# Accounting for Roll-over Risk in the Pricing of Caps and Floors

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# Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

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# Abstract

The peak of the global financial crisis necessitated practitioners to rethink the single curve approach to pricing interest-rate derivatives. This was as a result of a violation in spot-forward parity relationships thereby prompting markets to realise the presence of a new type of risk and subsequently the need for a multi-curve pricing framework. The *roll-over risk* framework is one that accounts for liquidity constraints and default risk thereby providing a cogent explanation for the spot-forward parity violation that led to the need for multiple curves. The primary objective of this work is to price XIBOR-based caps and floors under a framework which accounts for roll-over risk. This reformulation of interest-rate derivatives is achieved using Fourier Transform methods as well as Monte Carlo simulations for comparison. We found that the results obtained using the two approaches were comparable even though the two methods are different in nature. This agreement in prices is compelling evidence that the computations are correct.

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## Chapter 1

# Introduction

Interest-rates are the backbone of modern day finance and economics, both in theory and practice. Owing to fundamentals in risk-neutral contingent claim pricing theory, the pricing, valuation and hedging of effectively all financial derivative products relies heavily on understanding and modelling interest-rate dynamics. Of interest in this work is the modelling of interbank interest-rates (generally referred to as XIBOR) from short-rates in the presence of roll-over risk, to be described below. Thereafter, the focus switches to the pricing of interest-rate cap and floor derivative instruments when roll-over risk is accounted for.

When borrowing funds on an unsecured basis, an entity may choose to borrow funds ”in one go” (e.g. at the LIBOR or EURIBOR) or ”roll-over” borrowing (e.g. at the EONIA or SABOR rate, compounded daily) for a predetermined period. Given a bank with XIBOR panel average characteristics, this borrowing can be achieved at a rate equal to XIBOR. Alternatively, the bank may choose to borrow continuously at the prevailing interbank overnight rate. For equal tenors, the textbook no-arbitrage argument leads to the conclusion that the trader would be indifferent between the two borrowing strategies. On the other hand, post Global Financial Crisis (GFC) markets have consistently displayed the presence of a XIBOR-OIS spread shown in Figure 1.1. The XIBOR-OIS spread is effectively a reflection of the credit and liquidity risk taken by a counterparty that lends funds on an uncollateralised basis to a ”risky” counterparty in a market with liquidity constraints. The presence of the spread incorrectly leads to the conclusion that the trader would prefer to roll borrowing over at the prevailing overnight rate and hedge the stochasticity of the overnight rate with an OIS product. Furthermore, as an arbitrage strategy, the trader may choose to simultaneously lend out funds at the prevailing XIBOR in addition to the rolling over borrowing and using an OIS hedge. This would effectively lead to the trader pocketing the XIBOR-OIS spread at maturity, which according to Chang and Schlögl (2014), has been observed in financial markets since the GFC in 2008. Chang and Schlögl (2014) further point out that the aforementioned arbi-

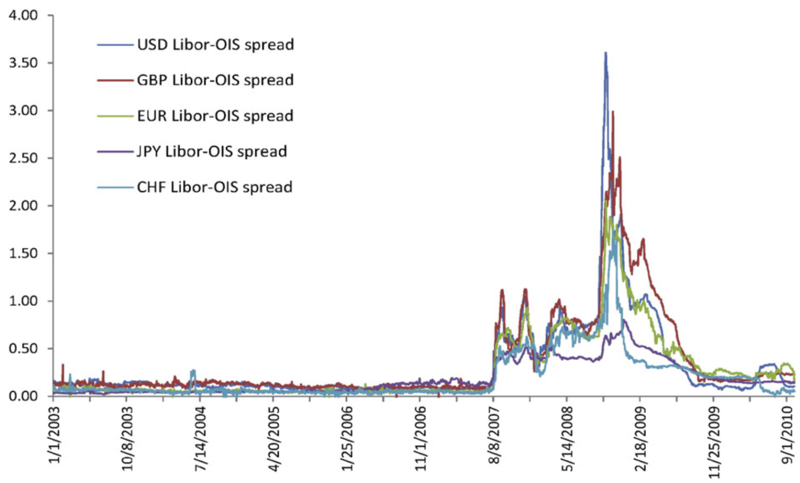


**Fig. 1.1:** LIBOR-OIS spreads in various markets at the height of the credit crunch (Cui *et al.*, 2016).

trage strategy is viable as market-observable XIBOR-OIS spreads exceed transaction costs. However, these conclusions are incorrect as they fail to account for the fact that the lending counterparty takes on credit risk and may experience liquidity-related premiums in funding their position. As such, the above-mentioned strategy does not fall within the standard textbook definition of arbitrage.

Loosening some of the assumptions, i.e., factoring in the borrowing entity's credit riskiness and market liquidity constraints, the work of Alfeus *et al.* (2020) serves as a stepping stone in understanding why the above-mentioned textbook no-arbitrage conclusions are incorrect and provides a framework in which these risks can be isolated and quantified using market-observable interest-rate instruments.

The aim of this dissertation is implement the Fourier Transform method in pricing interest-rate derivatives in which roll-over risk is accounted for. The dissertation closely follows the work of Alfeus *et al.* (2020) from which the square-root Cox-Ingersoll-Ross (CIR) model calibration parameters are obtained and used in the pricing of XIBOR-linked options. We view the dissertation as an extension of the work of Alfeus *et al.* (2020) into options as their primary focus is on the reformulation of XIBOR and swaps when roll-over risk is accounted for. To demonstrate the validity of prices obtained using the computationally intensive Fourier Transform method, a Monte Carlo pricing approach is also used.

The dissertation begins with review of background literature in the field of short rate modelling. This is done with the intention to layout key models in the modelling of interest-rates, demonstrate how bond prices are determined from the various one-factor models as well as provide justification for the selection of CIR dynamics as a model of choice in this work. This is then followed by an overview of pricing in the classical sense, i.e., where issues of credit/default and liquidity risk are not accounted for. Subsequently, we provide a review of the roll-over risk approach and how it impacts the pricing of cap and floor interest rate derivative products. This reformulation of interest-rates is then dissected in Chapter 4 and the results presented and discussed in Chapter 5. Final concluding remarks are then made together with recommendations for further work in Chapter 6.

# Chapter 2

# Interest-rate Modelling

## 2.1 Fundamentals – The Risk-free Approach

The coexistence of individuals and institutions with funding deficits and those with surplus funds is the primary driver for interest-rate markets. The lender gives up their liquidity for a finite period and is therefore compensated for this by earning interest whereas the borrower ”gains” liquidity at the price of the appropriate interest-rate.

Mathematically, let $A_t$ denote the loan balance of a lender who lends to a ”safe” entity such as a government in the domestic currency, then, the balance changes according to the differential equation $dA_t = r_t A_t dt$. Given the period from $t$ to $T$, the loan balance due to the lender amounts to

$$A_T = A_t e^{\int_t^T r_s\, ds}, \tag{2.1}$$

and would be paid at maturity. The interest rate $r_t$ is coined the ”risk-free” rate, hence the loan balance grows at a risk-free rate, i.e., the rate which does not price in default and liquidity risk. It is worth noting at this point that the risk-free rate is a theoretical abstraction of a rate such as the overnight rate. In practice, overnight rates such as EONIA, SABOR, etc. are the shortest tenors (one-day) for which lending and borrowing take place in interbank markets.

Expanding on the above, the zero-coupon bond (ZCB) can defined as a contract that does not pay coupons during the life of the bond but permits the holder to redeem one unit of currency at some future maturity time $T$. If the current time is $t$ and the ZCB matures at time $T$ (i.e., a $T$-bond), then the time $t$ $(< T)$ value of the contract is denoted $P(t, T)$. The price of a risk-free ZCB is given by the discounted future value of the bond under the risk neutral measure $\mathbb{Q}$ as

$$P(t, T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds}\big|\mathcal{F}_t\right]. \tag{2.2}$$

This denotes a standard conditional expectation on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\in[0,T]})$. It also then follows from the definition that the time $T$ value of the ZCB is given by $P(T,T) = 1$. Hence, the mark-to-market value of the ZCB will evolve with time as the risk-free rate changes but pays a guaranteed nominal of one currency unit at maturity.

