Quantitative Models for Prudential Credit Risk Management

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Declaration: This work has not been previously submitted in whole, or in part, for the award of any degree. It is my own work. Each significant contribution to, and quotation in, this dissertation from the work, or works, of other people has been attributed, and has been cited and referenced.

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Abstract

The thesis investigates the exogenous maturity vintage model (EMV) as a framework for achieving unification in consumer credit risk analysis. We explore how the EMV model can be used in origination modelling, impairment analysis, capital analysis, stress-testing and in the assessment of economic value. The thesis is segmented into five themes.

The first theme addresses some of the theoretical challenges of the standard EMV model – namely, the identifiability problem and the forecasting of the components of the model in predictive applications. We extend the model beyond the three time dimensions by introducing a behavioural dimension. This allows the model to produce loan-specific estimates of default risk. By replacing the vintage component with either an application risk or a behavioural risk dimension, the model resolves the identifiability problem inherent in the standard model. We show that the same model can be used interchangeably to produce a point-in-time probability forecast, by fitting a time series regression for the exogenous component, and a through-the-cycle probability forecast, by omitting the exogenous component. We investigate the use of the model for regulatory capital and stress-testing under Basel III, as well as impairment provisioning under IFRS 9. We show that when a Gaussian link function is used the portfolio loss follows a Vašíček distribution. Furthermore, the asset correlation coefficient (as defined under Basel III) is shown to be a function of the level of systemic risk (which is measured by the variance of the exogenous component) and the extent to which the systemic risk can be modelled (which is measured by the coefficient of determination of the regression model for the exogenous component).

The second theme addresses the problem of deriving a portfolio loss distribution from a loan-level model for loss. In most models (including the Basel-Vašíček regimes), this is done by assuming that the portfolio is infinitely large – resulting in a loss distribution that ignores diversifiable risk. We thus show that, holding all risk parameters constant, this assumption leads to an understatement of the level of risk within a portfolio – particularly for small portfolios. To overcome this weakness, we derive formulae that can be used to partition the portfolio risk into risk that is diversifiable and risk that is systemic. Using these formulae, we derive a loss distribution that better-represents losses under portfolios of all sizes.

The third theme is concerned with two separate issues: (a) the problem of model selection in credit risk and (b) the problem of how to accurately measure probability of insolvency in a credit portfolio. To address the first problem, we use the EMV model to study the theoretical properties of the Gini statistic for default risk in a portfolio of loans and derive a formula that estimates the Gini statistic directly from the model parameters. We then show that the formulae derived to estimate the Gini statistic can be used to study the probability of insolvency. To do this, we first show that when capital requirements are determined to target a specific probability of solvency on a through-the-cycle basis, the point-in-time

probability of insolvency can be considerably different from the through-the-cycle probability of insolvency – thus posing a challenge from a risk management perspective. We show that the extent of this challenge will be greater for more cyclical loan portfolios. We then show that the formula derived for the Gini statistic can be used to measure the extent of the point-in-time insolvency risk posed by using a through-the-cycle capital regime.

The fourth theme considers the problem of survival modelling with time varying covariates. We propose an extension to the Cox regression model, allowing the inclusion of time-varying macroeconomic variables as covariates. The model is specifically applied to estimate the probability of default in a loan portfolio, where the experience is decomposed the experience into three dimensions: (a) a survival time dimension; (b) a behavioural risk dimension; and (c) calendar time dimension. In this regard, the model can also be viewed as an extension of the EMV model – adding a survival time dimension. A model is built for each dimension: (a) the survival time dimension is modelled by a baseline hazard curve; (b) the behavioural risk dimension is modelled by a behavioural risk index; and (c) the calendar time dimension is modelled by a macroeconomic risk index. The model lends itself to application in modelling probability of default under the IFRS 9 regime, where it can produce estimates of probability of default over variable time horizons, while accounting for time-varying macroeconomic variables. However, the model also has a broader scope of application beyond the domains of credit risk and banking.

In the fifth and final theme, we introduce the concept of embedded value to a banking context. In long-term insurance, embedded value relates to the expected economic value (to shareholders) of a book of insurance contracts and is used for appraising insurance companies and measuring management's performance. We derive formulae for estimating the embedded value of a portfolio of loans, which we show to be a function of: (a) the spread between the rate charged to the borrower and the cost of funding; (b) the tenure of the loan; and (c) the level of credit risk inherent in the loan. We also show how economic value can be attributed between profits from maturity transformation and profits from credit and liquidity margin. We derive formulae that can be used to analyse the change in embedded value throughout the life of a loan. By modelling the credit loss component of embedded value, we derive a distribution for the economic value of a book of business.

The literary contributions made by the thesis are of practical significance. The thesis offers a way for banks and regulators to accurately estimate the value of the asset correlation coefficient in a manner that controls for portfolio size and intertemporal heterogeneity. This will lead to improved precision in determining capital adequacy – particularly for institutions operating in uncertain environments and those operating small credit portfolios – ultimately enhancing the integrity of the financial system. The thesis also offers tools to help bank management appraise the financial performance of their businesses and measure the value created for shareholders.

Table of Contents

Chapters and Sections

Abstract ... 3

Table of Contents ... 5

Acknowledgements ... 10

Chapter 1: Introduction ... 11

1.1 Credit Risk Analysis ... 11

1.1.1 An Overview ... 11

1.1.2 Unification in Credit Risk Analysis ... 12

1.1.3 Focus of the Thesis ... 13

1.1.4 Empirical Context ... 14

1.2 Research Question ... 15

1.2.1 Research Problems ... 15

1.2.2 Research Questions ... 17

1.3 Structure of the Thesis ... 18

Chapter 2: Literature Review ... 19

2.1 A Mathematical Statistics Basis for Credit Risk Analysis ... 19

2.2 Prudential Credit Risk Management ... 21

2.2.1 Impairment Modelling ... 21

2.2.2 Capital Requirements ... 22

2.2.3 Financial Valuation ... 23

2.3 Main Challenges in the Field ... 24

2.3.1 The Challenges of the EMV approach ... 24

2.3.2 IFRS 9 and Cox Regression ... 25

2.3.3 Weaknesses of the Basel-Vašíček Capital Framework ... 27

2.3.4 The Economic Value of a Loan Portfolio ... 29

2.3.5 Empirical Estimates of the Asset Correlation Coefficient ... 30

2.4 Contributions to the Field ... 31

Chapter 3: The Extended Exogenous Maturity Vintage Model ... 34

3.1 Overview of the Problem ... 34

3.2 The Exogenous Maturity Vintage Model ... 35

3.2.1 The Age Period Cohort Model ... 35

3.2.2 The Exogenous Maturity Vintage Model ... 36

3.3 Extending the Model ... 38

3.3.1 Extension: Addition of Behavioural Risk Grade 38
3.3.2 Extension: Addition of Application Risk Grade 39
3.3.3 Model Estimation 40
3.3.4 Predictive Modelling vs Explanatory Modelling 41
3.3.5 Extension: Macroeconomic Model 42
3.3.6 Extension: Quantile Function 44
3.3.7 Extension: Error Distribution and Capital Provisioning 46
3.3.8 Application in Impairment Modelling & Stress Testing 48
3.4 Case Studies 49
3.4.1 Identifiability Problem 50
3.4.2 Personal Loan Probability of Default 52
3.5 Summary of the Findings 64
Chapter 4: Diversifiable and Undiversifiable Risk 66
4.1 Overview of the Problem 66
4.2 Extending the Large Homogenous Portfolio Approximation 67
4.2.1 Implications of the LHP Assumptions on Variance 69
4.2.2 Formulae for the Variance Portfolio Loss Rate 70
4.2.3 The Variance Formulae under Special Cases 71
4.2.4 Extending the LHP Approximation: the Largeness Assumption 73
4.2.5 Extending the LHP Approximation: the Homogeneity Assumption 74
4.2.6 Applying the Extended LHP Approximation on an EMV Model 75
4.3 Case Study 76
4.4 Summary of the Findings 80
Chapter 5: The Gini Statistic as a Measure for Risk and Diversity 82
5.1 Overview of the Problem 82
5.2 The Gini Statistic 83
5.2.1 **The Gini Statistics in Credit Analytics** 83
5.2.2 Mathematical Specification of the Gini Statistic 86
5.2.3 The Gini Statistic under the Vašíček Distribution 86
5.2.4 Special Cases of the Gini Statistic 88
5.2.5 The Gini Statistic for the Exogenous Maturity Vintage Model 89
5.3 Probability of Insolvency 91
5.3.1 The Capital Buffer 92
5.3.2 Measuring the Risk Associated with a Through-the-Cycle Capital Buffer 94
5.3.3 Point-in-Time vs Through-the-Cycle 95
5.4 Case Study 96
5.5 Summary of the Findings 98

Chapter 6: The EMV Model for Survival Analysis 99

6.1 Overview of the Problem 99

6.2 Proposed Model 102

6.2.1 Lifetime Probability of Default 102

6.2.2 A Review of Cox Regression 103

6.2.3 A Discrete-Time Survival Model with Time-Varying Covariates 104

6.2.4 Attrition Model 110

6.3 Case Study 113

6.3.1 Fitting the Proposed Model 113

6.3.2 Fitting Cox Model 118

6.3.3 Validating the Models 119

6.3.4 Hazard Validation 119

6.3.5 Applying the model for IFRS 9 124

6.4 Summary of Findings 125

Chapter 7: The Concept of Economic Value for a Loan Portfolio 126

7.1 Overview of the Problem 126

7.2 A Basic Model of Embedded Value 129

7.2.1 A Review of EV for a Long-Term Insurer 129

7.2.2 A Basic Model for EV for a Lending Business 130

7.3 A More Detailed Model for Embedded Value 132

7.3.1 Refinement of Economic Value 132

7.3.2 Attribution of Economic Value 136

7.3.3 Evolution of EV 137

7.3.4 Limitations and Further Refinements of the EV Formulae 139

7.3.5 Comments on the Probability Distribution of EVk 140

7.4 Case Study 141

7.4.1 Simulation Case 141

7.4.2 Real-World Case 143

7.5 Summary of the Findings 148

Chapter 8: Conclusions 150

8.1 Review of the Research 150

8.2 Limitations of the Research 152

8.3 Areas of Further Research 153

Chapter 9: Appendices 154

9.1 Setting parameter values for the LHP approximation 154

9.2 Derivation of the portfolio value-at-risk 155

9.3 Derivation of the asset correlation coefficient 156

9.4 Regression for the exogenous component of the extended EMV model 156
9.5 Deriving the extended LHP approximation 157
9.6 Estimated parameters of the extended EMV model 158
9.7 Estimated parameters for the Survival Analysis EMV Model 160
9.8 Discounted mean term of a mortgage loan 162
9.9 Moments of the Vašíček Distribution 163
9.10 Simulations for the Identifiability Problem 163
9.11 Simulations for the Portfolio Default Rate 167
9.12 Sample Specifications 169
9.13 More on the components of the Survival Analysis EMV model 170
Chapter 10: References 172

Tables

Table 1: Default Rate and Asset Correlation Coefficients 47
Table 2: the AIC for Different Model Types Ranked by AIC 54
Table 3: Summary of Parameter Estimates for EMB Model 57
Table 4: Macroeconomic Variables for Exogenous Component 57
Table 5: Parameter Estimates for Exogenous Component 58
Table 6: Summary of Model Parameters 59
Table 7: Model Gini Statistic 60
Table 8: Mean and variance of R under the compound binomial and the Vašíček distributions 73
Table 9: Portfolio Default Rate Variance for Different Portfolio Sizes 77
Table 10: Macroeconomic Variables 115
Table 11: Fit Statistics for Macroeconomic Models 116
Table 12: Illustration of Simulation Rates and Parameters 142
Table 13: Illustration of Simulated Loan Values 142
Table 14: Economic value of a loan throughout its lifetime 143
Table 15: Pillar III disclosures for the largest Four South African banks 146
Table 16: Pillar III risk disclosures for the largest four South African banks 147
Table 17: Parameter inputs into estimating the economic value of the mortgage cohort 147
Table 18: The factors underlying our calculation of economic value 147
Table 19: Monthly calculations of the economic value of the cohort 148
Table 20: Detailed Parameter Estimates for the Extended EMV Model 160
Table 21: Detailed Parameter Estimated for the Survival Analysis EMV Model 162

Figures

Figure 1: IlluStrative Portfolio Loss Distribution 13
Figure 2: Asset Correlation Coefficient 47
Figure 3: Simulated EMV Effects and Default Rates 50
Figure 4: Estimated EMV Effects Under Standard Model & Simulated Distribution of PD under Extended Model 52
Figure 5: Estimated vs Simulated Effects under Standard and Extended EMV Models for the Same Data 52

Figure 6: Behavioural and Application Compnents Under EMBA Model 55
Figure 7: Estimated Components for EMB Model 56
Figure 8: Exogenous Component Model Fitness 58
Figure 9: Model Accuracy Across Range 60
Figure 10: Model Accuray over Calendar Time 61
Figure 11: Portfolio PD 95% Confidence Interval 62
Figure 12: Underestimation of Variance of Portfolio Default Rate 78
Figure 13: Ability of LHP and Extended LHP Models to Estimate the CDF in Small Portfolios 79
Figure 14:Ability of LHP and Extended LHP Models to Estimate the CDF in Large Portfolios 80
Figure 15: Hypothetical Lorenz curve of income distribution 83
Figure 16: Lorenz curve for observed defaults using observed defaults as the input 85
Figure 17: Lorenz curve for expected defaults using expected defaults as the input 85
Figure 18: Lorenz curve for economic scenarios 94
Figure 19: Simulated Gaussian systemic risk for a credit portfolio 97
Figure 20: Simulated portfolio default rates under different scenarios 97
Figure 21: Simulated quantiles and insolvency probabilities 97
Figure 22: The Gini statistic measuring the risk posed by through-the-cycle capital requirements 98
Figure 23: Illustration of Data Setup for Hazard Model 106
Figure 24: Illustration of Proposed Model 110
Figure 25: Baseline Hazard Functions 114
Figure 26: Behavioural Risk Index 114
Figure 27: Macroeconomic Risk Index 115
Figure 28: Macroeconomic Risk Index Model 116
Figure 29: Attrition Model Components 117
Figure 30: Cox Regression Baseline Hazard Function (Compared to IFRS 9 Model Baseline Hazard) 118
Figure 31: Cox Regression Parameter Estimates 119
Figure 32: Hazard Accuracy by Horizon (Holdout Sample) 120
Figure 33: Hazard Accuracy by Event Month (Holdout Sample) 121
Figure 34: Hazard Accuracy by Behavioural Risk Group (Holdout Sample) 121
Figure 35: Attrition Hazard by Remaining Term and Event Month (Holdout Sample) 122
Figure 36: Validating the cumulative PD by horizon 122
Figure 37: Validating 12-month PD over calendar month 123
Figure 38: Validating the PD across range 123
Figure 39: Assessing the discriminatory power of the PD model 124
Figure 40: Repo Rate, Prime Overdraft Rate and the interest yield for South African Mortgages 144
Figure 41: Estimating the market's equity risk premium and the South African banking sector's beta 145
Figure 42: Python Script for Simulating Loan Defaults 167
Figure 43: Python Script for Portfolio Default Rate Simulation 169

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For my mother.

Chapter 1: Introduction

Credit plays a central role in the modern financial system. Credit issued by banks constitutes a large portion of the credit in existence today. This has given rise to the discipline of credit risk analysis within the banking sector, which is the main focus of this thesis. We begin by providing an overview of credit risk analysis and management, before we set out the research aims and questions addressed in this thesis.

1.1 Credit Risk Analysis

Credit risk analysis involves the analysis of the creditworthiness of a borrower, either on existing credit arrangements or on potential new loans.

1.1.1 An Overview

At loan origination, credit risk analysis involves estimating the potential economic value the loan will create for the lender. This is generally done by estimating certain aspects of the expected lifetime credit loss on the loan – typically the probability of default. For ease of interpretation, the probability of default is often transformed into a credit score, e.g., the Fair, Isaac and Company (FICO) score in the USA and Experian Credit Score in South Africa[1].

Once a loan is disbursed, credit risk analysis involves monitoring the credit risk that underlies the individual loan and the portfolio of loans. The analyses carried out in this regard can be separated into impairment analysis and capital analysis. Impairment analysis involves estimating the expected credit loss to be incurred on a given loan once it has been disbursed. In banks, this analysis is mainly required for the preparation of published and management financial accounts. For published accounts, the principles underlying the calculation of expected credit loss are generally prescribed by accounting standards. These standards dictate specifics such as the definition of loss used and the horizon over which the potential loss is calculated. The IFRS 9 standard is the latest of these accounting standards (IFRS 9, 2014).

Capital analysis is concerned with estimating the amount of capital a lender ought to hold in order to withstand credit losses on a portfolio of loans, particularly losses in excess of those accounted for in impairment analysis. In order to allow for benefits of diversification and inter-loan correlations, such analysis tend to look at all loans within the portfolio, not just a single loan. The main output from capital analysis is a capital requirement amount. This amount can either be regulatory capital, in the case where the calculation is based on the regulator's prescribed (or approved) method for calculating the capital requirement, or economic capital, in the case where the calculation is the lender's internal best estimate

[1] For a more detailed discussion on the use of probability of default and credit scores, see Hand (2005).

of the capital requirement. Over the past two decades, regulators globally have generally subscribed to versions of the Basel accords. At present, Basel III forms the basis for many of the banking regulations applied globally (Basel III, 2010). In regulatory regimes that are based on Basel III (as well as its predecessor), the capital analysis often also involves other supplementary analyses such as stress-testing. Stress-testing aims to assess the resilience of the portfolio and adequacy of capital under macroeconomic stress scenarios[2].

Credit risk analysis is also carried out in cases where loans go into arrears and default. In this regard, the analysis involves assessing the potential for losses (as well as the recoverability these losses) that might result from a given delinquent account. Analysis concerned with measuring recoverability of losses is generally referred to as collections analysis[3].

Another area of credit risk analysis involves the estimation of the economic value of a loan portfolio. Historically, such analyses were generally conducted under the ambit of credit scoring any pricing. For example, risk-based pricing is concerned with estimating the appropriate interest rate to charge on a loan in order to optimise the economic value that is shared between the lender and the borrower through the loan[4]. Thus, in this domain, the focus was generally on understanding loan-level profitability. However, the estimation of economic value for a loan portfolio is also of interest in financial economics. Here, financial analysts have generally been interested in understanding how loan portfolios within a lending business or a bank contribute to the fair value of the entire business.

1.1.2 Unification in Credit Risk Analysis

In this thesis, we will be mainly concerned with modelling the loss distribution of loan portfolios and how this can be used for impairment modelling, capital analysis and measuring economic value. Generally, when a lender grants loans, only a portion of the total principal granted (allowing for interest) will be repaid to the lender. The amount that is not repaid is what we refer to as the loss on a loan portfolio. This loss amount is a random variable and, in one form or another, is the main item of interest in credit risk analysis.

In impairment analysis, the aim is to estimate the mean of the loss distribution. Capital analysis, on the other hand, is primarily concerned with estimating different points on the tail of this distribution. This is illustrated in Figure 1. The ultimate outcome from capital analysis is an estimate of how much capital to hold above the expected loss amount, which is already accounted for in impairment analysis – this

[2] A general discussion of stress-testing is provided by Foglia (2009).

[3] For a discussion of collections analysis and modelling, see Mays (2001).

[4] Edelberg (2006) provides an example of how lenders have used risk-based pricing in consumer loans.

amount is often referred to as *unexpected loss*[5]. The plot also illustrates how the base distribution used for impairment analysis or capital analysis might be altered when stress-testing analysis is carried out. The aim here is to understand how the distribution changes in response to the macroeconomic environment – particularly in cases of macroeconomic stress.

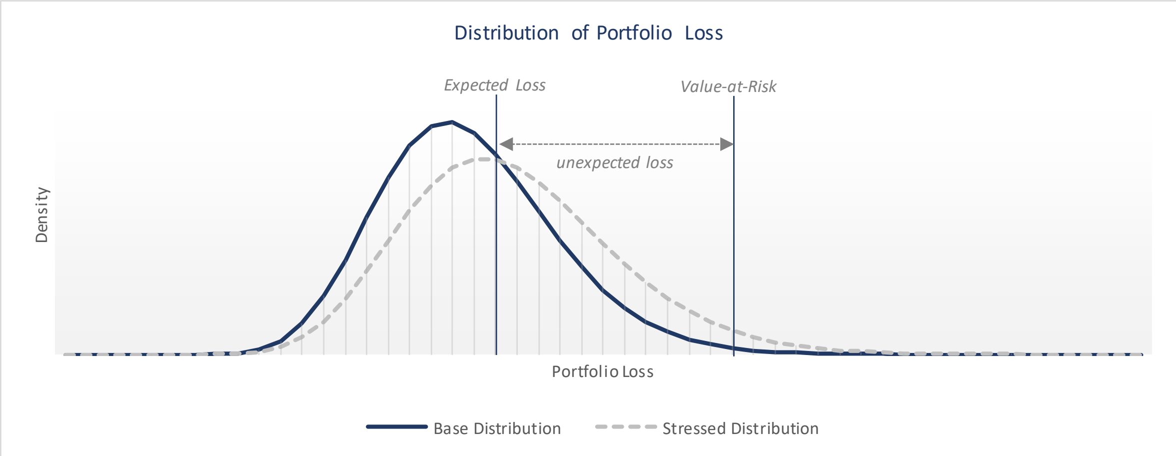

FIGURE 1: ILLUSTRATIVE PORTFOLIO LOSS DISTRIBUTION

The notion of *unification* in credit risk analysis refers to the aim to have one single framework through which the loss distribution can be derived, such that it is usable across all three applications. Moreover, such a framework should harmonise with the framework that is used at loan origination. After all, credit scoring is also concerned with understanding the loss distribution – the only distinction is that this loss distribution is analysed prior to the loan being disbursed and thus, generally, with less information. Therefore, we can say that unification is also interested in how the loss distribution evolves as a loan matures and as more information is gained.

In addition, considering the fact that providers of capital are ultimately concerned with earning a decent return on their capital, an additional aspect of unification should consider how the loss distribution affects the economic value of a portfolio.

1.1.3 Focus of the Thesis

The thesis investigates the exogenous maturity vintage (EMV) model as a candidate framework for achieving unification in credit risk analysis[6]. The first focus is on addressing some of the theoretical challenges of the standard EMV model. Once these have been addressed, we describe how the model can be used to incorporate information available at scoring stage and how this is information is supplemented with behavioural information as the account ages. We then discuss how, through the large

[5] Note that there are some differences between how expected loss is calculated for accounting purposes (under IFRS 9 impairment analysis) and how it is calculated for capital analysis (under Basel II). More details on these differences are given in Chapter 2.
[6] See Breeden (2007) for a description of the EMV model. We also provide more detail of the model in Chapter 2.

homogenous portfolio (LHP) assumptions[7], the EMV model can be used to derive a portfolio loss distribution. Following this, we demonstrate how the loss distribution can be used to determine the level of economic capital to hold against a loan portfolio.

The second focus is on extending the LHP assumptions that are used to derive the loss distribution (which are the same assumptions used under Basel III). In particular, we derive formulae that can be used to partition the risk inherent in the loss distribution into risk that is diversifiable (i.e., risk that reduces as the portfolio increases in size) and risk that is systemic (i.e., risk that does not reduce as the portfolio increases in size). These formulae are used to improve the LHP assumptions, leading to a loss distribution that better-captures the level of risk in portfolios of all sizes.

The third focus is on understanding how different specifications of the EMV model can be compared. This is done by deriving formulae that can be used to quantify the discriminatory power of a credit risk model. The chosen measure is the Gini statistic, which is in common use in the credit risk domain[8]. We illustrate how this measure can also be used to gauge the extent to which the loss distribution can change in response to changes in the macroeconomic environment.

The fourth focus is on understanding how the EMV model can be extended to accommodate the credit modelling requirements under IFRS 9. This leads to an extension of the EMV model into the survival time domain. In this regard, we show that the model addresses the challenge of incorporating time-varying covariates into a survival analysis model. The fifth and final focus is on understanding how the survival analysis version of the EMV model can be used to estimate the economic value of a loan portfolio.

1.1.4 Empirical Context

The thesis makes use of case studies to demonstrate and apply the ideas and techniques developed. The case studies will be based on both simulated and actual data. The actual data used relates to portfolios of credit issued in the South African credit market.

South Africa has a well-developed banking system, dominated by five large banks. These large banks account for 80% to 100% of most major banking and lending activities. Being systematically important, these banks are regulated by the Prudential Authority of the South African Reserve Bank, which subscribes to the Basel regulatory regimes. Nevertheless, non-bank lenders also play a meaningful role in lending, particularly for unsecured non-transactional lending space. However, as these play a less

[7] The LHP assumptions are used to derive the Basel III model for credit risk capital. See Malwandla (2016) for a discussion.

[8] See Mair, Reise and Bentler (2008) for applications of the Gini statistic in credit risk.

central role in the financial system, they are generally not scrutinised to the same extent by the Regulator from a capital adequacy perspective.

South African banks and non-bank lenders generally comply with international financial reporting standards. Therefore, for credit risk reporting purposes, South African lenders prepare their accounts in accordance with the IFRS 9 standard. However, it is mainly the large banks that operate with sophisticated models for calculated impairment provisions under IFRS 9.

Our focus on the South African market, which has a number of peculiarities (e.g., simultaneously being the most developed financial system on the African continent and one of the most unequal economies globally), may limit the generality of the case studies presented.

1.2 Research Question

The thesis addresses six research problem. We set these out below, before specifying the specific research questions.

1.2.1 Research Problems

The problems addressed in the thesis are as follows.

1. **Unification in credit risk analysis.**

   In the various areas of credit risk analysis (particularly, origination modelling, impairment analysis and capital analysis), probability of default plays a central role. However, there is a general lack of unification in the models used to measure probability of default across these areas. Generally, models developed for impairment analysis will differ from models developed for capital analysis. Moreover, within capital modelling, there is generally no link between the model that is used to calculate expected loss and the model that is used to calculate the quantiles of the loss distribution.

   The thesis will explore the exogenous maturity vintage modelling model as a potential framework for unifying credit risk analysis. However, the standard model has weaknesses that must be addressed before exploring the model further.

   a. **Weaknesses of the exogenous maturity vintage model.**

   The exogenous maturity vintage model is an adaptation of the age period cohort model from demographics into a credit risk context[9]. The model decomposes the experience on a credit portfolio (or on a population, from a demographic perspective) into a three time-related components: (a) an age (maturity) component; (b) a period (exogenous) component; and (c) a

[9] See Holford (1991) for a description of the age period cohort model.

cohort (vintage) component. These models suffer from the *identifiability problem*, which is discussed by Reither, Land, Jeon, Powers, Masters, Zheng, Hardy, Keyes, Fu, Hanson, Smith, Utz & Yang (2015). This problem arises because the three components of the model are structurally inter-related, which creates analytical problems when estimating the parameters of the model. While there have been several attempts to resolve this challenge, there has generally been no consensus on whether the solutions proposed are valid.

2. **Weaknesses of the Basel-Vašíček model framework.**
   The Vašíček model used within the Basel III framework (as well as its predecessor), henceforth the *Basel-Vašíček framework*, has a number of weaknesses, which were detailed by Malwandla (2016). Specifically relevant for this thesis are the following shortcomings: (a) the framework was derived for corporate credit, under the Merton (1974) framework, with no line of sight into the consumer credit environment; (b) some of the prescribed parameters are not specifically calibrated (or, at least, tested) for the specifics of a loan portfolio under consideration; (c) the derivation of the framework is asymptotic (i.e., assumes that the portfolio is infinitely large), which might compromise its applicability to smaller portfolios.
3. **Proportional hazard survival analysis with time-varying covariates.**
   Under some interpretation of IFRS 9, the ideal model used to calculate impairment provisions should take both account-specific data (which is generally static data at the point of calculation) and macroeconomic data (which is time-varying time series data at the point of calculation) into account and should be able to estimate probability of default over a 12-month period and over the lifetime of the account (unless separate models are to be used for the different time horizons). Logistic regression, which has been the benchmark for modelling probability of default in consumer credit, has the shortcoming that it generally only accommodates a single time horizon. Therefore, given that account lifetime will vary from one account to the next and from one period to the next, it is not ideally suited for IFRS 9 (at least in its standard form). Survival analysis is better suited for modelling over varying horizons. However, proportional hazards survival analysis models, which are the most common form of survival analysis, are generally incapable of accommodating for time-varying (macroeconomic) data. There are survival models that allow for time-varying covariates, but these are either generally intractable or inflexible.
4. **Measuring economic value in credit risk analysis.**
   As discussed in the introduction, within credit risk, economic value for loan portfolios is generally only carried out within the context of pricing and profitability analysis. Meanwhile, financial analysts are generally interested in estimating the economic value of an entire loan portfolio. Judging by the widespread use of embedded value in long-term insurance, the banking industry might benefit from developing an analogous concept: economic value.
5. **Analytical properties of the Gini statistic**

The Gini statistic is a popular concept within credit risk analysis. It is an adaptation of the Gini coefficient, in economics (Gini, 1921). It generally is used to measure the discriminatory power a model for the probability of default. However, unlike the Gini coefficient, the analytical properties of the Gini statistic in relation to the properties of a credit portfolio are not as well-studied.

1.2.2 Research Questions

These research problems are distilled into the following research questions, which have been grouped by the research problem they relate to.

Unification in credit risk analysis

1. Is there a model that can unify probability of default across the area of impairment analysis and capital analysis? *[Chapter 3]*
2. Are there ways of overcoming the identifiability problem in the exogenous maturity model? *[Chapter 3]*

Weaknesses of the Basel-Vašíček model framework

3. Are there ways of testing the Basel-Vašíček model parameters against the portfolio under consideration? *[Chapter 3]*
4. Are there ways of generalising the Basel-Vašíček model for applications to portfolios of finite size? *[Chapter 4]*

Analytical properties of the Gini statistic

5. Can we derive analytical properties for the Gini statistic through the parameters of the exogenous maturity vintage model? *[Chapter 5]*

Proportional hazard survival analysis with time-varying covariates

6. Can we extend the exogenous maturity vintage model to a survival analysis domain, allowing for the incorporation of time-varying variables? *[Chapter 6]*

Measuring economic value in credit risk analysis

7. Can an analogous concept to embedded value in long-term insurance be derived for a credit portfolio? *[Chapter 7]*

1.3 Structure of the Thesis

The rest of the thesis is structured as follows. Chapter 2 provides a brief literature review across the various areas explored in the thesis and highlights the contributions to literature being made by the thesis. Chapter 3 presents extensions to the EMV model, with an application to real-world data. Chapter 4 uses the EMV framework to introduce a concept of diversifiable and undiversifiable risk in a credit risk context, leading a generalisation of the Basel-Vašíček framework for finite portfolio sizes. Chapter 5 uses the EMV framework to derive a few theoretical properties for the Gini statistic of a credit portfolio. Chapter 6 presents an application of the decomposition approach used in the EMV model to survival analysis, leading to an extension of the Cox proportional hazard model. Chapter 7 presents a design for the concept of economic value (which is the analogue of embedded value in long-term insurance) for a portfolio of amortising loans. Chapter 8 provides a conclusion to the research, followed by appendices in Chapter 9 and references in Chapter 10.

Chapter 2: Literature Review

This chapter provides a brief literature review across the areas covered in the thesis, before setting out the main contributions to literature made by the thesis. Given the mathematical nature of the problems addressed by the thesis, we begin by setting up the mathematical statistical basis for credit risk analysis.

2.1 A Mathematical Statistics Basis for Credit Risk Analysis

Credit risk analysis generally employs techniques from mathematical statistics (e.g., generalised linear modelling and survival analysis), operations research (e.g., decision trees) and, recently, machine learning (e.g., neural networks and random forests). This thesis will mainly make use of techniques from mathematical statistics. Therefore, the thesis will generally adopt a mathematical and statistical characterisation of the credit risk profile of loans and loan portfolios.

Let X_k be a random variable equalling the loss on loan k in a portfolio of n loans. The loss can be decomposed into three parts, as follows:

$$X_k = D_k \times EAD_k \times LGD_k, \tag{2.1}$$

where D_k is the default outcome:

$$D_k = \begin{cases} 0 & if\ the\ loan\ never\ defaults \\ 1 & if\ the\ loan\ defaults \end{cases}, \tag{2.2}$$

EAD_k is the exposure at default, and LGD_k is the loss-rate given default (or just *loss given default*) i.e., how much of the exposure at default is ultimately written off. Each of the three components are random variables in their own right. The default outcome D_k is a Bernoulli random variable, with a parameter equal to pd_k, which is the probability of default on the loan. The exposure at default EAD_k is largely influenced by the time at which default occurs, especially for amortising loans. The loss X_k will generally follow a mixed distribution, with a probability mass equal to $1 - pd_k$ for zero loss and a probability density for positive loss.

In some applications, it is assumed that EAD_k and LGD_k are independent (in some cases, constant[10]), which allows for a simple formula for expected loss:

$$E[X_k] = PD_k \times E[EAD_k] \times E[LGD_k]. \tag{2.3}$$

[10] Of note, this is the case under the Basel-Vašíček framework.

The lender only originates a loan under the statistical expectation to make an economic profit from the loan. Let E_k be the economic profit expected from the loan. For simplicity, we write E_k as a function of two components:

$$E_k = G_k - X_k, \tag{2.4}$$

where G_k is the gross profit on the loan (i.e., before accounting for potential credit loss). The gross profit G_k is a random variable and will be a function of factors such as loan-related expenses (and the associated inflation rate), interest charged to the borrower and the lender's cost of funding the loan.

Let L be a random variable representing the potential loss at a portfolio level, defined as follows:

$$L = \sum_{k=1}^{n} X_k. \tag{2.5}$$

For a large portfolio of homogenous loans, assuming that loans are independent, the loss L can be approximated by the normal distribution, via the Central Limit Theorem (Hilhorst, 2009). In many applications, the assumption of homogeneity is held, but not the assumption of independence. Therefore, the normal distribution is generally not considered a good approximation for the loss distribution. Under the Basel III regime, the loss distribution is approximated by the Vašíček distribution, which assumes that individual losses are correlated through a single systemic risk factor. In particular, the derivation of the Basel-Vašíček framework assumes that the only source of randomness in the loss distribution comes from the probability of default (Vašíček, 1987) [11], i.e., it assumes that all loans default with the same exposure and ultimately result in the same loss given default.

Analytically, the portfolio loss can be represented in three components as well:

$$L = n \times \left[\sum_{k=1}^{n} \frac{D_k}{n}\right] \times \left[\frac{\sum_{k=1}^{n} D_k \times EAD_k}{\sum_{k=1}^{n} D_k}\right] \times \left[\frac{\sum_{k=1}^{n} D_k \times EAD_k \times LGD_k}{\sum_{k=1}^{n} D_k \times EAD_k}\right]$$

$$= n \times PD \times EAD \times LGD, \tag{2.6}$$

where $PD = \left[\sum_{k=1}^{n} \frac{D_k}{n}\right]$ is the portfolio default rate, $EAD = \left[\frac{\sum_{k=1}^{n} D_k \times EAD_k}{\sum_{k=1}^{n} D_k}\right]$ is the default-weighted EAD and $LGD = \left[\frac{\sum_{k=1}^{n} D_k \times EAD_k \times LGD_k}{\sum_{k=1}^{n} D_k \times EAD_k}\right]$ is the exposure-weighted LGD[12].

[11] However, we note that there is a difference between the theoretical derivation of the Basel-Vašíček model and how it is implemented. For instance, the models are sometimes applied at segment-level, despite the fact that the theoretical derivation is at a portfolio-level. This application allows different estimates for EAD and LGD to be applied for different accounts within the same portfolio.

[12] Note that the EAD and LGD components will only be defined in instances where there were defaults in the portfolio.

By assuming that EAD and LGD are constants, the loss distribution may be derived by looking at the distributional properties of PD only.

Aside from the losses, we can also estimate the economic profits at a portfolio level:

$$E = \sum_{k=1}^{n} G_k - \sum_{k=1}^{n} X_k$$

$$= G - L, \quad (2.7)$$

where $G = \sum_{k=1}^{n} G_k$ is gross profit on the portfolio.

2.2 Prudential Credit Risk Management

The thesis focuses on three aspects of credit risk analysis: impairment analysis, capital analysis and economic value analysis (however, we also touch on other areas such credit scoring at origination and stress-testing). We provide a closer mathematical statistical overview of these areas below.

2.2.1 Impairment Modelling

Impairment analysis is concerned with estimating the expected loss on the outstanding loans of a credit provider. In analytical terms, let A_t be the contractual value of credit contract at time t:

$$A_t = \sum_{k=1}^{T-t} p_k (1 + i_k)^{-(k-t)}, \quad (2.8)$$

where p_k is the contractual payment at time t, i_k is the effective interest rate applying between time t and time k and $T - t$ is the remaining contractual term. This contractual value assumes that all payments occur as stipulated in the contract, i.e., it ignores credit risk[13]. A more accurate valuation of the loan would be the carrying value, which allows credit risk:

$$B_t = A_t - E[X_t], \quad (2.9)$$

where $E[X_t]$ is the expected credit loss on the loan. It is the carrying value that ultimately sits on the balance sheet of the lender, not the contractual value. It is for this reason that the calculation of expected credit loss $E[X_t]$, for impairment purposes, is prescribed by the prevailing accounting standards. When applied in an impairment analysis context, the final expected credit loss number is generally referred to as an impairment provision.

Prior to 2018, IAS 39 (2004) was the accounting standard that stipulated the principles that were to be adhered to in calculating $E[X_t]$. The most important of these principles was that the expected loss

[13] The contractual value also discounts cashflows at the contractual interest rate, so that the value will equal the principal outstanding on the loan.

number should only allow for impairment events that had already occurred. An impairment event here is defined to mean any event, financial or otherwise, that occurs during the tenure of the loan that significantly *increases* the likelihood of losses being incurred. These events include, as examples, the loss of employment, divorce or death on the part of the borrower. As such, the expected loss number would often be segmented into two portions: (a) expected loss on identified impaired accounts; (b) expected loss on unidentified impaired accounts. The former would be calculated on accounts where the lender is reasonably certain that an impairment event has occurred, e.g., accounts that are in arrears. The latter would be calculated on accounts that are *performing* (up to date with payments), with the assumption that a proportion of these accounts had already undergone an impairment event which the lender is not yet aware of. For this reason, the latter are often referred to as IBNR provisions (incurred but not report) or general provisions.

Since 2018, expected credit losses are calculated under the prescription of IFRS 9. Under this standard, the expected credit loss must equal:

- Expected losses on defaults occurring over the next 12 months, for accounts that have not experienced a *significant increase in credit risk* (SICR); and
- Expected losses on defaults that may occur over the entire life of the loan, for accounts that have experienced a SICR.

The expectation is that IFRS 9 will lead to impairment provisions that are more reactive to changes in the credit profile of the portfolio, thereby increasing the credibility of the balance sheet numbers reported by credit providers. The standard was partly motivated by the perceived shortcoming of IAS 39 during the Great Recession, where it is believed that impairment provisions did not promptly react to the changing nature of credit risk in the months and years leading up to the recession.

In the interest of producing more reactive provisions, IFRS 9 has stricter requirements on data to be used in calculating expected loss, as well as the mechanism to be used. For instance, IFRS 9 requires that provisions should be estimated using forward-looking data (e.g., credit bureau data and macroeconomic data) and that the final provision number should be based on weighting of the provisions resulting under different macroeconomic scenarios. For more details on IFRS 9, please see PwC (2017).

2.2.2 Capital Requirements

Capital analysis is primarily concerned with estimating a particular quantile of the loss distribution. This is can be contrasted with impairment analysis, which is concerned with estimating the mean of the loss distribution. In analytical terms, if F is the distribution function[14] for portfolio loss, capital analysis seeks to estimate portfolio loss quantiles:

$$VaR_{\alpha} = F^{-1}(\alpha), \qquad (2.10)$$

for a given α, where F^{-1} is the inverse of F[15]. As discussed above, the calculation of regulatory capital in many jurisdictions is prescribed by the Basel accords. We also highlighted the fact that, if L is the random variable representing portfolio loss such that (as per Equation 2.6):

$$L = n \times PD \times EAD \times LGD, \qquad (2.11)$$

then Basel III is based on a model that assumes that EAD and LGD are constants. Analytically, this means that the quantile of portfolio loss is a multiple of the quantile for the portfolio default rate:

$$Q[L] = n \times Q[PD] \times EAD \times LGD, \qquad (2.12)$$

where Q is the quantile operator. Under this set of assumptions, the distribution of portfolio default rate is one of the key factors influencing portfolio default rate. Given that the portfolio default rate is an average over individual account default indicators (as shown in Equation 2.6), which are Bernoulli random variables, one might be tempted to assume that portfolio default rates follow a scaled binomial distribution, which in turn can be approximated by the normal distribution via the Central Limit Theorem. However, as discussed above, this is not the case, since (it is assumed) defaults across loans in the portfolio are not independent.

Under Basel III, portfolio default rates are assumed to follow the Vašíček distribution. This assumption is itself based on several assumptions, including: (a) the portfolio is infinitely large; and (b) the portfolio consists of homogenous accounts. For a more extensive discussion of the model assumed in Basel III, including a list of possible shortcomings, see Malwandla (2016).

2.2.3 Financial Valuation

Discussions of economic value are generally disjoint from discussions of credit risk analysis. This might be because the information available to financial analysts, who are most interested in estimating economic value, is far thinner than information available to credit risk analysts, who are best placed to estimate economic value. Financial analysts generally only work with information that is publicly available when estimating the economic value of a portfolio of loans or a lending business. Therefore,

[14] By distribution function we mean cumulative density function.
[15] Note that the quantile is generally referred to as value-at-risk in finance literature.

financial analysts in the banking sector mainly rely on accounting ratios such as *price-to-book-value*, *net interest margin* and *return on equity* as a way of ascribing a valuation to banks and credit providers.

By contrast, in the long-term insurance industry, financial analysts also make use what is referred to as embedded value. Embedded value is the estimate of the economic value of a book of insurance policies. This valuation is generally prepared by actuaries within an insurance company, to be used as a guide for financial analysts in appraising the company. It also provides valuable management information and assists shareholders in setting appropriate remuneration policies for the company's management team. It is particularly useful in insurance sector valuations due to the complex and long-term nature of insurance contracts, i.e., embedded value is a way of bridging the information asymmetry that might exist between management and investors in such a complex sector.

Therefore, it might be argued that the concept of embedded value (or, more generally, economic value) might be equally useful in the banking sector since contracts in this space are also long-term and sometimes fairly complex. It may also guide management teams in setting appropriate lending policies. However, this concept is yet to become popular in the banking sector. For a more comprehensive introduction to embedded value in long-term insurance, see Sarafeim (2011).

2.3 Main Challenges in the Field

Having set up a mathematical statistical basis for understanding problems credit risk analysis, we provide an overview of the literature on the problems addressed in this thesis.

2.3.1 The Challenges of the EMV approach

The exogenous maturity vintage (EMV) model derives from the age period cohort (APC) model, used in the field of demographics. Let $h(t,s)$ be the mortality rate during period t of lives belonging to cohort s, i.e, lives aged $t-s$ during period t. The standard APC model decomposes the experience according to the following equation:

$$h(t,s) = A_{t-s} + P_t + C_s, \tag{2.13}$$

where A_{t-s} represents the effect of age on mortality, P_t represents the effect of time period on mortality and C_s represents the effect of birth cohort on mortality. The modelling process involves estimating the values of A_{t-s}, P_t and C_s for every t and s represented in the sample. See Holford (1991) for some example of its application.

In consumer credit risk, early descriptions of the EMV model were offered by Breeden (2002) and Breeden (2007). In this model, Age relates to the maturity of the account, Cohort relates to the vintage of the account and Period relates to exogenous (period-related) effects. The EMV model has many advantageous features for modelling impairments under IFRS 9. Specifically, through the exogenous effect, it allows for an elegant way for incorporating macroeconomic variables into a logistic regression framework, which is a shortcoming of standard logistic regression model.

However, the standard EMV model has three major shortcomings that are relevant for this thesis: (a) the framework only caters for a single time horizon; (b) the framework does not readily account for behavioural variables, except through segmentation; and (c) the framework is subject to an *identifiability* problem.

The first shortcoming connects to the fact that IFRS 9 requires that models be able to estimate probability of default over both a 12-month horizon and the lifetime of the loan. This generally requires that the model should be able to accommodate different event horizons simultaneously. We discuss this shortcoming further below.

The second shortcoming also relates to IFRS 9, which requires that the probability of default should account for forward-looking behavioural data. The standard EMV model only operates along the three time dimensions described above. Therefore, there is no simple way of accounting for behavioural factors (such as the borrower's credit bureau profile or credit score), except through segmenting the portfolio according to risk group and building a separate model for each group.

The third challenge relates to relates to the fact that the three time effects of the age period cohort model (and the EMV model) are structurally dependent, according to the formula $Age = Period - Cohort$. This introduces an identifiability problem: there is no way to vary a single effect while holding the other effects constant, which is the basis for multivariate regression analysis and estimation (Glenn, 1976). Consequently, the relationship between mortality rate (or credit risk) and the individual effects might be misconstrued by the model.

Several attempts to address the identifiability problem have been made, including Fu (2008) and Yang (2006). However, there is still no consensus on whether these attempts have been successful, or whether the problem is even solvable (e.g., Reither, Land, Jeon, Powers, Masters, Zheng, Hardy, Keyes, Fu, Hanson, Smith, Utz & Yang (2015)). For the specific case of credit risk modelling, we discuss an approach for circumventing this problem.

2.3.2 IFRS 9 and Cox Regression

As discussed above, the modelling challenge presented by IFRS 9 can be interpreted as follows: we require a model for probability of default that incorporates both static and time-varying covariates, where the default event is observed over different event horizons.

The challenge is that standard logistic regression cannot handle time-varying covariates and variable time horizons (Crook, 2010). While EMV is somewhat able to allow for macroeconomic variable, it is not readily suited for varying time horizons. This challenge extends to other techniques based on modelling binary outcome, including decision trees and random forests.

The task of modelling outcome over a variable event horizon is generally fulfilled through survival analysis. Here, there are many techniques, including accelerated lifetime models (Bagdonavicius & Nikulin, 2001), frailty models (Wienke, 2010) and proportional hazard models (Breslow, 1975). However, these techniques still generally struggle to accommodate time-varying variables, i.e., variables that may change between the point of observation and the point of outcome.

For instance, consider Cox regression, which is one of the more widely applied forms of survival analysis in credit risk (Cox, 1972). Cox regression is a form proportional hazards survival model, operating in continuous time. The model assumes that likelihood of experiencing the event of interest at a given point is characterised by a single function for all members of the population, called the baseline hazard function. Let $b(t)$ be the baseline hazard function at time t. The hazard for each member within the population is proportional to the baseline hazard function at all points in survival time. Specifically, the hazard for member j in the population is given by:

$$h_j(t) = b(t)e^{r_j}, \tag{2.14}$$

where r_j is the risk factor for member j in the population. The risk factor is estimated through regression analysis:

$$r_j = \sum_{k=1}^{p} \beta_k X_{k,j}, \tag{2.15}$$

where β_k are regression parameters and $X_{k,j}$ are the factors relating to member j. Crucially, notice that r_j is not a function of time t. That means that the risk factors included in r_j cannot vary with time. For instance, in the context of credit risk, the factors may include variables such as the type of account, the credit score (as observed at the point of scoring), but not time-varying factors such as household debt as a ratio of gross domestic product or the level of interest rates. Strictly speaking, these can be included into r_j, but cannot be allowed to vary with time. For instance, if we were to apply the model to an account today, we would only be able to input today's interest rates not the interest rates expected in a year's time. This is equivalent to treating time-varying covariates as static covariates.

There have been attempts to overcome this challenge in Cox regression, i.e., a framework for Cox regression with time-varying covariates (e.g., Austin (2011) and Zhang, Reinikainen, Adeleke, Pieterse, Groothuis-Oudshoorn (2018)). However, the models are generally intractable and do not translate readily to where the time-varying covariates change frequently, as macroeconomic variables tend to.

The problem of survival analysis with time-varying variable is somewhat universal (i.e., not specific to proportional hazard models): the inclusion of time-varying covariates generally increases the mathematical complexity of the model. Consider threshold regression, which was shown to have some idyllic properties when modelling probability of default with static macroeconomic variables in capital analysis by Malwandla (2016). As soon as the macroeconomic variables are allowed to vary with time, the regression model ceases to have its closed-form mathematical properties (Lee, Whitmore & Rosner, 2010). Some accelerated lifetime models, which are a popular alternative to proportional hazard models, are able to accommodate time-varying covariates (e.g., Hernán, Cole, Margolick, Cohen, Robins (2005)), but lack the flexibility of semi-parametric proportional models.

Therefore, while this thesis is looking at the issue of survival analysis with time-varying variables within a credit risk analysis context, we note that the challenges addressed, and solutions proposed are more universal within the domain of survival analysis. However, we do note that alternative ways of conducting survival analysis with time-varying covariates do exist. Aside from the examples above, we note, for example, the novel use of survival trees, as discussed by Bou-Hamad, Larocque & Ben-Ameur (2011). Nevertheless, our aim was to discuss the potential for survival analysis with time-varying covariates under the EMV framework – in the interest of *unification*.

2.3.3 Weaknesses of the Basel-Vašíček Capital Framework

Malwandla (2016) identified a number of shortcomings to the Basel-Vašíček capital framework. These were:

1. The derivation of the framework assumes that the portfolio is infinitely large;
2. The derivation of the framework assumes that the risks within the portfolio are homogenous;
3. The derivation of the framework assumes that the exposures within the portfolio is homogenous;
4. The derivation of the framework assumes that the loss rates given default is non-random and homogenous;
5. The derivation of the framework assumes that the exogenous (systemic) risk is normally distribution;
6. The application of the framework assumes that the loss rates given default are not correlated to the probability of default; and
7. The application of the framework assumes that the credit risk cycle is not subject to structural discontinuities.

In addition to these, Malwandla (2016) also noted that, as a consequence of the above, the parameters prescribed in most applications of the Basel-Vašíček capital for the purposes of regulatory capital might lead to inadequate capital requirements.

This thesis will address the first shortcoming. This shortcoming relates to the estimation of the portfolio default rate. Let P be the default rate in a portfolio of homogenous loans:

$$P = \sum_{j=1}^{n} \frac{D_j}{n} \ , \tag{2.16}$$

where D_j is a Bernoulli random variable, with parameter p, representing the default rate on account j within the portfolio of size n. Assuming that p is a constant, P follows a scaled binomial distribution:

$$Prob[P \leq x] = \sum_{k=0}^{\lfloor nx \rfloor} \binom{n}{k} p^k (1-p)^{n-k}. \tag{2.17}$$

However, for credit risk analysis, p is generally[16] assumed to be a function of a random exogenous factor ε:

$$p = p(\varepsilon). \tag{2.18}$$

Therefore, the distribution function for P becomes more complex:

$$Prob[P \leq x] = \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\lfloor nx \rfloor} \binom{n}{k} p(e)^k \big(1 - p(e)\big)^{n-k} \right) g(e) de, \tag{2.19}$$

where g is the distribution function of ε. This formula has no closed-form solution for a general g, or where g is the Gaussian density function, as assumed in most applications. The Basel-Vašíček capital framework overcomes this challenge by assuming that the portfolio is infinitely large, so that $P|\varepsilon$ is constant, by the Law of Large Numbers. Under this assumption, if ε is normally distributed then P follows a Vašíček distribution, leading to the Basel-Vašíček capital framework. See Campolongo, Jönsson & Schoutens (2012) for a discussion on the derivation.

The challenge of overcoming this assumption has not generally been addressed in literature. The most widespread approach for handling this assumption is through the granularity adjustment, proposed by Gordy (2004) and later refined by Gordy & Lütkebohmert (2016). This approach relies on deriving a Taylor approximation to the value-at-risk for a given portfolio. Gordy & Lütkebohmert (2016) note that this approach is itself an asymptotic approximation, which means that it may be less applicable for smaller portfolios. Another noteworthy attempt was offered by Pimbley (2011), who applied an approximation to the factorial functions in Equation 2.19 in order to simplify the integral.

[16] For example, in the Merton (1974) model.

The thesis approaches this problem from a different perspective: by separating portfolio credit risk into a systemic and a diversifiable component. The thesis also re-addresses and refines some of the solutions derived by Malwandla (2016), for the other shortcomings listed above, by using the EMV model.

2.3.4 The Economic Value of a Loan Portfolio

The valuation of a credit operation is used by financial analysts in appraising a company, especially in the banking sector. It is also used by shareholders to set metrics for remunerating the company's management team. The most universal valuation approach is the dividend discount model:

$$P = \sum_{k=1}^{\infty} D_k (1 + r_k)^{-k}, \tag{2.20}$$

where P is the valuation, D_k is the dividend paid during period k and r_k is the shareholders' required return up to period k. In practice, a simpler version of the model is applied by assuming a long-run dividend growth rate of g and a flat required return of r over all periods k, leading to the following model, commonly referred to as the Gordon growth model (Gordon, 1959):

$$P = D_0 \times \left(\frac{1+g}{r-g}\right), \tag{2.21}$$

with $r > g$. In valuing a credit operation, the challenge is that the existing portfolio of business might be long-term in nature. Due to the complex and uncertain nature of credit operations, it is not always easy to estimate the dividends expected each year, or the dividend growth rate. For example, two portfolios of the same face value (principal amount) might have vastly different dividend profiles depending on, amongst other things: (a) the sensitivity of interest income to the central bank lending rate; (b) the funding method used by the provider; (c) the level of credit risk relative to the interest rate; and (d) the tenure mix of loans in the portfolio. These factors affect the size of the dividend, the growth rate (or run-off rate) of the dividends and the volatility of the dividend profile. Financial analysts thus tend use various ratios to help capture the nuances that exist within a loan portfolio (e.g., see Allayannis & Ramraika (2009)).

This challenge is not unique to banking – it also exists in the long-term insurance industry, where two books of policies with equal face value (asset-liability position) can have vastly different dividend profiles and, thus, economic valuations. In insurance, the financial analyst valuation challenge is ameliorated by the use of embedded value (Castellani, de Felice, Moriconi, Pacati, 2005). Embedded value is an insurer's internal estimate of the economic value to shareholders of a book of insurance contracts. It is used to aid analyst in valuing the company, provide management information, help appraise the management team and define the remuneration policy. As with insurance, the concept of embedded value is not expected to replace the use of traditional valuation approaches and ratios, but simply to augment it.

Therefore, the thesis will explore how the concept of embedded value can be translated to the valuation of a portfolio of loans.

2.3.5 Empirical Estimates of the Asset Correlation Coefficient

This thesis mainly seeks to offer theoretical approaches for dealing with various problems in consumer credit risk. However, in some instances – particularly in relation to the adequacy of capital requirements – the research will offer comparable empirical results.

As mentioned, the Basel-Vašíček capital frameworks has a number of deficits – some of which relate to the value of the asset correlation coefficient. In principle, the asset correlation coefficient should be determined by the nature of the credit portfolio concerned, the macroeconomic environment as well as the size of the portfolio (Malwandla, 2016). However, the Basel-Vašíček framework prescribes the value of the asset correlation coefficient to be used in determining capital requirements. As such, there is plenty of literature dealing with the question of whether the prescribed asset correlation coefficients are reflective of actual experience. A large portion of this literature deals with corporate credit risk (e.g., Düllmann, Scheicher & Schmieder (2007), Lopez (2004) and Chernih, Henrard & Vanduffel (2010)). We are mainly concerned with consumer credit – for which there is less empirical work.

Stoffberg & van Vuuren (2015) apply a distribution-based technique to estimate the asset correlation coefficient for various classes of credit risk. Initially proposed by Botha & van Vuuren (2010), the technique involves using observed default or credit loss data to estimate the asset correlation coefficient. The technique estimates the asset correlation coefficient based on two sample statistics: the mean and mode of the observed defaults or losses[17]. Using this technique, Stoffberg & van Vuuren (2015) estimated an asset correlation of around 7% for credit cards. Similarly, Mwamba & Mhlophe (2019) used this technique to obtain an asset correlation coefficient of between 5.8% and 7%, depending on the nature of the loss data used, i.e., the technique allows the asset correlation coefficient to be estimated on either default data or on loss data, which results in different estimates.

An important difficulty with distribution-based estimates of the asset correlation coefficient is described by Wunderer (2019): when a portfolio is composed of non-homogenous exposures and the nature of the inhomogeneity varies over time, each observed portfolio default rate is effectively being sampled from a different distribution. For corporate exposures, he offers that this is one of the reasons why asset correlations derived from default data tend to be lower than those derived from asset data (e.g., Ammari & Lakhnati (2017) and Curcio, Gianfrancesco & Malinconico (2011)).

[17] Precisely, the asset correlation coefficient ρ is found by solving the equation $\sqrt{\frac{1-\rho}{1-2\rho}} = \frac{\Phi^{-1}(m)}{\Phi^{-1}(p)}$, where p and m are the mean and mode of the observed losses.

Belloti & Crook (2012) follow a different approach for estimating the value of the asset correlation coefficient. They consider borrower-level data and macroeconomic data to model the probability of default at an individual-level, while assuming that the observable borrower-level and macroeconomic data leaves a portion of the variation in probability of default over unexplained, i.e., their model includes a time-dependent latent variable. The fact that the model accounts for heterogeneity in risk, through borrower-level data, addresses the problem posed by Wunderer (2019). This model is applied to credit card data to obtain an estimated asset correlation coefficient of 0.6% – considerably lower than the 4% prescribed under the Basel-Vašíček framework. They compare their work to the work of Rösch & Scheule (2004), who used a similar approach to find an asset correlation coefficient of around 1% for credit cards and 0.73% for other consumer loans. However, as noted by Rösch & Scheule (2004), the inclusion of macroeconomic variables in the model has the effect of reducing the asset correlation coefficient considerable.

