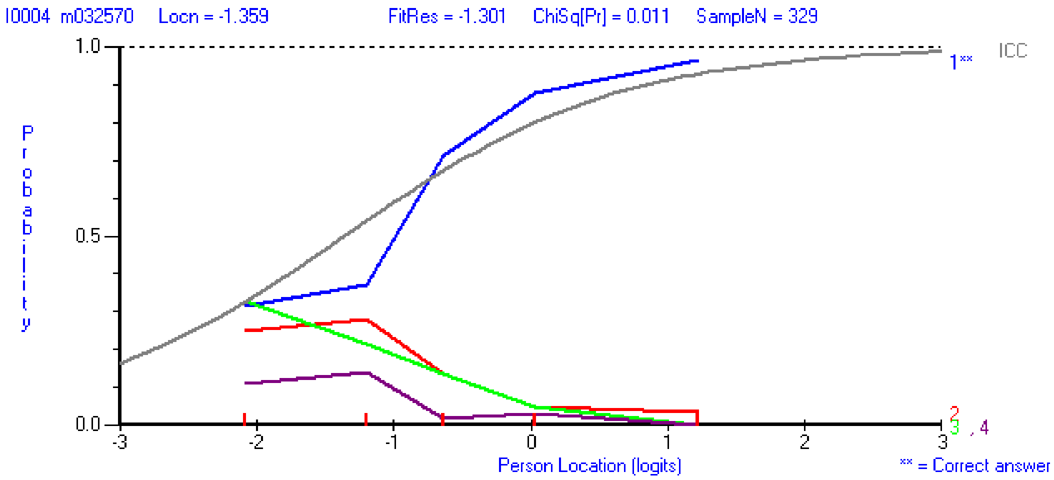


Table 35: Item 4 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quintile groups				
				-2.082	-1.196	-0.633	-0.028	1.223
A. 12%	63%	blue	$\frac{3}{25} = \frac{12}{100} = 12\%$	31%	37%	71%	81%	97%
B. 3%	15%	red	Append % sign	25%	28%	14%	5%	3%
C. 0.3%	17%	green	Insert decimal point, append % sign	33%	22%	14%	5%	0%
D. 0.12%	6%	purple	Correct calculation, append % sign	11%	14%	2%	3%	0%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Percent Item 8: location -0.060 (level 3)

Table 36: Item 8 description

Percent item	Mathematical structure			Response process
8. A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds? A. 640 zeds B. 900 zeds C. 960 zeds D. 1000 zeds	100% of 800 + 20% of 800 = 120% of 800	**M_1** 100 20 120	**M_2** 800 160 960	Calculate relative change, percent increase

Figure 21: Item 8: Multiple-choice response by quintile group

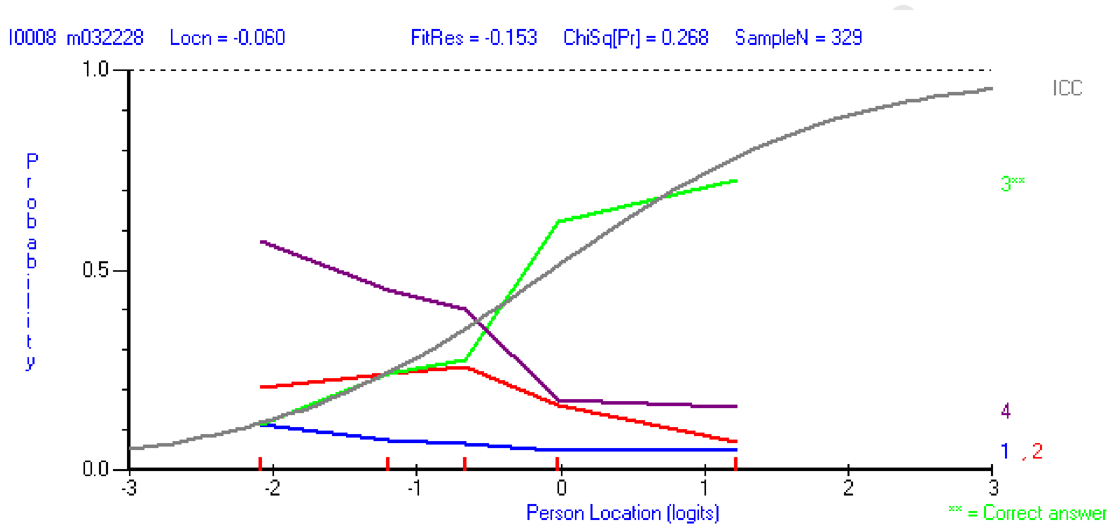


Table 37: Item 8 inferred procedures for multiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quintile groups				
				-2.09	**-1.19**	**-0.66**	**-0.02**	**1.23**
C. 960z	38%	green	Percentage increase	11%	24%	20%	62%	72%
D. 1000z	35%	purple	Part whole 800 to 1000 - increase represents 20% of final amount	57%	45%	40%	17%	16%
B. 900z	19%	red	Estimate a number greater than 800	21%	24%	26%	16%	7%
A. 640z	7%	blue	Decrease by 20% rather than increase	11%	7%	6%	5%	5%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Level 4 and Level 7 percent item analysis: Description and comparison

The context for the problem in **Item 7 (location 0.937)**(see Table 38), at **Level 4**, involves a time difference and a percent decrease. The problem solving process entails comparing the two time periods, finding the residual distance, and then finding the ratio of the residual distance to the original time. This ratio (in fraction form) is then transformed into a percentage. The increase in difficulty from **Item 8** at Level 3 to **Item 7** is substantial.

The Level 7 item, **Item 26 (location 5.069)**[3](see Table 40) requires two steps; with a first reading the two steps may not be obvious. The first step involves a percent calculation where a fraction meaning is implicit, *60% of 40*, and gives the number of girls. In the next phase of the question, there is a change of referent with 10 boys added. The final phase involves a new calculation of the ratio of girls to the whole; this ratio is converted to a percentage.

Only 23% of the group answered **Item 7** correctly, **Option C** (20%), and a mere 8% answered **Item 26** correctly. What is striking about **Item 7**, is that 64% of learners overall selected the **Option B** (5%), which suggests that only the first step, *25 subtract 20, gives 5*, was considered. Thereafter the % sign is added.

Analysis by quintile group

A finer analysis of **Item 7** at different proficiency levels (see Table 39) indicates that of the learners in the highest performing quintile, only 47% selected the correct response. The **Option B** (5%), which indicated that learners could just append the % sign, attracted 38% of the highest group, and 80% of the learners at the lowest performing quintile. For **Item 26**, (see Figure 26) only 30% of the top quintile attained the correct answer, down to less than five percent at the next highest level, and then a progressive decrease.

Critical features

The items that present greater difficulty are those items where more than one step is required. If the context is simple or the task has one step, the items are manageable. However, as the difficulty level, and complexity, increases, the need for problem solving skills increases.

---

[3] It will be observed in the item analyses that we have most item analyses based on the initial run (RUN 2), but with Item 19 slotted back at an estimated location based on the first run (RUN 1). But then for Items 26, 27, 29, 33 and 35, it was deemed better to use the estimation based on the later analyses (RUN 3) for reasons explained in Chapter 4. The graphs presented of these items are those generated with RUN 3, RESCORE A or B.

Percent Item 7: location 0.937 (level 4)

Table 38: Item 7 description

Percent item	Mathematical structure		Response process
7. When a new highway is built, the average time it takes a bus to travel from one town to another is reduced from 25 minutes to 20 minutes. What is the percent decrease? A. 4% B. 5% C. 20% D. 25%	25 – 20 $\frac{5}{25}$ convert to %	**M_1** **M_2** 5 x 25 100	Recognising the two-step process, identify referent in each case and relationship between the parts.

Figure 22: Item 7: Multiple-choice distractor plots by quintile group

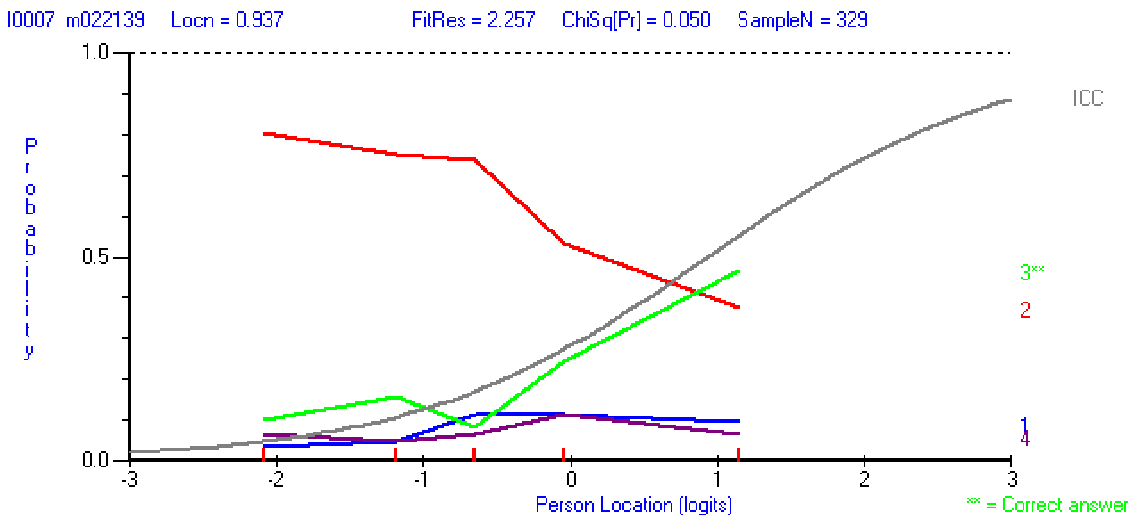


Table 39: Item 7 inferred procedures formultiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quintile groups				
				-2.087	**-1.182**	**-0.657**	**-0.042**	**1.147**
C. 20%	23%	green	$\frac{5}{25}$ converted to %	10%	16%	8%	24%	47%
B. 5%	64%	red	25-20 = 5 (append % sign)	80%	75%	74%	53%	38%
A. 4%	8%	blue	$\frac{5}{20} = \frac{1}{4} = 4\%$	3%	5%	11%	11%	9%
D. 25%	7%	purple	$\frac{5}{20}$ converted to %	7%	5%	7%	11%	6%
Total				**100%**	**100%**	**100%**	**100%**	**100%**

Percent: Item 26 (location 5.069) (level 7)

Table 40: Item 26 description

Percent item	**Mathematical structure**			**Response process**
26. A computer club had 40 members, and 60% of the members were girls. Later, 10 boys joined the club. What percentage of members now are girls?	60% of 40 = 24,	$\mathbf{M_1}$	$\mathbf{M_2}$	Proportion (reasoning)
	40 + 10 = 50	60	x	2-step
	$\frac{24}{50} = \frac{24\times 2}{50\times 2}$	100	40	
		x	24	
		100	50	

Figure 23: Item 26: Item characteristic curve indicating quintile scores

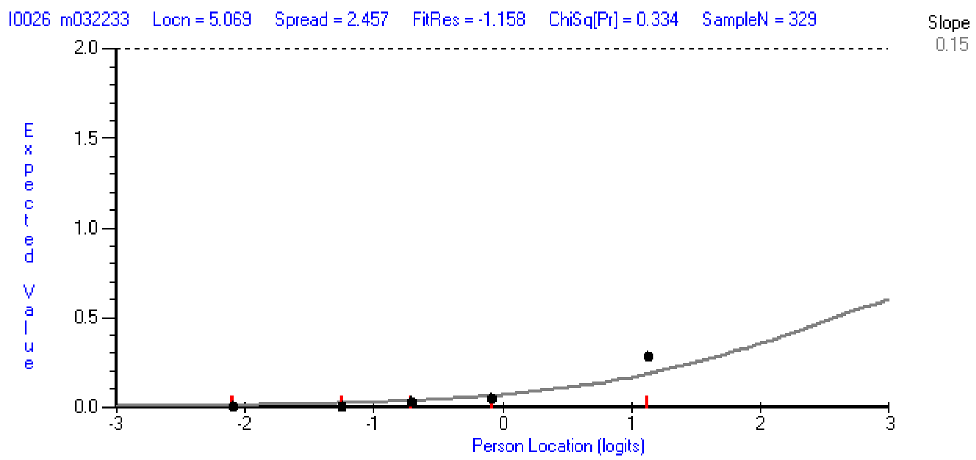
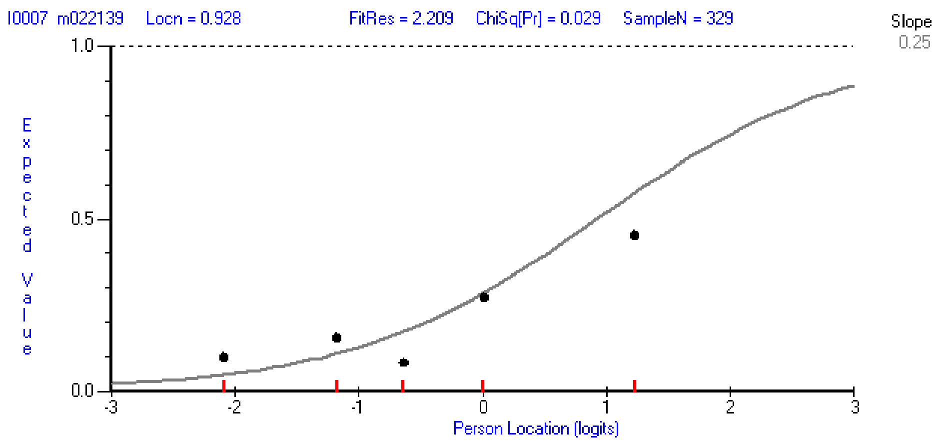


Figure 24: Item 7: Item characteristic curve indicating quintile scores[4].



[4]Item 7 is presented in comparison with Item 26 to show the difference between an item functioning as expected and an itemnot properly targeted for the population being tested.

B4: Probability item analysis (see Section 7.8, thesis document)

Level 2, 3 and 4 probability item analysis: Description and comparison

The problem in **Item 11 (location -1.402)**, is to identify the fraction part associated with the least likely probability (see Table 41). The first hidden question is "Which of the following fractions parts has the smallest area?" Once this answer has been established, the learner has to associate "least likely" with the smallest area. Unfamiliarity with a spinner means that the context may not support correct reasoning. The part-whole rational number sub-construct is required for this problem. Over half the group, 64% answered this item correctly.

Item 14 (location -0.416) requires learners to connect the notion of a probability with that of a fraction (see Table 43). The way that the problem is posed is more difficult than "Find $\frac{1}{5}$ of 30?", though it may be that without necessarily understanding the question, some learners may have correctly used this procedure. The underpinning rational number subconstruct is that of the operator. Of the group, 46% answered correctly.

Item 19 (location 0.073) included concepts of random sampling, ratio and proportion, in addition to a ratio understanding of rational number (see Table 45). Given a frequency count of 45 boys in a sample of 100, the problem was to project this probability on to the population of 1200. Some 32% of the group as a whole answered correctly.

