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University of Cape Town
Department of Mathematics and Applied
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*Quasi-pseudometric spaces and some
of their completions*

Makitu Kivuvu Charly

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Contents

Acknowledgements	i
Abstract	iii
1 Introduction and Basic concepts	1
1.1 Introduction	1
1.2 Basic concepts	5
2 Bicompletion of quasi-pseudometric spaces or Salbany's completion	6
2.1 Definition of a Cauchy sequence in a quasi-pseudometric space	6
2.2 Bicompletion of quasi-pseudometric spaces or Salbany's completion	8
2.3 Extension of quasi-uniformly continuous mappings	12
3 Bicompletion of weightable quasi-pseudometric spaces and completion of partial metric spaces	17
3.1 Some definitions in a partial metric space	17
3.2 Bicompletion of weightable quasi-pseudometric spaces and completion of partial metric spaces	22
3.3 Banach's fixed point theorem for a partial metric space	25
4 Bicompletion of an asymmetric normed linear space	28
4.1 Definitions in an asymmetric normed linear space	29
4.2 Bicompletion of an asymmetric normed linear space	31
4.3 Extension of mappings	35
4.4 The partial quasi-metric of a biBanach space	37
5 Construction of the bicompletion of quasi-uniform spaces using quasi-pseudometrics	39

5.1	Introduction to quasi-uniform spaces	40
5.2	Bicompletion of quasi-uniform spaces	41
5.3	Extension of mappings	45
6	The B-completion of quasi-pseudometric spaces	47
6.1	Definition of a balanced Cauchy filter pair in a quasi-pseudometric space	48
6.2	The B -completion of (X, d)	54
6.3	Extension of mappings	64
6.4	Connections with Doitchinov's work on balanced quasi-pseudometrics	72
6.5	Conclusion	75
	Bibliography	76

Abstract

In the first part we recall results from the theory of the bicompletion of quasi-pseudometric spaces and its applications in functional analysis and theoretical computer science.

In the second part we introduce the notion of a balanced Cauchy filter pair and construct a double completion of T_0 -quasi-pseudometric spaces which contains the bicompletion of the original space. Our aim is to extend Doitchinov's completion theory for balanced quasi-pseudometric spaces to arbitrary T_0 -quasi-pseudometric spaces.

Chapter 1

Introduction and Basic concepts

1.1 Introduction

In the last decade interest in the study of quasi-pseudometric spaces and complete quasi-pseudometric spaces has increased considerably. In particular the theory of quasi-pseudometric spaces has been applied successfully in theoretical computer science.

If we omit the symmetry condition in the definition of a pseudometric, we obtain the notion of a quasi-pseudometric which is called *an asymmetric distance function*.

Asymmetric distance functions had already been considered by Hausdorff in the beginning of the twentieth century when in his book on set-theory [13] he discussed what is now called *the Hausdorff metric* of a metric space. A.W. Wilson [34] introduced the term quasi-metric and noted that convergences in quasi-metric spaces arise in three natural ways. However, several generalized distance functions were already used by E.W. Chittenden and V.W. Niemytzki [29] in their study of the metrization problem.

Complete spaces in general play a fundamental role in analysis, because they are at the origin of important theorems like: Theorem of a contraction mapping or Banach fixed point theorem, the Baire category theorem etc...The notion of completeness is based on a concept of Cauchy sequence which generalizes the concept of a convergent sequence. Completeness guar-

antees the existence of the limit of a Cauchy sequence. More precisely a quasi-pseudometric space is **complete** provided that every Cauchy sequence converges.

There are various completion theories that have been developed for quasi-pseudometric spaces in the past century like : The Smyth completion and the Yoneda completion [22].

The work of Salbany in [31] introduced another notion of completion for quasi-pseudometric spaces that we call the **bicompletion** or Salbany's completion for quasi-pseudometric spaces. He showed that each T_0 -quasi-pseudometric space has a T_0 -bicompletion that is unique up to isometry.

In [8] Di Concilio studied quasi-pseudometric spaces with their associated topologies and investigated completeness of quasi-pseudometric spaces in an approach similar to that of Salbany.

In [22] H.-P. Künzi and M.P. Schellekens presented the Yoneda completion of a quasi-pseudometric space. They showed that the largest class idempotent under the Yoneda completion consists of the Smyth-completable spaces. The Yoneda completion of a Smyth-completable quasi-pseudometric space coincides with its bicompletion. In particular, these authors proved that the Yoneda completion is idempotent on the class of totally bounded spaces.

In [25] S.G. Matthews introduced the notion of a partial metric space and obtained among other results a nice relationship between partial metric spaces and the so-called **weightable quasi-pseudometric spaces**. Romaguera and his collaborators showed in [27] that the bicompletion of a weightable quasi-pseudometric space is a weightable quasi-pseudometric space.

From that result they deduced that any partial metric space has a unique partial metric completion up to isometry. A similar result was obtained by S.O' Neill [29].

Furthermore the study of asymmetric norms naturally leads to a theory of asymmetric functional analysis. It is well known that each asymmetric normed space is a quasi-pseudometric space. In [14] L.M. García, S. Romaguera and E.A. Sánchez-Pérez studied the bicompletion of an asymmetric normed linear space.

A very interesting completion of a quasi-pseudometric space was developed

by late Doitchinov in [11]. He replaced in his construction the d^* -Cauchy sequences (x_n) by so-called Cauchy pairs of sequences $\langle (x_n), (y_n) \rangle$.

For his study, he considered those T_0 -quasi-pseudometrics that satisfied an additional condition which he called balancedness. For balanced T_0 -quasi-pseudometrics he obtained a very convincing completion theory. Later he formulated an analogous completion theory for a quiet quasi-uniform space which is based on the concept of Cauchy pairs of nets [10]. Unfortunately the condition of balancedness turns out to be rather restrictive.

It is well known that a family of quasi-pseudometrics on a set generates a quasi-uniformity and each quasi-uniform space can be embedded into a product of quasi-pseudometric spaces. The study of bicompletion of quasi-uniform spaces can be done using the embedding approach and by bicompleting the quasi-pseudometric factor spaces.

This thesis is devoted to an investigation of the theory of the bicompletion of quasi-pseudometric spaces and finally constructs a double completion which we call the ***B-completion*** of quasi-pseudometric spaces. The thesis is divided into two main parts with six chapters.

Part A deals with the bicompletion and various applications for quasi-pseudometric spaces that are described in the literature.

Chapter 1 is a general introduction that covers basic definitions and explains in detail the notion of a quasi-pseudometric space with its associated topologies.

Chapter 2 constructs the bicompletion or Salbany's completion for a T_0 -quasi-pseudometric space. We show that each T_0 -quasi-pseudometric space has a bicompletion which is unique up to isometry and investigate the extension of quasi-uniformly continuous mappings between quasi-pseudometric spaces.

Chapter 3 presents the bicompletion of a weightable quasi-pseudometric space and the completion of partial metric spaces. We show that there is an algebraic equivalence between a partial metric and a weightable quasi-pseudometric and prove that the completion of a partial metric space can be obtained from the bicompletion of its associated weighted quasi-pseudometric space.

Chapter 4 deals with the bicompletion of an asymmetric normed linear space. It is generally known that each asymmetric normed linear space is a quasi-pseudometric space. The bicompletion of an asymmetric normed linear space can be obtained from results in Chapter 2. We also introduce the notion of the partial quasi-metric space associated with the asymmetric normed linear space. We show that there are connections between partial quasi-metrics and quasi-pseudometrics with compatible weight. In asymmetric functional analysis that seems to be a new idea that cannot be found in the literature.

Chapter 5 presents the bicompletion of a quasi-uniform space. Since each quasi-uniform space can be embedded into a product of quasi-pseudometric spaces, the embedding approach can be used to construct the bicompletion of a quasi-uniform space. With this method, we show that each T_0 -quasi-uniform space has a T_0 -bicompletion and investigate the extension theorem of mappings between bicomplete quasi-uniform T_0 -spaces.

Part B deals with the B -completion for quasi-pseudometric spaces and contains some original results.

Chapter 6 introduces a new notion of completeness for T_0 -quasi-pseudometric spaces. We define so-called balanced Cauchy filter pairs $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-pseudometric space and construct a double completion of T_0 -quasi-pseudometric spaces that we call the **B -completion** of quasi-pseudometric spaces. The B -completion contains the bicompletion of the original space and our construction can be applied to any T_0 -quasi-pseudometric space. In the case of balanced T_0 -quasi-pseudometrics, the B -completion yields up to isometry the Doitchinov completion.

1.2 Basic concepts

This section presents some of basic concepts and facts from the general theory of quasi-pseudometric spaces.

Definition 1.2.1 Let X be a set and let $d : X \times X \rightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of non-negative reals. Then d is called a **quasi-pseudometric** on X if

$$(d1) \quad d(x, x) = 0 \text{ whenever } x \in X;$$

$$(d2) \quad d(x, z) \leq d(x, y) + d(y, z) \text{ whenever } x, y, z \in X.$$

We say that d is a **T_0 -quasi-pseudometric** if d also satisfies the following condition: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.

The pair (X, d) is called **quasi-pseudometric space**. The condition (d2) is called the **triangle inequality**.

A quasi-pseudometric d is called **pseudometric** when it satisfies the symmetry condition : For all $x, y \in X$

$$(d3) \quad d(x, y) = d(y, x).$$

Let d be a quasi-pseudometric on X , we define the conjugate d^{-1} of d by : $d^{-1}(y, x) = d(x, y)$ whenever $x, y \in X$ and $d^* = \max\{d, d^{-1}\}$ is a pseudometric on X .

Every quasi-pseudometric d induces a topology $\mathcal{T}(d)$, where the basic open neighborhoods of a point $x \in X$ are the sets $V_{d, \varepsilon}[x] = \{y \in X : d(x, y) < \varepsilon\}$ where $\varepsilon > 0$.

Note that the topology $\mathcal{T}(d)$ is a **Hausdorff Topology** if we have for any sequence $(x_n) \in X$, $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ and $\lim_{n \rightarrow \infty} d(y, x_n) = 0$ then $x = y$ (That is, it is impossible to have $x_n \rightarrow x$ and $x_n \rightarrow y$ for $x \neq y$).

Chapter 2

Bicompletion of quasi-pseudometric spaces or Salbany's completion

In this chapter we shall investigate the bicompletion of T_0 -quasi-pseudometric spaces introduced by Salbany in [31]. We will first recall some of the definitions of Cauchy sequences in quasi-pseudometric spaces and secondly, we shall construct the bicompletion of a T_0 -quasi-pseudometric space (X, d) and show that each T_0 -quasi-pseudometric space has a bicompletion .

In the last section, we shall investigate the extension of quasi-uniformly continuous mappings and prove that the bicompletion of quasi-pseudometric spaces is unique up to isometry.

2.1 Definition of a Cauchy sequence in a quasi-pseudometric space

Normally a concept of completeness is based on a notion of a Cauchy sequence. There are several definitions of a Cauchy sequence in the literature which can be used to define several kinds of completeness of quasi-pseudometric spaces.

The one introduced by Salbany will be used throughout this chapter to construct the bicompletion of a quasi-pseudometric space.

In the study of metric spaces, the usual definition is the following:

Definition 2.1.1 Let (X, d) be a metric space, a sequence (x_n) is called **Cauchy sequence** if for every $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that if $m \geq n \geq n_\varepsilon$ then $d(x_n, x_m) < \varepsilon$.

The following definition is according to Pervin and Sieber (see [31]).

Definition 2.1.2 Let (X, d) be a quasi-pseudometric space. A sequence (x_n) is called a Cauchy sequence if for every $\varepsilon > 0$, there are $n_\varepsilon \in \mathbb{N}$ and x_ε such that $d(x_\varepsilon, x_n) < \varepsilon$ whenever $n \geq n_\varepsilon$.

Kelly has given the following definition (see [1]).

Definition 2.1.3 ([1]) If (X, d) is a quasi-pseudometric space, a sequence (x_n) is called :

(i) **Left-Cauchy** if for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n \geq k$.

(ii) **Right-Cauchy** if for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m \geq k$.

In [11] Doitchinov has used the following definition:

Definition 2.1.4 ([11]) We call a sequence (x_n) in the quasi-pseudometric space (X, d) a **D-Cauchy** sequence provided that there is a sequence $(y_k) \in X$ satisfying the property that for every $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \geq n_\varepsilon$, we have that $d(y_m, x_n) < \varepsilon$. In this case we call (y_m) a cosequence of (x_n) and we write $\lim_{m, n \rightarrow \infty} d(y_m, x_n) = 0$.

The following definition is due to Salbany (see [31]).

Definition 2.1.5 ([31]) Let (X, d) be a quasi-pseudometric space. The sequence (x_n) in X is a Cauchy sequence if $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$. A quasi-pseudometric space (X, d) is **bicomplete** if every Cauchy sequence (x_n) converges with respect to $T(d)$ and with respect to $T(d^{-1})$ to a point x_0 .

Note that (x_n) is a Cauchy sequence in (X, d) in this sense if and only if (x_n) is a Cauchy sequence in the pseudometric space (X, d^*) .

2.2 Bicompletion of quasi-pseudometric spaces or Salbany's completion

This section investigates the bicompletion of T_0 -quasi-pseudometric spaces which is also called Salbany's completion.

It will be constructed with a simple method that has been used by Salbany in [31].

The following proposition is immediate (see [31]).

Proposition 2.2.1 *A T_0 -quasi-pseudometric space (X, d) is bicomplete if and only if the metric space (X, d^*) is complete, where $d^* = d \vee d^{-1}$, that is $d^* = \max\{d, d^{-1}\}$.*

Definition 2.2.1 ([31]) *A bicompletion of a T_0 -quasi-pseudometric space (X, d) is a bicomplete T_0 -quasi-pseudometric space (Y, e) such that (X, d) is isometric to a $\mathcal{T}(e^*)$ -dense subspace of (Y, e) .*

In the following, we are going to construct the bicompletion of a given T_0 -quasi-pseudometric space.

Proposition 2.2.2 ([31]) *Let (X, d) be a quasi-pseudometric space. Define an equivalence relation \sim on X by $x \sim y$ if and only if $d(x, y) = 0 = d(y, x)$. Let \hat{X} be the set of all equivalence classes \hat{x} with respect to \sim where $x \in X$. Then the function \hat{d} on \hat{X} defined by $\hat{d}(\hat{x}, \hat{y}) = d(x, y)$ is a T_0 -quasi-pseudometric on \hat{X} .*

Proof. It is clear that \sim is reflexive and symmetric. We now show that \sim is transitive.

Let $x \sim y$ and $y \sim z$, so we have that $d(x, y) = 0 = d(y, x)$ and $d(y, z) = 0 = d(z, y)$. By using the triangle inequality as $d(x, z) \leq d(x, y) + d(y, z)$ and $d(z, x) \leq d(z, y) + d(y, x)$ we get that $d(x, z) = 0 = d(z, x)$. That is $x \sim z$.

Then \sim is transitive. The quotient set is denoted by \widehat{X} .

We next show that \widehat{d} is well-defined on \widehat{X} .

Suppose that $x, x', y, y' \in X$, $x \sim x'$ and $y \sim y'$. By the triangle inequality we see that $d(x', y') \leq d(x', x) + d(x, y) + d(y, y')$ thus, $d(x', y') \leq 0 + d(x, y) + 0$. Similarly, we get that $d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$, that is $d(x, y) \leq 0 + d(x', y') + 0$, hence $d(x, y) = d(x', y')$ and we have shown that \widehat{d} is well-defined.

We now show that \widehat{d} is T_0 .

If $\widehat{d}(\hat{x}, \hat{y}) = \widehat{d}(\hat{y}, \hat{x}) = 0$, then $d(x, y) = d(y, x) = 0$ which implies that $\hat{x} = \hat{y}$.

The following part will discuss the main theorem dealing with the bicompletion of a quasi-pseudometric space.

Theorem 2.2.1 ([31]) *Each T_0 -quasi-pseudometric space (X, d) has a bicompletion denoted by $(\widetilde{X}, \widetilde{d})$ which is a T_0 -quasi-pseudometric space.*

Proof. Let us denote by Y the set of all Cauchy sequences in the metric space (X, d^*) . For each pair $((x_n), (y_n)) \in Y \times Y$, define $d'((x_n), (y_n)) = \lim_{n \rightarrow \infty} d(x_n, y_n)$. Then (Y, d') is a quasi-pseudometric space.

Before showing that d' is a quasi-pseudometric on Y , we need the following lemma.

Lemma 2.2.1 *Let (X, d) be a quasi-pseudometric space. Then for all $a, b, x, y \in X$, we have that*

$$|d(x, y) - d(a, b)| \leq d^*(x, a) + d^*(y, b).$$

Proof. If $x, y, a, b \in X$, by the triangle inequality, we get that

$$d(x, y) - d(a, b) \leq d(x, a) + d(b, y)$$

and

$$d(a, b) - d(x, y) \leq d(a, x) + d(y, b)$$

that implies

$$|d(x, y) - d(a, b)| \leq d^*(x, a) + d^*(y, b).$$

Corollary 2.2.1 Let (X, d) be a quasi-pseudometric space and $(x_n), (y_n)$ sequences in (X, d) . If $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$ with respect to $\mathcal{T}(d^*)$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$.

Proof. Let (x_n) and (y_n) be sequences in (X, d^*) . It follows from Lemma 2.2.1 that

$$|d(x_n, y_n) - d(x, y)| \leq d^*(x_n, x) + d^*(y, y_n) \rightarrow 0.$$

Hence

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y).$$

Lemma 2.2.2 ([31]) The space (Y, d') is a quasi-pseudometric space.

Proof. Let $(x_n), (y_n) \in Y$ where (x_n) and (y_n) are two Cauchy sequences in X . We observe that $(d(x_n, y_n))$ is a Cauchy sequence of real numbers. For $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$, such that $d^*(x_n, x_m) < \frac{\varepsilon}{2}$ and $d^*(y_n, y_m) < \frac{\varepsilon}{2}$ whenever $n, m \geq n_\varepsilon$. It follows from the above lemma that

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d^*(x_n, x_m) + d^*(y_n, y_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n, m \geq n_\varepsilon$. Hence we get that $\lim_{n \rightarrow \infty} d(x_n, y_n)$ exists. Moreover we have that:

(a) $d'((x_n), (x_n)) = \lim_{n \rightarrow \infty} d(x_n, x_n) = 0.$

(b) Let $(x_n), (y_n)$ and $(z_n) \in Y$, suppose that $d'((x_n), (y_n)) = a$ and $d'((y_n), (z_n)) = b$. For any $\varepsilon > 0$, there are m_1, m_2 such that $d(x_n, y_n) < a + \frac{\varepsilon}{2}$ whenever $n \geq m_1$ and $d(y_n, z_n) < b + \frac{\varepsilon}{2}$ whenever $n \geq m_2$.

It follows that $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < a + \frac{\varepsilon}{2} + b + \frac{\varepsilon}{2} = a + b + \varepsilon$ whenever $n \geq m_1, m_2$. Hence, we get that $d'((x_n), (z_n)) \leq d'((x_n), (y_n)) + d'((y_n), (z_n))$, and so d' is a quasi-pseudometric on Y .

Now, let $(x_n), (y_n) \in Y$, we define an equivalence relation on Y by : $(x_n) \sim (y_n)$ if $(d')^*((x_n), (y_n)) = 0$. We denote the quotient set by \tilde{X} and $[(x_n)]$ denotes an equivalence class. For each pair $[(x_n)], [(y_n)] \in \tilde{X}$, let

$$\tilde{d}([(x_n)], [(y_n)]) = d'((x_n), (y_n))$$

that means

$$\tilde{d}([(x_n)], [(y_n)]) = \lim_{n \rightarrow \infty} d(x_n, y_n).$$

In the following step, we will show that \tilde{X} has a subspace which is isometric to X .

Lemma 2.2.3 ([31]) *The T_0 -quasi-pseudometric space (X, d) can be isometrically embedded into (\tilde{X}, \tilde{d}) .*

Proof. Let X_0 be the subspace of \tilde{X} consisting of those equivalence classes which contain a Cauchy sequence (x_n) for which $x_n = x$ where $x \in X$ whenever $n \in \mathbb{N}$. Denote by $[(\underline{x})]$ an element in X_0 . If $[(\underline{x})], [(\underline{y})] \in X_0$, define the map $i : X \rightarrow \tilde{X}$ by $i(x) = [(\underline{x})]$. Then $\tilde{d}(i(x), i(y)) = \tilde{d}([(x)], [(y)]) = d(x, y)$. It is then clear that i is an isometry from X into \tilde{X} .

Since (X, d) is T_0 space, i is injective because $x \neq y$ implies that $i(x) \neq i(y)$, that is we cannot find two different Cauchy sequences of this kind in the same equivalence class and $i(X) = X_0$ can be identified with X . Then X can be regarded as a subspace of (\tilde{X}, \tilde{d}) .

Lemma 2.2.4 ([31]) *X_0 is a $\mathcal{T}((\tilde{d})^*)$ -dense subspace of the quasi-pseudometric space (\tilde{X}, \tilde{d}) .*

Proof. Let $[(x_n)] \in \tilde{X}$ where (x_n) be a Cauchy sequence in (X, d) . For every $\varepsilon > 0$ there exists N such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$. Then $i(x_m) \in X_0$ for fixed m ; letting $m \rightarrow \infty$, we get that $i(x_m) \rightarrow [(x_n)]$ in (\tilde{X}, \tilde{d}) . Hence X_0 is $\mathcal{T}((\tilde{d})^*)$ -dense in \tilde{X} .