Effectively, the inclusion of macroeconomic variables converts the asset correlation coefficient from a through-the-cycle measure to a point-in-time measure, as discussed by Malwandla (2016). The fact that the Basel-Vašíček framework is a through-the-cycle regime might justify why the asset correlation coefficient estimated by Belloti & Crook (2012) and Rösch & Scheule (2004) are considerably lower than the prescribed 4%. Moreover, the distribution-based technique used by Stoffberg & van Vuuren (2015) and Mwamba & Mhlophe (2019) results in a through-the-cycle measure, which is why the estimated values are larger than in those found by Belloti & Crook (2012).

There are a number of important observations to be made from this brief empirical review. Firstly, more work exists on the empirical estimation of asset correlation coefficient for corporate exposures than for consumer credit. Secondly, there is considerable inconsistency the asset correlation coefficients found by different studies, with some studies producing point-in-time measures while other produce through-the-cycle measures. Therefore, care should be taken in comparing the estimates produced by different studies. Thirdly, some techniques are better than others in controlling for heterogeneity in credit exposure, so that not all estimates are equally credible. Finally, there is generally no attempt to address the impact of portfolio size on the variability in observed experience, which means that estimates from different studies might also differ due to the sample size.

2.4 Contributions to the Field

Following the literature review, we summarise the contributions to literature being made by the thesis.

1. **The thesis provides an extension to the exogenous maturity vintage to include behavioural covariates, which also circumvents the identifiability problem.**

The identifiability problem is one of the main shortcomings of the exogenous maturity vintage model. The extension presented in this thesis solves this shortcoming by replacing the vintage dimension with a behavioural risk dimension.

2. **The thesis shows how the exogenous maturity vintage model can be used to derive parameters that are analogous to those used in the Basel-Vašíček model, thereby providing a means to test the applicability of the regulatory capital model to a given portfolio.**

   The Basel-Vašíček framework prescribes the value of the asset correlation coefficient to be used in determining capital requirements. There is a growing literature questioning whether these prescribed values are correct, with some finding the actual asset correlation coefficient is much lower than that which is prescribed by the framework. However, within the literature there is inconsistency in how the actual asset correlation is measures (i.e., some estimate a point-in-time measure while some estimate a through-the-cycle measure; and some control for heterogeneity while other do not), leading to inconsistent results across different studies. Moreover, none of the techniques in literature are able to control for the impact of sample size on the actual asset correlation coefficient. The extension to the exogenous maturity vintage model provided in this thesis allows us to simultaneously estimate both the point-in-time and through-the-cycle asset correlation coefficient, while controlling for sample size and for heterogeneity.

3. **The thesis defines the notion of systemic and diversifiable risk for a credit portfolio, leading to a generalisation of the Basel-Vašíček model to portfolios of finite size.**

   The Basel-Vašíček framework makes the assumption that the portfolio being modelled is of infinite size, which ignores sampling error. This effectively means that the risk being considered is systemic risk only, while diversifiable risk (which reduces to zero as the portfolio size approaches infinity) is ignored. This thesis provides a way for estimating both the systemic and diversifiable components of portfolio risk.

4. **The thesis derives the analytical properties of the Gini statistic based on the parameters of the exogenous maturity vintage model.**

   The Gini statistic is a popular measure of the discriminatory power of a model. There is limited literature dealing with the theoretical properties of the Gini statistic as they pertain to the properties of a model for the probability of default. This thesis adds to this literature.

5. **The thesis generalizes the exogenous maturity vintage model into the survival time domain, leading to a generalisation of the Cox regression model to allow for time-varying covariates in discrete-time.**

   The standard exogenous maturity vintage model is applicable in situations where the outcome variable is measured over a fixed time horizon. This thesis offers an extension to this model by allowing the outcome variable to be measured over variable time horizon. This converts the model into a survival model which can accommodate time-varying covariates.

6. **The thesis derives formulae that can be used to estimate the economic value of a portfolio of amortising loans.**

   This is a novel contribution to literature.

Chapter 3: The Extended Exogenous Maturity Vintage Model

3.1 Overview of the Problem

Consumer credit risk modelling is made up of several focus area, the main ones being impairment modelling, regulatory and economic capital, stress-testing, application scoring and behavioural scoring. The theme that runs across these elements is the probability of default. Even though probability of default is fundamentally the same concept across these applications, the models used for its estimation are sometimes disconnected within the same institution.

Where this disconnect is more apparent is in the regulatory and economic capital, wherein the portfolio loss aggregation models (commonly, the Basel-Vašíček framework) lack a micro-foundation. This owes to the fact that the Vašíček (1987) model is derived for corporate credit (as opposed to consumer credit), under the Merton (1974) framework. The disconnect is also seen in the differences in modelling standards between the different functions. For example, impairment modelling under IFRS 9 (IFRS 9, 2014) requires losses (on accounts that have undergone significant increase in credit risk) to be estimated over a lifetime horizon, while Basel III (Basel III, 2010) requires a 12-month horizon. Furthermore, the Basel III regulatory framework is centred around the idea of through-the-cycle risk assessment, while IFRS 9 is based on point-in-time and forward-looking risk assessment.

In this chapter we propose a framework for moving towards unification in consumer credit risk modelling, with focus on the probability of default. The proposed framework is based on the exogenous maturity vintage (EMV) model, which attempts to decompose risk into three components: loan vintage, loan maturity and an exogenous component (e.g., see Forster & Sudjianto, 2013). The EMV model suffers from a fundamental problem arising from the fact that the three components are interdependent, making them difficult to estimate (Glenn, 1976).

We start by offering a solution to this identifiability problem. This is done by replacing the vintage component with a behavioural risk grade. This solution also improved the discriminatory power of the EMV model and ensures that it uses all available information. We also consider the possibility of including a behavioural risk dimension and an application risk dimension, to the extent that these are not co-linear. We demonstrate how the extended EMV model can be used in loan origination, stress testing, impairment modelling under IFRS 9 and economic capital.

This chapter focuses closely on economic capital, where the issue of risk aggregation is dealt with in detail. We offer a single formula for the estimating asset correlation coefficient (as defined in the Baseil-Vašíček framework) of a portfolio on both a point-in-time basis and a through-the-cycle basis. The inputs into the formula are shown to be the parameter of the exogenous component of the EMV model as well as coefficient of determination of the macroeconomic model for the exogenous component. The formula thus encapsulates the idea that unexpected loss is a function of the total level of systemic uncertainty (which is represented by the volatility of the exogenous component) and how well the model reduces the level of uncertainty (which is represented by the coefficient of determination). The equation further captures the idea that the systemic uncertainty is a function of the economic uncertainty and how reactive the portfolio is to the economic environment. The equation could thus assist practitioners in assessing the adequacy of regulatory capital, since the Basel III regulatory regime does not directly address how the value of the asset correlation coefficient can be estimated for a given portfolio – the regime instead prescribes a value for the asset correlation coefficient.

The rest of chapter is organised as follows. We begin by discussing the EMV model and the identifiability problem. We then offer a few extensions to the standard EMV model to accommodate behavioural and application risk. We subsequently show how the model can be applied to IFRS 9 impairment modelling, stress testing and economic capital. For applications in economic capital, we use the extended EMV model to derive the Vašíček distribution. Finally, we apply the model on simulated data, to demonstrate the identifiability problem, and real data, to demonstrate the workings of the model.

3.2 The Exogenous Maturity Vintage Model

The standard EMV model (as described by Forster & Sudjianto (2013)) derives from the Age Period Cohort (APC) model, which is widely applied in demography, sociology and epidemiology. The approach decomposes a given observed factor into three independent effects: an age effect (or maturity effect), a cohort effect (or vintage effect) and a period effect (or exogenous effect).

3.2.1 The Age Period Cohort Model

Formally, consider the experience of subjects of a mortality investigation. Let $h(t,s)$ be the mortality rate during calendar period t of lives belonging to cohort s, i.e, these are subjects aged $t-s$ during period t. The standard APC model decomposes the experience according to the following equation:

$$h(t,s) = A_{t-s} + P_t + C_s, \qquad (3.1)$$

where A_{t-s} represents the effect of age on mortality, P_t represents the effect of the time period on mortality and C_s represents the effect of birth cohort on mortality. The modelling process involves estimating the values of A_{t-s}, P_t and C_s for every t and s represented in the model sample.

In this model, variations in the age effect represents how mortality is influenced by the age of the life. For example, this can be compared to the Gompertz (1825) law in human mortality. The period effect aims to capture how the prevailing conditions during a particular period have influence on the mortality rate. For example, this can be compared to the effect of war during a particular year on the mortality (e.g., Glei, Bruzzone & Caselli, 2005). The vintage effect represents any variation that may exist in mortality rate by birth cohort. For example, this can be compared to epigenetic effects on mortality of humans born during a famine (e.g., Lumey, Stein, Kahn, van der Pal-de Bruin, Blauw, Zybert & Susser, 2007).

One of the weaknesses of the model is that is assumes no interaction between the three effects, which may not always be the case. Extending the examples in the previous paragraph, the effects of war on mortality may vary by age group. Another weakness of the model is that it does not allow for other covariates, except through segmentation. For instance, the model does not account for how weight at birth, gender or intelligence at point of observation affect the mortality rate.

From an estimation perspective, the main weakness of the APC model is that the three effects are intrinsically dependent on each other, i.e., given the Age and Period, the Cohort is known. This introduces an identifiability problem: there is no way to vary a single effect while holding the other effects constant (Glenn, 1976). As consequence, it is easy for the relationship between the mortality rate and the individual effects to be misconstrued.

There have been considerable efforts to address the identifiability problem. Fu (2008) proposes a cohort smoothing regime to allow independent variation in the age and period effects. Yang (2006) uses Bayesian inference to develop a hierarchical APC model to bypass for the identifiability problem. There is still debate on the extent to which these approaches truly correct for the identifiability problem (e.g., Reither, Land, Jeon, Powers, Masters, Zheng, Hardy, Keyes, Fu, Hanson, Smith, Utz & Yang 2015).

3.2.2 The Exogenous Maturity Vintage Model

The APC model has gained adaptation in credit risk analysis, where it has been used to model the probability of default. In this area, the model is referred to as the EMV model, where the maturity effect substitutes for Age, the vintage effect substitutes for the Cohort and the exogenous effect substitutes for Period (Breeden, 2016).

Let $P(t,s)$ be the probability of default during calendar time t on a loan originated during calendar time s, i.e., the loan is of maturity $t-s$ during time t. The EMV model may specified as follows:

$$P(t,s) = G(M_{t-s}, E_t, V_s), \tag{3.2}$$

where M_{t-s} is the maturity effect, E_t is the exogenous effect, V_s is the vintage effect and G is the link function. The chosen link function should ideally ensure that the model predictions are always bounded in the interval $[0,1]$, since the target variable is a probability.

We will mainly focus on the additive probit link function:

$$P(t,s) = \Phi(\alpha + M_{t-s} + E_t + V_s), \tag{3.3}$$

where Φ is the distribution function of the standard normal distribution.

The interpretation of the components of this model is similar to the interpretation of the components of the APC model. The maturity component represents how the likelihood of default changes as an account matures. The vintage component represents how different cohorts of accounts differ in terms of default rate. This is likely to be influenced by differences in application scorecards and lending policies across different cohorts[18]. The exogenous component represents how default rates change as influenced by conditions prevailing during a given period. It is expected to primarily represent economic conditions but may also capture changes in lending policies and consumer sentiments[19].

The exogenous component can be considered as a time series in t. Therefore, we can decompose it into a deterministic and stochastic component, as follows:

$$E_t = \mu_E(t) + \sigma_E \ddot{E}_t, \tag{3.4}$$

where $\ddot{E}_t$ is normally distributed with a mean of zero and a standard deviation of one; and μ_E and σ_E are the trend and volatility parameters of the process. Owing to the notion that credit markets exhibit cyclicality (Geanakoplos, 2009), as opposed to following a long-term trend (apart from structural dislocations), we assume the deterministic component to be constant[20]:

$$\mu_E(t) = \mu_E. \tag{3.5}$$

In a similar fashion, we can standardise the maturity component across age $t-s$ and the vintage component across cohort s, as follows:

$$M_{t-s} = \mu_M + \sigma_M \ddot{M}_{t-s} \text{ and } V_s = \mu_V + \sigma_V \ddot{V}_s. \tag{3.6}$$

[18] For example, if the lending policy during a particular period is lenient (resulting in unusually high acceptance rates on new loan applications), we would expect the vintages originated during that period to exhibit atypically high default rates – leading to a high vintage effect for that period.
[19] For example, default rates are generally expected to be higher during tough economic periods, as the financial positions of borrowers come under strain – leading to a higher exogenous effect during such periods.
[20] It is important to note here that the assumption that credit risk is cyclical is somewhat equivalent to assuming that either $\mu_E(t) = \mu_E$ or, more generally, $\mu_E(t) = \mu_E(t-k)$, where k is the length of the cycle. We adopt the former assumption for brevity and because it conforms to the data examined in this thesis. This assumption implies that the exogenous component is a driftless process: $E[\Delta E_t] = 0$.

The model equation can thus be rewritten as follows:

$$P(t,s) = \Phi\big(\mu + \sigma_M \ddot{M}_{t-s} + \sigma_E \ddot{E}_t + \sigma_V \ddot{V}_s\big), \tag{3.7}$$

where $\mu = \alpha + \mu_M + \mu_E + \mu_V$. Under this formulation, the standard deviations serve as regression coefficients for the three components, and their relative sizes can be interpreted as the relative contribution of the components to the overall default rate[21]. The mean $\Phi(\mu)$ can be viewed as type of *baseline* default rate, which is adjusted upwards or downwards depending on the combined effect of the components. The standard deviation of the exogenous component has a special meaning in the context of this model: it represents the amount of systemic risk that exists within the portfolio, as a result of exogenous factors. This will be discussed further below.

3.3 Extending the Model

The standard EMV model only segments the population by age and cohort (since period is a function of these two). Credit risk modelling is generally, through regression, interested in segmenting the experience across more dimensions. This is often done through application and behavioural scoring models. We thus consider the extension of the EMV model to incorporate macroeconomic variables. This aspect of the chapter thus aims to achieve the same goals as Breeden (2016), which is to incorporate behavioural data for purposes such as IFRS 9 impairment modelling. However, the specific approach taken in this chapter leads to an idyllic formulation for the probability of default – one which allows for simple formulae for the portfolio loss distribution (discussed in this and the next chapter) and certain other measures (discussed in a latter chapter).

3.3.1 Extension: Addition of Behavioural Risk Grade

Let $B_{j,t}$ be the behavioural score for account j on the book at time t such that, *ceteris paribus*, the higher the value of $B_{j,t}$ the higher the probability of default. Suppose that $B_{j,t}$ is used to derive η risk grades for the population, from lowest risk to highest risk. These grades can be used to enhance the EMV model as follows:

$$P(t,s,k) = G(M_{t-s}, E_t, V_s, B_k), \tag{3.8}$$

where B_k represents the effect of the k^{th} behavioural risk grade on probability of default. The linear version of the model can be written as follows:

[21] Specifically, the standard deviation of a component represents the change in change in $\Phi^{-1}(P)$ per unit change in the component. For example, $\sigma_E = \frac{\partial}{\partial E}\Phi^{-1}(P)$.

$$P(t,s,k) = \Phi\big(\mu + \sigma_M \ddot{M}_{t-s} + \sigma_E \ddot{E}_t + \sigma_V \ddot{V}_s + \sigma_B \ddot{B}_k\big), \tag{3.9}$$

where $\mu = \alpha + \mu_M + \mu_E + \mu_V + \mu_B$ and μ_B is the mean of the behavioural component, such that $B_k = \mu_B + \sigma_B \ddot{B}_k$. In this formulation, the risk grade is presented as an additional component, independent of the other three. However, it should be noted that this is equivalent to the following formulation:

$$P(t,s,k) = \Phi\big(\mu_k + \sigma_M \ddot{M}_{t-s} + \sigma_E \ddot{E}_t + \sigma_V \ddot{V}_s\big), \tag{3.10}$$

which has three components but allowing the baseline to change as a function of the risk grade, i.e., $\mu_k = \mu + \sigma_B \ddot{B}_k$. It should also be noted that this approach to incorporating behavioural risk into the model is different from that which was offered by Breeden (2016), to which this approach should be seen as an alternative.

3.3.2 Extension: Addition of Application Risk Grade

Generally, banks manage risk using both behavioural scorecards and application scorecards. Application scorecards are meant to be an assessment of the default risk as estimated at the point of application. Let $A_{j,t}$ be the application score for account j originated at time t such that, *ceteris paribus*, the lower the value of $A_{j,t}$ the higher the probability of default as estimated at point of application[22].

As with the behavioural score, suppose that the application score is used to derive λ risk grades. Since application risk grades are an assessment of the risk inherent in a new loan, they may be a suitable substitute for the vintage effect. As mentioned above, the application risk profile during a particular period may have an influence on the vintage effect for loans originated during that period. The model may be restated as follows:

$$P(t,s,l,k) = \Phi\big(\mu + \sigma_M \ddot{M}_{t-s} + \sigma_E \ddot{E}_t + \sigma_D \ddot{N}_l + \sigma_B \ddot{B}_k\big), \tag{3.11}$$

where $\ddot{N}_l$ represents the effect of the l^{th} application risk grade on probability of default, $\mu = \alpha + \mu_M + \mu_E + \mu_N + \mu_B$, and μ_N is the mean of the application risk component, i.e., the application risk component is given by $N_l = \mu_N + \sigma_N \ddot{N}_l$, as with the other components.

The replacement of the vintage component with application risk grade removes the identifiability problem. It also removes the need to forecast the vintage effect before the model is applied on new cohorts where the vintage effect would not have yet been estimated[23] – the application risk grade can be derived for every member of the portfolio, provided the scorecard inputs are available.

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[22] An example of this would be the FICO (Fair Isaac Corporation) score in the USA and the Experian Credit Score in South Africa.

[23] This refers to the fact that the vintage effect for a new vintage of loans is only knowns once the default experience is seen. Therefore, if the EMV model is estimating probability of default over a 12-month horizon, the vintage effect on a new loan would only be known a minimum of 12-months after the loan is originated.

In essence, the behavioural risk grade and the application risk grade are an attempt at estimating the same effect. The difference between them is that the application risk grade is tailored to assess risk of default earlier in the life of a loan, while behavioural risk grade specialises as the loan matures, i.e., the behavioural risk grade becomes stronger (more discriminatory) as an account matures, while the application risk grade becomes weaker. The model link function can be used to specify this principle more deliberately. For example:

$$P(t,s,k) = \Phi\big(E_t + U_{t-s,l,k}\big), \tag{3.12}$$

where $U_{t,s,l,k} = N_l \alpha_{t-s} + B_k(1-\alpha_{t-s})$ and α_{t-s} is monotonic decreasing with $t-s$.

This formulation allows the model to be used to assess the risk of default from the point of origination until maturity. Crucially, the assessment allows us to incorporate macroeconomic conditions. The model may thus go some way in solving an old problem in credit scoring: scoring models are overly sensitive to the period in which they are developed (Crook, Hamilton and Thomas, 1992). Part of the reason for this is that scoring models generally do not incorporate macroeconomic information, which plays a significant role in explaining default rates over time.

3.3.3 Model Estimation

The model components can be estimated via maximum likelihood estimation. Let $D_{j,t}$ be the default indicator for loan j in the portfolio during time t. The log-likelihood function for the estimation of the four-dimension model (exogenous, maturity vintage and behavioural component, EMVB) is as follows:

$$l = \sum_t \sum_j D_{j,t} \times \ln\big[P\big(t, s_{j,t}, k_{j,t}\big)\big] + \big(1 - D_{j,t}\big) \times \ln\big[1 - P\big(t, s_{j,t}, k_{j,t}\big)\big], \tag{3.13}$$

where $s_{j,t}$ and $k_{j,t}$ are the vintage and behavioural risk grades of account j during time t, and $P\big(t, s_{j,t}, k_{j,t}\big)$ is given by:

$$P\big(t, s_{j,t}, k_{j,t}\big) = \Phi\left(\alpha + \sum_i \delta_{\{t-s_{j,t}=i\}} m_{t-s_{j,t}} + \sum_i \delta_{\{t=i\}} e_t + \sum_i \delta_{\{s_{j,t}=i\}} v_{s_{j,t}} + \sum_i \delta_{\{k_{j,t}=i\}} b_{k_{j,t}}\right), \tag{3.14}$$

where δ is the indicator function. The parameters being optimised by this function are $\boldsymbol{M} = \{m_1, m_2, \dots\}$, $\boldsymbol{E} = \{e_1, e_2, \dots\}$ and $\boldsymbol{V} = \{v_1, v_2, \dots\}$ and $\boldsymbol{B} = \{b_1, b_2, \dots\}$. Notice then that $\boldsymbol{B}$ is simply the behavioural component in vector form, $\boldsymbol{E}$ is the exogenous component in vector form, $\boldsymbol{V}$ the vintage component in vector form and $\boldsymbol{M}$ is the maturity component in vector form.

In other words, each of the model dimensions (t, s, $t - s$ and k in Equation 3.9) are treated as dummy variables. The parameters along the t dimension, contained in $\boldsymbol{E}$, represent the exogenous effect; the estimates along the s dimension, contained in $\boldsymbol{V}$, represent the vintage effect; the estimates along the $t - s$ dimension, contained in $\boldsymbol{M}$, represent the maturity effect; and the estimates along the k dimension, contained in $\boldsymbol{B}$, represent the behavioural effect.

It is important to note that for the EMVB[24] model and the EMV model the maximum likelihood estimation approach described by these formulae does not produce reliable estimates. Specifically, depending on the starting value of the estimation procedure, the optimisation can yield different results. This is a consequence of the identifiability problem described above. Therefore, the above likelihood function is only reliable when either the E, V or M dimensions are excluded from the model.

3.3.4 Predictive Modelling vs Explanatory Modelling

The distinction between *explanatory* modelling and *predictive* modelling is relevant when discussing the EMV model and the extensions described above. The standard EMV model is best-suited for *explanatory* modelling, i.e., it helps explain already-observed default experience by decomposing it into the different dimensions. It is not as readily suitable for *predictive* modelling, i.e., it is not as useful in predicting the default experience of a portfolio where defaults are yet to be observed. This is because the exogenous component and vintage component are generally unknown until the default experience has already been observed.

Practically, this means that the model cannot inform us about the default risk inherent in a given vintage until the default experience on that vintage has been observed. Similarly, the model cannot inform us about the default risk inherent in a given time period until the default experience for that time period has been observed. The reason for this is that the period and vintage for a portfolio whose default is yet to be observed will not have been part of the model development sample and will thus be outside the *sample space* allowed for in the model. This means that, until such a time when the default experience has been observed, the vintage and cohort effects will effectively be random variables.

Notice also that the same does not apply for the maturity effect, unless the portfolio concerned consists of loans whose maturity is outside of the range of maturities observed in the model development sample. The problem also does not apply for the behavioural and application risk effects. This is part of the rationale for replacing the vintage effect with the behavioural or application risk effect.

Therefore, for *predictive* applications using an extended EMV model (where the vintage effect is replaced by a risk effect), the only unknown will generally be the exogenous effect. In order to apply

[24] EMVB is an acronym for the EMV model with a behavioural dimension (B) added.

the model, an assumption would need to be made concerning the distribution of the exogenous effect ($\ddot{E}_t$ in Equation 3.11) so that an expectation can be taken over the model equation ($P(t,s,l,k)$ in Equation 3.11).

An alternative approach is to develop a separate model to predict the exogenous effect for a given period prior to observing the default experience. This is another extension to the standard model, which we discuss in more detail below.

3.3.5 Extension: Macroeconomic Model

In a model with behavioural risk grade, the application risk grade (or vintage) and maturity effect, the exogenous component should be largely influenced by macroeconomic factors, as discussed above. Therefore, a macroeconomic regression model for the exogenous component would be expected to produce a decent fit.

Fitting a regression model to the exogenous component might be useful for two reasons. Firstly, the exogenous effect for a given time period can only be known once the default experience for that time period has been observed, as discussed above. Specifically, assuming that the model estimates a 12-month probability of default, the exogenous effect for a portfolio observed at time t can only be known at time $t + 12$ months. Therefore, when applying the model to loans where the default experience is yet to be observed (which will generally be the majority of the applications), the exogenous component will be a random variable, with some standard deviation. Fitting a regression model to the exogenous component serves the purpose of reducing the variance of this random variable, which means that there will be more certainty in the model predictions. Secondly, fitting a regression model helps with forecasting and scenario analysis, which are important for stress testing and impairment modelling under IFRS 9.

Let $\boldsymbol{Y}_t = \{Y_1(t), Y_2(t), \dots Y_q(t)\}$ be a vector of q macroeconomic factors observed by period t. In order to fit the regression model, we first standardise the exogenous component (which is estimated by maximising Equation 3.13), as per Equation 3.4. We then define the macroeconomic model as follows:

$$\ddot{E}_t = \bar{E}_t + \sigma_\varepsilon \varepsilon_t, \tag{3.15}$$

where $\bar{E}_t = \sum_{j=1}^{q} \beta_j Y_j(t)$ and ε_t has a mean of zero and a standard deviation of one[25]. Therefore, the model may be rewritten as follows:

$$P(t,s,l,k,\varepsilon_t) = \Phi\big(\mu + \sigma_M \ddot{M}_{t-s} + \sigma_E \bar{E}_t + \sigma_V \ddot{V}_l + \sigma_B \ddot{B}_k + \sigma_E \sigma_\varepsilon \varepsilon_t\big). \tag{3.16}$$

[25] These conditions for ε_t are not particularly onerous since they arise as general properties of a linear regression model.

Notice that in the original model form (in Equation 3.9), $\sigma_M \ddot{M}_{t-s}$ and $\sigma_B \ddot{B}_k$ are deterministic, while $\sigma_E \ddot{E}_t$ would be a random variable with a mean of zero and a standard deviation of σ_E[26]. In this new model form $\sigma_M \ddot{M}_{t-s}$, $\sigma_E \bar{E}_t$ and $\ddot{B}_k$ are deterministic, while $\sigma_E \sigma_\varepsilon \varepsilon_t$ is a random variable with a mean of zero and a standard deviation of $\sigma_E \sigma_\varepsilon$. As will be shown shortly, $\sigma_E \sigma_\varepsilon$ is always less or equal to σ_E, which means that the regression model reduces the variance of the model[27].

The above model can be reformulated in the following way:

$$P(t, s, l, k, e_t) = \Phi\big(U_{t,s,l,k} + \sigma_e e_t\big), \tag{3.17}$$

where $U_{t,s,l,k}$ is the deterministic component of the default risk and e_t is the random component. The value of $U_{t,s,l,k}$, σ_e and e_t depend on whether a regression model for the exogenous component is fitted, as we will explain below.

For convenience, we assume that the random component e_t is normally distributed[28]. This assumption leads to the following expression for the *expected*[29] probability of default (Owen,1980) [30]:

$$\bar{P}_{t,s,l,k} = E[P(t, s, l, k, e_t)] = \Phi\left(\frac{U_{t,s,l,k}}{\sqrt{1+\sigma_e^2}}\right). \tag{3.18}$$

If no regression model is fitted, then $U_{t,s,l,k}$ becomes a through-the-cycle (TTC) measure of risk and e_t becomes a random variable representing the level of the cycle at period t:

$$U_{t,s,l,k} = \mu + \sigma_M \ddot{M}_{t-s} + \sigma_V \ddot{V}_l + \sigma_B \ddot{B}_k,\ \sigma_e = \sigma_E \text{ and } e_t = \ddot{E}_t. \tag{3.19}$$

The TTC expected probability of default is given by:

$$\bar{P}_{TTC}(t, s, l, k, e_t) = \Phi\left(\frac{\mu + \sigma_M \ddot{M}_{t-s} + \sigma_V \ddot{V}_l + \sigma_B \ddot{B}_k}{\sqrt{1+\sigma_E^2}}\right). \tag{3.20}$$

If a regression model is fitted, then $U_{t,s,l,k}$ becomes a point-in-time (PiT) measure of risk and e_t becomes a random error representing the systemic risk left unexplained by the macroeconomic model:

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[26] As discussed above, $\sigma_V \ddot{V}_l$ would also be a random variable. However, we do not discuss the properties of $\sigma_V \ddot{V}_l$ since we argue that it may be replaced by $\sigma_B \ddot{B}_k$.

[27] This follows from the finding that $\sigma_\varepsilon^2 \leq 1$, discussed further below.

[28] This is also not a particularly onerous assumption since it arises as a property of linear regression. It is also possible to relax this assumption, as discussed by Malwandla (2016).

[29] The word expected emphasises the fact that $p(t, s, l, k, e_t)$ is a random variable, since one of its inputs e_t is also a random variable.

[30] Owen (1980) showed that if Z is a standard normal random variable then $E[\Phi(\mu + \sigma Z)] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$, for constant μ and σ.

$$U_{t,s,l,k} = \mu + \sigma_M \ddot{M}_{t-s} + \sigma_V \ddot{V}_l + \sigma_B \ddot{B}_t + \sigma_E \bar{E}_t,\ \sigma_e = \sigma_E \sigma_\varepsilon \text{ and } e_t = \varepsilon_t. \quad (3.21)$$

Therefore, the PiT expected probability of default becomes (Owen, 1980):

$$\bar{P}_{PiT}(t,s,l,k,e_t) = \Phi\left(\frac{\mu + \sigma_M \ddot{M}_{t-s} + \sigma_V \ddot{V}_l + \sigma_B \ddot{B}_t + \sigma_E \bar{E}_t}{\sqrt{1 + \sigma_E^2 \sigma_\varepsilon^2}}\right). \quad (3.22)$$

Notice that it is the presence or absence of $\sigma_E \bar{E}_t$ that determines whether $U_{t,s,l,k}$ is PiT or TTC.

Four properties of σ_ε should be noted:

- Since $Var(\ddot{E}_t) = Var(\bar{E}_t) + \sigma_\varepsilon^2$ and $Var(\ddot{E}_t) = 1$, it is necessary that $\sigma_\varepsilon^2 \leq 1$.
- The theoretical coefficient of determination of $\bar{E}_t$ as a model for $\ddot{E}_t$ is given by $r^2 = 1 - \sigma_\varepsilon^2$.
- To the extent that E_t characterises the macroeconomic or credit risk cycle, σ_E, which is the standard deviation of E_t, can be regarded as a measure of total systemic risk.
- Since the coefficient of determination is interpreted as the proportion of variation explained by the model, σ_e represents the portion of the systemic risk σ_E that is left unexplained by the model for the exogenous component:

$$\sigma_e^2 = \sigma_E^2 (1 - r^2). \quad (3.23)$$

3.3.6 Extension: Quantile Function

In credit risk modelling, specifically when determining capital requirements, we are concerned about more than just the expected default rate. We are also concerned about the distribution of possible portfolio default rates. This is important when determining the value-at-risk[31] for portfolio loss. Given the account-level default rate $P(t,s,l,k,e_t)$, a few simplifying assumptions are often made to arrive at a distribution of default rates. These assumptions are collectively referred to as the Large Homogenous Portfolio (LHP) assumptions. The derivation of the distribution is discussed by Malwandla (2016), for the case of a logistic regression model with a complementary log-log link function. Here we alter the derivation for the specific case of the EMV model.

Consider a portfolio of n accounts observable at time t. For a given random error ε_t, the portfolio loss is given by:

$$l(\varepsilon_t) = \frac{1}{n}\sum_{j=1}^{n} D_{j,t} \times EAD_{j,t} \times LGD_{j,t}, \quad (3.24)$$

[31] In mathematical statistical terms, the value-at-risk is defined as a specified quantile of the loss distribution, over some specified loss horizon. In credit risk application, this is generally the 99th or 99.5th percentile, over a one-year period.

where $D_{j,t}$ is the default indicator[32], $EAD_{j,t}$ is the exposure at default and $LGD_{j,t}$ the loss given default. The first part of the LHP assumptions is that the portfolio is homogenous in risk. This implies that all accounts have the same exposure at default EAD and the same loss given default LGD, so that:

$$l(\varepsilon_t) = \left[\frac{1}{n}\sum_{j=1}^{n} D_{j,t}\right] \times EAD \times LGD$$

$$= R_t \times EAD \times LGD, \tag{3.25}$$

where R_t is the portfolio default rate for time t, i.e., the portfolio loss under the LHP assumptions becomes a multiple of the portfolio default rate.

The homogeneity portion of the LHP assumption also implies that all accounts have the same default risk profile during period t, i.e.:

$$U_{t,s,l,k} = U_t. \tag{3.26}$$

We propose the following as a simple estimate of U_t:

$$\widehat{U}_t = \sqrt{1+\sigma_e^2} \times \Phi^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} \Phi\left(\frac{U_{t,s_j,l_j,k_j}}{\sqrt{1+\sigma_e^2}}\right)\right). \tag{3.27}$$

This estimate for U_t ensures that the expected (unconditional) portfolio default rate under the LHP assumption is equal to the expected portfolio default rate without the LHP assumption (see Appendix 9.1).

The consequence of the homogeneity assumption is that the number of portfolio defaults becomes a conditional binomial distribution, since all accounts have the same default rate $\Phi(U_t + \sigma_e e_t)$, i.e., conditioned on e_t.

The second part of the LHP assumptions is that the portfolio is infinitely large, i.e.:

$$R_t \to \Phi(U_t + \sigma_e e_t) \text{ as } n \to \infty,$$

by the Law of Large Numbers.

Notice, however, that although R_t converges to $\Phi(U_t + \sigma_e e_t)$, it is still a random variable since e_t is a random variable. Therefore, we evaluate the distribution function of the portfolio default rate as follows:

[32] $D_{j,t}$ is a Bernoulli random variable with parameter $P\left(t, s_j, l_j, k_j, e_{t_j}\right)$.

$$D(x) = Prob[R_t \leq x]$$

$$= Prob[\Phi(U_t + \sigma_e e_t) \leq x]$$

$$= \Phi\left(\frac{\Phi^{-1}(x) - U_t}{\sigma_e}\right), \tag{3.28}$$

We can thus evaluate the quantile function for the portfolio default rate as follows (see Appendix 9.2):

$$k(\alpha) = \Phi[\sigma_e \Phi^{-1}(\alpha) + U_t]. \tag{3.29}$$

The portfolio value-at-risk can thus be determined as a multiple of this quantile, since both exposure at default and loss given default are assumed to be constants:

$$c(\alpha, t) = \Phi[\sigma_e \Phi^{-1}(\alpha) + U_t] \times EAD \times LGD. \tag{3.30}$$

Note here that the $c(\alpha, t)$ might generally understated, for a number of reasons, as discussed by Malwandla (2016). One of these reasons is the assumption that the portfolio is infinitely large, which is the subject of the next chapter.

3.3.7 Extension: Error Distribution and Capital Provisioning

The value-at-risk formula derived above has close resemblance to the Basel III capital requirements value-at-risk formula, which is based on the quantile function from the Vašíček distribution:

$$c(\alpha, t) = \Phi\left[\sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(\alpha) + \sqrt{\frac{1}{1-\rho}}\Phi^{-1}(\bar{R}_t)\right] \times EAD \times LGD, \tag{3.31}$$

where ρ is the *asset correlation coefficient* and $\bar{R}_t$ is the mean portfolio default rate.

The value-at-risk from Equation 3.30 is:

$$c(\alpha, t) = \Phi\left[\sigma_e \Phi^{-1}(\alpha) + \sqrt{1 + \sigma_e^2} \times \Phi^{-1}(\bar{R}_t)\right] \times EAD \times LGD, \tag{3.32}$$

since $\frac{U_t}{\sqrt{1+\sigma_e^2}} = \Phi^{-1}(\bar{R}_t)$, in same manner as Equation 3.18. The asset correlation coefficient can thus be written as (see Appendix 9.3 for detailed proof):

$$\rho = \frac{\sigma_e^2}{1+\sigma_e^2}. \tag{3.33}$$

Therefore, the model results in the following formula for portfolio value-at-risk:

$$c(\alpha, t) = \Phi\left[\sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(\alpha) + \sqrt{\frac{1}{1-\rho}}\Phi^{-1}(\bar{R}_t)\right] \times EAD \times LGD. \tag{3.34}$$

The value of $\bar{R}_t$ and ρ depend on whether we estimate capital on a TTC basis (with no regression for the exogenous component) or a PiT basis (with a regression for the exogenous component), as described in Table 1 and discussed above.

Basis	Default Rate $\bar{\boldsymbol{P}}_{\boldsymbol{t}}$	Asset Correlation Coefficient $\boldsymbol{\rho}$
Point-in-Time	$\Phi\left(\mu + \sigma_M \ddot{M}_{t-s} + \sigma_V \ddot{V}_l + \sigma_B \ddot{B}_t\right)$	$\frac{\sigma_E^2 \sigma_\varepsilon^2}{1 + \sigma_E^2 \sigma_\varepsilon^2}$
Through-the-Cycle	$\Phi\left(\mu + \sigma_M \ddot{M}_{t-s} + \sigma_V \ddot{V}_l + \sigma_B \ddot{B}_t + \sigma_E \bar{E}_t\right)$	$\frac{\sigma_E^2}{1 + \sigma_E^2}$

TABLE 1: DEFAULT RATE AND ASSET CORRELATION COEFFICIENTS

When calculating capital requirements for regulatory purposes, the TTC asset correlation coefficient can be substituted with the one prescribed in the regulatory framework. Therefore, the model can calculate economic and regulatory capital on either a PiT or a TTC basis.

Note that, more generally, the asset correlation coefficient can be rewritten as a function of the coefficient of determination and measure for total systemic risk:

$$\rho = \frac{\sigma_E^2(1 - r^2)}{1 + \sigma_E^2(1 - r^2)}. \tag{3.35}$$

In the TTC model, the coefficient of determination equals zero since no model is fitted to explain the exogenous component. The graph below illustrates the sensitivity of the asset correlation coefficient to the level of systemic risk and the coefficient of determination.

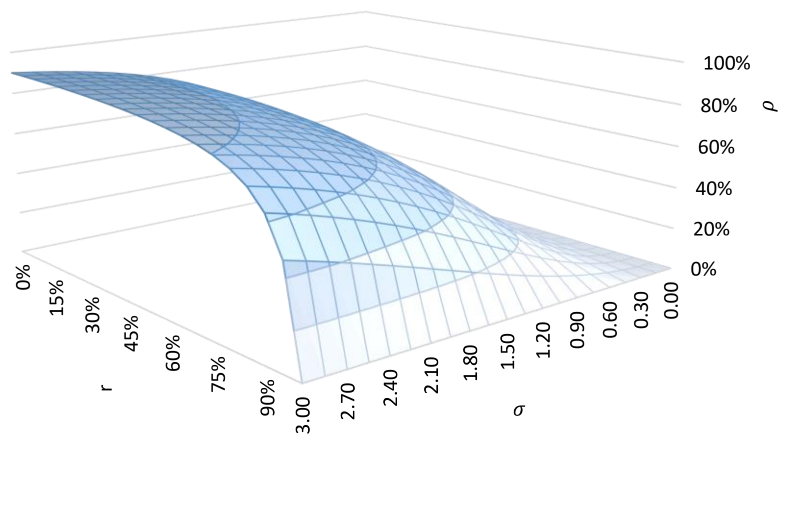

FIGURE 2: ASSET CORRELATION COEFFICIENT

3.3.8 Application in Impairment Modelling & Stress Testing

The discussion has so far shown that the EMV model can be applied to model default risk at an account-level and determine required capital for credit risk. We now briefly discuss how the model is applicable to determining impairment provisions under IFRS 9 and stress testing.

Under IFRS 9, impairments are held on either a 12-month expected loss basis or a lifetime expected loss basis:

$$EL = \begin{cases} PD_{12} \times EAD_{12} \times LGD & \text{if an account has not significantly deteriorated} \\ PD_L \times EAD_L \times LGD & \text{if an account has significantly deteriorated} \end{cases},$$

where L is the lifetime of the account, PD_{12} is the probability of default over a horizon of 12 months, PD_L is the probability of default over the life of the loan, and EAD_{12} and EAD_L are the EAD on loans defaulting within 12 months and within the lifetime of the loan, respectively. The estimation of loss given default LGD and exposure at default EAD are out of the scope of this chapter.

Under the EMV model discussed above, the 12-month probability of default as at calendar time t is given by:

$$p_{12}(t, U_{t,s,l,k}) = \Phi\left(\frac{U_{t,s,l,k}}{\sqrt{1+\sigma_e^2}}\right), \tag{3.36}$$

where $U_{t,s,l,k}$ is the risk profile of the account. In order to derive the lifetime probability of default, we require a separate attrition model $q_{12}(t, U_{t,s,l,k})$, which estimates the probability that the account closes within 12 months. This can be fitted as an EMV model or through a different modelling approach. The probability of default over any horizon $k \in [12,24,36, \ldots]$ can be estimates as follows[33]:

$$D_k(t, U_{t,s,l,k}) = \sum_{j=1}^{k} p_{12}(t+j, U_{t+j,s,l,k}) \times [1 - D_{k-12}(t, U_{t+j,s,l,k}) - A_{k-12}(t, U_{t+j,s,l,k})],$$

and:

$$A_k(t, U_{t,s,l,k}) = \sum_{j=1}^{k} q_{12}(t+j, U_{t+j,s,l,k}) \times [1 - D_{k-12}(t, U_{t+j,s,l,k}) - A_{k-12}(t, U_{t+j,s,l,k})], \tag{3.37}$$

with initial conditions $D_0(t,u) = A_0(t,u) = 0$.

[33] This can be seen as a form of discrete-time competing-risk survival analysis, which is discussed in more detail in Chapter 6.

The intuition behind this formula is as follows. $D_k(t,u)$ represents the probability that an account observed during calendar time t will have defaulted by calendar time $t+k$, while $A_k(t,u)$ represents the probability that the account will have closed by calendar time $t+k$. Therefore, $e_k(t,u) = 1 - D_k(t,u) - A_k(t,u)$, referred to as the *exposed-to-risk*, represents the probability that the account will still be exposed to the risk either defaulting or closing, having neither defaulted nor closed since between calendar time t and time $t+k$. Therefore, the exposed-to-risk $e_k(t,u)$ multiplied by the 12-month probability of default $p_{12}(t+k,u)$ gives the marginal probability of defaulting during between calendar time $t+k$ and calendar time $t+k+12$, i.e., starting with the proportion still exposed to risk at calendar time $t+k$ (given by $e_k(t,u)$), it measures how many will default during the subsequent 12 months. Similarly, multiplying the exposed-to-risk $e_k(t,u)$ by the 12-month probability of closing $q_{12}(t+k,u)$ gives the marginal probability of closing between calendar time $t+k$ and calendar time $t+k+12$. Notice now that the summands in Equation 3.37 are the marginal probability of default and the marginal probability of closing. The summations thus give the cumulative probability of defaulting and closing.

The probability of default over intermediate horizons can then be approximated via interpolation. Note that the formula above assumes that the risk profile of the account will only change due to the time components of the model (i.e., the maturity and exogenous components); the behavioural component k is assumed to be static over time. In reality, this component is likely to be mean reverting. This can be modelled by allowing k to converge towards the mean for each successive horizon, e.g.:

$$D_k\left(t, U_{t,s,l,k}\right) = \sum_{j=1}^{k} p_{12}\left(t+j, U_{t+j,s,l,\lfloor k_j \rfloor}\right) \times \left[1 - D_{k-12}\left(t, U_{t+j,s,l,k}\right) - A_{k-12}\left(t, U_{t+j,s,l,k}\right)\right], \tag{3.38}$$

where $k_{j+1} = k_j + \theta\left(k_j - \bar{k}\right)$, $\bar{k}$ is the central (e.g., long-run, average or median) behavioural risk group within the population, $k_0 = k$ and θ is a mean reversion parameter.

The model is also useful for stress testing, as it allows the user to test the impact of the macroeconomic environment (through the exogenous component) as well as credit strategy and policies (through the application and behavioural components). For instance, one can test the effect of switching to an aggressive origination strategy by assuming a lower cut-off for application scores in future periods. Since the behavioural and application scores can be catered for explicitly in the model, this would allow us to directly test the impact of origination strategy on capital requirements and impairments.

3.4 Case Studies

The extended EMV model described above was tested using two case studies. The first case aims to illustrate the extent of the identifiability problem, through a simulation. The second case is based on real-world data, where it is used to model the probability of default on a portfolio of personal loans for a South African bank.

3.4.1 Identifiability Problem

In order to demonstrate the identifiability problem, we simulated defaults under two default rate models. The first model is the standard EMV model. Here we assumed that default on a loan in calendar time t, with vintage v follows a Bernoulli distribution with the following parameter:

$$P_{t,v} = \Phi(\mu + \sigma_e E_t + \sigma_m M_{t-v} + \sigma_v V_v), \tag{3.37}$$

where E_t (the exogenous component) varies in t, M_{t-v} (the maturity component) varies in $t - v$ and V_t (the vintage component) varies in v. For the simulation, we set $\mu = -1$, $\sigma_e = 0.6$, $\sigma_m = 0.2$ and $\sigma_v = 0.2$. Further, E_t was simulated as a sine wave, M_{t-v} was simulated as a decreasing exponential and V_v was simulated as a Gaussian function. We show the simulated shape of each component of these in Figure 3. The simulation produced 100 thousand observations – a histogram of the resultant distribution of default probabilities is also given in Figure 3. See Appendix 9.10 for more details on the simulation process.

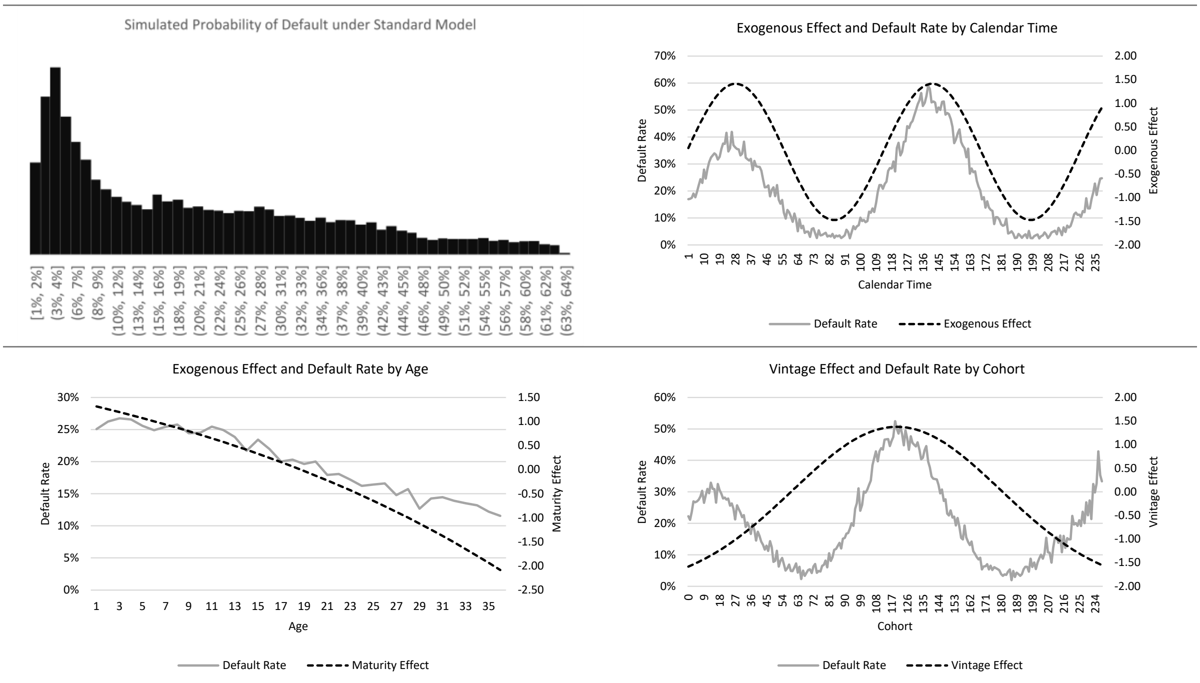

FIGURE 3: SIMULATED EMV EFFECTS AND DEFAULT RATES

The structural interdependence between the components serves to obfuscate the observed relationship between default rates and the three time dimensions. This is shown this in Figure 3, where we plot default rates by calendar time t, loan vintage v and loan maturity $t - v$.

In order to illustrate the identifiability problem, we used the EMV model to estimate the value of each of the three dimensions. We show these estimated in Figure 4. Alongside these, we plot the original simulated patterns for each effect, as shown in Figure 3. The extent of the difference between the simulated effect and the estimated effect highlights the inability of the EMV model to accurately identify the dimensions – in this case, this is most pronounced for the vintage component.

In order to demonstrate the ability of the extended EMV model to overcome this challenge, a different simulation was conducted. This time, we assumed that default on a loan in calendar time t, with maturity u and belonging to risk group r follows a Bernoulli distribution with the following parameter:

$$P_{t,a,b} = \Phi(\mu + \sigma_e E_t + \sigma_m M_a + \sigma_b B_r) \tag{3.37}$$

where E_t (the exogenous component) varies in t, M_a (the maturity component) varies in a and B_r (the vintage component) varies in r. As discussed in Section 3.2.2, the risk profile during a particular period may have an influence on the vintage effect for loans originated during that period. We thus assumed that the distribution of risk group is related to the vintage component. Therefore, the simulation was such that $r = \max(\min(\lfloor \ddot{r} \rfloor, 10), 0)$, where $\ddot{r} \sim N(\ddot{V}_v, \delta)$, for some δ, and $\ddot{V}_v$ is a non-standardised version of the vintage component simulated above. This is merely to ensure that there is a relationship between the vintage effect and the risk group, i.e., it ensures that variations in risk across vintages can ultimately be expressed in terms of variation in the risk profiles of the loans making up the vintages, as described in Section 3.2.2 (see Appendix 9.10 for more details on the simulation process).

We show a histogram of the distribution of simulated default probabilities in Figure 4.

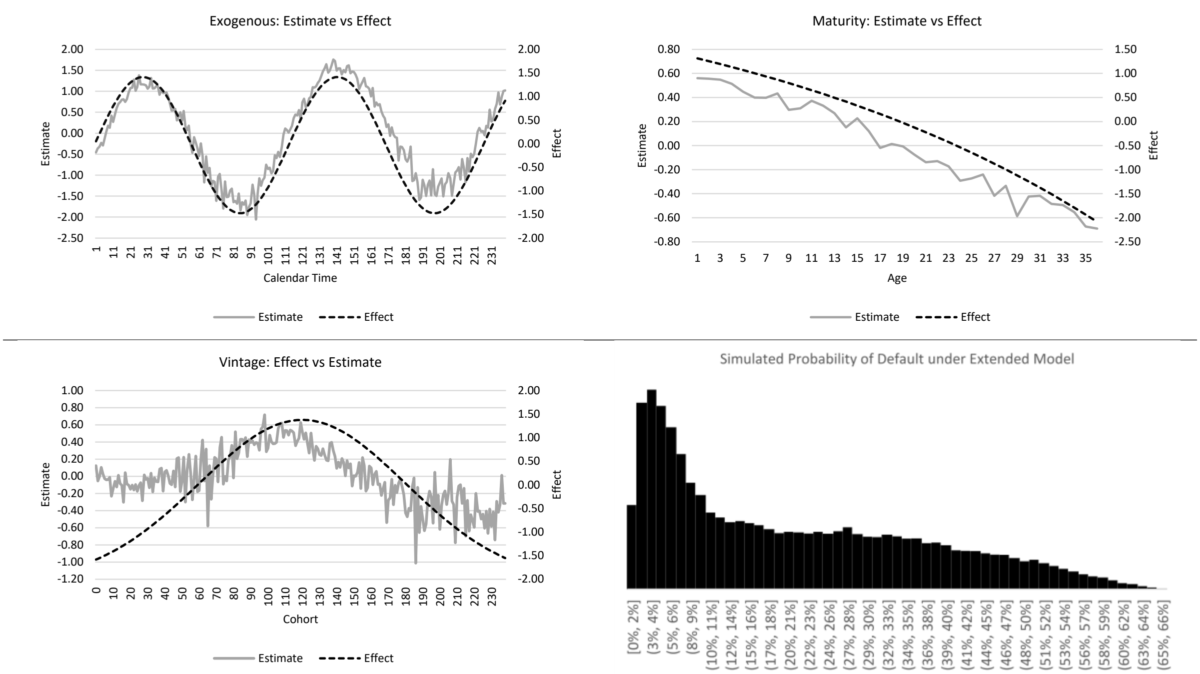

FIGURE 4: ESTIMATED EMV EFFECTS UNDER STANDARD MODEL & SIMULATED DISTRIBUTION OF PD UNDER EXTENDED MODEL

We used both the standard EMV model and the extended EMV model to estimate the value of each of the three dimensions. We show these estimated in Figure 5. Alongside these, we plot the simulated patterns. This shows that the dimensions of the extended EMV model (which uses behavioural risk in place of vintage) are more estimable than those of the standard EMV model.

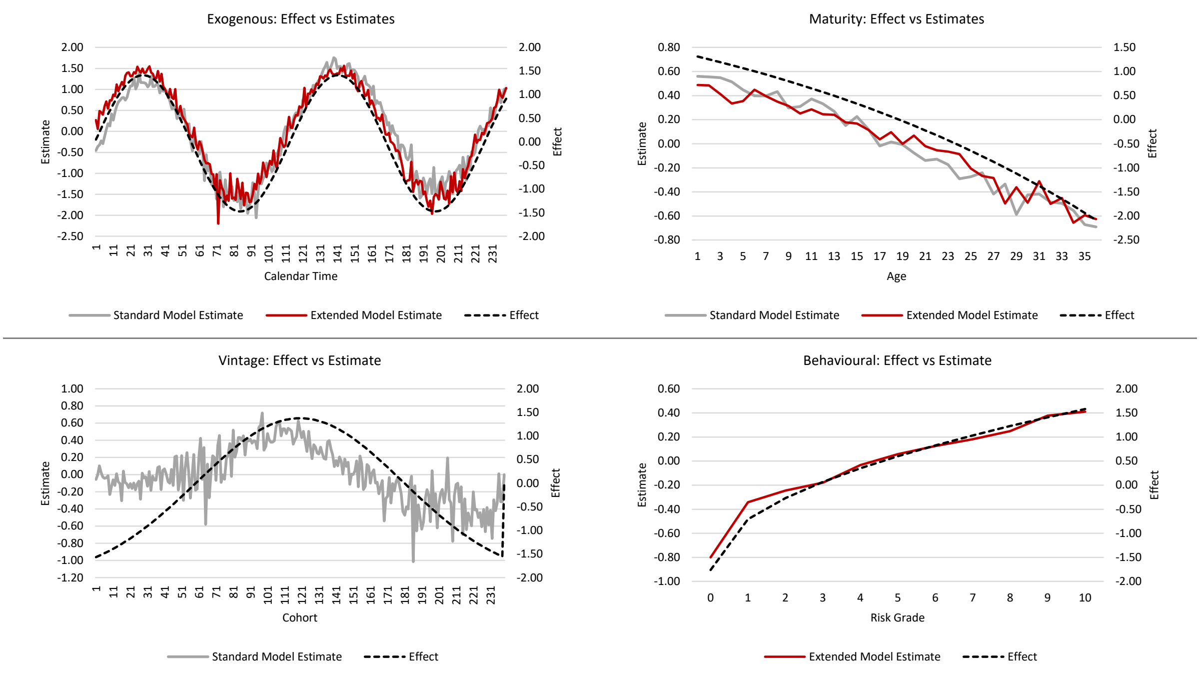

FIGURE 5: ESTIMATED VS SIMULATED EFFECTS UNDER STANDARD AND EXTENDED EMV MODELS FOR THE SAME DATA

3.4.2 Personal Loan Probability of Default

The second case studies a portfolio of personal loans from a South African bank, with the aim of estimating the probability of default. For this study, we adopted the 90-day default definition recommended under Basel III and IFRS 9. In addition, accounts were flagged as default for operational reasons, including write-off and litigation.

The sample consisted of 2 476 000 observations of performing accounts (i.e., accounts that are not in default), observed between September 2005 and June 2014. The horizon chosen for the probability of default is 12 months, which corresponds to the 12-month value-at-risk calculated under Basel III, as well as the expected loss calculation under IFRS 9 for *Stage 1* provisions (i.e., accounts that have not experienced a significant increase in credit risk since origination).

The portfolio was segmented into fixed-rate loans and variable-rate loans. Fixed rate loans are those where the interest rate charged is fixed for the duration of the loan, while variable rate loans have their interest rates vary with the benchmark interest rate (in South Africa, the benchmark rate is the Prime Overdraft Rate). More details on the sample as provided in Appendix 9.12.

3.4.2.1 Dimensional Specification

The extended EMV model proposed in this chapter has many different specifications. Firstly, there is a choice of which dimensions (variables) to include in the model. In this chapter, we discussed five potential dimensions: exogenous, maturity, vintage, behavioural and application. There are thus 31 possible model dimensional specifications, i.e., $2^5 - 1$ possible models with at least one dimension. In addition, there is also the choice of link functions to include. However, as noted above, models that have all three time-related dimensions will suffer from the identifiability problem, regardless of the link function chosen.