## 2.2 Affine Term Structure Models

**Definition 2.2.1.** *A short-rate model of the form $dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$ is said to possess an Affine Term Structure (ATS) if bond prices are given by the expression*

$$P(t,T) = F^T(t, r_t) = e^{\alpha(t,T) - \beta(t,T)\cdot r_t},$$

*where $\alpha(t,T)$ and $\beta(t,T)$ are deterministic functions.*

Even though not all short-rate models are affine, it is worth noting that the class of affine term structure models is very well understood in the field of quantitative finance. It can be shown that the drift and volatility components of the short-rate model are required to be affine in $r_t$, i.e., we require that the drift and volatility can be expressed in the forms

$$\mu(t, r_t) = A(t)r_t + B(t)$$

and

$$\sigma^2(t, r_t) = C(t)r_t + D(t),$$

where $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are deterministic functions of $t$. Thus, to compute bond prices, we would require for $\alpha(t,T)$ and $\beta(t,T)$ to be known. A standard approach used in determining $\alpha(t,T)$ and $\beta(t,T)$ is the derivation and solving of the weakly coupled complex-valued system of ordinary differential equations (ODEs). The system of ODEs can be derived via the use of the term structure partial differential equation (PDE).

Similar to the well-known Black-Scholes PDE, the term structure PDE may be derived by applying the Itô formula to the ATS bond price form. Instrumental in the derivation of the ATS PDE is the realisation that the market price of risk does not depend on maturity but rather only on time to maturity and the short-rate (Björk, 2004). It follows from the above considerations that when modelling the uncertainty in financial markets under the risk neutral measure $\mathbb{Q}$, the term structure PDE is given by

$$\frac{\partial}{\partial t}F^T(t,r) + \mu(t,r)\frac{\partial}{\partial r}F^T(t,r) + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2}{\partial r^2}F^T(t,r) - rF^T(t,r) = 0, \tag{2.3}$$

and has the terminal value $F^T(T, r) = 1$.

If the one-factor short-rate model has an ATS, the term structure PDE may then be solved analytically to yield a system of weakly coupled ODEs,

$$\frac{d}{dt}\beta(t,T) = -A(t)\beta(t,T) + \frac{1}{2}C(t)\beta^2(t,T) - 1,$$

with terminal value $\beta(T,T) = 0$ and

$$\frac{d}{dt}\alpha(t,T) = B(t)\beta(t,T) - \frac{1}{2}D(t)\beta^2(t,T),$$

with terminal value $\alpha(T,T) = 0$.

It is worth noting that ATS models have the main advantage of being tractable since they are relatively easy to compute. Depending on the complexity of the short-rate model used, the computations may be analytical solutions to the ODEs or via the short-rate distribution.

Alternatively, the term structure PDE may be solved via a simple application of the *Feynman-Kač Theorem*.

**Theorem 2.2.1** (Feynman-Kač Theorem)**.** *Suppose that $F^T(t,r)$ is a smooth solution to the boundary value problem represented by the term strucuture PDE in Equation* (2.3)*. Assume further that the process $g(t,r) = \sigma(t,r)\frac{\partial}{\partial r}F^T(t,r)$ is adapted to the $\mathcal{F}_t$-filtration and belongs to $\mathcal{L}^2$, i.e.,*

$$\int_0^t \mathbb{E}[g^2(s,r)]ds > \infty,\ \forall\, s > 0,$$

*the $F^T(t,r)$ has the representation*

$$F^T(t,r) = \mathbb{E}[F^T(T, r_T)|\mathcal{F}_t],$$

*whereby $r$ satisfies the SDE given by the short-rate mode $dr = \mu(t,r)dt + \sigma(t,r)dW_t$ under the risk-neutral measure $\mathbb{Q}$.*

## 2.3 A Review of Diffusion Short-rate Models

ZCB, short-rate, Heath-Jarrow-Morton (HJM) and Lognormal market models (LMM) are some of the most commonly used approaches to model the bond market (Björk, 2004). The focus of this paper is the modelling of short-rates as this is the form in which the modelling is most mathematically convenient when pricing interest-rate products, particularly, XIBOR-linked interest-rate derivatives.

The earliest approach to modelling the bond market dates back to Vasicêk (1977). The method attempts to fit the entire term structure of rates using a short-rate diffusion stochastic differential equation (SDE) given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^{\mathbb{Q}} \tag{2.4}$$

where $\theta$, $\kappa$ and $\sigma \in \mathbb{R}^+$ and $W_t^{\mathbb{Q}}$ is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. The model is mean reverting, with the rate of mean reversion being $\kappa$ and the mean reversion level being $\theta$. This means that when interest-rates satify $r_t > \theta$, the mean reversion characteristic of the model tends to cause the drift to be negative thereby resulting in a downward trend in interest-rates to the mean reversion level. The opposite trend is also true when interest-rates satisfy $r_t < \theta$. Solving the system of ODEs or implementing the short-rate distribution approach in the computing of ZCB's under the Vasicêk (1977) model yields

$$\beta(t,T) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-t)}\right]$$

and

$$\alpha(t,T) = -\frac{\sigma^2\beta^2(t,T)}{4\kappa} + \frac{\big(\beta(t,T) - (T-t)\big)\big(\kappa\theta - \frac{1}{2}\sigma^2\big)}{\kappa^2}.$$

A generalisation of the Vasicêk (1977) model is the Hull and White (1990) model which attempts to ensure the exact calibration to the bond, cap and swaption prices. The model sought to achieve this by introducing deterministic time varying parameters $g(t)$, $h(t)$ and $\sigma(t)$. As such, the model may be represented by the SDE

$$dr_t = (g(t) - h(t)r_t)dt + \sigma(t)dW_t^{\mathbb{Q}}. \tag{2.5}$$

A noteworthy feature, and arguable major limitation of both the Vasicêk (1977) and Hull and White (1990) models is that they permit negative interest-rates. It is primarily because of this disadvantage that Cox *et al.* (1985) suggested improving the Vasicêk (1977) model through the use of a square root short-rate diffusion model given by the SDE

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^{\mathbb{Q}}, \tag{2.6}$$

where $\theta$, $\kappa$ and $\sigma \in \mathbb{R}^+$ and $W_t^{\mathbb{Q}}$ is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. The model is mean reverting; with the rate of mean reversion being $\kappa$ and the mean reversion level being $\theta$. The added benefit of the model guaranteeing positivity for the short-rate is due to *Fellers Square Root Condition*.

**Theorem 2.3.1** (Fellers Square Root Condition)**.** *Given Equation* (2.6)*, let* $\theta$*,* $\kappa$ *and* $\sigma \in \mathbb{R}^+$ *and* $r_0 > 0$*. If Fellers condition,*

$$2\kappa\theta > \sigma^2$$

*is satisfied, then* $\exists$ *a unique positive solution to the Cox et al. (1985) diffusion model on each finite time interval* $t \in [0, \infty)$*.*

It can then be shown that the bond price parameters $\beta(t, T)$ and $\alpha(t, T)$ in the Cox *et al.* (1985) one-factor model can be computed as follows

$$\beta(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{2\gamma + (\kappa + \gamma)(e^{\gamma(T-t)} - 1)}$$

and

$$\alpha(t,T) = \frac{2\kappa\theta}{\sigma^2} \ln \left[ \frac{2\gamma e^{\frac{1}{2}(\kappa+\gamma)(T-t)}}{(\kappa+\gamma)(e^{\gamma(T-t)-1}) + 2\gamma} \right],$$

where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$. Furthermore, it is also possible to generalise the Cox *et al.* (1985) model to have deterministic time dependent functions in place of the constants $\kappa$, $\theta$ and $\sigma$.

A host of other short-rate models exist and are extensively covered in interest-rate modelling literature and more specifically by Filipović (2009) and Hull (2012).

# Chapter 3

# Pricing Interest-rate Instruments

## 3.1 Spot-forward Parity

Brigo and Mecurio (2006) and Hull (2012) provide an extensive overview of the literature behind the construction of a term structure in pre-GFC markets that permitted practitioners to construct and use a single curve to generate and discount future cash flows. The construction of the forward curve would involve the use of various market-observable vanilla financial instruments such as caps and swaps across different tenors. Spot-forward parity involves no-arbitrage arguments between borrowing at, say, a quoted 3-month XIBOR against daily rolling over borrowing at, say, the overnight rate.

For a practitioner borrowing one unit of currency at the current time $t$ for the period $\tau = T - t$ at the quoted $\tau$ month-XIBOR, denoted $L(t, T)$ – a simple rate, we have that the amount due for payment at $T$ amounts to

$$1 + \tau L(t, T).$$

Recall that pre-GFC markets viewed government and bank debt, as "safe", i.e., free from default-risk; effectively this meant that they could borrow at the risk-free rate $r_t$. This is however not the case in modern financial markets due to governments and banks being prone to default-risk. The work of Filipović and Trolle (2013) details the key considerations of pricing collateralised contracts where two counterparties post cash collateral on a continuous marking-to-market basis. The result is one that justifies the current practice of risky entities being able to borrow at the collateralised instantaneous equivalent rate $r_t^c$.