The following part will show that the quasi-pseudometric \tilde{d} is bicomplete.

Lemma 2.2.5 ([31]) *The space (\tilde{X}, \tilde{d}) is bicomplete.*

Proof. Let (ξ_n) be a Cauchy sequence in (\tilde{X}, \tilde{d}) . For each n , let us choose $i(z_n) \in X_0$ such that $(\tilde{d})^*(\xi_n, i(z_n)) < \frac{1}{n}$. We first need to show that

(z_n) is a Cauchy sequence in (X, d) .

We have that:

$$\begin{aligned} d(z_n, z_m) &= \tilde{d}(i(z_n), i(z_m)) \\ &\leq \tilde{d}(i(z_n), \xi_n) + \tilde{d}(\xi_n, \xi_m) + \tilde{d}(\xi_m, i(z_m)) \\ &\leq \frac{1}{n} + \frac{1}{m} + \tilde{d}(\xi_n, \xi_m). \end{aligned}$$

So (z_n) is a Cauchy sequence in (X, d) . Hence $[(z_n)] \in \tilde{X}$. It follows from the above lemma that $(\tilde{d})^*(i(z_n), [(z_n)]) \rightarrow 0$. We have that

$$\begin{aligned} (\tilde{d})^*(\xi_n, [(z_n)]) &\leq (\tilde{d})^*(\xi_n, i(z_n)) + (\tilde{d})^*(i(z_n), [(z_n)]) \\ &\leq \frac{1}{n} + (\tilde{d})^*(i(z_n), [(z_n)]) \end{aligned}$$

which implies that $(\tilde{d})^*(\xi_n, [(z_n)]) \rightarrow 0$, hence (ξ_n) converges in $(\tilde{X}, (\tilde{d})^*)$ and (\tilde{X}, \tilde{d}) is bicomplete.

The space (\tilde{X}, \tilde{d}) is a T_0 -quasi-pseudometric space by Proposition 2.2.2. Then the proof of the theorem is complete.

The quasi-pseudometric space (\tilde{X}, \tilde{d}) constructed in such a manner is a bicompletion of the quasi-pseudometric space (X, d) .

2.3 Extension of quasi-uniformly continuous mappings

In this section we shall investigate the extension of quasi-uniformly continuous mappings as one of many applications of Salbany's completion. The extension theorem will allow us to show the uniqueness of the T_0 -bicompletion. We first define the quasi-uniformly continuous mapping.

Definition 2.3.1 ([31]) *Let (X_1, d_1) and (X_2, d_2) be two quasi-pseudometric spaces. The map $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is called quasi-uniformly continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, $d_1(x, y) < \delta$ implies that $d_2(f(x), f(y)) < \varepsilon$.*

Proposition 2.3.1 ([31]) *Let $f : (X, d_1) \longrightarrow (Y, d_2)$ be a quasi-uniformly continuous map from a T_0 -quasi-pseudometric space (X, d_1) into a bicomplete T_0 -quasi-pseudometric space (Y, d_2) . Then f has a unique quasi-uniformly continuous extension map $\tilde{f} : (\tilde{X}, \tilde{d}_1) \longrightarrow (Y, d_2)$.*

Proof. Let $[(x_n)]$ be an equivalence class in \tilde{X} where (x_n) is a Cauchy sequence in (X, d) . Since f is quasi-uniformly continuous, the image of (x_n) is a Cauchy sequence $(f(x_n))$ in (Y, d_2) . Since Y is bicomplete and a T_0 -space, we have that the unique $\lim_{n \rightarrow \infty} f(x_n)$ exists in (Y, d_2^*) .

We define the map \tilde{f} by:

$$\tilde{f}([(x_n)]) = \lim_{n \rightarrow \infty} f(x_n)$$

and for any $x \in X$, we have that

$$\tilde{f}(i(x)) = f(x).$$

Hence \tilde{f} extends f .

We first show that \tilde{f} is well-defined:

Let $[(x_n)], [(y_n)] \in \tilde{X}$. If $[(x_n)] = [(y_n)]$, we have that $d_1(x_n, y_n) \longrightarrow 0$ and $d_1(y_n, x_n) \longrightarrow 0$. Since f is quasi-uniformly continuous, $d_2(f(x_n), f(y_n)) \longrightarrow 0$ and $d_2(f(y_n), f(x_n)) \longrightarrow 0$. Let us suppose that $\lim_{n \rightarrow \infty} f(x_n) = a$ and $\lim_{n \rightarrow \infty} f(y_n) = b$ in the space (Y, d_2^*) . We have that $\lim_{n \rightarrow \infty} d_2^*(a, f(x_n)) = 0$ and also, $\lim_{n \rightarrow \infty} d_2^*(b, f(y_n)) = 0$, hence

$$d_2(a, b) \leq d_2(a, f(x_n)) + d_2(f(x_n), f(y_n)) + d_2(f(y_n), b) \longrightarrow 0$$

and

$$d_2(b, a) \leq d_2(b, f(y_n)) + d_2(f(y_n), f(x_n)) + d_2(f(x_n), a) \longrightarrow 0.$$

Hence

$$d_2(a, b) = d_2(b, a) = 0.$$

Since d_2 is a T_0 -quasi-pseudometric, we get that

$$a = b,$$

that means

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n).$$

Thus

$$\tilde{f}([(x_n)]) = \tilde{f}([(y_n)])$$

and \tilde{f} is well-defined.

We next show that \tilde{f} is quasi-uniformly continuous.

For every $\varepsilon > 0$ there is $\delta > 0$ such that $d_1(x, y) < \delta$ implies that $d_2(f(x), f(y)) < \varepsilon$ whenever $x, y \in X$. Let $[(x_n)], [(y_n)] \in \tilde{X}$. If $\tilde{d}_1([(x_n)], [(y_n)]) < \delta$ then there exists N such that for all $n \geq N$, $d_1(x_n, y_n) < \delta$ and we have that $d_2(f(x_n), f(y_n)) < \varepsilon$. So for all $k \geq N$, because of $\tilde{f}([(x_n)]) = \lim_{k \rightarrow \infty} f(x_k)$, we get $d_2(\tilde{f}([(x_n)]), \tilde{f}([(y_n)])) \leq d_2(\tilde{f}([(x_n)]), f(x_k)) + d_2(f(x_k), f(y_k)) + d_2(f(y_k), \tilde{f}([(y_n)]))$.

Since $k \rightarrow \infty$, we see

$$d_2(\tilde{f}([(x_n)]), f(x_k)) \longrightarrow 0$$

and

$$d_2(f(y_k), \tilde{f}([(y_n)])) \longrightarrow 0,$$

hence

$$d_2(\tilde{f}([(x_n)]), \tilde{f}([(y_n)])) \leq \varepsilon.$$

So \tilde{f} is uniformly continuous.

Finally, it will be shown that \tilde{f} is unique.

Suppose that \tilde{g} is another quasi-uniformly continuous extension of f and (x_n) be a Cauchy sequence in X . We have that $\tilde{g} : (X, d_1^*) \rightarrow (Y, d_2^*)$ is continuous, and $d_1^*([(x_n)], i(x_n)) \rightarrow 0$ as $n \rightarrow \infty$. Thus we get that $\tilde{g}([(x_n)]) = \lim_{n \rightarrow \infty} \tilde{g}(i(x_n))$ in (Y, d_2^*) . Since \tilde{g} is an extension of f , we obtain that

$$\tilde{g}(i(x_n)) = f(x_n)$$

for each n . And so

$$\tilde{g}([(x_n)]) = \lim_{n \rightarrow \infty} f(x_n) = \tilde{f}([(x_n)]).$$

Hence

$$\tilde{g} = \tilde{f}.$$

We next introduce the extension of an isometric mapping.

Proposition 2.3.2 ([27]) *Let (X, d_1) be a quasi-pseudometric space and (Y, d_2) be a bicomplete T_0 -quasi-pseudometric space. If there is an isometry f from a dense subspace X_0 of (X, d_1^*) to Y , then f has a unique isometric extension $f^* : (X, d_1) \longrightarrow (Y, d_2)$.*

Proof. Let (x_n) be a Cauchy sequence in X_0 and $x \in X$ such that $d_1^*(x, x_n) \longrightarrow 0$. Then $(f(x_n))$ is a Cauchy sequence in the bicompletion space (Y, d_2) and converges to y^* in (Y, d_2^*) . That is $\lim_{n \rightarrow \infty} f(x_n) = y^*$. Similarly as before we define the map f^* by $f^*(x) = y^*$ for all $x \in X$. It follows from the above argument that f^* is well defined.

Since each isometry is quasi-uniformly continuous, we have that f^* is quasi-uniformly continuous.

We show that f^* is an isometry.

Let $x, y \in X$ and $(x_n), (y_n)$ be sequences in X_0 such that $d_1^*(x, x_n) \longrightarrow 0$ and $d_1^*(y, y_n) \longrightarrow 0$ as $n \longrightarrow \infty$. From the above lemma and by using the fact that d_2 is continuous with respect to $\mathcal{T}(d_2^*) \times \mathcal{T}(d_2^*)$, we get

$$d_2^*(f^*(x), f^*(y)) = \lim_{n \rightarrow \infty} d_2(f(x_n), f(y_n)).$$

Since f is isometric,

$$d_2(f(x_n), f(y_n)) = d_1(x_n, y_n)$$

and we have that

$$\lim_{n \rightarrow \infty} d_2(f(x_n), f(y_n)) = \lim_{n \rightarrow \infty} d_1(x_n, y_n) = d_1(x, y)$$

Hence

$$d_2(f^*(x), f^*(y)) = d_1(x, y).$$

So f^* is an isometry. Obviously, f^* is unique by the preceding result.

We next show that the bicompletion of a quasi-pseudometric space (X, d) is unique.

Corollary 2.3.1 *Each T_0 -quasi-pseudometric space (X, d) has a unique bicompletion up to isometry.*

Proof. We have to show that if (T, q) is a bicompletion of a T_0 -quasi-pseudometric space (X, d) , then there exists an isometry from (\tilde{X}, \tilde{d}) to (T, q) that leaves X pointwise fixed.

Let $i_1 : X \rightarrow \tilde{X}$ be the standard isometric embedding. Since X is dense in the space $(\tilde{X}, (\tilde{d})^*)$ and by the above proposition, there is an isometric extension $f_1 : (T, q) \rightarrow (\tilde{X}, \tilde{d})$ of i_1 .

Let $i_2 : X \rightarrow T$ be the standard isometric embedding. Let the map $f_2 : (\tilde{X}, \tilde{d}) \rightarrow (T, q)$ be an isometric extension of i_2 . The composite

$$f_1 \circ f_2 : (\tilde{X}, \tilde{d}) \rightarrow (\tilde{X}, \tilde{d})$$

and

$$f_2 \circ f_1 : (T, q) \rightarrow (T, q)$$

are such that for all $x \in X$,

$$f_1(f_2(x)) = x$$

and

$$f_2(f_1(x)) = x.$$

We have that $f_1 \circ f_2$ is an isometric extension of the identity map $i_X : X \rightarrow X$. But, the identity map $i_{\tilde{X}} : \tilde{X} \rightarrow \tilde{X}$ is also an isometric extension of the identity i_X . By uniqueness of the considered extension mappings, we get that

$$f_1 \circ f_2 = i_{\tilde{X}}$$

and similarly,

$$f_2 \circ f_1 = i_T.$$

Hence f_1 and f_2 are bijective and so, the bicompletion of (X, d) is unique up to isometry.

Chapter 3

Bicompletion of weightable quasi-pseudometric spaces and completion of partial metric spaces

The notion of partial metric spaces was introduced by S.G. Matthews in [25] as a part of his study of denotational semantics of dataflow networks.

In this chapter we will first introduce partial metric spaces and discuss their algebraic equivalence with weightable quasi-pseudometric spaces.

We will secondly construct the completion of a partial metric space and apply the bicompletion studied in the preceding chapter to show that the bicompletion of a weightable quasi-pseudometric space is a weightable quasi-pseudometric space. This result will be used to construct the completion of partial metric spaces.

The notion of completeness of a partial metric space will be applied in the last section to investigate the appropriate Banach fixed point theorem.

3.1 Some definitions in a partial metric space

We introduce the definition of a weightable quasi-pseudometric space and that of a partial metric space. We establish an equivalence between these two notions (see [25]).

Definition 3.1.1 ([25, 27]) A quasi-pseudometric space (X, d) is called **weightable** if there is a weighting function $w : X \rightarrow [0, \infty)$ such that for all $x, y \in X$,

$$d(x, y) + w(x) = d(y, x) + w(y) \quad (3.1)$$

We denote by $w(x)$ the weight of x and, if d is weightable by the weighting function w , the triple (X, d, w) is called a **weighted quasi-pseudometric space**.

We next define the concept of a partial metric space.

Definition 3.1.2 [25, 27]) A **partial metric or pmetric** on a nonempty set X is a function $p : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

- (P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
- (P2) $p(x, x) \leq p(x, y)$;
- (P3) $p(x, y) = p(y, x)$;
- (P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X .

Each partial metric on X generates a T_0 topology $\mathcal{T}(p)$ on X which has as a base the family of open p -balls $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$ (see [25]).

A sequence (x_n) in (X, p) converges to a point $x \in X$ if $\lim_{n \rightarrow \infty} p(x, x_n) = p(x, x)$.

We next recall the definition of a partial order and its associated quasi-pseudometric.

Definition 3.1.3 ([25]) A partial order \leq is a binary relation on a set X such that for all $x, y, z \in X$:

- (P01) $x \leq x$;
(P01) $x \leq y$ and $y \leq x$ imply that $x = y$;
(P02) $x \leq y$ and $y \leq z$ imply that $x \leq z$.

Remark 3.1.1 ([25]) For each partial metric p on X , the associated partial order \leq_p of p is defined by :
 $x \leq_p y$ if and only if $p(x, x) = p(x, y)$.
For each T_0 -quasi-pseudometric d on X , the associated partial order of d is given by: $x \leq_d y$ if and only if $d(x, y) = 0$. It is also called the specialization order.

We next introduce an algebraic equivalence between the partial metric space and the weightable quasi-pseudometric space.

Theorem 3.1.1 ([25, 27]) For each partial metric p on X , the function defined by $d_p : X \times X \rightarrow [0, \infty)$ where for all $x, y \in X$,

$$d_p(x, y) = p(x, y) - p(x, x)$$

is a weightable T_0 -quasi-pseudometric with the weighting function $w(x) = p(x, x)$ such that

$$\mathcal{T}(p) = \mathcal{T}(d_p)$$

and

$$\leq_p = \leq_{d_p}.$$

Proof. We first show that d_p is a T_0 -quasi-pseudometric.

For all $x, y, z \in X$, we have that:

$$(d1) \quad d_p(x, x) = p(x, x) - p(x, x) = 0.$$

$$(d2) \quad \begin{aligned} d_p(x, z) &= p(x, z) - p(x, x) \\ &\leq p(x, y) + p(y, z) - p(y, y) - p(x, x) \\ &= d_p(x, y) + d_p(y, z). \end{aligned}$$

Since $w(x) = p(x, x)$ is the weight of x , we have that

$$\begin{aligned}
d_p(x, y) + w(x) &= p(x, y) - p(x, x) + p(x, x) \\
&= p(y, x) - p(y, y) + p(y, y) \\
&= d_p(y, x) + w(y).
\end{aligned}$$

We show that d_p is T_0 .

From (P1), we have that

$$d_p(x, y) = p(x, y) - p(x, x) = 0 = p(y, x) - p(y, y) = d_p(y, x)$$

implies that

$$x = y.$$

Hence (X, d_p, w) is a weighted T_0 -quasi-pseudometric space.

It remains to show that $\mathcal{T}(d_p) = \mathcal{T}(p)$.

Let $x \in X$ and $\varepsilon > 0$, let $y \in B_{d_p}(x, \varepsilon)$. Then $d_p(x, y) = p(x, y) - p(x, x) < \varepsilon$ and hence, $p(x, y) < \varepsilon + p(x, x)$. Consequently $y \in B_p(x, \varepsilon)$ and $\mathcal{T}(d_p) \subseteq \mathcal{T}(p)$.

Conversely if $y \in B_p(x, \varepsilon)$, we have that $p(x, y) < \varepsilon + p(x, x)$. Thus $d_p(x, y) = p(x, y) - p(x, x) < \varepsilon$, $y \in B_{d_p}(x, \varepsilon)$ and $\mathcal{T}(p) \subseteq \mathcal{T}(d_p)$, hence

$$\mathcal{T}(p) = \mathcal{T}(d_p).$$

Finally, for any $x, y \in X$, $p(x, x) = p(x, y)$ if and only if $d_p(x, y) = 0$ thus $\leq_p = \leq_{d_p}$.

Theorem 3.1.2 ([25]) *For each weighted T_0 -quasi-pseudometric space (X, d) with weighting function w , the function defined by:*
 $p_d : X \times X \longrightarrow [0, \infty)$ *where for all $x, y \in X$,*

$$p_d(x, y) = d(x, y) + w(x)$$

is a partial metric on X such that

$$\mathcal{T}(p_d) = \mathcal{T}(d)$$

and

$$\leq_p = \leq_d .$$

Proof. We first prove that p_d is a partial metric.

Let $x, y, z \in X$.

(P1) If $x = y$, we get that: $p_d(x, x) = p_d(x, y)$.

Since d is a T_0 -quasi-pseudometric, we get that

$$p_d(x, y) = d(x, y) + w(x) = p_d(x, x) = d(y, x) + w(y) = p_d(y, y)$$

implies that

$$x = y.$$

(P2) Since $0 \leq d(x, y)$, we have that $w(x) \leq d(x, y) + w(x)$. By (d1), we get that $p_d(x, x) \leq p_d(x, y)$.

(P3) $d(x, y) + w(x) = d(y, x) + w(y)$ by (3.1), we conclude that $p_d(x, y) = p_d(y, x)$.

(P4) Since $d(x, z) \leq d(x, y) + d(y, z)$, we have that $d(x, z) + w(x) \leq (d(x, y) + w(x)) + (d(y, z) + w(y)) - w(y)$ and so,

$$p_d(x, z) \leq p_d(x, y) + p_d(y, z) - p_d(y, y).$$

Hence p_d is a partial metric.

It remains to show that $\mathcal{T}(p_d) = \mathcal{T}(d)$.

Indeed, let $x \in X$ and $\varepsilon > 0$. Let $y \in B_{p_d}(x, \varepsilon)$, then $p_d(x, y) = d(x, y) + w(x) < \varepsilon$ and hence $d(x, y) < \varepsilon - w(x)$ and therefore $y \in B_d(x, \varepsilon)$. Thus $\mathcal{T}(p_d) \subseteq \mathcal{T}(d)$.

Conversely if $y \in B_d(x, \varepsilon)$, we have that $d(x, y) < \varepsilon$, thus, $p_d(x, y) - w(x) < \varepsilon$ implies that $p_d(x, y) - p_d(x, x) < \varepsilon$ and $y \in B_{p_d}(x, \varepsilon)$. Hence $\mathcal{T}(d) \subseteq \mathcal{T}(p_d)$.

3.2 Bicompletion of weightable quasi-pseudometric spaces and completion of partial metric spaces

It will be shown in this section that the weighted T_0 -quasi-pseudometric space (X, d, w) has a bicompletion which is a weighted T_0 -quasi-pseudometric space. From this result, we will construct a completion of a partial metric space (see [27]). We will show that each partial metric space has a completion which is unique up to isometry.

We next introduce the definition of a Cauchy sequence in a partial metric space.

Definition 3.2.1 ([33, 27]) *A sequence (x_n) is called a **Cauchy sequence** in (X, p) if $\lim_{n,m \rightarrow \infty} p(x_n, x_m)$ exists. A partial metric space (X, p) is complete if every Cauchy sequence (x_n) converges to a point $x \in X$.*

Lemma 3.2.1 ([27]) *A partial metric space (X, p) is complete if and only if (X, d_p) is bicomplete. A weightable T_0 -quasi-pseudometric space (X, d) is bicomplete if and only if (X, p_d) is a complete partial metric space.*

Proof. We refer to [27] for the proof.