Distractor analysis by quartile

While 64% of learners answered **Item 11** correctly, **Option B**, the breakdown by quartiles indicates that 84% of the highest quartile answered correctly, but only 40% of the lowest performing quartile (see Table 42). The middle quartiles are unexpectedly inverted with the 75% of second lowest performing quartile answering correctly and 58% of the second highest performing quartile answering correctly. This unexpected response may be due to some chance factor in the relatively small group of 83 learners, or it may reflect an intuitive approach on the part of some learners who have not exhibited proficiency in other areas of mathematics. It has been noted that probabilistic thinking may differ in some respects from proportional reasoning. Interestingly the most selected incorrect **Option C** (19% of learners) was the largest fraction of the spinner, $\frac{1}{2}$, and the *most likely*, rather than the smallest fraction of the spinner, $\frac{1}{24}$, which was the *least likely*.

Item 14 was answered correctly by 46% of learners **Option E** (6) (see Table 44). However, a high 91% and 70% of learners in the highest, and second highest quartiles answered correctly in each of the lower two quartiles only 11% answered correctly.

It has been noted that **Item 19** (see Table 46) was removed from the calibration process for reasons given, however the information that can be gleaned from the item is worth considering. For all quartile groups the most selected response is the incorrect **Option A** (450). Some 48% of the highest quartile selected this option, while only 43% of the highest quartile selected the correct **Option C** (540).

Probability Item 11: location -1.40 (level 2)

Table 41: Item 11 description

Item 11 description	Mathematical Structure	Response process
The figure below shows a spinner with 24 sectors. When someone spins the arrow, it is equally likely to stop on any sector.  $\frac{1}{8}$ of the sectors are blue, $\frac{1}{24}$ are purple, $\frac{1}{2}$ are orange, and $\frac{1}{3}$ are red. If a person spins the arrow, on which colour sector is the spinner LEAST likely to stop? A. blue B. purple C. orange D. red	Equally likely $\frac{1}{24}$, therefore LEAST likely the lessor fraction. $\frac{1}{2} > \frac{1}{3} > \frac{1}{8} > \frac{1}{24}$	Part-whole concept Recognises that fraction area is related to probability

Figure 25: Item 11: Multiple-choice distractor plots by quartile group

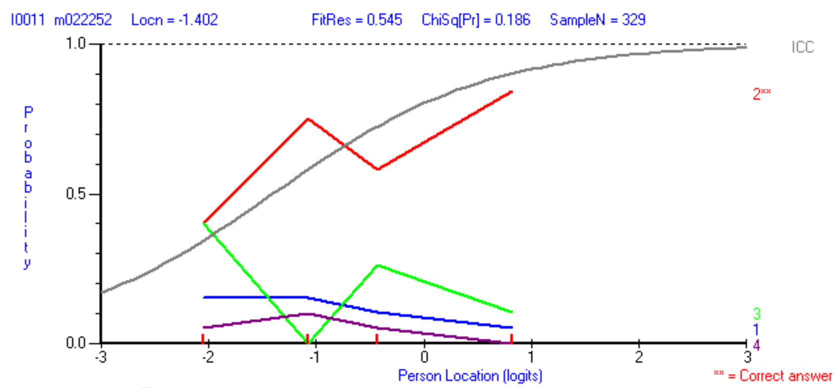


Table 42:Item 11 inferred procedures formultiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-2.048	-1.071	-0.443	0.825
B. purple	64%	red	Associated least likely with $\frac{1}{24}$	40%	75%	58%	84%
C. orange	19%	green	Selected $\frac{1}{2}$, confused *most likely* with *least likely*	40%	0%	26%	11%
A. blue	12%	blue	$\frac{1}{8}$	15%	15%	11%	5%
D. red	5%	purple	$\frac{1}{3}$	5%	10%	5%	0%
Total				**100%**	**100%**	**100%**	**100%**

Probability Item 14: location -0.416 (level 3)

Table 43: Item 14 description

Probability item	Mathematical Structure			Response process
14. In a Grade 8 class of 30 students, the probability that a student chosen at random will be less than 13 years old is $\frac{1}{5}$. How many students in the class are less than 13 years old? A. Two B. Three C. Four D. Five E. Six	$P = \frac{1}{5}$ Count = 30 $\frac{1}{5} = \frac{x}{30}$	**M_1** 1 5	**M_2** x *30*	Identifies fraction-probability link, applies the multiplicative operator

Figure 26: Item 14: Multiple-choice distractor plots by quartile group

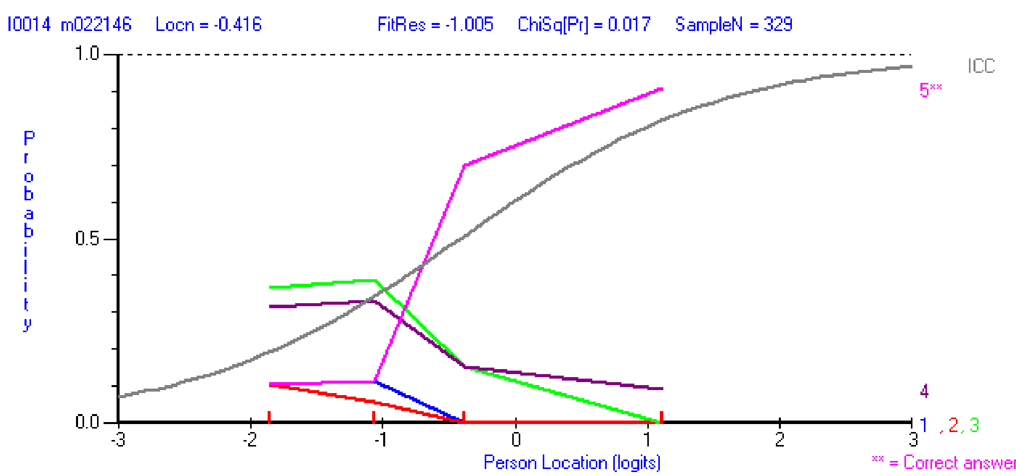


Table 44: Item 13 inferred procedures formultiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-1.852	**-1.062**	**-0.386**	**1.117**
E. 6	46%	pink	$\frac{1}{5} \times 30 = 6$	11%	11%	70%	91%
D. 5	22%	purple	Estimate	32%	33%	15%	9%
C. 4	23%	green	Estimate	37%	39%	15%	0%
A. 2	6%	blue	Guess	11%	11%	0%	0%
B. 3	5%	red	Guess	11%	6%	0%	0%
Total				**100%**	**100%**	**100%**	**100%**

Probability Item 19: location 0.073 (level 4)

Table 45 Item 19 description

Item 19	Mathematical Structure			Response process
19. In a school there were 1200 learners (boys and girls). A sample of 100 learners was selected at random, and forty five boys were found in the sample. Which of these is most likely to be the number of boys in the school? A. 450 B. 500 C. 540 D. 600	45: 100 :: x : 1200	M_1 45 100	M_2 x 1200	Recognises proportional relationship, relates this relationship to prospective probability

Figure 27: Item 19: Multiple-choice distractor plots by quartile group

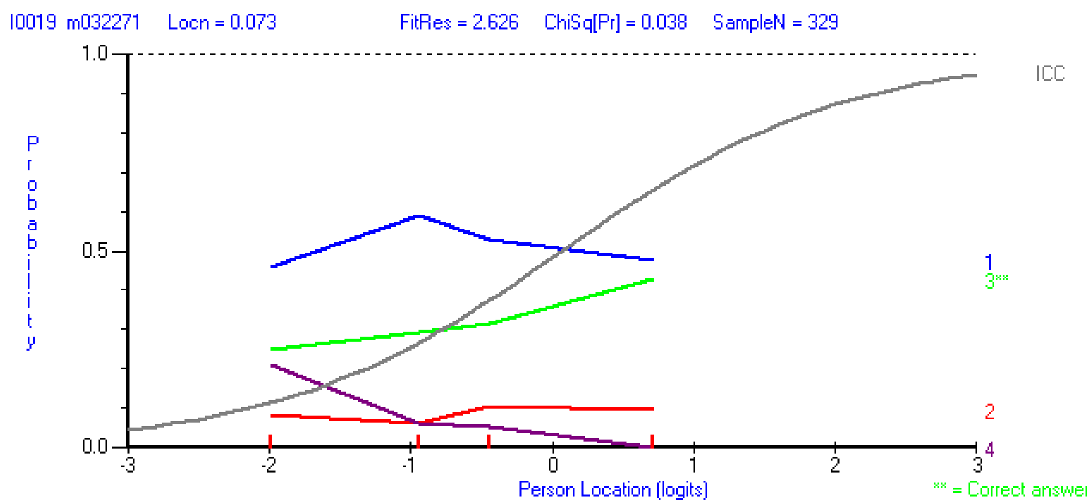


Table 46: Item 19 inferred procedures formultiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-1.993	**-0.946**	**-0.448**	**0.716**
C. 540	32%	green	45:100 :: x : 1200 45: 100, 90: 200, 450: 1000	25%	29%	32%	43%
A. 450	52%	blue	45 x 100	46%	59%	53%	48%
B. 500	9%	red	Less than half	8%	6%	11%	10%
D.600	8%	purple	About half	21%	6%	5%	0%
Total				**100%**	**100%**	**100%**	**100%**

B5: Pre - Algebra item analysis (see Section 7.8, thesis document)

Level 2 Pre-algebra item analysis: Description and comparison

Item 6 (-1.423) and **Item 12 (-1.412)** are located at almost the same difficulty level (see Table 47, and Table 49). Both items require reasoning about comparative relationships, however it is the direction of change, rather than a calculation that is required. **Item 6** uses terms such as "10 more than" and "more women than men", whereas in **Item 12** we have the terms "most often" or "fewer … than". Some 67% overall answered **Item 6** correctly and 63% **Item 12**. The use of natural language in **Item 6**, and the use of graphical representation in **Item 12** constitute a notable difference between the two items.

Item 23 (location -1.195) (see Table 51) requires reasoning about an equivalence relationship in the context of a scale balance. This problem can be solved by trial and error, substituting amounts for the unknown object. An algebraic approach is not necessary, though would be useful if the number range was higher.

Distractor analysis by quartile or quintile group

For **Item 6**, some 90% if the highest quartile answered correctly, **Option A,** with 74% in the second highest quartile, 61% in the second lowest quartile, and 43% for the lowest quartile (see Table 48). The incorrect option selected most frequently was that of **Option D** (*From the information given, you cannot tell whether there would be more women than men*). This selection suggests that this group of learners were confused by the comparative language. A similar pattern can be discerned for **Item 12**(see Table 50),and for **Item 23**(see Table 52).except that more of the lower two groups were distracted by the incorrect options.

Critical points and threshold concepts

The highest two quartiles of this group, show proficiency with problem situations, requiring pre-algebraic reasoning, though in these items the number range is low and the fraction values simple, thereby rendering the problems less difficult than they might be.

Pre-Algebra: Item 6: location -1.423 (level 2)

Table 47: Item 6 description

Pre-algebra	**Mathematical structure**	**Response process**
6.Two thirds of the people present at the beginning of a meeting are men. Nobody leaves but 10 more men and 10 more women arrive at the meeting. Which of the following statements are true? A. There would be more men than women at the meeting. B. There would be the same number of men as there are women at the meeting. C. There would be more women than men at the meeting. D. From the information given, you cannot tell whether there would be more women than men.	$\frac{2}{3}+\frac{1}{3}=1$ (Men + women is whole) $\frac{2}{3}>\frac{1}{3}$ $\frac{2}{3}(people)+10>\frac{1}{3}(people)+10$ $a+b=c$ If $a>b$ then $a+z>b+z$	Reasoning about the direction of change.

Figure 28: Item 6: Multiple-choice distractor plots by quartile group

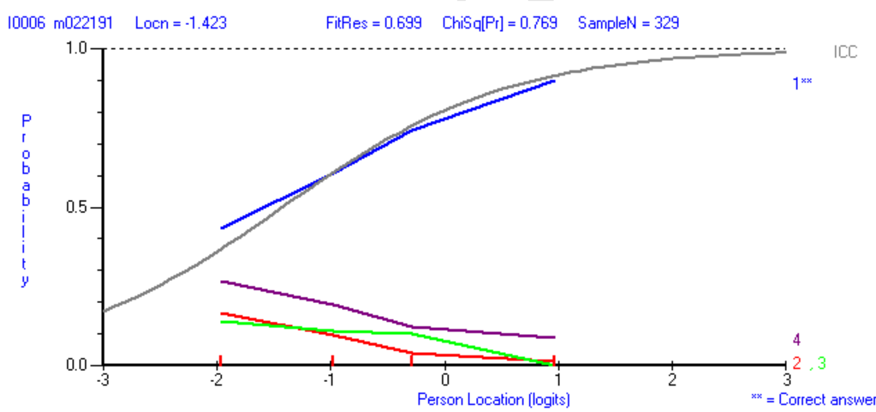


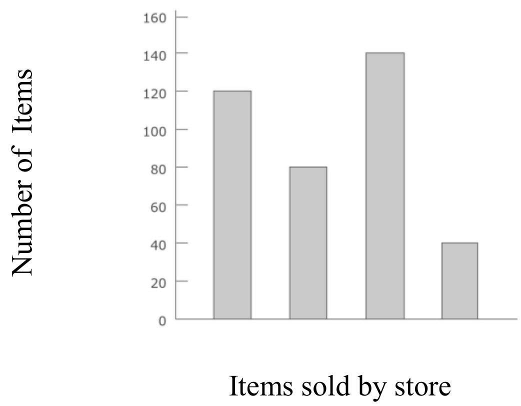
Table 48: Item 6 inferred procedures formultiple-choice responses

Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of 4 quartile groups**			
				-1.964	**-0.981**	**-0.291**	**0.969**
A.	67%	blue	$\frac{2}{3}(+10)>\frac{1}{3}(+10)$	43%	61%	74%	90%
D.	17%	purple	Confusion	27%	19%	12%	9%
B.	8%	red	Missing initial relationship of men to women	16%	10%	4%	1%
C.	9%	green	Guess	14%	11%	10%	0%
Total				**100%**	**100%**	**100%**	**100%**

Pre-Algebra: Item 12 (location -1.412) (level 2)

Table 49: Item 12 description

The graph shows the number of pens, pencils, rulers, and erasers sold by a store in one week.



The names of the items are missing from the graph. Pens were the most often sold, and fewer erasers than any other item were sold. More pencils than rulers were sold. How many pencils were sold?