Since most of the theory discussed above was based on the probit link function (mainly because this function leads to the Vašíček distribution for portfolio loss), we restrict the application discussion to this link function only. For selecting the dimensional specifications, we recommend using the common information criteria: Akaike Information Criterion (AIC), Bayesian Information Criterion and Schwartz Criterion (for the sake of practicality, these may be supplemented by measures such as the Gini statistic, which is in common use in credit risk modelling) – see Burnham & Anderson (2002) for a discussion of these measures. Table 2 shows the AIC for the different model specifications[34].

Model	**Fixed Rate Loans**	**Variable Rate Loans**		**Model**	**Fixed Rate Loans**	**Variable Rate Loans**
EMVBA	14023967	15088514		**MBA**	14096703	15178034
EMVA	14024038	15088633		**MA**	14096783	15178132
EMVB	14024061	15088682		**MB**	14096809	15178175
EMBA	14043687	15103096		**VBA**	14145114	15202691
EMA	14043758	15103204		**VA**	14145183	15202810
EMB	14043781	15103250		**VB**	14145203	15202857
EVBA	14044690	15104478		**BA**	14179438	15242878
EVA	14044760	15104603		**A**	14179509	15242990
EVB	14044782	15104653		**B**	14179531	15243036
MVBA	14063296	15141550		**EMV**	14867685	16650470
MVA	14063372	15141656		**EV**	14884307	16667967
MVB	14063395	15141700		**EM**	14903110	16675023
EBA	14087834	15169005		**MV**	14907904	16701158
EA	14087897	15169131		**E**	14922954	16708235

[34] We again note that, since these models are fitted via maximum likelihood, models that contain all three time-related dimensions are subject to the identifiability problem. We merely include these in the analysis for the sake of completeness, and to provide a sense of how much is lost by resolving the identifiability problem in the manner that we have in this thesis.

EB	14087917	15169180		**V**	14963533	16750295
				M	15010479	16759464

TABLE 2: THE AIC FOR DIFFERENT MODEL TYPES RANKED BY AIC

The table shows that the five-dimension model is the best-fitting for both fixed rate loans and variable rate loans. This means that all dimensions can explain a significant aspect of the default rate experience that is not explained by the other dimensions. However, this model is not the one we preferred above the other models for two reasons: (1) a model with the exogenous, maturity and vintage dimension at the same time will be subject to the identifiability problem; and (2) a model with both the vintage and application components may be over-specified and lead to multicollinearity[35].

The second reason may deserve further explanation. In our discussion above, we propose the application risk dimension as an alternative for the vintage dimension since the two are both attempting to assess the risk inherent in a particular cohort of customers. The difference between the two is that the application risk dimension is more granular, providing *bottom-up* aggregation of cohort risk. Additionally, the application risk dimension is easier to forecast than the vintage dimension since it can be directly linked to credit risk strategy. The weakness of the application risk dimension is that it is based on a model (the application scorecard), which may be subject to discontinuities when the model is changed or recalibrated. The fact that both application risk and vintage appear in the optimal model simply means that the application scorecard underlying the application risk dimension fails to capture material aspects of variations in risk within the portfolio; a portion of the uncaptured aspects are then represented by the vintage dimension.

Our reasoning above leads to the exclusion of the top three models in terms of AIC, leaving EMBA as the best-fitting. However, further investigation revealed strong co-linearity between the behavioural component and the application component. This is the result of a strong co-linearity between the behavioural scorecard and the application scorecard underlying the two components. The multicollinearity influences the model fitting in a manner that is similar to the effect of the identifiability problem – the results become difficult to interpret. This is illustrated in Figure 6, where we show the estimates of the behavioural and application dimensions under the EMBA model. These estimates are not monotonous in the two risk dimensions, which is unexpected given that the default rate is monotonous in both the application and behavioural scores[36].

[35] Generally, we would expect strong correlation between the application score and the behavioural score, especially in the durations of a loan, for reasons discussed above.

[36] Furthermore, Figure 7 reveals that once the application risk dimension is removed, the nature of the behavioural risk dimension changes drastically, adding credence to our suspicion of multicollinearity.

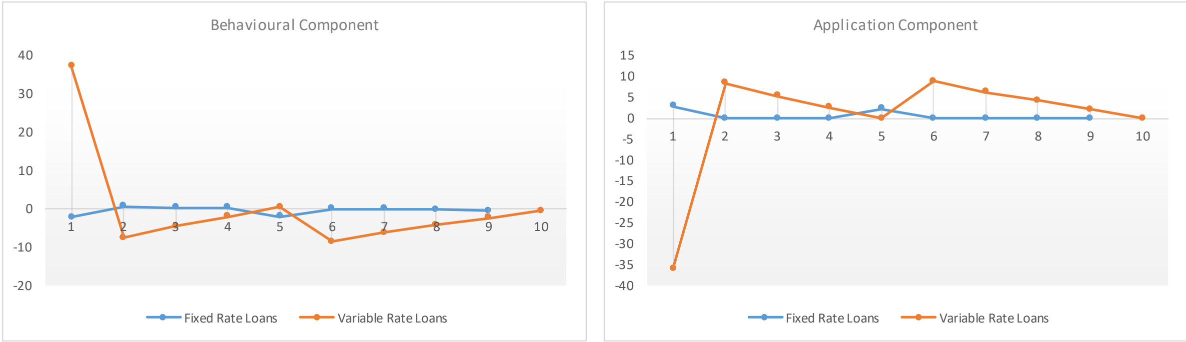

FIGURE 6: BEHAVIOURAL AND APPLICATION COMPNENTS UNDER EMBA MODEL

We thus reject the EMBA model, leaving the EMA model as our preference. However, we note that there is little difference between the AIC of the EMA model and the AIC of the EMB model. For practical reasons, we prefer the EMB model since the behavioural scorecard is generally based on more dynamic information than the application scorecard. Therefore, given the small difference in AICs, we select the EMB model as our final choice.

The model elimination process used to arrive at the EMB model as the model of choice deserves some clarification. The decision to exclude the first three models was based on principle, not empirical evidence. Meanwhile, the decision to select EMB and EMA over EMBA is based on empirical evidence and practical considerations. For example, had there been less correlation between the application and behavioural scorecards we would have preferred EMBA. Finally, the decision to choose EMB over EMA is based purely on reasons of practicality, in light of the marginal difference in AIC between the two models.

An additional point to note on the model selection is that it might lead to different results when a different criterion is used in place of AIC. One of the alternatives is the Gini statistic, which is mainly used to measure discriminatory power – especially for application scoring models. In a latter chapter, we will discuss how this measure can be estimated analytically under the extended EMV model.

3.4.2.2 Model Estimation

The EMB model was estimated using standard logistic regression. The resulting estimates for the three dimensions are shown in Figure 7 below (detailed parameter estimates are can be found in Appendix 0). The behavioural dimension is in line with the expectation that a higher behavioural (credit) score will lead to lower probability of default. Similarly, the maturity dimension shows that default likelihood decreases over the life of the account. The estimates of the exogenous component are also in line with expectation. The estimated time series are well explained by the macroeconomic environment.[37]

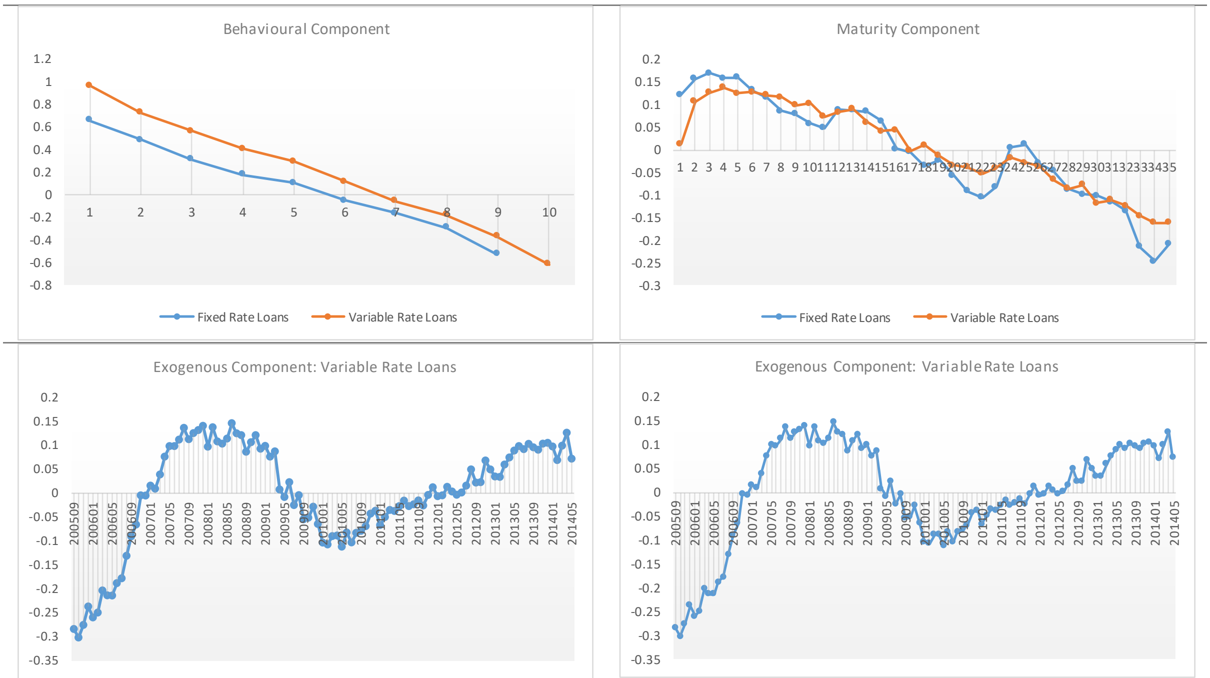

FIGURE 7: ESTIMATED COMPONENTS FOR EMB MODEL

We note that the lack of strict monotonicity in the maturity dimension is likely not a result of estimation error – as one might suspect. For example, notice that the default probability deviates from monotonicity at maturity 12 and 24. This is because the portfolio is composed of loans of different tenures, with tenures tending to be specified in multiples of 12 months. What the results are thus suggesting is that loans with longer tenure have a greater likelihood of default, e.g., loans surviving beyond maturity 12 are those that have longer tenure and have a higher rate of default than would be expected for a given maturity. A different way of accounting for these distinctions would be by including tenure as an input into the behavioural scorecard – this might lead to a more monotonic maturity effect.

Each of the components was standardised so that the mean is added to the intercept and the standard deviation is treated as a parameter estimate for the component. The results are given in Table 3. Note that the mean parameter is the sum of the intercept α and the means of the individual components $\mu = \alpha + \mu_M + \mu_E + \mu_B$.

[37] For example, the peak seen between 2007 and 2009 corresponds with the Great Recession, which had an impact on the South African economy.

Parameters	Fixed Rate Loans	Variable Rate Loans
α	-1.40292	-1.64782
μ_B	0.07809	0.18327
μ_E	-0.00048	-0.00034
μ_M	-0.00138	0.00495
μ	-1.32668	-1.45993
σ_B	0.37928	0.50060
σ_E	0.08707	0.10852
σ_M	0.11380	0.09351

TABLE 3: SUMMARY OF PARAMETER ESTIMATES FOR EMB MODEL

Therefore, the model equation is thus given by:

$$P(t,s,k) = \Phi\left(-1.32668 + 0.11380\,\ddot{M}_{t-s} + 0.08707\,\ddot{E}_t + 0.37928\,\ddot{B}_k\right), \tag{3.39}$$

for fixed-rate loans, and:

$$P(t,s,k) = \Phi\left(-1.45993 + 0.09351\,\ddot{M}_{t-s} + 0.10852\,\ddot{E}_t + 0.50060\,\ddot{B}_k\right), \tag{3.40}$$

for variable-rate loans. Notice that, since the exogenous components have not been modelled using macroeconomic variables at this stage, the models have no random component. However, in order to apply the model during period s we would have to know $\ddot{E}_s$. Therefore, if period s was not in the model development period (meaning that we do not have an estimate for $\ddot{E}_s$), $\ddot{E}_s$ would need to be treated as random variable (with a mean of zero and a standard deviation of one). The alternative is, of course, to model $\ddot{E}_s$ using macroeconomic data, which is done below.

3.4.2.3 Modelling the Exogenous Component

The exogenous components for the two sub-portfolios were regressed against macroeconomic variables. The variables we considered for the model are summarised in Table 5.

Variable Name	Description	Mean	Std.Dev	Min	Max
Prime	The Prime Overdraft Rate, which is the benchmark lending rate for South African banks.	10.57%	2.05%	8.5%	15.5%
CPI	The growth in the Consumer Price Index, as an indicator for price inflation.	5.25%	1.6%	0.4%	8.7%
GDP	The growth in nominal gross domestic product, as an indicator for aggregate economic activity.	9.25%	3.16%	4.15%	15.98%
SavingsToGDP	The ratio of household savings to GDP, as an indicator of the level of savings within the economy.	16.45%	1.28%	14.4%	18.90%
ConsumptionToGDP	The ratio of household consumption to GDP, as an indicator of the level of consumer spending.	60.17%	0.98%	58.2%	62.3%
CompensationToGDP	The ratio of total employee compensation to GDP.	50.93%	2.05%	47.6%	54.4%
DebtToIncome	The ratio of household debt to disposable income, as an indicator of the level of indebtedness.	79.8%	3.17%	72.3%	87.8%
Unemployment	The unemployment rate, as an indicator of economic conditions and the financial condition of the consumer.	24.4%	1.54%	21%	27.7%

TABLE 4: MACROECONOMIC VARIABLES FOR EXOGENOUS COMPONENT

The model estimates and fit statistics for the two sub-portfolios are given in Table 5 (see Appendix 9.4 for more details on the variable selection process). The variable-rate loans model was the better-fitting model, with a coefficient of determination equal to 73.2% regressed only on CPI, while the fixed-rate model only achieved a coefficient of determination equal to 64.9%.

Fixed-Rate Loans					
Variable	Lag	Estimate	Standard Error	P-Value	VIF
Intercept		-11.01364	5.76646	5.900%	0.00000
CPI	-7	0.17751	0.03087	0.000%	1.02611
ConsumptiontoGDP	15	0.26239	0.07964	0.140%	2.59047
SavingstoGDP	15	-0.35099	0.06772	0.000%	2.55178
Variable-Rate Loans					
Variable	Lag	Estimate	Standard Error	P-Value	VIF
Intercept		-2.76558	0.17275	0.000%	0.00000
CPI	-7	0.52549	0.03137	0.000%	1.00000
					***negative lags indicate leads**

TABLE 5: PARAMETER ESTIMATES FOR EXOGENOUS COMPONENT

Figure 8 shows plots for the exogenous effect alongside the fitted model for fixed-rate and variable-rate loans. The macroeconomic factors used are generally able to capture the fluctuations in the exogenous factors over time.

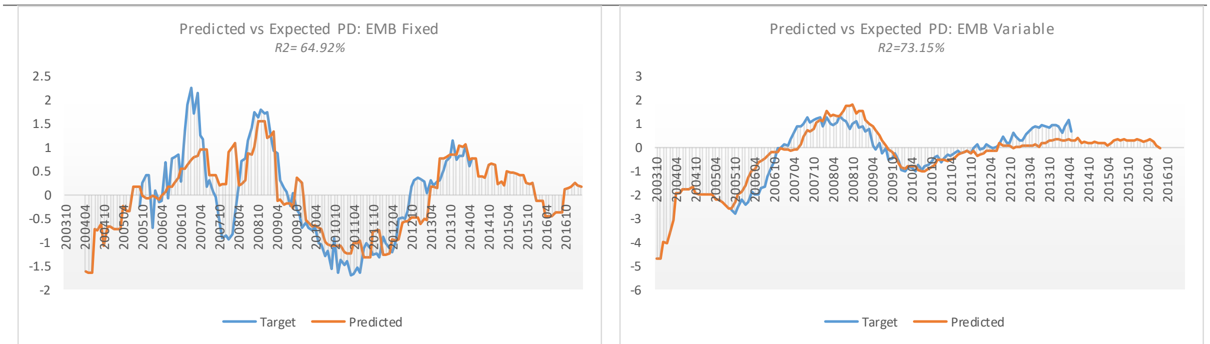

FIGURE 8: EXOGENOUS COMPONENT MODEL FITNESS[38]

Table 6 shows the parameter estimates including the random effect component. The table also shows the asset correlation coefficient of the model on a point-in-time and through-the-cycle basis. Note that, since exogenous components are standardised, the model standard error σ_ε is the square-root of one minus the coefficient of determination: $\sigma_\varepsilon = \sqrt{1 - r^2}$.

Parameters	**Fixed Rate Loans**	**Variable Rate Loans**
$\boldsymbol{\mu}$	-1.32668	-1.45993
$\boldsymbol{\sigma_B}$	0.37928	0.50060
$\boldsymbol{\sigma_E}$	0.08707	0.10852
$\boldsymbol{\sigma_M}$	0.11380	0.09351
$\boldsymbol{\sigma_\varepsilon}$	0.59225	0.51820

[38] We concede that better models for the exogenous component are possible. For example, the models fitted show some evidence for autocorrelation in errors, which suggests that time-series approaches (such as vector autoregression) may yield better fit. However, the suitability of model choice is generally circumstantial. Furthermore, the EMV modelling framework presented here can cope with any number of modelling approaches, provided the resultant prediction can be represented as the sum of a deterministic component and a random component.

$\boldsymbol{\rho_{PiT}}$	0.00265	0.00315
$\boldsymbol{\rho_{TTC}}$	0.00752	0.01164

TABLE 6: SUMMARY OF MODEL PARAMETERS

Therefore, the final point-in-time model for the two portfolios will be revised to include a random effect component:

$$P(t,s,k) = \Phi\left(-1.32668 + 0.11380\,\ddot{M}_{t-s} + 0.08707\,\bar{E}_t + 0.37928\,\ddot{B}_k + 0.05157\varepsilon_t\right), \tag{3.41}$$

for fixed-rate loans and:

$$P(t,s,k) = \Phi\left(-1.45993 + 0.09351\ddot{M}_{t-s} + 0.10852\bar{E}_t + 0.50060\ddot{B}_k + 0.05624\varepsilon_t\right), \tag{3.42}$$

for variable-rate loans. Note that the exogenous component has been split into a deterministic component $\bar{E}_t$ and a random component ε_t. The coefficient of the random component is thus a product of σ_E and σ_ε. Given that the model prediction is now a function of a random variable, the expected probability of default will be the expected value of the model formula, e.g., for fixed-rate loans the expected probability of default is given by:

$$\bar{P}_{PiT}(t,s,k) = \Phi\left(\frac{-1.32668+0.11380\,\ddot{M}_{t-s}+0.08707\,\bar{E}_t+0.37928\,\ddot{B}_k}{\sqrt{1+(0.05157)^2}}\right), \tag{3.43}$$

as described in previous Equation 3.18.

Finally, note that in the case where no macroeconomic model for the exogenous component exists, $\bar{E}_t = 0$ and $\sigma_\varepsilon = 1$. In words, this means that the random effect component has maximum volatility equal to σ_E. The expected probability under this scenario will be a through-the-cycle probability: e.g., for fixed-rate loans this will be:

$$\bar{P}_{TTC}(t,s,k) = \Phi\left(\frac{-1.32668+0.11380\,\ddot{M}_{t-s}+0.37928\,\ddot{B}_k}{\sqrt{1+(0.08707)^2}}\right). \tag{3.44}$$

3.4.2.4 Model Validation and Comparison

We validate the model across range and over calendar time, for accuracy and discriminatory power. The EMB model will generally perform better than the EMV model, but only to the extent that the behavioural scorecard is a better risk differentiator than vintage. Relative accuracy of the EMB model over the EMV model across the time dimension will be a function of the model fitted to the exogenous component.

Figure 9 shows the assessment of accuracy across range for the different models. The plots show that both models are accurate for both low-risk and high-risk predictions. However, the plots reveal how the

EMV model predicts probabilities of default in a narrower range than the EMB model. This confirms the idea that the EMB model will have greater discriminatory power than the EMV mode.

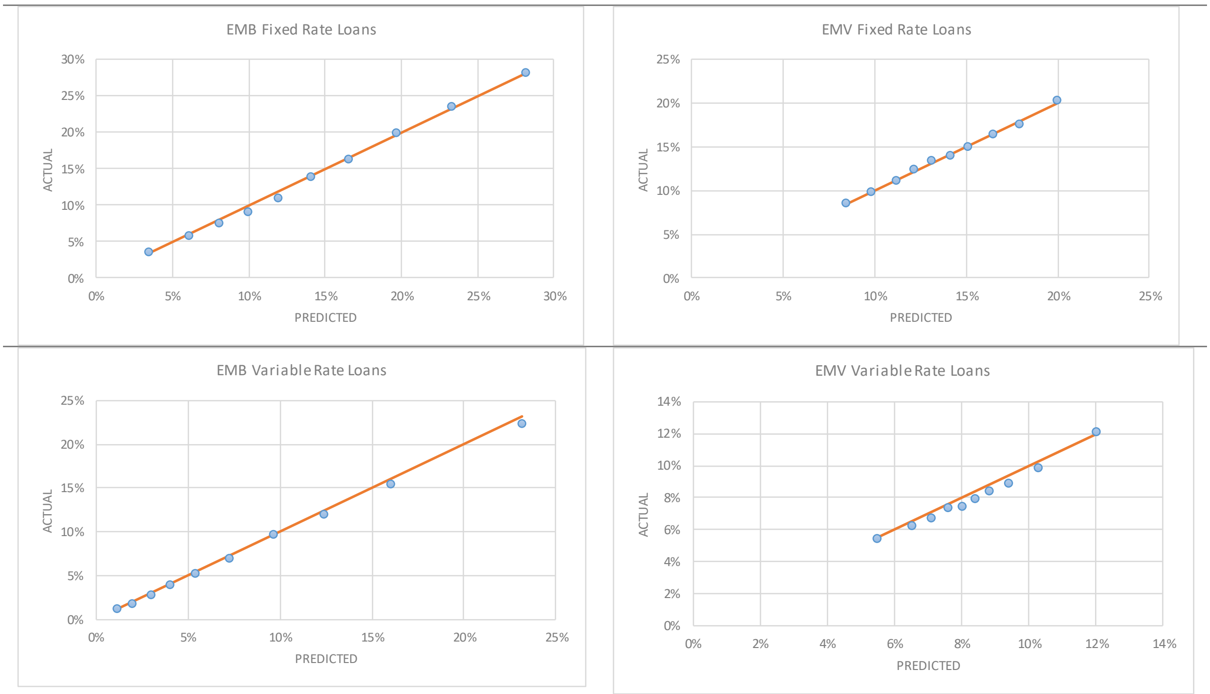

FIGURE 9: MODEL ACCURACY ACROSS RANGE

Table 7 further demonstrates the superiority of the EMB model in discriminatory power through the Gini statistic, where we see that the EMB model has a far greater Gini statistic on both portfolios. The Gini statistic here is used as a measure of the model’s ability to discriminate between accounts that will default over the 12-month horizon and those that will not, as per Mair, Reise and Bentler (2008). The Gini statistic, and how it behaves under the EMV model, are the subject of a latter chapter.

Gini Statistic	**EMB**	**EMV**
Fixed Rate Loans	37%	16%
Variable Rate Loans	47%	14%

TABLE 7: MODEL GINI STATISTIC

Figure 10 shows the assessment of accuracy over calendar time, which is mainly driven by the fitness of the exogenous component. This dimension influences how well the model will be able to form prediction through the cycle. Note that the fitness across this dimension will influence the level of unexpected loss – it is a function of the exogenous model’s coefficient of determination, which inputs into the asset correlation coefficient.

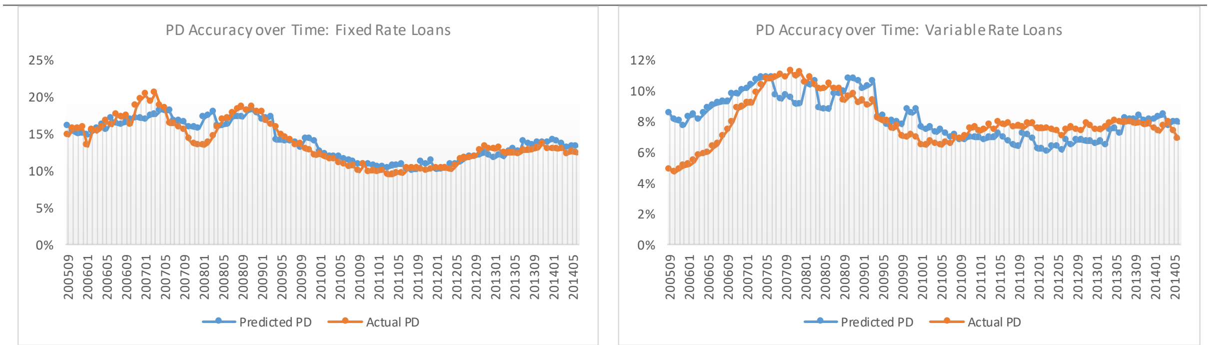

FIGURE 10: MODEL ACCURAY OVER CALENDAR TIME[39]

3.4.2.5 Economic Capital

The prediction error of the model over time is represented by the random component of the model equations described above (Equations 3.41 and 3.42). Using the Vašíček distribution, we form confidence intervals for the predicted probability of default (as per Equation 3.29). When the upper bound of the confidence interval is scaled up by the EAD multiplied by LGD, it yields portfolio value-at-risk, which is useable as credit risk economic capital requirement. The 95% confidence intervals are shown on both a TTC and PiT basis in Figure 11.

It should be noted that the size of the confidence interval will be driven by three factors: (1) the extent of systemic volatility, (2) the extent to which the portfolio is sensitive to systemic volatility and (3) how well the portfolio's sensitivity to systemic volatility is modelled through macroeconomic variables. This is illustrated by the equation of the asset correlation coefficient derived previously:

$$\rho = \frac{\sigma_E^2(1 - r^2)}{1 + \sigma_E^2(1 - r^2)}. \tag{3.45}$$

The volatility of the exogenous component σ_E^2 is determined by the level of systemic volatility within the economic environment concerned as well as how reactive the portfolio is to the economic environment, i.e., the first two factors in the previous paragraph. This means that two portfolios held within the same economic environment will have different level of exogenous volatility depending on how reactive their risks are to the economy. This is evidenced in this case study by the fact that variable-rate loans are about three times more sensitive to CPI than fixed-rate loans, as measured by the model parameters. Of course, this conclusion is not clear-cut since there are other variables within the models. However, a more general point can still be made that the variable-rate portfolio is expected be more sensitive to the economic environment since the loans are directly exposed to interest rate risk.

[39] The fact that the variable-rate loans model exhibits large errors in the earlier periods can be attributed to our observations on error autocorrelation, discussed in the previous footnote.

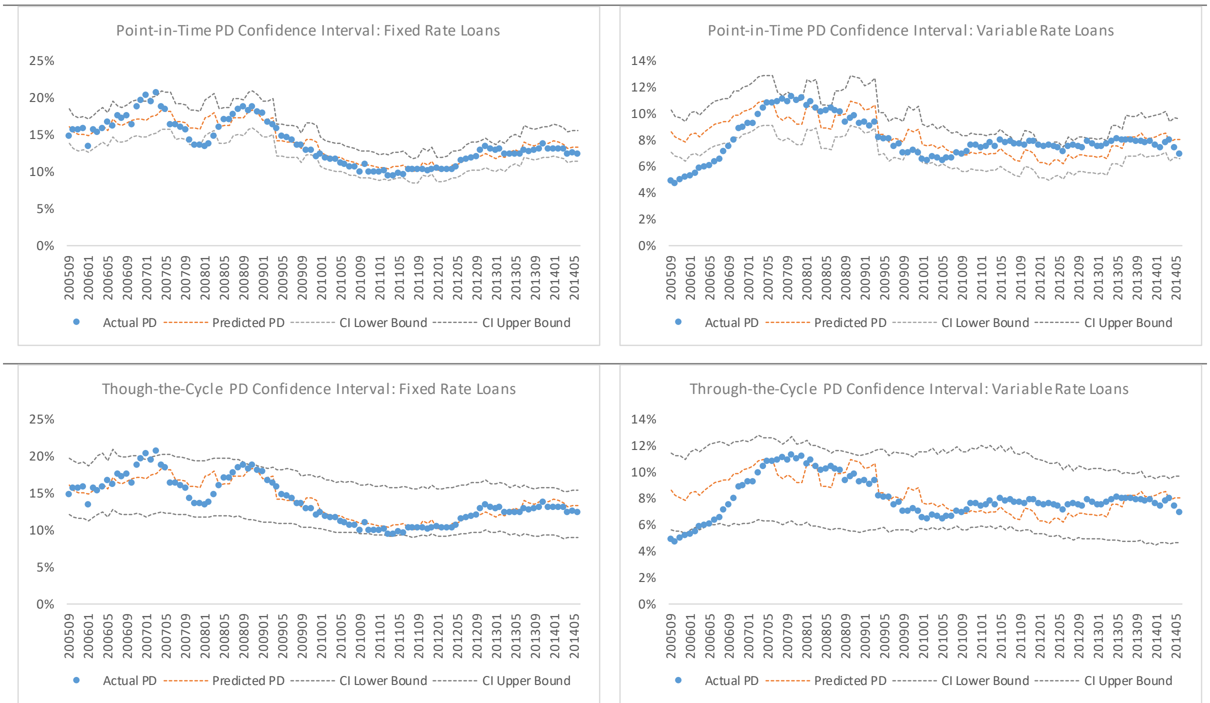

FIGURE 11: PORTFOLIO PD 95% CONFIDENCE INTERVAL

It is useful to think about the volatility of the exogenous component in this way, because one can have two portfolios that are both equally sensitive to the economic environment but are subject to different levels of systemic volatility through being concentrated in different economic environments. For instance, variable-rate loans in the low-income segment of the market may be exposed to different systemic volatility than variable-rate loans in the affluent segment of the market, say, to the extent that inflation on goods purchased by the low-income segment differs materially from inflation on goods purchased in the affluent segment. In this case, it would be the fact that the two markets are exposed to different levels of price (CPI) volatility that leads to different capital requirements, not the differences in the level of reactiveness in the portfolios. This difference is amplified when looking at portfolios that operate in different economic jurisdictions altogether. Both these points are important to consider in economic capital since they are not explicitly accounted for under Basel III regulatory capital.

How well the exogenous component is modelled is the third factor mentioned. As discussed previously, this factor also influences how well the predicted default rate matches the observed default rate. In this way, the size of the confidence interval can be thought of as a continuum, with the upper-bound being achieved when the coefficient of determination of the exogenous component model is zero – at which point the capital requirement is effectively TTC.

A final point to note is that the random component has been assumed to have no serial correlation. However, the plots suggest that serial correlation may exist, especially in the variable-rate model. Therefore, a possible extension to this model will be to incorporate serial correlation in the calculation of economic capital and expected PD.

Comparison to other studies

Table 6 shows the estimates values of the asset correlation coefficient for the two sub-portfolios, on both a point-in-time basis and a through-the-cycle basis. The through-the-cycle asset correlation coefficients are estimated at 0.8% and 1.2% for fixed rate loans and variable rate loans, respectively. The fact that variable rate loans have a higher coefficient makes intuitive sense, since these loans have default rates that are influenced by changes in interest rates.

These estimates asset correlation coefficients are generally much lower than those which are prescribed by regulators. For instance, the Basel regimes prescribe a coefficient of 4% for revolving retail exposures and 15% for mortgage exposures, while for other retail exposures (which include personal loans) the prescribed coefficient is a function of the probability of default. From Figure 11, we see that the default rates typically range from 10% to 20% for fixed rate loans and from 6% to 12% for variable rate loans. Using the prescribed formula, the asset correlation coefficient for the portfolios would be in the region of 3% to 4%. These are larger than the estimated values by at least a multiple.

Of course, that the prescribed asset correlation coefficients are generally larger than empirical estimation is well-known in literature. For instance, studying credit cards, Belloti & Crook (2012) and Rösch & Scheule (2004) find asset correlation coefficients in the region of 1%, compared to the prescribed 4% for revolving retail exposure. However, both these studies were conducted in developing countries where, following the discussion above, greater macroeconomic stability might be supportive of a lower asset correlation coefficient.

Based on South African experience, the work of Mwamba & Mhlophe (2019) suggests that the developing market experience might be different. They find asset correlation coefficients in the region of 5.8% to 7%, which are higher than the prescribed 4% for revolving retail exposures. However, as discussed in Section 2.3.5, the technique used by Mwamba & Mhlophe (2019) does not adjust for portfolio heterogeneity which, according to Wunderer (2019), might jeopardise the results. Indeed, even the estimates provided by Belloti & Crook (2012) and Rösch & Scheule (2004) do not directly correspond with the 0.8% and 1.2% estimates we find for the through-the-cycle coefficient. That is because both Belloti & Crook (2012) and Rösch & Scheule (2004) use models that incorporate macroeconomic variables, which would bring their estimates closer to a point-in-time coefficient than a through-the-cycle coefficient. Meanwhile, Mwamba & Mhlophe (2019) make no adjustment for the economic cycle, which results in a through-the-cycle coefficient.

In principle, the through-the-cycle coefficient is the most comparable across different studies. As discussed above, and summarised through Equation 3.45, the point-in-time coefficient is a function of the variables used to model the economic cycle. If the variables used result in a high coefficient of determination, then the point-in-time asset correlation coefficient will be lower. Consequently, studies that spend more time on modelling the economic cycle will automatically produce lower point-in-time coefficients, all else equal. Moreover, since the Basel-Vašíček framework has a through-the-cycle implementation, point-in-time asset correlation coefficients should not be benchmarked against the coefficients prescribed for use in the Basel-Vašíček framework since these are through-the-cycle.

To summarise, we are hesitant to make direct comparison between the asset correlation coefficients estimated here and those provided in other papers for two reasons: (a) some scholars estimate point-in-time asset correlation coefficients, which are generally not comparable across studies and are not comparable to those prescribed for the Basel-Vašíček framework; and (b) some scholars make no adjustments for portfolio heterogeneity, which distorts the estimation. Additionally, scholars generally make no adjustment for portfolio size, which makes the results even less comparable: smaller portfolios will generally produce larger asset correlation coefficients (this idea is developed further in the next chapter).

3.5 Summary of the Findings

We set out to provide an extension to the EMV model to incorporate behavioural data. We further described how to overcome the indefinability problem by incorporating application risk in place of the vintage component. Empirical tests showed how the extended EMV model (particularly the EMB model) outperforms the standard EMV model in terms of discriminatory power. This is owed to the fact that behavioural scorecards general outperform vintage in discriminating risk.

We showed how the EMV model can be used across most of the areas in credit risk analysis:

- When the probit link function is used, the model gives rise to the Vašíček distribution for the default rate distribution;
- The model can be used alongside an attrition model to produce a lifetime probability of default, for use in IFRS 9 modelling; and
- By allowing certain economic and behavioural parameters to be stressed, the model is usable as for stress-testing.

In the area of economic capital, the extended EMV model was used to derive a general formula for the asset correlation coefficient. The formula describes the asset correlation as a function of the total exogenous risk within a portfolio and the coefficient of determination of the macroeconomic model for the exogenous component. The formula can be used to further explain the fact the exogenous risk is a function of the total systemic volatility and the extent to which a given portfolio is sensitive to the level of systemic risk. This is particularly useful as it is not explicitly accounted for in the Basel III regulatory regime.

We also reveal areas where further research can be directed. Particularly, more work may be directed towards understanding the nature of the distribution of the exogenous risk. It was assumed that exogenous random error has no serial correlation. In practice, there are situations where serial correlation exists – properly modelling the error distribution may lead to different conclusions about the adequate level of economic capital.

Chapter 4: Diversifiable and Undiversifiable Risk

4.1 Overview of the Problem

The large homogenous portfolio (LHP) approximation is used in credit risk analysis to simplify the analytical probability distribution function of the loss rate on a credit portfolio. The approximation makes a few simplifying assumptions, which allow the distribution function to have a closed-form representation. It is applied to the components approach to credit risk analysis, consisting of probability of default, exposure at default and loss given default (see Thomas, Oliver & Hand (2005) for a discussion). The approximation assumes that the probability of default is the only source of randomness, i.e., it assumes the exposure at default and loss given default are non-random.

Even with the above simplifying assumption, the portfolio loss rate generally has a mathematically intractable probability distribution. To further simplify the distribution function, the approximation assumes that all loans within the portfolio have the same probability of default. However, this probability of default is unknown. It is assumed to take the form of a probit model, being influenced by a single, normally distributed exogenous factor. Consequently, the portfolio loss rate becomes a scaled compound binomial distribution[40].

The scaled compound binomial distribution is also mathematically intractable. Therefore, a final assumption is made that the portfolio is infinitely large. Under this final assumption, the distribution of portfolio loss becomes a scaled Vašíček distribution (see Campolongo, Jönsson, Schoutens (2012) for a more detailed discussion of the derivation). This derivation is what was followed in the previous chapter when deriving the portfolio loss distribution under the EMV model.

Important to our discussion is the fact that this approximation underlies the Basel III credit risk capital requirements and has thus become widely adopted. However, Malwandla (2016) discussed several shortcomings with this model as it is applied under the Basel III framework. One of the shortcomings was that the model, through its assumption that the portfolio is infinitely large, ignores sampling error as a source of risk and only focuses on systemic risk. Holding all model parameters equal, this leads to an understatement of the capital requirement for small portfolios, where sampling error plays a large role.

[40] Assuming that the portfolio has homogenous loans – with constant exposure-at-default (EAD) and loss given default (LGD) – the portfolio loss becomes $LGD \times EAD \times R$, where $R \sim Bn(n, p)$, n is the portfolio size and p is the default rate on a single loan. Therefore, the portfolio loss is a scaled binomial random variable. However, if p is itself unknown, the loss becomes a compound distribution (i.e., scaled compound binomial distribution).

Few attempts have been made to address this shortcoming have been made. Of note is the work done by Pimbley (2011), who attempted to overcome the challenge by using Stirling's approximation to the binomial function, which aids in simplifying the integral. Another noteworthy contribution is by Gordy (2004), later refined by Gordy & Lütkebohmert (2016), who introduced the *granularity adjustment*, based on a Taylor approximation of the portfolio loss distribution's value-at-risk measure. However, as noted by Gordy & Lütkebohmert (2016), the fact that the Taylor approximation is asymptotic means that the adjustment proposed may fail for smaller portfolios – the same would apply for Stirling's approximation, as used by Pimbley (2011)[41].

In this chapter we aim to address two problems. Firstly, we quantify the impact of the shortcoming identified by Malwandla (2016). This is done by deriving an analytical formula for the variance of the portfolio default rate (and thus, portfolio loss rate) and showing that this can be separated into an undiversifiable (systemic) component and a diversifiable component. We use this to show that, under the LHP assumption, the diversifiable component is assumed to be equal to zero. Secondly, we derive ways of adjusting the LHP approximation such that it does not understate the level of diversifiable risk. The adjustment discussed here is different from those which are proposed by Pimbley (2011) and Gordy (2004) in that: (a) it does not rely on asymptotic properties (which tends defeat the purpose of the adjustment); and (b) it can be used to adjust the entire distribution of losses, not just the quantile (VaR), as is the case for the contribution by Gordy (2004).

4.2 Extending the Large Homogenous Portfolio Approximation

The Large Homogenous Portfolio (LHP) approximation is used to estimate a portfolio's probability of default. Let P_k be the probability of default for loan k in a portfolio of n loans, such that[42]:

$$P_k = \Phi(\mu_k + \sigma\varepsilon), \tag{4.1}$$

where μ_k is the risk parameter associated with loan k, ε is the exogenous risk factor, common to all loans within the portfolio, and Φ is the distribution function of the standard normal distribution. Note that ε is a random variable, which means that P_k is also a random variable.

Let R be the default rate for the portfolio, defined as:

$$R = \frac{1}{n}\sum_{k=1}^{n} D_k, \tag{4.2}$$

[41] The fact that these approaches rely on asymptotic approximations somewhat defeats the purpose, since an adjustment is only needed for smaller portfolios.
[42] For continuity, this is identical to Equation 3.17 in the previous chapter.

where D_i is the default indicator, defined as:

$$D_k = \begin{cases} 0 & if\ account\ k\ does\ not\ default \\ 1 & if\ account\ k\ defaults \end{cases}. \tag{4.3}$$

The LHP assumptions are: (a) the portfolio is large; and (b) the portfolio is homogenous (i.e., all loans have the same default rate). The homogeneity assumption means that:

$$P_k = \Phi(\mu + \sigma\varepsilon) = P, \tag{4.4}$$

where P is the population probability of default and μ is the homogenous risk factor. Under the largeness assumption, we let $n \to \infty$ so that R tends towards its population mean, according to the Law of Large Numbers:

$$R \to P, \tag{4.5}$$

i.e., the sample mean R tends towards the population mean P. Note again that, due to ε, R is a random variable. Therefore, under the LHP assumptions, the distribution function of R is given by:

$$F(x) = Prob[R \leq x]$$

$$= G\left(\frac{\Phi^{-1}(x) - \mu}{\sigma}\right), \tag{4.6}$$

where G is the distribution function for ε. It is usually assumed that $\varepsilon \sim N(0,1)$, so that:

$$F(x) = \Phi\left(\frac{\Phi^{-1}(x) - \mu}{\sigma}\right), \tag{4.7}$$

which is the distribution function of the Vašíček distribution (see Appendix 9.9 for a summary of the properties of the Vašíček distribution). Under this assumption, the mean of R is given by (Owen, 1980):

$$E[R] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) = p, \tag{4.8}$$

where is the default rate parameter for the portfolio. In this case, the quantile function is given by:

$$k(\alpha) = \Phi(\sigma\Phi^{-1}(\alpha) + \mu). \tag{4.9}$$

For a portfolio with exposure of EAD (i.e., exposure at default) and loss rate of LGD (i.e., loss given default), both assumed to be non-random variables as per the LHP, the credit risk capital requirement can be set to equal[43]:

$$c(\alpha) = k(\alpha) \times EAD \times LGD. \tag{4.10}$$

This is the standard LHP assumption as discussed by Malwandla (2016). We now study this assumption further below.

4.2.1 Implications of the LHP Assumptions on Variance

The challenge with using $c(\alpha)$ is that the assumptions made in its derivation are generally not met. Of interest in this chapter is the assumption of largeness, i.e., $n \to \infty$. Therefore, we will be taking the homogeneity assumption for granted, i.e., we still assume that:

$$P_k = \Phi(\mu + \sigma\varepsilon), \tag{4.11}$$

for all loans in the portfolio. Under this construct, the formula for the portfolio probability of default can be derived explicitly (i.e., without needing further assumptions) through the Law of Total Probability, as follows:

$$\begin{aligned} F(x) &= Prob[R \le x] \\ &= \int_{-\infty}^{\infty} Prob[nP \le nx | \varepsilon = e] g(e) de \\ &= \int_{-\infty}^{\infty} \sum_{k=1}^{\lfloor nx \rfloor} \binom{n}{k} \Phi(\mu + \sigma\varepsilon)^k \big(1 - \Phi(\mu + \sigma\varepsilon)\big)^{n-k} g(e) de. \end{aligned} \tag{4.12}$$

where g is the density function for ε, i.e., R follows a scaled compound binomial distribution. However, this integral does not have a closed form solution, which is why the LHP assumption is often needed in practice. By letting $n \to \infty$, we get that:

$$F(x) = G\left(\frac{\Phi^{-1}(x) - \mu}{\sigma}\right), \tag{4.13}$$

as shown above.

The challenge with the largeness assumption is that it leads to an understatement of the variance of P, as we will show below, and, consequently, the VaR, as discussed by Gordy (2003). Specifically, consider that the distribution of P has three parameters (μ, σ and n) as shown in Equation 4.13. By

[43] Let EAD and LGD be constants and R be random, so that the loss is given by $L = R \times EAD \times LGD$, as discussed in Equation 3.25. The quantile of the loss distribution will be a multiple of the quantile of the distribution of R.

letting $n \to \infty$, n falls away as a parameter of the distribution. This has consequences on the analytical variance of R, i.e., for the same values for σ and μ, $Var[P]$ reduces as $n \to \infty$. The variance defines the *spread* of the distribution, which means that it is expected to have an influence on $c(\alpha)$.

To see how the largeness assumption influences the variance, consider that the variance of R can be decomposed into two parts, through the Law of Total Variance:

$$v^2(n) = Var\big[E[R|\varepsilon]\big] + E\big[Var[R|\varepsilon]\big]$$

$$= Var[\Phi(\mu+\sigma\varepsilon)] + E\left[\frac{\Phi(\mu+\sigma\varepsilon)\big(1-\Phi(\mu+\sigma\varepsilon)\big)}{n}\right]. \tag{4.14}$$

The first part of this formula can be thought of as systemic risk, which cannot be diversified away as the portfolio size grows larger, i.e., it is not a function of n. The second part can be thought of as diversifiable risk, since it tends to zero as the portfolio grows larger. Therefore, the LHP approximation will understate the variance of R by an amount equal to $E\big[Var[R|\varepsilon]\big]$, which is inversely proportional to n.

Unfortunately, unlike the variance, the quantile function (and, thus, the value-at-risk) does not simplify into a systemic and a diversifiable component. Therefore, we define a general quantile function for R as $k(\alpha, n, \sigma)$, which includes all the parameters of the distributions as its inputs. This does not have a closed form solution under the scaled compound binomial distribution. However, it has some interesting special cases. For $\sigma = 0$, the distribution tends to a scaled binomial distribution – so that:

$$k(\alpha, n, 0) = \frac{b\big(\alpha, n, \Phi(\mu)\big)}{n}, \tag{4.15}$$

where $b(\alpha, n, p)$ is the quantile function of the $Binomial(n, p)$ distribution. This means that the quantile is also a function of the portfolio size n. Another special case is where $n \to \infty$ (when the distribution of P tends to a Vašíček distribution), wherein the quantile becomes independent of n: $k(\alpha, \infty, \sigma) = k(\alpha)$, as shown in Equation 4.9.

4.2.2 Formulae for the Variance Portfolio Loss Rate

If we assume that $\varepsilon \sim Normal(0,1)$ then variance given by Equation 4.14 can be specified more explicitly[44]. We begin with the first factor:

$$Var_\varepsilon[\Phi(\mu+\sigma\varepsilon)] = E_\varepsilon[\Phi(\mu+\sigma\varepsilon)^2] - E_\varepsilon[\Phi(\mu+\sigma\varepsilon)]^2. \tag{4.16}$$

[44] For continuity, note that this is the same assumption made in Equation 3.17 in the previous chapter.

From Owen (1980), we have:

$$E[\Phi(\mu+\sigma\varepsilon)^2] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) - 2T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{1}{\sqrt{1+2\sigma^2}}\right)$$

$$= p - 2T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2s^2}}\right), \qquad (4.17)$$

where $T(h,a)$ is Owen's T function[45]. We also have:

$$E[\Phi(\mu+\sigma\varepsilon)] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) = p \qquad (4.18)$$

Therefore, combining Equations 4.17 and 4.18, the systemic variance factor (Equation 4.16) simplifies to:

$$Var[\Phi(\mu+\sigma\varepsilon)] = p(1-p) - 2T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2\sigma^2}}\right). \qquad (4.19)$$

Following from the above, the second variance factor in Equation 4.14 (the diversifiable risk) simplifies as follows:

$$E\left[\frac{\Phi(\mu+\sigma\varepsilon)\big(1-\Phi(\mu+\sigma\varepsilon)\big)}{n}\right] = \frac{1}{n}\left[E[\Phi(\mu+\sigma\varepsilon)] - E[\Phi(\mu+\sigma\varepsilon)^2]\right]$$

$$= \frac{1}{n}\left[\Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) - \left\{\Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) - 2T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{1}{\sqrt{1+2\sigma^2}}\right)\right\}\right]$$

$$= \frac{2}{n}T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{1}{\sqrt{1+2\sigma^2}}\right)$$

$$= \frac{2}{n}T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2\sigma^2}}\right). \qquad (4.20)$$

Therefore, the variance of R can be expressed as:

$$v^2(n,\sigma) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)\left(1-\Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)\right) - 2\left(\frac{n-1}{n}\right)T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{1}{\sqrt{1+2\sigma^2}}\right)$$

$$= p(1-p) - 2\left(\frac{n-1}{n}\right)T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2\sigma^2}}\right). \qquad (4.21)$$

4.2.3 The Variance Formulae under Special Cases

The above formula has some properties that are worth noting.

---

[45] Owen's T function is defined as: $T(h,a) = \phi(h)\int_0^a \frac{\phi(hx)}{1+x^2}dx$.

Case 1: as $n \to \infty$

As $n \to \infty$, the formula tends to the variance of the Vašíček distribution, which is the asymptotic distribution of R:

$$v^2(\infty, \sigma) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)\left(1 - \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)\right) - 2T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{1}{\sqrt{1+2\sigma^2}}\right). \tag{4.22}$$

Under this case, the variance is a function of μ and σ, with σ being the systemic risk parameter. Therefore, in cases where the portfolio is very large, the Vašíček distribution is a fair approximation to the portfolio loss distribution.

Case 2: $\sigma \to 0$

Also, for finite values of n, as $\sigma \to 0$, we have:

$$v^2(n, 0) \to \Phi(\mu)\big(1 - \Phi(\mu)\big) - 2\left(\frac{n-1}{n}\right)\left[\frac{1}{2}\Phi(\mu)\big(1 - \Phi(\mu)\big)\right]$$

$$= \frac{\Phi(\mu)\big(1-\Phi(\mu)\big)}{n}, \tag{4.23}$$

which is the variance of R under the binomial distribution. Therefore, in cases where the level of systemic risk is negligible, the binomial distribution is a fair approximation to the portfolio loss distribution.

Case 3: as $n \to \infty$ and $\sigma \to 0$

As the level of systemic risk reduces (i.e., as $\sigma \to 0$), the asymptotic variance will also tend towards zero. Firstly, observe that as $\sigma \to 0$, we have:

$$\Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) \to \Phi(\mu). \tag{4.24}$$

Also, from Own (1980), as $\sigma \to 0$, we have:

$$T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{1}{\sqrt{1+2\sigma^2}}\right) \to \frac{1}{2}\Phi(\mu)\big(1 - \Phi(\mu)\big). \tag{4.25}$$

Combining these, we have that $Var(R) \to 0$ as $n \to \infty$ and $\sigma \to 0$. Therefore, in cases where the level of systemic risk is negligible and the portfolio is large, the value-at-risk becomes negligible (since the variance tends to zero).

4.2.4 Extending the LHP Approximation: the Largeness Assumption

As shown in Equation 4.12, for finite n (with homogenous risk), R has a scaled compound binomial distribution. It is only as $n \to \infty$ where the distribution of R tends towards the Vašíček. Therefore, within the assumption that $n \to \infty$ we also have the assumption that $R \sim Vasicek$. For our purposes, we will retain the assumption that $P \sim Vasicek$, and only relax the assumption that $n \to \infty$ (i.e., we assume that even for smaller portfolios P still follows a Vašíček distribution).

In other words, there are essentially three parts to the LHP assumptions: (a) the portfolio is infinitely large; (b) the portfolio is homogenous; and (c) the portfolio default rate follows a Vašíček distribution. In the previous section we had only specified the two assumptions of homogeneity and largeness. This was done because the third assumption follows from the first one. Therefore, by relaxing the largeness assumption while retaining the assumption that $R \sim Vasicek$, we have weakened the LHP set of assumptions.

As a way of improving the LHP approximation, we propose *requiring* that the approximation produces the same mean and variance as the original distribution, i.e., we assume that R follows the Vašíček distribution, with the mean and variance equal to those under the original scaled compound binomial (as given in Equations 4.11 and 4.21, respectively).

Let $\hat{G}$ be distribution function for $R \sim Vasicek(q, s)$ under the LHP assumption:

$$\hat{G}(x, q, s) = \Phi\left(\frac{\Phi^{-1}(x) - \Phi^{-1}(q)\sqrt{1+s^2}}{s}\right). \tag{4.26}$$

In Table 8 we show the mean and variance of R under the distribution function $\hat{G}$, which is the Vašíček, alongside the analytical mean and variance for R under the scaled compound binomial distribution.

Moment	Scaled Compound Binomial $\boldsymbol{n, \sigma, p}$	Vašíček $\boldsymbol{s, q}$
Mean	$p = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$	q
Variance	$v^2(n,\sigma) = p(1-p) - 2\left(\frac{n-1}{n}\right) T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2\sigma^2}}\right)$	$v^2(\infty, s) = q(1-q) - 2T\left(\Phi^{-1}(q), \frac{1}{\sqrt{1+2s^2}}\right)$

TABLE 8: MEAN AND VARIANCE OF R UNDER THE COMPOUND BINOMIAL AND THE VAŠÍČEK DISTRIBUTIONS

Therefore, in order to meet our requirements, we set q equal to p and $v^2(\infty, s)$ equal to $v^2(n, \sigma)$. The second equation simplifies to (see Appendix 9.5):

$$T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2s^2}}\right) = \left(\frac{n-1}{n}\right) T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2\sigma^2}}\right). \tag{4.27}$$

In other words, meeting our *requirement* means that s must satisfy Equation 4.27. This ultimately means that s will be a function of n. Notice that under the standard LHP approximation, s would represent purely systemic risk, thereby being independent of n. However, in the extended LHP approximation s has essentially been modified to also include both systemic and diversifiable risk, thereby being a function of n. Let $s(n)$ be the value of s satisfying Equation 4.27 for a given n. For the sake of clarity, we can write $s(n)$ as:

$$s(n) = \sigma + \xi(\sigma, n, p), \tag{4.28}$$

where $\xi(\sigma, n, p)$ is our adjustment to the LHP approximation. Therefore, the distribution function for R under the extended LHP approximation is as follows:

$$F(x) = \Phi\left(\frac{\Phi^{-1}(x) - \Phi^{-1}(p)\sqrt{1+s(n)^2}}{s(n)}\right). \tag{4.29}$$

The corresponding quantile function is given by:

$$k(\alpha, n, \sigma) = \Phi\left(\Phi^{-1}(\alpha)s(n) + \Phi^{-1}(p)\sqrt{1+s(n)^2}\right). \tag{4.30}$$

For given values for p, n and σ, values for $s(n)$ can be estimated numerically[46].

4.2.5 Extending the LHP Approximation: the Homogeneity Assumption

In isolation, the homogeneity assumption is easier to relax. This is particularly the case when we impose a structure on the nature of the heterogeneity within the portfolio. As before, suppose that the probability of default on a single loan is given by:

$$P_k = \Phi(\mu_k + \sigma\varepsilon), \tag{4.31}$$

and we are interested in the distribution of the portfolio default rate:

$$R = \frac{1}{n}\sum_{k=1}^{n} D_k, \tag{4.32}$$

where D_i is the default indicator, defined as:

$$D_k = \begin{cases} 0 & \text{if account k does not default} \\ 1 & \text{if account k defaults} \end{cases}. \tag{4.33}$$

The expected value for R is given by:

$$E[R|\varepsilon] = \frac{1}{n}\sum_{k=1}^{n} \Phi(\mu_k + \sigma\varepsilon). \tag{4.34}$$

[46] We provide a Python script for calculating $s(n)$ in Appendix 9.11.

Let $n \rightarrow \infty$, as per the largeness assumption, and assume $\mu_k \sim N(m, \omega)$, i.e., we are imposing a Gaussian structure to the heterogeneity in risk. Under this structure, we have:

$$E[R|\varepsilon] = \int_{-\infty}^{\infty} \Phi(\mu + \omega\epsilon + \sigma\varepsilon)d\epsilon$$

$$= \Phi\left(\frac{\mu+\sigma\varepsilon}{\sqrt{1+\omega^2}}\right). \tag{4.35}$$

Similarly, the unconditional expectations for R is given by:

$$E[R] = \Phi\left(\frac{\mu}{\sqrt{1+\omega^2+\sigma^2}}\right). \tag{4.36}$$

By the Law of Large Numbers, we have that $R|\varepsilon \rightarrow^{(a.s)} E[R|\varepsilon]$ as $n \rightarrow \infty$, i.e., R converges *almost certainly* to $E[R]$[47]. Therefore, the distribution function for R becomes:

$$F(x) = Prob[R \leq x]$$

$$= \Phi\left(\frac{\Phi^{-1}(x)\sqrt{1+\omega^2}-\mu}{\sigma}\right), \tag{4.37}$$

from which the quantile function can be deduced:

$$k(\alpha) = \Phi\left(\frac{\sigma\Phi^{-1}(\alpha)+\mu}{\sqrt{1+\omega^2}}\right). \tag{4.38}$$

However, we emphasise that this operates under the assumption that the largeness assumption is met, i.e., this extension does operate in conjunction with the one discussed above.

4.2.6 Applying the Extended LHP Approximation on an EMV Model

The parameters of the model for the distribution of R in Equation 4.29 can be estimated using the exogenous maturity vintage model (EMV), discussed in Chapter 3. The EMV model is specified as follows:

$$P_k = \Phi\left(\sum \beta_k X_{kj} + \sum \alpha_k Y_k + \omega\varepsilon\right), \tag{4.39}$$

where $\boldsymbol{X}_k = (X_{k1}, X_{k2}, \dots)$ and $\boldsymbol{Y} = \{Y_1, Y_2, \dots\}$ are vectors for loan-specific for loan k and macroeconomic variables, respectively. Therefore, under the EMV model we have:

$$\mu_k = \sum \beta_k X_k + \sum \alpha_k Y_k, \tag{4.40}$$

[47] Since the default outcomes D_k are not identically distributed, we note that the additional constraint under the Law of Large Numbers that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{Var(D_k)}{n} < \infty$ is met, since $P_k(1-P_k) < 1$.