Therefore, for a practitioner who rolls over borrowing one unit of currency from the current time $t$ to $T$ for the period $\tau = T - t$ at the prevailing theoretically constructed collateralised interbank short-rate, denoted $r_t^c(t_i, t_{i+1})$ for $\mathcal{T} = \{t_0 = t, \ldots, t_n = T\}$, we have that the amount due for payment at $T$ amounts to

$$e^{\int_t^{t_1} r_s^c ds} e^{\int_{t_1}^{t_2} r_s^c ds} \ldots e^{\int_{t_{n-1}}^{T} r_s^c ds} = e^{\int_t^T r_s^c ds}.$$

Hence, to preclude arbitrage, pre-GFC markets demanded that

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s^c ds}\big(1+\tau L(t,T)\big)\big|\mathcal{F}_t\right] = 1.$$

The above considerations therefore implied that borrowers and lenders can fix interest-rates in advance or take the floating short-rate which may change. This would depend on their respective views on interest-rates within the pre-specified period.

However, according to Bianchetti (2008) and Chang and Schlögl (2014), the rudimentary spot-forward parity assumption that underpins the pricing and valuation approaches that were used before the GFC have been in question. Post-GFC markets have consistently exhibited what seems to be a violation of the spot-forward relationships described above. In particular, post-crisis markets show that at the current time $t$,

$$1 + \tau L(t,T) > \mathbb{E}_{\mathbb{Q}}[e^{\int_t^T r_s^c\, ds}|\mathcal{F}_t],$$

therefore suggesting the presence of a natural and consistent arbitrage in fixed income markets. This mismatch in XIBOR-linked loans with implied forward rates also suggests that the pre-GFC pricing mechanics are flawed, and hence one needs to rethink interest-rate pricing in modern financial markets. We now proceed to explore literature on the root-cause of this mismatch and hence its impact on pricing in post-GFC markets.

## 3.2 Roll-over Risk

There are three main sources of risk that are prevalent in a borrowing transaction. If rolling over borrowing, the borrowing entity faces liquidity and interest-rate risk whereas the lending entity faces credit and interest-rate risk.

Interest-rate risk refers to the risk that the overnight interest-rate at which the borrower rolls over borrowing is random. For the borrower, the stochasticity of the instantaneous overnight rate is a risk as it may increase during the period in question. This would lead to the borrower ultimately paying more interest. The opposite is true for the lender who faces the risk that interest-rates may decrease during the same period.

We can however turn to interest-rate swaps as a solution to hedge against the stochasticity of the overnight rate – this is done through the introduction of the overnight indexed swap (OIS). The introduction of a collateralised interest-rate swap that has the market-quoted overnight rate as the floating leg may be fixed resulting in the standard market OIS rate. Primarily due to the short tenor associated with borrowing overnight, Filipović and Trolle (2013) extensively cover the validity of considering the OIS rate as a "risk-free rate" proxy in the interbank market. As such, we can derive an OIS rate, which it turns out is the rate at which we can discount securities. It can be shown that the OIS rate at the current time $t$ is given by

$$OIS(t,T) = \frac{1 - \mathbb{E}_{\mathbb{Q}}[e^{\int_t^{t_n} r_s^c \, ds} | \mathcal{F}_t]}{\sum_{i=1}^n \tau_i \mathbb{E}_{\mathbb{Q}}[e^{\int_t^{t_i} r_s^c \, ds} | \mathcal{F}_t]},$$

where $t_n = T$. The above representation is further unpacked in the sections that follow.

Credit risk captures the possibility that that the borrowing entity might default in the future. Often due to the "too big to fail" mindset that existed in pre-GFC markets, the impact of this risk was severely underestimated and yet it was risk that needed to be priced into financial instruments. Implied by the definition of credit risk is that an entities credit rating might be downgraded, thereby causing the market to perceive the entity as more likely to default during the borrowing period. The implication is that market participants would be less willing to lend to the risky entity at the prevailing overnight rate without extra compensation for being exposed to the counterparty's credit risk. Evidently, as the global financial crisis unfolded, markets witnessed unprecedented default rates which effectively led to increased XIBOR-OIS spreads. The increased default rates in the interbank market meant that banks suddenly became unwilling to lend to other banks at OIS rates (i.e., where XIBOR $\approx$ OIS), rather, banks added spreads over and above OIS rates thereby resulting in the XIBOR-OIS spreads observed today. Hence, to protect themselves against default, entities lending to entities perceived as riskier would have to take out Credit Default Swaps (CDS).

Alfeus *et al.* (2020) and Filipović and Trolle (2013) refer to liquidity risk as the risk of a situation where funding in the market can only be accessed at an additional premium above the benchmark reference rates, over and above what is due to credit risk. This can include, for example, event risk at the time of pricing which may limit liquidity. Hence, the borrowing entity may find that they are not able to fund the entire notional at the reference overnight rate.

Roll-over risk is then defined by Alfeus *et al.* (2020) as the sum total of the credit and liquidity risk faced by an entity that decides not to borrow "in one go".

The presence of roll-over risk in post-GFC financial markets is evident in many different forms, including in markets where there has been clear presence of a basis spread in floating-for-floating IRS rates of different tenors. As detailed by Morini (2008), the presence of this basis spread in post-GFC markets in turn suggests that cash flows of different tenors embed different risk. That is, an entity is likely to default over, say, 12 months than within 3 months. Additionally, an entity rolling over borrowing might face liquidity challenges during the borrowing period due to various reasons. Some of the earliest work that attempts to explain roll-over risk is that of Bianchetti (2008) who suggested that a multi-curve framework would be needed since single-curve no arbitrage relations do not hold due to a "new" risk. This would then mean that only products of the same tenor may be used in constructing forward curves, i.e., there are many forward curves, each of which captures the (roll-over) risk premia associated with borrowing over different tenors. Hence, practitioners cannot use products of different tenors to construct one forward curve as it results in a mismatch of risk and therefore arbitrage opportunities. However, there still would only be one discount curve.

## 3.3 Modelling Roll-over Risk

### 3.3.1 Approach

We denote the instantaneous equivalent abstraction of the market overnight rate $r_t^c$. Given the aforementioned spread over the default-free rate $r_t$, the rate $r_t^c$ is given by

$$r_t^c = r_t + \Lambda_t q. \tag{3.1}$$

Alfeus *et al.* (2020) define the spread, $\Lambda_t q$, as the "average (market aggregated) credit spread across the XIBOR reference panel bank members"; with the terms $q$ and $\Lambda_t$ being the loss fraction through default and default intensity, respectively. Hence, the rate $r_t^c$ is one which a bank with panel member average characteristics is able to borrow at, i.e., it indicates the credit quality of the average of the panel member banks relative to the risk-free rate.

As mentioned in the previous section, roll-over risk comprises both credit and liquidity risks and is therefore modelled as such. For an individual XIBOR panel member, we let $\phi_t$ and $\lambda_t q$ be the instantaneous equivalents of pure the funding liquidity and credit spreads, relative to $r_t^c$, respectively. The roll-over risk premium, denoted $\pi_t$, can then be modelled as given by the equation

$$\pi_t = \phi_t + \lambda_t q. \tag{3.2}$$

Furthermore, if an entity is able to initially borrow at the rate $r_0^c$, then we can infer that $\pi_0 = \phi_0 + \lambda_0 q = 0$. Clearly, the quantity expressed below signifies the rate at which any member of the panel can access funding if rolling over borrowing during any $0 \leq t \leq T$ for the period 0 to $T$:

$$r_t + \Lambda_t q + \lambda_t q + \phi_t = r_t^c + \pi_t. \tag{3.3}$$

Hence, Equation (3.3) becomes the rate at which any panel member is able to roll borrowing over for the duration in question. For simplicity, we set $q = 1$. The result of borrowing from $t$ to $T$ is that the loan balance at maturity $T$ can be expressed as

$$e^{\int_t^T r_s + \Lambda_s + \lambda_s + \phi_s \, ds}. \tag{3.4}$$

Let $\delta$ be a random default time on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with with default intensity $(\Lambda_s + \lambda_s)_{s \geq t}$ that indicates the event that the counterparty defaults, we then introduce the following Lemma that is of importance to the subsequent derivations:

**Lemma 3.3.1.** *For an integrable random variable $X$ and default time $\delta > t \geq 0$ satisfying $\mathbb{Q}[\delta \leq t] = 0$ and $\mathcal{F}_t \supset \mathcal{G}_t$ such that the market sigma-algebra $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t \, \forall \, t \geq 0$ where $(\mathcal{H}_t)_{t \geq 0}$ is generated by the default indicator processes and all other processes are adapted to $(\mathcal{G}_t)_{t \geq 0}$ , one has*

$$\mathbb{E}_{\mathbb{Q}}[X \mathbb{I}_{\{\delta > T\}} | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[X e^{-\int_t^T \Lambda_s + \lambda_s \, ds} | \mathcal{G}_t].$$

We then pursue similar strategies to those presented in the preceding section in an attempt to better understand the effect of roll-over risk in fixed income markets. We can then formulate the time $t$ contract value for an entity that rolls borrowing by taking discounted conditional expectations of Equation (3.4) and using Lemma (3.3.1) as follows:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s \, ds} e^{\int_t^T r_s + \Lambda_s + \phi_s + \lambda_s \, ds} \mathbb{I}_{\{\delta > T\}} | \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{\int_t^T \phi_s \, ds} | \mathcal{G}_t\right]. \tag{3.5}$$

Note that the above equation models the effect of accounting for roll-over risk when a nominal amount of 1 is initially borrowed at $t$ and continuously rolled over until the terminal time $T$ at the instantaneous rate given by Equation (3.3).