A. 40 B. 80 C. 120 D 140

Figure 29: Item 12: Multiple-choice distractor plots by quartile group

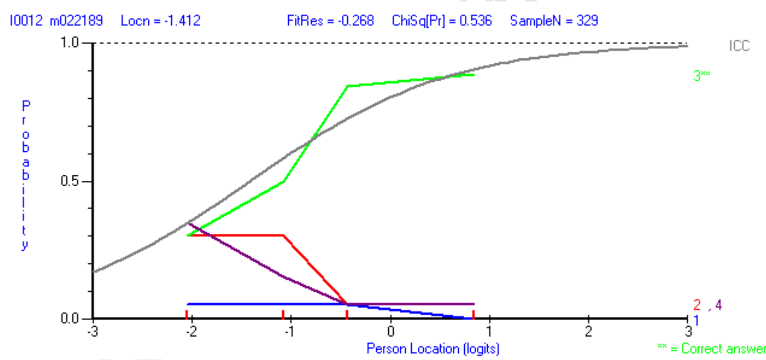


Table 50: Item 12 inferred procedures formultiple-choice responses

Option	Total	Line	Inferred mathematical procedure	Mean locations of quartile groups			
				-1.852	-1.062	-0.386	1.117
C.120	63%	green	Correct reading of relationships	30%	50%	84%	89%
B. 80	18%	red	Incomplete information	30%	30%	5%	6%
D.140	15%	purple	Guess	35%	15%	5%	6%
A.40	4%	blue	Guess	5%	5%	5%	0%
Total				**100%**	**100%**	**100%**	**100%**

Pre-Algebra: Item 23: location -1.195 (level 2)

Table 51: Item 23 description

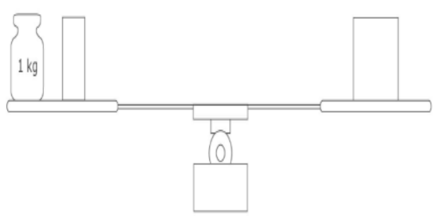
Pre-algebra item	**Mathematical structure**	**Response process**
23. The objects on the scale make it balance exactly. On the left pan there is a 1 kg weight (mass) and half a brick. On the right pan there is a brick. What is the weight (mass) of one brick? A. 0.5 B. 1 kg C. 2 kg D. 3 kg	1kg (mass) + $\frac{1}{2}$ brk (mass) = 1 brk (mass) $\frac{1}{2}$ brk + $\frac{1}{2}$ brk = 1 brk $\therefore$ $\frac{1}{2}$ brk = 1 kg $\therefore$ 1 brk = 2 kg	Reasoning about equivalence
	If $a + b = c$ and $d + b = c$, then $a = d$	

Figure 30: Item 23: Multiple-choice distractor plots by quartile group

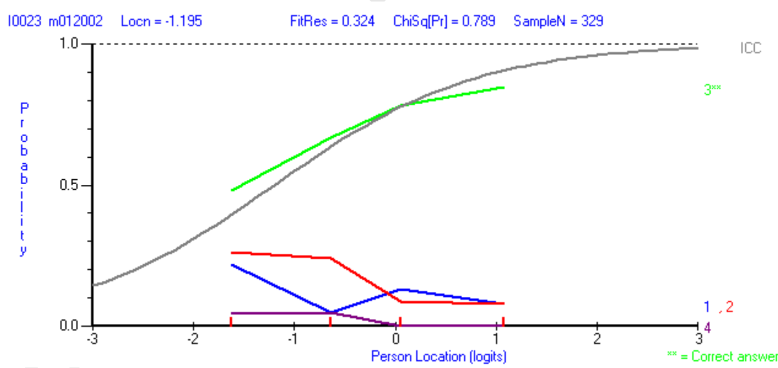


Table 52: Item 23 inferred procedures formultiple-choice responses

Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quartile groups**			
				-1.852	**-1.062**	**-0.386**	**1.117**
C. 2kg	70%	green	Reasoning with unknowns	48%	67%	78%	85%
B. 1 kg	17%	red	Considering only initial information	26%	24%	9%	8%
A.0.5 kg	12%	blue	Misunderstanding of balance scale	22%	5%	13%	8%
D.3 kg	4%	purple	Rough guess	4%	5%	0%	8%
Total				**100%**	**100%**	**100%**	**100%**

Level 3, 4 and 5 Pre-algebra item analysis: Description and comparison

Item 21 (location -0.877)(see Table 53) was calibrated as being slightly more difficult than **Item 23**, with 50% of learners answering correctly. The response process involved analysing the three figure pattern and then extending the pattern to the 10th figure. Simply counting and adding 3 matches for each additional figure would have achieved the correct answer. A more sophisticated algebraic approach may also have been used but was not required for this particular problem. A step up in estimated difficulty is **Item 25 (location -0.456)**(see Table 55). This problem requires finding common elements in two sequences. Again, because of the relatively low number range, this problem can also be solving by counting or adding. The multiplicative relationships though efficient were not necessary.

Items 27, 28 and 29(see Table 56) constitute consecutive questions on the same geometric figure. The questions require increasing levels of proficiency. The two part question could be answered through counting blocks (the first part), and then extending the drawing and counting again (the second part). However, for the third question an explanation is required which involves generalising the solution.

Analysis by quartile or quintile group

A distractor analysis of **Item 21** shows that 85% of the highest quartile answered correctly, **Option B**, but 0% of the lowest quartile (see Table 54). A similar pattern presents for **Item 25** with almost 80% of the highest quartile answering correctly, and with 10% of the lowest quartile answering correctly (see Figure 32). **Item 27**, **Items 28** and **Item 29** are linked to the same problem situation, with **Item 27** the most straightforward question and thereafter increasing in difficulty. This result is reflected in the learner responses (see Item characteristic curves, Figure 33, 34 and 35).

Critical findings

As these pre-algebra items increase in difficulty from Level 2 to Level 3, the highest quartile percentage correct decreases from 90% to 80%. However, for the lowest quartile there is a radical slide from 40% to close to zero. The ability to answer correctly for the highest group does not extend to **Item 28**, and **29**. Challenging items are needed to extend learners in the highest sectors.

Pre-Algebra: Item 21: location -0.877 (level 3)

Table 53: Item 21 description

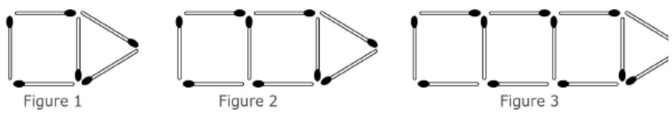
Pre-algebra item	**Mathematical structure**	**Response process**
Matchsticks are arranged as shown in the figures.  If the pattern is continued, how many matchsticks would be used to make figure 10? A. 30 B. 33 C. 36 D. 39 E. 42	**1 2 3 4 …. 10** **6 9 12** $3+(1\times 3)$ $3+(2\times 3)$ $3+(3\times 3)$ $\mathbf{3+(n\times 3)}$ $n=10$ $3+(10\times 3)=33$	Counting, or adding, reasoning multiplicatively or algebraically

Figure 31: Item 21: Multiple-choice distractor plots by quartile group

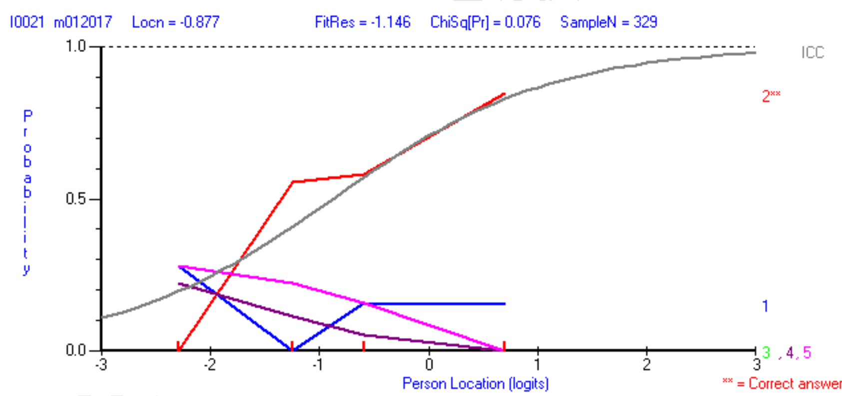


Table 54: Item 21 inferred procedures formultiple-choice responses

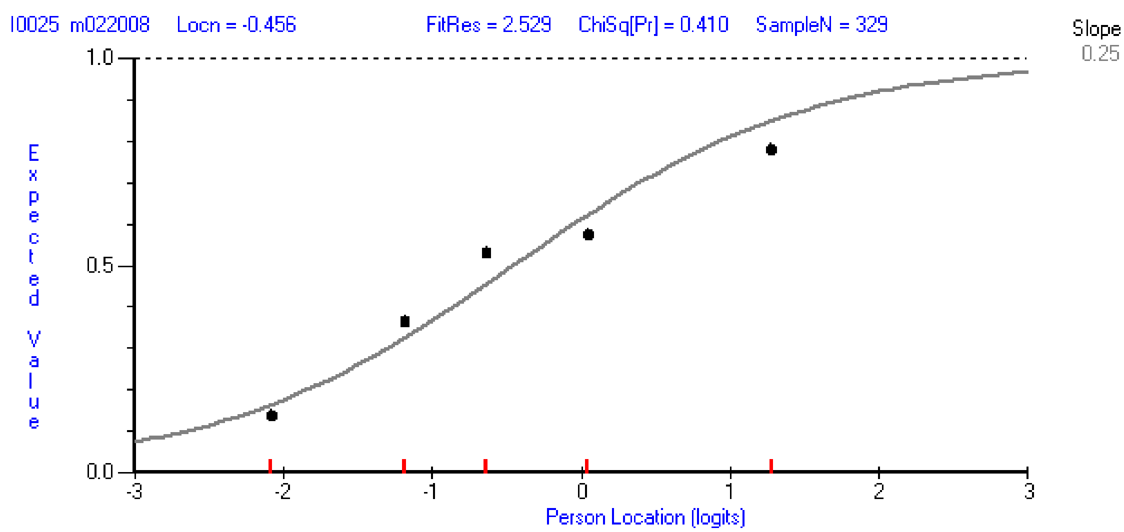
Option	**Total**	**Line**	**Inferred mathematical procedure**	**Mean locations of quartile groups**			
				-2.289	**-1.248**	**-0.596**	**0.695**
B	50%	red	Counting up in threes, or investigating mathematical structure	0%	56%	58%	85%
A	15%	blue	Miscounting	28%	0%	16%	15%
E	16%	pink	Miscounting	28%	22%	16%	0%
D	10%	purple	Miscounting	22%	11%	5%	0%
C	10%	green	Miscounting	22%	11%	5%	0%
Total				**100%**	**100%**	**100%**	**100%**

Pre-Algebra: Item 25: location -0.456 (level 3)

Table 55: Item 25 description

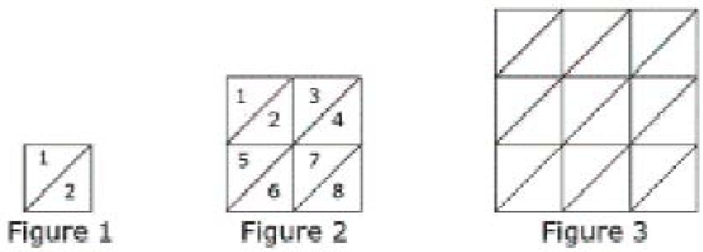
Pre-algebra	**Mathematical structure**	**Response process**
25. The numbers in the sequence 7, 11, 15, 19, 23, … increase by four. The numbers in the sequence 1, 10, 19, 28, 37, … increase by nine. The number nineteen is in both sequences. If the two sequences are continued, what is the next number that is in BOTH the first and the second sequences? Answer: …………………..	**7 11 15 19 23 …** (increase by 4) **1 10 19 28 37 . .** .(increase by 9) 19 + (4 × 9) - - - - - - - - - - Common multiple + ($r_1 \times r_2$) ??	Identifying the pattern, reasoning about common factors

Figure 32: Item 25: Item characteristic curve indicating quintile scores



Pre-Algebra: Items 27 (-0.251), 28 (1.296) and 29 (0.920) (Level 4, 5 and 7)

Table 56: Item 29 description

<table>
<tr><th>Pre-algebra Item 27, 28 and 29</th><th>Mathematical structure</th><th>Response process</th></tr>
<tr><td>The three figures below are divided into small congruent triangles,
</td><td></td><td></td></tr>
<tr><td>Complete the table below. First, fill in how many small triangles make up Figure 3. Then find the number of small triangles that would be needed for the fourth figure if the sequence of figures was extended.
<table><tr><th>Figure</th><th>Number of Small Triangles</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>8</td></tr><tr><td>3</td><td></td></tr><tr><td>4</td><td></td></tr></table></td><td>1 2 3 4 5 …
2 8 18 32

1 2(1 × 1)
2 2(2 × 2)
3 2(3 × 3)
4 2(4 × 4)

Number of triangles = 2($n \times n$)</td><td>Counting, identifying the pattern</td></tr>
<tr><td>B. The sequence of figures is extended to the 7th figure. How many small triangles would be needed for Figure 7?

Answer: ………………………</td><td>1 2 3 4 5 …
2 8 18 32

1 2(1 × 1)
2 2(2 × 2)
3 2(3 × 3)
4 2(4 × 4)
…
7 2(7 × 7)</td><td>Counting, identifying the pattern, Extending the pattern</td></tr>
<tr><td>C. The sequence of figures is extended to the 50th figure. Explain a way to find the number of small triangles in the 50th figure
that does not involve drawing and counting the number of triangles.</td><td>1 2 3 4 5 …
2 8 18 32

1 2(1 × 1)
2 2(2 × 2)
3 2(3 × 3)
4 2(4 × 4)
n 2($n \times n$) 50
2(50 × 50) = 5000</td><td>Counting, identifying the pattern, Extending the pattern Generalising</td></tr>
</table>

Figure 33: Item 27: Item characteristic curve indicating quintile scores

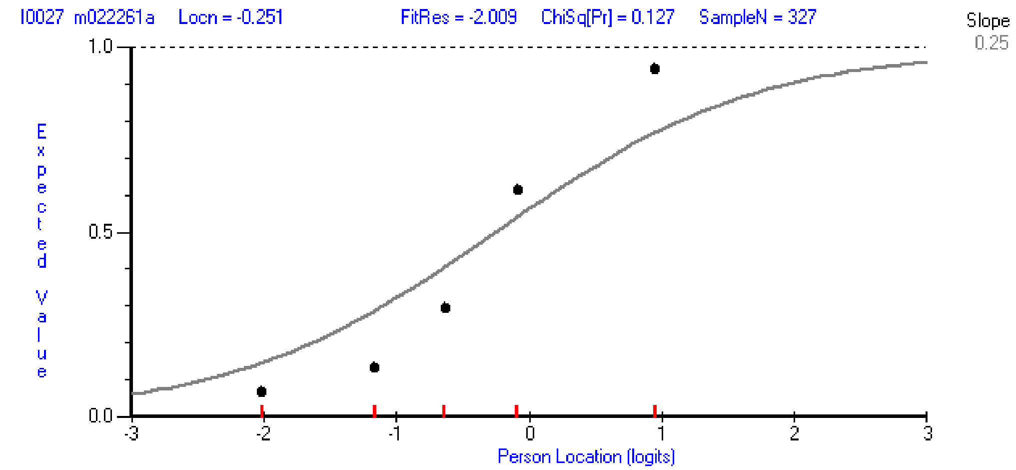


Figure 34: Item 28: Item characteristic curve indicating quintile scores

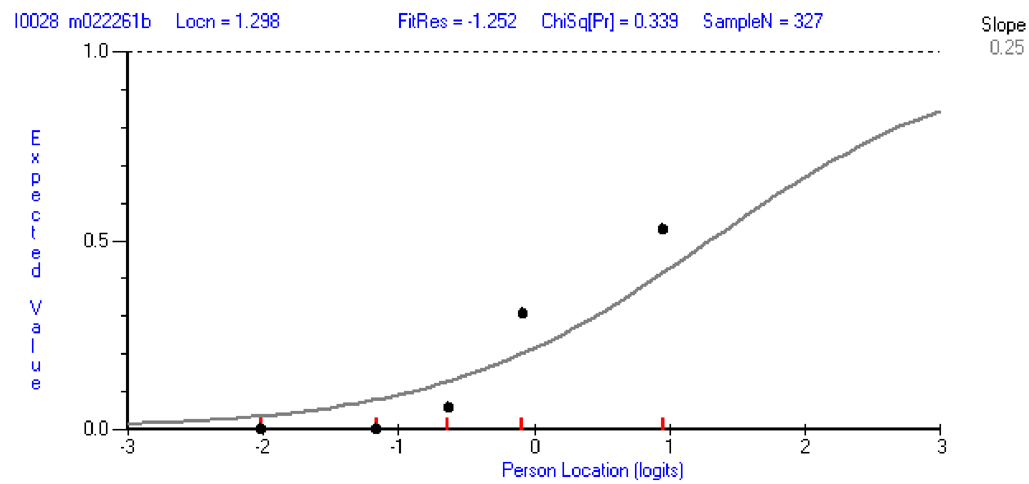
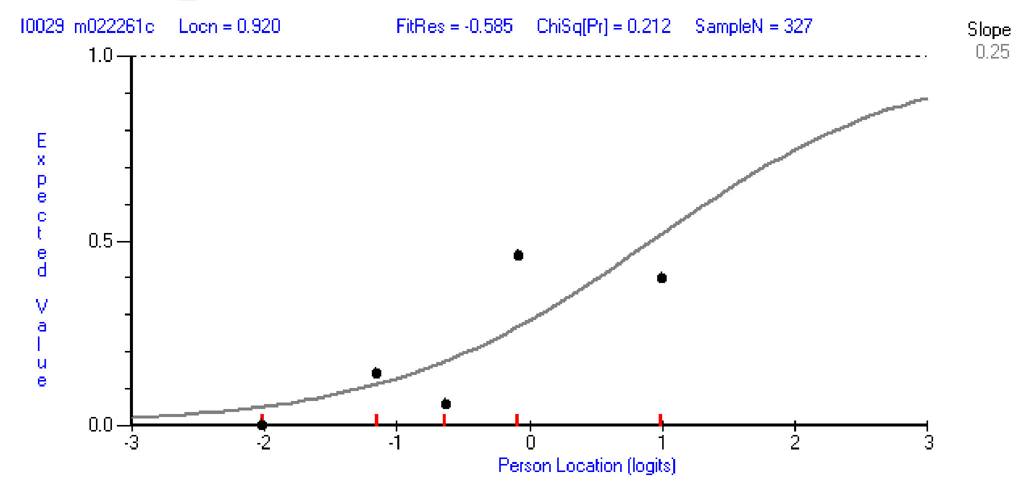