$\varepsilon \sim N(0,1)$ represents the residual exogenous risk after the inclusion of macroeconomic variables and ω is the standard deviation of the exogenous risk. The expected probability of default for each loan is thus given by:

$$E[P_k] = \Phi\left(\frac{\sum \beta_k X_k + \sum \alpha_k Y_k}{\sqrt{1+\omega^2}}\right) \coloneqq p_k, \tag{4.41}$$

as per Equation 4.8. The portfolio expected default rate is given by:

$$p = \sum_{j=1}^{n} \frac{p_k}{n}. \tag{4.42}$$

Under the LHP assumption, we also have that:

$$p = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right), \tag{4.43}$$

as per Equation 4.18. Combining Equation 4.34 and Equation 4.35, μ can be obtained as follows[48]:

$$\mu = \Phi^{-1}\left(\sum_{j=1}^{n} \frac{\bar{p}_k}{n}\right)\sqrt{1+\sigma^2}. \tag{4.44}$$

As mentioned above, the systemic component is given by ω, i.e., $\sigma = \omega$. Therefore, the distribution function of R under the LHP assumption is given by:

$$F(x) = \Phi\left(\frac{\Phi^{-1}(x) - \Phi^{-1}\left(\sum_{j=1}^{n} \frac{\bar{p}_k}{n}\right)\sqrt{1+\omega^2}}{\omega}\right), \tag{4.45}$$

as per Equation 4.7. This is extended into Equation 4.29 by substituting $s(n)$ for ω, where $s(n)$ is estimated through Equation 4.27.

4.3 Case Study

We now briefly discuss the practical implications of the extending the LHP approximation through a simulation study. We simulate the loss rate on a portfolio of n loans by assuming that the default rate on an account is as follows:

$$P_k = \Phi(\mu + \sigma\varepsilon), \tag{4.46}$$

where μ is the common risk factor and ε is the normally distributed exogenous variable. We simulate $\varepsilon \sim N(0,1)$ as $\Phi^{-1}(u)$, where u is sampled from a $U(0,1)$ distribution. We therefore simulate the default outcome as a Bernoulli random variable, as follows:

[48] For continuity, note that this is not much different from the formula provided in Equation 3.27.

$$D_k = \begin{cases} 0 & for\ P_k > u_k \\ 1 & for\ P_k \le u_k \end{cases}, \tag{4.47}$$

where each u_k is sampled from a $Uniform(0,1)$ distribution. Finally, the portfolio default:

$$R = \frac{1}{n}\sum_{k=1}^{n} D_k. \tag{4.48}$$

Since the loss rate is simply a multiple of the default rate (as described in Equation 4.10), we focus on studying the default rate only. We considered a number of scenarios for the portfolio size ranging from $n = 100$ to $n = 10000$, for each scenario creating 100 thousand simulations to produce a simulated distribution for R[49]. We also considered a range of mean portfolio default rates p and asset correlation coefficients ρ.

n	$p = 5\%$ and $\rho = 5\%$				$p = 20\%$ and $\rho = 5\%$			
	$\hat{v}(n,\sigma)$	$v(n,0)$	$v(\infty,\sigma)$	$v(n,\sigma)$	$\hat{v}(n,\sigma)$	$v(n,0)$	$v(\infty,\sigma)$	$v(n,\sigma)$
100	0.103%	0.048%	0.057%	0.104%	0.56%	0.16%	0.40%	0.55%
200	0.080%	0.024%	0.057%	0.080%	0.48%	0.08%	0.40%	0.48%
300	0.072%	0.016%	0.057%	0.072%	0.45%	0.05%	0.40%	0.45%
400	0.068%	0.012%	0.057%	0.069%	0.44%	0.04%	0.40%	0.44%
500	0.066%	0.010%	0.057%	0.066%	0.43%	0.03%	0.40%	0.43%
600	0.065%	0.008%	0.057%	0.065%	0.43%	0.03%	0.40%	0.42%
700	0.064%	0.007%	0.057%	0.064%	0.42%	0.02%	0.40%	0.42%
800	0.063%	0.006%	0.057%	0.063%	0.42%	0.02%	0.40%	0.42%
900	0.062%	0.005%	0.057%	0.062%	0.41%	0.02%	0.40%	0.42%
1000	0.061%	0.005%	0.057%	0.062%	0.42%	0.02%	0.40%	0.41%
2000	0.059%	0.002%	0.057%	0.059%	0.40%	0.01%	0.40%	0.41%
3000	0.059%	0.002%	0.057%	0.058%	0.40%	0.01%	0.40%	0.40%
4000	0.058%	0.001%	0.057%	0.058%	0.40%	0.00%	0.40%	0.40%
5000	0.058%	0.001%	0.057%	0.058%	0.40%	0.00%	0.40%	0.40%
6000	0.057%	0.001%	0.057%	0.058%	0.40%	0.00%	0.40%	0.40%
7000	0.057%	0.001%	0.057%	0.058%	0.40%	0.00%	0.40%	0.40%
8000	0.058%	0.001%	0.057%	0.057%	0.40%	0.00%	0.40%	0.40%
9000	0.057%	0.001%	0.057%	0.057%	0.40%	0.00%	0.40%	0.40%
10000	0.057%	0.000%	0.057%	0.057%	0.40%	0.00%	0.40%	0.40%

TABLE 9: PORTFOLIO DEFAULT RATE VARIANCE FOR DIFFERENT PORTFOLIO SIZES

We first demonstrate the inability of the standard LHP approximation to capture the variance of the portfolio default rate. For each simulation, we calculate $\hat{v}(n,\sigma)$, the sample standard deviation of the portfolio default rate. This is compared to the standard deviation implied by the LHP approximation, $v(\infty,\sigma)$, and the standard deviation under the scaled compound binomial distribution, $v(n,\sigma)$. We

[49] We chose 100 thousand scenarios because this ensured that the sample variances were stable – the coefficient of variation of the portfolio default rate for a portfolio with a probability of default equal to 5% and an asset correlation coefficient equal to 10% was approximately 0.8%, which we considered low enough to ensure for credible conclusions. See Appendix 9.11 for more details.

further compare this to the variance under the scaled binomial distribution, $v(n, 0)$. We show these in Table 9.

The results show that the LHP approximation $v(\infty, \sigma)$ leads to a sizeable understatement of the standard deviation for smaller portfolios. Also, the scaled binomial $v(n, 0)$ understates the standard deviation for large portfolios. The compound binomial $v(n, \sigma)$ leads to the closest approximation to the variance of the portfolio. In Figure 12 we show the underprediction for the LHP approximation (calculated as $\frac{v(\infty,\sigma)}{\hat{v}(n,\sigma)} - 1$) and the scaled binomial (calculated as $\frac{v(n,\sigma)}{\hat{v}(n,\sigma)} - 1$) which further highlights the observation that $v(n, \sigma)$ is a better approximation for the variance than $v(\infty, \sigma)$.

LHP

$n = 100$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	-93%	-71%	-37%	-3%
5%	-82%	-45%	-19%	-2%
10%	-74%	-35%	-15%	-4%
15%	-70%	-31%	-12%	-2%
20%	-67%	-28%	-11%	-3%

$n = 1\ 000$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	-58%	-19%	-6%	0%
5%	-30%	-8%	-2%	-1%
10%	-22%	-5%	-1%	-1%
15%	-20%	-5%	-1%	-1%
20%	-17%	-4%	-1%	-1%

$n = 5\ 000$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	-21%	-4%	0%	1%
5%	-8%	-2%	0%	1%
10%	-6%	-3%	1%	-1%
15%	-5%	-1%	-1%	1%
20%	-4%	-1%	0%	1%

$n = 10\ 000$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	-12%	-2%	-2%	-3%
5%	-3%	-1%	1%	-1%
10%	-3%	0%	0%	1%
15%	-2%	-1%	0%	0%
20%	-2%	0%	1%	0%

Extended LHP

$n = 100$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	-1%	-1%	1%	4%
5%	-1%	0%	0%	2%
10%	0%	0%	-1%	-1%
15%	0%	0%	0%	1%
20%	0%	0%	0%	0%

$n = 1\ 000$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	0%	1%	-1%	0%
5%	1%	0%	0%	0%
10%	0%	0%	0%	0%
15%	-1%	0%	0%	0%
20%	0%	0%	0%	-1%

$n = 5\ 000$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	0%	0%	1%	2%
5%	0%	-1%	1%	1%
10%	-1%	-1%	1%	-1%
15%	0%	0%	-1%	1%
20%	0%	0%	0%	1%

$n = 10\ 000$

p \ ρ	**1%**	**5%**	**15%**	**50%**
1%	-1%	0%	-1%	-3%
5%	1%	0%	1%	-1%
10%	-1%	1%	0%	1%
15%	0%	-1%	1%	0%
20%	0%	0%	1%	0%

FIGURE 12: UNDERESTIMATION OF VARIANCE OF PORTFOLIO DEFAULT RATE

What we are ultimately interested in is how well the different models are able to estimate the distribution function of the portfolio default rate, particularly at the tails of the distribution. Therefore, in Figure 13 we show the simulated distribution function plotted alongside the distribution functions under the LHP model (Equation 4.7) and under the extended LHP (ELHP) model (Equation 4.29). We show this for a sample size of $n = 100$, and for different mean portfolio default rates p and asset correlation coefficients ρ.

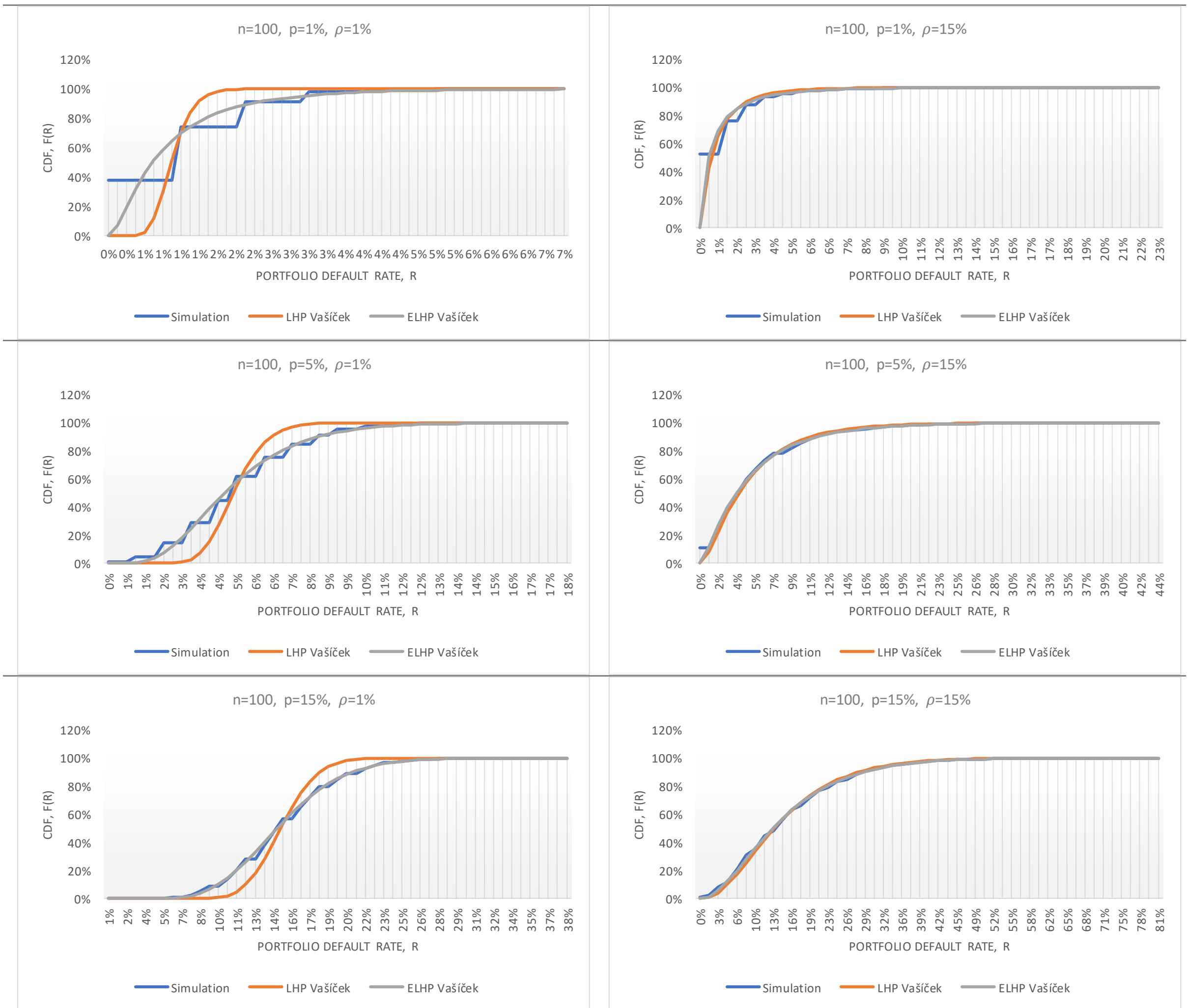

FIGURE 13: ABILITY OF LHP AND EXTENDED LHP MODELS TO ESTIMATE THE CDF IN SMALL PORTFOLIOS

The plot shows that the LHP assumption fits poorly for low levels of systemic risk, as measured by the asset correlation coefficient – with higher levels of systemic risk leading to better levels of fit. This is to be expected: as the total level of systemic risk increases, the proportion of total risk (the sum of systemic and diversifiable risk) explained by diversifiable risk decreases. Indeed, it is also the case that when diversifiable risk decreases (i.e., n increases), the level of fit will increase. This is shown in Figure 14. After all, the weakness of the LHP assumption, as discussed above, is that ignores diversifiable risk – not systemic risk – so that LHP model performs better when systemic risk contributes more to the total risk.

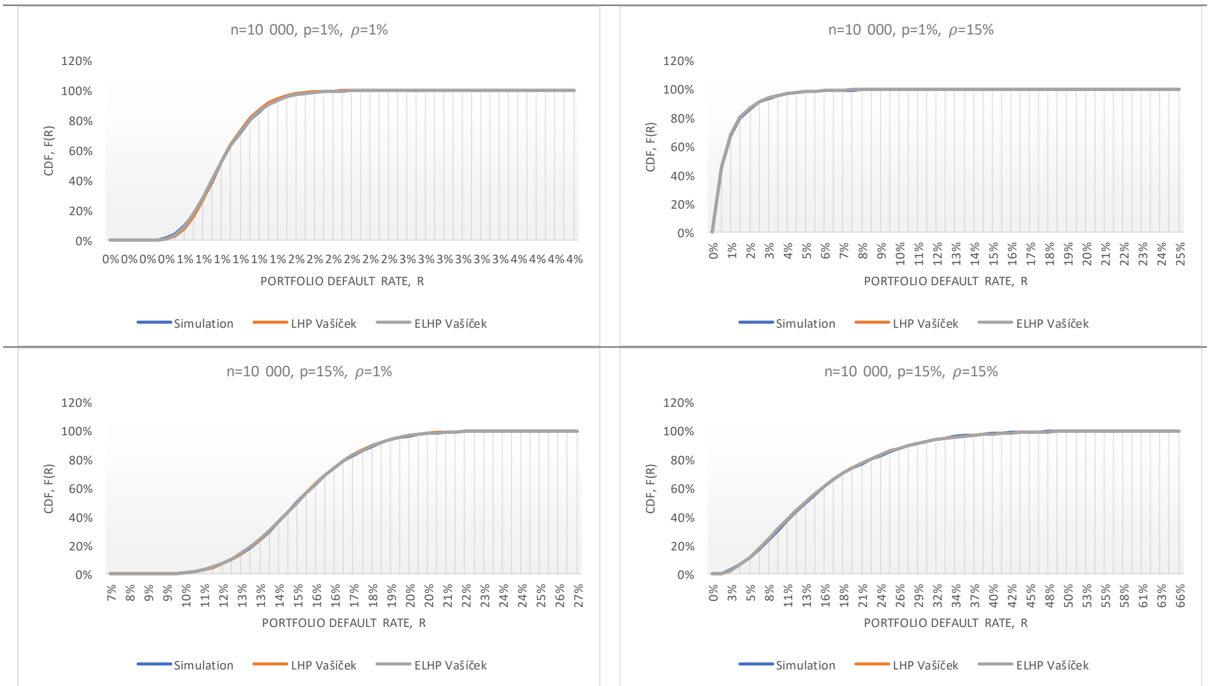

FIGURE 14:ABILITY OF LHP AND EXTENDED LHP MODELS TO ESTIMATE THE CDF IN LARGE PORTFOLIOS

Finally, note that in both Figure 13 and Figure 14, the extended LHP model provides a closer fit to the simulated data. This is most apparent for smaller portfolios, as expected from the theoretical discussion preceding this case study. Noting that, from Figure 13, the tendency if for the LHP model to underestimate the length of the tails of the distribution, the conclusion is that the LHP assumption can lead to an understatement of the portfolio value-at-risk – a problem that can be solved by the extended LHP assumption proposed above[50].

4.4 Summary of the Findings

This chapter elaborated on some of the shortcomings of the Basel III credit risk model, as discussed by Malwandla (2016). We showed the variance of a portfolio's loss rate can be decomposed into a systemic and a diversifiable component and that the LHP assumption essentially ignores the diversifiable component, which leads to an understatement of the capital requirement, all else equal. We then discussed an approach for improving the accuracy of the LHP approximation for smaller portfolios, by ensuring that the approximation does not understate the variance of the portfolio. We demonstrated that the extended LHP approximation leads to an improvement in the estimate of quantile function of the portfolio default rate, which is the main input into the Basel III capital formula for credit risk.

[50] A clear weakness of the extended LHP approximation proposed here is that it still retains the assumption that default rates follow a Vašíček distribution. To completely relax this assumption would require us to work with the compound binomial distribution, which is considerably less tractable. Nevertheless, two points should be noted in favour of this extension. Firstly, the extension improves upon the standard LHP approximation, which both assumes a Vašíček distribution and ignores sampling error, leading to an understatement of risk in smaller portfolio, as discussed above. Secondly, the extended LHP provides a better approximation for portfolios of all sizes.

An aspect of risk that we have not discussed in this chapter – something that is generally overlooked in literature and in practice – is the role of parameter estimation error in influencing the amount of capital required. Generally, a small portfolio (small n) will mean that the parameters of the model for probability of default (in this case, the EMV model) will be estimated on a small sample of loans – leading to more uncertainty about the parameter estimates themselves. Indeed, even in large portfolios, there can be significant parameter estimation error. Therefore, further research could investigate ways of addressing this source of uncertainty, either within the extended LHP framework described here or using a different framework altogether.

Chapter 5: The Gini Statistic as a Measure for Risk and Diversity

5.1 Overview of the Problem

Use of the Gini coefficient is commonplace in the study of the distribution of income and wealth. In recent decades it has also gained widespread adoption in the field of credit risk analysis and operations research (where it is referred to as the Gini statistic). In credit risk, it is used to measure the discriminatory power of a statistical model and is thus frequently use as a model selection criterion (see Mair, Reise, Bentler (2008))[51]. For instance, it was used in Chapter 3 as a criterion for selecting the model specifications for the exogenous maturity vintage (EMV) model. In this chapter, we add to the theory behind the Gini statistic, and show how to use in analysing the EMV model. Additionally, we extend its use to how a through-the-cycle (TTC) capital regime can lead to large differences in insolvency probability when risk is assessed on a point-in-time (PiT) basis.

The insolvency probability for lender (specifically, banks) is managed by holding capital. The amount of capital held is prescribed by the Basel III regime (Basel III, 2010). Under this regime, the amount of capital that a bank holds will be determined by a model for portfolio loss, where the capital is set to be enough to cover 99.5% of all loss scenarios across the credit risk cycle.

The fact that capital is held based on a through-the-cycle assessment of risk poses a challenge from a risk management perspective, as described by Malwandla (2016). While holding capital on a through-the-cycle basis is defensible on the basis that it makes for easier capital planning, it introduces the risk that the through-the-cycle capital assessment may be inaccurate if there is a structural change in the credit risk cycle. Furthermore, in this chapter we will show that at the bottom of the credit risk cycle, the probability of insolvency may increase well-beyond the 0.05% through-the-cycle benchmark targeted under Basel III. This means that for highly cyclical loan portfolios, the through-the-cycle approach may need to come with additional capital buffers in order to contain the solvency probability at the bottom of the cycle.

We set out to achieve two aims. Firstly, we derive a theoretical formula for the Gini statistic, which allows us to estimate the Gini statistic from the regression parameters of the probability of default model, i.e., without calculating it from observed defaults. We show that this is particularly convenient under the exogenous maturity vintage model. Secondly, we show that the Gini statistic can be applied to estimate the level of insolvency risk created by holding capital on a through-the-cycle basis.

[51] For instance, in Chapter 3 it is one of the measures that could have been used to choose between the different specifications of the extended EMV model.

5.2 The Gini Statistic

The Gini coefficient is largely used in economics literature to measure income inequality (Gini, 1912). It has a value of zero under conditions of perfect equality and a value of one under conditions of perfect inequality. Visually, it is often defined through the Lorenz (1905) curve, which we illustrate in Figure 15.

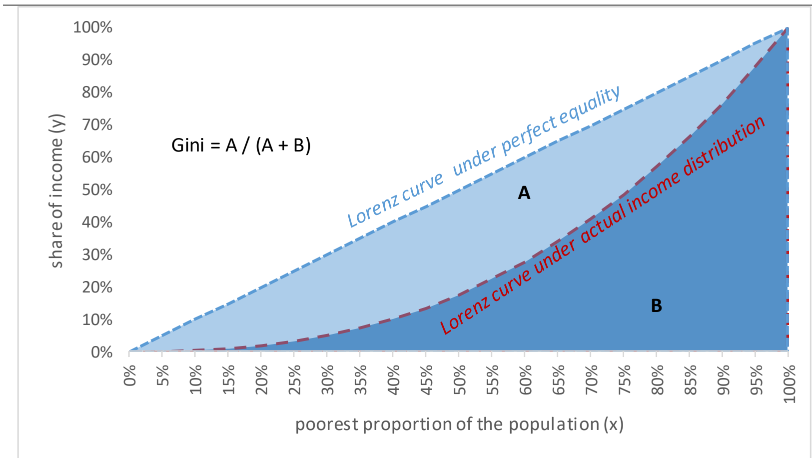

FIGURE 15: HYPOTHETICAL LORENZ CURVE OF INCOME DISTRIBUTION

The Lorenz curve represents the share of income going to the poorest $100x\%$ within the population, for a given x. In the graph above, we show two Lorenz curves – one under a hypothetical unequal income distribution and another under perfect income equality. The Gini statistic is thus defined as the area under the actual (or, in this case, hypothetical) curve divided by the area under perfect-equality curve. The area under the perfect-equality curve (given by area A plus area B in the graph) is equal to 0.5, since it is half the area of a unit square. Therefore, we can also calculate the Gini coefficient as one minus twice the area under the actual Lorenz curve:

$$Gini = 1 - 2B. \tag{5.1}$$

5.2.1 The Gini Statistics in Credit Analytics

In the domain of credit analytics, the Gini coefficient is defined in a similar manner, but often referred to as the Gini statistic – we adopt this distinction in this chapter, for disambiguation. In this domain, the Lorenz curve is defined to represent the proportion of defaults observed in the $100x\%$ least risky loans within a portfolio. Here, riskiness is defined according to a probability of default, or a similar measure. Therefore, the Gini statistic is an estimate of the ability to differentiate between good risks and bad risks through the use of a given risk measure.

There is a subtle conceptual difference between how the Lorenz curve is defined in these two domains. Let $L(x)$ be the Lorenz function. In economics, the input x into the Lorenz function is a measure of the poorest x according to income. The output of the curve is the share of income going to the poorest x. In other words, both the input and output are defined in terms of level of income. By contrast, in credit analysis, the input into the Lorenz function is the least risky x according to a (*expected)* probability of default, while the output is the proportion of default *observed* in the least risky x. Therefore, the credit risk analysis, there is a disconnect between input and output, i.e., the input is a measure of *expected* risk while the output is a measure of *observed* risk.

To the extent that income is measured objectively, the definition of the Lorenz curve in economics will be objective. However, in credit risk analysis, there is generally no single measure of *expected* default risk (i.e., probability of default) – there are many possible models that can be specified. Not only is the measure of risk subject to model design, it is also highly influenced by the data that is available to inform the model. For this reason, while the Gini coefficient can be said to measure the level of income dispersion within a population, the Gini statistic can only be said to measure the dispersion in risk *for a given model*, i.e., the Gini statistic cannot be said to measure the absolute level of dispersion in risk within the population. This is a consequence of the disconnect between input and output, as described above.

This challenge is difficult to resolve. Because the outcome being measured is binary (i.e., default versus non-default), it would not be very meaningful to have the input defined in terms of default outcome. Firstly, using the actual default outcome would ignore the fact that not everyone who default has the same propensity to default. Secondly, using the binary default outcome would mean that the Gini statistic does not provide a measure of the spread of risk within the portfolio, i.e., using a risk measure as the input into the Lorenz curve means that we can estimate the level of risk differentiation that can be achieved within the portfolio by differentiating using the risk measure under consideration. Measuring the level of risk differentiation is useful in many areas, such as when trying to estimate the scope for risk-based pricing in a portfolio. In Figure 16 we illustrate the Lorenz curve when the input is the default outcome (this is done for a portfolio with a 20% default observed default rate).

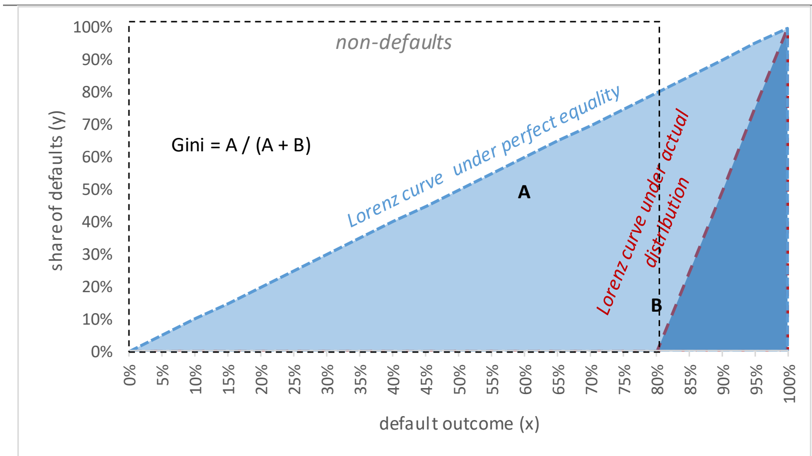

FIGURE 16: LORENZ CURVE FOR OBSERVED DEFAULTS USING OBSERVED DEFAULTS AS THE INPUT

Another way of trying to overcome the disconnect would be to change the output of the Lorenz curve to correspond with the input, i.e., define the curve such that the output $L(x)$ is the proportion of defaults *expected* (according to some probability of default model) in the least risky x within the population. However, this does not overcome the objectivity problem discussed above: different models, based on different sets of features and data, might potentially yield vastly different Gini statistics. Furthermore, this version of the Gini statistic would be based on the assumption that the probability model being used is an unbiased estimator of *actual* default rates for the given population. Mindful of these shortcomings, this is the approach we will take in this chapter. We illustrate this version of the Lorenz curve in Figure 16.

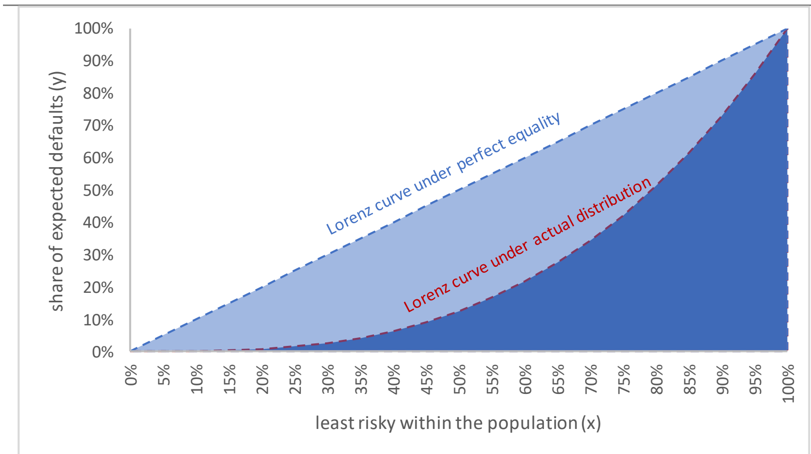

FIGURE 17: LORENZ CURVE FOR EXPECTED DEFAULTS USING EXPECTED DEFAULTS AS THE INPUT

In credit risk (specifically when assessing binary outcomes), the concept of the Gini statistic is identical to the concept of Somers' D, which is a measure of association in statistics (Somers, 1962).

5.2.2 Mathematical Specification of the Gini Statistic

In economics, the Gini coefficient can be deduced directly from the distribution of income, as discussed by Gastwirth (1972). Here we show that, in a similar manner, we can define the Gini statistics in credit analysis from the distribution of probability of default. Let $f(x)$ be the density function for the distribution of probability of default, so that $F(x)$ represents the proportion of the population with a probability of default less than or equal to x, where:

$$F(x) = \int_0^x f(u)du. \tag{5.2}$$

Therefore, the mean population probability of default is simply the expectation under the density function $f(x)$, as follows:

$$p = \int_0^1 uf(u)du. \tag{5.3}$$

Let $L(y)$ be the proportion of defaults expected from the least risky $100y\%$ within the portfolio, defined as:

$$L(y) = \frac{1}{p}\int_0^y uf(u)du, \tag{5.4}$$

i.e., $L(y)$ is the Lorenz curve for the portfolio. The Gini statistic for the portfolio is thus defined as:

$$G = 1 - 2\int_0^1 L(y)dy. \tag{5.5}$$

From Gastwirth (1972), we also have that the Gini statistic may be equivalently defined as:

$$G = \frac{1}{p}\int_0^1 F(y)[1 - F(y)]dy. \tag{5.6}$$

5.2.3 The Gini Statistic under the Vašíček Distribution

Let P be the probability of default for a randomly chosen loan in the portfolio. We assume that the probability of default is defined according to a probit model:

$$P = \Phi(m), \tag{5.7}$$

where m is the risk factor for the loan under consideration. We further assume that $m \sim N(\mu, \sigma)$. Consequently, the probability of default for loans within the portfolio follows a Vašíček (1987) distribution, with the following distribution function (the derivation of which was discussed in Chapter 3):

$$F(x) = \Phi\left(\frac{\Phi^{-1}(x)-\mu}{\sigma}\right). \tag{5.8}$$

The corresponding density function for the population is given by:

$$f(x) = \frac{\partial}{\partial x} F(x)$$

$$= \frac{\phi\left(\frac{\Phi^{-1}(x)-\mu}{\sigma}\right)}{\sigma\phi\left(\Phi^{-1}(x)\right)}. \tag{5.9}$$

The Lorenz curve for the population is thus given by:

$$L(y) = \int_0^y x \frac{\phi\left(\frac{\Phi^{-1}(x)-\mu}{\sigma}\right)}{\sigma\phi\left(\Phi^{-1}(x)\right)} dx. \tag{5.10}$$

This can be simplified through variable substitution, by letting $u = \frac{\Phi^{-1}(x)-\mu}{\sigma}$, so that $x = \Phi(\sigma u + \mu)$ and $du = \frac{dx}{\sigma\phi\left(\Phi^{-1}(x)\right)}$. The substitution produces:

$$L(y) = \int_{-\infty}^{\frac{\Phi^{-1}(y)-\mu}{\sigma}} \Phi(\sigma u + \mu)\Phi(u) du. \tag{5.11}$$

From Owen (1980), this integral resolves as:

$$L(y) = BvN\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\Phi^{-1}(y)-\mu}{\sigma}; \frac{-\mu}{\sqrt{1+\sigma^2}}\right), \tag{5.12}$$

where $BvN(x, y; \rho)$ is the cumulative density function of bi-variate normal distribution with correlation coefficient ρ, evaluated at (x, y).

From Gastwirth (1972), we define the Gini statistic as:

$$G = \frac{1}{P}\int_0^1 F(y)[1 - F(y)]dy$$

$$= \frac{1}{P}\int_0^1 \left[\Phi\left(\frac{\Phi^{-1}(y)-\mu}{\sigma}\right) - \Phi\left(\frac{\Phi^{-1}(y)-\mu}{\sigma}\right)^2\right] dy. \tag{5.13}$$

This can also be simplified through variable substitution, by letting $u = \Phi^{-1}(y)$, so that $y = \Phi(u)$, $du = \frac{dy}{\phi\left(\Phi^{-1}(y)\right)}$ and $dy = \phi(u)du$, leading to:

$$G = \frac{1}{P}\int_{-\infty}^{\infty} \left[\phi(u)\Phi\left(\frac{u-\mu}{\sigma}\right) - \phi(u)\Phi\left(\frac{u-\mu}{\sigma}\right)^2\right] du. \tag{5.14}$$

From Owen (1980), the first portion of the integral resolves to:

$$\int_{-\infty}^{\infty} \phi(u)\Phi\left(\frac{u-\mu}{\sigma}\right) du = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right), \tag{5.15}$$

and the second portion to:

$$\int_{-\infty}^{\infty} \phi(u)\Phi\left(\frac{u-\mu}{\sigma}\right)^2 du = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right) - 2T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\sigma}{\sqrt{2+\sigma^2}}\right), \tag{5.16}$$

where $T(h,a)$ is Owen's T function, which is defined as:

$$T(h,a) = \phi(h) \int_0^a \frac{\phi(hx)}{1+x^2} dx. \tag{5.17}$$

Substituting these into Equation 5.14, we have:

$$G = \frac{2}{P} T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\sigma}{\sqrt{2+\sigma^2}}\right). \tag{5.18}$$

Therefore, under the Vašíček distribution, the Gini statistic can be defined entirely in terms of the mean and variance parameters. In fact, even the mean probability of default P can be specified in terms of these:

$$P = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right). \tag{5.19}$$

Therefore, a complete formula for Gini statistic, in terms of μ and σ, under the probit model is[52]:

$$G = \frac{2}{\Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)} T\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\sigma}{\sqrt{2+\sigma^2}}\right). \tag{5.20}$$

The challenge with this formula is that Owen's T function has no closed form solution. However, Patefield & Tandy (2000) provide a few approaches for evaluating the function – the function can be approximated numerically.

5.2.4 Special Cases of the Gini Statistic

The derived formula for the Gini statistic has a few properties worth noting. Firstly, as $\sigma \to 0$, we have:

$$G \to \frac{2}{\Phi(\mu)} T(\mu, 0) = 0, \tag{5.21}$$

[52] We note that the formula is different from those derived in some literature. For instance, Řezáč & Řezáč (2011) derived a different formula for the Gini statistic. However, we note that the reason for the difference is that the underlying approach to measuring the risk is different. Řezáč & Řezáč (2011) start with a population of "Good" loans and a population of "Bad" loans, each with a separate distribution of risk scores. In our case, we consider only one population, in which each member of is subject to a risk of being "Good" or "Bad".

i.e., from Owen (1980), we have $T(\mu, 0) = 0$. Practically this means that if there is no heterogeneity in risk (i.e., every loan has the same default rate) then the Gini statistic will be zero, since the model will be unable to differentiate risk.

Another noteworthy case occurs as $\sigma \to \infty$, where we have:

$$G \to \frac{2}{\Phi(0)} T(0,1) = \frac{1}{2}. \tag{5.22}$$

However, since the probability of default also tends towards 50% as $\sigma \to \infty$, this not a very practical case. For a more useful case, we set $\mu = \tilde{\mu}\sqrt{1+\sigma^2}$, so that the Gini is given by:

$$G = \frac{2}{\Phi(\tilde{\mu})} T\left(\tilde{\mu}, \frac{\sigma}{\sqrt{2+\sigma^2}}\right). \tag{5.23}$$

Under this formulation, as $\sigma \to \infty$, we have:

$$G \to \frac{2}{\Phi(\tilde{\mu})} T(\tilde{\mu}, 1) = 1 - \Phi(\tilde{\mu}). \tag{5.24}$$

Note that setting $\mu = \tilde{\mu}\sqrt{1+\sigma^2}$ is equivalent to assuming that the mean $\Phi(\tilde{\mu})$ of the distribution is not a function of σ. This is a more useful case: for a portfolio where the loans have default probabilities that follow the Vašíček distribution, with an aggregate default rate of $\Phi(\tilde{\mu})$, the Gini statistic has an upper bound of $1 - \Phi(\tilde{\mu})$.

5.2.5 The Gini Statistic for the Exogenous Maturity Vintage Model

Under the extended exogenous maturity vintage model, the probability of default for loan k in a portfolio of n_s loans during period T is given by:

$$P_k = \Phi(U_k + e_T), \tag{5.25}$$

where U_k represents the loan-specific risk factors for loan k out of n_T loans and e_T represents period-specific risk factors during period T. The latter, e_T, is often taken to represent macroeconomic influences on default rates[53]. For a given period T we may treat e_T as a constant (although it will only be observable in retrospect), since it is the same for all loans within the portfolio. In order to apply the Gini statistic formula above, U_k needs to be normally distributed[54].

[53] In our discussion of the EMV model in Chapter 3, e_T represents the exogenous component.

[54] In practice, we can test this using a quantile-quantile plot for $\Phi^{-1}(P_k)$ for time T, for example.

Assuming that U_k is distributed normally, we can estimate the mean and variance parameters for input into the formula for the Gini statistic as follows:

$$\text{Mean: } \hat{\mu} = \frac{1}{n_T}\sum_{k=1}^{n_s} \Phi^{-1}(P_k), \tag{5.26}$$

$$\text{Standard deviation: } \hat{\sigma} = \sqrt{\sum_{k=1}^{n_T} \frac{(\Phi^{-1}(P_k)-\hat{\mu})^2}{n_T-1}}. \tag{5.27}$$

Under the EMV model, U_k may be specified as a regression formula, so that:

$$\Phi^{-1}(P_k) = \alpha + e_T + \sum_j \beta_j X_{kj}, \tag{5.28}$$

where $\{X_{kj}\}$ are loan-specific random variables. Consequently, the mean and variance may be alternatively specified as:

$$\text{Mean: } \hat{\mu} = (\alpha + e_T) + \sum_{j=1}^{p} \beta_{kj} m_j, \tag{5.29}$$

$$\text{Standard deviation: } \hat{\sigma} = \sqrt{\sum_{j=1}^{p} \beta_j s_j^2 + \sum_{j=1}^{p} \beta_j \beta_k s_j s_k \rho_{jk}}, \tag{5.30}$$

where m_j and s_j is the sample mean and standard deviation of X_j and ρ_{jk} is the correlation coefficient between X_k and X_j.

In both instances, the Gini statistic becomes:

$$G(e_T) = \frac{2}{\Phi\left(\frac{\hat{\mu}+e_S}{\sqrt{1+\hat{\sigma}^2}}\right)} T\left(\frac{\hat{\mu}+e_T}{\sqrt{1+\hat{\sigma}^2}}, \frac{\hat{\sigma}}{\sqrt{2+\hat{\sigma}^2}}\right). \tag{5.31}$$

There are a few points to note about e_T.

- In extended EMV model discussed in Chapter 3, e_T will generally be an unobservable random variable. This means that the Gini statistic will be a random variable in e_T.
- There are versions of the EMV model that do not include a latent variable, i.e., where $e_T = 0$. Under these models, the Gini statistic is deterministic.

In using Gini to measure discriminatory power, we are interested the case where U_k (in Equation 5.25) is the only random element in the model. However, if we assume that U_k is constant and study e_T instead, we can use the Gini statistic to measure the scope of differences in risk across the cycle. This is the aim in the remainder of the chapter.

5.3 Probability of Insolvency

The Gini statistic discussed above is an assessment of the default risk associated with loan-specific factors. It is also possible to use the Gini statistic as an assessment of the default risk associated with macroeconomic variables. In particular, we consider how the Gini statistic can be used to measure the risk of capital shortfall under the EMV model.

The default rate on a portfolio is calculated as follows:

$$R_T = \frac{1}{n_T}\sum_{j=1}^{n} D_{j,T}, \tag{5.32}$$

where n_T is the size of the portfolio and D_j is the default indicator on loan j within the portfolio during time T. In many capital models, the large homogenous portfolio (LHP) approximation is applied. This involves assuming that $n \to \infty$, so that

$$R_T = \Phi(\mu + \sigma e_T), \tag{5.33}$$

by the Law of Large numbers, where $\Phi(\mu + \sigma e_T)$ is a common probability of default across the portfolio. Here, μ represents the level of risk (which is assumed to be homogenous across loans in the portfolio and across time) and $e_T \sim N(0,1)$ represents exogenous risk. Some versions of the EMV model involve fitting a separate regression model for e_T, so that[55]:

$$e_T = r\bar{e}_T + \sqrt{1-r^2}\varepsilon_T, \tag{5.34}$$

where $\bar{e}_T$ is the fitted model, $\varepsilon_T \sim N(0,1)$ is the residual and r^2 is the coefficient of determination for the model. Therefore, the model may be equivalently written as[56]:

$$R_T = \Phi\left(\mu + \sigma r\bar{e}_T + \sigma\sqrt{1-r^2}\varepsilon_T\right). \tag{5.35}$$

There is a subtlety that should be noted in how e_T is perceived under point-in-time assessment of risk as compared to a through-the-cycle assessment. Under a through-the-cycle (TTC) assessment, T is treated as a random variable. Therefore, while $\bar{e}_T$ is a fitted model, it will be seen as a random variable since it fluctuates from one month to another, i.e., a TTC assessment is agnostic of T, and for an unknown T, $\bar{e}_T$ is unknown. Therefore, under a TTC assessment of risk $e_T \sim N(0,1)$, as in Equation 5.33. By contrast, when we perform a point-in-time (PiT) assessment of risk, we are conditioning on T, so that $\bar{e}_T$ can be treated as a constant, leaving ε_T as the only random component. Consequently, $e_T | T =$

[55] The derivation of this model was provided in Chapter 3.

[56] Notice that this is not unlike the form of the EMV model given by Equations 3.17 and 3.23, in Chapter 3.

$t \sim N\left(r\bar{e}_t, \sigma\sqrt{1-r^2}\right)$. In other words, a TTC assessment is interested in the unconditional distribution of risk across calendar time, while a PiT assessment is concerned with a conditional assessment of risk for a given calendar time.

This dichotomy[57] between PiT and TTC risk assessment also applies to the distribution of R_T. Under a TTC risk assessment, we do not condition on T, leading to the following distribution function for:

$$G_{TTC}(x) = Prob[R_T \leq x]$$

$$= \Phi\left(\frac{\Phi^{-1}(x)-\mu}{\sigma}\right). \tag{5.36}$$

Under a PiT assessment of risk, we condition on T, leading to the following distribution function:

$$G_{PiT}(x,t) = Prob[R_T \leq x | T = t]$$

$$= \Phi\left(\frac{\Phi^{-1}(x)-[\mu+\sigma r\bar{e}_T]}{\sigma\sqrt{1-r^2}}\right). \tag{5.37}$$

5.3.1 The Capital Buffer

The capital buffer is generally set to ensure that there is enough capital to withstand $100\alpha\%$ of all scenarios, for some chosen α. The capital buffer can be set either on a TTC basis or on a PiT basis. For the former, the EMV model can be used without fitting any macroeconomic variables to e_T, while the latter would require macroeconomic variables to be fitted. Under Basel III, $\alpha = 99.5\%$ and the capital requirement is set on a TTC basis. The capital requirement is calculated by inverting G_{TTC}, leading to the quantile function:

$$k_\alpha = \Phi(\Phi^{-1}(\alpha)\sigma + \mu). \tag{5.38}$$

Notice that this quantile function will not change with the stage of the cycle, i.e., it is independent of T. The PiT version is more dynamic, being the inverse of G_{PiT}:

$$l_{\alpha,T}(r) = \Phi\left(\Phi^{-1}(\alpha)\sqrt{1-r^2}\sigma + \mu + \sigma r\bar{e}_T\right), \tag{5.39}$$

i.e., through $\bar{e}_T$, the quantile will change as the cycle changes. Under Basel III, the capital requirement is set equal to:

[57] A characterisation of this dichotomy was discussed in Chapter 3. We merely offer a different version of the characterisation here to aid understanding – a version we believe better pre-empts section that follows.

$$c_\alpha = Q \times EAD \times LGD, \tag{5.40}$$

where Q is set equal to k_α, EAD is the exposure at default and LGD is the loss given default. The latter two variables are assumed to constants, which is why we only study the properties of Q. The coefficient of determination r^2 represents the extent to which the PiT model differs from the TTC model. Therefore, when $r^2 = 0$ (i.e., when there is no macroeconomic model) we have that $l_{\alpha,T}(r) = k_\alpha$.

Under a PiT assessment, the probability that the portfolio default rate R_T exceeds $l_{\alpha,T}$ will be $1-\alpha$, i.e.:

$$Prob[R_T > l_\alpha | T = t] = 1 - \alpha, \tag{5.41}$$

for any t, which is by definition. Consequently, the fact that k_α differs from $l_{\alpha,T}$ means that the probability that R_T exceeds k_α under a PiT assessment will differ from $1-\alpha$:

$$Prob[R_T > k_\alpha | T = t] = 1 - \alpha + \xi_t, \tag{5.42}$$

where ξ_t represents the imprecision created by setting capital requirements on a TTC basis and while insolvency risk on a PiT basis. By definition, the error term only falls away when we do not condition on T:

$$Prob[R_T > k_\alpha] = 1 - \alpha. \tag{5.43}$$

Therefore, we wish to evaluate $P[R_T > k_\alpha | T = t]$ more accurately:

$$\begin{aligned}
\pi_t &= Prob[R_T > k_\alpha | T = t] \\
&= Prob\left[\Phi\left(\mu + \sigma r \bar{e}_t + \sigma\sqrt{1-r^2}\varepsilon_t\right) > k_\alpha\right] \\
&= Prob\left[\varepsilon_T > \frac{\Phi^{-1}(k_\alpha) - \mu - \sigma r \bar{e}_t}{\sigma\sqrt{1-r^2}}\right] \\
&= 1 - \Phi\left(\frac{\mu + \sigma r \bar{e}_t - \Phi^{-1}(k_\alpha)}{\sigma\sqrt{1-r^2}}\right).
\end{aligned} \tag{5.44}$$

Substituting k_α, we obtain:

$$\pi_t = 1 - \Phi\left(\frac{r \bar{e}_t - \Phi^{-1}(\alpha)}{\sqrt{1-r^2}}\right). \tag{5.45}$$

Specifically, that π_T represents the probability of insolvency during period T, when capital is set on a TTC basis. We emphasise that the only reason this differs from $1-\alpha$ is that capital levels are set on a TTC basis while risk is being assessed on a PiT basis. We also emphasise the idea that coefficient of determination r^2 is a useful measure for the extent of difference that exists between the PiT assessment and the TTC assessment. Specifically, it characterises the explanatory power of the model fitted for $\bar{e}_T$. If the model is absolutely poor, so that $r^2 = 0$, notice that:

$$\pi_T = 1 - \alpha. \tag{5.46}$$

5.3.2 Measuring the Risk Associated with a Through-the-Cycle Capital Buffer

The use of the TTC capital buffer creates a risk management challenge in that for some periods π_T might be higher than an institution is willing to accept. There are at least a few ways of measuring the extent of risk created. One way is to simply calculate π_T for a few sampled values of $\bar{e}_T$. However, a way of summarising the risk across all levels of $\bar{e}_T$ is through the Gini statistic.

For this, we need to redefine the Lorenz curve. Let $L(x)$ represent the proportion of insolvencies observed in the $100x\%$ least severe economic scenarios. We illustrate this in Figure 18. Note here that the perfect-equality curve represents a case where the capital is determined on the same basis as risk is assessed.

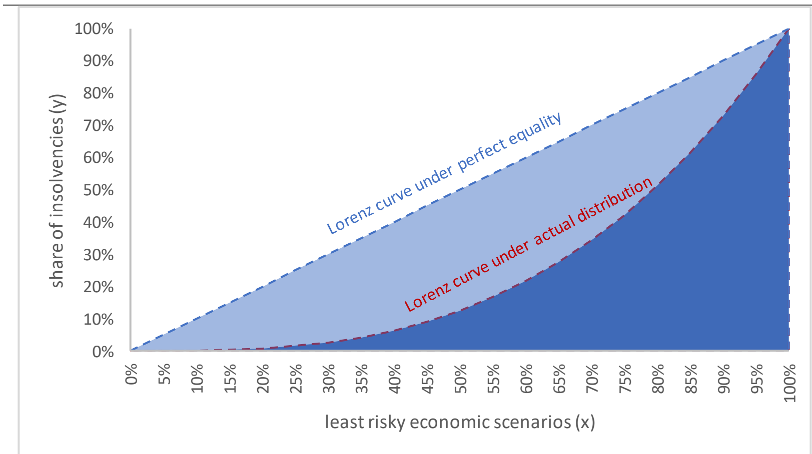

FIGURE 18: LORENZ CURVE FOR ECONOMIC SCENARIOS

Therefore, the Gini statistic associated with this Lorenz curve measures the risk created as the basis for measuring risk deviated from the basis for assessing capital. In particular, it measures the risk associated with having a TTC capital buffer as the PiT risk deviates significantly from the through-the-cycle average.

From Equation 5.45, notice that, for an unknown T, π_T follows a Vašíček distribution with location and scale parameters given by:

$$\mu = -\frac{\Phi^{-1}(\alpha)}{\sqrt{1-r^2}}; \sigma = \frac{r}{\sqrt{1-r^2}}. \tag{5.47}$$

Therefore, from Equation 5.20, the Gini statistic is given by:

$$G(r) = \frac{2}{1-\alpha} T\left(\Phi^{-1}(\alpha), \frac{r}{\sqrt{2-r^2}}\right). \tag{5.48}$$

This function has two obvious special cases. Firstly, when $r^2 = 0$ we have:

$$G(0) = 0, \tag{5.49}$$

i.e., when the fitted model is absolutely poor, there is not differentiation in the probability of insolvency, since $\pi_T = 1 - \alpha$ for all t. The second case is when $r^2 = 1$ (i.e., when the model fits perfectly), where we have:

$$G(1) = \alpha. \tag{5.50}$$

The way of interpreting this is that the larger the Gini statistic the more risk is created by setting capital on a TTC basis.

5.3.3 Point-in-Time vs Through-the-Cycle

The distinction between a TTC assessment of risk and a PiT assessment of risk is not a trivial one. The reality is that portfolio losses emerge on a PiT basis, not a TTC basis. For instance, if the losses over a one-year period (since the value-at-risk is over a one-year horizon) are higher than the available capital, the lender theoretically becomes insolvent – it does not matter whether the losses over the remainder of the cycle are more favourable.

The argument usually used to support a TTC regime for setting capital requirements is that it leads to better capital planning, since the capital requirement will not fluctuate with the cycle. However, we note that, in the same way that IFRS 9 requires lenders to incorporate macroeconomic forecasts into their assessment of losses, the capital regime can be designed to accommodate fluctuations in capital requirements in response to anticipated changes throughout the cycle. Nevertheless, regardless of the regime used to set the capital requirement, the fact that losses materialise on a PiT basis means that it is prudent to at least assess risk of insolvency on a point-in-time basis, even if the capital is set to TTC levels.

One conclusion that we draw from the preceding analysis is that two lenders that have the same TTC probability of insolvency may have two vastly different probabilities of insolvency at the bottom of the cycle, depending on how sensitive they are to the cycle. Additionally, if the down-phase of the cycle is protracted, relative to the up-phase, all lenders in the market will be exposed to a higher insolvency probability than budgeted for. This underpins the motivation for understanding the PiT risk posed by a TTC capital regime, as discussed above.

5.4 Case Study

We consider a case studies to demonstrate the use of the formulae developed above. We look at a simulation exercise to demonstrate how the insolvency probability can fluctuate over the course of the cycle.

Consider a loan portfolio with the following loss rate:

$$L_T = R_T \times EAD \times LGD, \tag{5.51}$$

where L_T is the loss rate during calendar month T and EAD and LGD are the exposure at default and loss given default parameters, respectively, which are assumed to be deterministic and known. We assume that R_T is a random variable defined as follows:

$$\begin{aligned} R_t &= \Phi(\mu + \sigma e_T) \\ &= \Phi\left(\mu + \sigma r \bar{e}_T + \sigma\sqrt{1 - r^2}\varepsilon_T\right), \end{aligned} \tag{5.52}$$

where μ is the mean parameter of the model, e_T is the systemic component, $\bar{e}_T$ is the deterministic (modelled) cyclical component of risk, ε_T is the random (unmodelled), σ is the sensitivity of the model to the cycle and r measures how much of the cycle has been modelled. Notice that these parameters correspond to those presented in the model in the previous section.

We require that for a randomly chosen $T = t$, the cyclical component e_T be normally distributed. We assume that the unmodelled component ε_T is normally distributed, which requires that the modelled component also be normally distributed. Since ε_T is unmodelled, we simulate its values by randomly sampling from a normal distribution. Simulating $\bar{e}_T$ is a bit trickier, since we require that it forms a cycle, i.e., over a short intervals of T, we require that $\bar{e}_T$ to have an approximately monotonous trend. Therefore, we simulate it as a normally distributed wave:

$$\theta_T \coloneqq m \times \Phi^{-1}\left(a \times \left(\frac{T}{w} - \left\lfloor \frac{T}{w} - \frac{1}{2} \right\rfloor\right) + \frac{1-a}{2}\right) + \sqrt{1 - m^2}\Phi^{-1}(u_T), \tag{5.53}$$

where w is the size of the cycle of θ_T (i.e., measured from trough to trough), a is the approximation accuracy (i.e., it truncates the cycle, since the full cycle would include $\Phi^{-1}(0) = -\infty$, and $\Phi^{-1}(1) = \infty$, which cannot be simulated), u_T is sampled from a $U(0,1)$ distribution and m is the amount of noise that exists within $\bar{e}_T$ (i.e., we assume that the cycle is not perfectly monotonic and includes some gaussian noise). In Figure 19 we show different simulated values for $\bar{e}_T$.

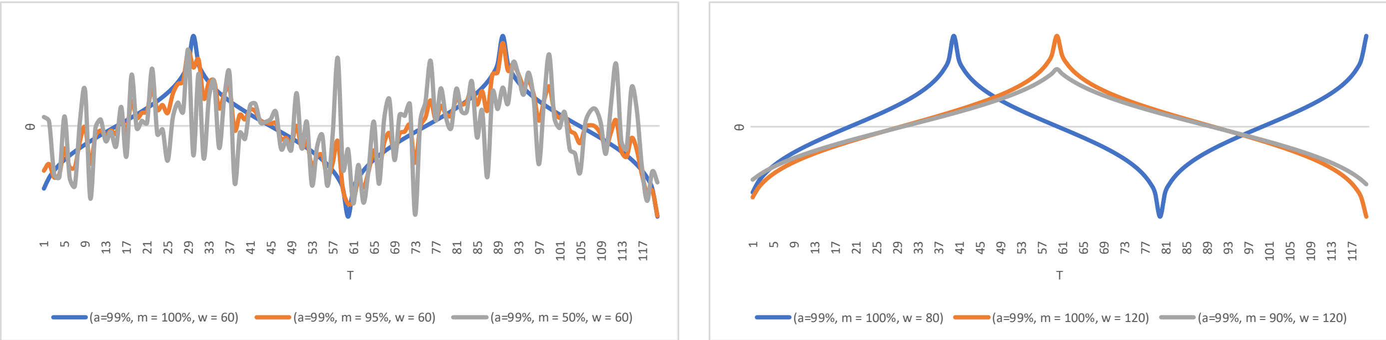

FIGURE 19: SIMULATED GAUSSIAN SYSTEMIC RISK FOR A CREDIT PORTFOLIO

Having simulated θ_T, ε_t is simulated by $\Phi^{-1}(\upsilon_T)$, where υ_T is sampled from a $U(0,1)$ distribution. We then calculate R_T through the formula in Equation 5.52, for a given μ, σ and r. In Figure 20 we show P_T for different values of μ, σ and r.

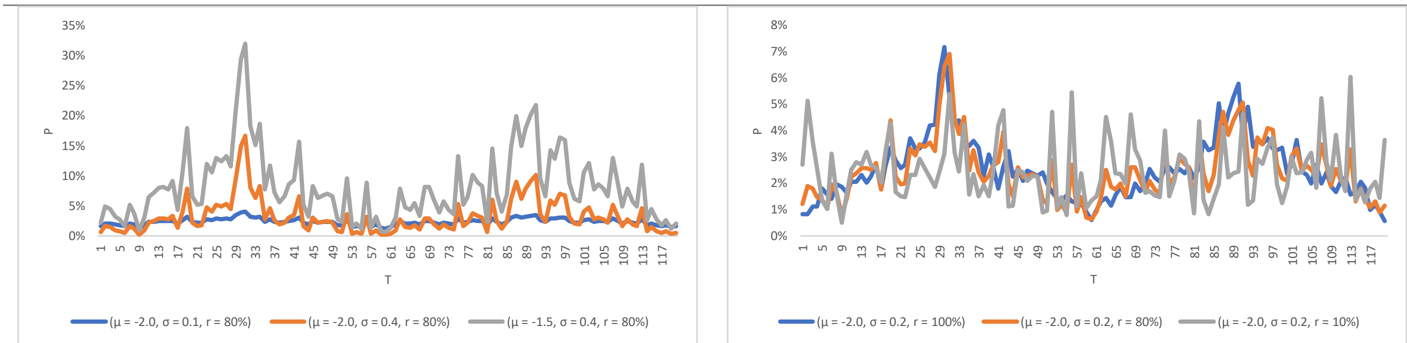

FIGURE 20: SIMULATED PORTFOLIO DEFAULT RATES UNDER DIFFERENT SCENARIOS

Using the formula in Equation 5.38, we can calculate the through-the-cycle capital requirement k_α required to maintain a $\alpha = 99.5\%$ probability of solvency. Similarly, we calculate the point-in-time equivalent $l_{\alpha,T}$ through Equation 5.39. We can further calculate a point-in-time assessment of insolvency under a through the cycle capital regime π_T using Equation 5.45. In Figure 21 we show these for two scenarios for μ, σ and r.