Theorem 3.2.1 ([27]) *Let (X, d) be a weighted quasi-pseudometric space with weighting function w . Then the bicompletion (\tilde{X}, \tilde{d}) of (X, d) is weightable with weighting function \tilde{w} given by:*

$$\tilde{w}([(x_n)_n]) = \lim_{n \rightarrow \infty} w(x_n).$$

Proof. We show first that \tilde{w} is well-defined on \tilde{X} . Let us suppose that (x_n) is a Cauchy sequence in (X, d^*) . For every $\varepsilon > 0$, there is N such that $d^*(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$. Since w is a weighting function for (X, d) , it follows from Lemma 2.2.1 that:

$$\begin{aligned} |w(x_n) - w(x_m)| &\leq |d(x_m, x_n) - d(x_n, x_m)| \\ &\leq d^*(x_m, x_n) + d^*(x_n, x_m) \end{aligned}$$

whenever $n, m \geq N$. Then $(w(x_n))$ is a Cauchy sequence in \mathbb{R} , so it converges. Let $[(x_n)_n], [(y_n)_n] \in \tilde{X}$. If $[(x_n)_n] = [(y_n)_n]$, we have that: $d(x_n, y_n) + w(x_n) = d(y_n, x_n) + w(y_n)$ for all $n \in \mathbb{N}$. Since $d^*(x_n, y_n) \rightarrow 0$, we get that $\lim_{n \rightarrow \infty} w(x_n) = \lim_{n \rightarrow \infty} w(y_n)$ and thus,

$$\tilde{w}([(x_n)_n]) = \tilde{w}([(y_n)_n]).$$

So, \tilde{w} is well defined.
Since for each $n \in \mathbb{N}$,

$$d(x_n, y_n) + w(x_n) = d(y_n, x_n) + w(y_n).$$

we get that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) + \lim_{n \rightarrow \infty} w(x_n) = \lim_{n \rightarrow \infty} d(y_n, x_n) + \lim_{n \rightarrow \infty} w(y_n)$$

and so

$$\tilde{d}([(x_n)_n], [(y_n)_n]) + \tilde{w}([(x_n)_n]) = \tilde{d}([(y_n)_n], [(x_n)_n]) + \tilde{w}([(y_n)_n]).$$

Hence \tilde{d} is weightable quasi-pseudometric.

In the light of the equivalence between partial metric spaces and weightable quasi-pseudometric spaces as shown in the first section, we have the following lemma:

Lemma 3.2.2 ([27]) (a) If (x_n) is a Cauchy sequence in a partial metric space (X, p) , then (x_n) is a Cauchy sequence in the metric space $(X, (d_p)^*)$.
(b) If (X, d) is a weightable quasi-pseudometric space, then every Cauchy sequence in (X, d^*) is a Cauchy sequence in the partial metric space (X, p_d) .

Proof. We again refer the reader to [27] for the proof.

We next define the completion of a partial metric space.

Definition 3.2.2 A partial metric space (Y, p_2) is called a **completion** of the partial metric space (X, p_1) if (Y, p_2) is a complete partial metric space such that (X, p_1) is isometric to a dense subspace of the partial metric space (Y, p_2) .

We next define isometric maps between partial metric spaces.

Definition 3.2.3 ([27]) Let (X, p_1) and (Y, p_2) be two partial metric spaces. A map $f : X \rightarrow Y$ is called an **isometry** from X to Y provided that $p_2(f(x), f(y)) = p_1(x, y)$ whenever $x, y \in X$.

We note that f is an isometry from (X, p_1) to (Y, p_2) if and only if f is an isometry from (X, d_{p_1}) to (Y, d_{p_2}) which is weight preserving, and we say that f is weight preserving if and only if $w(f(x)) = w(x)$ whenever $x \in X$.

We next introduce an extension theorem for partial metric spaces. To this end, we recall the following result that will be useful subsequently (compare Corollary 2.3.1).

Proposition 3.2.1 ([27]) Let (X, p_1) be a partial metric space and let (Y, p_2) be a complete partial metric space. Let A be a $\mathcal{T}((d_{p_1})^*)$ -dense subspace of (X, p_1) . If $f : A \rightarrow Y$ is an isometry between partial metric spaces, then it has a unique such extension $f^* : X \rightarrow Y$.

Proof. We observe that f is a weight preserving isometry from the T_0 -quasi-pseudometric space (A, d_{p_1}) to the bicomplete T_0 -quasi-pseudometric space (Y, d_{p_2}) . Let $a, b \in A$. We have that

$$d_{p_2}(f(a), f(b)) = d_{p_1}(a, b).$$

It follows from Proposition 2.3.2 that f has a unique isometric extension $f^* : (X, d_{p_1}) \rightarrow (Y, d_{p_2})$, that is $d_{p_2}(f^*(x), f^*(y)) = d_{p_1}(x, y)$ whenever $x, y \in X$. Furthermore f^* is readily seen to be weight preserving.

Indeed f^* is an isometry from (X, p_1) to (Y, p_2) , that is $p_2(f^*(x), f^*(y)) = p_1(x, y)$ whenever $x, y \in X$.

Theorem 3.2.2 ([27]) *Each partial metric space (X, p) has a completion which is unique up to isometry.*

Proof. Let (X, p) be a partial metric space. Consider the weightable T_0 -quasi-pseudometric space (X, d_p) . It follows from Theorem 3.2.1 that (\tilde{X}, \tilde{d}_p) is a weightable T_0 -bicompletion of (X, d_p) with a weighting function \tilde{w}_p given by

$$\tilde{w}_p([x_n]) = \lim_{n \rightarrow \infty} p(x_n, x_n).$$

It follows from Lemma 3.2.1 that the partial metric space $(\tilde{X}, p_{\tilde{d}_p})$ is complete. Therefore $(\tilde{X}, p_{\tilde{d}_p})$ is a completion of (X, p) which will be denoted by (\tilde{X}, \tilde{p}) .

Uniqueness of (\tilde{X}, \tilde{p}) follows from Proposition 2.3.2 in the usual way.

3.3 Banach's fixed point theorem for a partial metric space

In the following section we will investigate the contraction mapping theorem or Banach fixed point theorem for a given partial metric space. This important theorem gives us a sufficient condition for a unique fixed point to exist. The theorem will be applied in the case where (X, p) is complete and the map f from X into X is a contraction (see [25, 33]).

We first recall two definitions: The definition of a contraction mapping f and that of a fixed point which leads to the Banach fixed point theorem.

Definition 3.3.1 ([25]) *Let (X, p) be a partial metric space. The map $f : X \rightarrow X$ is called a **contraction** if there is a constant $0 < c < 1$ such that,*

$$p(f(x), f(y)) \leq c \cdot p(x, y)$$

whenever $x, y \in X$.

Definition 3.3.2 *Let X be a set and $f : X \rightarrow X$ a function. A point $x_0 \in X$ is called a **fixed point** of f if $f(x_0) = x_0$.*

Theorem 3.3.1 ([25]) *Let (X, p) be a complete partial metric space and $f : X \rightarrow X$ a contraction map. Then f has a unique fixed point $a \in X$ such that*

$$p(a, a) = w(a) = 0.$$

Proof. Suppose that $x \in X$. Then for all $n, k \in \mathbb{N}$, we have that:

$$\begin{aligned} p(f^{n+k+1}(x), f^n(x)) &\leq p(f^{n+k+1}(x), f^{n+k}(x)) + p(f^{n+k}(x), f^n(x)) \\ &\quad - p(f^{n+k}(x), f^{n+k}(x)) \\ &\leq c^{n+k} \cdot p(f(x), x) + p(f^{n+k}(x), f^n(x)). \end{aligned}$$

Thus, for $n, k \in \mathbb{N}$

$$\begin{aligned} p(f^{n+k+1}(x), f^n(x)) &\leq (c^{n+k} + \dots + c^n) \cdot p(f(x), x) + p(f^n(x), f^n(x)) \\ &\leq c^n \left(\frac{1 - c^{k+1}}{1 - c} \right) \cdot p(f(x), x) + c^n \cdot p(x, x) \\ &\leq c^n \cdot \left[\left(\frac{1}{1 - c} \right) \cdot p(f(x), x) + p(x, x) \right]. \end{aligned}$$

Since for all $n \in \mathbb{N}$,

$$p(f^n(x), f^n(x)) \leq c^n \cdot p(x, x),$$

then $(f^n(x))$ is a Cauchy sequence.

Since p is complete, there is $a \in X$ such that

$$(f^n(x)) \rightarrow a$$

and

$$p(a, a) = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} p(f^n(x), a) = 0.$$

That $p(f(a), a) = 0$, we have from the fact that for all $n \in \mathbb{N}$,

$$\begin{aligned} p(f(a), a) &\leq p(f(a), f^{n+k}(x)) + p(f^{n+k}(x), a) - p(f^{n+1}(x), f^{n+1}(x)) \\ &\leq c \cdot p(a, f^n(x)) + p(f^{n+k}(x), a). \end{aligned}$$

Hence, since $p(f(a), f(a)) \leq p(a, a) = 0$, it follows from (P1) that $f(a) = a$.

Finally we show that a is unique.

Suppose that $b \in X$ such that $b = f(b)$. Then,

$$p(a, b) = p(f(a), f(b)) \leq c \cdot p(a, b).$$

Thus, $c < 1$, $p(a, b) = 0$, and so $a = b$. Hence the fixed point of f is unique.

University of Cape Town

Chapter 4

Bicompletion of an asymmetric normed linear space

In this chapter we shall investigate the bicompletion of an asymmetric normed linear space. The bicompletion of an asymmetric normed linear space $(X, \|\cdot\|)$ will be obtained from its associated quasi-pseudometric space $(X, d_{\|\cdot\|})$. We shall prove that each asymmetric normed linear space has a bicompletion.

We will first introduce the notion of a linear space and show that each asymmetric normed linear space is a quasi-pseudometric space and then construct the bicompletion of a given asymmetric normed linear space.

We shall investigate an extension theorem for isometric mappings between asymmetric normed linear spaces and show that the bicompletion of an asymmetric normed linear space is unique up to isometric isomorphism.

In the last section we shall introduce the partial quasi-metric space $(X, p_{\|\cdot\|})$ associated with the asymmetric normed linear space and then we shall discuss connections between partial quasi-metrics and quasi-pseudometrics with compatible weight.

4.1 Definitions in an asymmetric normed linear space

We first introduce the notion of an asymmetric normed linear space and prove that each one is a quasi-pseudometric space.

In the following we shall only consider real linear spaces.

Definition 4.1.1 ([14]) *A vector space X is a set of objects called **vectors** with two binary operations such that:*

*$(x, y) \mapsto x + y$ from $X \times X$ into X is called the **addition** of vectors and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{R} \times X$ into X is called a the **multiplication** of vectors by scalars such that the following conditions are satisfied:*

For all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}^+$:

- (1) $x + 0 = x$;
- (2) $x + (-x) = 0$;
- (3) $x + y = y + x$;
- (4) $x + (y + z) = (x + y) + z$;
- (5) $\alpha(x + y) = \alpha x + \alpha y$;
- (6) $(\alpha + \beta)x = \alpha x + \beta x$;
- (7) $(\alpha(\beta x)) = (\alpha\beta)x$;
- (8) $1x = x$.

The triple $(X, +, \cdot)$ is a vector space, that is a real linear space.

We next define an asymmetric normed linear space.

Definition 4.1.2 ([14]) *Let $(X, +, \cdot)$ be a real linear space. An asymmetric norm (or a quasi-norm) on X is a nonnegative real-valued function $\| \cdot \|$ on X such that for all $x, y, z \in X$ and $\alpha \in \mathbb{R}^+$:*

- (A1) $\| x \| = \| -x \| = 0$ if and only if $x = 0$;
- (A2) $\| \alpha x \| = \alpha \| x \|$;
- (A3) $\| x + y \| \leq \| x \| + \| y \|$.

The pair $(X, \| \cdot \|)$ is called an **asymmetric normed linear space**. We observe that the function $\| x \|^- = \| -x \|$ is also an asymmetric norm on

X and $\|x\|^* = \max\{\|x\|, \|-x\|\}$ is a norm on X .

We next introduce the notions of an open ball in an asymmetric normed linear space.

Definition 4.1.3 ([14]) *Let $(X, \|\cdot\|)$ be an asymmetric normed linear space. If $x \in X$ and $\varepsilon > 0$, the set $B(x, \varepsilon) = \{y \in X : \|x - y\| < \varepsilon\}$ is the open ball with radius ε and center at x for the topology induced by $\|\cdot\|$. A sequence (x_n) in an asymmetric normed linear space $(X, \|\cdot\|)$ converges to a point $x \in X$ with respect to the topology induced by $\|\cdot\|$ if $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$.*

We next show that an asymmetric normed linear space is a quasi-pseudometric space.

Remark 4.1.1 ([14]) *Let $(X, \|\cdot\|)$ be an asymmetric normed linear space. The function defined by $d_{\|\cdot\|}(x, y) = \|y - x\|$ is a T_0 -quasi-pseudometric induced by the asymmetric norm on X that is called the **associated** quasi-pseudometric of $\|\cdot\|$.*

Proof. For any $x, y, z \in X$, we have that

$$(d1) \quad d_{\|\cdot\|}(x, x) = \|x - x\| = 0;$$

$$(d2) \quad \begin{aligned} d_{\|\cdot\|}(x, z) &= \|z - x\| = \|(z - y) + (y - x)\| \\ &\leq \|z - y\| + \|y - x\| \\ &= d_{\|\cdot\|}(x, y) + d_{\|\cdot\|}(y, z). \end{aligned}$$

To prove that $d_{\|\cdot\|}$ is T_0 -quasi-pseudometric.

Let $d_{\|\cdot\|}(x, y) = 0 = d_{\|\cdot\|}(y, x)$. It follows from definition of quasi-norm (A1) that $\|y - x\| = 0 = \|(y - x)\|$ implies that $x - y = 0$, and so $x = y$.

Hence $(X, d_{\|\cdot\|})$ is a quasi-pseudometric space and we note that the condition.

We start constructing the bicompletion of an asymmetric linear space.

4.2 Bicompletion of an asymmetric normed linear space

In this section we will construct the bicompletion of an asymmetric normed linear space. We shall use the result obtained in Chapter 2 according to which every T_0 -quasi-pseudometric has an essentially unique bicompletion. The bicompletion of an asymmetric normed linear space will be obtained from this result as a corollary.

A bicomplete asymmetric normed linear space is called **biBanach** space. It follows that the asymmetric normed linear space $(X, \| \cdot \|)$ is a biBanach space if and only if the space $(X, d_{\| \cdot \|})$ is bicomplete.

We next define a Cauchy sequence in an asymmetric normed linear space in the expected way.

Definition 4.2.1 ([14]) *Let $(X, \| \cdot \|)$ be an asymmetric normed linear space. The sequence $(x_n) \in X$ is a **Cauchy sequence** if for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\| x_n - x_m \| < \varepsilon$ whenever $n, m \geq n_0$.*

Definition 4.2.2 ([14]) *An isometry from an asymmetric normed linear space $(X, \| \cdot \|_X)$ to an asymmetric normed linear space $(Y, \| \cdot \|_Y)$ is a linear mapping $f : X \rightarrow Y$ such that $\| f(x) \|_Y = \| x \|_X$ for all $x \in X$.*

We note that if f is an isometric isomorphism from the asymmetric normed linear space $(X, \| \cdot \|_X)$ to the asymmetric linear space $(Y, \| \cdot \|_Y)$, then f is an isometric isomorphism from the normed linear space $(X, \| \cdot \|_X^*)$ to the normed linear space $(Y, \| \cdot \|_Y^*)$ and hence f is injective.

Two asymmetric normed linear spaces $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are called **isometrically isomorphic** if there is an isometric isomorphism from X to Y .

Definition 4.2.3 ([14, 15]) *Let $(X, \| \cdot \|_X)$ be an asymmetric normed linear space. We say that a biBanach space $(Y, \| \cdot \|_Y)$ is a completion of $(X, \| \cdot \|_X)$ if $(X, \| \cdot \|_X)$ is isometrically isomorphic to a subspace of $(Y, \| \cdot \|_Y)$ that is dense in the Banach space $(Y, \| \cdot \|_Y^*)$.*

Let \widehat{X} be the set of all Cauchy sequences in $(X, \|\cdot\|^*)$ and let $(x_n), (y_n) \in \widehat{X}$ such that $(x_n) \sim (y_n)$ if and only if $\lim_{n \rightarrow \infty} \|x_n - y_n\|^* = 0$. Then \sim is an equivalence relation on \widehat{X} and let us denote the quotient set by \widetilde{X} and the equivalence class of x by $[x]$ where $x := (x_n)$.

For each $[x], [y] \in \widetilde{X}$ and $\alpha \in \mathbb{R}$, we define

$$[x] + [y] = [x + y] \quad (4.1)$$

and

$$\alpha[x] = [\alpha x] \quad (4.2)$$

Note that these operations are well defined:

Let $(x_n), (x'_n), (y_n)$ and $(y'_n) \in \widehat{X}$, we have that $\|x_n - x'_n\|^* \rightarrow 0$ and $\|y_n - y'_n\|^* \rightarrow 0$ imply that $\|(x_n + y_n) - (x'_n + y'_n)\|^* \leq \|x_n - x'_n\|^* + \|y_n - y'_n\|^* \rightarrow 0$. In the same way we show that if $\|x_n - x'_n\|^* \rightarrow 0$, $\|\alpha x_n - \alpha x'_n\|^* = |\alpha| \|x_n - x'_n\|^* \rightarrow 0$ whenever $\alpha \in \mathbb{R}$. Hence (4.1) and (4.2) are well-defined.

Lemma 4.2.1 ([14]) *Let $(X, \|\cdot\|)$ be an asymmetric normed linear space and let $(x_n), (y_n) \in \widehat{X}$ where \widehat{X} is the set of all Cauchy sequences in $(X, \|\cdot\|)$. Then:*

- (1) $\lim_{n \rightarrow \infty} \|x_n\|$ exists.
- (2) $\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\|$ for each $y \in [x]$ where $x := (x_n)$ and $y := (y_n)$.
- (3) $(x_n + y_n)$ and $(\alpha \cdot x_n)$ are in \widehat{X} for $\alpha \in \mathbb{R}$.

Proof. (1) Let (x_n) be a Cauchy sequence in $(X, \|\cdot\|)$. For any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for all $n, m \geq n_0$. Therefore $|\|x_n\| - \|x_m\|| < \varepsilon$ and, thus $(\|x_n\|)$ is a Cauchy sequence in \mathbb{R} . Consequently, $\lim_{n \rightarrow \infty} \|x_n\|$ exists.

(2) Let (x_n) and (y_n) be Cauchy sequences in \widehat{X} . We have that: $\lim_{n \rightarrow \infty} \|x_n\| \leq \lim_{n \rightarrow \infty} \|x_n - y_n\| + \lim_{n \rightarrow \infty} \|y_n\|$. Since $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$, we get that $\lim_{n \rightarrow \infty} \|x_n\| \leq \lim_{n \rightarrow \infty} \|y_n\|$. Analogously, we find that $\lim_{n \rightarrow \infty} \|y_n\| \leq \lim_{n \rightarrow \infty} \|x_n\|$. Hence $\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\|$.

(3) Let $(x_n), (y_n)$ be in \widehat{X} . For any $\varepsilon > 0$, there are $n_1, n_2 \in \mathbb{N}$ such that $\|x_n - x_m\| < \frac{\varepsilon}{2}$ whenever $n, m \geq n_1$ and $\|y_n - y_m\| < \frac{\varepsilon}{2}$ whenever $n, m \geq n_2$. We have that for all $n, m \geq \max\{n_1, n_2\}$,

$$\begin{aligned} \|(x_n + y_n) - (x_m + y_m)\| &= \|(x_n - x_m) + (y_n - y_m)\| \\ &\leq \|x_n - x_m\| + \|y_n - y_m\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence $(x_n + y_n)$ is a Cauchy sequence. In the same way, we can show that $(\alpha x_n) \in \widehat{X}$.

Lemma 4.2.2 ([14]) *Let $(X, \|\cdot\|)$ be an asymmetric normed linear space. Then $(\widetilde{X}, +, \cdot)$ is a real linear space.*

Proof. Let $[x], [y], [z] \in \widetilde{X}$ and $\alpha, \beta \in \mathbb{R}$, we have that

- (1) $[x] + [0] = [x + 0] = [x]$;
- (2) $[x] + [-x] = [x - x] = [0]$;
- (3) $[x] + [y] = [x + y] = [y + x] = [y] + [x]$;
- (4) $[x] + ([y] + [z]) = [x] + ([y + z]) = [x + y + z] = ([x + y]) + [z] = ([x] + [y]) + [z]$;
- (5) $\alpha([x] + [y]) = [\alpha x + \alpha y] = \alpha[x] + \alpha[y]$;
- (6) $(\alpha + \beta)[x] = [\alpha x + \beta x] = \alpha[x] + \beta[x]$;
- (7) $(\alpha\beta)[x] = [(\alpha\beta)x] = \alpha(\beta[x]) = \alpha(\beta[x])$;
- (8) $1 \cdot [x] = [x]$.

Hence $(\widetilde{X}, +, \cdot)$ is a real linear space.

Proposition 4.2.1 ([14, 15]) *The sequence (x_n) is a Cauchy sequence in $(X, \|\cdot\|)$ if and only if (x_n) is a Cauchy sequence in $(X, d_{\|\cdot\|}^*)$.*

Proof. Let (x_n) be a Cauchy sequence in $(X, \|\cdot\|)$. We have that for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| \leq \varepsilon$ for all $n, m \geq n_0$. It follows from Remark 4.1.1 that

$$d_{\|\cdot\|}^*(x_n, x_m) = \|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq n_0$. Thus (x_n) is a Cauchy sequence in $(X, d_{\|\cdot\|}^*)$. The converse can be proved similarly.

We next show that each asymmetric normed linear space has a bicompletion.

Theorem 4.2.1 ([14]) *Each asymmetric normed real linear space $(X, \|\cdot\|)$ has a bicompletion $(\tilde{X}, \tilde{\|\cdot\|})$ which is an asymmetric normed linear space.*

Proof. Let $[x] \in \tilde{X}$, we define the function $\tilde{\|\cdot\|} : \tilde{X} \rightarrow [0, \infty)$ by $\tilde{\|[x]\|} = \lim_{n \rightarrow \infty} \|x_n\|$ for all $[x] \in \tilde{X}$. Then $\tilde{\|\cdot\|}$ is well defined.