Figure 35: Item 29: Item characteristic curve indicating quintile scores



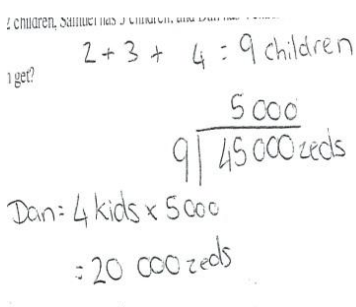
Appendix C: Chapter 8, Interview analyses and summaries

School A – High proficiency group

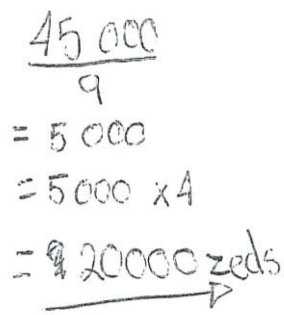
Table 57: Item 5, top group of School A

Item 5[5]: Three brothers Thabo, Samuel and Dan, receive a gift of 45 000 zeds from their father. The money is shared between the brothers in proportion to the number of children each one has. Thabo has two children, Samuel has three children, and Dan has 4 children. How many zeds does Dan get?

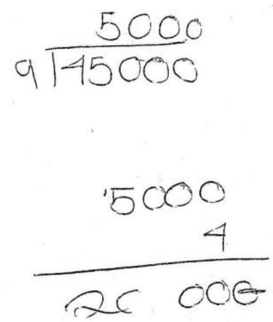
Adele, (4.13 logits)



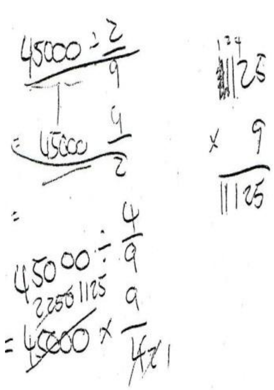
Kelly, (1.86 logits)



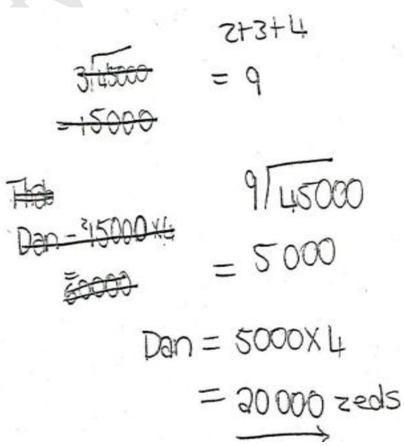
Jane, (1.41 logits)



Angela, (1.86 logits)



Carla, (1.38 logits)



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5 The numbers of the items in the scripts have been changed to align with the Rasch analysis numbers.

Table 58: Transcription, Item 5 top group of School A

Speaker	Transcript	Comments
Interviewer	*Let us look at number [5] … What was the answer? What did each of you get?*	
All:	20,000	
Angela	I didn't finish. I just guessed 15,000. *Ok, so why didn't you finish the sum? What did you find difficult about it?* I just forgot how to work out the ratios because we learnt how to do them. It was a long time ago. We knew about ratios, but I forgot. *It was a while ago.* It was last year this time. *So what do you understand about the word proportion?* It is the amount each person gets. It is like a fraction. *Likea fraction ...*	Angela was in the top group located at 1.86 logits according to the test. The question is whether she had learnt these procedures by rote or whether there is some other explanation for her difficulty in the interview situation.
Interviewer	*Who would like to volunteer? Who would like to explain how you worked this out?*	
Carla	You divide 40,000 by nine. And then you get 5000. *You mean 45,000. Why did you divide by nine?* Because you count all the children together, which is nine children, and then 45,000 is the amount the father has given them. *Then each child would get 5000 and what is the question?* How much will Dan get? He has four children, so four times 5000	Carla explains the procedure she uses.
Interviewer	*Is there another way to do this? Did anyone do this another way?*	
Kelly	I did the same thing. I first divided by nine then timesed by 4.	Written calculation shows sloppy use of the equals sign
Adele	I divided by nine and multiplied by four	Adele appears to have a schema for dealing with problems of this type. A next step might be to consolidate, "divide by nine and multiply by four into $\frac{4}{9} of 45000$

Table 59:Item 8, top group of School A

Item 8: A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds?		
Adele, (4.13 logits) 100% + 20% = 120% $\frac{120}{1\cancel{00}} \times \frac{8\cancel{00}}{1} = 960$ zeds	Kelly, (1.86 logits) $^{3}(800 \times 20) \div 100$ $= 16\,000 \div 100$ $= 160$ $= 160 + 800$ $= 960$ zeds	Jane, (1.41 logits) $\frac{100+20}{\cancel{100}} \times \frac{\cancel{800}}{1}$ 120 $\frac{\cancel{800}}{1} \times \frac{100+20}{\cancel{100}}$ $\frac{{}^{1}120}{\times 8}$ 960
Angela, (1.86 logits) ~~800 ÷ 20 × 100~~ = ~~400 × 100~~ ~~800 × 100 ÷ 20~~ = ~~400~~ = 4 $\frac{800}{1} \times \frac{20}{100}$ $= \frac{8}{1} \times \frac{20}{1}$ $= 160$ $= 800 + 160 = 960$ zeds	Carla, (1.38 logits) ~~$\frac{120}{100} \times \frac{800}{1}$~~ 7 $\frac{{}^{1}120}{\cancel{100}} \times \frac{\cancel{800}^{8}}{1}$ $= 960$ zeds	

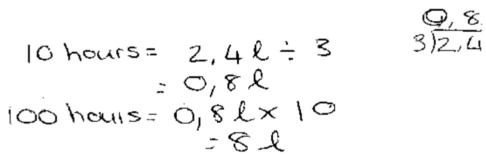
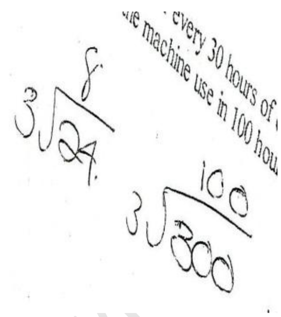
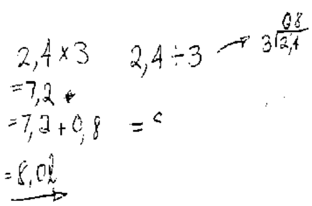
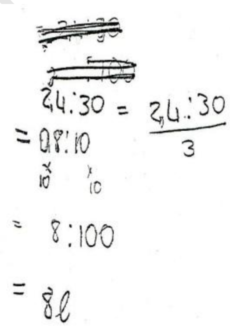
Table 60: Transcription, Item 8, top group of School A

Speaker	Transcript	Comments
Interviewer	*Let us go to the next one, Number [8]. Angela, did you find this one a little bit easier?*	
Angela	No this one was worse	
Interviewer	*We are looking at [8]. Did you get the answer 960?*	The group sat around the table. The interviewer was aware that this item was straightforward for all except Angela.
All, (except Angela)	Yes	
Angela	*What do you understand by percent? Let me start*	Angela has the procedure stored in her memory but is

Speaker	Transcript	Comments
	with you Angela. It is how much out of the hundred... The amount of something from a hundred *Okay, and then when there is an increase of 20%?* You work out 20% of the given amount ... the increase… and then to add to the amount you had before.	unable to recognise the concepts in the problem. The part-whole understanding is not adequate for this problem.
Interviewer	*And so did you all do that? Did you all find ....(the percentage first)?*	
Adele	No *Can you explain what you did Adele?*	
Adele	I said 120 over a hundred. I first added to get the percent. I added 20 onto the full price, the hundred, and then I timesed it by 800. *So what is the hundred percent?* Because you're increasing it. The amount is already a hundred over a hundred and then to increase it you add another 20 onto that to get the percentage.	"The amount is already a hundred over a hundred". You add another 20 on" to make it a ratio of 120 over a hundred. This concept would have been worth exploring.
Kelly	I've worked it out a different way. I worked out percentage first and then I divided by hundred, and then I plussedeight hundreds to that percentage *Can I have a look? You've multiplied 800 x 20 and then you divide it by hundred... OK I understand what you did. Why did you multiply by 20? Is this a method that you have learned?* That's how you get the percentage. If you multiply it by that (20) you get the percentage then you divide by a hundred. *So you are finding 20 parts out of a hundred.* Kelly (laughs) Am I?	The procedure used by Kelly (see above) may work in some instances, but doesn't make sense conceptually. The term percentage is being used for the amount obtained when multiplying by 20 (without dividing by 100) and is therefore incorrect. The term percentage is generally used for the target quantity.
Jane	I am sure mine is the same just written differently. *What you have done is multiply by 20 and divide by hundred. You've done the same. You've just done it in a different form*	Jane starts by finding 20% of 800, but then adapts her method to the more sophisticated, 120% of 800.

Table 61:Item 10, top group of School A

Item 10: A machine uses 2,4[6] litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use in 100 hours**?**

Adele, (4.13 logits) 		Jane, (1.41 logits) 
Kelly, (1.86 logits) 	Angela, (1.86 logits) 	Carla, (1.38 logits) 

6 The comma notation for decimals is used in South African schools. In the interviews this notation will be used.

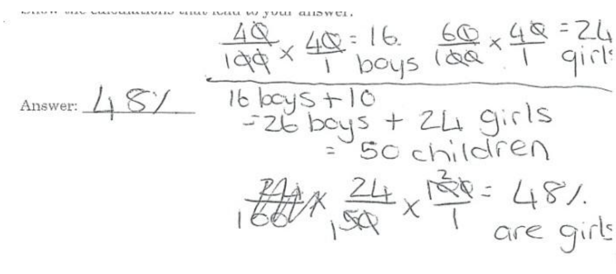
Table 62:Transcription, Item 10, top group of School A

Speaker	Transcript	Comments
Interviewer	*The next one - Number [10]. This one has decimals in it. Maybe this makes it more difficult? Who got the first answer? Who got 7,2?* Angela, you got 7,2 *The rest of you all got 8. 8 is correct.*	
Jane:	I moved the comma, and made it 24 and 300 and I divided by three to get 100 and then divided 24 by 3 to get eight. .... *So you said 24 litres for every 300 hours and then for 100 hours you divided 24 by 3, and you got eight?*	
Interviewer	*OK that is quite an efficient way of doing it. How did you do this, Adele?*	
Adele	I said 2,4 litres, divide by three (gives you 0,8) to get 10 hours. And then I multiplied 0,8 by 10 to get 100 hours	
Kelly	I worked it out kind of separately, I said 2, 4 times three because that's what be 90 hours and then I've got 7,2. Then I said 2, 4 divided by three which was 0, 8 so then it was 7,2 plus 0, 8 equals 8.0	Although less efficient Kelly understands what she is doing.
Interviewer	*Okay, Angela did you get stuck.*	
Angela	I did at totally differently, I found the percentage, don't ask me why.	

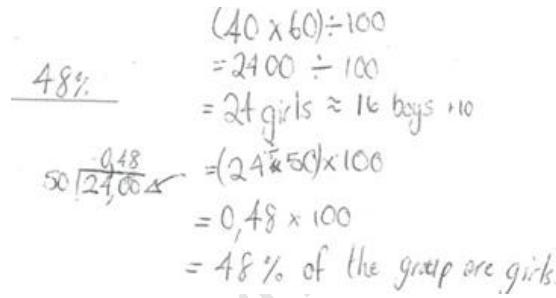
Table 63: Item 26, top group of School A

Item 26: A computer club had 40 members, and 60% of the members were girls. Later, 10 boys joined the club.

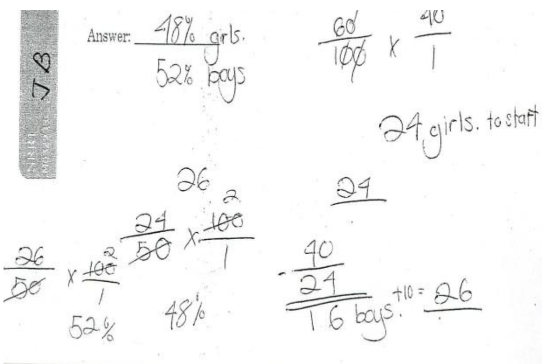
Adele, (4.13 logits)



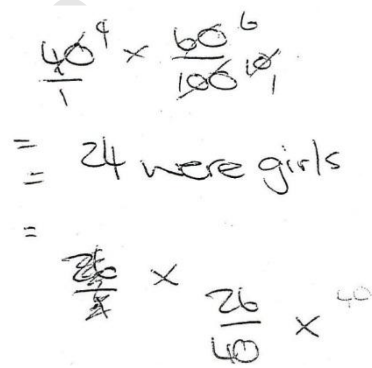
Kelly, (1.86 logits)



Jane, (1.41 logits)



Angela, (1.86 logits)



Carla, (1.38 logits)

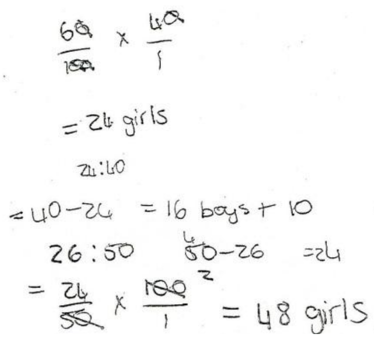


Table 64:Transcription, Item 26, top group of School A

Speaker	Transcript	Comments
Interviewer	*Number [26]. This problem had two steps to it and the first step ...... what was the answer to the first step?*	
Jane	24 *24 girls*	
Interviewer	*How did you get that?*	
Angela	I said forty (40) over one times by sixty (60) over a hundred (100) and then I got stuck. I didn't know how to do the boys.	
Interviewer	*You got 60% of the 40 and that was 24.* *And later 10 boys joined the club now, How does that change things?*	
Adele	40% boys at the beginning so that was 16 boys and then I added 10 that make 26. 24 were girls that made 50 children 24 girls in 50 children made 48%.	