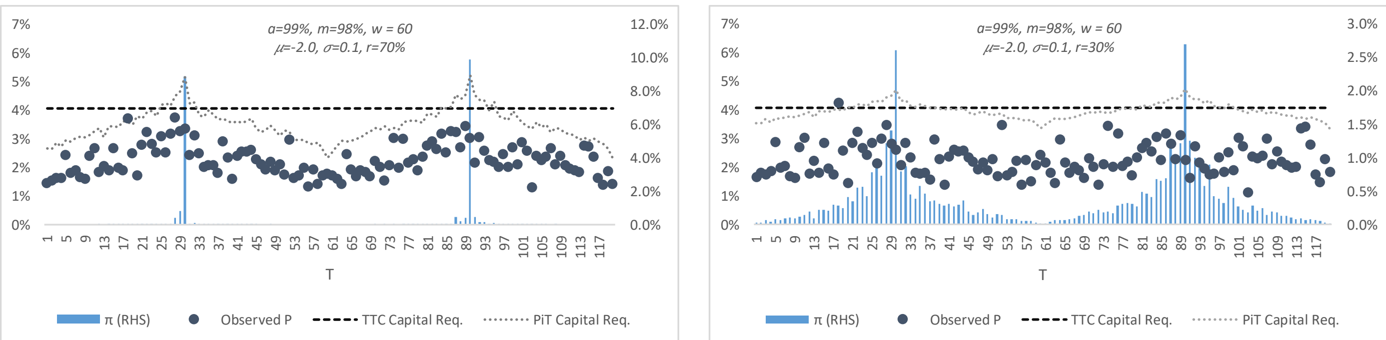

FIGURE 21: SIMULATED QUANTILES AND INSOLVENCY PROBABILITIES

From the graph above it is clear that the strength of the macroeconomic model fitted, as encapsulated in the coefficient of determination r, is what drives the insolvency probability estimate. This pattern can be summarised by calculating the Gini statistic for the probability of insolvency across the cycle, using Equation 5.48. We show this in Figure 22, where we have calculated the Gini statistic for different values of r.

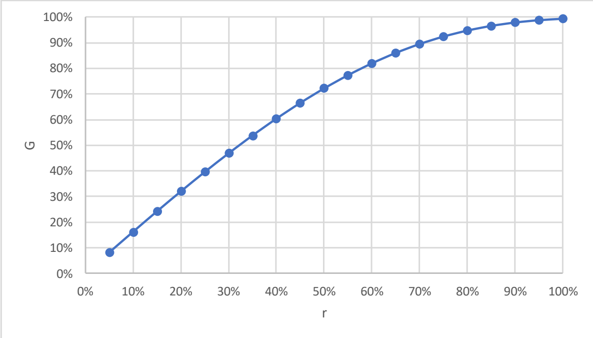

FIGURE 22: THE GINI STATISTIC MEASURING THE RISK POSED BY THROUGH-THE-CYCLE CAPITAL REQUIREMENTS

5.5 Summary of the Findings

In this chapter, we introduced and analysed a formula for estimating the theoretical Gini statistic for a portfolio of loans. This formula was studied further under the exogenous maturity vintage model, where we found that the Gini statistic is influenced by the mean and variances of the covariates of the model.

We derived a formula for analysing the point-in-time insolvency probability in a scenario where capital is held to maintain a through-the-cycle insolvency probability. We further showed that the Gini statistic can be used to measure the extent of insolvency risk that is created by holding capital on a point-in-time basis.

Chapter 6: The EMV Model for Survival Analysis

6.1 Overview of the Problem

Impairment modelling is one of the main focus areas within consumer credit risk analysis, alongside regulatory and economic capital modelling and credit scoring. In 2014 the International Accounting Standards Board published IFRS 9 (IFRS 9, 2014), a new accounting standard implemented in 2018. IFRS 9 prescribes the process of identifying loan impairments and estimating the expected loss on impaired and unimpaired loans, for the purpose of financial reporting.

In some ways, IFRS 9 represented a fundamental shift in the way impairment losses are estimated. Prior to IFRS 9, impairment provisioning was prescribed by IAS 39, which was based on an incurred loss philosophy (IAS 39, 2003). Under IAS 39, one only needed to raise impairment provisions with respect to loans where an impairment event (e.g., loss of employment) had already occurred, and the provision needed only to be sufficient to cover the specific loss event that had occurred (e.g., in the event that the borrower finds another job, the provision did not need to allow for the possibility of a subsequent loss of employment). However, the standard did not expect provisions to be held only when the lender had evidence that an impairment event had occurred, i.e., a lender would hold provisions for unidentified impairments, sometimes referred to as incurred but not reported (IBNR) or general provisions.

The perceived weakness of IAS 39 was that provisions did not anticipate potential losses from eminent or probable future events. For example, in a period where unemployment rates are expected to rise, one may expect to see higher impairment levels in the future, for which provisions can already be set aside. This weakness became most apparent during the Great Recession of 2008/09. IFRS 9 is thus seen as a remedy.

IFRS 9 is based on an expected loss philosophy, as opposed to an incurred loss philosophy. Firstly, the lender is expected to hold provisions on all credit facilities with a potential for loss. Where IAS 39 is based on the classification of loans into impaired and not impaired (although, in the case of IBNR, the classification is *fuzzy*), IFRS 9 requires loans to be classified into loans where significant increase in credit risk (SICR) has occurred and accounts where SICR has not occurred. For loans where SICR has not occurred, provisions are to reflect expected losses on all potential defaults that occur over a 12-month horizon. For loans where SICR has occurred, loan provisions are to reflect the expected losses on all potential defaults over the entire life of the loan. Therefore, at a minimum, provisions will cover the potential for loss over a 12-month horizon. Compare this to IAS 39, where, conceptually, no provisions were to be held with respect to loans where no impairment event has occurred. Furthermore, depending on the stringency of the definition for SICR adopted (which is not fully prescribed by the IFRS 9 standard), the provisions will start to increase in advance of the impairment event occurring (e.g., an anticipated rise in unemployment can already be used as a precursor for holding lifetime expected loss provisions on loans mostly likely to be affected by higher unemployment rates).

The concept of SICR and the estimation of lifetime expected loss provisions constitutes a shift in the way in which credit risk models tend to work. This chapter mainly focuses on modelling lifetime expected loss and does not address the challenges posed by SICR.

Loss modelling under Basel III (Basel III, 2010) and IAS 39 is generally based on the three-factor approach:

$$EL = PD \times EAD \times LGD, \tag{6.1}$$

where EL is the expected loss, PD is the probability of default, EAD is the expected exposure at default and LGD is the expected loss given default. For Basel III, the PD parameter is the probability of the loan defaulting over a 12-month horizon, while under IAS 39 the loss horizon was generally determined based on the nature of the portfolio, but was generally fixed for a given portfolio (or segment of the portfolio). Therefore, the common modelling approaches for the PD parameter were logistic regression (Bandyopadhyay, 2006) and decision trees (Safavian & Landgrebe, 1991) – see Thomas (2000) for a survey of the traditional approaches to credit scoring.

There were also more sophisticated approaches used to model PD, such as Cox regression (Cox, 1972) and neural networks (Yeh & Lien, 2009). Given that the remaining lifetime loans in a portfolio will generally differ, many of the approaches cited above no longer apply in their standard form. For example, logistic regression and decision trees are generally based on events that occur over a fixed horizon. Furthermore, in some cases (e.g., credit cards and overdraft), the remaining lifetime may follow a distribution of its own (in fact, even in mortgages and vehicle finance which generally have a fixed contractual term, there is the potential for early repayment which introduces randomness to the lifetime of the loan).

The second challenge with IFRS 9 is the fact that provisions for expected losses are expected to take anticipated economic conditions into account. In the strictest interpretation of the standard, this would mean that the PD model (assuming that a three-factor approach is adopted) is expected to take full time series of macroeconomic forecast as inputs, not just point estimates of macroeconomic variables at some fixed future horizon, for example. This requirement makes it difficult for even Cox regression and transition matrices (which can cope with variable event horizons) to meet the requirements for the PD model. This is because these cannot readily accommodate time-varying macroeconomic variables. While some accelerated lifetime models are able to accommodate time-varying covariates (e.g., Hernán, Cole, Margolick, Cohen, Robins (2005)), they do so with a lack flexibility (relative to semi-parametric proportional models).

This chapter therefore proposes an extension to the Cox regression framework through an extension of the exogenous maturity vintage model (EMV). The aim is for a simple survival model that can incorporate time-varying covariates. There are modelling techniques that currently allow for this. For instance, Zhang, Reinikainen, Adeleke, Pieterse, Groothuis-Oudshoorn (2018) as well as Austin (2011) apply Cox regression with time-varying covariates. The challenge with standard Cox regression with time-varying covariates is that it becomes less practical as the time-varying variables change with greater frequency.

There are also more novel techniques for survival analysis with time-varying covariates, including survival trees (Bou-Hamad, Larocque & Ben-Ameur, 2011) and threshold regression (Lee, Whitmore & Rosner, 2010), although some of these are become less tractable with the inclusion of time-varying variables. Some of these newer techniques can naturally handle multiple states, so that they can simultaneously deal with both default and attrition risk. Examples of these are the competition risk random forest model (Mogensen & Gerds, 2013) and the DeepHit multi-state model (Lee, Zame, Yoon & Schaar, 2018). Nevertheless, our aim was to discuss the potential for survival analysis with time-varying covariates under the EMV framework discussed in Chapter 3.

The EMV model is a form of logistic regression, where the inputs into the model are estimated via a decomposition of the data into a maturity component (representing the impact of account age on credit risk), a vintage component (representing the impact of the application date on credit risk) and an exogenous component (which is generally taken to represent the impact of macroeconomic conditions on credit risk). This model was the subject of Chapter 3. The obvious weakness of this model, for the purpose of IFRS 9 modelling, is the fact that it still a form of logistic regression, which is generally predicated on losses occurring over a fixed horizon. We thus extend the framework by introducing survival time as an additional dimension into the model. We adopt the three-component approach, where the PD component is modelled as a survival model with time-varying macroeconomic inputs. We show that the proposed model can be seen as an extension of Cox regression, where the baseline hazard is permitted to vary semi-parametrically. However, given that some loans may settle early, we model the probability of default through a competing risk framework, where a loan is modelled as being exposed to both the risk of default and the risk of closing.

The contribution to literature made in this chapter can be understood within three different contexts. In the credit risk modelling context, the chapter offers a viable model for implementing IFRS 9. In the broader mathematical statistical context, the chapter offers an extension to Cox regression, as described above. Concurrently, it may be seen as an extension to the standard EMV model into the survival time dimension. The rest of the chapter is organised as follows. Section 6.2 presents the theoretical framework for the proposed model. Section 6.3 demonstrates the use of the model using a portfolio unsecured loans issued by a South African bank and benchmarks the model's performance against Cox regression. Section 6.4 concludes with some implications of the model and areas for further research.

6.2 Proposed Model

The proposed methodology for developing a model to estimate IFRS 9 expected loss consists of three segments: a probability of default (PD) model, an expected exposure at default (EAD) model and an expected loss given default (LGD) model. This chapter details an approach for modelling the PD component only, which is perhaps the most analytically involved element of IFRS 9 implementation.

6.2.1 Lifetime Probability of Default

Consider a portfolio of loans observed at some calendar date s. Let $\boldsymbol{X}_{j,s} = \{X_{j,s,1}, \dots, X_{j,s,p}\}$ be the vector of p covariates for loan j in a portfolio of n loans. Let $T_{j,s}$ be the length of time loan j was observed before observation was terminated. Observation of a loan is terminated at the earliest of: (a) the first the point when the account moves into default; (b) the closure of the account; and (c) the point where the investigation ceases.

Let $D_{j,s}$ be the default indicator, indicating whether default is the reason for termination of observation:

$$D_{j,s} = \begin{cases} 1 & \text{if } \mathrm{T}_{\mathrm{j,s}} \text{ is the first time, since observation at time s, the loan enters default} \\ 0 & \text{otherwise} \end{cases}. \tag{6.2}$$

Let $C_{j,s}$ be the closure indicator, indicating whether the closure is the reason for termination of observation:

$$C_{j,s} = \begin{cases} 1 & \text{if } \mathrm{T}_{\mathrm{j,s}} \text{ is the time at which the account closes} \\ 0 & \text{otherwise} \end{cases}. \tag{6.3}$$

From Equation 6.2, notice that when $D_{j,s} = 0$ the survival time $T_{j,s}$ effectively corresponds to a censored observation. This censoring will either be a *genuine censoring* (i.e., corresponding to the end of the available data) or the closure of the account. In other words, when defining the default indicator, censorship corresponds to all survival times where default does not occur. Notice similarly, from Equation 6.3, that when $C_{j,s} = 0$ the survival time $T_{j,s}$ corresponds to either a genuine censoring or a default.

Let $\boldsymbol{Y}_s = \{Y_{s,1}, \dots, Y_{s,q}\}$ be a vector of q macroeconomic variables observable at time s. Finally, let $h_{j,s}(t)$ be the default hazard rate and $g_{j,s}(t)$ be the attrition hazard rate (which defines closure the risk of closure) for account j in the portfolio.

Using the competing risk approach, we use two simultaneous difference equations (see Stepanova & Thomas (2002)):

$$PD_{j,s}(t) = \sum_{k=1}^{t} h_{j,s}(k) S_{j,s}(k-1) \text{ and } S_{j,s}(k+1) = S_{j,s}(k) \times \left[1 - g_{j,s}(k) - h_{j,s}(k)\right], \tag{6.4}$$

where $S_{j,s}(k)$ is the probability of an account surviving both the risk of closure and the risk of default from calendar month s to calendar month $s+k$ and $PD_{j,s}(k)$ is the probability that an account observed in calendar month s has experienced its first default by calendar month $s+k$. These equations rely on the initial condition $S_{j,s}(0) = 1$, i.e., the survival function is equal to 1 at the point of observation. A third equation can be added to these in order to calculate the probability of closure:

$$PC_{j,s}(t) = \sum_{k=1}^{t} g_{j,s}(k) S_{j,s}(k-1), \tag{6.5}$$

where $PC_{j,s}(k)$ is the probability that an account observed in calendar month s has closed by calendar month $s+k$. For more discussion on this form of competing-risk survival analysis in banking, see Stepanova & Thomas (2002). In this chapter, we place more effort into modelling the default hazard than the attrition hazard, but note that the same approaches used for the former may be attempted for the latter.

6.2.2 A Review of Cox Regression

The most widely used survival analysis model for the type of data described above is Cox regression. This is a proportional hazards model that assumes that the default hazard for loan j at any horizon t is proportional to some baseline hazard curve, where the proportionality constant is obtained via a regression equation:

$$h_{j,s}(t) = b(t) e^{\sum_{k=1}^{p} \beta_k x_{s,j,k}}, \tag{6.6}$$

where $b(t)$ is the baseline default hazard function and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_p\}$ is a vector of p parameters. The parameter vector is estimated by maximising a partial likelihood function[58], which is independent of the

[58] A partial likelihood function exists when the likelihood function can be factorised. For example, consider a model with two sets of parameters $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. If the likelihood function can be specified as $L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = L_1(\boldsymbol{\theta}_1) L_2(\boldsymbol{\theta}_2)$ then L_1 and L_2 are both referred to as partial likelihood functions. The parameter estimates for $\boldsymbol{\theta}_1$ can be found by maximising L_1 and those for $\boldsymbol{\theta}_2$ by maximising L_2.

baseline hazard function (Cox, 1972). The baseline can thus be estimated non-parametrically in the context of the estimated parameter vector (Breslow, 1975).

The main weaknesses of the Cox regression model, for our purposes, are: (a) it does not allow for time varying covariates; and (b) it assumes a static baseline hazard function. The first weakness means that the proportionality constant is driven by the same set of variables for all survival times, i.e., as the definition implies, the proportionality constant is constant across all survival times. This is the reason why Equation 6.2 omitted the macroeconomic variables, since these are time-varying (however, we could have included them in the equation as static variables observed at survival time zero). The second weakness relates to the fact that the hazard function is not recalculated dynamically, instead being estimated within the context of the fitted parameters and thus on the same set of data used to fit estimate the parameters.

These two weaknesses are generally not problematic for most applications but become burdensome when modelling under IFRS 9. Recall that IFRS 9 requires the provisions to account for macroeconomic forecasts. Therefore, it would be ideal for the PD model to capture the full forecasted time series, not just the static values at the observation date. Secondly, we would expect the shape of the baseline hazard to reflect future economic conditions. For example, if we anticipate an increase in the base interest rate at horizon k (for products with variable interest rates), we would expect an increase in the baseline default hazard a few months after horizon k[59]. Therefore, as expectation of future macroeconomic variables change, the shape of the baseline default hazard should also change.

The last point above reveals the fact that the two weaknesses of the Cox regression model identified above are, to an extent, related. In the example above, we characterise an expected increase in interest rates as leading to a change in the baseline hazard function. An alternative approach for dealing with this situation, while retaining a constant baseline hazard, would be to simply include interest rates as a time-varying covariate into the model. This would have the effect of altering the proportionality factor applied to the baseline hazard to arrive at a final hazard, i.e., the proportionality *constant* would no longer be constant, but change with the forecasted level of interest rates. However, both these approaches are not easily achievable within the Cox regression framework. We thus derive an alternative model to address the challenges discussed above.

6.2.3 A Discrete-Time Survival Model with Time-Varying Covariates

The main elements of information that an IFRS 9 PD model should ideally account for are account-level behavioural data, macroeconomic data and structural information (by the latter, we mean

[59] An increase default rates would be expected since the cost of servicing a loan is higher when interest rates are high.

information that characterises a typical loan, independent of behaviour and economic environment). We thus propose a three-component model, consisting of:

- Baseline hazard: this component aims to capture the *distance to default*[60] component of a loan (for example, a loan that is up to date will generally have a different default pattern to a loan that is 30 days in arrears, since it will generally take an up-to-date loan longer to *reach* default).
- Behavioural risk index: this is a scorecard summarising the account-level data, insofar as it affects default risk (e.g., this could be based on a credit bureau score).
- Macroeconomic risk index: this is a time series index summarising the influence of the economic environment on default rates.

The model results in the following hazard function:

$$h_{j,s}(t) = f\left(b_t, \varphi_{j,s}, e_s\right), \qquad (6.7)$$

where b_t is the baseline hazard, $\varphi_{s,j}$ is the behavioural risk index, e_s is the macroeconomic index and f is the link function. In most of this chapter, we adopt an additive probit link function:

$$h_{j,s}(t) = \Phi\left(b_t + \varphi_{j,s} + e_s\right). \qquad (6.8)$$

6.2.3.1 Estimation of Model

From Equation 6.7, notice that each of the three components of the model vary across different dimensions: b_t varies across survival time, e_s varies across calendar time and $\varphi_{j,s}$ across loans. This poses a challenge in ensuring that the different components of the model are estimated independently from each other. Recall that in Cox regression the parameters can be estimated independently of the baseline, via the partial likelihood function (Breslow, 1975). This approach is no available here since the likelihood function cannot be factorised into partial likelihood functions[61]. We thus follow a binary decomposition approach, as used in the EMV model.

Although survival data does not have a binary outcome, the underlying hazard rate can be modelled as a binary outcome. The approach taken thus requires us to set up the data in a binary fashion. For each active account j in the dataset during time s we create $T_{j,s}$ observations, with $T_{j,s}$ being the length of time into the future that the account was observed before either defaulting or being censored. For each of the observations created, we also create four indicators:

[60] This component has some relationship with the notion of distance to default in the Merton (1974) model. This becomes more apparent when the model is translated into a survival analysis model, as per Malwandla (2016).
[61] If $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ represent two sets of parameters for the model, the likelihood function cannot be written in the form $L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = L_1(\boldsymbol{\theta}_1)L_2(\boldsymbol{\theta}_2)$.

- t: the time since observation, running from 0 to $T_{j,s}$.
- $d_{j,s,t}$: the default indicator, indicating whether account j defaulted at horizon t, i.e., whether account j defaulted during calendar time $s+t$, so that $d_{j,s,t} = \begin{cases} 0 & if\ D_{j,s} = 0\ or\ T_{j,s} \neq t \\ 1 & if\ D_{j,s} = 1\ and\ T_{j,s} = t \end{cases}$.
- u: the calendar month, which is $s+t$, representing the calendar month during which the loan was exposed to risk.
- $G_{j,s}$: the behavioural risk group to which account j belongs at the point of observation (according to a behavioural risk model, to be discussed below), which will be the same for all the records created[62].

An example of the creation of the data is provided in Figure 23[63].

Record ID	s	j	$T_{j,s}$	$D_{j,s}$	$G_{j,s}$
1	Jan-10	1	7	0	5
2	Jan-10	2	2	1	6
3	Jan-10	3	3	1	5
4	Feb-10	1	6	0	5
5	Feb-10	2	1	1	8
6	Feb-10	3	2	1	6
7	Mar-10	1	5	0	4
8	Mar-10	3	1	1	7

New Record ID	Old Record ID	s	u	j	t	$d_{j,s,t}$	$G_{j,s}$
1	1	Jan-10	Feb-10	1	1	0	5
2	1	Jan-10	Mar-10	1	2	0	5
3	1	Jan-10	Apr-10	1	3	0	5
4	1	Jan-10	May-10	1	4	0	5
5	1	Jan-10	Jun-10	1	5	0	5
6	1	Jan-10	Jul-10	1	6	0	5
7	1	Jan-10	Aug-10	1	7	0	5
8	2	Jan-10	Feb-10	2	1	0	6
9	2	Jan-10	Mar-10	2	2	1	6
10	3	Jan-10	Feb-10	3	1	0	5
11	3	Jan-10	Mar-10	3	2	0	5
12	3	Jan-10	Apr-10	3	3	1	5
13	4	Feb-10	Mar-10	1	1	0	5
14	4	Feb-10	Apr-10	1	2	0	5
15	4	Feb-10	May-10	1	3	0	5
16	4	Feb-10	Jun-10	1	4	0	5
17	4	Feb-10	Jul-10	1	5	0	5
18	4	Feb-10	Aug-10	1	6	0	5
19	5	Feb-10	Mar-10	2	1	1	8
20	6	Feb-10	Mar-10	3	1	0	6
21	6	Feb-10	Apr-10	3	2	1	6
22	7	Mar-10	Apr-10	1	1	0	4
23	7	Mar-10	May-10	1	2	0	4
24	7	Mar-10	Jun-10	1	3	0	4
25	7	Mar-10	Jul-10	1	4	0	4
26	7	Mar-10	Aug-10	1	5	0	4
27	8	Mar-10	Apr-10	3	1	1	7

FIGURE 23: ILLUSTRATION OF DATA SETUP FOR HAZARD MODEL

The $d_{j,s,t}$ column thus represents our target variable in the binary regression, with the covariates being t, $G_{j,s}$ and u. Averaging $d_{j,s,t}$ along t provides an estimate of how the baseline hazard varies by survival time, averaging $d_{j,s,t}$ along u provides an estimate of how the macroeconomic index varies by calendar time, while averaging along $G_{j,s}$ provides an estimate of how the behavioural risk index varies across risk groups. However, using simple univariate averages does not provide an accurate reflection as it does not control for the influence of the other variables. We thus adopt a decomposition approach, by simultaneously regressing all three dimensions against the $d_{j,s,t}$, to yield a point estimate of the effect that each dimension has on the default hazard while controlling for the other variables. This process is

[62] Note that we treat the risk group as a static variable, observed only at the point of observation. This is only because, in practice, it will be difficult to forecast the risk group that a loan will belong to in the future. If the borrower has a framework for forecasting risk grouping, in the same way as most banks have mechanisms for forecasting macroeconomic variables, the risk group can be allowed to vary with time, i.e., treated as time-varying variable.

[63] Notice that each record in the original table is treated as a separate observation, with a unique set of covariates and its own survival time. Each record in the new table represents a period during which the record in the original table was exposed to the risk of default. For example, averaging through the $d_{j,s,t}$ column in the new table provides an estimate of the monthly hazard rate. Also note that when $D_{j,s} = 0$ the survival time $T_{j,s}$ corresponds to either the closure of an account or the end of the observation period for the account

the same as the process followed in the EMV model – the only difference is the set of variables being regressed.

The log-likelihood function associated with this regression is as follows[64]:

$$l(\boldsymbol{b},\boldsymbol{e},\boldsymbol{\varphi}) = \sum_{j=1}^{n}\sum_{t=1}^{T_{j,s}}\left(d_{j,s,t} \times \ln\left[h_{j,s}(t)\right] + \left(1- d_{j,s,t}\right) \times \ln\left[1-h_{j,s}(t)\right]\right), \qquad (6.8)$$

where $h_{j,s}(t) = \Phi\left(\sum_h b_h \delta_{t=h} + \sum_g \varphi_g \delta_{G_{j,s}=g} + \sum_u e_u \delta_{s+t=u}\right)$ and δ is the indicator function. The parameters being estimated by maximising this log-likelihood function are $\boldsymbol{b} = \{b_1, b_2, \ldots\}$, $\boldsymbol{e} = \{e_1, e_2, \ldots\}$ and $\boldsymbol{\varphi} = \{\varphi_1, \varphi_2, \ldots\}$. Notice then that $\boldsymbol{b}$ is a vector of the baseline hazards, $\boldsymbol{e}$ is a vector of the macroeconomic index and $\boldsymbol{\varphi}$ is a vector of the behavioural risk index. We call the estimates for parameters $\boldsymbol{b}$, $\boldsymbol{e}$ and $\boldsymbol{\varphi}$ arising from this process *crude* estimates, denoted $\hat{\boldsymbol{b}} = \{\hat{b}_1, \hat{b}_2, \ldots\}$, $\hat{\boldsymbol{e}} = \{\hat{e}_1, \hat{e}_2, \ldots\}$ and $\hat{\boldsymbol{\varphi}} = \{\hat{\varphi}_1, \hat{\varphi}_2, \ldots\}$.

In other words, each of the model dimensions (u, t and G in Figure 23) are treated as dummy variables. The parameters along the u dimension (contained in $\boldsymbol{e}$) represent the macroeconomic risk index; the estimates along the t dimension (contained in $\boldsymbol{\varphi}$) represent the baseline hazard; and the parameters along the G dimension (contained in $\boldsymbol{b}$) represent the behavioural risk index. Also, notice that this is not much different from the maximum likelihood estimation described in Chapter 3.

Further note that other link functions are possible. The most important consideration in choosing a link function will be whether it satisfies the assumptions implied by the link function. For instance, using the *Coxian*[65] link function implies strict proportional hazards. Also note that, having obtained the crude estimates, it is possible to fit a regression to these in order to smooth and extrapolate each component. We discuss this below, where we go through each of the components in more detail.

6.2.3.2 Baseline Hazard

The baseline hazard is in many ways similar to the baseline hazard under Cox regression. In the context of modelling time-to-default, it is perhaps best understood as a measure of *distance-to-default*. This is especially evident when analysing consumer loans by delinquency. For example, under most definitions for default, an account that is up to date will generally take longer to reach default than an account that is 30 days in arrears.

This allows us to form an expectation for what the baseline hazard ought to look like. If we adopt a Merton-type interpretation of consumer credit loans (Merton, 1974), where we assume that default

[64] This is a standard log-likelihood function for binary regression.

[65] We define *Coxian* to mean a link function that resembles that which is used in Cox regression and reproduces some of the properties – such as proportionality. An example is defined in Equation 6.16.

occurs once the level of savings (as an analogue for net assets, in the case of a firm) fall below a particular threshold, we expect the distribution of time-to-default to be closely approximated by the inverse Gaussian distribution (Malwandla, 2016). Therefore, we would generally expect the baseline hazard for up-to-date loans resemble the inverse Gaussian hazard function:

$$b(t) = \frac{\frac{\alpha}{t}\phi\left(\frac{\mu t-\alpha}{\sqrt{t}}\right)}{1-\Phi\left(\frac{\mu t-\alpha}{\sqrt{t}}\right)-\Phi\left(\frac{-\mu t-\alpha}{\sqrt{t}}\right)e^{2\mu\alpha}}. \tag{6.9}$$

However, as will see in Section 6.3, loans that are in arrears tend to exhibit a different type of baseline hazard. The baseline hazard for such loans generally spikes at horizon $d-x$, where d is the default threshold in months and x is number of months in arrears. For example, if the default definition is 3 months in arrears then the hazard function for a loan that is 1 month past-due would spike at horizon 2, since that is how long it will take a typical loan to reach default. The reason why loans in arrears might not comply with the inverse Gaussian hazard function is that once loans are in arrears they may have a lower (often negative) *savings drift*[66] relative normal loans. Where this is not the case there will be a greater *curing* rate from arrears, and the hazard function will likely align closer with the inverse Gaussian process. The divergence from the inverse Gaussian hazard would also be caused by the fact that most default definitions are in terms of months in arrears – which means that default will generally occur only after $d-x$ months, regardless of the consumer's savings level.

The baseline hazard is, however, not meant to be influenced by behavioural data, macroeconomic data or survivorship bias, since these are to be captured by the other components of the model.

Actuarial graduation[67] can be applied to the estimated hazard function for smoothing and extrapolation:

$$\hat{b}_t \approx b(t), \tag{6.10}$$

where $b(t)$ is the graduation function, i.e., the baseline hazard applied in the final model would be:

$$b_t = b(t). \tag{6.11}$$

6.2.3.3 Behavioural Risk Index

The behavioural risk index is based on a behavioural scorecard that is designed to discriminate with respect to risk between different accounts within a portfolio. The index is thus meant to capture all account specific risk drivers, such as income, account age and delinquency on other loans. The

[66] Savings drift is defined by Malwandla (2016) within an adaptation of the Merton model for consumer loans. The savings drift determines consumer default in the same way that the drift in firm asset value defines corporate default in the Merton model, i.e., an account starts moving towards default when the accountholder's savings process becomes negative. The savings drift is simply the drift of the savings process.

[67] Actuarial graduation is the process of smoothing an empirical (crude) hazard function often by fitting a parametric curve, as described by Renshaw (1991).

behavioural risk index is analogous to the behavioural risk dimension of the EMV model described in Chapter 3.

The behavioural index is, however, not expected to capture any factors that are mainly driven by economic conditions (e.g., the interest rate being charged on a loan), since these will be introduced through the macroeconomic index, or by the delinquency on the specific loan under consideration, since this is accounted for in the baseline hazard. The behavioural scorecard can be fitted via conventional logistic regression. In practice, the behavioural scorecard underlying the index can be based on a credit bureau score, an internal behavioural score or even an application score.

In general, we can specify a behavioural scorecard as a regression formula:

$$v_{j,s} = z\left(\sum_{k=1}^{p} \beta_k X_{j,s,k}\right), \qquad (6.12)$$

where z is some link function. In order to define the behavioural risk index, we first define the risk groups arising from the behavioural scorecard:

$$G_{j,s} = g\left(v_{j,s}\right), \qquad (6.12)$$

where g is a function that maps each score $v_{j,s}$ into discrete risk groups, and $G_{j,s}$ is the risk group for account j during month s according to this grouping scheme. The behavioural risk index is therefore the default risk associated with a given group. We can thus rewrite $G_{j,s}$ as $\varphi_{G_{j,s}}$, i.e., φ_g is the behavioural risk index for accounts in risk group g.

6.2.3.4 Macroeconomic Risk Index

The macroeconomic risk index is a time series that represents the influence of economic conditions on the credit portfolio in a given month. It is analogous to the exogenous component of the EMV model described in Chapter 3. It is expected to only capture the influence of economic conditions and not, for example, the influence of portfolio composition, management action and policy changes.

Having estimated the crude macroeconomic index through Equation 6.8, we can apply a time series regression for smoothing and forecasting:

$$\hat{e}_s \approx \sum_{k=1}^{q} \beta_k Y_s, \qquad (6.14)$$

i.e., the index applied in final model would be[68]:

[68] Note that this regression formula does not have an error term since is purely used for smoothing. Including an error term in the regression would produce random effect on the model.

$$e_s = \sum_{k=1}^{q} \beta_k Y_s. \tag{6.15}$$

6.2.3.5 Link Function

The choice of link function is an important part of the fitting of the model. In this chapter, we assumed the probit link function given in Equation 6.8. However, alternatives such as the logit function and the complementary-log-log function are also viable. A noteworthy choice of link function is the following:

$$h_{j,s}(t) = \Phi(\alpha_1 b_t + \alpha_2 e_s) e^{\alpha_3 \varphi_{G_{j,s}}}. \tag{6.16}$$

Notice that this closely resembles the Cox regression model, where $b(t,s) = \Phi(\alpha_1 b_t + \alpha_2 e_s)$ is the baseline hazard and $e^{\alpha_3 \varphi_{G_{j,s}}}$ is the proportionality constant. The difference in this model is that the baseline will now be a function of the economic expectations contained in e_s as well as the default structure contained in b_t. This adds to the earlier comments made about the fact that a proportional hazard model with changing baseline can (depending on the chosen link function) be fashioned as a constant baseline proportional hazard model with time-varying covariates.

In Figure 24 we illustrate how the components of the model roll up to produce the final hazard function.

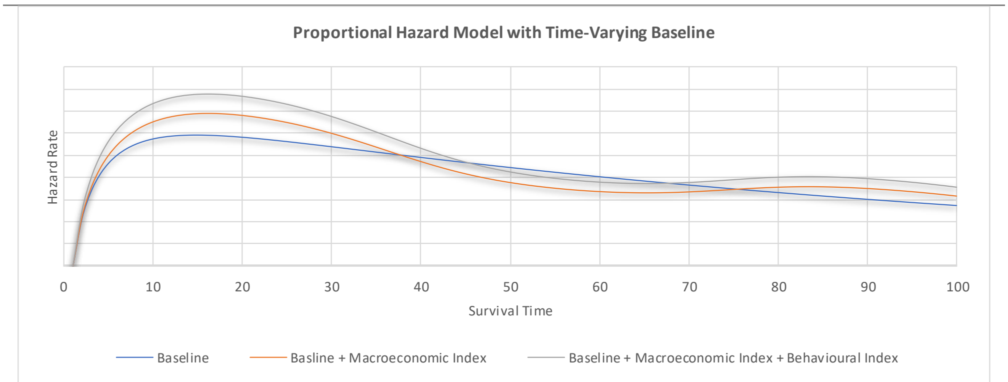

FIGURE 24: ILLUSTRATION OF PROPOSED MODEL

Given that we applied a graduation to the baseline hazard and a regression model to the macroeconomic index, which produced error terms, the model would need to be re-estimated using the graduated baseline and the macroeconomic index regression model as covariates, in place of the crude estimates. If this is not done (and the graduation and regression model are included in the original model, in place of the crude estimates) then the model essentially becomes a random effects model – so that an expectation would need to be taken in order to apply the most.

6.2.4 Attrition Model

We expect that fitting the attribution model will generally be simpler. What influences the attrition hazard will generally vary by type of product. For fixed-term amortising products such as mortgages, vehicle finance and amortising personal loans we would expect the attrition rate to be largely driven by the remaining term on the product (i.e., the closer an account is to the end of the term, the more likely it is to be settled). For revolving and transactional loan products, where there is generally no fixed term, the attrition rate will generally be more difficult to model, but may be influenced by factors such as the age of the loan or of the borrower.

There are different ways of thinking about the role of an attrition hazard model. In certain cases, it may be completely reasonable to use the remaining term on the loan as the loan's lifetime, which means that the attrition model is no longer required. This treatment is equivalent to having an attrition hazard equal to 100% at the end of the contractual term of the loan, and equal to 0% elsewhere. In this case, the PD model would be said to be based on contractual lifetime. What is described in the previous paragraph is what is called behavioural lifetime, as it models the actual repayment patterns observed on an account.

On certain products, the behavioural lifetime will be lower than the contractual lifetime. This may be the case, for instance, on mortgage loan or vehicle finance portfolios that experience a large number of early settlements. Therefore, the choice between contractual and behavioural term may lead to significant differences in results. Also note here that if contractual term is chosen, the PD model might not validate against observed data since contractual term might grossly overstate the amount of time accounts are exposed to the risk of default, i.e., the choice of contractual term might be based on a policy decision, in the interest of conservatism, rather than an attempt to optimise fitness.

The discussion of attrition is further complicated by a question of what constitutes a lifetime. For example, in cases where a term extension or re-advance is possible, should the attrition model only account for the lifetime under the prevailing loan terms, or must the implied lifetime be lengthened to allow for the possibility of further term extension? In a credit card portfolio, does a limit increase represent the inception of a new loan?

In practice, the choice between contractual and behavioural definitions of lifetime, and the question of what constitutes lifetime, will be answered by prevailing internal policy, external auditors and local regulations. In this thesis we do not focus on such policy considerations, but on the estimation of the hazard function itself, for a given definition of lifetime. Moreover, we only discuss behavioural lifetime, since an attrition model for contractual lifetime is generally trivial, as explained above.

In the above, we only mentioned the product-specific differentiators of attrition. There are many other possible differentiators. The economic environment can have an influence on attrition since it influences the extent to which borrowers can afford an early settlement or the reliance of consumers on overdrafts and credit cards. Borrower-specific factors may also driver attrition: for example, an accountholder that never uses his/her credit card may be more likely to close it than an accountholder that uses it regularly.

From a parameter estimation perspective, the attrition hazard model is not much different from the default hazard model described above. The main difference is that instead of modelling the time-to-default as we were in the previous instance, we model the time-to-closure. The dimensions chosen to include in the model will depend on all the factors highlighted above. For instance, if it is believed that the economic environment has an influence on attrition, the macroeconomic index can be included as a dimension of the model. Below we give an example of how to estimate the attrition model based on remaining term and a macroeconomic risk index:

$$g_{j,s}(t) = \Phi\left(r_{l_j-t} + \epsilon_s\right), \tag{6.17}$$

where l_j is the remaining contractual lifetime for account j as at calendar month s, r_{l_j-t} is the effect for remaining lifetime and ϵ_s is the macroeconomic risk index.

We start with the same data transformation described for the PD model. For each active account j in the dataset during time s we create $T_{j,s}$ observations, with $T_{j,s}$ being the length of time into the future that the account will was observed before either closing or being censored (e.g., censored through default). Let $R_{j,s}$ be the remaining term on account j during time s. For each of the observations created, we also create four indicators:

- t: the time since observation, running from 0 to $T_{j,s}$.
- $c_{j,s,t}$: the closure indicator, indicating whether account j closed at horizon t, i.e., whether account j closed during calendar time $s+t$, so that $c_{j,s,t} = \begin{cases} 0 & if\ C_{j,s} = 0\ or\ T_{j,s} \neq t \\ 1 & if\ C_{j,s} = 1\ and\ T_{j,s} = t \end{cases}$.
- u: the calendar month, which is $s+t$.
- r: the remaining term, which is $R_{j,s} - t$.

Notice that except for the choice of variables, this process is identical to the processed described for the PD model, as illustrated in Figure 23.

Similarly, the log-likelihood function associated with this regression is as follows:

$$l(\boldsymbol{r}, \boldsymbol{e}) = \sum_{j=1}^{n} \sum_{t=1}^{T_{j,s}} \left(c_{j,s,t} \times \ln\left[g_{j,s}(t)\right] + \left(1 - c_{j,s,t}\right) \times \ln\left[1 - g_{j,s}(t)\right]\right), \tag{6.18}$$

where $g_{j,s}(t) = \Phi\left(\sum_h r_h \delta_{R_{j,s}-t=h} + \sum_u \epsilon_u \delta_{s+t=u}\right)$. The parameters being optimised by this function are $\boldsymbol{r} = \{r_1, r_2, \dots\}$ and $\boldsymbol{e} = \{\epsilon_1, \epsilon_2, \dots\}$. In this model, $\boldsymbol{r}$ is a vector of the factors representing attrition differences by remaining term and $\boldsymbol{e}$ is a vector of the macroeconomic index. For example, the attrition rate of a loan with remaining term h during calendar month s will be given by $\Phi(r_h + \epsilon_u)$.

6.3 Case Study

The model was applied to a sample from a portfolio of personal loans issued by a South African bank. For this study, we adopted the 90-day default definition recommended under Basel III and IFRS 9.

The sample consisted of 2 476 000 observations of performing accounts (i.e., accounts that are not in default), observed between September 2005 and June 2014. The portfolio was segmented into four cycles: (a) Cycle 0: accounts current with payments, (b) Cycle 1: accounts in arrears by less than 1 payment, (c) Cycle 2: accounts in arrears by at least 1 payment but less than 2 payments, (c) Cycle 3: accounts in arrears at least 2 payments but less 3 payments. More details on the sample as provided in Appendix 9.12.

6.3.1 Fitting the Proposed Model

The default hazard model was fitted with an additive probit link function, and the three components were estimated via maximum likelihood – likewise for the attrition hazard model (detailed parameter estimates are can be found in Appendix 9.7).

6.3.1.1 Default Hazard

In Figure 25, we show the estimated baseline hazard function from the model for each of the three cycles. The baseline for Cycle 0 was graduated using a time-scaled inverse Gaussian hazard function, while the rest were fitted using a power function coupled with point estimates. Note that in the graph we show $\Phi(b_t)$ instead of b_t – we also graduated $\Phi(b_t)$ to ensure that the graduated values were positive.

What is worth noting about the baseline hazard is how the shape varies for the different cycles. This characterises the distance to default, as mentioned in Section 6.2.3.2. For example, accounts that are observed in Cycle 2 experience a spike in default in the second month since observation, owing to the fact that it will take a typical account 2 months to reach the three-payment default threshold.

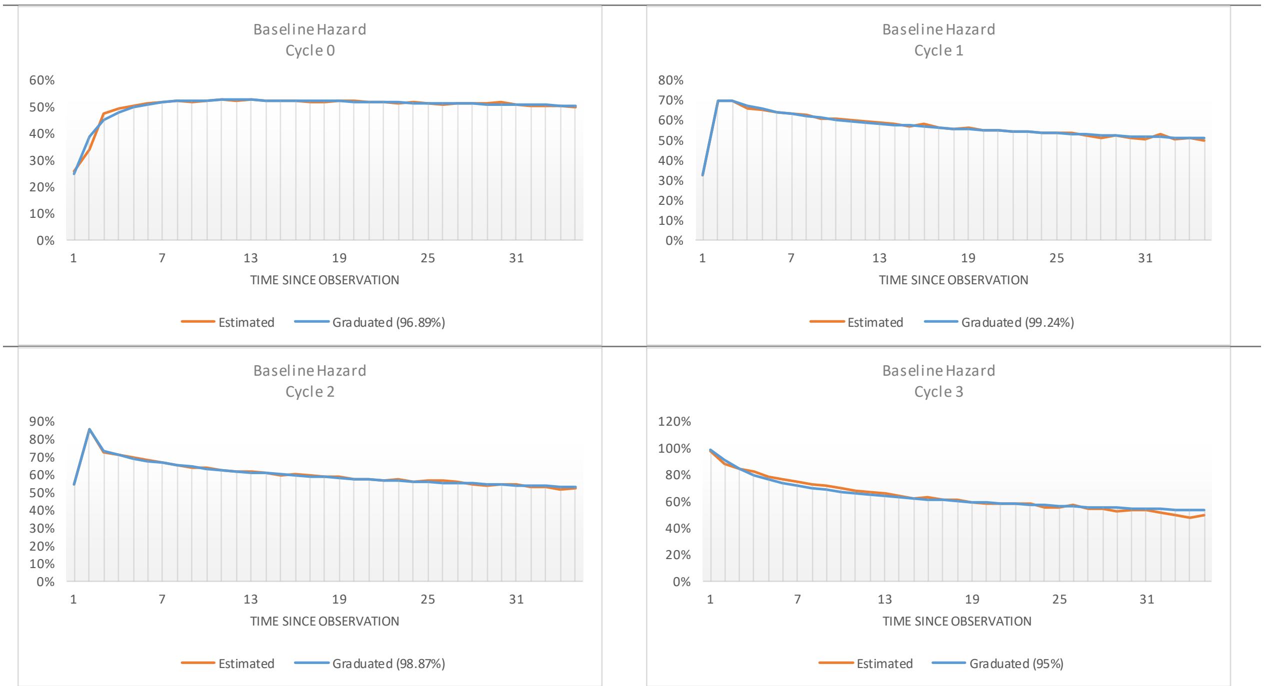

FIGURE 25: BASELINE HAZARD FUNCTIONS

Secondly, we look at the behavioural risk index as estimated for each cycle. This is given in Figure 26 below (see Appendix 9.13 for more details on the scorecard underlying the risk index). What is interesting to note with the estimated index is its range for the different cycles. Cycle 0 has the widest range, signifying greater risk differentiation within the sub-population. The range decreases with each cycle, leaving Cycle 3 with the least differentiation. The intuition behind this is that as an account rolls further into arrears, the power of the behavioural scorecard to differentiate between risks will reduce, i.e., this is what is sometimes referred to as *restriction of range* (Yamamoto, 1965). In other words, the difference in the range of the risk index informs us of the discriminatory power that the PD model will have for the different cycles.

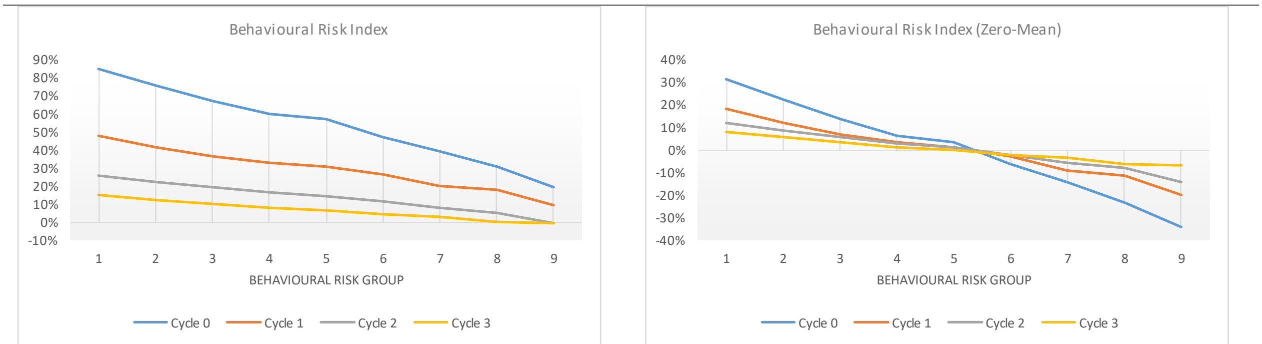

FIGURE 26: BEHAVIOURAL RISK INDEX

The macroeconomic index was estimated for the four cycles. These are shown in Figure 27 below. Our observation on these estimates is that they are correlated for the different cycles, which is what we would expect: all cycles are exposed to the same macroeconomic and thus the indexes should move in synchrony. Secondly, the macroeconomic indexes are quite volatile. Although part of the volatility could be assumed to be driven by true macroeconomic volatility, it is also possible that the volatility arises out of estimation error.

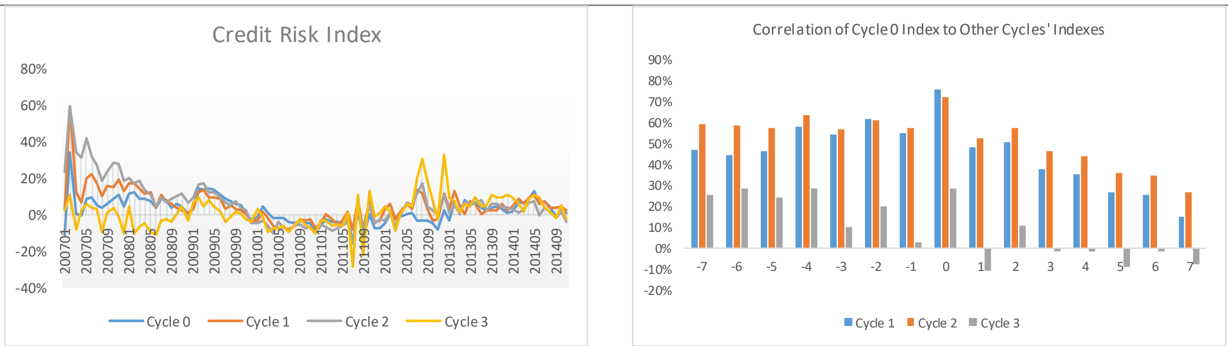

FIGURE 27: MACROECONOMIC RISK INDEX

Finally, ignoring the volatility, we see that the trend in the indexes corresponds to observed macroeconomic fluctuations. For example, in the period following 2008, the South African reserve bank cut interest rates considerably while banks began to implement stricter lending policies to curb the effects of the downturn. Both of these likely explain the decline in the indexes over this period.

Variable Name	Description	Mean	Std.Dev	Min.	Max.
Prime	The Prime Overdraft Rate, which is the benchmark lending rate for South African banks.	10.57%	2.05%	8.5%	15.5%
GVT10Yield	The yield on the 10-year South African government bond, as an early indicator of interest rates.	8.36%	0.61%	6.96%	10.35%
CPI	The growth in the Consumer Price Index, as an indicator for price inflation.	5.25%	1.6%	0.4%	8.7%
GDP	The growth in nominal gross domestic product, as an indicator for aggregate economic activity.	9.25%	3.16%	4.15%	15.98%
SavingsToGDP	The ratio of household savings to GDP, as an indicator of the level of savings within the economy.	16.45%	1.28%	14.4%	18.90%
ConsumptionToGDP	The ratio of household consumption to GDP, as an indicator of the level of consumer spending.	60.17%	0.98%	58.2%	62.3%
CompensationToGDP	The ratio of total employee compensation to GDP.	50.93%	2.05%	47.6%	54.4%
DebtToIncome	The ratio of household debt to disposable income, as an indicator of the level of indebtedness.	79.8%	3.17%	72.3%	87.8%
Unemployment	The unemployment rate, as an indicator of economic conditions and the financial condition of the consumer.	24.4%	1.54%	21%	27.7%

TABLE 10: MACROECONOMIC VARIABLES

A regression model was fitted to the crude macroeconomic index for the different cycles, for the model to be applicable prospectively. The variables we considered for the model are summarised in the following table[69].

[69] Variables were sourced from the South African Reserve Bank's quarterly bulletin.

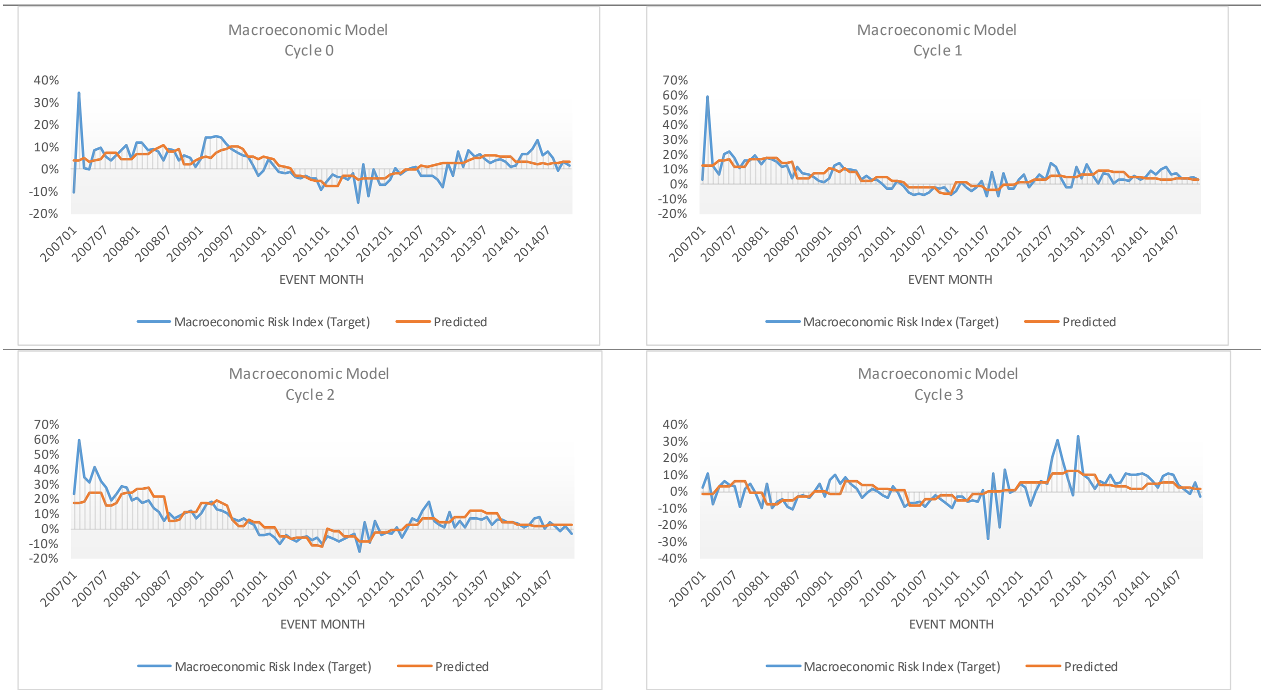

FIGURE 28: MACROECONOMIC RISK INDEX MODEL

Figure 28 below shows the estimated macroeconomic index and the macroeconomic regression used to smooth and forecast this index (see Appendix 9.13 for more information on the modelling process followed).

Model	R-Square	RMSE	Variable	Estimate	Standard Error	P-Value	Variance Inflation Factor
Cycle 0	**40.17%**	**5.51%**	Intercept	-2.04522	0.33443	0.00%	0.00000
			CPI	0.01912	0.00406	0.00%	1.00743
			ConsumptionToGDP	0.03264	0.00556	0.00%	1.00743
Cycle 1	**44.39%**	**6.82%**	Intercept	-3.02841	0.44050	0.00%	0.00000
			Prime	0.01046	0.00307	0.10%	1.08119
			ConsumptionToGDP	0.04930	0.00744	0.00%	1.08119
Cycle 2	**65.02%**	**7.41%**	Intercept	-4.94748	0.47290	0.00%	0.00000
			Prime	0.01999	0.00328	0.00%	1.04688
			ConsumptionToGDP	0.07962	0.00795	0.00%	1.04688
Cycle 3	**32.15%**	**7.45%**	Intercept	-3.01481	0.53193	0.00%	0.00000
			mev_GDP	-0.01219	0.00246	0.00%	1.14080
			mev_ConsumptiontoGDP	0.05237	0.00898	0.00%	1.14080
Attrition	**88.39%**	**4.16%**	Prime	0.36556	0.02033	0.00%	0.00000
			DebtToIncome	-0.04862	0.00183	0.00%	1.00000

TABLE 11: FIT STATISTICS FOR MACROECONOMIC MODELS

Table 11 provides the fitting statistics for these macroeconomic models. All four models have ConsumptionToGDP as a covariate, with two models also featuring Prime. This is owed to the fact the indices are correlated, being generated by the same credit portfolio.

6.3.1.2 Attrition Hazard

In Figure 29 we show the remaining term component, along with the graduation function. As one would expect, the likelihood of attrition increases as the account approaches the end of its term, reaching its peak at the end of the term. There are accounts that stay on book post their original term – due to term extensions and delinquency. It is due to the latter reason that we would expect attrition rates to remain high in the period after the expiry of the original term. However, for term extensions the attrition rate will eventually decrease, depending on the typical length of term extensions.

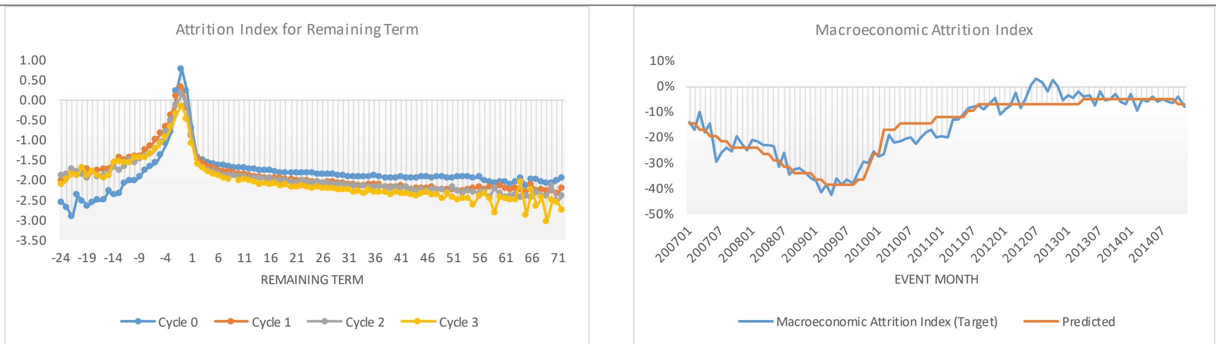

FIGURE 29: ATTRITION MODEL COMPONENTS

Figure 29 also shows the estimated macroeconomic risk index. Notice that the attrition macroeconomic index moves in an opposite direction to the default macroeconomic index. This is intuitive, since in periods of economic prosperity defaults would be expected to be low while pre-payments would likely be high (and vice versa). The period of lowest attrition rates thus corresponds with an economic downturn. In Figure 29, we also see the fitted macroeconomic model for the index. The model fitting details are given in Table 11.