On the other hand, given that OIS contracts are valued using

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s^c ds}\left([1 + \tau OIS(t,T)] - e^{\int_t^T r_s^c ds}\right) | \mathcal{G}_t\right] \tag{3.6}$$

for practitioners wanting to pay a floating rate and receive a fixed rate, we can use this to hedge the interest rate risk using OIS. This yields the following important expression:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\int_t^T \phi_s \, ds} - 1 + e^{-\int_t^T r_s^c \, ds}[1 + \tau OIS(t,T)] | \mathcal{G}_t\right].$$

Lending at the LIBOR rate yields the following:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}[1+\tau L(t,T)]\mathbb{I}_{\{\delta>T\}}|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s+\Lambda_s+\lambda_s\,ds}[1+\tau L(t,T)]|\mathcal{G}_t\right]. \tag{3.7}$$

Having accounted for roll-over risk in the model, it then follows that in order to preclude arbitrage we must have the following relation between borrowing in one-go and rolling over, given that they start with one unit of capital:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\int_t^T \phi_s\,ds} - 1 + e^{-\int_t^T r_s^c\,ds}[1+\tau OIS(t,T)]\big|\mathcal{G}_t\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s^c+\lambda_s\,ds}[1+\tau L(t,T)]\big|\mathcal{G}_t\right].$$

Hence, if we set $\lambda_s = 0$ for the period $t$ to $T$, then

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\int_t^T \phi_s\,ds} - 1|\mathcal{G}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s^c\,ds}[L(t,T) - OIS(t,T)]\tau\big|\mathcal{G}_t\right].$$

Additionally, if we also set $\phi_s = 0$ for the period $t$ to $T$, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s^c\,ds}[L(t,T) - OIS(t,T)]\tau\big|\mathcal{G}_t\right] = 0.$$

It is therefore trivial to see that in the absence of roll-over risk, the XIBOR-OIS spread is non-existent. Consequently, the above relationships further emphasise that $OIS(t,T) = L(t,T)$ if we do not account for roll-over risk.

Lastly, we define the OIS discount factor as $D^{OIS}(t,T) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s^c\,ds}|\mathcal{F}_t\right]$ and note that the value of the OIS is zero at inception and hence:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s^c\,ds}[1+\tau OIS(t,T)]\big|\mathcal{F}_t\right] = 1.$$

### 3.3.2 Cox-Ingersol-Ross Model Approach to Modelling Roll-over Risk

We now intend to model stochastic dynamics of $r_t^c$, $\lambda_t$ and $\phi_t$ under risk neutral dynamics. We achieve this by modelling the affine Markov state vector $Y_t = [y_t^1, \ldots, y_t^d]$ and expressing the variables of interest as follows:

$$r_t^c = a_0 + \sum_{i=1}^{d} a_i y_t^i,$$
$$\lambda_t = b_0 + \sum_{i=1}^{d} b_i y_t^i,$$
$$\phi_t = c_0 + \sum_{i=1}^{d} c_i y_t^i, ,$$

where $a_0$, $b_0$ and $c_0$ are deterministic scalars, $a$, $b$ and $c$ are arbitrary vectors. $Y_t = [y_t^1, \ldots, y_t^d]$ are processes described by Cox *et al.* (1985) (CIR) square-root process dynamics, i.e.,

$$dy_t^i = \kappa^i(\theta^i - y_t^i)dt + \sigma_i\sqrt{y_t^i}dW_t^i, \tag{3.8}$$

where $W_t^i$ for $i = \{1 \ldots d\}$ are independent Wiener processes modelled under the risk neutral measure. The CIR three-factor model presents itself as an attractive model since default and liquidity risk cannot be negative. For reasons that will become clear soon, Equation (3.8) may be re-written in its state vector form as

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \tag{3.9}$$

where:
– $\mu(Y_t) = K_0 + K_1Y_t$ for $K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$ ,
– $\left(\sigma(Y_t)\sigma(Y_t)^T\right)_{ij} = (H_0)_{ij} + (H_1)_{ij} \cdot Y_t$ for $H = (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$ ,
– $\varrho(Y_t) = \rho_0 + \rho_1 \cdot Y_t$ for $\varrho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n$, and $\varrho(Y_t)$ represents $r_t^c$, $\lambda_t$, $\phi_t$ or linear combinations thereof.

### 3.3.3 Pricing in Roll-over Risk – Fourier Transform Approach

We closely follow the method documented by Duffie *et al.* (2000) in the pricing of non-linear interest-rate derivatives. For convenience, we denote the $\mathbb{E}_{\mathbb{Q}}[\cdot|\mathcal{F}_t]$ as $\mathbb{E}_t[\cdot]$. For an affine jump-diffusion process such as that described by CIR square-root process dynamics, the time $t$ price of a zero-coupon bond paying an amount of $e^{u\cdot Y_T}$ at terminal time $T$ is given by

$$\begin{aligned} P(u, Y_t, t, T) &= \mathbb{E}_t[e^{-\int_t^T \varrho(Y_s)ds} e^{u\cdot Y_T}] \\ &= e^{\alpha(t)+\beta(t)\cdot Y_t}, \end{aligned} \tag{3.10}$$

where $\alpha(t)$ and $\beta(t)$ satisfy the following ordinary differential equations:

$$\begin{aligned} \frac{d}{dt}\beta(t) &= \rho_1 - K_1^T\beta(t) - \frac{1}{2}\beta(t)^T H_1\beta(t), \\ \frac{d}{dt}\alpha(t) &= \rho_0 - K_0 \cdot \beta(t)^T - \frac{1}{2}\beta(t)H_0\beta(t), \end{aligned} \tag{3.11}$$

with boundary conditions satisfying $\beta(T) = u$ and $\alpha(T) = 0$.

Our next goal is to price options under a framework where roll-over risk is accounted for. Work by Duffie *et al.* (2000) generalises the pricing of contingent claims using Fourier Transformations of state vector $Y_t$. It can be shown that the initial value of a contingent claim with the terminal payoff given by $(e^{d \cdot Y_T} - c)^+$ can be expressed as

$$\begin{aligned} C(d,c,T) &= \mathbb{E}_0\big[e^{-\int_0^T \varrho(s)ds}\big(e^{d\cdot Y_T} - c\big)^+\big] \\ &= G_{d,-d}(-\ln(c); Y_0, T) - cG_{0,-d}(-\ln(c); Y_0, T), \end{aligned} \tag{3.12}$$

where, if well-defined,

$$G_{a,b}(x; Y_0, T) = \frac{P(a, Y_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im[P(a + \imath vb, Y_0, 0, T)e^{-\imath vx}]}{v} dv,$$

where $\Im(c)$ is the imaginary part of $c \in \mathbb{C}$ and $\imath$ is one imaginary unit.

### 3.3.4 Pricing in Roll-over Risk – Monte Carlo Approach

Crude Monte Carlo integration primary relies on the *Strong Law of Large Numbers* and the *Central Limit Theorem*. The two results are fundamental in delivering a solution to the general problem of evaluating an intergral

$$\begin{aligned} I_A(fw) &= \int_A f(x)w(x)dx \\ &= \mathbb{E}[f(\mathbf{X})], \end{aligned}$$

where $A \subseteq \mathbb{R}^k$ for some $k$, $w$ is a probability density function with $\{x : w(x) \neq 0\} \subseteq A$ and $\mathbf{X} \sim w$ is a random vector. It can then be shown that the estimator

$$\hat{I}_{A,n}(fw) = \frac{1}{n}\sum_{i=1}^{n} f(X_i),$$

is an unbiased estimator of $I_A$ if $X_1, \ldots, X_n$ are independent and identically distributed random variables with a common probability density function $w$.