School B – High proficiency group

Table 65: Item 5, top group of School B

Item 5: Three brothers Thabo, Samuel and Dan, receive a gift of 45 000 zeds from their father. The money is shared between the brothers in proportion to the number of children each one has. Thabo has two children, Samuel has three children, and Dan has 4 children. How many zeds does Dan get?	
Anna (2.37 logits),	45 000 $\boxed{2 \quad 3 \quad 4}$ 9 $9\overline{)45000}$ = 5000 Dan = 4 4 x 5000 = 20 000
Thembani, (0.601 logits)	$9\overline{)45}$ 5 4 x 5 = 20 3 x 5 = 15 2 x 5 = 10
Sipho (0. 477 logits)	No attempt

Table 66:Transcription, Item 5, top group of School B

Speaker	Transcript	Comments
Interviewer	*Now that you have finished, I would like to discuss how you approached the problem. Anna?*	
Anna	I first thought about ratio and proportion, and then I thought about adding up all the children… So I could work out how much one child would get. And then I worked it out to be five thousand. And then they asked how many zeds would Dan get … so I multiplied this by 4, and I got 20 000.	Anna immediately categorises the problem as a ratio and proportion problem. And she has a developed schema and hence procedure to apply.
Interviewer	And did you get the same answer, Thembani?	
Thembani	First of all I added all the children of Samuel, Thabo and Dan, which added up to 9. So what I said was 45 000 divided by nine equal 5. So what I said was the children I timesed by five, how many children each one had I timesed it by five. So what I did was ask, how many did Dan have. I said 4 times 5 which equals 20.	In comparison with Anna, Thembani does not immediately identify the type of problem. However, he understands the problem and solves it correctly.
Interviewer	Good, and you, Sipho,	
Sipho*	I didn't answer it.	This was not followed up by the researcher.
	*Sipho's reluctance to participate is difficult to explain. A check of his script seven months earlier shows the correct choice of distractor for this item. He obtained 12 correct answers out of 17. In fact he gets 2 marks for one of the difficult items, where he has to explain the generalisation of a pattern, something not many children got correct. However, three of the incorrect answers are the next three interview items. A check of the Rasch analysis shows a fit residual of 0.032, the Guttman pattern is fine. It could be that he objected to staying after school.	

Table 67: Item 8, top group of School B

Item 8: A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds?	
Anna (2.37 logits),	$\frac{80\cancel{0}}{1} \times \frac{\cancel{1}20}{10\cancel{0}}$ $= 160$ $800 + 160$ $= 960$zeds
Thembani, (0.601 logits)	$800 \div \cancel{20}\,5 = 160$ $800 - 160 = 640$
Sipho (0. 477 logits)	No attempt

Table 68:Transcription, Item 8, top group of School B

Speaker	Transcript	Comments
Interviewer	*Let's go to the next item*	
Anna	"A shop increased its prices by 20% … I thought of interest and financial maths. So I thought how much 20% of 800 is, and that is 160, and I added 160 to the 800 and got R960.	In both Item 5 and Item 8 Anna calls on the topic category. She is very clear about what she has to do and uses appropriate strategies.
Interviewer	*And what was your answer, Thembani?*	
Thembani	I made a mistake. Instead of adding I subtracted. *Now why do you think you did that? Because that is what a lot of people did.* I don't know, maybe it's because of doing the sum. I just forgot that I had to add. *So what you did was find the discount, instead of the increase. That is an easy thing to do, because a lot of your sums at school are like that (discount sums). It was a case of not reading the sum properly. You understand the idea.*	Thembani converts 20% to a fraction 1/5 but then subtracts instead of adding.
Interviewer	*And you, Sipho?*	
Sipho	I needed to get the percentage so I divided 900 twice which gave me 50%, then I divided 500 twice which gave me 25%. This gave me 222. From 222… I subtracted this from 900.	Sipho divides 900 by two and gets 500 and then by 2 again and gets 222. Clearly Sipho is lacking basic skills, though still a mystery concerning his location

Table 69: Item 10, top group of School B

Item 10: A machine uses 2,4 litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use in 100 hours?	
Anna (2.37 logits),	(A) 7,2 (B) 8,0 (C) 8,4 (D) 9,6 2,4l = 2400ml (30hrs) 30hrs = 2400ml 2400 + 2400 + 2400 = 7200 30)2400 = 80 1hr = 80ml 10hrs = 800ml 90hrs = 7200ml 7200 + 800 = 8000ml 100hrs = 8000ml
Thembani, (0.601 logits)	2,4 x 3,3 = 24 x 33 72 720 7,92
Sipho (0. 477 logits)	No attempt

Table 70: Transcription, Item 10, top group of School B

Speaker	Transcript	Comments
Interviewer	*Let's go to the next item. Let's start with you Thembani.*	
Thembani	What I did was multiplication, because they said if like 2,4 litres of gasoline will be used for… 30 hours of operation uses 2,4 litres. So what I said since they asked for how much they used in 100 hours, I said 2,4 time 3, and what I ended to was 7,2. Since I was looking for what number could I round off, the nearest number I could find was 8.0. *So you were using estimation techniques. That is the right answer you got to, but if you hadn't had multiple-choice you would not have got it right?* No.	It appears that Thembani has learnt Anna's strategy of classifying the problem type. He brings his understanding of multiplication to bear on the problem. He is tripped up by the fact that there is no whole number factor with which to multiply 30 by in order to get 100. This teaching situation is at the right level, just within the zone in which Thembani could profitably benefit from instruction. The chance was not taken! The interviewer is being unfair here. Who knows? Thembani may have come up with a strategy.
Interviewer	*Let's look at how you (Maria) have done the problem. We can compare the two methods and we decide how …* *Maybe you (Thembani) can get a different idea of how to do it.*	Again, underestimating Thembani, and missing an opportunity of exploring his underlying thinking here.
Anna	I said 2,4 litres is 2400 millilitres which is 30 hours. I divided 2400 millilitres by 30 hours to get 80, which means 1 hour is equal to 80 millilitres, which means 10 hours is equal to 800 millilitres, And then for 90 hours I added them. I got seven thousand two hundred (7 200). Then I added my 800 to get my 8 000, which is 8 litres. *I am not quite sure … you divided that (2400) by 30 and then you got one hour was equal to this, then you worked out what ten hours was equal to.* Then I went back to the question to work out … I knew that if I multiplied this by 3 I would get …(90 hours and 7,2 litres) *Oh I see, you multiplied by 3 and you found …*	Anna manages this problem in two stages. Her first strategy is to find one hour's worth of gasoline. She does this correctly by dividing 2400 by 30 and gets 80 millilitres. She then calculates the amount of gasoline for ten hours by multiplying by ten. She then goes back to the problem and calculates the amount of gasoline for 90 hours, by adding 2,4 + 2,4 + 2,4 and gets 7,2. She then adds 800 (0,8) to 7,2 , making up the extra 10 hours. Anna realises very quickly that she could improve her strategy, and no doubt will in future problems of this sort. She had calculated the amount of gasoline per hour.

Speaker	Transcript	Comments
	That 90 hours *And then you added 10 hours. Good… good.* *Now see what you have done here. Could you have got from there 10 hours equals 800 millilitres to 100 hours.* I could have multiplied it by ten. And that would have been 100 hours is equal to … 8000.	
Interviewer	*And you, Sipho?* I think I got it wrong.(Sipho)	The problem here for the interviewer was that there was a perceived large difference between Thembani, Anna and Sipho. The interviewer didn't take the time with Sipho as the other two had obviously understood. In retrospect this was incorrectly judged and wrong.

Table 71: Item 26, top group of School B

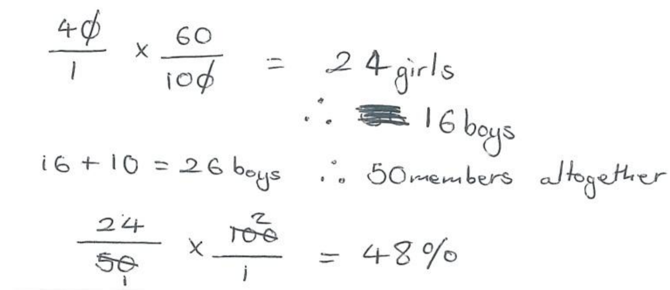
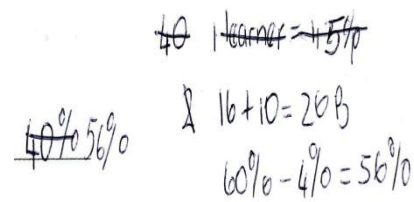
Item 26: A computer club had 40 members, and 60% of the members were girls. Later, 10 boys joined the club. What percentage of members now are girls?	
Anna (2.37 logits),	
Thembani, (0.601 logits)	
Sipho (0. 477 logits)	No attempt

Table 72: Transcription, Item 26, top group of School B

Speaker	Transcript	Comments
Interviewer	*Let's look at number [26].*	
	This was very hard.	
	It was one of the hardest questions in the paper. Very few people got this correct, and I see you (Anna) got it correc.t	
	"A computer club had 40 members …" (Interviewer reads)	
	Now how did you think about this problem?	
Thembani	First I mentioned that there were 50 boys and 50 girls, so there would be 20-20. So I said how would the girls end up being 60%.	
	So I take the forty members, right. I timessed it by 1,5	
	Why did you times by 1,5?	
	To get 100	

Speaker	Transcript	Comments
	Forty (40) by fifteen (1,5)? B: (speaking to himself) How can I do this? Forty by 1,5. I am a little bit confused, Miss. But I ended up with a percentage of 60. OK … (waits)	
Interviewer	*Let's look at how Anna did the sum and then we will come back and look at different parts of the question*.	
Anna	First of all I tried to find out how many girls and boys were in the class. So I said 60% of the forty members which gave me 24 girls and 16 boys. Then I added up …	
Interviewer	Can we just stop at that point, 24 girls and 16 boys …	
Thembani	That's what I did. So that's what I got 24. Then what I did 16 plus ten boys. *So did you get the 24?* I got 16 for the boys and 24 for the girls. *Carry on* *And then what you did ... She (Anna) has found the percentage of 40. She has found 60% of 40.*	
Anna	And then I added the ten boys that came into the class and that gave me 26 boys out of the 50 members altogether. And then I said the 24 girls over the 50 members altogether, then made that a percentage which gave me 48% *OK did you find this problem difficult?* No *Could you see the answer immediately?* Well, yes.	
Thembani	My original answer was correct. *16 is correct So where did you get the 24? You got the 24 girls and the 16 boys So then you got the 26 boys. That is perfectly right. Then how did you get from the 26 boys. ... You are on the right track. So what went wrong here?* I must admit difficulty with this. I was trying to find a … I said 40 times 1.5, *How did it end up to be 100%? ... 40% times 1,5 …You mean 2,5* No *You added 1,5 instead of multiplying by 1,5.* *OK thank you very much* *Did you think you did well on the test (Sipho). Did I muddle you up with someone else? Did I get the name*	

Speaker	Transcript	Comments
	wrong	
Sipho	But I didn’t finish.	
Interviewer	Did you work this one out? Do you mostly work things out in your head? So when someone tells you to write thing down, it throws you a bit?	
Anna	Sometime I get the answer, but when the teacher asks me, I wouldn’t be able to explain exactly how I got it	
Thembani	They give you a sum. You can do it in your head, so you don’t write it down. You don’t take any notes; you just know you can do it. You come up with the real answer but you don’t know how to explain.	

School A – Middle proficiency group

Table 73: Item 5, middle group of School A

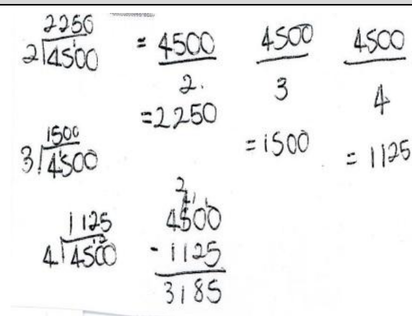
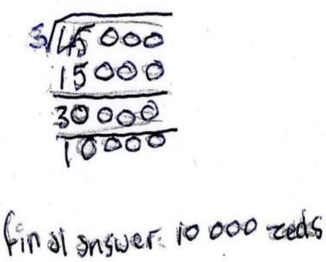
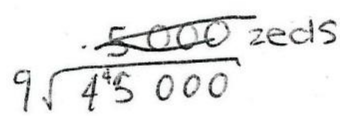
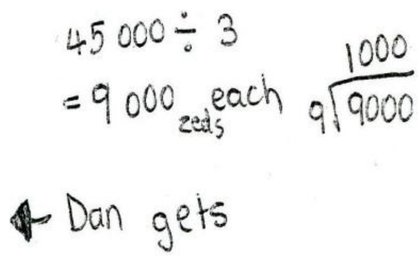
Item 5: Three brothers Thabo, Samuel and Dan, receive a gift of 45 000 zeds from their father. The money is shared between the brothers in proportion to the number of children each one has. Thabo has two children, Samuel has three children, and Dan has 4 children. How many zeds does Dan get?	
Carola, (0.18 logits),	
Shiluba, (0.26 logits)	
Linda, (0.12 logits)	
Kate,(0.12 logits)	

Table 74: Transcription, Item 5, middle group of School A

Speaker	Transcript	Comments
Interviewer	*Let us listen to Question number [5]. (Interviewer reads the question)* *How did you approach this problem? Let's start with Shiluba.*	The interviewer was aware that there were problems here and therefore decided to read the question.
Shiluba	I didn't really completely understand but I got a part answer. My logic behind it … I divided 3 into the R45 000 … And this gave me R15 000. … I think I then timesed by two over one which gave me 30, which I then divided by three which then gave me R10 000.	It appears that Shiluba unthinkingly plucks numbers from the text. However, observe her understanding at the end of this 6-minute discussion.
Interviewer	*OK, Alright. Anybody do something different? Carola?*	
Carola	Ja, I divided all of them *By what?* 4 500 divide by 2 … *45 000 divide by …* Ja 45 000 divide by … *Divide by …* Divide by 2 *Why divide by 2?* For Sam, for Thabo, and I did the same for Samuel (divide by 3) and Dan (divide by 4). *OK (waiting)* I don't know why I did that (laughs, and group laughs with her).	Carola's approach to the problem is systematic in a sense (see working above). She takes the amount 45 000, and divides first by 2 for Thabo, 3 for Sam and 4 for Daniel. She lacks the notion of proportional reasoning. She realises as she speaks that what she is doing does not make sense. The interviewer chose to leave her rather than intervene or scaffold at this point.
Interviewer	*OK, and Linda?*	
Linda	I realise I did it wrong … really bad … I worked out all of them together. All the children. And I divided them by the zeds. Then I got 5 000. Then I think you would divide the 5 000 by 4. *OK, good. So you got the first part right. Why did you add all the children up?* I did that wrong, I don't know why. *In fact what you have done is correct.*	The first part of the problem was correct. The interviewer chose to ignore the "divide by four" for the present, but to focus on the initial correct reasoning.
Interviewer	*What you two (Shiluba and Carola) did was get distracted by the final question.*	
Kate	I said 45 000 divide by 3 and I got 9 000. And then I don't know what I did and I got 5 000 *So far you are on the right track. You didn't read the last part of the question.*	While Kate makes an arithmetical error she has some unformed idea. While listening to the others in the group she has gained an insight. Like