We note that the attrition hazard model fitted in this case study is much simpler than the default hazard model. In practice, the attrition hazard model may be as complex as the default hazard model – consisting of a term structure component, a behavioural component and a macroeconomic component. For instance, there are a number of examples in literature of more elaborate models for attrition, including Marimo & Chimedza (2017) who model prepayment in vehicle finance, Ambrose & LaCour-Little (2001), who model prepayment risk in adjustable-rate mortgages and Hayre (2003), who studies prepayment in Dutch mortgages. The many factors that may be predictive of early repayment (such as loan-to-value ratios in mortgages or age of vehicle in vehicle finance) may be used as inputs into an *attrition or prepayment scorecard*, which can serve as the basis for the behavioural index of the attrition model. In such a case, the attrition hazard model fitted would be similar in complexity to the default hazard model.

6.3.2 Fitting Cox Model

Cox regression was also applied to the dataset described above to produce a model for the probability of default[70]. Unlike the proposed model, Cox regression does not accommodate macroeconomic variables. Therefore, the only outputs from the regression exercise are the baseline hazard function and the hazard ratios applying to each behavioural risk group. These can be compared to the proposed model's baseline hazard and behavioural risk index.

In Figure 30 we show the baseline hazards under Cox regression, plotted alongside the proposed model's baseline hazard. What is evident is the fact that the two models produced fairly similar baseline hazards, which we argued to be mainly influenced by the distance-to-default. However, it is also evident that the shapes do not line up precisely. For example, Cycle 2's baseline hazard for the proposed model decays faster than the baseline hazard under Cox regression. We partly attribute these differences to the choice of link functions (i.e., Cox regression is based on a Coxian link function, while the proposed model was fitted with a probit link function).

FIGURE 30: COX REGRESSION BASELINE HAZARD FUNCTION (COMPARED TO IFRS 9 MODEL BASELINE HAZARD)

In Figure 31 we show the parameter estimates under each of the models (taking the exponent of the parameter estimates would produce) the hazard ratio for that group. We also plot the parameter estimates with the mean of the estimates subtracted for each cycle. This shows that the variance of the

70 We opted to compare the proposed model to Cox regression on the basis that we see the proposed model as an extension to standard Cox regression, incorporating a variable baseline hazard. It might be insightful to compare the proposed technique to an approach that incorporates time-varying covariates. However, as discussed above, techniques that make use of time-varying covariates are less widespread – particularly in the consumer credit risk domain.

hazard rate decreases as the cycle increases. This means there is less diversity in risk as accounts become more delinquent, i.e., credit risk seems to exhibit the restriction of range phenomenon mentioned above.

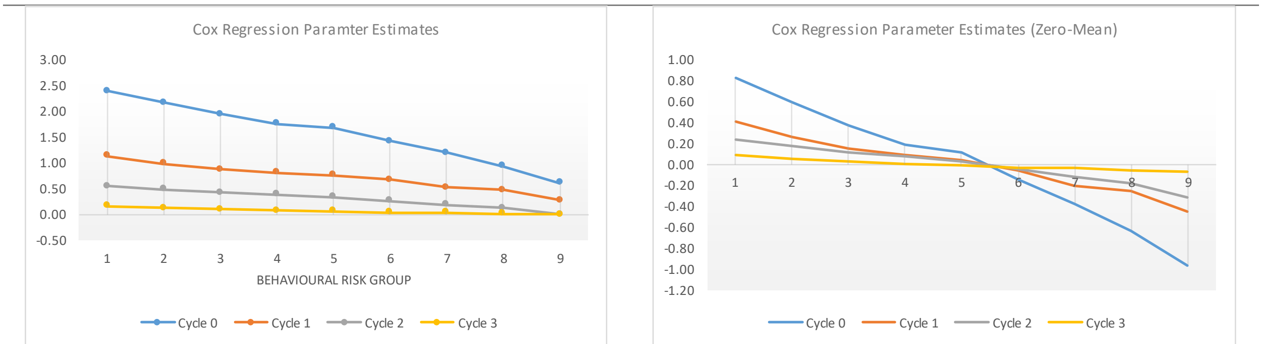

FIGURE 31: COX REGRESSION PARAMETER ESTIMATES

6.3.3 Validating the Models

The model is validated in two stages. Firstly, we validate the default and attrition hazard models separately, to identify whether each individual model predicts with sufficient accuracy. In this validation we are assessing accuracy of the hazards, survival functions. Afterwards, we combine the two models to form a probability of default, using the competing risk framework, as detailed by Equations 6.4 and 6.5. We validate this probability for accuracy and discriminatory power.

6.3.4 Hazard Validation

In Figure 32, we show the assessments for the default hazard, where we analyse accuracy of the default hazard across horizon and calendar month. Generally, both the proposed model and the Cox regression model are fit the data well by survival time, with the level of fitness improving with delinquency. However, these plots do not reveal how well the model fits across risk group and across calendar time.

In Figure 33, we show a similar assessment for the attrition hazard, across remaining term and calendar month. It should be unsurprising that the proposed model has greater levels of accuracy over time, given that the Cox regression does not take in time-varying covariates.

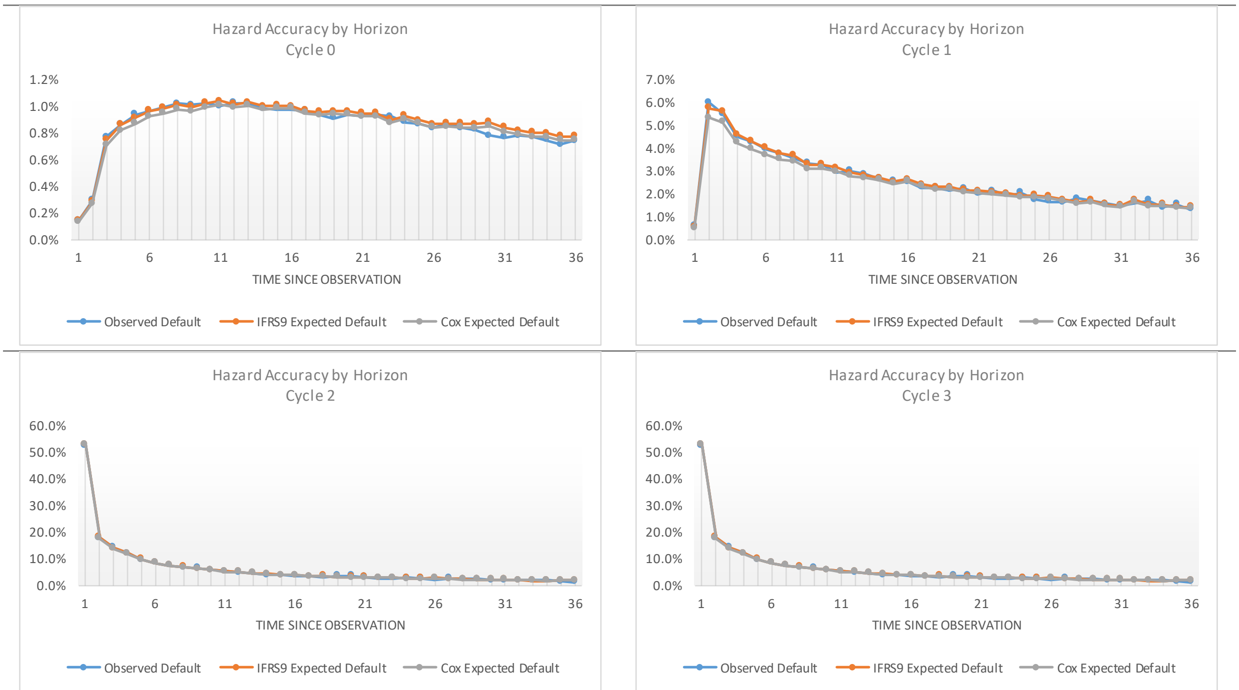

FIGURE 32: HAZARD ACCURACY BY HORIZON (HOLDOUT SAMPLE)

In Basel III terminology, the Cox regression can thus be seen as a through-the-cycle model. However, we acknowledge that one can also theoretically convert the proposed model into a through-the-cycle model. This is done by estimating the expected value of the hazard rate across the macroeconomic risk index, as follow:

$$\bar{h}_j(t) = E_{e_s}\left[\Phi\left(\alpha_1 b_t + \alpha_2 \varphi_{G_{j,s}} + \alpha_3 e_s\right)\right]. \tag{6.19}$$

Assuming that $e_s \sim N(\mu_e, \sigma_e)$, the expected hazard function becomes:

$$\bar{h}_j(t) = E_{e_s}\left[\Phi\left(\frac{\alpha_1 b_t + \alpha_2 \varphi_{G_{j,s}} + \alpha_3 \mu_e}{\sqrt{1+\alpha_3 \sigma_e}}\right)\right], \tag{6.20}$$

which follows from Owen (1980).

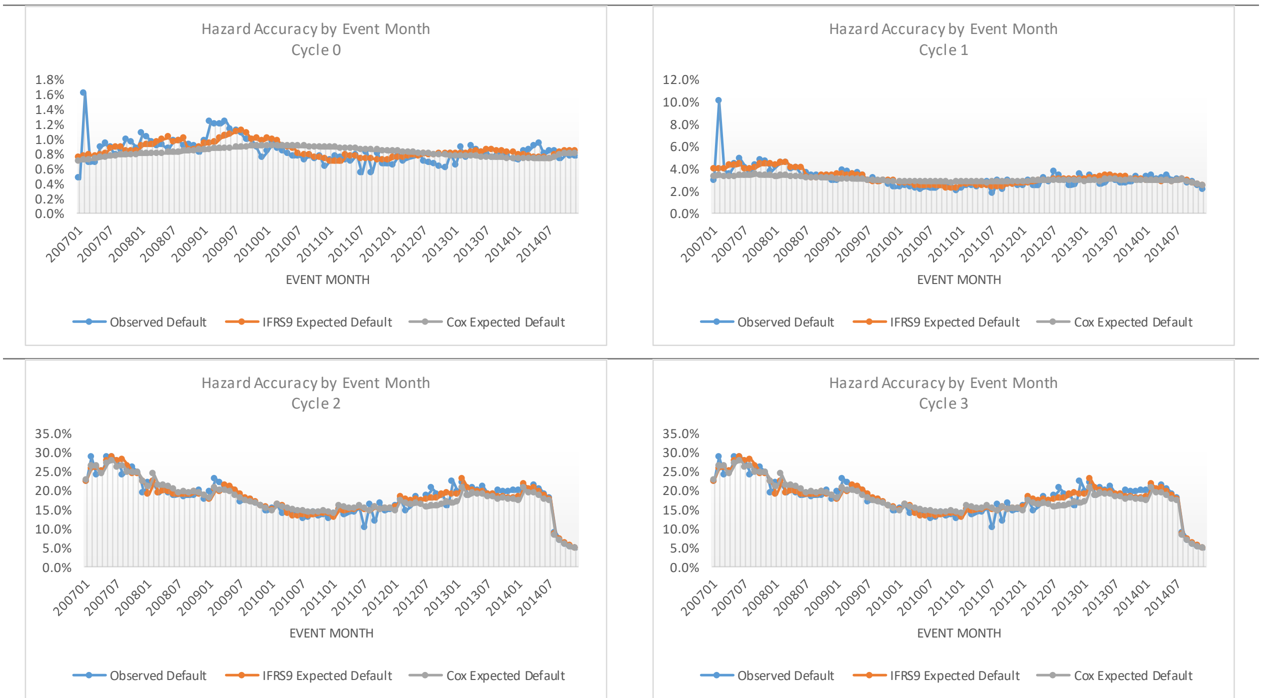

FIGURE 33: HAZARD ACCURACY BY EVENT MONTH (HOLDOUT SAMPLE)

In Figure 34 we show model accuracy across the behavioural risk components, where we also see that all three models fit fairly well. We also see evidence of our earlier observation that the amount of risk diversity decreases with the level of risk.

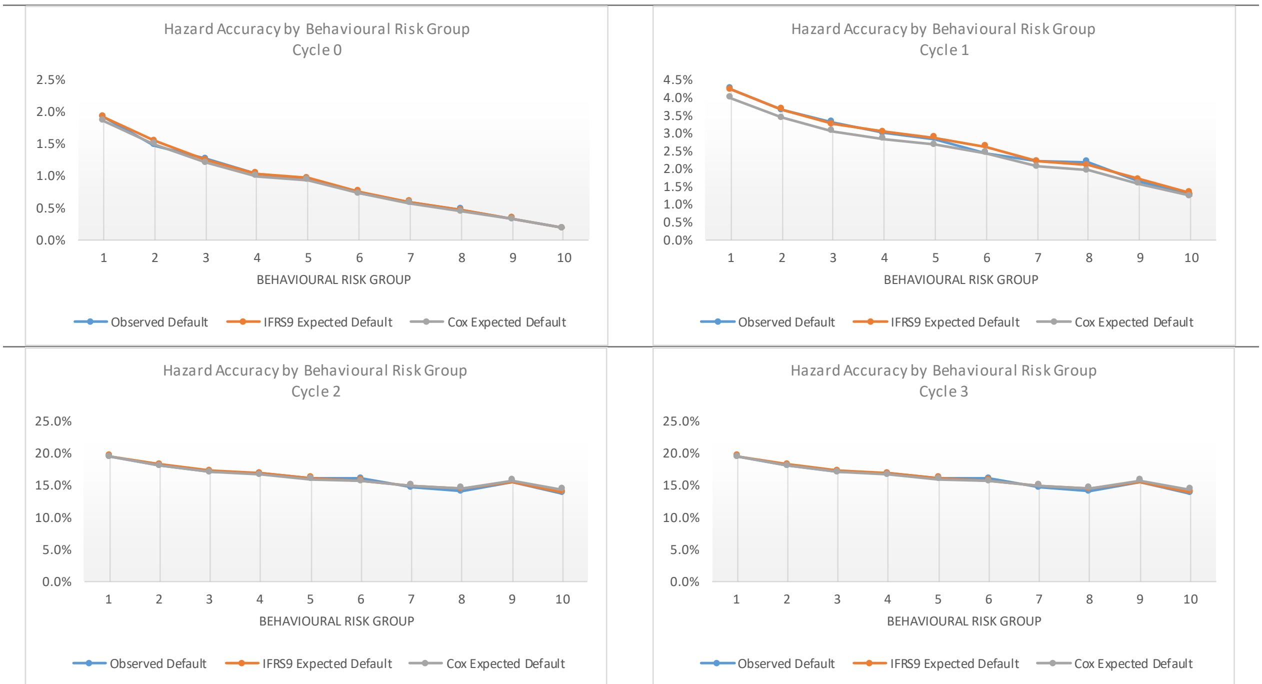

FIGURE 34: HAZARD ACCURACY BY BEHAVIOURAL RISK GROUP (HOLDOUT SAMPLE)

In Figure 35 we assess the fitness of the attrition hazard across remaining term and by calendar month. We see that the model fits well by remaining term but has some periods of over-prediction and underprediction by calendar time. One remedy for this would be to model the macroeconomic index with error autocorrelation.

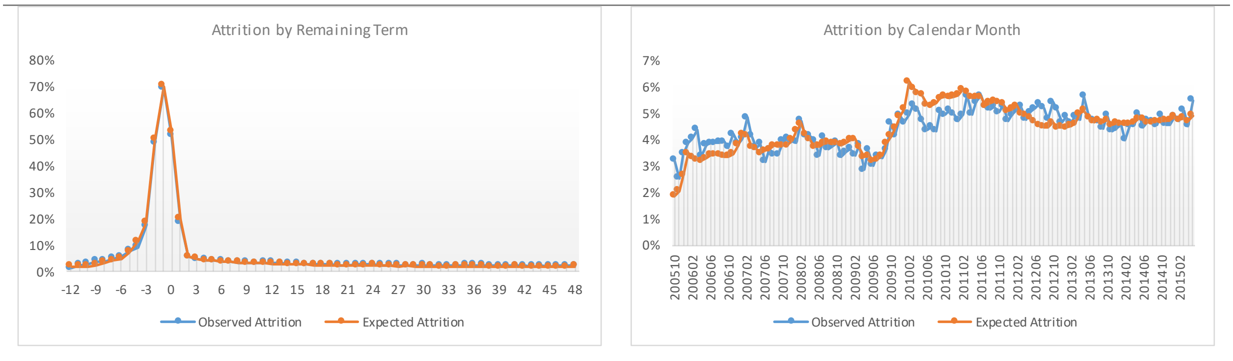

FIGURE 35: ATTRITION HAZARD BY REMAINING TERM AND EVENT MONTH (HOLDOUT SAMPLE)

6.3.4.1 PD Validation

The combined model is assessed for accuracy by outcome period and accuracy by calendar month. This assessment is shown in Figure 36, where we see the model performing well on both counts. Note that the fitness of the PD by horizon if largely influenced by the baseline hazard and attrition model.

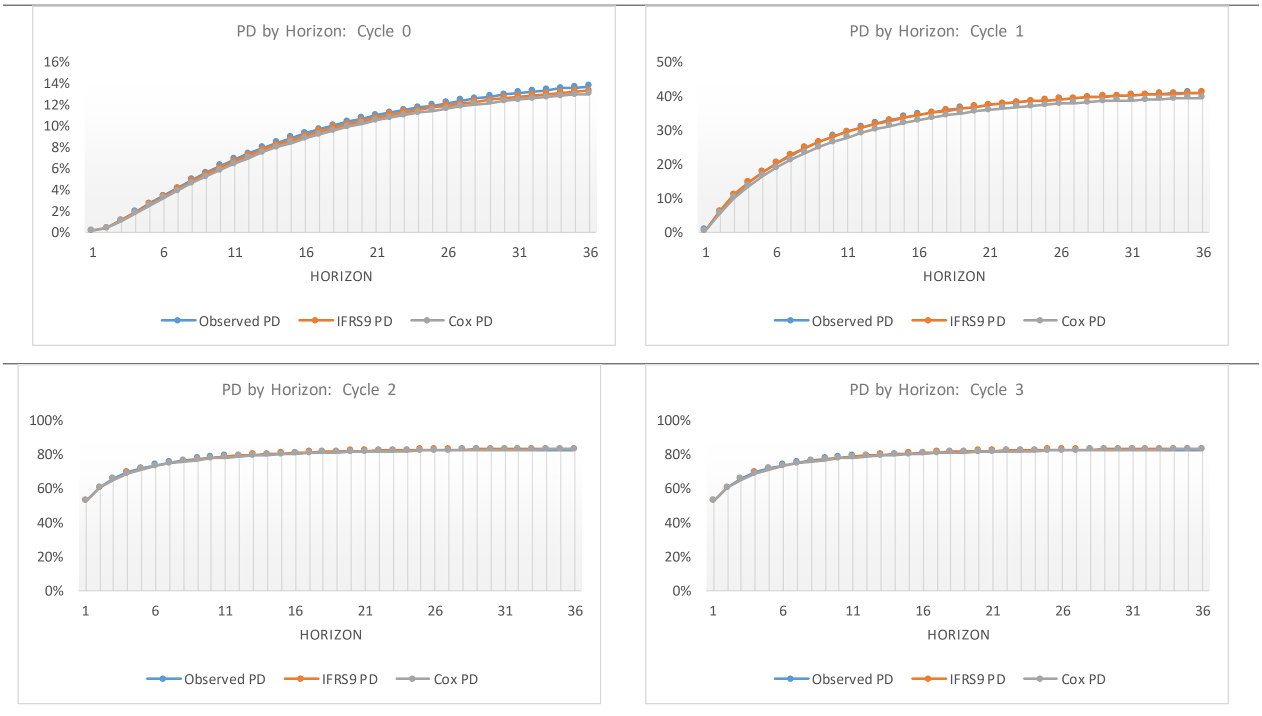

FIGURE 36: VALIDATING THE CUMULATIVE PD BY HORIZON

We also validate the PD model over time, by calendar month. In Figure 37 we specifically focus on the 12-month probability of default. Note that the fitness across time is largely influenced by the fitness of the macroeconomic index model. This is where it is evident that the proposed model performs better than the Cox model, which includes no macroeconomic data.

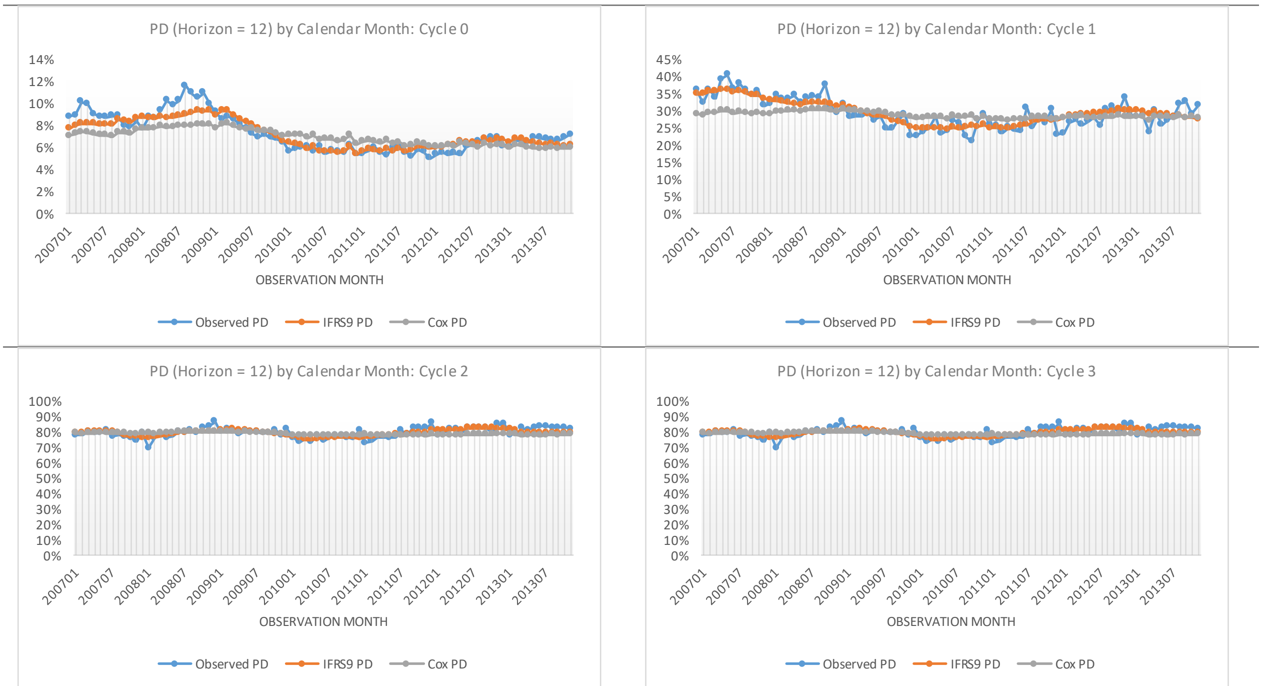

FIGURE 37: VALIDATING 12-MONTH PD OVER CALENDAR MONTH

Additionally, the model is assessed across range, i.e., across the different levels of loan-specific risk. We show this assessment in Figure 38. The level of fitness here is largely influence by the behavioural risk index model. Here the two models perform fairly well, as both include behavioural data.

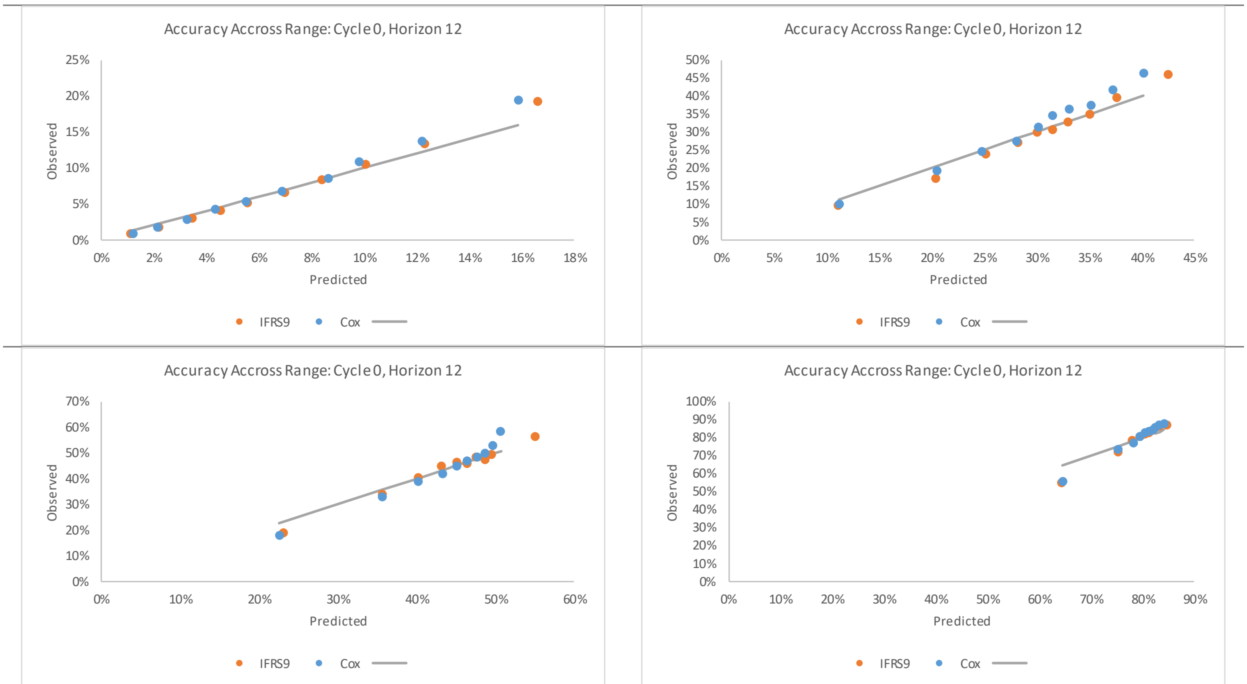

FIGURE 38: VALIDATING THE PD ACROSS RANGE

The model was also assessed for discriminatory power using the Gini statistic. This was done for different horizons, in Figure 39. Given that the Gini statistic is largely driven by behavioural data, there is the statistic between the two models.

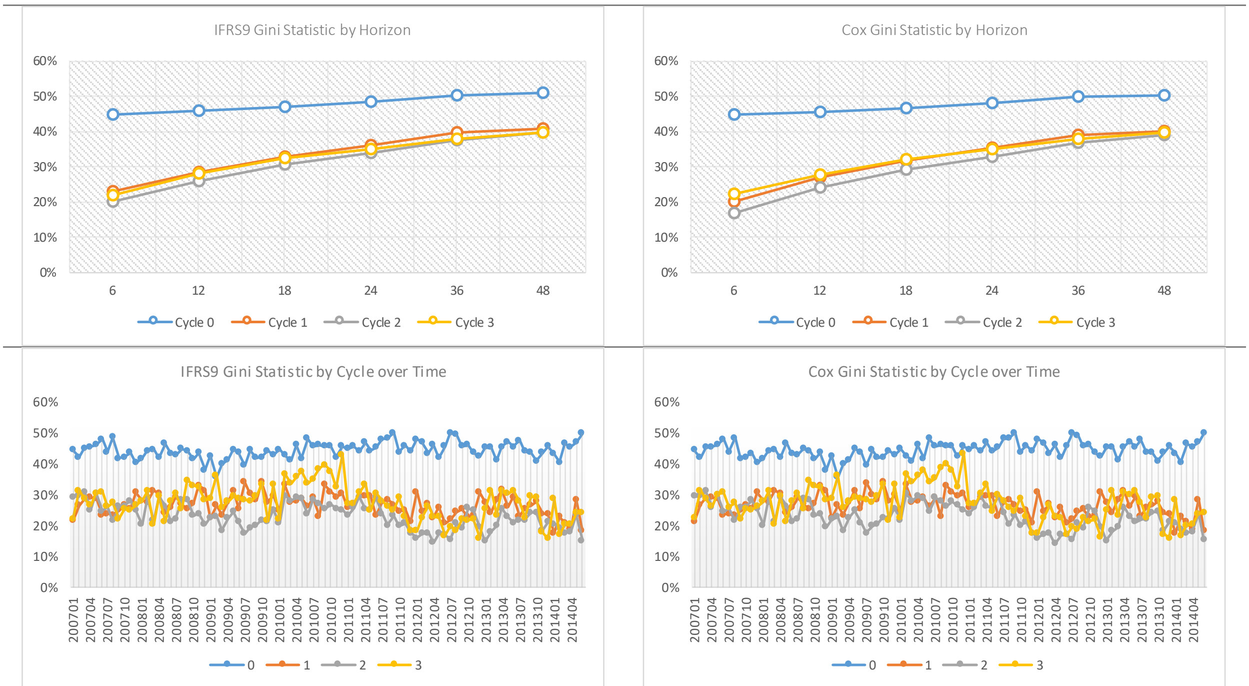

FIGURE 39: ASSESSING THE DISCRIMINATORY POWER OF THE PD MODEL

6.3.5 Applying the model for IFRS 9

In order to apply the model to estimate expected credit losses under IFRS 9, we need a model for EAD and LGD. These have not changed significantly between IAS 39 and IFRS 9. As such, the same modelling approaches can be applied as under IAS39. Through Equation 6.4, we are able to use the default model and attrition model to estimate the cumulative probability of default $PD_{j,s}(t)$. From this, we can calculate the marginal probability of default, which is:

$$pd_{j,s}(t) = PD_{j,s}(t) - PD_{j,s}(t-1). \tag{6.21}$$

This is a function of macroeconomic variables (as well as loan-specific variables) and thus can be rewritten as $pd_{j,s}(t, \boldsymbol{Y}_s)$, where $\boldsymbol{Y}_s$ is a vector of macroeconomic variables. The expected loss is thus given by:

$$EL_s(t, \boldsymbol{Y}_s) = \sum_{j=1}^{t} pd_{j,s}(j, \boldsymbol{Y}_s) \times EAD_t \times LGD_t, \tag{6.22}$$

i.e., $EL_s(t, \boldsymbol{Y}_s)$ is the t-month expected loss given macroeconomic scenario $\boldsymbol{Y}_s$. Therefore, for 12-month expected loss we set $t = 12$, while for lifetime expected loss we set $t = T$.

IFRS 9 requires that the expected loss by be a weighted average of the expected loss under a few different macroeconomic scenarios. Therefore, the final expected loss number will be:

$$EL_s(t) = \sum_{c=1}^{o} EL_s\left(t, \boldsymbol{Y}_{s,c}\right) \times \pi_c, \tag{6.23}$$

where $\boldsymbol{Y}_{s,c}$ is scenario c out of o macroeconomic scenario, with an associated probability π_c representing the likelihood of the scenario.

6.4 Summary of Findings

We proposed a discrete-time extension to the standard Cox regression model. The extension allows us to include time-varying macroeconomic variables into the model. The model is thus ideally suited for application in IFRS 9 PD modelling, which requires a lifetime PD that is influenced by macroeconomic data. The model is based on a decomposition approach, similar to that which is used under the EMV model. As such, it can be seen as an extension of the EMV approach, by adding an additional dimension, which is the horizon (or survival time) dimension.

The model was benchmarked against the standard Cox regression model. Here we find that Cox regression performs just as well across horizon (survival time) and across range (different levels of behavioural risk) but does not perform as well over time (calendar time). This is expected since the main differentiator of the proposed model is the inclusion of time varying macroeconomic variables, which operate across calendar time. However, this also means that the ability of the proposed model to improve the predictions of the standard Cox model is determined by the fitness of the macroeconomic model.

Chapter 7: The Concept of Economic Value for a Loan Portfolio

7.1 Overview of the Problem

In financial analysis, the valuation of a company is an attempt to capture the economic value of the net assets and profit streams of the company. Over time, several different valuation approaches have been developed and generally find differing levels of favour amongst financial analysts in different contexts. Most of these approaches are centred on the idea of discounted cashflow. In principle, the financial value of owning shares in a company is from the cashflow that shareholders receive from the company, in the form of dividends or share buy-backs. Hence, one of the most popular valuation approaches is the dividend discount model:

$$P = \sum_{k=1}^{\infty} D_k (1 + r_k)^{-k}, \tag{7.1}$$

where P is the valuation, D_k is the dividend paid during period k and r_k is the shareholders' required return up to period k. This general model is often simplified by assuming a perpetual dividend growth rate of g and a flat required return of r, which leads to the following model, commonly referred to as the Gordon growth model (Gordon, 1959):

$$P = D_0 \times \left(\frac{1+g}{r-g}\right). \tag{7.2}$$

There are various extensions to this standard model. For example, instead of assuming that the perpetual growth rate g applies from the valuation date onwards, one can assume a 2-stage (or, more generally, a n-stage model) where with different growth rates applying in the different time periods.

A modification of the Gordon growth model arises when we assume that dividends are a product of earnings E and the pay-out ratio p, so that the price-to-earnings ratio can be defined as:

$$\frac{P}{E} = p \times \left(\frac{1+g}{r-g}\right). \tag{7.3}$$

A further modification is to express the earnings as a product of the company's equity (or book value) B and the return on equity RoE, so that the price-to-book value defined as:

$$\frac{P}{B} = RoE \times p \times \left(\frac{1+g}{r-g}\right). \tag{7.4}$$

Both the price-to-book value (P/B) and price-to-earnings (P/E) ratios are popular valuation metrics amongst analysts. The P/E is the more widely used, especially for companies with a stable earnings outlook. P/B ratios generally differ by sector and are generally preferred for analysing financial services companies (see Cheng & McNamara (2000) for a review). They are especially useful for analysing banks, largely because banks are generally required to hold a lot of capital and the regulator prescribes the amount of capital that each bank needs to hold. Differences in P/B across banks, therefore, will be driven largely by how well the different banks are able to generate earnings (and pay out dividends) through the capital base. According to the model, in order to know the justifiable P/B range for a bank, one needs to have a view on the earnings growth g, the pay-out ratio p and the RoE. For this purpose, a bank's RoE can be decomposed in a few ways:

$$RoE = \frac{E_{lending}+\left(E-E_{lending}\right)}{B_{lending}+\left(B-B_{lending}\right)}$$

$$= \left(\frac{E_{lending}}{B_{lending}}\right) \times \left(\frac{B_{lending}}{B}\right) + \left(\frac{E-E_{lending}}{B-B_{lending}}\right) \times \left(\frac{B-B_{lending}}{B}\right)$$

$$= RoE_{lending} \times \left(\frac{B_{lending}}{B}\right) + RoE_{other} \times \left(\frac{B-B_{lending}}{B}\right), \tag{7.5}$$

where $E_{lending}$, $B_{lending}$ and $RoE_{lending}$ are the earnings, book value and RoE for the lending portion of the business, while RoE_{other} is the RoE from the remainder of the book. The lending business will generally be the largest portion and the book value $B_{lending}$ will be mainly made up of economic or regulatory capital. Some of the main determinants of $RoE_{lending}$ will be the net interest margin[71] and the capital requirement. If A denotes the total exposure on the lending business (i.e., gross advances), then the $RoE_{lending}$ can be understood as[72]:

$$RoE_{lending} = \frac{A}{B_{lending}} \times \frac{NII-CL+NIR-e}{A}$$

$$= \frac{A}{B_{lending}} \times \left(NIM - CLR + \frac{NIR-e}{A}\right), \tag{7.6}$$

where NII is net interest income, CL is the credit loss, NIR is non-interest revenue, e is expenses, NIM is the net interest margin and CLR is the credit loss ratio. NII will generally be significantly larger than NIR for a bank with a sizeable lending operation. Therefore, more focus is often paid to understanding the gearing ratio $\frac{A}{B_{lending}}$ (which basically represents the onerousness of capital requirements) and NIM which varies by type of business (see Choudhry (2018) for a discussion).

The P/B metric is equally applicable in the insurance industry. However, although a comparable metric to the NIM can be defined for insurance companies, it is not as widely understood as the NIM in the

[71] Net interest margin is equal to the interest the bank earns on loans to customers, net of the cost of funding the loans, divided by the book value (or exposure) of the loans.
[72] Note that the numerator $(NII - CL + NIR - e)$ is simply a decomposition of the earnings from a banking business.

banking sector. A common valuation approach in the long-term insurance sector centres on the concept of embedded value (EV) (see Diers, Eling, Kraus & Reuß (2012) for a discussion). The embedded value of an insurance company is made up to two components: the value in-force (VIF) and the adjusted net worth (ANW). The VIF is basically the present value of profits from future existing business, while the ANW is the book value of the insurer. In territories where it is used, EV is generally calculated and disclosed by insurance companies in their financial reports, along with a summary of the assumptions that underlie its calculation (e.g., ASSA, 2012).

EV attempts to directly measure the economic value of the insurance business already written; it does not capture future new business. As such, EV should theoretically be adjusted upwards to reflect the expected economic value of future new business, to arrive at an appraisal value. EV would thus not be as sensible for sectors where the nature of business is not based on term contracts (such as the retail sector) or where the tenure of the contracts is short (such as the short-term insurance sector). Given the contractual and long-term nature of lending business, the concept of EV should be as applicable in banking as it is in long-term insurance.

A P/B valuation approach for the banking sector generally must make some assumptions about the RoE, which translates to an assumption about the NIM, adjusted for credit loss. Holding the growth outlook constant, such valuations might fall into the trap of assuming a one-to-one relationship between $NIM - CLR$ and the economic value of a lending business. However, there are several factors affecting the economic value of lender, including the tenure profile of the loan (e.g., see Gross (2007)).

This chapter thus sets out to develop the concept of EV for a lending business. We offer an analytical formula for estimating the economic value of a loan book, accounting for the tenure of the book, the interest rate, the cost of funding, credit losses, opportunity cost of capital and shareholders' required return. We also show how the economic value can be attributed to the main sources of value in a lending business: maturity transformation[73] (including or excluding the impact of retail funding) and credit risk. We further show how the concept of lifetime expected credit loss under IFRS 9 (IFRS 9, 2014) is an important aspect of the calculation EV.

The rest of the chapter is organised as follows. In Section 2 we offer a brief review of EV for long-term insurers, before introducing a basic model of EV for a lending business. In Section 3 we offer a few refinements, leading a final model of EV. We also discuss how to analyse the changes in EV over the life of a lending book and how to attribute economic value between maturity transformation and credit

[73] Maturity transformation is the practice by banks of funding long-term lending to customers using short-term deposits and loans from customers and lenders, thus earning a *spread* profit from the difference between long-term interest rates and short-term interest rates.

risk. In Section 4 we demonstrate the use of the formula for a stylised portfolio and for an actual book of loans, before offering some concluding remarks in Section 5.

7.2 A Basic Model of Embedded Value

We define the economic value as a measure of the future profits to shareholders (or owners) of a lending business. This is similar to the concept of EV in insurance companies. We only focus on profits from lending activities, ignoring non-interest revenues and costs. Our analysis will focus on amortising products, where repayment amounts are equal and occur in regular intervals over a predetermined term. Therefore, revolving products are beyond the scope of the chapter.

7.2.1 A Review of EV for a Long-Term Insurer

In long-term insurance, EV is defined as the sum of two values: the value in-force (VIF) and the adjusted net worth, sometimes referred to as the net asset value (NAV). The VIF is a measure of the expected future profits to shareholders from existing insurance contracts. This considers premiums and investment returns on the revenue side and claims (and associated reserves), maintenance expenses (and associated reserves) and shareholder tax. It also includes an allowance for the cost of capital held to support the insurance business. However, this does not include any allowance for profits from future new business, or any future policy alterations.

Given the uncertain and long-term nature of insurance, the profit stream underlying the calculation of the VIF will be based on assumptions for future mortality and morbidity (which influence the claims experience), expenses and expense inflation as well as financial market performance (which influences investment returns). Furthermore, the VIF is calculated by discounting expected future profits. Therefore, it will be sensitive to the discount rate assumed.

The NAV is made up of two components: required capital and free assets. Required capital relates to economic capital that is required to run the existing insurance business, as determined by the insurer and regulator. Free assets are what is available for immediate distribution, but are generally held to fund new business, take advantage of opportunities that may arise or to act as a capital buffer.

Over the lifetime of a portfolio of policies, value slowly transfers from the VIF to the NAV, as the profits underlying the calculation of the VIF are actually *earned*. However, given the many assumptions underlying the VIF, actual experience will invariably deviate from that which was assumed in calculating the VIF. These deviations are called experience variances. In addition, the insurer's view on the assumptions themselves will generally change from one period to the next, which will lead to changes to the VIF. Therefore, each period the VIF will change due to the release of profits into the NAV, as well as experience variances and assumption changes. Additionally, as new policies are written (and existing policies are altered), the VIF will also change due to this new business. The total impact of the new business-related changes is referred to as the embedded value of new business (VNB).

The subject of EV is vast, and the discussion above only offers a glimpse into it. For instance, there are different concepts of EV used in different jurisdictions. European insurers tend to use either European EV or market-consistent EV, while South African insurers tend to use traditional EV (which is close to European EV). For more on the topic, see Castellani, de Felice, Moriconi & Pacati (2005), Sarafeim (2011) and Horton, Macve, Serafeim (2007).

7.2.2 A Basic Model for EV for a Lending Business

Under IFRS 9, loans are carried on the balance sheet of lenders at amortised costs. Before allowing for impairments, the amortised cost will generally be equal to the face value of the loan (i.e., the amount borrowed to the customer). Let A be the principal on a loan taken out over a term of n months with an effective monthly interest rate of i. The carrying amount at amortised cost will be equal to A, with the following implied relationship:

$$\begin{aligned} A &= \sum_{j=1}^{n} p(1+i)^{-j} \\ &= p\frac{1-(1+i)^{-n}}{i} \\ &= p\frac{1-v_{n}^{i}}{i} \\ &= pa_{\bar{n}}^{i}, \qquad (7.7) \end{aligned}$$

where p is the instalment amount on the loan, v_n^i is the discount factor over n periods at rate i and $a_{\bar{n}}^{i}$ is the annuity factor for equal payments over n periods at rate i. The value of p can be worked out from Equation 7.7 as:

$$p = \frac{A}{a_{\bar{n}}^{i}}. \qquad (7.8)$$

More generally, at duration k, we can express the carrying value of the loan as:

$$\begin{aligned} A_k &= \sum_{j=1}^{n-k} p(1+i)^{-j} \\ &= pa_{\overline{n-k}}^{i}. \qquad (7.9) \end{aligned}$$

The stream of instalments from the loan is the main form of revenue from the lending operation. In order to arrive at an estimate of economic value, we need to also consider the expenditure side of the business. The main expenditure under the loan arrangement will be the funding costs, i.e., the repayment of the money used by the lender to finance the loan. The main forms of funding for banks is retail funding, consisting of customer deposits, and wholesale funding, which is sourced from the wider institutional credit market. Let L_k be the principal amount of funding outstanding at duration k, with respect to the loan. As a necessity, L_k will exceed A_k, but the lender will generally aim to minimise the cost of funding by keeping L_k as close to A_k as practicable. Therefore, our derivation of economic value assumes that $L_k = A_k$ (however, this assumption can be relaxed by assuming that a buffer is held over A_k).

Let w be the weighted average cost of funding. The lending franchise can generally only be viable when w is below i, which allows the lender to earn a spread, i.e., maturity transformation profits. Where the instalment amounts under this loan are taken to be constant, the fact that i exceeds w will generally mean that the funding repayments are not constant. Therefore, let q_k be the funding repayment at duration k. The gross economic value of the loan at duration can thus be calculated as the present value of instalment revenue less funding repayments, discounted at the lender's required return (or cost of equity) r:

$$\widehat{VIF}_k = \sum_{j=1}^{n-k} (p - q_{j+k}) v_j^r \coloneqq M_k. \tag{7.10}$$

The measure above is only valid to the extent that the borrower is able to make all instalment payments as they fall due. In reality, the lender will be exposed to default risk, which means that the actual instalment payments may be lower than p. For this reason, the economic value is better seen as a random variable than a constant. Therefore, we estimate the expected economic value of the loan. One way of doing this is to deduct the expected shortfall between the present value of contractual instalment p and the expected present value of the actual payment stream $\boldsymbol{p} = \{\ddot{p}_1, \ddot{p}_2, \dots, \ddot{p}_n\}$, where $\ddot{p}_k$ is the payment actually received at duration k, i.e., allowing for the possibility of default. This is given by:

$$EL_k = \sum_{j=k}^{n-k} (p - \ddot{p}_j) v_j^r, \tag{7.11}$$

as at duration k. This is equivalent to the lifetime expected credit loss estimate required for IFRS 9 impairment provisioning (as discussed in the previous chapter), with the discount rate set to equal the required return. Therefore, we can deduct EL_k from $\widehat{VIF}_k$ above to allow for the credit losses in our calculation of economic value.

In the same way that we estimated the expected credit losses over the life of the contract L, we can also estimate the expected maintenance expenses attaching. However, some lenders include a recurring fee as part of the contract which is intended to cover operating expenses. In such a case, the presumption would be that the fees will be enough to cover all the operating expenses. Therefore, we opt to ignore the impact of expenses in our calculation of economic value.

The final step in our derivation is to account for the opportunity cost of the capital held to support the loan. Let EC_k be the economic capital held with respect to the loan at duration k. This capital is expected to be released throughout the life of the loan as the loan is paid down. Therefore, we calculate the expected opportunity cost of capital as the expected present value of the capital releases less EC_k, which is donated by CoC_k, discounted at the required return.

After allowing for instalment revenue, funding repayments, defaults and the opportunity cost of capital, the expected economic value of the loan, from the lender's perspective, is given by:

$$VIF_k = M_k - EL_k - CoC_k. \tag{7.12}$$

Note that this definition of economic value is comparable to the VIF in insurance, since it does not include the net assets held by the lending business. A more complete representation of economic value would thus be:

$$EV_k = (M_k - EL_k - CoC_k) + (EC_k + FA_k), \tag{7.13}$$

where $EC_k + FA_k$ represents the total assets held by the business and FA_k are free assets held in excess of economic capital.

7.3 A More Detailed Model for Embedded Value

We now further develop the concepts introduced above to arrive at an analytical formula for economic value. We also discuss how to attribute and assess economic value over time.

7.3.1 Refinement of Economic Value

Our definition of economic value makes use of the concept of lifetime expected credit loss, as introduced by the IFRS 9 standard. There are various modelling approaches used to estimate this quantity. However, the various estimation techniques will generally reproduce the same cashflow equation, albeit often implicitly. The basic cashflow formula for expected credit losses is as follows:

$$EL_k = E\left[\sum_{j=1}^{n-k}\left(p - \ddot{p}_{j+k}\right)v_j^r\right], \tag{7.14}$$

where $\ddot{p}_k$ is the actual payment received during time k. Note that $\ddot{p}_k$ is a random variable, while p is a known constant as determined by the terms of the contract. A different way of looking at this equation is to assume that p only differs from $\ddot{p}_k$ in the event of a default, so that the equation changes to:

$$
\begin{aligned}
EL_k &= \textstyle\sum_{j=1}^{n-k} pv_j^r - E\left[\sum_{j=1}^{n-k} \ddot{p}_{j+k} v_j^r\right] \\
&= \textstyle\sum_{j=1}^{n-k} pv_j^r - E\left[\sum_{j=1}^{n-k} \left(\left(1-d_{j+k}\right)p + d_{j+k}\ddot{p}_{j+k}\right) v_j^r\right] \\
&= \textstyle\sum_{j=1}^{n-k} pv_j^r - \sum_{j=1}^{n-k}\left(1-d_{j+k}\right)pv_j^r - E\left[\sum_{j=1}^{n-k} \left(\left(1-d_{j+k}\right)p + d_{j+k}\dot{p}_{j+k}\right) v_j^r\right] \\
&= \textstyle\sum_{j=1}^{n-k} pv_j^r\, d_{j+k} - E\left[\sum_{j=1}^{n-k}\left(d_{j+k}\dot{p}_{j+k}\right)v_j^r\right] \\
&= E\left[\textstyle\sum_{j=1}^{n-k} d_{j+k}\left(p-\dot{p}_{j+k}\right)v_j^r\right], \qquad (7.15)
\end{aligned}
$$

where d_k indicates whether the loan is in default during period k, i.e., we assume that $\ddot{p}_k$ can be broken up as follows:

$$\ddot{p}_k = p(1-d_k) + \dot{p}_k d_k. \qquad (7.16)$$

This formulation splits each payment by whether the account is in default or not. A different way of looking at payments is by separating the payment stream by whether the account has already been in default or not. This is as follows:

$$
\begin{aligned}
EL_k &= E\left[\textstyle\sum_{j=1}^{n-k}\left(p-\ddot{p}_{j+k}\right)v_j^r\right] \\
&= E\left[\textstyle\sum_{j=1}^{n-k}\left(c_j \sum_{l=j}^{n-k}(p-\check{p}_{l+k})v_l^r + \left(1-c_j\right)\sum_{l=j}^{n-k}(p-p)v_l^r\right)\right] \\
&= E\left[\textstyle\sum_{j=1}^{n-k}\left(c_j \sum_{l=j}^{n-k}(p-\check{p}_{l+k})v_l^r\right)\right] \\
&= \textstyle\sum_{j=1}^{n-k} pd_{j+k} E\left[\sum_{l=j}^{n-k}(p-\check{p}_{l+k})v_l^r\right], \qquad (7.17)
\end{aligned}
$$

where c_k indicates whether the account first entered default at time k, $pd_k = E[c_k]$ is the probability that the loan enters default at time k, i.e., the equation assumes independence between c_j and $\sum_{l=j}^{n-k}(p-\check{p}_{l+k})v_l^r$. Note that this equation calculates the expected loss on the remaining life of the loan following the first default event, so that pd_k is calculated assuming that default is an *absorbing state*. For example, if $pd_1 = 1$ then $pd_k = 0$ for all $k > 1$, which means that the loan defaults at time 1 (and only at time 1) with certainty. Under this scenario, the equation above converts to the initial equation (Equation 7.14):

$$
\begin{aligned}
EL_k &= E\left[\textstyle\sum_{j=1}^{n-k}\left(p-\check{p}_{j+k}\right)v_j^r\right] + \sum_{j=1}^{n-k} 0 \times E\left[\sum_{l=j}^{n-k}(p-p)v_l^r\right] \\
&= E\left[\textstyle\sum_{j=1}^{n-k}\left(p-\check{p}_{j+k}\right)v_j^r\right]. \qquad (7.18)
\end{aligned}
$$

Perhaps the most popular approach for estimating credit loss is the components approach, made up of probability of default (PD), expected exposure at default (EAD) and expected loss given default (LGD). By rearranging the loss formula above, we can reproduce the components-based formula:

$$EL_k = \sum_{j=1}^{n-k}\left(pd_{j+k}E\left[\sum_{l=j}^{n-k}(p-\check{p}_{l+k})v_l^r\right]\right)$$

$$= \sum_{j=1}^{n-k}\left(pd_{j+k}A_{j+k}E\left[\sum_{l=j}^{n-k}\left(\frac{(p-\check{p}_{l+k})}{A_{j+k}}v_{l-j}^r\right)\right]v_j^r\right)$$

$$= \sum_{j=1}^{n-k} pd_{j+k}EAD_{j+k}LGD_{j+k}v_j^r, \qquad (7.19)$$

where $EAD_j = A_j$ and LGD_j is given by:

$$LGD_{j+k} = E\left[\sum_{l=j}^{n-k}\left(\frac{(p-\check{p}_{l+k})}{A_j}v_{l-j}^r\right)\right]. \qquad (7.20)$$

Note that Equation 7.19 is nearly identical to Equation 6.23 from the previous chapter.

Due to the model's assumption that default is an absorbing state, the loss given default is best understood as loss given *first* default, i.e., it calculates future expected loss following the first default event from the point of observation.

Our definition of economic value also made use of the concept of cost of capital, which is the estimated opportunity cost of economic capital held to support the business. It is estimated as the expected present value of the release of capital as the loan is paid down. As a simplifying assumption, we assume that the capital held is always held in proportion to the value of the loan at amortising cost A_k. Letting c denote this proportion, the cost of capital can be calculated as:

$$CoC_k = EC_k - \sum_{j=1}^{n-k} c(-\Delta A_{k+j})v_j^r, \qquad (7.21)$$

i.e., a proportion c of the change in the loan's value is released each period. Following from Equation 7.7, the change in the carrying value of the loan at amortising cost, under contractual assumptions, is given by:

$$\Delta A_k = p\left[a_{\overline{n-k}|}^i - a_{\overline{n-k+1}|}^i\right]$$

$$= p\left[\frac{v_{n-k+1}^i - v_{n-k}^i}{i}\right]$$

$$= -pv_{n-k+1}^i. \qquad (7.22)$$

Thus, we can state the cost of capital more elaborately as:

$$CoC_k = EC_k - \sum_{j=1}^{n-k} cpv_{n-k-j+1}^i v_j^r$$

$$= EC_k - cpv_{n-k+1}^i \left[\sum_{j=k}^{n} \frac{(1+i)^j}{(1+r)^j}\right]$$

$$= EC_k - cpv_{n-k+1}^i a_{\overline{n-k}|}^h, \quad (7.23)$$

where h is defined as:

$$h = \frac{1+r}{1+i} - 1. \quad (7.24)$$

Note that the value of EC_k, or c, can be derived via the economic capital methodology described in Chapters 3 and 4.

Our definition of economic value also referred to q_k as the repayment made to providers of funding at duration k. Under our assumption that the lender aims to match the loan amount A_k to the amount of funding L_k, the repayment amount is given by:

$$q_k = (-\Delta A_k) + wA_{k-1}$$

$$= pv_{n-k+1}^i + \frac{wp}{i}\left(1 - v_{n-k+1}^i\right)$$

$$= pv_{n-k+1}^i \left[1 - \frac{w}{i}\right] + \frac{w}{i} p, \quad (7.25)$$

i.e., ΔA_k is the amount of principal repaid to the providers of funding and wA_{k-1} is the amount of interest paid during the period. Therefore, the expression for the expected present value of funding repayments simplifies as follows:

$$\sum_{j=1}^{n-k} q_{k+j} v_j^r = pv_{n-k+1}^i \sum_{j=k}^{n} \frac{(1+i)^j}{(1+r)^j} + \frac{wp}{i}\left(\sum_{j=k}^{n} v_j^r - v_{n+1}^i \sum_{j=k}^{n} \frac{(1+i)^j}{(1+r)^j}\right)$$

$$= pv_{n-k+1}^i a_{\overline{n-k}|}^h + \frac{wp}{i} a_{\overline{n-k}|}^r - \frac{wpv_{n+1}^i}{i} a_{\overline{n-k}|}^h$$

$$= pv_{n-k+1}^i a_{\overline{n-k}|}^h \left[1 - \frac{w}{i}\right] + \frac{wp}{i} a_{\overline{n-k}|}^r. \quad (7.26)$$

Finally, we can calculate the expected present value of the difference between instalment payments by the borrower and the lender's repayment to funding providers:

$$\sum_{j=1}^{n-k} (p - q_j) v_j^r = \sum_{j=k}^{n} pv_j^r - \sum_{j=k}^{n} q_k v_j^r$$

$$= pa_{\overline{n-k}|}^r - pv_{n-k+1}^i a_{\overline{n-k}|}^h \left[1 - \frac{w}{i}\right] - \frac{wp}{i} a_{\overline{n-k}|}^r$$

$$= pa^{r}_{\overline{n-k}}\left[1-\frac{w}{i}\right] - pv^{i}_{n-k+1}a^{h}_{\overline{n-k}}\left[1-\frac{w}{i}\right]. \quad (7.27)$$

Combining all the above refinements, we obtain the following equation for economic value, before allowing for economic capital and free assets:

$$VIF_k = \sum_{j=1}^{n-k}\left(p - q_{j+k}\right) \times v_j^r - EL_k - CoC_k$$

$$= \left\{pa^{r}_{\overline{n-k}}\left[1-\frac{w}{i}\right] - pv^{i}_{n-k+1}a^{h}_{\overline{n-k}}\left[1-\frac{w}{i}\right]\right\} - EL_k - \left\{EC_k - cpv^{i}_{n+1}a^{h}_{\overline{n-k}}\right\}$$

$$= \left\{pa^{r}_{\overline{n-k}}\left[1-\frac{w}{i}\right]\right\} + \left\{pv^{i}_{n+1}a^{h}_{\overline{n-k}}\left[c+\frac{w}{i}-1\right]\right\} - \{EL_k + EC_k\}. \quad (7.28)$$

The last two components of the equation ($EL_k + EC_k$) represent the sum of expected and unexpected loss from the book of business. Given that economic capital is held to withstand a 99.5% loss scenario, the sum of these two components would usually be interpreted as the 99.5% percentile of the loss distribution. However, while economic capital is generally calculated based on a 12-month expected credit loss, the expected loss in the equation is calculated over the entire lifetime of the loan. Therefore, the sum of the two components will exceed the 99.5% percentile of the loss distribution by an amount equal to difference between the lifetime expected credit loss and the 12-month expected credit loss.

Another point to note is that if we exclude the expected loss and economic capital components ($EL_k + EC_k$) and then set $c = 0$, we get an estimate of the expected present value of the net interest income from the existing book of business.

We have so far only dealt with the measurement of value for a homogenous portfolio of loans, with the same repayment period and interest rate. We now consider how the model can be extended to deal with a heterogeneous portfolio of loans. Let $VIF_k(i, n)$ be the economic value during period k for loans with a tenure of n with an interest rate of i. The economic value of the full portfolio can be derived by summing across all such heterogeneous sub-portfolios:

$$\overline{VIF}_k = \sum_j VIF_k\left(i_j, n_j\right). \quad (7.29)$$

7.3.2 Attribution of Economic Value

At the outset of a loan and throughout its lifetime, it may be useful to understand the sources of economic value for the loan. The three sources we describe in this thesis are maturity transformation, deposit funding and the credit spread.