We prove in the next three lemmas that $(\tilde{X}, \tilde{\|\cdot\|})$ is a biBanach asymmetric normed linear space.

Lemma 4.2.3 ([14]) *$(\tilde{X}, \tilde{\|\cdot\|})$ is an asymmetric normed linear space.*

Proof. It is clear that the function $(\tilde{\|\cdot\|})$ is a nonnegative real-valued function on \tilde{X} .

Let $x = (x_n)$ be an element in \tilde{X} and $\alpha \in \mathbb{R}$ such that

(A1) $\tilde{\|[x]\|} = \tilde{\|[-x]\|} = 0$, then $\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|-x_n\| = 0$, so $\lim_{n \rightarrow \infty} (\|x_n\|) = 0$, hence $[x] = [0] = 0$.

(A2) $\tilde{\|\alpha \cdot [x]\|} = \tilde{\|[\alpha \cdot x]\|} = \lim_{n \rightarrow \infty} \|\alpha x_n\| = \alpha \lim_{n \rightarrow \infty} \|x_n\| = \alpha \tilde{\|[x]\|}$.

(A3) $\tilde{\|[x] + [y]\|} = \tilde{\|[x + y]\|} = \lim_{n \rightarrow \infty} \|x_n + y_n\| \leq \lim_{n \rightarrow \infty} \|x_n\| + \lim_{n \rightarrow \infty} \|y_n\| = \tilde{\|[x]\|} + \tilde{\|[y]\|}$.

Thus $\tilde{\|\cdot\|}$ is an asymmetric norm on \tilde{X} and since the space $(\tilde{X}, +, \cdot)$ is a linear space, we conclude that $(\tilde{X}, \tilde{\|\cdot\|})$ is an asymmetric normed linear space.

Lemma 4.2.4 ([14]) *$(X, \|\cdot\|)$ is isometrically isomorphic to a subspace of $(\tilde{X}, \tilde{\|\cdot\|})$ that is dense in the Banach space $(\tilde{X}, (\tilde{\|\cdot\|})^*)$.*

Proof. Let us denote by $[\hat{x}]$ the element in $(\tilde{X}, \tilde{\|\cdot\|})$ which contains the constant sequence x, x, \dots, x, \dots for each $x \in X$.

Define the map $i : X \longrightarrow \tilde{X}$ by $i(x) = [\hat{x}]$ for all $x \in X$. We have that $\tilde{\|}i(x)\tilde{\|} = \|x\|$ whenever $x \in X$.

We next show that i is linear.

Let $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. We have

$i(\alpha x + \beta y) = [\widehat{\alpha x + \beta y}] = [\alpha \cdot \hat{x} + \beta \cdot \hat{y}] = \alpha \cdot [\hat{x}] + \beta \cdot [\hat{y}] = \alpha \cdot i(x) + \beta \cdot i(y)$. Thus, the space $(X, \|\cdot\|)$ is isometrically isomorphic to the linear subspace $i(X)$ of $(\tilde{X}, \tilde{\|\cdot\|})$.

Lemma 4.2.5 ([14]) $(\tilde{X}, \tilde{\|\cdot\|})$ is a biBanach space.

Proof. Let us consider the space $(X, d_{\|\cdot\|})$. It follows that $(X, d_{\|\cdot\|})$ has a quasi-pseudometric bicompletion $(\tilde{X}, \tilde{d}_{\|\cdot\|})$. If we denote by $(\tilde{X}, d_{\tilde{\|\cdot\|}})$ the quasi-pseudometric space associated with $(\tilde{X}, \tilde{\|\cdot\|})$, we have to show that $\tilde{d}_{\|\cdot\|} = d_{\tilde{\|\cdot\|}}$.

We have that

$$d_{\tilde{\|\cdot\|}}([x], [y]) = \lim_{n \rightarrow \infty} d_{\|\cdot\|}(x_n, y_n) = \tilde{\|}[y] - [x]\tilde{\|} = \tilde{\|}[y - x]\tilde{\|} = \lim_{n \rightarrow \infty} \|y_n - x_n\| = \tilde{d}_{\|\cdot\|}([x], [y]).$$

So, $\tilde{d}_{\|\cdot\|} = d_{\tilde{\|\cdot\|}}$. Hence $(\tilde{X}, \tilde{\|\cdot\|})$ is a biBanach space. Since $(\tilde{X}, \tilde{d}_{\|\cdot\|})$ is a bicompletion of $(X, d_{\|\cdot\|})$, it follows that $i(X)$ is dense in $(\tilde{X}, (\tilde{\|\cdot\|})^*)$.

Hence the statement in the theorem is verified and $(\tilde{X}, \tilde{\|\cdot\|})$ is a bicompletion of the asymmetric normed linear space $(X, \|\cdot\|)$.

4.3 Extension of mappings

In the following section we shall state the expected extension theorem of mappings in asymmetric normed linear spaces.

Proposition 4.3.1 ([14]) Let $(X, \|\cdot\|_X)$ be an asymmetric normed linear space and $(Y, \|\cdot\|_Y)$ be a biBanach space. Suppose that there is an (linear) isometry $f : A \longrightarrow Y$ and that A is dense in the normed linear space $(X, \|\cdot\|_X^*)$. Then f has a unique isometric extension \tilde{f} from X to Y .

Proof. Let $x \in X$ and (x_n) be a Cauchy sequence in A such that $\lim_{n \rightarrow \infty} \|x_n - x\|^* = 0$. Then f is an isometry and thus it is quasi-uniformly continuous. We have that the sequence $(f(x_n))$ is a Cauchy sequence in the Banach space $(Y, \|\cdot\|_Y^*)$. Therefore it converges to a point $x^* \in Y$. Then we define the map $\tilde{f} : X \rightarrow Y$ by $\tilde{f}(x) = \lim_{n \rightarrow \infty} f(x_n) = x^*$ for all $x \in X$. It is easy to show that \tilde{f} is independent of the choice of (x_n) .

We will show that \tilde{f} is an isometric map on X .

Let $x \in X$ and (x_n) be a Cauchy sequence in A such that $\lim_{n \rightarrow \infty} \|x_n - x\|^* = 0$. Then, since the quasi-norm $\|\cdot\|_Y$ is continuous and by the fact that f is an isometry, we get that

$$\|\tilde{f}(x)\|_Y = \lim_{n \rightarrow \infty} \|f(x_n)\|_Y = \lim_{n \rightarrow \infty} \|x_n\|_X = \|x\|_X.$$

Hence, \tilde{f} is an isometry.

We show that \tilde{f} is linear on X .

Let $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. Let (x_n) and (y_n) be sequences in A that converge to x and y respectively in the normed linear space $(X, \|\cdot\|_X^*)$. Then $(\alpha x_n + \beta y_n)$ converges to $\alpha x + \beta y$ with respect to $\mathcal{T}_{\|\cdot\|_X^*}$, so by definition of \tilde{f} , $f(\alpha x_n + \beta y_n)$ converges to $\tilde{f}(\alpha x + \beta y)$ with respect to $\mathcal{T}_{\|\cdot\|_Y^*}$. Thus $(\alpha f(x_n) + \beta f(y_n))$ converges to $(\alpha \tilde{f}(x) + \beta \tilde{f}(y))$ with respect to $\mathcal{T}_{\|\cdot\|_Y^*}$. Therefore $\tilde{f}(\alpha x + \beta y) = \alpha \tilde{f}(x) + \beta \tilde{f}(y)$.

Hence \tilde{f} is linear on X . So by definition, \tilde{f} is an isometric isomorphism.

It is easy to show that \tilde{f} is unique.

We next show that the bicompletion of an asymmetric normed linear space is unique up to isometry.

Lemma 4.3.1 ([14, 15]) *Each bicompletion of an asymmetric normed linear space $(X, \|\cdot\|)$ is isometrically isomorphic to $(\tilde{X}, \|\cdot\|)$.*

Proof. Let $(Y, \|\cdot\|_Y)$ be a bicompletion of $(X, \|\cdot\|)$ with $i_1 : X \rightarrow Y$ as an isometric embedding. Since $i_1(X)$ is dense in the Banach space $(Y, \|\cdot\|_Y)$, there is an isometric extension $g_1 : \tilde{X} \rightarrow Y$ of i_1 . Analogously, if $i_2 : X \rightarrow \tilde{X}$ is the isometric embedding discussed above, it has an isometric extension $g_2 : Y \rightarrow \tilde{X}$.

It follows from Corollary 2.3.1 that g_1 and g_2 are inverse to each other and so, $(\tilde{X}, \tilde{\|\cdot\|})$ and $(Y, \|\cdot\|)$ are isometrically isomorphic.

4.4 The partial quasi-metric of a biBanach space

In this section we shall introduce the partial quasi-metric space $(X, p_{\|\cdot\|})$ associated with the asymmetric normed linear space. We show that there are connections between partial quasi-metrics and quasi-pseudometrics with compatible weight.

In the following we consider the space $(X, \|\cdot\|)$ as a biBanach space. We shall first define a partial quasi-metric on a set X .

Definition 4.4.1 (see [23]) *A partial quasi-metric p on a set X is a function $p : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:*

- (1) $x = y$ if and only if $p(x, x) = p(x, y)$ and $p(y, y) = p(y, x)$;
- (2) $p(x, x) \leq p(x, y)$ and $p(y, y) \leq p(x, y)$;
- (3) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$.

Of course a partial quasi-metric satisfying the additional condition: $p(x, y) = p(x, y)$ is a partial metric.

Definition 4.4.2 *Let $(X, \|\cdot\|)$ be a biBanach space. The partial quasi-metric associated with $\|\cdot\|$ is given by the formula:*

$$p_{\|\cdot\|}(x, y) = \|y - x\| + \|x\|$$

whenever $x, y \in X$.

We next define the concept of a quasi-pseudometric with compatible weight.

Definition 4.4.3 (see [23]) *A quasi-pseudometric space with compatible weight on a set X is a triple (X, d, w) such that d is a quasi-pseudometric on X and $w : X \rightarrow [0, \infty)$ is a function satisfying the condition: $w(y) \leq d(x, y) + w(x)$ whenever $x, y \in X$.*

Remark 4.4.1 Let $(X, \|\cdot\|)$ be a biBanach space and $d_{\|\cdot\|}$ the associated quasi-pseudometric. Then $p(x, y) = d_{\|\cdot\|}(x, y) + w(x)$ is indeed a partial quasi-metric, where $w(x) = p(x, x) = \|x\|$ whenever $x, y \in X$ and the weight w is compatible with $d_{\|\cdot\|}$.

University of Cape Town

Chapter 5

Construction of the bicompletion of quasi-uniform spaces using quasi-pseudometrics

In this chapter we shall describe the bicompletion of a T_0 -quasi-uniform space. We will construct the bicompletion of a T_0 -quasi-uniform space by employing the approach that uses the embedding of a quasi-uniform space into a product of quasi-pseudometric spaces.

We will first introduce the notion of a quasi-uniform space (X, \mathcal{U}) and secondly construct the bicompletion of (X, \mathcal{U}) .

It will be shown that each T_0 -quasi-uniform space has a bicompletion which is a T_0 -quasi-uniform space.

The last section will deal with the extension of mappings between bicomplete quasi-uniform spaces and will show that the bicompletion of a T_0 -quasi-uniform space is unique up to quasi-uniform isomorphism.

5.1 Introduction to quasi-uniform spaces

In this section we introduce the concept of quasi-uniform spaces and discuss a connection with quasi-pseudometric spaces.

Definition 5.1.1 ([12]) *A quasi-uniformity \mathcal{U} on a set X is a filter on $X \times X$ such that:*

- (1) *Each member U of \mathcal{U} contains the diagonal $\Delta = \{(x, x) : x \in X\}$ of X ;*
- (2) *For each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = V \circ V = \{(x, z) \in X \times X : \text{there is } y \in X \text{ such that } (x, y) \in V, (y, z) \in V\}$.*

The members U of \mathcal{U} are called **entourages** of \mathcal{U} and the elements of X are called **points**.

The pair (X, \mathcal{U}) is called a **quasi-uniform space**.

If \mathcal{U} is a quasi-uniformity on a set X , then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ on $X \times X$ is also a quasi-uniformity on X . The quasi-uniformity \mathcal{U}^{-1} is called the **conjugate** of \mathcal{U} . A quasi-uniformity that is equal to its conjugate is called a **uniformity**.

If \mathcal{U}_1 and \mathcal{U}_2 are two quasi-uniformities on a set X , then we say that \mathcal{U}_1 is coarser than \mathcal{U}_2 (or \mathcal{U}_2 is finer than \mathcal{U}_1) provided that $\mathcal{U}_1 \subseteq \mathcal{U}_2$.

We denote by \mathcal{U}^* the coarsest uniformity finer than \mathcal{U} and its conjugate \mathcal{U}^{-1} , (i.e. $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$).

Below, we shall define the concept of a quasi-uniformity induced by a quasi-pseudometric.

Definition 5.1.2 ([12]) *Let d be a quasi-pseudometric on a set X . For each $\varepsilon > 0$, set $U_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$. The quasi-uniformity on $X \times X$ generated by the base $\{U_\varepsilon : \varepsilon > 0\}$ is called the quasi-pseudometric quasi-uniformity \mathcal{U}_d induced by d on X .*

Definition 5.1.3 ([12]) *Each quasi-uniformity \mathcal{U} on a set X induces a topology $\mathcal{T}(\mathcal{U})$ as follows:*

$\mathcal{T}(\mathcal{U}) = \{A \subseteq X : \text{For each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq A\}$ where $U(x) = \{y \in X : (x, y) \in U\}$.

The neighborhood filter of $x \in X$ with respect to the topology $\mathcal{T}(\mathcal{U})$ is given by $\mathcal{U}(x) = \{U(x) : U \in \mathcal{U}\}$.

Let (X, \mathcal{U}) be a quasi-uniform space. Then $\mathcal{T}(\mathcal{U})$ is a T_0 -topology if and only if $\cap \mathcal{U}$ is a partial order, and $\mathcal{T}(\mathcal{U})$ is a T_1 -topology if and only if $\cap \mathcal{U}$ is equal to the diagonal of X .

Proof. The proofs are obvious.

We next define the quasi-uniformly continuous mappings between two quasi-uniform spaces.

Definition 5.1.4 ([12, 31]) A map $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ between two quasi-uniform spaces (X, \mathcal{U}) and (Y, \mathcal{V}) is called **quasi-uniformly continuous** provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$. Here $f \times f$ is the product map from $X \times X$ into $Y \times Y$ defined by $(f \times f)(x_1, x_2) = (f(x_1), f(x_2))$ whenever $x_1, x_2 \in X$.

We observe that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are quasi-uniformly continuous maps, then the composite $g \circ f$ of f and g is quasi-uniformly continuous. A bijection $f : X \rightarrow Y$ is a quasi-uniform isomorphism if f and f^{-1} are quasi-uniformly continuous. It is also called a **quasi-unimorphism map**.

Proposition 5.1.1 ([12]) Let $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ be a quasi-uniformly continuous map. Let \mathcal{F} be a \mathcal{U}^* -Cauchy filter base. Then $f(\mathcal{F}) := \{f(F) : F \in \mathcal{F}\}$ is a \mathcal{V}^* -Cauchy filter base.

Proof. It is obvious that $f(\mathcal{F})$ is a filter base. Let $V \in \mathcal{V}$, there is an entourage $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x, y) \in U$. Since \mathcal{F} is a \mathcal{U}^* -Cauchy filter base, there is $F \in \mathcal{F}$ such that $F \times F \subseteq U$. Then $f(F) \times f(F) \subseteq V$ and $f(\mathcal{F})$ is a \mathcal{V}^* -Cauchy filter base.

5.2 Bicompletion of quasi-uniform spaces

In the following section we will construct the bicompletion for a given T_0 -quasi-uniform space. There are at least two ways in which we can do this

construction.

The first approach is embedding a quasi-uniform space into a product of a quasi-pseudometric spaces and using the idea of the bicompletion of quasi-pseudometric spaces.

The second approach consists of using the \mathcal{U}^* -Cauchy filters on X . This approach has been used by P. Fletcher and W.F. Lindgren in [12].

For the present discussion, we will use the first approach to construct the bicompletion of a T_0 -quasi-uniform space (X, \mathcal{U}) .

The next theorem shows that every quasi-uniformity on a set X can be generated by a family of quasi-pseudometrics.

Theorem 5.2.1 ([12]) *Let $\{U_n : n = 0, 1, \dots\}$ be a sequence of a binary relations on a set X such that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ and for each n, U_n contains the diagonal. Then there is a quasi-pseudometric d on X such that $U_{n+1} \subseteq \{(x, y) : d(x, y) < 2^{-n}\} \subseteq U_n$ for each non-negative integer n .*

Proof. The proof of this theorem can be found in the book of J.L.Kelly.

Let \mathcal{U} be a quasi-uniformity on X and \mathcal{D} be the family of all quasi-pseudometrics d on X such that $\mathcal{U}_d \subseteq \mathcal{U}$. We call \mathcal{D} the **family of quasi-pseudometrics associated** with \mathcal{U} and write $\mathcal{D}_{\mathcal{U}}$. For $d \in \mathcal{D}$ and $r > 0$, let $U_{d,r} = \{(x, y) : d(x, y) < r\}$. The family of all sets of the form $U_{d,r}$ is a subbase for a quasi-uniformity on X . By Theorem 5.2.1 we conclude that for any quasi-uniform space (X, \mathcal{U}) , the associated family of quasi-pseudometrics \mathcal{D} satisfies $\mathcal{U} = \bigvee \{\mathcal{U}_d : d \in \mathcal{D}\}$.

Definition 5.2.1 *Let $\{(X_i, \mathcal{U}_i) : i \in I\}$ be a family of quasi-uniform spaces and let $X = \prod_{i \in I} X_i$. The product quasi-uniformity $\prod_{i \in I} \mathcal{U}_i$ is the coarsest quasi-uniformity on X for which all the projections $p_i : X \rightarrow X_i$ are quasi-uniformly continuous.*

Proposition 5.2.1 *A map $f : (X, \mathcal{U}) \rightarrow (\prod_{i \in I} Y_i, \prod_{i \in I} \mathcal{V}_i)$ is quasi-uniformly continuous if and only if for each $i \in I$, $p_i \circ f : (X, \mathcal{U}) \rightarrow (Y_i, \mathcal{V}_i)$ is quasi-uniformly continuous.*

Proof. Let us suppose that f is quasi-uniformly continuous. It follows from the above definition that each p_i is quasi-uniformly continuous and so, the composite $p_i \circ f$ is quasi-uniformly continuous.

Conversely, suppose that each $p_i \circ f$ is quasi-uniformly continuous. Let $K \in \prod_{i \in I} \mathcal{V}_i$. Then $\bigcap_{i \in F} [(p_i \times p_i)^{-1}(K_i)] \subseteq K$ where $K_i \in \mathcal{V}_i$ and F is a finite subset of I .

Hence $\bigcap_{i \in F} [(p_i \circ f) \times (p_i \circ f)]^{-1}(K_i) \subseteq (f \times f)^{-1}(K)$. Since $p_i \circ f$ are all quasi-uniformly continuous, $\{[(p_i \circ f) \times (p_i \circ f)]^{-1}(K_i) : i \in I\}$ is a set of entourages of \mathcal{U} . Thus f is quasi-uniformly continuous.

Proposition 5.2.2 *Each T_0 -quasi-uniform space (X, \mathcal{U}) can be embedded into a product of T_0 -quasi-pseudometric spaces.*

Proof. We suppose that $\mathcal{D}_{\mathcal{U}} = \{d_i : i \in I\}$ in order to simplify the notation. For each $i \in I$, let (X_i, e_i) be the canonical T_0 -quotient space of (X, d_i) . Let us denote by (Y_i, ρ_i) the bicompletion of (X_i, e_i) .

We shall define $f : (X, \mathcal{U}) \longrightarrow (\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{U}_{e_i})$ as a quasi-uniform embedding map from (X, \mathcal{U}) into the product $(\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{U}_{e_i})$.

Indeed for each $i \in I$, let X_i be the family of all sets $A_x^i = \{y \in X : d_i^*(x, y) = 0\}$ where $x \in X$. Then $e_i(A_x^i, A_y^i) = d_i(x, y)$ whenever $x, y \in X$. We define the map $f : X \longrightarrow \prod_{i \in I} X_i$ by $(f(x))_i = A_x^i$.

We first show that f is one-to-one.

Let $x, y \in X$, and suppose that $x \neq y$. Then there exists $i \in I$ such that $d_i(x, y) \neq 0$, that means $(f(x))_i = A_x^i \neq (f(y))_i = A_y^i$, thus f is injective.

Let us show that f is quasi-uniformly continuous.

For each $i \in I$, let $p_i : (\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{U}_{e_i}) \longrightarrow (X_i, \mathcal{U}_{e_i})$ be the projection.

It follows from the above theorem and the uniform continuity of T_0 -quotient maps $p_i \circ f$ that the map f is quasi-uniformly continuous.

Let us consider the obvious map $f^{-1} : f(X) \longrightarrow X$ defined on the image of $f(X)$. For convenience, we call that map the inverse f^{-1} of f . It remains to show that f^{-1} is quasi-uniformly continuous.