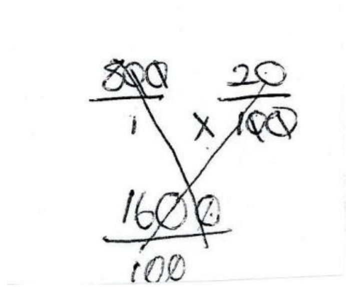
Speaker	Transcript	Comments
	I never worked out how much Dan alone gets. If I had divided 5 000 by 4, I would have got the answer. *You mean multiply by 4.* (Silence) I have no idea (and laughs)	Linda she uses the expression "divide by four" when she means multiply. This error is rectified later by the group as a whole.
Interviewer	*OK, so has anybody got a picture of what is happening here?*	
Shiluba	We were supposed to add up the children and divide it into 45 000. And then we would get 5 000. *And what does that 5 000 mean. 5 000 for … Dan?* No, 5 000 for each of the children *For each of the children …* (Murmurs of realisation from the group) You get 15 000. (The group corrects her.) You get 20 000.	Given that the entire discussion takes 6 minutes, Shiluba has indeed made progress. After a very unclear beginning, she has understood the requirements of the question. She is assisted by the group who also by this time have understood the question.
Interviewer	*OK. Did you get that?* (Confirmation and chuckling from the group) *So where was the problem? Was it in the reading?*	
Shiluba	In the "How many zeds does Dan get?" Because I didn't really, … I kind of shut off … Which is that part? *The middle part?* Ja, "The money is shared between the brothers in proportion to the number of children each one has". *OK,* So I disregarded the children. *That is very interesting.*	Shiluba also shows insight into her processing of the information. She identifies the specific point of misunderstanding.
Interviewer	*So did most of you get the answer 15 (thousand).* *You got five (Linda), You got five (on a second attempt) (Kate). You got fifteen (Carola).*	
Shiluba	I got ten (chuckles)	
Interviewer	*So what do you understand about the word proportion. What does proportion mean?* An amount. Ja, a certain amount (Shiluba).	Shiluba's very clear identification of the point of misunderstanding enabled the

Speaker	Transcript	Comments
	Others agree. *In your everyday lives, where do you come across the word proportion?* (Some discussion but nothing forthcoming)	interviewer to take the concept "proportion" a bit further. This could be the starting point for a next lesson.

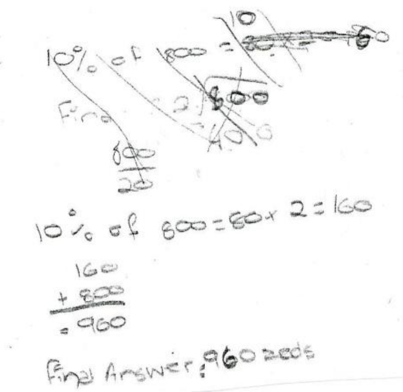
Table 75: Item 8, middle group of School A

Item 8: A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds?

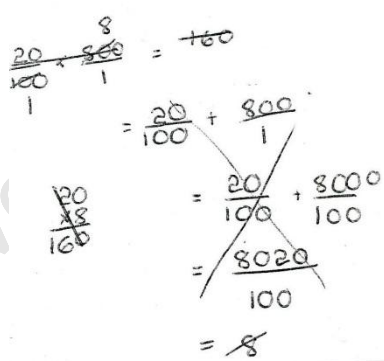
Carola, (0.18 logits),



Shiluba, (0.26 logits)



Linda, (0.12 logits)



Kate,(0.12 logits)

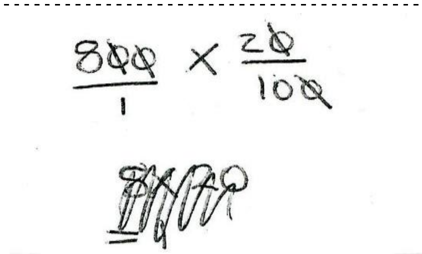


Table 76:Transcription, Item 8, middle group of School A

Speaker	Transcript	Comments
Carola	Missing data	Carola remembers the algorithm $\frac{800}{1} \times \frac{20}{100}$, but then is confused about the "cancelling process".
Shiluba	Missing data	After initial confusion (see scan) Shiluba realises she can find 10% of 800 quite easily. She then finds 2 x 80 to get 160, which she then ads to 800. This could be regarded as a primitive process but at least she is sure of what she is doing.
Linda	Missing data	Linda starts with the correct algorithm, $\frac{20}{100} \times \frac{800}{1}$ = 160 but then crosses this out and tries adding instead. $\frac{20}{100} + \frac{800}{1}$ = (confusion with addition of fractions)
Kate	Missing data	Kate starts with the algorithm $\frac{800}{1} \times \frac{20}{100}$, but is confused with "cancelling". She crosses out 3 zeroes at the top and one zero at the bottom. Still gets 8 x 20 = 1 (but then unsure how to proceed)

Table 77: Item 10, middle group of School A

Item 10: A machine uses 2,4 litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use in 100 hours?	
Carola, (0.18 logits),	2,4 = 30 4,8 = 60 7,6 = 90 8,0 = 100
Shiluba, (0.26 logits)	30 x 3 90 hours 2,4 x 3 x 3 7,2 +8,0
Linda, (0.12 logits)	2,4 x 3 7,2 + 10 hours = 8,0
Kate,(0.12 logits)	30 + 30 + 30 = 90 2,4 x 3 = 7,2 ℓ in 90 hours +10 8,2

Table 78:Transcription, Item 10, middle group of School A

Speaker	Transcript	Comments
Carola	Missing data	From Carola's working we can infer that she has used an additive strategy to get to 90. But she makes a mistake with adding decimals. She has arrived at 7,6 instead of 7,2. After that we cannot be sure. As this was a multiple-choice question she may have estimated the answer as being greater than 7,6 and less than 8,4.
Shiluba	Missing data	Shiluba uses the more advanced multiplication strategy. She multiplies 2,4 by 3 and 30 by 3, and arrives at 7,2 litres and 90 hours. She suspects that the additional amount of gasoline, equivalent to the ten hours is 8, but it perhaps doesn't really understand the decimal notation.
Linda	Missing data	Linda multiplies 2,4 by 3 to get 7,2. She writes 7,2 (litres) + 10 (hours). Her mathematics teacher may object! She gets the correct answer though it is not clear whether she made the calculation: 10 hours is equivalent to 0,8 litres.
Kate	Missing data	Kate combines an additive strategy for the hours: 30 + 30 + 30 = 90 and a multiplicative strategy for the gasoline; 2,4 $\times$ 3 = 7,2. She writes 7,2 in 90 hours, but is then at a loss and she adds 7,2 litres + 10 hours. And gets 8,2 "somethings".

Table 79: Item 26, middle group of School A

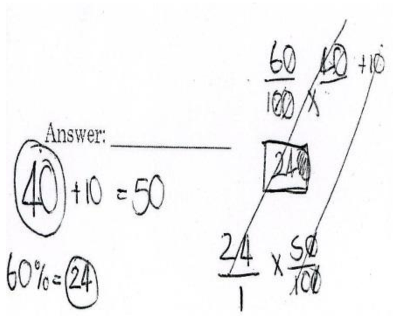
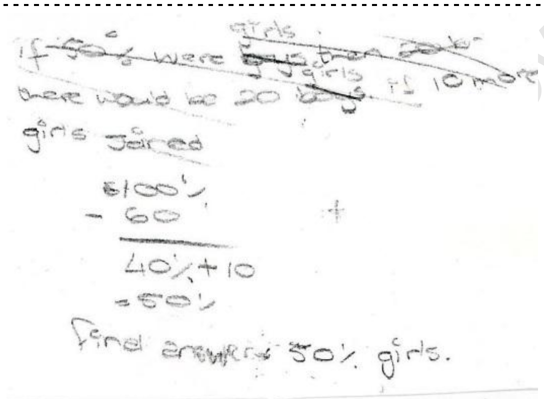
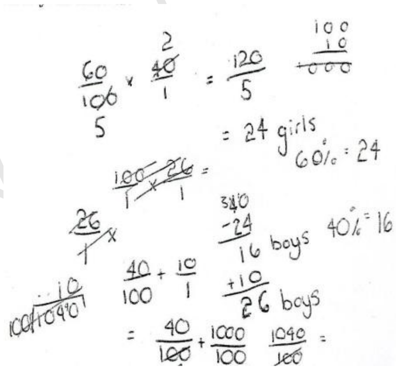
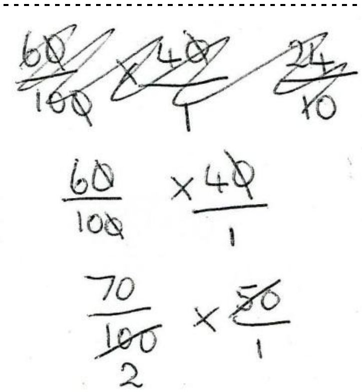
Item 26: A computer club had 40 members, and 60% of the members were girls. Later, 10 boys joined the club. What percentage of members now are girls?	
Carola, (0.18 logits),	
Shiluba, (0.26 logits)	
Linda, (0.12 logits)	
Kate, (0.12 logits)	

Table 80: Transcription, Item 26, middle group of School A

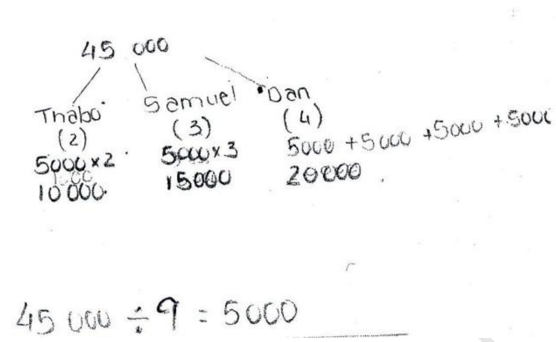
Speaker	Transcript	Comments
Carola	Missing data	Carola was able to solve the first part of the problem, 60% of 40, that is 24 girls. The reference to the whole then changed. 10 boys were added. In the second part of the problem, the requirement was to describe the ratio of girls to the whole and express as a percentage. This change of problem type confused her.
Shiluba	Missing data	Shiluba applied additive strategies which were not helpful.
Linda	Missing data	Linda was able to find the part amount when given a percentage and a whole.
Kate	Missing data	Kate managed the first part of the problem, though had difficulty with cancelling.

School B – Middle proficiencygroup

Table 81: Item 5, middle group of School B

Item 5: Three brothers Thabo, Samuel and Dan, receive a gift of 45 000 zeds from their father. The money is shared between the brothers in proportion to the number of children each one has. Thabo has two children, Samuel has three children, and Dan has 4 children. How many zeds does Dan get?

Maria, (-0.20 logits),



Phaphama. (-0.52 logits)

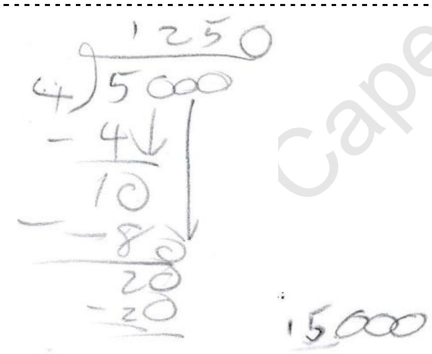


Table 82: Transcription, Item 8, middle group of School B

Speaker	Transcript	Comments
Interviewer	*OK, Let's look at that first item. Number [5]*	
Phaphama (reading)	. "Three brothers receive a gift of 45 000 zeds … " *Now how did you go about solving that (problem)*	It was considered important to read the problem again, even though each learner had read the problem and provided a written answer.
Phaphama	At first I thought about dividing by three, for Thabo, Samuel and Dan, and then they would give to the children. But then that is not what the question asks … which is where I got stuck … If I said like 5 000… 9 times 5 is forty five … but what would the others get … Thabo and Dan.	Phaphama has some insight into the problem. The interviewer mishears and attributes understanding where it is not warranted. He is using a multiplication strategy to solve a

Speaker	Transcript	Comments
	You are on the right track. Say that again. You said "Divide by nine" and you got 5 000	division problem.
	Yes.	
	What was the five thousand? Why did you divide by nine?	It could be that at this point the intervention by the interviewer confuses Phaphama. He is making sense in his own way.
	Because Dan has 4 children and Samuel …. 15 …30 …10 … I don't know.	
	It sounds to me you are on the right track. 45 thousand rand. How many children are there?	
	(Adds the children) …. Nine children Yes that is where I got the nine from.	
	And in order to find out what each one gets, what can you do?	
	Ummm. 5000 each.	Phaphama confuses multiply and divide.
	Now let's look again at the question.	
	5 000 divide by 4 … It is 45 000 divide by 9. … So now it is divide by 4.	The interviewer initially thought using the term "divide" rather than "multiply" was a slip, but clearly there was a conceptual problem.
	Dan has four children … so. 5 000 for each child so …	
	Divide by 4	
	I think you are on the right track so how much will that be.	The attempt by the interviewer to prompt was unsuccessful. It was thought preferable to leave this issue and come back to it
	(Thinking) One thousand two hundred and fifty (1 250).	
	Each child and how many children are there?	
	Four (4)	
	In total	
	Nine (9)	
	And how much money is there in total	
	Forty five thousand (45 000)	
	What is the relationship between number of children and the forty-five thousand rand (R45 000)?	
	(Silence)	
	OK let's leave it there …	
Interviewer	*Can you tell me how you, Maria, approached this problem?*	
Maria	I divided the 45 000 by nine and got five thousand.	Maria had understood the idea of proportion and clearly had a concept in "sharing in proportional
	Then I multiplied the 5 000 by 4. Each	

Speaker	Transcript	Comments
	child was going to get 5 000, so I added … so I multiplied 5 000 by 4 , that is Dan's children, and then 5 000 by 3, for Samuel and 5 000 by 2 for Thabo. Then I am getting 20 000 for Dan, 15 for Samuel and 10 000 for Thabo.	amounts". She says "added", then changes to "multiplied". Both strategies are evident in her working. See above.
Interviewer	*OK! Can you see what Maria did? Where did you make your mistake?*	
Phaphama	I divided by 4 *Yes, divided instead of multiplying.*	Maria's explanation is closer to Phaphama's developing understanding than that the interviewer it seems.
Interviewer	*When you first read the question, did it make sense to you? Did you understand it?*	
Maria	No, I had to read it again	