Maturity transformation relates to the mismatch between the tenure of the loan and the tenure of the liabilities used to fund the loan. The contribution of maturity transformation to economic value is estimated as the difference between the expected present value of repayments at the actual funding cost of w and the expected present value of repayments assuming funding was as implied by the risk-free yield curve:

$$
\begin{aligned}
EMaturity_k &= \sum_{j=k+1}^{n}\left(\Delta A_k + w_j A_{k-1}\right) - \sum_{j=k+1}^{n}(\Delta A_k + wA_{k-1}) \\
&= \sum_{j=k+1}^{n} w_j A_{k-1} - \sum_{j=k+1}^{n} wA_{k-1} \\
&= (\overline{w} - w) \times \left(\sum_{j=k+1}^{n} A_{k-1}\right),
\end{aligned}
\tag{7.30}
$$

where w_k is the market cost of funding at tenure k and $\overline{w}$ is the weighted average market cost of funding over the term of the loan:

$$
\overline{w} = \frac{\sum_{j=k+1}^{n} w_j A_{k-1}}{\sum_{j=k+1}^{n} A_{k-1}}. \tag{7.31}
$$

Note that the cost of funding w includes the impact of both maturity transformation and retail funding. Therefore, we can split the economic value above into that which arises from pure maturity transformation (i.e., funding mismatch) and that which arises from deposit funding. To calculate the former, we replace the actual cost of funding w with an estimate of what the cost of funding would be assuming no retail funding, $\ddot{w}$.

$$
\ddot{E}Maturity_k = (\overline{w} - \ddot{w}) \times \left(\sum_{j=k+1}^{n} A_{k-1}\right). \tag{7.32}
$$

After accounting for maturity transformation, we assume that the remainder of economic value is derived as a profit loading for credit and liquidity risk, which is given by:

$$
EMargin_k = \sum_{j=k+1}^{n}(p - q_k) \times (1 + r)^{-k+j} - EL_k - CoC_k - (\overline{w} - w) \times \left(\sum_{j=k+1}^{n} A_{k-1}\right). \tag{7.33}
$$

7.3.3 Evolution of EV

Over the life of the loan, the lender may wish to keep track of the expected release of economic value from the loan book and the variances that arise as actual experience deviates from expectation. We begin by examining how the economic value of a single loan changes from one period to the next. The change in economic value can be understood as the sum of (1) an expected conversion of economic value into cashflow profits, (2) the expected return on EV (referred to as the *unwind* in of EV in long-term insurance), (3) the impact of variances and changes in assumptions.

The simplest to measure is the expected return on economic value. Since shareholders require a return of r, which is the rate at which we discount the expected cash flows underlying the economic value, the economic value is expected to grow to produce a return of r each period. Therefore, the expected return over period k equals $rVIF_{k-1}$. Additionally, a rate of return of r is expected on the economic capital backing the business, equal to rEC_{k-1}.

During period k, the economic value is also expected to decrease by an amount equal to the cashflow expected to emerge as profits during period k:

$$CF_k = [(p - q_k) - (pd_k EAD_k LGD_k)]. \tag{7.34}$$

This is the surplus net interest margin after paying for expected losses incurred due to default. This can also be shown analytically from Equation 7.12. Before allowing for variances and assumption changes, the change in EV should be given by:

$$\begin{aligned}
\Delta\widehat{VIF}_k &= \Delta M_k - \Delta EL_k - \Delta CoC_k \\
&= [M_{k-i}r - (p - q_k)] - [EL_{k-1}r - (pd_k EAD_k LGD_k)] - [CoC_{k-1}r - EC_{k-1}r] \\
&= [(M_{k-i} - EL_{k-1} - CoC_{k-1})r + EC_{k-1}r] - [(p - q_k) - (pd_k EAD_k LGD_k)] \\
&= (VIF_{k-1} + EC_{k-1})r - [(p - q_k) - (pd_k EAD_k LGD_k)] \\
&= (VIF_{k-1} + EC_{k-1})r - CF_k. \qquad (7.35)
\end{aligned}$$

Variances will mainly impact the cash flow component of the formula above. The first of these variances occurs when net interest income, $p - q_k$, differs from expectation. This will occur when the cost of funding w differs from expectation or if the interest rate on the loan differs from initial expectation. The latter should only occur on floating rate loans. The other set of variances will occur due to losses, i.e., if the predicted PD, EAD and LGD components differ from those observed or *implemented* (As far as PD and EAD are concerned, it should be fairly easy to compare the observed default rate to the predicted default rate. However, since some loans, such as mortgages, can have a long workout period, it generally is difficult to measure variances associates with the LGD parameter. The closest approximation would be to compare the latest modelled LGD to the previous model). Let VCF_k denote the variances observed during period k.

Assumption changes will arise in the form of changes in cost of funding, interest rates and the cost of equity. In addition, changes in the PD, EAD and LGD parameters will also come through as assumption changes. Let AC_k be the impact of assumption changes for period k. An additional change to the economic value that has not been discussed so far is the impact of new loans originated, i.e., VNB in long-term insurance. Let VNB_k denote the impact of new loans originated during period k. Therefore, the change in economic value is given by:

$$\Delta VIF_k = (VIF_k + EC_{k-1})r - CF_k + VCF_k + AC_k + VNB_k. \tag{7.36}$$

Note that the cash flow component of change in economic value does not constitute a real reduction in economic value, only a release of economic value from the VIF_kinto the profit and loss account, or free assets[74]. It only reduces the economic value here since our definition of economic value is based on expected future profits from the loan book (in other words, it compares to the VIF component in long-term insurance – we did not take the value of net assets into account). While the cash flows (including variances) reduce VIF_k, they will increase the net assets value by an equivalent amount.

7.3.4 Limitations and Further Refinements of the EV Formulae

It is important to be careful of how to treat early-stage (pre-default) delinquencies in the above formulae. The loss component EL_k only includes the expected impact of defaults on cash flows, it does not allow for initial loan delinquency. This treatment assumes that a delinquency that does not lead to default has no impact on the economic value of the loan, so that we only need worry about losses from accounts that eventually default. This assumption is not expected to be material, since delinquent borrowers that do not default are contractually expected to pay the additional interest accumulated. However, since the interest rate charged on the loan i will generally differ from the cost of funding w, a delinquent loan that does not default theoretically generates additional net interest income over the duration of the delinquency, since the average account balance over the life of the loan would be higher due to the delinquency. The measure derived above does not account for this impact of delinquency on EV.

The other limitation of the formulae presented above is the fact that we ignore the impact of operating expenses. In a previous section we discussed the fact that some lenders include a regular maintenance fee, which we assume is generally used to cover maintenance costs. We thus ignored the impact of maintenance expenses on economic value since we also did not allow for maintenance fee revenue. However, we can allow for these explicitly, as follows:

$$E_k = \sum_{j=1}^{n-k} \left(f_{j+k} - e_{j+k}\right) v_j^r, \tag{7.37}$$

where E_k is the impact of adding of maintenance expenses and fees to EV_k, f_k is the fee revenue during period k and e_k are the expected expense during period k.

[74] In long-term insurance, it is often referred to as the expected transfer from VIF to NAV.

Finally, we did not discuss the impact of early settlements/repayment of loans on the economic value. These will have a negative impact on the economic value since they leave a shorter period over which the interest spread can be earned. Conversely, contract extensions have the impact of increasing the economic value. Since these are generally difficult to forecast, they can be allowed for through the assumption changes component of economic value. However, there are approaches of modelling default rates while allowing for early settlements – for example, see Marimo & Chimedza (2017).

7.3.5 Comments on the Probability Distribution of EV_k

Our discussion above has represented the economic value on a best estimate probability-weighted basis (i.e., it is the expected present value of cashflows). However, since the cashflows are uncertain, the economic value can also be equally viewed as a random variable:

$$V\ddot{I}F_k = M_k - \ddot{L}_k - CoC_k, \tag{7.38}$$

where $\ddot{L}_k$ is the random loss associated with the loan portfolio, defined as:

$$\ddot{L}_k = \sum_{j=1}^{n-k} \ddot{pd}_{j+k} EAD_{j+k} LGD_{j+k} v_j^r, \tag{7.39}$$

and $\ddot{pd}_{j+k}$ is a random variable representing the probability of default. For the purpose of the discussion, we rewrite the expression for $\ddot{L}_k$ as follows:

$$\begin{aligned}\ddot{L}_k &= \left[\frac{\sum_{j=1}^{n-k} \ddot{pd}_{j+k} EAD_{j+k} LGD_{j+k} v_j^r}{\sum_{j=1}^{n-k} \ddot{pd}_{j+k} EAD_{j+k}}\right] \times \left[\frac{\sum_{j=1}^{n-k} \ddot{pd}_{j+k} EAD_{j+k}}{\sum_{j=1}^{n-k} \ddot{pd}_{j+k}}\right] \times \left[\sum_{j=1}^{n-k} \ddot{pd}_{j+k}\right] \\ &= \overline{LGD}_k \times \overline{EAD}_k \times \ddot{PD}_k, \end{aligned} \tag{7.40}$$

where $\ddot{PD}_k$, $\overline{EAD}_k$ and $\overline{LGD}_k$ are the weighted lifetime PD, EAD and LGD for the portfolio during time k, i.e.,

$$\left\{\overline{LGD}_k, \overline{EAD}_k, \ddot{PD}_k\right\} = \left\{\left[\frac{\sum_{j=1}^{n-k} \ddot{pd}_{j+k} EAD_{j+k} LGD_{j+k} v_j^r}{\sum_{j=1}^{n-k} \ddot{pd}_{j+k} EAD_{j+k}}\right], \left[\frac{\sum_{j=1}^{n-k} \ddot{pd}_{j+k} EAD_{j+k}}{\sum_{j=1}^{n-k} \ddot{pd}_{j+k}}\right], \left[\sum_{j=1}^{n-k} \ddot{pd}_{j+k}\right]\right\}. \tag{7.41}$$

Note that the statement of the random economic value $V\ddot{I}F_k$ assumes that credit losses are the only source of randomness. In practice, randomness can arise also through the other components of economic value since the cost of funds and the interest rate charged to the borrower may change over time (the latter will generally only change in the case of floating rate loans). However, in Figure 40 we will show changes in the cost of funds (assuming wholesale funding is used) are generally highly correlated to changes in the cost interest rate charged to consumers (i.e., on floating rate loans). Consequently, the economic value will not be as sensitive to rate changes as it is to changes in credit risk.

A further assumption we make is that $\ddot{PD}_k$ is the only random component of the $\ddot{L}_k$, while $\overline{EAD}_k$ and $\overline{LGD}_k$ are constant. We do not believe this assumption to be particularly onerous, given that it underlies the derivation of the regulatory capital requirement for credit risk under Basel III, as discussed in Chapter 3.

We assume that $\ddot{PD}_k$ is represented by the following probit model:

$$\ddot{PD}_k = \Phi(\Phi^{-1}(\overline{PD}_k) - \sigma\varepsilon), \tag{7.42}$$

where $\overline{PD}_k$ and σ the parameters of the model and ε is the source of randomness, following a standard normal distribution, i.e., ε would normally represent an exogenous (often macroeconomic) source of randomness. Note that this characterisation is equivalent to the exogenous maturity vintage model discussed in Chapter 3. In order to arrive at a distribution for $\ddot{L}_k$, we make use of the other assumptions underlying the Basel II capital requirement, i.e., the large homogenous portfolio assumptions (LHP). These result in the Vašíček (1987) distribution for the probability of default:

$$Prob\left[\ddot{PD}_k \leq x\right] = \Phi\left(\frac{\Phi^{-1}(x) - \Phi^{-1}(\overline{PD}_k)}{\sigma}\right). \tag{7.43}$$

Under these assumptions, the α^{th} percentiles of the economic value $\ddot{VIF}_k$ are given by:

$$Q_{\ddot{VIF}_k}(\alpha) = M_k - CoC_k - \Phi\left(\sigma\Phi^{-1}(x) + \Phi^{-1}(\overline{PD}_k)\right) \times \overline{LGD}_k \times \overline{EAD}_k. \tag{7.44}$$

7.4 Case Study

To demonstrate the application of the formulae derived above, we consider two case studies. The first is based on a simulation of a typical loan. We also apply the model to a cohort of mortgage loans issued into the South African market.

7.4.1 Simulation Case

We consider a loan with a principal amount of 10 000, which is the carrying value at origination. The loan is repaid over $n = 36$ months at an interest rate of $i = 1\%$ per month. Funding for the loan is available to the lender at $w = 0.42\%$ per month, and shareholders require a return on capital of $r = 1.08\%$ on the lending business. The marginal probability of default on the loan starts at 1% and reduces to 0.7% by the end of the term. The is summarised in Table 12.

	Rates			Credit Risk Parameters			
Duration k	Interest Rate i	Cost of Equity r	Cost of Funding w	Marginal PD pd_k	Lifetime PD $\ddot{PD}_k$	Lifetime EAD $\overline{EAD}_k$	Lifetime LGD $\overline{LGD}_k$
1	1.00%	1.08%	0.42%	1.00%	30.36%	5 734.38	50.00%
2	1.00%	1.08%	0.42%	0.99%	29.36%	5 589.09	50.00%
3	1.00%	1.08%	0.42%	0.98%	28.37%	5 443.26	50.00%
4	1.00%	1.08%	0.42%	0.97%	27.39%	5 296.89	50.00%
5	1.00%	1.08%	0.42%	0.96%	26.42%	5 149.99	50.00%
6	1.00%	1.08%	0.42%	0.95%	25.46%	5 002.55	50.00%
7	1.00%	1.08%	0.42%	0.94%	24.51%	4 854.58	50.00%
8	1.00%	1.08%	0.42%	0.93%	23.57%	4 706.06	50.00%
9	1.00%	1.08%	0.42%	0.92%	22.63%	4 557.01	50.00%
10	1.00%	1.08%	0.42%	0.91%	21.71%	4 407.43	50.00%
11	1.00%	1.08%	0.42%	0.90%	20.80%	4 257.30	50.00%
12	1.00%	1.08%	0.42%	0.90%	19.89%	4 106.63	50.00%
							...
30	1.00%	1.08%	0.42%	0.75%	5.08%	1 302.46	50.00%
31	1.00%	1.08%	0.42%	0.74%	4.33%	1 141.54	50.00%
32	1.00%	1.08%	0.42%	0.73%	3.59%	980.08	50.00%
33	1.00%	1.08%	0.42%	0.72%	2.86%	818.09	50.00%
34	1.00%	1.08%	0.42%	0.72%	2.13%	655.55	50.00%
35	1.00%	1.08%	0.42%	0.71%	1.41%	492.47	50.00%
36	1.00%	1.08%	0.42%	0.70%	0.70%	-328.85	50.00%

TABLE 12: ILLUSTRATION OF SIMULATION RATES AND PARAMETERS

In Table 13 we illustrate the change in carrying value and the cash flows associated with the loan.

	Loan Values and Movements						
Duration k	Loan Carrying Value A_k	Instalment p	Interest Income $A_{k-1}i$	Funding Repayment ΔA_k	Interest Expense $A_{k-1}w$	Credit Loss $pd_k EAD_k LGD_k$	Release of Capital $c\Delta A_k$
1	10 000.00	332.14	100.00	273.81	-41.67	-50.00	23.21
2	9 767.86	332.14	97.68	275.16	-40.70	-48.35	23.45
3	9 533.39	332.14	95.33	276.53	-39.72	-46.72	23.68
4	9 296.58	332.14	92.97	277.91	-38.74	-45.10	23.92
5	9 057.41	332.14	90.57	279.31	-37.74	-43.50	24.16
6	8 815.84	332.14	88.16	280.72	-36.73	-41.92	24.40
7	8 571.85	332.14	85.72	282.14	-35.72	-40.35	24.64
8	8 325.43	332.14	83.25	283.58	-34.69	-38.80	24.89
9	8 076.54	332.14	80.77	285.03	-33.65	-37.26	25.14
10	7 825.16	332.14	78.25	286.50	-32.60	-35.74	25.39
11	7 571.27	332.14	75.71	287.98	-31.55	-34.24	25.64
12	7 314.84	332.14	73.15	289.47	-30.48	-32.75	25.90
30	2 234.72	332.14	22.35	319.11	-9.31	-8.35	30.98
31	1 924.93	332.14	19.25	320.91	-8.02	-7.12	31.29
32	1 612.03	332.14	16.12	322.74	-6.72	-5.90	31.60
33	1 296.01	332.14	12.96	324.58	-5.40	-4.70	31.92
34	976.83	332.14	9.77	326.44	-4.07	-3.51	32.24
35	654.45	332.14	6.54	328.33	-2.73	-2.33	32.56
36	328.85	332.14	-3.29	330.22	-1.37	-1.16	32.89

TABLE 13: ILLUSTRATION OF SIMULATED LOAN VALUES

A flat monthly instalment amount of 332.14 is paid throughout the life of the loan. Only 232.14 of the instalment goes to repaying the loan in the first month (duration $k = 1$), since an interest charge of 100.00 is paid to the lender. The 232.14 is thus used by the lender to repay the providers of funding. In addition to this, the finding providers require an interest amount of 41.67, leaving only 58.33 as net interest income. However, credit losses take off 50.00 for the period, leaving only 8.33 as cash flow profits.

In Table 14 we show the expected economic value on the loan throughout its lifetime. The loan starts off with an economic value of 45.74 at origination (duration $k = 0$). The value of the loan increases initially while the unwind of the economic value exceeds the release of cash flow. However, as the release of cash flow accelerates with lower credit losses. The economic value decreases gradually to reach zero at the end of the term (duration $k = 36$).

	Economic Value Components					Changes in Economic Value		
Duration k	PV of Contractual Cash Flows M_k	Expected Loss EL_k	Cost of Capital CoC_k	Economic Value VIF_k	Economic Capital EC_k	Change in Economic Value ΔVIF_k	Expected Return $(VIF_k + EC_{k-1})r$	Cash Flow CF_k
0	996.47	765.67	1 716.39	45.74	1 000.00			
1	948.93	723.96	1 624.15	48.74	976.79	3.00	11.33	-8.33
2	902.23	683.45	1 534.46	51.22	953.34	2.48	11.11	-8.63
3	856.39	644.14	1 447.32	53.21	929.66	1.99	10.88	-8.89
4	811.44	606.01	1 362.72	54.73	905.74	1.52	10.65	-9.13
5	767.40	569.08	1 280.67	55.80	881.58	1.07	10.41	-9.33
6	724.28	533.32	1 201.16	56.45	857.19	0.65	10.16	-9.51
7	682.13	498.75	1 124.18	56.70	832.54	0.25	9.90	-9.65
8	640.95	465.35	1 049.74	56.57	807.65	-0.13	9.63	-9.77
9	600.78	433.13	977.84	56.08	782.52	-0.49	9.36	-9.85
10	561.65	402.08	908.47	55.26	757.13	-0.82	9.08	-9.90
11	523.56	372.20	841.64	54.13	731.48	-1.13	8.80	-9.93
12	486.57	343.49	777.34	52.72	705.58	-1.41	8.51	-9.92
30	38.50	24.01	55.18	7.34	192.49	-2.16	2.52	-4.69
31	27.69	17.15	39.45	5.40	161.20	-1.94	2.16	-4.11
32	18.59	11.44	26.33	3.70	129.60	-1.70	1.80	-3.50
33	11.23	6.86	15.81	2.28	97.68	-1.42	1.44	-2.86
34	5.65	3.43	7.91	1.17	65.45	-1.11	1.08	-2.19
35	1.90	1.14	2.64	0.40	32.89	-0.77	0.72	-1.49
36	0.00	0.00	0.00	0.00	0.00	-0.77	0.00	-1.49

TABLE 14: ECONOMIC VALUE OF A LOAN THROUGHOUT ITS LIFETIME

7.4.2 Real-World Case

As a real-world application, we use the formulas derived above to estimate the economic value of the South African mortgage lending market. We start with an easier problem of estimating the economic value of a single cohort of loans. According to data issued by South Africa's National Credit Regulator, South African lenders originated 35.5 thousand mortgage loans during the first quarter of 2018. The average value of the loans originated was about R1.0m, implying an aggregate value of R35.5bn for the entire industry.

In order to estimate the economic value of these loans, we need to first estimate the inputs into the valuation formula produced above. These are (a) the interest rate charged on these loans; (b) the required return for the loans; (c) the lenders' cost of capital; (d) the expected loss; (e) economic capital; and (f) the tenure of the loans.

7.4.2.1 Estimating the interest rate and cost of funds

In South Africa, the benchmark lending rate used by lenders is the Prime Overdraft Rate, where customers are generally charged a premium or discount to the Prime Overdraft Rate depending on factors such as the risk profile of the customer and the loan-to-value ratio. The Prime Overdraft Rate itself is set by the South African Reserve Bank, with reference to the Repo rate. Over the past decade, the Prime Overdraft Rate has been set at 3.5% above the Repo rate. In Figure 40 we show South Africa's Repo rate over time, alongside the Prime Overdraft rate. For this exercise we will use the Repo rate (which is the rate at which the South African Reserve Bank lends to commercial banks) as the cost of capital. In reality, the cost of funding for lenders, particularly banks, will be lower since they use retail funding. However, for this exercise we ignore the value added by retail funding – in effect, we assume that this value should be used to value the deposit franchise of a bank, not the lending franchise. For the first quarter of 2018, the Repo rate was 6.5%.

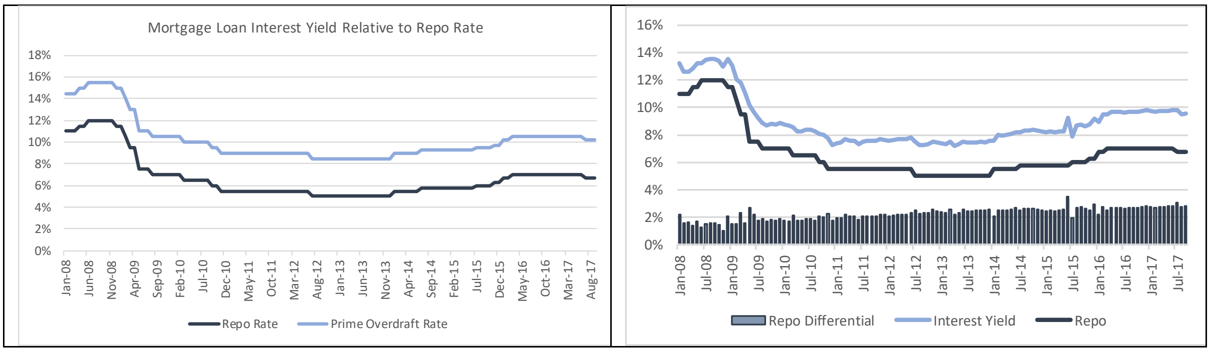

FIGURE 40: REPO RATE, PRIME OVERDRAFT RATE AND THE INTEREST YIELD FOR SOUTH AFRICAN MORTGAGES

In Figure 40 we also show the average interest yield on South African mortgage loans over time and compare this to the Repo rate (since the Repo rate and Prime Overdraft rate are absolutely correlated). Over the past two years, mortgage rates have yielded average of almost 3% above the Repo rate. Therefore, we assume an interest rate of 9.5% (made up of a 6.5% Repo rate plus a 3% premium).

7.4.2.2 Estimating the cost of capital

The lenders' required return is more difficult to estimate. To do this, we assume that the lending franchise is funded entirely through equity, so that the required return is equal to the cost of equity. The second assumption we make is that the cost of equity for the large four (in terms of mortgage lending) South African retail banks (Absa, Firstrand, Nedbank and Standard Bank) are representative of the industry's cost of equity. We believe this assumption to be fair, since these four banks account for most of the industry's mortgage lending.

We estimate the cost of equity for these four banks using the Capital Asset Pricing Model[75]. The inputs into the Capital Asset Pricing Model are the risk-free rate (r_f), the banking sector's beta parameter (β_{banks}) and the market's equity risk premium ($r_m - r_f$), so that the banking sector's cost of equity is given by:

$$r_{banks} = r_f + \beta_{banks} \times (r_m - r_f). \tag{7.45}$$

For the risk-free rate we use the yield on the South African Government's 10-Year bond, which averaged about 8.75% for the first quarter of 2018. The equity risk premium for the market is estimated as the average of the difference between the risk-free rate and the total return on the South African JSE All Share Index, over the past 20 years. This averaged to about 5.6%, as shown in Figure 41.

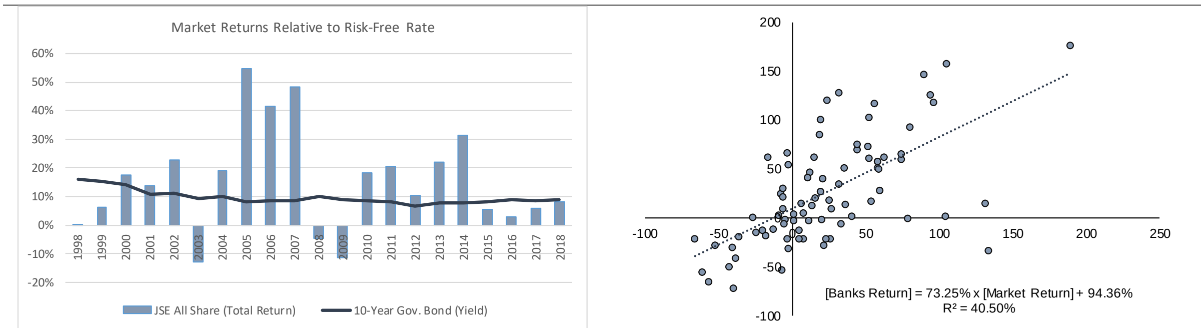

FIGURE 41: ESTIMATING THE MARKET'S EQUITY RISK PREMIUM AND THE SOUTH AFRICAN BANKING SECTOR'S BETA

The beta parameter is estimated through regression analysis. We regress the annualised quarterly total returns on the banking sector (consisting of the four banks above, with equal weighting) against the quarterly total returns on the All-Share Index. The regression yielded a beta parameter of 73%, as shown in Figure 41. Therefore, using the formula above, we estimate the lenders' cost of equity at about 12.75% for the first quarter of 2018.

7.4.2.3 Estimating the expected loss and economic capital

We estimate the economic capital on these loans, expressed as percentage of the carrying value, by looking at the Pillar III disclosures of the largest four South African retail banks, as required under Basel III. According to these disclosures, the large four banks had about R844bn in mortgage loans, producing around R254bn in risk-weighted assets. Under Basel III, banks are required to hold 12.5% of risk-weighted assets as required capital. We thus assume that economic capital for these banks amounted to R32bn (which is 12.5% of R254bn). Therefore, the banks hold an average of just below 4% of loan carrying value as economic capital. We summarise this in Table 15.

[75] Of course, we are mindful of the many imperfections in the Capital Asset Pricing Model, as discussed by Dayala (2012). However, we believe that the model serves well for illustrative purposes, since the estimation of the cost of equity is not the main subject of this chapter.

Bank	Loan Carrying Value	RWA	Economic Capital (at 12.5% of RWA)	Economic Capital as % of Carrying Value
Bank A	223 267	65 428	8 179	3.7%
Bank B	181 085	56 311	7 039	3.9%
Bank C	319 798	88 535	11 067	3.5%
Bank D	120 128	43 891	5 486	4.6%
Total	**844 278**	**254 165**	**31 771**	**3.8%**

TABLE 15: PILLAR III DISCLOSURES FOR THE LARGEST FOUR SOUTH AFRICAN BANKS

We also base our estimate of expected loss on the Pillar III disclosures. Using the components-based model, we show in Table 16 that the large four banks had a PD (over a 12-month horizon) of 7.5%, an EAD of 1.12 (relative to gross advances) and an LGD of 17.1%. This equates to a 12-month expected loss of 1.4%, as a proportion of carrying value. However, this is a 12-month expected loss and we require a lifetime expected loss for our purposes. The Pillar III disclosures do not require banks to disclose their estimates of the lifetime expected credit loss. We estimate this by adjusting the PD parameter from a 12-month horizon to lifetime horizon. A crude way of doing this is to assume that the hazard rate of default is constant throughout the lifetime of the loan. Under this assumption, the lifetime probability of default would be given by:

$$PD_{lifetime} = 1 - (1 - PD_{12month})^{n/12}, \tag{7.46}$$

where $PD_{12month}$ is the 12-month PD[76]. The typical mortgage tenure in South Africa is 20 years, although it sometimes goes up to 30 years. For the purpose of this exercise, we assume $n = 240$ months. The PD parameter is not the only one expected to change when the horizon increases from 12 months to lifetime; the EAD parameter should change. The LGD parameter would likely also change, but there are no obvious reasons why the change would be significant. Meanwhile, the EAD would almost certainly decrease. The longer the horizon, the lower the EAD. This is because mortgage loans amortise – meaning that the balance will gradually decrease over time, unlike revolving loans.

Instead of making a downward adjustment to the EAD, a simpler way is to adjust the PD downwards so that it reflects the discounted mean term (or duration) instead of the full tenure of the loan. The discounted mean term for a mortgage loan is given by (see Appendix 0 for a derivation):

$$d_n^i = \frac{a_{\overline{n}}^i(1+i) - nv_n^i}{ia_{\overline{n}}^i}. \tag{7.47}$$

At an interest rate of 9.5% over a tenure of 20 years, the discounted mean term works out to about 7 years. Therefore, substituting n for d_n in the equation above, we get a lifetime PD of 42% (compared

[76] Here we are essentially assuming that time-to-default follows a geometric distribution.

to the 7.5% 12-month PD). This yields a lifetime expected loss of 8%, as a proportion of gross advances. We summarise the details in Table 16.

Bank	Loan Carrying Value	PD (12-Month)	EAD	LGD	PD (Lifetime)	Expected Loss (12-Month)	12-Month Expected Loss as % of Carrying Value	Expected Loss (Lifetime)	Lifetime Expected Loss as % of Carrying Value
Bank A	223 267	10.0%	114.4%	11.5%	52.3%	2 940	1.3%	15 320	6.9%
Bank B	181 085	5.0%	112.6%	19.0%	30.0%	1 928	1.1%	11 642	6.4%
Bank C	319 798	7.2%	106.1%	21.3%	40.7%	5 201	1.6%	29 420	9.2%
Bank D	120 128	7.6%	118.6%	18.9%	42.5%	2 048	1.7%	11 454	9.5%
Total	**844 278**	**7.5%**	**111.5%**	**17.1%**	**41.8%**	**12 117**	**1.4%**	**67 837**	**8.0%**

TABLE 16: PILLAR III RISK DISCLOSURES FOR THE LARGEST FOUR SOUTH AFRICAN BANKS

7.4.2.4 Estimating the economic value of the 2018Q1 cohort of loans

Having estimated all the inputs into the formula, we are now in a position to estimate the economic value of the cohort of loans originated during the first quarter of 2018. We summarise the parameter inputs in Table 17.

Parameters	Estimate (Annual)	Estimate (Monthly)
Interest Rate (i)	9.50%	0.79%
Cost of Funding (w)	12.75%	0.54%
Cost of Capital (r)	6.50%	1.01%
Economic Capital (EC, c)	4.00%	4.00%
Expected Loss (EL)	4.00%	4.00%
Term (n)	20	240

TABLE 17: PARAMETER INPUTS INTO ESTIMATING THE ECONOMIC VALUE OF THE MORTGAGE COHORT

Additionally, Table 18 we summarise all the factors forming part of our derived formulae for economic value. These are calculated based on the parameter inputs above.

Factor	Formula	Estimate (Monthly)
p	${}^{1}/{}_{a_{\bar{n}}^{i}}$	0.93%
h	$\frac{1+r}{1+i}-1$	0.21%
$a_{\bar{n}}^{i}$	$\frac{1-v_{n}^{i}}{i}$	107.28
$a_{\bar{n}}^{r}$	$\frac{1-v_{n}^{r}}{i}$	90.47
$a_{\bar{n}}^{h}$	$\frac{1-v_{n}^{h}}{i}$	188.02
v_{n+1}^{i}	$(1+i)^{-(n+1)}$	0.15
$\boldsymbol{VIF_0}$	$\left\{\boldsymbol{pa_{\bar{n}}^{r}}\left[\mathbf{1}-\frac{\boldsymbol{w}}{\boldsymbol{i}}\right]\right\}+\left\{\boldsymbol{pv_{n+1}^{i}a_{\bar{n}}^{h}}\left[\boldsymbol{c}+\frac{\boldsymbol{w}}{\boldsymbol{i}}-\mathbf{1}\right]\right\}-\{\boldsymbol{EL}+\boldsymbol{EC}\}$	**7.41%**

TABLE 18: THE FACTORS UNDERLYING OUR CALCULATION OF ECONOMIC VALUE

The table above shows that the economic value of the cohort, at the loan inception, is 7.41% of the principal amount. This implies economic value of R2.63bn from mortgages originated during the period. Note that this excludes any costs associated with originating the loans (such as administration and marketing), under the presumption that these are recovered through fees charged to the customer at inception and throughout the life of the loan. In Table 19 we breakdown the monthly profit projections underlying the economic value of the cohort. Note that the EV of R5.47bn shown in the table is before the 8% allowance for expected loss.

Duration	Opening Balance	Instalment	Interest	Cost of Funding	Cost of Capital	Profit	PV Profit	Cumulative EV
1	35 500	-330.91	281.04	-192.29	-14.27	74.48	73.74	73.74
2	35 450	-330.91	280.65	-192.02	-14.25	74.37	72.90	146.64
3	35 400	-330.91	280.25	-191.75	-14.23	74.27	72.07	218.71
4	35 349	-330.91	279.85	-191.47	-14.21	74.16	71.25	289.97
5	35 298	-330.91	279.44	-191.20	-14.19	74.05	70.44	360.41
6	35 247	-330.91	279.04	-190.92	-14.17	73.95	69.64	430.05
7	35 195	-330.91	278.63	-190.64	-14.15	73.84	68.85	498.90
8	35 143	-330.91	278.21	-190.36	-14.13	73.73	68.06	566.96
9	35 090	-330.91	277.79	-190.07	-14.11	73.62	67.28	634.24
10	35 037	-330.91	277.37	-189.78	-14.09	73.51	66.51	700.75
11	34 983	-330.91	276.95	-189.49	-14.06	73.39	65.75	766.50
12	34 929	-330.91	276.52	-189.20	-14.04	73.28	64.99	831.49
24	34 247	-330.91	271.13	-185.51	-13.77	71.85	56.52	1 555.30
36	33 498	-330.91	265.19	-181.45	-13.47	70.28	49.03	2 183.93
48	32 674	-330.91	258.67	-176.99	-13.14	68.55	42.42	2 728.50
60	31 769	-330.91	251.50	-172.08	-12.77	66.65	36.58	3 198.83
120	25 700	-330.91	203.46	-139.21	-10.33	53.92	16.24	4 709.31
240	328	-330.91	2.60	-1.78	-0.13	0.69	0.06	5 468.79

TABLE 19: MONTHLY CALCULATIONS OF THE ECONOMIC VALUE OF THE COHORT

7.5 Summary of the Findings

We introduced the concept of embedded value for the lending business of a bank and discussed the analogies to a similar concept in long-term insurance. We showed how his can be used to estimate the value of a loan book as well as for estimating the attribution of value between value arising from maturity transformation and value arising from credit and liquidity pricing margins. We demonstrated how expected earnings and variances from expectations can be calculated throughout the life of the portfolio. We also showed that, under certain assumptions, the economic value of a loan portfolio follows a scaled Vašíček distribution.

The measure of value introduced in this chapter can be used as a way of appraising the value of a lending business, in the same way that embedded value has become a popular measure when appraising the value of a long-term insurer. We showed that our measure of value is more reflective than traditional metrics such as net interest margin, as it allows for the tenure and risk profile of the loan portfolio. Through the distributional assumptions made, it is also possible to estimate confidence interval for the economic value.

The chapter raises areas for further investigation. Our derived measure if only applicable to a portfolio of amortising loan products. Further research could investigate the measurement of economic value in a portfolio of revolving products. Our derived measure also has some limitation, highlighted in the chapter. For example, the measure does not allow for the impact of early repayments on economic value, to the extent that they can be modelled. Further, our distributional assumption for economic value ignored the fact that changes in interest rates and cost of funding may be an additional source of randomness in the measurement of economic value.

Chapter 8: Conclusions

8.1 Review of the Research

The thesis set out to answer several questions relating to consumer credit risk modelling and management.

Greater unification in consumer credit risk modelling

The first question was to explore the idea of *unification*: to provide a model that can be used across the various areas of consumer credit risk management. The thesis proposed the exogenous maturity vintage model. We started by resolving the perceived weaknesses of the standard model (i.e., the identifiability problem and its inability to provide account-level discrimination). This allows the model to be used in areas ranging from application scoring to behavioural risk modelling. We further developed the model by offering an approach to use this model to calculate the asset correlation coefficient. This means that the model can be used to determine both required capital, using prescribed values for the asset correlation coefficient, and economic capital, by using calculated values for the asset correlation coefficient. We further showed that the model proposed can be used to determine impairment provisions under IFRS 9 and for stress-testing.

The implication of this model is that a bank's credit management would be able to achieve greater levels of harmonisation by using a model that is consistent across all major areas.

Overcoming the identifiability problem

The second question related to finding a solution to the identifiability problem inherent in the standard exogenous maturity vintage model. This was altering the dimensions of the standard model, replacing the vintage dimension with a behavioural risk dimension.

Our extended model can thus be estimated without the need for any special adjustments or estimation technique (outside of standard maximum likelihood estimation, for instance). The parameter estimates are thus more reliable and easier to interpret than in the standard model. Our extension also means that the model is able to provide granular account-level predictions of risk.

Estimating the asset correlation coefficient

The third question surrounded accurate estimation of the asset correlation coefficient for a credit portfolio. The thesis showed that the parameters of the extended exogenous maturity vintage model can be used produce both a point-in-time and through-the-cycle estimates for the asset correlation. The thesis also described how the asset correlation coefficient changes as the size of a portfolio changes. Combining this with the extension to the exogenous maturity vintage model means that we are able to estimate the asset correlation coefficient relevant to a portfolio, given the cyclical properties of the portfolio, the operating environment and the size of the portfolio.

The research would thus permit banks and regulators to better understand the adequacy of bank capital. This has particular relevance for banks in developing countries – where the portfolios might be smaller and the cyclical nature of credit more atypical (thus warranting asset correlation coefficients that differ materially from those currently prescribed by regulators).

Discriminatory power of a model

The fourth question related to the measurement of the discriminatory power of a model. We showed that the Gini statistic of the exogenous maturity vintage model in relation to a given portfolio can be estimated as a function of the parameters of the model. We also showed how fundamental features of the population, particularly the overall average probability of default, impose an upper bound on the Gini statistic that may be achieved by the model.

A better understanding of the properties of the Gini statistic can help practitioners design better models and to understand the limitations of the Gini statistic.

Extending the exogenous maturity vintage model to a survival analysis domain

The fifth question related to the extension of the exogenous maturity vintage model to the survival time domain. The thesis demonstrated that by replacing the maturity dimension with a survival time dimension and the vintage dimension with a behavioural risk dimension, the exogenous maturity vintage model can be converted into a discrete time survival model. The model was shown to be a discrete-time analogue to the Cox proportional hazard survival model, but with the baseline hazard being time-varying instead of static.

The model developed would permit banks to accurately model probability of default under IFRS 9, where losses are sometimes measured over the entire remaining lifetime of an account rather than a fixed horizon. The model also permits us to incorporate time-varying macroeconomic forecasts, which is an important aspect of impairment modelling under IFRS 9.

Economic value of portfolio

The last question relates to the measurement of the economic value to shareholders of a portfolio of credit exposures. The thesis derived a series of formulae that may be used to estimate the economic value of a portfolio and how it is influenced by the pricing of the loans, the cost of funding, the loan tenure profile, the expected losses on the portfolio and the economic or regulatory capital requirements. The formula also allows us to understand how economic value changes as new loans are booked and as the loss experience unfolds.

A better understanding of how economic value is generated and how it changes would improve the ability of bank management to generate value for shareholders. It would further help financial analysts in appraising the fair value of a credit portfolio and help shareholders set appropriate incentives and remuneration structures for those who oversee the credit portfolios.

8.2 Limitations of the Research

The research has two sets of limitations.

Conceptual limitations

The conceptual limitations of the research mainly relate to the modelling of capital requirements and the measurement of economic value. In terms of capital requirements, the research mainly focused on the empirical estimation of the asset correlation coefficient, while controlling for portfolio size and heterogeneity. However, the research does not consider other aspects that may influence capital requirements, as discussed by Malwandla (2016), including the stochasticity of loss given default and how loss given default relates to probability of default.

The approach offered to measure economic value was mainly restricted to amortising products. We did not consider how this might be adopted to a portfolio of revolving loans or for amortising products with prepayment risk.

Empirical limitations

The empirical case studies presented in the thesis were based on South African data. Therefore, the research does not provide insight into how the models and approaches discussed might apply in other markets – particularly less developed markets.

For the survival analysis model presented in Chapter 6, the empirical case study only considered standard Cox regression as a comparative benchmark. This is because the proposed model is presented as an extension to standard Cox regression. Moreover, Cox regression is the most widely applied survival analysis technique in consumer credit risk – techniques incorporating time-varying covariates are less ubiquitous in this domain. Nevertheless, the fact that the model was not benchmarked against a model incorporating time-varying covariates is a limitation to the case study.

8.3 Areas of Further Research

The research left a few areas where future inquiry could be directed. We summarise some of these below.

1. **Understanding and modelling exogenous risk in a credit portfolio**
   In Chapter 3, we showed that the exogenous component of the exogenous maturity vintage model determines the level of systemic risk within a portfolio, which is a key influence on the capital requirement. The chapter mainly assumed that the exogenous risk follows a normal distribution, which leads to the Basel- Vašíček framework. However, as shown by Malwandla (2016), this assumption can be relaxed. Therefore, the model discussed here leaves scope to explore a wider set of distributions for the exogenous component, which might lead to significantly different results when calculating capital requirements. There is also scope to refine the techniques used for modelling the exogenous component, which was modelled via regression analysis in this thesis.
2. **Deriving formulae for analysing economic value for revolving loan portfolios**
   The analysis of economic value carried out in Chapter 7 was purely for amortising loan arrangements. However, analogous formulae may be derived for analysing economic value for revolving loan portfolios.
3. **Refining and extending the analysis of economic value for loan portfolios**
   The formulae derived for economic value, in Chapter 7, contained some simplifying assumptions. For example, the formulae did not explicitly account for the possibility and impact of early repayment. Additionally, the analyses provided only allowed for portfolio default rate as the only source of randomness a portfolio's economic value. In reality, multiple other factors, such as interest rates and cost of funding, are random influences on economic value. Therefore, Chapter 7 only provided an introductory look at the concept of economic value.

Chapter 9: Appendices

9.1 Setting parameter values for the LHP approximation

In Section 3.3.6 (Equation 3.27), we defined the probability of default for loan j out of a portfolio of n loans as:

$$P_j = \Phi\left(U_{t,s_j,l_j,k_j} + \sigma_e e_t\right), \tag{9.1}$$

where U_{t,s_j,l_j,k_j} is the deterministic aspect of the loan's risk profile and $e_t \sim N(0,1)$. Under the large homogenous portfolio (LHP) approximation, it is assumed that all loans within the portfolio have the same risk-profile:

$$U_t = U_{t,s_j,l_j,k_j}. \tag{9.2}$$

We proposed that U_t should be set equal to:

$$\widehat{U}_t = \sqrt{1+\sigma_e^2} \times \Phi^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} \Phi\left(\frac{U_{t,s_j,l_j,k_j}}{\sqrt{1+\sigma_e^2}}\right)\right), \tag{9.3}$$

on the basis that this would ensure that the expected probability of default is the same under the LHP as under the original portfolio. We show this below.

The portfolio probability of default under Equation 9.1 is as follows:

$$R_t = \frac{1}{n}\sum_{j=1}^{n} D_j, \tag{9.4}$$

where D_j is the default indicator. The expected portfolio probability of default is given by:

$$\begin{aligned} E[R_t] &= \frac{1}{n}\sum_{j=1}^{n} E\left[D_j\right] \\ &= \frac{1}{n}\sum_{j=1}^{n} E\left[E\left[D_j | e_t\right]\right] \\ &= \frac{1}{n}\sum_{j=1}^{n} E\left[\Phi\left(U_{t,s_j,l_j,k_j} + \sigma_e e_t\right)\right], \end{aligned} \tag{9.5}$$

where the second expectation is over e_t. From Owen (1980), this simplifies as follows:

$$E[R_t] = \frac{1}{n}\sum_{j=1}^{n} \Phi\left(\frac{U_{t,s_j,l_j,k_j}}{\sqrt{1+\sigma_e^2}}\right), \tag{9.6}$$

which is the expected portfolio default rate under the model proposed in Equation 9.1. Now consider the expected portfolio default rate under the model proposed in Equation 9.2. This is simply Equation 9.6, with U_{t,s_j,l_j,k_j} replaced by $\widehat{U}_t$:

$$E\left[\widehat{R}_t\right] = \frac{1}{n}\sum_{j=1}^{n} \Phi\left(\frac{\widehat{U}_t}{\sqrt{1+\sigma_e^2}}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \Phi\left(\frac{\sqrt{1+\sigma_e^2} \times \Phi^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} \Phi\left(\frac{U_{t,s_j,l_j,k_j}}{\sqrt{1+\sigma_e^2}}\right)\right)}{\sqrt{1+\sigma_e^2}}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \Phi\left(\frac{\sqrt{1+\sigma_e^2} \times \Phi^{-1}(E[R_t])}{\sqrt{1+\sigma_e^2}}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \Phi\left(\Phi^{-1}(E[R_t])\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} E[R_t]$$

$$= E[R_t]. \qquad (9.7)$$

Therefore, the expected portfolio default rate under the LHP assumption is preserved when U_t is set equal to Equation 9.3.

9.2 Derivation of the portfolio value-at-risk

In Section 3.3.6 (Equation 3.28), we derived the portfolio default rate to have the following distribution function:

$$D(x) = \Phi\left(\frac{\Phi^{-1}(x) - U_t}{\sigma_e}\right), \qquad (9.8)$$

where U_t is the deterministic aspect of the risk profile of the loan, which is assumed to be homogenous. The quantile function is found as the inverse of $D(x)$. Let α be the desired quantile of the distribution. We require $k(\alpha)$ such that:

$$D(k(\alpha)) = \alpha. \qquad (9.9)$$

This requires that:

$$\Phi\left(\frac{\Phi^{-1}(k(\alpha)) - U_t}{\sigma_e}\right) = \alpha$$

$$\Rightarrow \frac{\Phi^{-1}(k(\alpha)) - U_t}{\sigma_e} = \Phi^{-1}(\alpha)$$

$$\Rightarrow \Phi^{-1}(k(\alpha)) = \sigma_e \Phi^{-1}(\alpha) + U_t$$

$$\Rightarrow k(\alpha) = \Phi[\sigma_e \Phi^{-1}(\alpha) + U_t], \qquad (9.10)$$

as given in Equation 3.29.

9.3 Derivation of the asset correlation coefficient

In Section 3.3.7 (Equation 3.31), we provided the following formula for the value-at-risk under Basel II & III:

$$c(\alpha,t) = \Phi\left[\sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(\alpha) + \sqrt{\frac{1}{1-\rho}}\Phi^{-1}(\bar{R}_t)\right] \times EAD \times LGD, \tag{9.11}$$

where ρ is the asset correlation coefficient and $\bar{R}_t$ is the mean portfolio default rate.

This is similar to the value-at-risk under the Vašíček distribution, as provided in Equation 3.32:

$$c(\alpha,t) = \Phi\left[\sigma_e\Phi^{-1}(\alpha) + \sqrt{1+\sigma_e^2} \times \Phi^{-1}(\bar{R}_t\,)\right] \times EAD \times LGD. \tag{9.12}$$

For a special value of ρ, the Basel value-at-risk (Equation 9.11) and the Vašíček value-at-risk (Equation 9.12) are equivalent. In Equation 3.33, we specified this special value to be:

$$\rho = \frac{\sigma_e^2}{1+\sigma_e^2}. \tag{9.13}$$

This can be shown by substituting Equation 9.13 into Equation 9.11, as follows:

$$\begin{aligned} c(\alpha,t) &= \Phi\left[\sqrt{\frac{\left(\frac{\sigma_e^2}{1+\sigma_e^2}\right)}{1-\left(\frac{\sigma_e^2}{1+\sigma_e^2}\right)}}\Phi^{-1}(\alpha) + \sqrt{\frac{1}{1-\left(\frac{\sigma_e^2}{1+\sigma_e^2}\right)}}\Phi^{-1}(\bar{R}_t)\right] \times EAD \times LGD \\ &= \Phi\left[\sqrt{\frac{\left(\frac{\sigma_e^2}{1+\sigma_e^2}\right)}{\left(\frac{1}{1+\sigma_e^2}\right)}}\Phi^{-1}(\alpha) + \sqrt{\frac{1}{\left(\frac{1}{1+\sigma_e^2}\right)}}\Phi^{-1}(\bar{R}_t)\right] \times EAD \times LGD \\ &= \Phi\left[\sigma_e\Phi^{-1}(\alpha) + \sqrt{1+\sigma_e^2}\Phi^{-1}(\bar{R}_t)\right] \times EAD \times LGD, \end{aligned} \tag{9.14}$$

which is equivalent to Equation 9.12.

9.4 Regression for the exogenous component of the extended EMV model

In Section 3.4.2 we offered the results of a regression for the exogenous component, summarised in Table 5. The process followed to arrive at the models selected consists of three steps.

Lag Selection

The first step was to select the lag (or lead) at which each variable should be considered for the purpose of modelling. This was done by constructing correlograms of the correlation between the target variable (the exogenous component) and the independent variables at each lag (and lead), ranging from a lag of 18 months to a lead of 18 months. Each independent variable was included at a lag (or lead) corresponding to the maximum *absolute* correlation with the target variable.

Stepwise Regression

Once the variable lags were selected, model selection was done through stepwise regression, i.e., the regression procedure successively added or removed independent variables based on the AIC. For a more details on the stepwise selection procedure used, see SAS (1989).

Assessment

The models resulting from the stepwise regression were assessed for:

- **Multicollinearity**: to ensure no multicollinearity, we ensured that any model selected had a variance inflation factor below 3;
- **Sensibility**: as an additional check on multicollinearity, and to ensure that the model would produce sensible predictions, we ensured that the parameter estimate associated with each independent variable has the same sign as the correlation between the independent variable and the target variable; and
- **Significance**: we ensured that all variables included in the model have statistically significant parameters.

In the event that a model failed the three assessments above, the variable causing the failure (e.g., a variable with a variance inflation factor above 3, or a variable with an insignificant parameter) would be excluded from the stepwise regression process.

9.5 Deriving the extended LHP approximation

In Section 4.2.4, Table 8, we required that:

$$v^2(n,\sigma) = p(1-p) - 2\left(\frac{n-1}{n}\right)T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2\sigma^2}}\right) \tag{9.15}$$

equal:

$$v^2(\infty,s) = q(1-q) - 2T\left(\Phi^{-1}(q), \frac{1}{\sqrt{1+2s^2}}\right), \tag{9.16}$$

with $p = q$. Simplified, this means:

$$T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2s^2}}\right) = \left(\frac{n-1}{n}\right) T\left(\Phi^{-1}(p), \frac{1}{\sqrt{1+2\sigma^2}}\right), \tag{9.17}$$

as provided in Equation 4.27.

9.6 Estimated parameters of the extended EMV model

Table 20 below summarised the parameters estimates for the EMB model fitted in Section 3.4.2.