Option pricing can now be approached in the same way the integral estimator works. That is, option pricing can be reduced to an integration problem where the ultimate objective is to compute the discounted value of contingent claim

$$C(d,c,T) = \mathbb{E}_{\hat{\mathbb{Q}}}\big[e^{-\int_0^T \varrho(s)ds}\big(e^{d\cdot Y_T} - c\big)^+\big].$$

Knowing the distribution of the state vector $Y_t$, the integral may be computed by simply evaluating the inner function and taking an average to produce an estimation of the integral.

## 3.4 Transitioning from XIBOR into RFRs

Interbank offered rates have long been the benchmark interest-rate index referenced in fixed income markets. Following the LIBOR fixing scandal, detailed by

Ashton and Christophers (2015), financial industry regulators have since sought alternative benchmark rates to replace XIBOR based on a new set of overnight risk-free rates (RFRs). Various jurisdictions have opted for various alternatives; more generally, these alternative RFRs may be divided into secured and unsecured.

Regulators in the US have, through the Alternative Reference Rates Committee (ARRC), opted for the Secured Overnight Financing Rate (SOFR) as a replacement rate for USD-LIBOR. SOFR is argued to be a more robust rate that reflects day-to-day transactions in the US repo market. Additional key features of SOFR include the fact that it is collateralised, has a robust underlying volume, is available beyond the interbank market and is backward-looking (Schrimpf and Sushko, 2019).

Mathematically, the discrete overnight SOFR ($R_{d_i}$) can be represented by

$$R_{d_i}(t_i) = \frac{1}{d_i}\left(\frac{1}{P(t_i, t_i + d_i)} - 1\right)$$

where $d_i = \frac{1}{360}$ on business days not followed by a holiday and $d_i = \frac{3}{360}$ on Fridays. The construction of a replacement benchmark for LIBOR that is based on SOFR is then made possible and is given by

$$R^{\tau}(t, T) = \frac{1}{\tau}\left(\prod_{i=1}^{n}(1 + d_i R_{d_i}(t_i)) - 1\right) \tag{3.13}$$

for $\tau$-month SOFR, $\tau = T - t$. The work of Skov and Skovmand (2021) expands on how this reconstruction of the benchmark reference rate impacts SOFR-linked interest-rate derivatives, specifically futures and caps.

# Chapter 4

# Research Objectives and Model Implementation

Having dealt with the literature on pricing pre- and post-GFC markets, this dissertation ultimately aims to address the pricing of contingent claims; particularly caps and floors in the presence of roll-over risk. The following list of objectives allows us to achieve this aim:

## 4.1 Objectives

1. We intend to price cap and floor derivative instruments under a model in which roll-over risk is accounted for. The work herein achieves this through an application of the Fourier Transform approach to option pricing.

2. The second objective is to compare prices obtained per objective one above to those obtained through an application of crude Monte Carlo methods.

## 4.2 Model Implementation

### 4.2.1 Model Parameters

The pricing of caps and floors in this dissertation relied on the calibration parameters provided by Alfeus *et al.* (2020) in Table 4.1 below. These parameters were calibrated using market data quoted on 31/10/2017 for USD interest rate swaps, overnight index swaps, 1-month/3-month, 3-month/6-month basis swaps as well as prevailing credit default swap price which were imperative in the isolation of liquidity from credit risk. Furthermore, although we previously set $q = 1$ in Chapter 3, we revert to setting $q = 0.6$ as this is the appropriate setting that is consistent with the parameters calibrated by Alfeus *et al.* (2020).

**Tab. 4.1:** Basis model parameters adapted from Alfeus *et al.* (2020).

| $y(0)$ | $a$ | $b$ | $c$ | $q$ | $\sigma$ | $\kappa$ | $\theta$ |
|---|---|---|---|---|---|---|---|
| 0.773084 | 3.34e-03 | 5.39e-05 | 3.72e-05 | | 0.264573 | 0.260876 | 0.798057 |
| 0.013896 | 0.00000 | 0.113603 | 0.008913 | 0.6 | 0.004227 | 0.397512 | 0.0002119 |
| 0.065454 | 0.00000 | 7.94-05 | 4.09e-06 | | 0.403512 | 0.903787 | 0.805810 |

### 4.2.2 Analytical Solution – Fourier Transform Approach

The expression below was derived earlier in Chapter 3 when comparing rolling-over borrowing with borrowing in one go:

$$\mathbb{E}_t\left[e^{\int_t^T \phi_s\,ds} - e^{-\int_t^T \lambda_s q\,ds} + e^{-\int_t^T r_s^c+\lambda_s q\,ds}[1+\tau OIS(t,T)]\right] = \mathbb{E}_t\left[e^{-\int_t^T r_s^c+\lambda_s q\,ds}[1+\tau L(t,T)]\right].$$

Simplifying, we obtain that

$$\mathbb{E}_t\left[e^{-\int_t^T r_s^c+\lambda_s q\,ds}[1+\tau L(t,T)]\right] = \mathbb{E}_t\left[e^{\int_t^T \phi_s\,ds}\right],$$

and hence

$$\begin{aligned}
[1+\tau L(t,T)]\mathbb{E}_t\left[e^{-\int_t^T r_s^c+\lambda_s q\,ds}\right] &= \mathbb{E}_t\left[e^{\int_t^T \phi_s\,ds}\right],\\
L(t,T) &= \frac{1}{\tau}\left[\frac{\mathbb{E}_t\left[e^{\int_t^T \phi_s\,ds}\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_s^c+\lambda_s q\,ds}\right]} - 1\right]\\
L_{tT} &= \frac{1}{\tau}[e^{\alpha_t^1-\alpha_t^2+(\beta_t^1-\beta_t^2)\cdot Y_t} - 1],
\end{aligned}$$

where $\alpha_t^1$ , $\alpha_t^2$, $\beta_t^1$ and $\beta_t^2$ are given by the ODEs in Equation (3.11), each with different specifications for the discount rate that is integrated over in Equation (3.10). In the absence of roll-over risk, the expression for $L_{tT}$ and $OIS_{tT}$ simplifies to

$$\begin{aligned}
OIS_{tT} &= \frac{1}{\tau}\left[\frac{1}{\mathbb{E}_t\left[e^{-\int_t^T r_s^c\,ds}\right]} - 1\right]\\
&= \frac{1}{\tau}[e^{\alpha_t^3+\beta_t^3\cdot Y_t} - 1].
\end{aligned}$$

Step 1: Define state vectors for the expectations and their parameters.
a. For $\mathbb{E}_t[e^{\int_t^T \phi_s\,ds}]$, we would like to model $\phi_t = c_0 + \sum_{i=1}^d c_i y_t^i$, and hence:

$$\begin{aligned}
\varrho(Y_t) := -\phi_t = -c_0 - \sum_{i=1}^d c_i y_t^i,\\
\implies \rho_0 = -c_0 \text{ and } \rho_1 = -[c_1, c_2, c_3]^T.
\end{aligned}$$

b. For $\mathbb{E}_t[e^{-\int_t^T r_s^c + \lambda_s q\, ds}]$, we would like to model both $r_t^c = a_0 + \sum_{i=1}^d a_i y_t^i$ and $\lambda_t = b_0 + \sum_{i=1}^d b_i y_t^i$, hence:

$$\varrho(Y_t) := r_t^c + \lambda_t q = a_0 + b_0 q + \sum_{i=1}^d (a_i + b_i q) y_t^i,$$
$$\implies \rho_0 = a_0 + b_0 q \text{ and } \rho_1 = [a_1 + b_1 q, a_2 + b_2 q, a_3 + b_3 q]^T.$$

c. For the case where roll-over risk is not accounted for i.e., $\mathbb{E}_t[e^{-\int_t^T r_s^c\, ds}]$, we would like to model $r_t^c = a_0 + \sum_{i=1}^d a_i y_t^i$, hence:

$$\varrho(Y_t) := r_t^c = a_0 + \sum_{i=1}^d a_i y_t^i,$$
$$\implies \rho_0 = a_0 \text{ and } \rho_1 = [a_1, a_2, a_3]^T.$$