Table 83: Item 8, middle group of School B

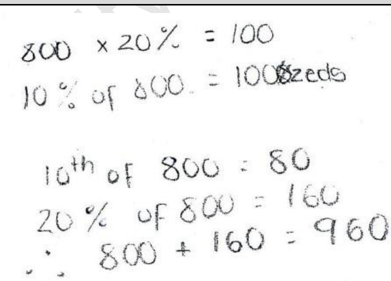
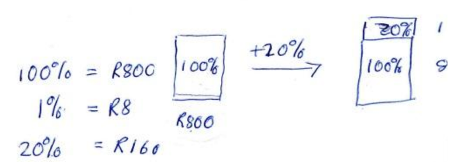
Item 8: A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds?	
Maria, (-0.20 logits),	
Phaphama. (-0.52 logits)	

Table 84: Transcription, Item 5, middle group of School B

Speaker	Transcript	Comments
Interviewer	*Let's look at the next one. Maria, will you read the question?*	
Maria (reading)	"A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds? *OK so how did you do that?*	
Maria	I thought 10% of 800 is a 100 zeds. And then I added to the hundred 200 zeds *Say that again. You found 10% of 800?* Yes *And you got 100?* I did this in my head. I am not sure it is the right one. *What does 10% mean? If you tell me you got 10% for a test, what does this mean?* It means I got a tenth of the marks. *And a tenth of that (pointing to 800).* One hundred (100) *What is one tenth?* (Thinking a long time). I think it is 80. *It is 80. Then if 10% is 80, what is 20%?* One hundred and sixty (160) *Now would you change your answer?* It is C (referring to one of the multiple-choice distractors). *How do you know it is C?* Because 800 and 160 is 960. *So you had the right idea but you made the miscalculation here (pointing to the 10% of 800 equal to 100).*	The amount of 800 seems to seduce learners into adding 20 ….. and getting 1000.
Interviewer	*How did you approach the problem, Phaphama?*	
Phaphama	What I first did … it said the shop increased. … I have never been really good with percentages, so that is why I got it wrong. *What percentage is 800?* It is 80% *You thought 800 is 80%, then you thought what is 100%. Is that right?* Yes	

Speaker	Transcript	Comments
	What does 100% mean?	
	It means the whole thing.	
	Eight hundred (800) is what you start with. Then you have to add the 20% on top of that.	
	What is not clear?	
	How you get to 960?	
Interviewer	*Can you (Maria) explain how you worked this out?*	
Maria	I found a tenth which is 10% which is 80. and then I added another 10% which is 160. I added 160 to the 800.	
Phaphama	*OK, I am just going to draw something here to show you.*	
	You have got to find 20% of this (800). What is 20% of this?	
	20% is a fifth. So you could find a fifth of that?	
	Or you could find a 10% and then another 10%.	
	20 parts of 100. How many parts of 800?	
	4?… 4 000?	
	100% = 800 Right?	
	Now we want to know what 1% is. How are we going to find that? What is 1%.	
	Isn't it 8?	
	Good. Now we say, how much is 20%?	
	20% of a 100?	
	You have 1%. How do you get from 1 to 20?	
	You add 19.	
	You add 19 or …. That is one possibility … adding 19 …or	
	Multiply … (tentatively)	
	Multiply by ….	
	By 20	
	Good, multiply by 20. We multiply this (1%) by	

Speaker	Transcript	Comments
	20. We also multiply that (8) by 20.	
	Eight (8) times twenty (20) is 160. OK	
	So now you are adding that 160 .. that is you are adding 20%	
	OK.	
	(Pointing to the diagram) you have your 800 here. You add your 160. You have your 100 here ... you add your 20%	
Notes	This impromptu lesson was an attempt to provide a visual picture for Phaphama. It is evident that the Interviewer made some false starts that didn't gain traction. The method of finding 1% is noted in Parker &Leinhardt (1995). Will discuss this in the body of the research. It is interesting to note that Maria uses this technique confidently in a later problem. Phaphama initially uses an additive strategy to get from 1 to 20. But then is able to recall that he can multiply.	

Table 85: Item 10, middle group of School B

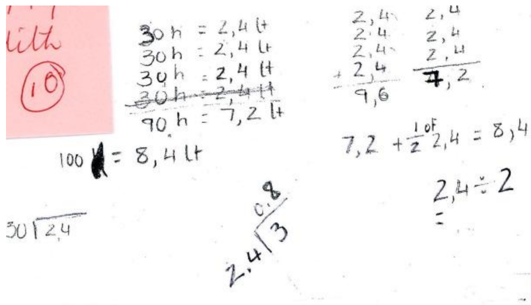
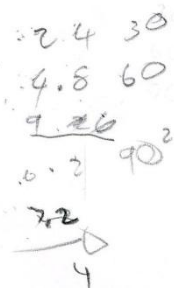
Item 10: A machine uses 2,4 litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use in 100 hour?	
Maria, (-0.20 logits),	
Phaphama. (-0.52 logits)	

Table 86:Transcription, Item 10, middle group of School B

Speaker	**Transcript**	**Comments**
Interviewer	*Let's look at the next one.*	
Phaphama(reading)	"A machine uses ….."	
	Now how did you go about that problem?	
Phaphama	I did 3 plus 2 is 6 … what I basically wanted to do was … 3 + 6 is nine … 30 hours is 2,4 litres … 60 hours would be 4.8 …. And 90 got up to 6.2 …	Phaphama uses *plus* when he means *multiply*.
	How did you get 6.2?	He makes an error with adding. This trips him up.
	I did a miscalculation somewhere here.	
	Now, what was it?	
	4 plus 4 is 8, and 8 plus 4 is 12.	
	You forgot about carrying the 1. That would have given you 7,2. That is for ninety. You said 30 …60 …90. They are asking you for 100. So how would you get to 100?	The Interviewer prompts, perhaps unnecessarily.
	7,2 for 90 …. (Thinking) I think I would end up with 8,4 (one of the distractors).	
	We'll come back to that.	

Speaker	Transcript	Comments
Interviewer	*And you, Maria?*	
Maria	I also did what he did. I add the 30's and the 2,4's (in parallel). At 90 I got 7,2. And I added to the 7,2, half of 2,4. *Why did you add half of 2,4?* Because I ………wanted to get to a 100 because I only had ninety here. *Why did you add half? Let's try and think about that. Why did you find half of 2,4? …..* *Silence for a while.* *If 30 hours gives you 2,4 … then for ten hours … do you see that?* *She has divided 2,4 by 2. Is she right?*	
Phaphama	If I divide by 2 then I get 15 *So what can you do instead? Divide the thirty by … ?*	
Maria	By 3	
	Good, and then divide this (the gasoline) by three. And what would your answer be if you divide 2,4 by 3? …. *No the other way around … divide 3 into 2,4.*	Phaphama divides 3 by 2,4
Phaphama (counting)	(Phaphama counts 3 … 6 … 9 …) *You are counting this up …3 ….6 …9….12.* (Phaphama counts) … 12 15 18 21 24 … that will be 8. *Is it 8 or point 8?* It will be comma eight. I was thinking about this. 3 can't go into 2. *OK let's do this another way. 30 is equivalent to 2,4. We could say one hour but let's say ten hours. How do we get from here (30) to here (10). We go three. … and then* 0.8 (Maria) *Then to get 100 hours?* ….. *Ok, do you think you have got that one now? If you got another one like that in the exam, would you be able to do it?.* *Let's do it another way* *30 hours … what would it be for one hour …2,4 litres, what would be for one hour … you divide 2,4*	Scan diagram

Speaker	Transcript	Comments
	by 30 . . .	
	2,4 by 30? (Phaphama)	
	. . . to get one hour.	
	You get 0.08 and then for 100 hours ….	
	You could do this with a calculator ….	
	You have got 2,4 divided by 30, which will equal 0.08. (Phaphama)	
	Check here. .	

Table 87Item 26, middle group of School B

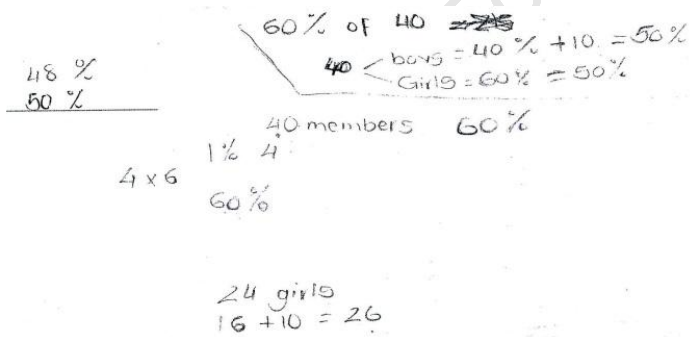
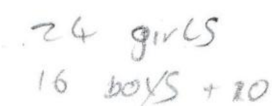
Item 26: A computer club had 40 members, and 60% of the members were girls. Later, 10 boys joined the club. What percentage of members now are girls?	
Maria, (-0.20 logits),	
Phaphama. (-0.52 logits)	

Table 88:Transcription, Item 26, middle group of School B

Speaker	Transcript	Comments
Interviewer	*Let's go to the next question …*	
Phaphama	This one had percentages again.	
	And you don't like percentages. Why don't you like percentages?	
	I like more algebraic expressions	
	You like algebra but you don't like percentage?	

Speaker	Transcript	Comments
	Yes and geometry … the angles. I like better working with letters and numbers.	
	OK, let's look at the last one	
Maria	1% of 800 is 8, so 1% of 40 is ….	
Phaphama	40 divided by 4, first we've got to read the question	
	(taking the lead) … first we have to read the questions	
	(reading the question) "A computer club ….	
	60. % of the members were girls … this means the majority was … "later 10 boys joined the club. What % (of the club) are now girls?	
	This is a dumb question.	
Interviewer	*Ok read this to me.*	
Maria	"A computer club has forty members. 60% are girls. Later 10 boys joined the club. What % of the club now are girls? Show the calculations that led to your answer."	
	So the first part. How did you do that? Or how are you thinking about that now? Even if you haven't got to the answer	
	EL: I am thinking about what you said previously, and I just said 1% of 40 is 4. Now for me to get the 60 I think I need to times it by …..6 … the answer is 24. Right? 24 of them are girls. The rest are boys.	
	Good. 24 are girls, the rest are boys. So you have got the first part right. 24 are girls and 16 are boys. Now what happens?	
	Now I give the ten to the boys. Then I will have 26 boys and 24 girls is..	
Phaphama	How did she get to 24 … girls?	
Maria	I timesed the four by 6.	
	How did you get the 4?	
	The 4 … 1% of 40 is 4	
Phaphama	Is she right?	
	1% ….	
	She basically said 40% divide by … 4	
Maria	No	
Phaphama	40 by 4 is one	
Interviewer	*If she is to get 1% she has to divide 40 by 100. 100% is 40.*	
	So in fact you got the right answer, but your working was a little bit wrong. In fact what you should have said is (40 divide by 100 is equal to 0,4.	

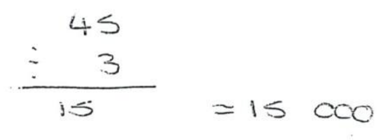
Speaker	Transcript	Comments
	Then times 0,4 times 60 will give you 24.	
	This isn't necessarily the easiest way of doing this. You could find 60% of 40. 60 out of 100. How can you simplify this?	
	6 out of 10.	
	Now what is 6 out of ten of 40? 6/10s of 40? What is 1/10th of 40? 1/10 of 40 is 4. And what is 6/10s	
Maria	24.	
	24 … You've got 24. Now what happens?	
Phaphama	24 and 16 makes 40.	
	Now what happens 10 more boys join. That makes …	
	50.	
	Now 24 of 50 what is that … out of 100?	
Phaphama	*Let me show you here 24 out of 50, is going to be what out of 100.You get from there to there …. You multiply by two so from there to there.*	
	24 times 2	
	Yes	
	So that is the percentage	

School A –Lower proficiencygroup

Table 89: Item 5, lower group of School A

Item 5: Three brothers Thabo, Samuel and Dan, receive a gift of 45 000 zeds from their father. The money is shared between the brothers in proportion to the number of children each one has. Thabo has two children, Samuel has three children, and Dan has 4 children. How many zeds does Dan get?

Zanele(-0.82 logits)



Cheryl(-0,93 logits)

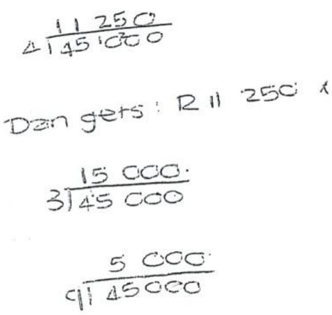


Table 90:Transcription, Item 5, lower group of School A

Speaker	Transcription	Comments
Interviewer	*Let's go to Number [5]. How did you approach this problem? Can you read this?* Cheryl reads: "Three brothers …" *There are two steps to this question …*	Looking at their scripts it was evident these learners had struggled with this item. By asking Cheryl to read, the Interviewer expected them to start to engage with the problem.
Zanele	I got this wrong because I said you divide this by 4, because he had four children. And then it is wrong *And now how do you think you should do it.* Divide by three … [does calculation] This is how much Samuel will get from his dad.	It appears that Zanele has not understood the problem.

Speaker	Transcription	Comments
	Why do you say that?	
	Because he has three children, and if you divide 45 000 by three you get 15 000	Some confusion here. Also for the interviewer. The right answer for the wrong reasons.
	You're on the way …. And if Samuel is going to get 15 thousand how much is Dan going to get?	
	I don't know how to work this out, because if I divide by 4 … he gets 11 thousand … 11 thousand 3 hundred and something …	
nterviewer	*And how did you think about this problem (Cheryl)?*	
	I think you have it now ... Can you explain?	
Cheryl	What I did is I tried to divide the number into all of these ….I thought to myself … um … if I divide this … Then each child will get 5 000.	Cheryl works backwards. Possibly because this item has multiple-choice responses, but a more likely hypothesis is that she uses her number sense to reason qualitatively and then applies the numbers.
	And how did you get each child to get 5 000?	
	I don't know … I thought it was a perfect amount that would go into that (45 000) Thabo will get 10 000, Samuel will get 15 000 and Dan has four children, he will get 20 000, and this makes 45000	
	OK, good. You solved the problem more intuitively ... You didn't really know how you got the 5 000?	
	No, I didn't	
	Add the two, the three and the four (pointing to the text).	
	Seven .	
	No	
	Oh that's nine!	
	And if you divided the 9 into the 45.	
Zanele	Ohhh! (sudden realisation of understanding)	Zanele, through listening to the reasoning of Cheryl, comes to a realisation of the problem.
Cheryl	I know what I am doing but I can't say it.	Cheryl could benefit from articulating the problem and then translating her articulated thoughts into mathematics.
	You are on the edge of understanding this (concept).	

Table 91Item 8, lower group of School A

Item 8: A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds?	
Zanele(-0.82 logits)	*Zanele calculated 20% of 800 and then "got stuck". With help she completed the problem* $\frac{\cancel{800}^{8}}{1} \times \frac{20}{\cancel{100}_{1}}$ $\quad$ $20 \times 8 = 16.0$ $= 160$ $+800$ 960 zeds...
Cheryl(-0,93 logits)	$\frac{2\cancel{0}}{10\cancel{0}} \times \frac{80\cancel{0}}{1} = 160$ 800 $+160$ $= 960$

Table 92: Transcription, Item 8, lower group of School A

Speaker	Transcription	Comments
Interviewer	*Let's look at Number [8]*	
Cheryl	What I did here was put 20 over a 100 and then I times by 800 over 1. And then I crossed out the nought then I got 160. And then what he previously sold for was 800, and so I added the 160.	Compared with the learners in the middle group it appears that Cheryl is quite happy with a routine procedure even though she may not know why it works. On the other hand she may have an intuitive understanding that is not expressed.
Interviewer	*And you Zanele?*	
Zanele	I put 20 over 100 and then multiplied by 80 over 1 and then I got 160. And then I got stuck. *Let's have a look, why did you get lost? The shop sold the product for 20% more. ... so you found the 20% and then you add this on.*	Zanele remembers the first part of the algorithm, but is not sure what the result means.