Variable	Level ID	Level Value	Fixed Rate Loans			Variable Rate Loans		
			Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value
Intercept			-1.4029	0.0007	0.0000	-1.6478	0.1264	0.0000
E	1	200509	-0.0214	0.0085	0.0122	-0.2842	0.0058	0.0000
E	2	200510	0.0222	0.0078	0.0044	-0.3019	0.0055	0.0000
E	3	200511	0.0354	0.0074	0.0000	-0.2756	0.0051	0.0000
E	4	200512	0.0362	0.0071	0.0000	-0.2372	0.0050	0.0000
E	5	200601	-0.0613	0.0070	0.0000	-0.2599	0.0047	0.0000
E	6	200602	0.0085	0.0066	0.1953	-0.2496	0.0044	0.0000
E	7	200603	-0.0145	0.0064	0.0227	-0.2034	0.0042	0.0000
E	8	200604	-0.0125	0.0064	0.0484	-0.2135	0.0040	0.0000
E	9	200605	0.0590	0.0060	0.0000	-0.2140	0.0039	0.0000
E	10	200606	-0.0069	0.0058	0.2345	-0.1883	0.0038	0.0000
E	11	200607	0.0673	0.0055	0.0000	-0.1779	0.0037	0.0000
E	12	200608	0.0678	0.0056	0.0000	-0.1309	0.0036	0.0000
E	13	200609	0.0736	0.0053	0.0000	-0.0903	0.0034	0.0000
E	14	200610	0.0235	0.0051	0.0000	-0.0657	0.0033	0.0000
E	15	200611	0.1167	0.0048	0.0000	-0.0049	0.0031	0.1203
E	16	200612	0.1648	0.0045	0.0000	-0.0051	0.0031	0.0959
E	17	200701	0.1975	0.0045	0.0000	0.0154	0.0030	0.0000
E	18	200702	0.1485	0.0042	0.0000	0.0093	0.0030	0.0020
E	19	200703	0.1879	0.0041	0.0000	0.0387	0.0029	0.0000
E	20	200704	0.1081	0.0039	0.0000	0.0763	0.0029	0.0000
E	21	200705	0.1012	0.0038	0.0000	0.0985	0.0028	0.0000
E	22	200706	0.0156	0.0038	0.0000	0.0982	0.0029	0.0000
E	23	200707	0.0252	0.0038	0.0000	0.1118	0.0029	0.0000
E	24	200708	0.0046	0.0037	0.2209	0.1362	0.0029	0.0000
E	25	200709	-0.0032	0.0036	0.3738	0.1125	0.0029	0.0000
E	26	200710	-0.0525	0.0037	0.0000	0.1255	0.0029	0.0000
E	27	200711	-0.0803	0.0036	0.0000	0.1318	0.0029	0.0000
E	28	200712	-0.0755	0.0034	0.0000	0.1406	0.0030	0.0000
E	29	200801	-0.0836	0.0034	0.0000	0.0972	0.0030	0.0000
E	30	200802	-0.0728	0.0033	0.0000	0.1377	0.0031	0.0000
E	31	200803	-0.0353	0.0031	0.0000	0.1078	0.0031	0.0000
E	32	200804	0.0189	0.0030	0.0000	0.1034	0.0031	0.0000
E	33	200805	0.0618	0.0028	0.0000	0.1139	0.0032	0.0000
E	34	200806	0.0673	0.0028	0.0000	0.1459	0.0032	0.0000
E	35	200807	0.1006	0.0027	0.0000	0.1249	0.0032	0.0000
E	36	200808	0.1236	0.0027	0.0000	0.1210	0.0032	0.0000
E	37	200809	0.1507	0.0027	0.0000	0.0857	0.0033	0.0000
E	38	200810	0.1417	0.0027	0.0000	0.1062	0.0033	0.0000
E	39	200811	0.1568	0.0027	0.0000	0.1212	0.0034	0.0000
E	40	200812	0.1484	0.0026	0.0000	0.0926	0.0034	0.0000
E	41	200901	0.1524	0.0027	0.0000	0.0987	0.0034	0.0000
E	42	200902	0.1033	0.0027	0.0000	0.0761	0.0035	0.0000
E	43	200903	0.0802	0.0027	0.0000	0.0872	0.0035	0.0000
E	44	200904	0.0761	0.0028	0.0000	0.0076	0.0036	0.0333
E	45	200905	0.0269	0.0028	0.0000	-0.0081	0.0036	0.0245
E	46	200906	0.0118	0.0029	0.0000	0.0229	0.0037	0.0000
E	47	200907	0.0044	0.0030	0.1409	-0.0252	0.0037	0.0000
E	48	200908	-0.0215	0.0030	0.0000	-0.0043	0.0038	0.2574
E	49	200909	0.0032	0.0031	0.2935	-0.0548	0.0038	0.0000
E	50	200910	-0.0290	0.0031	0.0000	-0.0515	0.0039	0.0000
E	51	200911	-0.0358	0.0032	0.0000	-0.0283	0.0039	0.0000
E	52	200912	-0.0606	0.0032	0.0000	-0.0646	0.0039	0.0000
E	53	201001	-0.0518	0.0033	0.0000	-0.1034	0.0039	0.0000
E	54	201002	-0.0609	0.0034	0.0000	-0.1073	0.0040	0.0000
E	55	201003	-0.0663	0.0035	0.0000	-0.0898	0.0040	0.0000
E	56	201004	-0.0593	0.0036	0.0000	-0.0885	0.0039	0.0000
E	57	201005	-0.0901	0.0036	0.0000	-0.1119	0.0040	0.0000
E	58	201006	-0.0926	0.0037	0.0000	-0.0822	0.0039	0.0000
E	59	201007	-0.1128	0.0038	0.0000	-0.1032	0.0039	0.0000

Variable	Level ID	Level Value	Fixed Rate Loans			Variable Rate Loans		
			Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value
E	60	201008	-0.1039	0.0039	0.0000	-0.0832	0.0038	0.0000
E	61	201009	-0.1371	0.0039	0.0000	-0.0789	0.0038	0.0000
E	62	201010	-0.0801	0.0040	0.0000	-0.0693	0.0037	0.0000
E	63	201011	-0.1431	0.0040	0.0000	-0.0428	0.0037	0.0000
E	64	201012	-0.1208	0.0041	0.0000	-0.0370	0.0037	0.0000
E	65	201101	-0.1288	0.0041	0.0000	-0.0667	0.0037	0.0000
E	66	201102	-0.1215	0.0041	0.0000	-0.0500	0.0037	0.0000
E	67	201103	-0.1476	0.0042	0.0000	-0.0352	0.0037	0.0000
E	68	201104	-0.1458	0.0043	0.0000	-0.0376	0.0037	0.0000
E	69	201105	-0.1339	0.0043	0.0000	-0.0272	0.0037	0.0000
E	70	201106	-0.1432	0.0043	0.0000	-0.0157	0.0037	0.0000
E	71	201107	-0.0994	0.0043	0.0000	-0.0270	0.0037	0.0000
E	72	201108	-0.0886	0.0044	0.0000	-0.0234	0.0037	0.0000
E	73	201109	-0.0980	0.0044	0.0000	-0.0154	0.0037	0.0000
E	74	201110	-0.1113	0.0044	0.0000	-0.0258	0.0038	0.0000
E	75	201111	-0.1092	0.0045	0.0000	-0.0040	0.0038	0.2945
E	76	201112	-0.1160	0.0046	0.0000	0.0123	0.0039	0.0013
E	77	201201	-0.0778	0.0046	0.0000	-0.0065	0.0039	0.0948
E	78	201202	-0.0895	0.0046	0.0000	-0.0049	0.0039	0.2072
E	79	201203	-0.0995	0.0047	0.0000	0.0129	0.0040	0.0012
E	80	201204	-0.1065	0.0047	0.0000	0.0032	0.0040	0.4352
E	81	201205	-0.0842	0.0046	0.0000	-0.0035	0.0040	0.3921
E	82	201206	-0.0445	0.0046	0.0000	0.0010	0.0041	0.8027
E	83	201207	-0.0415	0.0045	0.0000	0.0157	0.0041	0.0001
E	84	201208	-0.0445	0.0044	0.0000	0.0493	0.0041	0.0000
E	85	201209	-0.0311	0.0045	0.0000	0.0221	0.0041	0.0000
E	86	201210	0.0139	0.0044	0.0015	0.0230	0.0040	0.0000
E	87	201211	0.0262	0.0044	0.0000	0.0680	0.0040	0.0000
E	88	201212	0.0308	0.0044	0.0000	0.0496	0.0041	0.0000
E	89	201301	0.0296	0.0044	0.0000	0.0345	0.0040	0.0000
E	90	201302	0.0234	0.0045	0.0000	0.0337	0.0040	0.0000
E	91	201303	0.0018	0.0045	0.6839	0.0596	0.0040	0.0000
E	92	201304	0.0255	0.0045	0.0000	0.0744	0.0041	0.0000
E	93	201305	0.0163	0.0046	0.0004	0.0887	0.0040	0.0000
E	94	201306	0.0233	0.0046	0.0000	0.0984	0.0040	0.0000
E	95	201307	0.0264	0.0047	0.0000	0.0919	0.0040	0.0000
E	96	201308	0.0458	0.0047	0.0000	0.1028	0.0041	0.0000
E	97	201309	0.0616	0.0047	0.0000	0.0957	0.0041	0.0000
E	98	201310	0.0688	0.0047	0.0000	0.0907	0.0040	0.0000
E	99	201311	0.0989	0.0048	0.0000	0.1035	0.0041	0.0000
E	100	201312	0.0646	0.0048	0.0000	0.1052	0.0041	0.0000
E	101	201401	0.0702	0.0048	0.0000	0.0975	0.0041	0.0000
E	102	201402	0.0711	0.0049	0.0000	0.0691	0.0041	0.0000
E	103	201403	0.0903	0.0050	0.0000	0.0988	0.0041	0.0000
E	104	201404	0.0513	0.0049	0.0000	0.1261	0.0041	0.0000
E	105	201405	0.0658	0.0049	0.0000	0.0720	0.0042	0.0000
M	1	1	0.1224	0.0012	0.0000	0.0132	0.0013	0.0000
M	2	2	0.1573	0.0016	0.0000	0.1072	0.0018	0.0000
M	3	3	0.1701	0.0016	0.0000	0.1269	0.0017	0.0000
M	4	4	0.1593	0.0017	0.0000	0.1382	0.0017	0.0000
M	5	5	0.1610	0.0017	0.0000	0.1250	0.0017	0.0000
M	6	6	0.1322	0.0017	0.0000	0.1287	0.0018	0.0000
M	7	7	0.1162	0.0018	0.0000	0.1215	0.0018	0.0000
M	8	8	0.0869	0.0018	0.0000	0.1172	0.0018	0.0000
M	9	9	0.0789	0.0019	0.0000	0.0991	0.0018	0.0000
M	10	10	0.0580	0.0019	0.0000	0.1025	0.0019	0.0000
M	11	11	0.0479	0.0020	0.0000	0.0735	0.0019	0.0000
M	12	12	0.0886	0.0021	0.0000	0.0838	0.0020	0.0000
M	13	13	0.0871	0.0022	0.0000	0.0905	0.0020	0.0000
M	14	14	0.0846	0.0022	0.0000	0.0609	0.0021	0.0000
M	15	15	0.0633	0.0023	0.0000	0.0413	0.0021	0.0000
M	16	16	0.0024	0.0024	0.3201	0.0442	0.0022	0.0000
M	17	17	-0.0048	0.0025	0.0511	-0.0017	0.0022	0.4519
M	18	18	-0.0360	0.0026	0.0000	0.0103	0.0023	0.0000
M	19	19	-0.0230	0.0027	0.0000	-0.0125	0.0023	0.0000
M	20	20	-0.0565	0.0028	0.0000	-0.0340	0.0024	0.0000
M	21	21	-0.0911	0.0029	0.0000	-0.0378	0.0025	0.0000
M	22	22	-0.1058	0.0030	0.0000	-0.0526	0.0025	0.0000
M	23	23	-0.0819	0.0032	0.0000	-0.0390	0.0026	0.0000
M	24	24	0.0046	0.0036	0.1980	-0.0168	0.0027	0.0000
M	25	25	0.0121	0.0037	0.0012	-0.0284	0.0029	0.0000
M	26	26	-0.0299	0.0039	0.0000	-0.0372	0.0029	0.0000
M	27	27	-0.0472	0.0040	0.0000	-0.0656	0.0030	0.0000
M	28	28	-0.0867	0.0042	0.0000	-0.0862	0.0031	0.0000
M	29	29	-0.0985	0.0044	0.0000	-0.0772	0.0031	0.0000

Variable	Level ID	Level Value	Fixed Rate Loans			Variable Rate Loans		
			Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value
M	30	30	-0.1010	0.0046	0.0000	-0.1179	0.0033	0.0000
M	31	31	-0.1148	0.0049	0.0000	-0.1112	0.0033	0.0000
M	32	32	-0.1341	0.0050	0.0000	-0.1238	0.0034	0.0000
M	33	33	-0.2139	0.0053	0.0000	-0.1458	0.0035	0.0000
M	34	34	-0.2461	0.0055	0.0000	-0.1609	0.0037	0.0000
M	35	35	-0.2096	0.0060	0.0000	-0.1619	0.0038	0.0000
B	1		0.6592	0.0009	0.0000	0.9654	0.1264	0.0000
B	2		0.4866	0.0010	0.0000	0.7242	0.1264	0.0000
B	3		0.3133	0.0011	0.0000	0.5609	0.1264	0.0000
B	4		0.1756	0.0012	0.0000	0.4044	0.1264	0.0014
B	5		0.1048	0.0012	0.0000	0.2943	0.1264	0.0199
B	6		-0.0507	0.0013	0.0000	0.1130	0.1264	0.3712
B	7		-0.1646	0.0015	0.0000	-0.0562	0.1264	0.6563
B	8		-0.2937	0.0017	0.0000	-0.1822	0.1264	0.1495
B	9		-0.5277	0.0021	0.0000	-0.3735	0.1264	0.0031

TABLE 20: DETAILED PARAMETER ESTIMATES FOR THE EXTENDED EMV MODEL

Notice here that a high p-values for certain levels of a variable merely means that for a given level the parameter estimate was close to zero – it does not mean that the variable, overall, is insignificant. For instance, for fixed-rate loans, the exogenous effect during March 2013 (201303) was close to the mean, which produced an insignificant p-value for that period.

9.7 Estimated parameters for the Survival Analysis EMV Model

Table 21 summarises the parameter estimates for the survival model fitted in Section 6.3.1.1.

Variable	Level Value	Cycle 0			Cycle 1			Cycle 2			Cycle 3		
		Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value
Intercept		-3.0003	0.0043	0.0000	-2.6504	0.0242	0.0000	-2.4817	0.0253	0.0000	-2.3572	0.0580	0.0000
Base	1	-0.5304	0.0039	0.0000	-0.3454	0.0203	0.0000	0.2333	0.0229	0.0000	2.3132	0.0480	0.0000
Base	2	-0.2927	0.0038	0.0000	0.6279	0.0200	0.0000	1.1843	0.0229	0.0000	1.3196	0.0481	0.0000
Base	3	0.0479	0.0037	0.0000	0.6114	0.0200	0.0000	0.7180	0.0229	0.0000	1.1622	0.0481	0.0000
Base	4	0.1011	0.0037	0.0000	0.5173	0.0200	0.0000	0.6901	0.0229	0.0000	1.0613	0.0481	0.0000
Base	5	0.1290	0.0037	0.0000	0.4891	0.0200	0.0000	0.6471	0.0229	0.0000	0.9454	0.0482	0.0000
Base	6	0.1500	0.0037	0.0000	0.4593	0.0200	0.0000	0.5961	0.0229	0.0000	0.8648	0.0482	0.0000
Base	7	0.1609	0.0037	0.0000	0.4350	0.0201	0.0000	0.5555	0.0230	0.0000	0.7929	0.0483	0.0000
Base	8	0.1731	0.0037	0.0000	0.4215	0.0201	0.0000	0.5254	0.0230	0.0000	0.7505	0.0483	0.0000
Base	9	0.1702	0.0037	0.0000	0.3817	0.0201	0.0000	0.4958	0.0230	0.0000	0.7139	0.0484	0.0000
Base	10	0.1807	0.0038	0.0000	0.3810	0.0201	0.0000	0.4760	0.0230	0.0000	0.6635	0.0484	0.0000
Base	11	0.1844	0.0038	0.0000	0.3557	0.0201	0.0000	0.4568	0.0230	0.0000	0.6122	0.0485	0.0000
Base	12	0.1825	0.0038	0.0000	0.3394	0.0202	0.0000	0.4299	0.0231	0.0000	0.5873	0.0486	0.0000
Base	13	0.1878	0.0038	0.0000	0.3205	0.0202	0.0000	0.4204	0.0231	0.0000	0.5601	0.0487	0.0000
Base	14	0.1780	0.0038	0.0000	0.3018	0.0202	0.0000	0.4156	0.0231	0.0000	0.5027	0.0488	0.0000
Base	15	0.1804	0.0038	0.0000	0.2812	0.0203	0.0000	0.3675	0.0232	0.0000	0.4594	0.0489	0.0000
Base	16	0.1818	0.0038	0.0000	0.2981	0.0203	0.0000	0.3769	0.0232	0.0000	0.4548	0.0490	0.0000
Base	17	0.1724	0.0038	0.0000	0.2603	0.0204	0.0000	0.3563	0.0232	0.0000	0.4122	0.0491	0.0000
Base	18	0.1692	0.0038	0.0000	0.2482	0.0204	0.0000	0.3488	0.0233	0.0000	0.4089	0.0492	0.0000
Base	19	0.1696	0.0038	0.0000	0.2476	0.0204	0.0000	0.3409	0.0233	0.0000	0.3984	0.0494	0.0000
Base	20	0.1740	0.0038	0.0000	0.2366	0.0205	0.0000	0.3197	0.0234	0.0000	0.3835	0.0495	0.0000
Base	21	0.1694	0.0038	0.0000	0.2217	0.0206	0.0000	0.3070	0.0234	0.0000	0.3683	0.0497	0.0000
Base	22	0.1721	0.0038	0.0000	0.2225	0.0206	0.0000	0.2909	0.0235	0.0000	0.3391	0.0500	0.0000
Base	23	0.1581	0.0039	0.0000	0.2063	0.0207	0.0000	0.3107	0.0235	0.0000	0.3282	0.0502	0.0000
Base	24	0.1618	0.0039	0.0000	0.2009	0.0208	0.0000	0.2815	0.0236	0.0000	0.3055	0.0505	0.0000
Base	25	0.1501	0.0039	0.0000	0.1855	0.0209	0.0000	0.2852	0.0237	0.0000	0.2948	0.0507	0.0000
Base	26	0.1397	0.0039	0.0000	0.1753	0.0210	0.0000	0.2808	0.0238	0.0000	0.2942	0.0509	0.0000
Base	27	0.1482	0.0039	0.0000	0.1552	0.0212	0.0000	0.2596	0.0239	0.0000	0.2709	0.0513	0.0000
Base	28	0.1456	0.0039	0.0000	0.1503	0.0213	0.0000	0.2417	0.0241	0.0000	0.2551	0.0516	0.0000
Base	29	0.1434	0.0040	0.0000	0.1612	0.0214	0.0000	0.2364	0.0242	0.0000	0.2321	0.0521	0.0000
Base	30	0.1439	0.0040	0.0000	0.1264	0.0216	0.0000	0.2374	0.0243	0.0000	0.2039	0.0526	0.0001
Base	31	0.1295	0.0040	0.0000	0.1088	0.0218	0.0000	0.2251	0.0244	0.0000	0.2048	0.0529	0.0001
Base	32	0.1260	0.0040	0.0000	0.1617	0.0218	0.0000	0.2024	0.0247	0.0000	0.1765	0.0536	0.0010
Base	33	0.1210	0.0041	0.0000	0.1345	0.0220	0.0000	0.2072	0.0248	0.0000	0.1463	0.0544	0.0072
Base	34	0.1168	0.0041	0.0000	0.1192	0.0223	0.0000	0.1664	0.0252	0.0000	0.1050	0.0554	0.0582
Base	35	0.1022	0.0042	0.0000	0.1097	0.0226	0.0000	0.1678	0.0254	0.0000	0.1215	0.0557	0.0293
Base	36	0.1070	0.0042	0.0000	0.0899	0.0229	0.0000	0.1547	0.0256	0.0000	0.0992	0.0566	0.0797
Base	37	0.0987	0.0042	0.0000	0.0887	0.0231	0.0001	0.1560	0.0259	0.0000	0.1386	0.0565	0.0142
Base	38	0.0877	0.0043	0.0000	0.1043	0.0232	0.0000	0.1791	0.0260	0.0000	0.1274	0.0572	0.0260

Variable	Level Value	Cycle 0			Cycle 1			Cycle 2			Cycle 3		
		Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value
Base	39	0.0877	0.0043	0.0000	0.0833	0.0236	0.0004	0.1563	0.0264	0.0000	0.0799	0.0585	0.1721
Base	40	0.0800	0.0044	0.0000	0.0666	0.0240	0.0056	0.1184	0.0269	0.0000	-0.0044	0.0608	0.9430
Base	41	0.0710	0.0045	0.0000	0.0404	0.0245	0.0989	0.0757	0.0275	0.0060	0.0059	0.0614	0.9239
Base	42	0.0660	0.0045	0.0000	-0.0251	0.0255	0.3244	0.0787	0.0279	0.0048	-0.0544	0.0637	0.3937
Base	43	0.0461	0.0046	0.0000	0.0693	0.0250	0.0056	0.0295	0.0288	0.3049	-0.0245	0.0636	0.6999
Base	44	0.0499	0.0047	0.0000	0.0011	0.0261	0.9673	0.0890	0.0287	0.0019	-0.0322	0.0649	0.6195
Base	45	0.0442	0.0048	0.0000	-0.0141	0.0267	0.5963	0.0525	0.0296	0.0761	-0.0875	0.0675	0.1945
Base	46	0.0211	0.0049	0.0000	-0.0345	0.0274	0.2083	0.0917	0.0297	0.0020	-0.0528	0.0674	0.4339
Base	47	0.0159	0.0050	0.0016	0.0082	0.0274	0.7662	0.0418	0.0310	0.1768	0.0326	0.0657	0.6204
Macro	200701	-0.1640	0.0033	0.0000	0.0210	0.0139	0.1323	0.2189	0.0137	0.0000	0.0061	0.0260	0.8149
Macro	200702	0.2975	0.0026	0.0000	0.6076	0.0128	0.0000	0.5866	0.0119	0.0000	0.1083	0.0250	0.0000
Macro	200703	-0.0427	0.0030	0.0000	0.1273	0.0134	0.0000	0.3113	0.0127	0.0000	-0.0817	0.0250	0.0011
Macro	200704	-0.0462	0.0029	0.0000	0.0722	0.0134	0.0000	0.3112	0.0122	0.0000	0.0261	0.0247	0.2899
Macro	200705	0.0436	0.0028	0.0000	0.1981	0.0130	0.0000	0.4023	0.0114	0.0000	0.0590	0.0241	0.0145
Macro	200706	0.0579	0.0027	0.0000	0.2228	0.0129	0.0000	0.3134	0.0113	0.0000	0.0382	0.0238	0.1085
Macro	200707	0.0204	0.0027	0.0000	0.1680	0.0129	0.0000	0.2642	0.0112	0.0000	0.0337	0.0237	0.1558
Macro	200708	-0.0069	0.0027	0.0113	0.1029	0.0129	0.0000	0.2022	0.0110	0.0000	-0.0956	0.0236	0.0000
Macro	200709	0.0214	0.0027	0.0000	0.1605	0.0128	0.0000	0.2188	0.0107	0.0000	0.0146	0.0234	0.5329
Macro	200710	0.0549	0.0026	0.0000	0.1729	0.0127	0.0000	0.2724	0.0104	0.0000	0.0392	0.0233	0.0924
Macro	200711	0.0667	0.0026	0.0000	0.1966	0.0126	0.0000	0.2650	0.0102	0.0000	-0.0163	0.0232	0.4826
Macro	200712	0.0119	0.0026	0.0000	0.1206	0.0127	0.0000	0.1797	0.0103	0.0000	-0.1192	0.0233	0.0000
Macro	200801	0.0885	0.0025	0.0000	0.1710	0.0126	0.0000	0.2064	0.0101	0.0000	0.0377	0.0232	0.1042
Macro	200802	0.0853	0.0025	0.0000	0.1781	0.0126	0.0000	0.1569	0.0100	0.0000	-0.1220	0.0231	0.0000
Macro	200803	0.0540	0.0025	0.0000	0.1597	0.0126	0.0000	0.1755	0.0098	0.0000	-0.0866	0.0231	0.0002
Macro	200804	0.0493	0.0025	0.0000	0.1236	0.0126	0.0000	0.1370	0.0098	0.0000	-0.0554	0.0230	0.0160
Macro	200805	0.0412	0.0025	0.0000	0.1322	0.0126	0.0000	0.1175	0.0097	0.0000	-0.0996	0.0230	0.0000
Macro	200806	0.0061	0.0026	0.0162	0.0430	0.0127	0.0007	0.0383	0.0097	0.0000	-0.1103	0.0230	0.0000
Macro	200807	0.0550	0.0025	0.0000	0.1124	0.0126	0.0000	0.0950	0.0095	0.0000	-0.0422	0.0230	0.0663
Macro	200808	0.0525	0.0025	0.0000	0.0757	0.0126	0.0000	0.0655	0.0095	0.0000	-0.0441	0.0229	0.0546
Macro	200809	0.0112	0.0025	0.0000	0.0684	0.0126	0.0000	0.0842	0.0093	0.0000	-0.0489	0.0229	0.0328
Macro	200810	0.0291	0.0025	0.0000	0.0452	0.0127	0.0004	0.0968	0.0093	0.0000	-0.0172	0.0228	0.4518
Macro	200811	0.0165	0.0025	0.0000	0.0406	0.0127	0.0013	0.1093	0.0092	0.0000	0.0368	0.0228	0.1068
Macro	200812	-0.0271	0.0025	0.0000	0.0143	0.0127	0.2588	0.0653	0.0093	0.0000	-0.0410	0.0229	0.0735
Macro	200901	0.0201	0.0025	0.0000	0.0349	0.0127	0.0059	0.0893	0.0091	0.0000	0.0595	0.0228	0.0091
Macro	200902	0.1179	0.0024	0.0000	0.1314	0.0125	0.0000	0.1564	0.0089	0.0000	0.0873	0.0227	0.0001
Macro	200903	0.1124	0.0024	0.0000	0.1401	0.0125	0.0000	0.1686	0.0088	0.0000	0.0464	0.0227	0.0410
Macro	200904	0.1169	0.0024	0.0000	0.1012	0.0126	0.0000	0.1229	0.0088	0.0000	0.0592	0.0227	0.0090
Macro	200905	0.1190	0.0024	0.0000	0.1041	0.0126	0.0000	0.1164	0.0088	0.0000	0.0223	0.0227	0.3245
Macro	200906	0.0869	0.0024	0.0000	0.0923	0.0126	0.0000	0.1033	0.0089	0.0000	0.0105	0.0227	0.6442
Macro	200907	0.0692	0.0024	0.0000	0.0337	0.0127	0.0079	0.0593	0.0089	0.0000	-0.0532	0.0228	0.0194
Macro	200908	0.0556	0.0024	0.0000	0.0614	0.0127	0.0000	0.0454	0.0089	0.0000	-0.0131	0.0228	0.5647
Macro	200909	0.0378	0.0024	0.0000	0.0354	0.0127	0.0054	0.0554	0.0089	0.0000	0.0078	0.0228	0.7331
Macro	200910	0.0362	0.0024	0.0000	0.0347	0.0127	0.0064	0.0357	0.0089	0.0000	-0.0082	0.0228	0.7208
Macro	200911	-0.0083	0.0024	0.0007	0.0078	0.0128	0.5447	0.0150	0.0089	0.0925	-0.0293	0.0229	0.2008
Macro	200912	-0.0607	0.0025	0.0000	-0.0358	0.0129	0.0055	-0.0482	0.0090	0.0000	-0.0497	0.0229	0.0304
Macro	201001	-0.0342	0.0025	0.0000	-0.0306	0.0129	0.0178	-0.0492	0.0090	0.0000	0.0192	0.0230	0.4022
Macro	201002	0.0210	0.0024	0.0000	0.0143	0.0129	0.2662	-0.0343	0.0089	0.0001	-0.0142	0.0230	0.5354
Macro	201003	-0.0106	0.0025	0.0000	-0.0214	0.0129	0.0984	-0.0607	0.0089	0.0000	-0.1061	0.0231	0.0000
Macro	201004	-0.0399	0.0025	0.0000	-0.0502	0.0131	0.0001	-0.0861	0.0090	0.0000	-0.0657	0.0231	0.0044
Macro	201005	-0.0471	0.0025	0.0000	-0.0724	0.0131	0.0000	-0.0404	0.0089	0.0000	-0.0766	0.0231	0.0009
Macro	201006	-0.0479	0.0025	0.0000	-0.0516	0.0132	0.0000	-0.0684	0.0090	0.0000	-0.0627	0.0232	0.0068
Macro	201007	-0.0651	0.0025	0.0000	-0.0676	0.0132	0.0000	-0.0873	0.0090	0.0000	-0.0993	0.0233	0.0000
Macro	201008	-0.0778	0.0025	0.0000	-0.0564	0.0132	0.0000	-0.0657	0.0090	0.0000	-0.0644	0.0233	0.0056
Macro	201009	-0.0567	0.0025	0.0000	-0.0156	0.0132	0.2351	-0.0577	0.0090	0.0000	-0.0211	0.0232	0.3646
Macro	201010	-0.0735	0.0025	0.0000	-0.0221	0.0132	0.0952	-0.0710	0.0090	0.0000	-0.0572	0.0233	0.0142
Macro	201011	-0.0667	0.0025	0.0000	-0.0217	0.0133	0.1017	-0.0601	0.0090	0.0000	-0.0680	0.0234	0.0036
Macro	201012	-0.1265	0.0026	0.0000	-0.0777	0.0135	0.0000	-0.0987	0.0091	0.0000	-0.1009	0.0235	0.0000
Macro	201101	-0.0874	0.0026	0.0000	-0.0423	0.0134	0.0016	-0.0560	0.0091	0.0000	-0.0261	0.0235	0.2667
Macro	201102	-0.0579	0.0025	0.0000	0.0064	0.0133	0.6331	-0.0656	0.0090	0.0000	-0.0340	0.0234	0.1460
Macro	201103	-0.0697	0.0026	0.0000	-0.0197	0.0134	0.1437	-0.0885	0.0091	0.0000	-0.0789	0.0235	0.0008
Macro	201104	-0.0721	0.0026	0.0000	-0.0357	0.0136	0.0086	-0.0744	0.0091	0.0000	-0.0595	0.0235	0.0113
Macro	201105	-0.0818	0.0026	0.0000	-0.0133	0.0136	0.3266	-0.0667	0.0091	0.0000	-0.0615	0.0235	0.0088
Macro	201106	-0.0463	0.0026	0.0000	0.0290	0.0135	0.0317	-0.0265	0.0090	0.0032	0.0010	0.0234	0.9661
Macro	201107	-0.1858	0.0027	0.0000	-0.0997	0.0140	0.0000	-0.1620	0.0093	0.0000	-0.2915	0.0239	0.0000
Macro	201108	-0.0217	0.0025	0.0000	0.0734	0.0135	0.0000	0.0450	0.0089	0.0000	0.1000	0.0234	0.0000
Macro	201109	-0.1613	0.0027	0.0000	-0.0760	0.0140	0.0000	-0.1084	0.0093	0.0000	-0.2220	0.0239	0.0000
Macro	201110	-0.0358	0.0026	0.0000	0.0670	0.0136	0.0000	0.0416	0.0090	0.0000	0.1195	0.0234	0.0000
Macro	201111	-0.1075	0.0026	0.0000	-0.0150	0.0140	0.2825	-0.0449	0.0092	0.0000	-0.0164	0.0237	0.4893
Macro	201112	-0.1062	0.0027	0.0000	-0.0212	0.0140	0.1318	-0.0464	0.0092	0.0000	0.0061	0.0237	0.7965
Macro	201201	-0.0936	0.0026	0.0000	0.0174	0.0140	0.2136	-0.0286	0.0092	0.0018	0.0531	0.0237	0.0250
Macro	201202	-0.0405	0.0026	0.0000	0.0689	0.0138	0.0000	0.0019	0.0091	0.8308	0.0283	0.0236	0.2299
Macro	201203	-0.0711	0.0026	0.0000	-0.0229	0.0142	0.1069	-0.0587	0.0092	0.0000	-0.0918	0.0238	0.0001
Macro	201204	-0.0522	0.0026	0.0000	0.0132	0.0141	0.3499	-0.0065	0.0091	0.4798	0.0004	0.0237	0.9874

Variable	Level Value	Cycle 0			Cycle 1			Cycle 2			Cycle 3		
		Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value	Estimate	Std. Error	P-Value
Macro	201205	-0.0390	0.0026	0.0000	0.0843	0.0139	0.0000	0.0695	0.0090	0.0000	0.0909	0.0236	0.0001
Macro	201206	-0.0223	0.0026	0.0000	0.0396	0.0141	0.0051	0.0504	0.0091	0.0000	0.0483	0.0237	0.0412
Macro	201207	-0.0645	0.0027	0.0000	0.1719	0.0138	0.0000	0.1232	0.0090	0.0000	0.2014	0.0236	0.0000
Macro	201208	-0.0667	0.0027	0.0000	0.1335	0.0140	0.0000	0.1762	0.0090	0.0000	0.3034	0.0236	0.0000
Macro	201209	-0.0711	0.0027	0.0000	0.0402	0.0143	0.0050	0.0542	0.0093	0.0000	0.1796	0.0240	0.0000
Macro	201210	-0.0923	0.0027	0.0000	-0.0260	0.0146	0.0761	0.0226	0.0094	0.0162	0.0937	0.0242	0.0001
Macro	201211	-0.1176	0.0028	0.0000	-0.0126	0.0147	0.3910	-0.0063	0.0095	0.5059	-0.0219	0.0245	0.3703
Macro	201212	-0.0153	0.0027	0.0000	0.1310	0.0142	0.0000	0.1033	0.0093	0.0000	0.3072	0.0240	0.0000
Macro	201301	-0.0758	0.0027	0.0000	0.0409	0.0145	0.0049	0.0027	0.0094	0.7775	0.1081	0.0244	0.0000
Macro	201302	0.0303	0.0026	0.0000	0.1230	0.0142	0.0000	0.0477	0.0092	0.0000	0.0768	0.0241	0.0015
Macro	201303	-0.0335	0.0027	0.0000	0.0582	0.0144	0.0000	0.0089	0.0094	0.3405	0.0049	0.0243	0.8417
Macro	201304	0.0531	0.0026	0.0000	0.0020	0.0147	0.8909	0.0597	0.0093	0.0000	0.0650	0.0242	0.0072
Macro	201305	0.0276	0.0026	0.0000	0.0557	0.0144	0.0001	0.0693	0.0093	0.0000	0.0443	0.0242	0.0670
Macro	201306	0.0254	0.0027	0.0000	0.0884	0.0144	0.0000	0.0758	0.0093	0.0000	0.0970	0.0242	0.0000
Macro	201307	0.0109	0.0027	0.0000	0.0211	0.0147	0.1506	0.0781	0.0094	0.0000	0.0400	0.0243	0.0999
Macro	201308	-0.0078	0.0027	0.0037	0.0311	0.0147	0.0340	0.0164	0.0095	0.0858	0.0476	0.0244	0.0509
Macro	201309	0.0024	0.0027	0.3828	0.0253	0.0147	0.0848	0.0495	0.0095	0.0000	0.1018	0.0244	0.0000
Macro	201310	0.0120	0.0027	0.0000	0.0221	0.0147	0.1334	0.0514	0.0095	0.0000	0.0919	0.0244	0.0002
Macro	201311	0.0074	0.0027	0.0062	0.0720	0.0145	0.0000	0.0539	0.0095	0.0000	0.0985	0.0245	0.0000
Macro	201312	-0.0173	0.0027	0.0000	0.0337	0.0147	0.0219	0.0333	0.0096	0.0005	0.1064	0.0245	0.0000
Macro	201401	-0.0112	0.0027	0.0000	0.0587	0.0147	0.0000	0.0333	0.0095	0.0005	0.0965	0.0246	0.0000
Macro	201402	0.0400	0.0027	0.0000	0.1011	0.0145	0.0000	0.0237	0.0093	0.0108	0.0540	0.0243	0.0266
Macro	201403	0.0457	0.0027	0.0000	0.0659	0.0146	0.0000	0.0400	0.0094	0.0000	0.0150	0.0244	0.5390
Macro	201404	0.0704	0.0027	0.0000	0.0987	0.0145	0.0000	0.0752	0.0094	0.0000	0.0867	0.0242	0.0003
Macro	201405	0.1103	0.0026	0.0000	0.1225	0.0145	0.0000	0.0973	0.0094	0.0000	0.1027	0.0243	0.0000
Macro	201406	0.0463	0.0027	0.0000	0.0702	0.0147	0.0000	0.0106	0.0096	0.2707	0.0877	0.0245	0.0003
Macro	201407	0.0673	0.0027	0.0000	0.0835	0.0147	0.0000	0.0502	0.0096	0.0000	0.0366	0.0246	0.1376
Macro	201408	0.0424	0.0027	0.0000	0.0559	0.0149	0.0002	0.0243	0.0097	0.0124	0.0317	0.0262	0.2274
Macro	201409	-0.0137	0.0028	0.0000	0.0468	0.0153	0.0022	-0.0006	0.0101	0.9499	-0.0083	0.0275	0.7621
Macro	201410	0.0290	0.0028	0.0000	0.0614	0.0155	0.0000	0.0262	0.0103	0.0109	0.0320	0.0282	0.2569
Macro	201411	0.0089	0.0029	0.0021	0.0402	0.0160	0.0121	-0.0206	0.0107	0.0555	-0.0241	0.0298	0.4191
Behav	1	0.8257	0.0009	0.0000	0.4828	0.0076	0.0000	0.2819	0.0079	0.0000	0.1474	0.0244	0.0000
Behav	2	0.7287	0.0010	0.0000	0.4149	0.0076	0.0000	0.2450	0.0079	0.0000	0.1214	0.0245	0.0000
Behav	3	0.6573	0.0010	0.0000	0.3678	0.0076	0.0000	0.2166	0.0080	0.0000	0.1007	0.0245	0.0000
Behav	4	0.5824	0.0010	0.0000	0.3328	0.0077	0.0000	0.1918	0.0080	0.0000	0.0847	0.0245	0.0006
Behav	5	0.5547	0.0010	0.0000	0.3084	0.0077	0.0000	0.1683	0.0080	0.0000	0.0688	0.0246	0.0051
Behav	6	0.4647	0.0010	0.0000	0.2604	0.0078	0.0000	0.1351	0.0082	0.0000	0.0517	0.0248	0.0371
Behav	7	0.3867	0.0010	0.0000	0.2064	0.0080	0.0000	0.0936	0.0083	0.0000	0.0284	0.0250	0.2574
Behav	8	0.3075	0.0010	0.0000	0.1862	0.0082	0.0000	0.0780	0.0086	0.0000	0.0021	0.0255	0.9330
Behav	9	0.1928	0.0011	0.0000	0.0966	0.0088	0.0000	0.0082	0.0092	0.3760	-0.0010	0.0273	0.9698

TABLE 21: DETAILED PARAMETER ESTIMATED FOR THE SURVIVAL ANALYSIS EMV MODEL

Notice here that a high p-values for certain levels of a variable merely means that for a given level the parameter estimate was close to zero – it does not mean that the variable, overall, is insignificant. For instance, for Cycle 1, the macroeconomic risk index during December 2008 (200812) was close to the mean, which produced an insignificant p-value for that period.

9.8 Discounted mean term of a mortgage loan

In Section 7.4.2, Equation 7.47, we provided the following formula for the discounted mean term of a loan with constant instalments:

$$d_n^i = \frac{a_{\bar{n}}^i(1+i) - n v_n^i}{i a_{\bar{n}}^i}, \tag{9.18}$$

where i is the interest rate, n is the number of instalments, v_k^i is the discount factor:

$$v_n^i = (1+i)^{-k}, \tag{9.19}$$

and $a_{\bar{k}}^i$ is the annuity factor:

$$a_{\bar{k}}^{i} = \frac{1-v_k^i}{i}. \tag{9.20}$$

9.9 Moments of the Vašíček Distribution

In the thesis, we refer to the Vašíček a number of times. We briefly describe the properties of the distribution below.

Let $X{\sim}N(\mu,\sigma)$ be a Gaussian random variable and Z be defined as follows:

$$Z = \Phi(Z), \tag{9.21}$$

then Z follows a Vašíček distribution. The distribution function was shown in Equation 5.8 to be:

$$F(x) = \Phi\left(\frac{\Phi^{-1}(x)-\mu}{\sigma}\right), \tag{9.22}$$

and the density function was shown in Equation 5.9 to be:

$$f(x) = \frac{\phi\left(\frac{\Phi^{-1}(x)-\mu}{\sigma}\right)}{\sigma\phi\left(\Phi^{-1}(x)\right)}. \tag{9.23}$$

The mean of the distribution was shown in Equation 4.8 to equal:

$$E[\Phi(\mu+\sigma\varepsilon)] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right), \tag{9.24}$$

and the variance was shown in Equation 4.22 to equal:

$$v^2(\infty,\sigma) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)\left(1-\Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)\right) - 2T\left(\frac{\mu}{\sqrt{1+\sigma^2}},\frac{1}{\sqrt{1+2\sigma^2}}\right). \tag{9.25}$$

These can be specified in terms of the mean and variance parameters, μ and σ as above, or in terms of a mean probability of default p and an asset correlation coefficient ρ. In the latter, we apply the following substitutions:

$$\sigma = \sqrt{\frac{\rho}{1-\rho}} \text{ and } \mu = \frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}. \tag{9.26}$$

9.10 Simulations for the Identifiability Problem

The simulations for loan defaults discussed in Section 3.4.1 were generated through Python. We provide the script used in Figure 42. The algorithm encoded in this script is composed of three sections.

1) **Simulating model components**

The case study simulated consisted of simulating $n = 100\ 000$ observations. For each observation, we first simulated the values of the four dimensions: exogenous, maturity, vintage and behavioural.

Exogenous component

To simulate the exogenous component, we first simulated n uniform random variables for $T \in [1; 240]$, where T represents calendar time, i.e., we simulated n observations of T, where T could only take on 240 possible months.

The exogenous component E was simulated as a function of T:

$$E = \sin\frac{T}{18}.$$

This value was then standardised to obtain the final exogenous component:

$$\tilde{E} = \frac{E-\mu_E}{\sigma_E},$$

where μ_E and σ_E denote the mean and standard deviation of E.

Vintage component

To simulate the vintage component, we first simulated n uniform values for $C \in [\max(0, T-36)\,; T]$, where C represents the calendar month on which the loan was originated, i.e., C was simulated such that all accounts would be aged 36 or less, where age is calculated as $T - C$.

The vintage component V was simulated as a function of C:

$$V = \phi(C; 120{,}60),$$

where $\phi(x; \mu, \sigma)$ is the density function of the normal distribution, with a mean of μ and a standard deviation of σ.

This value was then standardised to obtain the final vintage component:

$$\tilde{V} = \frac{V-\mu_V}{\sigma_V},$$

where μ_V and σ_V denote the mean and standard deviation of V.

Maturity component

The maturity component was simulated from the cohort C and time T:

$$M = e^{\frac{T-C}{36}}.$$

This value was then standardised to obtain the final maturity component:

$$\widetilde{M} = \frac{M-\mu_M}{\sigma_M},$$

where μ_M and σ_M denote the mean and standard deviation of M.

Behavioural component

To simulate the behavioural component, we first simulated n random values for the risk group G from the simulated vintage V:

$$G = \max(\min(\lfloor 1000V + 2.5Z \rfloor, 1)\,, 0),$$

where Z is a simulated stnadnard normal random variable, while V is the simulated (non-standardised) vintage, from above. In other words, we assume that the distribution of risk group G is a function of the cohort (or vintage) that the account belongs to.

The behavioural component was simulated from the risk group G:

$$B = \sqrt{G}.$$

This value was then standardised to obtain the final behavioural component:

$$\tilde{B} = \frac{B-\mu_B}{\sigma_B},$$

where μ_B and σ_B denote the mean and standard deviation of B.

2) <u>Simulating defaults</u>

The default data was generated under two models. The first default indicator $D_{standard}$ was generated under the standard EMV model, according to the following default probability:

$$p_{standard} = \Phi\big(\mu + \beta_E\tilde{E} + \beta_M\widetilde{M} + \beta_V\tilde{V}\big),$$

while the second default indicator $D_{extended}$ was generated under the extended EMV model replacing the vintage dimension with the behavioural risk dimension.:

$$p_{extended} = \Phi\big(\mu + \beta_E\tilde{E} + \beta_M\widetilde{M} + \beta_B\tilde{B}\big).$$

The model parameters were set as: $\mu = -1$, $\beta_E = 0.6$, $\beta_M = 0.2$, $\beta_V = 0.2$ and $\beta_B = 0.2$.

Therefore, $D_{standard}$ was generated as a Bernoulli random variable with parameter $p_{standard}$. Similarly, $D_{extended}$ was generated as a Bernoulli random variable with parameter $p_{extended}$.

Therefore, the simulation produced n observations with two sets of outcome variables $D_{standard}$ and $D_{extended}$.

3) <u>Modelling defaults</u>

In Section 3.4.1, in order to test the identifiability problem, we use the data generated by the standard model (i.e., with outcome $D_{standard}$) to fit a standard EMV model. The estimates of the three components produced by the model, the exogenous component $\hat{E}$, the maturity component $\widehat{M}$ and the vintage component $\hat{V}$, are standardised and compared to the simulated values $\ddot{E}$, $\ddot{M}$ and $\ddot{V}$, respectively. Assuming no identifiability problem, we would expect the estimated values to align with the simulated values – which is not what we see in Section 3.4.1.

We repeat this analysis on the data generated by the extended model (i.e., with outcome $D_{extended}$), this time fitting the extended EMV model. The estimates of the three components produced by the model, the exogenous component $\hat{E}$, the maturity component $\widehat{M}$ and the behavioural component $\hat{B}$ are standardised and compared to the simulated values $\ddot{E}$, $\ddot{M}$ and $\ddot{B}$, respectively. Assuming no identifiability problem, we would expect the estimated values to align with the simulated values – which is what we see in Section 3.4.1.

```
import pandas as pd
import numpy as np
from sklearn.preprocessing import StandardScaler
from scipy.stats import norm
from scipy import random
from sklearn.linear_model import LogisticRegression

#generate random simulations
CalendarTime=random.randint(1,240,100000)
VintageTime=[random.randint(max(0,x-36),x,1)[0] for x in CalendarTime]
MaturityTime=CalendarTime-VintageTime
```

```
simdata=pd.DataFrame()
simdata['CalendarTime']=CalendarTime
simdata['VintageTime']=VintageTime
simdata['MaturityTime']=MaturityTime

#define and standardise pattern of components
simdata['Exogenous']=np.sin(simdata['CalendarTime']/18)
simdata['Maturity']=-np.exp(simdata['MaturityTime']/36)
simdata['Vintage']=norm.pdf(simdata['VintageTime'],120,60)
simdata['RiskGroup']=simdata.apply(lambda x:max(min(np.floor(random.normal(1000*x['Vintage'],2.5)),10),0),axis=1)
simdata['Behavioural']=np.sqrt(simdata['RiskGroup'])

simdata['Exogenous']=(simdata['Exogenous']-simdata['Exogenous'].mean())/simdata['Exogenous'].std()
simdata['Maturity']=(simdata['Maturity']-simdata['Maturity'].mean())/simdata['Maturity'].std()
simdata['Vintage']=(simdata['Vintage']-simdata['Vintage'].mean())/simdata['Vintage'].std()
simdata['Behavioural']=(simdata['Behavioural']-simdata['Behavioural'].mean())/simdata['Behavioural'].std()

simdata[['CalendarTime','Exogenous']].drop_duplicates().plot.scatter(x='CalendarTime',y='Exogenous')
simdata[['MaturityTime','Maturity']].drop_duplicates().plot.scatter(x='MaturityTime',y='Maturity')
simdata[['VintageTime','Vintage']].drop_duplicates().plot.scatter(x='VintageTime',y='Vintage')
simdata[['RiskGroup','Behavioural']].drop_duplicates().plot.scatter(x='RiskGroup',y='Behavioural')

#define parameters of probability of default
mu=-1
s_exogenous=0.6
s_maturity=0.2
s_vintage=0.2
s_behavioural=0.2

simdata['PD']=norm.cdf(mu+s_exogenous*simdata['Exogenous']+s_maturity*simdata['Maturity']+s_vintage*simdata['Vintage'])
simdata['PD_Extended']=norm.cdf(mu+s_exogenous*simdata['Exogenous']+s_maturity*simdata['Maturity']+s_behavioural*simdata['Be
havioural'])

simdata['PD'].hist(bins=25)
simdata['PD_Extended'].hist(bins=25)
print(simdata['PD'].mean())
print(simdata['PD_Extended'].mean())

#simulate defaults - standard EMV
simdata['D']=simdata.apply(lambda x:random.binomial(1,x['PD'],1)[0],axis=1)

simdata.groupby('CalendarTime')['D'].mean().plot(kind='line')
simdata.groupby('MaturityTime')['D'].mean().plot(kind='line')
simdata.groupby('VintageTime')['D'].mean().plot(kind='line')
print(simdata['PD'].mean())

#simulate defaults - extended EMV
simdata['D_Extended']=simdata.apply(lambda x:random.binomial(1,x['PD_Extended'],1)[0],axis=1)

simdata.groupby('CalendarTime')['D_Extended'].mean().plot(kind='line')
simdata.groupby('MaturityTime')['D_Extended'].mean().plot(kind='line')
simdata.groupby('RiskGroup')['D_Extended'].mean().plot(kind='line')

simdata.to_csv('emv_simulation.csv')
e=simdata[['CalendarTime','Exogenous']].drop_duplicates().rename({'CalendarTime':'Time','Exogenous':'Value'},axis=1).assign(Effect='Ex
ogenous')
m=simdata[['MaturityTime','Maturity']].drop_duplicates().rename({'MaturityTime':'Time','Maturity':'Value'},axis=1).assign(Effect='Maturi
ty')
v=simdata[['VintageTime','Vintage']].drop_duplicates().rename({'VintageTime':'Time','Vintage':'Value'},axis=1).assign(Effect='Vintage')
b=simdata[['RiskGroup','Behavioural']].drop_duplicates().rename({'RiskGroup':'Time','Behavioural':'Value'},axis=1).assign(Effect='Behavio
ural')
e.append(m).append(v).append(b).to_csv('effects.csv', index=False)


#fit EMV model to EMV data
x=pd.get_dummies(simdata[['CalendarTime','MaturityTime','VintageTime']], columns=['CalendarTime','MaturityTime','VintageTime'])
y=simdata['D']
emv_model = LogisticRegression(random_state=0).fit(x,y)
parameters=pd.DataFrame()
parameters['Parameter']=x.columns
parameters['Feature']=parameters.apply(lambda x:x['Parameter'].split('_')[0],axis=1)
```

```
parameters['Level']=pd.to_numeric(parameters.apply(lambda x:x['Parameter'].split('_')[1],axis=1))
parameters['Estimate']=emv_model.coef_[0]

parameters[parameters['Feature']=='CalendarTime'].plot.scatter(x='Level',y='Estimate')
parameters[parameters['Feature']=='MaturityTime'].plot.scatter(x='Level',y='Estimate')
parameters[parameters['Feature']=='VintageTime'].plot.scatter(x='Level',y='Estimate')

parameters.to_csv('emv_data_std_emv_model_estimates.csv')

#fit standard EMV model to the extended EMV data
x=pd.get_dummies(simdata[['CalendarTime','MaturityTime','VintageTime']], columns=['CalendarTime','MaturityTime','VintageTime'])
y=simdata['D_Extended']
ext_std_emv_model = LogisticRegression(random_state=0).fit(x,y)
ext_std_parameters=pd.DataFrame()
ext_std_parameters['Parameter']=x.columns
ext_std_parameters['Feature']=ext_std_parameters.apply(lambda x:x['Parameter'].split('_')[0],axis=1)
ext_std_parameters['Level']=pd.to_numeric(ext_std_parameters.apply(lambda x:x['Parameter'].split('_')[1],axis=1))
ext_std_parameters['Estimate']=ext_std_emv_model.coef_[0]

ext_std_parameters[ext_std_parameters['Feature']=='CalendarTime'].plot.scatter(x='Level',y='Estimate')
ext_std_parameters[ext_std_parameters['Feature']=='MaturityTime'].plot.scatter(x='Level',y='Estimate')
ext_std_parameters[ext_std_parameters['Feature']=='VintageTime'].plot.scatter(x='Level',y='Estimate')
ext_std_parameters.to_csv('ext_emv_data_std_emv_model_estimates.csv')

#fit extended EMV model to the extended EMV data
x=pd.get_dummies(simdata[['CalendarTime','MaturityTime','RiskGroup']], columns=['CalendarTime','MaturityTime','RiskGroup'])
y=simdata['D_Extended']
ext_emv_model = LogisticRegression(random_state=0).fit(x,y)
ext_parameters=pd.DataFrame()
ext_parameters['Parameter']=x.columns
ext_parameters['Feature']=ext_parameters.apply(lambda x:x['Parameter'].split('_')[0],axis=1)
ext_parameters['Level']=pd.to_numeric(ext_parameters.apply(lambda x:x['Parameter'].split('_')[1],axis=1))
ext_parameters['Estimate']=ext_emv_model.coef_[0]

ext_parameters[ext_parameters['Feature']=='CalendarTime'].plot.scatter(x='Level',y='Estimate')
ext_parameters[ext_parameters['Feature']=='MaturityTime'].plot.scatter(x='Level',y='Estimate')
ext_parameters[ext_parameters['Feature']=='RiskGroup'].plot.scatter(x='Level',y='Estimate')

ext_parameters.to_csv('ext_emv_data_ext_emv_model_estimates.csv')
```

FIGURE 42: PYTHON SCRIPT FOR SIMULATING LOAN DEFAULTS

9.11 Simulations for the Portfolio Default Rate

The simulations for portfolio default rate discussed in Section 4.3 were generated through Python. We provide the script used in Figure 43. This includes a function (*solve*) for approximating, in Equation 4.27. The algorithm encoded in this script can be summarised as follows.

1) **Generating portfolio default rate**

   We wish to generate k simulations of the default rate on a portfolio of size n. We begin by generating k simulations of standard normal random variables ε. The probability of default on the portfolio is then given by the following probit model:

   $$p = \Phi(\mu + \sigma\varepsilon),$$

   where μ and σ are the parameters of the model.

   From this, we then generate k portfolio default rates $\frac{X}{n}$, where X is simulated as a Binomial random variable with rate parameter p and n trials.

   In summary, the inputs into this simulation are the parameters μ and σ, the portfolio size n and the number of simulations required k. The output will be k simulated portfolio default rates.

2) **Choosing the number of simulations k**

We chose $k = 100\ 000$ simulations because this ensured that the sample variances were stable. This was determined by setting μ and σ such that the expected value of the portfolio default rate p would be 5% and the asset correlation coefficient would be 10%. These values were selected arbitrarily to correspond to reasonable real-world values. For instance, the asset correlation coefficient of 10% is somewhere between the 4% prescribed for revolving consumer loans and the 15% prescribed for mortgage loans under the Basel regulatory regimes. For this purpose, the portfolio size was kept at $n = 100$, which is a moderate portfolio size.

Using this, we generated 100 *different sets* of k simulations, i.e., a total of $100k$ simulations, divided into k sets. The average portfolio default rate was calculated for each of the k sets of 100 simulations, i.e., resulting in 100 sample means. The coefficient of variation was then calculated as the standard deviation over the mean of the 100 sample means. The resulting coefficient is approximately 0.8%, which we considered low enough to ensure for credible conclusions.

```
import pandas as pd
from scipy.stats import norm
from scipy import random
from scipy.special import owens_t
from scipy.optimize import brentq
import numpy as np

def port(n_vector,epd,rho,k):
    R=pd.DataFrame()
    s=np.sqrt(rho/(1-rho))
    m=norm.ppf(epd)*np.sqrt(1+s**2)
    for n in n_vector:
        vdd=[]
        for z in random.normal(m,s,k):
            p=norm.cdf(z)
            vdd.append(random.binomial(n,p,1)[0]/n)
        add=pd.DataFrame()
        add['n']=[n]
        add['rho']=[rho]
        add['epd']=[epd]
        add['sigma']=[s]
        add['v']=[np.var(vdd)]
        add['v_bin']=[epd*(1-epd)/n]
        add['v_vas']=[epd*(1-epd)-2*owens_t(m/np.sqrt(1+(s**2)),1/np.sqrt(1+2*(s**2)))]
        add['v_sbd']=[epd*(1-epd)-2*((n-1)/n)*owens_t(m/np.sqrt(1+(s**2)),1/np.sqrt(1+2*(s**2)))]
        R=R.append(add)
    return R

def f(x,*pars):
    n,m,s=pars
    orig_v=((n-1)/n)*owens_t(m,1/np.sqrt(1+2*(s**2)))
    modf_v=owens_t(m,1/np.sqrt(1+2*(x**2)))
    return 1000*(orig_v-modf_v)

#function for estimating s
def solve(epd,n,s):
    m=norm.ppf(epd)
    root, info = brentq(f, 0, 100, full_output=True, args=(n,m,s))
    return root

def simulate(n,epd,rho,k):
    R=pd.DataFrame()
    s=np.sqrt(rho/(1-rho))
    m=norm.ppf(epd)*np.sqrt(1+s**2)
    vdd=[]
```

```
    for z in random.normal(m,s,k):
        p=norm.cdf(z)
        vdd.append(random.binomial(n,p,1)[0]/n)
    R=pd.DataFrame()
    R['R']=vdd
    R['n']=n
    R['rho']=rho
    R['epd']=epd
    R['sigma']=s
    return R[['n','epd','rho','sigma','R']]

#test accuracy of 100 simulations
sim_100=port(list(np.repeat(10000,100)),0.05,0.15,100)
print(sim_100['v'].std()/sim_100['v'].mean())

#test accuracy of 1 000 simulations
sim_1000=port(list(np.repeat(10000,100)),0.05,0.15,1000)
print(sim_1000['v'].std()/sim_1000['v'].mean())

#test accuracy of 10 000 simulations
sim_10000=port(list(np.repeat(10000,100)),0.05,0.15,10000)
print(sim_10000['v'].std()/sim_10000['v'].mean())

#test accuracy of 100 000 simulations
sim_100000=port(list(np.repeat(10000,100)),0.05,0.15,100000)
print(sim_100000['v'].std()/sim_100000['v'].mean())

#run full comparison of different estimates of variance
surface=pd.DataFrame()
for p in [0.01,0.05,0.10,0.15,0.20]:
    for acc in [0.01,0.05,0.15,0.5]:
        surface=surface.append(port(list(range(100,1000,100))+list(range(1000,10100,1000)),p,acc,100000))

surface['modified sigma']=surface.apply(lambda x: solve(x['epd'],x['n'],x['sigma']), axis=1)
surface.to_csv(r'modifiedLHP.csv')

n_sim=10000
sims=pd.DataFrame()
for n,epd,rho in zip(surface['n'],surface['epd'],surface['rho']):
    sims=sims.append(simulate(n,epd,rho,n_sim))

sims.to_csv(r'sims.csv')
```
FIGURE 43: PYTHON SCRIPT FOR PORTFOLIO DEFAULT RATE SIMULATION

9.12 Sample Specifications

In Chapters 3 and 6 we made use of a sample of loans data to demonstrate the models discussed. We briefly discuss the sample properties here.

The sample consists of loans issued by a South African retail bank. The loans were unsecured personal loans. The loan sample relates contains two types of products – fixed-rate loans and variable-rate loans. Fixed-rate loans relate to loans where the interest rate charged to the borrower stays constant over the entire life of the loan. Variable-rate loans relate to loans where the interest rate charged to the borrower varies periodically with changes in the Prime Overdraft rate, which is the benchmark consumer lending rate in South Africa.

The sample consists of accounts that were randomly sampled from a population of loans disbursed between September 2005 and June 2014. Performance on these loans was monitored monthly between

October 2005 and June 2015. Loans were only monitored for the period while they were active (not closed) and not in default. There were 2 476 000 observations meeting this criterion.

The models discussed in Chapters 3 and 5 were focused on probability of default. The definition of default used in generating the sample was chosen to align closely with that of the bank where the loans were originated – which in turn aligns with prevailing norms within the South African industry. Specifically, loans were classified as being in default if: (a) they were more than 3-month in default; or (b) they were moved into litigation due to operational reasons (e.g., if the borrower became deceased). Under this definition of default, the observed 12-month default rate on the portfolio was generally between 10% and 15%, depending on the time period.

The models applied to the data segmented the data into delinquency cycles. These cycles were defined to align with the originating bank's own cycle definitions. Generally, the following definition is used: (a) Cycle 0: accounts current with payments, (b) Cycle 1: accounts in arrears by less than 1 payment, (c) Cycle 2: accounts in arrears by at least 1 payment but less than 2 payments, (c) Cycle 3: accounts in arrears at least 2 payments but less 3 payments.

The models applied made used of behavioural and application scores / risk grades. These were provided to the Bank by an external vendor (a credit bureau). The application score is designed and used to assess the level of credit risk on new loan applications, while the behavioural score is designed and used for ongoing credit risk measurement. Both scores make use of data provided by the bank and other credit providers in the South African market, considering factors such as *the delinquency profile of the borrower on various categories of loans* and *the length and breadth of available credit history on the borrower*. The credit scores take on numeric variables, typically from 300 to 800, with higher scores being associated with lower credit risk. As described in Chapters 3 and 5, the scores were not used in their raw form – they were generally grouped into quantiles, mainly to enhance the credibility of the model.

9.13 More on the components of the Survival Analysis EMV model

In Section 6.3 we provided an empirical case study of the survival analysis EMV model, which involved fitting a regression model to the macroeconomic risk index and creating a behavioural risk index from a scorecard. We provide more details on these below.

Regression model for macroeconomic risk index

The variables used in the regression were included with no lags or leads, as they were expected to have a contemporaneous relationship with the default hazard rate, i.e., unlike a 12-month probability of default, which is an average of a year's worth of experience, a one-month hazard rate is expected to be impacted current macroeconomic circumstances. However, in principle, lags and leads might be considered as part of the modelling process, in a manner similar to that which was described in Appendix 9.4.

The model selection was done through stepwise regression, i.e., the regression procedure successively added or removed independent variables based on the AIC. For a more details on the stepwise selection procedure used, see SAS (1989).

The models resulting from the stepwise regression were assessed for:

- **Multicollinearity**: to ensure no multicollinearity, we ensured that any model selected had a variance inflation factor below 3;
- **Sensibility**: as an additional check on multicollinearity, and to ensure that the model would produce sensible predictions, we ensured that the parameter estimate associated with each independent variable has the same sign as the correlation between the independent variable and the target variable; and
- **Significance**: we ensured that all variables included in the model have statistically significant parameters.

In the event that a model failed the three assessments above, the variable causing the failure (e.g., a variable with a variance inflation factor above 3, or a variable with an insignificant parameter) would be excluded from the stepwise regression process.

Behavioural scorecard

The scorecard used in the to construct the behavioural risk index was provided to the Bank by an external vendor (a credit bureau). The scorecard produces behavioural scores designed and used for ongoing credit risk measurement. The scorecard makes use of data provided by the bank and other credit providers in the South African market, considering factors such as *the delinquency profile of the borrower on various categories of loans* and *the length and breadth of available credit history on the borrower*. The credit scores take on numeric variables, typically from 300 to 800, with higher scores being associated with lower credit risk.

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