Step 2: Solve the resulting ODE's

$$\frac{d}{dt}\beta(t) = \rho_1 - K_1^T \beta(t) - \frac{1}{2}\beta(t)^T H_1 \beta(t),$$
$$\frac{d}{dt}\alpha(t) = \rho_0 - K_0 \cdot \beta(t)^T - \frac{1}{2}\beta(t) H_0 \beta(t),$$

using the ODE45 function on Matlab where $\beta(T) = 0$ and $\alpha(T) = 0$.
Step 3: Simplifying and pricing a caplet proceeds as follows:

$$\begin{aligned}
C_0 &= \mathbb{E}_t\left[e^{-\int_0^T r_s^c ds}\tau\big(L_{tT} - K\big)^+\right] \\
&= \mathbb{E}_t\left[e^{-\int_0^t r_s^c ds} e^{-\int_t^T r_s^c ds}\tau\big(L_{tT} - K\big)^+\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_t\big[e^{-\int_0^t r_s^c ds} e^{-\int_t^T r_s^c ds}\tau\big(L_{tT} - K\big)^+\big]\right] \\
&= \mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}\tau\big(L_{tT} - K\big)^+ \mathbb{E}_t\big[e^{-\int_t^T r_s^c ds}\big]\right] \\
&= \mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}\tau\big(L_{tT} - K\big)^+ e^{\alpha_t^3 + \beta_t^3 \cdot Y_t}\right] \\
&= \mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}\big(e^{\alpha_t^1 - \alpha_t^2 + (\beta_t^1 - \beta_t^2)\cdot Y_t} - 1 - K\tau\big)^+ e^{\alpha_t^3 + \beta_t^3 \cdot Y_t}\right].
\end{aligned}$$

Therefore we require that:

$$\big(e^{\alpha_t^1 - \alpha_t^2 + (\beta_t^1 - \beta_t^2)\cdot Y_t} - 1 - K\tau\big) \geq 0$$
$$(\beta_t^1 - \beta_t^2) \cdot Y_t \geq \ln[1 + K\tau] + \alpha_t^2 - \alpha_t^1.$$

Define the following:

$$
\begin{aligned}
\aleph_t^1 &:= \beta_t^1 - \beta_t^2 + \beta_t^3,\\
\aleph_t^2 &:= \beta_t^2 - \beta_t^1,\\
x &:= \alpha_t^1 - \alpha_t^2 - \ln[1 + K\tau].
\end{aligned}
$$

Hence:

$$
\begin{aligned}
C_0 &= \mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}\left(e^{\alpha_t^1-\alpha_t^2+(\beta_t^1-\beta_t^2)\cdot Y_t} - 1 - K\tau\right)^+ e^{\alpha_t^3+\beta_t^3\cdot Y_t}\right]\\
&= e^{\alpha_t^3}\mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}\left(e^{\alpha_t^1-\alpha_t^2+(\beta_t^1-\beta_t^2)\cdot Y_t}\right)e^{\beta_t^3\cdot Y_t}\mathbb{I}_{(\beta_t^1-\beta_t^2)\cdot Y_t\geq\ln[1+K\tau]+\alpha_t^2-\alpha_t^1}\right]\\
&- e^{\alpha_t^3}(1+K\tau)\mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}e^{\beta_t^3\cdot Y_t}\mathbb{I}_{(\beta_t^1-\beta_t^2)\cdot Y_t\geq\ln[1+K\tau]+\alpha_t^2-\alpha_t^1}\right]\\
&= e^{\alpha_t^3+\alpha_t^1-\alpha_t^2}\mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}e^{\aleph_t^1\cdot Y_t}\mathbb{I}_{-\aleph_t^2\cdot Y_t\geq -x}\right] - e^{\alpha_t^3}(1+K\tau)\mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}e^{\beta_t^3\cdot Y_t}\mathbb{I}_{-\aleph_t^2\cdot Y_t\geq -x}\right]\\
&= e^{\alpha_t^3+\alpha_t^1-\alpha_t^2}\mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}e^{\aleph_t^1\cdot Y_t}\mathbb{I}_{\aleph_t^2\cdot Y_t\leq x}\right] - e^{\alpha_t^3}(1+K\tau)\mathbb{E}_t\left[e^{-\int_0^t r_s^c ds}e^{\beta_t^3\cdot Y_t}\mathbb{I}_{\aleph_t^2\cdot Y_t\leq x}\right]\\
&= e^{\alpha_t^3+\alpha_t^1-\alpha_t^2}\left[\frac{P(\aleph_t^1, Y_0, 0, t)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im[P(\aleph_t^1 + \imath v\aleph_t^2, Y_0, 0, t)e^{-\imath vx}]}{v}dv\right]\\
&- e^{\alpha_t^3}(1+K\tau)\left[\frac{P(\beta_t^3, Y_0, 0, t)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im[P(\beta_t^3 + \imath v\aleph_t^2, Y_0, 0, t)e^{-\imath vx}]}{v}dv\right],
\end{aligned}
$$

where the last step follows from Equation (3.12). A similar procedure was then followed so as to come to a numerical solution for floorlets.

Lastly, cap and floor prices were computed by fixing the time to maturity as well as the compounding frequency and thereafter summing up the various caplet and floorlet prices.

# Chapter 5

# Results and Analysis

## 5.1 Inter-bank Offered Rates

We begin by computing XIBOR based on the Alfeus *et al.* (2020) model and parameters in Table 4.1. The reformulation of XIBOR to account for liquidity constraints and credit risk is displayed in Figure 5.1.

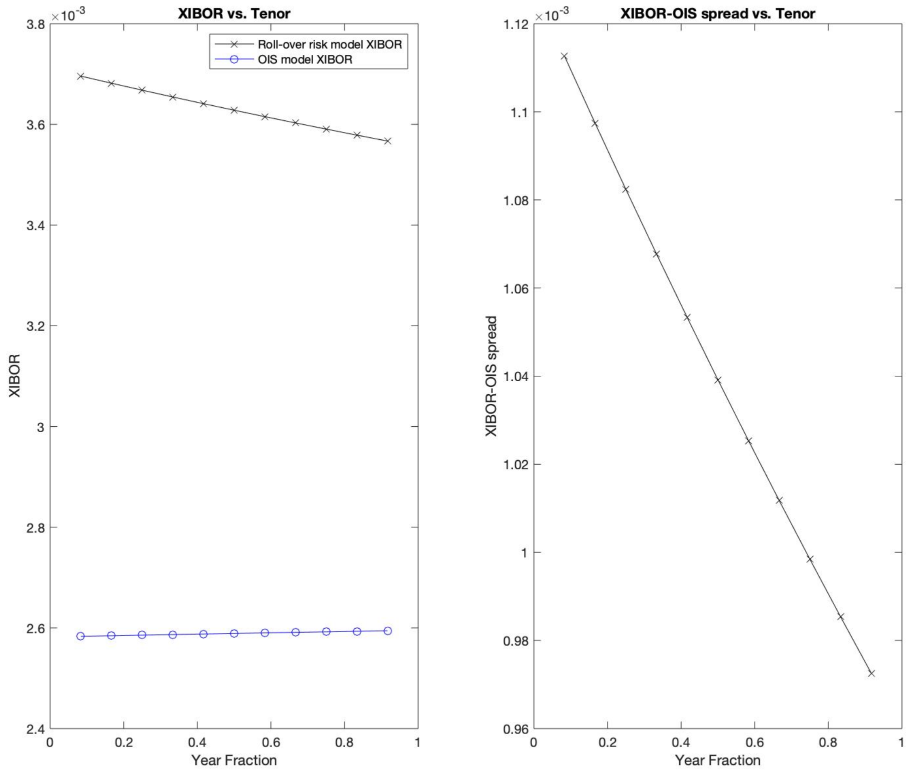


**Fig. 5.1:** XIBOR and OIS vs. tenor.

The implied roll-over risk XIBOR was calculated for different tenors using the equation $L_{tT} = \frac{1}{\tau}[e^{\alpha_t^1 - \alpha_t^2 + (\beta_t^1 - \beta_t^2) \cdot Y_t} - 1]$ whereas the expression OIS-based expression for XIBOR is $L_{tT} = \frac{1}{\tau}[e^{\alpha_t^3 + \beta_t^3 \cdot Y_t} - 1]$. The implied roll-over risk XIBOR required solving of the ODEs so as to attain liquidity and credit risk parameters $\alpha_t^1$, $\alpha_t^2$, $\beta_t^1$ and $\beta_t^2$. On the other hand, OIS-based XIBOR required the parameters linked with the collateralisation rate discount factor.

In the roll-over risk formulation, XIBOR exhibited a decreasing trend with an increase in tenor and ultimately a XIBOR-OIS spread that diminishes with tenor. We attribute this to the data available at the time which led to this trend where it seemed that the market was had cash deficit in the short-term and was therefore willing to borrow at higher rates in the short-end than for longer tenors. However, this would require further investigation.