Let $U \in \mathcal{U}$. Then $U = \bigvee_{i \in I} \mathcal{U}_{d_i}$. We can assume that there is $j \in I$ such that $U \in \mathcal{U}_{d_j}$. Therefore, there is $\varepsilon > 0$ such that $U_{d_j, \varepsilon} \subseteq U$. Then $[(p_j \times p_j)^{-1}(U_{e_j, \varepsilon}) \in \prod_{i \in I} \mathcal{U}_{e_i}$. We have that $(f^{-1} \times f^{-1})[(p_j \times p_j)^{-1}(U_{e_j, \varepsilon})] = [((p_j \circ f) \times (p_j \circ f))^{-1}(U_{e_j, \varepsilon})] \subseteq U_{d_j, \varepsilon} \subseteq U$.

Thus, f^{-1} is a quasi-uniformly continuous and so, f is a quasi-uniform embedding map.

Definition 5.2.2 *A T_0 -quasi-uniform space (X, \mathcal{U}) is called **bicomplete** if each \mathcal{U}^* -Cauchy filter converges with respect to the topology $\mathcal{T}(\mathcal{U}^*)$.*

Proposition 5.2.3 *Let $\{(X_i, \mathcal{V}_i) : i \in I\}$ be a nonempty family of quasi-uniform spaces and let (X, \mathcal{U}) denote the product quasi-uniform space of this family. Then (X, \mathcal{U}) is bicomplete if and only if (X_i, \mathcal{V}_i) is bicomplete for each $i \in I$.*

Proof. We first suppose that for each $i \in I$, (X_i, \mathcal{V}_i) is bicomplete and let \mathcal{F} be a \mathcal{U}^* -Cauchy filter in the product space. Put $\mathcal{F}_i = p_i(\mathcal{F})$ for each $i \in I$. Since p_i is quasi-uniformly continuous for each $i \in I$, $p_i(\mathcal{F})$ is a \mathcal{V}_i^* -Cauchy filter and thus converges to some x_i with respect to $\mathcal{T}(\mathcal{V}_i^*)$. It follows that \mathcal{F} converges to $(x_i)_{i \in I}$ with respect to $\mathcal{T}(\mathcal{U}^*)$, and so the product space is bicomplete.

Conversely, suppose that (X, \mathcal{U}) is bicomplete. Fix $j \in I$, let \mathcal{F}_j be a \mathcal{V}_j^* -Cauchy filter on X_j . For each $i \in I$ and $i \neq j$ choose a point $x_i \in X_i$ such that $\mathcal{F}_i := \mathcal{U}^*(x_i)$. Let us consider the filter \mathcal{F} generated by the set $\{\prod_{i \in I} F_i : F_i \in \mathcal{F}_i, i \in I\}$. By the definition of the product quasi-uniformity, \mathcal{F} is a \mathcal{U}^* -Cauchy filter. Since \mathcal{F} converges to some (y_i) with respect to $\mathcal{T}(\mathcal{U}^*)$, then the filter $\mathcal{F}_j = p_j(\mathcal{F})$ converges to y_j with respect to $\mathcal{T}(\mathcal{V}_j^*)$. Hence the space (X_j, \mathcal{V}_j) is bicomplete.

Remark 5.2.1 *If (X, d) is a T_0 -quasi-pseudometric space, then the space (X, d^*) is bicomplete if and only if (X, \mathcal{U}_{d^*}) is a complete uniform space.*

This is a well-known result about pseudometric spaces. The proof is left to the reader.

Definition 5.2.3 ([31]) *A bicompletion of a T_0 -quasi-uniform space (X, \mathcal{U}) is a T_0 -bicomplete quasi-uniform space $(\tilde{X}, \tilde{\mathcal{U}})$ which has a $\mathcal{T}((\tilde{\mathcal{U}})^*)$ -dense subspace quasi-uniformly isomorphic to (X, \mathcal{U}) .*

Lemma 5.2.1 *Every $\mathcal{T}(\mathcal{U}^*)$ -closed subset of a bicomplete quasi-uniform space (X, \mathcal{U}) is bicomplete.*

Proof Let T be a $\mathcal{T}(\mathcal{U}^*)$ -closed subset of X and let \mathcal{F}_T be a Cauchy filter in the subspace $(T, \mathcal{U}|T)$. Then \mathcal{F}_T is a filter base for a \mathcal{U}^* -Cauchy filter \mathcal{F} on X . Since (X, \mathcal{U}) is bicomplete, we get that \mathcal{F}_T converges to x with respect to the topology $\mathcal{T}(\mathcal{U}^*)$ and $x \in T$, hence $(T, \mathcal{U}|T)$ is bicomplete.

Theorem 5.2.2 ([31]) *Each T_0 -quasi-uniform space has a T_0 -bicompletion.*

Proof. Let (X, \mathcal{U}) be a T_0 quasi-uniform space. It follows from Proposition 6.3.3 that (X, \mathcal{U}) can be embedded into the product space $(\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{U}_{e_i})$. Let $(\prod_{i \in I} Y_i, \prod_{i \in I} \mathcal{U}_{\rho_i})$ be the bicomplete product space described above. Let $f : (X, \mathcal{U}) \longrightarrow (\prod_{i \in I} Y_i, \prod_{i \in I} \mathcal{U}_{\rho_i})$ be the quasi-uniform embedding map from (X, \mathcal{U}) into the bicomplete space $(\prod_{i \in I} Y_i, \prod_{i \in I} \mathcal{U}_{\rho_i})$. Let $S = cl_{T(\prod_{i \in I} \mathcal{U}_{\rho_i})} f(X)$ is in $(\prod_{i \in I} Y_i, \prod_{i \in I} \mathcal{U}_{\rho_i})$. It follows from the above lemma that $(S, \prod_{i \in I} \mathcal{U}_{\rho_i}|S)$ is bicomplete and so $(S, (\prod_{i \in I} \mathcal{U}_{\rho_i})|S)$ is a bicompletion of the space (X, \mathcal{U}) which is denoted as $(\tilde{X}, \tilde{\mathcal{U}})$.

5.3 Extension of mappings

In the following section we shall state an extension theorem of mappings into a bicomplete quasi-uniform space which makes it possible to show that the bicompletion of (X, \mathcal{U}) is unique up to isomorphism.

Proposition 5.3.1 ([12]) *Let (X, \mathcal{U}) be a quasi-uniform space and (Y, \mathcal{V}) be a bicomplete T_0 -quasi-uniform space. Let S be a dense subset of $(X, \mathcal{T}(\mathcal{U}^*))$ and let $f : (S, \mathcal{U}|S) \longrightarrow (Y, \mathcal{V})$ be a quasi-uniformly continuous map. Then there exists a unique quasi-uniformly continuous extension $g : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ of f .*

For the proof we refer the reader to [12], [Theorem 3.29, p. 61].

Proposition 5.3.2 *Let (X, \mathcal{U}) and (Y, \mathcal{V}) be bicomplete T_0 -quasi-uniform space and S, T be dense subsets of $(X, \mathcal{T}(\mathcal{U}^*))$ and $(Y, \mathcal{T}(\mathcal{V}^*))$ respectively. Then a quasi-uniform isomorphism $f : (S, \mathcal{U}|S) \longrightarrow (T, \mathcal{V}|T)$ can be extended to a quasi-uniform isomorphism $g : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$.*

Proof. It follows from the above proposition that f has a quasi-uniformly continuous extension $g_1 : X \longrightarrow Y$ and f^{-1} has a quasi-uniformly continuous extension $g_2 : Y \longrightarrow X$. Then $g_2 \circ g_1$ is a quasi-uniformly continuous extension of the identity map on S .

Similarly, $g_1 \circ g_2$ is a quasi-uniformly continuous extension map of the identity map on T . In either case it is the unique quasi-uniformly continuous extension of the identity on S or T respectively. Hence we have that $g_2 \circ g_1 = id_X$ and $g_1 \circ g_2 = id_Y$.

Hence, g_1 is a quasi-unimorphism from (X, \mathcal{U}) to (Y, \mathcal{V}) .

Corollary 5.3.1 *If (X, \mathcal{U}) is a T_0 -quasi-uniform space, any T_0 -bicompletion of (X, \mathcal{U}) is quasi-uniformly isomorphic to $(\tilde{X}, \tilde{\mathcal{U}})$.*

Chapter 6

The B -completion of quasi-pseudometric spaces

In this main chapter we investigate the B -completion of T_0 -quasi-pseudometric spaces which is a double completion for a T_0 -quasi-pseudometric space (X, d) . In [11] Doitchinov studied a very interesting completion theory for T_0 -quasi-pseudometric spaces. He considered those quasi-pseudometric spaces that satisfy an additional condition called *balancedness*. He replaced the d^* -Cauchy sequences by so-called Cauchy pairs of sequences. We will throughout this chapter extend Doitchinov's theory of completion for balanced quasi-pseudometric spaces to arbitrary T_0 -quasi-pseudometric spaces.

We will first introduce the notion of a balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ and construct a completion of any T_0 -quasi-pseudometric space which we call the *B -completion* of T_0 -quasi-pseudometric spaces. Our completion will contain the bicompletion of the original space.

We will define balanced (quasi-uniformly continuous) maps between T_0 -quasi-pseudometric spaces and investigate an extension theorem for these maps into the B -complete spaces.

In the last section we will establish the connections with Doitchinov's work on balanced quasi-pseudometric spaces developed in [11].

6.1 Definition of a balanced Cauchy filter pair in a quasi-pseudometric space

In the following section we introduce the concept of a balanced Cauchy filter pair on a quasi-pseudometric space. We also define the distance between two Cauchy filter pairs.

The next lemma deals with the T_0 -quotient of a quasi-pseudometric space. Implicitly this construction has been of course used before when constructing the bicompletion but it is necessary to make it functional for the following more sophisticated investigations.

Lemma 6.1.1 *Let (X, d) be a quasi-pseudometric space. Define an equivalence relation \sim on X by $x \sim y$ if $d(x, y) = 0 = d(y, x)$. Let \hat{X} be the set of equivalence classes $q_X(x)$ with respect to \sim where $x \in X$. Then \hat{d} on \hat{X} defined by $\hat{d}(q_X(x), q_X(y)) = d(x, y)$ defines a T_0 -quasi-metric \hat{d} on \hat{X} . In the following, $q_X : X \rightarrow \hat{X}$ will denote the isometric quotient map defined by $x \mapsto q_X(x)$ whenever $x \in X$.*

Let $f : (X, d) \rightarrow (Y, e)$ be a quasi-uniformly continuous map between quasi-pseudometric spaces (X, d) and (Y, e) . Then $\hat{f} : (\hat{X}, \hat{d}) \rightarrow (\hat{Y}, \hat{e})$ defined by $\hat{f}(q_X(x)) := (q_Y \circ f)(x)$ whenever $x \in X$ is a well-defined quasi-uniformly continuous map. It is an isometry provided that f is an isometry.

Proof. For the first part of the proof we refer the reader to the proof of Proposition 2.2.2.

Suppose that for $x_1, x_2 \in X$ we have $d(x_1, x_2) = d(x_2, x_1) = 0$. Then by quasi-uniform continuity of f we have $e(f(x_1), f(x_2)) = e(f(x_2), f(x_1)) = 0$. Thus $q_Y(f(x_1)) = q_Y(f(x_2))$ and \hat{f} is well-defined.

If f is quasi-uniformly continuous, for each $\epsilon > 0$ there is $\delta > 0$ such that for any $x, y \in X$, $d(x, y) < \delta$ implies that $e(f(x), f(y)) < \epsilon$. Thus for any $x, y \in X$, $d(x, y) = \hat{d}(q_X(x), q_X(y)) < \delta$ implies that $e(f(x), f(y)) = \hat{e}((q_Y \circ f)(x), (q_Y \circ f)(y)) = \hat{e}(\hat{f}(q_X(x)), \hat{f}(q_X(y))) < \epsilon$. Thus \hat{f} is quasi-uniformly continuous if f is quasi-uniformly continuous.

Similarly if f is an isometry we have $\widehat{d}(q_X(x), q_X(y)) = d(x, y) = e(f(x), f(y)) = \widehat{e}((q_Y \circ f)(x), (q_Y \circ f)(y)) = \widehat{e}((\widehat{f} \circ q_X)(x), (\widehat{f} \circ q_X)(y))$ whenever $x, y \in X$. Hence \widehat{f} is an isometry if f is an isometry.

Definition 6.1.1 Let (X, d) be a quasi-pseudometric space and let A, B be nonempty subsets of X . Then we define the **2-diameter** from A to B by $\Phi_d(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$.

Note that usually $\Phi_d(A, A)$ is called the diameter of A . Of course ∞ is a possible value of a 2-diameter. For singleton $\{x\}$ we write $\Phi_d(x, A)$ and $\Phi_d(B, x)$ instead of $\Phi_d(\{x\}, A)$ and $\Phi_d(B, \{x\})$, respectively.

Note that if d^{-1} is the conjugate of d , then $\Phi_{d^{-1}}(A, B) = \Phi_d(B, A)$.

Let X be a set. For each $x \in X$ we shall denote the filter on X generated by the filter base $\{\{x\}\}$ on X by \underline{x} .

We next define the Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) . (Compare [11]).

Definition 6.1.2 Let (X, d) be a quasi-pseudometric space. We shall say that a pair $\langle \mathcal{F}, \mathcal{G} \rangle$ of filters \mathcal{F} and \mathcal{G} on X is a **Cauchy filter pair** on (X, d) if $\inf_{F \in \mathcal{F}, G \in \mathcal{G}} \Phi_d(F, G) = 0$.

Lemma 6.1.2 Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two Cauchy filter pairs on a quasi-pseudometric space (X, d) . Then $\inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \Phi_d(F, G')$ is a non-negative real number.

Proof. There are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\Phi_d(F, G) \leq 1$ and there are $F' \in \mathcal{F}'$ and $G' \in \mathcal{G}'$ such that $\Phi_d(F', G') \leq 1$. Let $f \in F$, $g \in G'$ arbitrary and choose some fixed $f'_0 \in F'$, and $g_0 \in G$. Then by the triangle inequality we have

$$d(f, g') \leq d(f, g_0) + d(g_0, f'_0) + d(f'_0, g') \leq 1 + d(g_0, f'_0) + 1.$$

Thus

$$\Phi_d(F, G') \leq 1 + d(g_0, f'_0) + 1.$$

Hence

$$\inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \Phi_d(F, G')$$

is a non-negative real number.

We next define the distance between two Cauchy filter pairs.

Definition 6.1.3 *Let (X, d) be a quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two Cauchy filter pairs on X . Then we define the distance from $\langle \mathcal{F}, \mathcal{G} \rangle$ to $\langle \mathcal{F}', \mathcal{G}' \rangle$ by:*

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = \inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \Phi_d(F, G') = \inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \sup_{f \in F, g' \in G'} d(f, g').$$

Remark 6.1.1 *Note that this distance belongs to \mathbb{R}^+ according to Lemma 6.1.2 and that a filter pair $\langle \mathcal{G}, \mathcal{H} \rangle$ on a quasi-pseudometric space (X, d) is a Cauchy filter pair if and only if $d^+(\langle \mathcal{G}, \mathcal{H} \rangle, \langle \mathcal{G}, \mathcal{H} \rangle) = 0$.*

Example 6.1.1 *Let (X, d) be a quasi-pseudometric space and $x \in X$. Then $\alpha_X(x) = \langle \underline{x}, \underline{x} \rangle$ is a Cauchy filter pair on (X, d) . Furthermore for any $x, y \in X$ we have $d^+(\alpha_X(x), \alpha_X(y)) = d(x, y)$.*

Proof. The first statement is obvious, since $d(x, x) = 0$. The second assertion is evident, too.

We next introduce the notion of a balanced Cauchy filter pair on (X, d) that will be used to construct the B -completion.

Definition 6.1.4 *Let (X, d) be a quasi-pseudometric space. A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) is said to be **balanced** on (X, d) if for each $x, y \in X$ we have*

$$d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y).$$

Remark 6.1.2 *Let (X, d) be a quasi-pseudometric space. A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-pseudometric space (X, d) is balanced if and only if for each $x, y \in X$ we have*

$$d^+(\alpha_X(x), \alpha_X(y)) \leq d^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) + d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y)).$$

Proof. The assertion is evident.

Remark 6.1.3 Note that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a balanced Cauchy filter pair on the quasi-pseudometric space (X, d) if and only if $\langle \mathcal{G}, \mathcal{F} \rangle$ is a balanced Cauchy filter pair on the quasi-pseudometric space (X, d^{-1}) .

Proof. The assertion is obvious, because $\Phi_d(A, B) = \Phi_{d^{-1}}(B, A)$ whenever $A, B \subseteq X$.

Remark 6.1.4 Observe that balancedness of a Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ in a quasi-pseudometric space (X, d) depends not only on the pair, but also on the space (X, d) in which it is embedded, since the definition mentions arbitrary points $x, y \in X$.

We next define the relation coarser between Cauchy filter pairs.

Definition 6.1.5 Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two filter pairs on a set X . Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is called **coarser** than $\langle \mathcal{F}', \mathcal{G}' \rangle$ provided that both $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$.

Remark 6.1.5 Let (X, d) be a quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two Cauchy filter pairs on (X, d) such that $\langle \mathcal{F}, \mathcal{G} \rangle$ is coarser than $\langle \mathcal{F}', \mathcal{G}' \rangle$. Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced if $\langle \mathcal{F}', \mathcal{G}' \rangle$ is balanced.

Proof. The proof is straightforward.

Example 6.1.2 Let (X, d) be a quasi-pseudometric space. Then for each $x \in X$, $\alpha_X(x)$ and $\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$ are balanced Cauchy filter pairs on (X, d) .

Proof. By the triangle inequality, $\alpha_X(x)$ is clearly balanced, since $d(a, b) \leq d(a, x) + d(x, b)$ whenever $a, b \in X$. Furthermore we see that $\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$ is a Cauchy filter pair on (X, d) , since $\inf_{n \in \mathbb{N}} \Phi_d(U_{2^{-n}}^{-1}(x), U_{2^{-n}}(x)) = 0$. It is balanced by Remark 6.1.5, since it is coarser than $\alpha_X(x)$.

We next define B-completeness.

Definition 6.1.6 Let (X, d) be a quasi-pseudometric space. An arbitrary Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on X is said to converge to $x \in X$ provided that

$$\inf_{G \in \mathcal{G}} \Phi_d(x, G) = 0$$

and

$$\inf_{F \in \mathcal{F}} \Phi_d(F, x) = 0.$$

A quasi-pseudometric space (X, d) is called **B-complete** if each balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ converges in X .

Remark 6.1.6 Note that for any quasi-pseudometric space (X, d) , (X, d^{-1}) is B-complete if and only if (X, d) is B-complete.

Remark 6.1.7 If $\langle \mathcal{F}', \mathcal{G}' \rangle$ is a balanced Cauchy filter pair on a quasi-pseudometric space (X, d) that converges to $x \in X$ and $\langle \mathcal{F}, \mathcal{G} \rangle$ is any Cauchy filter pair on (X, d) such that $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = 0 = d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle)$, then $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to x too.

Proof. Let $g \in G \in \mathcal{G}$. Then $d(x, g) \leq \inf_{G' \in \mathcal{G}'} \Phi_d(x, G') + \inf_{F' \in \mathcal{F}'} \Phi_d(F', g)$ by balancedness of $\langle \mathcal{F}', \mathcal{G}' \rangle$. Thus $\Phi_d(x, G) \leq \inf_{G' \in \mathcal{G}'} \Phi_d(x, G') + \inf_{F' \in \mathcal{F}'} \Phi_d(F', G)$. Furthermore $\inf_{G' \in \mathcal{G}'} \Phi_d(x, G') = 0$ and $\inf_{G \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \Phi_d(F', G) = 0$, because $\langle \mathcal{F}', \mathcal{G}' \rangle$ is a Cauchy filter pair on (X, d) . We conclude that $\inf_{G \in \mathcal{G}} \Phi_d(x, G) = 0$. Analogously we can prove that $\inf_{F \in \mathcal{F}} \Phi_d(F, x) = 0$.

Proposition 6.1.1 In a T_0 -quasi-pseudometric space (X, d) the limit of a balanced Cauchy filter pair is unique if it exists.

Proof. Assume that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a balanced Cauchy filter pair on (X, d) . Let $x, y \in X$. Suppose that $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to x as well as to y .

Then $\inf_{F \in \mathcal{F}} \Phi_d(F, x) = 0$, $\inf_{G \in \mathcal{G}} \Phi_d(x, G) = 0$, $\inf_{F \in \mathcal{F}} \Phi_d(F, y) = 0$ and $\inf_{G \in \mathcal{G}} \Phi_d(y, G) = 0$. Therefore by balancedness of $\langle \mathcal{F}, \mathcal{G} \rangle$ we see that

$$0 \leq d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y) = 0 + 0 = 0,$$

as well as

$$0 \leq d(y, x) \leq \inf_{G \in \mathcal{G}} \Phi_d(y, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, x) = 0 + 0 = 0.$$

Since (X, d) is a T_0 -quasi-pseudometric space, we conclude that $x = y$.

The following lemma makes it possible to see that the B -completion contains the bicompletion of the original space.