Speaker	Transcription	Comments
	If the question said % decrease, would you have known what to do with it then?	

Table 93: Item 10, lower group of School A

Item 10: A machine uses 2,4 litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use in 100 hours?	
Zanele(-0.82 logits)	No attempt made initially
Cheryl(-0,93 logits)	No attempt made initially

Table 94:Transcription, Item 10, lower group of School A

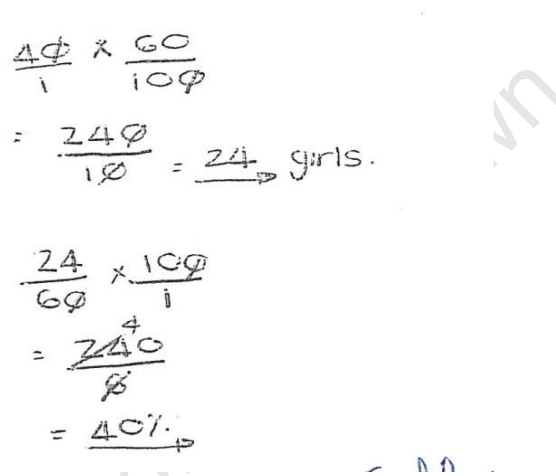
Speaker	Transcription	Comments
Interviewer	*Now we are going to look at number [10].* What's going on here? A machine uses 2,4 litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use for 100 hours? .... *Now what is happening here?* .... *Have you got a picture of this in your mind?*	The interviewer had seen that the two learners had not attempted this problem. The purpose of continuing was to judge just how far the learners schemas extended and what would be required for further development.
Zanele	I know that for 30 hours it will use 2,4 litres, then for 100 hours ….. I will know the answer, but I won't know exactly how to work it out. *Which is the answer* I think it is 7,2. I don't know why.	It was decided not to pursue this further, but as is evident further in the interview Zanele has continued with the problem.
Interviewer	*And you, Cheryl?*	
Cheryl	I was thinking it was 8,4. I was thinking if I could halve this, half of 30 and then try and figure out what could go into 100. *You are on the right track. Do you think you could work this out for ten hours? How much gasoline for 10 hours?* Divide by three. *That's good, do you think you could work out (the amount) for 100 hours?*	Cheryl was on track. However, the introduction of a division procedure was necessary to take her further.
(Cheryl explains to Zanele)	You divide 30 by 3 to get 10, then you must also divide 2,4 by 3., then you get 0.8.	
Zanele	Ah… and then you must multiply by 10 (to get 100)….Ah, I get it	
	Would (this problem) have been easier for you if you	Concerns about the

Speaker	Transcription	Comments
	had said 2,4 litres of coke for 30 children?	unfamiliarity of the term gasoline have been raised.
Cheryl	No, it's the same	

Table 95: Item 26, lower group of School A

Item 26: A computer club had 40 members, and 60% of the members were girls. Later, 10 boys joined the club. What percentage of members now are girls?

Zanele(-0.82 logits)



Cheryl(-0,93 logits)

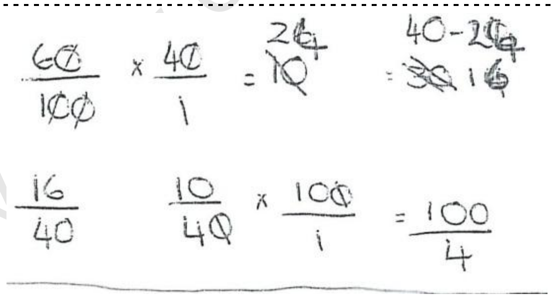


Table 96:Transcription, Item 26, lower group of School A

Speaker	Transcription	Comments
Interviewer	*You have both got the first part of it,* *Why did you put 60?* *What does percent mean?*	
Zanele	Out of 100 *If you had a maths test and you got 40 out of 40, what would be your percent?*	

Speaker	Transcription	Comments
	100%	
Cheryl	*So if the computer club had 40 members what would be the 100%?* 40.	
Interviewer	Now the next part, if 10 members join the club what is the 100% going to be. (Both girls laugh)	
Interviewer	*Let's say this is the club ... it has 40 people in it. The total amount in the club is 40.* *What % is girls/ ... and we work this out… what percentage is girls? We worked it out to be 24. And that should be 16 (see drawing) and that should be 40%. Am I right?* *Now we have a different situation. How many are in here now? 10 boys joined. Now 50 represents the total. Now the girls come to here ... a little bit less than half …24 and the boys come to 26…. And if we have to convert this to a percentage …. What is it going to be 24 out of 50 .. and what is this ...*	Scan drawing
Cheryl	48 (Cheryl). *Does this make sense? 48% are girls and 52% are boys*	
Cheryl and Zanele	Not really, ()	
Cheryl	But I understand what you have done. …. I thought the 10 members were part of the 40	

School B – Lower proficiency group

Table 97: Item 5, lower group of School B

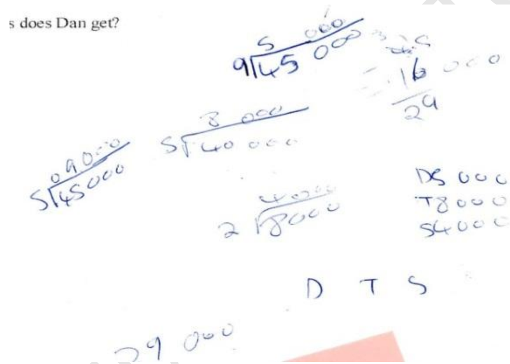
Item 5: Three brothers Thabo, Samuel and Dan, receive a gift of 45 000 zeds from their father. The money is shared between the brothers in proportion to the number of children each one has. Thabo has two children, Samuel has three children, and Dan has 4 children. How many zeds does Dan get?	
Amukelani(-2.72 logits)	Marks three of the distractors 5000 15 000 20 000
Mishack (-2.19 logits)	
Mahesh (-2.60 logits)	

Table 98:Transcription, Item 5, lower group of School A

Speaker	Transcript	Comments
Interviewer	*Can you tell me how you thought about this problem*	Amukelanihad marked distractors.
Amukelani	The number children counted altogether is 9, so I divided this 45 into the number of children So the number of children is nine … and the	The interview with Amukelaniwas interesting. Clearly a good measure of reasoning is evident. It is not

Speaker	Transcript	Comments
	amount of money 45, so 45 divided by nine … the answer is five. *So just read through the question again* (After about 2 minutes) Ohhhh now I get it … okay I get it … *So what is the answer?* Oh Miss let me see ….. (reads through again) *How many children does Dan have?* He has four children *So you see Dan has four children. What does the R5000 represent?* The one child. *And so the four children …* The four children altogether make 20 … *So which is the correct answer* Oh, for Dan it is 20. Okay, now I get it.	clear whether his reading ability is poor. He seemed to require a great deal of prompting. There were long silent gaps where it appeared he was trying to work things out. However, Amukelani, though lacking foundational skills, partly attains the concept.

Table 99:Item 8, lower group of School B

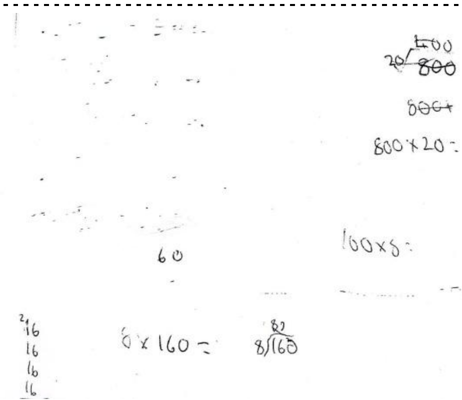
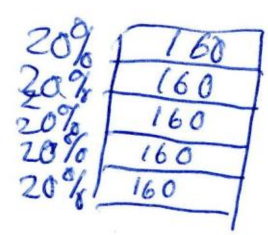
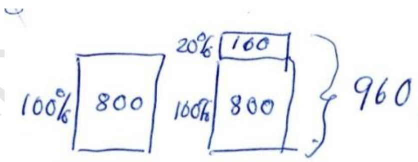
Item 8: A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds?	
Amukelani(-2.72 logits)	No attempt made
Mishack (-2.19 logits)	No attempt made
Mahesh (-2.60 logits)	

Table 100: Transcription, Item 8, lower group of School B

Speaker	Transcript	Comments
Interviewer	*Let's read the item*. "A shop increased its price by 20%. What is the new price of an item which previously sold for 800 zeds?"	
Amukelani	It's going to be …	Amukelaniwas very anxious to get a quick answer to the problem.
	Now first try it and then we'll discuss it.	
	Increase the price by 20% meaning miss that the price would be a thousand.(1000)	The answer of 1000 for this problem was an insight. Like Phaphama, Amukelanimakes the mistake of equating 800 with 80%.
	Can you explain to me how you did that?	
	They increased the price by 20% and the original price was R800 so an increase of 20% …	
	What is 20% of R800?	Amukelanihas some facility with decimals.
	It is R200. Oh … of R800. … It is R200	
	Is there another way? Do you know how to write 20% as a fraction?	
	Miss, I can write it in decimal form.	
	OK, write it in decimal form.	
	Miss, it is still going to give you the same thing.	This answer revealed to the Interviewer that Amukelanihas serious gaps in early mathematical knowledge. The concept of percent is confused.
	Show me	
	It's going to be 0,20	
	So you are going to find 0,20 of R800?	
	(Long gap)	
	Let me start like this. 50% is 400. 40% then is 2 … then is 1	
	Ok you have fifty percent. Do you think you can work out what is 10%?	
	10% is 100.	
	Ok what does 10% mean?	
	It means the number 800 is already 100%. When you take it back it is supposed to go lower it is supposed to decrease.	
	Is 800 a 100%?	
	Yes, from the product.	
	And then … If 800 is 100%, what is 10% going to be. Is it going to be one tenth?	
	Yes miss	

Speaker	Transcript	Comments
	So what is one tenth of 800?	
	800 is 100%, so we want to know what is 10%.	
	Can you work out what 1% is going to be?	
	Ten	
	Is one percent going to be ten?	
	Ya miss	
	What would you say to me if I said 1% is 8.	
	No miss one percent would be a hundred. It would be a hundred	
	You're going a bit off track now. Can I just explain something to you?	Amukelaniresponded to a visual image. The learning when attached to a diagrammatic representation was fruitful.
	This is 100%. Right? Now we are going to increase this amount by 20%. This is a 100 and this is 20%. This is 800 and what do I have to add there.	
	KM: oh …	
	Is that alright? Do you get it? So what is your answer now.	
	Nine sixty (960)	
	This would be easier if you did the sum from graphs.	
	A picture or a graph. What is difficult about this?	
	When you put the question in your mind there is too much to think about. It comes all mixed up.	
	OK but when you get a picture it helps you.	

Comment on Mishack and Mahesh's working.

Mishack	No information
Mahesh	Mahesh knows that he has to multiply 80 by 20. And then divides by 100. He gets 160. But is not sure what to do after that.

Table 101: Item 10, lower group of School B

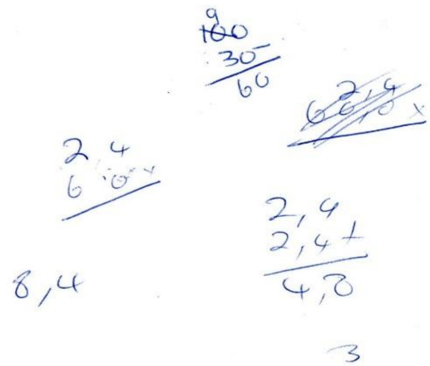
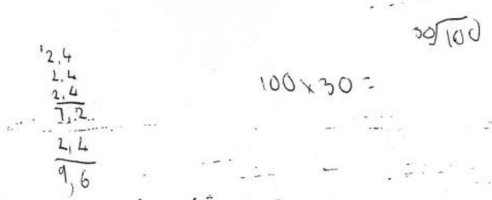
Item 10: A machine uses 2,4 litres of gasoline for every 30 hours of operation. How many litres of gasoline will the machine use in 100 hours?	
Amukelani(-2.72 logits)	No attempt made
Mishack (-2.19 logits)	
Mahesh (-2.60 logits)	

Table 102Transcription, Item 8, lower group of School B

Speaker	Transcript	Comments
Interviewer	*Ok let's do one more. This one has got decimals in it which you seem to know. You know about decimals, don't you?* *It is this one here.* *Read it out to me. (Amukelanireads)*	Item 10 is about 2.5 logits above the location of Amukelani, which if the instrument in functioning well and is indeed measuring proficiency means that Amukelanimay have difficulty here.
Amukelani	KM: The machine uses 2,4 litres of gasoline for every 30 hours … of operation. *How many litres for 100 hours?* 30 times 2 will be 4,8 4,8 times 2 will be (works out on paper) So far I am at 90. *You are on the right track. You have got 90.* I am at ninety because here we have 30.	However, the question remains, "What is the "zone of proximal development" in which Amukelaniwill benefit from instruction? Perhaps this depends on the patience of the teacher! Item 10 (location

Speaker	Transcript	Comments
	So how are you going to find the other ten?	
	It will be 120.	
	Let's forget about the 100 for the moment. Let's read the question again.	
	The machine uses 2,4 litres for every 30 hours of operation.	
	How will you find out how much it uses for ten hours?	The interviewer was a bit out of touch with the learner who may have been on another track.
	Don't worry about the hundred, we have found for the 90. We are worrying about the 10 hours.	
	Still using this, miss? I'll times it by 5.	
	Ok, let's try and make a picture of this. Because this is the way you find it helpful. Now imagine your dad. He knows about engines, doesn't he? Imagine you have a generator at home that provides electricity. Every time he puts in 2,4 litres of gasoline/petrol into his machine, you get light for thirty hours. Now if you have light for only ten hours then how many litres were in the machine. Do you understand? Does that make sense?	30 30 2,4 2,4 7,
	Would it make sense to you to divide 30 into three equal parts, you get 10, 10, 10 (draws picture) …then you divide 2,4 into 3 equal parts how much will you get?	÷3 (30 hours = 2,4 li 10 hours = 0,8
	It will be... When you say 2,4 divide by 2 you say 1,2. and the 1,2 …	
	Let's make a picture 30 hours starts here and goes on to there.	
	If you divide this up you get. You said to divide by 2, then we would get 1,2. But we need to divide by 3. Let's leave the decimal comma for the moment. If it didn't have a decimal, if it was 24 and you divided by 3, could you work that out.	
	(Amukelanispends about 50 seconds) It will be 8.	
	But what will it be, will it be 8 litres? Remember this is 2,4.	
	I don't know miss.	
	Could it be 0,8?	
	Yes.	
	Yes you must write ...	

Speaker	Transcript	Comments
	7,2 and 0,8. Yes this one. 8,0.	