## 5.2 Interest-rate Derivative Pricing

The pricing of interest-rate derivatives is based on the derivations in Chapter 4. A major advantage of the Fourier Transform method employed is that it permitted the simultaneous pricing of caplets and floorlets with various strikes. As per the preceeding section, pricing in the roll-over risk framework required a reformulation of XIBOR into a 'risky' rate as opposed to pre-GFC financial markets.

### 5.2.1 Caplets and Floorlets

Figure 5.2 and 5.5 display caplet and floorlet pricing modelled under the three-factor CIR model, as mentioned in the preceeding sections. The prices were computed for a range of strike rates, 3- and 6-month tenors, respectively, using the Fourier Transform approach as well as Monte Carlo estimates. As expected, both methods exhibit the expected behavior with respect to how caplet and floorlet prices change with a change in strike as well as a change in tenor.

For both 3- and 6-month tenors, caplet and floorlet prices obtained using the roll-over risk model reflected the changes in simulated XIBOR. That is, a comparison between XIBOR computed using the OIS model vs. the roll-over risk model, as explained above has a spread due to the added risk. As such, this affected the payoff for caplets (floorlets) in the sense that the caplets priced using the roll-over risk model priced higher (lower) than those priced using the OIS model. The main reason for this is the payoff for the two securities, i.e., the payoff for a caplet relies on $(L_{tT} - K)^+$ whereas the floorlet is based on a discounted version of $(K - L_{tT})^+$.

The error between the analytical solution and the Monte Carlo estimate is displayed in Figure 5.3 and 5.4 for the 3-month tenor as well as Figure 5.6 and 5.7 for the 6-month tenor. Even though the Fourier Transform and Monte Carlo methods are different in nature, the errors realised were extremely small. The fact that prices obtained using these two methods effectively coincide is strong evidence that our calculations are correct. Lastly, although not shown in the trends, the error between the closed solution and the Monte Carlo sample size did reduce with an increase the sample size.