Lemma 6.1.3 *Let \mathcal{F} be a d^* -Cauchy filter on a quasi-pseudometric space (X, d) . Then $\langle \mathcal{F}, \mathcal{F} \rangle$ is a balanced Cauchy filter pair on (X, d) .*

Proof. Let $x, y \in X$ and $\epsilon > 0$. Observe that $\langle \mathcal{F}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, d) , since $\inf_{F \in \mathcal{F}} \Phi_d(F, F) = 0$. Note next that there is $F \in \mathcal{F}$ such that $\Phi_d(x, F) \leq d^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{F} \rangle) + \frac{\epsilon}{3}$ and $\Phi_d(F, y) \leq d^+(\langle \mathcal{F}, \mathcal{F} \rangle, \alpha_X(y)) + \frac{\epsilon}{3}$. Furthermore there is $F' \in \mathcal{F}$ such that $\Phi_d(F', F') \leq \frac{\epsilon}{3}$ and $F' \subseteq F$. Let $f_1, f_2 \in F'$. Then by the triangle inequality we have that

$$\begin{aligned} d(x, y) &\leq d(x, f_1) + d(f_1, f_2) + d(f_2, y) \leq \Phi_d(x, F') + \frac{\epsilon}{3} + \Phi_d(F', y) \leq \\ &\Phi_d(x, F) + \frac{\epsilon}{3} + \Phi_d(F, y) \leq d^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{F} \rangle) + \epsilon + d^+(\langle \mathcal{F}, \mathcal{F} \rangle, \alpha_X(y)). \end{aligned}$$

Thus $\langle \mathcal{F}, \mathcal{F} \rangle$ is a balanced Cauchy filter pair on (X, d) .

Recall that a quasi-pseudometric space (X, d) is called **bicomplete** provided that each d^* -Cauchy filter converges in (X, d^*) . It is well known from the theory of pseudometric spaces that the latter condition is equivalent to the condition that each d^* -Cauchy sequence converges in (X, d^*) (compare with Remarque 5.2.1).

The following remark shows that each B -complete space is bicomplete.

Remark 6.1.8 *Each quasi-pseudometric space (X, d) that is B -complete is bicomplete.*

Proof. Let \mathcal{F} be a d^* -Cauchy filter on (X, d) . Then by Lemma 6.1.3 $\langle \mathcal{F}, \mathcal{F} \rangle$ is a balanced Cauchy filter pair on (X, d) . By B -completeness of (X, d) there is $x \in X$ such that $d^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{F} \rangle) = 0$ and $d^+(\langle \mathcal{F}, \mathcal{F} \rangle, \alpha_X(x)) = 0$. Hence \mathcal{F} converges in (X, d^*) to x and therefore (X, d) is bicomplete.

6.2 The B -completion of (X, d)

In this section we construct the B -completion of a T_0 -quasi-pseudometric space (X, d) . Let X^+ denote the set of all balanced Cauchy filter pairs on (X, d) and let (X^b, d^b) denote the T_0 -quasi-pseudometric quotient of (X^+, d^+) . We shall call (X^b, d^b) **the standard B -completion** of (X, d) .

Proposition 6.2.1 *Let (X, d) be a quasi-pseudometric space and let X^+ be the set of all balanced Cauchy filter pairs $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) . Define $d^+ : X^+ \times X^+ \rightarrow [0, \infty)$ as above. Then (X^+, d^+) is a quasi-pseudometric space.*

Proof. As we have noted above, $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) = 0$ whenever $\langle \mathcal{F}, \mathcal{G} \rangle \in X^+$. It remains to verify that d^+ satisfies the triangle inequality. Let $\epsilon > 0$. We find $F_\epsilon \in \mathcal{F}, G'_\epsilon \in \mathcal{G}'$ such that $\Phi_d(F_\epsilon, G'_\epsilon) \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) + \frac{\epsilon}{2}$ and similarly $F'_\epsilon \in \mathcal{F}', G''_\epsilon \in \mathcal{G}''$ such that $\Phi_d(F'_\epsilon, G''_\epsilon) \leq d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) + \frac{\epsilon}{2}$.

For each $f \in F_\epsilon, g'' \in G''_\epsilon$ we have

$$d(f, g'') \leq \Phi_d(f, G'_\epsilon) + \Phi_d(F'_\epsilon, g'') \leq \Phi_d(F_\epsilon, G'_\epsilon) + \Phi_d(F'_\epsilon, G''_\epsilon),$$

because $\langle \mathcal{F}', \mathcal{G}' \rangle$ is balanced on (X, d) . It follows that

$$d(f, g'') \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) + d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) + \epsilon$$

whenever $f \in F_\epsilon$ and $g'' \in G''_\epsilon$.

Therefore $\Phi_d(F_\epsilon, G''_\epsilon) \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) + d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) + \epsilon$.

Hence $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) = \inf_{F \in \mathcal{F}, G'' \in \mathcal{G}''} \Phi_d(F, G'') \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) + d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle)$, since $\epsilon > 0$ was arbitrary. We have verified the triangle inequality.

Remark 6.2.1 *By Remark 6.1.2 we see that if Y is a set of Cauchy filter pairs containing the set $\alpha_X(X)$ and d^+ satisfies the triangle inequality on $Y \times Y$, then Y can contain only balanced Cauchy filter pairs.*

For the following it is useful to be aware of the following simple auxiliary result.

Lemma 6.2.1 *An isometry $g : (X, d) \rightarrow (Y, e)$ from a T_0 -quasi-pseudometric space (X, d) to a quasi-pseudometric space (Y, e) is injective.*

Proof. For any $x, y \in X$ $g(x) = g(y)$ implies that $e(g(x), g(y)) = 0 = e(g(y), g(x))$ and thus $d(x, y) = 0 = d(y, x)$, since g is an isometry. Hence $x = y$, because (X, d) is a T_0 -quasi-pseudometric space.

Remark 6.2.2 *It follows from Lemma 6.2.1 that if (X, d) is a T_0 -space, then $\alpha_X : X \rightarrow X^+$ is an isometric embedding of (X, d) into (X^+, d^+) . Indeed, as we have noted above, $d(x, y) = d^+(\alpha_X(x), \alpha_X(y))$ whenever $x, y \in X$.*

According to Lemma 6.1.1 we obtain an equivalence relation \cong on X^+ among the balanced Cauchy filter pairs on X by defining two balanced Cauchy filter pairs $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ to be *equivalent* if

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = 0$$

and

$$d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) = 0.$$

Observe that if (X, d) is a quasi-pseudometric space and $\langle \mathcal{F}_1, \mathcal{G} \rangle$ and $\langle \mathcal{F}_2, \mathcal{G} \rangle$ are both balanced Cauchy filter pairs on (X, d) , then $\langle \mathcal{F}_1, \mathcal{G} \rangle$ and $\langle \mathcal{F}_2, \mathcal{G} \rangle$ are equivalent.

If $\langle \mathcal{F}_i, \mathcal{G}_i \rangle_{i \in I}$ is a family of filter pairs on a set X , we define their *2-intersection* as the filter pair $\langle \bigcap_{i \in I} \mathcal{F}_i, \bigcap_{i \in I} \mathcal{G}_i \rangle$ on X .

It is obvious that the 2-intersection of two balanced Cauchy filter pairs $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ belonging to distinct equivalence classes of \cong in a quasi-pseudometric can never be a (balanced) Cauchy filter pair, since comparable

balanced Cauchy filter pairs are clearly always equivalent and thus $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ were equivalent too.

Lemma 6.2.2 *Let (X, d) be a quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . Then there exists a unique minimal (balanced) Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) . It can be described as the 2-intersection of all balanced Cauchy filter pairs belonging to the equivalence class of $\langle \mathcal{F}, \mathcal{G} \rangle$. Moreover it belongs to the equivalence class of $\langle \mathcal{F}, \mathcal{G} \rangle$ and has a countable base.*

Proof. Let us assume that the family $\langle \mathcal{F}_i, \mathcal{G}_i \rangle_{i \in I}$ of filter pairs denotes all balanced Cauchy filter pairs on (X, d) equivalent to $\langle \mathcal{F}, \mathcal{G} \rangle$. Let $n \in \mathbb{N}$. Then for each $i \in I$ choose $F_i^n \in \mathcal{F}_i$, and $K_i^n \in \mathcal{G}_i$ such that $\Phi_d(F_i^n, K_i^n) < \frac{1}{n}$.

Similarly for each $i \in I$ we can choose $H_i^n \in \mathcal{F}$ and $G_i^n \in \mathcal{G}_i$ such that $\Phi_d(H_i^n, G_i^n) < \frac{1}{n}$.

Consider arbitrary elements $f_i \in F_i^n$ and $g_j \in G_j^n$ with $i, j \in I$. Then $d(f_i, g_j) \leq \Phi_d(f_i, K_i^n) + \Phi_d(H_j^n, g_j) < \frac{2}{n}$ by balancedness of $\langle \mathcal{F}, \mathcal{G} \rangle$.

Consequently $\Phi_d(\bigcup_{i \in I} F_i^n, \bigcup_{j \in I} G_j^n) \leq \frac{2}{n}$.

Let \mathcal{H} be the filter on X generated by $\{\bigcup_{i \in I} F_i^n : n \in \mathbb{N}\}$ and let \mathcal{K} be the filter on X generated by $\{\bigcup_{i \in I} G_i^n : n \in \mathbb{N}\}$. We have proved that $\langle \mathcal{H}, \mathcal{K} \rangle$ is a Cauchy filter pair on (X, d) . By construction $\langle \mathcal{H}, \mathcal{K} \rangle$ is coarser than $\langle \bigcap_{i \in I} \mathcal{F}_i, \bigcap_{i \in I} \mathcal{G}_i \rangle$, which is coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. Thus by Remark 6.1.5 $\langle \mathcal{H}, \mathcal{K} \rangle$ is balanced. It is clearly equivalent to $\langle \mathcal{F}, \mathcal{G} \rangle$, because it is coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. Therefore $\langle \mathcal{H}, \mathcal{K} \rangle = \langle \mathcal{F}_i, \mathcal{G}_i \rangle$ for some $i \in I$ and $\langle \mathcal{H}, \mathcal{K} \rangle = \langle \bigcap_{i \in I} \mathcal{F}_i, \bigcap_{i \in I} \mathcal{G}_i \rangle$. Since any Cauchy filter pair on (X, d) coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced and equivalent to $\langle \mathcal{F}, \mathcal{G} \rangle$, the pair $\langle \mathcal{H}, \mathcal{K} \rangle$ is the unique minimal Cauchy filter pair contained in $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) .

Example 6.2.1 *Each minimal balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-pseudometric space (X, d) is round, that is, \mathcal{F} is equal to the filter $\mathcal{U}_d^{-1}(\mathcal{F})$ generated by the base $\{U^{-1}(F) : U \in \mathcal{U}_d, F \in \mathcal{F}\}$ and \mathcal{G} is equal to*

the filter $\mathcal{U}_d(\mathcal{G})$ generated by the base $\{U(G) : U \in \mathcal{U}_d, G \in \mathcal{G}\}$.

For each $x \in X$ the coarsest Cauchy filter pair equivalent to a balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ converging to x in (X, d) is $\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$.

Proof. Note that $\langle \mathcal{U}_d^{-1}(\mathcal{F}), \mathcal{U}_d(\mathcal{G}) \rangle$ is certainly a Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. Thus it is balanced and equal to $\langle \mathcal{F}, \mathcal{G} \rangle$ if $\langle \mathcal{F}, \mathcal{G} \rangle$ is also balanced and minimal Cauchy. Hence the first statement is proved.

To prove the second statement, we note first that $\langle \mathcal{F}, \mathcal{G} \rangle$ and the coarser balanced filter pair $\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$ are equivalent. Hence it suffices to show that any Cauchy filter pair belonging to the equivalence class of $\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$ converges to x . However this follows from Remark 6.1.7.

Given a quasi-pseudometric space (X, d) we shall now consider the associated T_0 -quasi-pseudometric quotient space $(\widehat{X^+}, \widehat{d^+})$ of (X^+, d^+) which we shall denote for simplicity by (X^b, d^b) .

According to Lemma 6.2.2 we can identify X^b with the subspace of all balanced Cauchy filter pairs on (X, d) that are minimal elements in the space (X^+, d^+) of all balanced Cauchy filter pairs on (X, d) where the order on X^+ is determined by the relation coarser.

Definition 6.2.1 *Let (X, d) be a T_0 -quasi-pseudometric space. Then the T_0 -quasi-pseudometric space (X^b, d^b) will be called the (standard) **B-completion** of (X, d) .*

We set $\beta_X = q_{X^+} \circ \alpha_X$ where $q_{X^+} : (X^+, d^+) \rightarrow (X^b, d^b)$ is the T_0 -quotient map according to Lemma 6.1.1. (Of course, for each $x \in X$, $\beta_X(x) = \langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$, using the convention formulated above.)

Corollary 6.2.1 *If (X, d) is a T_0 -quasi-pseudometric space, then $\beta_X : (X, d) \rightarrow (X^b, d^b)$ is an (isometric) embedding.*

Proof. We have that $\beta_X = q_{X^+} \circ \alpha_X$, where α_X and q_{X^+} both are isometries. The statement follows from Lemma 6.2.1.

We next define when a quasi-uniformly continuous map is called balanced.

Definition 6.2.2 A quasi-uniformly continuous map $f : (X, d) \longrightarrow (Y, e)$ between quasi-pseudometric spaces (X, d) and (Y, e) is called **balanced** provided that for each balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) , the Cauchy filter pair $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is balanced on (Y, e) . (Note first that $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is a Cauchy filter pair on (Y, e) , because f is quasi-uniformly continuous.)

Lemma 6.2.3 Let (X, d) and (Y, e) be quasi-pseudometric spaces and let $f : (X, d) \longrightarrow (Y, e)$ be a surjective isometry.

(a) Then f is balanced.

(b) If $\langle \mathcal{F}', \mathcal{G}' \rangle$ is a balanced Cauchy filter pair on (Y, e) , then $\langle f^{-1}\mathcal{F}', f^{-1}\mathcal{G}' \rangle$ is a balanced Cauchy filter pair on (X, d) .

Proof. (a) Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . Since f is quasi-uniformly continuous, $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is a Cauchy filter pair on (Y, e) . Let $y_1, y_2 \in Y$. By surjectivity of f choose $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then

$$e(y_1, y_2) = d(x_1, x_2) \leq \inf_{G \in \mathcal{G}} \Phi_d(x_1, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, x_2) = \inf_{G \in \mathcal{G}} \Phi_e(y_1, fG) + \inf_{F \in \mathcal{F}} \Phi_e(fF, y_2),$$

because f is an isometry and $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced. Hence $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is a balanced Cauchy filter pair on (Y, e) . Therefore f is balanced.

(b) Since f is an isometry, we have that $\Phi_e(F', G') = \Phi_d(f^{-1}F', f^{-1}G')$ where $F' \in \mathcal{F}'$ and $G' \in \mathcal{G}'$. Hence $\langle f^{-1}\mathcal{F}', f^{-1}\mathcal{G}' \rangle$ is a Cauchy filter pair on (X, d) . Let $x, y \in X$. Then $d(x, y) = e(f(x), f(y)) \leq \inf_{G' \in \mathcal{G}'} \Phi_e(f(x), G') + \inf_{F' \in \mathcal{F}'} \Phi_e(F', f(y)) = \inf_{G' \in \mathcal{G}'} \Phi_d(x, f^{-1}G') + \inf_{F' \in \mathcal{F}'} \Phi_d(f^{-1}F', y)$ because f is an isometry. Thus the pair $\langle f^{-1}\mathcal{F}', f^{-1}\mathcal{G}' \rangle$ is a balanced Cauchy filter pair on (X, d) .

Corollary 6.2.2 Let (X, d) be a quasi-pseudometric space and $(\widehat{X}, \widehat{d})$ its T_0 -quasi-pseudometric quotient space. Then (X, d) is B -complete if and only if $(\widehat{X}, \widehat{d})$ is B -complete.

If $f : (X, d) \longrightarrow (Y, e)$ is balanced map between two quasi-pseudometric spaces (X, d) and (Y, e) , then $\widehat{f} : (\widehat{X}, \widehat{d}) \longrightarrow (\widehat{Y}, \widehat{e})$ is balanced, where $(\widehat{X}, \widehat{d})$ denotes the T_0 -quasi-pseudometric quotient of (X, d) and $(\widehat{Y}, \widehat{e})$ denotes the T_0 -quasi-pseudometric quotient of (Y, e) .

Proof. The proof makes use of Lemma 6.2.3. Suppose that (X, d) is B -complete. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on $(\widehat{X}, \widehat{d})$. Therefore $\langle q_X^{-1}\mathcal{F}, q_X^{-1}\mathcal{G} \rangle$ is a balanced Cauchy filter pair on (X, d) . Thus for some $x \in X$, $\langle q_X^{-1}\mathcal{F}, q_X^{-1}\mathcal{G} \rangle$ converges to x . Then $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to $q_X(x)$. So $(\widehat{X}, \widehat{d})$ is B -complete.

For the converse suppose that $(\widehat{X}, \widehat{d})$ is B -complete. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . Then $\langle q_X\mathcal{F}, q_X\mathcal{G} \rangle$ is a balanced Cauchy filter pair on $(\widehat{X}, \widehat{d})$. Thus $\langle q_X\mathcal{F}, q_X\mathcal{G} \rangle$ converges to $q_X(x)$ for some $x \in X$. Then $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to x . Hence (X, d) is B -complete.

In order to prove the second statement, suppose that $f : (X, d) \longrightarrow (Y, e)$ is balanced. By Lemma 6.1.1 we know that \widehat{f} is quasi-uniformly continuous, since f is quasi-uniformly continuous. Let $\langle \mathcal{F}', \mathcal{G}' \rangle$ be a balanced Cauchy filter pair on $(\widehat{X}, \widehat{d})$.

Then by Lemma 6.2.3 $\langle q_X^{-1}\mathcal{F}', q_X^{-1}\mathcal{G}' \rangle$ is a balanced Cauchy filter pair on (X, d) . Therefore $\langle f q_X^{-1}\mathcal{F}', f q_X^{-1}\mathcal{G}' \rangle$ is a balanced Cauchy filter pair on (Y, e) , because f is balanced.

By Lemma 6.2.3 it follows that $\langle q_Y f q_X^{-1}\mathcal{F}', q_Y f q_X^{-1}\mathcal{G}' \rangle$ is a balanced Cauchy filter pair on $(\widehat{Y}, \widehat{e})$. But the latter Cauchy filter pair is equal to $\langle \widehat{f}\mathcal{F}', \widehat{f}\mathcal{G}' \rangle$, since for any $x \in X$, we have that $\widehat{f}(q_X(x)) = q_Y(f(x))$ and $\{q_Y(f(x))\} = q_Y(f(q_X^{-1}\{q_X(x)\}))$, because for all $y \in X$, $d(y, x) = d(x, y) = 0$ implies that $e(f(y), f(x)) = e(f(x), f(y)) = 0$. Hence $\langle \widehat{f}\mathcal{F}', \widehat{f}\mathcal{G}' \rangle$ is balanced on $(\widehat{Y}, \widehat{e})$. Then, \widehat{f} is balanced.

We next prove a lemma that turns out to be useful in the proof that for any quasi-pseudometric space (X, d) , (X^+, d^+) is indeed B -complete.

Lemma 6.2.4 *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on a quasi-pseudometric space (X, d) . Then for each $x \in X$ and $m \in \mathbb{N}$ there is $g \in X$ such that*

$$d^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) \leq d(x, g) + \frac{1}{m}$$

and

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(g)) < \frac{1}{m}.$$

Proof. There are $F_m \in \mathcal{F}$ and $G_m \in \mathcal{G}$ such that $\Phi_d(F_m, G_m) < \frac{1}{m}$. Furthermore for some $g \in G_m$, $d^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) = \inf_{G \in \mathcal{G}} \Phi_d(x, G) \leq \Phi_d(x, G_m) \leq d(x, g) + \frac{1}{m}$. Here we have used that $\Phi_d(x, G_m)$ is bounded, which can be seen as follows: Fix $f_m \in F_m$. Then for any $g \in G_m$, we have $d(x, g) \leq d(x, f_m) + d(f_m, g)$ and thus $\Phi_d(x, G_m) \leq d(x, f_m) + 1$. Furthermore $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(g)) < \frac{1}{m}$, since $G_m \in \underline{g}$ and $F_m \in \mathcal{F}$. Hence the assertion holds.

Corollary 6.2.3 *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on a quasi-pseudometric space (X, d) . Then for each $y \in X$ and $m \in \mathbb{N}$ there is $f \in X$ such that*

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y)) \leq d(f, y) + \frac{1}{m}$$

and

$$d^+(\alpha_X(f), \langle \mathcal{F}, \mathcal{G} \rangle) < \frac{1}{m}.$$

Proof. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, d) . There are $F_m \in \mathcal{F}$ and $G_m \in \mathcal{G}$ such that $\Phi_d(F_m, G_m) < \frac{1}{m}$. For some $f \in F_m$ and $y \in X$, we have that $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y)) = \inf_{F \in \mathcal{F}} \Phi_d(F, y) \leq \Phi_d(F_m, y) \leq d(f, y) + \frac{1}{m}$. Since $\Phi_d(F_m, y)$ is bounded, fix $g_m \in G_m$, then for any $f \in F_m$, we have that $d(f, y) \leq d(f, f_m) + d(f_m, y)$ and thus, we have that $\Phi_d(F_m, y) \leq d(f_m, y) + 1$. Furthermore, $d^+(\alpha_X(f), \langle \mathcal{F}, \mathcal{G} \rangle) < \frac{1}{m}$.

We next show that (X^b, d^b) is B -complete.

Theorem 6.2.1 *Let (X, d) be a T_0 -quasi-pseudometric space. Then (X^b, d^b) is B -complete.*

Proof. By Corollary 6.2.2 it suffices to show that (X^+, d^+) is B -complete. Suppose that $\langle \Xi, \Upsilon \rangle$ is a balanced Cauchy filter pair on (X^+, d^+) . For each $n \in \mathbb{N}$ choose $X_n \in \Xi$ and $Y_n \in \Upsilon$ such that $\Phi_{d^+}(X_n, Y_n) < \frac{1}{n}$. Without loss of generality we can assume that both sequences $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ are decreasing. For each $n \in \mathbb{N}$ and $x \in X$ we find $\eta_x^n \in Y_n$ such that $\Phi_{d^+}(\alpha_X(x), Y_n) \leq d^+(\alpha_X(x), \eta_x^n) + \frac{1}{n}$. Here we have used boundedness of $\Phi_{d^+}(\alpha_X(x), Y_n)$ which

is established in the same way as boundedness of $\Phi_d(x, G_n)$ in the proof of Lemma 6.2.4.

Similarly for each $y \in X$ and $n \in \mathbb{N}$ choose $\xi_y^n \in X_n$ such that $\Phi_{d^+}(X_n, \alpha_X(y)) \leq d^+(\xi_y^n, \alpha_X(y)) + \frac{1}{n}$.

For all $x \in X$ and $n \in \mathbb{N}$ each η_x^n is a balanced Cauchy filter pair on (X, d) . Similarly for all $y \in X$ and $n \in \mathbb{N}$ each ξ_y^n is a balanced Cauchy filter pair on (X, d) .

By Lemma 6.2.4 for each $n \in \mathbb{N}$ and $x \in X$ we find $h^{x,n} \in X$ such that $d^+(\alpha_X(x), \eta_x^n) \leq d(x, h^{x,n}) + \frac{1}{n}$ and $d^+(\eta_x^n, \alpha_X(h^{x,n})) < \frac{1}{n}$.

Similarly by Corollary 6.2.3 for each $n \in \mathbb{N}$ and $y \in X$ we find $g^{n,y} \in X$ such that $d^+(\xi_y^n, \alpha_X(y)) \leq d(g^{n,y}, y) + \frac{1}{n}$ and $d^+(\alpha_X(g^{n,y}), \xi_y^n) < \frac{1}{n}$.

For each $n \in \mathbb{N}$ set $G_n = \{g^{m,y} : m \geq n, m \in \mathbb{N} \text{ and } y \in Y\}$ and for each $n \in \mathbb{N}$ set $H_n = \{h^{m,x} : m \geq n, n \in \mathbb{N} \text{ and } x \in X\}$.

Note that the sequences $(G_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ of X are decreasing. Let \mathcal{G} be the filter on X generated by the filter base $\{G_n : n \in \mathbb{N}\}$ and \mathcal{H} be the filter on X generated by the filter base $\{H_n : n \in \mathbb{N}\}$.

One checks that $\langle \mathcal{G}, \mathcal{H} \rangle$ is a Cauchy filter pair on (X, d) as follows:

Let $n \in \mathbb{N}$. We recall that $\Phi_{d^+}(X_n, Y_n) < \frac{1}{n}$. Hence for all $m_1, m_2 \in \mathbb{N}$ such that $m_1, m_2 \geq n$ and $x, y \in X$ by the triangle inequality we have

$$\begin{aligned} d(g^{m_1,y}, h^{m_2,x}) &= d^+(\alpha_X(g^{m_1,y}), \alpha_X(h^{m_2,x})) \leq \\ &d^+(\alpha_X(g^{m_1,y}), \xi_y^{m_1}) + d^+(\xi_y^{m_1}, \eta_x^{m_2}) + d^+(\eta_x^{m_2}, \alpha_X(h^{m_2,x})) < \frac{3}{n}. \end{aligned}$$

We conclude that $\langle \mathcal{G}, \mathcal{H} \rangle$ is a Cauchy filter pair on (X, d) .

Let $x, y \in X$.

Since $\langle \Xi, \Upsilon \rangle$ is balanced on (X^+, d^+) , we have

$$\begin{aligned} d(x, y) &= d^+(\alpha_X(x), \alpha_X(y)) \leq \\ &\inf_{n \in \mathbb{N}} \Phi_{d^+}(\alpha_X(x), Y_n) + \inf_{n \in \mathbb{N}} \Phi_{d^+}(X_n, \alpha_X(y)). \end{aligned}$$

Consequently

$$d(x, y) \leq \left(\inf_{n \in \mathbb{N}} d^+(\alpha_X(x), \eta_x^n) + \frac{1}{n} \right) + \inf_{n \in \mathbb{N}} \left(d^+(\xi_y^n, \alpha_X(y)) + \frac{1}{n} \right)$$

by our choices of the Cauchy filter pairs η_x^n and ξ_y^n on X .

It follows that $d(x, y) \leq \inf_{n \in \mathbb{N}} (d(x, h^{n,x}) + \frac{2}{n}) + \inf_{n \in \mathbb{N}} (d(g^{n,y}, y) + \frac{2}{n})$. We conclude that $d(x, y) \leq \inf_{n \in \mathbb{N}} \Phi_d(x, H_n) + \inf_{n \in \mathbb{N}} \Phi_d(G_n, y)$, because $h^{n,x} \in H_n$ and $g^{n,y} \in G_n$. Hence $\langle \mathcal{G}, \mathcal{H} \rangle$ is a balanced Cauchy filter pair on (X, d) .

It remains to show that $\langle \Xi, \Upsilon \rangle$ converges to the point $\langle \mathcal{G}, \mathcal{H} \rangle$ in X^+ .

Let $n \in \mathbb{N}$ and let $\xi = \langle \mathcal{A}, \mathcal{B} \rangle \in X_n \subseteq X^+$. There are $A_n \in \mathcal{A}$ and $B_n \in \mathcal{B}$ such that $\Phi_d(A_n, B_n) < \frac{1}{n}$. Let $a \in A_n$. Then $d^+(\alpha_X(a), \xi) = \inf_{B \in \mathcal{B}} \Phi_d(a, B) < \frac{1}{n}$. Furthermore for each $m \in \mathbb{N}$ with $m \geq n$ and each $x \in X$, $d^+(\xi, \eta_x^m) < \frac{1}{n}$ and $d^+(\eta_x^m, \alpha_X(h^{m,x})) < \frac{1}{n}$. Thus for each $a \in A_n$, any $m \in \mathbb{N}$ with $m \geq n$ and any $x \in X$ we have $d(a, h^{m,x}) < \frac{3}{n}$ by the triangle inequality applied to d^+ . Hence $\Phi_d(A_n, H_n) \leq \frac{3}{n}$ and $d^+(\xi, \langle \mathcal{G}, \mathcal{H} \rangle) \leq \frac{3}{n}$. Therefore $\Phi_{d^+}(X_n, \langle \mathcal{G}, \mathcal{H} \rangle) \leq \frac{3}{n}$.

Analogously, we conclude that $\Phi_{d^+}(\langle \mathcal{G}, \mathcal{H} \rangle, Y_n) \leq \frac{3}{n}$. Hence $\langle \Xi, \Upsilon \rangle$ converges on (X^+, d^+) to the point $\langle \mathcal{G}, \mathcal{H} \rangle$ in X^+ . We have shown that (X^+, d^+) is B -complete.

We next show that the isometric embedding from X into the set X^b is bijective if (X, d) is B -complete.

Corollary 6.2.4 *Let (X, d) be a B -complete T_0 -quasi-pseudometric space. Then the isometric embedding $\beta_X : (X, d) \rightarrow (X^b, d^b)$ is bijective. (Therefore (X, d) and (X^b, d^b) can be identified under these conditions.)*

Proof. Since (X, d) is a T_0 -space, β_X is injective (see Lemma 6.2.1). Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . By B -completeness of (X, d) it converges to some $x \in X$. Then $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\alpha_X(x)$ are equivalent balanced Cauchy filter pairs on (X, d) . Thus $q_{X^+}(\langle \mathcal{F}, \mathcal{G} \rangle) = q_{X^+}(\alpha_X(x)) = \beta_X(x)$ and we have shown that β_X is also surjective.

Corollary 6.2.5 *Let (X, d) be a T_0 -quasi-pseudometric space. Then its B -completion (X^b, d^b) is a bicomplete T_0 -space. In particular it contains the bicompletion of (X, d) .*

Proof. Bicompleteness of (X^b, d^b) follows from Remark 6.1.3 and Theorem 6.2.1. We can extend the isometric embedding $\beta_X : (X, d) \rightarrow (X^b, d^b)$ to a quasi-uniformly continuous map $\beta_{\tilde{X}} : (\tilde{X}, \tilde{d}) \rightarrow (X^b, d^b)$. Indeed $\beta_{\tilde{X}}$ is an isometric embedding. Thus, if (X, d) is a subspace of a bicomplete quasi-pseudometric T_0 -space (X^b, d^b) then it contains the bicompletion of (X, d) . (see [31]).

Corollary 6.2.6 *Let (X, d) be a T_0 -quasi-pseudometric space. Then the B -completion of (X, d^{-1}) is isometric to the conjugate space of the B -completion of (X, d) . (Hence the constructed B -completion completes both (X, d) and (X, d^{-1}) at the same time. That is why we speak of a double completion of (X, d)).*

Proof. The assertion is obvious.

Definition 6.2.3 *Let (x_n) and (y_n) be two sequences in a set X . Let \mathcal{F} be the filter generated by the filter base $\{\{x_k : k \geq n, k \in \mathbb{N}\} : n \in \mathbb{N}\}$ and let \mathcal{G} be the filter generated by the filter base $\{\{y_k : k \geq n, k \in \mathbb{N}\} : n \in \mathbb{N}\}$ on X . Then we shall say that $\langle \mathcal{F}, \mathcal{G} \rangle$ is the filter pair generated by the pair $\langle (x_n), (y_n) \rangle$ of sequences on X .*

Example 6.2.2 *Let $X = \{-\frac{1}{n+1}, \frac{1}{n+1} : n \in \mathbb{N}\}$. For each $x, y \in X$, set $d(x, y) = 1$ if $y < 0 < x$ and $d(x, y) = \min\{|x - y|, 1\}$ otherwise. It is readily checked that (X, d) is a T_0 -quasi-pseudometric space.*

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be the filter pair on X generated by $\langle (-\frac{1}{n+1}), (\frac{1}{n+1}) \rangle$. Observe that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, d) , which is not balanced, since

$$1 = d\left(\frac{1}{4}, -\frac{1}{4}\right) \not\leq \inf_{G \in \mathcal{G}} \Phi_d\left(\frac{1}{4}, G\right) + \inf_{F \in \mathcal{F}} \Phi_d\left(F, -\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

On the other hand the non-convergent Cauchy filter pair $\langle \mathcal{F}, \mathcal{F} \rangle$ and $\langle \mathcal{G}, \mathcal{G} \rangle$ show that (X, d) is not bicomplete.

We leave it to the reader to check the following additional facts:

The B -completion of (X, d) , which evidently can be identified with the bicompletion of (X, d) is obtained by adding two new distinct points 0^- and 0^+ to X which represent the equivalence classes of $\langle \mathcal{F}, \mathcal{F} \rangle$ resp. $\langle \mathcal{G}, \mathcal{G} \rangle$. Then, d^b extends d as follows:

$d^b(0^-, x) = d^b(x, 0^+) = |x|$ if $x \in X$; $d^b(x, 0^-) = 1$ if $x > 0$; $d^b(x, 0^-) = |x|$ if $x < 0$; $d^b(0^+, x) = 1$ if $x < 0$; $d^b(0^+, x) = x$ if $x > 0$; and $d^b(0^-, 0^+) = 0$; $d^b(0^+, 0^-) = 1$. Of course $d^b(0^-, 0^-) = d^b(0^+, 0^+) = 0$.

6.3 Extension of mappings

In this section we shall investigate an extension of quasi-uniformly continuous (balanced) mappings. We will first prove a few technical lemmas which will be useful in the proof of the extension theorem of balanced maps. We also show that the natural isometric embedding map α_X is balanced.

Lemma 6.3.1 *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be Cauchy filter pairs on a quasi-pseudometric space (X, d) . Then*

$$\inf_{G' \in \mathcal{G}'} \Phi_{d^+}(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(G')) = \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g').$$

Similarly

$$\inf_{F \in \mathcal{F}} \Phi_{d^+}(\alpha_X(F), \langle \mathcal{F}', \mathcal{G}' \rangle) = \inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d(f, g').$$

Proof. Using the definitions of Φ_{d^+} and Φ_d , we have

$$\inf_{G' \in \mathcal{G}'} \Phi_{d^+}(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(G')) = \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(g')) =$$

$$\inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \Phi_d(F, \{g'\}) = \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g').$$

The second statement is proved analogously.

Lemma 6.3.2 Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two Cauchy filter pairs on a quasi-pseudometric space (X, d) . Then

$$\inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d(f, g') \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle)$$

and

$$\inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g') \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle).$$

Proof. Let $f \in F \in \mathcal{F}$ and $g' \in G' \in \mathcal{G}'$.

We have

$$\inf_{G' \in \mathcal{G}'} \Phi_d(f, G') \leq \Phi_d(f, G').$$

Therefore

$$\sup_{f \in F} \inf_{G' \in \mathcal{G}'} \Phi_d(f, G') \leq \sup_{f \in F} \Phi_d(f, G').$$

Consequently

$$\inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \Phi_d(f, G') \leq \inf_{F \in \mathcal{F}} \sup_{f \in F} \Phi_d(f, G').$$

Finally

$$\inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \Phi_d(f, G') \leq \inf_{G' \in \mathcal{G}'} \inf_{F \in \mathcal{F}} \sup_{f \in F} \Phi_d(f, G') = d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle)$$

by definition of d^+ . Hence we have established the first inequality. The second inequality is proved analogously.

Corollary 6.3.1 Let (X, d) be a quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . Then $\langle \alpha_X \mathcal{F}, \alpha_X \mathcal{G} \rangle$ converges in (X^+, d^+) to the point $\langle \mathcal{F}, \mathcal{G} \rangle$.

Proof. Set $\langle \mathcal{F}', \mathcal{G}' \rangle = \langle \mathcal{F}, \mathcal{G} \rangle$ in Lemma 6.3.1 and use Lemma 6.3.1 and the fact that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair.

Lemma 6.3.3 Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on a quasi-pseudometric space (X, d) and let $\langle \mathcal{F}', \mathcal{G}' \rangle$ be any Cauchy filter pair on (X, d) . Then

$$d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) = \inf_{F' \in \mathcal{F}'} \sup_{f' \in F'} \inf_{G \in \mathcal{G}} \sup_{g \in G} d(f', g).$$

Similarly

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g').$$

Proof. Let $f' \in F' \in \mathcal{F}'$ and let $g \in G \in \mathcal{G}$. Then

$$d(f', g) \leq \inf_{G \in \mathcal{G}} \Phi_d(f', G) + \inf_{F \in \mathcal{F}} \Phi_d(F, g)$$

by balancedness of $\langle \mathcal{F}, \mathcal{G} \rangle$. Thus

$$\Phi_d(F', G) \leq \sup_{f' \in F'} \inf_{G \in \mathcal{G}} \Phi_d(f', G) + \sup_{g \in G} \inf_{F \in \mathcal{F}} \Phi_d(F, g).$$

Consequently

$$\begin{aligned} d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) &= \inf_{F' \in \mathcal{F}'} \inf_{G \in \mathcal{G}} \Phi_d(F', G) \leq \\ &\inf_{F' \in \mathcal{F}'} \sup_{f' \in F'} \inf_{G \in \mathcal{G}} \Phi_d(f', G) + \inf_{G \in \mathcal{G}} \sup_{g \in G} \inf_{F \in \mathcal{F}} \Phi_d(F, g) = \\ &\inf_{F' \in \mathcal{F}'} \sup_{f' \in F'} \inf_{G \in \mathcal{G}} \sup_{g \in G} \Phi_d(f', g) + 0, \end{aligned}$$

because $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter on (X, d) and because of Lemma 6.3.2. Therefore

$$d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) \leq \inf_{F' \in \mathcal{F}'} \sup_{f' \in F'} \inf_{G \in \mathcal{G}} \sup_{g \in G} d(f', g).$$

The reverse inequality and thus equality hold because of Lemma 6.3.2. The second equality is proved analogously.

Corollary 6.3.2 *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two balanced Cauchy filter pairs on a quasi-pseudometric space (X, d) . Then*

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g') = \inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d(f, g').$$

Proof. The assertion is a consequence of Lemma 6.3.1.

We next show that α_X is balanced.

Corollary 6.3.3 *Let (X, d) be a quasi-pseudometric space. Then the map $\alpha_X : (X, d) \rightarrow (X^+, d^+)$ is balanced.*

Proof. Since α_X is an isometry, α_X is quasi-uniformly continuous. It remains to show that for any $\langle \mathcal{F}, \mathcal{G} \rangle \in X^+$ we have that the Cauchy filter pair $\langle \alpha_X \mathcal{F}, \alpha_X \mathcal{G} \rangle$ is balanced in (X^+, d^+) .

Consider any $\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle \in X^+$.

Then by the triangle inequality, Corollary 6.3.2 and Lemma 6.3.1 we have that,

$$\begin{aligned} d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) &\leq d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) + d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) = \\ &\inf_{G \in \mathcal{G}} \sup_{g \in G} \inf_{F' \in \mathcal{F}'} \sup_{f' \in F'} d(f', g) + \inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G'' \in \mathcal{G}''} \sup_{g'' \in G''} d(f, g'') = \\ &\inf_{G \in \mathcal{G}} \Phi_{d^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, \alpha_X(G)) + \inf_{F \in \mathcal{F}} \Phi_{d^+}(\alpha_X(F), \langle \mathcal{F}'', \mathcal{G}'' \rangle). \end{aligned}$$

Hence $\langle \alpha_X \mathcal{F}, \alpha_X \mathcal{G} \rangle$ is balanced and we have shown that the map α_X is balanced.

Corollary 6.3.4 *Let (X, d) be a quasi-pseudometric space. Then $\beta_X : (X, d) \rightarrow (X^b, d^b)$ is balanced.*

Proof. By the preceding result we know that $\alpha_X : (X, d) \rightarrow (X^+, d^+)$ is balanced. From Lemma 6.2.3 (a) it follows that $\beta_X = q_{X^+} \circ \alpha_X$ is balanced.

Proposition 6.3.1 *Let $f : (X, d) \rightarrow (Y, e)$ be a balanced map between T_0 -quasi-pseudometric spaces (X, d) and (Y, e) . Then there is a unique balanced map $\tilde{f} : (X^b, d^b) \rightarrow (Y^b, e^b)$ such that $\tilde{f} \circ \beta_X = \beta_Y \circ f$. If f is also an isometry, then \tilde{f} is an isometry.*

Proof. For each $\langle \mathcal{F}, \mathcal{G} \rangle \in X^+$ we set $f^+(\langle \mathcal{F}, \mathcal{G} \rangle) = \langle f\mathcal{F}, f\mathcal{G} \rangle$. This equation define a map $f^+ : (X^+, d^+) \rightarrow (Y^+, e^+)$, because f is balanced.

Suppose first that f is an isometry. Then

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = \inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \Phi_d(F, G') =$$

$$\inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \Phi_e(fF, fG') = e^+(\langle f\mathcal{F}, f\mathcal{G} \rangle, \langle f\mathcal{F}', f\mathcal{G}' \rangle) = e^+(f^+(\langle \mathcal{F}, \mathcal{G} \rangle), f^+(\langle \mathcal{F}', \mathcal{G}' \rangle))$$

whenever $\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle \in X^+$. Hence in the case under consideration f^+ is an isometry.

Assume now that f is quasi-uniformly continuous. Therefore for each $\epsilon > 0$ there is $\delta > 0$ such that for each $x, y \in X$, $d(x, y) < \delta$ implies that $e(f(x), f(y)) < \epsilon$. Consider any $\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle \in X^+$ such that $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) < \delta$. Then there are $F \in \mathcal{F}$ and $G' \in \mathcal{G}'$ such that $\Phi_d(F, G') < \delta$. Thus by quasi-uniform continuity of f , we get that $\Phi_e(fF, fG') \leq \epsilon$ and hence $e^+(f^+(\langle \mathcal{F}, \mathcal{G} \rangle), f^+(\langle \mathcal{F}', \mathcal{G}' \rangle)) = e^+(\langle f\mathcal{F}, f\mathcal{G} \rangle, \langle f\mathcal{F}', f\mathcal{G}' \rangle) \leq \epsilon$. We have shown that f^+ is quasi-uniformly continuous.

Obviously for each $x \in X$, $(f^+ \circ \alpha_X)(x) = \langle \underline{f(x)}, \underline{f(x)} \rangle = (\alpha_Y \circ f)(x)$.

We next show that f^+ is balanced provided that f is balanced. Note first that f^+ is quasi-uniformly continuous, since f is quasi-uniformly continuous.

To this end let $\langle \Xi, \Upsilon \rangle$ be a balanced Cauchy filter pair on (X^+, d^+) . We want to show that $\langle f^+\Xi, f^+\Upsilon \rangle$ is a balanced Cauchy filter pair on (Y^+, e^+) .