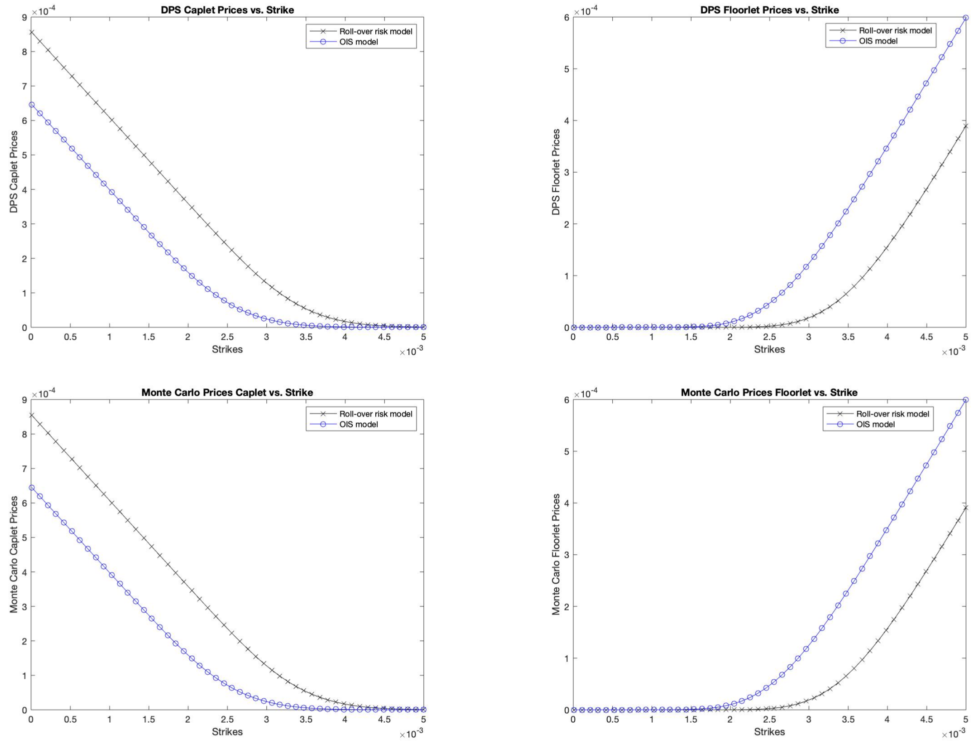


**Fig. 5.2:** 3 month Caplet and Floorlet Prices.

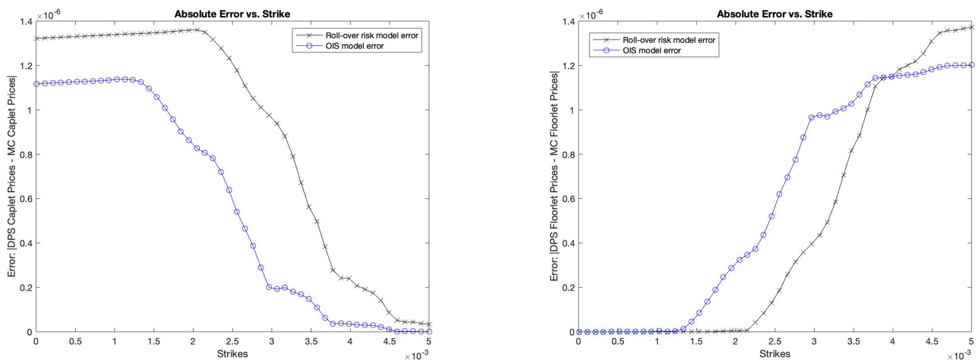


**Fig. 5.3:** Absolute Error in Caplet and Floorlet Prices (3 month).

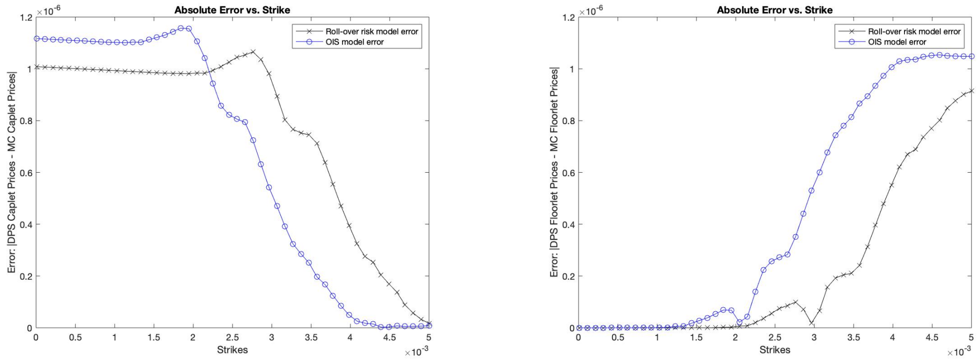


**Fig. 5.4:** Second Run Absolute Error in Caplet and Floorlet Prices (3 month).

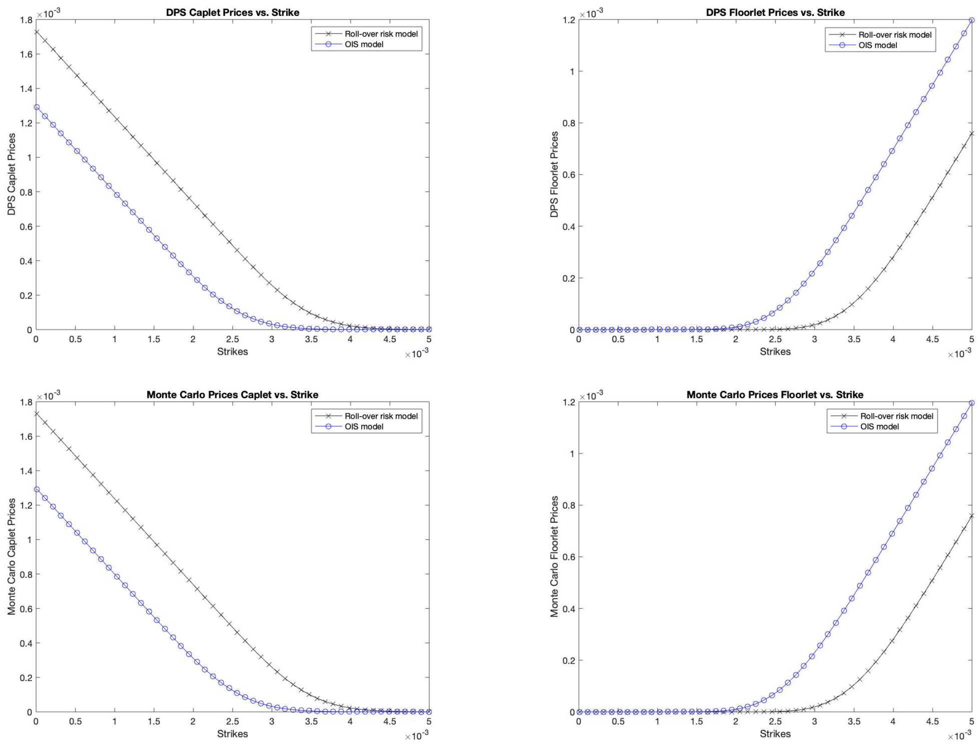


**Fig. 5.5:** 6 month Caplet and Floorlet Prices.

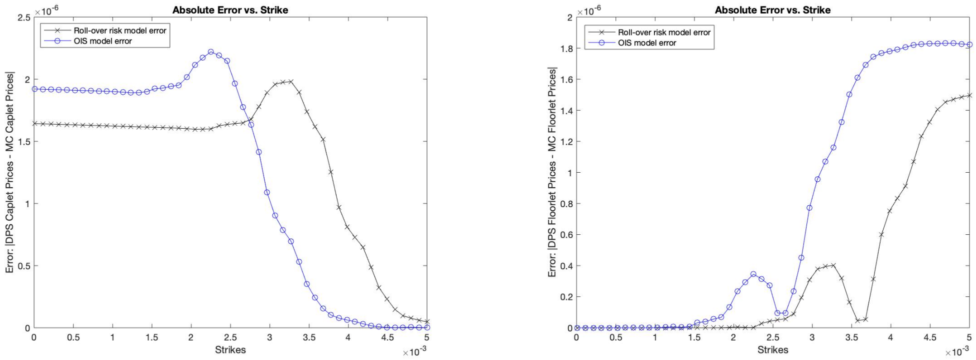


**Fig. 5.6:** Absolute Error in Caplet and Floorlet Prices (6 month).

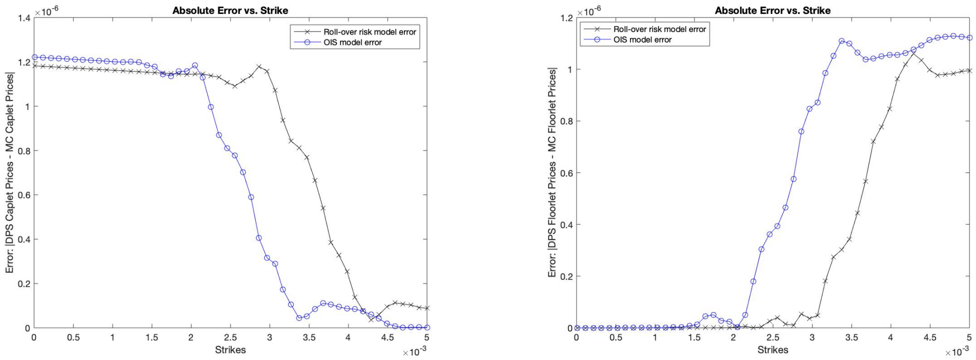


**Fig. 5.7:** Second Run Absolute Error in Caplet and Floorlet Prices (6 month).

### 5.2.2 Caps and Floors

Having computed caplet and floorlet prices using both the Fourier Transform approach documented by Duffie *et al.* (2000) and Monte Carlo simulations, the extension into caps and floors was made possible through the summation of the computed caplet and floorlet prices. The computation required fixing the maturity as well as the compounding frequency (i.e., tenor). The resultant trend in cap and floor prices was similar to that observed in the caplet and floorlet pricing as the strike changes. Figure 5.8 is a graphical representation of the prices for various strikes for 1-year and 5-year maturities for both 3-month and 6-month tenors.

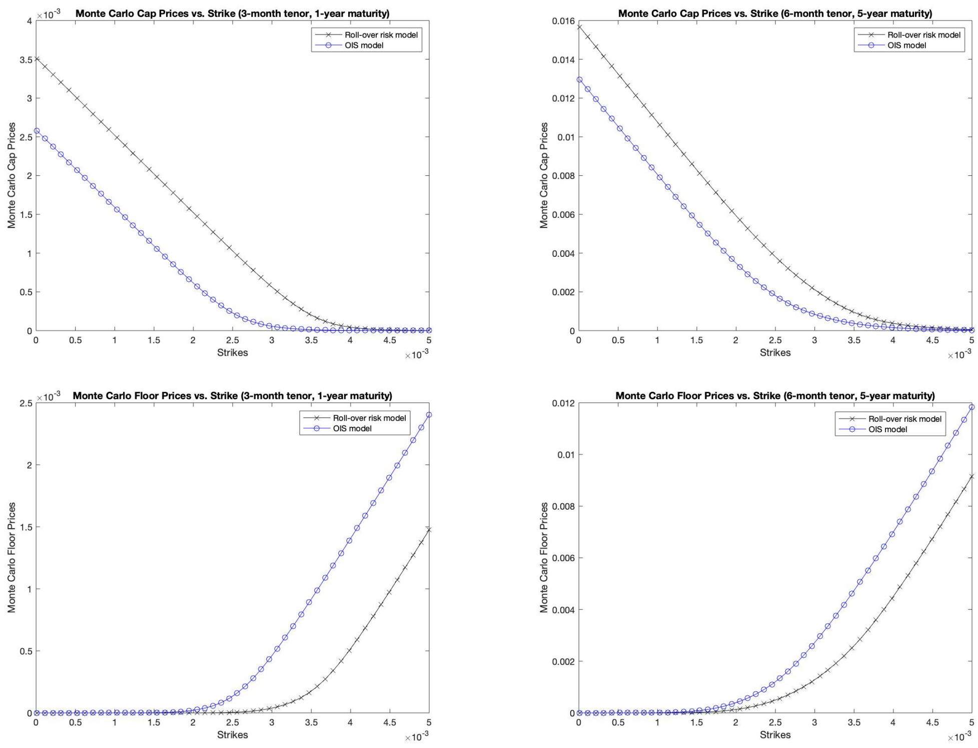


**Fig. 5.8:** Monte Carlo Cap and Floor Prices.

Lastly, cap prices were computed for 3-month and 6-month tenors for maturities ranging from 1-year to 30-years. The results are shown in Figure 5.9.

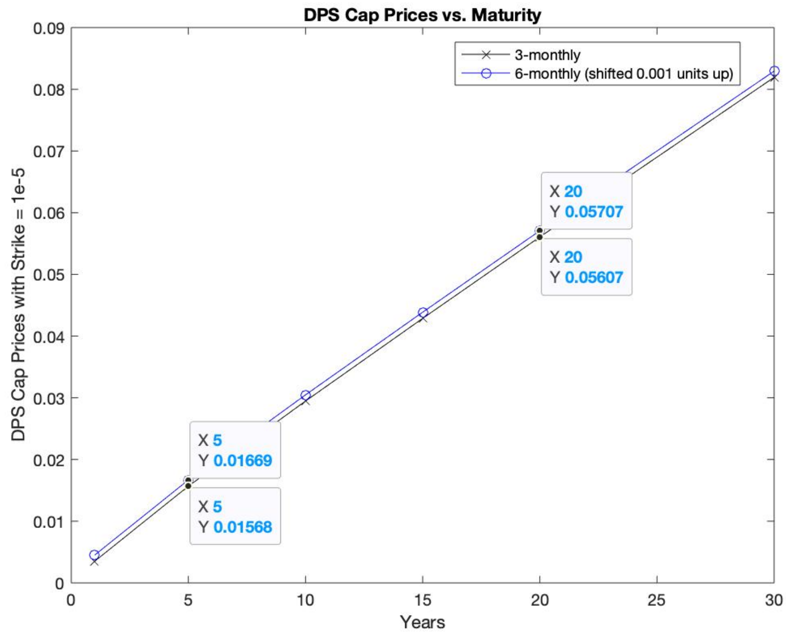


**Fig. 5.9:** DPS Cap Prices for different maturities.

# Chapter 6

# Conclusion

The pricing of caplets and floorlets was successfully achieved using the Fourier Transform approach in combination with the three-factor CIR model. Prices obtained using the Fourier Transform method, which we regard as the analytical solution, compared well with those computed using Monte Carlo simulations. The error between the two methods varied with each run of the code due to the minor variation resulting from the random number generation with each run. Even though the Monte Carlo is faster in calculating one option price, the method has the major disadvantage that the algorithm needs to be rerun if the initial state variables change; on the other hand, the Fourier Transform allows you to solve the ODEs once per parameter set, and then use these solutions for any initial state values.

Further work could improve on the pricing of interest-rate derivatives by calibrating to both caps and floors so as to compute more accurate floor prices. Additionally, a volatility model could also be employed to better simulate the underlying market dynamics. Lastly, the methods employed herein could also be applied to the pricing of non-vanilla interest-rate derivatives such as European-style barrier options and potentially American-style options, amongst others.

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