Suppose the contrary. Then there are $a, b \in Y$ and $X \in \Xi$ and $Y \in \Upsilon$ such that

$$e^+(a, b) > \Phi_{e^+}(a, f^+Y) + \Phi_{e^-}(f^+X, b).$$

There are decreasing sequences (X_n) and (Y_n) such that for each $n \in \mathbb{N}$, $(X_n) \in \Xi$ and $(Y_n) \in \Upsilon$ such that $\Phi_{d^+}((X_n), (Y_n)) < \frac{1}{n}$ and $X_1 \subseteq X$ and $Y_1 \subseteq Y$.

Exactly as in the proof of Theorem 6.2.1 we construct a balanced Cauchy filter pair $\langle \mathcal{G}, \mathcal{H} \rangle$ on (X, d) , where \mathcal{G} has the countable filter base $\{G_n : n \in \mathbb{N}\}$ and \mathcal{H} has a countable filter base $\{H_n : n \in \mathbb{N}\}$ defined there.

By our assumption on the map f we deduce that $\langle f\mathcal{G}, f\mathcal{H} \rangle$ is a balanced Cauchy filter pair on (Y, e) . Thus $\langle (\alpha_Y \circ f)(\mathcal{G}), (\alpha_Y \circ f)(\mathcal{H}) \rangle$ is a balanced

Cauchy filter pair on (Y^+, e^+) , because α_Y is balanced by Corollary 6.3.3.

In particular there is $n_0 \in \mathbb{N}$ such that $\Phi_{e^+}((\alpha_Y \circ f)(G_{n_0}), (\alpha_Y \circ f)(H_{n_0})) \leq 1$ and we have that

$$e^+(a, b) \leq \inf_{n \in \mathbb{N}} \Phi_{e^+}(a, (\alpha_Y \circ f)(H_n)) + \inf_{n \in \mathbb{N}} \Phi_{e^+}((\alpha_Y \circ f)(G_n), b).$$

Then for each $n \in \mathbb{N}$ with $n \geq n_0$ there are $k_n \in \mathbb{N}$ with $k_n \geq n$ and $x_n \in X$ such that $\Phi_{e^+}(a, (\alpha_Y \circ f)(H_n)) \leq e^+(a, (\alpha_Y \circ f)(h^{k_n, x_n})) + \frac{1}{n}$. Here we have used the fact that for each $n \in \mathbb{N}$ with $n \geq n_0$ we have that $\Phi_{e^+}(a, (\alpha_Y \circ f)(H_n)) \leq \infty$, which can be established analogously to boundedness of $\Phi_d(x, G_n)$ in the proof of Lemma 6.2.4.

Consequently by the triangle inequality applied to e^+ , we obtain that for each $n \in \mathbb{N}$ such that $n \geq n_0$,

$$\Phi_{e^+}(a, (\alpha_Y \circ f)(H_n)) \leq e^+(a, f^+(\eta_{x_n}^{k_n})) + e^+(f^+(\eta_{x_n}^{k_n}), (f^+ \circ \alpha_X)(h^{k_n, x_n})) + \frac{1}{n}.$$

Observe also that

$$\lim_{n \in \mathbb{N}, n \geq n_0} e^+(f^+(\eta_{x_n}^{k_n}), (f^+ \circ \alpha_X)(h^{k_n, x_n})) = 0,$$

since $d^+(\eta_{x_n}^{k_n}, \alpha_X(h^{k_n, x_n})) < \frac{1}{k_n} \leq \frac{1}{n}$ whenever $n \in \mathbb{N}$ and $n \geq n_0$, and the map f^+ is quasi-uniformly continuous.

Because $\eta_{x_n}^{k_n} \in Y_{k_n} \subseteq Y$ whenever $n \in \mathbb{N}$ and $n \geq n_0$, it follows that

$$\inf_{n \in \mathbb{N}} \Phi_{e^+}(a, (\alpha_Y \circ f)(H_n)) \leq \Phi_{e^+}(a, f^+(Y)).$$

Analogously we conclude that

$$\inf_{n \in \mathbb{N}} \Phi_{e^+}((\alpha_Y \circ f)(G_n), b) \leq \Phi_{e^+}(f^+(X), b).$$

Hence altogether we get that

$$e^+(a, b) \leq \Phi_{e^+}(a, f^+(Y)) + \Phi_{e^+}(f^+(X), b).$$

We have reached a contradiction and deduce that $\langle f^+\Xi, f^+\Upsilon \rangle$ is a balanced Cauchy filter pair on (Y^+, e^+) . Thus $f^+ : X^+ \rightarrow Y^+$ is balanced.

Finally set $\tilde{f} = \widehat{f^+}$. This completes the definition of the map $\tilde{f} : (\tilde{X}, \tilde{d}) \rightarrow (\tilde{Y}, \tilde{e})$.

We conclude by Lemma 6.1.1 and Corollary 6.2.2 that \tilde{f} is an isometry (balanced, respectively) provided that f is an isometry (balanced, respectively).

Now suppose that $g : (\tilde{X}, \tilde{d}) \rightarrow (\tilde{Y}, \tilde{e})$ is another balanced map such that $\beta_Y \circ f = g \circ \beta_X$.

Let $\langle \mathcal{F}, \mathcal{G} \rangle \in X^+$. Then $\langle \alpha_X \mathcal{F}, \alpha_X \mathcal{G} \rangle$ converges to $\langle \mathcal{F}, \mathcal{G} \rangle$ in X^+ by Corollary 6.3.1.

Thus with the help of quasi-uniform continuity of q_{X^+} and g we see that $\langle (g \circ \beta_X) \mathcal{F}, (g \circ \beta_X) \mathcal{G} \rangle$ converges to $g(q_{X^+}(\langle \mathcal{F}, \mathcal{G} \rangle))$.

Similarly $\langle (\tilde{f} \circ \beta_X) \mathcal{F}, (\tilde{f} \circ \beta_X) \mathcal{G} \rangle$ converges to $\tilde{f}(q_{X^+}(\langle \mathcal{F}, \mathcal{G} \rangle))$.

Because $\tilde{f} \circ \beta_X = g \circ \beta_X$ we have $\langle (g \circ \beta_X) \mathcal{F}, (g \circ \beta_X) \mathcal{G} \rangle = \langle (\tilde{f} \circ \beta_X) \mathcal{F}, (\tilde{f} \circ \beta_X) \mathcal{G} \rangle$.

Since (\tilde{Y}, \tilde{e}) is a T_0 -quasi-pseudometric space and $\tilde{f} \circ \beta_X$ is a balanced map, we conclude that $g(q_{X^+}(\langle \mathcal{F}, \mathcal{G} \rangle)) = \tilde{f}(q_{X^+}(\langle \mathcal{F}, \mathcal{G} \rangle))$. Hence $g = \tilde{f}$.

Proposition 6.3.2 *Let (X, d) be a subspace of the B -complete T_0 -quasi-pseudometric space (Y, e) . Suppose that the embedding $i : (X, d) \rightarrow (Y, e)$ is balanced and that for each $y \in Y$ there is a balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) such that the filter pair $\langle i\mathcal{F}, i\mathcal{G} \rangle$ converges to y . Then the B -completion (\tilde{X}, \tilde{d}) of (X, d) is isometric to (Y, e) under the isometric balanced extension \tilde{i} of i to \tilde{X} .*

Proof. Clearly by Proposition 6.3.1 the map $i : (X, d) \rightarrow (Y, e)$ has an isometric balanced extension $\tilde{i} : (\tilde{X}, \tilde{d}) \rightarrow (\tilde{Y}, \tilde{e})$, where by Corollary 6.2.4 (\tilde{Y}, \tilde{e}) can be identified with (Y, e) , because (Y, e) is B -complete. Since (\tilde{X}, \tilde{d}) is a T_0 -quasi-pseudometric space, \tilde{i} is injective by Lemma 6.2.1. Let $y \in Y$. By the density assumption formulated in Corollary 6.3.1 there is a balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) such that the filter pair $\langle i\mathcal{F}, i\mathcal{G} \rangle$ converges to y . Then $\tilde{i}(q_{X^+}(\langle \mathcal{F}, \mathcal{G} \rangle)) = (q_{Y^+} \circ i^+)(\langle \mathcal{F}, \mathcal{G} \rangle) = q_{Y^+}(\langle i\mathcal{F}, i\mathcal{G} \rangle) = q_{Y^+}(\alpha_Y(y)) = \beta_y(y) = y$. The last equality makes use of our identification by Corollary 6.2.4 between (\tilde{Y}, \tilde{e}) and (Y, e) . Hence we conclude that \tilde{i} is an

isometric bijection.

We next illustrate the importance of balanced of the natural isometric embedding of our theory by an example. We leave the details of the proof to the reader. We introduce the terminology that will turn out to be useful in the next section.

Definition 6.3.1 *Let (X, d) be a quasi-pseudometric space. A pair of sequences $\langle (x_n), (y_n) \rangle$ in (X, d) will be called a **balanced** Cauchy pair of sequences provided that the filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ generated by $\langle (x_n), (y_n) \rangle$ is a (balanced) Cauchy filter pair on (X, d) .*

Example 6.3.1 *Let (Y, d) be a T_0 -quasi-pseudometric subspace $\{-\frac{1}{n+1}, \frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}$ of the rational T_0 -quasi-pseudometric Sorgenfrey line. Set $Z = Y \cup \{0^-\}$ by adding a new point 0 to Y . Extend d from $Y \times Y$ to $Z \times Z$ as follows: Set $d(-\frac{1}{n+1}, 0^-) = \frac{1}{n+1}$ whenever $n \in \mathbb{N}$. $d(0^-, 0^-) = 0$ and $d = 1$ otherwise. It is readily verified that (Z, d) is a T_0 -quasi-pseudometric space.*

One checks that on the subspace $H = Z \setminus \{0\}$ of Z the pair $\langle (-\frac{1}{n+1}), (\frac{1}{n+1}) \rangle$ of sequences generates a balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ that is not convergent in H . Therefore H is not B -complete.

In order to prove balancedness of $\langle \mathcal{F}, \mathcal{G} \rangle$, since $d \leq 1$ it suffices to consider the case that, $a, b \in H$, $\inf_{G \in \mathcal{G}} \Phi_d(a, G) < 1$ and $\inf_{F \in \mathcal{F}} \Phi_d(F, b) < 1$. It follows that $a < 1$, and $b > 0$ or $b = 0^-$, which implies that $d(a, b) \leq \inf_{G \in \mathcal{G}} \Phi_d(a, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, b) = 1$. We have shown that $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced. On the other hand one verifies that the T_0 -quasi-pseudometric space Z is B -complete. However it is not the B -completion H^b of H , since the convergent Cauchy filter pair $\langle \mathcal{F}', \mathcal{G}' \rangle$ on Z generated by $\langle (-\frac{1}{n+1}), (\frac{1}{n+1}) \rangle$ is not balanced in Z : We have that

$$1 = d(0, 0^-) \not\leq \inf_{G' \in \mathcal{G}'} \Phi_d(0, G') + \inf_{F' \in \mathcal{F}'} \Phi_d(F', 0^-) = 0 + 0.$$

If we replace d on Z by changing $d(0, 0^-) = 1$ to $d'(0, 0^-) = 0$. and setting $d'(x, y) = d(y, x)$ otherwise, we obtain a space (Z, d') isometric to the B -completion H^b of H .

We have shown that the isometric, but non-balanced embedding i of H into the B -complete space Z cannot be extended to an isometric embedding \tilde{i} of H^b into Z , because by injectivity such an embedding \tilde{i} would have to map the limit of $\langle \beta_X \mathcal{F}, \beta_X \mathcal{G} \rangle$ to 0 and therefore cannot be isometric. because $d' \neq d$.

6.4 Connections with Doitchinov's work on balanced quasi-pseudometrics

In the following section, we explain the connections between our completion and the Doitchinov completion developed in [11].

In [11] Doitchinov studied the completion for balanced T_0 -quasi-pseudometric spaces. He has shown that each balanced quasi-pseudometric space has a completion. It will be shown in this section that for balanced T_0 -quasi-pseudometric spaces our completion is isometric to the Doitchinov completion.

Definition 6.4.1 ([11]) *A quasi-pseudometric space (X, d) is called balanced if (x'_n) and (x''_m) are two sequences in (X, d) and $x', x'' \in X$, then from $d(x', x'_n) \leq r'$ for each n , and $d(x''_m, x'') \leq r''$ for each m and $\lim_{m,n} d(x''_m, x'_n) = 0$ it follows that $d(x', x'') \leq r' + r''$.*

Proposition 6.4.1 *The following conditions are equivalent for a quasi-pseudometric space (X, d) :*

- (a) (X, d) is balanced.
- (b) Every Cauchy pair of sequences in (X, d) is balanced.
- (c) Every Cauchy filter pair on (X, d) is balanced.

Proof. (a) \implies (b): Let $\langle (x_n), (y_n) \rangle$ be a Cauchy pair of sequences in (X, d) and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be its generated filter pair on X . Suppose that there are $x', x'' \in X$ and $n \in \mathbb{N}$ such that

$$d(x', x'') > \Phi_d(x', \{y_k : k \geq n, k \in \mathbb{N}\}) + \Phi_d(\{x_k : k \geq n, k \in \mathbb{N}\}, x'').$$

But the latter condition contradicts the assumption that d is balanced. We conclude that $\langle \mathcal{F}, \mathcal{G} \rangle$ and, thus, $\langle (x_n), (y_n) \rangle$ are balanced.

(b) \implies (c): Suppose the contrary. Then there is a Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) with $a, b \in X$ and $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $d(a, b) > \Phi_d(a, G) + \Phi_d(F, b)$. Find a pair of sequences $\langle (f_n), (g_n) \rangle$ in X as follows: Choose inductively decreasing sequences (F_n) and (G_n) such that $F_1 \subseteq F$, $G_1 \subseteq G$, $F_n \in \mathcal{F}$, $G_n \in \mathcal{G}$ and $\Phi_d(F_n, G_n) < \frac{1}{n}$ whenever $n \in \mathbb{N}$. Furthermore for each $n \in \mathbb{N}$ find $f_n \in F_n$ and $g_n \in G_n$. By our hypothesis the Cauchy filter pair $\langle \mathcal{F}', \mathcal{G}' \rangle$ on X generated by $\langle (f_n), (g_n) \rangle$ is balanced, and thus $d(a, b) \leq \inf_{G' \in \mathcal{G}'} \Phi_d(a, G') + \inf_{F' \in \mathcal{F}'} \Phi_d(F', b) \leq \Phi_d(a, G) + \Phi_d(F, b)$ — a contradiction. We conclude that each Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) is balanced.

(c) \implies (a): Suppose that (x'_n) and (x''_m) are two sequences in (X, d) such that $\lim_{m,n} d(x''_m, x'_n) = 0$. Consider the filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ generated on X by the Cauchy pair $\langle (x'_n), (x''_m) \rangle$ of sequences. Assume now that $x', x'' \in X$, $d(x', x'_n) \leq r'$ for each n , and $d(x''_m, x'') \leq r''$ for each m . Since by our assumption $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced, we conclude that $d(x', x'') \leq r' + r''$. Therefore d is balanced by Definition 6.4.1.

Remark 6.4.1 *Let (X, d) be a balanced quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, d) . Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is equivalent to a Cauchy filter pair generated by a balanced Cauchy pair of sequences. Hence the B -completion of a balanced T_0 -quasi-pseudometric space can be built with the help of balanced Cauchy pairs of sequences only, since such sequences can represent all equivalence classes of X^+ .*

Proof. By Proposition 6.4.1 the given Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced. Similarly as above we choose decreasing sequences (F_n) and (G_n) such that $\Phi_d(F_n, G_n) < \frac{1}{n}$, $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}$ whenever $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ find $f_n \in F_n$ and $g_n \in G_n$. Then according to Proposition 6.4.1 the Cauchy filter pair $\langle \mathcal{F}', \mathcal{G}' \rangle$ generated by $\langle (f_n), (g_n) \rangle$ is balanced, since d is balanced. Furthermore $\langle \mathcal{F}', \mathcal{G}' \rangle$ and $\langle \mathcal{F}, \mathcal{G} \rangle$ are clearly equivalent by the construction of $\langle \mathcal{F}', \mathcal{G}' \rangle$.

Remark 6.4.2 *Let (X, d) be a balanced quasi-pseudometric space*

and let $\langle (x_n), (y_n) \rangle$ be a Cauchy pair of sequences in (X, d) . Then if for some $x \in X$, we have that $\lim_{n \rightarrow \infty} d(x, y_n) = 0$, then $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.

Proof Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be the Cauchy filter pair generated by $\langle (x_n), (y_n) \rangle$. Fix $n \in \mathbb{N}$. Then $d(x_n, x) \leq \inf_{G \in \mathcal{G}} \Phi_d(x_n, G) + \inf \Phi_d(x, x)$, because the Cauchy filter pair $\langle \underline{x}, \mathcal{G} \rangle$ is balanced by Proposition 6.4.1. Thus $d(x_n, x)$ converges to 0, since $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair.

Proposition 6.4.2 *A balanced quasi-pseudometric space (X, d) is B -complete if and only if each Cauchy pair $\langle (x_n), (y_n) \rangle$ of sequences converges. (That is, there is $x \in X$ such that the sequences $(d(x, y_n))_n$ and $(d(x_n, x))_n$ both converge to 0.)*

Proof. Throughout the proof we shall make use of Proposition 6.4.1. Let (X, d) be a B -complete balanced quasi-pseudometric space. For a given Cauchy pair $\langle (x_n), (y_n) \rangle$ of sequences in (X, d) , we consider its generated (balanced) Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on X . Since (X, d) is B -complete, then $\langle \mathcal{F}, \mathcal{G} \rangle$ converges in X . Obviously $\langle (x_n), (y_n) \rangle$ converges in X .

For the converse suppose that each Cauchy pair of sequences converges on a balanced quasi-pseudometric space (X, d) .

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be an arbitrary balanced Cauchy filter pair on (X, d) . Exactly as in the proof of Remark 6.4.1 we choose a Cauchy pair of sequences $\langle (f_n), (g_n) \rangle$ and let $\langle \mathcal{F}', \mathcal{G}' \rangle$ be its generated (balanced) Cauchy filter pair on X .

By assumption $\langle (f_n), (g_n) \rangle$ converges to $x \in X$. Then $d^+(\alpha(x), \langle \mathcal{F}, \mathcal{G} \rangle) \leq d^+(\alpha(x), \langle \mathcal{F}', \mathcal{G}' \rangle) + d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) = 0 + 0$, since $\langle (x), \mathcal{G}' \rangle$ and $\langle \mathcal{F}', \mathcal{G} \rangle$ are Cauchy filter pairs. Then $\inf_{G \in \mathcal{G}} \Phi_d(x, G) = 0$. Similarly $\inf_{F \in \mathcal{F}} \Phi_d(F, x) = 0$ and hence $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to x . Hence (X, d) is B -complete.

Remark 6.4.3 *Each quasi-uniformly continuous map $f : (X, d) \rightarrow (Y, e)$ from any quasi-pseudometric space (X, d) into a balanced quasi-pseudometric space (Y, e) is balanced.*

Proof. The assertion follows directly from Proposition 6.4.1 and the definition of a balanced map and a balanced quasi-pseudometric.

Proposition 6.4.3 *Let (X^D, d^D) be the Doitchinov completion of a balanced T_0 -quasi-pseudometric space (X, d) . Then the B -completion (X^b, d^b) of (X, d) is isometric to (X^D, d^D) .*

Proof. The proof is obvious, we refer the reader to [11] and to our characterization of the B -completion in Proposition 6.3.2

6.5 Conclusion

In this last chapter of the thesis, we have discussed the B -completion of a T_0 -quasi-pseudometric space (X, d) . We introduced a concept of balanced Cauchy filter pair and an interesting distance function d^+ on the set X^+ of all balanced Cauchy filter pairs of (X, d) . We defined the B -completeness condition which says that each balanced Cauchy filter pair converges. We then showed that each T_0 -quasi-pseudometric space (X, d) has an essentially unique (standard) B -completion (X^b, d^b) .

We introduced a concept of balanced map and proved that the natural isometric embedding of a T_0 -quasi-pseudometric space (X, d) into its B -completion (X^b, d^b) is balanced. We used this result to extend the balanced maps. We have established the connections with Doitchinov's work in [11] by showing that the B -completion of a balanced T_0 -quasi-pseudometric space is isometric to Doitchinov's completion.

Our conclusion leads us to list some open problems encountered throughout the present investigation. We hope to study these questions in future work.

Problem 6.5.1 Let (X, d) be a quasi-pseudometric space. If each filter pair generated by a balanced Cauchy pair of sequences converges in (X, d) , is (X, d) B -complete?, if not, is there a reasonable completion theory for the latter completeness property?

Problem 6.5.2 Could the present construction of the B -completion also be obtained with the help of Cauchy pairs of sequences instead of Cauchy pairs of filters?

Problem 6.5.3 Doitchinov has extended his theory of balanced T_0 -quasi-pseudometric spaces to a related theory of quiet T_0 -quasi-uniform spaces. Can many of the ideas presented in this investigation for quasi-pseudometric spaces be generalized to quasi-uniform spaces?

Problem 6.5.4 Find a natural example of a T_0 -quasi-pseudometric space for which its B -completion contains strictly the bicompletion.

We will next list all those articles that we have consulted during the completion of this thesis.

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